I.S.A.S. - International School of Advanced Studies

Collective phenomena in socio-economic systems: from interacting voters to financial markets instabilities

Thesis submitted for the degree of
PhilosophiæDoctor

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October 2005

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Introduction

Or, to change the metaphor slightly, professional investment may be likened to those newspaper competitions in which the competitors have to pick out the six prettiest faces from a hundred photographs, the prize being awarded to the competitor whose choice most nearly corresponds to the average preferences of the competitors as a whole; so that each competitor has to pick, not the faces which he himself finds prettiest, but those which he thinks likeliest to catch the fancy of the other competitors, all of whom are looking at the problem from the same point of view. It is not a case of choosing those which, to the best of one's judgement, are really the prettiest, nor even those which average opinion genuinely thinks the prettiest. We have reached the third degree where we devote our intelligences to anticipating what average opinion expects average opinion to be. And there are some, I believe, who practice the fourth, fifth and higher degrees.

John Maynard Keynes [1]

In this Thesis, I will address two general topics about socio-economic systems, relating to social choice theory (Part I) and to the properties of fluctuations in financial markets (Part II). Before describing the problems we will encounter, let us discuss the general context in which these issues arise and also, why a physicist should care at all.

The study of human behavior has always been a challenge. In the history, most of the greatest minds in literature, economy, biology and philosophy have spent time and dedicated works to understanding what are the basis of rational thinking and what its consequences and to study what motivations guide human activity and how they arise. In particular, economists and sociologists have tried to bring this subject in the realm of science, thus to develop disprovable theories of human behavior. When one attempts such a demanding task, many assumptions have to be made in order to treat a system as complex as that composed of many agents acting each with its own beliefs and expectations. Such complexity did not scare economists, who have tried and succeeded in developing rich theories of human behavior and to derive from them economic laws.

Often, the emphasis of these studies has been on motivating the assumptions about people's behavior. Of course one can devise more and more refined models of human rationality, describing in details the agents' expectations, their beliefs, their strategies. The derivation of laws for the collective behavior may be a challenging task in these cases, and it is legitimate to ask whether all this sophistication is indeed needed. Since Boltzmann, physicists have realized that the laws that govern aggregates of objects are due to statistical regularities that arise only at the macroscopic level, and that the detailed information about the microscopic constituents is often unnecessary. One clear and famous example is the Ising Model [2], which was proposed to explain the properties of ferromagnetic materials. The model is very simple, replacing each atom with a spin variable, thus forgetting all the details about the structure of the atoms, the nucleus, the electrons and so on. Yet, macroscopic properties such as phase transitions and magnetization are captured and reproduced. Such collective properties can be shown to depend only on a few key quantities, such as for example the dimensionality of the model, but not on the details at an atomic scale. Hence the lesson is that macroscopic laws emerge as a result of statistical regularities, and do not depend on the microscopic variables.

The idea is then to apply this approach to socio-economic systems, to model the emerging collective properties of such systems on a coarse-grained level, without resting on hardly testable behavioral assumptions about the individuals. To this scope let us introduce briefly some key ideas in economic literature, as well as the problems they generate and how statistical physics has helped and can help tackling these problems.

One of the basic assumptions of neoclassical economics has been that of being able to model an economy through the *representative agent approach*. This rests on the hypothesis that each group of agents can be modeled by some function reflecting the average behavioral motifs of the members of the group. The typical example is the way in which the price forming process of supply and demand is analyzed in classical economy textbooks. One assumes the existence of some *demand function*, which represents the quantity of a given good the agents acting in a particular market are willing to buy as the price of that good varies. Though it has given rise to many important theories and results, this approach misses some key features of systems as complex as the one under study.

First and foremost it misses the fact that the collective properties of a group of items may, and actually do, differ dramatically from that of single items taken separately. Going back to the previous example, the demand function is itself a collective property of economic systems, and it may have nothing to do with the characteristics of individuals. In an empirical study [3], Hardle and Kirman have analyzed the price and quantity details for trades in the Marseille fish market, deriving from the data demand curves. They found indeed that at the aggregate level a downward relationship exists, while in general this is not true at the individual level, and concluded that ”regularities are generated by aggregation rather than derived from individual behavior”. Then, if this is the case, one has to derive macroscopic laws for socio-economic systems from carefully aggregating the individuals’ ”microscopic” properties.

Moreover, direct interaction has often been neglected in the economics literature. For example, traders in a market are thought to interact only through the price of goods: they receive a signal (the price) and react to it. They are not influenced by each other, they are not even aware of each others’ existence. Models where there is a direct, conscious, interaction between the agents are termed models of ”imperfect competition”. Thus, the standard picture is a ”competitive equilibrium” where the signal induces the agents to make choices that clear the market (i.e. the offer matches the demand). This is somehow a benchmark and other models are often judged in respect of how they deviate from this one. We will come back to this issue later in this Introduction. Here we just note that in practical contexts there are many situations where the behavior of agents is directly influenced by other agents’ actions. One clear example is the widespread occurrence of conformism, where people imitate each other, with which we will deal in the first Part of this Thesis.

To tackle these problems, and keeping in mind that we want to develop theories for the collective behavior without entering in too many details about the individuals’ motivation, the general plan is to take into account the interaction among simplified agents and then derive macroscopic laws for the aggregate. This kind of situation, as we said, is quite common in statistical physics, where it is at the basis of the theory of phase transitions. It has long been known that studying the properties of a single iron atom, however deep the study may be, does not help much when one is interested in ”emerging” macroscopic properties such as the magnetization or, say, the fact that at high temperature a solid will melt into liquid state. These macroscopic properties are the result of the interaction of a large number of microscopic constituents and cannot be inferred from the analysis of such constituents. Just as anyone wishing to study thought processes should not simply generalize the average neuron, or one considering the motion of flocks should not concentrate on the ”average bird”, an economist should not model market behavior by ”modeling markets as individuals”[4], and it is precisely here that statistical physics can help.

Given this general critique that can be moved to economic models, we can be more specific by looking in detail at some ideas that were developed to describe socio-economic systems. In particular, the notion of rationality is central in economic theory, where one wishes to model purposeful behavior. The reason is readily understood by thinking for example about financial markets. There we find speculators of many different kinds, like for example trend-followers, investors wishing to minimize risk or fundamental traders who take trading decisions based on their belief of what is the expected ”fundamental” value of a given stock. How can we model the agents acting in this environment, if not by resorting to some assumptions on their expectations?

A tool that was devoloped in this direction is that of game theory [5]. Game theory is designed to take into account explicitly each individual's expectations and strategies, that are focused on anticipating other players' moves in an attempt to take the better decision. The assumption upon which the theory rests is that the players know everything about the other players' strategies, beliefs and expectations, that is, there is ”common knowledge” about what everybody knows. Then one looks for static equilibria, that is, situations in which each agent has no incentive to change strategy. The richness of such models is indeed vast, but this idea of rationality can be criticized in many aspects. The fundamental point is to realize that this notion of infinite rationality can be thought of as a zero-th order approximation. Indeed, there is a unique way of being perfectly rational, and it is the one discussed above. On the other hand, there are infinitely many ways of limiting the agents' rationality. This relates to what we mentioned above about competitive equilibrium. There is one way to deal with markets and derive a general equilibrium, but there are many ways of introducing interaction or imperfect competition. The idea behind the general equilibrium is that agents learn and react very quickly, so that their deviations from full rationality can be disregarded.

However, it turns out that if one wishes to model accurately individuals' behavior, many refinements of the perfect rationality are appropriate. For example, real world players can not be asked to store and process such a large amount of information. The notion of *bounded rationality* [6] has had some success in trying to capture the need to model economic agents as actors having not only a finite computational capacity, but, most importantly, as players that learn from the past events and whose rationality is shaped by the very same games they play. An example is that of markets of perishable goods, where the same agents trade the same goods day after day, and each day their decisions will be shaped by their previous experience. Realizing that an agent acting in an evolving environment (the market or in general the economic system under study) will necessarily have evolving beliefs and strategies, and thus that it can not be modeled as a static entity with a fixed plan of action has also been a concern to eminent economists [7]. This type of questions have been addressed quite seriously in recent years (see [8, 4]), but their history dates back to the keynesian beauty contest [1], that exemplifies the idea that investment is driven by expectations about what other investors think, rather than about the fundamental profitability of a particular investment, and the explicit strategy of each player with respect to her expectations need in principle to be taken into account.

This chain of models is what we were referring to in the first part of the Introduction. As statistical physicists, then, we will try to capture only very essential assumptions about the agents and work out the statistical laws that emerge at a coarse-grained level. Moreover, it becomes an extremely hard task to handle game-theoretic models with more than a few players. Tools capable of dealing with large number of microscopic components to derive the collective resulting behavior are needed, and again statistical physics can help providing this tools.

The physics community began to address this questions in the 90s, and one of the most famous approaches was that developed at the Santa Fe Institute, which rests on the idea of modeling markets in a way much more similar to how one models biological systems, that is, as evolving complex systems [9], composed of interacting heterogeneous agents. One of the first to introduce this view of economic systems was Brian Arthur [10, 11]. This gave rise to a wealth of papers, one of the most famous being the ”Santa Fe stock market model” [12] (an artificial model for stock markets, but see also [13]). The interest has since then focused mainly on financial markets, and the reason is twofold. There is certainly a practical interest in developing testable theories of financial markets (as easily understood!), and there are now some books covering this aspect in depth[14, 15]. But there is something more, namely the fact that social theories are very difficult to analyze on the basis of empirical evidence due to the scarce availability of data. This problem was eased in recent years by the availability of a great amount of data regarding financial markets. These data concern trades, and trades are the result of the individuals' decision process. Hence, financial markets are a prototype system to study for they provide a mean to test theories of collective human behavior as well as general assumptions on beliefs and expectations.

The detailed statistical analysis of financial markets has revealed important aspects that now go under the name *stylized facts*. These mainly concern the anomalous properties of price fluctuations [16]. The price of stocks was shown to udergo a process characterized by anomalously large fluctuations and non-trivial time dependence, with bursts of high volatility followed by periods of relative ”quiet”. This excess volatility, and its ubiquitous presence in different markets throughout the world, is hardly explained in terms of economic fundamentals, and it is now widely accepted that it stems instead from some internal market dynamics. One benchmark model for this is El Farol's bar problem [17], which led to the development of the Minority Game [18]: a model that is capable of reproducing and explaining the anomalous features that arise at a macroscopic level (aggregate) as resulting from the interaction of many heterogeneous agents with heterogeneous strategies, which, as we said, is precisely the goal of statistical physicists studying socio-economic systems.

Within this generic panorama of current research, this Thesis addresses problems that exemplify well the above discussion. In the first Part of this Thesis, and in particular in the first Chapter, we will deal with a problem of social choice. This amounts to the aggregation of the preferences of N individuals over a group of S items, or *alternatives*, which is a good example of a situation where one needs to derive the properties of the aggregate, given an heterogeneous group of agents, as discussed in the first part of this Introduction. Each agent, or *voter*, will have her own *ranking* of the S alternatives, from the preferred one to the worst. What are then the collective property of the "majority"? In particular, is it possible to define a preference ranking for the majority? Certainly, one must carefully deduce such properties, and the question of how to better aggregate individuals' preferences has been long since debated. Many *voting rules* have been proposed and studied, but it turns out that each of these rules has its flaws, as stated in Arrow's theorem [19]. In Chapter 1 we will discuss this issue in connection to one of the most studied voting rules (Pairwise Majority). The problem with this rule is that it can define a majority that prefers alternative A to alternative B, alternative B to C, and alternative C to A. This *intransitivity* is however not certain, and we will study the probability for the occurrence of such situation in a random population. This issue is also of practical interest towards determining how often one expects to encounter such problems in typical situations.

In Chapter 2 we generalize the above issue to the case of an interacting population. In particular, the type of interaction that we will introduce among voters is one that mimicks conformism, with agents trying to align their rankings of the alternatives to each other. We will ask how this interaction affects the findings of Chapter 1. Thus, we will face a system with many interacting agents and we will study its collective properties, along the lines drawn in this Introduction. We will borrow concepts and tools from statistical physics, introducing an order parameter that gives a measure of the collective order reached and, just like in standard statistical mechanics, a phase diagram will characterize the various "states" of the system.

This analogy will be pushed further in Chapter 3, where we will analyze the above-mentioned properties for real data. In particular, we will study the preferences of a group of voters over movies. In our view, this is a good example to test the theories developed for two reasons. First comes the fact that in priciple any ranking of a group of movies can be a legitimate and reasonable choice by anybody. Also, with all the advertisement for movies and with the fact that movies are almost always seen with other people, there is certainly a good deal of interaction upon the subject. This will conclude the first Part and our dealing with social science. In the second Part of the Thesis we will devote our attention to financial markets.

We have said how many anomalous statistical properties of asset prices have been revealed, and how these properties suggest to model financial markets as systems with endogenous feedbacks on the price dynamics. Our idea in the second Part of this Thesis will be that of generalizing these issues to a multi-asset market.

Hence at first we shall describe, in Chapter 4, the properties of such markets with a focus on the inter-asset correlations. Clearly, in a multi-asset setting, these are important quantities to characterize the market, and they have been widely studied. We will show that also here we can find interesting non-trivial behavior, and we will find some hints for the influence of internal feedbacks on the temporal evolution of the correlations.

This will lead us to the final Chapter, the fifth, where we will develop a model to explain and understand typical patterns in the temporal evolution of inter-asset correlations in term of an influence of financial investment strategies on the price process. The main point lies in the fact that investment strategies are determined by the historical properties of the system, but there is a feedback of these strategies on the price process itself. Hence this often-neglected interaction makes financial markets systems that feed on their same output. This feedback loop will be shown to be partly responsible for the observed fluctuation properties of the inter-asset correlations.

Part I

Emergence of consensus

Chapter 1

Probabilistic approach to Condocet's paradox

1.1 Introduction

Social systems have in the last decades attracted the attention of statistical physicists as a wealth of papers concerning social questions have appeared on physics journals. In close analogy with what happened for econophysics, the definition *sociophysics* has been proposed [20], though it hasn't catched as much. Among the problems studied one of the first, and most famous, is the neighbourhood problem put forth by Thomas Schelling [21]. This amounts to studying how an heterogeneous population, initially distributed randomly among the possible ”houses”, evolves to a state where homogeneous neighborhoods emerge through a dynamic where the agents prefer to have neighbours similar to themselves. The clear analogy with the physics of ferromagnets is striking and has been exploited in several models (see for example [22]). Other benchmark problems concern social choice and voting theory as for example, the voter model[23], that has been proposed to study how the vote's outcome between two alternatives is affected when voters influence each other. The reason for this outburst is that all this issues involve collective phenomena (such as the emergence of a common opinion in a large population) that are reminiscent of the collective phenomena in statistical physics. This approach to socio-econimic systems is thus very general and stems from realizing that the emergence of a ”macro-behaviour” can be the result of the interaction of many agents, each with different beliefs and expectations[24, 21].

Social choice and voting theory address the generic problem of how the individual preferences of N agents over a number S of alternatives can be aggregated into a social preference. In the case of two alternatices ($S = 2$) the voter model represents a clear link between the social problem and statistical physics. Also the random field Ising model has been proposed [25] in this context. In both cases the idea is that each agent in the population is represented by a spin variable s_i, whose value corresponds to the opinion of the voter. Then the interaction is a ferromagnetic one so as to mimick conformism: each "spin" tends to be aligned with its neighbors. The outcome of the "elections" is dictated simply by the usual order parameter, the magnetization $m = \sum_i s_i$.

This "majority" rule can be naturally extended to $S > 2$ alternatives by considering the social preferences stemming from majority voting on any pair of alternatives, i.e. pairwise majority rule (PMR). To be more precise, let us consider the case of three alternatives, and let us label the alternatives A, B, C. Each agent in the population has a transitive ranking of the alternatives, where *transitivity* means that if A is preferred to B and B to C, then A has to be preferred to C. The population is asked to vote between A and B, each agent votes for the alternative resulting higher in her ranking, and we get a statement for the preferred alternative like $A \succ B$, if the majority of the voters prefers A to B. Then we repeat this procedure for each pair of alternatives. The outcome $A \succ C$ and $B \succ C$ would lead to the social ordering of the preferences ranking A over B over C.

This extension however is problematic, as observed back in 1785 by Marie-Jean-Antoine-Niclas Caritat, Marquis de Condorcet [26]. He observed that the PMR among three alternatives may exhibit an irrational behavior, with the majority preferring alternative A to B, B to C and C to A, even though each individual has transitive preferences. To see how this can happen, consider the case of three voters and three alternatives. Let the first voter have the transitive ranking $A \succ B \succ C$, the second one $B \succ C \succ A$ and the third $C \succ A \succ B$. Then, in PMR voting, A would be indeed preferred to B, with two votes against one. Similarly, B would be preferred to C and C to A. These so-called Condorcet's cycles result in the impossibility to determine a socially preferred alternative or a complete ranking of the alternatives by pairwise majority voting (see also Ref. [27] for a relation with statistical mechanics of dynamical systems).

PMR is not the only way to aggregate individual rankings into a social preference [19, 28]. However the situation does not improve much considering other rules. For example, the transitivity of social preferences is recovered by resorting to voting rules like Borda count. This rule was proposed by a contemporary of Condorcet, Borda, in 1784. Actually, there was a fierce argument between the two mathematicians over which rule was the most fit, as both had their own strengths and drawbacks. It is interestiong to note that this harsh rivalry did not prevent Borda to risk his own life in pleading for clemency for Condorcet while the latter was listed to be tried as an enemy of the state. As to the Borda rule, here each voter assigns a score to each alternative, with high scores corresponding to preferred alternatives. Then the scores of each alternative are summed to give the total score, which is used to determine the ranking. It is clear that in this way, a transitive ranking for the majority is always generated, but other flaws emerge when looking carefully. As Condorcet immediatly pointed out, this rule is incompatible with PMR, since there is a chance that Borda rule will not elect as first choice the winner (when there is one) of PMR elections, that is, the candidate that would defeat all the others in pairwise elections. Indeed, this rule violates one of the basic requirements of a social choice rule.

The basic desiderata of a social choice rule are that it should be able to rank all alternatives for whatever individual preferences (*unrestricted domain*), it should be *transitive*, it should be *monotonous*, i.e. the social rank of an alternative A cannot decrease when an individual promotes A to a higher rank, and it should be *independent of irrelevant alternatives*, i.e. the social preference between A and B cannot depend on the preferences for other alternatives. For example We have seen that PMR fails to satisfy the transitivity requirement, while Borda rule clearly violates the last one, the independence of irrelevant alternatives, since I can assign the lowest score to an alternative which is perhaps my second-best, if I think that by doing this I increase the chances of my preferred alternative. Independence of irrelevant alternatives is important because it rules out the possibility of manipulating the election's outcome by falsely reporting individual preferences (see [29]). Another example of a system that fails to satisfy this requirement is plurality voting, which is used nowadays in most presidential elections, where each individual casts one vote for his top candidate and candidates are ranked according to the number of votes they receive. Indeed, plurality voting satisfies all requirement but the last one, as vividly illustrated by recent U.S. election outcomes [28]. To be more clear on this issue, there have reportedly been supporters of the independent candidate Nader that have cast their vote for Gore, given that their preferred alternative had no chance of winning. Hence, by falsely reporting their preferences (voting for Gore instead that for Nader), they can alter the relative ranking of other two alternatives (Bush and Gore).

The discomfort of social scientists with the impossibility to find a reasonable voting rule has been formalized by Arrow's celebrated theorem [19]. This states that a social choice rule that satisfies all of the above requirements has to be *dictatorial*, that is there exists an agent – the dictator – such that the social preference between any two alternatives is the preference of that agent.

A way to circumvent the impasse of this result is to study the properties of social choice rules on a restricted domain of possible individual preferences. For example in politics, it may be reasonable to rank all candidates from extreme left to extreme right. If the preferences of each individual has a "single peak" when candidates are ranked in this order (or any other order), then it has been shown that pairwise majority is transitive [30]. It has recently been shown that pairwise majority turns out to be the rule which satisfies all requirements in the largest domain [28], thus suggesting that pairwise majority is the best possible social choice rule.

Another possible approach to this problem relies on estimating the probability that these problems, for example Condorcet's cycles, occur in real situations. Actually, one could argue that the above discussed difficulties are just an academic problem, while in a typical situation one would not encounter them. In this view, it becomes important to quantify how good is majority rule by estimating the probability that pairwise majority yields a transitive preference relation in a typical case where individual preferences are drawn at random. This issue has been addressed by several authors [31, 32, 33, 34]. Here we will give a solution to this question.

1.2 A solution for Condorcet's paradox

Let us consider a population of N individuals with preferences over a set of S choices or candidates. We shall mainly be interested in the limit $N \to \infty$ of an infinite population. We limit attention to strict preferences, i.e. we rule out the case where agents are indifferent between items. Hence preference relations are equivalent to rankings of the S alternatives. It is convenient to represent rankings with matrices $\hat{\Delta}_i$ for each agent $i = 1, \ldots, N$, whose elements take values $\Delta_i^{ab} = +1$ or -1 if i prefers choice a to $b \neq a$ or vice-versa, with $a, b = 1, \ldots, S$. Notice that $\Delta_i^{ba} = -\Delta_i^{ab}$. Let $\mathcal{R}$ be the set of matrices $\hat{\Delta}$ which correspond to a transitive preference relation. Clearly the number of such matrices equals the number $|\mathcal{R}| = S!$ of rankings of the S alternatives. Hence not all the $2^{S(S-1)/2}$ possible asymmetric matrices with binary elements $\Delta_i^{ab} = \pm 1$ correspond to acceptable preference relations. For example, if $\Delta^{1,2} = \Delta^{2,3} = \Delta^{3,1}$ then $\hat{\Delta} \notin \mathcal{R}$. Below are two examples of matrices $\Delta \in \mathcal{R}$.

Ranking: ABC	**Ranking: BAC**
$A \succ B; B \succ C; A \succ C$	$B \succ A; B \succ C; A \succ C$
$\Downarrow$	$\Downarrow$
$\hat{\Delta} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$	$\hat{\Delta} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$

We use the term ranking to refer to matrices $\hat{\Delta} \in \mathcal{R}$ in order to avoid confusion later, when we will introduce preferences over rankings, i.e. over elements of $\mathcal{R}$. We assume that each agent i is assigned a ranking $\hat{\Delta}_i$ drawn independently at random from $\mathcal{R}$.

In order to compute the probability $P(S)$ that pairwise majority yields a transitive preference relation, in the limit $N \to \infty$, let us introduce the matrix

$$\hat{x} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \hat{\Delta}_i. \tag{1.1}$$

The assumption on $\hat{\Delta}_i$, that is, that they are randomly drawn from $\mathcal{R}$ implies that they are i.i.d. variables drawn from the same distribution, which has a finite variance. Hence we can apply the central limit theorem to obtain that the distribution of x^{ab} is Gaussian for $N \to \infty$ and it is hence completely specified by the first two moments $\left\langle x^{ab} \right\rangle = 0$ and $\left\langle x^{ab} x^{cd} \right\rangle = \{\mathcal{G}^{-1}\}^{ab,cd}$.

Let us address the problem of the profile chosen by each voter by defining an utility function $u_i(a)$, which defines the utility of voter i with respect to the choice a. The preference relations of each voter are then expressed by the condition:

$$a \succ_i b \Leftrightarrow u_i(a) > u_i(b). \tag{1.2}$$

To realize the condition known as *Impartial culture*, namely, that every profile be equiprobable, it is sufficient to state that the $u_i(a)$ be chosen randomly in the unit interval. One should note that by doing so, we ignore the possibility of *indifference* between two choices (i.e. the $\geq$ sign in equation 1.2). This is so since the set of such points is of zero measure.

Now, the preference relations of the majority can be computed by first calculating the quantities

$$\hat{x}^{ab} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left(2\theta(u_i(a) - u_i(b)) - 1\right), \tag{1.3}$$

where $\theta(x)$ stands for Heavyside's step function. Now the condition for the majority to prefer choice a to choice b can be stated as

$$a \succ b \Leftrightarrow \hat{x}^{ab} > 0.$$

The case where the equality holds can be disregarded with certainty if we restrict to an odd number of voters. There are $S(S-1)/2$ independent $\hat{x}$, since that is the number of couples out of S elements.

Now we can proceed to calculating the correlation matrix $\{\mathcal{G}^{-1}\}^{ab,cd} = \langle x^{ab} x^{cd} \rangle$ from the uniform distribution of the $u_i(a)$. We must note that the indices do not scan all the possible couples (a, b), but only those with $b > a$, hence the matrix $\mathcal{G}^{-1}$ is a $S(S-1)/2$ times $S(S-1)/2$ symmetric matrix. The $u_i(a)$ are iid random variables such that

$$\int_0^1 f(u_i(a)) \mathrm{du_i(a)} = 1$$

and

$$f(x) = 1,$$

but we note that the results are independent from the distribution of the $u_i(a)$. We thus write for the correlations

$$\langle x^{ab} x^{cd} \rangle = \int_0^1 \prod_{i=1}^{N} \mathrm{du_i(a)du_i(b)du_i(c)du_i(d)\frac{1}{N}\sum_{j=1}^{N}\phi_j(a,b)\sum_{k=1}^{N}\phi_k(c,d)}, \quad (1.4)$$

where

$$\phi_j(a,b) = 2\theta(u_j(a) - u_j(b)) - 1 \quad (1.5)$$

which gives, after some algebra,

$$\{\mathcal{G}^{-1}\}^{ab,cd} = \begin{cases} 1 & a = c,\ b = d \\ \frac{1}{3} & a = c,\ b \neq d \text{ or } a \neq c,\ b = d \\ -\frac{1}{3} & a = d,\ b \neq c \text{ or } a \neq d,\ b = c \\ 0 & \text{otherwise} \end{cases} \quad (1.6)$$

where we have introduced the notation $\mathcal{M}$ for matrices with elements $M^{ab,cd}$. The result of Eq. (1.6) can be obtained even without the direct calculation, but simply looking at the structure of the integral in Eq.(1.4). First, we note that the off-diagonal terms ($j \neq k$) do not play any role, by symmetry, and the result depends only on the relative value of the utilities. Then, the result $\{\mathcal{G}^{-1}\}^{ab,ab} = 1$ is trivial, as well as $\{\mathcal{G}^{-1}\}^{ab,cd} = 0$ when $a \neq c$ and $b \neq d$. As to $\{\mathcal{G}^{-1}\}^{ab,ad}$, with $b \neq d$, let us fix the utilities of a and b such that $u(a) > u(b)$. Then, there are three possibilities. If $u(a) > u(d)$ and $u(d) > u(b)$ the contribution of this configuration to the integral is positive, and the same applies when $u(a) > u(d)$ and $u(d) < u(b)$. On the other hand, the contribution is negative when $u(a) < u(d)$. Since the three possibilities here presented are equally weighted in (1.4), the result is $1/3$. Then, again by symmetry, $\{\mathcal{G}^{-1}\}^{ab,da}$ is $-1/3$.

The matrix $\mathcal{G}^{-1}$ can be inverted by a direct computation, making the ansatz

$$\mathcal{G}_{ijkl} = g_0\delta_{ik}\delta_{jl} + g_1[\delta_{ik}(1-\delta_{jl}) + \delta_{jl}(1-\delta_{ik}) - \delta_{il}(1-\delta_{jk}) - \delta_{jk}(1-\delta_{il})] \quad (1.7)$$

and imposing the condition $\mathcal{G}^{-1}_{ijab}\mathcal{G}_{abkl} = \delta_{ik}\delta_{jl}$. With a few manipulations we obtain

$$g_0 = 3\frac{S-1}{S+1} \quad (1.8)$$

$$g_1 = -\frac{3}{S+1}. \quad (1.9)$$

and we find that the matrix $\mathcal{G}$ has the same structure of $\mathcal{G}^{-1}$ but with $\mathcal{G}^{ab,ab} = 3\frac{S-1}{S+1}$, $\mathcal{G}^{ab,ad} = -\frac{3}{S+1} = \mathcal{G}^{ab,cb} = -\mathcal{G}^{ab,bd} = -\mathcal{G}^{ab,ca}$.

Let us first compute the probability $P_{CW}(S)$ that one of the alternatives, is better than all the others. This means that there is a consensus over the winner, while nothing is assumed for the relations between the other choices. The preferred alternative is known in social choice literature as the Condorcet winner, and much interest has been devoted to it, since the presence of such a preferred alternative saves at least the possibility of electing a favorite choice. $P_{CW}(S)$ is just the probability that $x^{1,a} > 0$ for all $a > 1$ multiplied by S. In this way, we recover a known results [32], which can be conveniently casted in the form

$$P_{CW}(S) = S\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} e^{-2y^2+(S-1)\log[\mathrm{erfc}(y)/2]}. \tag{1.10}$$

Notice that $P_{CW}(S)$ is much larger than the naïve guess $S/2^{S-1}$, derived assuming that $x^{ab} > 0$ occurs with probability $1/2$ for all ab. Indeed asymptotic expansion of Eq. (1.10) shows that

$$P_{CW}(S) \simeq \sqrt{\frac{\pi}{2}} \frac{\sqrt{\log S}}{S} \left[1 + O(1/\sqrt{\log S})\right]$$

decays extremely slowly for $S \gg 1$.

The probability P_+ that the majority ranking is equal to the cardinal one ($1 \succ 2 \succ \ldots \succ S$) is given by the probability that $x^{ab} > 0$ for all $a < b$. This is only one of the $S!$ possible orderings, then the probability of a transitive majority can be written as

$$P(S) = S! P_+ \tag{1.11}$$

hence

$$P(S) = S! \frac{[3/(2\pi)]^{\frac{S(S-1)}{4}}}{(S+1)^{\frac{S-1}{2}}} \int_0^{\infty} d\hat{x} \exp\left[-\frac{1}{2}\hat{x} \cdot \mathcal{G} \cdot \hat{x}\right] \tag{1.12}$$

where $\int_0^\infty d\hat{x} \equiv \int_0^\infty dx_{1,2} \ldots \int_0^\infty dx_{S-1,S}$ and we defined the product $\hat{r} \cdot \hat{q} = \sum_{a<b} r^{ab} q^{ab}$ and its generalization to matrices $\hat{r} \cdot \mathcal{M} \cdot \hat{q} = \sum_{a<b} \sum_{c<d} r^{ab} M^{ab,cd} q^{cd}$. The normalization factor is computed from the spectral analysis of $\mathcal{G}$. The matrix $\mathcal{G}$ has $S-1$ eigenvectors of the form $z_{1,k}^{ab} = \delta_{a,k-1}\mathrm{sign}(b-k+1)$ with $k = 2, \ldots, S$, and eigenvalue $\lambda = 3/(S+1)$. This can be verified by direct calculation. Note that the vectors z_k^{ab} are not orthogonal, but are linearly independent. Direct substitution shows that all vectors of the form $z_{j,k}^{ab} = \delta_{a,1}(\delta_{b,j} - \delta_{b,k}) + \delta_{a,j}\delta_{b,k}$ are also eigenvectors of $\mathcal{G}$ for $1 < j < k$, with eigenvalue $\lambda = 3$. Then $\lambda = 3$ has degeneracy $(S-1)(S-2)/2$. The set of $S(S-1)/2$ linearly independent vectors $z_{j,k}^{ab}$ with $1 \le j < k \le S$ allows us to build a complete orthonormal basis of eigenvectors and to compute $\det \mathcal{G}$.

1.3 Numerical simulations

We were not able to find a simpler form for the probability of Eq. (1.12), however one can perform numerical simulations to confirm analytical calculations of the previous section. The two approaches that we take here are, first, the numerical integration of Eq. (1.12) via Monte Carlo simulations, secondly, the *ab initio* simulations of the system with a finite but large number of voters N.

The numerical integration can be performed in many ways. The first step in this direction is always to draw a random $S*(S-1)/2$ components vector x from the distribution $\frac{[3/(2\pi)]^{\frac{S(S-1)}{4}}}{(S+1)^{\frac{S-1}{2}}}\left[-\frac{1}{2}\hat{x}\cdot\mathcal{G}\cdot\hat{x}\right]$. Then the simplest idea is just to evaluate the probability $P_+ = Pr\{x_i > 0 \forall i\}$ by calculating the frequency with which we draw a vector $\hat{x}$ with all the components positive. Then one has the estimate $P(S) = S!P_+$. This simple method of integration is adequate for small S, but as S increases it is clear that the probability of drawing a vector x in the subspace $x_i > 0$ decreases very rapidly, hence one ends up drawing many vectors that are then not used, with great waste of computational time and reachable precision. One way to tackle this issue is that of adding a vector with components equal to λ to the vector $\hat{x}$, so that there is a higher probability of obtaining a vector in the desired subspace. Then we reweight the frequency by an appropriate factor which depends on λ. The price we pay for this reweighting is that the error increases with λ , so that also this method has its limits for high S.

The method which turns out to be the less expensive, and thus the one we finally use, is the following. We still draw the random vector $\hat{x}$ as usual, but then we proceed to evaluate $P(S)$ directly, instead of going through the evaluation of P_+. To do this, we write the majority matrix:

$$M_{ab} = \theta(x_{ab}) \tag{1.13}$$

such that $M_{ab} = 1$ if the majority prefers a to b (that is, $x_{ab} > 0$), 0 otherwise. Then we search for loops (Condorcet's cycles) in this matrix. This is most efficently done by writing the vector $c_i = \sum_j M_{ij}$ which counts the number of ones in each row. The majority is transitive if the vector c contains once and only once *each* of the integers $0, 1, ..., S-1$. This method turns out to be quite efficient, the results are plotted below.

As to the simulations of the system, one simply sets up N voters, assigns to each of them a random ranking of the preferences and then checks for transitivity of the majority.

Fig. 1.1 reports Montecarlo estimates of $P(S)$. For $S = 3$ we recover the result [31]

$$P(3) = P_{CW}(3) = \frac{3}{4} + \frac{3}{2\pi}\sin^{-1}\frac{1}{3} \cong 0.91226\ldots \tag{1.14}$$

Again the naïve guess $P(S) \approx S!/2^{S(S-1)/2}$ based on the fraction of acceptable rankings largely underestimates this probability. This means that the collective behavior of the majority hinges upon the (microscopic) transitivity of individual rankings.

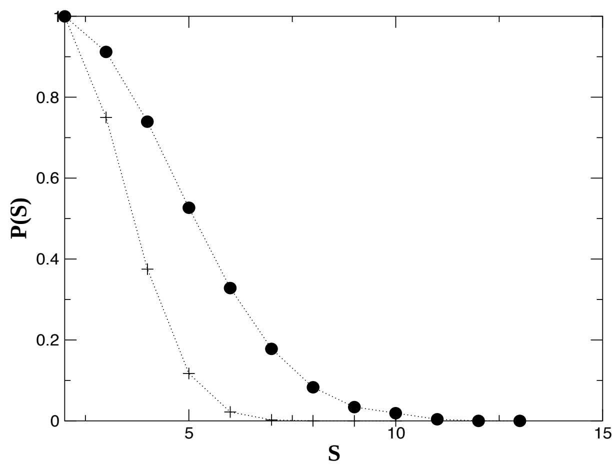

Figure 1.1: Probability $P(S)$ of a transitive majority ($\bullet$) compared to the naïve guess $S!/2^{S(S-1)/2}$ ($+$).

Chapter 2

Statistical physics of interacting voters

2.1 Introduction

When studying human behaviour it is often convenient to treat, as we did in the previous chapter, the agents as non-interacting. Actually, traditional economics relies on the assumption that one can model a population through the representative agent approach. This amounts to replacing individual preferences with average functions (like for example demand function when studying markets), a feature that of course simplifies the treatment of the model, but certainly misses one of the key features of human dynamics, interaction among individuals. Such approach has become the subject of debate among sociologists and economists in recent years, and models where interaction is explicitly taken into account have been put forth. The introduction of many kind of interactions among individuals has been one of the main reasons for the success of game theory in economics. The paradigm of classical economics has been questioned and investigated by economists using the tools of game theory as a mean to explicitly account for the interaction among agents since the seminal work of Nash [35]. This approach, although very interesting, has a limitation in the fact that one is forced to study systems with a finite number of participants, and the extension to the limit of large populations can be problematic. However, it was understood that studying systems composed of many (possibly heterogeneous) agents, like for example the system studied in the previous Chapter, one needs tools to explicitly account for agents' interaction.

The idea of interacting agents where the state of an agent is determined by the explicit form of the interaction with other agents is central in statistical physics, and there it gives rise to a wide range of interesting phenomena. Here, the emergence of macroscopic properties which differ dramatically from the microscopic nature of the agents suggests that this approach can be successfully applied to the study of socio-economic systems and that this can lead to the derivation of some ”thermodinamic” laws for the bahavior of aggregates that could differ dramatically from the expectations of the representative agents theories, as discussed in the general Introduction.

In this view, we want to study how to generalize the analysis of the previous Chapter to the case when agents influence each other. In particular we restrict to the relevant case where the interaction arises from conformism [21]. This case is one of the most studied since its history dates back to Keynes' work on beauty contests [1], and a quite general treatment can be found in [36]. Basically conformism can stem from three different reasons [37]. It can be *pure* or *imitative*, because people simply want to be like others. It can be due to the fact that in some cases conforming facilitates life (*instrumental conformism*), which is probably one of the mechanisms at the basis of the emergence of languages and culture. Or it can be due to people deriving information about the value of a choice from other people's behavior (*informational conformism*): if many people behave in a certain way, then I may be induced to thinking that it is the ”correct” behavior. An effect of this kind is thought to be present in political elections, when there is a fraction of the population that seems to vote for the alternative that is most likely to win.

Whatever the case, one has to explicitly take into account the interaction among individuals to treat phenomena where conformism clearly plays a central role. Among these, we cite for example fashions or fanaticism, where it may lead to the rise and spread of broadly accepted systems of values [38], crashes and rallies in financial markets [39] and motion of crowds [40]. Another clear case where conformism is at the basis of an emergent property of a system of agents is Schelling's neighborhood problem [21], which is a special case of systems where one studies the conditions for the emergence and maintenance of cooperation among individuals.

A model for this phenomenon that comes from statistical physics and has attracted much attention is the Random Field Ising Model, originally studied as a model for disordered ferromagnets [41]. Here N spin variables s_i interact with each other with direct couplings J_{ij} and a random magnetic field ϕ_i is present at each site. The analogy with social systems is carried out by mapping the two states of the spin into two opposite opinions on an alternative (choice or issue), the random fields into *a priori* opinion, and the interaction into the influence that agents have onto each other. Then the macroscopic property of interest is the emergence of a common opinion (magnetization) [25, 42]

We will introduce interaction in the system studied in the previous chapter with preferences of the agents being shaped by the interaction among them. Namely, to mimick conformism, we will assume that agents will want to agree with the majority. Then we study the emerging properties of the resulting model by addressing several questions. Is there the emergence of a common ranking of S alternatives? How much information should the agents share in order to achieve consensus on S items? What is the probability of having a transitive majority when agents conform their preferences to each other? Will it differ significantly from the non-interacting case? At any rate, our discussion will focus on the consequence of conformism on the collective behavior, without entering into details as to where this conformism stems from.

We show that the occurrence of a transitive social choice on a number S of alternatives for any choice of the individual preferences, is related to the emergence of spontaneous magnetization in a multi-component random field Ising model. We find a phase diagram similar to that of the single component model [43] with a ferromagnetic phase and a tricritical point separating a line of second order phase transitions from a first order one. The ferromagnetic state describes the convergence of a population to a common and transitive preference ranking of alternatives, due to social interaction.

Remarkably, we find that the ferromagnetic region expands as S increases. Hence while without interaction the probability $P(S)$ of a transitive majority vanishes rapidly as S increases, if the interaction strength is large enough, the probability of a transitive majority increases with S and it reaches one for S large enough. In other words, an interacting population may reach more likely consensus when the complexity of the choice problem (S) increases.

We finally contrast these findings with the case where agents need not express a transitive vote (e.g. they may vote for A when pitted against B, for B agains C and for C against A). This is useful because we find that then the probability of finding a transitive majority is much lower. In other words, individual coherence is crucial for conformism to enforce a transitive social choice.

2.2 Interacting voters

So let us now introduce interaction among voters. We assume that agents have an *a-priori* transitive preference over the alternatives, specified by a ranking $\hat{\Delta}_i \in \mathcal{R}$. We allow however agents to have a voting behavior which does not necessarily reflect their *a-priori* ranking, that is, we introduce a new matrix $\hat{v}_i$ such that $v_i^{ab} = +1$ (-1) if agent i, in a context between a and b, votes for a (b). We will first study the case when $\hat{v}_i \in \mathcal{R}$, which corresponds to agents having a rational voting behavior. This means that even though an agent is influenced by others, she will maintain a coherent choice behavior (transitivity). We will contrast this case with that where the constraint on individual coherence $\hat{v}_i \in \mathcal{R}$ is removed.

To account for interaction, the matrix $\hat{v}_i$ depends not only on agents' preferences $\hat{\Delta}_i$, but also on the interaction with other agents. Within economic literature, this dependence is usually introduced by means of an utility function u_i which agents tend to maximize. In our case this utility function will depend on agents' choice, that is, on their possible preference profiles. A voter will have the profile that maximizes her utility. Notice that this utility function represents a preference over preferences (rankings). In fact, each ranking is itself a preference relation between the alternatives, and it specifies what alternatives are better in the agent's view. By introducing an utility function, that is a function that maps each ranking a voter can have into a real number, we are actually "ranking the rankings" to determine which preference profile is best suited for each voter.

Formally, this utility function depends both on an idiosyncratic term $\hat{\Delta}_i \in \mathcal{R}$ describing the *a priori* ranking, and on the behavior of other agents, $\hat{v}_{-i} \equiv \{\hat{v}_j,\ \forall j \neq i\}$, through the majority matrix

$$\hat{m} = \frac{1}{N} \sum_{i=1}^{N} \hat{v}_i. \tag{2.1}$$

More precisely, we define an utility function

$$u_i(\hat{v}_i, \hat{v}_{-i}) = (1 - \epsilon)\hat{\Delta}_i \cdot \hat{v}_i + \epsilon \hat{m} \cdot \hat{v}_i. \tag{2.2}$$

where the last term captures conformism as a diffuse preference for aligning to the majority. For $\epsilon = 0$ maximal utility in Eq. (2.2) is attained when agents vote as prescribed by their *a priori* rankings, i.e. $\hat{v}_i = \hat{\Delta}_i\ \forall i$. On the contrary, for $\epsilon = 1$ agents totally disregard their rankings and align on the same ranking $\hat{v}_i = \hat{m}\ \forall i$, which can be any of the $S!$ possible ones.

2.2.1 Nash equilibria

In economic literature, when one is presented with a problem defined by an utility function $u_i(s_i, s_{-i})$ for each agent i playing strategy s_i when all the other players play strategies defined by s_{-i}, to find the solution one has to find equilibrium points. These equilibrium points are termed *Nash equilibria*, and are defined as follows. At equilibrium, each agent will not have any incentive to switch strategy, hence her optimal strategy s_i^* will be the "best response" to other players' strategies s_{-i}. But this has to be true for each player, hence s_i^* is defined by being the best response to other players' optimal strategies, that is one needs to maximize $u_i(s_i^*, s_{-i}^*)$. Since we are dealing with a problem that arises in the context of social choice theories, we will adopt this formalism to tackle it.

Let us characterize the possible stable states, i.e. the Nash equilibria of the game defined by the payoffs of Eq. (2.2). These are states $\hat{v}_i^*$ such that each agent has no incentives to change his behavior, if others stick to theirs, i.e. $u_i(v_i, v^*_{-i}) \leq u_i(v^*_i, v^*_{-i})$ for all i. The *random* state $\hat{v}^*_i = \hat{\Delta}_i$ is (almost surely) a Nash equilibrium $\forall \epsilon < 1$, because the payoff of aligning to the majority $\hat{m} = \hat{x}/\sqrt{N}$ is negligible with respect to that of voting according to own ranking $\hat{\Delta}_i$. Then we have $u_i(\hat{\Delta}_i, \hat{\Delta}_{-i}) = \frac{S(S-1)}{2}[1 - \epsilon + \epsilon O(1/\sqrt{N})]$. This Nash equilibrium is characterized by a majority which is not necessarily transitive, i.e. which is transitive with probability $P(S) < 1$ for $N \gg 1$.

Also *polarized* states with $\hat{v}_i = \hat{m}$ for all i are Nash equilibria for $\epsilon > 1/2$. Indeed, with some abuse of notation, when all agents take $\hat{v}_j = \hat{m}$ for some $\hat{m}$, agent i receives an utility $u_i(\hat{m}, \hat{m}) = (1-\epsilon)\hat{\Delta}_i \cdot \hat{m} + \frac{S(S-1)}{2}\epsilon$. The agents who are worse off are those with $\hat{\Delta}_i = -\hat{m}$ for whom $u_i(\hat{m}, \hat{m}) = \frac{S(S-1)}{2}[2\epsilon - 1] \cong -u_i(\hat{\Delta}_i, \hat{m}) + O(1/N)$. Then as long as $\epsilon > 1/2$, even agents with $\hat{\Delta}_i = -\hat{m}$ will not profit from abandoning the majority. Therefore $\hat{v}_i = \hat{m}$ for all i is a Nash equilibrium. Notice that whether the majority is transitive ($\hat{m} \in \mathcal{R}$) or not depends on whether agents express transitive preferences ($\hat{v}_i \in \mathcal{R}$) or not. In the former case the majority will be transitive whereas if non transitive voting is allowed there is no need to have $\hat{m} \in \mathcal{R}$ and there are $2^{S(S-1)/2}$ possible polarized Nash equilibria. Only in $S!$ of them the majority is transitive (i.e. when $\hat{m} \in \mathcal{R}$).

It is easy to check that there are no other Nash equilibria. Indeed, let us assume that there exists a finite magnetization, that is a finite fraction of agents choose the same ranking, while the others choose their *a priori* ranking Δ. Then, for $\epsilon > 1/2$ every agent playing Δ will benefit from switching to the ranking preferred by this fraction, hence Δ is not her best response. Summarizing, for $\epsilon > 1/2$ there are two Nash equilibria. Depending on the dynamics by which agents adjust their voting behavior one or the other of these states will be selected. Here, statistical physics methods can be applied since the situation is analogous to determining which stable state (minima of a free energy) a hamiltonian dynamics will select.

2.3 Statistical Mechanics of interacting voters

Strict utility maximization leads to the presence of multiple equilibria, leaving open the issue of which equilibrium will the population select. It is useful to generalize the strict utility maximization into a stochastic choice behavior which allows for mistakes (or experimentation) with a certain probability [44]. This on one side may be realistic in modelling many socio-economic phenomena [24, 45, 37]. On the other hand this rescues the uniqueness of the solution, in terms of the probability of occurrence of a given state $\{\hat{v}_i\}$, under some ergodicity hypothesis. Here, as in [45], we assume that agents have the following probabilistic choice behavior: agents are asynchronously given the possibility to revise their voting behavior. When agent i has a revision opportunity, he picks a voting profile $\hat{w}$ ($\in \mathcal{R}$ when voters are rational) with probability

$$P\{\hat{v}_i = \hat{w}\} = Z_i^{-1} e^{\beta u_i(\hat{w},\hat{v}_{-i})} \tag{2.3}$$

where Z_i is a normalization constant. Without entering into details, for which we refer to Ref. [45], let us mention that Eq. (2.3) does not necessarily assume that agents randomize their behavior on purpose. It models also cases where agents maximize a random utility with a deterministic term u_i and a random component. Then the parameter β is related to the degree of uncertainty (of the modeler) on the utility function.

When agent i revises his choice the utility difference $\delta u_i = u_i(\hat{v}_i, \hat{v}_{-i}) - u_i(\hat{v}_i', \hat{v}_{-i})$ for a change $\hat{v}_i \to \hat{v}_i'$ is equal to the corresponding difference in $-H$, where

$$H\{\hat{v}_i\} = -(1-\epsilon)\sum_{i=1}^{N} \hat{\Delta}_i \cdot \hat{v}_i - \frac{\epsilon}{2N}\sum_{i,j=1}^{N} \hat{v}_j \cdot \hat{v}_i. \tag{2.4}$$

hence in the long run, the state of the population will be described by the Gibbs measure $e^{-\beta H}$ because the dynamics based on Eq. (2.3) satisfies detailed balance with the Gibbs measure.

H in Eq. (2.4) is the Hamiltonian of a multi-component random field Ising model (RFIM) where each component v_i^{ab} with $a < b$ is a component of the spin, $\hat{\Delta}_i$ represents the random field and the term $\frac{\epsilon}{2N}\sum_{i,j=1}^{N} \hat{v}_j \cdot \hat{v}_i$ is a mean field interaction. Indeed $\hat{v}_i$ has $S(S-1)/2$ components which take values $v_i^{ab} = \pm 1$. The peculiarity of this model is that the components of the fields $\hat{\Delta}$ are not independent. Indeed not all the $2^{S(S-1)/2}$ values of $\hat{\Delta}_i$ are possible but only those that correspond to a transitive ranking of the S alternatives ($\hat{\Delta}_i \in \mathcal{R}$) which are $S!$. The same applies to the spin components $\hat{v}_i$ when rational voting behavior is imposed. Were it not for this constraint, the model would just correspond to a collection of $S(S-1)/2$ uncoupled RFIM.

The statistical mechanics approach of the RFIM [41, 43] can easily be generalized to the present case. The partition function can be written as

$$Z(\beta) = \mathrm{Tr}_{\{\hat{v}_i\}} e^{-\beta H} = \int d\hat{m} e^{-N\beta f(\hat{m})} \tag{2.5}$$

where the trace $\mathrm{Tr}_{\{\hat{v}_i\}}$ over spins runs on all $\hat{v}_i \in \mathcal{R}$ when voting behavior is rational, or over all $\hat{v}_i$ otherwise. The free energy $f(\hat{m})$ is given by

$$f(\hat{m}) = \frac{\epsilon}{2}\hat{m}^2 - \frac{1}{N\beta}\sum_{i=1}^{N} \log\left[\sum_{\hat{v}} e^{\beta[(1-\epsilon)\hat{\Delta}_i + \epsilon\hat{m}]\cdot\hat{v}}\right] \tag{2.6}$$

where once again the sum over the $\hat{v}_i$ runs inside $\mathcal{R}$ if agents are rational, or is not limited otherwise. It is evident that f is self averaging. Hence in the limit $N \to \infty$ we can replace $\frac{1}{N}\sum_i \ldots$ with the expected value $\frac{1}{S!}\sum_{\hat{\Delta}\in\mathcal{R}} \ldots \equiv \langle \ldots \rangle_\Delta$ on $\hat{\Delta}_i$. It is also clear that the integral over $\hat{m}$ of Eq. (A.8) in this limit is dominated by the saddle point value. The saddle point equation

$$\frac{\partial f}{\partial \hat{m}} = 0 \tag{2.7}$$

yields

$$\hat{m} = \left\langle \frac{\sum_{\hat{v}} \hat{v} e^{\beta[(1-\epsilon)\hat{\Delta}+\epsilon\hat{m}]\cdot\hat{v}}}{\sum_{\hat{v}} e^{\beta[(1-\epsilon)\hat{\Delta}+\epsilon\hat{m}]\cdot\hat{v}}} \right\rangle_\Delta \tag{2.8}$$

This equation can be solved from direct iteration and shows that for large enough values of $\beta > \beta_c$ there is a transition, as ϵ increases, from a paramagnetic state with $\hat{m} = 0$ to a polarized (ferromagnetic) state where $\hat{m} \neq 0$. In Fig. (2.1) we plot the result of such iterative solution for the RFIM case, i.e. when $S = 2$. Since for some values T, ϵ both the ferromagnetic and the paramagnetic state can be stable, we have solved for the magnetization $\hat{m}$ starting both from a $\hat{m} = 0$ and from $\hat{m} = \hat{1}$ states. Then we selected the correct equilibrium state by comparing the free energy of the different solutions. The stability of the paramagnetic solution $\hat{m} = 0$ can be inferred from the expansion of Eq. (2.8) around $\hat{m} = 0$, which reads

$$\hat{m} = \beta\epsilon\mathcal{J}\cdot\hat{m} + O(\hat{m}^3) \tag{2.9}$$

where

$$J^{ab,cd} = \left\langle \left\langle v^{ab}v^{c,d}|\hat{\Delta}\right\rangle_v - \left\langle v^{ab}|\hat{\Delta}\right\rangle_v \left\langle v^{c,d}|\hat{\Delta}\right\rangle_v \right\rangle_\Delta . \tag{2.10}$$

Here averages $\left\langle \ldots |\hat{\Delta}\right\rangle_v$ over $\hat{v}$ are taken with the distribution

$$P(\hat{v}|\hat{\Delta}) = \frac{e^{\beta(1-\epsilon)\hat{\Delta}\cdot\hat{v}}}{\sum_{\hat{u}} e^{\beta(1-\epsilon)\hat{\Delta}\cdot\hat{u}}}. \tag{2.11}$$

When the largest eigenvalue Λ of $\beta\epsilon\mathcal{J}$ is larger than one, the paramagnetic solution $\hat{m} = 0$ is unstable and only the polarized solution $\hat{m} \neq 0$ is possible. In Fig.2.1 the line that marks the region of instability of the paramagnetic solution is plotted at the bottom.

2.3.1 Constrained case, $\hat{v}_i \in \mathcal{R}$

Let us specialize to the case where a rational voting behavior is imposed on voters. That is, agents will influence each other, but not to the point of picking an intransitive preference profile v_i. Hence both the individual *a-priori* rankings $\hat{\Delta}_i$ and the voting behavior $\hat{v}_i$ of each agent are transitive. Results for the numerical iteration of Eq. (2.8) are shown in the inset of Fig.2.2 for different values of β and for $S = 5$. Fig. 2.2 shows the phase diagram for $S = 2,\ 3$ and for $S = 5$. The transition from the paramagnetic phase to the ferromagnetic one is continuous for intermediate values of β ($\beta_t < \beta < \beta_c$) but becomes discontinuous when $\beta > \beta_t$. The transition point β_t (•) generalizes the tricritical point of the RFIM [43] ($S = 2$).

The condition $\Lambda = 1$ on the largest eigenvalue Λ of $\beta\epsilon\mathcal{J}$ reproduces the second order transition line. The line $\Lambda = 1$ continues beyond the tricritical point and it marks the border of the region where the paramagnetic solution $\hat{m} = 0$ is unstable (dotted line in Fig. 2.2). Below the lower branch of the $\Lambda = 1$ line the paramagnetic solution is locally stable but it is not the most probable. Indeed the polarized state $\hat{m}^*$ which is the non-trivial solution of Eq. (2.8) has a lower free energy $f(\hat{m}^*) < f(0)$. Still in numerical simulation the state $\hat{m} = 0$ can persist for a very long time in this region. The polarized solution $\hat{m}^*$ becomes metastable and then disappears to the left of the transition line in Fig. 2.2.

With respect to the dependence on S of the phase diagram, we observe that at $\beta \to \infty$ the phase transition takes place at $\epsilon = 2/3$ independent of S. At the other extreme, for $\epsilon = 1$ we find that $\mathcal{J} = \mathcal{G}^{-1}$, since with $\epsilon = 1$, $J^{ab,cd} = \frac{1}{S!}\sum_{\hat{v}\in\mathcal{R}} v^{ab}v^{c,d}$. So $J^{ab,ab} = 1$ and $J^{ab,cd} = 0$ by symmetry if $a \neq c, d$ and $b \neq c, d$. Furthermore $J^{ab,ad}$ depends only on the relative ordering between a, b and d in the permutation $\hat{v}$. The permutations where a is between b and d, which are $1/3$, give $v^{ab}v^{ad} = -1$, whereas the remaining $\hat{v}$ give $v^{ab}v^{ad} = 1$. Hence $J^{ab,ad} = \frac{1}{3}$. Likewise we find $J^{ab,cb} = -J^{ab,ca} = -J^{ab,bd} = \frac{1}{3}$. Thus we can apply the spectral analysis carried out in the previous chapter for the matrix $\mathcal{G}$, to find that the largest eigenvalue of $\beta\mathcal{J}$ is $\Lambda = \beta\frac{S+1}{3}$ and the condition $\Lambda = 1$ implies that

$$\beta_c(\epsilon = 1) = \frac{3}{S+1} \tag{2.12}$$

Hence as S increases the region where the polarized phase is stable becomes larger and larger. In other words it becomes more and more easy for a population of agents who influence each other to become polarized on the same opinion. This is somewhat at odd with naïve expectation, because as S increases the complexity of the choice problem also increases and reaching consensus becomes more difficult. Indeed the probability $P(S)$ to find consensus on S choices in a random population drops very rapidly to zero as S increases. Nevertheless, the effects of interaction toward conformism becomes stronger. We attribute this to the fact that for large S the fraction of allowed spin configurations $\hat{v} \in \mathcal{R}$ is greatly reduced, thus inducing a strong interaction among the different spin components. This results in the fact that ordering becomes easier and easier when S increases. We will see later, in Section 2.4, what effect this ordering has on the probability of finding a transitive majority.

2.3.2 Unconstrained case

Here the constraint $\hat{v}_i \in \mathcal{R}$ is not imposed, while we keep $\hat{\Delta} \in \mathcal{R}$. This means that an agent can be influenced by other agents' preferences to the point of picking an intransitive preference. In this case all the traces over the $\hat{v}_i$ in the above equations can be computed component-wise, independently, as in a multi-component random field Ising model. A direct computation of the matrix $J^{ab,cd}$ is possible, and yields

$$J^{ab,cd} = \delta_{ac}\delta_{bd}\left[1 - \tanh^2(\beta(1-\epsilon)\right]. \tag{2.13}$$

Notice that, for any β and ϵ, the maximum eigenvalue $\Lambda = \beta\epsilon[1-\tanh^2(\beta(1-\epsilon)]$ of the matrix $\beta\epsilon\mathcal{J}$ is independent of S and it coincides with that of the RFIM ($S = 2$). Hence the phase diagram is that of the RFIM for all $S \geq 2$. The different spin components behave independently. The correlation induced by the constraints on the *a-priori* preferences $\Delta - i \in \mathcal{R}$ does not influence the thermodynamics properties. Note that for $\epsilon \to 1$ the condition $\Lambda = 1$ implies $\beta = 1$ and for $\beta \to \infty$ the phase transition takes place at $\epsilon = 2/3$, independent of S.

2.4 $P(S)$ with interacting voters

The main result of the previous section, that is, the fact that ordering becomes easier as S increases when rational voting behavior is assumed for each agent, has interesting effects on the probability of finding a transitive majority. To investigate this, we analyze the probability $P_{\beta,\epsilon}(S)$ of a transitive majority in an interacting population. We will ask if this probability differs much from that of the non-interacting case and, if it differs, in what direction. Of course one is led to think that when a magnetization is present, it will be easier to have a transitive majority, but to make a precise statement and to understand to what extent and in what range of the parameters this is true we will derive an expression for $P_{\beta,\epsilon}(S)$.

The calculation is a generalization of the one presented for the non-interacting population. Let

$$\hat{z} = \frac{1}{\sqrt{N}}\sum_i^N \hat{v}_i.$$

We want to compute, at a fixed ϵ and β, the probability distribution of $\hat{z}$. We shall first compute average quantities keeping fixed the realization of the disorder $\hat{\Delta}_i$ (quenched disorder), and then average over the realizations of the disorder. This is correct procedure to take into account separately the in-sample fluctuations and the differences due to disorder, since averaging at the same time disorder and in-sample quantities (like the v_i) (annealed calculation) leads to wrong results. The point is that thermal fluctuations are governed by a time-scale much faster than that of the disorder, like what happens in the physics of disordered systems.

$$
\begin{aligned}
&P\left(\hat{z}|\{\hat{\Delta}_i\}\right) = \mathcal{N}\mathrm{Tr}_{\hat{v}_i} e^{-\beta\mathcal{H}\hat{v}_i}\delta\left(\hat{z} - \tfrac{1}{\sqrt{N}}\textstyle\sum_i^N \hat{v}_i\right)\\
&= \mathcal{N} e^{\frac{\beta\epsilon}{2}\hat{z}\cdot\hat{z}} \int d\hat{\lambda} e^{i\hat{\lambda}\cdot\hat{z}} \textstyle\prod_{i=1}^N \mathrm{Tr}_{\hat{v}_i} e^{[\beta(1-\epsilon)\hat{\Delta}_i - i\hat{\lambda}/\sqrt{N}]\cdot\hat{v}_i}
\end{aligned}
$$

now the term $\hat{\lambda}/\sqrt{N}$ is small compared to the other one and we can expand it

$$
\begin{aligned}
& \mathrm{Tr}_{\hat{v}_i} e^{[\beta(1-\epsilon)\hat{\Delta}_i - i\hat{\lambda}/\sqrt{N}]\cdot\hat{v}_i} = \\
= \; & \mathrm{Tr}_{\hat{v}_i} e^{\beta(1-\epsilon)\hat{\Delta}_i\cdot\hat{v}_i}\left[1 - \frac{i}{\sqrt{N}}\hat{\lambda}\cdot\hat{v}_i - \frac{1}{2N}(\hat{\lambda}\cdot\hat{v}_i)^2 + \dots\right] \\
= \; & \mathrm{Tr}_{\hat{v}_i} e^{\beta(1-\epsilon)\hat{\Delta}_i\cdot\hat{v}_i}\left[1 - \frac{i}{\sqrt{N}}\hat{\lambda}\cdot\left\langle\hat{v}|\hat{\Delta}_i\right\rangle - \right. \\
- \; & \left. \frac{1}{2N}\sum_{ab,cd}\lambda^{ab}\lambda^{cd}\left\langle\hat{v}^{ab}\hat{v}^{cd}|\hat{\Delta}_i\right\rangle + \dots\right]
\end{aligned}
$$

where, again, averages over the $\hat{v}$ are taken with the distribution (2.11). The factor $Z_i = \mathrm{Tr}_{\hat{v}_i} e^{\beta(1-\epsilon)\hat{\Delta}_i\cdot\hat{v}_i}$ can be absorbed in the normalization constant, so that if we re-exponentiate the terms, we find

$$
\begin{aligned}
&\mathrm{Tr}_{\hat{v}_i} e^{[\beta(1-\epsilon)\hat{\Delta}_i - i\hat{\lambda}/\sqrt{N}]\cdot\hat{v}_i} \cong \\
&\cong Z_i e^{-\frac{i}{\sqrt{N}}\hat{\lambda}\cdot\left\langle\hat{v}|\hat{\Delta}_i\right\rangle - \frac{1}{2N}\sum_{ab,cd}\lambda^{ab}\lambda^{cd}\mathcal{J}^{ab,cd}}
\end{aligned}
$$

This gives

$$
\begin{aligned}
P\left(\hat{z}|\{\hat{\Delta}_i\}\right) &= \mathcal{N}' e^{\frac{\beta\epsilon}{2}\hat{z}\cdot\hat{z}} \int d\hat{\lambda} e^{i\hat{\lambda}\cdot(\hat{z}-\hat{y}) - \frac{1}{2}\hat{\lambda}\cdot\mathcal{J}\cdot\hat{\lambda}} \\
&= \mathcal{N}'' e^{\frac{\beta\epsilon}{2}\hat{z}\cdot\hat{z} - \frac{1}{2}(\hat{z}-\hat{y})\cdot\mathcal{J}^{-1}\cdot(\hat{z}-\hat{y})} \\
&= \mathcal{N}'' e^{-\frac{1}{2}\hat{z}\cdot[\mathcal{J}^{-1} - \beta\epsilon\mathcal{I}]\cdot\hat{z} + \hat{z}\cdot\mathcal{J}^{-1}\cdot\hat{y} - \frac{1}{2}\hat{y}\cdot\mathcal{J}^{-1}\cdot\hat{y}} \\
\text{where} \qquad & \hat{y} = \frac{1}{\sqrt{N}}\sum_i^N\left\langle\hat{v}|\hat{\Delta}_i\right\rangle
\end{aligned}
$$

and $\mathcal{J}$ given by Eq. (2.10). Now one needs to take the average over $P(\hat{y})$. In general this is a Gaussian distribution

$$
P(\hat{y}) \propto e^{-\frac{1}{2}\hat{y}\cdot\mathcal{A}\cdot\hat{y}} \tag{2.14}
$$

and, considering the $\hat{y}$ dependence of the normalization $\mathcal{N}$"

$$\mathcal{N}\text{"} \propto e^{\frac{1}{2}\hat{y}\cdot\mathcal{J}^{-1}\cdot\hat{y}-\frac{1}{2}\hat{y}\cdot\frac{1}{\mathcal{J}-\beta\epsilon\mathcal{J}^2}\cdot\hat{y}}$$

we get

$$P(\hat{z}) \propto e^{-\frac{1}{2}\hat{z}\cdot\mathcal{K}\cdot\hat{z}} \tag{2.15}$$

where

$$\mathcal{K} = \mathcal{J}^{-1} - \beta\epsilon\mathcal{I} - \frac{1}{\mathcal{J}\mathcal{A}\mathcal{J} + \frac{1}{\mathcal{J}^{-1}-\beta\epsilon\mathcal{I}}} \tag{2.16}$$

As before, this probability can be computed to the desired level of accuracy with the Montecarlo method once one has an expression for the matrix $\mathcal{K}$. The derivation is different in the case when we impose a transitivity constraint on the voting profiles v_i or not. Let us see how.

2.4.1 Constrained case

When $\hat{v}_i \in \mathcal{R}$ we have

$$\left\{\mathcal{A}^{-1}\right\}^{ab,cd} = \left\langle \langle v^{ab}|\Delta\rangle\langle v^{cd}|\Delta\rangle \right\rangle_{\Delta}. \tag{2.17}$$

Fig. 2.4 ($\diamond$) shows that the resulting $P_{\beta,\epsilon}(S)$ may exhibit a non-monotonic behavior with S: first it decreases as $P(S)$ and then, as the point (β, ϵ) approaches the phase transition line it starts increasing. If $\epsilon > 2/3$, there is a value S^* beyond which the system enters in the polarized phase and $P_{\beta,\epsilon}(S) = 1\ \forall S \geq S^*$.

2.4.2 Unconstrained case

In this case $\left\langle \hat{v}|\hat{\Delta}_i \right\rangle = t\hat{\Delta}_i$ where we introduce the shorthand $t = \tanh[\beta(1-\epsilon)]$. Then $\mathcal{A} = \mathcal{G}/t^2$ or

$$P(\hat{y}) \propto e^{-\frac{1}{2t^2}\hat{y}\cdot\mathcal{G}\cdot\hat{y}} \tag{2.18}$$

in addition

$$\mathcal{J} = (1-t^2)\mathcal{I} \tag{2.19}$$

hence setting $f = 1 - \beta\epsilon(1-t^2)$

$$\mathcal{K} = \left[\frac{1}{1-t^2} - \beta\epsilon\right]\mathcal{I} - \frac{t^2}{1-t^2}\frac{f}{t^2\mathcal{I} + f(1-t^2)\mathcal{G}}. \tag{2.20}$$

The behavior of the probability can be understood in some interesting limits. For $\beta \to \infty$ we get

$$\mathcal{K} \simeq \mathcal{G} + O(1-t^2)$$

which simply states that as the temperature goes to zero the probability reduces to that of the constrained case, as it should. Note that $\mathcal{K} \to \mathcal{G}$ also as we approach the critical line where $1 - \beta\epsilon(1-t^2) \to 0$.

Instead for $\epsilon \to 0$ we have

$$\mathcal{K} \to \frac{1}{1-t^2}\left[\mathcal{I} - \frac{1}{\mathcal{I} + (t^{-2}-1)\mathcal{G}}\right]$$

The high T limit $\beta \to 0$ reads

$$\mathcal{K} \simeq \mathcal{I} - \beta^2 \mathcal{G}^{-1} + \dots$$

that is, since the matrix $\mathcal{K}$ is diagonal the probability of finding a transitive majority drops to the trivial one, namely $S!2^{-S(S-1)/2}$. So without the constraint of rational voting the probability of a transitive outcome can be greatly reduced. Again, Monte Carlo simulations are shown in Fig.2.4 ($*$). Note the marked decrease of the probability of finding a transitive majority with respect to the constrained and to the non-interacting case.

2.5 Conclusions

In conclusion we have studied the properties of pairwise majority voting in random populations of heterogeneous interacting agents. The heterogeneity is given by the endogenous preferences each voter would give in the absence of interaction. We have shown that the properties of pairwise majority in a random interacting population are related to the properties of a multi-component RFIM, whith a constraint on the components which reflects the transitivity of individual preferences. The model is interesting both for its applications to social since, but also because of the novelty of the constraint we impose on the components of the spin.

This model can be solved exactly and features a ferromagnetic phase where the population reaches a consensus (i.e. a transitive majority) with probability one. The transition can be first or second order, with a tricritical point that generalizes the one found in the standard RFIM with dicotomous field. Thus we can have, depending on the parameters, a continuous transition or an abrupt emergence of consensus.

As to the dependance on the number of voters, we find that the ferromagnetic phase gets larger and larger as S increases, meaning that consensus is reached more easily when the complexity of the problem (i. e. the number of alternatives) is large enough.

We can also compute the probability of having a transitive majority in an interacting population for any value of the interaction strength or of the temperature. With respect to the case when rational voting behavior is not imposed, we note the strikingly different effect that intereaction can have, dependant on how this interaction is introduced. In fact, if we impose a transitive voting behavior, the probability to find a transitive majority is increased, while relaxing this constraint can result in a decrease of this probability.

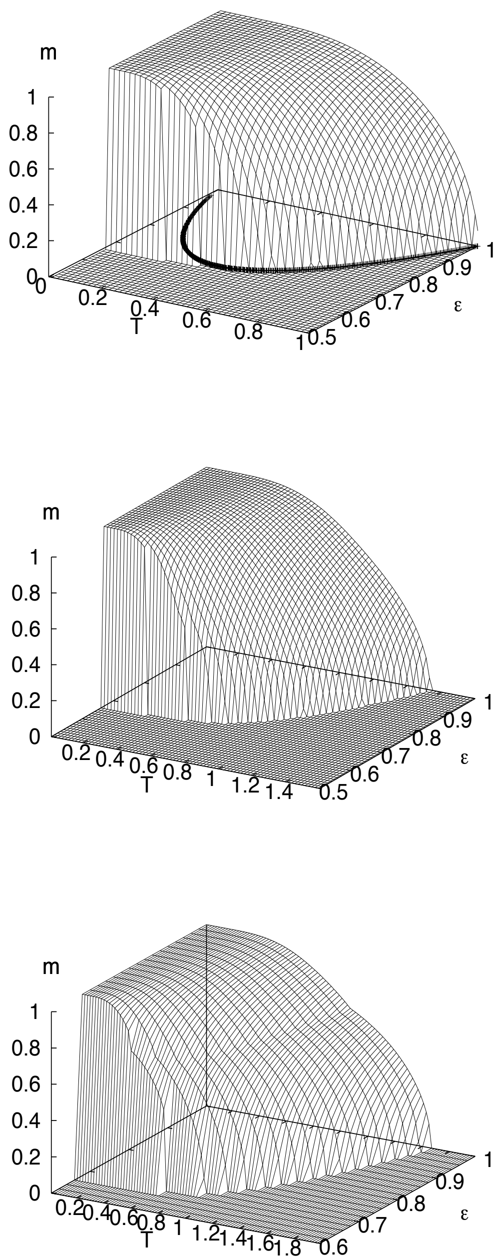

Figure 2.1: Phase diagram for $S = 2$ (top) $S = 3$ (center) and $S = 4$ (bottom). The plot shows the magnetization, while on the $S = 2$ graph we have also drawn the line that marks the instability of the paramagnetic solution. Note the increase of the ferromagnetic region

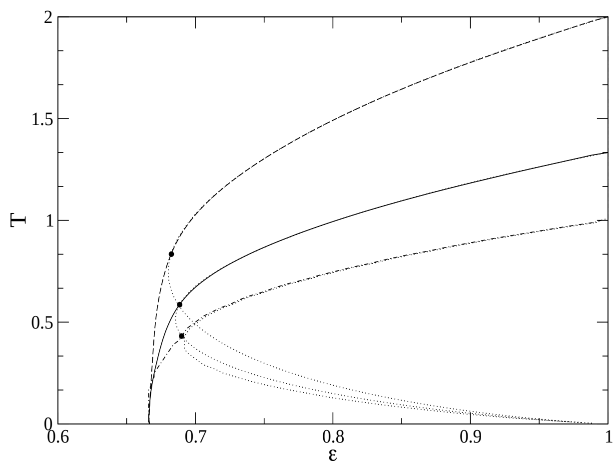

Figure 2.2: Plot of the magnetization for $S = 2,\ 3$ and 5 (dot-dashed, straight and dashed lines). Dotted lines mark the region where the $\hat{m} = 0$ phase is unstable. These meet the lines across which the transition takes place, at the tricritical point (•). Inset: magnetization for $1/\beta = 0.25,\ 0.5, \ldots, 1.75$ and $S = 5$ as a function of ϵ.

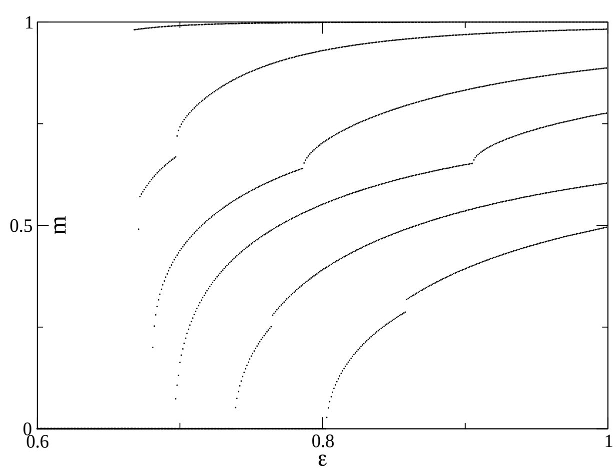

Figure 2.3: Magnetization for $1/\beta = 0.25,\ 0.5, \ldots, 1.75$ and $S = 5$ as a function of ϵ.

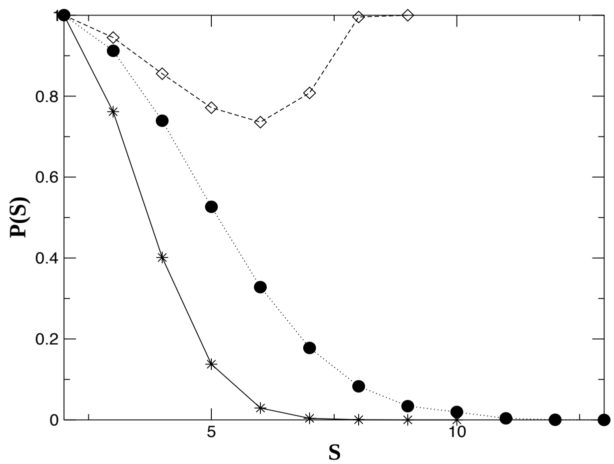

Figure 2.4: Probability $P(S)$ of a transitive majority. (•) shows the case of a non-interacting population, ($\diamond$) shows the case of an interacting population with $\beta = 0.45$ and $\epsilon = 0.8$, $*$ show the same case for the unconstrained case.

Chapter 3

Transitivity in preferences over movies

3.1 Introduction

As we said, one of the problems with social theory is the difficulty of having real-world data to test any theory. This is also the case for voting theory, where it is difficult to find complete rankings of alternatives for a large number of voters. Think for example at presidential elections. There, each voter is simply asked to cast a vote for its preferred alternative (candidate), and the winner will be the candidate receiving the largest number of votes. Indeed, PMR is not used much. Nevertheless, one would like to compare the statistical analysis of the previous Chapters with some real data to answer a couple of questions.

The issue of preference aggregation in real situations is very interesting and we can ask how PMR works in practice. More precisely, can we calculate the probability $P(S)$ of finding transitive majority in a real case? And how does this compare with the theoretical findings of Chapter 1? Indeed, the main criticism to PMR is related to the intransitivity issue, hence being able to quantify the frequency of occurrence of the Condorcet Paradox is an important goal, with a practical application, namely, estimating the ”badness” of PMR's failure.

As to the theory developed in Chapter 2, comparing the theory with real data could shed light on whether conformism plays any role. This issue could be addressed either by the analysis of $P(S)$ or, more directly, by looking for the existence of a magnetization in the preferences.

In this Chapter we will try to answer this questions resorting to data taken from the web. This data describes ratings people give to movies and we think that these ratings constitute a good tool to test the findings of the previous Chapters. A first reason for this is the fact that *a priori* we don't expect any particular ranking of the alternatives (movies) to be largely preferred. That is, we face a situation where in principle all the alternatives are equivalent. This is so, since movies are in general produced in order to have the best possible result in terms of income, and to this scope producers are always looking for niches of possible users to satisfy. This process will result in the production of heterogeneous movies designed to fit the preferences of large classes of persons. Since of people's attitudes are in general very heterogeneous, it can be thought that given a group of movies, it will be possible to find people who rank these movies in any order. That is, there is no reason why there should be any *a priori* ranking of the alternatives.

Moreover, the issue of interaction treated in Chapter 2 can be fruitfully discussed here, since there is certainly a great deal of interaction on movies. That of movies is an often-discussed topic with friends. The first reason for this is that there is a lot of advertisement about movies on television, on the newspapers and so on, and this advertisement brings its subject more into the discussions (it is precisely its goal), and hence increases the interaction. Then, movies in cinemas are rarely seen by one's self. Rather, often people watch movies in groups, and discuss about the movie as soon as they finish watching them. With all this interaction, people's preferences can certainly be shaped by each other, and the ratings that we will analyze are reported *after* the interaction has occurred. Thus these ratings represent preferences affected by the interaction, and not the *a priori* preferences that would have been given without interaction. That is, to use the terminology of the Chapter 2, the ratings give us the $\hat{v}$'s of each voter, and not the Δ's.

In the next Section we will describe the data we analyze and how we treat it, and in the following two Sections we will look first (Section 3.3) at the behavior of $P(S)$ and then (Section 3.4) at the existence of a magnetization, to investigate the presence and type of (imitative) interaction. Our findings suggest the existence of a macroscopic magnetization. More interestingly, this magnetization becomes more and more evident, when we restrict the analysis to the most popular movies. For what stated above, we believe that the more popular a movie is, the more interaction there will be regarding it. Hence the fact that magnetization increases with the popularity allows us to conclude that this magnetization actually emerges as a result of an interaction which could be of the type studied in Chapter 2.

3.2 Data analyzed

The data we analyze was downloaded from the web page of GroupLens Research (http:www.cs.umn.edu/Research/GroupLens/index.html). This consists of ratings given by users to movies, precisely we have 1 million ratings for 3090 movies given by a total of 6040 users. Each user gives an integer rating to movies according to the scale from 1 (the worst), to 5 (the best). This is a large enough dataset to analyze, and to look for example at the intransitivity of these preferences. Moreover, the ratings are given by a very heterogeneous group, with people of both sexes ranging from 18 to 60 years old, and coming from very different professions. This point is very important for allowing us to expect heterogeneous *a priori* rankings or, to use the notation of Chapters 1 and 2, equiprobable Δ's.

First of all, we have to get rid of some problems. This very large dataset contains votes from people, but not everybody has rated all the movies, since there are persons that have rated around 20 movies and persons that have rated more than 2000. Similarly, there are movies that have been rated only by a small fraction of users, and movies that have been rated by a lot of persons. Then, we can have different approaches to these problems that arise when using all the data, that is when working with the *unrestricted set*. The first idea is that of analyzing the 33 most rated movies. There are 415 users that rated at least 29 of these movies, and we refer to this data as the *restricted set*. Within this set, almost all the movies have been rated by everybody.

Then, we have for these movies the ratings of each user i given by $r_i(a)$, and we can construct easily the vote cast by each user, and hence the corresponding matrix $\hat{v}$, setting

$$v_i^{ab} = 1 \Leftrightarrow r_i(a) > r_i(b) \quad (3.1)$$

$$v_i^{ab} = -1 \Leftrightarrow r_i(a) < r_i(b) \quad (3.2)$$

$$v_i^{ab} = 0 \Leftrightarrow r_i(a) = r_i(b) \quad (3.3)$$

Here we will have many draws, both because the ratings take discrete values and because some users do not rate all the movies (hence the number of voters is not always odd). To deal with this additional problem we have two alternatives. The first is to consider only those v_i which are different from zero, which amounts to considering only *strict* preferences. Otherwise, we can complete the preferences by adding to each $r_i(a)$ a real number drawn uniformly in the open interval $[0, 1]$. In this way, if a movie has a higher rating than another, their relative ordering will nor be affected, while we will be able to decide a relative ordering also for movies that are rated equally by a voter.

3.3 Transitivity

The first property that we look at is of course intransitivity. We want to derive results similar to those of Fig.1.1. To do so, we select S movies at random, and then we have for these movies the ratings of each user i given by $r_i(a)$. Then we can construct easily the vote cast by each user, and hence the corresponding matrix $\hat{v}$, as stated in the previous Section.

Now we construct a matrix $\hat{M}$ which contains the properties of the majority of the voters by writing:

$$M^{ab} = \sum_{i=1}^{415} v_i^{ab}. \tag{3.4}$$

The matrix element M^{ab} is positive if the majority prefers alternative a to b, negative viceversa, and zero in case of a draw. We will show the results obtained in the restricted set (415 voters) both for the strict preferences and for the completed preferences.

We analyze the probability of having a transitive majority as the frequency with which transitivity happens, when repeating the above procedure for many different choices of the S movies and for increasing values of S. The results are plotted in Fig.3.1 and can be compared to the random non interacting case. We see that the probability of having a transitive majority is much greater here.

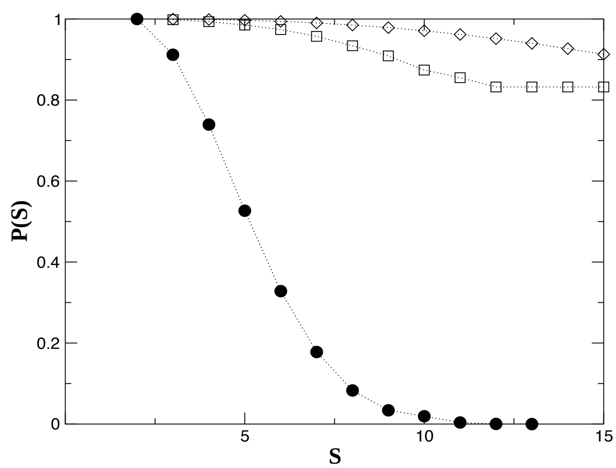

Figure 3.1: Probability $P(S)$ of a transitive majority. ($\bullet$) shows the case of a non-interacting random population, ($\diamond$) shows the probability obtained by sampling S movies, while ($\square$) shows the same probability, but when we add a random (real) term to each voter's (integer) rating of movies, so as to obtain strict preferences.

The randomization procedure introduced in completing the preferences will of course add some intransitivity, but we see from Fig.3.1 that $P(S)$ is still very high.

This results seem to suggest the existence of some order, given the very high values of $P(S)$ both in the "pure" and in the completed case, and we are tempted to say that there is an emerging order. We will directly investigate this issue in the next Section, by directly looking at the magnetization.

3.4 Magnetization

To look for the existence of an order, we analyze the distribution of the magnetization in the realizations studied above. For a given realization the magnetization reads:

$$P(m) = \frac{2}{S(S-1)} \sum_{a>b} \delta(m - M^{ab}) \tag{3.5}$$

and is then averaged on different realizations and choices of the S movies.

The result for the restricted set (33 movies) is plotted in Fig.3.2, and we see the emergence of two small peaks close to $m = 0$ for both cases of completed and strict preferences. The fact that the distribution is broad is explained, in the theoretical framework of the previous chapters, as the temperature being greater than zero. The two peaks suggest the existence of an ordering responsible for the emergence of the macroscopic magnetization, in clos analogy with the model studied in the previous Chapter.

To shed some light on the above finding, and have a more sharp evidence for the presence of ordering, we analyze another quantity of interest, the overlap. This is defined as

$$O_i = \frac{2}{S(S-1)} \sum_{a>b} \mathrm{sign}(M^{ab}) v_i^{ab}, \tag{3.6}$$

and describes how much each voter is aligned with the majority. Indeed, for each voter i, the contribution of each couple of movies a and b to the overlap will be positive only if the corresponding v_i^{ab} is aligned to the majority, M^{ab}. Hence a positive overlap is a signature of ordering. The overlap distribution, given by

$$P(O) = \frac{1}{N} \sum_{i} \delta(O - O_i) \tag{3.7}$$

is plotted in Fig.3.3. In a ferromagnetic state, we expect that a vast fraction of voters will have a positive overlap. As is seen from the graph, 96% of the overlaps are positive, which is a very good indicator that there is indeed an emerging magnetization.

To see this more clearly, we can refine the data in Fig.3.2 by restricting to couple of movies for which we have more than a certain number of votes. If we construct the magnetization in the unrestricted set, each term m^{ab} will be the result of a certain (variable) number of individual votes v_i^{ab} since, as we explained above, non every voter rates every movie. Let the number of votes being cast on each couple of alternatives be n^{ab}. We will look at how the distribution of the magnetization evolves when we consider only the couples for which n^{ab} is above

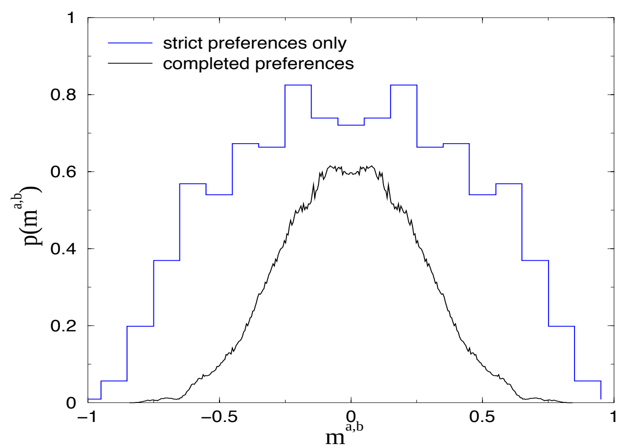

Figure 3.2: Distribution of the magnetization.

a given threshold value n_t, that is

$$P_t(m) = \frac{2}{S(S-1)} \sum_{ab|n^{ab}>n_t} \delta(m - M^{ab}) \tag{3.8}$$

The idea behind this analysis is that the distribution of the magnetization will be sharper if we restrict to the most popular movies for two reasons. First, more voted movies will be less noisy for statistical reasons. Then, one expects that most popular movies are the ones for which there is more information exchange, hence interaction with other people (also induced by advertisements). This turns out to be correct, as seen in Fig,3.4, where we plot the distribution of the magnetization for different thresholds on the number of votes. As we increase the threshold, the two peaks become sharper, reminiscent of what happens in the ferromagnetic state.

3.5 Conclusions

Here we found an example were the theoretical framework of the previous chapter can be applied to a real-world case. That of movies seems to be a particularly suited dataset for many reasons. First, the large number of data points allows us to compare the findings with the analytical results which, strictly speaking, would be valid only in the thermodynamic limit. This is indeed an important part, since not much insight can be gained by analyzing small datasets, and since this problem is one of the most troubling to social scientists wishing to describe real data.

Then, the movie dataset is also a good choice for the particular model we developed in the second chapter since we think that not only there is a certain

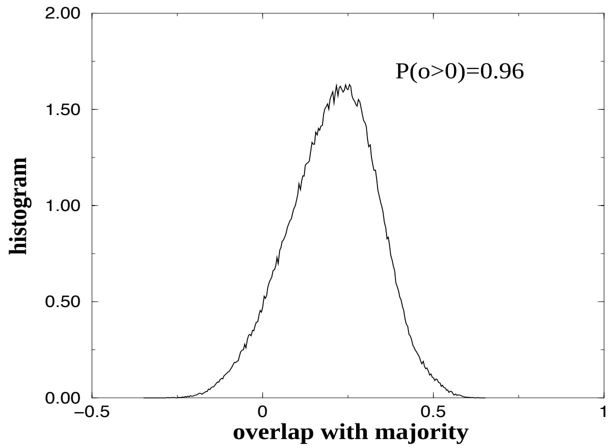

Figure 3.3: Distribution of the overlap.

amount of interaction between people about movies (that of movies is an often debated subject), but most importantly because this interaction can be of the type we analyzed, that is, imitative. This is so because there is a common opinion that typically arises through newspapers reviews, movie trailers, and to the fact that the more a movie earns (i.e. the more it is seen), the more media of any kind are going to talk about it. This generates a positive feedback that can be the origin of the emergence of a macroscopic ordering, and is confirmed by the analysis we carry out restricting to the most popular movies. As we select more and more popular movies, advertisement, discussions and information about these movies gets to play an increasingly important role, and hence the interaction becomes stronger. Indeed the presence of an ordering becomes more evident.

We can thus conclude that this interaction leads to the formation of a common opinion, hence making voters' rankings conform to each other. The emergence of a magnetization, the existence of a positive overlap are consequences of this ordering, and the final result is a marked decrease of the probability of finding Condorcet's cycles, that is, intransitivity.

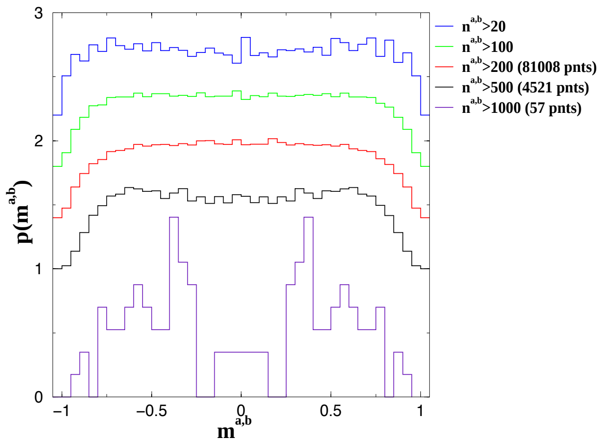

Figure 3.4: Distribution of the magnetization for different thresholds in the number of votes. Curves are shifted for better visualization.

Part II

Correlations in finance

Introduction

The rise of information technology has provided scientists with a wealth of data that is somehow related to people's activities, like for example data on the internet and e-mail usage [46, 47, 48]. Parallel to this, physicists have realized that finance is a field where GBs of information is easily accessible, and this information regards prices of stocks and, ultimately, buy and sell orders. Since this orders are initiated by some subject (be it a person or a society) on the basis of some decision process, it is evident that financial markets constitute a prototype system to study empirically motivations and rationality that guide human behavior.

In his 1900 dissertation[49], Louis Bachelier (1870-1946) anticipated much of what is now a standard in financial theory, from random walks of prices to martingales. The basic observation is about the prices of assets, p_t, on a given day t. Bachelier suggested to look at the daily increments in prices δp , defined as

$$\delta p_t = p_t - p_{t-1} \tag{3.9}$$

and to model them as a random process. He thus derived the probability distribution of the Wiener process, actually before Einstein's 1905 paper, and from that he calculated a formula for the barrier price of an option. His work was unfortunately largely ignored by his contemporaries, and it was only in the late XXth century that these considerations attracted again the attention of scientists. More than seventy years after Bachelier's work, Black and Scholes wrote their famous paper [50] on option pricing, by modeling the price as a Geometric Brownian Motion. This means that the relative increments of the price, or *returns*, are independent random variables, rather than the price itself. From this, more accurate option pricing formulas can be deduced, and since then financial mathematicians and physicists have developed more and more refined models for the price process (see [15] for a review) and now these models extend to a wide class of stochastic processes [51, 52]. This accurate analysis and description has become a must, since the availability of high frequency (tick-by-tick) data made it possible to detect many important and quite striking properties of financial time series. Among these *stylized facts*, the ones that drew much attention where the existence of large fluctuations and the presence of scale-invariant distributions. There exists now a vast literature on the statistical description of financial markets, and we refer to [16, 15] for a review.

Clearly, the detailed statistical analysis and characterization of financial time series is of great practical interest for anybody willing to invest money on the basis of scientific theories. The two objects which are of greatest importance are the expected return R and the volatility σ, defined by:

$$R = E[x] \tag{3.10}$$
$$\sigma^2 = E[x^2] - E[x]^2, \tag{3.11}$$

where $E[.]$ denotes the expected value and

$$x_t = \frac{p_t - p_{t-1}}{p_t} \tag{3.12}$$

defines the return. These two quantities give a measure of, respectively, how much one can gain from investing on a given asset, and of the risk associated with that investment and are thus deeply studied in investment analysis books [53].

The empirical analysis of these and related quantities from financial markets has revealed a wealth of interesting features, in particular fluctuation phenomena of great complexity [15]. For example, the distribution of returns is characterized by anomalous, ”fat” tails, that can be described by an power-law behavior. This means that large variations are far more probable than one would expect from a gaussian model, and this observation has implications in the evaluation of investment’s risks. Another striking phenomenon is that of *volatility clustering*, namely the observation that financial time series are characterized by periods of high fluctuations and periods of low fluctuations, and again this property becomes important for investment decisions. It is thus clear why a wealth of theories have been proposed in order to model the price behavior. Roughly, we can divide the models proposed into two classes: there are those that attempt to give a phenomenological description of the price process, thus trying to reproduce as accurately as possible the statistical features of the empirical time series, and models that rely instead on some behavioral or motivational assumptions about the traders in the market to derive the resulting price process.

Among the first class, discrete time ARCH-GARCH models [54, 55] or continuous time stochastic volatility models [56] constitute an attempt to describe phenomenologically via a system of stochastic equations the price process. These equations have become more and more accurate in describing stylized facts such as volatility clustering or leverage effect [57], and, as mentioned in the previous Chapter, are actually used in practical contexts, for example for option pricing when the underlying asset exhibits non-gaussian behavior [52] as compared to the classical Black-Scholes formula [50].

If these models are oriented to the characterization and possibly to the prediction of the statistical behavior of financial time series, a second class of models has been developed. Here the emphasis is shifted towards understanding the origin of the observed anomalous features through the study of equations that take into account motivational arguments. According to standard economic theories, the price of a stock should be the discounted value of all future dividends. Since the actual dividend process is not known, one will have certain expectations about it, and we can think that the volatility of the prices derives from this uncertainty. Can excess volatility and excess trade volume be explained by this fundamental view? It was found [58], for foreign exchange markets, that roughly 90% of the daily transactions cannot be accounted for in this scheme, and are instead due to speculative trading. If this is true, then one has to model explicitly different trading strategies and their impact on the market price.

This line of research was further stimulated by the development of agent-based models, where the market is depicted as a system of heterogeneous agents that interact via various mechanisms (minority/majority games) [18, 59]. Here the rationale is to try to derive the stylized facts from models that incorporate behavioral rules for the agents. Indeed it has been shown that many stylized facts of financial market fluctuations, such as fat tails in the return distribution or volatility clustering, can be reproduced by such models of interacting traders in a market [13].

The approach that emerges is that of explaining financial fluctuations as an endogenously-generated phenomenon stemming from the interaction of many agents and strategies, rather than an exogenous component, in line with the general arguments that we discussed in the Introduction to this Thesis. In this second Part I will adopt this approach for multi-asset financial markets, first (Chapter 4) by describing the correlations that emerge between different assets and in the final Chapter by developing a model that tries to describe the observed fluctuation phenomena as stemming from a particular type of feedback on the price dynamics induced by investment strategies.

Chapter 4

The structure of correlations in financial markets

4.1 Introduction

Most of the models discussed in the Introduction are focused on the properties of a single asset. From a practical point of view, though, a financial market is composed of many assets, whose dynamics influence each other. Hence, apart from the study of the statistics of prices of single stocks, one can be interested in characterizing those properties of financial markets that derive from the interdependence of the assets it is composed of. The idea to model stock returns as random walks, or, in general, stochastic processes, offers us a first important tool to the scope. Indeed, when one is faced with many noisy time series, a typical quantity of interest is the correlation between them. Hence one can ask for the correlation properties of group of assets.

This information is of great importance also towards determining investment strategies. This is easily understood from a simple example. Suppose we have two stocks, whose returns follow a random walk. Investing $1/2$ of one's wealth on a stock and the rest on a stock that is anticorrelated with the first one is a simple way of trading some risk at the expense of expected return. However, the importance of inter-asset correlations for determining the risk of investments is highlighted also with heuristic arguments. Think of investing all our wealth in a given stock. This will result in a complete failure in case that stock price falls abruptly. Since the abrupt fall is an extreme event, investing in two uncorrelated (or anticorrelated) stocks diminishes the risk of such a dramatic crash. Hence the general rule: diversification reduces risk.

It becomes thus clear that the detailed characterization of stocks' co-movement is essential for planning effective investment strategies. In fact, the very structure of the correlation matrix determines how much risk can be diversified away and how this should be done. The precise mechanism by which this is achieved is explained in *portfolio theory*, whose details are presented in Section 5.2. For the present scope, it is sufficient to note the importance for practical purposes of studying the properties of the correlation matrix.

Many studies have been carried out to this scope, and it turns out that the correlation matrix of a group of assets exhibits properties largely independent from the particular assets under study. The models we will describe in this Chapter have been developed to explain the observed patterns and predict the future correlations among assets. Let x_i be the price of stock i. Then the covariance of stocks i and j is measured by

$$C_{ij} = \langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle , \tag{4.1}$$

where $\langle . \rangle$ denotes temporal average.

It is important to note here that there are two kinds of correlations one can be interested in. There is the historical correlation, which is observed from past financial time series, and the expected future correlations. These two need not be the same, and actually the idea behind the theories discussed here is that of deriving the expected future correlations from the historical ones. To this scope, one has to understand what *factors* enter into the determination of C_{ij}, then devise techniques to extract this information from the observed time series. Finally, we need to use this very information to make predictions on the expected movements of stock prices.

The question that emerges is how these correlations between different assets build up. Are they the result of some endogenous properties of the market? Is the correlation due to the specific activity of the firms? Is there a component that arises due to the price impact of a particular type of investment? As we will see, it turns out that the answer is often yes, and the task one is faced with is that of determining what effect is most important in different situations, and try to quantify all these components.

In the next Section, we review the classical theories developed by economists to explain the observed correlations and predict future ones. Economists have focused on the idea of common *factors* that influence the dynamics of assets' prices, that is, they explain the correlations as a result of some pre-existing common behavior dictated by fundamentals. For example, stocks can be correlated because they belong to the same sector. The models that translate this idea into mathematics are briefly presented. Then, in Section 4.3 we will describe the approach physicists have had with this issue. This approach has been more empirical, and we will review the detailed statistical analysis of the inter-assets correlations and in particular of the spectral properties of the correlation matrix. Moreover, we will see how techniques have been devised, to obtain a ”network structure” from financial data. The idea is that of translating the correlations between assets into links between companies. Those that are mostly correlated are joined by a link, and if carefully done this can reveal interesting structures. Then, we will present a new model that we have developed in order to construct a network structure from financial data based on very general assumptions. Finally, as an example of the various type of information that can contribute to the coefficients C_{ij}, we will show that there is an interesting link between the covariance C_{ij} and the composition of the board of directors of the two companies i and j, at least for the Milan Stock Exchange.

4.2 An economist's view

In addition to the empirical study of correlation matrices, one has to devise some model to predict future inter-asset correlations. Indeed, we have already said that the historical correlations can be used for predicting future ones, but they are not the same. The typical investor will have to follow more than 150 stocks, and it certainly a hard task. Traditional methods, namely having a security analyst predicting the future expected returns, will not in general be useful towards the determination of the correlation structure. This is so due both to the very high number of correlation coefficients that are to be determined, and for the fact that security analysts are typically specialized in a particular sector, say steel or oil, and will encounter difficulties in estimating the co-movement of stocks belonging to different sectors. Hence a simple testable model is needed in order to make investment decisions based on some sound analysis.

An empirical model that accounts for this idea is perhaps the oldest and most widely used simplification scheme to describe assets' returns: the single index model[53]. This model stems from the casual observation that when the market goes up, most stocks go up, and viceversa, thus suggesting the common response to market changes as a major reason fo stocks' co-movement. One can write the return on any given stock i as:

$$x_i = a_i + \beta_i R_m \tag{4.2}$$

where a_i is the component of the return independent of the market, R_m is the market return and β_i is a stock-specific constant that measures the expected change in x_i given R_m. Given this model, one can observe the price changes over a certain period, the fit the model to tha data to obtain the parameters and the distributions of the random variables a_i and R_m, and then use this very information to make investment decisions. There is a wealth of techniques devised to get better and more accurate estimates of the parameters, but it would lead us far from our point to discuss them. In Appendix B we briefly describe how this model is used in portfolio analysis.

A simple generalization of this model, the multi-index model. It is clear that, apart from market return, there are othe factors that influence the co-movement of assets. For example, there will be a positive correlation between two stocks belonging to the same industrial sector, as they will both go up when the sector is in a positive trend. Hence more refined models have been devised in order to take into account many different factors. The general multi-index model can be written as:

$$R_i = a_i + \sum_{j=1}^{N} b_{ij} I_j \tag{4.3}$$

where again a_i is independent of all the indices, while b_{ij} measures the expected change in R_i relative to the value of index I_j. The single-index model is a simple case of this, with $N = 1$ and $I_1 = R_m$. Does one gain much from this added modeling depth? To answer this question one has to make comparisons based on the performance of these models. That is, one model is better than another if, when used to predict future correlation matrices, it gives better (more accureate) results. Elton and Gruber [60] found that while multi-index model are generally better in reproducing the historical correlation matrix, they are significantly outperformed by the simple one factor model when it comes to prediction, if one uses as indices the industry classification. This means, as they clearly state, that the added indices ”introduce more random noise than real information in the forecasting process”. However, these models can indeed outperform the simple single-index model, but one has to be very careful with how assets are grouped and what indices to include.

4.3 A physicist’s view

Physicists have had perhaps a more empirical approach to this problem. Instead of devising models for inter-asset correlations, the first step has been the detailed statistical characterization of correlation matrices.

Let us suppose that a group of N assets undergoes a random process, such that the returns on each asset are drawn from independent normal distributions. In this case, the correlation matrix spectrum can be calculated analytically, and it turns out that the distribution of the eigenvalues is [61]

$$\rho(\lambda) = \frac{q}{2\pi\sigma^2}\frac{\sqrt{(\lambda_{max} - \lambda)(\lambda - \lambda_{min})}}{\lambda} \tag{4.4}$$

$$\lambda_{min}^{max} = \sigma^2(1 + 1/q \pm 2\sqrt{1/q}) \tag{4.5}$$

where $q = N/T$ and T is the number of observations for each data set, while σ^2 is the variance of each dataset. Notice that this distribution has a lower and an upper bound (λ_{min} and λ_{max}, respectively), beyond which the probability of finding an eigenvalue is zero (in the $N \rightarrow \infty$ limit). The authors of Ref.[62] have shown that there is a remarkabe agreement between the predictions of RMT and empirical data extracted from 406 S&P stocks, with 94% of the total number of eigenvalues falling in the bulk of the RMT region, as shown in Fig. 4.1 . Similar results have been found for the Tokyo Stock Exchange [63].

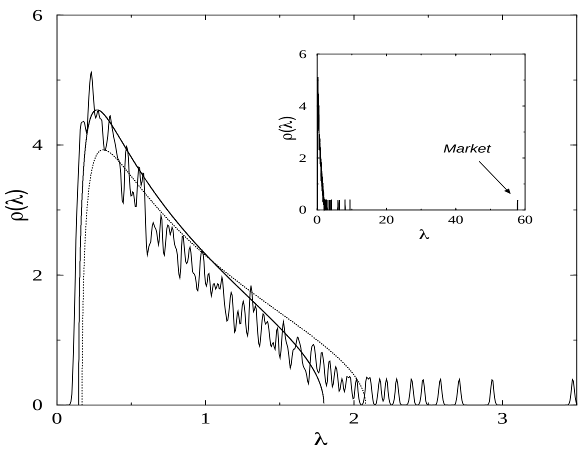

Figure 4.1: Smoothed density of eigenvalues of C. Taken from [62], courtesy M. Potters.

One might be tempted to conclude that no useful information can be extracted from the correlation matrices, but the situation is not so bad. Actually it has been shown [64, 65] that in the classical portfolio problem noise hasn't the relevant role one might expect (the displacement due to noise is relatively small, 5% to 15% according to [64])[1]. More importantly, the eigenvectors corresponding to the highest eigenvalues have been shown to carry much information. In particular, it has been shown [66], that these eigenvectors are centered around well-identified industrial sectors, hence somehow enforcing the picture of multi-factor models: each eigenvector corresponds to the collective motion of a group of assets belonging to the same sectors. This information can actually be used for practical purposes. For example, filtering out the eigenvectors corresponding to eigenvalues in the bulk, and retaining the large deviating ones, one can have an estimate for the correlation matrix $\hat{C}$ that is much more useful since, having removed the noisy component, it contains only effective information. Indeed the use of filtered covariance matrices

[1]Note however that the situation can be dramatically different when introducing non-linear constraints, see [64]

has become a must, while this kind of questions have attracted much attention, and many different filtering techniques have been devised to get more accurate estimates, see [67, 68].

If correlation matrices contain useful information, it becomes important to study the stability of their properties in time. The time dependence of the large eigenvalues has also been studied in some detail [69], and it was found that sudden decreases of the stock prices are related to an increase in the largest eigenvalue (that is, they are more collective), while this is not the case when prices rise. The temporal evolution of correlations was studied also in Ref. [70], where the authors found that there exist a directed network of influence between companies, with the time dependent cross correlations showing a peak (maximum correlation) at nonzero time shift. We will also study issues related to the time evolution of the correlation matrix in the next Chapter.

4.3.1 The network structure of financial correlations

Another interesting approach physicists have developed is that of visualizing the correlations among assets by constructing networks out of financial data. In these networks, companies would be joined by a link when they are highly correlated. One such approach is that of the Minimum Spanning Tree[71, 72, 73]. A spanning tree is a graph without loops connecting all the N nodes with $N-1$ links. To construct it, one calculates the matrix $\hat{C}$, then introduces a metric distance between pairs of stocks by defining:

$$d_{ij} = \sqrt{2(1 - C_{ij})}. \tag{4.6}$$

Then the MST selects the shortest $N-1$ links spanning all the nodes.

An example is shown in Fig.4.2. Some important features can be extracted also in this way, the most emergent being again the sector classification.

Two basic extension of this framework have been devised. The first one is the study of how the emerging tree structure evolves in time [74]. The topological properties of the tree have been related to market crashes. Another approach is that of using the same technique to construct trees of world markets, of volatility, and so on[72].

Clustering the assets is useful not only in visualization of correlations, but also to determine the optimal portfolio. In particular, the authors of Ref. [75] have shown that the use of clustering algorithms can improve the ratio between expected and realized risk of the portfolio.

Although all these techniques work quite well, in the sense that they give reasonable and interesting results, the procedure can be questionable. In fact, there is no reason why one should construct a network of assets as a tree. Forcing a

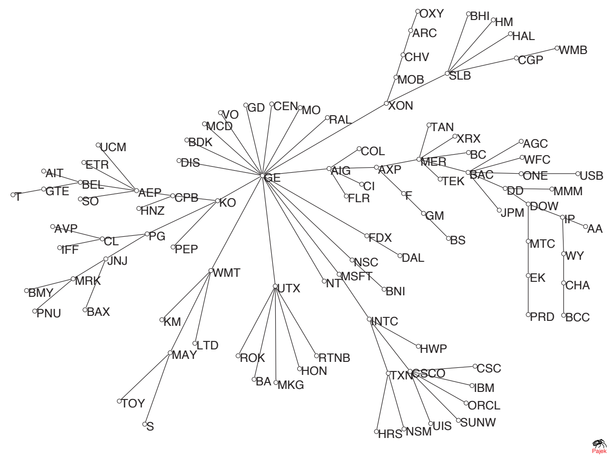

Figure 4.2: MST of 100 highly capitalized stocks traded in the US market Taken from [72], courtesy G. Caldarelli.

particular network topology can be misleading, and most of all, any loop structure is missed by construction. In the next section we introduce a technique to reconstruct a network structure out of financial data that does not rest upon any *a priori* topological assumption.

4.4 Gaussian model

In what we have seen, the correlations are identified with interactions among companies. Pairs of assets that are most highly correlated are joined by a link, to represent their interaction. However, there is a more subtle link between the two properties, since it is the interaction, present *a priori*, that induces correlations. This is the starting point of our analysis in this Section.

Let us assume that the observed time series of the system under study are the result of some stochastic process evolving on an underlying network structure, not known *a priori*. Then the existing network structure will be responsible for the interaction, and hence of the correlations that we will find. In particular, we will assume a gaussian model of harmonic oscillators interacting with each other. The (unknown) adjacency matrix A_{ij} specifies which node interacts with which, being $A_{ij} = 1$ if nodes i and j are connected, 0 otherwise. The equation of motion of this network of harmonic oscillators is

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_i = -J\sum A_{ij}(x_i - x_j) \tag{4.7}$$

where m is the mass of each oscillators and J is the coupling strength. The Hamiltonian for such a system is

$$H = \frac{1}{2m}\sum_i p_i^2 + \frac{1}{2}m\omega_0^2 \sum_{ij} x_i L_{ij} x_j, \tag{4.8}$$

where $\hat{L}$ is the Laplacian matrix of the graph, given by

$$L_{ij} = \delta_{ij} s_i - A_{ij} \tag{4.9}$$

and

$$s_i = \sum_j A_{ij}. \tag{4.10}$$

In (4.8) $\omega_0^2 = J/m$. If we consider the network of oscillators in thermal equilibrium with a heat bath at temperature T, we can calculate averages with the Boltzmann weight, and the correlation matrix C will be the inverse of the laplacian matrix of the graph, $\hat{L}$:

$$C_{ij} = \langle x_i x_j \rangle = (L^{-1})_{ij}. \tag{4.11}$$

These considerations have been developed in the context of generalizing Peierls result [76] for the thermodynamic instability of low-dimensional crystalline structures to structurally disordered materials [77].

Here we borrow this idea to develop a procedure aimed at reconstructing the network topology (that is, obtaining values for the adjacency matrix A_{ij}) by exploiting the relation (4.11) between the expected observed fluctuations and the Laplacian $\hat{L}$. The first problem we encounter is that in (4.11) we have that $\hat{L} = \hat{C}^{-1}$, hence in reconstructing the laplacian $\hat{L}$ the importance of each eigenvalue of $\hat{C}$ is inversely proportional to the eigenvalue itself. This means that the high eigenvalues, which we have seen carry a lot of information, will have a small weight as compared to the noisy small eigenvalues. Then, we proceed as follows. We first construct the correlation matrix $\hat{C}$ from the empirical time series x_i and then diagonalize it to obtain the eigenvalues λ and eigenvectors $|\lambda\rangle$. The resulting eigenvalue spectrum will in general have a lower part described by Wigner's semicircle law, with the highest eigenvalues deviating from it. These eigenvalues, and the corresponding eigenvectors, are the ones that carry the relevant information, as we have seen in the previous section. We can define a threshold value λ_{th} that defines the boundary between the "RMT" region and the informative part of the spectrum. This can be done in several ways. One possibility is that of defining $\lambda_{th} = \lambda_{max}$ with λ_{max} defined by Eq. (4.4), so that any eigenvalue above the threshold would be above the high cutoff of RMT spectra. We will follow a more phenomenological approach, and define λ_{th} graphically, as will be clear in the following. So, we construct a matrix $\tilde{C}$ by keeping the eigenvalues in the deviating tail and replacing all the eigenvalues in the semi-circle part with their mean value, so as to keep the trace constant, $\mathrm{Tr}C = \mathrm{Tr}\tilde{C}$. Hence by definition the eigenvalues $\tilde{\lambda}$ of $\tilde{C}$ are

$$\tilde{\lambda} = \lambda \Leftrightarrow \lambda \geq \lambda_{th} \tag{4.12}$$
$$\tilde{\lambda} = l \Leftrightarrow \lambda \leq \lambda_{th} \tag{4.13}$$

and

$$l = \langle \lambda \rangle_{\lambda \leq \lambda_{th}}, \tag{4.14}$$

while

$$|\lambda\rangle = |\tilde{\lambda}\rangle \tag{4.15}$$

We state now that

$$\hat{L} = \sum_i \frac{1}{\tilde{\lambda}} |\tilde{\lambda}\rangle\langle\tilde{\lambda}| \tag{4.16}$$

is the laplacian of the graph. Again, this matrix will have a central part (of mean zero) given by the random part of C, and thus bearing no information. But there will be a negative tail corresponding to the important part of the correlations. We will thus fix a threshold value L_{th} and construct the adjacency matrix according to

$$A_{ij} = 1 \Leftrightarrow -L_{ij} > L_{th} \tag{4.17}$$
$$A_{ij} = 0 \, \mathrm{otherwise.} \tag{4.18}$$

In this context, the maximum eigenvalue λ_{max} simply corresponds to motion the center of mass. Hence we can also exclude this eigenvalue since it carries no relevant information about the structure of the correlation matrix. This observation is important in particular for small systems, where the number of non-trivial eigenvalues is small, since then adding one large eigenvalue that tends to link all the companies can easily disrupt the structure. This is the case in the example studied in the next Section.

4.4.1 The Dow Jones companies network

Let us see in a real case how this works. We take as our time serie the returns for the stocks that compose the Dow Jones Index starting from March 1991to March 1997. The resulting correlation matrix has a spectrum that is shown in Fig. 4.3

In this case we see clearly some eigenvalues deviating from the semi-circle law. The largest one, however, simply corresponds to the collective motion of the assets. It is known as the *market mode* and, although it is interesting for other purposes, it should not reveal information about the network structure, so

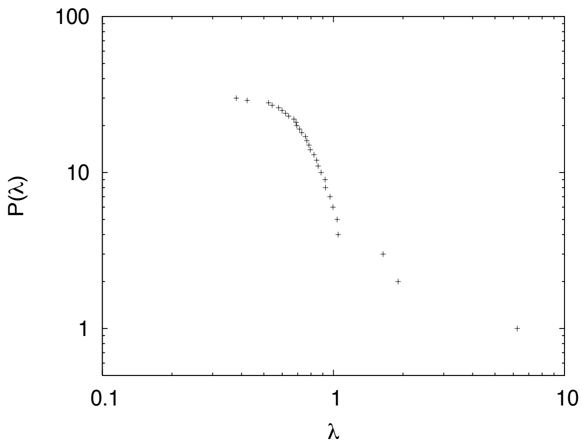

Figure 4.3: Integral distribution of the eigenvalues of the correlation matrix for Dow Jones Companies.

we decide to exclude also this eigenvalue. By inverting the resulting $\tilde{C}$ we obtain a proxy for the laplacian of the graph and, by retaining only the non-random part of it, we reconstruct the network structure, which is plotted in Fig. 4.4

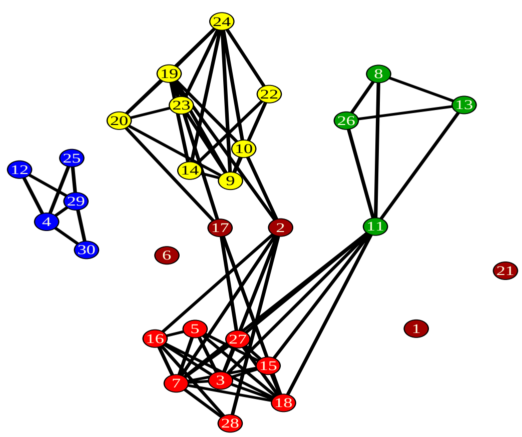

Figure 4.4: Graphical representation of the network obtained with our procedure for assets comprised in the Dow Jones Index.

To validate our procedure, we run a gaussian model on this network. We simulate a system of coupled oscillators via the set of Langevin equations

$$\dot{x}_i = \sum_j L_{ij} x_j + \eta_i, \tag{4.19}$$

which is the equation resulting from the Hamiltonian (4.8) where η_i is a Wigner process such that

$$\langle \eta \rangle = 0 \tag{4.20}$$

$$\sigma_\eta = \sqrt{\langle \eta^2 \rangle - \langle \eta \rangle^2} = \alpha. \tag{4.21}$$

We then construct from these artificial time series x_i the covariance matrix $C_{ij} =< x_i x_j > - < x_i >< x_j >$. Then on this ”artificial” covariance matrix we repeat the extraction of the network to test if the results are consistent with the starting network structure. If the procedure is correct, we should get a good agreement between the two networks obtained, the one extracted from real data and the artificial one. For adequate values of the parameters the similarity is striking, as is clear from Fig. 4.5, upper graph, with a 10000 time-steps run, and even in a smaller simulation, Fig. 4.5, lower graph. In these cases the maximum eigenvalue was retained in the analysis (i.e. it was not included in mean of the ”noisy” eigenvalues), since we can subtract the collective motion directly when simulating the Langevin Equations.

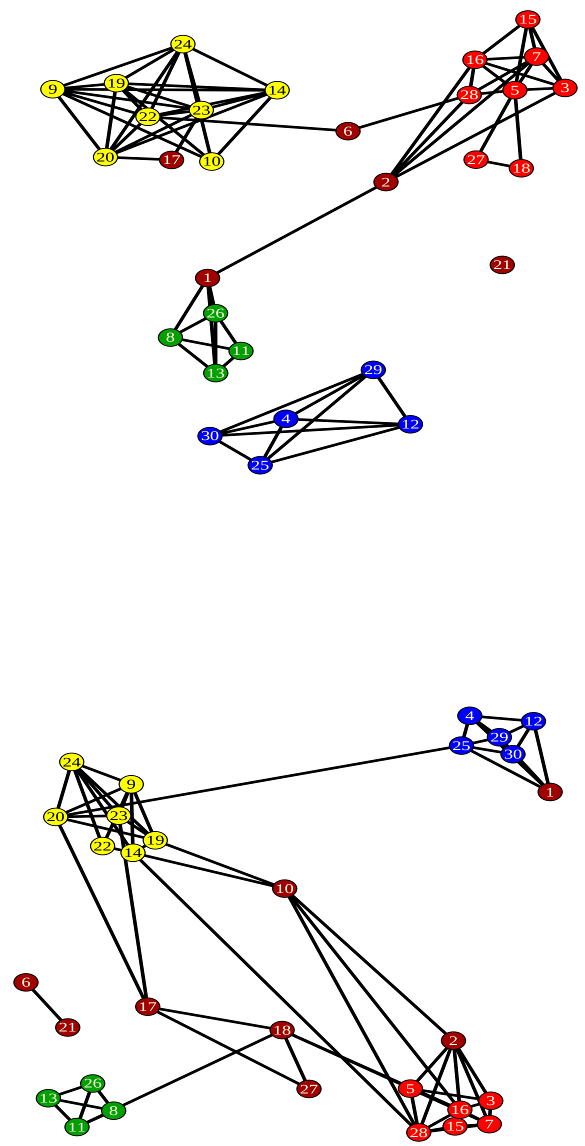

Figure 4.5: Upper graph: network obtained by running a gaussian model on the network of Fig. 4.4 for 10000 time steps, with $\alpha = 0.001$. Bottom graph: network obtained by running a gaussina model on the network of Fig. 4.4 for 3000 time steps, with $\alpha = 0.001$.

4.4.2 NYSE

Here we use dayly data from 4000 NYSE stocks. The tail is stronger in this case, and including the largest eigenvalue is no longer a big problem.

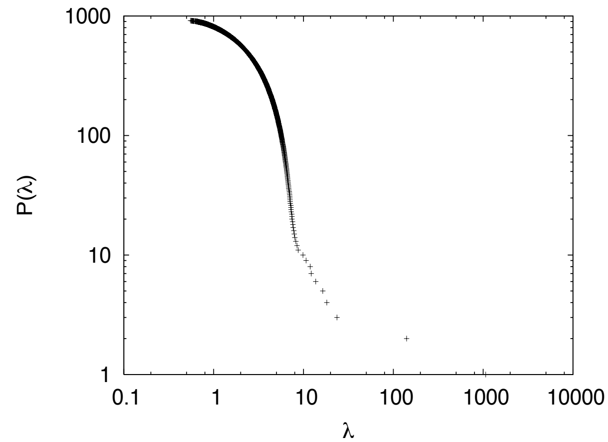

Figure 4.6: Eigenvalue distribution for 4000 NYSE assets

The network obtained with our filtering procedure displays a wealth of interesting features. The first thing to note is that we get a very neat community structure, corresponding to the sectors. For example, the light cyan cluster on the bottom-right is composed solely of oil companies, while the yellow one just above it corresponds to the electric sector. On the left of this lies the pharmaceutic industry. The big cluster on the top-left is composed mainly of banks (green) and investment companies (cyan). On the bottom lies the sector of gold (golden colour). The company that is linked both to this sector and to the bank sector is De Beers. Now we start to note that we also get information on each individual assets' connections, and these seem to be quite precise. For example, a striking fact is that the two big companies heavily linked to both the pharmaceutic industry and financial sector (they seem to have many common interests) are Glaxo Wellcome and Smith-Kline. The data analysed is from the 90's, and indeed the two companies merged in 2000. Also, we can see Halliburton's link to the oil sector companies, not surprisingly since it is an oil-services company.

Thus we have developed a technique to construct networks from financial data. This technique can be very helpful in visualizing the structure of financial correlations, and its results agree with the finding that highest eigenvalues contain information about the sectors. But there are two advantages with our procedure. First, with respect to other visualization procedures aimed at constructing networks, we base our procedure upon a very simple model of correlations. We make no *a priori* assumption on the topology of the network, and we think that the resulting

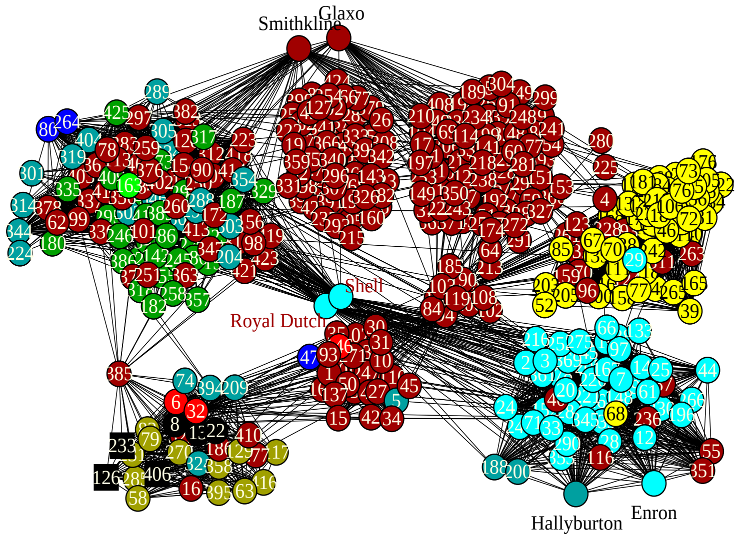

Figure 4.7: Network obtained with our procedure with 4000 NYSE stocks.

structure will be more stable, since it accounts by construction for the presence of noise [2]. Moreover, taking into account all the possible links we highlight many properties that are more detailed than the mere belonging to a sector. We can have information about a single company and its specific links, and this gives us important clues that could be exploited when taking investment decisions. For example, in the NYSE network we have seen two such cases: Halliburton, a services company linked almost exclusively to the oil sector, and Glaxo and Smith-Kline. Having to invest, one could certainly exclude simultaneous investments on these two companies, or similarly, exclude investing heavily both on the oil sector and on Halliburton. Of course this could be detected also via standard portfolio selection procedures but though preliminary, these results suggest the possibility of exploiting these information systematically. The development of this idea is left as a subject for future research.

4.5 Board of directors

Among the various informations that one can extract form financial time series, one of interest is that concerning the board of directors of companies. Each company has a certain number of people in its board of directors, and one might think that people involved in the board of a company are experts of the type of market that that company belongs to. Since a given person can belong to boards of more than one company, one is led to think that companies sharing a common market will have a higher probability of having one or more people in common in the

[2] It is not clear, for example, how noise affects the overall structure of minimum spanning trees.

board of directors. Then it is straightforward to ask if the financial correlations reflect, at least in part, this information.

To test this hipothesys, we used data from companies traded at the Italian exchange in Milan. The data available on the board of directors is from 2002, then, to compare with financial data, we collected daily time series from Jan. 1^{st}, 2001 to Dec. 31^{st}, 2003. Of all the companies, we selected only those that were traded for at least 300 days in this period. Moreover, we had to disregard some companies for which we could not find historical quotes, like companies that merged with others or died due to bankruptcy. Of the remaining 69 companies we constructed the correlation matrix $\hat{C}$. We then divided the matrix elements C_{ij} into two groups: $\mathcal{C}_0$ and $\mathcal{C}_1$. To do this, we calculate the matrix $\hat{B}$ according to $B_{ij} = 1$ if companies i and j share at least one person in their boards of directors, 0 otherwise. Then

$$C_{ij} \in \mathcal{C}_1 \Leftrightarrow B_{ij} = 1 \tag{4.22}$$

$$C_{ij} \in \mathcal{C}_0 \Leftrightarrow B_{ij} = 0 \tag{4.23}$$

We can the run a statistical test to check if the two groups can come from the same distribution. Using the Kolmogorov-Smirnov test [78, 79], we found that the probability that the two data sets are drawn from the same distribution is 7.2510^{-4}. The conclusion is clearly that financial correlations bear a trace of this information.

This conclusion can be seen also by Fig.4.8. This is derived by merging together data in $\mathcal{C}_1$ with data in $\mathcal{C}_0$. Then we rank all the correlation coefficients from higher to lowest. Dots in Fig. 4.8 indicate that at that rank there is a coefficient from the group $\mathcal{C}_1$, that is, a covariance between two companies that share at least one member of the board. From the resulting histogram one can see how data in $\mathcal{C}_1$ is grouped at higher values with respect to data in $\mathcal{C}_0$. Indeed, 15 correlation coefficients in $\mathcal{C}_1$ rank amongst the highest 10% of the whole group of coefficients. We note that there is also a peak at low values of the rank, but the Figure confirms the statistical test in neglecting the possibility that the two groups are drawn from the same distribution.

4.6 Conclusions

In this Chapter we briefly reviewed the tools developed by economists and physicists to determine and predict the structure of financial correlations. We have seen both an attempt to model correlations in order to have good forecasts of the coefficients that enter the portfolio optimization problem, and a detailed characterization of the statistical properties of such correlations. These techniques turn

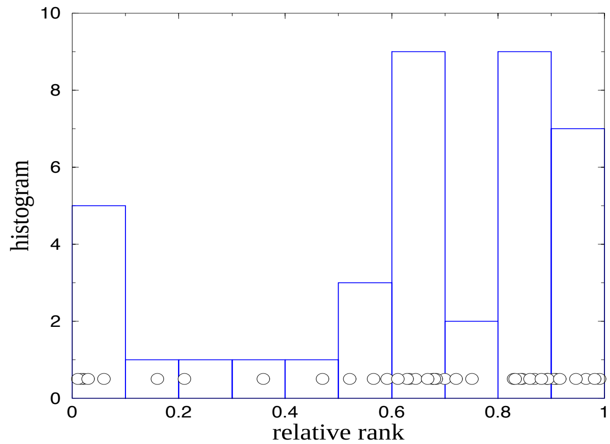

Figure 4.8: Histogram of the rank distribution of the correlation indices for couples of companies that share at least a director in the board. The empty dots represent one such correlation coefficient, while the rank is with respect to the whole group of coefficients.

out to be very useful in having precise estimates of risk of investments, as well as being of theoretical interest. We have also shown the existing methods used for the topological analysis of financial markets and presented a new procedure to cast financial data into a network form that can be used as a graphical aid to visualize the existing correlations. This procedure reveals microscopic properties, such as the detailed links of each company, that are hidden to the macroscopic observation of the spectrum of the correlation matrix. We hope to be able to extract from this type of information new tools for portfolio optimization, but research is still ongoing in this area.

Chapter 5

Dynamics of financial correlations

5.1 Introduction

The statistical properties of ensembles of assets have been shown in the previous Chapter to display very interesting features, for example the existence in the covariance matrix of large, stable eigenvalues carrying information about the market dynamics. Also these features can be thought of as deriving from internal market dynamics since certainly the *bare* economic factors are not the only contribution to financial correlations. Moreover, these large eigenvalues exhibit themselves a temporal pattern of fluctuation phenomena of great interest. One characteristic is the presence of instabilities, or peaks of correlations, evident when one looks at the evolution of the largest eigenvalue of the covariance matrix, as clearly shown in Fig. 5.1, where we can see abrupt peaks emerging often. These peaks then decay with a characteristic time-scale, but we can see that the height of the peaks seems to some extent to be scale-free, with similar patterns being developed for small as well as high jumps. That is, it seems that the dynamics responsible for the high jumps that we observe in Fig. 5.1 is responsible also for the many small peaks that are present. This is what happens in many phenomena in statistical physics, where the origin of this scale-free behavior lies in the collective nature of the processes under study. This intuition is indeed confirmed by the direct investigation of the dynamics of the largest eigenvalue (Λ) of the covariance matrix. The broad distribution of the day-to-day differences $\delta\Lambda = \Lambda(t) - \Lambda(t-1)$ is shown in Fig. 5.2 for group of companies belonging to 4 different world indices, and all show a similar ”fat tailed” behavior.

It is difficult to explain these features as an effect of exogenous shocks to the system and we will try to resort to an internal mechanism to generate this dynamics. What can this mechanism be? Among the other contributions to trading activity one can consider those of speculators and investors. An investment

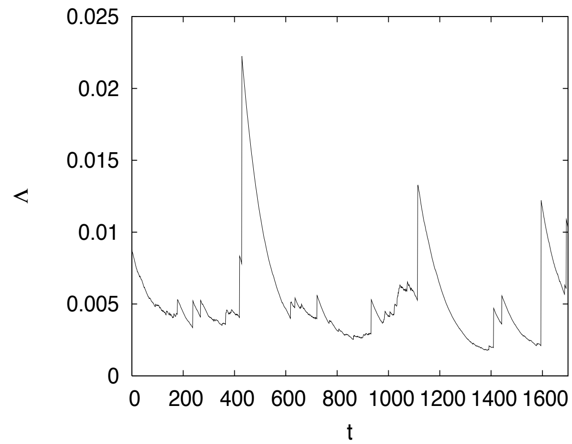

Figure 5.1: Maximum eigenvalue of the covariance matrix as a function of time for Toronto Stock exchange (see [80] for details on the data). The covariance matrix for each time t is obtained via $C_{ij} = \langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle$, where $\langle . \rangle$ denotes an exponentially weighted average with $\tau = 50$.

component which has a particular significance for financial correlations is that described by portfolio theory, as explained in the previous Chapter. Large traders such as institutions or banks certainly adopt such theories when taking positions in a market, and these subjects can move a great deal of assets. In the 80's and 90's, another large player emerged in the market: *hedge funds*. These are funds that try to take advantage of various investment strategies, and they have indeed obtained great success. The assets under management by hedge funds grew from a value of 193 millions U.S. dollars in 1980 to the considerable level of $109,576$ millions of U.S. dollars in 1997 [81]. These investment funds are often quick to react to news, given their light structure with respect to financial institutions. Moreover, their managers have among investors a reputation for astuteness, and their moves are often followed by others. All this can lead to thinking of a possible positive feedback. That is, the rumor that hedge funds are taking a position may encourage other investors to react, and if this effect is large enough, it may well have an effect on the price. It is thus evident that prices will tend to move along the direction determined by investment theories, and that fluctuation phenomena in the dynamics of prices or of correlations could be due partly due to the feedback effect of strategic investment.

Our idea is then to take into account this component (financial investment) as an internal influence on the market dynamics and derive the resulting effect on the correlations and thus, ultimately, on risk. In particular, we will specialize to investments done on the basis of standard portfolio optimization theories, that

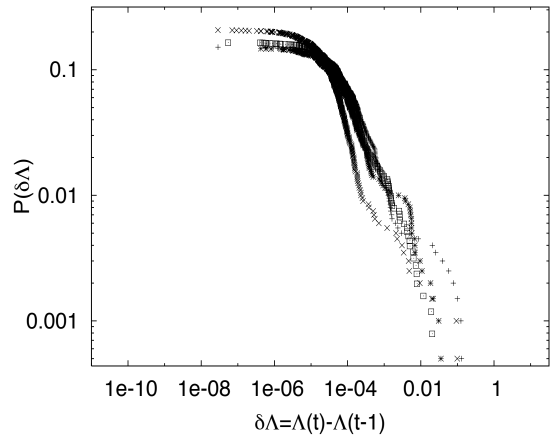

Figure 5.2: Cumulative distribution of the day-to day change in the maximum eigenvalue for different indices: DAX (+), TSX($\star$), DOW($\Box$), ASX($\times$).

allow to decrease the risk of investments through diversification. One of the key functions of financial markets is indeed that of allowing companies to “trade” their risk for return, by spreading it across financial investors. Investors on their side, diversify (i.e. spread) their strategies across stocks so as to minimize risk, as postulated by portfolio optimization theory [82, 53]. The ability of a financial market to absorb risk depends (see for example Section 5.2 and Appendix B) on the cross correlation between the assets it is composed of. On the other hand, it is also reasonable to expect that trading of the optimal portfolio induces correlations in the financial returns, i.e. that it affects the ability of the market to absorb risk. This suggests that financial correlations enter into a closed feedback because they determine in part those trading strategies which contribute to the price dynamics, i.e. to the financial correlations themselves. This feedback loop is indeed implicit in Capital Asset Pricing Model (CAPM) [83] which assumes that all traders will invest in the same way (according to the optimal portfolio) . Thus, the correlations among assets can be described by a single *factor*. This intuition is confirmed by the fact that the largest eigenvalue of empirical correlation matrices is very well separated from all the other ones [69].

The issue of the feedback loop discussed above has been recently addressed also in the economic literature [84, 85], with a detailed characterization of investor’s expectations about future prices. From this expectations the trading decisions are derived, and then the price is changed via a market clearing mechanism (offer = demand). In the following, we will not deal with so detailed models, motivated by the idea that the laws that govern the collective properties we are interested in are not very sensitive to the small-scale details, given their statistical origin.

The purpose of this research is to show, by resorting to a simple model, that a non-trivial dynamics of correlations can indeed result from the internal dynamics of a financial market were agents follow optimal portfolio strategies. This, however, occurs only close to a critical point where a dynamic instability occurs. Not only we find very realistic dynamics of correlations close to the critical point, but maximum likelyhood parameter estimation from real data suggests that markets are indeed close to the instability.

In the next Section we will review the standard assumptions and results of portfolio theory, and we will use this results as a source of feedback for the stock prices in the model presented in Section 5.3. We will then give an equilibrium solution of the model and characterize the resulting emergence of a ”phase transition” from a stable to an unstable phase. We will show how the dynamics of correlations in real markets closely resembles that of the model close to the phase transition. Then in Section 5.4 we will devise a technique to fit real data from financial markets with our model, and we will find that indeed financial markets seem to lie very close to the transition.

5.2 Portfolio theory

Let us start with a simple example. Let us suppose that we have two assets and let us denote their daily returns by x_1 and x_2, the respective variances as σ_1^2 and σ_2^2 and their covariance as σ_{12}. Then we want to calculate the expected return and the risk (variance) associated to the portfolio obtained by investing a fraction $z_1 = 1/2$ of one’s wealth into the first stock, and $z_2 = 1/2$ on the other. If the expected returns are R_1 and R_2, then expected value of the return R_p on the portfolio will be

$$E[R_p] = \bar{R}_p = \frac{1}{2}R_1 + \frac{1}{2}R_2 \tag{5.1}$$

since the return on a portfolio of assets is simply a weighted average of the return on the individual assets, where the weight for each asset is just the fraction of the total wealth invested on that asset. As to the variance, one has:

$$\begin{aligned} \sigma_p^2 &= E[(R_p - \bar{R}_p)^2] = E[(z_1 x_1 + z_2 x_2 - (z_1 R_1 + z_2 R_2))^2] = \\ &= z_1^2\sigma_1^2 + z_2^2\sigma_2^2 + 2 z_1 z_2 \sigma_{12} = \frac{1}{4}\sigma_1^2 + \frac{1}{4}\sigma_2^2 + \frac{1}{2}\sigma_{12}. \end{aligned}$$

Now, it is evident that choosing two anti-correlated assets, with $\sigma_{12} < 0$, one actually *decreases* the variance, hence the risk.

This is precisely the mechanism that one wishes to exploit to decrease risk, and portfolio theory deals with how to allocate one’s resources optimally to this scope. The theory has been extended to general systems with an unlimited number of assets, with the presence of a risk-free asset, when short sales are allowed or not and so on [53]. The original theory was derived by Markovitz [82]. Let us briefly recall its main conclusions. Call $\hat{z}_i$ the fraction of one's wealth to be invested on asset i and let C_{ij} be the covariance matrix of the returns x_i:

$$C_{ij} = E[x_i x_j] - E[x_i]E[x_j]. \tag{5.2}$$

Then one tries to minimize the variance of the portfolio, which amounts to minimizing $\sigma_p^2 = \sum_{ij} z_i C_{ij} z_j$ with the constraint $\sum_i z_i = 1$. Introducing a Lagrange multiplier λ one seeks

$$|z\rangle = \mathrm{argmin}_{|z\rangle} \left(\frac{1}{2}\langle z|C|z\rangle - (\lambda - \langle z|1\rangle) \right) \tag{5.3}$$

where we have introduced bra-ket notation[1]. The result, after some simple algebra, reads

$$|z\rangle = \langle z|C^{-1}|z\rangle^{-1} C^{-1}|1\rangle. \tag{5.4}$$

The basic extension of this framework requires one additional constraint in (5.3) by fixing the total return on the portfolio $R_p = \sum_i z_i R_i$. The calculations are a little more involved (see next chapter), but the idea is the same.

In the following, we shall develop a model to test the feedback that this type of investment strategies can have on the dynamics of assets prices.

5.3 Model

Modelling such a complex systems as a financial market inevitably implies dealing with many complications. For example, portfolio investment decisions are in principle based on expected returns and covariances which may be different from the historical ones and different from trader to trader, also because traders may have different time horizons. A market mechanism needs to be specified to describe how investment decisions influence price dynamics.

Our aim here is to focus on the financial market as an interacting system, trying to capture in a simple yet plausible way the interaction among assets, and the resulting correlations, induced by portfolio investment. Hence we will rely on very rough approximations: we shall assume that *i)* all traders have the same trading horizon and follow Markovitz optimal portfolio strategy and that *ii)* that they use historical data to estimate correlations and expected returns. Finally, rather than

[1] $|x\rangle$ should be considered as a column vector, wereas $\langle x|$ is a row vector. Hence $\langle x|y\rangle$ is the scalar product and $|x\rangle\langle y|$ is the direct product, i.e. the matrix with entries $a_{i,j} = x_i y_j$.

deriving price impacts from a specific market mechanism, *iii)* we shall postulate a simple linear impact function for the price dynamics. Such rough approximations will leave us with a simple model of interacting assets whose dynamics accounts, in a schematic way, for fundamentals, portfolio investment and speculation. The model allows us to study the interplay between these contributions to trading both analytically and by numerical simulations. Rather than insisting on the validity of the approximations on theoretical a ground, we shall draw the main conclusions and then compare them with empirical data from real financial markets.

Let us consider a set of N assets. We denote by $|x\rangle$ the vector of log-prices and use bra-ket notation. We focus on daily time-scale and assume that $|x(t)\rangle$ undergoes the dynamics

$$|x(t+1)\rangle = |x(t)\rangle + |\beta(t)\rangle + \xi(t)|z(t)\rangle. \tag{5.5}$$

Our focus is on the properties of correlations, so we shall keep things simple and neglect issues related to fat tails or volatility clustering by assuming that all terms in the right hand side can be well modelled by Gaussian vectors, uncorrelated in time. The rationale behind Eq.(5.5) is to write a phenomenological equation to describe the dynamics of the market. Hence the first thing one would write is a process were the difference in the log-prices is given by some random (possibly gaussian) process, and this is the term $|\beta(t)\rangle$, which is the vector of *bare* returns, i.e. it describes the fundamental economic processes which drive the prices. This is assumed to be a Gaussian random vector with

$$E[|\beta(t)\rangle] = |b\rangle, \qquad E[|\beta(t)\rangle\langle\beta(t')|] = |b\rangle\langle b| + \hat{B}\delta_{t,t'} \tag{5.6}$$

$|b\rangle$ and $\hat{B}$ will be considered as parameters in what follows.

With the other term we try to capture inter-asset correlations emerging from portfolio investments. To do so, we will assume that the vector of log-prices is affected by fluctuations in the direction $|z(t)\rangle$, which represents the direction selected by portfolio investors. The vector gives the direction in the N dimensional assets space, and the fluctuations are given by some random process. Here, $\xi(t)$ is an independent Gaussian variable with mean fixed ϵ and variance Δ. These parameters describe fluctuations in the number and impact[2] of portfolio investors. The vector $|z(t)\rangle$, which describes the direction along which portfolio investment affects the inter-asset correlations, has to be the optimal portfolio calculated with standard portfolio optimization techniques. This is so, since this is the only direction in space which is preferred with respect to portfolio investments. So, the vector $|z(t)\rangle$ will be determined by a risk minimization procedure for fixed expected return R (expressed in money units, not in percentage, so that $|z(t)\rangle$ itself

[2]Fluctuations can be either in the number of investors or in the impact that the investments have on the price. In fact, the variables ξ are the product of a population term and a liquidity term.

has the meaning of money invested in each asset, not of percentage of one investor's wealth). We also fix the total wealth $\langle z|1\rangle = W$ invested, and keep it fixed (in the framework of the previous Chapter W was set to 1), so that $|z\rangle$ will be the solution of

$$\min_{|z\rangle,\nu}\left[\frac{1}{2}\langle z|\hat{C}(t)|z\rangle - \nu\left(\langle z|r(t)\rangle - R\right) - \sigma\left(\langle z|1\rangle - W\right)\right] \tag{5.7}$$

where $\hat{C}(t)$ is the correlation matrix at time t. In principle one can consider several component of portfolio investors, each with a different return R_k and with different impact parameters ϵ^k, Δ^k. This would lead to each investor having a different vecor $|z_k\rangle$. However, it can be shown that this problem is under general assumptions equivalent to the one above, with effective parameters $\tilde{R}$ and $\tilde{\Delta}$ that can be exactly computed (see Appendix C for details).

Both the expected returns and the correlation matrix are computed from historical data over a characteristic time τ:

$$\begin{aligned}
|r(t+1)\rangle &= \mu|r(t)\rangle + (1-\mu)|\delta x(t)\rangle && (5.8)\\
\hat{C}_1(t+1) &= \mu\hat{C}_1(t) + (1-\mu)|\delta x(t)\rangle\langle\delta x(t)| && (5.9)\\
\hat{C}(t+1) &= \hat{C}_1(t+1) - |r(t)\rangle\langle r(t)| && (5.10)
\end{aligned}$$

where $|\delta x(t)\rangle \equiv |x(t+1)\rangle - |x(t)\rangle$ and $\mu = \exp(-1/\tau)$. This makes the set of equations above a self-contained dynamical stochastic system. In a nutshell, it describes a simple way in which the economic bare correlated fluctuations $|\beta(t)\rangle$ are *dressed* by the actions of different financial investors. Other more complex choices for this interaction are possible, but Eq. (5.5) can be thought of as the lowest order of a phenomenological equation, similar to Landau's theory of critical phenomena [86]. Higher orders, e.g. $|z_{t+1}\rangle - |z_t\rangle$, will in principle be present, but we assume that their effect is small. The system of equations above represent an interestingly complex system, which, as we shall see, has a non-trivial behavior worth being studied and it sheds some light on empirical findings.

With respect to the economics literature, we note that our model can be viewed as a single-index model, given that the dynamics of each of the assetts is governed by a stochastic variable ($\xi(t)$), to which an asset-dependent index is attached (z_i). However, our index is not given exogenously, but rather is endogenously determined from the history of the system itself. To be more precise in this important issue, the index (or *factor*) that links the return on asset i to the market return (which in some sense is to me mapped on $\xi(t)$ in our model) is nothing but the investment on asset i given by the optimal portfolio. Since this portfolio is determined by the return process itself, we see why we have a model where the indeces need not to be specified *a priori*, but can instead be determined from the model itself.

An issue one could address is how the *bare* economic correlations described by the matrix $\hat{B}$ are *dressed* by trading activity to yield the matrix $\hat{C}$ of financial correlations. Notice that returns $|r(t)\rangle$ and correlations $\hat{C}(t)$ are directly measurable, whereas their *bare* counterparts $|b\rangle$ and $\hat{B}$ are not easily accessible. The effect of financial activity will be mostly that of modifying correlations along a single direction and will manifest in the largest eigenvalue of $\hat{C}$. The structure of $\hat{B}$ along other directions will be weakly affected. We shall assume in most of the following that $\hat{B} = B\hat{I}$ is proportional to the identity matrix. This does not affect much our main conclusions but it will make our analysis much simpler.

In what follows, we shall first study analytically the limit $\tau \to \infty$ and then confirm its predictions with numerical simulations.

5.3.1 The equilibrium solution

Taking the expected value of equation (5.5) we get

$$|r\rangle = |b\rangle + \epsilon|z\rangle \tag{5.11}$$

while setting to zero the derivative of the argument of Eq. (5.7) we get, again at equilibrium,

$$\hat{C}|z\rangle = \nu|r\rangle + \sigma|1\rangle. \tag{5.12}$$

By combining the above equations we can derive for $|z\rangle$

$$|z\rangle = \frac{1}{\hat{C} - \nu\epsilon}|q\rangle \tag{5.13}$$

where $|q\rangle = \nu|b\rangle + \sigma|1\rangle$. Since

$$\begin{aligned}
\hat{C} &= E[|\delta x\rangle\langle\delta x|] - E[|\delta x\rangle]E[\langle\delta x|] = \\
&= \hat{B} + \Delta|z\rangle\langle z| = \\
&= \hat{B} + \Delta\frac{1}{\hat{C} - \nu\epsilon}|q\rangle\langle q|\frac{1}{\hat{C} - \nu\epsilon}
\end{aligned}$$

and since $\hat{B} = B\hat{I}$, the matrix $\hat{C}$ will have $N-1$ eigenvalues equal to those of $\hat{B}$, and one eigenvalue, which we call Λ, parallel to the state $|q\rangle$ This equation, along with the constraints $\langle z|1\rangle = W$ and $\langle z|r\rangle = R$, must be solved to find Λ. Since $\Lambda = \langle q|\hat{C}|q\rangle$, we get the system of equations:

$$\begin{aligned}
\Lambda^2 &= \Lambda B + \Delta(\nu R + \sigma W) && (5.14)\\
W &= \langle z|1\rangle && (5.15)\\
R &= \langle z|r\rangle && (5.16)
\end{aligned}$$

Let us first describe what happens for $\epsilon = 0$ and $\langle b|1\rangle = 0$. Then, the solution is

$$\Lambda = B + \Delta \left(\frac{W^2}{\langle 1|1\rangle} + \frac{R^2}{\langle b|b\rangle} \right) \tag{5.17}$$

We see an inverse dependence of the correlations on the market return (note that since $\epsilon = 0$ $|r\rangle = |b\rangle$), which means that portfolio investment will contribute significantly to the observed correlations mainly in stagnant phases.

In the general case when $\epsilon \neq 0$ and $\langle b|1\rangle \neq 0$, the algebra is a bit more involved, and we get to a second order equation. The correct solution can be choosen by looking at the $R \to 0$, $W \to 0$ limit, and it reads

$$\begin{aligned} \Lambda = B \quad + \quad & \frac{\Delta\langle b|b\rangle}{\epsilon^2} \left\{ \frac{\gamma^2}{2} + \frac{\epsilon R}{\langle b|b\rangle} - \frac{\langle b|1\rangle \epsilon W}{N\langle b|b\rangle} \right. \\ & \left. - \frac{\gamma}{2} \sqrt{1 + \frac{4\epsilon R}{\langle b|b\rangle} - \frac{(\langle b|1\rangle - 2\epsilon W)^2}{N\langle b|b\rangle}} \right\} \end{aligned} \tag{5.18}$$

where $\gamma = \sqrt{1 - \langle b|1\rangle^2/(N\langle b|b\rangle)}$. Notice that, $\langle b|1\rangle/N$ is the average bare return and, if total return R and portfolio investment W are both proportional to N, then the contribution to Λ due to portfolio investment is also proportional to N. This is indeed the order of Λ in empirical data.

In Fig.5.3 we plot Λ as a function of $\langle b|b\rangle$, the inverse dependence of the maximum eigenvalue on the mean squared drift in numerical simulations follows precisely the theoretical line, both when $\epsilon = 0$ and when $\epsilon \neq 0$ (here we have set $\langle b|1\rangle = 0$ in the simulations).

In figure 5.4 we plot Λ as a function of ϵ along with the analytic result.

It is worthwhile noting that the strength of the feedback-induced correlation depends on the properties of the noise ξ in an interesting way. The inverse dependence on the mean ϵ shows that these correlations decrease when the impact of portfolio trading becomes stronger with respect to the fluctuations of the impact Δ. On the other side, when Δ increases the correlations grow. This is not surprising, since by construction the term describing portfolio investment contributes to the covariance matrix proportionally to Δ.

5.3.2 Transition to instability

Even more interesting is the fact that for W greater that a value W^* the system has no solution, as evident also from Eq.(5.18). Correspondingly, for $W \to W^*$, we see that $\frac{\partial\Lambda}{\partial W} \to \infty$. The critical value is given by

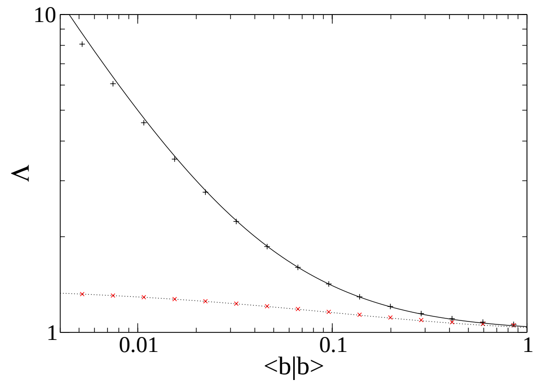

Figure 5.3: Maximum eigenvalue of the correlation matrix as a function of $\langle b|b\rangle$ for $N = 20$ $\Delta = 0.04$, $R = 1$, $\tau = 20000$. Line is the theoretical curve, Eq.(5.18).

$$W^* = \frac{N}{2\epsilon}\left[\sqrt{\frac{\langle b|b\rangle + 4\epsilon R}{N}} + \frac{\langle b|1\rangle}{N}\right]. \tag{5.19}$$

As $W \to W^*$ the solution of Eq. (5.18) develops a singularity with infinite slope $\frac{\partial\Lambda}{\partial W} \to \infty$. This is reminiscent of the divergence of susceptibility χ close to a phase transition, signalling that the response $\delta\Lambda = \chi\delta W$ to a small perturbation δW diverges as $W \to W^*$. In Fig. 5.5 we plot the maximum eigenvalue as a function of W along with the theoretical line from Eq.(5.18),

The importance of this result lies in the fact that we get two distinct phases. The first one, which is the one we solved, corresponds to $W < W^*$ and is a phase were the system admits a stationary solution of the equations in the $\tau \to \infty$ limit. Here, when performing numerical simulations at finite τ, we will get a convergence to the analytic solution as we increase τ. On the other hand, for $W > W^*$, no equilibrium solution exhists for large τ, fluctuations become larger and larger. This behavior is evident from Fig.5.6, were we plot the normalized standard deviation of the maximum eigenvalue versus W for increasing values of τ.

To better understand what happens as $W \to W^*$, we can look at Fig. 5.7. There we show the temporal evolution of Λ for different W. We see that as $W \to W^*$ the dynamic becomes unstable. Many singularities emerge, where correlations tend to become very large, due to the fact that no solution to our equation for Λ can be found. In this view, one may think that we are studying a type of market instabilities that manifest when the investment grows close to the absorption threshold W^*, and indeed it seems that the temporal evolution of Λ in

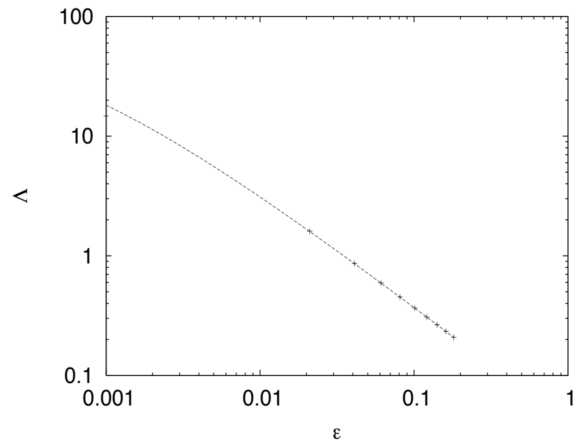

Figure 5.4: Maximum eigenvalue of the correlation matrix as a function of ϵ for $N = 20$ $\Delta = 0.04$, $R = 1$, $\tau = 20000$. Line is the theoretical curve, Eq.(5.18).

our model close to criticality closely resembles that of real markets, as emerges clearly from Fig. 5.8, where we plot Λ for the Toronto stock Exchange companies (a) and for a simulation of the model close to the critical point (b).

The idea that an internal mechanism like the one we are studying can be responsible for the observed fluctuation properties of the covariance matrix is enforced by the results shown in Fig.5.9, where we show that the broad cumulative distribution of the daily differences in Λ can be reproduced by the model close to the critical point.

To test this hypothesis we will develop in Section 5.4 a technique to fit real data with our model, and derive thus the values of the parameters W and W^* for real markets. Before moving on, let us discuss the issue of temporal fluctuations at finite τ.

5.3.3 Fluctuations

Given the solution in the $\tau \to \infty$ limit, the natural question arises of how that solution is affected by fluctuations at finite τ.

To study this interesting issue, we will analyze a simpler version of the model of Sec. 5.32. In particular, we relax the assumption on the wealth. Setting $\sigma = 0$ is equivalent to allowing for infinite wealth, that is, to saying that investments are based on fixing the return one wants, and then investing the wealth needed. The algebra becomes simpler, and here there is no critical parameter. The equilibrium solution reads

$$\Lambda = B + \frac{\Delta R^2}{\langle r|r\rangle}. \tag{5.20}$$

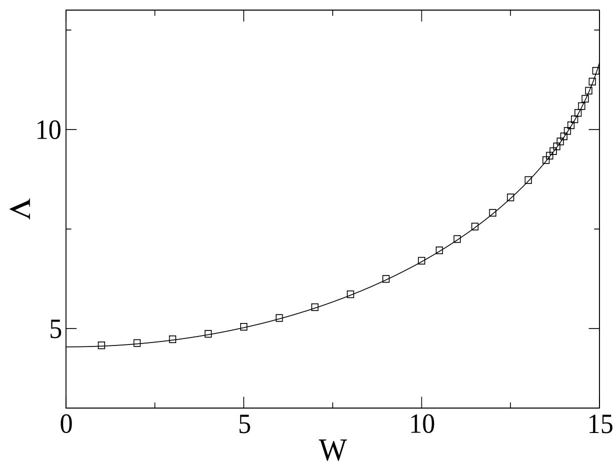

Figure 5.5: Maximum eigenvalue of the correlation matrix as a function of W for $N = 20$. $\Delta = 1$, $R = 1$,$\tau = 20000$, $\epsilon = 0.1$.

Let us define $\beta = 1/\tau$ and write

$$|r(t)\rangle = |r_0\rangle + \beta|r_1(t)\rangle \tag{5.21}$$

$$\hat{C}(t) = \hat{C}_0 + \beta\hat{C}_1(t) \tag{5.22}$$

where $|r_0\rangle$ and $\hat{C}_0$ are the equilibrium solutions. We are interested in finding the fluctuations with a perturbative expansion around these solutions. We thus have for small β

$$|z(t)\rangle = \frac{R}{\langle r(t)|\hat{C}(t)^{-1}|r(t)\rangle}\hat{C}^{-1}(t)|r(t)\rangle \tag{5.23}$$

from which we get

$$|z(t)\rangle = A\frac{|r_0\rangle + \beta|r_1(t)\rangle}{\hat{C}_0 + \beta\hat{C}_1(t)} \tag{5.24}$$

with

$$A = \frac{R}{\left[\langle r_0|\hat{C}_0^{-1}|r_0\rangle + \beta(2\langle r_1(t)|\hat{C}_0^{-1}|r_0\rangle - \langle r_0|\hat{C}_0^{-2}\hat{C}_1(t)|r_0\rangle)\right]^{-1}} \tag{5.25}$$

up to order β^2. Setting $\langle r_0|\hat{C}_0^{-1}|r_0\rangle = L_0$ we get after some algebra

$$|z(t)\rangle = |z_0\rangle + \beta|z_1(t)\rangle \tag{5.26}$$

with

$$|z_0\rangle = \frac{R}{L_0}\hat{C}_0^{-1}|r_0\rangle$$

$$|z_1(t)\rangle = \frac{R\hat{C}_0^{-1}}{L_0}\left[|r_1(t)\rangle - \hat{C}_0^{-1}\hat{C}_1(t)|r_0\rangle + \frac{2\langle r_1(t)|\hat{C}_0^{-1}|r_0\rangle - \langle r_0|\hat{C}_0^{-2}\hat{C}_1(t)|r_0\rangle}{L_0}|r_0\rangle\right]$$

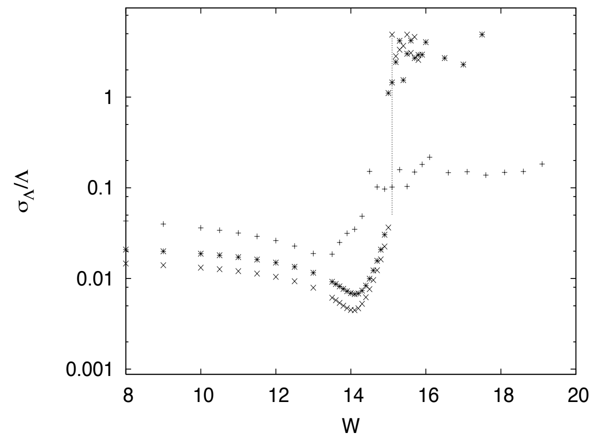

Figure 5.6: Relative fluctuation of the maximum eigenvalue as a function of W in a simulation of the model with $N = 20$, $\epsilon = 0.1$, $R = 1$, $\Delta = 1$,$B = 10^{-2}$,$\tau = 1000$ (+), $\tau = 20000$ ($\times$) and $\tau = 50000$ ($\star$). Vertical line is the theoretical critical value of W.

Now, since we can always write $\hat{C}(t) = \hat{B} + \Delta|z(t)\rangle\langle z(t)|$, we get up to first order

$$\hat{C}(t) = \hat{C}_0 + \beta\hat{C}_1(t) = \hat{B} + \Delta(|z_0\rangle\langle z_1(t)| + |z_1(t)\rangle\langle z_0|). \tag{5.27}$$

Assuming that the eigenvectors of $\hat{C}$ are not changed to first order in β we can project Eq.(5.27) along $|z_0\rangle$ and compute the maximum eigenvalue Λ of the correlation matrix. After some algebra we find

$$\Lambda(t) \approx \frac{\langle z_0|\hat{C}(t)|z_0\rangle}{\langle z_0|z_0\rangle} = \Lambda_0 + \beta\lambda_1(t) \tag{5.28}$$

where

$$\Lambda_1(t) = \frac{\langle z_0|\hat{C}_1(t)|z_0\rangle}{\bar{z}} = \tag{5.29}$$

$$= \frac{\Delta 3R^2\langle r_1(t)|r_0\rangle}{1 + \frac{2R^2}{\Lambda_0\langle r_0|r_0\rangle}}\frac{1}{\langle r_0|r_0\rangle} \tag{5.30}$$

We note here that the term $\langle r_1(t)|r_0\rangle$ is the first term in the fluctuations σ_r of $\langle r(t)|r(t)\rangle$. Hence if we are interested in the fluctuations $\sigma_\Lambda^2 = \langle\Lambda_1^2\rangle$ at high Λ, that is, when the model predicts a collective motion of the assets induced by portfolio investments we find a relation with σ_r. In particular we can write $\Lambda \approx \langle r|r\rangle^{-1}$ and, since $\bar{z} = R^2/\langle r|r\rangle$,

$$\frac{\sigma_\Lambda^2}{\sigma_r^2} \propto \Lambda^4 \tag{5.31}$$

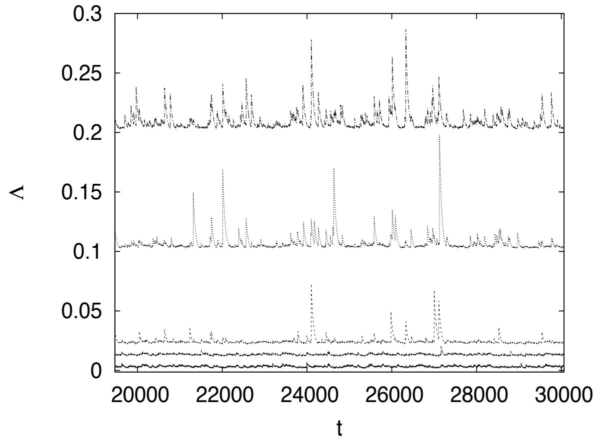

Figure 5.7: Maximum eigenvalue of the covariance matrix as a function of time for $\tau = 50$ $N = 20$, $R = 10^{-2}$, $B = 10^{-6}$,$\Delta = 0.04$,$\epsilon = 10^{-1}$, and, from bottom to top (curves are shifted for clarity), $W = 1.2$, $W = 1.32$, $W = 1.42$ and $W = 1.48$. Components of $|b\rangle$ where generated uniformly in the interval $[0, 2 \cdot 10^{-3}]$, resulting in $W^* \approx 1.41$.

This result is confirmed by numerical simulations of the model. In Fig. 5.10 we plot the quantity $\sigma^2_\Lambda/\sigma^2_{\langle r|r\rangle}$ as a function of Λ for increasing values of the time constant $\tau = 500$, 1000 and 5000. The line is the curve Λ^4, and we see an increasing accuracy of this approximation as we increase τ, i. e. decrease β.

5.4 Fitting the dataset

Here we will try to fit real data with our model. The idea, as stated in the previous Section, comes from the similar anomalous behavior of the temporal evolution of correlations in simulations and in real markets, as in Fig. 5.8. Moreover, the wide distribution of day-to-day differences in Λ can also be reproduced by the model.

Let us rewrite The model as

$$|\delta x(t)\rangle = |b\rangle + |\eta(t)\rangle + [1 + \zeta(t)]|y(t)\rangle \tag{5.32}$$

where $|\delta x(t)\rangle$ is the vector of daily returns, $|\eta\rangle$ is a zero average gaussian vector with i.i.d. components of variance B, $\zeta(t)$ is gaussian i.i.d. with zero average and variance $D = \Delta/\epsilon^2$ and $|y(t)\rangle$ is the portfolio with $\langle y(t)|r(t)\rangle = \rho = \epsilon R$ and $\langle y(t)|1\rangle = \omega = \epsilon W$. Notice that $|y(t)\rangle$ depends on ρ and ω and it is computed from the past data set's correlation matrix and returns. We compute the likelihood

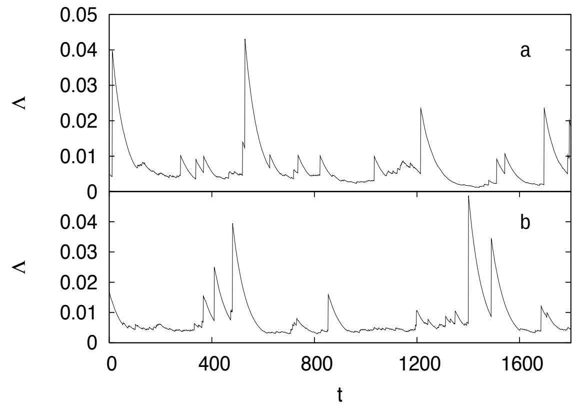

Figure 5.8: Maximum eigenvalue of the correlation matrix as a function of time for $\tau = 50$. a) Toronto Stock exchange [80]. Here the correlation matrix is obtained using Eq. (5.9) with $|\delta x_t\rangle$ taken from historical data. b) simulation of Eq.(5.5) with $N = 20$, $R = 10^{-2}$, $B = 10^{-6}$,$\Delta = 0.04$,$\epsilon = 10^{-1}$,$W = 1.4$. Components of $|b\rangle$ where generated uniformly in the interval $[0, 2\cdot 10^{-3}]$, resulting in $W^* \approx 1.41$.

$$P\{|\delta x\rangle||b\rangle, |y\rangle\} = \langle\prod_{i,t}\delta\left(b_i + \eta_i(t) + [1+\zeta(t)]y_i(t) - \delta x_i(t)\right)\rangle_{\eta,\zeta} \tag{5.33}$$

The gaussian integrals are easily carried out, and we get

$$\mathcal{L} \equiv \log P\{|\delta x\rangle||b\rangle, |y\rangle\} = -\frac{NT}{2}\log(2\pi B) + \frac{1}{2B}\sum_t F(t) \tag{5.34}$$

with

$$F(t) = \left\{\frac{\langle g(t)|y(t)\rangle^2}{1/\mu + \langle y(t)|y(t)\rangle} - \langle g(t)|g(t)\rangle - B\log(1+\mu\langle y(t)|y(t)\rangle)\right\} \tag{5.35}$$

where we defined $|g(t)\rangle = |b\rangle + |y(t)\rangle - |\delta x(t)\rangle$ and $\mu \equiv D/B = \Delta/(B\epsilon^2)$. Setting to zero the partial derivative of $\mathcal{L}$ wrt b_i, we find

$$|b\rangle = \frac{1}{T}\sum_t\left[\frac{-1+\mu(\langle b|y\rangle - \langle\delta x|y\rangle)}{1+\mu\langle y(t)|y(t)\rangle}|y(t)\rangle + |\delta x(t)\rangle\right] \tag{5.36}$$

Substituting this back in $\mathcal{L}$ leaves us with a function of the parameters $B,\ \mu,\ \rho,\ \omega$ which has to be maximized. The ratio $\mu = B/D$ satisfies the equation

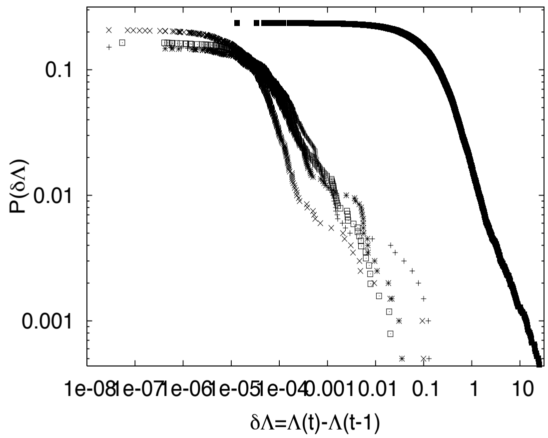

Figure 5.9: Cumulative distribution of the day-to day change in the maximum eigenvalue for different indices: DAX (+), TSX($\star$), DOW($\Box$), ASX($\times$). Also shown is the same distribution for a numerical simulation of the model with $N = 20$, $\epsilon = 0.1$, $R = 1$, $W = 14.7$, $\Delta = 1$,$B = 10^{-2}$,$\tau = 100$.

$$\mu = \frac{\sum_t \frac{\langle g|y\rangle^2 - B\langle y|y\rangle}{(1+\mu\langle y|y\rangle)^2}}{\sum_t \frac{B\langle y|y\rangle^2}{(1+\mu\langle y|y\rangle)^2}} \tag{5.37}$$

Notice that the solution[3] $|y\rangle$ of portfolio optimization problem can be written explicitly in terms of ρ and ω as

$$|y\rangle = \rho|\rho\rangle + \omega|\omega\rangle \tag{5.38}$$

where

$$|\rho\rangle = \hat{C}^{-1}\frac{\chi_{11}|r\rangle - \chi_{r1}|1\rangle}{\chi_{rr}\chi_{11} - \chi_{r1}^2}, \quad |\omega\rangle = \hat{C}^{-1}\frac{-\chi_{r1}|r\rangle + \chi_{rr}|1\rangle}{\chi_{rr}\chi_{11} - \chi_{r1}^2} \tag{5.39}$$

where $\chi_{ab} \equiv \langle a|\hat{C}^{-1}|b\rangle$. The two vectors $|\rho(t)\rangle$ and $|\omega(t)\rangle$ can explicitly be computed directly from the data-set. This means in practice that it is enough, for each t to solve two equations of the type $\hat{C}(t)|x\rangle = |a(t)\rangle$ for all values of ρ and ω. The likelihood can be expressed explicitly in terms of ρ and ω and the partial derivatives $\partial\mathcal{L}/\partial\rho$ and $\partial\mathcal{L}/\partial\omega$ can be computed. In order to do this, it is convenient to write $|g\rangle = |h\rangle + |y\rangle$, so that $\mathcal{L}$ depends on ρ and ω only through the combinations

[3]The index t is suppressed, when not needed.

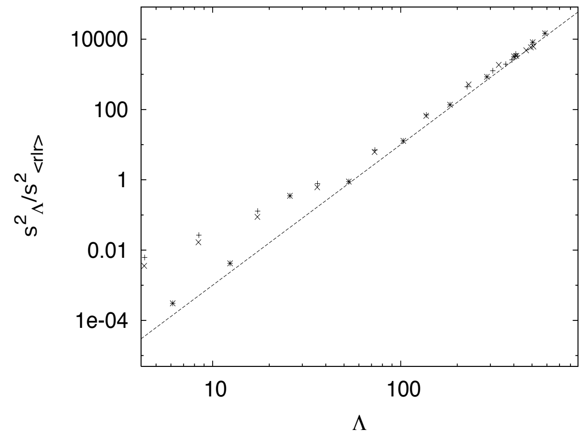

Figure 5.10: Fluctuations in Λ relative to the fluctuations in $\langle r|r\rangle$ are plotted against Λ for $N = 20$, $\Delta = 4$, $R = 20$, $\epsilon = 0.001$. The time constant is $\tau = 500$ ($+$), $\tau = 1000$ ($\times$) and $\tau = 5000$ ($*$).

$\langle h|y\rangle$ and $\langle y|y\rangle$ so that, e.g.

$$\frac{\partial \mathcal{L}}{\partial \rho} = \sum_t \frac{\partial \mathcal{L}}{\partial \langle h|y\rangle} \langle h|\rho\rangle + 2 \sum_t \frac{\partial \mathcal{L}}{\partial \langle y|y\rangle} \langle y|\rho\rangle \tag{5.40}$$

where we used that $\partial\langle h|y\rangle/\partial\rho = \langle h|\rho\rangle$ and $\partial\langle y|y\rangle/\partial\rho = 2\langle y|\rho\rangle$. Defining

$$H_a(\rho,\omega) = \sum_t \frac{\partial \mathcal{L}}{\partial \langle h|y\rangle} \langle h|a\rangle, \quad L_{ab}(\rho,\omega) = \sum_t \frac{\partial \mathcal{L}}{\partial \langle y|y\rangle} \langle a|b\rangle, \quad a, b = \rho, \omega$$

we have

$$\rho = \frac{1}{2} \frac{H_\omega L_{\rho\omega} - H_\rho L_{\omega\omega}}{L_{\rho\rho} L_{\omega\omega} - L_{\rho\omega}^2}, \qquad \omega = \frac{1}{2} \frac{H_\rho L_{\rho\omega} - H_\omega L_{\rho\rho}}{L_{\rho\rho} L_{\omega\omega} - L_{\rho\omega}^2}$$

Note that the indices $\rho,\ \omega$ are labels in H and L. Both coefficients depend on ρ and ω. Hence the left hand side depends on ρ and ω.

Setting to zero the partial derivative of $\mathcal{L}$ with respect to B yields a simple linear equation for B itself:

$$B = -\frac{2\pi}{NT} \sum_t \left(\frac{\mu \langle g|y\rangle^2}{1 + \mu \langle y|y\rangle} - \langle g|g\rangle \right) \tag{5.41}$$

Now to avoid part of the difficulty of fitting a model with so many parameters, we can rewrite Eq.(5.36) into a simpler form, given that the equation is linear in $|b\rangle$. We thus get

$$|b\rangle = (\hat{I} - \hat{A})^{-1} |a\rangle \tag{5.42}$$

with

$$A_{ij} = \mu T^{-1} \sum_t \frac{y_i y_j}{1 + \mu \langle y|y \rangle}$$

and

$$|a\rangle = \frac{1}{T} \sum_t \left(\frac{-1 - \mu \langle x|y \rangle}{1 + \mu \langle y|y \rangle} |y\rangle + |x\rangle \right).$$

To do the fit we now proceed as follows. We first calculate, as starting values, $|b\rangle$ as the average drift of the assets in the period considered. Similarly, B is taken to be the mean variance. Next we perform a maximization procedure on the function $\mathcal{L}$, for example using a simplex algorithm. This is not too difficult, since we are taking fixed $|b\rangle$ and B, while the parameters (i.e. the dimensionality of the space) are just three: μ, ω, ρ. After the simplex has converged, we recalculate $|b\rangle$ and B from equations (5.42) and (5.41), respectively. Now we perform again the simplex to get new values for μ, ω, ρ, and so on until convergence. Actually, a smoothing parameter α is introduced, so that at each iteration the parameters are taken to be α times their new value plus $(1 - \alpha)$ times their old value.

Since the procedure is quite complex, we performed some tests. We generate an artificial time series according to our model, with known values of the parameters, and then we try to fit this data. In Fig. 5.11 we plot the result of such a fit. On the horizontal axis are the 100-days periods. To each point corresponds one such period, that is, 100 data points, that are fitted to give the parameters. The results shown are for ω, ρ and ω_c, and they are quite close to the real values. In this case both the time scale used to generate the data (τ) and the one used to calculate the exponentially weighted correlation matrix (τ') were the same, $\tau = \tau' = 100$, but tests with different time scales have also been performed. The stability is quite good as long as $\tau' \approx \tau$ (that is, the two time scales do not differ by an order of magnitude), as expected since for example when $\tau' << \tau$ the correlation matrix does not contain all the information used by the investors to calculate the portfolio.

Another test is to use again artificial time series with different values of ω, and run the fitting algorithm on this series, without varying the initial conditions, that is, the starting point of the simplex. In Fig.5.12 we plot the results of such a test. On the horizontal axis is the real value of ω that was used to generate the data (ω_r), while on the vertical axis we plot the fitted value, ω_f. Each point corresponds to a 500 days period, over which the fit was performed.

In Figure 5.13 we plot the same quantities, but obtained with a slightly modified version of the algorithm. This was devised in order to test the robustness of our fitting procedure. Having at our disposal two methods can be helpful in understanding how stable the results that we find are, and this is important for the application to real data that we will perform in the next Section. The modified algorithm goes as follows. We use the simplex method only for the 2 variables ω

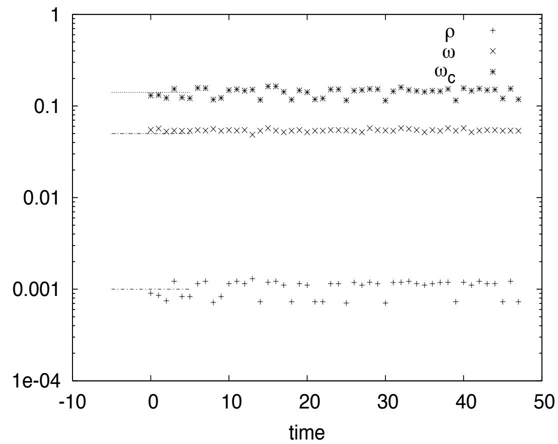

Figure 5.11: Each point is a 100-days period. The data were generated with $\omega = 0.05$, $\rho = 0.001$, $\omega_c = 0.14$, $\tau = 100$, $\epsilon = 0.1$, $\Delta = 0.04$.

and ρ. For μ we use instead the same iterative procedure used for B and $|b\rangle$ using Eq.(5.37). Note that in all the cases the iterative procedure involves some smoothing. In this case, results on artificial data are quite robust, but there is a sistematic error, the cause of which we were still not able to identify. This error can be calculated by fitting these artificial sequences, and it turns out that $\omega_f/\omega_r = 1.18 \pm 0.01$. The plot (5.13) is drawn after correcting ω_f for this factor.

5.5 Real Markets

The first thing to do if we want to fit real data with our model is to test the two methods discussed in the previous section. Here, having at our disposal two methods, even if clearly correlated, helps us to check the sensitivity and the errors that we do. Hence we try to fit dayly data for stocks composing the DAX index in the german stock exchange, for the period 1998-2005. We use $\tau = 100$, while the period over which we fit is a 300 days long window. The choice of the time scale is such that we are at an intermediate level between the month and the year (250 trading days), which is a reasonable time scale for many investment decisions. In Fig. 5.14 we plot the results for two quantities: B and ω, while on the horizontal axis we have the 300 days periods. Results are given for both methods, and the agreement is accepptable.

Next we proceed to fit real datasets for four world indexes: the DAX, the Dow Jones, the Toronto Stock exchange and the australia stock exchange. We use again $\tau = 100$ and 300 days windows. In Fig. 5.15 we plot the results, that show how these indexes are quite close to the critical line $\omega = \omega_c$. For each index, each

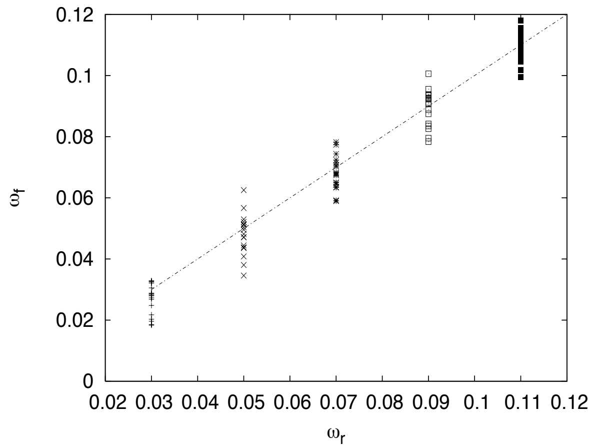

Figure 5.12: Each point is a 500-days period. The data were generated with $\omega = 0.03$ (+), $\omega = 0.05$ (×), $\omega = 0.07$ (□), $\omega = 0.09$ (■). $\rho = 0.001$, $\omega_c = 0.14$, $\tau = 100$, $\epsilon = 0.1$, $\Delta = 0.04$. The starting point for the simplex was the same for all the realizations, even at different ω.

point represents one time window, the fit is done with method 1, which seems a bit more stable and does not require *a posteriori* corrections.

The picture that emerges once again, from the analysis of this section, is that of financial markets, in particular world market indexes, that lie very close to the transition point from a stable to an unstable phase.

5.6 Conclusions

In summary, we have studied a simple model of multi-asset market which takes into account the feedback of risk minimization strategies on the dynamics of assets prices. The model is very stylized and misses many important aspects, but relies on the idea that, when modeling complex systems to look for the collective properties, statistical regularities emerge, that are not closely linked to the microscopic details. Rather than insisting on the validity of the approximations on a theoretical ground, we have compared the main findings of the model with empirical data from real financial markets. Since our interest was in the correlations among assets, we looked at the time dependence of these correlations. We found that the model reproduces realistic time series of correlations along with the statistics of the maximum eigenvalue of the correlation matrix. We then analyzed the equilibrium solution of the model, and found the existence of two phases, a stable and an unstable one. The critical point for the transition between the two is dictated by a parameter that measures the scale of the impact of the feedback.

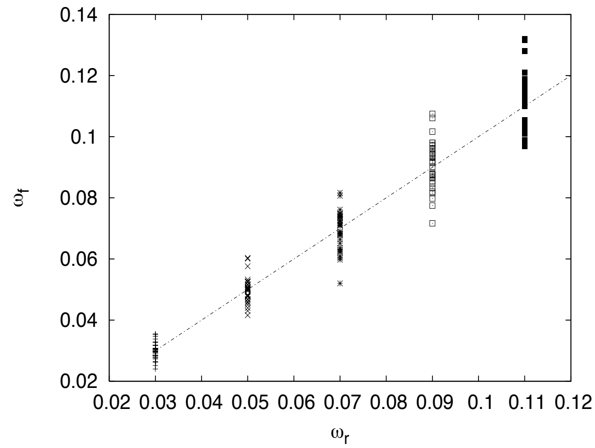

Figure 5.13: Result of the fit, performed with the second version of the algorithm, of artificial time series. Each point is a 500-days period. The data were generated with $\omega = 0.03$ ($+$), $\omega = 0.05$ ($\times$), $\omega = 0.07$ ($\square$), $\omega = 0.09$ ($\blacksquare$). $\rho = 0.001$, $\omega_c = 0.14$, $\tau = 100$, $\epsilon = 0.1$, $\Delta = 0.04$. The starting point for the simplex was the same for all the realizations, even at different ω.

Moreover, the analysis of real markets within our framework predicts that real markets are close to this phase transition.

This does not mean that external influences on the market do not play any role. Indeed, the scatter of points in Fig. 5.15 for the same market implies a time dependence of the parameters that the model cannot explain. Furthermore, it is undeniable that global events have an effect on financial markets (similar patterns in the dynamics of Λ_t for different markets can indeed be found). Our point is that there is a sizeable contribution to the collective behaviour of the market which arises from its internal dynamics and which is a potential cause of instability.

Furthermore, the finding that real markets "self-organize" close to a critical point where correlations become unstable lends itself to a simple interpretation. As long as the impact of risk minimization strategies is small, market correlations are stable. This means that these strategies are reliable and efficient, which attracts even more investment on these strategies (i.e. W increases). However, as W increases, the market approaches the instability line, where correlations respond to further investment, thus making risk minimization strategies less efficient. This suggests that W should self-organize close to the critical value W^*. Such a scenario is reminiscent of the picture which Minority Games [18] provide of single asset markets as systems close to a critical point. There speculators are attracted by an information rich, predictable market which gets less and less predictable because of the very impact of their trading. This pushes the (Minority Game's) market close to the critical point where it becomes information-efficient. It is in-

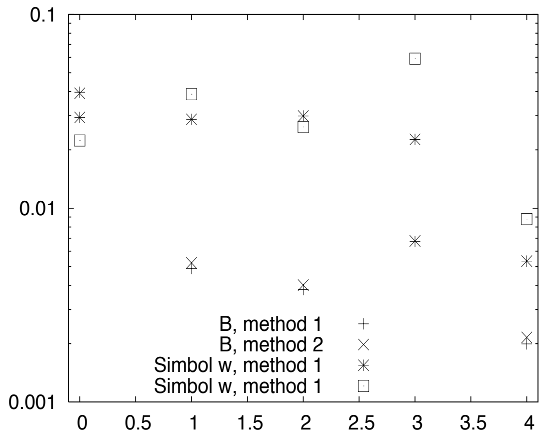

Figure 5.14: Results of the fit on Dax companies. The starting values of the simplex are $\omega = 10^{-2}$ and $\rho = 10^{-3}$ with simplex amplitudes (see text) $\delta\omega = 10^{-2}$ and $\delta\rho = 10^{-4}$. Smoothing parameter is $\alpha = 0.1$

deed exactly there that Minority Game models exhibit statistical properties very similar to those of real financial markets [87].

Our findings provide a further example of a case where, when the scale of human activity reaches a critical point, the collective properties of the system change dramatically. In all these cases, the assumption that the individual intervention has no impact on the aggregate – the so-called price taking behavior in markets – fails because the system reaches a point of infinite susceptibility. These problems naturally arise where agents rely on the exploitation of a *public good* [88]. Their appearance in financial markets – the prototype examples of perfect competition – is somewhat paradoxical.

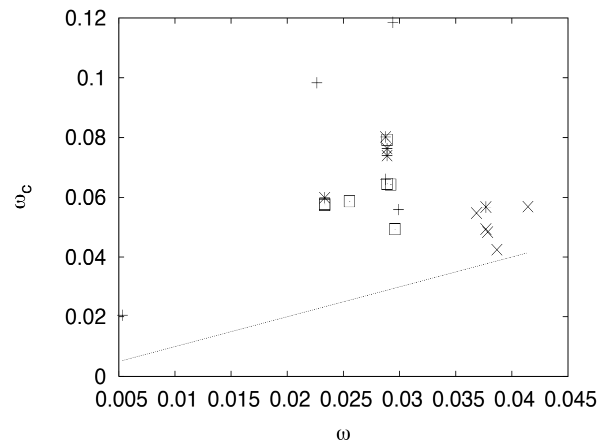

Figure 5.15: Results of the fit on world indexes: DAX (+), DOW (□), TSX(⋆) and ASX(×). $\alpha = 0.2$, $\tau = 100$.

Appendix A

Appendix to Chapter 1

In this Appendix we give the details of the calculation of the free energy for the model of interacting voters of Chapter 2.

The partition function can be written as

$$Z(\beta) = \mathrm{Tr}_{\{\hat{v}_i\}} e^{-\beta H} \tag{A.1}$$

where the trace $\mathrm{Tr}_{\{\hat{v}_i\}}$ over spins runs on all $\hat{v}_i \in \mathcal{R}$ when voting behavior is rational, or over all $\hat{v}_i$ otherwise. Thus we have

$$Z(\beta) = \mathrm{Tr}_{\{\hat{v}_i\}} e^{-\beta(1-\epsilon)\sum_i \hat{\Delta}_i \hat{v}_i + \frac{\beta\epsilon}{2N}\sum_{ij} \hat{v}_i \hat{v}_j} \tag{A.2}$$

and linearizing the last term yields

$$\begin{aligned} Z(\beta) &= \mathrm{Tr}_{\{\hat{v}_i\}} e^{-\beta(1-\epsilon)\sum_i \hat{\Delta}_i \hat{v}_i + \frac{\beta\epsilon}{2N}(\sum_i \hat{v}_i)^2} && \text{(A.3)} \\ &= \int d\hat{y} \mathrm{Tr}_{\{\hat{v}_i\}} \exp\left[-\beta(1-\epsilon)\sum_i \hat{\Delta}_i \hat{v}_i + \sqrt{\frac{\beta\epsilon}{N}}\sum_i \hat{v}_i \hat{y} - \frac{1}{2}y^2\right] && \text{(A.4)} \\ &= \int d\hat{y} e^{-\frac{1}{2}y^2} e^{\sum_i \log \mathrm{Tr} \exp f_i} && \text{(A.5)} \end{aligned}$$

where

$$f_i = \sqrt{\frac{\beta\epsilon}{N}} \hat{v}_i \hat{y} + \beta(1-\epsilon)\hat{\Delta}_i \hat{v}_i \tag{A.6}$$

and we have used the relation

$$e^{\lambda a^2} = \frac{1}{(2\pi)^1/2} \int_{-\infty}^{\infty} \mathrm{d}x \exp(\frac{1}{2}x^2 + ax\sqrt{2\lambda}). \tag{A.7}$$

Now setting $\hat{y} = \sqrt{\beta\epsilon N}\hat{m}$ we find

$$Z(\beta) = \int d\hat{m} e^{-N\beta f(\hat{m})} \tag{A.8}$$

with the free energy $f(\hat{m})$ given by

$$f(\hat{m}) = \frac{\epsilon}{2}\hat{m}^2 - \frac{1}{N\beta}\sum_{i=1}^{N}\log\left[\sum_{\hat{v}} e^{\beta[(1-\epsilon)\hat{\Delta}_i+\epsilon\hat{m}]\cdot\hat{v}}\right] \tag{A.9}$$

Appendix B

The single-index model

Here we will briefly review the single-index model, with a particular emphasis on how it is used towards portfolio optimization. This is an example where the mechanism of diversification and portfolio investment is made very clear by the structure of the equations. Of course in real markets the situation is not so simple, there are many types of correlations among stocks, but the general idea is the same.

Recall the single-index equation:

$$x_i = a_i + \beta_i R_m \tag{B.1}$$

where a_i and R_m are random variables. Let us recast this in the form

$$x_i = \alpha_i + e_i + \beta_i R_m \tag{B.2}$$

with $\alpha_i = E[a_i]$, e_i is a random variable such that

$$\begin{aligned} E[e_i] &= 0 && \text{(B.3)} \\ E[(e_i)^2] &= \sigma_{ei}^2. && \text{(B.4)} \end{aligned}$$

We also have by definition that $E[(R_m - \bar{R}_m)^2] = \sigma_m^2$ and by assumption the e_i are uncorrelated with each other and with the market return. Simple algebra yields the following:

$$\begin{aligned} \bar{R}_i &= E[x_i] = \alpha_i + \beta_i \bar{R}_m && \text{(B.5)} \\ \sigma_i^2 &= E[(x_i - \bar{R}_i)^2] = \beta_i^2 \sigma_m^2 + \sigma_{ei}^2 && \text{(B.6)} \\ \sigma_{ij} &= E[(x_i - \bar{R}_i)(x_j - \bar{R}_j)] = \beta_i \beta_j \sigma_m^2, && \text{(B.7)} \end{aligned}$$

which highlights how the correlations between assets emerge only through the common link to the market return.

The expected return on a portfolio of assets is

$$\bar{R}_p = \sum_{i=1}^{N} z_i \bar{R}_i. \tag{B.8}$$

The variance of the portfolio is given by

$$\sigma_p^2 = \sum_i z_i^2 \sigma_i^2 + \sum_{i \neq j} z_i z_j \sigma_{ij} \tag{B.9}$$

which gives, after substituting for σ_i^2 and σ_{ij},

$$\sigma_p^2 = \sum_{ij} z_i z_j \beta_i \beta_j \sigma_m^2 + \sum_i z_i^2 \sigma_{ei}^2. \tag{B.10}$$

If we now define

$$\beta_p = \sum_i z_i \beta_i, \tag{B.11}$$

we get

$$\sigma_p^2 = \beta_p^2 \sigma_m^2 + \sum_i z_i^2 \sigma_{ei}^2. \tag{B.12}$$

Now suppose we invest equal amounts of money into each of the N stocks, that is, $z_i = 1/N \forall i$. Then the variance of the portfolio can be written as

$$\sigma_p^2 = \beta_p^2 \sigma_m^2 + \frac{1}{N} \left(\frac{1}{N} \sigma_{ei}^2 \right) \tag{B.13}$$

The last term decreases very rapidly as we increase the number of stocks held in the portfolio, while the risk that is not eliminated is the one associated to the term β_p. Thus the single index-model allows to divide the risk of each security into a *diversifiable* component (or *unsystematic risk*) and a nondiversifiable one, the *systematic risk*.

Appendix C

Many investors

Here we will show how the question of many portfolio investors, each with its own parameters W_k and R_k can be remapped, under suitable assumptions, to the general case discussed so far. The first step is to generalize Eq.(5.5) and Eq.(5.13), thus we write:

$$|\delta x\rangle = |\beta\rangle + \sum_k \xi_k(t)|z_k\rangle. \tag{C.1}$$

Here, each $|z_k\rangle$ is defined by the solution of Eq.(5.7), and it is the portfolio for investor (type) k characterized by parameters R_k, ϵ_k and Δ_k, with obvious meaning. For simplicity, we have set $W_k = 1 \forall k$.

The difficulty is due to the fact that we cannot write an equation for $\hat{C}$ with a single preferred direction, since the direction of each $|z_k\rangle$ will depend on the lagrange multipliers ν_k and σ_k associated with each investor. Taking the expected value of $|\delta x\rangle$ is not a problem, since we can write

$$E[\sum_k \epsilon_k |z_k\rangle] = \epsilon_T[\tilde{\nu}\hat{C}^{-1}|r\rangle + \tilde{\sigma}\hat{C}^{-1}|1\rangle]$$

with $\epsilon_t = \sum_k \epsilon_k$, $\tilde{\nu} = \sum_k \epsilon_k \nu_k / \epsilon_T$ and $\sigma\nu = \sum_k \epsilon_k \sigma_k / \epsilon_T$, so that the average return is still affected by portfolio investment along a single direction, determined by the weighted average of the vectors $|z_k\rangle$.

The quadratic relationship becomes instead

$$E[|\delta x\rangle\langle\delta x|] = \hat{B} + \sum_k \Delta_k |z_k\rangle\langle z_k|,$$

which does not allow to state that the correlation matrix $\hat{C}$ will have all eigenvectors corresponding to those of B, except one. In fact, each term $|z_k\rangle\langle z_k|$ has components along two directions, namely along $\hat{C}^{-1}|r\rangle$ and along $\hat{C}^{-1}|1\rangle$, as is clear from Eq.(5.12), since we have

$$|z_k\rangle = \nu_k \hat{C}^{-1}|r\rangle + \sigma_k \hat{C}^{-1}|1\rangle. \tag{C.2}$$

The coefficients are determined as usual from the minimization of risk under constraints, and they are[1]

$$\nu_k = \frac{\chi_{11} R_k - \chi_{1r}}{\chi_{11}\chi_{rr} - \chi_{1r}^2} \tag{C.3}$$

$$\sigma_k = \frac{\chi_{rr} - R_k \chi_{1r}}{\chi_{11}\chi_{rr} - \chi_{1r}^2}. \tag{C.4}$$

In the plane defined by the two directions $\hat{C}^{-1}|r\rangle$ and $\hat{C}^{-1}|1\rangle$, let us define θ_r the angle between them. Geometrical considerations show that the direction of each vector $|z_k\rangle$ forms with the vector $\hat{C}^{-1}|1\rangle$ an angle

$$\theta_k = \mathrm{Arccot}[\mathrm{Cot}(\theta_r) + \frac{A_k}{\mathrm{Sin}(\theta_r)}]$$

with

$$A_k = \frac{\sigma_k}{\nu_k} = \frac{\langle r|\hat{C}^{-1}|r\rangle - R_k \langle r|\hat{C}^{-1}|1\rangle}{R_k\langle 1|\hat{C}^{-1}|1\rangle - \langle r|\hat{C}^{-1}|r\rangle}.$$

This means that the direction is determined solely by A_k. The derivative dA/dR reads

$$\frac{\partial A_k}{\partial R_k} = \frac{\chi_{1r} - \chi_{11}\chi_{rr}}{(\chi_{11} R_k - \chi 1r)^2} \tag{C.5}$$

and it tends to zero as R grows above a value R^*, and this is fundamental since if dA/dR goes to zero, differences in R_k will not correspond to different A_k, and hence investors will have different $|z_k\rangle$, but all parallel. The single direction argument works in this case, and the analysis carried out in Section 5.32, assuming a single eigenvector deviating from those of the matrix $\hat{B}$, is correct. To get an idea of the value R^* we can look at what happens in the completely random case, that is when we assume that $|\delta x\rangle$ are uncorrelated random variables. Here the matrix elements of $\hat{C}^{-1}$ can be calculated. If $|r\rangle$ has elements of order s_r then

$$R^* = \frac{s_r}{\sqrt{N}}$$

We thus see that our monodimensional analysis will be more and more valid as the system size (number of assets N) grows. This is a quite attractive feature since, as we have seen in Appendix B, it is in the investors' interest to have N large, so as to diversify away any nonsystematic risk.

As a test of these calculations, we performed numerical simulations with 4 different values of R_k (that is, 4 portfolios). We expect that the theoretical line of

[1] We use the notation $\chi_{ab} = \langle a|\hat{C}^{-1}|b\rangle$

equation (5.18) holds, since we are using (without loss of generality) the same ϵ and Δ for the 4 components. In a first simulation, to simplify the analysis, we use different investors bu fix the random variable ξ, that is we write:

$$|\delta x\rangle = |\beta\rangle + \xi(t) \sum_k |z_k\rangle.$$

In this case the analysis is very simple, there is no need to introduce the angles θ_k and, after simple algebra, we get the effective parameter

$$\tilde{R} = \sum_k R_k \tag{C.6}$$

Curves obtained from Eq.(5.18) with the effective parameter $\tilde{R}$ in place of R perfectly fits the data, Fig C.1

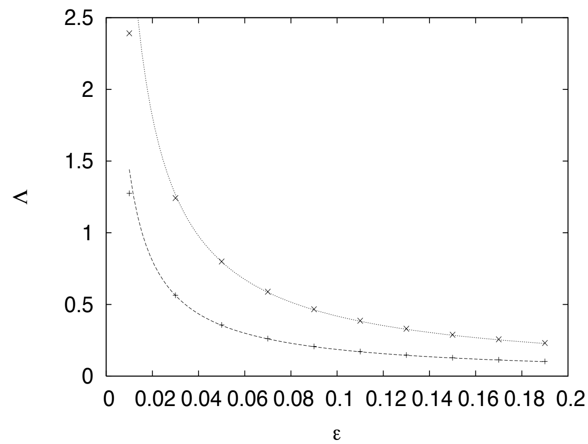

Figure C.1: Maximum eigenvalue of the correlation matrix as a function of ϵ for $N = 20$, with 4 different portfolio investment components. The different components are generated by setting $R_k = 0.1 + \rho$ with ρ uniformly drawn in the interval $[0, 0.1]$. $\Delta = 0.04$ (lower curve) and $\Delta = 0.09$ (upper curve), $\tilde{R} = 0.648141$, $\tau = 1000$, $W = 10$. Note that here each R_k is of order 0.1, while we have $s_r \sim 10^{-2}$ and hence $R* \sim 10^{-3}$. That is, $R_k >> R*$.

If instead we write

$$|\delta x\rangle = |\beta\rangle + \sum_k \xi_k(t)|z_k\rangle. \tag{C.7}$$

we have to set

$$\tilde{R} = \sqrt{\sum_k R_k^2 \frac{\Delta_k}{\tilde{\Delta}}} \tag{C.8}$$

$$\tilde{\Delta} = \sum_k \Delta_k \tag{C.9}$$

$$\tilde{\epsilon} = \sum_k \epsilon_k \tag{C.10}$$

and again the theory fits numerical simulations, Fig. C.2.

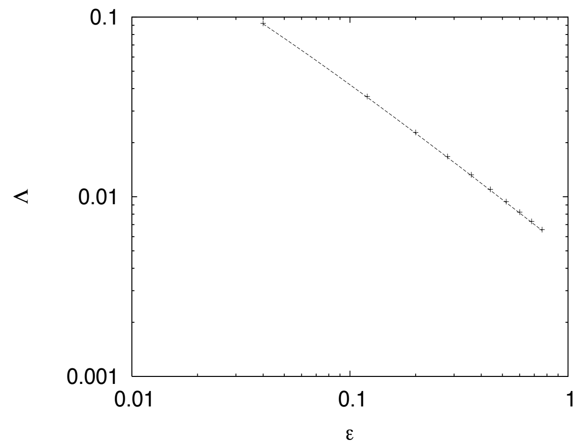

Figure C.2: Maximum eigenvalue of the correlation matrix as a function of $\tilde{\epsilon}$ for $N = 20$, with 4 different portfolio investment components simulated according to Eq.(C.7). $\Delta = 0.04$, $\tilde{R} = 0.648141$, $\tau = 1000$, $W = 1$. Note that here each R_k is of order 0.1, while we have $s_r \sim 10^{-2}$ and hence $R* \sim 10^{-3}$. That is, $R_k >> R*$.

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Acknowledgements

During the three years spent working in this Sector at SISSA, many people have helped me, in many different ways. First and foremost, I must deeply thank my Supervisor Matteo Marsili. His experience has been priceless to me, and his help and example allowed me to fight through the unavoidable difficulties that periodically arise in this work. Most importantly, he taught me what really means *capire le cose*. I have benefited from many discussions with all the people in Matteo's group at ICTP, but a particular thank goes to Ginestra Bianconi, with whom part of the ideas that gave rise to the second Part of this Thesis were developed. She has always been present when I have asked for her time. In the years, there have been many opportunities to interact with visiting scientists, and I need to thank Fernando Vega-Redondo and Giulia Iori for sharing with me their views and opinions on some of the topics presented here. Also, I wish to thank Raffaella Burioni for her suggestions about network theory. Thanks also to Guido Caldarelli for providing the data on the board of directors and for permission to use Fig. 4.2 and to Marc Potters for Fig. 4.1. Thanks also to all the people in the Statistical and Biological Physics Sector at SISSA, and to everybody in the School. This is a beautiful place where to work.