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Semistability and Decorated Bundles

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> There once lived a man who learned how to slay dragons and gave all he possessed to master the art. After three years he was fully prepared, but he found no opportunity to use his skill.
>
> - Dschuang Dsi -

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“Don’t believe in yourself, believe in me who believes in you!”
- Kamina -

”Don’t believe in the you who believes in me. Don’t believe in the me who believes in you. Believe in the you who believes in yourself.”
- Kamina -

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Chapter 1

Introduction

Historical background

According to Hartshorne [13], one of the guiding problems in Algebraic Geometry is the classification of algebraic varieties up to isomorphism. Two variants of the problem are the classification of complex projective varieties up to isomorphism and classify all closed subvarieties of a given projective space $\mathbb{P}^n$ up to projective equivalences. We remind the reader that two subvarieties of $\mathbb{P}^n$ are called *projectively equivalent*, if there is an automorphism of this ambient space which carries the first variety onto the second one. As Hartshorne also describes, these classifications problems usually fall into two parts. First, has some discrete numerical invariants such as the Hilbert polynomial of the (polarized) variety, which yields a first subdivision of the class of all objects. Second, the objects with fixed numerical invariants usually come in families of positive dimension and one has to construct a **moduli space** for them, which is an algebraic variety whose points are in natural correspondence with the set of isomorphism classes of the objects with fixed numerical data. Mumford has conceived his Geometric Invariant Theory as a major tool for constructing such moduli spaces.

Another important variant of this guiding problem is in the classification, up to isomorphism, of vector and principal bundles over smooth, or maybe singular, varieties. This problem is closely related to problem of classifying algebraic varieties up to isomorphism. In fact a vector or principal bundle over, let's say, a smooth projective variety X is nothing else that an algebraic variety E with a surjective morphism $E \to X$ and some other compatibility conditions such as local triviality. Of couse the first problem to treat is the problem of classifying vector bundles over an algebraic smooth, projective curves over $\mathbb{C}$. The basic invariants of a vector bundle E in this case are the rank, i.e., roughly speaking, the dimension of the fibre, and the degree. They determine E as a topological $\mathbb{C}$-vector bundle. The problem

of classifying all vector bundles of fixed degree d and rank r is not easy at all. The first, and simplest, cases are the following: If $r = 1$ this problem is covered by the theory of Jacobian varieties, if $X = \mathbb{P}^1$ Grothendieck's splitting theorem [12] provides the classification. Finally, if the genus of X is one the classification has been worked out by Atiyah [1] which gives a (1-1) correspondence between irreducible vector bundles of degree d and rank r and X. All other cases are not trivial in the sense that vector bundles of degree d and rank r cannot be parametrized by discrete data. Therefore one seeks a variety parameterizing them with some "good" property, replacing, in some sense, the universal property of the Jacobian.
This good property was formulated by Mumford in his definition of **coarse moduli space** in [24]. We recall that a **fine moduli space** for a given (algebraic) moduli problem is an algebraic variety M with a family U parametrised by M having the following (universal) property: for every family E parametrised by a base space S, there exists a unique (up to isomorphism) map $f : S \to M$ such that $E \simeq f^*U$. A **coarse moduli space** for a given moduli problem is an algebraic variety M with a bijection $\alpha : A/_\sim \to M$ (where A is the set of objects to be classied up to the equivalence relation $\sim$) from the set $A/_\sim$ of equivalence classes in A to M such that:

i) for every family E with base space S, the composition of the given bijection $\alpha : A/_\sim \to M$ with the function $\nu_E : S \to A/_\sim$, which sends $s \in S$ to the equivalence class $[E_s]$ of the object E_s with parameter s in the family E, is a morphism;

ii) when N is any other variety with $\beta : A/_\sim \to N$ such that the composition $\beta \circ \nu_E : S \to N$ is a morphism for each family E parametrised by a base space S, then $\beta \circ \alpha^{-1} : M \to N$ is a morphism.

However, in the costruction of the moduli space of vector bundles over curves one checks that the family of vector bundles is not bounded which implies that a coarse moduli space cannot exist. For this reason one has to consider suitable (bounded) subfamilies of the family of vector bundles: the families of **semistable** and **stable** vector bundles. We recall that a vector bundle E over a curve is (semi)stable if for any proper subbundle $F \subset E$

$$\mu(F) \doteqdot \frac{\deg F}{\operatorname{rk}(F)} \underset{(=)}{<} \mu(E).$$

Therefore large part of the theory focusses on semistable vector bundles.

The costruction of the moduli space of stable vector bundles was made by Seshadri in [28]. Unfortunately the moduli space is only quasi-projective. To compactify one has to consider **S-equivalence** classes of semistable vector bundles. S-equivalence has the following important property: two *stable*

vector bundles are S-equivalent if and only if they are isomorphic, while isomorphic strictly *semistable* vector bundles are S-equivalent but the converse is not true in general. To be more precise, given a semistable vector bundle E one constructs a particular filtration $0 \subset E_1 \subset \cdots \subset E_s \subset E_{s+1} = E$ of E, called the **Jordan-Hölder** filtration, and defines a **graded object** $\mathrm{gr}(E) \doteqdot \bigoplus_{i=1}^{s+1} E_{i+1}/E_i$ associated with the given filtration. It turns that the filtration is not unique in general but the graded objects, up to isomorphisms, are. Therefore one says that two vector bundles are S-equivalent if and only if their associated graded are isomorphic. So the moduli space of S-equivalence classes of semistable bundles exists by the same construction and is a normal projective variety compactifying the moduli space of stable bundles. The generalization of this result over higer-dimensional varieties is due to Gieseker, Maruyama and Simpson: [7], [22] and [37] respectively.
These works revealed that, on higher-dimensional manifolds, one has to include semistable torsion free sheaves in order to obtain projective moduli spaces, namely the moduli space of locally free sheaves (or vector bundles) is not projective and so one has to include also torsion free sheaves to get a projective manifold.
The same holds true for principal bundles: also for principal bundles one has to include some kind of "degenerate" objects in order to obtain a projective moduli space. In [26] Ramanathan gave a GIT costruction of the moduli space of semistable principal G-bundles on smooth projective curve. Hyeon, in [16], has generalized Ramanathan's construction to give the moduli spaces of stable principal bundles over higher-dimensional base schemes, but the resulting moduli spaces are only quasi-projective. The necessary singular generalizations of principal bundles have been considered only in the case of classical groups $O(n,\mathbb{C})$, $SO(n,\mathbb{C})$ and $Sp(n,\mathbb{C})$. Indeed, for G one of these groups, the principal G-bundles have natural interpretation as vector bundles with additional structures, and these can be extended in the context of torsion free sheaves.
In [31] Schmitt proposes a more general approach for this problem: given a principal G-bundle, with G a reductive group, and a representation $\rho : G \to \mathrm{Sl}(V)$, he constructs a *singular principal bundle* imitating the construction of a principal G-bundle from a principal $\mathrm{Gl}(V)$-bundle E over a smooth variety X and a section $X \to E/G$. Therefore, from a principal G-bundle and a representation of G in $\mathrm{Sl}(V)$, he obtained a pair (E, τ) where E is a vector bundle and τ is a morphism induced by a section $\sigma : X \to E/G$ which "remembers" the principal bundle structure. Therefore, also in this case, these vector bundles with an addittional structure can be generalized to the setting of torsion free sheaves. Schmitt then gives a notion of semistability for such singular principal bundles which generalizes the one of principal bundles and proves that there exists a (projective) coarse moduli space for the families of equivalence classes of semistable singular principal bundles. Therefore the next step is to consider vector bundles with extra structure,

the so-called "decorated" bundles.

Decorated bundles

In the framework of bundles with a decoration we recall two types of objects which incorporate all others: **decorated sheaves** and the so called **tensors**. The former were introduced by Schmitt while the latter by Gomez and Sols. We recall breafly the definitions of such objects. A *decorated sheaf* of type (a, b, c, N) over X is a pair $(\mathcal{E}, \varphi)$ where $\mathcal{E}$ is a torsion free sheaf over X and

$$\varphi : (\mathcal{E}^{\otimes a})^{\oplus b} \otimes (\det \mathcal{E}^{\vee})^{\otimes c} \longrightarrow \mathsf{N},$$

while a *tensor* of type (a, b, c, D_u) is a pair $(\mathcal{E}, \varphi)$ where $\mathcal{E}$ is a torsion free sheaf and

$$\varphi : (\mathcal{E}^{\otimes a})^{\oplus b} \otimes \longrightarrow (\det \mathcal{E})^{\otimes c} \otimes D_u,$$

where D_u is a locally free sheaf belonging to a fixed family $\{D_u\}_{u \in R}$ parametrized by a scheme R. As one can easily see these two objects are quite similar and both incorporates many types of bundles with a morphism, such as framed bundles, Higgs bundles, quadratic, orthogonal and symplectic bundles, and many others. The problem to classify decorated sheaves up to equivalence is therefore related to many classification problems in algebraic geometry. In order to solve this classification problem by establishing the existence of a coarse moduli space, one needs to introduce a notion of semistability. The notion of semistability for decorated sheaves and tensors is the same. In both cases one tests the (semi)stability of an object $(\mathcal{E}, \varphi)$ against **saturated weighted fitrations** of $\mathcal{E}$, namely against pairs $(\mathcal{E}^\bullet, \underline{\alpha})$ consisting of a filtration

$$\mathcal{E}^\bullet : \quad 0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_s \subset \mathcal{E}_{s+1} = \mathcal{E}$$

of saturated sheaves of $\mathcal{E}$, i.e., such that $\mathcal{E}/\mathcal{E}_i$ is torsion free, and a tuple

$$\underline{\alpha} = (\alpha_1, \ldots, \alpha_s)$$

of positive rational numbers. Then one says that a decorated sheaf or tensor $(\mathcal{E}, \varphi)$ is (semi)stable with respect to δ if and only if for any weighted filtration

$$P(\mathcal{E}^\bullet, \underline{\alpha}) + \delta\, \mu(\mathcal{E}^\bullet, \underline{\alpha}; \varphi) \underset{(-)}{\succ} 0.$$

Here δ is a fixed polynomial of degree $\dim(X) - 1$ and, denoting by $\mathsf{P}_{\mathcal{E}}$ the Hilbert polynomial of a sheaf, $P(\mathcal{E}^\bullet, \underline{\alpha})$ is defined as

$$\sum_i \alpha_i \left(\mathsf{P}_{\mathcal{E}} \cdot \mathrm{rk}(\mathcal{E}_i) - \mathrm{rk}(\mathcal{E}) \cdot \mathsf{P}_{\mathcal{E}_i} \right)$$

Finally $\mu(\mathcal{E}^\bullet, \underline{\alpha}; \varphi)$ is

$$- \min_{i_1, \ldots, i_a} \{ \gamma^{(i_1)} + \cdots + \gamma^{(i_a)} \mid \varphi_{|_{(\mathcal{E}_{i_1} \otimes \cdots \otimes \mathcal{E}_{i_a})^{\oplus b}}} \not\equiv 0 \}$$

where

$$\begin{aligned}\gamma &= (\gamma^{(1)}, \ldots, \gamma^{(r)}) \\ &\doteqdot \sum_i \alpha_i (\underbrace{\mathrm{rk}(\mathcal{E}_i) - r, \ldots, \mathrm{rk}(\mathcal{E}_i) - r}_{\mathrm{rk}(\mathcal{E}_i)\text{-times}}, \underbrace{\mathrm{rk}(\mathcal{E}_i), \ldots, \mathrm{rk}(\mathcal{E}_i)}_{r-\mathrm{rk}(\mathcal{E}_i)\text{-times}}). \end{aligned} \tag{1.1}$$

The notation "$\underset{(-)}{\succ}$" means that "$\succ$" has to be used in the definition of stable and "$\succeq$" in the definition of semistable. Moreover we recall that, given two polynomials p and q in $\mathbb{Q}[x]$, we write $p \underset{(-)}{\succ} q$ if and only if there exists x_0 such that $p(x) \underset{(=)}{>} q(x)$ for any $x \geq x_0$.
The classification problem for δ-semistable decorated sheaves (or tensors) with fixed Hilbert polynomial $\mathsf{P}_{\mathcal{E}} = \mathsf{P}$ is abstractly solved by a projective moduli space M^δ. The existence of M^δ was established by Schmitt over curves [33] and by Gomez and Sols over manifolds of arbitrary dimension [9]. Then Schmitt proved that these moduli spaces M^δ do not depend "too much" on δ in the sense that there are only finitely many different moduli spaces occurring among the M^δ [34].

Our objectives

As one can easily convinces oneself looking at the definition of $\mu(\mathcal{E}^\bullet, \underline{\alpha}; \varphi)$, the semistability for decorated sheaves is not easy to handle and in general is quite complicated to calculate. This fact affects the possibility of generalizing many basic tools that instead exists for vector bundles. For example, until now, there is no notion of a maximal destabilizing object for decorated sheaves nor a notion of Jordan-Hölder or Harder-Narasimhan filtration. This thesis is devoted to the study of semistability condition of decorated bundles in order to better understand and simplify it in the hope this will be useful in the study of decorated sheaves. We approach the problem in two different ways: on one side we "enclose" the above semistability condition between a stronger semistability condition and a weaker one, on the other side we try, and succeed for the case of $a = 2$, to bound the length of weighted filtrations on which one checks the semistability condition.

To be more precise: in the former approach we say that a decorated sheaf $(\mathcal{E}, \varphi)$ of type (a, b, c, N) is ε**-(semi)stable** with respect to a fixed polynomial δ of degree $\dim X - 1$ if for any subsheaf $\mathcal{F} \subset \mathcal{E}$

$$\mathrm{rk}(\mathcal{E}) \left(\mathsf{P}_{\mathcal{F}} - a\,\delta\,\varepsilon(\varphi_{|\mathcal{F}})\right) \underset{(-)}{\prec} \mathrm{rk}(\mathcal{F}) \left(\mathsf{P}_{\mathcal{E}} - a\,\delta\,\varepsilon(\varphi)\right),$$

where $\varepsilon(\varphi) = 1$ if $\varphi \neq 0$ and zero otherwise. Similarly, given a sheaf $\mathcal{E}$ and a subsheaf $\mathcal{F} \subset \mathcal{E}$, we define a function $\mathsf{k}_{\mathcal{F},\mathcal{E}}$ with values in the set $\{0, 1, \ldots, a\}$ and depending on the behaviour of φ on $\mathcal{F}$ as subsheaf of $\mathcal{E}$ (see Equation

(2.16)). Then we say that a decorated sheaf $(\mathcal{E},\varphi)$ is **k-(semi)stable** if for any subsheaf $\mathcal{F}\subset\mathcal{E}$ one has that

$$\mathrm{rk}(\mathcal{E})\,(\mathsf{P}_{\mathcal{F}}-\delta\,\mathsf{k}_{\mathcal{F},\mathcal{E}}))\ \underset{(-)}{\prec}\ \mathrm{rk}(\mathcal{F})\,(\mathsf{P}_{\mathcal{E}}-\delta\,\mathsf{k}_{\mathcal{E},\mathcal{E}})\,.$$

What happens is that

$$\varepsilon\text{-(semi)stable}\ \Rightarrow\ \text{(semi)stable}\ \Rightarrow\ \mathsf{k}\text{-(semi)stable}$$

and, if $\mathrm{rk}(\mathcal{E})=2$, (semi)stability is equivalent to k-(semi)stability. In this respect, we generalize some known results (in the case of vector bundles) to the case of ε-semistable decorated sheaf and, to a lesser extent, to the k-(semi)stable case. In fact, using ε-semistability, we find the ε-maximal destabilizing subsheaf, prove a Mehta-Ramanathan's like theorem about the behavior of slope ε-semistability under restriction to curves, find a moduli space for ε-semistable decorated sheaves and define an Uhlenbeck-Donaldson compactification for them. Since k-semistability is a little bit more complicated to handle with we managed to find a k-maximal destabilizing subsheaf and prove a Mehta-Ramanathan theorem only for rank ≤ 3.

In the latter approach we study the effective semistability condition asking ourselves which filtrations are really essential to check semistability. Let $(\mathcal{E},\varphi)$ be a decorated sheaf (or tensor), $\mathcal{E}^\bullet_{\mathtt{I}}:\ 0\subset\mathcal{E}_1\subset\cdots\subset\mathcal{E}_s\subset\mathcal{E}$ be a filtration of $\mathcal{E}$ indexed by the set $\mathtt{I}\doteqdot\{1,2,\dots,s\}$ and finally let $\underline{\alpha}_{\mathtt{I}}=(\alpha_1,\dots,\alpha_s)$ be a weight vector for the filtration. Then let us point out that, in general, it is *not* true that

$$\mu(\mathcal{E}^\bullet,\underline{\alpha};\varphi)=\sum_{i=1}^{s}\mu(0\subset\mathcal{E}_i\subset\mathcal{E},\alpha_i;\varphi).$$

We say that a filtration is **non-critical** if the above equality holds and **critical** otherwise. For any pair of disjoint subfiltrations $E^\bullet_{\mathtt{I}'}$ and $E^\bullet_{\mathtt{I}''}$ of $E^\bullet_{\mathtt{I}}$ (i.e., $\mathtt{I}',\mathtt{I}''\subset\mathtt{I}$ and $\mathtt{I}'\cap\mathtt{I}''=\emptyset$) one easily checks that

$$\mu(\mathcal{E}^\bullet_{\mathtt{I}},\underline{\alpha}_{\mathtt{I}};\varphi)\leq\mu(\mathcal{E}^\bullet_{\mathtt{I}'},\underline{\alpha}_{\mathtt{I}'};\varphi)+\mu(\mathcal{E}^\bullet_{\mathtt{I}''},\underline{\alpha}_{\mathtt{I}''};\varphi).$$

Therefore the length of a filtration is important for checking semistability condition in the sense that it could happen that

$$(P_{\mathtt{I}'}+\delta\mu_{\mathtt{I}'})+(P_{\mathtt{I}''}+\delta\mu_{\mathtt{I}''})\succeq 0\succ P_{\mathtt{I}}+\delta\mu_{\mathtt{I}},$$

where with $P_{\mathtt{I}}$ and $\mu_{\mathtt{I}}$ we mean $P(\mathcal{E}^\bullet_{\mathtt{I}},\underline{\alpha}_{\mathtt{I}})$ and $\mu(\mathcal{E}^\bullet_{\mathtt{I}},\underline{\alpha}_{\mathtt{I}};\varphi)$ respectively.
At least when $a=2$ we proved that, given a weighted filtration of length ≥ 2, there always exist two weighted filtrations such that

$$(P_{\mathtt{I}'}+\delta\mu_{\mathtt{I}'})+(P_{\mathtt{I}''}+\delta\mu_{\mathtt{I}''})=P_{\mathtt{I}}+\delta\mu_{\mathtt{I}}.\tag{1.2}$$

This result clearly implies that in this case the semistability condition can be checked only over filtrations of length ≤ 2. As we remarked before, orthogonal bundles are examples of decorated bundles of type $(2, 1, 0, \mathcal{O}_X)$ and therefore they inherit the (semi)stability condition of decorated vector bundles. However orthogonal bundles have already a notion of (semi)stability: an orthogonal bundle E over a smooth curve is sayed (semi)stable if every proper isotropic subbundle F of E has degree zero. As an application of this result we will show that, at least in the case of orthogonal bundles over curves, the two (semi)stability conditions coincide. Finally, as a futher application, we implement a java code which finds the sets $\mathtt{I}'$ and $\mathtt{I}''$ for which equality 1.2 holds.

Contents and results chapter by chapter

* In Chapter 2 we recall the definition of decorated bundles and sheaves and we present their notion of (semi)stability and slope (semi)stability. In Section 2.2.1 we introduce the notions of ε-(semi)stability and k-(semi)stability. After that we explain in which sense principal bundles, Higgs bundles, quadric, orthogonal, symplectic and framed bundles can be regarded as decorated vector bundles or sheaves (Section 2.3).

* Chapter 3 is devoted to generalizing the Mehta-Ramanathan theorem to the case of ε-(semi)stable and k-(semi)stable decorated sheaves. In both cases, to do this, one needs some preliminar results. After introducing the notion of decorated coherent sheaves and semistability notions for these objects (Section 3.1), we treat the ε-semistability and k-semistability cases separately (Section 3.2 and 3.3 respectively). In the former case, we first prove, using quite standard arguments, the existence of a (unique) ε-maximal destabilizing subsheaf for decorated sheaves (Section 3.2.1). Then we recall the definition of families of decorated sheaves flat over the fibre of a morphism (Section 3.2.2).
  In many constructions quotiens are easier to handle than subobjects. However quotients of decorated sheaves (or tensors) are not well defined in general as decorated sheaves. In fact, given a quotient $\mathcal{Q}$ of a decorated sheaf $(\mathcal{E}, \varphi)$, i.e., a quotient $\mathcal{E} \twoheadrightarrow \mathcal{Q}$, it is not always possible to induce a non-zero morphism $\overline{\varphi}$ on $\mathcal{Q}$. Despite this there is a one-to-one correspondence between decorated subsheaves of a given decorated sheaf and its quotients. In Section 3.2.3 we face this problem and "translate" the semistability condition, which is checked over subsheaves, to a condition over quotients. After recalling in Section 3.2.4 the notion of the quot functor $\underline{\mathsf{Q}_{\text{uot}}}_{X/S}(\mathcal{E}, \mathsf{P})$, we prove the opennes of the ε-semistability condition and the existence of a relative ε-maximal destabilizing subsheaf for families of decorated sheaves (Section 3.2.5).

Finally, in Section 3.2.7, we state and prove a restriction theorem for slope ε-semistable decorated sheaves:

(**Theorem 51**). *Let X be a smooth projective surface and $\mathcal{O}_X(1)$ a very ample line bundle. Let $(\mathcal{E},\varphi)$ be a slope ε-semistable decorated sheaf. There is an integer a_0 with the following property: for all $\mathsf{a} \geq \mathsf{a}_0$ there is a dense open subset $U_{\mathsf{a}} \subset |\mathcal{O}_X(\mathsf{a})|$ such that for all $D \in U_{\mathsf{a}}$ the divisor D is smooth and $(\mathcal{E},\varphi)_{|D}$ is slope ε-semistable.*

In Section 3.3 we prove for the k-semistability case almost the same results that we proved for the ε-semistability but we restrict our attention only to rank ≤ 3 decorated bundles since we were not be able to prove the existence of a k-maximal destabilizing subsheaf for rank greater than 3.
Finally in Section 3.4 we give some important remarks about the previous results.

∗ Chapter 4 is devoted to the costruction of the moduli space of ε-semistable decorated sheaves and the costruction of the Uhlenbeck-Donaldson compactification for slope ε-semistable decorated sheaves. In the first section of this chapter we recall the definition of families of decorated sheaves and prove that the families of ε-semistable decorated sheaves and slope ε-semistable decorated sheaves with fixed Hilbert polynomial are bounded (Lemma 62 and Corollary 63). We prove that the ε-semistability condition "cames from" a GIT semistability condition (Lemma 64 and Proposition 65). Thanks to this result we are able to construct a moduli space for such objects as a GIT quotient of an appropriate scheme:

(**Teorem 66**). *Let δ a rational polynomial of degree $\dim X - 1$ with positive leading coefficient. There is a projective scheme $\mathcal{M}_{\delta}^{\varepsilon\text{-}ss}(\mathsf{P},\underline{\mathsf{t}})$ that corepresents the moduli functor $\underline{\mathfrak{M}}_{\delta}^{\varepsilon\text{-}ss}(\mathsf{P},\underline{\mathsf{t}})$. Moreover there is an open subscheme $\mathcal{M}_{\delta}^{\varepsilon\text{-}ss}(\mathsf{P},\underline{\mathsf{t}})$ that represents the subfunctor $\underline{\mathfrak{M}}_{\delta}^{\varepsilon\text{-}ss}(\mathsf{P},\underline{\mathsf{t}})$.*

The classical Mehta-Ramanathan theorems are used in the algebro-geometric construction of the Uhlenbeck-Donaldson compactification of moduli space of slope stable vector bundles on a nonsingular projective surface X. In the same way we construct the Uhlenbeck-Donaldson compactification of moduli space of slope ε-stable decorated sheaves on X (Section 4.2). More precisely, in Section 4.2.1, we recall some definitions and results about the Grothendieck group of coherent sheaves over a scheme; in Section 4.2.2 we re-adapt the techniques used in [4] in order to construct a line bundle $\mathcal{L}(n_1,n_2)$ and prove that it is (finitely) generated by its invariant sections (Proposition 70). This fact is the core of the constructions of Section 4.2.3 and essentially leads, together with some technical results, to the construction of the moduli space of slope ε-semistable decorated sheaves:

$$\mathcal{M}_{\overline{\delta}}^{\text{slope-}\varepsilon\text{-ss}}(\mathsf{c},\underline{\mathsf{t}}) \doteqdot \operatorname{Proj}\left(\bigoplus_{k\geq 0} H^0(\mathtt{R}_{\overline{\delta}}^{\text{slope-}\varepsilon\text{-}ss}, \mathcal{L}(n_1,n_2)^{\otimes kN})^{\mathrm{Sl}(\mathsf{P}(m))}\right).$$

The closure inside $\mathcal{M}_{\overline{\delta}}^{\text{slope-}\varepsilon\text{-ss}}(\mathsf{c},\underline{\mathsf{t}})$ of the slope ε-stable decorated sheaves is called the Uhlenbeck-Donaldson compactification.

∗ In Chapter 5 we investigate the semistability condition of decorated sheaves in order to simplify it. In particular, in Section 5.1, we prove the following reduction theorem:

(**Theorem 77**). *Let $(\mathcal{E},\varphi)$ be a decorated sheaf of type $(2,b,c,\mathsf{N})$. It is enough to check the semistability condition on subbundles and critical weighted filtrations of length two.*

Thanks to this result we can prove that, in order to check the semistability condition of $a=2$ decorated sheaves, we can restrict our attention not only to length ≤ 2 weighted filtrations but also to length ≤ 2 filtrations with weight vector identically 1 (Lemma 81). Therefore, making some futher calculations, we are able to prove that a decorated sheaf $(\mathcal{E},\varphi)$ of type $(2,b,c,\mathsf{N})$ is (semi)stable with respect to δ if it is k-(semi)stable and, for any critical filtration $0\subset\mathcal{E}_i\subset\mathcal{E}_j\subset\mathcal{E}$ of length 2,

$$(r_{\mathcal{E}_i}+r_{\mathcal{E}_j})\mathsf{P}_{\mathcal{E}} - r_{\mathcal{E}}(\mathsf{P}_{\mathcal{E}_i}+\mathsf{P}_{\mathcal{E}_j}) - 2\delta(r_{\mathcal{E}_i}+r_{\mathcal{E}_j}-r) \underset{(-)}{\succ} 0.$$

In section 5.2 we state (and prove) the reduction theorem (Theorem 77) in some more general context. Indeed, if (E,φ) is a decorated bundle of type (a,b,N) and rank r and $(E^\bullet,\underline{\alpha})$ is a weighted filtration of (E,φ) indexed by $\mathtt{I}$ we construct a $\underbrace{r\times\cdots\times r}_{a\text{-times}}$ "matrix" $M_{\mathtt{I}}(E^\bullet;\varphi)$ which represent the behavior of φ (to be equal or different from zero) over the given filtration. Defining $R_{\mathtt{I}}(l)\doteqdot\sum_{i\in\mathtt{I},\,i\geq l}\alpha_i$ and setting

$$R_{\mathtt{I}}\doteqdot\max_{i,j\in\overline{\mathtt{I}}}\{R_{\mathtt{I}}(i_1)+\cdots+R_{\mathtt{I}}(i_a)\,|\,m_{i_1\dots i_a}\neq 0\}.$$

We prove the following theorem

(**Theorem 84**). *Let $a=2$. Fix a well-ordered set $\mathtt{I}$, a vector $\alpha_{\mathtt{I}}$ of real numbers and a symmetric matrix $M_{\mathtt{I}}(E^\bullet;\varphi)$ as before. Denote by $|\cdot|$ the cardinality of a set. Then exists $t\in\mathbb{N}$, sets $\mathtt{J}^{(1)},\dots,\mathtt{J}^{(t)}$ and positive real vectors $\alpha_{\mathtt{J}^{(1)}},\dots,\alpha_{\mathtt{J}^{(t)}}$ such that*

i) $|\mathtt{J}^{(s)}|\leq 2$ *for any* $s=1,\dots,t$*;*

ii) $\mathtt{J}^{(1)} \cup \cdots \cup \mathtt{J}^{(t)} = \mathtt{I}$;

iii) $\sum_{s=1}^{t} \alpha_i^{(s)} = \alpha_i$, *where is to be understood that* $\alpha_i^{(s)} = 0$ *if* $i \notin \mathtt{J}^{(s)}$;

iv) $\sum_{s=1}^{t} R_{\mathtt{J}^{(s)}} = R_{\mathtt{I}}$.

In Section 5.3, as an application of Theorem 77, we show that the usual notion of (semi)stability for orthogonal bundles over curves coincides with the (semi)stability condition induced by considering orthogonal bundles as decorated bundles. Finally, in Section 5.4 we briefly discuss a java implementation of an algorithm which "decomposes" matrices in the sense of Theorem 84; while in Section 5.5 we discuss the possibility of extending the above result to $a \geq 3$.

Chapter 2

Decorated Sheaves

2.1 Definition and first properties

Let X be a smooth projective variety over $\mathbb{C}$ of dimension n and fix $\mathcal{O}_X(1)$ an ample line bundle. Let $\delta = \delta(x) \doteqdot \delta_{n-1}x^{n-1} + \cdots + \delta_1 x + \delta_0$ be a fixed polynomial with positive leading coefficient and $\overline{\delta} \doteqdot \delta_{n-1}$.

Definition 1. Let N be a line bundle over X and let a, b, c be nonnegative integers. A **decorated vector bundle** of type $\underline{\mathfrak{t}} = (a, b, c, \mathsf{N})$ over X is the datum of a vector bundle E over X and a morphism

$$\varphi : E_{a,b,c} \doteqdot (E^{\otimes a})^{\oplus b} \otimes (\det E)^{\otimes -c} \longrightarrow \mathsf{N} \tag{2.1}$$

A **decorated sheaf** of type $\underline{\mathfrak{t}}$ is instead a pair $(\mathcal{E}, \varphi)$ such that $\mathcal{E}$ is a torsion free sheaf and φ is a morphism as in (2.1). Sometimes we will call these objects simply decorated sheaves (respectively bundles) instead of decorated sheaves (resp. bundles) of type $\underline{\mathfrak{t}} = (a, b, c, \mathsf{N})$ if the input data are understood.

Remark 2. Note that, although $\mathcal{E}$ is torsion free, the sheaf $\mathcal{E}_{a,b,c}$ may have torsion.

Now we want to define morphisms between such objects. Let $(\mathcal{E}, \varphi)$ and $(\mathcal{E}', \varphi')$ be decorated sheaves (resp. bundles) of the same type $\underline{\mathfrak{t}}$. A morphism of sheaves (resp. bundles) $f : \mathcal{E} \to \mathcal{E}'$ is a **morphism of decorated sheaves** (resp. **bundles**) if exists a scalar morphism $\lambda : \mathsf{N} \to \mathsf{N}$ making the following diagram commute:

$$\begin{array}{ccc} \mathcal{E}_{a,b,c} & \xrightarrow{f_{a,b,c}} & \mathcal{E}'_{a,b,c} \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi'} \\ \mathsf{N} & \xrightarrow{\lambda} & \mathsf{N}. \end{array} \tag{2.2}$$

We will say that a morphism of decorated sheaves (bundles) $f:(\mathcal{E},\varphi)\to(\mathcal{E}',\varphi')$ is **injective** if exists an injective morphism of sheaves (bundles) $f:\mathcal{E}\hookrightarrow\mathcal{E}'$ and a non-zero scalar morphism λ such that the above diagram commutes. Analogously we will say that a morphism of decorated sheaves (bundles) $f:(\mathcal{E},\varphi)\to(\mathcal{E}',\varphi')$ is a **surjective** if exists a surjective morphism of sheaves (bundles) $f:\mathcal{E}\to\mathcal{E}'$ and a scalar morphism λ making diagram (2.2) commute. Finally we will say that $(\mathcal{E},\varphi)$ and $(\mathcal{E}',\varphi')$ are **equivalent** if exists an injective and surjective morphism of decorated sheaves (bundles) between them.

2.2 Semistability conditions

Let $(\mathcal{E},\varphi)$ be a decorated sheaf of type $\underline{\mathfrak{t}}=(a,b,c,\mathsf{N})$ and let $r=\mathrm{rk}(\mathcal{E})$. We want to introduce a notion of semistability. To this end let

$$0\subsetneq\mathcal{E}_{i_1}\subsetneq\cdots\subsetneq\mathcal{E}_{i_s}\subsetneq\mathcal{E}_r=\mathcal{E} \tag{2.3}$$

be a filtration of saturated subsheaves of $\mathcal{E}$ such that $\mathrm{rk}(\mathcal{E}_{i_j})=i_j$ for any $j=1,\dots,s$, let $\underline{\alpha}=(\alpha_{i_1},\dots,\alpha_{i_s})$ be a vector of positive rational numbers and finally let $\mathtt{I}=\{i_1,\dots,i_s\}$ be the set of indexes appearing in the filtration. We will refer to the pair $(\mathcal{E}^\bullet,\underline{\alpha})_{\mathtt{I}}$ as **weighted filtration** of $\mathcal{E}$ indexed by $\mathtt{I}$ or simply weighted filtration if the set of indexes is clear from the context, moreover we will denote by $|\mathtt{I}|$ the cardinality of the set $\mathtt{I}$. Such a weighted filtration defines the polynomial

$$P_{\mathtt{I}}(\mathcal{E}^\bullet,\underline{\alpha})\doteqdot\sum_{i\in\mathtt{I}}\alpha_i\left(\mathsf{P}_{\mathcal{E}}\cdot\mathrm{rk}(\mathcal{E}_i)-\mathrm{rk}(\mathcal{E})\cdot\mathsf{P}_{\mathcal{E}_i}\right), \tag{2.4}$$

and the rational number

$$L_{\mathtt{I}}(\mathcal{E}^\bullet,\underline{\alpha})\doteqdot\sum_{i\in\mathtt{I}}\alpha_i\left(\deg\mathcal{E}\cdot\mathrm{rk}(\mathcal{E}_i)-\mathrm{rk}(\mathcal{E})\cdot\deg\mathcal{E}_i\right). \tag{2.5}$$

Moreover we associate with $(\mathcal{E}^\bullet,\underline{\alpha})_{\mathtt{I}}$ the following rational number depending also on φ:

$$\mu_{\mathtt{I}}(\mathcal{E}^\bullet,\underline{\alpha};\varphi)\doteqdot-\min_{i_1,\dots,i_a\in\overline{\mathtt{I}}}\{\gamma_{\mathtt{I}}^{(i_1)}+\cdots+\gamma_{\mathtt{I}}^{(i_a)}\,|\,\varphi|_{(\mathcal{E}_{i_1}\otimes\cdots\otimes\mathcal{E}_{i_a})^{\oplus b}}\not\equiv 0\} \tag{2.6}$$

where $\overline{\mathtt{I}}\doteqdot\mathtt{I}\cup\{r\}$, $\mathsf{P}_{\mathcal{E}}$ (respectively $\mathsf{P}_{\mathcal{E}_i}$) is the Hilbert polynomial of $\mathcal{E}$ (resp. $\mathcal{E}_i$) and

$$\begin{aligned}\gamma_{\mathtt{I}}&=(\gamma_{\mathtt{I}}^{(1)},\dots,\gamma_{\mathtt{I}}^{(r)})\\&\doteqdot\sum_{i\in\mathtt{I}}\alpha_i(\underbrace{\mathrm{rk}(\mathcal{E}_i)-r,\dots,\mathrm{rk}(\mathcal{E}_i)-r}_{\mathrm{rk}(\mathcal{E}_i)\text{-times}},\underbrace{\mathrm{rk}(\mathcal{E}_i),\dots,\mathrm{rk}(\mathcal{E}_i)}_{r-\mathrm{rk}(\mathcal{E}_i)\text{-times}}).\end{aligned} \tag{2.7}$$

Definition 3 (**Semistability**). A decorated sheaf $(\mathcal{E},\varphi)$ of type (a,b,c,N) is δ-**(semi)stable** if for any weighted filtration $(\mathcal{E}^\bullet,\underline{\alpha})$ the following inequality holds:

$$P_{\mathtt{I}}(\mathcal{E}^\bullet,\underline{\alpha})+\delta\mu_{\mathtt{I}}(\mathcal{E}^\bullet,\underline{\alpha};\varphi)\underset{(-)}{\succ}0. \tag{2.8}$$

It is **slope** $\overline{\delta}$**-(semi)stable** if

$$L_{\mathtt{I}}(\mathcal{E}^\bullet,\underline{\alpha})+\overline{\delta}\mu_{\mathtt{I}}(\mathcal{E}^\bullet,\underline{\alpha};\varphi)\underset{(=)}{>}0. \tag{2.9}$$

Remark 4.

1. The morphism $\varphi:\mathcal{E}_{a,b,c}\to\mathsf{N}$ induces a morphism $\mathcal{E}_{a,b}\to(\det\mathcal{E})^{\otimes c}\otimes\mathsf{N}$. With abuse of notation, we still refer to the former by φ. In this context is easy to see that a decorated sheaf of type (a,b,c,N) corresponds (uniquely up to isomorphism) to a decorated sheaf of type $(a,b,0,(\det\mathcal{E})^{\otimes c}\otimes\mathsf{N})$. Therefore the category of decorated sheaves (with fixed determinant $=\mathcal{L}$) of type (a,b,c,N) is equivalent to the category of decorated sheaves (with fixed determinant $=\mathcal{L}$) of type $(a,b,0,\mathcal{L}^{\otimes c}\otimes\mathsf{N})$. From now on we will always identify such decorated sheaves.

2. Let $(\mathcal{E}^\bullet,\underline{\alpha})$ be a weighted filtration and suppose that $\mu_{\mathtt{I}}=-(\gamma_{\mathtt{I}}^{(i_1)}+\cdots+\gamma_{\mathtt{I}}^{(i_a)})$, then there exists at least one permutation $\sigma:\{i_1,\dots,i_a\}\to\{i_1,\dots,i_a\}$ such that $\varphi|_{(\mathcal{E}_{\sigma(i_1)}\otimes\cdots\otimes\mathcal{E}_{\sigma(i_a)})^{\oplus b}}\not\equiv0$. We can say that, although the morphism φ is not symmetric, the semistability condition has a certain symmetric behavior.

3. From now on we will write

$$\varphi|_{(\mathcal{E}_{i_1}\diamond\cdots\diamond\mathcal{E}_{i_a})^{\oplus b}}\not\equiv0$$

if there exists at least one permutation $\sigma:\{i_1,\dots,i_a\}\to\{i_1,\dots,i_a\}$ such that $\varphi|_{(\mathcal{E}_{\sigma(i_1)}\otimes\cdots\otimes\mathcal{E}_{\sigma(i_a)})^{\oplus b}}\not\equiv0$.

Fix now a weighted filtration $(\mathcal{E}^\bullet,\underline{\alpha})$ indexed by $\mathtt{I}$, define $r_s\doteqdot\mathrm{rk}(\mathcal{E}_s)$ and suppose that the minimum of $\mu_{\mathtt{I}}(\mathcal{E}^\bullet,\underline{\alpha};\varphi)$ is attained in $i_1,\dots,i_a$. Then

$$\begin{aligned}\mu_{\mathtt{I}}(\mathcal{E}^\bullet,\underline{\alpha};\varphi)&=-\left(\gamma_{\mathtt{I}}^{(i_1)}+\cdots+\gamma_{\mathtt{I}}^{(i_a)}\right)\\&=-\left(\sum_{s\in\mathtt{I}}\alpha_sr_s-\sum_{s\geq i_1}\alpha_sr+\cdots+\sum_{s\in\mathtt{I}}\alpha_sr_s-\sum_{s\geq i_a}\alpha_sr\right)\\&=-a\sum_{s\in\mathtt{I}}\alpha_sr_s+r\left(\sum_{s\geq i_1}\alpha_s+\cdots+\sum_{s\geq i_a}\alpha_s\right).\end{aligned}$$

Then define

$$R_{\mathtt{I}}(l) \doteqdot \sum_{s\geq l, s\in \mathtt{I}} \alpha_s \text{ for } l\in \mathtt{I} \text{ and } R_{\mathtt{I}}(r)\doteqdot 0,$$

$$R_{\mathtt{I}} = R_{\mathtt{I}}(\mathcal{E}^\bullet, \underline{\alpha}; \varphi) \doteqdot \max_{i_1,\dots,i_a\in \overline{\mathtt{I}}} \left\{ R_{\mathtt{I}}(i_1)+\cdots+R_{\mathtt{I}}(i_a) \,|\, \varphi|_{(\mathcal{E}_{i_1}\otimes\cdots\otimes\mathcal{E}_{i_a})^{\oplus b}} \not\equiv 0 \right\} \tag{2.10}$$

and finally fix, for any $i\in \mathtt{I}$, the following quantities

$$C_i \doteqdot r_i \mathsf{P}_{\mathcal{E}} - r\mathsf{P}_{\mathcal{E}_i} - ar_i, \tag{2.11}$$
$$c_i \doteqdot r_i \deg \mathcal{E} - r\deg \mathcal{E}_i - ar_i. \tag{2.12}$$

Therefore the semistability condition (2.8) is equivalent to the following

$$\sum_{i\in \mathtt{I}} \alpha_i C_i + r\delta R_{\mathtt{I}} \underset{(=)}{>} 0, \tag{2.13}$$

while the slope semistability condition (2.9) is equivalent to the following

$$\sum_{i\in \mathtt{I}} \alpha_i c_i + r\overline{\delta} R_{\mathtt{I}} \underset{(=)}{>} 0. \tag{2.14}$$

Sometimes, for convenience's sake, we will write $R_{\mathtt{I}}(i_1,\dots,i_a)$ instead of $R_{\mathtt{I}}(i_1)+\cdots+R_{\mathtt{I}}(i_a)$.

2.2.1 Others notions of semistabilities

Let $(\mathcal{E},\varphi)$ be as before and let $\mathcal{F}$ be a subsheaf of $\mathcal{E}$ then define

$$\varepsilon_{\mathcal{F}} = \varepsilon(\mathcal{F},\varphi) \doteqdot \begin{cases} 1 \text{ if } \varphi|_{\mathcal{F}_{a,b}} \not\equiv 0 \\ 0 \text{ otherwise,} \end{cases} \tag{2.15}$$

and

$$\mathsf{k}_{\mathcal{F},\mathcal{E}} = \mathsf{k}(\mathcal{F},\mathcal{E},\varphi) \doteqdot \begin{cases} a \text{ if } \varphi|_{\mathcal{F}_{a,b}} \not\equiv 0 \\ a-s \text{ if } \varphi|_{\mathcal{F}^{\diamond(a-s)}\diamond\mathcal{E}^{\diamond s}} \not\equiv 0 \text{ and } \varphi|_{\mathcal{F}^{\diamond(a-s+1)}\diamond\mathcal{E}^{\diamond s-1}} \equiv 0 \\ 0 \text{ otherwise,} \end{cases} \tag{2.16}$$

Here with the notation $\mathcal{F}^{\diamond(a-s)}\diamond\mathcal{E}^{\diamond s}$ we mean any tensor product between $\mathcal{E}$ and $\mathcal{F}$ where $\mathcal{E}$ appears exactly s-times while $\mathcal{F}$ appears $a-s$-times, and when we write $\varphi|_{(\mathcal{F}^{\diamond a-s}\diamond\mathcal{E}^{\diamond s})^{\oplus b}} \not\equiv 0$ we mean that there exists at least one tensor product between $\mathcal{F}$ and $\mathcal{E}$ over which φ is not identically zero (see Remark 4 point (3)).

Let $(\mathcal{E},\varphi)$ be a decorated sheaf; we will say that $(\mathcal{E},\varphi)$ is **ε-(semi)stable**, **slope ε-(semi)stable**, **k-(semi)stable** or **slope k-(semi)stable** if and only if for any subsheaf $\mathcal{F}$ the following inequalities hols:

$$\varepsilon\text{-(semi)stable} \qquad \mathsf{p}_{\mathcal{F}} - \frac{a\delta\varepsilon_{\mathcal{F}}}{\mathrm{rk}(\mathcal{F})} \underset{(-)}{\prec} \mathsf{p}_{\mathcal{E}} - \frac{a\delta}{\mathrm{rk}(\mathcal{E})} \tag{2.17}$$

$$\text{slope } \varepsilon\text{-(semi)stable} \qquad \mu(\mathcal{F}) - \frac{a\overline{\delta}\varepsilon_{\mathcal{F}}}{\mathrm{rk}(\mathcal{F})} \underset{(=)}{<} \mu(\mathcal{E}) - \frac{a\overline{\delta}}{\mathrm{rk}(\mathcal{E})} \tag{2.18}$$

$$\mathsf{k}\text{-(semi)stable} \qquad \mathsf{p}_{\mathcal{F}} - \frac{\delta\mathsf{k}_{\mathcal{F},\mathcal{E}}}{\mathrm{rk}(\mathcal{F})} \underset{(-)}{\prec} \mathsf{p}_{\mathcal{E}} - \frac{a\delta}{\mathrm{rk}(\mathcal{E})} \tag{2.19}$$

$$\text{slope } \mathsf{k}\text{-(semi)stable} \qquad \mu(\mathcal{F}) - \frac{\overline{\delta}\mathsf{k}_{\mathcal{F},\mathcal{E}}}{\mathrm{rk}(\mathcal{F})} \underset{(=)}{<} \mu(\mathcal{E}) - \frac{a\overline{\delta}}{\mathrm{rk}(\mathcal{E})} \tag{2.20}$$

where $\mathsf{p}_{\mathcal{F}} \doteqdot \frac{\mathsf{P}_{\mathcal{F}}}{\mathrm{rk}(\mathcal{F})}$ is the reduced Hilbert polynomial.

The above conditions are equivalent to the following:

$$(2.17) \Leftrightarrow \mathsf{P}_{\mathcal{E}}\mathrm{rk}(\mathcal{F}) - \mathrm{rk}(\mathcal{E})\mathsf{P}_{\mathcal{F}} + a\delta(\mathrm{rk}(\mathcal{E})\varepsilon_{\mathcal{F}} - \mathrm{rk}(\mathcal{F})) \underset{(-)}{\succ} 0$$

$$(2.18) \Leftrightarrow \deg\mathcal{E}\mathrm{rk}(\mathcal{F}) - \mathrm{rk}(\mathcal{E})\deg\mathcal{F} + a\overline{\delta}(\mathrm{rk}(\mathcal{E})\varepsilon_{\mathcal{F}} - \mathrm{rk}(\mathcal{F})) \underset{(=)}{>} 0$$

$$(2.19) \Leftrightarrow \mathsf{P}_{\mathcal{E}}\mathrm{rk}(\mathcal{F}) - \mathrm{rk}(\mathcal{E})\mathsf{P}_{\mathcal{F}} + \delta(\mathrm{rk}(\mathcal{E})\mathsf{k}_{\mathcal{F},\mathcal{E}} - a\mathrm{rk}(\mathcal{F})) \underset{(-)}{\succ} 0$$

$$(2.20) \Leftrightarrow \deg\mathcal{E}\mathrm{rk}(\mathcal{F}) - \mathrm{rk}(\mathcal{E})\deg\mathcal{F} + \overline{\delta}(\mathrm{rk}(\mathcal{E})\mathsf{k}_{\mathcal{F},\mathcal{E}} - a\mathrm{rk}(\mathcal{F})) \underset{(=)}{>} 0.$$

Moreover note that k-(semi)stability is equivalent to the usual (semi)stability applied to filtrations of length one. In fact let $\mathcal{F}$ be a subsheaf of $\mathcal{E}$ and consider the filtration $0 \subset \mathcal{F} \subset \mathcal{E}$ with weight vector $\underline{\alpha} = \underline{1}$. An easy calculation shows that

$$\begin{aligned} P(0 \subset \mathcal{F} \subset \mathcal{E}, \underline{1}) + \delta\,\mu(0 \subset \mathcal{F} \subset \mathcal{E}, \underline{1}; \varphi) \ = \\ = \mathsf{P}_{\mathcal{E}}\mathrm{rk}(\mathcal{F}) - \mathrm{rk}(\mathcal{E})\mathsf{P}_{\mathcal{F}} \ + \ \delta\,(\mathrm{rk}(\mathcal{E})\mathsf{k}_{\mathcal{F},\mathcal{E}} - a\mathrm{rk}(\mathcal{F})). \end{aligned}$$

More precisely, these three notions of semistability are related in the following way:

Proposition 5. *ε-(semi)stable $\Rightarrow$ (semi)stable $\Rightarrow$ k-(semi)stable.*

Proof. Let $(\mathcal{E},\varphi)$ be a ε-(semi)stable decorated sheaf of rank $\mathrm{rk}(\mathcal{E}) = r$, let $(\mathcal{E}^\bullet, \underline{\alpha})$ be a weighted filtration indexed by $\mathtt{I}$ and let r_i be the rank of $\mathcal{E}_i$. For any $i \in \mathtt{I}$,

$$\mathsf{P}_{\mathcal{E}} r_i - r\mathsf{P}_{\mathcal{E}_i} + a\delta(r\varepsilon_{\mathcal{E}_i} - r_i) \underset{(-)}{\succ} 0,$$

therefore

$$\sum_{i\in\mathtt{I}} \alpha_i \left(\mathsf{P}_{\mathcal{E}} r_i - r\mathsf{P}_{\mathcal{E}_i} + a\delta(r\varepsilon_{\mathcal{E}_i} - r_i)\right) = P_{\mathtt{I}} + a\delta(r\sum_{i\in\mathtt{I}}\alpha_i\varepsilon_{\mathcal{E}_i} - \sum_{i\in\mathtt{I}}\alpha_i r_i) \underset{(-)}{\succ} 0.$$

We want to show that $P_{\mathtt{I}} + \delta\mu_{\mathtt{I}} \underset{(-)}{\succ} 0$. Denote by $\varepsilon_i \doteqdot \varepsilon_{\mathcal{E}_i}$ and let $j_0 \doteqdot \min\{k \in \mathtt{I} \,|\, \varepsilon_k \neq 0\}$. Therefore

$$\begin{aligned}
\mu_{\mathtt{I}} &= -\min\{\gamma_{\mathtt{I}}^{(i_1)} + \cdots + \gamma_{\mathtt{I}}^{(i_a)} \,|\, \varphi|_{(E_{i_1}\otimes\cdots\otimes E_{i_a})^{\oplus b}} \neq 0\} \\
&\geq -a\gamma_{\mathtt{I}}^{(j_0)} \\
&= a\left(\sum_{i\geq j_0, i\in\mathtt{I}} \alpha_i r - \sum_{i\in\mathtt{I}} \alpha_i r_i\right) \\
&= a\left(\sum_{i\in\mathtt{I}} \alpha_i \varepsilon_i r - \sum_{i\in\mathtt{I}} \alpha_i r_i\right) \\
&= a\left(\sum_{i\in\mathtt{I}} \alpha_i(\varepsilon_i r - r_i)\right).
\end{aligned}$$

So

$$\begin{aligned}
P_{\mathtt{I}} + \delta\mu_{\mathtt{I}} \underset{(-)}{\succ} P_{\mathtt{I}} + a\delta\left(\sum_{i\in\mathtt{I}} \alpha_i(\varepsilon_i r - r_i)\right) = \\
= \sum_{i\in\mathtt{I}} \alpha_i \left(\mathsf{P}_{\mathcal{E}} r_i - r\mathsf{P}_{\mathcal{E}_i} + a\delta(r\varepsilon_{\mathcal{E}_i} - r_i)\right) \underset{(-)}{\succ} 0,
\end{aligned}$$

and we are done.

Finally, given a (semi)stable decorated sheaf, we want to show that is k-(semi)stable. Let $\mathcal{F}$ be a subsheaf of $\mathcal{E}$ of rank $r_{\mathcal{F}}$; if we consider the filtration $0 \subset \mathcal{F} \subset \mathcal{E}$ with weights identically 1, after small calculation ones get that

$$0 \underset{(-)}{\prec} P(0 \subset \mathcal{F} \subset \mathcal{E}, \underline{1}) + \delta\mu(0 \subset \mathcal{F} \subset \mathcal{E}, \underline{1}; \varphi) = \mathsf{P}_{\mathcal{E}} r_{\mathcal{F}} - r\mathsf{P}_{\mathcal{F}} + \delta(r\mathsf{k}_{\mathcal{F},\mathcal{E}} - ar_{\mathcal{F}})$$

and we have done. ♦

Note that $\mu(\mathcal{E}^\bullet, \underline{\alpha}; \varphi)$ is not additive for all filtrations, i.e., it is not always true that

$$\mu(\mathcal{E}^\bullet, \underline{\alpha}; \varphi) = \sum_{i\in\mathtt{I}} \mu(0 \subset \mathcal{E}_i \subset \mathcal{E}, \alpha_i; \varphi). \tag{2.21}$$

We will call **non-critical** a filtration for which (2.21) holds and **critical** otherwise. Finally we will say that φ is **additive** if equality (2.21) holds for any weighted filtration, i.e., there are no critical filtrations.

Remark 6. 1. It easy to see that for any filtration (indexed by $\mathtt{I}$)

$$\mu_{\mathtt{I}}(\mathcal{E}^\bullet, \underline{\alpha}; \varphi) \leq \sum_{i\in\mathtt{I}} \mu(0 \subset \mathcal{E}_i \subset \mathcal{E}, \alpha_i; \varphi).$$

Therefore any subfiltration of a non-critical one is still non-critical. Indeed suppose that $\mathcal{E}^\bullet$ is a non critical filtration indexed by $\mathtt{I}$ and $\mathtt{J} \subset \mathtt{I}$ indexes a critical subfiltration of $\mathcal{E}^\bullet$. Then $\mu_{\mathtt{I}}(\mathcal{E}^\bullet, \underline{\alpha}; \varphi) = \sum_{i\in\mathtt{I}} \mu(0 \subset \mathcal{E}_i \subset \mathcal{E}, \alpha_i; \varphi)$ (since the whole filtration is non critical) and $\mu_{\mathtt{J}}(\mathcal{E}^\bullet, \underline{\alpha}; \varphi) < \sum_{i\in\mathtt{J}} \mu(0 \subset \mathcal{E}_i^\bullet \subset \mathcal{E}^\bullet, \alpha_i; \varphi)$. Therefore $\mu_{\mathtt{I}\smallsetminus\mathtt{J}}(\mathcal{E}^\bullet, \underline{\alpha}; \varphi) > \sum_{i\in\mathtt{I}\smallsetminus\mathtt{J}} \mu(0 \subset \mathcal{E}_i \subset \mathcal{E}, \alpha_i; \varphi)$ which is absurd.

2. If φ is additive k-(semi)stability implies (semi)stability and therefore the two conditions are equivalent

3. Checking semistability conditions over non-critical filtrations is the same to check them over subbundles.

4. Thanks to the previous considerations, the following conditions are equivalent:

   (a) $(\mathcal{E}, \varphi)$ is δ-(semi)stable;

   (b) For any subsheaf $\mathcal{F}$ and for any <u>critical</u> filtration $(\mathcal{E}^\bullet, \underline{\alpha})$ the following inequalities hold

$$0 \underset{(-)}{\prec} (\mathrm{rk}(\mathcal{F})\mathsf{P}_{\mathcal{E}} - r\mathsf{P}_{\mathcal{F}}) - \delta(r\mathsf{k}_{\mathcal{F},\mathcal{E}} - a\mathrm{rk}(\mathcal{F})),$$
$$0 \underset{(-)}{\prec} P(\mathcal{E}^\bullet, \underline{\alpha}) + \delta\, \mu(\mathcal{E}^\bullet, \underline{\alpha}; \varphi).$$

   Observe that the first part of condition (2) is just requiring that (E, φ) is k-(semi)stable.

5. Note that Proposition 5 and points 2, 3 and 4 above hold also for slope semistability.

2.3 Examples of decorated sheaves and their semistability

2.3.1 Principal Bundles

Let G a reductive algebraic group, $\rho : G \to \mathrm{Gl}(V)$ be a faithful representation with $\rho(G) \subseteq \mathrm{Sl}(V)$ and let $\mathcal{P} \to X$ be a G-principal bundle.

Remark 7. i) If G is semisimple, i.e., if the radical of G is trivial, one can take the adjoint representation;

ii) any representation from a semisimple group to $\mathrm{Gl}(V)$ lands in $\mathrm{Sl}(V)$ since semisimple groups no have non-trivial character.

iii) If $\rho : G \to \mathrm{Gl}(V)$ is any representation then the representation

$$\begin{aligned} \rho \oplus (\det \circ \rho)^{-1} : G &\longrightarrow \mathrm{Gl}(V \oplus \mathbb{C}) \\ g &\longmapsto \rho(g) \oplus \frac{1}{\det(\rho(g))} \end{aligned}$$

obviously lands in $\mathrm{Sl}(V \oplus \mathbb{C})$.

We will show how $\mathcal{P}$ corresponds to a pair (E, τ) consisting of a vector bundle with trivial determinant and a morphism of bundles. Define E as the vector bundle associated with $\mathcal{P}$ via ρ, i.e., $E = (\mathcal{P} \times_\rho V)$ is the quotient of $(\mathcal{P} \times V)$ with respect to the action $(p, v) \cdot g \doteqdot (p \cdot g, \rho(g)^{-1} \cdot v)$ for all $p \in \mathcal{P}$, $v \in V$ and $g \in G$. Note that $\det E \simeq \mathcal{O}_X$ since the representation ρ lands in $\mathrm{Sl}(V)$. The morphism τ is instead defined as follows: G acts via ρ on the principal $\mathrm{Gl}(V)$-bundle $\mathcal{P} \times_\rho \mathrm{Gl}(V) \simeq \underline{\mathrm{Isom}}(V \otimes \mathcal{O}_X, E)$, therefore we can quotient for this action and get

$$\begin{array}{ccc} \mathcal{P} & \longrightarrow & \underline{\mathrm{Isom}}(V \otimes \mathcal{O}_X, E) \\ \downarrow & & \downarrow \\ X = \mathcal{P}/G & \xrightarrow{\sigma} & \underline{\mathrm{Isom}}(V \otimes \mathcal{O}_X, E)/G. \end{array} \tag{2.22}$$

Here σ is the morphism induced to the quotient by the inclusion of $\mathcal{P}$ in $\underline{\mathrm{Isom}}(V \otimes \mathcal{O}_X, E)$.
Since, for any $x \in X$, $\mathrm{Isom}(V, E_x) = \{f \in \mathrm{Hom}(V, E_x) \mid \det(f) \neq 0\}$ and the det is a G invariant morphism, the inclusion $\underline{\mathrm{Isom}}(V \otimes \mathcal{O}_X, E) \hookrightarrow \underline{\mathrm{Hom}}(V \otimes \mathcal{O}_X, E)$ lands in the open subscheme of stable points of $\underline{\mathrm{Hom}}(V \otimes \mathcal{O}_X, E)$ and therefore induces an inclusion between the quotients

$$\begin{array}{ccc} \underline{\mathrm{Isom}}(V \otimes \mathcal{O}_X, E) & \hookrightarrow & \underline{\mathrm{Hom}}(V \otimes \mathcal{O}_X, E) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Isom}}(V \otimes \mathcal{O}_X, E)/G & \hookrightarrow & \underline{\mathrm{Hom}}(V \otimes \mathcal{O}_X, E)//G. \end{array} \tag{2.23}$$

Finally one has that

$$\underline{\mathrm{Isom}}(V \otimes \mathcal{O}_X, E)/G \simeq \underline{\mathrm{Isom}}(V \otimes \mathcal{O}_X, E)^G$$
$$\underline{\mathrm{Hom}}(V \otimes \mathcal{O}_X, E)/G \simeq \underline{\mathrm{Hom}}(V \otimes \mathcal{O}_X, E)^G$$

where $\underline{\mathrm{Isom}}(V \otimes \mathcal{O}_X, E)^G$ (resp. $\underline{\mathrm{Hom}}(V \otimes \mathcal{O}_X, E)^G$) is the sheaf of G-invariant isomorphisms (resp. morphism). From the fact that $\underline{\mathrm{Hom}}(V \otimes \mathcal{O}_X, E) \simeq \underline{\mathrm{Spec}}(\mathrm{Sym}^\star(V \otimes E^\vee))$ one can prove that

$$\underline{\mathrm{Hom}}(V \otimes \mathcal{O}_X, E)^G \simeq \underline{\mathrm{Spec}}(\mathrm{Sym}^\star(V \otimes E^\vee)^G),$$

(see [31]), and therefore the section σ (composed with the inclusion) induces a morphism of sheaves $\tau : \mathrm{Sym}^\star(V \otimes E^\vee)^G \to \mathcal{O}_X$.

Conversely if $(\mathcal{E}, \tau)$ is a pair consisting of a torsion free sheaf $\mathcal{E}$ over X and a non-trivial homomorphism $\tau : \mathrm{Sym}^\star(V \otimes E^\vee)^G \to \mathcal{O}_X$ of $\mathcal{O}_X$

algebras, then, as before, giving τ is equivalent to giving a section $\sigma : X \to \underline{\mathrm{Hom}}(V \otimes \mathcal{O}_X, \mathcal{A}^\vee)//G$. Such a pair is called also **pseudo G-bundle**. Let $U_\mathcal{E} \subset X$ be an open subset where $\mathcal{E}$ is locally free. If there exist an open subset $U \subset U_\mathcal{E}$ such that $\sigma(U) \subset \underline{\mathrm{Isom}}(V \otimes \mathcal{O}_{X|_U}, \mathcal{A}^\vee{}_{|_U})//G$, then the pseudo G-bundle is called **singular principal G-bundle** and defines a principal G-bundle $\mathcal{P}(\mathcal{E}, \tau) \doteqdot \sigma^*(\underline{\mathrm{Isom}}(V \otimes \mathcal{O}_X, \mathcal{E}^\vee))$ over U via pullback:

$$\begin{array}{ccc} \mathcal{P}(\mathcal{E},\tau)_{|_U} & \longrightarrow & \underline{\mathrm{Isom}}(V \otimes \mathcal{O}_X, \mathcal{E}^\vee)_{|_U} \\ \downarrow & & \downarrow \\ U & \xrightarrow{\sigma_{|_U}} & \underline{\mathrm{Isom}}(V \otimes \mathcal{O}_X, \mathcal{E}^\vee)_{|_U}//G. \end{array} \tag{2.24}$$

If moreover $\mathcal{E}$ has degree zero and $\sigma(U_\mathcal{E}) \subset \underline{\mathrm{Isom}}(V \otimes \mathcal{O}_{X|_U}, \mathcal{A}^\vee{}_{|_U})//G$, then we say that $(\mathcal{E}, \tau)$ is a **honest singular principal G-bundle**; since we are supposing X to be smooth these two notions coincide, i.e., every singular principal G-bundle is honest (see [17] and [31] Section 2.1 for further details).

Remark 8. Our notation is consistent with the notation of Schmitt in [35], while in [32] there is a slight different notation: Schmitt calls *singular principal G-bundle* what we call pseudo G-bundle. In [32] there is no notion of singular principal G-bundle, as the one we introduced before.

Finally we relate honest singular principal bundles with decorated sheaves. First recall that

Definition 9. Let $\rho : \mathrm{Gl}(V) \to \mathrm{Gl}(W)$ be a representation and let V, W be finite-dimensional $\mathbb{C}$-vector spaces. The representation ρ is **homogeneous** if $\mathbb{C}^* \subset \mathrm{Gl}(V)$ acts by $z \cdot \mathrm{id}_V \mapsto z^\alpha \cdot \mathrm{id}_W$ for some $\alpha \in \mathbb{Z}$.

Proposition 10 (Corollary 1.2 in [33])**.** *Let $\rho : Gl_r(\mathbb{C}) \to Gl(V)$ be a homogeneous representation on the finite-dimensional $\mathbb{C}$-vector space V. Then there exist $a, b, c \in \mathbb{N}$, $c > 0$, such that ρ is a direct summand of the natural representation $\rho_{a,b,c} : Gl_r(\mathbb{C}) \to Gl(V_{a,b,c})$.*

Let $(\mathcal{E}, \tau)$ be an honest singular principal bundle. We recall that $\tau : \mathrm{Sym}^\star(V \otimes \mathcal{E}^\vee)^G \to \mathcal{O}_X$. Since G is reductive, the $\mathcal{O}_X$-algebra $\mathrm{Sym}^\star(V \otimes \mathcal{E}^\vee)^G$ is finitely generated (Hilbert finiteness theorem, see for example [29] Theorem 1.2.1.4) and therefore exist an integer s such that the following map is surjective

$$\bigoplus_{i=1}^{s} \mathrm{Sym}^i(V \otimes \mathcal{E}^\vee)^G \twoheadrightarrow \mathrm{Sym}^\star(V \otimes \mathcal{E}^\vee)^G. \tag{2.25}$$

The morphism τ induces a morphism

$$\tau' : \bigoplus_{i=1}^{s} \mathrm{Sym}^i(V \otimes \mathcal{E}^\vee)^G \twoheadrightarrow \mathrm{Sym}^\star(V \otimes \mathcal{E}^\vee)^G \xrightarrow{\tau} \mathcal{O}_X \tag{2.26}$$

Remark 11. Recall that $r = \dim(V)$ and consider the representation

$$R : G \times \mathrm{Gl}_r(\mathbb{C}) \to \mathrm{Gl}(V \otimes \mathbb{C}^r)$$
$$(g, g') \mapsto R(g, g') : v \otimes w \mapsto \rho(g)(v) \otimes g' \cdot w$$

for $g \in G$, $g' \in \mathrm{Gl}_r(\mathbb{C})$ $v \in V$ and $w \in \mathbb{C}^r$. This representation yields a rational representation on the algebra $\mathrm{Sym}^\star(V \otimes \mathbb{C}^r)^G$, respecting the grading and therefore we have a representation

$$\mathrm{Gl}_r(\mathbb{C}) \to \mathrm{Gl}\left(\bigoplus_{i=1}^{s} \mathrm{Sym}^i(V \otimes \mathbb{C}^r)^G\right).$$

Unfortunately this representation is <u>not</u> homogeneous (it is homogeneous iff the sum over s consists of only one term), and so we can not use the above proposition. We need first to pass to the induced homogeneous representation:

$$t(s) : \mathrm{Gl}_r(\mathbb{C}) \to \mathrm{Gl}(\mathbb{U}(s))$$

where

$$\mathbb{U}(s) \doteqdot \bigoplus_{\substack{\underline{h}=(h_1,\dots,h_s) \\ h_i \geq 0,\ \sum i \cdot h_i = s!}} \mathbb{S}^{\underline{h}} \qquad \text{and} \qquad \mathbb{S}^{\underline{h}} \doteqdot \bigotimes_{i=1}^{s} \mathrm{Sym}^{h_i}\left(\mathrm{Sym}^i(V \otimes \mathbb{C}^r)^G\right).$$

Remark 12. The above fact holds in general. In fact, every representation ρ of $\mathrm{Gl}_r(\mathbb{C})$ obviously splits into a direct sum of homogeneous representations, say $\rho_1, \dots, \rho_s$. If any such representation has positive degree α_i, then we can pass to the following homogeneous representation

$$\rho' : \bigoplus_{\nu_1\alpha_1 + \cdots + \nu_s\alpha_s = \mathrm{d}} \mathrm{Sym}^{\nu_1}\rho_1 \otimes \cdots \otimes \mathrm{Sym}^{\nu_s}\rho_s,$$

where d is a common multiple of the α_i.

Therefore, thanks to the representation $t(s)$ (Remark 11), the morphism τ' induces a morphism

$$\varphi_\tau : \mathbb{U}(s) \to \mathcal{O}_X \tag{2.27}$$

and, by Proposition 10, exists a, b, c and a sheaf $\mathcal{E}'$ such that

$$\varphi : \mathcal{E}_{a,b,c} = \mathbb{U}(s) \oplus \mathcal{E}' \stackrel{\varphi_\tau \oplus 0}{\longrightarrow} \mathcal{O}_X$$

Indeed, consider now $\underline{h} = (h_1, \dots, h_s)$ such that $\sum i \cdot h_i = s!$, then the natural homomorphism

$$\bigotimes_{i=1}^{s} (V \otimes \mathcal{E}^\vee)^{\otimes i \cdot h_i} \to \bigotimes_{i=1}^{s} \mathrm{Sym}^{h_i}\left(\mathrm{Sym}^i(V \otimes \mathcal{E}^\vee)\right) \to \mathbb{S}^{\underline{h}} \to \mathcal{O}_X$$

induces a morphism

$$\left((V \otimes \mathcal{E}^\vee)^{\otimes s}\right)^{\oplus N} \longrightarrow \mathbb{U}(s) \longrightarrow \mathcal{O}_X.$$

For further details see [31] and [32].

Semistability

Let $(\mathcal{E}, \tau)$ be a singular principal bundle and let $U \subseteq X$ be a non-empty open set over which $\mathcal{E}$ trivializes and the section σ, induced by τ, is non-zero. Over U the above morphisms φ and φ_τ induce the following maps

$$\begin{aligned}\overline{\sigma}_{|_U}(s) :& U \to \mathbb{P}(V \otimes \mathcal{E}_{|_U})//G \hookrightarrow \mathbb{P}(\mathbb{U}(s)) \times U \to \mathbb{P}(\mathbb{U}(s))\\ \widetilde{\sigma}_{|_U}(s) :& U \stackrel{\overline{\sigma}_{|_U}(s)}{\longrightarrow} \mathbb{P}(\mathbb{U}(s)) \hookrightarrow \mathbb{P}\left(((V \otimes \mathcal{E})^{\otimes s!})^{\oplus N}\right).\end{aligned}$$

The representation $t(s)$ gives rise to an action of $\mathrm{Gl}_r(\mathbb{C})$ on $\mathbb{U}(s)$ and so to an action on $\mathbb{P}(\mathbb{U}(s))$ with a linearization on the line bundle $\mathcal{O}_{\mathbb{P}(\mathbb{U}(s))}(1)$. Any weighted filtration $(\mathcal{E}^\bullet, \underline{\alpha})$ of $\mathcal{E}$ defines a one-parameter subgroup $\lambda : \mathbb{C}^* \to \mathrm{Gl}(\mathbb{U}(s))$. For any point $x \in \mathbb{P}(\mathbb{U}(s))$ the point $x_\infty = \lim_{z\to\infty} \lambda(z) \cdot x$ is a fixed point for the action of $\mathbb{C}^*$ induced by λ. So the linearization provides a linear action of $\mathbb{C}^*$ on the one dimensional vector space. This action is of the form $z \cdot v = z^\gamma v$ for some $\gamma \in \mathbb{Z}$, and finally one defines $\mu_{t(s)}(\lambda; x) = -\gamma$. Eventually one defines

$$\mu(\mathcal{E}_{|_U}, \underline{\alpha}; \overline{\sigma}_{|_U}(s)) \doteqdot \max_{x \in \mathrm{Im}\overline{\sigma}_{|_U}(s)} \mu_{t(s)}(\lambda; x).$$

Then, for any weighted filtration $(\mathcal{E}^\bullet, \underline{\alpha})$ of $\mathcal{E}$, one defines

$$\mu(\mathcal{E}^\bullet, \underline{\alpha}; \tau) \doteqdot \frac{1}{s!}\mu(\mathcal{E}_{|_U}, \underline{\alpha}; \overline{\sigma}_{|_U}(s)) = \frac{1}{s!}\mu(\mathcal{E}_{|_U}, \underline{\alpha}; \widetilde{\sigma}_{|_U}(s)).$$

We say that an honest singular G-bundle is δ-(semi)stable if and only if for any weighted filtration $(\mathcal{E}^\bullet, \underline{\alpha})$, indexed by $\mathtt{I}$, the following inequality holds

$$P_{\mathtt{I}}(\mathcal{E}^\bullet, \underline{\alpha}) + \delta\mu(\mathcal{E}^\bullet, \underline{\alpha}; \tau) \underset{(-)}{\succ} 0.$$

Lemma 13 ([31] Lemma 2.2.3)**.** *1. There is a constant polynomial δ_{Gies}, such that, for every polynomial $\delta' \preceq \delta_{Gies}$ and every δ'-semistable singular principal bundle $(\mathcal{E}, \tau)$, the sheaf $\mathcal{E}$ is itself Gieseker-semistable.*

2. There is a polynomial δ_μ of degree exactly $\dim(X) - 1$*, such that, for every polynomial $\delta' \preceq \delta_\mu$ and every δ'-semistable singular principal bundle $(\mathcal{E}, \tau)$, the sheaf $\mathcal{E}$ is itself Mumford-semistable.*

Since honest singular G-bundles come from principal bundles, one might expect that the semistability conditions for the two kinds of objects coincide. However, this is not the case. To see this we first introduce another notion of semistability which is closer to semistability for principal bundles.
Let $(\mathcal{E}, \tau)$ be an honest singular G-bundle and $\lambda : \mathbb{C}^* \to G$. A **reduction** of $(\mathcal{E}, \tau)$ to λ is a pair (U, β) which consists of a big open subset $U \subset X$ over which $\mathcal{E}$ is locally free, and a section

$$\beta : U \to \mathcal{P}(\mathcal{E}, \tau)_{|_U}/Q_G(\lambda)$$

where
$$Q_G(\lambda) \doteqdot \{g \in G \,|\, \exists \lim_{z\to\infty} \lambda(z)\cdot g \cdot \lambda(z)^{-1}\}$$
is the parabolic subgroup in G induced by λ.

Remark 14. Let $\rho : G \to Sl(V) \hookrightarrow Gl(V)$ a faithful representation.

1. Since G is reductive, if Q' is a parabolic subgroup of $Gl(V)$, then $Q' \cap G$ is a parabolic subgroup of G. [Sketch: if B (B_G) is a Borel subgroup of $Gl(V)$ (resp. of G) then, up to conjugacy class, $B \cap G = B_G$ and so $Q' \cap G \subseteq B_G$]
2. Given a parabolic subgroup Q of G and a representation ρ, we can construct a parabolic subgroup of $Gl(V)$; in fact, given Q, there exists a one-parameter subgroup $\lambda : \mathbb{C}^* \to G$ such that $Q = Q_G(\lambda)$, then the set $Q_{Gl(V)}(\rho \circ \lambda)$ is a parabolic subgroup of $Gl(V)$.
3. $Q_{Gl(V)}(\rho \circ \lambda)$, or simply $Q_{Gl(V)}(\lambda)$, is the stabilizer of the flag induced by λ in $\mathrm{Gl}(V)$.
4. Given $\lambda' : \mathbb{C}^* \to Gl(V)$, or equivalently the parabolic subgroup Q' associated with λ', there always exists $\lambda : \mathbb{C}^* \to G$ such that $Q' \cap G = Q_G(\lambda)$ (see (1)). The following diagram however is **not** in general commutative:
$$\begin{array}{ccc} \mathbb{C}^* & \xrightarrow{\lambda'} & Gl(V) \\ {\scriptstyle\lambda}\downarrow & \nearrow_{\rho} & \\ G & & \end{array}$$
5. Given a parabolic subgroup $Q' \subset Gl(V)$ and fixing a representation $\rho : G \to G(V)$, it is possible to define a parabolic subgroup $Q = Q' \cap G \subset G$ (see (1)) and from Q we can obtain a parabolic subgroup $Q'' \subset Gl(V)$ as explained in (2). Therefore, fixing a basis of $Gl(V)$, we have a map
$$\zeta : \{\text{Parabolic subgroups of } Gl(V)\} \to \{\text{Parabolic subgroups of } Gl(V)\}.$$
6. We will call **stable** the parabolic subgroups of $Gl(V)$ such that $Q' = \zeta(Q')$, with respect to the same basis of $Gl(V)$.

Given a reduction (U, β) of $(\mathcal{E}, \tau)$ to λ, consider the following composition:
$$\beta' : U \xrightarrow{\beta} \mathcal{P}(\mathcal{E}, \tau)/Q_G(\lambda) \hookrightarrow \underline{\mathrm{Isom}}(V \otimes \mathcal{O}_U, \mathcal{E}_{|_U}{}^\vee)/Q_{\mathrm{Gl}(V)}(\lambda).$$
Therefore the section β induces a section β' of the $Q_{\mathrm{Gl}(V)}(\lambda)$-bundle $\underline{\mathrm{Isom}}(V \otimes \mathcal{O}_U, \mathcal{E}_{|_U}{}^\vee) \to \underline{\mathrm{Isom}}(V \otimes \mathcal{O}_U, \mathcal{E}_{|_U}{}^\vee)/Q_{\mathrm{Gl}(V)}(\lambda)$; since λ induces a flag $V^\bullet$ of V

with weights $\alpha_i(\lambda) = \frac{\gamma_{i+1}-\gamma_i}{r}$ (where γ_i are the weights of the action of $\mathbb{C}^*$ on $\mathrm{Gl}(V)$), it is easy to see that section β' corresponds to a filtration $0 \subset \mathcal{E}'_1 \subset \cdots \subset \mathcal{E}'_s \subset {\mathcal{E}_{|U}}^\vee$ with the same weight vector $\underline{\alpha}(\lambda) = (\alpha'_1, \ldots, \alpha'_s)$. By dualizing, one gets a filtration of $\mathcal{E}_{|U}$ by subbundles which extends by a filtration $\mathcal{E}^\bullet_\beta$ of $\mathcal{E}$ by saturated sheaves. Finally define $\underline{\alpha} = (\alpha_1, \ldots, \alpha_s) \doteqdot (\alpha'_s, \ldots, \alpha'_1)$. Therefore, $(\mathcal{E}, \tau)$ is **β-(semi)stable** (respectively **slope β-(semi)stable**) if and only if for any β reduction one has that

$$P(\mathcal{E}^\bullet_\beta, \underline{\alpha}_\beta) \underset{(-)}{\succ} 0 \qquad \big(L(\mathcal{E}^\bullet_\beta, \underline{\alpha}_\beta) \underset{(=)}{>} 0 \text{ respectively}\big)\,.$$

Remark 15. 1. The following result holds:

Proposition 16 (Proposition 1. in [20])**.** *A filtration $\mathcal{E}^\bullet$ of $\mathcal{E}$ is a β-filtration if and only if the parabolic subgroup Q' associated with it is stable.*

2. If G is semisimple, one has the following implications ([32] Remark 1.1):

$$\begin{aligned} P(\mathcal{E},\tau) \text{ is Ramanathan-stable } &\Rightarrow (\mathcal{E},\tau) \text{ is stable} \\ &\Rightarrow (\mathcal{E},\tau) \text{ is semistable} \\ &\Rightarrow P(\mathcal{E},\tau) \text{ is Ramanathan-semistable} \end{aligned}$$

We recall that a principal G-bundle P is **Ramanathan-(semi)stable** if and only if for any parabolic subgroup H of G, any reduction $\sigma_H : X \to P(G/H)$ and any dominant character χ of H one has that

$$\deg \sigma^*_H(L_\chi) \underset{(=)}{<} 0,$$

where we denoted by L_χ the line bundle induced by χ.

3. Let G be a semisimple group, $\rho : G \to Sl(V) \subset Gl(V)$ a faithful representation and $\lambda : \mathbb{C}^* \to G$ a one-parameter subgroup such that $Q_G(\lambda)$ is a maximal parabolic subgroup of G. Then the parabolic subgroup $Q_{Gl(V)}(\rho \circ \lambda)$ of $Gl(V)$ is **not** maximal. Thanks to this observation we obtain that every β-filtration $E^\bullet_\beta :\ 0 \subset E_1 \subset \cdots \subset E_s \subset E_{s+1} = E$ has length greater then or equal to 2, i.e., $s \geq 2$.
Notice that the parabolic subgroup of G associated with a β-filtration is always a proper subgroup. Therefore, according to the definition of Ramanathan, the (semi)stability condition is checked only for maximal *proper* parabolic subgroups of G. If G is reductive but not semisimple the above consideration does not hold in general as the following example shows:

Example 17. Consider $G = Gl(k)$, $\rho : Gl(k) \to Gl(n)$ $(k < n)$ the inclusion (in the left up corner) and $\lambda : \mathbb{C}^* \to G$ given by

$$\lambda(z) \doteqdot \begin{pmatrix} z^\gamma & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}.$$

Therefore $\lambda' = \rho \circ \lambda$ is given by

$$\lambda'(z) \doteqdot \begin{pmatrix} z^\gamma & 0 & \dots & & & \dots & 0 \\ 0 & 1 & \ddots & & & & \vdots \\ \vdots & \ddots & \ddots & & & & \\ & & & 1 & & & \\ & & & & 1 & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & 0 \\ 0 & \dots & & & \dots & 0 & 1 \end{pmatrix}.$$

give arise to a maximal parabolic subgroup.

Finally the following results, due to Schmitt, holds:

Lemma 18 ([32] Lemma 4.4 and Proposition 4.5)**.** *Let $(\mathcal{E}, \tau)$ be an honest singular G-bundle, then*

1. *for any reduction β of $(\mathcal{E}, \tau)$ to the one-parameter subgroup λ of G*

$$\mu(\mathcal{E}^\bullet_\beta, \underline{\alpha}_\beta; \varphi) = 0,$$

where $(\mathcal{E}^\bullet_\beta, \underline{\alpha}_\beta)$ is the weighted filtration of $\mathcal{E}$ associated with λ and β.

2. *If $(\mathcal{E}^\bullet, \underline{\alpha})$ is a weighted filtration of $\mathcal{E}$ with $\mu(\mathcal{E}^\bullet, \underline{\alpha}; \varphi) = 0$, there exists a reduction β to a one-parameter subgroup $\lambda : \mathbb{C}^* \to G$ with*

$$(\mathcal{E}^\bullet, \underline{\alpha}) = (\mathcal{E}^\bullet_\beta, \underline{\alpha}_\beta).$$

Proposition 19 ([32] Proposition 5.3)**.** *Let $\delta \in \mathbb{Q}[x]$ be a positive polynomial of degree exactly* $\dim(X) - 1$. *Then the following properties holds true.*

1. *An honest singular G-bundle $(\mathcal{E}, \tau)$ is (semi)stable if and only if the associated decorated bundle $(\mathcal{E}, \varphi)$ is δ-(semi)stable.*
2. *$(\mathcal{E}, \tau)$ is a (semi)stable honest singular G-bundle, then it is δ-(semi)stable.*

2.3.2 Higgs Bundles

Notation. In this section C will denote, unless otherwise stated, a smooth, irreducible projective curve over $\mathbb{C}$ of genus $g > 0$, while Ω^1_C will denote the canonical line bundle over C. Denote by G a semisimple algebraic group.

A Higgs bundle is a holomorphic vector bundle together with a Higgs field. To be more precise

Definition 20. A **Higgs vector bundle** is a pair (E, ϕ) where E is a vector bundle over C, while ϕ is a morphism

$$\phi : E \to E \otimes \Omega^1_C.$$

A **principal Higgs G-bundle** is a pair (P, ϕ) consisting of a principal G-bundle P over C and a section

$$\phi : X \to Ad(P) \otimes \Omega^1_C,$$

where $Ad(P) = P \times_{Ad} \mathfrak{g}$ and Ad is the adjoint representation of G on its Lie algebra $\mathfrak{g}$.

Two Higgs vector bundles (E, ϕ) and (F, ψ) are **isomorphic** if there is an isomorphism $f : E \to F$ such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E \otimes \Omega^1_C \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f\otimes \mathrm{id}} \\ F & \xrightarrow{\psi} & F \otimes \Omega^1_C \end{array}$$

Similarly two principal Higgs G-bundles (P, ϕ) and (P', ψ) are **isomorphic** if there is an isomorphism $f : P \to P'$ of principal bundles such that the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\phi} & Ad(P) \otimes \Omega^1_C \\ & \searrow^{\psi} & \downarrow{\scriptstyle Ad(f)\otimes \mathrm{id}} \\ & & Ad(P') \otimes \Omega^1_C \end{array}$$

Now we want to discuss in wich sense a Higgs vector, or principal, bundle over a curve can be regarded as a decorated bundle. We begin with the vector bundle case.

Let (E, ϕ) a Higgs vector bundle over C of rank $\mathrm{rk}(E) = r$. The morphism $\phi : E \to E \otimes \Omega^1_C$ corresponds to a morphism $E \otimes E^\vee \to \Omega^1_C$, that, with abuse of notation, we still denote by ϕ. From the isomorphism

$$\bigwedge^k E \simeq \left(\bigwedge^{r-k} E \right)^\vee \otimes \bigwedge^r E, \tag{2.28}$$

one gets that $\bigwedge^{r-1} E \simeq E^\vee \otimes \det E$. So there is an injective morphism of vector bundles

$$i : E \otimes E^\vee \simeq E \otimes \bigwedge^{r-1} E \otimes (\det E)^\vee \hookrightarrow E^{\otimes r} \otimes (\det E)^\vee$$

Therefore (E, ϕ) corresponds to a decorated bundle (E, φ) of type $(r, 1, 1, \Omega^1_C)$ where we set $\varphi|_{i(E\otimes E^\vee)} \doteqdot \phi$ and zero otherwise. Note that, for any $\lambda \in \mathbb{C}^*$, (E, ϕ) and $(E, \lambda\phi)$ are not isomorphic as Higgs vector bundles while the associated decorated vector bundles (E, φ) and $(E, \lambda\varphi)$ are isomorphic.

Consider now a Higgs bundle (E, ϕ) of rank r such that $\det E \simeq \mathcal{O}_C$. Fix a section $\omega : \mathcal{O}_C \to \Omega^1_C$ and consider the morphism $\phi \oplus \omega : E \otimes E^\vee \oplus \mathcal{O}_C \to \Omega^1_C$. As before, the isomorphism $\bigwedge^{r-1} E \simeq E^\vee \otimes \det E \simeq E^\vee$ induces an inclusion

$$i : E \otimes E^\vee \oplus \mathcal{O}_C \simeq E \otimes \bigwedge^{r-1} E \oplus \det E \hookrightarrow E^{\otimes r} \oplus E^{\otimes r}.$$

Therefore, if we fix a section $\omega : \mathcal{O}_C \simeq \det E \to \Omega^1_C$, (E, ϕ) corresponds to a decorated bundle (E, φ) of type $(r, 2, 0, \Omega^1_C)$ where we set $\varphi|_{i(E\otimes E^\vee)} \doteqdot \phi \oplus \omega$ and zero otherwise. Note that, in this case, the (non isomorphic) Higgs bundles (E, ϕ) and $(E, \lambda\phi)$ are non isomorphic also as decorated vector bundles, i.e., (E, φ) and (E, φ'), where φ is induced by $\phi \oplus \omega$ and φ' by $\lambda\phi \oplus \omega$, are non-isomorphic.

Let now (P, ϕ) be a principal Higgs G-bundle and let $\rho : G \to \mathrm{Gl}(V)$ be a (fixed) faithful representation. Since G is semisimple, ρ lands in $\mathrm{Sl}(V)$. As we explained in the previous section, the principal G-bundle P corresponds to a pair (E, τ), where E is the vector bundle associated with the principal $\mathrm{Gl}(V)$-bundle $P_\rho = P \times_\rho \mathrm{Gl}(V)$ and $\tau : \mathrm{Sym}^\star(E \otimes V)^G \to \mathcal{O}_C$ is the morphism associated with $\sigma : C \to \underline{\mathrm{Isom}}(\mathcal{O}_C \otimes V, E^\vee)^G$, which, in turn, is induced by the morphism $P \to \underline{\mathrm{Isom}}(\mathcal{O}_C \otimes V, E^\vee) \simeq P \times_\rho \mathrm{Gl}(V)$. With this notation it is easy to see that the morphism $\phi : C \to \mathrm{Ad}(P) \otimes \Omega^1_C$ induces a morphism $\phi : C \to \mathrm{End}(E) \otimes \Omega^1_C$. Therefore a principal Higgs G-bundle (P, ϕ) corresponds to a triple (E, τ, ϕ) where (E, ϕ) is a vector Higgs bundle while (E, τ) is equivalent, roughtly speaking, to the principal bundle P. As showed in the previous section the pair (E, τ) induces a decorated vector bundle (E, φ_1) of type $(a_1, b_1, c_1, \mathsf{N}_1)$, while (E, ϕ) induces a decorated vector bundle (E, φ_2) of type $(a_2, b_2, c_2, \mathsf{N}_2)$. So, in some sense, a principal Higgs G-bundles corresponds to a **double-decorated vector bundle** $(E, \varphi_1, \varphi_2)$. Observe that, since the representation ρ lands in $\mathrm{Sl}(V)$, $\det E \simeq \mathcal{O}_C$ and so $c_1 = c_2 = 0$. Let $\rho_{a_1,b_1} : \mathrm{Sl}(V) \to \mathrm{Sl}(V_{a_1,b_1})$ and $\rho_{a_2,b_2} : \mathrm{Sl}(V) \to \mathrm{Sl}(V_{a_2,b_2})$ the obvious representations, then, for $i = 1, 2$, E_{a_i,b_i} coincide with $E_{\rho_{a_i,b_i}} = E \times_{\rho_{a_i,b_i}} Gl(V)$ and the morphisms $\phi_i : E_{\rho_{a_i,b_i}} \to \mathsf{N}_i$

correspond to morphisms $\sigma_i : C \to \mathbb{P}(E_{\rho_{a_i,b_i}})$. Then set $\chi \doteqdot \rho_{a_1,b_1} \otimes \rho_{a_2,b_2}$ and define $\phi : E_\chi \to \mathsf{N} \doteqdot \mathsf{N}_1 \otimes \mathsf{N}_2$ as the morphism induced by the following composition:

$$\sigma : C \stackrel{(\sigma_1,\sigma_2)}{\longrightarrow} \mathbb{P}(E_{\rho_{a_1,b_1}}) \times \mathbb{P}(E_{\rho_{a_2,b_2}}) \longrightarrow \mathbb{P}(E_\chi).$$

Observe that $\chi \simeq \rho_{a_1+a_2,b_1b_2}$. Therefore the double-decorated vector bundle (E,φ_1,φ_2) of type $((a_1,b_1,\mathsf{N}_1),(a_2,b_2,\mathsf{N}_2))$ corresponds to a decorated vector bundle (E,φ) of type $(a_1+a_2, b_1b_2, \mathsf{N}_1 \otimes \mathsf{N}_2)$.

Suppose now that $\hat{C}$ is a nodal curve with a simple node x_0. Let $\nu : \widetilde{C} \to \hat{C}$ be the normalization and let $\{y_1,y_2\} = \nu^{-1}(x_0)$. Then Bhosle shows in [2] that there is a correspondence between torsion free sheaves over a nodal curve $\hat{C}$ and *generalized parabolic vector bundles* over the normalization $\widetilde{C}$ of the nodal curve. We recall that a **generalized parabolic vector bundle** with support the divisor D, is a pair (E,q) where E is a vector bundle over $\widetilde{C}$ and $q : E_{|D} \to R$ is a surjective homomorphism of vector spaces. In the case of nodal curves the correspondence is between torsion free sheaves over $\hat{C}$ and generalized parabolic vector bundles over $\widetilde{C}$ supported on the divisor $D = x_1 + x_2$. Therefore the surjective morphism of vector spaces q goes from $E_{x_1} \oplus E_{x_2}$ to an $\mathrm{rk}(E)$-dimensional vector space R. More precisely, if (E,q) is a generalized parabolic vector bundle over $\widetilde{C}$, Bhosle shows that the sheaf $\mathcal{E}$

$$\mathcal{E} \doteqdot \ker[\nu_* E \longrightarrow \nu_*(E_{x_1} \oplus E_{x_2}) \simeq E_{x_1} \oplus E_{x_2} \stackrel{q}{\longrightarrow} R]$$

is a torsion free sheaf over $\hat{C}$ such that $\nu^*\mathcal{E} = (E,q)$.

Thanks to this result and to the previous considerations one can convince oneself that there is a correspondence between principal Higgs G-bundles (P,ϕ) over a nodal curve $\hat{C}$ and quadruples $(E,q,\widetilde{\tau},\widetilde{\varphi})$, called **descending principal Higgs G-bundles**, where (E,q) is a generalized parabolic vector bundle over $\widetilde{C}$, $\widetilde{\tau} : \mathrm{Sym}^\star(E \otimes V)^G \to \mathcal{O}_{\widetilde{C}}$ is a homomorphism of $\mathcal{O}_{\widetilde{C}}$-algebras and $\widetilde{\varphi} : \widetilde{C} \to \mathrm{End}(E) \otimes \Omega^1_{\widetilde{C}}$ is a section such that:

1. The pair $(E,\widetilde{\tau})$ defines a principal G-bundle $\mathcal{P}(E,\widetilde{\tau})$ on $\widetilde{C}$ (therefore $\det E \simeq \mathcal{O}_{\widetilde{C}}$);
2. The image of the morphism τ from the triple $(\mathcal{E},\tau,\phi) = \nu_*(E,q,\widetilde{\tau},\widetilde{\varphi})$ lies in the subalgebra $\mathcal{O}_{\hat{C}}$ of $\nu_*\mathcal{O}_{\widetilde{C}}$;
3. The image of the morphism ϕ from the triple $(\mathcal{E},\tau,\phi) = \nu_*(E,q,\widetilde{\tau},\widetilde{\varphi})$ lies in $\mathrm{End}(\mathcal{E}) \otimes \Omega^1_{\hat{C}}$.

Finally, following the constructions above, it is easy to see that a descending principal Higgs G-bundle could be injected into a double decorated vector bundle with a parabolic structure. The latter could in turn be injected into a decorated parabolic bundle over $\widetilde{C}$, i.e., a triple (E, q, φ) where $q : E_{x_1} \oplus E_{x_2} \to R$ is a surjective morphism and $\varphi : E_{a,b} \to \mathsf{N}$ is the usual decoration morphism.

Semistability

All objects we introduced before have their own notion of (semi)stability, in this section we want to recall the various notions of (semi)stabilities and give an idea of why coincide.

We recall that a subbundle $F \subseteq E$ is sayed **ϕ-invariant** if and only if $\phi(F) \subseteq F \otimes \Omega^1_C$. A Higgs bundle (E, ϕ) over a smooth irreducible curve C is **(semi)stable** if an only if for any ϕ-invariant subbundles of $F \subseteq E$ the following inequality holds:

$$\mu(F) \underset{(=)}{<} \mu(E). \tag{2.29}$$

If (E, φ) is the decorated vector bundle associated with a Higgs bundle (E, ϕ), then (E, φ) is (semi)stable as decorated bundle if an only if (E, ϕ) is (semi)stable as Higgs bundle. To be more precise:

Lemma 21. *There is a positive rational number δ_∞, such that for all $\delta \geq \delta_\infty$ and all pairs (E, φ) induced by a Higgs bundle (E, ϕ) the following conditions are equivalent:*

1. *(E, φ) is (semi)stable decorated with respect to δ;*
2. *for every nontrivial subbundle F of E with $\phi(F) \subset F \otimes \Omega^1_C$ inequality* (2.29) *holds.*

Proof. The proof is as in [33], Lemma 3.13. ♦

Let K be a closed subgroup of G, and $\sigma : X \to E(G/K) \simeq E/K$ a reduction of the structure group of E to K. So one has a principal K-bundle F_σ on X and a principal bundle morphism $i_\sigma : F_\sigma \to E$ inducing an injective morphism of bundles $\mathrm{Ad}(F_\sigma) \to \mathrm{Ad}(E)$. Let Π_σ : $\mathrm{Ad}(E) \otimes \Omega^1_C \to (\mathrm{Ad}(E)/\mathrm{Ad}(F_\sigma)) \otimes \Omega^1_C$ be the induced projection. Then a section $\sigma : X \to E/K$ is a Higgs reduction of (E, ϕ) if $\phi \in \ker \Pi_\sigma$. A principal Higgs G-bundle (E, ϕ) over C is **(semi)stable** if for every parabolic subgroup $P \subset G$ and every Higgs reduction σ one has $\deg \sigma^*(T_{E/P,X}) \underset{(=)}{>} 0$.

In the previous section we saw that the (semi)stability of principal bundles coincide with the semistability of decorated vector bundles. Therefore, thanks to the previuos results, we get that a principal Higgs G-bundle over a smooth irreducible curve C is (semi)stable if and only if the associated decorated bundle is (semi)stable. Analogously, one can check that a principal Higgs G-bundle over a nodal curve $\hat{C}$ is (semi)stable if and only if the associated parabolic decorated bundle over the normalization $\widetilde{C}$ of $\hat{C}$ is (semi)stable. For a more detailed explanation of these facts see [20].

2.3.3 Quadric, Orthogonal and Symplectic Bundles

Perhaps the simplest examples of decorated vector bundle are provided by quadric, orthogonal and symplectic bundles. First of all we recall what these objects are. A **quadric bundle** over a smooth projective variety X is a pair (E, Q) where E is a vector bundle over X, while

$$Q : \mathrm{Sym}^2 E \to L$$

is a morphism between the vector bundle $\mathrm{Sym}^2 E$ and a fixed line bundle L. An **orthogonal bundle** is a quadric bundle (E, Q) with $L = \mathcal{O}_X$, such that the bilinear form $Q : \mathrm{Sym}^2 E \to \mathcal{O}_X$ induces an isomorphism $Q : E \to E^\vee$. Finally a **symplectic vector bundle** over X is a pair (E, ω), where E is a real vector bundle over X and ω is a smooth section of $E^\vee \wedge E^\vee$ such that for each $x \in X$, (E_x, ω_x) is a symplectic vector space. The section ω is called a **symplectic bilinear** form on E.

It is easy to see that all these objects are decorated bundles of type $(2, 1, 0, \mathsf{N})$, for appropriate N.

In Section 5.3 we will recall the semistability condition for orthogonal bundles over curves and show that coincides with semistability of decorated bundles, at least in the case of bundles over curves. The general case is a straightforward generalization.

2.3.4 Framed Bundles

Let $(X, \mathcal{O}_X(1))$ be, as usual, a polarized smooth projective variety. We want to introduce the notion of framed bundles and framed sheaves over X. These objects are probably one of the first examples of bundles "decorated" by an additional structure given by a morphism. Fix a coherent sheaf F, called **framing sheaf**, over X. Then a **framed sheaf** is a pair (E, α) where E is a coherent sheaf on X and $\alpha : E \to F$ is a morphism of coherent sheaves. A **framed bundle** on X is instead a pair (E, α) such that E is a torsion free sheaf on X, $D \subset X$ is a effective divisor, F is a vector bundle on D and $\alpha : E_{|D} \to F$ is an isomorphism.

For any framed sheaf (E, α), we define the function $\varepsilon(\alpha)$ by

$$\varepsilon(\alpha) \doteqdot \begin{cases} 1 \text{ if } \alpha \neq 0 \\ 0 \text{ if } \alpha = 0. \end{cases}$$

Let n denote the dimension of X and $\delta(x) = \delta_{n-1}x^{n-1} + \cdots + \delta_1 x + \delta_0 \in \mathbb{Q}[x]$ denote a fixed polynomial with positive leading coefficient $\delta_{n-1} > 0$. We define the **framed degree** of (E, α)

$$\deg(E, \alpha) \doteqdot \deg E - \delta_{n-1}\varepsilon(\alpha),$$

and the **framed Hilbert polynomial**

$$\mathsf{P}_{(E,\alpha)} \doteqdot \mathsf{P}_E - \delta\varepsilon(\alpha).$$

Finally a framed sheaf is **(semi)stable** if and only if for any framed subsheaf $(F, \alpha_{|_F})$ of (E, α)

$$\mathrm{rk}(E)\mathsf{P}_{(F,\alpha_{|_F})} \underset{(-)}{\prec} \mathrm{rk}(F)\mathsf{P}_{(E,\alpha)}$$

and **slope (semi)stable** if

$$\mathrm{rk}(E)\deg(F, \alpha_{|_F}) \underset{(=)}{<} \mathrm{rk}(F)\deg(E, \alpha).$$

If, in the definition of decorated bundles of type (a, b, c, N), ones admits the line bundle N to be a coherent sheaf then a framed bundle with framing sheaf F is nothing else that a decorated bundle of type $(1, 1, 0, F)$ and framed semistability coincide with decorated ε-semistability. Similar considerations can be made for framed sheaves.

Chapter 3

Mehta-Ramanathan theorems

Notation. Let $(X, \mathcal{O}_X(1))$ be, as usual, a polarized projective smooth variety of dimension n, $\delta = \delta(x) \doteqdot \delta_{n-1}x^{n-1} + \cdots + \delta_1 x + \delta_0$ be a fixed polynomial with positive leading coefficient and let $\overline{\delta} = \delta_{n-1}$.

Decorated sheaves were introduced by Schmitt and provide a useful a machinery to study principal bundles or more generally vector bundles with additional structures. However in general it is quite hard to check semistability for decorated sheaves because one has to verify inequality (2.8) and therefore calculate μ_{I} for any filtration. For this reason we introduce ε-semistability, a more computable notion of semistability, stronger than the usual semistability given by Schmitt. ε-semistability is quite similar to the semistability condition for framed sheaves given by Huybrechts and Lehn in [14].

3.1 Decorated coherent sheaves

With the expression "**decorated coherent sheaf**" we mean a decorated sheaf $(\mathcal{A}, \varphi)$ such that $\mathcal{A}$ is just a coherent sheaf (and not necessarily torsion free).
Before proceeding we recall what a decorated coherent *subsheaf* is. If $i\colon (\mathcal{F}, \psi) \to (\mathcal{A}, \varphi)$ is an injective morphism of decorated sheaves we get immediately from condition (2.2) that $\lambda \cdot \psi = i^*\varphi$. From now on we will say that the tripe $((\mathcal{F}, \psi), i)$ is a **decorated subsheaf** of $(\mathcal{A}, \varphi)$ and we will denote it just by $(\mathcal{F}, \varphi_{|\mathcal{F}})$. Note moreover that, if $\mathcal{F}$ is a subsheaf of $\mathcal{A}$ and $i : \mathcal{F} \to \mathcal{A}$ is the inclusion, then it defines a decorated subsheaf; in fact defining $\psi = \varphi_{|_{i_{a,b}(\mathcal{F}_{a,b})}}$ the triple $((\mathcal{F}, \psi), i)$ is a decorated subsheaf of $(\mathcal{A}, \varphi)$.

For an arbitrary decorated coherent sheaf $(\mathcal{A},\varphi)$ define the **ε-decorated degree**

$$\deg(\mathcal{A},\varphi) \doteqdot \deg(\mathcal{A}) - a\overline{\delta}\varepsilon(\mathcal{A},\varphi),$$

where $\deg\mathcal{A} \doteqdot c_1(\mathcal{A})\cdot\mathcal{O}_X(1)^{n-1}$, and the **$\varepsilon$-decorated Hilbert polynomial**

$$\mathsf{P}_{(\mathcal{A},\varphi)}(m) \doteqdot \mathsf{P}_{\mathcal{A}}(m) - a\delta(m)\varepsilon(\mathcal{A},\varphi)$$

If moreover rank $\mathrm{rk}(\mathcal{A})>0$ we define the **ε-decorated slope** and, respectively, the **reduced ε-decorated Hilbert polynomial** as:

$$\mu(\mathcal{A},\varphi) \doteqdot \frac{\deg(\mathcal{A},\varphi)}{\mathrm{rk}(\mathcal{A})} \qquad \mathsf{p}_{(\mathcal{A},\varphi)}(m) \doteqdot \frac{\mathsf{P}_{(\mathcal{A},\varphi)}(m)}{\mathrm{rk}(\mathcal{A})}.$$

Sometimes, if the morphism φ is clear from the context, we will write $\deg_\varepsilon(\mathcal{A})$ instead of $\deg(\mathcal{A},\varphi)$, $\mu_\varepsilon(\mathcal{A})$ instead of $\mu(\mathcal{A},\varphi)$ and $\mathsf{P}^\varepsilon_{\mathcal{A}}$ (respectively $\mathsf{p}^\varepsilon_{\mathcal{A}}$) instead of $\mathsf{P}_{(\mathcal{A},\varphi)}$ (resp. $\mathsf{p}_{(\mathcal{A},\varphi)}$).

Recall that given two polynomials $p(m)$ and $q(m)$ then $p\preceq q$ if and only if there exists $m_0\in\mathbb{N}$ such that $p(m)\le q(m)$ for any $m\ge m_0$.

Definition 22. Let $(\mathcal{A},\varphi)$ be a decorated coherent sheaf of positive rank, than we will say that $(\mathcal{A},\varphi)$ is **ε-(semi)stable** or, respectively, **slope ε-(semi)stable** with respect to δ (resp. $\overline{\delta}$) if and only if for any proper non trivial subsheaf $\mathcal{F}\subset\mathcal{A}$ the following inequality holds:

$$\mathsf{P}_{(\mathcal{F},\varphi|_{\mathcal{F}})}\,\mathrm{rk}(\mathcal{A}) \underset{(-)}{\prec} \mathsf{P}_{(\mathcal{A},\varphi)}\,\mathrm{rk}(\mathcal{F}). \tag{3.1}$$

or, respectively,

$$\deg(\mathcal{F},\varphi)\,\mathrm{rk}(\mathcal{A}) \underset{(=)}{<} \mathrm{rk}(\mathcal{F})\,\deg(\mathcal{A},\varphi) \tag{3.2}$$

If $\mathrm{rk}(\mathcal{A})=0$ we say that $(\mathcal{A},\varphi)$ is semistable (resp. slope semistable) or stable (resp. slope stable) if moreover $\mathsf{P}_{\mathcal{A}}=\delta$ (resp. $\deg\mathcal{A}=\overline{\delta}$).

In particular this ε-(semi)stability extends ε-semistability, defined in Section 2.2.1, to decorated *coherent* sheaves.

Remark 23. Note that

slope ε-stable $\Rightarrow$ ε-stable $\Rightarrow$ ε-semistable $\Rightarrow$ slope ε-semistable;

and recall that ε-semistability (slope ε-semistability) is *strictly stronger* than the usual semistability (resp. slope semistability) introduced in Chapter 2.

The kernel of φ lies in $\mathcal{A}_{a,b}$, so for our purpose we need to define a subsheaf of $\mathcal{A}$ that plays a similar role to the kernel of φ. Therefore we let

$$\mathsf{K} \doteqdot \max\{0 \subseteq \mathcal{F} \subseteq \mathcal{A} \mid \mathcal{F}_{a,b} \subseteq \ker\varphi\}$$

where the maximum is taken with respect to the partial ordering given by the inclusion of sheaves. Note that K is unique, indeed if K' is another maximal element, then $\mathsf{K} \cup \mathsf{K}'$ is a subsheaf of $\mathcal{A}$ such that $(\mathsf{K} \cup \mathsf{K}')_{a,b} \subset \ker\varphi$ and this is absurd.

Remark 24. 1. Let $T(\mathcal{A})$ be the torsion part of $\mathcal{A}$. The torsion part $T(\mathcal{A}_{a,b})$ lies in $\ker\varphi$, otherwise there would be a non zero morphism between a sheaf of pure torsion and the torsion free sheaf N and this is impossible. Therefore also the twisted torsion part $T(\mathcal{A})_{a,b} \subset T(\mathcal{A}_{a,b})$ lies in the kernel of φ.

2. $T(\mathcal{A}) \subseteq \ker\varphi$ (point (1)) therefore $T(\mathcal{A}) \subseteq \mathsf{K}$;

3. $\mathcal{A}$ is torsion free if and only if K is torsion free. Indeed, suppose that K is torsion free and that $T(\mathcal{A}) \neq \emptyset$, then $\mathsf{K} \subsetneq \mathsf{K} \cup T(\mathcal{A})$ which is absurd for maximality of K. The converse is obvious.

4. $\mathcal{A}_{a,b}$ is torsion free if and only if $\ker\varphi$ is torsion free.

5. If $(\mathcal{A},\varphi)$ is semistable and $\mathrm{rk}(\mathcal{A}) > 0$ then K is a torsion free sheaf. Indeed if $T(\mathsf{K})$ is the torsion part of K, $\mathrm{rk}(T(\mathsf{K})) = 0$ and than, for the semistability condition, we get that:

$$\mathrm{rk}(\mathcal{A})\mathsf{P}_{(T(\mathsf{K})_{a,b},\varphi|_{T(\mathsf{K})_{a,b}})} \overset{\preceq}{(-)} 0.$$

Therefore $T(\mathsf{K})$ is zero and K is torsion free.

6. If $(\mathcal{A},\varphi)$ is semistable and $\mathrm{rk}(\mathcal{A}) > 0$ then $\mathcal{A}$ is pure of dimension $\dim X$ and therefore torsion free. Indeed let $\mathcal{F}$ a subsheaf of $\mathcal{A}$ of pure torsion, then by the semistability condition we get that

$$\mathrm{rk}(\mathcal{A})\left(\mathsf{P}_{\mathcal{F}} - a\delta\varepsilon(\varphi_{|\mathcal{F}})\right) \preceq \mathrm{rk}(\mathcal{F})\left(\mathsf{P}_{\mathcal{A}} - a\delta\right) = 0.$$

Moreover, for point (1), $\mathcal{F}_{a,b} \subset \ker\varphi$ and so $\mathsf{P}_{\mathcal{F}} \preceq 0$, this immediately implies $\mathcal{F} = 0$.

Remark 25. Note that Remark 24 holds also in the slope ε-semistable case.

Proposition 26. *Let $(\mathcal{A},\varphi)$ be a decorated coherent sheaf of positive rank and let $T \doteqdot T(\mathcal{A})$ be the torsion of $\mathcal{A}$. Then the following statements hold.*

1. *$(\mathcal{A},\varphi)$ is ε-semistable with respect to $\delta \Longrightarrow \mathcal{A}/T$ is ε-semistable with respect to δ.*

2. *$(\mathcal{A}/T, \varphi)$ is ε-semistable with respect to $\delta \Longrightarrow (\mathcal{A}, \varphi)$ is ε-semistable with respect to δ or T is the maximal destabilizing subsheaf of $\mathcal{A}$ in the sense of Remark 30.*

Proof. First of all note that, since $T \subset \ker\varphi$, the pair $(\mathcal{A}/T, \varphi)$ is a well-defined decorated (torsion free) sheaf of the same type of $(\mathcal{A}, \varphi)$.

1. If $(\mathcal{A}, \varphi)$ ε-semistable with respect to δ, then as in Remark 24 one can prove that $T = 0$ and so obviously $(\mathcal{A}/T, \varphi)$ is semistable.
2. Suppose that $(\mathcal{A}/T, \varphi)$ ε-semistable with respect to δ. If T does not destabilize, then $\mathsf{P}^{\varepsilon}_{T} = \mathsf{P}_{T} \preceq 0$ and so $T = 0$ and $(\mathcal{A}, \varphi)$ is ε-semistable. Otherwise $\mathsf{P}^{\varepsilon}_{T} = \mathsf{P}_{T} \succeq 0$ and Remark 30 shows that is the maximal destabilizing subsheaf.

♦

3.2 Mehta-Ramanathan for slope ε-semistable decorated sheaves

In this section we want to prove a Mehta-Ramanathan theorem for slope ε-semistable decorated sheaves. Before we proceed we need some notation and preliminary results.

Notation. Let k be an algebraic closed field of characteristic 0, S an integral k-scheme of finite type. X will be a smooth projective variety over k, $\mathcal{O}_X(1)$ an ample line bundle on X and $f : X \to S$ a projective flat morphism. Note that $\mathcal{O}_X(1)$ is also f-ample. In this section we will suppose that any decorated sheaf is of type (a, b, N). If $(\mathcal{W}, \varphi)$ is a decorated coherent sheaf over X we denote by $\mathcal{W}_s$ the restriction $\mathcal{W}_{|X_s}$, where $X_s \doteqdot f^{-1}(s)$, and φ_s the restriction $\varphi_{|\mathcal{W}_s}$. Finally, if $\mathcal{F}$ is a sheaf, we will denote by $r_{\mathcal{F}}$ the quantity $\mathrm{rk}(\mathcal{F})$.

3.2.1 Maximal destabilizing subsheaf

Proposition 27. *Let $(\mathcal{E}, \varphi)$ be a decorated sheaf over a nonsingular projective smooth variety X. If $(\mathcal{E}, \varphi)$ is not ε-semistable there is a unique, ε-semistable, proper subsheaf $\mathcal{F}$ of $\mathcal{E}$ such that:*

1. *$\mathsf{p}^{\varepsilon}_{\mathcal{F}} \succeq \mathsf{p}^{\varepsilon}_{\mathcal{W}}$ for all subsheaf $\mathcal{W}$ of $\mathcal{E}$.*
2. *If $\mathsf{p}^{\varepsilon}_{\mathcal{F}} = \mathsf{p}^{\varepsilon}_{\mathcal{W}}$ then $\mathcal{W} \subset \mathcal{F}$.*

*The subsheaf $\mathcal{F}$, with the induced morphism $\varphi_{|\mathcal{F}}$, is called **maximal destabilizing subsheaf** of $(\mathcal{E}, \varphi)$.*

Proof. First we recall that by definition $\mathcal{E}$ is torsion free and therefore of positive rank.

We define a partial ordering on the set of decorated subsheaves of a given decorated sheaf $(\mathcal{E}, \varphi)$. Let $\mathcal{F}_1, \mathcal{F}_2$ two subsheaves of $\mathcal{E}$, then

$$\mathcal{F}_1 \preccurlyeq \mathcal{F}_2 \quad \Longleftrightarrow \quad \mathcal{F}_1 \subseteq \mathcal{F}_2 \quad \wedge \quad \mathsf{P}_{(\mathcal{F}_1,\varphi_1)} \operatorname{rk}(\mathcal{F}_2) \preceq \mathsf{P}_{(\mathcal{F}_2,\varphi_2)} \operatorname{rk}(\mathcal{F}_1) \tag{3.3}$$

where $\varphi_i = \varphi_{|\mathcal{F}_i}$. Note that the set of the subsheaves of a sheaf $\mathcal{E}$ with this order relation $\preccurlyeq$ satisfies the hypothesis of Zorn's Lemma, so there exists a maximal element (non unique in general). Let

$$\mathcal{F} \doteqdot \min_{\operatorname{rk}(\mathcal{G})} \{\mathcal{G} \subset \mathcal{E} \,|\, \mathcal{G} \text{ is } \preccurlyeq\text{-maximal}\} \tag{3.4}$$

i.e., $\mathcal{F}$ is a $\preccurlyeq$-maximal subsheaf with minimal rank among all $\preccurlyeq$-maximal subsheaves. Then we claim that $(\mathcal{F}, \varphi_{|\mathcal{F}})$ has the asserted properties.

Suppose that exists $\mathcal{G} \subset \mathcal{E}$ such that

$$\mathsf{p}^{\varepsilon}_{\mathcal{G}} \succeq \mathsf{p}^{\varepsilon}_{\mathcal{F}} \tag{3.5}$$

First we show that we can assume $\mathcal{G} \subset \mathcal{F}$ by replacing $\mathcal{G}$ by $\mathcal{F} \cap \mathcal{G}$. Indeed if $\mathcal{G} \not\subset \mathcal{F}$, $\mathcal{F}$ is a proper subsheaf of $\mathcal{F} + \mathcal{G}$ in fact $\mathcal{F} \not\subset \mathcal{G}$ (otherwise $\mathcal{F} \preccurlyeq \mathcal{G}$ which is absurd for maximality of $\mathcal{F}$). By maximality one has that

$$\mathsf{p}^{\varepsilon}_{\mathcal{F}} \succ \mathsf{p}^{\varepsilon}_{\mathcal{F}+\mathcal{G}}. \tag{3.6}$$

Using the exact sequence

$$0 \longrightarrow \mathcal{F} \cap \mathcal{G} \longrightarrow \mathcal{F} \oplus \mathcal{G} \longrightarrow \mathcal{F} + \mathcal{G} \longrightarrow 0$$

one finds $\mathsf{P}_{\mathcal{F}} + \mathsf{P}_{\mathcal{G}} = \mathsf{P}_{\mathcal{F}\oplus\mathcal{G}} = \mathsf{P}_{\mathcal{F}\cap\mathcal{G}} + \mathsf{P}_{\mathcal{F}+\mathcal{G}}$ and $\operatorname{rk}(\mathcal{F}) + \operatorname{rk}(\mathcal{G}) = \operatorname{rk}(\mathcal{F} \oplus \mathcal{G}) = \operatorname{rk}(\mathcal{F} \cap \mathcal{G}) + \operatorname{rk}(\mathcal{F} + \mathcal{G})$. Hence

$$r_{\mathcal{F}\cap\mathcal{G}}(\mathsf{p}_{\mathcal{G}} - \mathsf{p}_{\mathcal{F}\cap\mathcal{G}}) = r_{\mathcal{F}+\mathcal{G}}(\mathsf{p}_{\mathcal{F}+\mathcal{G}} - \mathsf{p}_{\mathcal{F}}) + (r_{\mathcal{G}} - r_{\mathcal{F}\cap\mathcal{G}})(\mathsf{p}_{\mathcal{F}} - \mathsf{p}_{\mathcal{G}}). \tag{3.7}$$

where we denote by $r_{\mathcal{F}}$, $r_{\mathcal{G}}$, $r_{\mathcal{F}+\mathcal{G}}$ and $r_{\mathcal{F}\cap\mathcal{G}}$ the rank of $\operatorname{rk}(\mathcal{F})$, $\operatorname{rk}(\mathcal{G})$, $\operatorname{rk}(\mathcal{F} + \mathcal{G})$ and $\operatorname{rk}(\mathcal{F} \cap \mathcal{G})$ respectively.

If the morphism φ is zero ε-semistability coincides with the usual semistability for torsion-free sheaves and the existence of the maximal destibilizing subsheaf is a well-known fact that one can find, for example, in [15] Lemma 1.3.6. So we suppose that $\varepsilon(\varphi) = 1$. From the above inequalities between reduced decorated Hilbert polynomial of $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{F} + \mathcal{G}$ one can easily obtain:

$$\mathsf{p}_{\mathcal{F}+\mathcal{G}} - \mathsf{p}_{\mathcal{F}} \prec a\delta \left(\frac{\varepsilon_{\mathcal{F}+\mathcal{G}}}{r_{\mathcal{F}+\mathcal{G}}} - \frac{\varepsilon_{\mathcal{F}}}{r_{\mathcal{F}}} \right)$$
$$\mathsf{p}_{\mathcal{F}} - \mathsf{p}_{\mathcal{G}} \preceq a\delta \left(\frac{\varepsilon_{\mathcal{F}}}{r_{\mathcal{F}}} - \frac{\varepsilon_{\mathcal{G}}}{r_{\mathcal{G}}} \right),$$

therefore, using equation (3.7) and after some easy computations, one gets

$$\begin{aligned}
&r_{\mathcal{F}\cap\mathcal{G}}(\mathsf{p}^{\varepsilon}_{\mathcal{G}} - \mathsf{p}^{\varepsilon}_{\mathcal{F}\cap\mathcal{G}}) = \\
&\quad = r_{\mathcal{F}+\mathcal{G}}(\mathsf{p}_{\mathcal{F}+\mathcal{G}} - \mathsf{p}_{\mathcal{F}}) + (r_{\mathcal{G}} - r_{\mathcal{F}\cap\mathcal{G}})(\mathsf{p}_{\mathcal{F}} - \mathsf{p}_{\mathcal{G}}) - a\delta\varepsilon_{\mathcal{G}}\frac{r_{\mathcal{F}\cap\mathcal{G}}}{r_{\mathcal{G}}} + a\delta\varepsilon_{\mathcal{F}\cap\mathcal{G}} \\
&\quad \prec a\delta r_{\mathcal{F}\cap\mathcal{G}}\left(\frac{\varepsilon_{\mathcal{F}+\mathcal{G}}}{r_{\mathcal{F}+\mathcal{G}}} - \frac{\varepsilon_{\mathcal{F}}}{r_{\mathcal{F}}}\right) + a\delta(r_{\mathcal{G}} - r_{\mathcal{F}\cap\mathcal{G}})\left(\frac{\varepsilon_{\mathcal{F}}}{r_{\mathcal{F}}} - \frac{\varepsilon_{\mathcal{G}}}{r_{\mathcal{G}}}\right) - a\delta\varepsilon_{\mathcal{G}}\frac{r_{\mathcal{F}\cap\mathcal{G}}}{r_{\mathcal{G}}} + a\delta\varepsilon_{\mathcal{F}\cap\mathcal{G}} \\
&\quad = a\delta\left(\varepsilon_{\mathcal{F}+\mathcal{G}} - \varepsilon_{\mathcal{F}} - \varepsilon_{\mathcal{G}} + \varepsilon_{\mathcal{F}\cap\mathcal{G}}\right) \\
&\quad \preceq 0.
\end{aligned}$$

Therefore we can suppose both $\mathcal{G} \subset \mathcal{F}$ and $\mathsf{p}^{\varepsilon}_{\mathcal{G}} \succeq \mathsf{p}^{\varepsilon}_{\mathcal{F}}$, and, up to replacing $\mathcal{G}$, we can suppose that $\mathcal{G}$ is maximal in $\mathcal{F}$ with respect to $\preccurlyeq$. Let $\mathcal{G}'$ be a $\preccurlyeq$-maximal in $\mathcal{E}$ among all subsheaves (of $\mathcal{E}$) containing $\mathcal{G}$. Then

$$\mathsf{p}^{\varepsilon}_{\mathcal{F}} \preceq \mathsf{p}^{\varepsilon}_{\mathcal{G}} \preceq \mathsf{p}^{\varepsilon}_{\mathcal{G}'}.$$

Note that neither $\mathcal{G}'$ is contained in $\mathcal{F}$, because $\mathcal{F}$ has minimal rank between all $\preccurlyeq$-maximal subsheaves of $\mathcal{E}$, nor $\mathcal{F}$ is contained in $\mathcal{G}'$, for maximality of $\mathcal{F}$; therefore $\mathcal{F}$ is a proper subsheaf of $\mathcal{F} + \mathcal{G}'$ and, for maximality, $\mathsf{p}^{\varepsilon}_{\mathcal{F}} \succeq \mathsf{p}^{\varepsilon}_{\mathcal{F}+\mathcal{G}'}$. As before one gets

$$\mathsf{p}^{\varepsilon}_{\mathcal{F}\cap\mathcal{G}'} \succ \mathsf{p}^{\varepsilon}_{\mathcal{G}'} \succeq \mathsf{p}^{\varepsilon}_{\mathcal{G}},$$

but $\mathcal{G} \subseteq \mathcal{F} \cap \mathcal{G}' \subseteq \mathcal{F}$ and this contradicts the assumpions on $\mathcal{G}$. Therefore $\mathcal{F}$ satisfies the required properties. The uniqueness and the ε-semistability of $\mathcal{F}$ easily follow from properties (1) and (2). ♦

Lemma 28. *Let $(\mathcal{E}, \varphi)$ be as before. If it is not slope ε-semistable there is a unique proper subsheaf $\mathcal{F}$ of $\mathcal{E}$ such that:*

1. *$\mu_{\varepsilon}(\mathcal{F}) \geq \mu_{\varepsilon}(\mathcal{W})$ for all subsheaves $\mathcal{W}$ of $\mathcal{E}$.*
2. *If $\mu_{\varepsilon}(\mathcal{F}) = \mu_{\varepsilon}(\mathcal{W})$ then $\mathcal{W} \subset \mathcal{F}$.*

Proof. The proof is the same of Proposition 27: it is sufficient to replace p^{ε} with μ_{ε}, P^{ε} with $\deg_{\varepsilon}$ and δ with $\overline{\delta}$. ♦

Remark 29. Note that, if $(\mathcal{E}, \varphi)$ is ε-semistable, or, respectively, slope ε-semistable, then the maximal decorated destabilizing (resp. slope destabilizing) subsheaf coincides with $\mathcal{E}$.

Proposition 30. *Let $(\mathcal{A}, \varphi)$ be a decorated coherent sheaf of positive rank, then Proposition 27 and Lemma 28 hold true, in the sense that if $(\mathcal{A}, \varphi)$ is not ε-semistable (slope ε-semistable respectively) there is a unique, ε-semistable, proper subsheaf $\mathcal{F}$ of $\mathcal{E}$ such that:*

1. *$\mathsf{P}^{\varepsilon}_{\mathcal{F}} rk(\mathcal{E}) \succeq \mathsf{P}^{\varepsilon}_{\mathcal{W}} rk(\mathcal{F})$ for all subsheaves $\mathcal{W}$ of $\mathcal{E}$.*
2. *If $\mathsf{P}^{\varepsilon}_{\mathcal{F}} rk(\mathcal{W}) = \mathsf{P}^{\varepsilon}_{\mathcal{W}} rk(\mathcal{F})$ then $\mathcal{W} \subset \mathcal{F}$.*

or, respectively

1'. $\deg_\varepsilon(\mathcal{F})rk(\mathcal{W}) \geq \deg_\varepsilon(\mathcal{W})rk(\mathcal{F})$ *for all subsheaves* $\mathcal{W}$ *of* $\mathcal{E}$.

2'. *If* $\deg_\varepsilon(\mathcal{F})rk(\mathcal{W}) = \deg_\varepsilon(\mathcal{W})rk(\mathcal{F})$) *then* $\mathcal{W} \subset \mathcal{F}$.

Proof. Indeed, let $\mathcal{F}$ be a minimal rank sheaf between all $\preccurlyeq$-maximal sheaves as in the proof of Proposition 27. Suppose that $\text{rk}(\mathcal{F}) = 0$, then $\mathcal{F} \subset \mathsf{K} = T(\mathcal{A})$, and so, by maximality $\mathcal{F} = T(\mathcal{A})$. If $\mathcal{G}$ is such that $\text{rk}(\mathcal{G}) > 0$ and $T(\mathcal{A}) \preccurlyeq \mathcal{G}$ then $\mathsf{P}_{T(\mathcal{A})} \preceq 0$ but, by hypothesis, $T(\mathcal{A})$ destabilize and so $\mathsf{P}_{T(\mathcal{A})} \succ 0$. Finally $T(\mathcal{E})$ is clearly unique and semistable.
Otherwise, if $\text{rk}(\mathcal{F}) > 0$, then $\mathcal{A}$ has no nontrivial rank zero subsheaves, in particular is torsion free. Indeed if exists a subsheaf $\mathcal{G} \subset \mathcal{A}$ with $r_\mathcal{G} = 0$ then, by the above considerations exists $\mathcal{G}'$ with $\text{rk}(\mathcal{G}') = 0$, $\mathcal{G} \subseteq \mathcal{G}'$ and $\mathcal{G}'$ $\preccurlyeq$-maximal which is absurd by the assumptions on $\mathcal{F}$. Then the proof continues as the proof of Proposition 27.
The proof in the case of slope ε-semistability is the same. ♦

3.2.2 Families of decorated sheaves

Let $f : Y \to S$ be a morphism of finite type of Noetherian schemes. Recall that a **flat family** of coherent sheaves **on the fibre of the morphism** f is a coherent sheaf $\mathcal{A}$ over Y, which is flat over S, i.e., for any $y \in Y$ $\mathcal{A}_y$ is flat over the local ring $\mathcal{O}_{S,f(y)}$. If $\mathcal{A}$ is flat over the fibre of f the Hilbert polynomial $\mathsf{P}_{\mathcal{A}_s}$ is locally constant as a function of s. The converse is not true in general, but, if S is reduced, then the two assertions are equivalent.

Definition 31. Let $(\mathcal{E}, \varphi)$ be a *decorated sheaf* over Y of type (a, b, N) and $f : Y \to S$ be a morphism of finite type between Noetherian schemes. Then $(\mathcal{E}, \varphi)$ is a **flat family over the fibre of** f if and only if

- $\mathcal{E}$ and N are flat families of coherent sheaves over the fibre of $f : Y \to S$;
- $\mathcal{E}_s = \mathcal{E}_{|_{f^{-1}(s)}}$ is torsion free for all $s \in S$;
- $\mathsf{N}_s = \mathsf{N}_{|_{f^{-1}(s)}}$ is locally free for all $s \in S$;
- $\varepsilon(\varphi_s) = \varepsilon(\varphi_{|_{\mathcal{E}_s}})$ is locally constant as a function of s.

Note that the above conditions imply that the ε-Hilbert polynomials $\mathsf{P}^\varepsilon_{\mathcal{E}_s}$ are locally constant for $s \in S$.

Definition 32. Let $(\mathcal{A}, \varphi)$ be a *decorated coherent sheaf of positive rank.* Then $(\mathcal{A}, \varphi)$ is a **flat family over the fibre of** f if and only if

- $\mathcal{A}$ and N are flat families of sheaves over the fibre of $f : Y \to S$;
- N_s is locally free for all $s \in S$;

- $\varepsilon(\varphi_s)$ is locally constant as a function of s;
- $\mathrm{rk}(\mathcal{A}_s) > 0$ for any $s \in S$.

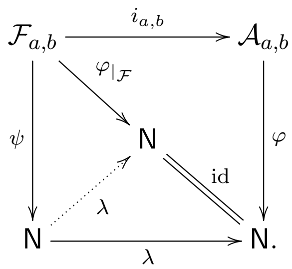
As in Definition 31 the above conditions imply that the ε-Hilbert polynomials $\mathsf{P}^{\varepsilon}_{\mathcal{A}_s}$ are locally constant for $s \in S$.

3.2.3 Families of quotients

Let $(\mathcal{A}, \varphi)$ be a decorated coherent sheaf over X and let $q : \mathcal{A} \twoheadrightarrow \mathcal{Q}$ be surjective morphism of sheaves. Let $\mathcal{F}$ be the subsheaf of $\mathcal{A}$ defined by $\ker q$, so the following succession of sheaves is exact:

$$0 \longrightarrow \mathcal{F} \stackrel{i}{\longrightarrow} \mathcal{A} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Note that $\mathcal{F}$ is uniquely determined by $\mathcal{Q}$ and therefore also $(\mathcal{F}, \varphi_{|\mathcal{F}})$ is uniquely (up to isomorphism of decorated sheaves) determined by $\mathcal{Q}$. Indeed, let $(\mathcal{F}, \psi)$ be another decorated subsheaf of $(\mathcal{A}, \varphi)$, then, by definition of decorated subsheaf, there exists a non-zero scalar morphism $\lambda : \mathsf{N} \to \mathsf{N}$ such that $\lambda \circ \psi = \varphi$, then



Since the big square and the upper triangle commute, the entire diagram commutes and so it easy to see that $(\mathcal{F}, \psi)$ and $(\mathcal{F}, \varphi_{|\mathcal{F}})$ are isomorphic as decorated sheaves.

Suppose now that $(\mathcal{F}, \varphi_{|\mathcal{F}})$ de-semistabilizes a decorated sheaf $(\mathcal{E}, \varphi)$ (with respect to the slope ε-semistability), then $\mu_\varepsilon(\mathcal{F}) > \mu_\varepsilon(\mathcal{E})$ and so

$$\mu(\mathcal{F}) - \frac{a\overline{\delta}\varepsilon_{\mathcal{F}}}{r_{\mathcal{F}}} > \mu(\mathcal{E}) - \frac{a\overline{\delta}\varepsilon_{\mathcal{E}}}{r_{\mathcal{E}}}$$

$$\deg(\mathcal{F}) > r_{\mathcal{F}} \left[\mu(\mathcal{E}) + a\overline{\delta} \left(\frac{\varepsilon_{\mathcal{F}}}{r_{\mathcal{F}}} - \frac{\varepsilon_{\mathcal{E}}}{r_{\mathcal{E}}} \right) \right]$$

Recalling that $\deg(\mathcal{E}) = \deg(\mathcal{F}) + \deg(\mathcal{Q})$,

$$\deg(\mathcal{Q}) < \deg(\mathcal{E}) - r_{\mathcal{F}} \left[\mu(\mathcal{E}) + a\overline{\delta} \left(\frac{\varepsilon_{\mathcal{F}}}{r_{\mathcal{F}}} - \frac{\varepsilon_{\mathcal{E}}}{r_{\mathcal{E}}} \right) \right]$$

$$= \mu(\mathcal{E}) r_{\mathcal{Q}} - a\overline{\delta} \left(\varepsilon_{\mathcal{F}} - \varepsilon_{\mathcal{E}} \frac{r_{\mathcal{F}}}{r_{\mathcal{E}}} \right)$$

and therefore, if $\varepsilon_{\mathcal{E}} = 1$,

$$\mu(\mathcal{Q}) < \mu(\mathcal{E}) - a\overline{\delta}\left(\frac{\varepsilon_{\mathcal{F}}}{r_{\mathcal{Q}}} - \frac{r_{\mathcal{F}}}{r_{\mathcal{Q}} r_{\mathcal{E}}}\right) = \mu(\mathcal{E}) + a\overline{\delta} \cdot \begin{cases} \frac{r_{\mathcal{F}}}{r_{\mathcal{Q}} r_{\mathcal{E}}} \doteqdot C_0 & \text{if } \varepsilon_{\mathcal{F}} = 0 \\ -\frac{1}{r_{\mathcal{E}}} \doteqdot -C_1 & \text{if } \varepsilon_{\mathcal{F}} = 1, \end{cases} \tag{3.8}$$

otherwise, if $\varepsilon_{\mathcal{E}} = 0$ then also $\varepsilon_{\mathcal{F}} = 0$ and so we get that $\mu(\mathcal{Q}) < \mu(\mathcal{E})$.

Remark 33. Defining $\deg_{\varepsilon}(\mathcal{Q}) \doteqdot \deg_{\varepsilon}(\mathcal{E}) - \deg_{\varepsilon}(\mathcal{F})$ and $\mu_{\varepsilon}(\mathcal{Q}) = \deg_{\varepsilon}(\mathcal{Q})/r_{\mathcal{Q}}$ one easily gets that $\mu_{\varepsilon}(\mathcal{F}) > \mu_{\varepsilon}(\mathcal{E})$ if and only if $\mu_{\varepsilon}(\mathcal{Q}) \leq \mu_{\varepsilon}(\mathcal{E})$. Note that in general it is not possible to define a morphism ψ over $\mathcal{Q}$ such that $(\mathcal{Q}, \psi)$ is a decorated sheaf and $\varepsilon(\mathcal{Q}, \psi) + \varepsilon(\mathcal{F}, \varphi_{|\mathcal{F}}) = \varepsilon(\mathcal{E}, \varphi)$. In fact it is possible to define a morphism to the quotient satisfying such properties if and only if $\mathsf{k}_{\mathcal{F},\mathcal{E}} = 0$ or $\mathsf{k}_{\mathcal{F},\mathcal{E}} = a$. This is because only in these two cases the morphism t:

$$\begin{array}{ccccc} \mathcal{F}_{a,b} & \hookrightarrow & \mathcal{E}_{a,b} & \xrightarrow{\varphi} & \mathsf{N} \\ & & \downarrow & & \uparrow \\ & & \mathcal{E}_{a,b}/\mathcal{F}_{a,b} & \xrightarrow[t]{} & (\mathcal{E}/\mathcal{F})_{a,b} \end{array}$$

is well defined and so it is possible to give a well-defined structure of decorated sheaf to $(\mathcal{E}/\mathcal{F})$ defining a morphism $\overline{\varphi} : (\mathcal{E}/\mathcal{F})_{a,b} \to \mathsf{N}$.

Analogously, if $(\mathcal{F}, \varphi_{|\mathcal{F}})$ de-semistabilizes $(\mathcal{E}, \varphi)$ with respect to the ε-semistability, i.e., if

$$\mathsf{p}_{\mathcal{F}} - a\delta\frac{\varepsilon_{\mathcal{F}}}{r_{\mathcal{F}}} \succ \mathsf{p}_{\mathcal{E}} - \frac{a\delta}{r_{\mathcal{E}}},$$

then similar calculations show that

$$\mathsf{p}_{\mathcal{Q}} \prec \mathsf{p}_{\mathcal{E}} + a\delta \cdot \begin{cases} C_0 & \text{if } \varepsilon_{\mathcal{F}} = 0 \\ -C_1 & \text{if } \varepsilon_{\mathcal{F}} = 1 \end{cases} \tag{3.9}$$

Note that condition (3.8) implies condition (3.9), conversely, if $\mathsf{p}_{\mathcal{Q}} \prec \mathsf{p}_{\mathcal{E}} + \delta C$ then $\mu(\mathcal{Q}) \leq \mu(\mathcal{E}) + \overline{\delta} C$.

Let $(\mathcal{E}, \varphi)$ be a flat family of decorated sheaves over the fibre of a projective morphism $f : X \to S$. Let $\mathsf{P} = \mathsf{P}_{\mathcal{E}_s}$ and $\mathsf{p} = \mathsf{p}_{\mathcal{E}_s}$ the Hilbert polynomial and, respectively, the reduced Hilbert polynomial of $\mathcal{E}$ (which are constant because the family is flat over S). Define:

1. $\mathfrak{F}$ as the family (over the fibre of f) of saturated subsheaves $\mathcal{F} \hookrightarrow \mathcal{E}_s$ such that the induced torsion free quotient $\mathcal{E}_s \twoheadrightarrow \mathcal{Q}$ satisfy $\mu(\mathcal{Q}) \leq \mu(\mathcal{E}_s) + a\overline{\delta} C_0$;

2. $\mathfrak{F}_{\mathsf{o}}$ as the family of decorated subsheaves $(\mathcal{F}, \varphi_{|\mathcal{F}}) \hookrightarrow (\mathcal{E}_s, \varphi_{|\mathcal{E}_s})$ such that:

- $\varepsilon(\mathcal{F}, \varphi_{|\mathcal{F}}) = 0$;
- $\mathsf{p}^{\varepsilon}_{\mathcal{F}} = \mathsf{p}_{\mathcal{F}} \succ \mathsf{p}^{\varepsilon}_{\mathcal{E}}$ (i.e., $\mathsf{p}_{\mathcal{Q}} \prec \mathsf{p} + a\delta C_0$ with $\mathcal{Q} \doteqdot \mathrm{coker}(\mathcal{F} \hookrightarrow \mathcal{E}_s)$);
- $\mathcal{F}$ is a saturated subsheaf of $\mathcal{E}_s$;

3. $\mathfrak{F}_1$ as the family of decorated subsheaves $(\mathcal{F}, \varphi_{|\mathcal{F}}) \hookrightarrow (\mathcal{E}_s, \varphi_{|\mathcal{E}_s})$ such that:

   - $\varepsilon(\mathcal{F}, \varphi_{|\mathcal{F}}) = 1$;
   - $\mathsf{p}^{\varepsilon}_{\mathcal{F}} \succ \mathsf{p}^{\varepsilon}_{\mathcal{E}}$ (i.e., $\mathsf{p}_{\mathcal{Q}} \prec \mathsf{p} - a\delta C_1$ with $\mathcal{Q} \doteqdot \mathrm{coker}(\mathcal{F} \hookrightarrow \mathcal{E}_s)$);
   - $\mathcal{F}$ is a saturated subsheaf of $\mathcal{E}_s$;

We want to prove that the set of Hilbert polynomials of destabilizing decorated subsheaves of a flat family $(\mathcal{E}, \varphi)$ of decorated sheaves over the fibre of a projective morphism $f : X \to S$ is a finite set. From this we conclude that the semistability condition is an open condition, i.e., the set $\{s \in S \,|\, (\mathcal{E}_s, \varphi_s) \text{ is slope } \varepsilon\text{-semistable}\}$ is open in S. In order to prove this result we first need to recall some facts.

Definition 34. A family of isomorphism classes of coherent sheaves on a projective scheme Y over k is **bounded** if there is a k-scheme S of finite type and a coherent $\mathcal{O}_{S\times Y}$-sheaf $\mathcal{G}$ such that the given family is contained in the set $\{\mathcal{G}_{|\mathrm{Spec}(k(s))\times Y} \,|\, s \text{ is a closed point in } S\}$.

Definition 35. A sheaf $\mathcal{A}$ over Y is said m**-regular** if

$$H^i(Y, \mathcal{A}(m-i)) = 0 \text{ for all } i > 0.$$

Define the **Mumford-Castelnuovo regularity** of $\mathcal{A}$ as

$$\mathrm{reg}(\mathcal{A}) \doteqdot \inf\{m \in \mathbb{Z} \,|\, \mathcal{A} \text{ is } m\text{-regular}\}$$

Then the following statements hold:

Lemma 36 (Lemma 1.7.2 [15])**.** *If $\mathcal{A}$ is m-regular, then*

i) $\mathcal{A}$ is m'-regular for all integers $m' \geq m$.

ii) $\mathcal{A}(m) = \mathcal{A} \otimes \mathcal{O}_X(m)$ is globally generated.

iii) For all $n \geq 0$ the natural homomorphisms

$$H^0(X, \mathcal{A}(m)) \otimes H^0(X, \mathcal{O}_X(n)) \longrightarrow H^0(X, \mathcal{A}(m+n))$$

are surjective.

Lemma 37 (Lemma 1.7.6 of [15])**.** *The following properties of families of sheaves $\{\mathcal{A}_i\}_{i\in I}$ are equivalent:*

i) the family is bounded;

ii) the set of Hilbert polynomials $\{\mathsf{P}_{\mathcal{A}_i}\}_{i\in I}$ is finite and there is a uniform bound for reg$(\mathcal{A}_i) \leq C$ for all $i \in I$

iii) the set of Hilbert polynomials $\{\mathsf{P}_{\mathcal{A}_i}\}_{i\in I}$ is finite and there is a coherent sheaf $\mathcal{A}$ such that all $\mathcal{A}_i$ admit surjective morphisms $\mathcal{A} \to \mathcal{A}_i$.

Definition 38. Let $\mathcal{A}$ be a coherent sheaf. We call **hat-slope** the rational number

$$\hat{\mu}(\mathcal{A}) \doteqdot \frac{\beta_{\dim\mathcal{A}-1}(\mathcal{A})}{\beta_{\dim\mathcal{A}}(\mathcal{A})},$$

where $\beta_i(\mathcal{A})$ is defined as the coefficient of x^i of the Hilbert polynomial of $\mathcal{A}$ multiplied by $i!$, i.e., if $\mathsf{P}_{\mathcal{E}}(x) = \sum_{i=0}^{\dim\mathcal{A}} \beta_i \frac{x^i}{i!}$ then $\beta_i(\mathcal{A}) \doteqdot \beta_i$.

Lemma 39 (Lemma 2.5 in [11])**.** *Let $f : Y \to S$ be a projective morphism of Noetherian schemes and denote by $\mathcal{O}_Y(1)$ a line bundle on Y, which is very ample relative to S. Let $\mathcal{A}$ be a coherent sheaf on Y and $\mathfrak{Q}$ the set of isomorphism classes of quotients sheaves $\mathcal{Q}$ of $\mathcal{A}_s$ for s running over the points of S. Suppose that the dimension of Y_s is $\leq r$ for all s. Then the coefficient $\beta_r(\mathcal{Q})$ is bounded from above and below, and $\beta_{r-1}(\mathcal{Q})$ is bounded from below. If $\beta_{r-1}(\mathcal{Q})$ is bounded from above, then the family of sheaves $\mathcal{Q}/T(\mathcal{Q})$ is bounded.*

Proposition 40. *Let $\mathcal{A}$ be a flat family of coherent sheaves on the fibres of a projective morphism $f : Y \to S$ of Noetherian schemes. Then the family of torsion free quotient $\mathcal{Q}$ of $\mathcal{A}_s$ for $s \in S$ with hat slope bounded from above is a bounded family.*

Proof. It is an easy corollary of Lemma 2.5 in [11]. ♦

Thanks to Proposition 40 the family $\mathfrak{F}$ is bounded. Due the previous considerations both families $\mathfrak{F}_0$ and $\mathfrak{F}_1$ can be regarded as subfamilies of $\mathfrak{F}$ and therefore $\mathfrak{F}_0$ and $\mathfrak{F}_1$ are bounded families as well. Thanks to Proposition 37 the sets $\{\mathsf{P}_{\mathcal{F}} \,|\, \mathcal{F} \in \mathfrak{F}_0\}$ and $\{\mathsf{P}_{\mathcal{F}} \,|\, \mathcal{F} \in \mathfrak{F}_1\}$ are finite.

3.2.4 Quot schemes

Let $\mathcal{A}$ be a coherent sheaf over X flat over the fibres of $f : X \to S$. Let $P \in \mathbb{Q}[x]$ be a polynomial. Define a functor

$$\underline{\mathsf{Q}} \doteqdot \underline{\mathsf{Q}_{\text{uot}}}_{X/S}(\mathcal{A}, P) : (\mathrm{Sch}/S) \to (\mathrm{Sets})$$

as follows: if $T \to S$ is scheme over S let $\underline{\mathsf{Q}}(T)$ be the set of all T-flat coherent quotient sheaves $\mathcal{A}_T \twoheadrightarrow \mathcal{Q}$ with Hilbert polynomial P, where $\mathcal{A}_T$ denotes the sheaf over $X_T = X \times_S T$ induced by $\mathcal{A}$. If $g : T' \to T$ is an S-morphism, let $\underline{\mathsf{Q}}(g) : \underline{\mathsf{Q}}(T) \to \underline{\mathsf{Q}}(T')$ be the map that sends $\mathcal{A}_T \twoheadrightarrow \mathcal{Q}$ to $\mathcal{A}_{T'} \twoheadrightarrow g_X^* \mathcal{Q}$, where $g_X : X_{T'} \to X_T$ is the map induced by g.

Theorem 41 (Theorem 2.2.4 in [15]). *The functor* $\underline{\mathsf{Quot}}_{X/S}(\mathcal{A}, P)$ *is represented by a projective* S*-scheme* $\pi : \mathsf{Quot}_{X/S}(\mathcal{A}, P) \to S$.

Consider now a decorated coherent sheaf $(\mathcal{A}, \varphi)$ over X, flat over the fibre of $f : X \to S$ and let $P \in \mathbb{Q}[x]$ be a polynomial. Define the functor

$$\underline{\mathsf{Q}}^{\mathrm{o}} \doteqdot \underline{\mathsf{Quot}}^{\mathrm{o}}_{X/S}(\mathcal{A}, \varphi, P) : (\mathrm{Sch}/S) \to (\mathrm{Sets})$$

as follows: if $T \to S$ is scheme over S let $\underline{\mathsf{Q}}^{\mathrm{o}}(T)$ be the set of all T-flat coherent quotient sheaves $\mathcal{A}_T \twoheadrightarrow \mathcal{Q}$ with Hilbert polynomial P such that $\varepsilon(\varphi_T|_{\ker(\mathcal{A}_T \twoheadrightarrow \mathcal{Q})}) = 0$, where $\mathcal{A}_T$ denotes the sheaf over $X_T = X \times_S T$ induced by $\mathcal{A}$ and $\varphi_T : (\mathcal{A}_T)_{a,b} \to \mathcal{A}_{a,b} \xrightarrow{\varphi} \mathsf{N}$ is the morphism induced by φ. If $g : T' \to T$ is an S morphism, let $\underline{\mathsf{Q}}^{\mathrm{o}}(g) : \underline{\mathsf{Q}}^{\mathrm{o}}(T) \to \underline{\mathsf{Q}}^{\mathrm{o}}(T')$ be the map that sends $\mathcal{A}_T \twoheadrightarrow \mathcal{Q}$ to $\mathcal{A}_{T'} \twoheadrightarrow g_X^* \mathcal{Q}$, note that $g_X^* \varphi_T$ is zero if restricted on $\ker(\mathcal{A}_{T'} \twoheadrightarrow g_X^* \mathcal{Q})$.

Theorem 42. *The functor* $\underline{\mathsf{Quot}}^{\mathrm{o}}_{X/S}(\mathcal{A}, \varphi, P)$ *is represented by a projective* S*-scheme* $\pi^{\mathrm{o}} : \mathsf{Quot}^{\mathrm{o}}_{X/S}(\mathcal{A}, \varphi, P) \to S$ *that is a closed subscheme of* $\mathsf{Quot}_{X/S}(\mathcal{A}, P)$.

Proof. The additional property is closed and therefore, using the same arguments of the proof of Theorem 1.6 in [36], one can prove that $\mathsf{Quot}^{\mathrm{o}}_{X/S}(\mathcal{A}, \varphi, P) = \{q \in \mathsf{Quot}_{X/S}(\mathcal{A}, P) \mid \varepsilon(\varphi|_{\ker(q)}) = 0\}$ is a closed projective subscheme of $\mathsf{Quot}_{X/S}(\mathcal{A}, P)$. ♦

3.2.5 Openness of semistability condition

Proposition 43. *Let* $f : X \to S$ *be a projective morphism of Noetherian schemes and let* $(\mathcal{E}, \varphi)$ *be a flat family of decorated sheaves over the fibre of* f*. The set of points* $s \in S$ *such that* $(\mathcal{E}_s, \varphi_s)$ *is* ε*-(semi)stable with respect to* δ *is open in* S.

Proof. Let $\mathsf{P} = \mathsf{P}_{\mathcal{E}_s}$ and $\mathsf{p} = \mathsf{p}_{\mathcal{E}_s}$ the Hilbert polynomial and, respectively, the reduced Hilbert polynomial of $\mathcal{E}$. We first consider the semistable case. Let

$$A \doteqdot \{P'' \in \mathbb{Q}[x] \mid \exists\, s \in S, \exists\, q : \mathcal{E}_s \twoheadrightarrow \mathcal{Q} \text{ such that } \mathsf{P}_{\mathcal{Q}} = P'' \text{ and } \ker(q) \in \mathfrak{F}\} \tag{3.10}$$

and, for $i = 0, 1$, let

$$A_i \doteqdot \{P'' \in \mathbb{Q}[x] \mid \exists\, s \in S, \exists\, q : \mathcal{E}_s \twoheadrightarrow \mathcal{Q} \text{ such that } \mathsf{P}_{\mathcal{Q}} = P'' \text{ and } \ker(q) \in \mathfrak{F}_{\mathrm{i}}\}$$

The sets A, A_0 and A_1 are finite because the families $\mathfrak{F}$, $\mathfrak{F}_{\mathrm{o}}$ and $\mathfrak{F}_1$ are bounded as proved in Section 3.2.3. For any $P'' \in A_1$ consider the Quot scheme $\pi : \underline{\mathsf{Quot}}_{X/S}(\mathcal{E}, P'') \to S$, while for $P'' \in A_0$ consider the Quot scheme $\pi^{o} : \underline{\mathsf{Quot}}^{o}_{X/S}(\mathcal{E},\varphi,P'') \to S$. Both images $S(P'')$ of π (for $P'' \in A_1$) and $S^{o}(P'')$ of π^{o} (for $P'' \in A_0$) are closed sets of S. Therefore the union

$$\left(\bigcup_{P''\in A_0} S^{o}(P'')\right) \cup \left(\bigcup_{P''\in A_1} S(P'')\right)$$

is a closed subset of S, in fact it is finite union of closed sets. Finally is easy to see that $(\mathcal{E}_s,\varphi_s)$ is semistable if and only if s is *not* in the above union. The proof of the stable case is similar to the semistable case, it is indeed sufficient to consider, for $i=0,1$, the sets

$$A_i^{\mathrm{st}} \doteqdot \{P'' \in A \,|\ \text{with } \mathsf{p}_{\mathcal{Q}} \preceq \mathsf{p} + (1-i)(-a\delta C_0) + i(a\delta C_1)\}$$

and continue as in the semistable case. ♦

3.2.6 Relative maximal destabilizing subsheaf

Theorem 44. *Let $(X,\mathcal{O}_X(1))$, S, $f : X \to S$ and $(\mathcal{E},\varphi)$ as before. Then there is an integral k-scheme T of finite type, a projective birational morphism $g : T \to S$, a dense open subset $U \subset T$ and a flat quotient $\mathcal{Q}$ of $\mathcal{E}_T$ such that for all points $t \in U$, $\mathcal{F}_t \doteqdot \ker(\mathcal{E}_t \twoheadrightarrow \mathcal{Q}_t)$ with the induced morphism $\varphi_{t|\mathcal{F}_t}$ is the maximal destabilizing subsheaf of $(\mathcal{E}_t,\varphi_t)$ or $\mathcal{Q}_t = \mathcal{E}_t$.*
Moreover the pair $(g,\mathcal{Q})$ is universal in the sense that if $g' : T' \to S$ is any dominant morphism of k-integral schemes and $\mathcal{Q}'$ is a flat quotient of $\mathcal{E}_{T'}$, satisfying the same property of $\mathcal{Q}$, there is an S-morphism $h : T' \to T$ such that $h_X^*(\mathcal{Q}) = \mathcal{Q}'$.

Proof. In the proof we apply the same arguments as in [27]. Define $B_1 = A_1$ and $B_0 = A_0'$, i.e.,

$$B_0 = \{P'' \in A \,|\, \mathsf{p}_{\mathcal{Q}} \preceq \mathsf{p} - a\delta C_0\}$$
$$B_1 = \{P'' \in A \,|\, \mathsf{p}_{\mathcal{Q}} \prec \mathsf{p} + a\delta C_1\}$$

Then define

$$\check{B}_0 \doteqdot \{P'' \in B_0 \,|\, \pi^{o}(\mathsf{Quot}^{o}_{X/S}(\mathcal{E},\varphi,P'')) = S\}$$
$$\check{B}_1 \doteqdot \{P'' \in B_1 \,|\, \pi(\mathsf{Quot}_{X/S}(\mathcal{E},P'')) = S \text{ and } \forall s \in S\ \ \pi^{-1}(s) \not\subset \mathsf{Quot}^{o}_{X/S}(\mathcal{E},\varphi,P''))\}$$

Note that $B_0 \cup B_1$ and $\check{B}_0 \cup \check{B}_1$ are nonempty. We want to define an order relation on $B_0, \check{B}_0, B_1$ and $\check{B}_1$ but first we need the following costruction: let P_1'', P_2'' be polynomials in $B_0, \check{B}_0, B_1$ or $\check{B}_1$; then there exist surjective morphisms $q_i : \mathcal{E}_s \to \mathcal{Q}_i$ $(i=1,2)$ such that $P_i'' = \mathsf{P}_{\mathcal{Q}_i}$. Define, for $i=1,2$, $P_i \doteqdot \mathsf{P}_{\ker(q_i)}$, $r_i \doteqdot \mathrm{rk}(\ker(q_i))$ and $p_i = P_i/r_i$. We will say that the polynomials P_i are *associated* with the polynomials P_i''.

If $P_i'' \in B_0$ or $\check{B}_0$ define the following ordering relation:

$$P_1'' \lhd P_2'' \Longleftrightarrow p_1 \succ p_2 \quad \text{or} \quad p_1 = p_2 \text{ and } r_1 > r_2,$$

otherwise, if $P_i'' \in B_1$ or $\check{B}_1$, define:

$$P_1'' \lhd P_2'' \Longleftrightarrow p_1 - \frac{a\delta}{r_1} \succ p_2 - \frac{a\delta}{r_2} \quad \text{or} \quad p_1 - \frac{a\delta}{r_1} = p_2 - \frac{a\delta}{r_2} \text{ and } r_1 > r_2$$

Let $P_-''^i$, for $i = 0, 1$, be a $\lhd$-minimal polynomial among all polynomials in $\check{B}_i$ and P_-^i the associated polynomials. Then consider the following cases:

Case 1: $p_-^0 \succ p_-^1 - \frac{a\delta}{r_-^1}$;

Case 2: $p_-^0 \prec p_-^1 - \frac{a\delta}{r_-^1}$;

Case 3: $p_-^0 = p_-^1 - \frac{a\delta}{r_-^1}$ and $r_-^0 > r_-^1$;

Case 4: $p_-^0 = p_-^1 - \frac{a\delta}{r_-^1}$ and $r_-^0 < r_-^1$;

In the first and third case define $P_-'' = P_-''^0$, in the second and fourth case put $P_-'' = P_-''^1$. Note that the set

$$U_-' \doteqdot \left(\bigcup_{P'' \in B_0, P'' \lhd P_-''^0} \pi^{\circ}(\mathsf{Q}\text{uot}^{\circ}_{X/S}(\mathcal{E}, \varphi, P'')) \right) \cup \left(\bigcup_{P'' \in B_1, P'' \lhd P_-''^1} \pi(\mathsf{Q}\text{uot}_{X/S}(\mathcal{E}, P'')) \right)$$

is a proper closed subscheme of S. In fact it is proper and closed because it is a finite union of closed proper subschemes of S. Call U_- its complement in S.

Suppose that $P_-'' \in \check{B}_0$. By definition the projective morphism

$$\pi^{\circ}(\mathsf{Q}\text{uot}^{\circ}_{X/S}(\mathcal{E}, \varphi, P_-'')) \to S$$

is surjective and for any point $s \in S$ the fibre of π° at s parametrizes possible quotients with Hilbert polynomial P_-''. The associated subsheaf of any such quotient is, by costruction, the maximal decorated destabilizing subsheaf. The case that $P_-'' \in \check{B}_1$ is similar. Finally by re-adapting the techniques used in the proof of the corresponding result in [27], one concludes. ♦

3.2.7 Restriction theorem

Let X be a smooth projective variety and $\mathcal{O}_X(1)$ be a fixed ample line bundle. Let $(\mathcal{E}, \varphi)$ be a decorated sheaf of type (a, b, N) over X with non-zero decoration morphism. For a fixed positive integer $\mathsf{a} \in \mathbb{N}^+$, we define:

- $\Pi_{\mathsf{a}} \doteqdot |\mathcal{O}_X(\mathsf{a})|$ the complete linear system of degree a in X;

- $Z_{\mathsf{a}} \doteqdot \{(D,x) \in \Pi_{\mathsf{a}} \times X \mid x \in D\}$ the incidence variety with projections

$$\begin{array}{ccccc} \Pi_{\mathsf{a}} \times X & \hookleftarrow & Z_{\mathsf{a}} & \xrightarrow{q_{\mathsf{a}}} & X \\ & & \downarrow{\scriptstyle p_{\mathsf{a}}} & & \\ & & \Pi_{\mathsf{a}} & & \end{array}$$

One can prove (see Section 2 of [23]) that:

$$\mathrm{Pic}(Z_{\mathsf{a}}) = q_{\mathsf{a}}^{*}\mathrm{Pic}(X) \oplus p_{\mathsf{a}}^{*}\mathrm{Pic}(\Pi_{\mathsf{a}}). \tag{3.11}$$

For any sheaf $\mathcal{G}$ over X one has $P_{\mathcal{G}_{|D}}(n) = P_{\mathcal{G}}(n) - P_{\mathcal{G}}(n-\mathsf{a})$, therefore, given a decorated sheaf $(\mathcal{E},\varphi)$ over X with decoration of type $\underline{\mathfrak{t}} = (a,b,\mathsf{N})$, for all $D \in \Pi_{\mathsf{a}}$ the restrictions $\mathcal{E}_{|D}$ and $\mathsf{N}_{|D}$ have constant Hilbert polynomials. Since Π_{a} is reduced, as remarked at the beginning of Section 3.2.2, it follows that $q_{\mathsf{a}}^{*}\mathsf{N}$ and $q_{\mathsf{a}}^{*}\mathcal{E}$ are flat families of sheaves on the fibre of $p_{\mathsf{a}} : Z_{\mathsf{a}} \to \Pi_{\mathsf{a}}$.

Remark 45. If for any $D \in \Pi_{\mathsf{a}}$ $\varphi_{\mathsf{a}|_{(q_{\mathsf{a}}^{*}\mathcal{E})_{|_{p_{\mathsf{a}}^{-1}(D)}}}} = \varphi_{|_{\left(\mathcal{E}_{|D}\right)}} \neq 0$, the family of decorated sheaves $(q_{\mathsf{a}}^{*}\mathcal{E}, q_{\mathsf{a}}^{*}\varphi)$ is flat. Otherwise, since to be nonzero is open condition, there exists a dense open subset of Π_{a} over which $(q_{\mathsf{a}}^{*}\mathcal{E}, q_{\mathsf{a}}^{*}\varphi)$ is flat.

Thanks to this remark and Theorem 44, there exist a dense open subset V_{a} of Π_{a} and a torsion-free sheaf $\mathcal{Q}_{\mathsf{a}}$ over $Z_{V_{\mathsf{a}}} \doteqdot Z_{\mathsf{a}} \times_{\Pi_{\mathsf{a}}} V_{\mathsf{a}}$ such that:

- $(\mathcal{E}_{\mathsf{a}}, \varphi_{\mathsf{a}}) \doteqdot (q_{\mathsf{a}}^{*}\mathcal{E}, q_{\mathsf{a}}^{*}\varphi)$ is flat over V_{a};
- $\mathcal{Q}_{\mathsf{a}}$ is flat over V_{a};
- $\mathcal{F}_{\mathsf{a}} \doteqdot \ker(\mathcal{E}_{\mathsf{a}} \to \mathcal{Q}_{\mathsf{a}})$, with the induced morphism $\varphi_{\mathsf{a}|_{\mathcal{F}_{\mathsf{a}}}}$, is the relative maximal decorated destabilizing subsheaf of $(\mathcal{E}_{\mathsf{a}}, \varphi_{\mathsf{a}})$; i.e., for any $D \in V_{\mathsf{a}}$ $\mathcal{F}_{\mathsf{a}|_{p_{\mathsf{a}}^{-1}(D)}}$ (with the induced morphism) de-semistabilize $(\mathcal{E}_{\mathsf{a}}, \varphi_{\mathsf{a}})_{|_{p_{\mathsf{a}}^{-1}(D)}}$.

Recall that:

- by construction of the relative maximal decorated destabilizing subsheaf, the quantity

$$\varepsilon\left(\mathcal{F}_{\mathsf{a}|_{p_{\mathsf{a}}^{-1}(D)}}, \varphi_{\mathsf{a}|_{\left(\mathcal{F}_{\mathsf{a}|_{p_{\mathsf{a}}^{-1}(D)}}\right)}}\right)$$

  depends only on a and not on $D \in V_{\mathsf{a}}$ and for this reason from now on we will denote by $\varepsilon(\mathsf{a})$;

- $(\mathcal{E}_{\mathsf{a}}, \varphi_{\mathsf{a}})$, $(\mathcal{F}_{\mathsf{a}}, \varphi_{\mathsf{a}|_{\mathcal{F}_{\mathsf{a}}}})$ and $\mathcal{Q}_{\mathsf{a}}$ are flat families of decorated sheaves (resp. sheaves) over V_{a}.

Let $\mathcal{G}_{\mathsf{a}}$ be a line bundle which extends $\det(\mathcal{Q}_{\mathsf{a}})$ to all Z_{a}; in view of (3.11) the line bundle $\mathcal{G}_{\mathsf{a}}$ can be uniquely decomposed as $\mathcal{G}_{\mathsf{a}} = q_{\mathsf{a}}^* L_{\mathsf{a}} \otimes p_{\mathsf{a}}^* M_{\mathsf{a}} = L_{\mathsf{a}} \boxtimes M_{\mathsf{a}}$ with $L_{\mathsf{a}} \in \mathrm{Pic}(X)$ and $M_{\mathsf{a}} \in \mathrm{Pic}(\Pi_{\mathsf{a}})$. Note that $\deg(\mathcal{Q}_{\mathsf{a}|_{p_{\mathsf{a}}^{-1}(D)}}) = \mathsf{a}\deg(L_{\mathsf{a}})$.

For a general divisor $D \in \Pi_{\mathsf{a}} = |\mathcal{O}_X(\mathsf{a})|$, let $\deg(\mathsf{a})$, $\mathrm{rk}(\mathsf{a})$ and $\mu(\mathsf{a})$ denote the degree, rank and slope of the maximal decorated destabilizing subsheaf $(\mathcal{F}_{\mathsf{a}}, \varphi_{\mathsf{a}|_{\mathcal{F}_{\mathsf{a}}}})_{|_{p_{\mathsf{a}}^{-1}D}}$ of $(\mathcal{E}_{\mathsf{a}|_{p_{\mathsf{a}}^{-1}D}}, \varphi_{\mathsf{a}|_{p_{\mathsf{a}}^{-1}D}})$. Let $\mu_{\varepsilon}(\mathsf{a}) = \mu(\mathsf{a}) - \frac{a\overline{\delta}\varepsilon(\mathsf{a})}{\mathrm{rk}(\mathsf{a})}$, $\deg^q(\mathsf{a}) = \deg(\mathcal{E}_{\mathsf{a}|_{p_{\mathsf{a}}^{-1}D}}) - \deg(\mathsf{a})$, $\mathrm{rk}^q(\mathsf{a}) = \mathrm{rk}(\mathcal{E}_{\mathsf{a}|_{p_{\mathsf{a}}^{-1}D}}) - \mathrm{rk}(\mathsf{a})$ and $\varepsilon^q(\mathsf{a}) = \varepsilon_{\mathcal{E}_{\mathsf{a}|_{p_{\mathsf{a}}^{-1}D}}} - \varepsilon(\mathsf{a})$. Finally $\mu^q(\mathsf{a}) = \frac{\deg^q(\mathsf{a})}{\mathrm{rk}^q(\mathsf{a})}$, $\deg^q_{\varepsilon}(\mathsf{a}) = \deg(\mathsf{a}) - a\overline{\delta}\varepsilon^q(\mathsf{a})$ and $\mu^q_{\varepsilon}(\mathsf{a}) = \mu^q(\mathsf{a}) - a\delta\frac{\varepsilon^q(\mathsf{a})}{\mathrm{rk}^q(\mathsf{a})} = \frac{\deg^q_{\varepsilon}(\mathsf{a})}{\mathrm{rk}^q(\mathsf{a})}$.

Let $U_{\mathsf{a}} \subset V_{\mathsf{a}}$ denote the dense open set of points $D \in V_{\mathsf{a}}$ such that D is smooth.

Lemma 46 (Lemma 7.2.3 in [15])**.** *Let* $\mathsf{a}_1, \ldots, \mathsf{a}_l$ *be positive integers,* $\mathsf{a} = \sum_i \mathsf{a}_i$ *and* $D_i \in U_{\mathsf{a}_i}$ *divisors such that* $D = \sum_i D_{\mathsf{a}_i}$ *is a divisor with normal crossing. Then there is a smooth locally closed curve* $C \subset \Pi_{\mathsf{a}}$ *containing the point* D *such that* $C \smallsetminus \{D\} \subset U_{\mathsf{a}}$ *and* $Z_C \doteqdot C \times_{\Pi_{\mathsf{a}}} Z_{\mathsf{a}}$ *is smooth in codimension* 2*.*

Lemma 47. *Let* $\mathsf{a}_1, \ldots, \mathsf{a}_l$ *be positive integers and* $\mathsf{a} = \sum_i \mathsf{a}_i$*. Then*

- $\mu(\mathsf{a}) \leq \sum_i \mu(\mathsf{a}_i)$,
- $\mu^q(\mathsf{a}) \geq \sum_i \mu^q(\mathsf{a}_i)$,
- $\mu^q_{\varepsilon}(\mathsf{a}) \geq \sum_i \mu^q_{\varepsilon}(\mathsf{a}_i)$

and in case of equality $rk^q(\mathsf{a}) \leq \min_i rk^q(\mathsf{a}_i)$*, or equivalently* $rk(\mathsf{a}) \geq \max_i rk(\mathsf{a}_i)$*.*

Proof. Let $D_i \in U_{\mathsf{a}_i}$, for $i = 1, \ldots, l$, be divisors satisfying the requirements of Lemma 46, be $D \doteqdot \sum_i D_i$ and let C be a curve with the properties of Lemma 46. There exists over V_{a} a maximal decorated destabilizing subsheaf $\mathcal{F}_{\mathsf{a}}$ with the associated torsion free quotient $\mathcal{E}_{\mathsf{a}|_{Z_{V_{\mathsf{a}}}}} \to \mathcal{Q}_{\mathsf{a}}$. Recall that both sheaves are flat over V_{a}. Its restriction to $V_{\mathsf{a}} \cap C$ can uniquely be extended to a C flat quotient $\mathcal{E}_{\mathsf{a}|_{Z_C}} \to \mathcal{Q}_C$ and let $\mathcal{F}_C = \ker(\mathcal{E}_{\mathsf{a}|_{Z_C}} \to \mathcal{Q}_C)$, then also $\mathcal{F}_C$ extends $\mathcal{F}_{\mathsf{a}|_{V_{\mathsf{a}} \cap C}}$ to all C. Note that also $\mathcal{F}_C$ is flat over C and so $\mathsf{P}_{\mathcal{F}_{C|_D}} = \mathsf{P}_{\mathcal{F}_{C|_c}}$ for any $c \in C$. Therefore $\mu(\mathcal{F}_{C|_D}) = \mu(\mathsf{a})$, $\mathrm{rk}(\mathcal{F}_{C|_D}) = \mathrm{rk}(\mathsf{a})$ and $\varepsilon(\mathcal{F}_{C|_D}) = \varepsilon(\mathsf{a})$. Let

- $\overline{\mathcal{Q}}_D \doteqdot \mathcal{Q}_{C|_D}/T(\mathcal{Q}_{C|_D})$ and $\overline{\mathcal{F}}_D \doteqdot \ker(\mathcal{E}_{\mathsf{a}|_D} \to \overline{\mathcal{Q}}_D)$, i.e., they fit in the exact sequence
$$0 \longrightarrow \overline{\mathcal{F}}_D \longrightarrow \mathcal{E}_{\mathsf{a}|_D} \longrightarrow \frac{\mathcal{Q}_{C|_D}}{T(\mathcal{Q}_{C|_D})} \longrightarrow 0;$$
- $\mathcal{Q}_i \doteqdot \overline{\mathcal{Q}}_{D|_{D_i}}/T(\overline{\mathcal{Q}}_{D|_{D_i}})$ and $\mathcal{F}_i \doteqdot \ker((\mathcal{E}_{\mathsf{a}|_D})_{|_{D_i}} \to \mathcal{Q}_i)$, i.e., they fit in the exact sequence
$$0 \longrightarrow \mathcal{F}_i \longrightarrow \mathcal{E}_{\mathsf{a}|_{D_i}} \longrightarrow \frac{\overline{\mathcal{Q}}_{D|_{D_i}}}{T(\overline{\mathcal{Q}}_{D|_{D_i}})} \longrightarrow 0;$$

Then one gets

- $\operatorname{rk}(\mathsf{a}) = \operatorname{rk}(\mathcal{F}_{C|_D}) = \operatorname{rk}(\overline{\mathcal{F}}_D) = \operatorname{rk}(\overline{\mathcal{F}}_{D|_{D_i}}) = \operatorname{rk}(\mathcal{F}_i)$ and $\operatorname{rk}^q(\mathsf{a}) = \operatorname{rk}(\mathcal{Q}_{C|_D}) = \operatorname{rk}(\overline{\mathcal{Q}}_D) = \operatorname{rk}(\overline{\mathcal{Q}}_{D|_{D_i}}) = \operatorname{rk}(\mathcal{Q}_i)$;
- $\mu^q(\mathsf{a}) = \mu(\mathcal{Q}_{C|_D}) \geq \mu(\overline{\mathcal{Q}}_D)$ and $\mu(\mathsf{a}) = \mu(\mathcal{F}_{C|_D}) \leq \mu(\overline{\mathcal{F}}_D)$;
- $\mu(\overline{\mathcal{Q}}_{D|_{D_i}}) \geq \mu(\mathcal{Q}_i)$ and $\mu(\overline{\mathcal{F}}_{D|_{D_i}}) \leq \mu(\mathcal{F}_i)$.

Since $\mathcal{E}_{|_D}$ and $\overline{\mathcal{Q}}_D$ are pure, and the sequences
$$0 \longrightarrow \overline{\mathcal{Q}}_D \longrightarrow \bigoplus_i (\overline{\mathcal{Q}}_D)_{|_{D_i}} \longrightarrow \bigoplus_{i<j} (\overline{\mathcal{Q}}_D)_{|_{D_i \cap D_j}} \longrightarrow 0$$
$$0 \longrightarrow \mathcal{E}_{|_D} \longrightarrow \bigoplus_i (\mathcal{E}_{|_D})_{|_{D_i}} \longrightarrow \bigoplus_{i<j} (\mathcal{E}_{|_D})_{|_{D_i \cap D_j}} \longrightarrow 0$$

are exact modulo sheaves of dimension $n-3$, following the same calculations of Lemma 7.2.5 in [15], one gets that
$$\mu(\overline{\mathcal{Q}}_D) = \sum_i \left(\mu((\overline{\mathcal{Q}}_D)_{|_{D_i}}) - \frac{1}{2} \sum_{j \neq i} \left(\frac{\operatorname{rk}((\overline{\mathcal{Q}}_D)_{|_{D_i \cap D_j}})}{\operatorname{rk}^q(\mathsf{a})} - 1 \right) \mathsf{a}_i \mathsf{a}_j \right)$$
$$\mu(\mathcal{E}_{|_D}) = \sum_i \left(\mu((\mathcal{E}_{|_D})_{|_{D_i}}) - \frac{1}{2} \sum_{j \neq i} \left(\frac{\operatorname{rk}((\mathcal{E}_{|_D})_{|_{D_i \cap D_j}})}{\operatorname{rk}(\mathcal{E}_{|_D})} - 1 \right) \mathsf{a}_i \mathsf{a}_j \right).$$

and
$$\mu(\mathcal{Q}_i) \leq \mu((\overline{\mathcal{Q}}_D)_{|_{D_i}}) - \frac{1}{2} \sum_{j \neq i} \left(\frac{\operatorname{rk}((\overline{\mathcal{Q}}_D)_{|_{D_i \cap D_j}})}{\operatorname{rk}(\overline{\mathcal{Q}}_D)} - 1 \right) \mathsf{a}_i \mathsf{a}_j$$

Therefore $\mu(\overline{\mathcal{Q}}_D) \geq \sum_i \mu(\mathcal{Q}_i)$, $\deg(\mathcal{E}_{|_D}) \leq \sum_i \deg((\mathcal{E}_{|_D})_{|_{D_i}})$ and so easy calculations show that $\mu(\overline{\mathcal{F}}_D) \leq \sum_i \mu(\mathcal{F}_i)$.

Note that, since $T_F \doteqdot \overline{\mathcal{F}}_D/(\mathcal{F}_C)_{|D}$ is pure torsion, $\varphi_{|T_F} = 0$ (see Remark 24) and so $\varepsilon(\varphi_{|(\mathcal{F}_C)_{|D}}) = \varepsilon(\varphi_{|\overline{\mathcal{F}}_D})$. For the same reason $\varepsilon(\varphi_{|(\overline{\mathcal{F}}_D)_{|D_i}}) = \varepsilon(\varphi_{|\mathcal{F}_i})$. Moreover, if $\varepsilon(\mathsf{a}) = \varepsilon(\mathcal{F}_{C|D}) = 0$ then obviously also $\varepsilon_{\mathcal{F}_i} = 0$ for all i; conversely if $\varepsilon(\mathsf{a}) = 1$ then there exists at least one i such that $\varepsilon_{\mathcal{F}_i} = 1$. Therefore $\sum_i \varepsilon_{\mathcal{F}_i} \geq \varepsilon(\mathsf{a}) \geq \varepsilon_{\mathcal{F}_i}$.

Therefore, defining $\varepsilon_{\mathcal{Q}_i} = (1-\varepsilon_{\mathcal{F}_i})$ and $\varepsilon_{\overline{\mathcal{Q}}_D} = (1-\varepsilon_{\overline{\mathcal{F}}_D})$ as in Remark 33, thanks to the previous inequalities and considerations, one gets $\sum_i \varepsilon_{\mathcal{Q}_i} \geq \varepsilon_{\overline{\mathcal{Q}}_D} \geq \varepsilon_{\mathcal{Q}_i}$ and so

$$\mu^q_\varepsilon(\mathsf{a}) \geq \mu_\varepsilon(\overline{\mathcal{Q}}_D) \geq \sum_i \mu_\varepsilon(\mathcal{Q}_i) \geq \sum_i \mu^q_\varepsilon(\mathsf{a}_i).$$

If $\mu^q_\varepsilon(\mathsf{a}) = \sum_i \mu^q_\varepsilon(\mathsf{a}_i)$ it follows that $\mu^q_\varepsilon(\mathcal{Q}_i) = \mu^q_\varepsilon(\mathsf{a}_i)$. Since $\mu^q_\varepsilon(\mathsf{a})$ is the decorated slope of the minimal destabilizing quotient (i.e., its kernel is the maximal decorated destabilizing subsheaf), we have $\mathrm{rk}^q(\mathsf{a}) = \mathrm{rk}(\mathcal{Q}_i) \geq \mathrm{rk}^q(\mathsf{a}_i)$ for all i. ♦

Corollary 48. *$rk^q(\mathsf{a})$, $\frac{\varepsilon^q(\mathsf{a})}{\mathsf{a}}$, $\frac{\mu^q(\mathsf{a})}{\mathsf{a}}$, $\frac{\mu^q_\varepsilon(\mathsf{a})}{\mathsf{a}}$, $\frac{\mu(\mathsf{a})}{\mathsf{a}}$, $\mu_\varepsilon(\mathsf{a})$, $rk(\mathsf{a})$ and $\varepsilon(\mathsf{a})$ are constant for $\mathsf{a} \gg 0$.*

Proof. The quantities $\mathrm{rk}^q(\mathsf{a})$ and $\frac{\mu^q(\mathsf{a})}{\mathsf{a}}$ are constant as proved in [15] Corollary 7.2.6. The same arguments show that $\frac{\mu^q_\varepsilon(\mathsf{a})}{\mathsf{a}}$ is constant as well. Therefore $\frac{\varepsilon^q(\mathsf{a})}{\mathsf{a}}$ has to be constant too and easy calculations show that also $\varepsilon(\mathsf{a})$, $\frac{\mu(\mathsf{a})}{\mathsf{a}}$ and $\mu_\varepsilon(\mathsf{a})$ are constant. ♦

Corollary 49. *For $\mathsf{a} >> 0$ or $\varepsilon^q(\mathsf{a}) = 0$ and $\varepsilon(\mathsf{a}) = 1$ or $\varepsilon(\mathcal{E}_\mathsf{a}, \varphi_\mathsf{a}) = 0$.*

Proof. Since $\frac{\varepsilon^q(\mathsf{a})}{\mathsf{a}}$ is definitively constant, $\varepsilon^q(\mathsf{a}) = 0$ for $\mathsf{a} >> 0$. Since $\varepsilon^q(\mathsf{a}) = \varepsilon(\mathcal{E}_\mathsf{a}, \varphi_\mathsf{a}) - \varepsilon(\mathsf{a})$ or they are (definitively) both zero or both one. ♦

Lemma 50 (Lemma 7.2.7 [15])**.** *There exist $\mathsf{a}_0 \in \mathbb{N}$ and a line bundle $L \in Pic(X)$ such that $L_\mathsf{a} \simeq L$ for any $\mathsf{a} > \mathsf{a}_0$.*

In this way we have proved that for $\mathsf{a} >> 0$ an extension of $\det(\mathcal{Q}_\mathsf{a})$ is of the form $L \boxtimes M_\mathsf{a}$ with $L \in \mathrm{Pic}(X)$ and $\deg(\mathcal{Q}_{\mathsf{a}|D}) = \mathsf{a}\deg(L)$ for any $D \in V_\mathsf{a}$. Now we can state and prove the main theorem of this section:

Theorem 51. *Let X be a smooth projective surface and $\mathcal{O}_X(1)$ be a very ample line bundle. Let $(\mathcal{E}, \varphi)$ be a slope ε-semistable decorated sheaf. Then there is an integer a_0 such that for all $\mathsf{a} \geq \mathsf{a}_0$ there is a dense open subset $U_\mathsf{a} \subset |\mathcal{O}_X(\mathsf{a})|$ such that for all $D \in U_\mathsf{a}$ the divisor D is smooth and $(\mathcal{E}, \varphi)_{|D}$ is slope ε-semistable.*

Proof. We proof the theorem by reduction to absurd. Suppose the theorem is false: thanks to the previous constructions there exists a line bundle L_{a} such that

$$\frac{\deg(L_{\mathsf{a}}) - a\overline{\delta}\varepsilon^q(\mathsf{a})}{\mathrm{rk}^q(\mathsf{a})} < \mu_\varepsilon(\mathcal{E})$$

and $1 \leq \mathrm{rk}^q(\mathsf{a}) \leq \mathrm{rk}(\mathcal{E})$. We recall that $\mathrm{rk}^q(\mathsf{a})$ and L_{a} are constant for a greater than a certain constant a_0, so from now on we suppose that a is so and we call $L_{\mathsf{a}} = L$ and $\mathrm{rk}^q(\mathsf{a}) = r^q$. We want to construct a rank r^q quotient $\mathcal{Q}$ of $\mathcal{E}$ such that $\det(\mathcal{Q}) = L$.
Let a be a sufficiently large integer, $D \in U_{\mathsf{a}}$ and let $(\mathcal{F}_D, \varphi_{|_{\mathcal{F}_D}})$ be the maximal decorated destabilizing subsheaf of $(\mathcal{E}, \varphi)_{|_D}$ and $\mathcal{Q}_D \doteqdot \mathrm{co}\ker(\mathcal{F}_D \hookrightarrow \mathcal{E}_{|_D})$ the associated minimal decorated destabilizing quotient. Put $L_D \doteqdot \det Q_D$ and note that $L_D = L_{|_D}$ (by uniqueness of the maximal destabilizing subsheaf and so of the minimal destabilizing quotient). The surjective morphism $\mathcal{E}_{|_D} \to Q_D$ induces a surjective homomorphism $\sigma_D \colon \Lambda^{r^q}\mathcal{E}_{|_D} \to L_D$ and morphisms

$$i_{|_D} \colon D \longrightarrow \mathrm{Grass}(\mathcal{E}_{|_D}, r^q) \longrightarrow \mathbb{P}(\Lambda^{r^q}\mathcal{E}_{|_D}).$$

Consider the exact sequence

$$\mathrm{Hom}(\Lambda^{r^q}\mathcal{E}, L(-\mathsf{a})) \to \mathrm{Hom}(\Lambda^{r^q}\mathcal{E}, L) \xrightarrow{f} \mathrm{Hom}(\Lambda^{r^q}\mathcal{E}_{|_D}, L_{|_D}) \to \mathrm{Ext}^1(\Lambda^{r^q}\mathcal{E}, L(-\mathsf{a})).$$

By Serre's theorem and Serre duality one has that for $i = 0, 1$ and $\mathsf{a} \gg 0$

$$\mathrm{Ext}^i(\Lambda^{r^q}\mathcal{E}, L(-\mathsf{a})) = H^{n-i}(X, \Lambda^{r^q}\mathcal{E} \otimes L^\vee \otimes \omega_X(\mathsf{a})) = 0.$$

Hence if a is big enough f is bijective and σ_D extends uniquely to a homomorphism $\sigma \in \mathrm{Hom}(\Lambda^{r^q}\mathcal{E}, L)$. Using the same arguments of the final part of the proof of Theorem 7.2.1 in [15], σ induces a morphism $i \colon X \to \mathbb{P}(\Lambda^{r^q}\mathcal{E})$ that factorize thorough $\mathrm{Grass}(\mathcal{E}, r^q)$ and so we obtain a quotient $q \colon \mathcal{E} \to \mathcal{Q}$. Since $\det \mathcal{Q}_{|_D} \equiv L_D = L_{|_D}$ for all $D \in U_{\mathsf{a}}$, by Lemma 7.2.2 [15], we get $L = \det \mathcal{Q}$. Define $\mathcal{F} \doteqdot \ker(\mathcal{E} \to \mathcal{Q})$ and note that $\mathcal{F}_{|_D} = \mathcal{F}_D$. Finally, thanks to Fujita's vanishing theorem ([19] pg 66),

$$H^i(X, \mathcal{F}_{a,b} \otimes \mathsf{N}^\vee \otimes \omega_X(\mathsf{a})) = 0$$

for $i > 0$ and a big enough. Therefore

$$\mathrm{Ext}^j(\mathcal{F}_{a,b}, \mathsf{N}(-\mathsf{a})) = H^{n-j}(X, \mathcal{F}_{a,b} \otimes \mathsf{N}^\vee \otimes \omega_X(\mathsf{a})) = 0$$

for $j = 0, 1$. The same holds also for $\mathcal{E}$ and so the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{F}_{a,b}, \mathsf{N}) & \longleftrightarrow & \mathrm{Hom}(\mathcal{F}_{a,b_{|_D}}, \mathsf{N}_{|_D}) \\ \uparrow & & \uparrow \\ \mathrm{Hom}(\mathcal{E}_{a,b}, \mathsf{N}) & \longleftrightarrow & \mathrm{Hom}(\mathcal{E}_{a,b_{|_D}}, \mathsf{N}_{|_D}) \end{array}$$

which proves that we can extend to all $\mathcal{F}_{a,b}$ the morphism we have over $\mathcal{F}_{a,b|_D}$ in such a way that $\varepsilon(\varphi_{|\mathcal{F}}) = \varepsilon(\mathsf{a})$. By construction $(\mathcal{F}, \varphi_{|\mathcal{F}})$ destabilizes, with respect to the slope ε-semistability, the decorated sheaf $(\mathcal{E}, \varphi)$ and this contradicts the hypothesis. ♦

3.3 Mehta-Ramanathan theorem for slope k-semistable decorated sheaves of rank 2 and 3

Notation. Let X be a smooth projective variety, $\mathcal{O}_X(1)$ a fixed ample line bundle over X, k an algebraic closed field of characteristic 0, S an integral k-scheme of finite type and $f : X \to S$ a projective flat morphism. Note that $\mathcal{O}_X(1)$ is also f-ample.

Proposition 52 (**Properties of $\mathsf{k}_{\mathcal{F},\mathcal{E}}$**). *Let $(\mathcal{E}, \varphi)$ be a decorated sheaf of type (a, b, N) and rank r. Let $\mathcal{G}, \mathcal{F}$ be subsheaves of $\mathcal{E}$. Then the following statements hold:*

1. *There exist an open subset $U \subseteq X$ and complex vector spaces V' and V of dimension rk$(\mathcal{F})$ and r (respectively) such that $\mathcal{F}_{|_U} \simeq V' \otimes \mathcal{O}_U$, $\mathcal{E}_{|_U} \simeq V \otimes \mathcal{O}_U$ and $\mathsf{k}_{\mathcal{F},\mathcal{E}} = \mathsf{k}_{\mathcal{F}_{|_U},\mathcal{E}_{|_U}}$.*
2. *If there exists an open subset U of X such that $\mathcal{F}_{|_U}$ is isomorphic to $\mathcal{G}_{|_U}$, then $\mathsf{k}_{\mathcal{F},\mathcal{E}} = \mathsf{k}_{\mathcal{G},\mathcal{E}}$.*
3. *$\mathsf{k}_{\mathcal{F}+\mathcal{G},\mathcal{E}} \geq \max\{\mathsf{k}_{\mathcal{F},\mathcal{E}}\ ,\ \mathsf{k}_{\mathcal{G},\mathcal{E}}\}$.*
4. *$\mathsf{k}_{\mathcal{F}\cap\mathcal{G},\mathcal{E}} \leq \min\{\mathsf{k}_{\mathcal{F},\mathcal{E}}\ ,\ \mathsf{k}_{\mathcal{G},\mathcal{E}}\}$.*
5. *$\mathsf{k}_{\mathcal{F},\mathcal{E}} + \mathsf{k}_{\mathcal{G},\mathcal{E}} \geq \mathsf{k}_{\mathcal{F}+\mathcal{G},\mathcal{E}}$.*
6. *If $\mathsf{k}_{\mathcal{F}\cap\mathcal{G},\mathcal{E}} = \mathsf{k}_{\mathcal{F}\cap\mathcal{G},\mathcal{F}+\mathcal{G}}$ then*

$$\mathsf{k}_{\mathcal{F}+\mathcal{G},\mathcal{E}} + \mathsf{k}_{\mathcal{F}\cap\mathcal{G},\mathcal{E}} \leq \mathsf{k}_{\mathcal{F},\mathcal{E}} + \mathsf{k}_{\mathcal{G},\mathcal{E}} \tag{3.12}$$

   in particular

$$\mathsf{k}_{\mathcal{F}+\mathcal{G},\mathcal{F}+\mathcal{G}} + \mathsf{k}_{\mathcal{F}\cap\mathcal{G},\mathcal{F}+\mathcal{G}} \leq \mathsf{k}_{\mathcal{F},\mathcal{F}+\mathcal{G}} + \mathsf{k}_{\mathcal{G},\mathcal{F}+\mathcal{G}}.$$

Proof. 1. Let $U_{\mathcal{F}}$ be a maximal open subset where $\mathcal{F}$ is locally free and admits a trivialization. Suppose that $\mathsf{k}_{\mathcal{F},\mathcal{E}} = k$, then there exist an open subset $U' \subseteq X$, k local sections $f_1, \ldots, f_k \in H^0(U', \mathcal{F}_{|_{U'}})$ and $a - k$ local sections $e_1, \ldots, e_{a-k} \in H^0(U', \mathcal{E}_{|_{U'}})$ such that

$$\varphi((f_1, \ldots, f_t, e_1, \ldots, e_{a-t})^{\oplus b}) \neq 0.$$

Let $U \doteqdot U' \cap U_{\mathcal{F}}$, then $\mathcal{E}_{|_U} \simeq V \otimes \mathcal{O}_U$ and $\mathsf{k}_{\mathcal{F},\mathcal{E}} = \mathsf{k}_{\mathcal{F}_{|_U},\mathcal{E}_{|_U}}$.

2. The statement follows directly from (1).

3. Follows from the fact that $\mathcal{F}, \mathcal{G} \subseteq \mathcal{F} + \mathcal{G}$.

4. Since $\mathcal{F} \cap \mathcal{G} \subseteq \mathcal{F}, \mathcal{G}$, it is easy to see that the statement holds.

5. Let $t = \mathsf{k}_{\mathcal{F}+\mathcal{G},\mathcal{E}}$ and $k = \mathsf{k}_{\mathcal{F},\mathcal{E}}$. Thanks to the first point, we can suppose that $\mathcal{F} \cap \mathcal{G}, \mathcal{F}, \mathcal{G}$ and $\mathcal{F} + \mathcal{G}$ are trivial sheaves. Let $f_1 + g_1, \dots f_t + g_t$ and $e_1, \dots e_{a-t}$ be sections of $\mathcal{F} + \mathcal{G}$ and $\mathcal{E}$ respectively such that f_i are sections of $\mathcal{F}$, g_i are sections of $\mathcal{G}$ and $\varphi((f_1 + g_1, \dots f_t + g_t, e_1, \dots e_{a-t})^{\oplus b}) \neq 0$. Let $f = \sum f_i$, $g = \sum g_i$ and $e = \sum e_i$, then also $\varphi(((f+g)^{\otimes t} \otimes e^{\otimes a-t})^{\oplus b}) \neq 0$. But
$$(f+g)^{\otimes t} \otimes e^{\otimes a-t} = \left(\sum_{i=0}^{t} \binom{t}{i} f^{\otimes t-i} \otimes g^{\otimes i} \right) \otimes e^{\otimes a-t}.$$
Since $k \leq t$ there exists $i_0 \geq 0$ such that $t - i_0 = k$. Then for any $0 \leq i < i_0$ one has $\varphi((f^{\otimes t-i} \otimes g^{\otimes i} \otimes e^{\otimes a-t})^{\oplus b}) = 0$, since $\mathsf{k}_{\mathcal{F},\mathcal{E}} = k$ and $t - i > k$. Therefore
$$\varphi\left(\left(\sum_{i=i_0}^{t} \binom{t}{i} f^{\otimes t-i} \otimes g^{\otimes i} \otimes e^{\otimes a-t} \right)^{\oplus b} \right) \neq 0$$
and so $\mathsf{k}_{\mathcal{G},\mathcal{E}} \geq i_0 = t - k = \mathsf{k}_{\mathcal{F}+\mathcal{G},\mathcal{E}} - \mathsf{k}_{\mathcal{F},\mathcal{E}}$.

6. Let $s = \mathsf{k}_{\mathcal{F}\cap\mathcal{G},\mathcal{E}} = \mathsf{k}_{\mathcal{F}\cap\mathcal{G},\mathcal{F}+\mathcal{G}}$. If $s = 0$ there is nothing to prove. Otherwise, similarly to the proof of the previous point, we can choose a section h of $\mathcal{F} \cap \mathcal{G}$ and sections f and g of $\mathcal{F}$ and $\mathcal{G}$ respectively such that $f + g$ is a section of $\mathcal{F} + \mathcal{G}$ and $\varphi((h^{\oplus s} \oplus (f+g)^{\oplus a-s})^{\oplus b}) \neq 0$. In particular note that $\mathsf{k}_{\mathcal{F}+\mathcal{G},\mathcal{E}} = \mathsf{k}_{\mathcal{F}+\mathcal{G},\mathcal{F}+\mathcal{G}} = a$. Then is easy to see that $\mathsf{k}_{\mathcal{G},\mathcal{E}} \geq a - \mathsf{k}_{\mathcal{F},\mathcal{E}} + s$.

♦

Let $(\mathcal{E}, \varphi)$ be a decorated sheaf and $\mathcal{F}$ a subsheaf of $\mathcal{E}$. As usual denote
$$\mathsf{P}^{\mathsf{k}}_{\mathcal{F}} \doteqdot \mathsf{P}_{\mathcal{F}} - \delta \mathsf{k}_{\mathcal{F},\mathcal{E}},$$
$$\mathsf{p}^{\mathsf{k}}_{\mathcal{F}} \doteqdot \mathsf{P}^{\mathsf{k}}_{\mathcal{F}} / \mathrm{rk}(\mathcal{F}),$$
$$\deg^{\mathsf{k}}(\mathcal{F}) \doteqdot \deg(\mathcal{F}) - \overline{\delta} \mathsf{k}_{\mathcal{F},\mathcal{E}},$$
$$\mu^{\mathsf{k}}(\mathcal{F}) \doteqdot \deg(\mathcal{F})^{\mathsf{k}} / \mathrm{rk}(\mathcal{F}).$$
We recall that $(\mathcal{E}, \varphi)$ is **k-(semi)stable**, respectively **slope k-(semi)stable**, if and only if for any $\mathcal{F} \subset \mathcal{E}$
$$\mathsf{p}^{\mathsf{k}}_{\mathcal{F}} \underset{(-)}{\prec} \mathsf{p}^{\mathsf{k}}_{\mathcal{E}},$$
or
$$\mu^{\mathsf{k}}(\mathcal{F}) \underset{(=)}{<} \mu^{\mathsf{k}}(\mathcal{E}),$$
respectively.

3.3.1 Maximal destabilizing subsheaf

Notation. In this section, unless otherwise stated, any decorated sheaf will have rank $r \leq 3$.

Proposition 53. *Let $(\mathcal{E}, \varphi)$ be a decorated sheaf of type (a, b, N) and rank $r = 2$ or $r = 3$. If $(\mathcal{E}, \varphi)$ is not slope k-semistable then there exists a unique, k-slope-semistable subsheaf $\mathcal{F}$ of $\mathcal{E}$ such that:*

1. *$\mu^{\mathsf{k}}(\mathcal{F}) \geq \mu^{\mathsf{k}}(\mathcal{W})$ for any $\mathcal{W} \subset \mathcal{E}$.*
2. *If $\mu^{\mathsf{k}}(\mathcal{F}) = \mu^{\mathsf{k}}(\mathcal{W})$ then $\mathcal{W} \subseteq \mathcal{F}$.*

*The subsheaf $\mathcal{F}$, with the induced morphism $\varphi_{|\mathcal{F}}$, is called the **maximal slope k-destabilizing subsheaf**.*

Proof. Define the following partial ordering on the set of decorated subsheaves of $\mathcal{E}$. Let $\mathcal{F}_1, \mathcal{F}_2$ be two subsheaves of $\mathcal{E}$; then

$$\mathcal{F}_1 \preccurlyeq^{\mathsf{k}} \mathcal{F}_2 \iff \mathcal{F}_1 \subseteq \mathcal{F}_2 \text{ and } \mu^{\mathsf{k}}(\mathcal{F}) \leq \mu^{\mathsf{k}}(\mathcal{E}).$$

The set of subsheaves of $\mathcal{E}$ with this ordering relation satisfies the hypotheses of Zorn's Lemma, so there exists a maximal element (not unique in general). Choose an element $\mathcal{F}$ in the following set:

$$\min_{\mathrm{rk}(\mathcal{G})} \{\mathcal{G} \subset \mathcal{E} \mid \mathcal{G} \text{ is } \preccurlyeq^{\mathsf{k}} \text{-maximal}\}.$$

Then we claim that $(\mathcal{F}, \varphi_{|\mathcal{F}})$ has the asserted properties.

By contradiction, suppose that there exists $\mathcal{G} \subset \mathcal{E}$ such that $\mu^{\mathsf{k}}(\mathcal{G}) \geq \mu^{\mathsf{k}}(\mathcal{F})$, i.e.,

$$\mu(\mathcal{G}) - \frac{\overline{\delta}\,\mathsf{k}_{\mathcal{G},\mathcal{E}}}{r_{\mathcal{G}}} \geq \mu(\mathcal{F}) - \frac{\overline{\delta}\,\mathsf{k}_{\mathcal{F},\mathcal{E}}}{r_{\mathcal{F}}}.$$

Claim. We can assume $\mathcal{G} \subseteq \mathcal{F}$ by replacing $\mathcal{G}$ by $\mathcal{G} \cap \mathcal{F}$.

Indeed, if $\mathcal{G} \not\subseteq \mathcal{F}$, $\mathcal{F}$ is a proper subsheaf of $\mathcal{F}+\mathcal{G}$ since (by the assumpions we made on the k-slope of $\mathcal{G}$ and by maximality of $\mathcal{F}$) $\mathcal{F} \not\subset \mathcal{G}$. By maximality

$$\mu^{\mathsf{k}}(\mathcal{F}) > \mu^{\mathsf{k}}(\mathcal{F} + \mathcal{G}).$$

Using the exact sequence

$$0 \longrightarrow \mathcal{F} \cap \mathcal{G} \longrightarrow \mathcal{F} \oplus \mathcal{G} \longrightarrow \mathcal{F} + \mathcal{G} \longrightarrow 0$$

one finds, following calculations we made in the proof of Proposition 27, that

$$r_{\mathcal{F}\cap\mathcal{G}}(\mu^{\mathsf{k}}(\mathcal{G}) - \mu^{\mathsf{k}}(\mathcal{F} \cap \mathcal{G})) < \overline{\delta}(\mathsf{k}_{\mathcal{F}+\mathcal{G},\mathcal{E}} + \mathsf{k}_{\mathcal{F}\cap\mathcal{G},\mathcal{E}} - \mathsf{k}_{\mathcal{F},\mathcal{E}} - \mathsf{k}_{\mathcal{G},\mathcal{E}}). \tag{3.13}$$

Therefore if $(\mathsf{k}_{\mathcal{F}+\mathcal{G},\mathcal{E}} + \mathsf{k}_{\mathcal{F}\cap\mathcal{G},\mathcal{E}} - \mathsf{k}_{\mathcal{F},\mathcal{E}} - \mathsf{k}_{\mathcal{G},\mathcal{E}}) \leq 0$ then $\mu^{\mathsf{k}}(\mathcal{F}\cap\mathcal{G}) \geq \mu^{\mathsf{k}}(\mathcal{G})$ and the claim holds true.

First suppose that $r = 2$.
Consider $\mathcal{F}\cap\mathcal{G}$. If $\mathrm{rk}(\mathcal{F}\cap\mathcal{G}) = 0$ then $\mathsf{k}_{\mathcal{F}\cap\mathcal{G},\mathcal{E}} = 0$ and, thanks to point (5) of Proposition 52, the right part of equation (3.13) is less or equal to zero, and the claim holds true. If $\mathrm{rk}(\mathcal{F}\cap\mathcal{G}) = 2$ then $r_{\mathcal{E}} = r_{\mathcal{G}} = r_{\mathcal{F}} = r_{\mathcal{F}\cap\mathcal{G}} = r_{\mathcal{F}+\mathcal{G}}$, $\mathcal{F}$ coincides, up to a rank zero sheaf $\mathcal{T} = \mathcal{E}/\mathcal{F}$, with $\mathcal{E}$ and $\mathsf{k}_{\mathcal{F}+\mathcal{G},\mathcal{E}} = \mathsf{k}_{\mathcal{F},\mathcal{E}} = \mathsf{k}_{\mathcal{G},\mathcal{E}} = \mathsf{k}_{\mathcal{F}\cap\mathcal{G},\mathcal{E}} = \mathsf{k}_{\mathcal{E},\mathcal{E}} = a$. Since $\deg(\mathcal{E}) = \deg(\mathcal{F}) + \deg(\mathcal{T})$ and $\mathsf{k}_{\mathcal{F},\mathcal{E}} = a = \mathsf{k}_{\mathcal{E},\mathcal{E}}$ one get that $\mu^{\mathsf{k}}(\mathcal{F}) \leq \mu^{\mathsf{k}}(\mathcal{E})$ and $\mathcal{F}$ is not $\preccurlyeq^{\mathsf{k}}$-maximal, which is absurd. Therefore $\mathrm{rk}(\mathcal{F}\cap\mathcal{G}) = 1$. If $r_{\mathcal{F}}$ of $r_{\mathcal{G}}$ are equal to 2 then, as before, one easily gets that $\mathcal{F}$ is not maximal, against the assumptions. The only chance is that $r_{\mathcal{F}} = r_{\mathcal{G}} = r_{\mathcal{F}\cap\mathcal{G}} = r_{\mathcal{F}+\mathcal{G}} = 1$ and so all these sheaves coincide with each other up to rank zero sheaves. Thus $\mathsf{k}_{\mathcal{G},\mathcal{E}} = \mathsf{k}_{\mathcal{F},\mathcal{E}} = \mathsf{k}_{\mathcal{F}\cap\mathcal{G},\mathcal{E}} = \mathsf{k}_{\mathcal{F}+\mathcal{G},\mathcal{E}}$. Therefore the inequality 3.12 holds true and $\mu^{\mathsf{k}}(\mathcal{F}\cap\mathcal{G}) \geq \mu^{\mathsf{k}}(\mathcal{G}) \geq \mu^{\mathsf{k}}(\mathcal{F})$.

Now suppose that $r = 3$.
If $\mathrm{rk}(\mathcal{F}\cap\mathcal{G}) = 0$ then $\mathsf{k}_{\mathcal{F}\cap\mathcal{G},\mathcal{E}} = 0$ and the right part of equation (3.13) is less or equal to zero and the claim holds true. If $\mathrm{rk}(\mathcal{F}\cap\mathcal{G}) = 3$ as before we easily fall in contradiction. If $\mathrm{rk}(\mathcal{F}) = 3$ then $\mathsf{k}_{\mathcal{F},\mathcal{E}} = a$ and $\mathcal{F}$ coincides, up to a rank zero sheaf, with $\mathcal{E}$; so $\mu^{\mathsf{k}}(\mathcal{F}) \leq \mu^{\mathsf{k}}(\mathcal{E})$ and $\mathcal{F}$ is not maximal, that is absurd. Similarly if $\mathrm{rk}(\mathcal{G}) = 3$, then $\mu^{\mathsf{k}}(\mathcal{F}) \leq \mu^{\mathsf{k}}(\mathcal{G}) \leq \mu^{\mathsf{k}}(\mathcal{E})$, that is again absurd. Therefore the possible cases are the following:

$\mathrm{rk}(\mathcal{F}\cap\mathcal{G})$	$\mathrm{rk}(\mathcal{F})$	$\mathrm{rk}(\mathcal{G})$	$\mathrm{rk}(\mathcal{F}+\mathcal{G})$	implies
1	1	1	1	$\mathsf{k}_{\mathcal{F}\cap\mathcal{G},\mathcal{E}} = \mathsf{k}_{\mathcal{F},\mathcal{E}} = \mathsf{k}_{\mathcal{G},\mathcal{E}} = \mathsf{k}_{\mathcal{F}+\mathcal{G},\mathcal{E}}$
1	1	2	2	$\mathsf{k}_{\mathcal{F}\cap\mathcal{G},\mathcal{E}} = \mathsf{k}_{\mathcal{F},\mathcal{E}}$ and $\mathsf{k}_{\mathcal{G},\mathcal{E}} = \mathsf{k}_{\mathcal{F}+\mathcal{G},\mathcal{E}}$
1	2	1	2	$\mathsf{k}_{\mathcal{F}\cap\mathcal{G},\mathcal{E}} = \mathsf{k}_{\mathcal{G},\mathcal{E}}$ and $\mathsf{k}_{\mathcal{F},\mathcal{E}} = \mathsf{k}_{\mathcal{F}+\mathcal{G},\mathcal{E}}$
1	2	2	3	$\mathsf{k}_{\mathcal{F}+\mathcal{G},\mathcal{E}} = \mathsf{k}_{\mathcal{E},\mathcal{E}} = a$
2	2	2	2	$\mathsf{k}_{\mathcal{F}+\mathcal{G},\mathcal{E}} = \mathsf{k}_{\mathcal{E},\mathcal{E}} = a$

The non-trivial cases are the following: $\mathrm{rk}(\mathcal{F}\cap\mathcal{G}) = 1$ and $\mathrm{rk}(\mathcal{F}) = \mathrm{rk}(\mathcal{G}) = 2$ or $\mathrm{rk}(\mathcal{F}\cap\mathcal{G}) = \mathrm{rk}(\mathcal{F}) = \mathrm{rk}(\mathcal{G}) = 2$. In the first case $\mathrm{rk}(\mathcal{F}+\mathcal{G}) = 3$ and so $\mathsf{k}_{\mathcal{F}+\mathcal{G},\mathcal{E}} = a$ and $\mathsf{k}_{\mathcal{F}\cap\mathcal{G},\mathcal{F}+\mathcal{G}} = \mathsf{k}_{\mathcal{F}\cap\mathcal{G},\mathcal{E}}$, in the second case $\mathrm{rk}(\mathcal{F}+\mathcal{G}) = 2$ and so $\mathsf{k}_{\mathcal{F}\cap\mathcal{G},\mathcal{E}} = \mathsf{k}_{\mathcal{F}+\mathcal{G},\mathcal{E}} = \mathsf{k}_{\mathcal{F},\mathcal{E}} = \mathsf{k}_{\mathcal{G},\mathcal{E}}$. Therefore in both cases equation (3.12) holds true and equation (3.13) holds with the less or equal than zero. Then $\mu^{\mathsf{k}}(\mathcal{F}\cap\mathcal{G}) \geq \mu^{\mathsf{k}}(\mathcal{G}) \geq \mu^{\mathsf{k}}(\mathcal{F})$ and the claim holds true.

Since we have proved the claim, the proof may continue as the proof of Proposition 27. ⧫

Proposition 54. *Let $(\mathcal{E},\varphi)$ be a decorated sheaf of type (a,b,N) and rank*

$r = 2$ or $r = 3$. If $(\mathcal{E}, \varphi)$ is not k-semistable then exists a unique, k-semistable subsheaf $\mathcal{F}$ of $\mathcal{E}$ such that:

1. *$\mathsf{p}^k_{\mathcal{F}} \preceq \mathsf{p}^k_{\mathcal{F}}$ for any $\mathcal{W} \subset \mathcal{E}$.*
2. *If $\mathsf{p}^k_{\mathcal{F}} = \mathsf{p}^k_{\mathcal{W}}$ then $\mathcal{W} \subseteq \mathcal{F}$.*

*The subsheaf $\mathcal{F}$, with the induced morphism $\varphi_{|\mathcal{F}}$, is called the **k-maximal destabilizing subsheaf**.*

Proof. The proof is similar to the proof of Proposition 53. ♦

Remark 55. As in the ε-semistable case, if $(\mathcal{E}, \varphi)$ is k-semistable (resp. slope k-semistable) the maximal k-destabilizing (resp. slope k-destabilizing) subsheaf coincide with $\mathcal{E}$.

3.3.2 Restriction theorem

In the previous section we proved that, given a decorated sheaf $(\mathcal{E}, \varphi)$ of rank less or equal to 3, there exists a unique maximal k-destabilizing subsheaf $(\mathcal{F}, \varphi_{|\mathcal{F}})$. Since, as we noticed in Section 3.2.3, there is a one-to-one correspondence between decorated subsheaves of $(\mathcal{E}, \varphi)$ and quotients of $\mathcal{E}$, we will call **minimal k-destabilizing quotient** the (unique) sheaf $\mathcal{Q} \doteqdot \text{coker}(\mathcal{F} \hookrightarrow \mathcal{E})$ such that $\mathcal{F}$ is the maximal k-destabilizing subsheaf.

In analogy with Section 3.2, we will say that a decorated sheaf $(\mathcal{E}, \varphi)$ over a Noetherian scheme Y is **flat over the fibre of a morphism** $f : Y \to S$ of finite type between Noetherian schemes if and only if

- $\mathcal{E}$ and N are flat families of sheaves over the fibre of $f : Y \to S$;
- $\mathsf{k}_{\mathcal{E}_s, \mathcal{E}_s}$ is locally constant as a function of s, where $\mathcal{E}_s \doteqdot \mathcal{E}_{|_{f^{-1}(s)}}$.

Note that the above conditions imply that the k-Hilbert polynomials $\mathsf{P}^{\mathsf{k}}_{\mathcal{E}_s}$ are locally constant for $s \in S$. The converse holds only if S is irreducible: i.e., asking that $\mathsf{P}^{\mathsf{k}}_{\mathcal{E}_s}$ is locally constant as function of s is equivalent to ask that $\mathsf{P}_{\mathcal{E}_s}$ and $\mathsf{k}_{\mathcal{E}_s, \mathcal{E}_s}$ are locally constant as functions of s.

If $(\mathcal{F}, \varphi_{|\mathcal{F}})$ slope de-semistabilizes (resp. de-semistabilizes) $(\mathcal{E}, \varphi)$ then $\mu^{\mathsf{k}}(\mathcal{F}) > \mu^{\mathsf{k}}(\mathcal{E})$ (resp $\mathsf{p}^{\mathsf{k}}_{\mathcal{F}} \succ \mathsf{p}^{\mathsf{k}}_{\mathcal{E}}$). Let $\mathcal{Q} \doteqdot \text{coker}(\mathcal{F} \hookrightarrow \mathcal{E})$; then

$$\mu(\mathcal{Q}) < \mu(\mathcal{E}) - \overline{\delta} \left(\frac{\mathsf{k}_{\mathcal{F},\mathcal{E}}}{r_{\mathcal{Q}}} - \frac{a r_{\mathcal{F}}}{r_{\mathcal{E}} r_{\mathcal{Q}}} \right),$$

or

$$\mathsf{p}_{\mathcal{Q}} \prec \mathsf{p}_{\mathcal{E}} - \delta \left(\frac{\mathsf{k}_{\mathcal{F},\mathcal{E}}}{r_{\mathcal{Q}}} - \frac{a r_{\mathcal{F}}}{r_{\mathcal{E}} r_{\mathcal{Q}}} \right),$$

respectively.

Define $C_{\mathsf{k},i} \doteqdot \left(\frac{i}{r_{\mathcal{Q}}} - \frac{ar_{\mathcal{F}}}{r_{\mathcal{E}}r_{\mathcal{Q}}}\right)$ for $i = 0, \ldots, a$. Let $(\mathcal{E}, \varphi)$ be a flat family of decorated sheaves over the fibre of a projective morphism $f : X \to S$. Let $\mathsf{P} = \mathsf{P}_{\mathcal{E}_s}$ and $\mathsf{p} = \mathsf{p}_{\mathcal{E}_s}$ the Hilbert polynomial and, respectively, the reduced Hilbert polynomial of $\mathcal{E}_s$ (which are constant because the family is flat over S). Define:

1. $\mathfrak{F}_{\mathsf{k}}$ as the family on X parameterized by S of saturated subsheaves $\mathcal{F} \hookrightarrow \mathcal{E}_s$ such that the induced torsion free quotient $\mathcal{E}_s \twoheadrightarrow \mathcal{Q}$ satisfy $\mu(\mathcal{Q}) \leq \mu(\mathcal{E}_s) + \overline{\delta} C_{\mathsf{k},0}$;
2. $\mathfrak{F}_{\mathsf{k},i}$ as the family of decorated subsheaves $(\mathcal{F}, \varphi_{|\mathcal{F}}) \hookrightarrow (\mathcal{E}_s, \varphi_{|\mathcal{E}_s})$ such that:
   - $\mathsf{k}_{\mathcal{F},\mathcal{E}_s} = i$;
   - $\mathsf{p}^{\mathsf{k}}_{\mathcal{F}} \succ \mathsf{p}^{\mathsf{k}}_{\mathcal{E}}$ (i.e., $\mathsf{p}_{\mathcal{Q}} \prec \mathsf{p} + \delta C_{\mathsf{k},i}$ with $\mathcal{Q} \doteqdot \operatorname{coker}(\mathcal{F} \hookrightarrow \mathcal{E}_s)$);
   - $\mathcal{F}$ is a saturated subsheaf of $\mathcal{E}_s$,

   for $i = 0, \ldots, a$.

It is easy to see that these families are bounded and therefore, using the same techniques used in Section 3.2.5 one can prove that k-semistability is open. More precisely

Proposition 56. *Let $f : X \to S$ be a projective morphism of Noetherian schemes and let $(\mathcal{E}, \varphi)$ be a flat family of decorated sheaves over the fibre of f. The set of points $s \in S$ such that $(\mathcal{E}_s, \varphi_{|\mathcal{E}_s})$ is k-(semi)stable with respect to δ is open in S.*

Proof. Let

$$A \doteqdot \{P'' \in \mathbb{Q}[x] \,|\, \exists\, s \in S, \exists\, q : \mathcal{E}_s \twoheadrightarrow \mathcal{Q} \text{ such that } \mathsf{P}_{\mathcal{Q}} = P'' \text{ and } \ker(q) \in \mathfrak{F}_{\mathsf{k}}\} \tag{3.14}$$

and, for $i = 0, \ldots, a$,

$$A_{\mathsf{k},i} \doteqdot \{P'' \in \mathbb{Q}[x] \,|\, \exists\, s \in S, \exists\, q : \mathcal{E}_s \twoheadrightarrow \mathcal{Q} \text{ such that } \mathsf{P}_{\mathcal{Q}} = P'' \text{ and } \ker(q) \in \mathfrak{F}_{\mathsf{k},i}\}.$$

Then, using the same techniques used in the proof of Proposition 43, one concludes the proof. ♦

Thanks to the previous results and using the same arguments as in the proof of Theorem 44, it is easy to see that the following theorem holds true:

Theorem 57 (Relative maximal k-destabilizing subsheaf). *Let X, $\mathcal{O}_X(1)$, S and $f : X \to S$ as before. Let $(\mathcal{E}, \varphi)$ be a decorated sheaf of rank $r \leq 3$. Then there is an integral k-scheme T of finite type, a projective birational morphism $g : T \to S$, a dense open subset $U \subset T$ and a flat quotient $\mathcal{Q}$ of $\mathcal{E}_T$ such that for all points $t \in U$, $\mathcal{F}_t \doteqdot \ker(\mathcal{E}_t \twoheadrightarrow \mathcal{Q}_t)$ with the*

induced morphism $\varphi_{t|_{\mathcal{F}_t}}$ *is the maximal* k*-destabilizing subsheaf of* $(\mathcal{E}_t, \varphi_t)$ *or* $\mathcal{Q}_t = \mathcal{E}_t$.
Moreover the pair $(g, \mathcal{Q})$ *is universal in the sense that if* $g' : T' \to S$ *is any dominant morphism of* k*-integral schemes and* $\mathcal{Q}'$ *is a flat quotient of* $\mathcal{E}'_T$*, satisfying the same property of* $\mathcal{Q}$*, there is an* S*-morphism* $h : T' \to T$ *such that* $h^*_X(\mathcal{Q}) = \mathcal{Q}'$.

Finally, following the costructions made in Section 3.2.7 and replacing k with ε, one can prove the following

Theorem 58 (**Mehta-Ramanathan for slope k-semistable decorated sheaves**)**.** *Let* X *be a smooth projective surface and* $\mathcal{O}_X(1)$ *be, as usual, a very ample line bundle. Let* $(\mathcal{E}, \varphi)$ *be a slope* k*-semistable decorated sheaf of rank* $r \leq 3$*. Then there is an integer* a_0 *such that for all* $\mathsf{a} \geq \mathsf{a}_0$ *there is a dense open subset* $U_{\mathsf{a}} \subset |\mathcal{O}_X(\mathsf{a})|$ *such that for all* $D \in U_{\mathsf{a}}$ *the divisor* D *is smooth and* $(\mathcal{E}, \varphi)_{|_D}$ *is slope* k*-semistable.*

3.3.3 Decorated sheaves of rank 2

Lemma 59. *Let* $(\mathcal{E}, \varphi)$ *be a decorated sheaf of rank* $r = 2$*. Then the following conditions are equivalent:*

- $(\mathcal{E}, \varphi)$ *is (semi)stable (in the sense of Definition 3);*
- $(\mathcal{E}, \varphi)$ *is* k*-(semi)stable.*

Proof. Since the rank of $\mathcal{E}$ is equal to 2 all filtrations of $\mathcal{E}$ are non-critical and of lengh one. Then the statement follows from the fourth point of Remark 6. ♦

Thanks to the previous Lemma all results in the previous section holds true for semistable decorated sheaves. In particular, if $(\mathcal{E}, \varphi)$ is a rank 2 decorated sheaf, then

- we have found the maximal destabilizing subsheaf and the relative maximal destabilizing subsheaf of $\mathcal{E}$;
- we have provided the Harder-Narasimhan filtration and the relative Harder-Narasimhan filtration of $\mathcal{E}$;
- we have proved that the semistability condition is open;
- we have proved a Mehta-Ramanathan theorem for such objects.

3.4 Remarks

1. In Section 3.2 we never used that N is of rank 1 nor that it is a vector bundle. We only used that it is a pure dimensional torsion free sheaf (of positive rank). Therefore all results in this chapter can be easily generalized for pairs $(\mathcal{E},\varphi)$ of type (a,b,c,N) where $\mathcal{E}$ and N are torsion free sheaves over X, a,b,c are positive integers and

$$\varphi\colon \mathcal{E}_{a,b} \longrightarrow \det(\mathcal{E})^{\otimes c} \otimes \mathsf{N}.$$

2. Let $(\mathcal{A},\varphi)$ be a decorated sheaf of rank $r>0$. Define $\widetilde{\mathcal{A}} \doteqdot \mathcal{A}_{a,b}$; then the pair $(\widetilde{\mathcal{A}},\varphi)$ can be regarded as a framed sheaf. Recall that a framed sheaf (A,α) of positive rank and with nonzero morphism α is slope semistable with respect to $\widetilde{\delta}$ if and only if for any $F\subset A$

$$\mu(F) - \frac{\widetilde{\delta}\,\varepsilon(\alpha_{|_F})}{\mathrm{rk}(F)} \leq \mu(A) - \frac{\widetilde{\delta}}{\mathrm{rk}(A)}$$

Suppose now that $(\widetilde{\mathcal{A}},\varphi)$ is frame semistable with respect to $\widetilde{\delta}$, then $(\mathcal{A},\varphi)$ is slope ε-semistable with respect to $\hat{\delta} \doteqdot \frac{\widetilde{\delta}}{a^2\,b\,r^{a-1}}$. In fact if

$$\mathrm{rk}(\mathcal{A}_{a,b}) = b\,r^a \qquad \deg(\mathcal{A}_{a,b}) = a\,b\,r^{(a-1)}\deg(\mathcal{A})$$

and so if $\mathcal{F}$ is a subsheaf of $\mathcal{A}$ then

$$\mu(\mathcal{F}_{a,b}) - \frac{\widetilde{\delta}\,\varepsilon(\varphi_{|_{\mathcal{F}_{a,b}}})}{r_{\mathcal{F}_{a,b}}} \leq \mu(\widetilde{\mathcal{A}}) - \frac{\widetilde{\delta}}{r_{\widetilde{\mathcal{A}}}}$$

which implies that

$$a\mu(\mathcal{F}) - \frac{\widetilde{\delta}\,\varepsilon(\varphi_{|_{\mathcal{F}_{a,b}}})}{b(r_{\mathcal{F}})^a} \leq a\mu(\mathcal{A}) - \frac{\widetilde{\delta}}{br^a}$$

and so

$$\mu_\varepsilon(\mathcal{F}) = \mu(\mathcal{F}) - a\underbrace{\frac{\widetilde{\delta}}{a^2b(r_{\mathcal{F}})^{a-1}}}_{=\hat{\delta}}\;\frac{\varepsilon(\varphi_{|_{\mathcal{F}_{a,b}}})}{r_{\mathcal{F}}} \leq \mu(\mathcal{A}) - a\underbrace{\frac{\widetilde{\delta}}{a^2br^{a-1}}}_{=\hat{\delta}}\;\frac{1}{r} = \mu_\varepsilon(\mathcal{E}).$$

Since the subsheaves of $\mathcal{A}$ correspond to subsheaves of $\mathcal{A}_{a,b}$ but this correspondence is *not* surjective, the converse does not hold in general but only if $a=1$ (b and c generic). Thanks to the previous calculations and to Proposition 5, one has that

$$\begin{aligned}(\mathcal{A}_{a,b},\varphi)\;\; \widetilde{\delta} \text{ frame slope (semi)stable} &\Rightarrow (\mathcal{A},\varphi)\;\; \hat{\delta} \text{ slope } \varepsilon\text{-(semi)stable}\\ &\Rightarrow \hat{\delta} \text{ slope (semi)stable}\\ &\Rightarrow \hat{\delta}\;\; \mathsf{k}\text{-(semi)stable.}\end{aligned}$$

Replacing $\deg_\varepsilon$ by P^ε and μ_ε by p^ε, similar calculations show that the same result hols also for semistability and not only for slope semistability.

3. Let $(\mathcal{E}, \varphi)$ be a decorated sheaf. If it is not semistable with respect to definition (2.9) then it is not slope ε-semistable (see Proposition 5). Let $\mathcal{F} \subset \mathcal{E}$ be the maximal (slope ε) destabilizing subsheaf and suppose that $\varepsilon(\varphi_{|\mathcal{F}}) = 1$, then $\mathcal{F}$ destabilize $(\mathcal{E}, \varphi)$. Indeed in this case $a\varepsilon_{\mathcal{F}} = \mathsf{k}_{\mathcal{F},\mathcal{E}}$ and so

$$\frac{\deg(\mathcal{F}) - a\overline{\delta}\varepsilon_{\mathcal{F}}}{r_{\mathcal{F}}} > \frac{\deg(\mathcal{E}) - a\overline{\delta}}{r_{\mathcal{E}}},$$

multiplying by $r_{\mathcal{E}}\, r_{\mathcal{F}}$ one gets that

$$\deg(\mathcal{E}) r_{\mathcal{F}} - \deg(\mathcal{F}) r_{\mathcal{E}} - a\delta r_{\mathcal{F}} + r_{\mathcal{E}}\delta \mathsf{k}_{\mathcal{F},\mathcal{E}} < 0.$$

Chapter 4

Moduli spaces

Notation. Again P denotes a fixed numerical polynomial of degree $\dim X$. Any flat family of decorated sheaves will be supposed to have constant polynomial P. $(X, \mathcal{O}_X(1))$ will be, as usual, a smooth projective variety with a fixed ample line bundle.

4.1 Moduli space for ε-semistable decorated sheaves

We are interested in families of decorated sheaves over X parametrized by a Noetherian scheme S. Therefore with the expression "flat family of decorated sheaves" we mean a flat family of decorated sheaves over the fibres of $\pi_S \colon X \times S \to S$ in the sense of Definition 31. Namely

Definition 60 (**Families of decorated sheaves**)**.** Let S be a scheme and $\underline{\mathfrak{t}} = (a, b, \mathsf{N})$, where N in this chapter denotes a vector bundle over X. A family of decorated sheaves of type $\underline{\mathfrak{t}}$ parametrized by S is a pair $(\mathsf{E}, \varphi_{\mathsf{E}})$ where:

- E is a sheaf over $X \times S$;
- $\varphi_{\mathsf{E}} \colon \mathsf{E}_{a,b} \to \pi_X^* \mathsf{N}$;
- $\varphi_{\mathsf{E}_s} \colon (\mathsf{E}_s)_{a,b} \to (\pi_X^* \mathsf{N})_s \simeq \mathsf{N}$ is not zero for all $s \in S$;

with $\pi_X \colon X \times S \to X$ the projection. We will say that the family is **flat** over S if the induced morphism $\mathsf{E} \to X \times S \xrightarrow{\pi_S} S$ is flat. Finally we will say that two families $(\mathsf{E}, \varphi_{\mathsf{E}})$ and $(\mathsf{E}', \varphi_{\mathsf{E}}')$ of decorated sheaves of type $\underline{\mathfrak{t}}$ are **isomorphic** if there exists an isomorphism of sheaves $f \colon \mathsf{E} \to \mathsf{E}'$ for which exists $\lambda \in \mathcal{O}_S^*$ such that the following diagram commutes

$$\begin{CD} \mathsf{E}_{a,b} @>{f_{a,b}}>> \mathsf{E}'_{a,b} \\ @V{\varphi}VV @VV{\varphi_{\mathsf{E}}'}V \\ \pi_X^* \mathsf{N} @>{\pi_S^* \lambda}>> \pi_X^* \mathsf{N}. \end{CD}$$

We recall the following fundamental result.

Theorem 61 (Maruyama [21]). *Let* P *be a polynomial and* C *a constant. Then the family of torsion free coherent* $\mathcal{O}_X$*-modules* $\mathcal{A}$ *with Hilbert polynomial* $\mathsf{P}_{\mathcal{A}} = \mathsf{P}$ *and* $\mu(\mathcal{F}) \leq C$ *for any* $\mathcal{F} \subseteq \mathcal{A}$*, is bounded in the sense of Definition 34.*

As a consequence

Lemma 62. *The family* $\mathsf{S}_{\delta}^{\varepsilon\text{-}ss}(\mathsf{P}, \underline{\mathsf{t}})$ *of* ε*-semistable decorated sheaves (with respect the parameter* δ*) of type* $\underline{\mathsf{t}}$ *over* X *with fixed Hilbert polynomial* P *is bounded.*

Proof. Let $(\mathcal{E}, \varphi) \in \mathsf{S}_{\delta}^{\varepsilon\text{-}ss}(\mathsf{P}, \underline{\mathsf{t}})$ and $\mathcal{F}$ a subsheaf of $\mathcal{E}$. By the semistability condition we get

$$\mathrm{rk}(\mathcal{E}) \left(\mathsf{P}_{\mathcal{F}} - a\delta\varepsilon(\varphi_{|\mathcal{F}})\right) \preceq \mathrm{rk}(\mathcal{F}) \left(\mathsf{P}_{\mathcal{E}} - a\delta\right).$$

Recall that $\mathcal{E}$ is pure and $\mathrm{rk}(\mathcal{F}) > 0$ (see Remark 24) so that one has

$$\mathsf{p}_{\mathcal{F}} \preceq \mathsf{p}_{\mathcal{E}} + a\delta \left(\frac{\varepsilon(\varphi_{|\mathcal{F}})}{\mathrm{rk}(\mathcal{F})} - \frac{1}{\mathrm{rk}(\mathcal{E})}\right)$$

which implies

$$\mu(\mathcal{F}) \leq \mu(\mathcal{E}) + a\overline{\delta}$$

and so, since $\mu(\mathcal{E}) \doteqdot d/r$ is constant for every $\mathcal{E} \in \mathsf{S}_{\delta}^{\varepsilon\text{-}ss}(\mathsf{P}, \underline{\mathsf{t}})$, the family is bounded. ♦

Lemma 63. *The family of* $\mathsf{S}_{\overline{\delta}}^{slope\text{-}\varepsilon\text{-}ss}(\mathsf{P}, \underline{\mathsf{t}})$ *of slope* ε*-semistable decorated sheaves (with respect the parameter* $\overline{\delta}$*) of type* $\underline{\mathsf{t}}$ *over* X *with fixed Hilbert polynomial* P *is bounded.*

Proof. The proof is similar to the proof of Lemma 62 ♦

Therefore the family of ε-semistable decorated sheaves with fixed type and Hilbert polynomial is bounded. Thanks to Lemma 37 and Lemma 36, there exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$ any $\mathcal{E} \in \mathsf{S}_{\delta}^{\varepsilon\text{-}ss}$ is m**-regular**, in particular

- $\mathcal{E}(m)$ is globally generated;
- $H^i(X, \mathcal{E}(m-i)) = 0$ for any $i \geq 1$;
- $h^0(X, \mathcal{E}(m)) \doteqdot \dim\left(H^0(X, \mathcal{E}(m))\right) = \mathsf{P}(m)$.

Fix $m \geq m_0$, let H be a vector space of dimension $\mathsf{P}(m)$ and define $\mathcal{H} \doteqdot H \otimes \mathcal{O}_X(-m)$. Since $\mathcal{E}(m)$ is globally generated there is a surjective morphism

$$H \twoheadrightarrow \mathcal{E}(m) = \mathcal{E} \otimes \mathcal{O}_X(m)$$

which induces a morphism

$$\mathcal{H} = H \otimes \mathcal{O}_X(-m) \twoheadrightarrow \mathcal{E}.$$

Applying Theorem 41 one sees that there exists a projective scheme $\mathsf{Q}_{\mathrm{uot}}(\mathcal{H}, \mathsf{P})$, parameterizing all quotients of $\mathcal{H}$, whose closed points are the morphisms $[\mathsf{q} : \mathcal{H} \twoheadrightarrow \mathcal{E}] \in \mathsf{Q}_{\mathrm{uot}}(\mathcal{H}, \mathsf{P})$. For l big enough the standard map

$$\begin{array}{ccc} \mathsf{Q}_{\mathrm{uot}}(\mathcal{H}, \mathsf{P}) & \hookrightarrow & \mathrm{Grass}\left(H \otimes H^0(X, \mathcal{O}_X(m-l)), \mathsf{P}(l)\right) \\ & & \downarrow \\ & & \mathbb{P}\left(\bigwedge^{\mathsf{P}(l)}\left(H \otimes H^0(X, \mathcal{O}_X(m-l))\right)\right) \end{array}$$

is a well-defined closed immersion. Let $\mathcal{O}_{\mathsf{Quot}}(1)$ be the corresponding very ample line bundle on $\mathsf{Q}_{\mathrm{uot}}(\mathcal{H}, \mathsf{P})$.

Let $\mathsf{q} : \mathcal{H} \to \mathcal{E}$ be an element in $\mathsf{Q}_{\mathrm{uot}}(\mathcal{H}, \mathsf{P})$ representing a decorated sheaf $(\mathcal{E}, \varphi)$. Then a morphism $\varphi\colon \mathcal{E}_{a,b} \to \mathsf{N}$ induces a morphism $\mathcal{H}_{a,b} \to \mathsf{N}$, in fact

$$\mathcal{H}_{a,b} = \left((H \otimes \mathcal{O}_X(-m))^{\otimes a}\right)^{\oplus b} = H_{a,b} \otimes \mathcal{O}_X(-am),$$

and so

$$\begin{array}{ccc} H \otimes \mathcal{O}_X(-m) & \xrightarrow{\mathsf{q}} & \mathcal{E} \\ \downarrow & & \downarrow \\ H_{a,b} \otimes \mathcal{O}_X(-am) & \xrightarrow{\mathsf{q}_{a,b}} & \mathcal{E}_{a,b} \\ \downarrow & \swarrow_{\varphi} & \\ \mathsf{N}. & & \end{array} \tag{4.1}$$

Since a morphism $H_{a,b} \otimes \mathcal{O}_X(-am) \to \mathsf{N}$ corresponds to a morphism $H_{a,b} \to H^0(X, \mathsf{N}(am))$ and the latter can be parametrized by the projective space $\mathbb{P} \doteqdot \mathbb{P}\left(\mathrm{Hom}(H_{a,b}, H^0(X, \mathsf{N}(am)))^\vee\right)$ any decoration φ is represented by an element in $\mathbb{P}$.

Let $\mathsf{Q}_{\mathrm{uot}}(\mathcal{H}, \mathsf{P}, \mathsf{N})$ be the closed subscheme of $\mathsf{Q}_{\mathrm{uot}}(\mathcal{H}, \mathsf{P}) \times \mathbb{P}$ consisting of pairs

$$([\mathsf{q}\colon H \otimes \mathcal{O}_X(-m) \to \mathcal{E}], [\mathsf{f}\colon H_{a,b} \otimes \mathcal{O}_X(-am) \to \mathsf{N}]) \tag{4.2}$$

such that there exists a decorated sheaf $(\mathcal{E}, \varphi)$ of type $\underline{\mathfrak{t}}$ such that the morphism $\varphi\colon \mathcal{E}_{a,b} \to \mathsf{N}$ makes the diagram (4.1) commute.

Let $\mathcal{O}_{\mathbb{P}}(1)$ be an ample line bundle in $\mathbb{P}$ and recall that $\mathcal{O}_{\mathsf{Quot}}(1)$ is an ample line bundle on $\mathsf{Q}_{\mathrm{uot}}(\mathcal{H},\mathsf{P})$ induced by the closed immersion $\mathsf{Q}_{\mathrm{uot}}(\mathcal{H},\mathsf{P}) \longrightarrow \mathbb{P}(\bigwedge^{\mathsf{P}\varepsilon(l)} H \otimes H^0(X,\mathcal{O}_X(l-m)))$.

Let $p_1 : \mathsf{Q}_{\mathrm{uot}}(\mathcal{H},\mathsf{P}) \times \mathbb{P} \to \mathsf{Q}_{\mathrm{uot}}(\mathcal{H},\mathsf{P})$ and $p_2 : \mathsf{Q}_{\mathrm{uot}}(\mathcal{H},\mathsf{P}) \times \mathbb{P} \to \mathbb{P}$ be the projections, then consider the line bundle

$$\begin{aligned}\mathcal{L}'(n_1,n_2) &= p_1^*\mathcal{O}_{\mathsf{Quot}}(n_1) \otimes p_2^*\mathcal{O}_{\mathbb{P}}(n_2)\\ &\doteqdot \mathcal{O}_{\mathsf{Quot}}(n_1) \boxtimes \mathcal{O}_{\mathbb{P}}(n_2)\end{aligned}$$

defined over $\mathsf{Q}_{\mathrm{uot}}(\mathcal{H},\mathsf{P}) \times \mathbb{P}$.

The action of $\mathrm{Sl}(H)$ on H induces actions on $\mathsf{Q}_{\mathrm{uot}}(\mathcal{H},\mathsf{P})$ and on $\mathbb{P}$ which are compatible and so induces a well-defined action on $\mathsf{Q}_{\mathrm{uot}}(\mathcal{H},\mathsf{P},\mathsf{N})$ and a (natural) linearization on

$$\mathcal{L}(n_1,n_2) \doteqdot \mathcal{L}'(n_1,n_2)_{|\mathsf{Quot}(\mathcal{H},\mathsf{P},\mathsf{N})}$$

Recall that for a decorated sheaf $(\mathcal{W},\psi)$ we denote

$$\mathsf{P}_{(\mathcal{W},\psi)} \doteqdot \mathsf{P}_{\mathcal{W}} - a\delta\varepsilon(\psi) \doteqdot \mathsf{P}^{\varepsilon}_{\mathcal{W}}$$

and define

$$\frac{n_2}{n_1} \doteqdot a\delta(m)\frac{\mathsf{P}^{\varepsilon}(l)}{\mathsf{P}^{\varepsilon}(m)} - a\delta(l).$$

In order to construct the moduli space of ε-semistable decorated sheaves a fundamental step is to prove that the points in $\mathsf{Q}_{\mathrm{uot}}(\mathcal{H},\mathsf{P},\underline{\mathsf{t}})$ are semistable as points of a projective scheme with respect to the linearization on $\mathcal{L}(n_1,n_2)$ if and only if the associated decorated sheaf that they represent is ε-semistable as decorated sheaves. Then, roughly speaking, the GIT quotient of $\mathsf{Q}_{\mathrm{uot}}(\mathcal{H},\mathsf{P},\underline{\mathsf{t}})$, that is well defined and projective by general GIT theory, will be the desired moduli space.

The proof of the equivalence between GIT semistability and ε-semistability is composed by the following subsequent results.

Lemma 64. *For sufficiently large l a point $([\mathsf{q}],[\mathtt{f}]) \in \mathsf{Q}_{uot}(\mathcal{H},\mathsf{P},\mathsf{N})$ is (semi)stable with respect to the linearization of $\mathcal{L}(n_1,n_2)$ if and only if the following holds: if H' is a nontrivial proper subspace of H and $\mathcal{F} \subset \mathcal{E} = \mathsf{q}(\mathcal{H})$ the subsheaf generated by $H' \otimes \mathcal{O}_X(-m)$, then*

$$\dim H' \,(n_1\mathsf{P}(l)+n_2) \underset{(=)}{<} \dim H \,\left(n_1\mathsf{P}_{\mathcal{F}}(l) + n_2\varepsilon(\varphi_{|\mathcal{F}})\right), \tag{4.3}$$

where φ is the morphism induced by $\mathtt{f}$.

Proposition 65. *For sufficiently large l a point $([\mathtt{q}],[\mathtt{f}]) \in \mathsf{Q}_{uot}(\mathcal{H},\mathsf{P},\mathsf{N})$ is (semi)stable with respect to the linearization of $\mathcal{L}(n_1,n_2)$ if and only if the corresponding decorated sheaf $(\mathcal{E},\varphi)$ is ε-(semi)stable with respect to δ and the morphism $H \to H^0(X,\mathcal{E}(m))$ induced by $\mathtt{q}$ is an isomorphism.*

Proof. (Lemma 64). The proof is similar to the proof of Proposition 3.1 in [14]. Let $\mathtt{q} : H \to H^0(X,\mathcal{E}(m))$ and $\mathtt{f} : H_{a,b} \to \mathsf{N}(am)$ be homomorphisms representing the point $([\mathtt{q}],[\mathtt{f}])$ and let $W \doteqdot H^0(X,\mathcal{O}_X(l-m))$. Define $h^0_l \doteqdot h^0(X,\mathcal{E}(l))$. The morphism $\mathtt{q}$ induces homomorphisms $\mathtt{q}' : H\otimes W \to H^0(X,\mathcal{E}(l))$ and $\mathtt{q}'' : \bigwedge^{h^0_l}(H\otimes W) \to \det H^0(X,\mathcal{E}(l))$. If $\{w_1,\dots,w_t\}$ is a basis for W and $\{v_1,\dots,v_h\}$ is a basis for H, then a basis for $\bigwedge^{h^0_l}(H\otimes W)$ is given by elements of the form

$$u_{IJ} = (v_{i_1}\otimes w_{j_1})\wedge\cdots\wedge(v_{i_{h^0_l}}\otimes w_{j_{h^0_l}})$$

where I,J are multi-indices satisfying $i_k \le i_{k+1}$ and $j_k < j_{k+1}$ if $i_k = i_{k+1}$. Given a one-parameter subgroup λ of $\mathrm{Sl}(H)$ with weight vector $\underline{\xi} = (\xi_1,\dots,\xi_h)$, then $\mathbb{C}^*$ acts on $\bigwedge^{h^0_l}(H\otimes W)$ by

$$\lambda(t)\cdot u_{IJ} = t^{\xi_I}u_{IJ}, \qquad \xi_I \doteqdot \sum_{i_k\in I}\xi_{i_k}.$$

Now let

$$\mu(\mathtt{q}'';\lambda) = -\min\{\xi_I \mid \exists I,J \text{ with } \mathtt{q}''(u_{IJ})\neq 0\}.$$

This number can be computed as follows. Let ϖ denote the function $t \longmapsto \dim \mathtt{q}'(< u_1,\dots,u_t > \otimes W)$. It is easy to see that

$$\mu(\mathtt{q}'';\lambda) = -\sum_{i=1}^{h}\xi_i\,(\varpi(i)-\varpi(i-1)).$$

Similarly if we set

$$\mu(\mathtt{f};\lambda) = -\min\{\xi_i \,|\, \mathtt{f}((\underbrace{v_i\otimes\cdots\otimes v_i}_{a\text{-times}})^{\oplus b})\neq 0\}.$$

then $\mu(\mathtt{f};\lambda) = -\xi_\tau$ where $\tau = \min\{i \,|\, \mathtt{f}_{|(<v_1,\dots,v_i>)_{a,b}} \neq 0\}$.

By the Hilbert-Mumford criterion ([24] Theorem 2.1) we have:

- $([\mathtt{q}],[\mathtt{f}])$ *is a (semi)stable point of* $\mathsf{Q}_{uot}(\mathcal{H},\mathsf{P},\mathsf{N})$ *if and only if for all one-parameter subgroups λ one has*

$$n_1\,\mu(\mathtt{q}'';\lambda) + n_2\,\mu(\mathtt{f};\lambda) \underset{(=)}{>} 0,$$

or equivalently,

$$n_1\sum_{i=1}^{h}\xi_i\,(\varpi(i)-\varpi(i-1)) + n_2\,\xi_\tau \underset{(=)}{<} 0. \tag{4.4}$$

The left hand side is a linear form of weight vectors whose coefficients are determined only by the choice of the basis. Keeping such a basis fixed for a moment, it is enough to check the inequality for the special one-parameter subgroups $\lambda^{(j)}$ giving the weight vectors

$$\underline{\xi}^{(j)} = (\underbrace{j-h,\dots,j-h}_{j\text{-times}},\ \underbrace{j,\dots,j}_{(h-j)\text{-times}}) \qquad j = 1,\dots,h-1.$$

where we recall that we put $h = \dim H$.

For $\underline{\xi}^{(j)}$ the inequality (4.4) is equivalent to

$$j\ (n_1 h_l^0 + n_2) \underset{(=)}{<} h(n_1 \varpi(i) + n_2 \varepsilon(j)),$$

where

$$\varepsilon(j) = \begin{cases} 1 & \text{if } \mathtt{f}_{|(<v_1,\dots,v_j>)_{a,b}} \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then the following holds:

- $([\mathtt{q}],[\mathtt{f}])$ *is a (semi)stable point of* $\mathsf{Q}_{uot}(\mathcal{H},\mathsf{P},\mathsf{N})$ *if and only if for all nontrivial proper subspaces* H' *of* H *one has*

  $$\dim H'\ (n_1\, h_l^0 + n_2) \underset{(=)}{<} \dim H\ (n_1\ \dim \mathtt{q}'(H' \otimes W) + n_2\, \varepsilon(Y')),$$

  where

  $$\varepsilon(H') = \begin{cases} 1 & \textit{if } \mathtt{f}_{|H'_{a,b}} \neq 0 \\ 0 & \textit{otherwise.} \end{cases}$$

Let $\mathcal{F}$ be the subbundle $\mathtt{q}(H' \otimes \mathcal{O}_X(-m))$. In this case the decoration $\varphi : \mathcal{E}_{a,b} \to \mathsf{N}$ vanishes when restricted to $\mathcal{F}$ if and only if $\mathtt{f}_{|H'_{a,b}\otimes \mathcal{O}_X(-am)} = 0$. Hence $\varepsilon(H') = \varepsilon(\varphi_{|\mathcal{F}})$ and recalling that (for l big enough) $\mathsf{P}(l) = h_l^0$ and $\mathsf{P}_{\mathcal{F}}(l) = \dim \mathtt{q}'(H' \otimes W)$ we are done. ♦

Proof. (Proposition 65). Observe that if $([\mathtt{q}],[\mathtt{f}])$ is a semistable point the homomorphism $H \to H^0(X,\mathcal{E}(m))$ must be injective. Indeed if H' is its kernel than $\mathtt{q}'(H' \otimes W) = 0$ and $\varepsilon(H') = 0$, so the previous proposition shows that $\dim(H') = 0$. Hence, since $\dim H = h^0(X,\mathcal{E}(m))$, the morphism $H \to H^0(X,\mathcal{E}(m))$ is an isomorphism.
Substituting

$$\frac{n_2}{n_1} \doteqdot a\delta(m)\frac{\mathsf{P}^{\varepsilon}(l)}{\mathsf{P}^{\varepsilon}(m)} - a\delta(l) \tag{4.5}$$

in the inequality (4.3) and setting $H' \doteqdot H \cap H^0(X,\mathcal{F}(m))$ for any nontrivial proper subbundle $\mathcal{F}$ of $\mathcal{E}$ we can rewrite the stability criterion (4.3) as follows:

- $([\mathtt{q}], [\mathtt{f}])$ *is a (semi)stable point of* $\mathsf{Q}_{uot}(\mathcal{H}, \mathsf{P}, \mathsf{N})$ *if and only if for all non trivial proper subsheaf* $\mathcal{F}$ *of* $\mathcal{E}$ *with the induced decoration the following holds:*

$$\dim H' \left(1 + \frac{a\delta(m)}{\mathsf{P}^\varepsilon(m)}\right) \mathsf{P}^\varepsilon(l) \underset{(=)}{<} \dim H \left(\mathsf{P}^\varepsilon_{\mathcal{F}}(l) + \frac{\mathsf{P}^\varepsilon(l)}{\mathsf{P}^\varepsilon(m)}\, a\delta(m)\varepsilon_{\mathcal{F}}\right) \tag{4.6}$$

where as usual we denote by $\varepsilon_{\mathcal{F}}$ *the quantity* $\varepsilon(\mathcal{F}, \varphi_{|\mathcal{F}})$.

Recalling that $\mathsf{P}_{\mathcal{F}}(m) = h^0(X, \mathcal{F}(m))$, $\mathsf{P}(m) = \mathsf{P}^\varepsilon(m) + a\delta(m)$ and that

$$\dim H' = \mathsf{P}^\varepsilon_{\mathcal{F}}(m) + a\delta(m)\varepsilon_{\mathcal{F}}$$

the previous inequality is equivalent to the following:

$$(\mathsf{P}^\varepsilon_{\mathcal{F}}(m) + a\delta(m)\varepsilon_{\mathcal{F}})\, \mathsf{P}^\varepsilon(l) \left(\frac{\mathsf{P}(m)}{\mathsf{P}^\varepsilon(m)}\right) \underset{(=)}{<} \mathsf{P}(m) \left(\frac{\mathsf{P}^\varepsilon(m)\mathsf{P}^\varepsilon_{\mathcal{F}}(l) + \mathsf{P}^\varepsilon(l)\, a\delta(m)}{\mathsf{P}^\varepsilon(m)}\right)$$

and after some simplifications we get

$$\mathsf{P}^\varepsilon_{\mathcal{F}}(m)\mathsf{P}^\varepsilon(l) \underset{(=)}{<} \mathsf{P}^\varepsilon(m)\mathsf{P}^\varepsilon_{\mathcal{F}}(l)$$

Since the inequality (4.6) holds for every l large enough the same inequality holds also for the coefficients of polynomials in l. Then we derive the inequality:

$$r_{\mathcal{E}}\mathsf{P}^\varepsilon_{\mathcal{F}}(m) \underset{(=)}{<} r_{\mathcal{F}}\mathsf{P}^\varepsilon(m), \tag{4.7}$$

and then $\mathcal{E}$ is (semi)stable.

Conversely, if $(\mathcal{E}, \varphi)$ is ε-(semi)stable as decorated sheaf, then for any nontrivial proper subsheaf $\mathcal{F}$ one has $r_{\mathcal{E}}\mathsf{P}^\varepsilon_{\mathcal{F}}(m) \underset{(=)}{<} r_{\mathcal{F}}\, \mathsf{P}^\varepsilon(m)$. Since the previous inequality is equivalent to (4.6), we are done. ♦

Using GIT machinery one can prove the following result:

Theorem 66. *Let* δ *a rational polynomial of degree* $\dim X - 1$ *with positive leading coefficient. There is a projective scheme* $\mathcal{M}^{\varepsilon\text{-}ss}_{\delta}(\mathsf{P}, \underline{\mathsf{t}})$ *that corepresents the moduli functor* $\underline{\mathfrak{M}}^{\varepsilon\text{-}ss}_{\delta}(\mathsf{P}, \underline{\mathsf{t}})$ *which to a scheme* S *associates the equivalence classes of families of* ε*-semistable decorated sheaves with Hilbert polynomial* P *parametrised by* S*. Moreover there is an open subscheme* $\mathcal{M}^{\varepsilon\text{-}s}_{\delta}(\mathsf{P}, \underline{\mathsf{t}})$ *that represents the subfunctor* $\underline{\mathfrak{M}}^{\varepsilon\text{-}s}_{\delta}(\mathsf{P}, \underline{\mathsf{t}})$ *of* ε*-stable decorated sheaves.*

For the proof of the previous statements is enough to re-adapt the techniques used in the proof of the corresponding results in [14] (Proposition 3.3).

4.2 U-D compactification for slope ε-semistable decorated sheaves

4.2.1 General theory and preliminary results

Notation. In this section X will be a smooth projective variety of dimension $n = \dim X$ with a fixed ample line bundle $\mathcal{O}_X(1)$ and Y a quasi-projective variety. Finally S will denote a Noetherian scheme.

Let $\mathcal{C}$ be an additive full subcategory of an abelian category $\mathcal{C}_{\mathcal{A}}$. Let $\mathrm{Ob}(\mathcal{C})$ denote the class of objects in the category $\mathcal{C}$ and let $F(\mathcal{C})$ be the free abelian group generated by the objects in $\mathrm{Ob}(\mathcal{C})$ modulo isomorphisms. An element of $F(\mathcal{C})$ is a finite formal sum of elements $[T] \in \mathrm{Ob}(\mathcal{C})/_{\mathrm{isom}}$, i.e.,

$$\sum_{T \in \mathrm{Ob}(\mathcal{C})} n_T [T]$$

where n_T are integers and all n_T, except for a finite number, are zero. Given an exact sequence

$${}_{T'}T_{T''}: \qquad 0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0$$

of elements in $\mathrm{Ob}(\mathcal{C})$ define the element $Q({}_{T'}T_{T''}) \doteqdot [T] - [T'] - [T''] \in F(\mathcal{C})$. Let $H(\mathcal{C})$ be the subgroup of $F(\mathcal{C})$ generated by all elements of the form $Q({}_{T'}T_{T''})$. Finally define the **Grothendieck group** of $\mathcal{C}$ as

$$K(\mathcal{C}) \doteqdot F(\mathcal{C})/H(\mathcal{C}).$$

Let Y be a quasi-projective variety. Then the category $\mathrm{Coh}(Y)$ of coherent sheaves on Y is an abelian category and the category $\mathrm{Vect}(Y)$ of vector bundles over Y is an additive full subcategory category of $\mathrm{Coh}(Y)$. Then the **Grothendieck group of coherent sheaves** over Y is defined as

$$K_0(Y) \doteqdot K\left(\mathrm{Coh}(Y)/_{\simeq}\right),$$

and the **Grothendieck group of vector bundles** over Y is defined as

$$K^0(Y) \doteqdot K\left(\mathrm{Vect}(Y)/_{\simeq}\right).$$

Note that the morphism of categories $\mathrm{Vect}(Y) \longrightarrow \mathrm{Coh}(Y)$ induces a natural homomorphism $K^0(Y) \longrightarrow K_0(Y)$. The following result is due to Borel and Serre [3].

Theorem 67. *If Y is nonsingular, quasi-projective and irreducible, the canonical homomorphism*

$$K^0(Y) \longrightarrow K_0(Y)$$

is an isomorphism of groups.

Let now X be a smooth projective variety of dimension $n = \dim X$. We will denote by $K(X) = K_0(X) = K^0(X)$. Since the tensor product of vector bundles is a vector bundle, the map

$$\begin{aligned} \mathrm{Vect}(X) \times \mathrm{Vect}(X) &\longrightarrow \mathrm{Vect}(X) \\ (F, E) &\longmapsto F \otimes E \end{aligned}$$

is well defined. Moreover the tensor product is associative, commutative and $\mathcal{O}_X$ is clearly the identity element. One can show that $H(\mathrm{Vect}(X))$ is an ideal of $F(\mathrm{Vect}(X))$ and therefore $(K(X), \otimes)$ inherits a commutative ring structure.

Finally two classes u, u' in $K(X)$ are said to be **numerically equivalent** $u' \equiv u''$ if their difference is contained in the radical of the quadratic form

$$\begin{aligned} \chi : K(X) \times K(X) &\longrightarrow \mathbb{R} \\ (u, v) &\longmapsto \chi(u \otimes v) \doteqdot \sum_{i=0}^{n} (-1)^i \ h^i(X, u \otimes v) \end{aligned}$$

Denote by $K(X)_{\mathrm{num}} \doteqdot K(X)/_{\equiv}$. Note that by Hirzebruch-Riemann-Roch theorem $\chi(u \otimes v) = \int_X \mathrm{ch}(u)\mathrm{ch}(v)\mathrm{td}(X)$ and therefore the numerical behavior of $\mathsf{c} \in K(X)_{\mathrm{num}}$ is determined by its associated rank and Chern classes $c_i(\mathsf{c})$. For any class c in $K(X)_{\mathrm{num}}$ and any $m \in \mathbb{N}$ denote by $\mathsf{c}(m) \doteqdot \mathsf{c} \otimes [\mathcal{O}_X(m)]$ and with $\mathsf{P}_{\mathsf{c}}(m) \doteqdot \chi(\mathsf{c}(m))$ the associated Hilbert polynomial.

A flat family A of coherent sheaves on X parametrized by S defines an element $[\mathsf{A}] \in K^0(X \times S)$ and since the projection $p : X \times S \to S$ is a projective morphism of Noetherian schemes it induces a (well-defined) morphism

$$\begin{aligned} p_! : K^0(X \times S) &\longrightarrow K^0(S) \\ [\mathsf{A}] &\longmapsto \sum_{i \geq 0} (-1)^i [R^i p_* \mathsf{A}]. \end{aligned}$$

Definition 68. Let A be a flat family of coherent sheaves on X parametrized by S. Define $\lambda_{\mathsf{A}} : K(X) \to \mathrm{Pic}(S)$ to be the following composition of morphisms

$$\lambda_{\mathsf{A}} : K(X) \xrightarrow{\pi_X^*} K^0(X \times S) \xrightarrow{\cdot \otimes [\mathsf{A}]} K^0(X \times S) \xrightarrow{(\pi_S)_!} K^0(S) \xrightarrow{\det} \mathrm{Pic}(S).$$

Lemma 69 (Lemma 8.1.2 in [15])**.** *Let A, A' and A'' be S-flat families of coherent sheaves over X. The following statements hold.*

i) If $0 \longrightarrow \mathsf{A}' \longrightarrow \mathsf{A} \longrightarrow \mathsf{A}'' \longrightarrow 0$ *is exact, for any* $u \in K(X)$

$$\lambda_{\mathsf{A}}(u) \cong \lambda_{\mathsf{A}'}(u) \otimes \lambda_{\mathsf{A}''}(u).$$

ii) If $f : S' \to S$ *is a morphism then, for any* $u \in K(X)$,

$$f^*(\lambda_{\mathsf{A}}(u)) = \lambda_{f^*\mathsf{A}}(u).$$

iii) If G is an algebraic group, S a scheme with a G-action and A a G-linearized S-flat family of coherent sheaves on X, then λ_{A} factors through the group $Pic^G(S)$ of isomorphism classes of G-linearized line bundles on S.

iv) Let N be a locally free $\mathcal{O}_S$-sheaf. For any class $v \in K(X)_{num}$

$$\lambda_{\mathsf{A}\otimes\pi_S^*(N)}(u) \cong \lambda_{\mathsf{A}}(u)^{rk(N)} \otimes \det(N)^{\chi(u\otimes v)}.$$

Let us denote by $\mathtt{H}_X$ the divisor associated with $\mathcal{O}_X(1)$, let $\mathtt{h}_X = [\mathcal{O}_X(1)]$ its class in $K(X)$. Fix a point $x \in X$ and let $\mathtt{c} \in K(X)_{\mathrm{num}}$, then define

$$u_i(\mathtt{c}) \doteqdot -\mathrm{rk}(\mathtt{c}) \cdot \mathtt{h}_X^i + \chi(\mathtt{c} \otimes \mathtt{h}_X^i) \cdot [\mathcal{O}_x]$$

4.2.2 Construction of the line bundle

From now on we fix $\dim X = 2$ and a numerical class $\mathtt{c} \in K(X)_{\mathrm{num}}$. Denote by $\mathsf{P} \doteqdot \mathsf{P}_{\mathtt{c}}$ the associated Hilbert polynomial. Recall that, since $\dim X = 2$, numerical classes are uniquely determined by their rank and by their first and second Chern classes. Let $(r, \mathsf{d}, c_2) \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$ the triple which determines $\mathtt{c}$. Let $\delta(m) = \overline{\delta} \cdot m + \delta_0$ the fixed stability polynomial.

Let $\mathtt{S}_{\overline{\delta}}^{\text{slope-}\varepsilon\text{-}ss}(\mathsf{P}, \underline{\mathfrak{t}})$ be the family of slope ε-semistable (with respect to δ) decorated sheaves of type $\underline{\mathfrak{t}} = (a, b, \mathsf{N})$ with Hilbert polynomial P. Recall that the family $\mathtt{S}_{\overline{\delta}}^{\text{slope-}\varepsilon\text{-}ss}(\mathsf{P}, \underline{\mathfrak{t}})$ is bounded (see Lemma 63) and therefore there exists an integer m_0 such that for any $m \geq m_0$, any slope ε-semistable decorated bundle $(\mathcal{E}, \varphi)$ is m-regular, and so

- $\mathcal{E}(m)$ is globally generated;
- $H^i(X, \mathcal{E}(m - i)) = 0$ for any $i > 1$;
- $h^0(X, \mathcal{E}(m)) \doteqdot \dim \big(H^0(X, \mathcal{E}(m))\big) = \mathsf{P}(m)$.

Let H be a vector space of dimension $\mathsf{P}(m) = \dim H^0(X, \mathcal{E}(m))$ and define $\mathcal{H} \doteqdot H \otimes \mathcal{O}_X(-m)$. Let $\mathsf{Q}_{\mathrm{uot}}(\mathcal{H}, \mathsf{P})$, $\mathbb{P} = \mathbb{P}(\mathrm{Hom}(H_{a,b}, H^0(X, \mathsf{N}(am)))^\vee)$ and $\mathsf{Q}_{\mathrm{uot}}(\mathcal{H}, \mathsf{P}, \mathsf{N})$ as in Section 4.1. We define $\check{\mathtt{R}}_{\overline{\delta}}^{\text{slope-}\varepsilon\text{-}ss}$ as the locally closed subscheme of the scheme $\mathsf{Q}_{\mathrm{uot}}(\mathcal{H}, \mathsf{P}, \mathsf{N})$ formed by the pairs

$$([\mathtt{q} \colon H \otimes \mathcal{O}_X(-m) \to \mathcal{E}], [\mathtt{f} \colon H_{a,b} \otimes \mathcal{O}_X(-am) \to \mathsf{N})])$$

such that $(\mathcal{E},\varphi)\in \mathtt{S}_{\overline{\delta}}^{\text{slope-}\varepsilon\text{-}ss}$ (where φ is the morphism induced by $\mathtt{f}$) and $\mathtt{q}$ induces an isomorphism $H\to H^0(X,\mathcal{E}(m))$. Since $\mathtt{S}_{\overline{\delta}}^{\varepsilon\text{-}ss}\subset \mathtt{S}_{\overline{\delta}}^{\text{slope-}\varepsilon\text{-}ss}$, $\check{\mathtt{R}}_{\overline{\delta}}^{\text{slope-}\varepsilon\text{-}ss}$ contains a subset $\mathtt{R}_{\overline{\delta}}^{\varepsilon\text{-}ss}$ consisting of ε-semistable pairs $(\mathcal{E},\varphi)$, and it is known that $\mathtt{R}_{\overline{\delta}}^{\varepsilon\text{-}ss}$ is open in $\check{\mathtt{R}}_{\overline{\delta}}^{\text{slope-}\varepsilon\text{-}ss}$. We denote by $\mathtt{R}_{\overline{\delta}}^{\text{slope-}\varepsilon\text{-}ss}$ the closure of $\mathtt{R}_{\overline{\delta}}^{\varepsilon\text{-}ss}$ in $\check{\mathtt{R}}_{\overline{\delta}}^{\text{slope-}\varepsilon\text{-}ss}$.

In order to construct the U-D compactification for decorated sheaves we need a line bundle with "enough" $Sl(H)$-invariant sections. In order to define such a line bundle we need first to recall the notion of universal quotient.

On $\mathsf{Q}_{\text{uot}}(\mathcal{H},\mathsf{P})\times X$ there is a universal quotient $\widetilde{\mathsf{E}}^{\mathrm{u}}$ and a morphism

$$\mathcal{O}_{\mathsf{Q}_{\text{uot}}}\boxtimes\mathcal{H}\xrightarrow{\widetilde{\mathrm{q}}^{\mathrm{u}}}\widetilde{\mathsf{E}}^{\mathrm{u}}.$$

Let E^{u} the restriction to $\mathsf{Q}_{\text{uot}}(\mathcal{H},\mathsf{P},\mathsf{N})\times X$ of the pullback of $\widetilde{\mathsf{E}}^{\mathrm{u}}$ via the projection $\mathsf{Q}_{\text{uot}}(\mathcal{H},\mathsf{P})\times\mathbb{P}\times X\to\mathsf{Q}_{\text{uot}}(\mathcal{H},\mathsf{P})\times X$, namely

$$\begin{array}{ccccc}
\mathsf{E}^{\mathrm{u}}\doteqdot i^*\pi^*\widetilde{\mathsf{E}}^{\mathrm{u}} & & \pi^*\widetilde{\mathsf{E}}^{\mathrm{u}} & & \mathcal{O}_{\mathsf{Q}_{\text{uot}}}\boxtimes\mathcal{H}\twoheadrightarrow\widetilde{\mathsf{E}}^{\mathrm{u}}\\
\downarrow & & \downarrow & & \downarrow\\
\mathsf{Q}_{\text{uot}}(\mathcal{H},\mathsf{P},\mathsf{N})\times X & \xhookrightarrow{i} & \mathsf{Q}_{\text{uot}}(\mathcal{H},\mathsf{P})\times\mathbb{P}\times X & \xrightarrow{\pi} & \mathsf{Q}_{\text{uot}}(\mathcal{H},\mathsf{P})\times X
\end{array}$$

There is a locally defined morphism $\Phi_{\mathsf{E}^{\mathrm{u}}}\colon \mathsf{E}^{\mathrm{u}}_{a,b}\to\pi_X^*\mathsf{N}$ where

$$\pi_X\colon \mathsf{Q}_{\text{uot}}(\mathcal{H},\mathsf{P},\mathsf{N})\times\mathbb{P}\times X\to X$$

is the natural projection.

According to notation of Section 4.2.1 and Section 4.1, let

$$\mathcal{L}(n_1,n_2)\doteqdot\left(\lambda_{\widetilde{\mathsf{E}}^{\mathrm{u}}}(u_1)^{\otimes n_1}\boxtimes\mathcal{O}_{\mathbb{P}}(n_2)\right)_{|_{\mathtt{R}_{\overline{\delta}}^{\text{slope-}\varepsilon\text{-}ss}}}$$

i.e., denoting by i, p_1 and p_2 the following morphisms

$$\begin{array}{ccccc}
\mathtt{R}_{\overline{\delta}}^{\text{slope-}\varepsilon\text{-}ss}\times X & \xhookrightarrow{i} & \mathsf{Q}_{\text{uot}}(\mathcal{H},\mathsf{P})\times\mathbb{P}\times X & \xrightarrow{p_2} & \mathbb{P}\\
& & \downarrow{\scriptstyle p_1} & & \\
& & \mathsf{Q}_{\text{uot}}(\mathcal{H},\mathsf{P}) & &
\end{array}$$

then

$$\mathcal{L}(n_1,n_2)=i^*\left(p_1^*\lambda_{\widetilde{\mathsf{E}}^{\mathrm{u}}}(u_1)^{\otimes n_1}\otimes p_2^*\mathcal{O}_{\mathbb{P}}(n_2)\right)_{|_{\mathtt{R}_{\overline{\delta}}^{\text{slope-}\varepsilon\text{-}ss}}}$$

where we set, as in Section 4.1,

$$\frac{n_2}{n_1}\doteqdot a\delta(m)\frac{\mathsf{P}^{\varepsilon}(l)}{\mathsf{P}^{\varepsilon}(m)}-a\delta(l) \tag{4.8}$$

for $m\geq m_0$ and l big enough.

Proposition 70. *For $\nu \gg 0$ the line bundle $\mathcal{L}(n_1, n_2)^{\otimes \nu}$ on $\mathsf{R}_{\overline{\delta}}^{slope\text{-}\varepsilon\text{-}ss}$ is generated by its $Sl(W)$-invariant sections.*

Proof. We readapt to our case the techniques used in [4]. The main idea is the following: if S is a scheme parameterizing a flat family $(\mathsf{E}, \varphi_{\mathsf{E}})$ of slope ε-semistable decorated sheaves on X, and if $C \in |\mathcal{O}_X(\mathsf{a})|$ is a general smooth curve with a big enough (see Chapter 3), then restricting $(\mathsf{E}, \varphi_{\mathsf{E}})$ to $S \times C$ produces a family of generically slope ε-semistable decorated sheaves over C and therefore a rational map $S \dashrightarrow M_C$ from S to the moduli space M_C of semistable decorated sheaves on the curve C. The ample line bundle $\mathcal{L}_0(n_1, n_2)$ on M_C pullbacks to a power of $\mathcal{L}(n_1, n_2)$ and in this manner we can produce sections in the latter line bundle.

In detail, let $S = \mathsf{R}_{\overline{\delta}}^{\text{slope-}\varepsilon\text{-}ss}$. The pullback of the universal quotient $(\mathsf{E}^{\mathrm{u}}, \varPhi_{\mathsf{E}^{\mathrm{u}}})$ of $Y \doteqdot \mathsf{Q}\text{uot}(\mathcal{H}, \mathsf{P}, \mathsf{N})$ to S gives a flat family $(\mathsf{E}, \varphi_{\mathsf{E}})$ of slope ε-semistable decorated sheaves on X with numerical class $\mathsf{c} \doteqdot (r, \mathsf{d}, c_2)$, Hilbert polynomial P and decoration of type $\underline{\mathsf{t}} = (a, b, \mathsf{N})$.

For $C \in U_{\mathsf{a}} \subset |\mathcal{O}_X(\mathsf{a})|$ a general curve, the restriction of the family to $S \times C$ gives a family $(\mathsf{E}_C, \varphi_{\mathsf{E}C}) \doteqdot (\mathsf{E}, \varphi_{\mathsf{E}})_{|C}$ of decorated sheaves on C. By Theorem 51 the general element of this family is ε-semistable with respect to $\overline{\delta}$. Let

$$i : C \hookrightarrow X$$

be the inclusion, then the class $i^*\mathsf{c}$ is uniquely determined by r and $i^*\mathsf{d} = \mathsf{d}_{|C}$. Let m' be a large positive integer, H_C a vector space of dimension $\mathsf{P}_{i^*\mathsf{c}}(m')$ and $\mathcal{H}_C = H_C \otimes \mathcal{O}_C(-m')$. Let $Q_C \subset \mathsf{Q}\text{uot}_C(\mathcal{H}_C, \mathsf{P}_{i^*\mathsf{c}})$ be the closed subset parameterizing quotients of $\mathcal{H}_C$ with determinant $\mathsf{d}_{|C}$. Let us denote by $(\widetilde{\mathsf{E}}^{\mathrm{u}}_C, \varPhi_{\mathsf{E}^{\mathrm{u}}C})$ the universal quotient $\mathcal{O}_{Q_C} \boxtimes \mathcal{H}' \to \widetilde{\mathsf{E}}^{\mathrm{u}}_C$ and with $\mathbb{P}_C \doteqdot \mathbb{P}(\mathrm{Hom}[(H_C)_{a,b}, H^0(C, \mathsf{N}_C(am'))]^{\vee})$ so that, as usual, a point of $\mathbb{P}$ corresponds to a morphism $(\mathcal{H}_C)_{a,b} = (\mathcal{H}_C^{\otimes a})^{\oplus b} \to \mathsf{N}_{|C}$. Consider the subscheme

$$Y_C \doteqdot \mathsf{Q}\text{uot}_C(\mathcal{H}_C, \mathsf{P}_{i^*\mathsf{c}}, \mathsf{N}_{|C}) \subset Q_C \times \mathbb{P}_C \subset \mathsf{Q}\text{uot}_C(\mathcal{H}_C, \mathsf{P}_{i^*\mathsf{c}}) \times \mathbb{P}_C$$

defined similarly to the scheme Y above (see also 4.2). Denote by $p_{1,C} : Y_C \to Q_C$ and $p_{2,C} : Y_C \to \mathbb{P}_C$ the projections, recall that $\mathtt{H}_X$ denote the divisor in X associated with the (fixed) ample line bundles $\mathcal{O}_X(1)$ and let $\deg C = C \cdot \mathtt{H}_X$. Consider the line bundle

$$\begin{aligned} \mathcal{L}'_0(n_1, \mathsf{a}n_2) &\doteqdot p_{1,C}^*(\lambda_{\widetilde{\mathsf{E}}^{\mathrm{u}}_C}(u_0(i^*\mathsf{c})))^{\otimes n_1 \deg C} \otimes p_{2,C}^*\mathcal{O}_{\mathbb{P}_C}(\mathsf{a}n_2) \\ &\doteqdot (\lambda_{\widetilde{\mathsf{E}}^{\mathrm{u}}_C}(u_0(i^*\mathsf{c})))^{\otimes n_1 \deg C} \boxtimes \mathcal{O}_{\mathbb{P}_C}(\mathsf{a}n_2) \end{aligned} \quad (4.9)$$

where n_1 and n_2 are defined in equation 4.8.

In analogy with Proposition 3.5 in [4] one can prove:

Lemma 71. *Given a point* $([\mathtt{q}_C : \mathcal{H}_C \to E], [\mathtt{f}_C : (\mathcal{H}_C)_{a,b} \to \mathsf{N}_{|C}(am')]) \in Y_C$ *the following assertions holds:*

1. (E, φ) *is* ε*-semistable and* $H_C \to H^0(C, E(m'))$ *is an isomorphism, where we denoted by* φ *the morphism induced by* $[\mathtt{f}_C]$.
2. $([\mathtt{q}_C], [\mathtt{f}_C])$ *is a semistable point in* Y_C *for the action of* $Sl(H_C)$ *with respect to the canonical linearization of* $\mathcal{L}_0'(n_1, \mathsf{a}n_2)$.
3. *There is an integer* ν *and a* $Sl(H_C)$ *invariant section* σ *of* $\mathcal{L}_0'(n_1, \mathsf{a}n_2)^\nu$ *such that* $\sigma([\mathtt{q}_C], [\mathtt{f}_C]) \neq 0$.
4. *Let* $s \in S = \mathsf{S}_{\overline{\delta}}^{slope\text{-}\varepsilon\text{-}ss}$ *a point such that* $(\mathsf{E}_{s|_C}, \varphi_{\mathsf{E}_{s|_C}} : (\mathsf{E}_{s|_C})_{a,b} \to \mathsf{N}_{|_C})$ *is* ε*-semistable with respect to* $\overline{\delta}_C$. *There is a* $Sl(H)$*-invariant section* $\overline{\sigma} \in H^0(\mathsf{R}_{\overline{\delta}}^{slope\text{-}\varepsilon\text{-}ss}, \mathcal{L}(n_1, n_2)^{\mathsf{a}\nu})^{Sl(H)}$ *such that* $\overline{\sigma}(s) \neq 0$.

Now Proposition 70 follows from the fourth point of Lemma 71. ♦

4.2.3 Construction of the Uhlenbeck-Donaldson compactification

By Proposition 70, the sheaf $\mathcal{L}(n_1, n_2)^\nu$ is generated by its invariant sections. Let $W \subset W^\nu \doteqdot H^0(\mathsf{R}_{\overline{\delta}}^{\text{slope-}\varepsilon\text{-}ss}, \mathcal{L}(n_1, n_2)^\nu)^{\mathrm{Sl}(H)}$ be a finite dimensional vector space which generates $\mathcal{L}(n_1, n_2)^\nu$. Let $w_1, \dots w_t$ be a basis for W and let

$$
\begin{aligned}
j_W : \mathsf{R}_{\overline{\delta}}^{\text{slope-}\varepsilon\text{-}ss} &\to \mathbb{P}(W) \\
s &\mapsto [w_1(s), \dots, w_t(s)]
\end{aligned}
$$

be the induced $\mathrm{Sl}(\mathsf{P}_{\mathsf{c}}(m))$-invariant morphism.

The following Lemma is a generalization of a classical result by Langton [18].

Lemma 72. *Let* $(R, \mathfrak{m})$ *be a discrete valuation ring with residue field* k *and quotient field* K *and let* X *be a smooth projective surface over* k. *Let* $(\mathcal{E}, \varphi)$ *be a flat family of decorated sheaf over* X *such that* $\mathcal{E}_K = \mathcal{E} \otimes_R \mathsf{E}$, *with the induced decoration* φ_K, *is a slope* ε*-semistable decorated sheaf. Then there is a decorated sheaf* $(\mathsf{E}, \varphi_{\mathsf{E}})$ *such that* $(\mathsf{E}_K, \varphi_{\mathsf{E}K}) = (\mathcal{E}_K, \varphi_K)$ *and* $(\mathsf{E}_k, \varphi_{\mathsf{E}k})$ *is slope* ε*-semistable.*

Proof. We have already noticed (see Section 3.2.3) that there is a one-to-one correspondence between decorated subsheaves $(\mathcal{F}, \varphi_{|_\mathcal{F}})$ of a fixed decorated sheaf $(\mathcal{E}, \varphi)$ and its quotients $\mathcal{Q}$ (without morphisms). Moreover, defining $\varepsilon(\mathcal{Q}) \doteqdot \varepsilon(\mathcal{E}, \varphi) - \varepsilon(\mathcal{F}, \varphi_{|_\mathcal{F}})$ and $\mathsf{P}_\mathcal{Q}^\varepsilon \doteqdot \mathsf{P}_\mathcal{Q} - a\delta\varepsilon(\mathcal{Q})$, one immediately has that $\mathsf{P}_{(\mathcal{E},\varphi)} = \mathsf{P}_{(\mathcal{F},\varphi_{|_\mathcal{F}})} + \mathsf{P}_\mathcal{Q}^\varepsilon$, i.e., the decorated Hilbert polynomial is additive on short exact sequences. Since the proof of Proposition 4.2 in [4] does not depend on the morphisms involved in the proof but only on their behavior ($\varepsilon = 0$ or $\varepsilon \neq 0$) and on the additivity of the Hilbert polynomials on short exact sequences of framed sheaves, the proof in this case is the same of Proposition 4.2 in [4]. ♦

The above result immediately implies that the image of the morphism j_W is proper (see Proposition 8.2.5 in [15]) and so the following results holds:

Proposition 73. *$M_W \doteqdot j_W(\mathsf{R}_{\overline{\delta}}^{slope\text{-}\varepsilon\text{-}ss})$ is a projective scheme.*

By using Proposition 73 and proceeding as in [15], Proposition 8.2.6, we can prove the following result:

Proposition 74. *There is an integer $N > 0$ such that $\bigoplus_{l\geq 0} W_{lN}$ is a finitely generated graded ring.*

Let N be as in the above proposition. Define the moduli space of slope ε-semistable decorated sheaves as

$$\mathcal{M}^{\text{slope}} \doteqdot \mathcal{M}_{\overline{\delta}}^{\text{slope-}\varepsilon\text{-}ss}(\mathsf{c}, \underline{\mathsf{t}}) \doteqdot \text{Proj}\left(\bigoplus_{k\geq 0} H^0(\mathsf{R}_{\overline{\delta}}^{\text{slope-}\varepsilon\text{-}ss}, \mathcal{L}(n_1, n_2)^{\otimes kN})^{\mathrm{Sl}(\mathsf{P}(m))}\right),$$

and let $\pi : \mathsf{R}_{\overline{\delta}}^{\text{slope-}\varepsilon\text{-}ss} \to \mathcal{M}^{\text{slope}}$ be the canonically induced morphism.

Let $\mathcal{M} \doteqdot \mathcal{M}_{\overline{\delta}}^{\varepsilon\text{-}ss}(\mathsf{c}, \underline{\mathsf{t}})$ be the moduli space of ε-semistable decorated sheaves on X of type $\underline{\mathsf{t}} = (a, b, \mathsf{N})$ introduced in Section 4.1 and let $\mathcal{M}^{\text{slope-}\varepsilon\text{-}s}$ be the open subset of $\mathcal{M}$ corresponding to slope ε-stable pairs (E, φ) with E locally free. Then, following [4], we can prove that there is a regular morphism of moduli spaces $\gamma : \mathcal{M} \to \mathcal{M}^{\text{slope}}$ such that $\gamma|_{\mathcal{M}^{\text{slope-}\varepsilon\text{-}s}} : \mathcal{M}^{\text{slope-}\varepsilon\text{-}s} \to \mathcal{M}^{\text{slope}}$ is an embedding. By analogy with the non-decorated case we define the **Uhlenbeck-Donaldson compactification** of $\mathcal{M}^{\text{slope-}\varepsilon\text{-}s}$ as the closure of $\gamma(\mathcal{M}^{\text{slope-}\varepsilon\text{-}s})$ inside $\mathcal{M}^{\text{slope}}$.

We skech the costruction of the morphism γ. Let $(\mathsf{E}, \varphi_{\mathsf{E}})$ be a flat family of decorated sheaves parametrised by S (see Definition (60)) and let $\pi_X : X \times S \to X$ and $\pi_S : X \times S \to S$ be the projections. Consider the schemes

$$\widetilde{S} \doteqdot \underline{\text{Isom}}(H \otimes \mathcal{O}_S, \pi_{S*}\mathsf{E}) \xrightarrow{\tau} S$$
$$\breve{S} \doteqdot \underline{\text{Isom}}(H_{a,b} \otimes \mathcal{O}_S, \pi_{S*}\mathsf{E}_{a,b}) \xrightarrow{\tau_1} S$$

together with the projections $\pi_{\widetilde{S}} : X \times \widetilde{S} \to \widetilde{S}$ and $\pi_{\breve{S}} : X \times \breve{S} \to \breve{S}$. Note that there is an injective morphism $i : \widetilde{S} \to \breve{S}$ such that $\tau_1 \circ i = \tau$. Recall that an isomorphism between two decorated sheaves $(\mathcal{E}, \varphi)$ and $(\mathcal{E}', \varphi')$ is a pair $(g, \lambda) \in \text{Isom}(\mathcal{E}, \mathcal{E}') \times \mathbb{C}^*$ such that $\lambda \circ \varphi = \varphi' \circ g_{a,b}$. The morphism

$\varphi_{\mathsf{E}} : \mathsf{E}_{a,b} \to \mathcal{O}_S \boxtimes \mathsf{N}$ induces a morphism $\varphi_{\mathsf{E}} : \mathcal{O}_{X\times S} \to \underline{\mathrm{Hom}}(\mathsf{E}_{a,b}, \mathcal{O}_S \boxtimes \mathsf{N})$ and therefore a morphism $\varphi_{\mathsf{E}} : \mathcal{O}_S \to \pi_{S*}\underline{\mathrm{Hom}}(\mathsf{E}_{a,b}, \mathcal{O}_S \boxtimes \mathsf{N})$, moreover, if $\lambda \in \mathcal{O}_S^*$ than the pair $(\mathsf{E}, \lambda \cdot \varphi_{\mathsf{E}})$ is isomorphic to the pair $(\mathsf{E}, \varphi_{\mathsf{E}})$. Since we are interested in isomorphism classes of families of decorated sheaves, for this section, accordingly to the notation in [4], we will say that a family of decorated sheaves parametrized by S is a triple $(\mathsf{E}, \mathsf{L}, \varphi_{\mathsf{E}})$ where E is as before, L is a line bundle over S and

$$\varphi_{\mathsf{E}} : \mathsf{L} \longrightarrow \pi_{S*}\underline{\mathrm{Hom}}(\mathsf{E}_{a,b}, \mathcal{O}_S \boxtimes \mathsf{N}).$$

Let $(\widetilde{\mathsf{E}}, \widetilde{\mathsf{L}}, \varphi_{\widetilde{\mathsf{E}}})$ and $(\breve{\mathsf{E}}, \breve{\mathsf{L}}, \varphi_{\breve{\mathsf{E}}})$ be the lifted families over $\widetilde{S}$ and $\breve{S}$ respectively, i.d.,

$$\begin{array}{ccccccc}
(id\times\tau)^*\mathsf{E} \doteqdot \widetilde{\mathsf{E}} & \longrightarrow & (id\times\tau_1)^*\mathsf{E} \doteqdot \breve{\mathsf{E}} & \longrightarrow & \mathsf{E} & & \\
\downarrow & & \downarrow & & \downarrow & & \\
X\times\widetilde{S} & \xrightarrow{id\times i} & X\times\breve{S} & \xrightarrow{id\times\tau_1} & X\times S & \xrightarrow{\pi_X} & X \\
\downarrow{\scriptstyle \pi_{\widetilde{S}}} & & \downarrow{\scriptstyle \pi_{\breve{S}}} & & \downarrow{\scriptstyle \pi_S} & & \\
\widetilde{S} & \xrightarrow{i} & \breve{S} & \xrightarrow{\tau_1} & S & &
\end{array}
\qquad \widetilde{S} \xrightarrow{\tau} S$$

where $\varphi_{\widetilde{\mathsf{E}}} = (id\times\tau)^*\varphi_{\mathsf{E}}$ and $\varphi_{\breve{\mathsf{E}}} = (id\times\tau_1)^*\varphi_{\mathsf{E}}$. Denoting by $\widetilde{\mathsf{L}} = (id\times\tau)^*\mathsf{L}$, $\widetilde{\mathsf{N}} = (id\times\tau)^*(\mathcal{O}_S \boxtimes \mathsf{N})$, $\breve{\mathsf{L}} = (id\times\tau_1)^*\mathsf{L}$ and $\breve{\mathsf{N}} = (id\times\tau_1)^*(\mathcal{O}_S \boxtimes \mathsf{N})$, then

$$\varphi_{\widetilde{\mathsf{E}}} : \widetilde{\mathsf{L}} \to \pi_{\widetilde{S}*}\underline{\mathrm{Hom}}(\widetilde{\mathsf{E}}_{a,b}, \widetilde{\mathsf{N}})$$
$$\varphi_{\breve{\mathsf{E}}} : \breve{\mathsf{L}} \to \pi_{\breve{S}*}\underline{\mathrm{Hom}}(\breve{\mathsf{E}}_{a,b}, \breve{\mathsf{N}}).$$

Let the morphism $\widetilde{\varphi_{\mathsf{E}}}$ be the following composition:

$$\begin{array}{ccc}
\pi_S^*\mathsf{L} \otimes \mathsf{E}_{a,b} & \xrightarrow{\pi_S^*\varphi_{\mathsf{E}}\otimes id} & \pi_S^*\pi_{S*}\underline{\mathrm{Hom}}(\mathsf{E}_{a,b}, \mathcal{O}_S \boxtimes \mathsf{N}) \otimes \mathsf{E}_{a,b} \\
 & & \downarrow{\scriptstyle \mathrm{ev}} \\
 & & \underline{\mathrm{Hom}}(\mathsf{E}_{a,b}, \mathcal{O}_S \boxtimes \mathsf{N}) \otimes \mathsf{E}_{a,b} \\
 & \searrow{\scriptstyle \widetilde{\varphi_{\mathsf{E}}}} & \downarrow{\scriptstyle \mathrm{can}} \\
 & & \mathcal{O}_S \boxtimes \mathsf{N}.
\end{array}$$

Applying the functor π_{S*} to the morphism $\widetilde{\varphi_{\mathsf{E}}}$ we obtain a morphism $\widehat{\varphi_{\mathsf{E}}} : \mathsf{L} \otimes \pi_{S*}\mathsf{E}_{a,b} \to H^0(X, \mathsf{N}) \otimes \mathcal{O}_S$. Let taut : $H_{a,b} \otimes \mathcal{O}_{\breve{S}} \xrightarrow{\sim} \tau_1^*\pi_{S*}\mathsf{E}_{a,b} = \pi_{\breve{S}*}\breve{\mathsf{E}}$ be the tautological morphism and consider the composition

$$\breve{\mathsf{L}} \otimes H_{a,b} \otimes \mathcal{O}_{\breve{S}} \xrightarrow{id\otimes\mathrm{taut}} \breve{\mathsf{L}} \otimes \tau_1^*\pi_{S*}\mathsf{E}_{a,b} \xrightarrow{\tau_1^*\widehat{\varphi_{\mathsf{E}}}} H^0(X, \mathsf{N}) \otimes \mathcal{O}_{\breve{S}}.$$

Finally we pullback this morphism over $\widetilde{S}$ via the map $i : \widetilde{S} \to \breve{S}$ and so we get a morphism $\widetilde{\mathsf{L}} \otimes H_{a,b} \otimes \mathcal{O}_{\widetilde{S}} \longrightarrow H^0(X, \mathsf{N}) \otimes \mathcal{O}_{\widetilde{S}}$ and therefore a morphism

$$f'_{\widetilde{\mathsf{E}}} : \widetilde{\mathsf{L}} \to \underline{\mathrm{Hom}}(H_{a,b}, H^0(X, \mathsf{N})) \otimes \mathcal{O}_{\widetilde{S}}.$$

By the universal property of the projective space $\mathbb{P}$ the morphism $f_{\widetilde{\mathsf{E}}}$ defines a morphism $f_{\widetilde{\mathsf{E}}} : \widetilde{S} \to \mathbb{P}$.

Now we begin the construction of γ and explain in which sense $\mathcal{M}^{\mathrm{slope}}$ is the moduli space of slope ε-semistable sheaves. Let $\underline{\mathfrak{M}}$, respectively, $\underline{\breve{\mathfrak{M}}}^{\mathrm{slope}}$ denote the functor which associates with S the set of isomorphism classes of S-flat families of ε-semistable, respectively, slope ε-semistable decorated sheaves of class c on X. Consider an open subfunctor $\underline{\mathfrak{M}}^{\mathrm{slope}}$ of $\underline{\breve{\mathfrak{M}}}^{\mathrm{slope}}$ which associates with S the set of isomorphism classes of those families $[(\mathsf{E}, \mathsf{L}, \varphi_{\mathsf{E}})] \in \underline{\breve{\mathfrak{M}}}^{\mathrm{slope}}(S)$ for which there exists a dense open subset S' of S such that $[(\mathsf{E}, \mathsf{L}, \varphi_{\mathsf{E}})_{|_{X \times S'}}] \in \underline{\mathfrak{M}}(S')$. Clearly $\underline{\mathfrak{M}}$ is an open subfunctor of $\underline{\mathfrak{M}}^{\mathrm{slope}}$.

For any scheme S and any family $[(\mathsf{E}, \mathsf{L}, \varphi_{\mathsf{E}})] \in \underline{\mathfrak{M}}^{\mathrm{slope}}(S)$ the principal $Gl(H)$-bundle $\tau : \widetilde{S} \to S$, by the universality of the Quot-scheme $\mathsf{Q}_{\mathrm{uot}}(\mathcal{H}, \mathsf{P}_{\mathsf{c}})$, defines a morphism $\Psi_{\widetilde{\mathsf{E}}} : \widetilde{S} \to \mathsf{Q}_{\mathrm{uot}}(\mathcal{H}, \mathsf{P}_{\mathsf{c}})$ and hence a morphism $\Phi^{\mathrm{slope}}_{\widetilde{\mathsf{E}}} = (\Psi_{\widetilde{\mathsf{E}}}, f_{\widetilde{\mathsf{E}}}) : \widetilde{S} \to \mathsf{R}^{\mathrm{slope}} \doteqdot \mathsf{R}_{\delta}^{\mathrm{slope}\text{-}\varepsilon\text{-}ss}$. This morphism is $Gl(H)$-invariant, and $\tau : \widetilde{S} \to S$ is a categorical quotient, so that there is a morphism $\Phi^{\mathrm{slope}}_{\mathsf{E}} : S \to \mathcal{M}^{\mathrm{slope}}$ making the following diagram commutative:

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\Phi^{\mathrm{slope}}_{\widetilde{\mathsf{E}}}} & \mathsf{R}^{\mathrm{slope}} \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\pi} \\ S & \xrightarrow[\Phi^{\mathrm{slope}}_{\mathsf{E}}]{} & \mathcal{M}^{\mathrm{slope}}. \end{array}$$

We thus obtain a natural transformation of functors $\Phi^{\mathrm{slope}} : \underline{\mathfrak{M}}^{\mathrm{slope}} \to \mathrm{Mor}(\cdot, \mathcal{M}^{\mathrm{slope}})$ given by $\Phi^{\mathrm{slope}}(S) : \underline{\mathfrak{M}}^{\mathrm{slope}}(S) \to \mathrm{Mor}(S, \mathcal{M}^{\mathrm{slope}})$, which to a triple $[(\mathsf{E}, \mathsf{L}, \varphi_{\mathsf{E}})]$ associates $\Phi^{\mathrm{slope}}_{\mathsf{E}}$.
We recall that the moduli space $\mathcal{M}$ corepresents the moduli functor $\underline{\mathfrak{M}}$ (Theorem 66) and so we have a natural transformation of functors $\Phi : \underline{\mathfrak{M}} \to \mathrm{Mor}(\cdot, \mathcal{M})$, $\Phi(S) : \underline{\mathfrak{M}}(S) \to \mathrm{Mor}(S, \mathcal{M})$, $[(\mathsf{E}, \mathsf{L}, \varphi_{\mathsf{E}})] \mapsto \Phi^{\mathrm{slope}}_{\mathsf{E}}$. Therefore we

have the following commutative diagram

$$\begin{array}{ccccc} & & \Phi^{\text{slope}}_{\widetilde{\mathsf{E}}} & & \\ \widetilde{S} & \xrightarrow{\Phi_{\widetilde{\mathsf{E}}}} & \mathtt{R}^{\varepsilon\text{-}ss} & \hookrightarrow & \mathtt{R}^{\text{slope}} \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ S & \xrightarrow{\Phi_{\mathsf{E}}} & \mathcal{M} & \overset{\gamma}{\dashrightarrow} & \mathcal{M}^{\text{slope}}. \\ & & \Phi^{\text{slope}}_{\mathsf{E}} & & \end{array}$$

Since $\pi : \mathtt{R}^{\varepsilon\text{-}ss} \to \mathcal{M}$ is a categorical quotient, it follows that there exists a morphism $\gamma : \mathcal{M} \to \mathcal{M}^{\text{slope}}$ such that $\Phi^{\text{slope}}_{\mathsf{E}} = \gamma \circ \Phi_{\mathsf{E}}$. Note that, by costruction, the morphism γ is dominant and projective, hence it is surjective.

Chapter 5

Reduction of the semistability condition

Let $(\mathcal{E}, \varphi)$ be a decorated (torsion free) sheaf of type (a, b, c, N) and let $\mathcal{E}^\bullet$ be a filtration of $\mathcal{E}$ indexed by $\mathtt{I}$. We construct a tensor $M_{\mathtt{I}}(\mathcal{E}^\bullet, \varphi) = (m_{i_1 \ldots i_a})_{i_1, \ldots, i_a \in \overline{\mathtt{I}}}$, associated with the given filtration and to the decoration morphism φ. Let $\Sigma \doteqdot \{\sigma : \mathtt{I}^{\times a} \to \mathtt{I}^{\times a} \,|\, \sigma \text{ is a permutation}\}$, then:

$$m_{i_1 \ldots i_a} \doteqdot \begin{cases} 1 \text{ if } \varphi|_{(\mathcal{E}_{i_1} \diamond \cdots \diamond \mathcal{E}_{i_a})^{\oplus b}} \not\equiv 0 \\ 0 \text{ otherwise.} \end{cases}$$

Note that

- $M_{\mathtt{I}}(\mathcal{E}^\bullet, \varphi)$ is symmetric, i.e., $m_{i_1 \ldots i_a} = m_{\sigma(i_1) \ldots \sigma(i_a)}$ for any permutation $\sigma \in \Sigma$ of a-terms;
- It is easy to see that

  $$\begin{aligned} \mu(\mathcal{E}^\bullet, \underline{\alpha}; \varphi) &= - \min_{i_1, \ldots, i_a \in \overline{\mathtt{I}}} \{\gamma_{\mathtt{I}}^{(i_1)} + \cdots + \gamma_{\mathtt{I}}^{(i_a)} \,|\, m_{i_1 \ldots i_a} \neq 0\} \\ &= r \max_{i_1, \ldots, i_a \in \overline{\mathtt{I}}} \{R_{\mathtt{I}}(i_1) + \cdots + R_{\mathtt{I}}(i_a) \,|\, m_{i_1 \ldots i_a} \neq 0\} - a \sum_{i \in \mathtt{I}} \alpha_i r_i, \end{aligned}$$

  where $R_{\mathtt{I}}(i_j)$ is defined in (2.10) and, as usual, we put $r_i = \mathrm{rk}(\mathcal{E}_i)$.
- Define the following (partial) ordering on the set $\{(i_1, \ldots, i_a) \,|\, \forall s \;\; i_s \in \mathtt{I} \text{ and } i_1 \leq \cdots \leq i_a\}$ of ordered a-tuple: we will say that $(i_1, \ldots, i_a) \preccurlyeq (j_1, \ldots, j_a)$ if and only if $i_s \leq j_s$ for any $s = 1, \ldots, a$.
  Then, if $m_{i_1 \ldots i_a} = 1$ for a certain a-tupla $(i_1, \ldots, i_a)$, it is easy to see that $m_{j_1 \ldots j_a} = 1$ for any $(j_1 \ldots j_a) \succcurlyeq (i_1, \ldots, i_a)$. Conversely, if $m_{i_1 \ldots i_a} = 0$, then $m_{j_1 \ldots j_a} = 0$ for any $(j_1 \ldots j_a) \preccurlyeq (i_1 \ldots i_a)$.

Therefore, in order to calculate $\mu(\mathcal{E}^\bullet, \underline{\alpha}; \varphi)$, it is enough to know the vector $\gamma_{\mathtt{I}}$ and the tensor $M_{\mathtt{I}}(\mathcal{E}^\bullet, \varphi)$. This lead us to the following remark:

suppose that φ and φ' are two "decorations" of the same sheaf $\mathcal{E}$ such that for any filtration $\mathcal{E}^\bullet$ of $\mathcal{E}$ one has that $M_{\mathtt{I}}(\mathcal{E}^\bullet, \varphi) = M_{\mathtt{I}}(\mathcal{E}^\bullet, \varphi')$. Then the decorated sheaf $(\mathcal{E}, \varphi)$ is (semi)stable if and only if $(\mathcal{E}, \varphi')$ is (semi)stable.

Thanks to the previous considerations, in order to simplify the semistability condition, one can consider only to $M_{\mathtt{I}}(\mathcal{E}^\bullet, \varphi)$ and $\gamma_{\mathtt{I}}$. We want to find an algorithm to "split" the filtration $\mathcal{E}^\bullet$ indexed by $\mathtt{I}$ with weight vector $\underline{\alpha} = (\alpha_i)_{i\in\mathtt{I}}$ in a certain number of weighted subfiltrations indexed by $\mathtt{J}^{(1)}, \ldots, \mathtt{J}^{(t)} \subset \mathtt{I}$ with weight vectors $\underline{\alpha}^{(1)}, \ldots, \underline{\alpha}^{(t)}$ in such a way that:

i) $|\mathtt{J}^{(s)}| \leq a$ for any $s = 1, \ldots, t$;

ii) $\mathtt{J}^{(1)} \cup \cdots \cup \mathtt{J}^{(t)} = \mathtt{I}$;

iii) $\sum_{s=1}^t \alpha_i^{(s)} = \alpha_i$, where is to be understood that $\alpha_i^{(s)} = 0$ if $i \notin \mathtt{J}^{(s)}$;

iv) $\sum_{s=1}^t R_{\mathtt{J}^{(s)}} = R_{\mathtt{I}}$.

In fact, if we manage to do this, from (ii) and (iii) one gets that

$$\sum_{s=1}^{t} \sum_{i\in \mathtt{J}^{(s)}} \alpha_i^{(s)} C_i = \sum_{\mathtt{I}} \alpha_i C_i,$$

where C_i were defined in (2.11), and so, thanks to (iv) and to (2.13),

$$\begin{aligned} P_{\mathtt{I}} + \delta\mu_{\mathtt{I}} &= \sum_{\mathtt{I}} (\alpha_i C_i) + r\delta R_{\mathtt{I}} \\ &= \sum_{s=1}^{t} \left(\sum_{\mathtt{J}^{(s)}} \alpha_i^{(s)} C_i + r\delta R_{\mathtt{J}^{(s)}} \right) \\ &= \sum_{s=1}^{t} \left(P_{\mathtt{J}^{(s)}} + \delta\mu_{\mathtt{J}^{(s)}} \right). \end{aligned}$$

From the above considerations, the positivity of $P_{\mathtt{J}^{(s)}} + \delta\mu_{\mathtt{J}^{(s)}}$, for any $s = 1, \ldots, t$, implies the one of $P_{\mathtt{I}} + \delta\mu_{\mathtt{I}}$ and so, in order to check semistability condition, ones can check it only over weighted filtration of length $\leq a$. We managed to prove this result at least in the case $a = 2$.

5.1 Reduction of the semistability condition for decorated sheaves of type $(2, b, c, \mathsf{N})$

Let $(\mathcal{E}, \varphi)$ be a decorated (torsion free) sheaf of type $(2, b, c, \mathsf{N})$. Replacing $\mathsf{N}' = (\det E)^{\otimes c} \otimes \mathsf{N}$ we can assume, without loss of generality, $c = 0$. The main result states that it is enough to check the semistability condition over (non weighted) filtrations of length 2. In order to prove this result we first need some notation and preliminary results.

Remark 75 (Notation). For convenience's sake we introduce the following notation: if $(\mathcal{E}^\bullet, \underline{\alpha})$ is a weighted filtration as before, indexed by $\mathtt{I} = \{i_1, \ldots, i_t\}$, we define $\overline{\mathtt{I}} = \mathtt{I} \cup \{r\}$, where it is always understood that $\mathcal{E}_r = \mathcal{E}$. Given a filtration $(\mathcal{E}^\bullet, \underline{\alpha})$ indexed by $\mathtt{I}$, we will denote by $\mu_{\mathtt{I}}$ the number $\mu(\mathcal{E}^\bullet, \underline{\alpha}; \varphi)$ if is clear from the context which filtration, weights and morphism we are considering. Moreover if $\mathcal{E}^\bullet$ is a filtration indexed by $\mathtt{I}$ ($\mathcal{F}$ is a subbundle of $\mathcal{E}$) then we denote by r_i, d_i and P_i (resp. $r_{\mathcal{F}}$, $d_{\mathcal{F}}$ and $\mathsf{P}_{\mathcal{F}}$) the rank, the degree and the Hilbert polynomial of $\mathcal{E}_i$ for any $i \in \mathtt{I}$ (resp. of $\mathcal{F}$). Finally, if we write "filtration" instead of "weighted filtration", we mean that all weights are equal to one.

Given a filtration $\mathcal{E}^\bullet$ indexed by $\mathtt{I} = \{i_1, \ldots, i_s\}$ ($i_s \leq r-1$) using local sections we can construct the (symmetric) matrix $M_{\mathtt{I}}(\mathcal{E}^\bullet, \varphi) = (m_{ij})_{ij \in \overline{\mathtt{I}}}$ where

$$m_{ij} = \begin{cases} 1 \text{ if } \varphi_{|\mathcal{E}_i \mathcal{E}_j} \neq 0 \\ 0 \text{ if } \varphi_{|\mathcal{E}_i \mathcal{E}_j} = 0. \end{cases}$$

Proposition 76. *Let* $(0 \subset \mathcal{E}_i \subset \mathcal{E}_j \subset \mathcal{E}\,, \alpha_i, \alpha_j)$ *be a critical weighted filtration of length two, then*

$$-\mu_{\{i,j\}} = 2(\alpha_i r_i + \alpha_j r_j) - r \max\{\alpha_i + \alpha_j, 2\alpha_j\}.$$

Proof. We consider the matrix $M_{\{i,j\}} = (m_{lk})_{l,k \in \overline{\{i,j\}}}$ representing φ with respect to the given filtration. One can check that the only critical case is represented by the following matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

and in this case

$$-\mu_{\{i,j\}} = \begin{cases} \gamma^{(i)}_{\{i,j\}} + \gamma^{(r)}_{\{i,j\}} \text{ if } \alpha_i \geq \alpha_j = 2(\alpha_i r_i + \alpha_j r_j) - r(\alpha_i + \alpha_j) \\ \\ \gamma^{(j)}_{\{i,j\}} + \gamma^{(j)}_{\{i,j\}} \text{ if } \alpha_i \leq \alpha_j = 2(\alpha_i r_i + \alpha_j r_j) - 2r\alpha_j \end{cases}$$

and this finishes the proof. ♦

Theorem 77. *Let* $(\mathcal{E}, \varphi)$ *be a decorated sheaf of type* $(2, b, c, \mathsf{N})$. *It is enough to check the semistability condition on subbundles and critical weighted filtrations of length two.*

Proof. We will prove the statement by induction on the cardinality of $\mathtt{I}$. If $|\mathtt{I}| \leq 2$ there is nothing to prove. Otherwise we will prove that, given a weighted filtration $(\mathcal{E}^\bullet, \underline{\alpha})$ indexed by $\mathtt{I}$ with $|\mathtt{I}| \geq 3$, there exist two weighted (proper) subfiltrations (if necessary with different weights) such that

$$P_{\mathtt{I}} + \delta\mu_{\mathtt{I}} = P_{\mathtt{J}} + \delta\mu_{\mathtt{J}} + P_{\mathtt{K}} + \delta\mu_{\mathtt{K}}. \tag{5.1}$$

More precisely we will show that the two subfiltrations indexed by $\mathtt{J}$ and $\mathtt{K}$ satisfy conditions *(ii)*, *(iii)* and *(iv)* of the previous section and $|\mathtt{J}|, |\mathtt{K}| < |\mathtt{I}|$.

Without loss of generality we can assume that the set $\mathtt{I}$ is well ordered and that the first (last) element of the set is indexed by 1 ($r-1$ respectively). Moreover if $i \in \mathtt{J}$ (where $\mathtt{J}$ is any well ordered set of indexes) with the notation $i+1$ we mean the successor of i inside $\mathtt{J}$. Let $M = (m_{ij})_{i,j \in \mathtt{I}}$ be the matrix representing the morphism φ with respect to the given filtration $(\mathcal{E}^\bullet, \underline{\alpha})$. If the first row of the matrix is zero then we can split the filtration as $(\mathcal{E}_1, \alpha_1)$ and $(\mathcal{E}^\bullet, \underline{\alpha})_{\mathtt{I} \smallsetminus \{1\}}$ and so we are done. Therefore we can assume that this is not the case.
Let us denote j_i, for any $i \in \mathtt{I}$, the minimum of the set $\{j \in \overline{\mathtt{I}} \,|\, m_{ij} \neq 0\}$. We will distinguish the cases $j_1 < r$ or $j_1 = r$. In the former case we split the filtration in two subfiltrations: $(\mathcal{E}^\bullet, \underline{\alpha})_{\mathtt{I} \smallsetminus \{r-1\}}$ and $(\mathcal{E}_{\{r-1\}}, \alpha_{\{r-1\}})$. With these choices equality (5.1) holds, in fact

$$R_{\mathtt{I}} \doteqdot \max_{s,t \in \overline{\mathtt{I}}} \{R_{\mathtt{I}}(s) + R_{\mathtt{I}}(t) \,|\, \varphi_{|\mathcal{E}_s \mathcal{E}_t} \neq 0\} = \max_{i \in \overline{\mathtt{I}}} \{R_{\mathtt{I}}(i) + R_{\mathtt{I}}(j_i)\}$$

and so, calling k the index such that (k, j_k) realizes the maximum, one has that

$$\begin{aligned} R_{\mathtt{I}} &= R_{\mathtt{I}}(k) + R_{\mathtt{I}}(j_k) = \alpha_k + \cdots + \alpha_{j_k - 1} + 2\alpha_{j_k} + \cdots + 2\alpha_{r-1} \\ &= R_{\mathtt{I} \smallsetminus \{r-1\}} + 2\alpha_{r-1} \\ &= R_{\mathtt{I} \smallsetminus \{r-1\}} + R_{\{r-1\}}, \end{aligned}$$

where the second equality holds since the maximum in the subfiltration indexed by $\mathtt{I} \smallsetminus \{r-1\}$ is still achieved in (k, j_k) if $j_k \neq r-1$, otherwise in $(1, r) = (k, r)$ if $j_k = r-1$. The last equality holds from the assumption $j_1 \neq r$ which implies $m_{1r-1} \neq 0$ and consequently $m_{r-1r-1} \neq 0$. Finally, recalling that $P_{\mathtt{I}} + \delta\mu_{\mathtt{I}} = \sum_{s \in \mathtt{I}} \alpha_s C_s + \delta r R_{\mathtt{I}}$, we get the thesis.
Suppose now that $j_1 = r$ and that $\max_{i \in \overline{\mathtt{I}}} \{R_{\mathtt{I}}(i) + R_{\mathtt{I}}(j_i)\}$ is gained in k. Therefore, for all $s \in \mathtt{I}$ such that $j_s \neq j_{s-1}$ and $s \leq j_s$, we have a set of inequalities in the variables α_i given from the inequalities $R_{\mathtt{I}}(k) + R_{\mathtt{I}}(j_k) \geq R_{\mathtt{I}}(s) + R_{\mathtt{I}}(j_s)$, i.e.,

$$\alpha_s + \cdots + \alpha_{k-1} \leq \alpha_{j_k} + \cdots + \alpha_{j_s - 1} \qquad s \leq k-1 \tag{5.2}$$

$$\alpha_k + \cdots + \alpha_{s-1} \geq \alpha_{j_s} + \cdots + \alpha_{j_k - 1} \qquad s \geq k+1. \tag{5.3}$$

If an index t is missing in the previous set of inequalities we can as before split the main filtration in two subfiltrations $(\mathcal{E}^\bullet, \underline{\alpha})_{\mathtt{I}\smallsetminus\{t\}}$ and $(\mathcal{E}_{\{t\}}, \alpha_{\{t\}})$. Since $R_{\mathtt{I}}(1) + R_{\mathtt{I}}(r) = \sum_{i\in\mathtt{I}} \alpha_i$ and we are supposing that the index t does not appear in the inequalities, this forces the coefficient of α_t to present (and equal to one) in all expressions of the form $R_{\mathtt{I}}(i) + R_{\mathtt{I}}(j_i)$. Therefore $j_k > t$, otherwise the coefficient of α_t in the expression $R_{\mathtt{I}}(k) + R_{\mathtt{I}}(j_k)$ should be two and so there would be an inequality in which α_t would appear with a non-zero coefficient, which is absurd. In particular $R_{\{t\}} = \alpha_t$ and $P_{\mathtt{I}} + \delta\mu_{\mathtt{I}} = P_{\mathtt{I}\smallsetminus\{t\}} + \delta\mu_{\mathtt{I}\smallsetminus\{t\}} + P_{\{t\}} + \delta\mu_{\{t\}}$. Indeed, if $k \neq t$, the maximum is still achieved in (k, j_k), otherwise $k = t$ and $j_s = j_k$ for all $s \geq k$ (otherwise α_k would appear in some inequality of type (5.2) and (5.3)) and so the maximum of the filtration indexed by $\mathtt{I}\smallsetminus\{t\}$ is realized in $(k+1, j_{k+1})$.

The last case we have to consider is when all indexes appear in inequalities (5.2) and (5.3). Note that the set of indexes that appear in inequalities (5.2), that we will call $\mathtt{J}$, does not intersect the set of indexes of inequalities (5.3), that we will denote $\mathtt{J}'$; moreover all indexes appearing in inequalities of type (5.2) ((5.3) respectively) appear in the first inequality of the same type (the last respectively). If k is such that the set J and J' are both not empty, then an easy calculation shows that $P_{\mathtt{I}} + \delta\mu_{\mathtt{I}} = P_{\mathtt{J}} + \delta\mu_{\mathtt{J}} + P_{\mathtt{J}'} + \delta\mu_{\mathtt{J}'}$. Indeed, in the filtration indexed by $\mathtt{J}$, the maximum is achieved in (j_k, j_k) while the maximum for the filtration indexed by $\mathtt{J}'$ is achieved in (k, r).

If $\mathtt{J} = \emptyset$ then $R_{\mathtt{I}} = R_{\mathtt{I}}(1) + R_{\mathtt{I}}(r)$, i.e., $k = 1$ and $j_k = r$. Let $S \doteqdot \{s \in \mathtt{I} \mid s \leq j_s \text{ and } j_s \neq j_{s-1}\} = \{s_1 = 1, \ldots, s_t\}$ (we are assuming $t \geq 3$), then we have the following inequalities:

$$R_{\mathtt{I}}(1) + R_{\mathtt{I}}(r) \geq R_{\mathtt{I}}(s_2) + R_{\mathtt{I}}(j_{s_2})$$
$$\vdots$$
$$R_{\mathtt{I}}(1) + R_{\mathtt{I}}(r) \geq R_{\mathtt{I}}(s_t) + R_{\mathtt{I}}(j_{s_t}).$$

or equivalently

$$\alpha_1 + \cdots + \alpha_{s_2-1} \geq \alpha_{j_{s_2}} + \cdots + \alpha_{r-1}$$
$$\vdots$$
$$\alpha_1 + \cdots + \alpha_{s_t-1} \geq \alpha_{s_t} + \cdots + \alpha_{r-1}.$$

In this case the subfiltrations we consider are the subfiltrations indexed by $\mathtt{K}' = \{1, \ldots, s_2-1, j_{s_2}, \ldots, r-1\}$ and $\mathtt{K}'' = \{1, 2, \ldots, j_{s_2}-1\}$ with weight vectors $(\alpha'_1, \ldots, \alpha'_{s_2-1}, \alpha_{j_{s_2}}, \ldots, \alpha_{r-1})$ and $(\alpha''_1, \ldots, \alpha''_{s_2-1}, \alpha_{s_2}, \ldots, \alpha_{j_{s_2}-1})$ respectively.
Thanks to the previous inequalities it easy to see that the weights α'_i and

α''_i can be chosen in a such way that $\alpha'_i + \alpha''_i = \alpha_i$ and

$$\alpha'_1 + \cdots + \alpha'_{s_2-1} \geq \alpha_{j_{s_2}}, \ldots, \alpha_{r-1}$$
$$\alpha''_1 + \cdots + \alpha''_{s_2-1} + \alpha_{s_2} + \cdots + \alpha_{s_t-1} \geq \alpha_{s_t} + \cdots + \alpha_{j_{s_2}-1}.$$

Therefore

$$P_{\mathtt{I}} + \delta\mu_{\mathtt{I}} = (P_{\mathtt{K}'} + \delta\mu_{\mathtt{K}'}) + (P_{\mathtt{K}''} + \delta\mu_{\mathtt{K}''}).$$

Finally, if $t = 2$, we have only the inequality $\alpha_1 + \cdots + \alpha_{s_t-1} \geq \alpha_{s_t} + \cdots + \alpha_{r-1}$. Then we call $\beta = \alpha_{s_t} + \cdots + \alpha_{r-1}$, we write $\beta = \beta_1 + \cdots + \beta_{s_t-1}$ such that, for any $i = 1, \ldots, s_t - 1$, $\alpha_i \geq \beta_i$. Then we consider weighted filtrations $0 \subset \mathcal{E}_i \subset \mathcal{E}_{s_t} \subset \cdots \subset \mathcal{E}_{r-1}$ with weights $(\alpha_i, \beta_{i,s_t}, \ldots, \beta_{i,r-1})$ where $\beta_{i,h}$ satisfy $\sum_{h=s_t}^{r-1} \beta_{i,h} = \beta_i$ and $\beta_{i,h} \geq \beta_{i,h'}$ if and only if $\alpha_h \geq \alpha_{h'}$, for any $i = 1, \ldots, s_t - 1$. With such choices is easy to see that

$$P_{\mathtt{I}} + \delta\mu_{\mathtt{I}} = \sum_{i=1}^{s_t-1} P_{\mathtt{K}_i} + \delta\mu_{\mathtt{K}_i},$$

where we denote $\mathtt{K}_i$ the set $\{i, s_t, \ldots, r-1\}$.

The case in which $\mathtt{J}' = \emptyset$ is similar and so we are done. ♦

Remark 78. As a consequence of the proof of Theorem 77 we have that every critical filtration splits as a certain number of length two critical filtrations and a non-critical one (which obviously can be decomposed as the union of length one filtrations).

Example 79. Let us fix $r = 5$ and let $0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \mathcal{E}_3 \subset \mathcal{E}_4 \subset \mathcal{E}$ be a filtration with weight vector $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ such that the matrix representing φ with respect the filtration is the following:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

In this case the filtration is critical, in fact, denoting by $\mathtt{I}$ the set $\{1, 2, 3, 4\}$, we have that

$$\begin{aligned} R_{\mathtt{I}} &= \max\{R_{\mathtt{I}}(1) + R_{\mathtt{I}}(r), R_{\mathtt{I}}(2) + R_{\mathtt{I}}(4), R_{\mathtt{I}}(3) + R_{\mathtt{I}}(3)\} \\ &= \max\{A \doteqdot \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, B \doteqdot \alpha_2 + \alpha_3 + 2\alpha_4, C \doteqdot 2\alpha_3 + 2\alpha_4\}, \end{aligned}$$

while $\sum_{i\in\mathtt{I}} R_{\{i\}} = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4$.

If the maximum is A then we have the following inequalities:

$$A \geq B \Rightarrow \alpha_1 \geq \alpha_4$$
$$A \geq C \Rightarrow \alpha_1 + \alpha_2 \geq \alpha_3 + \alpha_4.$$

So in this case we are in the situation $\mathtt{J} = \emptyset$ considered in the proof of Theorem 77; therefore we can find α_1', α_1'' such that $\alpha_1' + \alpha_1'' = \alpha_1$, $\alpha_1' \geq \alpha_4$ and $\alpha_1'' + \alpha_2 \geq \alpha_3$. Let us consider the filtrations $0 \subset \mathcal{E}_1 \subset \mathcal{E}_4 \subset \mathcal{E}$, $0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \mathcal{E}_3 \subset \mathcal{E}$ with weights (α_1', α_4) and $(\alpha_1'', \alpha_2, \alpha_3)$ respectively. Proceeding as before we split the filtrations $\{14\}$ and $\{123\}$ and we obtain that:

$$\begin{aligned} P_{\mathtt{I}} + \delta\mu_{\mathtt{I}} &= P_{\{14\}} + \delta\mu_{\{14\}} + P_{\{123\}} + \delta\mu_{\{123\}} \\ &= P_{\{14\}} + \delta\mu_{\{14\}} + P_{\{13\}} + \delta\mu_{\{13\}} + P_{\{23\}} + \delta\mu_{\{23\}} \end{aligned}$$

and we are done.

If the maximum is B then we have the following inequalities:

$$B \geq A \Rightarrow \alpha_4 \geq \alpha_1$$
$$B \geq C \Rightarrow \alpha_2 \geq \alpha_3.$$

Therefore $\mathtt{J} = \{1, 4\}$ and $\mathtt{J}' = \{2, 3\}$ are disjoint and an easy calculation shows that $R_{\mathtt{J}} = R_{\mathtt{J}}(4) + R_{\mathtt{J}}(4) = 2\alpha_4$ and $R_{\mathtt{J}}' = R_{\mathtt{J}}'(2) + R_{\mathtt{J}}'(5) = \alpha_2 + \alpha_3$, therefore $P_{\mathtt{I}} + \delta\mu_{\mathtt{I}} = P_{\{14\}} + \delta\mu_{\{14\}} + P_{\{23\}} + \delta\mu_{\{23\}}$ and we finish.

Finally, if the maximum is C, $\mathtt{J}' = \emptyset$ and calculations are similar to the case in which A is the maximum.

Thanks to previous results, in order to check the semistability condition, we can focus our attention only on subbundles and critical filtrations of length 2. More precisely:

Proposition 80. *Let $(\mathcal{E}, \varphi)$ as before, then the following statements are equivalent:*

1. *$(\mathcal{E}, \varphi)$ is δ-(semi)stable;*
2. *For any subsheaf $\mathcal{F}$ and for any critical weighted filtration $(0 \subset \mathcal{E}_i \subset \mathcal{E}_j \subset \mathcal{E}\,,\,(\alpha_i, \alpha_j))$ of length two the following inequalities hold:*

   - $(\mathsf{P}_{\mathcal{E}} r_{\mathcal{F}} - r\mathsf{P}_{\mathcal{F}}) - \delta(r\, k_{\mathcal{F},\mathcal{E}} - 2\, r_{\mathcal{F}}) \underset{(=)}{>} 0,$

   - $P_{\{i,j\}} - \delta\left(2(\alpha_i r_i + \alpha_j r_j) - r \max\{\alpha_i + \alpha_j, 2\alpha_j\}\right) \underset{(=)}{>} 0.$

Proof. The arrow (1) $\Rightarrow$ (2) is trivial. So suppose that (2) holds, then, as noticed before, semistability can be checked only on subsheaves and critical filtrations and, thanks to Theorem 77, we can consider only critical weighted filtrations of length two. Finally, due to Proposition 76, we get that

$$P_{\{i,j\}} + \delta\mu_{\{i,j\}} = P_{\{i,j\}} - \delta\left(2(\alpha_i r_i + \alpha_j r_j) - r\max\{\alpha_i + \alpha_j, 2\alpha_j\}\right)$$

and so we are done. ♦

Lemma 81. *Let $(\mathcal{E}, \varphi)$ be a decorated sheaf as before such that $P(\mathcal{E}^\bullet) + \delta\mu(\mathcal{E}^\bullet; \varphi) \geq 0$ for any length ≤ 2 filtration $\mathcal{E}^\bullet$ with weight vector identically one. Then $(\mathcal{E}, \varphi)$ is δ-(semi)stable.*

Proof. Clearly $(\mathcal{E}, \varphi)$ is k-(semi)stable since weights do not affect the semistability condition for subsheaves. Moreover by Theorem 77 we can check semistability only on critical weighted filtrations of length two. Let $(0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \mathcal{E}\,, \alpha_1, \alpha_2)$ be such a filtration. We want to show that

$$P + \delta\mu = \alpha_1 C_1 + \alpha_2 C_2 + r\delta\max\{\alpha_1 + \alpha_2, 2\alpha_2\} \underset{(=)}{>} 0.$$

If the maximum is $\alpha_1 + \alpha_2$, then $\alpha_1 \geq \alpha_2$ and the previous inequality becomes:

$$\alpha_1 C_1 + \alpha_2 C_2 + r\delta(\alpha_1 + \alpha_2) = \alpha_2(C_1 + C_2 + 2r\delta) + (\alpha_1 - \alpha_2)(C_1 + r\delta) \underset{(=)}{>} 0$$

where the last inequality holds since by hypothesis $C_1 + C_2 + 2r\delta$ and $C_1 + r\delta$ are non-negative.
Otherwise, if the maximum is $2\alpha_2$, then $\alpha_1 \leq \alpha_2$ so

$$\alpha_1 C_1 + \alpha_2 C_2 + r\delta(2\alpha_2) = \alpha_1(C_1 + C_2 + 2r\delta) + (\alpha_2 - \alpha_1)(C_2 + 2r\delta) \underset{(=)}{>} 0,$$

and $(\mathcal{E}, \varphi)$ is (semi)stable. ♦

In view of the previous results we can state the following (equivalent) definition of (semi)stability of decorated sheaves of type $(2, b, c, \mathsf{N})$:

Definition 82. A decorated sheaf $(\mathcal{E}, \varphi)$ of type $(2, b, c, \mathsf{N})$ is δ-(semi)stable if the following conditions hold:

1. If $\mathcal{F}$ is a proper subsheaf of $\mathcal{E}$, then
$$\frac{\mathsf{P}_{\mathcal{F}} - \delta\mathsf{k}_{\mathcal{F},\mathcal{E}}}{\mathrm{rk}(\mathcal{F})} \underset{(-)}{\prec} \frac{\mathsf{P}_{\mathcal{E}} - 2\delta}{r}.$$

2. If $0 \subset \mathcal{E}_i \subset \mathcal{E}_j \subset \mathcal{E}$ is a critical filtration, then
$$(r_i + r_j)\mathsf{P}_{\mathcal{E}} - r(\mathsf{P}_{\mathcal{E}_i} + \mathsf{P}_{\mathcal{E}_j}) - 2\delta(r_i + r_j - r) \underset{(-)}{\succ} 0.$$

Example 83. Let (E, φ) be a rank 5 decorated vector bundle of type $(2, 1, 0, \mathsf{N})$. Let $0 \subset E_1 \subset E_2 \subset E_3 \subset E_4 \subset E_5 = E$ be a filtration of E indexed by $\mathtt{I} = \{1, 2, 3, 4\}$ (with weight vector $\underline{\alpha} = \underline{1}$). Then all possible matrices $M_{\mathtt{I}}(E^{\bullet}; \varphi)$ are the following.

$$(1)\begin{pmatrix} 1&1&1&1&1\\ &1&1&1&1\\ &&1&1&1\\ &&&1&1\\ &&&&1 \end{pmatrix}(2)\begin{pmatrix} 0&1&1&1&1\\ &1&1&1&1\\ &&1&1&1\\ &&&1&1\\ &&&&1 \end{pmatrix}\quad(3)\begin{pmatrix} 0&0&1&1&1\\ &1&1&1&1\\ &&1&1&1\\ &&&1&1\\ &&&&1 \end{pmatrix}(4)\begin{pmatrix} 0&0&1&1&1\\ &0&1&1&1\\ &&1&1&1\\ &&&1&1\\ &&&&1 \end{pmatrix}$$

$$(5)\begin{pmatrix} 0&0&0&1&1\\ &1&1&1&1\\ &&1&1&1\\ &&&1&1\\ &&&&1 \end{pmatrix}(6)\begin{pmatrix} 0&0&0&1&1\\ &0&1&1&1\\ &&1&1&1\\ &&&1&1\\ &&&&1 \end{pmatrix}\quad(7)\begin{pmatrix} 0&0&0&1&1\\ &0&0&1&1\\ &&1&1&1\\ &&&1&1\\ &&&&1 \end{pmatrix}(8)\begin{pmatrix} 0&0&0&1&1\\ &0&0&1&1\\ &&0&1&1\\ &&&1&1\\ &&&&1 \end{pmatrix}$$

$$(9)\begin{pmatrix} 0&0&0&0&1\\ &1&1&1&1\\ &&1&1&1\\ &&&1&1\\ &&&&1 \end{pmatrix}(10)\begin{pmatrix} 0&0&0&0&1\\ &0&1&1&1\\ &&1&1&1\\ &&&1&1\\ &&&&1 \end{pmatrix}\quad(11)\begin{pmatrix} 0&0&0&0&1\\ &0&0&1&1\\ &&1&1&1\\ &&&1&1\\ &&&&1 \end{pmatrix}(12)\begin{pmatrix} 0&0&0&0&1\\ &0&0&1&1\\ &&0&1&1\\ &&&1&1\\ &&&&1 \end{pmatrix}$$

$$(13)\begin{pmatrix} 0&0&0&0&1\\ &0&0&0&1\\ &&1&1&1\\ &&&1&1\\ &&&&1 \end{pmatrix}(14)\begin{pmatrix} 0&0&0&0&1\\ &0&0&0&1\\ &&0&1&1\\ &&&1&1\\ &&&&1 \end{pmatrix}\quad(15)\begin{pmatrix} 0&0&0&0&1\\ &0&0&0&1\\ &&0&0&1\\ &&&1&1\\ &&&&1 \end{pmatrix}(16)\begin{pmatrix} 0&0&0&0&1\\ &0&0&0&1\\ &&0&0&1\\ &&&0&1\\ &&&&1 \end{pmatrix}$$

$$(17)\begin{pmatrix} 0&0&0&0&0\\ &1&1&1&1\\ &&1&1&1\\ &&&1&1\\ &&&&1 \end{pmatrix}(18)\begin{pmatrix} 0&0&0&0&0\\ &0&1&1&1\\ &&1&1&1\\ &&&1&1\\ &&&&1 \end{pmatrix}\quad(19)\begin{pmatrix} 0&0&0&0&0\\ &0&0&1&1\\ &&1&1&1\\ &&&1&1\\ &&&&1 \end{pmatrix}(20)\begin{pmatrix} 0&0&0&0&0\\ &0&0&1&1\\ &&0&1&1\\ &&&1&1\\ &&&&1 \end{pmatrix}$$

$$(21)\begin{pmatrix} 0&0&0&0&0\\ &0&0&0&1\\ &&1&1&1\\ &&&1&1\\ &&&&1 \end{pmatrix}(22)\begin{pmatrix} 0&0&0&0&0\\ &0&0&0&1\\ &&0&1&1\\ &&&1&1\\ &&&&1 \end{pmatrix}\quad(23)\begin{pmatrix} 0&0&0&0&0\\ &0&0&0&1\\ &&0&0&1\\ &&&1&1\\ &&&&1 \end{pmatrix}(24)\begin{pmatrix} 0&0&0&0&0\\ &0&0&0&1\\ &&0&0&1\\ &&&0&1\\ &&&&1 \end{pmatrix}$$

$$(25)\begin{pmatrix} 0&0&0&0&0\\ &0&0&0&0\\ &&1&1&1\\ &&&1&1\\ &&&&1 \end{pmatrix}(26)\begin{pmatrix} 0&0&0&0&0\\ &0&0&0&0\\ &&0&1&1\\ &&&1&1\\ &&&&1 \end{pmatrix}\quad(27)\begin{pmatrix} 0&0&0&0&0\\ &0&0&0&0\\ &&0&0&1\\ &&&1&1\\ &&&&1 \end{pmatrix}(28)\begin{pmatrix} 0&0&0&0&0\\ &0&0&0&0\\ &&0&0&1\\ &&&0&1\\ &&&&1 \end{pmatrix}$$

$$(29)\begin{pmatrix} 0&0&0&0&0\\ &0&0&0&0\\ &&0&0&0\\ &&&1&1\\ &&&&1 \end{pmatrix}(30)\begin{pmatrix} 0&0&0&0&0\\ &0&0&0&0\\ &&0&0&0\\ &&&0&1\\ &&&&1 \end{pmatrix}\quad(31)\begin{pmatrix} 0&0&0&0&0\\ &0&0&0&0\\ &&0&0&0\\ &&&0&0\\ &&&&1 \end{pmatrix}(32)\begin{pmatrix} 0&0&0&0&0\\ &0&0&0&0\\ &&0&0&0\\ &&&0&0\\ &&&&0 \end{pmatrix}$$

All possible subsets of $\mathtt{I} = \{1, 2, 3, 4\}$ are $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$, $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$ $\{2, 3\}$, $\{2, 4\}$, $\{3, 4\}$, $\{1\}$, $\{2\}$, $\{3\}$ and $\{4\}$. Let $\mathtt{J} \subseteq \mathtt{I}$, we recall in this case

$$\begin{aligned} \mu_{\mathtt{J}} &= -\min_{i,j \in \mathtt{J}} \{\gamma_{\mathtt{J}}^{(i)} + \gamma_{\mathtt{J}}^{(j)} \mid m_{ij} \neq 0\} \\ &= -2 \sum_{i \in \mathtt{J}} r_{E_i} + r \max_{i,j \in \mathtt{J}} \{R_{\mathtt{J}}(i) + R_{\mathtt{J}}(j) \mid m_{ij} \neq 0\} \end{aligned}$$

The following are the values of $\mu_{\mathtt{J}}$ as $\mathtt{J}$ varies between all possible subsets of $\mathtt{I}$.

	μ_{1234}	μ_{123}	μ_{124}	μ_{134}	μ_{234}	μ_{12}	μ_{13}	μ_{14}	μ_{23}	μ_{24}	μ_{34}	μ_{1}	μ_{2}	μ_{3}	μ_{4}
1	20	18	16	14	12	14	12	10	10	8	6	8	6	4	2
2	15	13	11	9	12	9	7	5	10	8	6	3	6	4	2
3	10	8	6	9	12	4	7	5	10	8	6	3	6	4	2
4	10	8	6	9	7	4	7	5	5	3	6	3	1	4	2
5	10	8	6	4	12	4	2	5	10	8	6	3	6	4	2
6	5	3	6	4	7	4	2	5	5	3	6	3	1	4	2
7	5	3	6	4	2	4	2	5	0	3	6	3	1	4	2
8	5	3	6	4	2	4	2	5	0	3	1	3	1	-1	2
9	10	8	6	4	12	4	2	0	10	8	6	3	6	4	2
10	5	3	1	4	7	4	2	0	5	3	6	3	1	4	2
11	0	3	1	4	2	4	2	0	0	3	6	3	1	4	2
12	0	3	1	-1	2	4	2	0	0	3	1	3	1	-1	2
13	0	3	1	4	2	4	2	0	0	-2	6	3	1	4	2
14	0	3	1	-1	-3	4	2	0	0	-2	1	3	1	-1	2
15	0	3	1	-1	-3	4	2	0	0	-2	-4	3	1	-1	2
16	0	3	1	-1	-3	4	2	0	0	-2	-4	3	1	-1	-3
17	10	8	6	4	12	4	2	0	10	8	6	-2	6	4	2
18	5	3	1	4	7	-1	2	0	5	3	6	-2	1	4	2
19	0	-2	1	4	2	-1	2	0	0	3	6	-2	1	4	2
20	0	-2	1	-1	2	-1	-3	0	0	3	1	-2	1	-1	2
21	0	-2	-4	4	2	-1	2	0	0	-2	6	-2	1	4	2
22	-5	-2	-4	-1	-3	-1	-3	0	0	-2	1	-2	1	-1	2
23	-5	-2	-4	-6	-3	-1	-3	0	0	-2	-4	-2	1	-1	2
24	-5	-2	-4	-6	-3	-1	-3	-5	0	-2	-4	-2	1	-1	-3
25	0	-2	-4	4	2	-6	2	0	0	-2	6	-2	-4	4	2
26	-5	-7	-4	-1	-3	-6	-3	0	-5	-2	1	-2	-4	-1	2
27	-10	-7	-4	-6	-8	-6	-3	0	-5	-2	-4	-2	-4	-1	2
28	-10	-7	-9	-6	-8	-6	-3	-5	-5	-7	-4	-2	-4	-1	-3
29	-10	-12	-4	-6	-8	-6	-8	0	-10	-2	-4	-2	-4	-6	2
30	-15	-12	-9	-11	-13	-6	-8	-5	-10	-7	-9	-2	-4	-6	-3
31	-20	-12	-14	-16	-18	-6	-8	-10	-10	-12	-14	-2	-4	-6	-8

5.2 The general setting

- Let a an integer ≥ 1.
- Let r be an integer ≥ 2.
- $(\mathtt{I}, \leq)$ be a well-ordered set of cardinality $r-1$ and let $\overline{\mathtt{I}} = \mathtt{I} \cup \{r\}$.
- Let $\alpha_i \in \mathbb{R}_{\geq 0}$ for any $i \in \mathtt{I}$ and let $\underline{\alpha}_{\mathtt{I}}$ be the corresponding vector of length $r-1$.
- Let $A \doteqdot \{(i_1, \ldots, i_a) \,|\, i_1, \ldots, i_a \in \mathtt{I} \text{ and } i_1 \leq \cdots \leq i_a\}$. Define over

the set A the following partial-ordering relation: we will say that $(i_1,\dots,i_a) \preccurlyeq (j_1,\dots,j_a)$ if and only if $i_1 \le j_1,\dots,i_a \le j_a$.

- Let $M_{\mathtt{I}} \doteqdot (m_{ij})_{i,j\in\overline{\mathtt{I}}}$ be a $r^{\times a}$ "matrix" with the following properties:
  - $M_{\mathtt{I}}$ is symmetric, i.e., $m_{i_1\dots i_a} = m_{\sigma(i_1)\dots\sigma(i_a)}$ for any $i_1,\dots,i_a \in \mathtt{I}$ and any permutation $\sigma : \{i_1,\dots,i_a\} \to \{i_1,\dots,i_a\}$;
  - $m_{ij} \in \{0,1\}$ for any $i,j \in \mathtt{I}$;
  - if $m_{i_1\dots i_a} = 1$ then $m_{j_1\dots j_a} = 1$ for any $(j_1,\dots,j_a) \succcurlyeq (i_1,\dots,i_a)$;
  - if $m_{i_1\dots i_a} = 0$ then $m_{j_1\dots j_a} = 0$ for any $(j_1,\dots,j_a) \preccurlyeq (i_1,\dots i_a)$.
- For any $l \in \overline{\mathtt{I}}$ define
$$R_{\mathtt{I}}(l) \doteqdot \sum_{i\in\mathtt{I},\, i\ge l} \alpha_i$$
if $l \in \mathtt{I}$ and $R_{\mathtt{I}}(l) = 0$ otherwise. We set
$$R_{\mathtt{I}} \doteqdot \max_{i,j\in\overline{\mathtt{I}}} \{R_{\mathtt{I}}(i_1) + \cdots + R_{\mathtt{I}}(i_a) \,|\, m_{i_1\dots i_a} \neq 0\}.$$

With this notation the following result holds:

Theorem 84. *Let $a = 2$. Fix a well-ordered set $\mathtt{I}$, a vector $\underline{\alpha}_{\mathtt{I}}$ of real numbers and a symmetric "boolean" matrix $M_{\mathtt{I}}$ as before. Denote by $|\cdot|$ the cardinality of a set. Then exist $t \in \mathbb{N}$, sets $\mathtt{J}^{(1)},\dots,\mathtt{J}^{(t)}$ and positive real vectors $\underline{\alpha}_{\mathtt{J}^{(1)}},\dots,\underline{\alpha}_{\mathtt{J}^{(t)}}$ such that*

i) $|\mathtt{J}^{(s)}| \le 2$ *for any* $s = 1,\dots,t$;

ii) $\mathtt{J}^{(1)} \cup \dots \cup \mathtt{J}^{(t)} = \mathtt{I}$;

iii) $\sum_{s=1}^t \alpha_i^{(s)} = \alpha_i$, *where is to be understood that* $\alpha_i^{(s)} = 0$ *if* $i \notin \mathtt{J}^{(s)}$;

iv) $\sum_{s=1}^t R_{\mathtt{J}^{(s)}} = R_{\mathtt{I}}$.

Proof. The proof of Theorem 77 provides us an algorithm to find $\mathtt{K}^{(1)}, \mathtt{K}^{(2)}$ satisfying points (ii), (iii) and (iv). Since $|\mathtt{K}^{(1)}|, |\mathtt{K}^{(2)}| \lneqq |I|$ iterating this process, after a finite number of steps, we obtain the thesis. ♦

Corollary 85. *Let r, $\mathtt{I}$, $\mathtt{J}^{(1)},\dots,\mathtt{J}^{(t)}$ and $\underline{\alpha}_{\mathtt{J}^{(1)}},\dots,\underline{\alpha}_{\mathtt{J}^{(t)}}$ as before and let δ be a fixed numerical polynomial. For any $i \in \mathtt{I}$ let C_i be a constant. Then*
$$\sum_{i\in\mathtt{I}} \alpha_i\, C_i + r\,\delta\, R_{\mathtt{I}} = \sum_{s=1}^t \left(\sum_{i\in\mathtt{J}^{(s)}} \alpha_i^{(s)}\, C_i + r\,\delta\, R_{\mathtt{J}^{(s)}} \right)$$

Proof. Condition (ii) and (iii) imply that $\sum_{i\in\mathtt{I}} \alpha_i\, C_i = \sum_{s=1}^t \sum_{i\in\mathtt{J}^{(s)}} \alpha_i^{(s)}\, C_i$. ♦

Corollary 86. *Let $(\mathcal{E}, \varphi)$ be a decorated sheaf of type $(2, b, c, \mathsf{N})$, then it is slope semistable with respect to $\overline{\delta}$ if and only if the following conditions hold:*

1. *If $\mathcal{F}$ is a proper subsheaf of $\mathcal{E}$, then*

$$\mu(\mathcal{F}) - \frac{\overline{\delta} k_{\mathcal{F},\mathcal{E}}}{rk(\mathcal{F})} \underset{(=)}{<} \mu(\mathcal{E}) - \frac{2\delta}{r_{\mathcal{E}}}.$$

2. *If $0 \subset \mathcal{E}_i \subset \mathcal{E}_j \subset \mathcal{E}$ is a critical filtration, then*

$$(r_{\mathcal{E}_i} + r_{\mathcal{E}_j}) \deg(\mathcal{E}) - r_{\mathcal{E}}(\deg(\mathcal{E}_i) + \deg(\mathcal{E}_j)) - 2\delta(r_{\mathcal{E}_i} + r_{\mathcal{E}_j} - r_{\mathcal{E}}) \underset{(=)}{>} 0.$$

Corollary 87. *Let $\mathcal{N}$ be a torsion free sheaf of positive rank, $(X, \mathcal{O}_X(1))$ a projective variety with an ample line bundle and $(\mathcal{A}, \varphi)$ a decorated coherent sheaf of type $(2, b, c, \mathcal{N})$. Then it is enough to check the semistability condition* (2.8) *over length ≤ 2 weighted filtrations.*

5.3 Quadric and orthogonal bundles over curves

As we remarked in Section 2.3.3, quadric and orthogonal bundles are decorated vector bundles of type $(2, 1, 0, \mathsf{N})$ and, respectively, $(2, 1, 0, \mathcal{O}_X)$. Therefore them inherids the (semi)stability condition of decorated vector bundles. Orthogonal bundles have already a notion of (semi)stability. For example in [25], an orthogonal bundle E over a smooth curve is sayed (semi)stable if every proper isotropic subbundle F of E has degree zero. As an application of Theorem 84 we will show that, at least in the case of orthogonal bundles over curves, the two (semi)stability conditions coincide.

We would like to point you out two papers by scar Garca-Prada, Peter B. Gothen and Ignasi Mundet i Riera, where they prove some results very much related (and in some cases essentially equivalent) to some results of this section. The simplification of the (semi)stability condition for quadratic pairs (i.e., the statement of Theorem 77 applied to non-degenerate quadric bundles) is contained in Theorem 4.9 of [5], although the setting of Theorem 4.9 refers to symplectic Higgs bundles of which quadric bundles are particular cases. While the relation between stability of nondegenerate quadric bundles and the corresponding (generalized) orthogonal bundles extends part of the contents of Theorem 4.2 in [6].

Let X be a smooth projective complex curve of genus g and N a line bundle over X. Let us fix $\delta \in \mathbb{R}^+$ and integers $r > 0$, d.

Definition 88 (**Quadric Bundles**)**.** A quadric bundle on X of type (r, d, N) is a pair (E, Q) where E is a vector bundle of rank r and degree d on X, and

$$Q : Sym^2 E \to \mathsf{N},$$

is a morphism of vector bundles.
A morphism between quadric bundles $f : (E, Q) \to (E', Q')$ is a morphism $f : E \to E'$ of vector bundles such that there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Sym}^2 E & \xrightarrow{\mathrm{Sym}^2 f} & \mathrm{Sym}^2 E' \\ {\scriptstyle Q}\Big\downarrow & & {\scriptstyle Q'}\Big\downarrow \\ \mathsf{N} & \xrightarrow{\lambda} & \mathsf{N} \end{array}$$

where λ is a scalar multiple of the identity.
We will say that a quadric bundle is **δ-(semi)stable** if and only if is δ-(semi)stable as decorated vector bundle.

The term "quadric" comes from the fact that for every $x \in X$ the morphism Q restricted to the fibre E_x defines a bilinear symmetric form and so a quadric in $\mathbb{P}^{r-1}$.

If the morphism Q is the zero morphism, a quadric bundle is just an ordinary vector bundle, so from now on we suppose that Q is not identically zero. Note that even if the map Q is non-zero it could happen that restricted to a subbundle it vanishes.

An **orthogonal bundle** is a vector bundle associated with a principal bundle with (complex) orthogonal structure group. Equivalently, it is a quadric bundle (E, Q) with $\mathsf{N} = \mathcal{O}_X$, such that the bilinear form $Q : \mathrm{Sym}^2 E \to \mathcal{O}_X$ induces an isomorphism $Q : E \to E^\vee$. In this case Q gives a smooth quadric C_x for each point $x \in X$. Note that the isomorphism $Q : E \to E^\vee$ forces the degree of E to be zero.

There is a notion of stability for orthogonal bundles (see [25] Ramanan): an orthogonal bundle E is (semi)stable if and only if for every proper isotropic subbundle F (i.e., $\mathsf{k}_{F,E} \leq 1$),

$$\deg(F) \underset{(=)}{<} 0 = \deg(E).$$

We will prove that an orthogonal bundle is (semi)stable if and only if it is δ-(semi)stable as a quadric bundle. We start with the following useful result:

Lemma 89. *Let (E, Q) be an orthogonal bundle, and let F be a proper vector subbundle of E. Then*

1. *There is an exact sequence*

$$0 \to F^\perp \to E \to F^\vee \to 0,$$

$\deg(F) = \deg(F^\perp)$ *and* $rk(F^\perp) + rk(F) = r$.

2. $k_{F,E} \geq 1$.
3. *If F is isotropic (i.e $k_{F,E} \leq 1$), then*

$$1 \leq rk(F) \leq \lfloor \frac{r}{2} \rfloor,$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x and $F^{\perp}$ denote as usual the orthogonal of F with respect to the non degenerate bilinear form Q.

Proof. For the proof of (1) see Lemma 3.3 of [8]. (2) and (3) depend on the fact that we are assuming the matrix to be non-degenerate. ♦

Lemma 90. *Let (E, Q) be a quadric bundle such that Q is non-degenerate. Let F be a proper isotropic vector subbundle of E. Then the filtration $0 \subset F \subsetneq F^{\perp} \subset E$ is critical.*

Proof. Let F' be the maximal isotropic subbundle of E containing F and let r' denote its rank. Let $A = (a_{ij})$ the matrix representing Q with respect to a basis of E subordinated to F'. Then $a_{r'r'} = 0$, $a_{r'+1r'+1} = 1$ and

$$A = \left(\begin{array}{ccc|ccc} 0 & & & & & \\ & \ddots & & & B & \\ & & 0 & & & \\ \hline & & & 1 & & \\ & B^t & & & \ddots & \\ & & & & & 1 \end{array}\right)$$

The matrix B is a $r' \times (r - r')$-matrix, every row contains at least a 1 and two different rows must be independent. This forces the matrix B to be of the following form:

$$\begin{pmatrix} 0 & \dots & 0 & \star \\ \vdots & \iddots & \iddots & \bullet \\ 0 & \iddots & \iddots & \vdots \\ \star & \bullet & \dots & \bullet \end{pmatrix} \text{ if } r \text{ is even, or } \begin{pmatrix} 0 & 0 & \dots & 0 & \star \\ \vdots & \vdots & \iddots & \iddots & \bullet \\ \vdots & 0 & \iddots & \iddots & \vdots \\ 0 & \star & \bullet & \dots & \bullet \end{pmatrix} \text{ if } r \text{ is odd,}$$

where "$\star$" is any non-zero complex number while "$\bullet$" denote any complex number.
In both cases, since $F' \subseteq F^{\perp}$, we have that

- $Q_{|FF} = Q_{|_{FF^{\perp}}} = 0$
- $Q_{|FE}, Q_{|_{F^{\perp}E}}, Q_{|EE}, Q_{|_{F^{\perp}F^{\perp}}} \neq 0$

and the filtration $0 \subset F \subset F^{\perp} \subset E$ is critical. ♦

Remark 91. With the same notation as before, if $F = F^{\perp}$, then $2r_F = r$ (in particular r is even); moreover, if (E, Q) is a δ-semistable quadric bundle, condition (1) of Definition 82 tells us that

$$dr_F - rd_F + \delta\,(2r_F - r) \geq 0$$

and so $\mu(F) \leq \mu(E)$.

Theorem 92. *An orthogonal bundle is (semi)stable if and only if it is δ-(semi)stable as a quadric bundle.*

Proof. We prove the assertion for semistability; is the proof for stability very similar.

Let (E, Q) be a quadric bundle such that $Q : Sym^2E \to \mathcal{O}_X$ is non-degenerate, and assume that it is δ-(semi)stable. Let F be an isotropic vector subbundle. If $F \neq F^{\perp}$, the filtration $0 \subset F \subsetneq F^{\perp} \subset E$ is critical by Lemma 90. Then the semistability condition with weights identically 1 tells us that

$$(r_F + r_{F^\perp})\,d - r\,(d_F + d_{F^\perp}) + 2\delta(r - r_F - r_{F^\perp}) \geq 0.$$

By the first point of Lemma 89 $d_F = d_{F^\perp}$ and $r = r_F + r_{F^\perp}$. Since E is an orthogonal bundle $\deg(E) = 0$ and the above inequality tells us that $\deg(F) \leq 0$. If $F = F^{\perp}$, by Remark 91 we still have $\mu(F) \leq 0$ which proves that E is semistable as an orthogonal bundle.

Conversely, let E be a semistable orthogonal bundle. Let F be any vector subbundle. Following Ramanan (see [25]) let $N = F \cap F^{\perp}$, and let N' be the vector subbundle generated by N. We have an exact sequence

$$0 \to N' \to F \oplus F^{\perp} \to M' \to 0 \tag{5.4}$$

where M' is the subbundle of E generated by $F + F^{\perp}$. We have $M' = (N')^{\perp}$. If $N' = 0$, then $E = F \oplus F^{\perp}$, $\mathsf{k}_{F,E} = 2$, and $\deg(F) = \deg(E) = 0$ (Lemma 89). Then

$$\frac{\deg(F) - \mathsf{k}_{F,E}\delta}{\mathrm{rk}(F)} = \frac{-2\delta}{\mathrm{rk}(F)} < \frac{-2\delta}{r} = \frac{\deg(E) - 2\delta}{r}.$$

If $N' \neq 0$, $\deg(F) = \deg(N')$ (by Lemma 89 and the exact sequence (5.4)), and then $\deg(F) \leq 0$ (because E is a semistable orthogonal bundle and N' is isotropic). Recalling that if $\mathsf{k}_{F,E} \leq 1$ then $1 \leq \mathrm{rk}(F) \leq \lfloor \frac{r}{2} \rfloor$, if $\mathsf{k}_{F,E} = 2$ there are no conditions on the rank of F but in any case we have:

$$\frac{\deg(F) - \delta\,\mathsf{k}_{F,E}}{\mathrm{rk}(F)} \leq -\frac{\delta\,\mathsf{k}_{F,E}}{\mathrm{rk}(F)} \leq -\frac{2\delta}{r} = \frac{\deg(E) - 2\delta}{r}.$$

Now let $0 \subset E_i \subset E_j \subset E$ be a critical filtration. Then E_i is isotropic and $E_j \subseteq E_i^{\perp}$(otherwise the filtration would not be critical). Therefore $\mathrm{rk}(E_j) \leq \mathrm{rk}(E_i^{\perp})$ and thanks to previous calculations $\deg(E_i), \deg(E_j) \leq 0$. Finally we have

$$P_{\{i,j\}} + \delta\mu_{\{i,j\}} = -r(\deg(E_i) + \deg(E_j)) + 2\delta(r - \mathrm{rk}(E_i) - \mathrm{rk}(E_j)) \geq 0,$$

and so by Definition 82 (E, Q) is δ-semistable as a quadric bundle. ♦

Remark 93. In *"Orthogonal and spin bundles over hyperelliptic curves"* Ramanan shows that an orthogonal bundle is semistable if and only if it is semistable as a vector bundle (Proposition 4.2). So as a corollary of Theorem 92 we obtain that a non-degenerate quadric bundle of degree zero (E, Q) is semistable if and only if E is a semistable vector bundle.

5.3.1 Generalized orthogonal bundles

We will call *generalized orthogonal bundle* a quadric bundle (E, Q) such that the morphism $Q\colon Sym^2E \to \mathsf{N}$ induces an isomorphism $E \to E^{\vee} \otimes \mathsf{N}$. This isomorphism connect the degree d of E with the degree n of N, in fact one has that $d = -d + rn$ and so $n = 2\mu(E)$.

For these objects a similar result to Lemma 89 holds:

Lemma 94. *Let (E, Q) be a generalized orthogonal bundle, and let F be a proper vector subbundle of E. Then there is an exact sequence*

$$0 \to F^{\perp} \to E \to F^{\vee} \otimes \mathsf{N} \to 0,$$

so $rk(F^{\perp}) + rk(F) = r$ and

$$\deg(F) = \deg(F^{\perp}) - d\left(1 - \frac{2r_F}{r}\right).$$

Thanks to the previous lemma, one can easily prove that Theorem 92 holds also for generalized orthogonal bundles.

5.4 The splitting algorithm: a java program

We have implemented a java program to actually find a decomposition of a given weighted filtration. Using notation of Section 5.2, if (E, φ) is a decorated bundle of type (a, b, N) and rank r and $(E^{\bullet}, \underline{\alpha})$ is a weighted filtration of (E, φ) we constructed a $r^{\times a}$ "matrix" $M_{\mathtt{I}}(E^{\bullet}; \varphi)$ which represent the behavior (to be equal or different from zero) of φ over the given filtration. All such matrices have the property that:

- if $m_{i_1 \ldots i_a} = 1$ then $m_{j_1 \ldots j_a} = 1$ for any $(j_1, \ldots, j_a) \succcurlyeq (j_1, \ldots, j_a)$ (i.e., the bottom right corner of the matrix is 1);

- if $m_{i_1\dots i_a}=0$ then $m_{j_1\dots j_a}=0$ for any $(j_1,\dots,j_a)\preccurlyeq(j_1,\dots,j_a)$ (i.e., the top left corner of the matrix is 0).

Thus the elements $m_{i_1,\dots,i_a}$ such that $(i_1,\dots,i_a)$ is $\preccurlyeq$-maximal, characterize the matrix. In other words a boolean matrix satisfying the above properties is uniquely determined by the set of its $\preccurlyeq$-maximal elements. Moreover it is easy to see that the maximum of a given weighted filtration varies between the maximal elements representing the matrix, on varying the weight vector $\underline{\alpha}$. Let $(E^\bullet,\underline{\alpha})_{\mathtt{I}}$ be a weighted filtration indexed by $\mathtt{I}$ and $M_{\mathtt{I}}=((i_1^{(1)},\dots,i_a^{(1)}),\dots,(i_1^{(s)},\dots,i_a^{(s)}))$ be the matrix associated with such filtration with respect to a fixed decorated bundle (E,φ). Suppose that the maximum $R_{\mathtt{I}}$ is attained in the a-tupla $(i_1^{(h)},\dots,i_a^{(h)})$, then $R_{\mathtt{I}}(i_1^{(h)},\dots,i_a^{(h)})\geq R_{\mathtt{I}}(i_1^{(h')},\dots,i_a^{(h')})$ for any other $h'=1,\dots,s$. Since

$$R_{\mathtt{I}}(i_1^{(t)},\dots,i_a^{(t)})=\sum_{j=1}^{a}R_{\mathtt{I}}(i_j^{(t)})=\sum_{j=1}^{a}\left(\sum_{l\in\mathtt{I},\,l\geq i_j^{(t)}}\alpha_l\right)$$

the inequalities $R_{\mathtt{I}}(i_1^{(h)},\dots,i_a^{(h)})\geq R_{\mathtt{I}}(i_1^{(h')},\dots,i_a^{(h')})$, on varying of h', imply conditions on the weight vector $\underline{\alpha}$, namely:

$$\sum_{j=1}^{a}\left(\sum_{l\in\mathtt{I},\,l\geq i_j^{(h)}}\alpha_l\right)\geq\sum_{j=1}^{a}\left(\sum_{l\in\mathtt{I},\,l\geq i_j^{(h')}}\alpha_l\right)\qquad\text{for any } h'=1,\dots,s. \tag{5.5}$$

Denote by $\Theta_{M_{\mathtt{I}}}(i_1^{(h)},\dots,i_a^{(h)})$ the set of all possible weight vectors for which the maximum of the filtation is still attained in $(i_1^{(h)},\dots,i_a^{(h)})$, i.e.,

$$\Theta_{M_{\mathtt{I}}}(i_1^{(h)},\dots,i_a^{(h)})\doteqdot\{\underline{\alpha}=(\alpha_1,\dots,\alpha_r)\,|\,\text{inequalities (5.5) are fulfilled}\}.$$

Then the idea of the algorithm is the following:

- first it creates all possible matrices of rank r represented as a number of maximal elements, i.e., a certain number of vectors of length a.
- then it sets a matrix $M=((i_1^{(1)},\dots,i_a^{(1)}),\dots,(i_1^{(s)},\dots,i_a^{(s)}))$ and try to decompose it in the following way:
  - generates all possible pairs of distinguish subsets $\mathtt{J},\mathtt{K}$ of $\mathtt{I}$ such that $\mathtt{J}\cup\mathtt{K}=\mathtt{I}$.
  - Extracts the submatrixes $M'_{\mathtt{J}}=((j_1^{(1)},\dots,j_a^{(1)}),\dots,(j_1^{(s')},\dots,j_a^{(s')}))$ and $M''_{\mathtt{K}}=((k_1^{(1)},\dots,k_a^{(1)}),\dots,(k_1^{(s'')},\dots,k_a^{(s'')}))$ of M with indexes $j_i^{(s)}\in\mathtt{J}$ and $k_i^{(s)}\in\mathtt{K}$.

- Fix a $\preccurlyeq$-maximal element $m = (i_1^{(h)}, \dots, i_a^{(h)})$ of the matrix, namely fix a possible maximum of the matrix.
- Generates all possible couples (m', m'') such that $m' \in \{(j_1^{(1)}, \dots, j_a^{(1)}), \dots, (j_1^{(s')}, \dots, j_a^{(s')})\}$ and $m'' \in \{(k_1^{(1)}, \dots, k_a^{(1)}), \dots, (k_1^{(s'')}, \dots, k_a^{(s'')})\}$
- Verifies if $R_{\mathtt{I}}(m) = R_{\mathtt{J}}(m') + R_{\mathtt{K}}(m'')$ or not.
- For the couples (m', m'') such that $R_{\mathtt{I}}(m) = R_{\mathtt{J}}(m') + R_{\mathtt{K}}(m'')$, it verifies if there exists weight vectors $\underline{\alpha}' \in \Theta_{M'_{\mathtt{J}}}(m')$ and $\underline{\alpha}'' \in \Theta_{M''_{\mathtt{K}}}(m'')$ such that $\alpha' + \alpha'' \in \Theta_{M_{\mathtt{I}}}(m)$.

I tested the algorithm for $r \leq 10$ and $a = 2$ and it works. Instead if $a = 3$ and for example $r = 5$ the total number of matrixes is 2079, the number of possible cases, i.e., the number of possible maximum multiplied by the number of matrixes is 7128. In 5992 cases the algorithm succeded in finding two sets $\mathtt{J}$ and $\mathtt{K}$ such that $\mathtt{J} \cup \mathtt{K} = \mathtt{I}$, $R_{\mathtt{I}} = R_{\mathtt{J}} + R_{\mathtt{K}}$ and $\alpha' + \alpha'' \in \Theta_{M_{\mathtt{I}}}(m)$, while in 1136 cases the algorithm fails.

5.5 The case $a \geq 3$

First we consider the following example.

Example 95 ($a = 3$ $r = 5$: a counterexample?)**.** Let (E, φ) be a decorated vector bundle of type $(3, 1, 0, \mathsf{N})$, let $(E^\bullet, \underline{\alpha})$ be a weighted filtration indexed by $\mathtt{I} = \{1, 2, 3, 4\}$ and suppose that $M_{\mathtt{I}} \doteqdot M_{\mathtt{I}}(\mathcal{E}^\bullet, \varphi)$ is the following "matrix":

$$M_{\mathtt{I}}=(m_{ijk})=\underset{i=1}{\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&1&1\\0&0&1&1&1\end{pmatrix}}\underset{i=2}{\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&1\\0&0&1&1&1\\0&0&1&1&1\end{pmatrix}}\underset{i=3}{\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&1&1&1\\0&0&1&1&1\\0&0&1&1&1\end{pmatrix}}$$

Finally suppose that $R_{\mathtt{I}} = R_{\mathtt{I}}(2) + R_{\mathtt{I}}(3) + R_{\mathtt{I}}(4) = \alpha_2 + 2\alpha_3 + 3\alpha_4$. Then we have the following inequalities:

$$\begin{aligned} R_{\mathtt{I}}(2) + R_{\mathtt{I}}(3) + R_{\mathtt{I}}(4) &\geq R_{\mathtt{I}}(1) + R_{\mathtt{I}}(3) + R_{\mathtt{I}}(r) \longleftrightarrow \alpha_4 \geq \alpha_1 \\ R_{\mathtt{I}}(2) + R_{\mathtt{I}}(3) + R_{\mathtt{I}}(4) &\geq R_{\mathtt{I}}(1) + R_{\mathtt{I}}(4) + R_{\mathtt{I}}(4) \longleftrightarrow \alpha_3 \geq \alpha_4 \\ R_{\mathtt{I}}(2) + R_{\mathtt{I}}(3) + R_{\mathtt{I}}(4) &\geq R_{\mathtt{I}}(3) + R_{\mathtt{I}}(3) + R_{\mathtt{I}}(3) \longleftrightarrow \alpha_2 \geq \alpha_3. \end{aligned}$$

Consider now the subfiltrations indexed by $\mathtt{J}^{(1)} = \{1, 2, 3\}$, $\mathtt{J}^{(2)} = \{1, 2, 4\}$ and $\mathtt{J}^{(3)} = \{1, 3, 4\}$ with weights vectors $(\alpha_1^{(1)}, \alpha_2^{(1)}, \alpha_3^{(1)})$, $(\alpha_1^{(2)}, \alpha_2^{(2)}, \alpha_4^{(2)})$

and $(\alpha_1^{(3)}, \alpha_3^{(3)}, \alpha_4^{(3)})$ respectively. An easy calculation shows that

$$R_{\mathtt{J}^{(1)}} = \begin{cases} R_{\mathtt{J}^{(1)}}(1,3,r) \text{ if } \alpha_1^{(1)} \geq \alpha_3^{(1)} \\ R_{\mathtt{J}^{(1)}}(3,3,3) \text{ otherwise} \end{cases}$$

$$R_{\mathtt{J}^{(2)}} = R_{\mathtt{J}^{(2)}}(1,4,4)$$

$$R_{\mathtt{J}^{(3)}} = \begin{cases} R_{\mathtt{J}^{(3)}}(1,3,r) \text{ if } \alpha_3^{(3)} \geq \alpha_4^{(3)} \text{ and } \alpha_1^{(3)} \geq \alpha_3^{(3)} + \alpha_4^{(3)} \\ R_{\mathtt{J}^{(3)}}(1,4,4) \text{ if } \alpha_3^{(3)} \leq \alpha_4^{(3)} \text{ and } \alpha_1^{(3)} \geq 2\alpha_3^{(3)} \\ R_{\mathtt{J}^{(3)}}(3,3,3) \text{ otherwise.} \end{cases}$$

In any case, for any $s = 1, 2, 3$ and for any other subfiltration indexed by $\mathtt{K}$ such that $\mathtt{J}^{(s)} \cup \mathtt{K} = \mathtt{I}$, one can check that the index 1 or 3 appear too much times, i.e., $R_{\mathtt{I}} \neq R_{\mathtt{J}^{(s)}} + R_{\mathtt{K}}$. Therefore is not possible to split the filtration $(E^{\bullet}, \underline{\alpha})_{\mathtt{I}}$ into two subfiltrations $(E^{\bullet}, \underline{\alpha}')_{\mathtt{I}'}$, $(E^{\bullet}, \underline{\alpha}'')_{\mathtt{I}''}$ in such a way that $\underline{\alpha}_{\mathtt{I}} = \underline{\alpha}'_{\mathtt{I}'} + \underline{\alpha}''_{\mathtt{I}''}$ and $R_{\mathtt{I}} = R_{\mathtt{I}'} + R_{\mathtt{I}''}$.

The above example provides a counter example to Theorem 84 in the case of $a = 3$. In ather words, using the notation of Section 5.2, let $\mathtt{I}$ be a well-ordered set, $\alpha_{\mathtt{I}}$ be a vector of positive real numbers and $M_{\mathtt{I}} = (m_{ijk})_{i,j,k \in \mathtt{I}}$ be a "boolean" simmetric cube, i.e., $m_{ijk} = m_{\sigma(i)\sigma(j)\sigma(k)} = 0$ or $= 1$ for any permutation $\sigma : \{i,j,k\} \to \{i,j,k\}$ and any $i,j,k \in \mathtt{I}$. Then it is not true that there exists $t \in \mathbb{N}$, sets $\mathtt{J}^{(1)}, \ldots, \mathtt{J}^{(t)}$ and positive real vectors $\alpha_{\mathtt{J}^{(1)}}, \ldots, \alpha_{\mathtt{J}^{(t)}}$ such that

i) $|\mathtt{J}^{(s)}| \leq 3$ for any $s = 1, \ldots, t$;

ii) $\mathtt{J}^{(1)} \cup \cdots \cup \mathtt{J}^{(t)} = \mathtt{I}$;

iii) $\sum_{s=1}^{t} \alpha_i^{(s)} = \alpha_i$, where is to be understood that $\alpha_i^{(s)} = 0$ if $i \notin \mathtt{J}^{(s)}$;

iv) $\sum_{s=1}^{t} R_{\mathtt{J}^{(s)}} = R_{\mathtt{I}}$.

The above conditions clearly imply, as showed in Section 5.2, that the semistability of a decorated bundle (E, φ) could be checked only over subfiltration of length ≤ 3 but the converse is not clear that holds true. Therefore the above example does not provide a counterexample to the following statement:

Let $\underline{\mathfrak{t}} \div (a, b, \mathsf{N})$ be a fixed type of decorated bundles. Then there exist a fixed natural number $s_{\underline{\mathfrak{t}}} \in \mathbb{N}$, depending on $\underline{\mathfrak{t}}$, such that for any decorated bundle (E, φ) of type $\underline{\mathfrak{t}}$ the following conditions are equivalent:

- $P_{\mathtt{I}} + \delta\mu_{\mathtt{I}} \geq 0$ *for any weighted filtration* $(E^{\bullet}, \underline{\alpha})_{\mathtt{I}}$*;*
- $P_{\mathtt{I}} + \delta\mu_{\mathtt{I}} \geq 0$ *for any weighted filtration* $(E^{\bullet}, \underline{\alpha})_{\mathtt{I}}$ *of length* $\leq a + s_{\underline{\mathfrak{t}}}$.

Until now we have not been able to prove the above statement.

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