Mathematical Physics Sector
SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI
INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

Some topics on Higgs bundles over projective varieties and their moduli spaces

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Submitted in partial fulfillment of the
requirements for the degree of
"Doctor Philosophiæ"

Academic Year 2012/2013

to my mother and my father,
to my sister and my aunt,
to Valentina,
to my pets...

Archimedes will be remembered when Aeschylus is forgotten,
because languages die and mathematical ideas do not.
"Immortality" may be a silly word,
but probably a mathematician has the best chance of whatever it may mean.

G. H. Hardy

Acknowledgment

First of all, I would like to thank my supervisor Ugo Bruzzo, who first suggested the problem of this thesis and then helped in numerous ways during the course of the research. He taught me the most that I have learned about this topic and even more.

Then I would like to thank Giulio Battaglia, Jamir Marino, Alex Massarenti and Andrea Pustetto, the time spent in Trieste wouldn't be the same without them.

I want to thank Valentina for her love and smiles and for coffee every morning!

Finally I am grateful to my family for helping me become who I am now.

Contents

Abstract

In this thesis we study vector bundles on projective varieties and their moduli spaces. In Chapters 2, 3 and 4 we recall some basic notions as Higgs bundles, decorated bundles and generalized parabolic sheaves and introduce the problem we want to study. In chapter 5, we study Higgs bundles on nodal curves. After moving on the normalization of the curve, starting from a Higgs bundle we obtain a generalized parabolic Higgs bundle. Using similar techniques of Schmitt ([47]) we are able to construct a projective moduli space which parametrizes equivalence classes of Higgs bundles on a nodal curve X. This chapter is an extract of a joint work with Andrea Pustetto ([22])

Later on Chapter 6 is devoted to the study of holomorphic pairs (or twisted Higgs bundles) on elliptic curve. Holomorphic pairs were introduced by Nitsure ([40]) and they are a natural generalization of the concept of Higgs bundles. In this Chapter we extend a result of E. Franco, O. García-Prada And P.E. Newstead ([16]) valid for Higgs bundles to holomorphic pairs.

Finally the last Chapter describes a joint work with Professor Ugo Bruzzo. We study Higgs bundles over varieties with nef tangent bundle. In particular generalizing a result of Nitsure we prove that if a Higgs bundle $\mathcal{E} = (E, \phi)$ over the variety X with T_X nef satisfies the condition : $f^*\mathcal{E}$ semistable for any morphism $f\colon C \to X$ where C is a smooth projective curve, then we must have $2rc_2(E) - (r-1)c_1^2(E) = 0$. In final part using similar results we show that the underlying vector bundle of a μ-semistable Higgs bundle on a Calabi-Yau manifold is μ-semistable.

Chapter 1

Introduction

1.1 Historical background

The notion of holomorphic vector bundle is common to some branches of mathematics and theoretical physics. In particular, this notion plays a fundamental role in complex differential geometry, algebraic geometry and Gauge theory. In this thesis we study more general objects than holomorphic vector bundles over algebraic varieties, "i.e." Higgs bundles, and some of their main properties. A Higgs bundle is a pair (E, ϕ) where E is a vector bundle over a projective variety X and $\phi \in \mathrm{H}^0(X, \mathrm{End}(E) \otimes \Omega^1_X)$ is a section satisfying the integrability condition $\phi \wedge \phi = 0$. For these objects the Narasimhan-Seshadri correspondence asserts that a flat Higgs bundle over X, is stable if and only if it corresponds to an irreducible representation of the fundamental group of X.

Higgs bundles were introduced in 1987 by Hitchin ([20]) in order to study Yang-Mills equation on Riemann surfaces. Eventually they have found applications in many areas of mathematics and mathematical physics. In particular, Hitchin showed that their moduli spaces give examples of Hyper-Kähler manifolds and that they provide an interesting example of integrable systems. The moduli space $\mathcal{M}_G$ of polystable principal G-Higgs bundles over a compact Riemann surface X, for G a real form of a complex semisimple Lie group G^c, may be identified through non-abelian Hodge theory with the moduli space of representations of the fundamental group of X (or certain central extensions of it) into G. Motivated partially by this identification, the moduli space of G-Higgs bundles has been studied by various researchers. Following the ideas of Hitchin, Simpson ([53]) has defined Higgs bundles on higher dimensional varieties and related them with representations of the fundamental group of the base variety.

A main problem in the theory of vector bundles on curves is to classify all of them in terms of discrete or continuous data. This was done for genus 0 curves by Grothendieck and for genus 1 curves by Atiyah in [1] where the author classifies indecomposable vector bundles over an elliptic curve. However the case of genus at least 2 has proved to be much more difficult. We will concentrate on the complex affine space of holomorphic structures on a fixed C^∞ complex vector bundle E over X of rank r and degree d. Actually to find an algebraic variety (or at least a scheme) which parametrizes all vector bundles is impossible, so one has to consider a sub family of the family of all vector bundles, namely, (*semi*)*stable* bundles. Mumford has proved that, on the space $\mathcal{M}^s(r,d)$ of (isomorphic classes) stable vector bundles on X of rank r and degree d, there is a natural structure of a non-singular quasi-projective variety ([34]). In [45] Seshadri gave a structure of normal projective variety to the space of unitary vector bundles, hence, using the Narashiman-Seshadri correspondence he gave a natural compactification of the space of stable vector bundles of degree d and rank r on X. This space is called the moduli space of semistable vector bundles and we denote it by $\mathcal{M}^{ss}(r,d)$. In particular $\mathcal{M}^{ss}(1,d)$ is the Jacobian of the curve X which is an Abelian variety of dimension $g(X)$ the genus of the curve.
In general $\mathcal{M}^{ss}(r,d)$ is a projective variety of dimension $r^2(g-1)+1$. Moreover if $(r,d)=1$ then $\mathcal{M}^{ss}(r,d)$ is a smooth variety isomorphic to $\mathcal{M}^s(r,d)$.

Hitchin showed that the moduli space of solutions to Hitchin's self-duality equations is isomorphic to the moduli space $\mathcal{M}^s_H(2,d)$ of rank 2 stable Higgs bundles with trace-free Higgs field and fixed determinant of odd degree. $\mathcal{M}^s_H(2,d)$ is a noncompact, smooth complex manifold of complex dimension $6g-6$ containing $T^*_{\mathcal{M}^s(2,d)}$ as a dense open set. The non compactness of $\mathcal{M}^s_H(2,d)$ is due essentially to the $\mathbb{C}^*$ action on it, defined by $z\cdot(E,\phi)=(E,z\cdot\phi)$. Here we give an algebraic compactification of the moduli space of principal Higgs G-bundles on nodal curves following the idea of Schmitt ([47]). The strategy is in some sense to embed the affine piece of the moduli space into a projective space and then consider its closure. When G is the general linear group and the curve is smooth one obtains a compact space which classifies pairs $\mathcal{E}=(E,[z,\phi])$ where z is a complex number and two pairs $\mathcal{E}_1$ and $\mathcal{E}_2$ are equivalent if there exists some nonzero complex number $\lambda\in\mathbb{C}^*$ such that $\lambda\cdot\mathcal{E}_1=\mathcal{E}_2$, where the action of $\mathbb{C}^*$ on such triples is given by $\lambda\cdot(E,[z,\phi])=(E,[\lambda z,\lambda\cdot\phi])$. So if $z\neq 0$ a pair $(E,[z,\phi])$ can be identify with a classical Higgs bundle while for $z=0$ ones get a Higgs bundle with a degenerating Higgs field.

Given a vector bundle E on a polarized smooth projective variety (X,H) there is a numerical invariant which allows us to predict if it could be semistable or not, namely its discriminant $\Delta(E) = c_2(E) - \frac{(r-1)}{2r} c_1^2(E)$. In fact one has that if E is semistable with respect to H then $\Delta(E) \cdot \mathrm{H}^{n-2} \geq 0$, this is called the Bogomolov's inequality. It may be interesting to classify the semistable bundles that satisfy the stronger equality $\Delta(E) = 0$. It was proved in [12] (see also [36, 8]) that these bundles are those whose normalized tautological divisor is numerically effective, and this condition turns out to be equivalent to the fact that for any morphism $f\colon C \to X$ from a smooth projective curve, the pullback f^*E is semistable. In [11] the authors proved that the pullback $f^*\mathcal{E}$ of a semistable Higgs bundles $\mathcal{E} = (E, \phi)$ with vanishing discriminant is semistable for any morphism $f\colon C \to X$. The other implication — i.e., if $f^*\mathcal{E}$ is semistable for all morphism $f\colon C \to X$ then $\mathcal{E}$ is semistable and $\Delta(E) = 0$ — is only conjectured (see also [56]). Here we give a proof of this fact for some special classes of varieties X.

1.2 About this thesis

This thesis is organized as follows.

In Chapter 2 we recall some basic results about Higgs bundles on smooth curves. In particular we give the definitions of semistability for vector and principal Higgs bundles, and introduce the concept of holomorphic pair or Twisted Higgs bundle ([40] and [18]). Then we describe the main properties of the moduli space of Higgs bundles and explain how it is related to representations of the fundamental group of the curve.

In Chapter 3 we define decorated vector bundles. This notion will help us to study the moduli space of principal Higgs bundles. Decorated bundles were introduced by Schmitt in [49] to compactify the moduli space of principal bundles. Roughly speaking, a decorated bundle is a pair (E, φ), where E is a vector bundle and

$$\varphi\colon (E^{\otimes a})^{\oplus b} \otimes (\det E)^c \to L,$$

a morphism of vector bundles, where a, b, c are integers and L is a line bundle. It is possible to give a notion of semistability for decorated bundles in a such way that the moduli space of semistable decorated bundles is a projective variety. For suitable choices of φ the pair (E, ϕ) can encode the structure of a principal bundle, "i.e.", the datum of (E, φ) is equivalent to giving a principal bundle P. Moreover, in this case the notion of semistability for decorated bundles is equivalent to the notion of semistability for principal bundles. Finally one can choose φ so that the decorated bundle (E, φ) is precisely a Higgs bundle and also in this case the notions of semistability agree.

Chapter 4 is devoted to the study of vector bundles on singular curves with more emphasis on the nodal case. Seshadri in 1982 showed that vector bundles on a nodal curve X are related to particular vector bundles on $\widetilde{X}$, the normalization of the curve. These objects are called generalized parabolic vector bundles and consist of pairs (E, q), where E is a vector bundle on $\widetilde{X}$ and $q\colon \bigoplus E_{x_i} \to R$ is a morphism from the direct sum of the fibre of E on the preimages of the singular points to a vector space R of dimension $\mathrm{rk}(E)$. In this correspondence semistable torsion-free sheaves on the curve X are related to semistable generalized parabolic vector bundles on $\widetilde{X}$ (see [5]).

In Chapter 5 we study the moduli space of principal Higgs bundles on nodal curves. Using the theory of decorated bundles we are able to 'transform' a principal Higgs bundle in a triple $(\mathcal{E}, \varphi_1, \varphi_2)$, where the morphism φ_1 encodes the principal bundle structure, while φ_2 corresponds to the Higgs field. Later on, we consider the normalization of the curve so that, using the results of Seshadri and Bhosle, we obtain a quadruple $(E, \varphi_1, \varphi_2, q)$. Combining the semistability condition for decorated bundles with the semistability condition given for generalized parabolic bundles we define a notion of semistability for such quadruples. The main result now is that the condition we give is equivalent to the semistability of principal Higgs bundles. Thanks to this we are able to construct a projective moduli space for such objects. Let us observe that in order to construct a decorated bundle starting from a principal G-bundle we have to fix a representation $\rho\colon G \to SL(V)$. If the group G is isomorphic to $SL(V)$ and we consider the identity representation then the induced morphism φ_1 is identically zero. In this case our moduli space is exactly the moduli space of generalized parabolic Higgs vector bundles.

In Chapter 6 we introduce new objects which extend the notion of holomorphic pairs, namely, semistable t-uples, "i.e.", t-uples $(E, \phi_1, \dots, \phi_n)$ with $\phi_i\colon E \to E \otimes L_i$ and we give a semistability condition for them. We extend some results of Nitsure to these objects and as an application we give a description of (semi)stable t-uples on curves when $\deg(L_i) = 0$ for all i. Finally we show that, if the line bundle L has degree zero, a pair (E, ϕ) over an elliptic curve is stable if and only if the underlying vector bundle is stable in the classical way, extending the same result for Higgs bundles given by E. Franco, O. García-Prada and P.E. Newstead in [16]. If the genus of the curve is grater then 1 we show that a stable pair (E, ϕ) with $\phi\colon E \to E \otimes L$ and $\deg(L) = 0$ is an extension of stable vector bundles.

Finally in Chapter 7 we study Higgs vector bundles over higher dimensional varieties. It is known (see [12]) that a μ-semistable vector bundle E on a variety X remains semistable when pulled-back to any smooth curve $f\colon C \to X$ if and only if satisfies

$$c_2(E) - \frac{(r-1)}{2r} c_1^2(E) = 0 \qquad \text{in } \mathrm{H}^4(X, \mathbb{R}).$$

For Higgs bundle it is known only the 'if' part while the other direction is only conjectured ([11]). Using similar techniques to Chapter 6 we prove this fact for a certain classes of classical varieties such as rationally connected varieties and abelian varieties. Moreover we extend this result to any finite quotient and rationally connected fibration of these varieties, and so we are able to give a proof for Higgs bundles on varieties with nef tangent bundle.
In the last part of this chapter we relate the μ-semistability of a Higgs bundle on a Calabi-Yau to the μ-semistability of the underlying vector bundle extending the same result for abelian varieties given in [7].

Chapter 2

General results on Higgs bundles

Notation

In this thesis by a variety X we mean a projective, irreducible, reduce scheme over the complex field. We will denote by $E \to X$ vector bundles on X, "i.e." locally free sheaves over X, while we will use the script character $\mathcal{E}$ to indicate any coherent sheaf or vector bundles with extra structures.
As usual the degree d of a vector bundle of rank r is defined as the degree of its determinant bundle $\bigwedge^r E \cong \det(E) \in \mathrm{Pic}(X)$, while the degree of any coherent sheaf is defined using free resolutions.
After fixing an ample line bundle $\mathcal{O}_X(1)$, for any sheaf $\mathcal{E}$ and integer $m \in \mathbb{Z}$ we define

$$\mathcal{E}(m) \colon = \mathcal{E} \otimes \mathcal{O}_X(m).$$

Finally whenever the word '(semi)stable' appears in a statement with the symbol '$(\leq)$', two statements should be read. The first with the word 'stable' and strict inequality, and the second with the word 'semistable' and the relation '$\leq$'.

2.1 Higgs vector bundles on smooth curves

Let X be a compact Riemann surface of genus g. We denote by K the canonical bundle of X. For a holomorphic vector bundle E we denote by d and r respectively its degree and rank and we define $\mu(E) := \frac{d}{r}$ the slope of E.

Definition 2.1. A Higgs vector bundle (respectively sheaf) on X is a pair $\mathcal{E} = (E, \phi)$ consisting of a holomorphic vector bundle (respectively sheaf) E on X, and a section $\phi : X \to \mathrm{End}(E) \otimes \mathrm{K}$, called the Higgs field.

The section ϕ can be viewed as a Higgs field of the dual bundle of E, so given a Higgs bundle $\mathcal{E}$ there is a natural notion of the dual $\mathcal{E}^\vee = (E^\vee, \phi)$. Moreover ϕ

induces in a natural way an element of $\mathrm{Hom}(E, E \otimes \mathrm{K})$ which we will denote still by ϕ.

Definition 2.2. A morphism $f : (E_1, \phi_1) \to (E_2, \phi_2)$ between two Higgs bundles is a homomorphism of vector bundles $f \in \mathrm{Hom}(E_1, E_2)$ such that the following diagram commutes:

$$\begin{tikzcd} E_1 \arrow[r, "\phi_1"] \arrow[d, "f"'] & E_1 \otimes \mathrm{K} \arrow[d, "f\otimes \mathrm{id}_{\mathrm{K}}"] \\ E_2 \arrow[r, "\phi_2"] & E_2 \otimes \mathrm{K} \end{tikzcd} \tag{2.1}$$

Moreover we say that E_1 is a Higgs subsheaf of E_2 if f is injective. We denote this by $E_1 \subset E_2$. In this case we can easily construct the quotient Higgs sheaf $(E_2/E_1, \overline{\phi})$ together with a surjective morphism of Higgs vector bundles $q\colon E_2 \to E_2/E_1$ whose kernel is exactly E_1.

Let us observe that if F is a Higgs subsheaf of (E, ϕ) then the pair $(F, \phi|_F)$ is actually a Higgs sheaf since $\phi(F) \subset F \otimes \mathrm{K}$. In a such situation we will say that F is a ϕ-invariant subsheaf of E.
As in the case of vector bundles, given two Higgs vector bundles $\mathcal{E}_1 = (E_1, \phi_1)$ and $\mathcal{E}_2 = (E_2, \phi_2)$ one can construct the direct sum

$$\mathcal{E}_1 \oplus \mathcal{E}_2\colon = (E_1 \oplus E_2, \phi_1 \oplus \phi_2)$$

and the tensor product

$$\mathcal{E}_1 \otimes \mathcal{E}_2\colon = (E_1 \otimes E_2, \phi_1 \otimes \mathrm{id}_{E_2} + \mathrm{id}_{E_1} \otimes \phi_2).$$

Definition 2.3. A Higgs bundle (E, ϕ) is (semi)stable if and only if for any ϕ-invariant subsheaf $F \subset E$ one has:

$$\mu(F)(\leq)\mu(E) \tag{2.2}$$

(E, ϕ) is called polystable if it is semistable as a Higgs bundle and it is isomorphic to a direct sum of stable Higgs bundles.

Example 2.4. Let X be a smooth projective curve of genus $g(X) > 1$ and let $E = \mathrm{K}^{\frac{1}{2}} \oplus \mathrm{K}^{-\frac{1}{2}}$, where $\mathrm{K}^{\frac{1}{2}}$ is a complex line bundle whose square is K. We obtain a family of Higgs fields on E parametrized by quadratic differentials, "i.e.", sections of the line bundle $\mathrm{K}^2 \simeq \mathrm{Hom}(\mathrm{K}^{-\frac{1}{2}}, \mathrm{K}^{\frac{1}{2}} \otimes \mathrm{K})$, by setting

$$\phi = \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix}$$

where 1 is the identity section of the trivial bundle $\mathrm{Hom}(\mathrm{K}^{\frac{1}{2}}, \mathrm{K}^{-\frac{1}{2}} \otimes \mathrm{K})$.

(E, ϕ) is a stable Higgs bundle since $\mathrm{K}^{\frac{1}{2}}$ is not ϕ-invariant and there are no subsheaves of positive degree preserved by ϕ. However the underlying vector bundle is not semistable.

We have the following results of semistability for Higgs vector bundles on curves.

Proposition 2.5. *The direct sum of two semistable Higgs bundles of the same slope is semistable.*

Proposition 2.6. *The tensor product of two semistable Higgs bundles is semistable.*

Proposition 2.7. *([12]) Let C and C' be two smooth connected projective curves and $f\colon C' \to C$ a finite unramified map. Let $\mathcal{E} = (E, \phi)$ be a Higgs vector bundle over C. Then $\mathcal{E}$ is semistable if and only if $f^*(\mathcal{E})\colon = (f^*(E), f^*(\phi))$ is semistable.*

2.1.1 Filtrations

Theorem 2.8. *Given a Higgs vector bundle (E, ϕ) over X, there is a unique strictly increasing filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{t-1} \subset E_t = E \tag{2.3}$$

of Higgs subsheaves such that for each $i = 1, \ldots, t$, the quotient E_i/E_{i-1}, equipped with the Higgs field induced by ϕ is Higgs semistable, and furthermore,

$$\mu(E_1) > \mu(E_2/E_1) > \cdots > \mu(E_j/E_{j-1}) > \cdots > \mu(E/E_{t-1}) > \mu(E).$$

This filtration is known as the Higgs Harder-Narasimhan filtration of (E, ϕ). So semistable Higgs bundles are fundamental blocks for Higgs vector bundles. Now we want to show that any semistable Higgs bundle admits a filtration in which every element is a stable Higgs vector bundle.

Let us consider a semistable Higgs vector bundle (E, ϕ). If it is stable then we finish, otherwise there exists a minimal ϕ-invariant subbundle $F_1 \subset E$ with $\mu(F_1) = \mu(E)$. The Higgs bundle (F_1, ϕ) is clearly stable and the quotient $(E/F_1, \overline{\phi})$ is semistable where $\overline{\phi}$ is induced by ϕ. Iterating this process at the end one obtains a filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{s-1} \subset F_s = E$$

where, for any $i = 1, \ldots, s$, $\mu(F_i/F_{i-1}) = \mu(E)$ and the induced Higgs bundle $(F_i/F_{i-1}, \overline{\phi_i})$ is stable.

This is the Jordan-Hölder filtration, whenever it is not unique, the graded Higgs sheaf

$$\mathrm{Gr}(E,\phi)\colon=\bigoplus_{i=1}^{r}(F_i/F_{i-1},\overline{\phi_i})$$

is unique up to isomorphism (see [53]).

Definition 2.9. Two semistable Higgs vector bundles (E_1,ϕ_1) and (E_2,ϕ_2) are said to be S-equivalent if $\mathrm{Gr}(E_1,\phi_1)\simeq\mathrm{Gr}(E_2,\phi_2)$.

Remark 2.10. If (E,ϕ) is stable then $\mathrm{Gr}(E,\phi)\simeq(E,\phi)$ and so two stable Higgs vector bundles are S-equivalent if and only if they are isomorphic.

2.1.2 Semistable pairs

A holomorphic pair (or L-Twisted Higgs bundles) is a pair (E,ϕ), where E is a vector bundle on X and $\phi\colon E\to E\otimes L$ a morphism of vector bundles with $L\in\mathrm{Pic}(X)$. These objects where introduced by N. Nitsure ([40]) and consequently studied by O. Garcìa-Prada, P. B. Gothen and I. Mundet i Riera ([18]). The notions of morphism between two holomorphic pairs, of semistability for them, of Harder-Narasimhan and Jordan-Hölder filtrations are the same given for Higgs bundles.

We have this useful result for semistable pairs (see [40]).

Proposition 2.11. *Let (E,ϕ) be a semistable pair with $\phi\not\equiv 0$, then we have $\deg L\geq 0$, and for all the successive quotients E_i/E_{i-1} of the Harder-Narasimhan filtration of E, the following inequality holds,*

$$\mu(E)-\frac{(r-1)^2}{r}\deg L\leq\mu(E_i/E_{i-1})\leq\mu(E)+\frac{(r-1)^2}{r}\deg L,$$

in particular if $\deg L=0$ then E is semistable.

The previous Proposition implies that the slopes which can occur as slopes of elements of the Harder-Narasimhan filtration of a semistable pair (E,ϕ) are bounded from below. The following Proposition shows that actually this condition implies that the same fact holds for any subsheaf of E.

Proposition 2.12. *Let E be a vector bundle on X and let $0\subset E_1\cdots\subset E_t=E$ be its Harder Narasimhan filtration. If there exists $\alpha\in\mathbb{R}$ such that*

$$\mu(E_i)\leq\mu(E)+\alpha\frac{(r-r_i)}{r_i},\qquad \textit{for any } i=1\dots t$$

then for any $F \subset E$,

$$\mu(F) \leq \mu(E) + \alpha \frac{(r - r_F)}{r_F},$$

where $r_i = rk(E_i)$ and $r_F = rk(F)$.

Proof. see [27] □

Corollary 2.13. If E is a semistable holomorphic pair then for any $F \subset E$ one has

$$\mu(F) \leq \mu(E) + \frac{(r-1)^2}{r} \deg L.$$

2.2 Higgs Principal G-bundles

Let G be a reductive complex algebraic group, and let $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ be the adjoint representation of G on its Lie algebra $\mathfrak{g}$. .

Definition 2.14. A principal Higgs G-bundle over X is a pair $\mathcal{P} = (P, \phi)$, where P is a principal G-bundle over X, and ϕ is a global section of $\mathrm{Ad}(P) \otimes \mathrm{K}$.

When G is the general linear group, under the identification $\mathrm{Ad}(P) \simeq \mathrm{End}(E)$, where E is the vector bundle corresponding to P, a principal Higgs G-bundle is a Higgs vector bundle in the sense of Definition 2.1.

A morphism between two principal Higgs bundles (P_1, ϕ_1) and (P_2, ϕ_2) is a principal bundle morphism $f\colon P_1 \to P_2$ such that $(\tilde{f} \otimes \mathrm{id})(\phi_1) = \phi_2$, where $\tilde{f}\colon \mathrm{Ad}(P_1) \to \mathrm{Ad}(P_2)$ is the induced morphism between the adjoint bundles.

If K is a closed subgroup of G, and $\sigma\colon X \to P(G/K) \simeq P/K$ a reduction of the structure group of P to K, σ induces a principal K-bundle $P_\sigma := \sigma^*(P)$ on X. The morphism $i_\sigma\colon P_\sigma \to P$ yields an injective morphism of vector bundles $\mathrm{Ad}(P_\sigma) \to \mathrm{Ad}(P)$. Let $\Pi_\sigma\colon \mathrm{Ad}(P) \otimes \mathrm{K} \to (\mathrm{Ad}(P)/\mathrm{Ad}(P_\sigma)) \otimes \mathrm{K}$ be the projection morphism.

Definition 2.15. A section $\sigma\colon X \to P/K$ is called a Higgs reduction of (P, ϕ) if $\phi \in \ker \Pi_\sigma$.

When this happens, the reduced bundle P_σ is equipped with a Higgs field ϕ_σ compatible with ϕ "i.e.", $(P_\sigma, \phi_\sigma) \to (P, \phi)$ is a morphism of principal Higgs bundles.

Remark 2.16. Let us again consider the case when G is the general linear group $GL(n, \mathbb{C})$, and let us assume that K is a maximal parabolic subgroup, so that G/K is the Grassmann variety $\mathrm{Gr}_k(\mathbb{C}^n)$ of k-dimensional quotients of $\mathbb{C}^n$ for some k. If V is the vector bundle corresponding to E, a reduction σ of G to K corresponds to a rank $n-k$ subbundle W of V, while σ is a Higgs reduction means that W is ϕ-invariant.

Definition 2.17. [10] Let X be a smooth projective curve. A principal Higgs G-bundle (P,ϕ) is (semi)stable if for every parabolic subgroup $K \subset G$ and every Higgs reduction $\sigma\colon X \to P/K$ one has

$$\deg \sigma^*(T_{P/K,X})(\geq)0.$$

Here, $T_{P/K,X}$, the vertical tangent bundle over P/K, is defined to be the vector bundle $E(\mathfrak{g}/\mathfrak{k}) = (E \times (\mathfrak{g}/\mathfrak{k}))/K$ over P/K associated to the K-bundle $P \to P/K$ and the K action on $\mathfrak{g}/\mathfrak{k}$ ($\mathfrak{k}$ is the Lie algebra of K) induced by the adjoint representation.

Remark 2.18. If the Higgs field is zero the previous definition is equivalent to the usual definition of semistability for principal bundles due to A. Ramanathan ([42])

Remark 2.19. If G is the general linear group $GL(n,\mathbb{C})$ then we saw that a principal Higgs bundle is essentially a Higgs vector bundle. We show that in this case definitions 2.3 and 2.17 agree.
If $\mathcal{P} = (P,\phi)$ is a principal Higgs bundle with structure group $GL(n,\mathbb{C})$, we denote by $(P(\mathbb{C}^n),\phi)$ the associated Higgs vector bundle. Let $K \subset GL(n,\mathbb{C})$ be the parabolic subgroup consisting of matrices of the form:

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where $A \in GL(r,\mathbb{C})$ and $C \in GL(n-r,\mathbb{C})$. Let $\sigma\colon X \to P/K$ be a Higgs reduction of the structure group, and let

$$\begin{pmatrix} a_{\alpha,\beta} & b_{\alpha,\beta} \\ 0 & c_{\alpha,\beta} \end{pmatrix}$$

be the transition functions for the K-bundle $\sigma^*(P)$ with respect to a trivializing cover $\{U_\alpha\}$. Let F be the rank r subbundle of $P(\mathbb{C}^n)$ corresponding to $(a_{\alpha,\beta})$. Then the quotient bundle $P(\mathbb{C}^n)/F$ corresponds to $c_{\alpha,\beta}$ and we have the following natural isomorphism

$$\sigma^*(T_{P/K}) \simeq F^* \otimes P(\mathbb{C}^n)/F.$$

So we get the following result

Proposition 2.20. *A principal Higgs $GL(n,\mathbb{C})$-bundle $\mathcal{P} = (P,\phi)$ over X is (semi)stable if and only if for every ϕ-invariant subsheaf F of $P(\mathbb{C}^n)$,*

$$\mu(F)(\leq)\mu(P(\mathbb{C}^n)).$$

Proof. We shall prove the statement for the "semistable" part only. One proves the "stable" part by simply replacing every inequality by a strict inequality. Suppose $(P(\mathbb{C}^n), \phi)$ is semistable in the sense of Definition 2.3, and let $\sigma\colon X \to P/K$ be a Higgs reduction of the structure group to a maximal parabolic subgroup K of G. Then K corresponds to a two-step flag

$$0 = V_0 \subset V_1 \subset V_2 = \mathbb{C}^n.$$

Let $F = \sigma^*(P(V_1))$, we have that $\sigma^*(T_{P/K}) \simeq F^* \otimes P(\mathbb{C}^n)/F$. Since $P(\mathbb{C}^n)$ is semistable and F is ϕ-invariant, we have

$$\mu(F) \leq \mu(P(\mathbb{C}^n)),$$

which is equivalent to

$$\mu(F) \leq \mu(P(\mathbb{C}^n)/F),$$

and so

$$\deg(F^* \otimes (P(\mathbb{C}^n)/F)) \geq 0.$$

Finally

$$\deg(F^* \otimes (P(\mathbb{C}^n)/F)) =$$
$$= -\deg(F)\mathrm{rk}(P(\mathbb{C}^n)/F) + \mathrm{rk}(F)\deg(P(\mathbb{C}^n)/F =$$
$$= (\mu(P(\mathbb{C}^n)/F) - \mu(F))\mathrm{rk}(P(\mathbb{C}^n)/F)\mathrm{rk}(F),$$

hence we obtain that $\mathcal{P}$ is semistable as a principal Higgs $GL(n, \mathbb{C})$-bundle.
Conversely, let $\mathcal{P}$ be a semistable principal Higgs $GL(n, \mathbb{C})$-bundle. Any ϕ-invariant vector subbundle F of $P(\mathbb{C}^n)$ is of the form $\sigma^*(P(\mathbb{C}^n))$ for some Higgs reduction of the structure group to a parabolic subgroup K corresponding to a flag $0 = V_0 \subset V_1 \subset V_2 = \mathbb{C}^n$. Since $\mathcal{P}$ is semistable as a principal Higgs $GL(n, \mathbb{C})$-bundle, we have

$$\deg(\sigma^*(T_{P/K})) = \deg(F^* \otimes (P(\mathbb{C}^n)/F)) \geq 0.$$

and this implies $\mu(F) \leq \mu(P(\mathbb{C}^n))$. □

Definition 2.21. Let P be a principal G bundle on X and σ be a reduction of structure group of P to a parabolic subgroup K of G, then this reduction is called Harder-Narasimhan reduction if the following two conditions hold :

1. If L is the Levi factor of K then the principal L bundle $P \times L$ over X is a semistable L bundle.

2. For any dominant character χ of K with respect to some Borel subgroup $B \subset K$ of G, the associated line bundle $L_\chi = \sigma^*(P \times_\chi \mathbb{C})$ over X has positive degree.

The existence and uniqueness of Harder-Narasimhan reduction for principal Higgs bundles on nonsingular projective variety was shown in [14].

Theorem 2.22. *Let (P, ϕ) be a principal Higgs G-bundle on a nonsingular projective variety X. Then there exists a canonical Harder-Narasimhan reduction (K, σ) where K is a parabolic subgroup of G and $\sigma \colon X \to P/K$ is a Higgs section of the associated fibre bundle P/K over X. The Harder-Narasimhan reduction is unique.*

While the analogue of Jordan Hölder filtration for principal Higgs bundles on curves is due to B.Graña Otero ([41])

Theorem 2.23. *Let G be a reductive algebraic group over $\mathbb{C}$, if $\mathcal{P} = (P, \phi)$ is a semistable principal Higgs G-bundle, there exists a parabolic subgroup K of G and an admissible reduction of the structure group of P to K such that the principal Higgs bundle obtained by extending the structure group to the Levi factor $L(K)$ of K is a stable principal Higgs bundle.*

Recall that a reduction σ is admissible if for any character χ on K which is trivial on the center of G, the line bundle L_χ has degree zero.

Remark 2.24. The last part of the previous Theorem tells us that the graded object which one can construct starting from a semistable principal Higgs bundle is polystable.

2.3 The moduli space of Higgs bundles

Recall that a *family* of vector bundles on X is given by a parameter scheme S and a vector bundle $\mathfrak{F}$ on $X \times S$. $\mathfrak{F}$ is flat over S if for any $(x, s) \in X \times S$, $\mathfrak{F}_{(X,s)}$ is flat over the local ring $\mathcal{O}_{S,s}$. If S is a reduced scheme then $\mathfrak{F}$ is flat over S if and only if the Hilbert polynomial of $\mathcal{E}_s$ is locally constant as a function of s. Moreover we say that $\mathfrak{F}$ is of type (d, r) is there exist an open dense subset $U \subset S$ such that for any $s \in U$, $\mathfrak{F}_s$ is of type (d, r).

2.3.1 Bounded families of vector bundles

A flat family $\mathfrak{F}$ of isomorphism classes of vector bundles on X of type (d, r) is said to be **bounded** if S can be chosen of finite type over $\mathbb{C}$.

Proposition 2.25. *$\mathfrak{F}$ is bounded if and only if there is a natural number m_0 such that for every vector bundle $E \in \mathfrak{F}$ and every $m \geq m_0$, E is m-regular, that is, the following conditions hold :*

- *$E(m)$ is globally generated and*
- *$h^i(E(m-i)) = 0$.*

Remark 2.26. By the Riemann-Roch theorem and the vanishing of $\mathrm{H}^1(E(m))$, one obtains

$$h^0(E(m)) = rm + d + r(1-g).$$

Let V be a vector space of dimension $rm + d + r(1-g)$. Since $E(m)$ is globally generated, for any $E \in \mathfrak{F}$ we have a surjective morphism $f\colon V \otimes \mathcal{O}_X \to E(m)$, and so any element of $\mathfrak{F}$ is a quotient of the sheaf $V \otimes \mathcal{O}_X(-m)$.

Grothendieck's quot scheme theorem tells us that there exists a projective scheme which parametrizes all quotients of a given coherent sheaf.

Theorem 2.27. *Fix d and $r > 0$, and let G be any coherent sheaf. Then there exist a projective scheme $\overline{\mathfrak{Q}}$, a $\overline{\mathfrak{Q}}$-flat family $\mathfrak{F}_{\overline{\mathfrak{Q}}}$ and a universal quotient*

$$q_{\overline{\mathfrak{Q}}}\colon \pi_X^*(G) \to \mathfrak{F}_{\overline{\mathfrak{Q}}}$$

on $\overline{\mathfrak{Q}} \times X$, such that for every quotients $G \to F$ with $\deg F = d$ and $rkF = r$ there is a point $t \in \overline{\mathfrak{Q}}$ with

$$q \sim q_{\overline{\mathfrak{Q}}_{|\{t\}\times X}},$$

where two quotients are equivalent if they have the same kernel and $\pi_X : \overline{\mathfrak{Q}} \times X \to X$ is the projection on the second factor.

Proof. see [21] $\square$

Proposition 2.28. *A family $\mathfrak{F}$ of isomorphism classes of vector bundles of type (d, r) is bounded if and only if there exists a constant C such that for any $E \in \mathfrak{F}$ we have $\mu(F) \leq \frac{d}{r} + C$ for any subbundle $F \subset E$.*

Proof. Let us assume that the family $\mathfrak{F}$ is bounded. There exist an integer m_0 such that $h^1(E(m_0) = 0$ for any $E \in \mathfrak{F}$. If the constant C did not exist, we would find a subbundle of E with slope large as we want and so an extension

$$0 \to F \to E \to Q \to 0$$

such that $\mu(Q) < -m_0 + \frac{2(g-1)}{\mathrm{rk}(Q)}$ "i.e." such that $Q^\vee(-m_0) \otimes \mathrm{K}$ has positive degree and so admits non zero sections. Since Q is a quotient of E, $Q^\vee$ is a subbundle of $E^\vee$ and so

$$h^0(E^\vee) \geq h^0(Q^\vee),$$

we get

$$\frac{h^1(E(m_0))}{\mathrm{rk}(Q)} \geq \frac{h^0(Q^\vee(-m_0) \otimes \mathrm{K})}{\mathrm{rk}(Q)} > 0,$$

which contradicts our assumption.
Conversely, let m such that $\mathrm{H}^1(E(m)) \neq 0$ and let $\eta \colon E(m) \to \mathrm{K}$ be a nontrivial homomorphism. Then we get an extension

$$0 \to F \colon = \ker(\eta) \to E(m) \to L \colon = \eta(E(m)) \to 0,$$

and $r \cdot m = \deg(F) + \deg(L) \leq (r-1)\frac{d}{r} + (r-1)m + (r-1)C + 2g - 2$. Hence if $m \geq \mu(E) + (r-1)C + 2g - 1$ we obtain that $h^1(E(m) = 0$. Similarly one can show that $(E(m))$ is globally generated and so the family is bounded. □

Remark 2.29. The previous Proposition and Corollary 2.13 tell us that the family of semistable holomorphic pairs and so the family of semistable Higgs bundle is bounded.

The Dolbeaut moduli space of stable Higgs bundle of rank n is the space

$$\mathrm{Dol}^n \colon = \{\text{stable rank } n \text{ Higgs bundle of degree } 0\} / \simeq .$$

It is a very interesting geometric object. We will describe now some of its most important features.

2.3.2 The rank one case

Let (E, ϕ) be a Higgs line bundle. For any $\phi \in \mathrm{H}^0(X, \mathrm{K} \otimes \mathrm{End}(E))$ the pair (E, ϕ) is stable. In particular we have that $\phi \in \mathrm{H}^0(X, \mathrm{End}(E) \otimes \mathrm{K}_X) \simeq \mathrm{H}^1(X, \mathrm{End}(E))^\vee$, and one has that:

$$\mathrm{Dol}^1 \simeq T^*\mathrm{Jac}(X),$$

"i.e." the cotangent bundle of the Jacobian of X. Since $\mathrm{Jac}(X)$ is a complex group

$$T^*\mathrm{Jac}(X) \simeq \mathrm{Jac}(X) \times \mathrm{H}^1(X, \mathcal{O}_X) \simeq U(1)^{2g} \times \mathbb{R}^{2g}.$$

So we have proved that $\mathrm{Dol}^1 \simeq (\mathbb{C}^*)^{2g}$. This is an isomorphism of algebraic group if one defines the multiplication law in Dol^1 as the tensor product of two Higgs line bundles,

$$(L_1, \phi_1) \otimes (L_2, \phi_2) \colon = (L_1 \otimes L_2, \phi_1 \otimes \mathrm{id}_{L_2} + \mathrm{id}_{L_1} \otimes \phi_2).$$

2.3.3 Higher rank case

For semistable pairs we have the following results due to Nitsure,

Theorem 2.30. *There exists a scheme $\mathcal{M}\colon = \mathcal{M}^{ss}(r,d,L)$ which is a coarse moduli space for S-equivalence classes of semistable pairs (E,ϕ) on X, where $rk(E) = r$, $\deg(E) = d$, and $\phi\colon E \to E \otimes L$. The isomorphism classes of stable pairs form an open subscheme $\mathcal{M}^s(r,d,L)$ of $\mathcal{M}$. Moreover $\mathcal{M}$ is a quasi-projective separated noetherian scheme of finite type over $\mathbb{C}$.*

The characteristic polynomial of a pair (E,ϕ) is an element of the vector space $A(r,L)\colon = \mathrm{H}^0(X,L) \oplus \cdots \oplus \mathrm{H}^0(X,L^r)$. For any family (E_T,ϕ_T) of semistable pairs parametrized by a scheme T, the characteristic polynomial gives a morphism from T to $A(n,L)$. Since S-equivalent semistable twisted Higgs bundles have the same characteristic polynomial, this defines a morphism from the moduli scheme $\mathcal{M}^{ss}(r,d,L)$ to $A(r,L)$.

Proposition 2.31. *The characteristic polynomial morphism from the moduli scheme $\mathcal{M}$ to the affine space $A(r,L)$ is a proper morphism.*

Remark 2.32. Let $L = \mathrm{K}$. The contangent bundle $T^*\mathcal{N}^s$ of the moduli space $\mathcal{N}^s$ of stable vector bundles of rank r and degree d is an open subscheme of $\mathcal{M}(r,d,\mathrm{K})$. By a result of Beauville, Narasimhan and Ramanan [4], the characteristic polynomial morphism from $T^*\mathcal{N}^s$ to $A(r,\mathrm{K})$ is dominant and so the morphism from $\mathcal{M}$ to $A(r,\mathrm{K})$ is surjective.

Proposition 2.33. *[40] Consider the following three special cases for the line bundle L:*

i) $L = K$*, the canonical bundle on* X*;*

ii) $\deg L = \deg K$ *but* L^r *is not isomorphic to* K^r *for any* r*;*

iii) $\deg L > \deg K$.

Then for any stable pair (E,ϕ) with $\phi \in H^0(X, L \otimes End(E))$, the dimension of $T_{(E,\phi)}(\mathcal{M})$ is independent of the specific pair (E,ϕ), and equals

i) $r^2(2g-2)+2$,

ii) $r^2(2g-2)+1$,

iii) $r^2(\deg L)+1$,

respectively.

Remark 2.34. In all the three cases above, we have $\dim T_{(E,\phi)}(\mathcal{M}) = r^2 \deg L + 1 + \dim \mathrm{H}^1(X, L)$, whenever (E, ϕ) is a stable pair.

If (E, ϕ) is a pair such that the underlying bundle E is itself stable then considering the pairs $(E, t \cdot \phi)$ with $t \to 0$ we see that (E, ϕ) is connected by a segment to $(E, 0)$. Since the moduli space of stable vector bundle is connected we obtain that all pairs such that the underlying bundle E is stable occur in a single connected component $\mathcal{M}^0$. $\mathcal{M}^0$ is a smooth quasi-projective variety (in particular irreducible and reduced) of dimension equal to

$$\dim(\mathcal{N}) + \dim \mathrm{H}^0(X, \mathrm{End}(E) \otimes L) = r^2 \deg L + 1 + \dim \mathrm{H}^1(X, L).$$

If $L = \mathrm{K}$ we have $\dim(\mathcal{M}) = 2r^2(g-1) + 2$. A simple calculation shows that $\dim A(r, \mathrm{K}) = g + 3g - 3 + \cdots + (2r-1)g - (2r-1) = r^2(g-1) + 1$, and so we have

$$\frac{\dim(\mathcal{M})}{2} = \dim A(r, \mathrm{K}).$$

Hence the generic fibre of the Hitchin fibration has dimension $r^2(g-1) + 1$.
Now we give some results about the moduli space of rank 2 semistable Higgs bundles ([20],[40]).

Proposition 2.35. *Let $\mathcal{M}(2, d, K)$ be the moduli space of stable Higgs bundles (E, ϕ) where E is a rank-2 bundle of odd degree over a Riemann surface X of genus $g > 1$, and $\phi \in H^0(X, End(E) \otimes K)$. Then $\mathcal{M}(2, d, K)$ is connected and simply connected.*

Proposition 2.36. *$\mathcal{M}(2, d, K)$ is a Hyper-Kähler manifold with natural symplectic form ω defined on the infinitesimal deformations $(\dot{E}, \dot{\phi})$ of a Higgs bundle (E, Φ), for $\dot{E} \in \Omega^{01}(\mathrm{End}_0 E)$ and $\dot{\phi} \in \Omega^{10}(\mathrm{End}_0 E)$, by*

$$\omega((\dot{E}_1, \dot{\phi}_1), (\dot{E}_2, \dot{\phi}_2)) = \int_X \mathrm{tr}(\dot{E}_1 \dot{\phi}_2 - \dot{E}_2 \dot{\phi}_1) \tag{2.4}$$

2.4 Representations of the fundamental group

Let us fix a point $x_0 \in X$. We denote by $\pi_1(X, x_0)$, or simply $\pi_1(X)$, the fundamental group of X,

$$\pi_1(X) = \langle a_1, \dots a_g, b_1, \dots, b_g \mid \Pi[a_i, b_i] = 1 \rangle,$$

and by $\widetilde{X}$ the universal cover of X.
Denote by $\mathrm{Hom}^{Irr}(\pi_1(X), SL(n, \mathbb{C}))$ the set of irreducible representations from $\pi_1(X)$ to $SL(n, \mathbb{C})$. The group $SL(n, \mathbb{C})$ acts on representations by conjugation,

$g \cdot \rho(l) \colon = g\rho(l)g^{-1}$ for $g \in SL(n,\mathbb{C})$, $\rho \in \pi_1(X)$ and $l \in \pi_1(X)$. The Betti groupoid is the category having $\mathrm{Hom}^{Irr}(\pi, SL(n,\mathbb{C}))$ as the set of objects and morphisms

$$g \colon \rho_1 \longrightarrow \rho_2$$

where $g \in SL(n,\mathbb{C})$, $\rho_1, \rho_2 \in \mathrm{Hom}(\pi_1(X), SL(n,\mathbb{C}))$ and

$$\rho_2 = \iota_g \circ \rho_1$$

where $\iota_g \colon SL(n,\mathbb{C}) \longrightarrow SL(n,\mathbb{C})$ is the inner automorphism defined by conjugation by g.

The map $\mathrm{Hom}(\pi_1(X), SL(n,\mathbb{C})) \longrightarrow SL(n,\mathbb{C})^{2g}$ which sends a representation

$$\rho \to (\rho(A_1), \rho(B_1), \dots, \rho(A_k), \rho(B_k))$$

embeds $\mathrm{Hom}^{Irr}(\pi_1(X), SL(n,\mathbb{C}))$ as the Zariski-closed subset of $SL(n,\mathbb{C})^{2g}$ defined by

$$[\alpha_1, \beta_1] \dots [\alpha_k, \beta_k] = 1. \tag{2.5}$$

Given an irreducible representation $\rho \colon \pi_1(X) \to SL(n,\mathbb{C})$ one can easily construct a flat vector bundle

$$E_\rho \colon = \widetilde{X} \times_\rho \mathbb{C}^n.$$

Conversely given a flat vector bundle one can obtain a canonical irreducible representation, namely, the holonomy representation. So we have a one-to-one correspondence

$$\{\rho \colon \pi_1(X) \to SL(n,\mathbb{C})\} \overset{holonomy}{\longleftrightarrow} \{\text{flat } SL(n,\mathbb{C})\text{-bundle over } X\}.$$

Let now (E, D) be a flat bundle over X. We want to obtain a stable Higgs bundle (E, ϕ) over X. Let h be a Hermitian metric on E. We can decompose D in its $(1,0)$ and $(0,1)$ components

$$D = D' + D''$$

and consider the unique operators D''_h and D'_h so that $D' + D''_h$ and $D'_h + D''$ become h-unitary connections. Let

$$\partial_h = \frac{D' + D'_h}{2} \quad \bar{\partial}_h = \frac{D'' + D''_h}{2} \quad \phi_h = \frac{D' - D'_h}{2} \quad \phi^*_h = \frac{D'' - D''_h}{2}.$$

It is not difficult to see that $D^2 = 0$ implies $\phi_h \wedge \phi_h = 0$ and $F_h + [\phi_h, \phi^*_h] = 0$, where F_h is the curvature of $\bar{\partial}_h + \partial_h$. Of course $\bar{\partial}_h$ defines a holomorphic structure on E, but ϕ_h need not be holomorphic with respect to it, "i.e." there is no reason why $\bar{\partial}_h \phi_h = 0$. This happens precisely when the metric is harmonic. In conclusion we have that

Theorem 2.37. *There is an equivalence of categories between the category of stable Higgs bundles over* (X, ω) *of degree 0 and the category of irreducible flat bundles which, in turn, is equivalent to the category of irreducible complex representations of the fundamental group of* X.

Chapter 3

Decorated vector bundles

Let X be a smooth projective curve over the field of complex numbers, and fix a representation $\rho\colon GL(r,\mathbb{C}) \to GL(V)$. Thank to ρ one can associate to every vector bundle E of rank r over X a vector bundle

$$E_\rho\colon = E \times_\rho V,$$

with fibre V. We would like to study triples (E, L, φ) where E is a vector bundle of rank r over X, L is a line bundle over X, and $\varphi\colon E_\rho \to L$ is a surjective homomorphism. This set-up comprises well-known objects such as framed vector bundles, Higgs bundles, and conic bundles. If the representation ρ satisfies some conditions one can reduce the problem to the study of triples as before where $\varphi\colon (E^{\otimes a})^{\oplus b} \otimes (\det(E))^{\otimes -c} \to L$

3.1 Definition and stability.

In this section we recall basic definitions and main properties of decorated vector bundles. Most of this section can be found in [50].
Let $\rho\colon GL(r,\mathbb{C}) \to GL(V)$ be an irreducible representation on the finite dimensional $\mathbb{C}$-vector space V.

Theorem 3.1. *There are integers $a_1, ..., a_r$ with $a_i \geq 0$ for $i = 1, ..., r-1$, such that ρ is a direct summand of the natural representation of $GL(r,\mathbb{C})$ on*

$$Sym^{a_1}(\mathbb{C}^r) \otimes \cdots \otimes Sym^{a_{r-1}}\left(\bigwedge^{r-1}\mathbb{C}^r\right) \otimes \left(\bigwedge^{r}\mathbb{C}^r\right)^{\otimes a_r}.$$

Proof. See [17], Proposition 15.47. $\square$

For any vector space W, the representations of $GL(W)$ on $\mathrm{Sym}^i(W)$ and $\bigwedge^i W$ are direct summands of the representation of $GL(W)$ on $W^{\otimes i}$. Setting $a\colon = a_1 + \cdots + a_{r-1}(r-1)$ and $c\colon = a_n$, we see that ρ is a direct summand of the representation $\rho_{a,c}\colon GL(r,\mathbb{C}) \to (\mathbb{C}^r)^{\otimes a} \otimes (\bigwedge^r \mathbb{C}^r)^{\otimes c}$.
It is possible in some sense to extend the previous theorem to more general representations than the irreducible ones. A representation $\rho\colon GL(r,\mathbb{C}) \to GL(V)$ is said to be *homogeneous* of degree $h \in \mathbb{Z}$ if for any $z \in \mathbb{C}^*$

$$\rho(z \cdot \mathrm{Id}_{GL(r,\mathbb{C})}) = z^h \cdot \mathrm{Id}_{GL(V)}.$$

Corollary 3.2. Let $\rho\colon GL(r,\mathbb{C}) \to GL(V)$ be a homogeneous representation, then, there exist $a,b,c \in \mathbb{Z}_{\geq 0}$, $c > 0$, such that ρ is a direct summand of the natural representation $\rho_{a,b,c}$ of $GL(r,\mathbb{C})$ on

$$V_{a,b,c}\colon = \left((\mathbb{C}^r)^{\otimes a}\right)^{\oplus b} \left(\bigwedge^r \mathbb{C}^r\right)^{\otimes -c}.$$

Proof. We can decompose $\rho = \rho_1 \oplus \cdots \oplus \rho_b$ where the ρ_i's are irreducible representations. By Theorem 3.1, there are integers a_i, c_i, $i = 1, ..., b$, with $a_i \geq 0$, $i = 1, ..., b$, such that ρ is a direct summand of $\rho_{a_1,c_1} \oplus \cdots \oplus \rho_{a_b,c_b}$. Our assumption on the action of $\mathbb{C}^*$ implies that $a_1 + rc_1 = \cdots = a_b + rc_b$. Let c be a positive integer which is so large that $c_i + c > 0$ for $i = 1, ..., b$. Then, ρ_{a_i,c_i} is the natural representation of $GL(r,\mathbb{C})$ on

$$(\mathbb{C}^r)^{\otimes a_i} \otimes \left(\bigwedge^r \mathbb{C}^r\right)^{\otimes c_i + c} \otimes \left(\bigwedge^r \mathbb{C}^r\right)^{\otimes -c},$$

Now, the $GL(r,\mathbb{C})$-module

$$(\mathbb{C}^r)^{\otimes a_i} \otimes \left(\bigwedge^r \mathbb{C}^r\right)^{\otimes c_i + c}$$

is a direct summand of $(\mathbb{C}^r)^{\otimes a}$, where $a\colon = a_1 + r(c_1 + c) = \cdots = a_b + r(c_b + c)$, and we are done. $\square$

So any homogeneous representation $\rho\colon GL(r,\mathbb{C}) \to GL(V)$ is a direct summand of the representation $\rho_{a,b,c}\colon GL(r,\mathbb{C}) \to GL(V_{a,b,c})$. This motives the following definition

Definition 3.3. Let us fix non-negative integer a, b, c and a line bundle L. Then a *decorated vector bundle* of type (a, b, c, L) is a pair (E, φ) where E is a vector bundle on X and

$$\varphi\colon E_{a,b,c} \to L,$$

is a surjective morphism of vector bundles, with $E_{a,b,c}\colon = (E^{\otimes a})^{\oplus b} \otimes (\det(E))^{\otimes -c}$.

Decorated vector bundles were defined by A.W. Schmitt in [49] in order to compactify the moduli space of principal bundles over projective varieties. Actually they are very general objects whose category contains the category of Higgs bundles, conic bundles, Bradlow pairs and so on.
We recall the definition of semistable decorated vector bundle given by Schmitt.
Let (E,φ) be a decorated vector bundle of type (a,b,c,L) and let $d=\deg(E), r=\mathrm{rk}(E)$ be integers.
A weighted filtration of E, $(E^\bullet,\underline{\alpha})$ indexed by $\mathtt{I}\colon=\{r_1,\dots,r_t,r\}$, is a pair consisting of a filtration $0\subset E_1\subset\cdots\subset E_t\subset E_r=E$ where $\mathrm{rk}(E_i)=r_i$ and a weight vector $\underline{\alpha}=(\alpha_1,\dots,\alpha_t)$. For any such filtration we define

$$\gamma_{\mathtt{I}}\colon=(\gamma_{\mathtt{I}}^1,\dots,\gamma_{\mathtt{I}}^r)=\sum_{i\in\mathtt{I}}\alpha_i(\underbrace{\mathrm{rk}E_i-r,\dots,\mathrm{rk}E_i-r}_{\mathrm{rk}E_i\text{-times}},\underbrace{\mathrm{rk}E_i,\dots,\mathrm{rk}E_i}_{r-\mathrm{rk}E_i\text{-times}}).$$

and

$$\mu(E^\bullet,\underline{\alpha};\varphi)\colon=-\min_{i_1,\dots,i_a\in\mathtt{I}}\{\gamma_{\mathtt{I}}^{(i_1)}+\cdots+\gamma_{\mathtt{I}}^{(i_a)}\mid\varphi|_{(E_{i_1}\otimes\cdots\otimes E_{i_a})^{\oplus b}}\not\equiv 0\}. \tag{3.1}$$

Definition 3.4. Fix a positive rational number δ. We say that (E,φ) is δ-*(semi)stable* if for any weighted filtration $(E^\bullet,\underline{\alpha})$ of E the following inequality holds

$$P(E^\bullet,\underline{\alpha})+\delta\,\mu(E^\bullet,\underline{\alpha};\varphi)(\geq)0$$

where

$$P(E^\bullet,\underline{\alpha})\colon=\sum_{i\in\mathtt{I}}\alpha_i(\deg(E)\mathrm{rk}E_i-\mathrm{rk}E\deg(E_i)).$$

Remark 3.5. If $\varphi=0$ then we set $\mu(E^\bullet,\underline{\alpha};0)=0$ and we obtain the classical semistability condition for vector bundles.

3.1.1 GIT interpretation of semistability

We want to relate the definition of semistability given for decorated vector bundles to the classical GIT semistability for points of varieties endowed with a group action (see [35] and [39]).
Let G be a reductive algebraic group and $\theta\colon G\times F\to F$ an action of G on a projective scheme F. Let $\pi\colon L\to F$ be an ample line bundle on F. A linearization of the given action in L is a lifting of the action θ to an action $\bar{\theta}\colon G\times L\to L$, such that

- For all $l\in L$ and $g\in G$, one has $\pi(\bar{\theta}(g,l))=\theta(g,\pi(l))$.
- For all $x\in F$, $g\in G$, the map $L_x\to L_{\theta(g,x)}$ given by the rule $x\to\theta(g,x)$ is linear.

Taking tensor powers, $\bar{\theta}$ provides us with linearizations of the action on any tensor power $L^{\otimes k}$ and actions of G on $H^0(F, L^{\otimes k})$ for any $k > 0$.
A point $x_0 \in F$ is called *semistable* if there exist an integer $k > 0$ and a G-invariant section $\sigma \in H^0(F, L^{\otimes k})$ not vanishing in x_0. If, moreover, the action of G on the set $\{ x \in F \mid \sigma(x) \neq 0 \}$ is closed and $\dim G \cdot x_0 = \dim G$, x_0 is called *stable*. Finally, a point $x \in F$ is called *polystable*, if it is semistable and its G-orbit is closed in F^{ss}. The sets F^{ss} and F^s of semistable and stable points are open G-invariant subsets of F.

Definition 3.6. Let X and Y be varieties and assume that on X acts an algebraic group G. A G-invariant map $p\colon X \to Y$ is called a good categorical quotient (of the G-action), if p satisfies

1. For all $U \subset Y$ open, $p^*\colon \mathcal{O}_Y(U) \to \mathcal{O}_X(p^{-1}(U))$ is an isomorphism onto the subring $\mathcal{O}_X(p^{-1}(U))^G$ of G-invariant functions.
2. If $W \subset X$ is closed and G-invariant, then $p(W) \subset Y$ is closed.
3. If $V_1, V_2 \subset X$ are closed, G-invariant, and $V_1 \cap V_2 = \emptyset$, then $p(V_1) \cap p(V_2) = \emptyset$.

A (good) categorical quotient is called a (good) geometric quotient if $\mathrm{Im}(g_X, \pi_X) = X \times_Y X$, where $g_X\colon G \times X \to X$ is the action of G on X and $\pi_X\colon G \times X \to X$ is the projection onto X.

Remark 3.7. Satisfying the property for being a geometric quotient means that the preimage under p of any closed point in Y is exactly one G-orbit in X.

The main result of Mumford's Geometric Invariant Theory is that the categorical quotients $F^{ss}//G$ and $F^s//G$ do exist and that $F^{ss}//G$ is a projective scheme whose closed points are in one to one correspondence to the orbits of polystable points, so that $F^s//G$ is in particular an orbit space.

Theorem 3.8 (Mumford)**.** *[35] Let X be a projective G-variety and L be an ample G-linearized line bundle on X. There exists a good categorical quotient*

$$p\colon X^{ss} \to X^{ss}//G.$$

Moreover there is an open subset $U \subset X^{ss}(L)//G$ such that $\pi^{-1}(U) = X^s(L)$ and $\pi\colon X^s \to X^s//G$ is a good geometric quotient. Furthermore $X^{ss}(L)//G$ is a projective variety.

Remark 3.9. Since G is a reductive group the the ring of G invariant functions

$$R = \bigoplus_{d \geq 0} \mathrm{H}^0(X, L^d),$$

is finitely generated, and one has

$$X^{ss}//G = Proj(R).$$

Decorated bundles as semistable points.

With reference to our previous discussion, a representation $\rho\colon G \to GL(V)$ gives rise to an action of G on V and so to an action on $\mathbb{P}(V)$ with a linearization on the line bundle $L = \mathcal{O}_{\mathbb{P}(V)}(1)$. Now for any $x \in \mathbb{P}(V)$, and any one-parameter subgroup $\lambda\colon \mathbb{C}^* \to G$, the point $x_\infty = \lim_{z\to\infty} \lambda(z)x$ is a fixed point for the action of $\mathbb{C}^*$ induced by λ. So the linearization provides a linear action of $\mathbb{C}^*$ on the one dimensional vector space L_{x_∞}. This action is of the form $z \cdot v = z^\gamma v$ for some $\gamma \in \mathbb{Z}$, finally we define

$$\mu_\rho(\lambda; x) = -\gamma.$$

More generally, if we have an action χ of an algebraic group G on a projective variety Y and a linearization of the action on an ample line bundle L, we can define in the same way $\mu_\chi(\lambda; y)$ for any one-parameter subgroup λ and any $y \in Y$. Finally, if we have a morphism of projective varieties $\sigma\colon X \to Y$, we define

$$\mu_\chi(\lambda; \sigma)\colon = \max_{x\in X} \mu_\chi(\lambda; \sigma(x)).$$

In this setting one can define a point $x \in X$ to be (semi)stable if and only if $\mu_\chi(\lambda; \sigma)(\geq)0$. This definition turns out to be equivalent to definition of semistability given before.

Now we want to relate this notion of semistability for points on projective varieties with the semistability condition we gave for decorated vector bundles. Given a representation $\rho\colon GL(V) \to GL(W)$, a surjective map $\varphi\colon E_\rho \to L$ provides a section $\sigma\colon \widetilde{X} \to \mathbb{P}(E_\rho)$. Indeed the surjective map $\varphi\colon E_\rho \to L$ induces an injective map $\varphi^\vee\colon L^\vee \to E_\rho^\vee$ hence a section $\sigma\colon \widetilde{X} \to \mathbb{P}(E_\rho)$ setting $\sigma(x) = [\varphi^\vee(l)]$ for some $l \in L$ over the point $x \in \widetilde{X}$ (is easy to see that the map is well defined and does not depend on the choice of l in the fibre of x). Conversely, a section σ determines a (unique) line bundle L and a map $\varphi\colon E_\rho \to L$, up to scalars, as follows; let $l_x \in E_\rho^\vee$ a representative for the class $\sigma(x)$, then, by letting x vary, the l_x "span" a line subbundle L' of $E_\rho^\vee$. This defines an immersion $j\colon L' \hookrightarrow E_\rho^\vee$ that is well defined up to the multiplication by a constant. Therefore we set $\varphi\colon = j^\vee$ and $L\colon = L'^\vee$ and we are done.

Given a decorated bundle (E, φ), the morphism $\varphi\colon E_{a,b,c} \to L$ induces a section $\sigma\colon X \to \mathbb{P}(E_{a,b,c})$ and so for any weighted filtration $(E^\bullet, \underline{\alpha})$ one can define

$$\mu(E^\bullet, \underline{\alpha}; \varphi)\colon = \mu_{a,b,c}(\lambda; \sigma),$$

where λ is the one parameter subgroup corresponding to $(E^\bullet, \underline{\alpha})$. The key point is that this definition agrees with the previous definition of $\mu(E^\bullet, \underline{\alpha}; \varphi)$ given in (3.1).

3.2 Principal bundles as decorated bundles

As we mentioned in the introduction, decorated bundles were introduced by Schmitt in order to give a compactification of the moduli space of semistable principal bundles over higher dimensional varieties. Given a principal bundle P one can define a decorated bundle (E, τ) where the morphism τ encodes the principal bundle structure of P. Here we only give the construction and the main results (see [49] for more details).

3.2.1 Singular principal bundles

Let $\rho\colon G \to GL(V)$ be a faithful representation. Any principal G-bundle P gives rise to a vector bundle $E\colon = P(V)$, by extending the structure group from G to $GL(V)$. Conversely given a vector bundle E one can consider the principal bundle $\mathrm{Iso}(V \otimes \mathcal{O}_X, E)$. So we have the following commutative diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d, "G"] & \mathrm{Iso}(V \otimes \mathcal{O}_X, E) \arrow[r] \arrow[d, "G"] & \mathrm{Hom}(V \otimes \mathcal{O}_X, E) \arrow[d, "G"] \\ X \arrow[r, "\sigma"] & \mathrm{Iso}(V \otimes \mathcal{O}_X, E)/G \arrow[r] & \mathrm{Hom}(V \otimes \mathcal{O}_X, E)//G \end{tikzcd} \tag{3.2}$$

where σ is the section associated with the reduction of $\mathrm{Iso}(V \otimes \mathcal{O}_X, E)$ to P. The section σ allows us to recover the principal G-bundle $P_\sigma\colon = \sigma^*(\mathrm{Iso}(V \otimes \mathcal{O}_X, E)$. So, thanks to the representation ρ, we obtain a one-to-one correspondence among principal G-bundles and pairs (E, σ) where E is a vector bundle with fibre V and $\sigma\colon X \to \mathrm{Iso}(V \otimes \mathcal{O}_X, E)/G$ a section.

Remark 3.10. Note that

$$\mathrm{Hom}(V \otimes \mathcal{O}_X, E)//G = \mathit{Spec}(\mathrm{Sym}^*(V \otimes E^\vee)^G).$$

Hence the datum of a section σ as before is equivalent to the datum of a homomorphism

$$\tau\colon \mathrm{Sym}^*(V \otimes E^\vee)^G \to \mathcal{O}_X,$$

with some descendent properties which tell us that the image of σ lies in $\mathrm{Iso}(V \otimes \mathcal{O}_X, E)/G \subset \mathrm{Hom}(V \otimes \mathcal{O}_X, E)//G$. If we want to stress the dependence on τ we will denote by P_τ the principal bundle P_σ.

So a principal G-bundle is determined by a pair (E, τ) consisting of a vector bundle E with fibre V and a homomorphism $\tau\colon \mathrm{Sym}^*(V \otimes E^\vee)^G \to \mathcal{O}_X$ of $\mathcal{O}_X$-algebras induced by σ.

Since $\mathrm{Sym}^*(E \otimes V)^G$ is a finitely generated $\mathcal{O}_X$-algebra, we get a surjective morphism

$$\bigoplus_{i=1}^{s} \mathrm{Sym}^i(E \otimes V)^G \longrightarrow \mathrm{Sym}^*(E \otimes V)^G,$$

for some $s \in \mathbb{N}$, so that τ induces a map

$$\varphi''\colon \bigoplus_{i=1}^{s} \mathrm{Sym}^i(E \otimes V)^G \longrightarrow \mathrm{Sym}^*(E \otimes V)^G \xrightarrow{\tau} \mathcal{O}_X.$$

The representation of $GL(r)$ on the algebra $\bigoplus_{i=1}^{s} \mathrm{Sym}^i(\mathbb{C}^r \otimes V)^G$ is not homogeneous, therefore we have to pass to the induced homogeneous representation

$$t(s)\colon GL(r) \to GL(U(s))$$

where

$$U(s)\colon = \bigoplus_{\underline{d}=(d_1,\dots d_s)} S^{\underline{d}} \text{ and}$$

$$S^{\underline{d}}\colon = \bigotimes_{i=1}^{s} \left(\mathrm{Sym}^{d_i}(\mathrm{Sym}^i(\mathbb{C}^r \otimes V))^G\right).$$

So we get a morphism

$$\varphi'\colon T\colon = \bigoplus_{\underline{d}=(d_1,\dots d_s)} \bigotimes_{i=1}^{s} \left(\mathrm{Sym}^{d_i}(\mathrm{Sym}^i(E \otimes V))^G\right) \to \mathcal{O}_X$$

induced by φ''. Finally, thanks to Corollary 3.2, there exist integers a, b, c such that $t(s)$ is a subrepresentation of $\rho_{a,b,c}$. Therefore we can extend φ' to a morphism

$$\varphi\colon E_{a,b,c} \longrightarrow \mathcal{O}_X,$$

such that $\varphi_{|T} = \varphi'$ (see [52] Section 3 for more details).
Conversely, if $E_{a,b,c}$ decomposes as $T \oplus W$ for some vector bundles W, and $\varphi_{|W} \equiv 0$, then φ induces a morphism $\tau\colon \mathrm{Sym}^*(E \otimes V)^G \to \mathcal{O}_X$.

To relate the semistability conditions for decorated bundles and principal bundles we need to recall some facts about parabolic reductions and representation theory.

Let $\rho\colon G \to SL(V) \hookrightarrow GL(V)$ a faithful representation, thanks to ρ we can identify G with a subgroup of $SL(V)$. Given a one-parameter subgroup $\lambda\colon \mathbb{C}^* \to G$ we denote

$$Q_G(\lambda)\colon = \{g \in G \mid \exists \lim_{z\to\infty} \lambda(z)\cdot g \cdot \lambda(z)^{-1}\}$$

the parabolic subgroup of G induced by λ.

Remark 3.11.

1. Since G is reductive, if Q' is a parabolic subgroup of $GL(V)$, then $Q'\cap G$ is a parabolic subgroup of G. Indeed, if B (B_G) is a borelian subgroup of $GL(V)$ (resp. of G) then, up to conjugacy class, $B \cap G = B_G$ and so $Q' \cap G \supseteq B_G$.

2. Given a parabolic subgroup Q of G and a representation ρ, we can construct a parabolic subgroup $Q_{GL(V)}$ of $GL(V)$ as follows; given Q, there exists a one-parameter subgroup $\lambda\colon \mathbb{C}^* \to G$ such that $Q = Q_G(\lambda)$, then one defines $Q_{GL(V)}$ the parabolic subgroup of $GL(V)$ induced by the one parameter subgroup $(\rho\circ\lambda)\colon \mathbb{C}^* \to GL(V)$.

3. Given $\lambda'\colon \mathbb{C}^* \to GL(V)$, or equivalently the parabolic subgroup Q' associated to λ', there always exists $\lambda\colon \mathbb{C}^* \to G$ such that $Q' \cap G = Q_G(\lambda)$.

4. Given a parabolic subgroup $Q' \subset GL(V)$ and fixed a representation $\rho\colon G \to GL(V)$, it is possible to define a parabolic subgroup $Q = Q' \cap G \subset G$ and from Q we can obtain a parabolic subgroup $Q'' \subset GL(V)$ as explained in point (2). Therefore we have a map

$$f\colon \{\text{Parabolic subgroups of } GL(V)\} \to \{\text{Parabolic subgroups of } GL(V)\}.$$

   We will call G-stable a parabolic subgroup of $GL(V)$, Q', such that $Q' = f(Q')$, with respect to the same basis of $GL(V)$. Clearly a subgroup Q' which comes from a parabolic subgroup of G is G-stable.

Now we want to construct a weighted filtration of a singular G-bundle (E,τ) starting from a given one-parameter subgroup of G, $\lambda\colon \mathbb{C}^* \to G$, and a section $\beta\colon X \to P_\tau/Q_G(\lambda)$. Consider the principal $Q_G(\lambda)$-bundle β^*P_τ. We define

$$E_i^\vee\colon = \beta^* P_\tau \times_\rho V_i^\vee, \qquad \text{for } i = 1,\dots,s,$$

this gives a filtration of $E^\vee$. Dualizing the inclusions $E_i^\vee \subset E^\vee$ and defining $E_i = \ker(E^\vee \to E^\vee_{s+1-i})$ we get a filtration $E^\bullet_\beta$ of E. Moreover, setting $\alpha_i \colon = (\gamma_{i+1} - \gamma_i)/r$ (where γ_i are the weights corresponding to the action of λ) and $\underline{\alpha}_\beta \colon = (\alpha_s, \dots, \alpha_1)$, we obtain the desired weighted filtration $(E^\bullet_\beta, \underline{\alpha}_\beta)$.

Conversely let $(E^\bullet, \underline{\alpha})$ be a weighted filtration of E and let $(E^{\bullet\vee}, \underline{\alpha}^\vee)$ the corresponding weighted filtration of $E^\vee$ where $\underline{\alpha}^\vee_\beta = (\alpha_s, \dots, \alpha_1)$ if $\underline{\alpha} = (\alpha_1, \dots, \alpha_s)$. To this filtration one can associate the morphisms

$$\begin{aligned} &\lambda' \colon \mathbb{C}^* \to GL(V) \\ &\beta' \colon X \to \mathrm{Iso}(V \otimes \mathcal{O}_X, E^\vee)/Q', \end{aligned}$$

where $Q' \colon = Q_{GL(V)}(\lambda')$. Indeed the filtration $(E^\bullet, \underline{\alpha})$ induces a weighted flag of V and so a one-parameter subgroup of $GL(V)$. Moreover the inclusion of principal bundles induces a section β' as follows

$$\begin{tikzcd} \mathrm{Iso}(V^\bullet \otimes \mathcal{O}_X, (E^\bullet)^\vee) \arrow[r, hook] \arrow[d] & \mathrm{Iso}(V \otimes \mathcal{O}_X, E^\vee) \arrow[d] \\ X \arrow[r, dashed, "\beta'"] & \mathrm{Iso}(V \otimes \mathcal{O}_X, E^\vee)/Q'. \end{tikzcd}$$

Then we will say that $E^\bullet$ is a **β-filtration**, and we will write $E^\bullet_\beta$ (instead of $E^\bullet$), if there exists $\beta \colon X \to P_\tau/Q$ such that the following diagram commutes

$$\begin{tikzcd} & P_\tau \arrow[r, hook] \arrow[d] & \mathrm{Iso}(V \otimes \mathcal{O}_X, E^\vee) \arrow[d] \\ & P_\tau/Q \arrow[r, "\bar{i}"] & \mathrm{Iso}(V \otimes \mathcal{O}_X, E^\vee)/Q' \\ X \arrow[ur, dashed, "\beta"] \arrow[urr, bend right, "\beta'"'] & & \end{tikzcd}$$

where $Q = Q' \cap G$.

Proposition 3.12. *Let $E^\bullet$ be a filtration of E. Then $E^\bullet$ is a β-filtration if and only if the parabolic subgroup Q' associated to such filtration is G-stable (in the sense of Remark 3.11 point (4)).*

Proof. The condition of being a β-filtration clearly implies that the parabolic subgroup is G-stable. Let us prove the opposite arrow.
The filtration $E^\bullet$ gives rise to a subbundle $I_{Q'}$ of $\mathrm{Iso}(V \otimes \mathcal{O}_X, E^\vee)$, and so the inclusion $I_{Q'} \hookrightarrow \mathrm{Iso}(V \otimes \mathcal{O}_X, E^\vee)$ induces a section $\beta' \colon X \to \mathrm{Iso}(V \otimes \mathcal{O}_X, E^\vee)/Q'$. We consider now the groups Q and Q'' constructed as in Remark 3.11 point (4)

and the one-parameter subgroups λ, λ' (respectively) associated to Q and Q'. The following diagram commutes

$$\begin{tikzcd} \mathbb{C}^* \arrow[r, "\lambda'"] \arrow[d, "\lambda"'] & GL(V) \\ G. \arrow[ur, "\rho"'] & \end{tikzcd} \tag{3.3}$$

In fact by hypothesis Q' is stable and so $Q' = Q''$.
Consider now the following diagram

$$\begin{tikzcd} & P_\tau \arrow[r, "i"] \arrow[d] & \mathrm{Iso}(V \otimes \mathcal{O}_X, E^\vee) \arrow[d] \\ & P_\tau/Q \arrow[r, "\bar{i}"] & \mathrm{Iso}(V \otimes \mathcal{O}_X, E^\vee)/Q' \\ X \arrow[urr, bend right, "\beta'"'] & & \end{tikzcd}$$

Note that the map $\bar{i}$ is well defined. In fact, denoting by $[\,\cdot\,]_{Q'}$ the class modulo Q', the map i induces $\bar{i}$ if and only if for any $q \in Q$ and for any $a' = q.\,a$ one has $[i(a')]_{Q'} = [i(a)]_{Q'}$. $i(a') = i(q.\,a) = \rho(q).\,i(a)$, $[i(a')]_{Q'} = [i(a)]_{Q'} \iff \rho(q) \in Q'$ but $\rho(Q) \subseteq Q'' = Q'$ and we are done.
Since $\rho(Q) \subseteq Q''$ stabilizes the filtration $E^\bullet$ we can consider the subbundle I_Q of P_τ, for which $I_Q \times_\rho Q'' = I_{Q'}$. The inclusion $I_Q \hookrightarrow P_\tau$ induces a morphism $\beta\colon X \to P_\tau/Q$ that makes the following diagram commute

$$\begin{tikzcd} I_{Q'} \arrow[rrr, bend left] \arrow[ddr, bend right] & & & \\ & I_Q \arrow[ul] \arrow[r] \arrow[dd] & P_\tau \arrow[r, "i"] \arrow[d] & \mathrm{Iso}(V \otimes \mathcal{O}_X, E^\vee) \arrow[d] \\ & & P_\tau/Q \arrow[r, "\bar{i}"] & \mathrm{Iso}(V \otimes \mathcal{O}_X, E^\vee)/Q' \\ & X \arrow[ur, "\beta"] \arrow[urr, bend right, "\beta'"'] & & \end{tikzcd} \tag{3.4}$$

and so we finish. Equivalently, we could show that $\mathrm{Im}(\beta') \cap \mathrm{Im}(\bar{i}) = \mathrm{Im}(\bar{i})$ and define $\beta = (\bar{i})^{-1} \circ \beta'$.

$\square$

Remark 3.13. In the previous proposition we have shown that the following conditions are equivalent :

1. $E^\bullet = E^\bullet_\beta$.

2. The diagram (3.3) commutes

3. The diagram (3.4) commutes.

4. Q' is G-stable.

Remark 3.14. Let G be a semisimple group, $\rho\colon G \to SL(V) \subset GL(V)$ a faithful representation and $\lambda\colon \mathbb{C}^* \to G$ a one-parameter subgroup such that $Q_G(\lambda)$ is a maximal parabolic subgroup of G. Then the parabolic subgroup $Q_{GL(V)}(\rho\circ\lambda)$ of $GL(V)$ is **not** maximal. Thanks to this observation, we see that every β-filtration $E^\bullet_\beta\colon\ 0 \subset E_1 \subset \cdots \subset E_s \subset E_{s+1} = E$ has length greater or equal than 2, "i.e." $s \geq 2$.

The previous remark tells us that the parabolic subgroup of G associated with a β-filtration is always a proper subgroup. Therefore, according to the definition of Ramanathan, the (semi)stability condition needs to be checked only for maximal *proper* parabolic subgroups of G. If G is reductive but not semisimple Remark 3.14 does not hold in general as the following example shows

Example 3.15. Consider $G = GL(k)$, $\rho\colon GL(k) \to GL(n)$ $(k < n)$ the inclusion (in the left up corner) and $\lambda\colon \mathbb{C}^* \to G$ given by

$$\lambda(z)\colon = \begin{pmatrix} z^\gamma & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & z^\gamma \end{pmatrix}.$$

Clearly the parabolic subgroup associated to λ is all G, however $\lambda' = \rho\circ\lambda$ is given by

$$\lambda'(z)\colon = \begin{pmatrix} z^\gamma & 0 & \dots & & & \dots & 0 \\ 0 & \ddots & \ddots & & & & \vdots \\ \vdots & \ddots & z^\gamma & & & & \\ & & & 1 & & & \\ & & & & 1 & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & 0 \\ 0 & \dots & & & \dots & 0 & 1 \end{pmatrix}$$

hence the corresponding filtration has length one.

Let β be a reduction of (E,τ) to the one parameter subgroup λ of G and $(E^\bullet_\beta,\underline{\alpha}_\beta)$ the associated weighted filtration of E. Then we have

Lemma 3.16.
$$\mu(E^\bullet_\beta,\underline{\alpha}_\beta;\varphi)=0,$$
where φ is the morphism induce by τ.

Proof. See [51] □

Our next goal is to show that the notion of semistability given for singular principal bundles is equivalent to semistability condition given by Ramanathan when the group G is semisimple.

Proposition 3.17. *[48] Let us fix a faithful representation $\rho\colon G\to GL(r)$ with G a semisimple group. Then a principal G-bundle P is (semi)stable if and only if the corresponding singular principal bundle (E,τ) is δ-(semi)stable as a decorated bundle for any $\delta\in\mathbb{Q}$.*

Proof. Let us prove that if P is semistable as a principal bundle then for any weighted filtration of $E=P(\mathbb{C}^r)$ we have
$$P(E^\bullet,\underline{\alpha})+\delta\,\mu(E^\bullet,\underline{\alpha};\varphi)\geq 0. \tag{3.5}$$
A. Ramanathan and S. Subramanian showed that if a principal G-bundles is semistable then the vector bundle $P(\mathbb{C}^r)$ is semistable ([44]) and so $P(E^\bullet,\underline{\alpha})\geq 0$. It remains to show that $\mu(E^\bullet,\underline{\alpha};\varphi)\geq 0$. This follows from the fact that the section σ corresponding to the morphism τ lands in $\mathrm{Iso}(V\otimes\mathcal{O}_X,E)/G\subset \mathrm{Hom}(V\otimes\mathcal{O}_X,E)//G$ and so the section determinant does not vanish on $\mathrm{Im}(\sigma)$, which implies that points in the image are semistable hence we get the thesis.
Now let us assume that P is stable, and consider any weighted filtration $(E^\bullet,\underline{\alpha})$ of E.
If $(E^\bullet,\underline{\alpha})$ is a β-filtration then it comes from a proper parabolic subgroup of G and consequently $P(E^\bullet,\underline{\alpha})>0$ and $\mu(E^\bullet,\underline{\alpha};\varphi)=0$.
Otherwise $P(E^\bullet,\underline{\alpha})$ could be 0 however $\mu(E^\bullet,\underline{\alpha})$ is strictly positive. In both cases we conclude that
$$P(E^\bullet,\underline{\alpha})+\mu(E^\bullet,\underline{\alpha};\varphi)>0.$$
The other arrow follows from the fact that for any subbundle $F\subset E$ which comes from a maximal parabolic subgroup of G one has by Lemma 3.16
$$\mu(0\subset F\subset E,(1);\tau)=0,$$
so inequality (3.5) implies
$$P(E^\bullet,\underline{\alpha})(\geq)0,$$
and we are done. □

3.3 Higgs vector bundles as decorated bundles

In this section we want to show that the theory of Higgs bundles can be reduced to the study of decorated vector bundles ([47],[48]).
As before, we fix integers d and $r > 0$, a line bundle L, and consider the representation

$$\rho\colon GL(r,\mathbb{C}) \to GL(\mathrm{End}(\mathbb{C}^r) \oplus \mathbb{C}).$$

where the action of $GL(r,\mathbb{C})$ to $\mathbb{C}$ is given by the identification $\mathbb{C} \simeq \bigwedge^r \mathbb{C}^r$. In this case, a decorated bundle of type (d,r,L) is a triple (E,ϕ,ω) consisting of a vector bundle E of degree d and rank r, a twisted endomorphism $\phi\colon E \to E \otimes L$, and a section $\omega\colon \det(E) \to L$.

Let $f\colon \mathbb{C}^r \to \mathbb{C}^r$ be a homomorphism of vector spaces. We call a sub vector space $V \subset \mathbb{C}^r$ f-superinvariant, if $V \subset \ker f$ and $f(\mathbb{C}^r) \subset V$. Then we have:

Lemma 3.18. *Let $[f,\epsilon] \in \mathbb{P}(Hom(\mathbb{C}^r,\mathbb{C}^r) \oplus \mathbb{C})$. Given a basis $\underline{v} = (v_1,...,v_r)$ of $\mathbb{C}^r = V$ and $i \in \{1,...,r-1\}$, set $V_{\underline{v}}^{(i)} \colon = \langle v_1,...,v_i \rangle$. Then*

- $\mu_\rho\left(\lambda(\underline{v},\gamma^{(i)}),[f,\epsilon]\right) = r$, *if $V_{\underline{v}}^{(i)}$ is not f-invariant.*
- $\mu_\rho\left(\lambda(\underline{v},\gamma^{(i)}),[f,\epsilon]\right) = -r$, *if $V_{\underline{v}}^{(i)}$ is f-superinvariant and $\epsilon = 0$.*
- $\mu_\rho\left(\lambda(\underline{v},\gamma^{(i)}),[f,\epsilon]\right) = 0$ *in all the other cases.*

In particular if (E,ϕ,ω) is as before, for any subbundle F of E with $\phi(F) \subset F \otimes L$, we find $\mu_\rho(F,(\phi,\sigma)) \leq 0$. This fact implies that if (E,ϕ,ω) is a δ-(semi)stable decorated bundle then $\mu(F)(\leq)\mu(E)$ for every nontrivial proper subbundle F of E with $\phi(F) \subset F \otimes L$, "i.e." it is semistable as a Higgs bundle. This condition implies that for every $\delta > 0$, every δ-semistable decorated bundles (E,ϕ,ω) of type (d,r,L), and every subbundle $F \subset E$

$$\mu(F) \leq \max\left\{\mu(E), \mu(E) + \frac{(r-1)^2}{r}\deg L\right\}. \tag{3.6}$$

See Proposition 2.13. Therefore, the set of isomorphism classes of decorated semistable bundles (E,ϕ,ω) is bounded.
Now we look at $\phi\colon \mathrm{End}(E) \to \mathrm{K}$. We fix $\det(E) \simeq \mathcal{O}_X$, if the genus of X is greater than 0 we can choose a section $\omega\colon \det E \simeq \mathcal{O}_X \to \mathrm{K}$ not identically zero. Let $\rho'\colon GL(V) \to GL(\mathrm{End}(V) \oplus \mathbb{C})$ be the natural representation obtained by identifying $\mathbb{C}$ with $\bigwedge^{\dim V} V$. The pair (ϕ,ω) induces a map

$$\psi'\colon E_{\rho'} = E \times_{\rho'} (\mathrm{End}(V) \oplus \mathbb{C}) \to \mathrm{K};$$

indeed $E_{\rho'} \simeq \mathrm{End}(E) \oplus \mathcal{O}_X$ and so $\psi'(e,z) = \phi(e) + \omega(z)$ for any $e \in \mathrm{End}(E)$ and $z \in \mathcal{O}_X$ over the same point $x \in X$.
In view the isomorphism

$$\bigwedge^k E \simeq \left(\bigwedge^{r-k} E\right)^\vee \otimes \bigwedge^r E,$$

the vector bundle $E \otimes E^\vee \oplus \det E$ is a subbundle of $E^{\otimes r} \oplus E^{\otimes r}$ and so a Higgs bundle is a particular decorated bundle with $a = r$ and $b = 2$. Conversely if we have a decorated bundle (E, ψ) such that $\psi_{|_{E\otimes E^\vee \oplus \det E}} = (\phi, \omega)$ then we can construct a Higgs bundle.

Remark 3.19. Suppose the genus of X is strictly positive and fix a morphism $\omega \in \mathrm{Hom}(\mathcal{O}_X, \mathrm{K})$ not identically zero. There is a natural inclusion of Higgs fields into a projective space given as follows

$$\begin{array}{ccc} \mathrm{Hom}(E, \mathrm{K} \otimes E) & \hookrightarrow & \mathbb{P}\left(\mathrm{Hom}(E, \mathrm{K} \otimes E) \oplus <\omega>\right) \\ v & \hookrightarrow & [v \colon 1] \end{array}$$

where by $<\omega>$ we denote the linear subspace of $\mathrm{Hom}(\mathcal{O}_X, \mathrm{K})$ generated by ω. Note that $[v \colon 1]$ and $[z \cdot v \colon 1]$ are different points for any $z \in \mathbb{C} \smallsetminus \{1\}$ (see [47] for more details).

Remark 3.20. For any line bundle L with $\mathrm{H}^0(X, L) \neq 0$ we can extend the previous construction to holomorphic pairs (E, ϕ) where E is a vector bundle with fiber V and $\phi \colon E \to E \otimes L$ is a morphism of vector bundles.

Proposition 3.21. *[50] There is a positive rational number δ_∞, such that for all $\delta \geq \delta_\infty$ and all decorated bundles (E, ϕ, ω), with $\sigma \colon \mathcal{O}_X \to L$ different from zero, the following conditions are equivalent :*

1. *(E, ϕ, ω) is a δ-(semi)stable decorated bundle;*
2. *for every nontrivial subbundle $F \subset E$ with $\phi(F) \subset F \otimes L$*

$$\mu(F)(\leq)\mu(E),$$

Proof. We have already shown that, if (1) holds, then $\mu(F)(\geq)\mu(E)$ for any ϕ-invariant subbundle. It remains to show that condition (2) implies that (E, ϕ, σ) is a semistable decorated bundle. Let $l \colon = \max\{\, 0, \deg L(r-1)^2/r\,\}$. Then, as before, $\mu(F)(\leq)\mu(E) + l$ for every nontrivial proper subbundle $F \subset E$, "i.e.",

$$d\mathrm{rk}(F) - r\deg(F)(\geq) - lr\mathrm{rk}(F)(\geq) - l(r-1)r.$$

Consider a weighted filtration $(E^\bullet, \underline{\alpha})$ such that, say, $E_{j_1}, ..., E_{j_t}$ are not invariant under ϕ, "i.e.", $\phi(E_{j_i}) \not\subset E_{j_i} \otimes L$, $i = 1, ..., t$, and $t > 0$. Let $\alpha\colon = \max\{\alpha_{j_1}, ..., \alpha_{j_t}\}$. One readily verifies $\mu_\rho(E^\bullet, \underline{\alpha}; (\phi, \sigma))(\geq)\alpha \cdot r$. We thus find

$$\begin{aligned} P(E^\bullet, \underline{\alpha}) + \delta\mu_\rho\left(E^\bullet, \underline{\alpha}\right) &\geq \sum_{i=1}^{t} \alpha_{j_i}\left(d \cdot \mathrm{rk} E_{j_i} - r\deg(E_{j_i})\right) + r\alpha\delta \\ &\geq -(r-1)rl\sum_{i=1}^{t}\alpha_{j_i} + r\alpha\delta \\ &\geq \big(-(r-1)^2 rl + r\delta\big)\alpha, \end{aligned}$$

so that $P(E^\bullet, \underline{\alpha}) + \delta\mu_\rho\big(E^\bullet, \underline{\alpha})$ will be positive if we choose $\delta > (r-1)^2 l$. $\square$

Chapter 4

Generalized parabolic vector bundles

4.1 The moduli space of generalized parabolic bundles

We recall the definition of generalized parabolic vector bundle on a curve X (see [5])

Definition 4.1. Let E be a vector bundle on X and $D = \{x_1, \dots, x_s | x_i \in X\}$. A parabolic structure of E at D is giving a flag (that is a parabolic subgroup of the structure group of E) on the fibre E_{x_i} of E at x_i for any $i = 1, \dots, s$,

$$E_{x_i}^{\bullet} \colon 0 \subset E_{x_i}^1 \subset \cdots \subset E_{x_i}^{t_i} = E_{x_i},$$

and a weight vector $\underline{\alpha} = (\alpha_{x_1}^1, \dots, \alpha_{x_i}^{t_i})$ where $\alpha_{x_i}^j$'s are nonnegative rational numbers for $j = 1, \dots, t_i$. A generalized parabolic vector bundle (GPB)with support D is the datum of a vector bundle E and a parabolic structure at D.

Given two generalized parabolic vector bundles E and F such that for any $x_i \in D$ and $j = 1, \dots t_i$ the corresponding flags $E_{x_i}^{\bullet}$ and $F_{x_i}^{\bullet}$ are isomorphic we will say that an isomorphism $f \colon E \to F$ of vector bundles is a generalized parabolic isomorphism if for any $i = 1, \dots, s$ and $j = 1, \dots, t_i$, $f(E_{x_1}^j) = F_{x_i}^j$ and $\alpha_{x_i}(E) = \alpha_{x_i}(F)$. Given a subsheaf $F \subset E$ we say that F is a generalized parabolic subsheaf if the inclusion is compatible with the parabolic structure.

Definition 4.2. Let E be a generalized parabolic vector bundle. The parabolic degree of E is defined by

$$\deg_{\mathrm{par}}(E) \colon = \deg(E) + \sum \alpha_i$$

Moreover we denote by $\mu_{\mathrm{par}}(E)$ the number $\frac{\deg_{\mathrm{par}} E}{\mathrm{rk} E}$.

Definition 4.3. Let E be a vector bundle on X with parabolic structures at a finite number of points of X. We say that E is parabolic (semi)stable if for any proper parabolic subsheaf F of E, we have

$$\mu_{\text{par}}(F)(\leq)\mu_{\text{par}}(E).$$

Remark 4.4. Let us observe that giving a subspace of the vector space E_x is equivalent to give a quotient $q\colon E_x \to R$ for R a complex vector space. So in the case the flag is just a subbundle, one considers a generalized parabolic vector bundle as a morphism $q\colon E_x \to R$. In this case one defines $\deg_{\text{par}}(E)\colon = \deg(E) - \alpha \dim(q(E_x))$.

We have the following theorem due to V.B. Mehta and C.S. Seshadri (see [31])

Theorem 4.5. *The moduli space $\mathcal{M}(r, d, \alpha)$ of gr_{par}-equivalence classes of semistable parabolic vector bundles of rank r an degree d with fixed type of parabolic structure at x_i $i = 1, \ldots, l$ on X is a normal projective varieties with dimension $r^2(g-1) + \sum_{i=1}^{l} f_i$, where f_i is the dimension of the of flag variety determines by the parabolic structure over x_i.*

4.2 Torsion free sheaves on nodal curves

Let X be an irreducible reduced curve with one ordinary double point x_0 as singularity. According to [46], for any torsion-free $\mathcal{O}_{x_0}$-module M of rank r there is a uniquely determined non-negative integer a such that $M \simeq \mathcal{O}_{x_0}^a \oplus \mathfrak{m}_{x_0}^{r-a}$, where $\mathfrak{m}_{x_0}$ is the maximal ideal of $\mathcal{O}_{x_0}$. In particular, for any torsion-free sheaf $\mathcal{E}$ of rank r and degree d on X, there is an integer a, uniquely determined such that $\mathcal{E}_{x_0} \simeq \mathcal{O}_{x_0}^a \oplus \mathfrak{m}_{x_0}^{r-a}$.

This gives a surjective homomorphism $\mathcal{E}_{x_0} \to k_{x_0}^a$, where k_{x_0} denotes the residue field at the point x_0. We have an exact sequence

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$$

where $\mathcal{G}$ is the the skyscraper sheaf concentrated at x_0 with fibre $k_{x_0}^a$. This implies that $\mathcal{E}$ is an extension of $\mathcal{G}$ by a torsion-free sheaf $\mathcal{F}$ with $\mathcal{F}_{x_0} \simeq m_{x_0}^r$.

Recall that a torsion free sheaf $\mathcal{E}$ over a curve X is said to be (semi)stable if and only if for every nontrivial subsheaf $\mathcal{F} \subset \mathcal{E}$ the following inequality holds

$$\frac{\deg \mathcal{F}}{\mathrm{rk}\mathcal{F}}(\leq)\frac{\deg \mathcal{E}}{\mathrm{rk}\mathcal{E}}$$

where the degree of a torsion free sheaf is defined by the equality

$$\chi(\mathcal{E})\colon = h^0(Y, \mathcal{E}) - h^1(Y, \mathcal{E}) = \deg \mathcal{E} + \mathrm{rk}\mathcal{E}(1-g).$$

Now consider the normalization map

$$\nu\colon \widetilde{X} \to X,$$

and denote the two points of $\nu^{-1}(x_0)$ by x_1 and x_2. For a torsion free sheaf $\mathcal{E}$ of rank r and degree d on X there is a vector bundle E on $\widetilde{X}$ such that

$$\mathcal{E} = \nu_*(E)$$

if and only if $\mathcal{E}_{x_0} = m^r_{x_0}$. In this case E is uniquely determined by $\mathcal{E}$ and

$$\deg(E) = \deg(\mathcal{E}) - r.$$

See [46] for the construction.

Given a parabolic vector bundle (E, q) on $\widetilde{X}$ where $q\colon E_{x_1} \oplus E_{x_2} \to R$ is a morphism of vector spaces being R is a vector space of dimension $\mathrm{rk}(E)$ (see Remark 4.4), the sheaf

$$\mathcal{E}\colon = Ker[\nu_* E \longrightarrow \nu_*(E_{x_1} \oplus E_{x_2}) \simeq E_{x_1} \oplus E_{x_2} \xrightarrow{q} R] \tag{4.1}$$

is a torsion free sheaf over X. So we obtain a map g which sends a generalized parabolic vector bundle (E, q) to the torsion free sheaf $\mathcal{E}$ defined as above. In particular, if we denote by $K\colon = \ker(q) \subset E_{x_1} \oplus E_{x_2}$ and by $p_1\colon K \to E_{x_1}$ and $p_2\colon K \to E_{x_2}$ the two projections we have that $\mathcal{E}$ is locally free if and only if p_1 and p_2 are both isomorphisms

In section 4.1 we gave a notion of semistability for generalized parabolic bundles. We want now to relate that semistability condition for parabolic with the semistability of the corresponding torsion free sheaf on X.

Let (E, q) be a generalized parabolic bundle of rank r on $\widetilde{X}$. If $\mathcal{E}$ denotes the corresponding torsion free sheaf on X, we have the exact sequence

$$0 \to \mathcal{E} \to \nu_*(E) \to \mathcal{G} \to 0,$$

where G denotes the skyscraper sheaf supported in x_0 with fibre a vector space of dimension $\dim q(E_{x_1} \oplus E_{x_2})$. So we obtain that

$$\deg \nu_*(E) = \deg \mathcal{E} + \dim(E_{x_1} \oplus E_{x_2}).$$

Recall that in this case a parabolic vector bundle (E, q) is said to be (semi)stable if and only if for every subbundle $F \subset E$ the following inequality holds

$$\frac{\alpha\text{-}\deg_{\mathrm{par}} F}{\mathrm{rk} F} (\leq) \frac{\alpha\text{-}\deg_{\mathrm{par}} E}{\mathrm{rk} E}, \tag{4.2}$$

where the α-parabolic degree of $F \subset E$ is defined as follows

$$\alpha\text{-}\deg_{\mathrm{par}} F\colon = \deg F - \alpha \ \dim q(F_{x_1} \oplus F_{x_2}).$$

From now on we fix the stability parameter $\alpha = 1$ and we write $\deg_{\mathrm{par}}$ for 1-$\deg_{\mathrm{par}}$ and (semi)stable instead of 1-(semi)stable.

Proposition 4.6. *A parabolic vector bundle (E, q) on $\widetilde{X}$ is (semi)stable if and only if the corresponding torsion free sheaf $\mathcal{E}$ on X is (semi)stable.*

Proof. We follow the notation of [5] where the author proves the same statement for the rank 2 case and uses a slight different notion of parabolic degree, which turns out to be equivalent to ours.
Let $F \subset E$ be a subbundle of E we have an exact sequence

$$0 \to \mathcal{F} \to \nu_*(F) \to G_{\mathcal{F}} \to 0$$

where $G\colon = \nu^*(F)/\mathcal{F}$ is a skyscraper sheaf supported on x_0 and $\deg G = \dim q(E_{x_1} \oplus E_{x_2})$. We have the following equalities

- $\deg \nu_*(E) = \deg \mathcal{E} + \dim(E_{x_1} \oplus E_{x_2})$.
- $\deg(\nu_*(F)) - \mathrm{rk}(F) = \deg F$.

Using the previous relations we get

$$\mu(F) = \mu(\mathcal{F}) + \dim q(F_{x_1} \oplus F_{x_2}) - 1$$

and

$$\mu(E) = \mu(\mathcal{E}).$$

So since $\mathcal{E}$ is (semi)stable we have $\mu(\mathcal{F})(\leq)\mu(\mathcal{E})$ and the last inequality becomes

$$\mu(F) - \frac{\dim(q(F_{x_1} \oplus F_{x_2})}{\mathrm{rk}(F)}(\leq)\mu(E) - 1,$$

which is exactly condition (4.2) $\square$

Chapter 5

The moduli space of principal Higgs bundles on nodal curves

In this section we want to study the moduli space of principal Higgs bundles over a nodal curve X. The basic idea is to relate these objects to *generalized parabolic decorated bundles* using the results of Section 4. It will turn out that the moduli space we get is a projective scheme and so we obtain a compactification of the moduli space of principal Higgs bundles.

Sheaves over singular varieties

We start by giving the definition of principal G bundle and dualizing sheaf on singular varieties.

Principal bundles on singular varieties

Definition 5.1. Let X be a projective scheme. A principal G bundle $\pi\colon P \to X$ with structure group G is a variety P with a free right G-action such that $P/G = X$ and π is G-invariant. Further, the bundle P is locally isotrivial, i.e, locally trivial in the 'fppf' topology.

Recall that the fppf topology on X is defined as follows; the coverings of X are collections $U_i \to X$ of flat maps locally of finite presentation, such that $\coprod_i Ui \to X$ is surjective as a map of sets.

Remark 5.2. If X is smooth the previous condition is equivalent to the classical definition of principal bundle ("i.e." requiring local isotriviality, namely, local triviality in the étale topology, see [3])

Dualizing sheaf

Definition 5.3. Let X a projective scheme of dimension n. A dualizing sheaf for X is a coherent sheaf $\mathring{\omega}_X$ equipped with a morphism $t\colon \mathrm{H}^n(X, \mathring{\omega}_X) \to \mathbb{C}$, called trace, such that for all coherent sheaves $\mathcal{F}$ on X, the composition

$$\mathrm{Hom}(F, \mathring{\omega}_X) \times \mathrm{H}^n(X, \mathcal{F}) \to \mathrm{H}^n(X, \mathring{\omega}_X) \to \mathbb{C},$$

of the natural pairing with the trace morphism induces an isomorphism

$$\mathrm{Hom}(F, \mathring{\omega}_X) \simeq \mathrm{H}^n(X, \mathcal{F})^\vee.$$

Let X as before. We have the following results (see [19])

1. There exists a unique dualizing sheaf for X.
2. Suppose that X is smooth and irreducible over $\mathbb{C}$. Then the canonical sheaf K is the dualizing sheaf $\mathring{\omega}_X$.

5.1 Singular principal Higgs G-bundles

Let X be an irreducible reduced nodal curve over $\mathbb{C}$ with a simple node $x_0 \in X$. We denote by $\nu\colon \widetilde{X} \to X$ the normalization map and by $D\colon = \{x_1, x_2\} = \nu^{-1}(x_0)$. Finally let us fix G to be a semisimple algebraic group over $\mathbb{C}$ and $\rho\colon G \to GL(V)$ a faithful representation of G.

Remark 5.4. Since G is semisimple every faithful representation $\rho\colon G \to GL(V)$ lands in $SL(V)$, indeed $\det\colon \rho(G) \to \mathbb{C}^*$ is a morphism of group and so, since $\rho(G)$ is semisimple and $\mathbb{C}^*$ is abelian, ρ is trivial.

Let $P \xrightarrow{G} X$ be a principal G-bundle and $\phi\colon X \to \mathrm{Ad}(P) \otimes \mathring{\omega}_X$ a Higgs field. Using the representation ρ we can associate with P a vector bundle $E\colon = P_\rho = P \times_\rho V$ over X; the inclusion of P in the $GL(V)$-bundle $\mathrm{Iso}(V \otimes \mathcal{O}_X, E^\vee)$ associated to E gives a section $\sigma\colon X \to \mathrm{Iso}(V \otimes \mathcal{O}_X, E^\vee)/G$ (see Section 3.2.1). Moreover the section σ induces a morphism of $\mathcal{O}_X$-algebras

$$\tau\colon \mathrm{Sym}^*(E \otimes V)^G \to \mathcal{O}_X.$$

The Higgs field $\phi\colon X \to \mathrm{Ad}(P) \otimes \mathring{\omega}_X$ induces a section $\phi\colon X \to \mathrm{End}(E) \otimes \mathring{\omega}_X$, that we call again ϕ for simplicity.
Conversely, given a vector bundle E with trivial determinant and morphisms $\tau\colon \mathrm{Sym}^*(E \otimes V)^G \to \mathcal{O}_X$ and $\phi\colon X \to \mathrm{End}(E) \otimes \mathring{\omega}_X$ such that the corresponding morphisms $\sigma\colon X \to \mathrm{Hom}(V \otimes \mathcal{O}_X, E^\vee)//G$ and $\phi\colon X \to \mathrm{Ad}(\mathrm{Iso}(V \otimes \mathcal{O}_X, E^\vee)) \otimes \mathring{\omega}_X$ have image in $\mathrm{Iso}(V \otimes \mathcal{O}_X, E^\vee)/G$ and in $\mathrm{Ad}(\sigma^*\mathrm{Iso}(V \otimes \mathcal{O}_X, E^\vee)) \otimes \mathring{\omega}_X$ respectively, $\sigma^*\mathrm{Iso}(V \otimes \mathcal{O}_X, E^\vee)$ is a principal Higgs G-bundle over X. So, for any fixed faithful representation $\rho\colon G \to SL(V)$, there is a one to one correspondence between

1. Principal Higgs G-bundles over X;
2. Triples (E, τ, ϕ) where
   - E is a locally free sheaf with $\det E \simeq \mathcal{O}_X$;
   - $\tau\colon \mathrm{Sym}^*(E \otimes V)^G \to \mathcal{O}_X$ is morphism of $\mathcal{O}_X$-algebras such that the induced section $\sigma\colon X \to \mathrm{Hom}(V \otimes \mathcal{O}_X, E^\vee)//G$ has image in $\mathrm{Iso}(V \otimes \mathcal{O}_X, E^\vee)/G$;
   - $\phi\colon X \to \mathrm{End}(E) \otimes \mathring{\omega}_X$ is a Higgs field which induces a morphism $X \to \mathrm{Ad}(\sigma^*\mathrm{Iso}(V \otimes \mathcal{O}_X, E^\vee)) \otimes \mathring{\omega}_X$.

The previous correspondence leads us to give the following definition

Definition 5.5 (**Singular principal Higgs G-bundles**)**.** A singular principal Higgs G-bundle is a triple $(\mathcal{E}, \tau, \phi)$ where

- $\mathcal{E}$ is a torsion free sheaf;
- $\tau\colon \mathrm{Sym}^*(\mathcal{E} \otimes V)^G \to \mathcal{O}_X$ is a morphism of $\mathcal{O}_X$-algebras;
- $\phi\colon X \to \mathrm{End}(\mathcal{E}) \otimes \mathring{\omega}_X$ is a section.

Let $\sigma\colon X \to \mathrm{Hom}(V \otimes \mathcal{O}_X, \mathcal{E}^\vee)//G$ be the section induced by τ and let $U_{\mathcal{E}}$ be the open subset of X in which $\mathcal{E}$ is locally free. If $\sigma(U_{\mathcal{E}}) \subseteq \mathrm{Iso}(V \otimes \mathcal{O}_X, \mathcal{E}^\vee{}_{|U_{\mathcal{E}}})/G$ we will say that the singular principal Higgs G-bundle is **honest**.

Remark 5.6. If the singular principal Higgs G-bundle $(\mathcal{E}, \tau)$ is honest and $U_{\mathcal{E}} = X$ then $\det \mathcal{E} \simeq \mathcal{O}_X$.

5.2 Double-decorated bundles

Descending principal Higgs G-bundles

Observe that the sections of the dualizing sheaf $\mathring{\omega}_X$ of X are given by meromorphic differential forms on the normalization of X such that they only have poles at most of order one at the preimages of the nodes of X. In particular this sheaf is the push-forward of the sheaf $\mathrm{K}(D)$, where K is the canonical bundle of $\widetilde{X}$.

Let $(\mathcal{E}, \tau, \phi)$ be a singular principal Higgs G-bundle over X, starting from the torsion free sheaf $\mathcal{E}$ we obtain a generalized parabolic vector bundle (E, q) over $\widetilde{X}$; moreover, we set $\widetilde{\phi}\colon = \nu^*\phi$ and $\widetilde{\tau}\colon = \nu^*\tau$.

Remark 5.7. From the inclusion $i\colon \mathcal{E}\otimes \mathrm{K}(D)\to \nu_* E\otimes \mathring{\omega}_X$ and the exactness of sequence

$$0\to \mathcal{G}_{x_0}\to \nu_* E\otimes \mathring{\omega}_X\to \nu_*(E\otimes \mathrm{K}(D))\to 0$$

we get an inclusion $j\colon \mathcal{E}\otimes \mathrm{K}(D)\to \nu_*(E\otimes \mathrm{K}(D))$, being $\mathcal{E}\otimes \mathrm{K}(D)$ torsion free and G_{x_0} a sheaf of pure torsion.

Definition 5.8. We say that a quadruple $(E,q,\widetilde{\tau},\widetilde{\phi})$ is a descending principal Higgs G-bundle if it comes from a singular principal bundle G-bundle.

Singular Higgs G-Bundles as decorated bundles

Definition 5.9. A double-decorated parabolic bundle is a quadruple $(E,q,\varphi_1,\varphi_2)$ where (E,q) is a generalized parabolic bundle of rank r and degree d, while

$$\begin{aligned}\varphi_1\colon E_{a_1,b_1,c_1}&\longrightarrow L_1\\ \varphi_2\colon E_{a_2,b_2,c_2}&\longrightarrow L_2,\end{aligned}$$

are morphisms of vector bundles being L_1 and L_2 line bundles.

Let $(E,q,\widetilde{\tau},\widetilde{\phi})$ a singular Higgs G-bundle with a GPS, then using the constructions given in Section 3.2 one can see the morphisms $\widetilde{\tau}$ and $\widetilde{\phi}$ as particular cases of decorations and so any such quadruple induces a double-decorated parabolic bundle.

Remark 5.10. If a double-decorated parabolic bundle $(E,q,\varphi_1,\varphi_2)$ is induced by a descending principal Higgs G-bundle, then $\det E\simeq \mathcal{O}_{\widetilde{X}}$ and so $E_{a,b,c,}=(E^{\otimes a})^{\oplus b}\otimes(\det E)^{\otimes -c}\simeq E_{a,b}$.

5.2.1 Stability for the double-decorated parabolic bundles

In this section we want to define a notion of (semi)stability for double-decorated parabolic bundles, the idea is to combine the notion of semistability for decorated bundles (Section 3) with the notion of semistability for generalized parabolic bundles (Section 4).

Given a double-decorated parabolic bundle $(E,q,\varphi_1,\varphi_2)$, the maps $\varphi_i\colon E_{a_i,b_i,c_i}\to L_i$ provide sections $\sigma_i\colon \widetilde{X}\to \mathbb{P}(E_{a_i,b_i,c_i})$ for $i=1,2$ (see Section 3.1.1).
Let $\lambda\colon \mathbb{C}^*\to G$ be a one-parameter subgroup of G, or equivalently let $(E^\bullet,\underline{\alpha})$ be the corresponding weighted filtration of E. We denote by

$$\mu_{\rho_{a_i,b_i,c_i}}(\lambda;\varphi_i)=\mu_{\rho_{a_i,b_i}}(E^\bullet,\underline{\alpha};\varphi_i)\colon =\mu_{\rho_{a_i,b_i,c_i}}(\lambda;\sigma_i)$$

where, with some abuse of notation, we denote by λ also the induced one-parameter subgroup

$$\mathbb{C}^* \xrightarrow{\lambda} G \xrightarrow{\rho} GL(V) \xrightarrow{\rho_{a_i,b_i,c_i}} GL(V_{a_i,b_i,c_i}),$$

obtained by composing λ with the representations ρ and ρ_{a_i,b_i,c_i}, and we denote by ρ_{a_i,b_i,c_i} also the composition $\rho_{a_i,b_i,c_i} \circ \rho$.

Remark 5.11. For $i = 1, 2$ the following equalities hold

1. $\mu_{\rho_{a_i,b_i,c_i}}(\lambda; \sigma_i(x)) = \mu_{\rho_{a_i,b_i,c_i}}(\lambda; \sigma_i(y))$ for all x, y belonging to the same irreducible component of $\widetilde{X}$ ([50] Remark 1.5).

2. $$\mu_{\rho_{a_i,b_i,c_i}}(\lambda; \sigma_i(x)) = -\min_j \{\gamma_j^{(i)} \mid (\sigma_i(x))(v_j^{(i)}) \neq 0\}$$

   where $\{v_j^{(i)}\}_j$ is a base of eigenvectors for the action of λ over V_{a_i,b_i,c_i} and by writing $(\sigma_i(x))(v_j^{(i)})$ we mean that we have chosen a representative of the class $\sigma_i(x) \in \mathbb{P}(E_{a_i,b_i,c_i})$, and so we can think of $\sigma_i(x)$ as an element of $V^\vee_{a_i,b_i,c_i}$.

3. $$\mu_{\rho_{a_i,b_i,c_i}}(E^\bullet, \underline{\alpha}; \varphi_i) = -\min\{\gamma_{j_1} + \cdots + \gamma_{j_{a_i}} \mid \varphi_i|_{(E_{j_1} \otimes \cdots \otimes E_{j_{a_i}})^{\oplus b_i}} \not\equiv 0\}$$

   where $0 \subset E_1 \subset \cdots \subset E_s \subset E_{s+1} = E$ is the weighted filtration with weights $\underline{\alpha} = (\alpha_j)_{j \leq s}$ induced by the one-parameter subgroup λ, and

   $$\underline{\gamma} = (\gamma_1, \ldots, \gamma_r) := \sum_{j=1}^{s} \alpha_j (\underbrace{\mathrm{rk}E_j - r, \ldots, \mathrm{rk}E_j - r}_{\mathrm{rk}E_j\text{-times}}, \underbrace{\mathrm{rk}E_j, \ldots, \mathrm{rk}E_j}_{r-\mathrm{rk}E_j\text{-times}}).$$

   See [52] Remark 3.1.1.

4. If the pair $(\widetilde{\phi}, \omega)$ induces φ_2 as in Section 3.3 we have

   $$\mu_{\rho_{a_2,b_2,c_2}}(E^\bullet, \underline{\alpha}; \varphi_2) = \mu_{\rho'}(E^\bullet, \underline{\alpha}; [\widetilde{\phi}, \omega]).$$

5.2.2 Equivalence among the semistability conditions

Our initial goal was to study principal Higgs G-bundles over a nodal curve X. To do that we generalized these objects to singular principal Higgs G-bundles over X. For the latter there is a natural condition of (semi)stability. We showed that dealing with such objects is the same as dealing with descending principal Higgs bundles over the normalization $\widetilde{X}$ of the curve X, and finally we saw that descending principal Higgs bundles are a special case of double-decorated vector bundles.

For all such objects one has a notion of (semi)stability, see Definitions 5.17 and 5.15. In this section we prove that the previous definitions of (semi)stability are equivalent, see Proposition 5.20 and Theorem 5.24.

On the other hand, if $G = SL(V)$, a descending principal Higgs G-bundle over $\widetilde{X}$ ([52] pg. 218) corresponds to a generalized parabolic Higgs vector bundles over $\widetilde{X}$ and the latter corresponds to a torsion free sheaf over X with a Higgs field. Bhosle in [5] gives a notion of (semi)stability for generalized parabolic vector bundles over $\widetilde{X}$ which we adapt to the case of generalized parabolic Higgs vector bundles. We will show that the two notions of (semi)stability are, in this special case, the same (Remark 5.18). Moreover we also show that the Definition 5.15 is equivalent to the definition of (semi)stability for torsion free sheaves with a Higgs field (see Proposition 5.19).

Definition 5.12 (**(Semi)stable honest singular principal Higgs G-bundle**). A honest singular principal Higgs G-bundle $(\mathcal{E}, \tau, \phi)$ over X is (semi)stable if and only if

$$P(\mathcal{E}^{\bullet}_{\beta}, \underline{\alpha}_{\beta})\colon = \sum_{i=1}^{s} \alpha_i \left(\deg \mathcal{E} \ \mathrm{rk}\mathcal{E}_i - \deg \mathcal{E}_i \ \mathrm{rk}\mathcal{E}\right) (\geq) 0$$

for every ϕ-invariant weighted β-filtration $(\mathcal{E}^{\bullet}_{\beta}, \underline{\alpha}_{\beta}) = 0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_s \subset \mathcal{E}$.

We will now recall the notion of Higgs reduction ([10] Definition 2.3).
Let $(E, q, \widetilde{\tau}, \widetilde{\phi})$ be a descending principal Higgs bundle, K a closed subgroup of G and $\beta\colon \widetilde{X} \to \mathcal{P}(E, \widetilde{\tau})/K$ a reduction of the structure group to K.
The morphism $i_\beta\colon \mathcal{P}(E, \widetilde{\tau})_\beta \to \mathcal{P}(E, \widetilde{\tau})$ induces an injective morphism of bundles $\mathrm{Ad}(\mathcal{P}(E, \widetilde{\tau})_\beta) \to \mathrm{Ad}(\mathcal{P}(E, \widetilde{\tau}))$.
Let $\Pi_\beta\colon \mathrm{Ad}(\mathcal{P}(E, \widetilde{\tau}) \otimes \mathrm{K}_{\widetilde{X}} \longrightarrow (\mathrm{Ad}(\mathcal{P}(E, \widetilde{\tau})/\mathrm{Ad}(\mathcal{P}(E, \widetilde{\tau})_\beta) \otimes \mathrm{K}_{\widetilde{X}}$ be the induced projection, we can now give the following

Definition 5.13 (**Higgs reduction**). A section $\beta\colon \widetilde{X} \to \mathcal{P}(E, \widetilde{\tau})/K$ is a Higgs reduction of $(E, q, \widetilde{\tau}, \widetilde{\phi})$ if $\widetilde{\phi} \in \ker \Pi_\beta$.

Remark 5.14. In the Definition 5.12, instead of requiring that $P(\mathcal{E}^\bullet_\beta, \underline{\alpha}_\beta)(\geq)0$ for every ϕ-invariant weighted filtration $(\mathcal{E}^\bullet_\beta, \underline{\alpha}_\beta)$, we could requiring the same inequality holds for every one-parameter subgroup $\lambda\colon \mathbb{C}^* \to G$ and for every Higgs reduction $\beta\colon X \to \mathcal{P}(X,\tau)/Q_G(\lambda)$.

Observe that if β is a Higgs reduction and K is a parabolic subgroup of G the filtration $(E^\bullet_\beta, \underline{\alpha}_\beta)$ is $\widetilde{\phi}$-invariant, "i.e.", $\widetilde{\phi}(E_i) \subseteq E_i$ for all i.

Definition 5.15 (**(Semi)stable descending principal Higgs G-bundle**)**.** Let $\mathfrak{E} = (E, q, \widetilde{\tau}, \widetilde{\phi})$ be a descending principal Higgs G-bundle over $\widetilde{X}$. We say that $\mathfrak{E}$ is (semi)stable if and only if for all $\lambda\colon \mathbb{C}^* \to G$ and for all Higgs reduction $\beta\colon \widetilde{X} \to \mathcal{P}(E, \widetilde{\tau})/Q_G(\lambda)$ the following inequality holds

$$P(\mathcal{E}^\bullet_\beta, \underline{\alpha}_\beta)\colon = \sum_{i=1}^{s} \alpha_i(\mathrm{rk}E_i \deg_{\mathrm{par}}(E) - \mathrm{rk}E \deg_{\mathrm{par}}(E_i)) \geq 0.$$

Definition 5.16 (**(Semi)stable generalized parabolic Higgs vector bundle**)**.** A generalized parabolic Higgs vector bundles $(E, q, \widetilde{\phi})$ over $\widetilde{X}$ is (semi)stable if for all subsheaves F of E such that $\phi_{|_F}\colon F \to F \otimes \mathrm{K}$ the condition

$$\mu_{\mathrm{par}}(F) \leq \mu_{\mathrm{par}}(E)$$

holds.

Definition 5.17 (**(Semi)stable double decorated parabolic bundle**)**.** We will say that the decorated bundle $(E, q, \varphi_1, \varphi_2)$ is (δ_1, δ_2)-(semi)stable, for $\delta_i \in \mathbb{Q}_{>0}$, if for all weighted filtrations $(E^\bullet, \underline{\alpha})$ the following inequality holds

$$P(E^\bullet, \underline{\alpha}) + \delta_1\, \mu_{\rho_{a_1,b_1,c_1}}(E^\bullet, \underline{\alpha}; \varphi_1) + \delta_2\, \mu_{\rho_{a_2,b_2,c_2}}(E^\bullet, \underline{\alpha}; \varphi_2) \geq 0$$

Remark 5.18. If $G = SL(V)$ Definitions 5.15 and 5.16 are equivalent, indeed if $G = SL(V)$ all filtrations are β-filtrations and requiring that β is a Higgs reduction is equivalent to require that the associated filtration is $\widetilde{\phi}$-invariant.

Proposition 5.19. *A generalized parabolic Higgs vector bundle $\mathfrak{H} = (E, q, \widetilde{\phi})$ over $\widetilde{X}$ is (semi)stable if and only if the corresponding Higgs torsion free sheaf $(\mathcal{E}, \phi)$ on X is (semi)stable.*

Proof. We already now that the notion of (semi)stability for generalized parabolic vector bundles over $\widetilde{X}$ is equivalent to the (semi)stability for the associated torsion free sheaf over X (Proposition 4.6). It remains to show that $F \subset E$ is $\widetilde{\phi}$-invariant

if and only if the corresponding torsion free sheaf $\mathcal{F}$ over X is ϕ-invariant.
Let us suppose that F is $\widetilde{\phi}$-invariant, "i.e." $\widetilde{\phi}_{|F}\colon F \to F \otimes \mathrm{K}(D)$. We can consider

$$\phi_{|\mathcal{F}}\colon \mathcal{F} \longrightarrow \nu_*\nu^*\mathcal{F} \overset{\nu_*(\phi'|_{\mathcal{F}})}{\longrightarrow} \nu_* F \otimes \mathring{\omega}_X.$$

Observing that $\mathcal{F}$ is the torsion free part of $\nu_*(F)$ we have that $\nu_* F \cap \mathcal{E} = \mathcal{F}$ and we finish.

Conversely given a ϕ-invariant subsheaf $\mathcal{F}$, since $\widetilde{\phi} = \nu^*\phi$, $E = \nu^*\mathcal{E}$ and $F = \nu^*\mathcal{F} \subseteq \nu^*\mathcal{E} = E$, we obtain that

$$\widetilde{\phi}_{|F} = \nu^*\phi_{|_{\nu^*\mathcal{F}}}\colon \nu^*\mathcal{F} \to \nu^*\mathcal{F} \otimes \mathrm{K}(D),$$

and we are done. □

Proposition 5.20 (**Equivalence between Definitions 5.12 and 5.15**). *A honest singular principal Higgs bundle $(\mathcal{E}, \tau, \phi)$ over X is (semi)stable if and only if the corresponding descending principal Higgs bundle $(E, q, \widetilde{\tau}, \widetilde{\phi})$ over $\widetilde{X}$ is (semi)stable.*

Proof. Observing that a β-filtration $(\mathcal{E}^\bullet_\beta, \underline{\alpha}_\beta)$ of $\mathcal{E}$ corresponds to a β-filtration $(E^\bullet_\beta, \underline{\alpha}_\beta)$ of E, the results follows immediately from Proposition 5.19. □

We recall that a double-decorated parabolic bundle $(E, q, \varphi_1, \varphi_2)$ is (δ_1, δ_2)-semistable if for any weighted filtration $(E^\bullet, \underline{\alpha})$ of E we have

$$P(E^\bullet, \underline{\alpha}) + \delta_1\, \mu_{\rho_{a_1,b_1,c_1}}(E^\bullet, \underline{\alpha}; \varphi_1) + \delta_2\, \mu_{\rho_{a_2,b_2,c_2}}(E^\bullet, \underline{\alpha}; \varphi_2) \geq 0.$$

Proposition 5.21. *The family of (δ_1, δ_2)-semistable double-decorated parabolic vector bundles $(E, q, \varphi_1, \varphi_2)$ of type (d, r, L) is bounded.*

Proof. We know ([50] Lemma 1.8) that for a generic morphism $\varphi\colon E_{a,b} \to L$ we have $|\mu_{\rho_{a,b}}(F, E)| \leq a(r-1)$ for any subbundle $F \subset E$, and so, thanks to the semistability of E,

$$P(F, E) \geq -(a_1(r-1)) - (a_2(r-1))\colon = C.$$

Then, recalling that $\deg_{\mathrm{par}}(F) = \deg(F) - \dim q(F_{N_1} \oplus F_{N_2})$ we have

$$\begin{aligned} d\,\mathrm{rk}F - r(\deg(F) - r) &\geq (d-r)\mathrm{rk}F - r(\deg(F) - \dim q(F_{N_1} \oplus F_{N_2})) \\ &= \deg_{\mathrm{par}}(E)\mathrm{rk}F - \deg_{\mathrm{par}}(F)\mathrm{rk}E \\ &= P(F, E) \geq C \end{aligned}$$

and so

$$\mu(F) \leq \frac{d + r^2 - C}{r}.$$

□

Given a descending principal Higgs bundle $\mathfrak{E} = (E, \phi, q)$ and a section $\sigma\colon \mathcal{O}_X \to \mathring{\omega}_X$, we obtain a double-decorated bundle on X considering $\varphi_1\colon = \tau$ and $\varphi_2\colon E_\rho \to \mathring{\omega}_X$ induced by the pair (ϕ, σ), where ρ is the representation $GL(r, \mathbb{C}) \to GL(Hom(\mathbb{C}^r, \mathbb{C}^r) \oplus \mathbb{C})$. Conversely with a given double decorated bundle, with the previous properties, we can associate a descending principal Higgs bundle with a section $\sigma\colon \mathcal{O}_X \to \mathring{\omega}_X$.

Lemma 5.22. *Given a generalized parabolic vector bundle (E, q) with trivial determinant and morphisms $\varphi_1\colon E_{a_1,b_1} \to \mathcal{O}_{\widetilde{X}}$, $\varphi_2\colon E_{a_2,b_2} \to K_{\widetilde{X}}$ induced respectively by morphisms $\widetilde{\tau}\colon Sym^*(E \otimes V)^G \to K_{\widetilde{X}}$, $\widetilde{\phi}\colon E \to E \otimes K_{\widetilde{X}}$ and $\omega\colon \mathcal{O}_{\widetilde{X}} \to K_{\widetilde{X}}$ as in Section 5.2, one has*

$$\mu_{\rho_{a_1,b_1}}(E^\bullet, \underline{\alpha}; \varphi_1) = 0 \Leftrightarrow (E^\bullet, \underline{\alpha}) \text{ is a } \beta\text{-filtration}$$

and

$$\mu_{\rho_{a_2,b_2}}(E^\bullet, \underline{\alpha}; \varphi_2) = 0 \Leftrightarrow (E^\bullet, \underline{\alpha}) \text{ is } \widetilde{\phi}\text{-invariant “i.e.” } \widetilde{\phi}(E_i) \subset E_i \text{ for any } i.$$

Proof. See [52] Proposition 4.2.2 for the first equivalence and [52] Section 3.6 for the second equivalence. □

Proposition 5.23. *The family of semistable descending principal Higgs bundles is bounded.*

Proof. Since the family of semistable Higgs vector bundles is bounded, then, following the idea of the proof of Proposition 4.12 in [43], one sees that the family of semistable Principal Higgs G-bundles is bounded when G is semisimple. □

Theorem 5.24 (**Equivalence between Definition 5.17 and 5.15**). *Given a descending principal Higgs bundle and a nonzero section $\omega\colon \mathcal{O}_{\widetilde{X}} \to K(D)$ there exists δ such that for any $\delta_1, \delta_2 \geq \delta$ the following conditions are equivalent :*

i) For any $\widetilde{\phi}$-invariant β-filtration $(E^\bullet_\beta, \underline{\alpha})$ one has

$$P(E^\bullet_\beta, \underline{\alpha}) \geq 0$$

ii) For any filtration $(E^\bullet, \underline{\alpha})$,

$$P(E^\bullet, \underline{\alpha}) + \delta_1 \mu_{\rho_{a_1,b_1}}(E^\bullet, \underline{\alpha}; \varphi_1) + \delta_2 \mu_{\rho_{a_2,b_2}}(E^\bullet, \underline{\alpha}; \varphi_2) \geq 0, \qquad (5.1)$$

where φ_1 and φ_2 are defined as in Lemma 5.22.

Proof. Let $(E^\bullet, \underline{\alpha})$ be a $\widetilde{\phi}$-invariant β-filtration. By Lemma 5.22

$$\mu_{\rho_{a_1,b_1}}(E^\bullet, \underline{\alpha}; \varphi_1) = \mu_{\rho_{a_2,b_2}}(E^\bullet, \underline{\alpha}; \varphi_2) = 0,$$

so, since by hypothesis $P(E^\bullet, \underline{\alpha}) + \delta_1 \mu_{\rho_{a_1,b_1}}(E^\bullet, \underline{\alpha}; \varphi_1) + \delta_2 \mu_{\rho_{a_2,b_2}}(E^\bullet, \underline{\alpha}; \varphi_2) \geq 0$, we get

$$(E^\bullet, \underline{\alpha}) \geq 0.$$

and inequality (5.1) holds.

Conversely, by Proposition 5.23 there exists a constant C such that $\mu(F) \leq \mu(E) + C$ for any $F \subset E$, and so for any weighted filtration $(E^\bullet, \underline{\alpha})$,

$$P(E^\bullet, \underline{\alpha}) = \sum_{i=1}^{s} \alpha_i (\deg_{\text{par}}(E)\, \mathrm{rk} E_i - r \deg_{\text{par}}(E_i)) \geq -\alpha r(r-1)C,$$

where α is $\max\{\alpha_i | i = 1 \ldots s\}$. If the filtration is a $\widetilde{\phi}$-invariant β-filtration then the conditions are clearly equivalent. Otherwise, if the filtration is not $\widetilde{\phi}$-invariant, one has $\mu_{\rho_{a_2,b_2}}(E^\bullet, \underline{\alpha}; \varphi_2) \geq r\alpha$, and if it is not a β-filtration, one has $\mu_{\rho_{a_1,b_1}}(E^\bullet, \underline{\alpha}; \varphi_1) \geq 1$ ([50] Lemma 3.14). So if we choose $\delta = \max\{-Cr\alpha(r-1), -C(r-1)\}$ we obtain

$$\begin{aligned} P(E^\bullet, \underline{\alpha}) &+ \delta_1 \mu_{\rho_{a_1,b_1}}(E^\bullet, \underline{\alpha}; \varphi_1) + \delta_2 \mu_{\rho_{a_2,b_2}}(E^\bullet, \underline{\alpha}; \varphi_2) \\ &\geq -\alpha r(r-1)C + \delta_1 \epsilon_1(E^\bullet, \underline{\alpha}) + \delta_2 \epsilon_2(E^\bullet, \underline{\alpha}) \geq 0 \end{aligned}$$

where

$$\epsilon_1(E^\bullet, \underline{\alpha}) = \begin{cases} 0 & \text{if } (E^\bullet, \underline{\alpha}) \text{ is a } \beta\text{-filtration} \\ 1 & \text{otherwise} \end{cases}$$

and

$$\epsilon_2(E^\bullet, \underline{\alpha}) = \begin{cases} 0 & \text{if } (E^\bullet, \underline{\alpha}) \text{ is a } \widetilde{\phi}\text{-invariant} \\ 1 & \text{otherwise.} \end{cases}$$

Since $(E^\bullet, \underline{\alpha})$ is not a $\widetilde{\phi}$-invariant β-filtration, ϵ_1 and ϵ_2 cannot both be zero, the inequality (5.1) holds. □

5.3 Moduli space

Given a descending principal Higgs bundle $\mathfrak{E} = (E, q, \widetilde{\tau}, \widetilde{\phi})$ on $\widetilde{X}$, if $\widetilde{\tau}\colon \mathrm{Sym}^*(E \otimes V)^G \to \mathcal{O}_{\widetilde{X}}$ is zero then $\mathfrak{E}$ is nothing but a generalized parabolic Higgs vector bundle. These objects are very close to the Higgs vector bundles studied in [9], [55]

and elsewhere; on the other hand if $\widetilde{\phi}\colon E \to E \otimes \mathrm{K}(D)$ is zero we get a parabolic principal bundle on a smooth curve and the moduli space of these objects was studied by Schmitt (see [52]). So the nontrivial case is when $\widetilde{\tau}$ and $\widetilde{\phi}$ are both non-zero.
As said before, we can consider the family of descending principal Higgs bundles as a subfamily of double decorated parabolic vector bundles with $\det E \simeq \mathcal{O}_{\widetilde{X}}$, and for what we saw above, we can assume that φ_1 and φ_2 are non-zero morphisms.
If we fix $D\colon = \det E$, a morphism $\varphi\colon E_{a,b,c} \to L$ induces a morphism, that we still call φ, from $E_{a,b}$ to $L \otimes D^{\otimes c}\colon = L_D$.
Let $\mathfrak{t}\colon = (D, r, \underline{a}, \underline{b}, \underline{c}, L_1, L_2)$. The objects we want to classify are quadruples $(E, q, \varphi_1, \varphi_2)$ of type $\mathfrak{t}$ where $\varphi_i\colon E_{a_i,b_i} \to L_{iD}$ are non-zero morphisms for $i = 1, 2$.

The last step consist in proving that a double decorated bundle can be viewed as a decorated bundle. Recall that giving a non-zero morphism $\varphi\colon E_\rho \to L$ is the same as giving a section $\sigma\colon X \to \mathbb{P}(E_\rho)$. Therefore the morphisms $\varphi_i\colon E_{a_i,b_i} \to L_i \otimes D^{\otimes c_i}$ correspond to morphisms $\sigma_i\colon X \to \mathbb{P}(E_{a_i,b_i})$ for $i = 1, 2$.
Now consider the Segre embedding

$$\sigma\colon X \xrightarrow{(\sigma_1,\sigma_2)} \mathbb{P}(E_{a_1,b_1}) \times \mathbb{P}(E_{a_2,b_2}) \to \mathbb{P}(E_\chi) \tag{5.2}$$

where $\chi = \rho_{a_1,b_1} \otimes \rho_{a_2,b_2}$ is a homogeneous representation. Observing that $\chi = \rho_{a_1+a_2,b_1b_2}\colon = \rho_{a,b}$ for $a = a_1 + a_2$ and $b = b_1b_2$, one sees that σ induces a morphism $\varphi\colon E_{a,b} \to L$ for $L = L_{1D} \otimes L_{2D} \simeq L_1 \otimes L_2 \otimes D^{\otimes(c_1+c_2)}$. So a quadruple $(E, q, \varphi_1, \varphi_2)$ can be viewed as a decorated parabolic vector bundle, with just one decoration. These objects were widely studied by Schmitt in [52], where he constructed their moduli spaces with respect to the following definition of (semi)stability

Definition 5.25 ((Semi)stable decorated bundles). Fix $\delta \in \mathbb{Q}_{>0}$. A decorated parabolic vector bundle (E, q, φ) where $\varphi\colon E_{a,b} \to L$ is δ-(semi)stable if and only if

$$P(E^\bullet, \underline{\alpha}) + \delta\mu_{\rho_{a,b}}(E^\bullet, \underline{\alpha}; \varphi)(\geq)0.$$

Now thanks to (5.2) a double-decorated parabolic bundle can be viewed as a decorated parabolic bundle and so we have two definitions of (semi)stability and we have to show that they agree.

Theorem 5.26. *For $(E, q, \varphi_1, \varphi_2)$ and φ defined as before the following conditions are equivalent :*

1. *For any weighted filtration $(E^\bullet, \underline{\alpha})$*
   $P(E^\bullet, \underline{\alpha}) + \delta_1\mu_{\rho_{a_1,b_1}}(E^\bullet, \underline{\alpha}; \varphi_1) + \delta_2\mu_{\rho_{a_2,b_2}}(E^\bullet, \underline{\alpha}; \varphi_2) \geq 0$
2. *For any weighted filtration $(E^\bullet, \underline{\alpha})$*
   $P(E^\bullet, \underline{\alpha}) + \delta\mu_{a,b}(E^\bullet, \underline{\alpha}; \varphi) \geq 0$,

where $a = a_1 + a_2$, $b = b_1 b_2$ *and* $\delta = \delta_1 = \delta_2 \in \mathbb{Q}_{>}0$.

Proof. We denote V_{a_1,b_1} and V_{a_2,b_2} by V^1 and V^2, respectively. We choose bases $\{v_i^1\}_{i\in I}$ and $\{v_j^2\}_{j\in J}$ of V^1 and V^2 such that the action of λ is diagonal, "i.e." $\lambda(z).(v_i^1) = z^{\gamma_i^1} v_i^2$ and $\lambda(z).(v_j^2) = z^{\gamma_j^2} v_j^2$ for any $z \in \mathbb{C}^*$, $i = 1, \dots, \dim(V^1)$ and $j = 1, \dots, \dim(V^2)$. Moreover we suppose that $\gamma_1^1 \leq, \dots, \leq \gamma_{s_1}^1$ and $\gamma_1^2 \leq \dots, \leq \gamma_{s_2}^2$. By construction the fiber of $E_{a,b}$ is $V^1 \otimes V^2$ and we have $\lambda(z).(v_i^1 \otimes v_j^2) = z^{\gamma_i^1 + \gamma 2_j} v_i^1 \otimes v_j^2$. Suppose now that $\mu_{\rho_{a_1,b_1}}(E^\bullet, \underline{\alpha}; \varphi_1) = -\gamma_{i_0}^1$ and $\mu_{\rho_{a_2,b_2}}(E^\bullet, \underline{\alpha}; \varphi_2) = -\gamma_{j_0}^2$, using the notations of section 5.2.1 this means that $\sigma_1(x)(v_i^1) = 0$ for any $i < i_0$ and $\sigma_2(x)(v_j^2) = 0$ for any $j < j_0$. Of course $\sigma(x)(v_{i_0}^1 \otimes v_{j_0}^2) \neq 0$ and so $\mu_{a,b} \geq -(\gamma_{i_0} + \gamma'_{j_0})$, if $mu_{a,b} > -(\gamma_{i_0} + \gamma'_{j_0})$ then since γ_i and γ'_j are ordered there exist i_1, j_1 such that $\sigma(x)(v_{i_1}^1 \otimes v_{j_1}^2) \neq 0$ with either $i_1 < i_0$ or $j_1 < j_0$, which is impossible. □

Corollary 5.27. A double-decorated parabolic bundle is (semi)stable if and only if its associated decorated bundle is so.

Hence, in order to study the moduli space of double-decorated parabolic bundles, we can consider the moduli space of decorated bundles (E, q, φ) where the morphism φ lands in $L = L_1 \otimes L_2 \otimes D^{\otimes(c_1+c_2)}$.

5.3.1 Families of decorated parabolic bundles

We want to define the concept of family for decorated parabolic bundles. We start with the notion of isomorphism.

Definition 5.28. Let (E, q, φ) and (E', q', φ'_1) be decorated parabolic bundles. They are isomorphic if and only if there exists an isomorphism of vector bundles $f\colon E \to E'$ and a pair $(g, h) \in Isom(R, R') \times Isom(L, L')$ such that the following diagrams commute :

$$\begin{tikzcd} E_{x_1} \oplus E_{x_2} \arrow[r, "q"] \arrow[d, "f"'] & R \arrow[d, "g"] \\ E'_{f(x_1)} \oplus E'_{f(x_2)} \arrow[r, "q'"] & R' \end{tikzcd} \qquad \begin{tikzcd} E_{a,b,c} \arrow[r, "\varphi"] \arrow[d, "f_{a,b,c}"'] & L \arrow[d, "h"] \\ E'_{a,b,c} \arrow[r, "\varphi'"] & L' \end{tikzcd}$$

Definition 5.29. A family of decorated parabolic bundles of type (d, r, a, b, c, L) parametrized by a scheme S is a triple (E_S, q_S, φ_S) such that

- E_S is a vector bundle over $\widetilde{X} \times S$;
- $q_S\colon \overline{\pi}_{S*}(E_S)_{|\{x_1,x_2\}\times S} \longrightarrow R_S$, where $\overline{\pi}_S\colon \{x_1, x_2\} \times S \longrightarrow \{x_0\} \times S$ and R_S is a vector bundle over S of rank r;

- $\varphi_S\colon (E_S)_{a,b,c} \to \pi_{\widetilde{X}}^* L$ is a homomorphism such that $\varphi_{S|_{\{s\}\times\widetilde{X}}} \not\equiv 0$.

Moreover the pair (E_S, q_S) is called a family of parabolic vector bundles parametrized by S. For more details see [52] Section 2.3.

We will say that two families (E_S, q_S, φ_S) and (E'_S, q'_S, φ'_S) are isomorphic if there exists an isomorphism of vector bundles $f_S\colon E_S \to E'_S$ such that the following diagrams commute

$$\begin{tikzcd} \overline{\pi}_{S*}(E_S)_{\{x_1,x_2\}\times S} \arrow[r, "q_S"] \arrow[d, "\overline{\pi}_{S*}(f_S)"'] & R_S \\ \overline{\pi}_{S*}(E'_S)_{f_S(\{x_1,x_2\}\times S)} \arrow[ur, "q'_S"'] & \end{tikzcd} \qquad \begin{tikzcd} (E_S)_{a,b,c} \arrow[r, "\varphi_S"] \arrow[d, "(f_S)_{a,b,c}"'] & \pi_{\widetilde{X}}^* L \\ (E'_S)_{a,b,c} \arrow[ur, "\varphi'_S"'] & \end{tikzcd}$$

In [52] Schmitt constructs a projective scheme $\mathcal{M}(\rho)^{(\delta)\text{-}ss}_{(d,r,a,b,L)}$ and an open subscheme $\mathcal{M}(\rho)^{(\delta)\text{-}s}_{(d,r,a,b,L)}$ which are moduli spaces for the following functors

$$\underline{\mathfrak{M}}(\rho)^{(\delta)\text{-}(s)s}_{d,r,a,b,L}\colon \mathrm{Sch}_{\mathbb{C}} \to \mathrm{Sets}$$
$$S \longmapsto \left\{ \begin{array}{c} \text{Isomorphism classes of families of} \\ \delta\text{-(semi)stable decorated parabolic} \\ \text{vector bundles on } \widetilde{X} \\ \text{of type } (d,r,a,b,L) \\ \text{parametrized by } S \end{array} \right\}$$

The previous discussion tells us that we can see the moduli space of semistable principal bundles on a nodal curve as a subset of the projective moduli space constructed by Schmitt. Taking the closure inside this projective space one is able to compactify the original moduli space.

Example 5.30. Let us consider the moduli space of vector Higgs bundles with trivial determinant on a nodal curve X. After moving the problem on the normalization $\widetilde{X}$ we get triples (E, q, ϕ). Following our construction we obtain that elements of the moduli space are objects of the form $(E, q, [\phi, \omega])$ where $\omega \det(E) \cong \mathcal{O}_{\widetilde{X}} \to \mathrm{K}(D)$. On the boundary of the moduli space we have triples $(L, q, [\phi, 0])$ and we will denote this point as (L, q, ϕ_∞). We compactify the moduli space adding to any line $\{(L, z\cdot\phi, \omega) | z \in \mathbb{C}^*\}$ the point $(L\phi_\infty$ for $z \in \mathbb{C}^*$ so that the boundary has codimension one in the moduli space.

Chapter 6

Vector bundles over elliptic curves and stable pairs

6.1 Stable t-uples

In this chapter we extend the notion of semistable pair given in Section 2.1.2 to stable t-uples "i.e." vectors $(E, \phi_1, \ldots, \phi_t)$ where as usual E is a vector bundle on X and the ϕ_i's are morphisms from E to $E \otimes L_i$, where for any $i = 1, \ldots, t$, L_i is a line bundle. Moreover we give a description of stable pairs when the degree of the line bundle L is zero, giving another interpretation of proposition 2.11. If the underlying curve is an elliptic curve then a stable Higgs bundle is actually a stable bundle [16], using our description we prove that the same result holds for stable pairs.

Recall that a holomorphic pair (E, ϕ) is a pair consisting of a vector bundle E of degree d and rank r on a Riemann surface X and a morphism of vector bundles $\phi\colon E \to E \otimes L$ where L is a fixed line bundle on X. A pair (E, ϕ) is said to be (semi)stable if and only if $\mu(F)(\leq)\mu(E)$ for any non-zero ϕ-invariant subsheaf $F \subset E$.

We want to generalize this notion to t-uples $(E, \phi_1, \ldots, \phi_t)$ (also simply denoted as (E, Φ)), where E is a vector bundle on X and $\phi_i\colon E \to E \otimes L_i$ for all $i = 1, \ldots, t$, moreover we require that $[\phi_i, \phi_j] = 0$ holds for any i and j.
We say that a t-uple $(E, \phi_1, \ldots, \phi_t)$ is (semi)stable if and only if $\mu(F) \leq \mu(E)$ for any non-zero ϕ_i-invariant subsheaf $F \subset E$, with $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$ and $\phi_i(F) \subset F \otimes L_i$ for any $i = 1, \ldots, t$.

Definition 6.1. A morphism $f\colon (E, \phi_1, \ldots, \phi_t) \to (F, \psi_1, \ldots, \psi_t)$ between two t-uples is a homomorphism of vector bundles $f \in \mathrm{Hom}(E, F)$ such that for any

$i = 1, \ldots, t$ the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{\phi_i} & E \otimes L_i \\ f \downarrow & & \downarrow f \otimes \lambda \cdot \mathrm{id}_{L_i} \\ F & \xrightarrow{\psi_i} & F \otimes L_i \end{array}$$

with $\lambda \in \mathbb{C}^*$.

The definitions of subsheaves and quotients are similar to the same definitions for Higgs bundles.

Lemma 6.2. *Let $(E, \phi_1, \ldots, \phi_t)$ and $(F, \psi_1, \ldots, \psi_t)$ be two semistable t-uples. If $Hom(((E, \phi_1, \ldots, \phi_t)), (F(\psi_1, \ldots, \psi_t))) \neq 0$, "i.e." there exists a non zero morphism of t-uples between E and F then*

$$\mu(E) \leq \mu(F).$$

Proof. The proof follows from the fact the $\ker(f)$ and $\mathrm{Im}(f)$ are invariant subsheaves of E and F respectively. □

Lemma 6.3. *Let $(E, \phi_1 \ldots, \phi_t)$ be a semistable pair, and N a line bundle on X, then the t-uple $(E \otimes N, \phi_1 \otimes id_N, \ldots, \phi_t \otimes id_N)$ is semistable.*

The Harder-Narasimhan filtration for t-uples

Let $(E, \phi_1, \ldots, \phi_t)$ be a not semistable t-uple. Then there exists a Φ-invariant subsheaf E_1 such that $\mu(E_1) \geq \mu(F)$, for any other Φ-invariant subsheaf F of E, and has maximal rank among all subsheaves with this property. Clearly (E_1, Φ) is semistable. If E/E_1 is semistable we finish, otherwise, we iterate this process at the end we obtain a filtration

$$0 \subset E_1 \subset \cdots \subset E_s \subset E$$

with $\mu(E_1) > \mu(E_2/E_1) > \cdots > \mu(E/E_s) > \mu(E)$. The filtration $0 \subset E_1 \subset \cdots \subset E_s \subset E$ is the *Harder-Narasimhan* filtration of (E, Φ).

In a similar way one constructs the *Jordan-Hölder* filtration for semistable t-uples. Given a semistable t-uple $(E, \phi_1, \ldots, \phi_t)$ one obtains a filtration

$$0 \subset F_1 \subset \cdots \subset F_l \subset E,$$

such that for any $j = 1, \ldots, l$, the quotients F_j/F_{j-1} are stable t-uples and $\mu(F_j) = \mu(E)$.

For semistable t-uples we have results very similar to Proposition 2.11, more precisely

Proposition 6.4. *Let $(E, \phi_1, \dots, \phi_t)$ be a semistable t-uple, then*

- *If $\phi_i = 0$ for any i then E is semistable.*
- *If $\phi_i \neq 0$ then $\deg L_i \geq 0$.*
- *If $\deg L_i = 0$ then the t-uple $(E, \phi_1, \dots, \overset{\circ}{\phi_i}, \dots, \phi_t)$ obtained from the original one by deleting the morphism ϕ_i is semistable.*

Proof. The first point is obvious. Let us assume that $\phi_i \neq 0$, so that we have a nonzero morphism $\phi_i \colon E \to E \otimes L_i$, and so, by Lemma 6.2, $\deg L_i \geq 0$.
We prove the result for $i = 1$, the general case is similar. Let us assume that $\deg L_1 = 0$, we want to show that $(E, \phi_2, \dots, \phi_t)$ is semistable. If not, let us consider its Harder-Narasimhan filtration, $0 \subset E_1 \subset \cdots \subset E_s$. Then $\mu(E_j) > \mu(E)$ and so, for $0 < j < s$, E_j is not $(\phi_1, \dots, \phi_t)$-invariant, since it is invariant with respect to $(\phi_2, \dots, \phi_t)$ it is not ϕ_1-invariant "i.e." $\phi_1(E_j) \not\subseteq E_j$. Hence the induced map $\phi_1^j \colon E_j \to (E/E_j) \otimes L_1$ is nonzero. Choose the largest j_0 with $0 \leq j_0 \leq j$ such that $\phi_1^j(E_{j_0}) = 0$ and the smallest l with $j + 1 \leq l \leq s$ such that $\phi_1(E_j) \subset (E_l/E_j) \otimes L_1$. Then ϕ_1 induces a non zero map from E_{j_0+1}/E_{j_0} to $(E_l/E_{l-1}) \otimes L_1$. Therefore we have

$$\mu(E_{j_0+1}/E_{j_0}) \leq \mu(E_l/E_{l-1}) + \deg L_1.$$

Now, as $\mu(E_1) > \mu(E_2/E_1) > \cdots > \mu(E_s/E_{s-1})$ and as $j_0 + 1 \leq j < j + 1 \leq l$, we have $\mu(E_l/E_{l-1}) > \mu(E_{j_0+1})/E_{j_0})$, but since $\deg L_1 = 0$ this is impossible and so $(E, \phi_2, \dots, \phi_t)$ is semistable. □

Remark 6.5. Following Nitsure's proof and considering the composition

$$\phi_t \circ \cdots \circ \phi_1 \colon E \to E \otimes L_1 \otimes \cdots \otimes L_t,$$

one obtains that for all the successive quotients E_i/E_{i-1} of the Harder-Narasimhan filtration of E, the following inequality holds,

$$\mu(E) - \frac{(r-1)^2}{r} d \leq \mu(E_i/E_{i-1}) \leq \mu(E) + \frac{(r-1)^2}{r} d,$$

where $d \colon = \deg(L_1 \otimes \cdots \otimes L_t) = \deg L_1 + \cdots + \deg L_t$. In particular by Proposition 2.12 the family of semistable t-uples is bounded.

If a pair (E, ϕ), with $\deg L = 0$ is semistable, then E is semistable as a vector bundle. If it is stable t-uples, then E has a rigid structure, as the following result proves

Proposition 6.6. *Let (E, Φ) be a stable t-uple, where $\phi_i \colon E \to E \otimes L_i$ for any i and $\deg L_1 = 0$. Then $(E, \phi_2, \dots, \phi_n)$ is obtained as a successive extension of stable t-uples.*

Proof. Since (E, Φ) is stable and $\deg L_1 = 0$ then by Proposition 6.4 $(E, \phi_2, \dots, \phi_n)$ is semistable. Let us consider its Jordan-Hölder filtration

$$0 \subset E_1 \subset \dots E_t = E.$$

$(E_1, \phi_2 \dots, \phi_n)$ is a stable t-uple with $\mu(E_1) = \mu(E)$, so E_1 cannot be ϕ-invariant and there is a nonzero morphism $\phi_1 \colon E_1 \to E_j/E_{j-1} \otimes L_1$ for some $j = 2, \dots, t$. Since both sides are stable t-uples, we have $E_1 \simeq E_j/E_{j-1} \otimes L_1$.
Now we look at E_2. As before it is not ϕ_1-invariant and we have two cases

- $\phi(E_2) \subset E_j \otimes L_1$,
- $\phi(E_2) \subset E_l \otimes L_1$ with $l > j$.

In the first case we have that $E_1 \simeq E_j/E_{j-1} \otimes L_1$ is a subbundle and a quotient of E_2 and so $E_2 = E_1 \oplus E_2/E_1$. Now stability of (E, Φ) implies that E_2/E_1 is not ϕ_1-invariant in particular we have a nonzero morphism

$$\phi \colon E_2/E_1 \to E_j/E_{j-1} \otimes L_1.$$

As before one concludes that ϕ is an isomorphism and so $E_2 = E_1 \oplus E_1$.

In the second case we have directly a non zero morphism between E_2/E_1 and $E_j/E_{j-1} \otimes L_1$ induced by ϕ_1, and we conclude that E_2 is an extension of E_1.
Iterating this process we obtain that E_l is an extension of E_{l-1} with E_1. In particular E is a successive extension of E_1, with $(E_1, \phi_2, \dots, \phi_n)$ a stable pair. □

Application to Higgs bundles

Let us fix the following objects :

- X a smooth projective curve;
- V the vector bundle $\mathrm{K} \oplus \mathcal{O}_X$.

Then a Higgs bundle (E, Φ) with $\Phi \in H^0(X, \mathrm{End}(E) \otimes V)$ is equivalent to a triple (E, ϕ_1, ϕ_2) with $\phi_1 \colon X \to \mathrm{End}(E) \otimes \mathrm{K}$ and $\phi_2 \colon X \to \mathrm{End}(E)$ holomorphic sections such that $[\phi_1, \phi_2] = 0$.
We say that (E, Φ) is semistable if for any Φ-invariant subbundle F one has $\mu(F) \leq \mu(E)$, or equivalently if the same inequality holds for any ϕ_1 and ϕ_2 invariant subsheaf F.
Applying Proposition 6.4, we obtain the following

Proposition 6.7. *The Higgs bundle (E, Φ) is semistable if and only if the Higgs bundle (E, ϕ_1) is semistable. Moreover if (E, Φ) is stable then E is a successive extension of stable Higgs bundles.*

Proof. If (E, ϕ_1) is semistable then also (E, Φ) is semistable.
Since $\deg \mathcal{O}_X = 0$ applying Proposition 6.4 we have the converse. The final part is exactly the statement of Proposition 6.6 $\square$

6.2 Stable pairs on elliptic curves

The theory of vector bundles over elliptic curve has been widely studied in the last years (see [1] or [57]). We start the section by recalling some important results about them.

- If $\gcd(n, d) = 1$, a stable vector bundle E of rank n and degree d satisfies $E \otimes L \cong E$ if and only if L is a line bundle in $\mathrm{Pic}^0(C)[n]$ ("i.e." L is such that $L^{\otimes n} \cong \mathcal{O}_C$);
- There exists a unique indecomposable bundle F_n (called the Atiyah bundle) of degree 0 and rank n such that $H^0(X, F_n) \neq 0$. Moreover $\dim H^0(X, F_n) = 1$ and F_n is a multiple extension of copies of $\mathcal{O}_C$. In particular F_n is semistable.
- Every indecomposable bundle of degree 0 and rank n is of the form $F_n \otimes L$ for a unique line bundle L of degree 0.
- If $\gcd(n, d) = h > 1$,
  - every indecomposable bundle of rank n and degree d is of the form $E' \otimes F_h$ for a unique stable bundle E' of rank $n' = \frac{n}{h}$ and degree $d' = \frac{d}{h}$;
  - every semistable bundle of rank n and degree d is of the form $\bigoplus_{j=1}^{s}(E'_j \otimes F_{h_j})$, where each E'_j is stable of rank n' and degree d' and $\sum_{j=1}^{s} h_j = h$;
  - every polystable bundle of rank n and degree d is of the form $E'_1 \oplus \ldots \oplus E'_h$, where each E'_i is stable of rank n' and degree d';
- If E is stable, $\mathrm{End}(E) \cong \bigoplus_{L_i \in \mathrm{Pic}^0(C)[n]} L_i$.
- $F_n \cong F_n^*$ and $F_n \otimes F_m$ is a direct sum of various F_l. In particular $\mathrm{End}(F_n) \cong F_1 \oplus F_3 \oplus \ldots \oplus F_{2n-1}$.

If C is an elliptic curve K_C is the trivial bundle and so a Higgs vector bundle (E, ϕ) is semistable as a Higgs bundle if and only if the underlying vector bundle E is semistable. In [16] the authors prove the same result for stability.

Proposition 6.8. *Let (E, ϕ) be a Higgs bundle on the elliptic curve C. Then (E, ϕ) is stable if and only if E is stable.*

Now we want to generalize this fact to stable pairs. We discuss the case $\mathrm{rk}(E) = 2$ the general case being similar.

Proposition 6.9. *Let (E, ϕ) be a rank 2 stable pair on the elliptic curve C where $\phi\colon E \to E \otimes L$ with $\deg(L) = 0$ and $L \neq \mathcal{O}_C$, Then E is stable as vector bundle.*

Proof. We already know that E is given by an extension of stable vector bundles

$$0 \to E_1 \to E \to E_2 \to 0$$

where $\mu(E) = \mu(E_1) = \mu(E_2)$. Since C is an elliptic curve E_1 and E_2 are line bundles. The non zero morphism $\phi\colon E_1 \to E/E_2 \otimes L$ between line bundles of the same slope must be an isomorphism. Let us suppose that $E_1 \ncong E_2$. In this case $\mathrm{H}^1(C, E_1 \otimes E_2^\vee) = 0$ and so $E = E_1 \oplus E_2$. Since (E, ϕ) is stable we have the following nonzero morphisms

$$\phi\colon E_1 \to E_2 \otimes L$$

and

$$\phi\colon E_2 \to E_1 \otimes L$$

hence $E_1 \cong E_1 \otimes L^2$. So $L \in \mathrm{Pic}^0(C)[n]$. But from $E_1 \cong E_2 \otimes L$ we get $E_1 \cong E_2$ which contradicts our assumption.
So we have $E_1 \cong E_2$, in particular $E \cong F_2 \otimes E_1$ or $E \cong E_1 \otimes (\mathcal{O}_C \oplus \mathcal{O}_C)$.In the first case we have

$$\mathrm{End}(E) \otimes L \cong (F_1 \oplus F_3) \otimes L.$$

So

$$\phi = \mathrm{id}_{E_1} \otimes \psi, \quad \psi \in \mathrm{H}^0\left(C, (F_1 \oplus F_3)\right).$$

Now $F_1 \oplus F_3$ has a unique subbundle $\mathcal{O}_C^2$ and $\mathrm{H}^0(\mathcal{O}_C^2) \overset{\cong}{\hookrightarrow} \mathrm{H}^0(F_1 \oplus F_3)$. In this case $\mathcal{O}_C^2$ is ψ-invariant and $E_1 \otimes \mathcal{O}_C^2$ is ϕ-invariant. This implies that (E, ϕ) is not stable. We have $E \cong E_1 \otimes \mathcal{O}_C^2$. In this case $\psi \in \mathrm{End}\mathcal{O}_C^2 = \{2 \times 2 \text{ matrices}\}$. Choose an eigenvector v for ψ. Then $E_1 \otimes v$ contradicts the stability of (E, ϕ) and E is stable.

□

Remark 6.10. If X is not an elliptic curve, the result about stability expressed in Proposition 6.9 does not hold. Let us indeed consider a nontrivial extension $0 \to L \to E \to L \to 0$ with $\deg L = 0$ and ϕ which sends L to $N \otimes \mathrm{K}$ with N a line subbundle of E. Given a line bundle S of E then we must have a nonzero map $S \to L$ and so $\deg S < \deg L$. Since the only subsheaves of degree 0 are not ϕ-invariant we have that the pair (E, ϕ) is stable but E is just semistable. In the case X is an elliptic curve with the previous notation we have that $\deg N < 0$ and there are no nonzero morphism $L \to N \otimes \mathcal{O}_X$.

Chapter 7

Higgs bundles over projective varieties

In this section we apply some results of previous sections to study Higgs vector bundles over higher dimensional projective varieties. Before we recall some basic facts about vector bundles over projective varieties.

7.1 Vector bundles over projective varieties

7.1.1 Semistable vector bundles

Let X be a smooth n-dimensional projective variety and let H be an ample line bundle on X. For any rank $r > 0$ torsion free sheaf E we denote by $c_i \in \mathrm{H}^{2i}(X, \mathbb{R})$ its Chern classes and define its slope by

$$\mu(E) \colon = \frac{c_1(E) \cdot \mathrm{H}^{n-1}}{r}.$$

The Hilbert polynomial P_E is defined by $P_E(m) = \chi(E \otimes \mathcal{O}_X(m\mathrm{H}))$.

Definition 7.1. A coherent sheaf E of rank $r > 0$ over the polarized variety (X, H) is called Gieseker (semi)stable if it is torsion-free and for all subsheaves $F \subset E$ with $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$ we have

$$\frac{P_F}{\mathrm{rk}(F)} (\leq) \frac{P_E}{\mathrm{rk}(E)}.$$

It is called μ-(semi)stable if

$$\mu(F) (\leq) \mu(E)$$

Remark 7.2. We have the following chain of implications

$$\mu\text{-}stable \Rightarrow Gieseker\ stable \Rightarrow Gieseker\ semistable \Rightarrow \mu\text{-semistable}.$$

Indeed the reduced Hilbert polynomial of E is

$$p_E \colon = \frac{P_E(m)}{\mathrm{rk}E} = m^n + \frac{c_1(E)\cdot \mathrm{H}^{n-1}}{\mathrm{rk}E} m^{n-1} + \text{low degree terms}$$

if F is a subsheaf of E, μ-stability implies that

$$\frac{c_1(F)\cdot \mathrm{H}^{n-1}}{\mathrm{rk}F} < \frac{c_1(E)\cdot \mathrm{H}^{n-1}}{\mathrm{rk}E}$$

hence it is Gieseker stable. By definition stability is a stronger condition then semistability and this implies μ-semistability since the condition $p_F \leq p_E$ forces the leading coefficients of the two polynomials to have the same relation.

7.1.2 Changing the polarization

In the definition of semistability, we fixed an ample line bundle H on X in order to give a definition of degree. Next example shows that actually there exist vector bundles which are μ-stable with respect to a given polarization but fail to be μ-semistable with respetc to an other polarization.

Example 7.3. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ be a quadric surface. We denote by l and m the standard basis of $Pic(X) \cong \mathbb{Z}^2$. So, $\mathrm{K}_X = -2l - 2m$, $l^2 = m^2 = 0$ and $ml = 1$. Let E be a rank 2 vector bundle on X given by a nontrivial extension :

$$0 \to \mathcal{O}_X(l-3m) \to E \to \mathcal{O}_X(3m) \to 0.$$

The nonzero extension $e \in \mathrm{Ext}^1(\mathcal{O}_X(3m), \mathcal{O}_X(l-3m))$ exists because

$$\mathrm{Ext}^1(\mathcal{O}_X(3m); \mathcal{O}_X(l-3m)) \cong \mathrm{H}^1(X, \mathcal{O}_X(l-6m)) \neq 0.$$

Moreover $c_1(E) = l$ and $c_2(E) = 3$. We now consider the ample line bundles $L = l + 5m$ and $N = l + 7m$. We have

1. E is not μ-semistable with respect to N, and
2. E is μ-stable with respect to L.

E is not μ-stable with respect to N because $\mathcal{O}_X(l-3m)$ is a rank 1 subbundle of E and

$$c_1(\mathcal{O}_X(l-3m))\cdot N = 4 > \frac{7}{2} = \frac{c_1(E)\cdot N}{\mathrm{rk}(E)}.$$

Let us check that E is μ-stable with respect to L. Let $\mathcal{O}_X(D)$ be any line subbundle of E. Since E sits in an exact sequence

$$0 \to \mathcal{O}_X(l-3m) \to E \to \mathcal{O}_X(3m) \to 0$$

we have

1. $\mathcal{O}_X(D) \hookrightarrow \mathcal{O}_X(l-3m)$, or
2. $\mathcal{O}_X(D) \hookrightarrow \mathcal{O}_X(3m)$

In the first case $\mathcal{O}_X(l-3m-D)$ is an effective divisor. Since L is an ample line bundle on X,

$$(l-3m-D) \cdot L \geq 0$$

hence

$$D \cdot L \geq (l-3m)(l+5m) = 2 < \frac{5}{2} = \frac{c_1(E) \cdot L}{\mathrm{rk}(E)}.$$

In a similar way one can prove that $\mathcal{O}_X(D) \hookrightarrow \mathcal{O}_X(3m)$ does not destabilize E.

7.1.3 Filtrations

For torsion-free sheaves on projective varieties there exist the analogue of Harder-Narasimhan and Jordan-Hölder filtrations.

Proposition 7.4 (Harder-Narasimhan filtartion)**.** *Every torsion-free coherent sheaf E has an increasing filtration by nonzero sub sheaves*

$$0 \subset E_1 \subset \cdots \subset E_s \subset E,$$

such that $E^i := E_i/E_{i-1}$ is Gieseker semistable and $p_i > p_j$ for $i < j$ where p_i is the reduced Hilbert polynomial of E^i.

Proposition 7.5 (Jordan-Hölder filtration)**.** *Let E be a Gieseker semistable sheaf with reduced Hilbert polynomial p_E, then E admits a (not necessary unique) filtration*

$$0 \subset E_1 \subset \cdots \subset E_l \subset E,$$

such that $E^i := E_i/E_{i-1}$ is Gieseker stable and $p_i = p_E$ for any i.

Remark 7.6. The same statements hold also for μ-semistability by replacing the reduced Hilbert polynomial with the slope in the previous constructions.

7.1.4 Operations on semistable vector bundles

Given two μ-semistable sheaves E and F over X, one can construct the direct sum $E \oplus F$ and the tensor product $E \otimes F$. A natural question is whenever this sheaves are still μ-semistable.
If the slopes are different, let us say $\mu(E) > \mu(F)$ then the subsheaf $E \subset E \oplus F$ has slope bigger then the slope of $E \oplus F$ hence the latter is not μ-semistable. By other hand it is easy to see that if $\mu(E) = \mu(F)$ then $E \oplus F$ is μ-semistable.
If E is μ-(semi)stable sheaf then $E \otimes L$ is μ-(semi)stable for any line bundle L. For higher rank case we have the following result

Proposition 7.7. *Let E and F be two μ-semistable vector bundles of slope $\mu(E)$ and $\mu(F)$ on a smooth curve C. Then the tensor product $E \otimes F$ is μ-semistable of slope*

$$\mu(E \otimes F) = \mu(E) + \mu(F).$$

Proof. Tensorizing with suitable line bundles L_E and L_F and eventually passing to a finite cover of C and applying Proposition 2.7, we can assume that $\mu(E) = \mu(F) = 0$. Finally by considering Jordan-Hölder filtrations we can suppose E and F are stable. So E and F are stable vector bundles of degree zero, hence by a theorem of Narasimhan and Seshadri, they correspond to irreducible unitary representations $\rho_E \colon \pi(C) \to U(\mathrm{rk}E)$ and $\rho_F \colon \pi(C) \to U(\mathrm{rk}F)$. The tensor product of unitary representations is still unitary so the vector bundle $E \otimes F$ is semistable (actually it is polystable). □

For higher dimensional case one can apply the Metha-Ramanathan restriction Theorem

Theorem 7.8 (Metha-Ramanathan)**.** *Let X be a nonsingular, projective, irreducible variety of dimension d, endowed with a very ample line bundle H, Let E be a μ-semistable vector bundle X then there is a positive integer a_0 such that for all $a \geq a_0$ there is a dense open subset $U_a \subset |aH|$ such that for all $D \in U_a$ the divisor D is smooth and $E|_D$ is μ-semistable.*
The same statement holds with "μ-semistable" replaced by "μ-stable".

Mehta-Ramanathan restriction theorem is very useful as it often allows one to reduce a problem from a higher-dimensional variety to a small dimension variety or even to a curve, as for example happens with the proof of Hitchin-Kobayashi correspondence (see [25]). For the proof we refer to the original papers of Metha and Ramanathan [29] and [30].

Let E and F be μ-semistable vector bundles over a projective surface X equipped with a polarization in an ample line bundle H. Applying the previous theorem we obtain that for a general element C of the linear system $|a\mathrm{H}|$ the restrictions $E|_C$ and $F|_C$ are semistable hence the tensor product $E|_C \otimes F|_C$ is semistable. Since the last condition holds for a general curve in the linear system $|a\mathrm{H}|$ then $E \otimes F$ must be μ-semistable and we are done. By induction on the dimension of X one obtains the result for any projective variety.

Remark 7.9. Let us observe that in general the tensor product of two μ-stable vector bundles is only μ-semistable. Indeed $\mathrm{End}(E) = E \otimes E^\vee$ has a nonzero section corresponding to the identity map of E, and $c_1(\mathrm{End}(E)) = 0$ so it is not μ-stable.

7.1.5 Bogomolov's inequality

Let E be a torsion-free sheaf on the polarized variety (X, H) with Chern classes $c_i \in \mathrm{H}^{2i}(X, \mathbb{R})$. In general semistability condition could be very difficult to check. One could hope to have a numerical way to predict whenever E is semistable or not; Bogomolov's inequality gives a partial answer to this question.

Let us define the class

$$\Delta(E) \colon = c_2(E) - \frac{(r-1)}{2r} c_1^2(E) \in \mathrm{H}^4(X, \mathbb{R}).$$

$\Delta(E)$ is called the discriminant of E. The discriminant satisfies the following properties :

1. If E is an invertible sheaf then $\Delta(E) = 0$.
2. If E and F are vector bundles then

$$\Delta(E \otimes F) = \Delta(E) + \Delta(F).$$

   In particular if L is a line bundle then $\Delta(E \otimes L) = \Delta(E)$.

Theorem 7.10 (Bogomolov)**.** *Let X be a smooth complex projective surface and E a μ-semistable torsion free sheaf on X, then the following inequality holds*:

$$\Delta(E) \geq 0.$$

Proof. We can assume that E is locally free. Indeed the bidual E^{**} is a μ-semistable torsion free sheaf with $\Delta(E^{**}) \leq \Delta(E)$. Finally replacing E with $\mathrm{End}(E)$ we can also assume that $c_1(E) = 0$. Let $k >> 1$ so that $k \cdot \mathrm{H}^2 > \mathrm{H} \cdot \mathrm{K}_X$ and that there is a smooth curve $C \in |k\mathrm{H}|$. Since for any integer m the symmetric power $S^m(E)$ of a μ-semistable bundle with $c_1(E) = 0$ is also a μ-semistable

bundle with the same first Chern class we have $\mathrm{H}^0(S^m E(-C)) = 0$.
Therefore, the standard exact sequence

$$0 \to S^m(E) \otimes \mathcal{O}_X(-C) \to S^m(E) \to S^m(E)|_C \to 0,$$

and Serre's duality lead to the estimates

$$h^0(S^m(E)) \leq h^0(S^m(E(-C))) + h^0(S^m(E)|_C) = h^0(S^m(E)|_C).$$

Considering $Y = \mathbb{P}(E|_C) \to C$ we see that

$$h^0(S^m(E)|_C) = h^0(Y, \mathcal{O}_Y(m))$$

by projection formula. Since $\dim Y = r$ there exists a constant A such that

$$h^0(S^m(E) \leq h^0(Y, \mathcal{O}_Y(m)) \leq A \cdot m^r,$$

for all $m > 0$. Similarly using restriction to $C \in |k\mathrm{H}|$ for large k, one can see that there exists a constant B such that

$$h^2(S^m(E)) = h^0(S^m(E) \otimes \mathrm{K}_X) \leq B \cdot m^r.$$

Therefore

$$\chi(S^m(E)) \leq h^o(S^m(E)) + h^2(S^m(E)) \leq (A + B)m^r.$$

However we have

$$\chi(S^m(E)) = -\frac{\Delta(E)}{2r}\frac{m^{r+1}}{(r+1)!} + O(m^r),$$

so $\Delta(E) \geq 0$.

$\square$

The case of μ-semistable torsion-free sheaves E on higher dimensional varieties follows from the case of μ-semistable torsion-free sheaves on surfaces taking into account that the restriction of E to a general complete intersection $Y\colon = D_1 \cap \cdots \cap D_d$ with $D_i \in |aH|$ and a as in Theorem 7.8, then $E|_Y$ is again μ-semistable and

$$a^{n-2}\Delta(E) = \Delta(E|_Y),$$

where n is the dimension of X. So a μ-semistable vector bundle E over the polarized variety (X, H) satisfies the inequality $\Delta(E) \cdot \mathrm{H}^{n-2} \geq 0$

7.1.6 Vector bundles with $\Delta(E) = 0$

In the previous section we showed that a μ-semistable torsion free sheaf must have non-negative discriminant. Now we want to investigate μ-semistable torsion free sheaves with vanishing discriminant. It will turn out that such objects are precisely semistable vector bundles on X such that for any morphism $f\colon C \to X$ from a smooth projective curve the pull-back $f^*(E)$ remains semistable.

Definition 7.11. A line bundle $L \in \mathrm{Pic}(X)$ is called nef if $\deg f^*(L) \geq 0$ for every morphism $f\colon C \to X$ from a compact Riemann surface C. A vector bundle E is called nef if $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$ is nef.

Definition 7.12. We say that a vector bundle E admits a projectively flat Hermitian structure if $\mathbb{P}(E)$ is defined by a representation

$$\pi_1(X) \to PU(r)$$

Clearly on curves a degree zero vector bundle is semistable if and only if it is nef, on higher dimensional varieties one has the following result (see [12] and [36])

Theorem 7.13. *Let E be a vector bundle of rank r on a projective variety X. Let H be an ample line bundle on X. Then the following conditions are equivalent:*

1. *For every morphism $f\colon C \to X$ from a smooth projective curve, $f^*(E)$ is semistable.*
2. *$E \otimes \frac{\det E^*}{r}$ is nef.*
3. *E is μ-semistable with respect to H and*
$$\Delta(E) = 0.$$
4. *E admits a filtration into subbundles*
$$0 = E_0 \subset E_1 \subset \cdots \subset E_t = E,$$
such that each E_i/E_{i-1} admits a projective flat Hermitian structure and
$$\mu(E_i/E_{i-1}) = \mu(E),$$
for every i.

In particular the previous proposition tells us that a semistable vector bundle E with $\Delta(E) = 0$ remains semistable when restricted to any smooth curve $C \subset X$. In some sense this is a different version of the Metha-Ramanathan restriction theorem for vector bundles on surfaces with the hypothesis $(r-1)c_1^2(E) = 2rc_2(E)$.

7.2 Higgs varieties

Given a semistable Higgs vector bundle $\mathcal{E} = (E, \phi)$ such that $\Delta(E) = 0$ then for any morphism $f: C \to S$, where C is a smooth projective curve, the pull-back $f^*\mathcal{E}$ is semistable as a Higgs bundle [11]. The converse is only conjectured. Here we prove this fact for varieties with nef tangent bundle. The idea is to connect Higgs semistability to classical semistability and so apply Theorem 7.13.
At the first we need to extend Proposition 2.11 to pairs (E, ϕ) where E is a holomorphic vector bundle on a smooth projective variety and ϕ is a section of $\mathrm{End}(E) \otimes M$ being M a vector bundle.
Recall that the definition of μ-semistability for Higgs bundles and holomorphic pairs on projective variety is essentially the same given for vector bundle with the difference that the condition on the slopes has to be checked only for invariant sheaves. We have the following result

Theorem 7.14. *Let X be a polarized variety and H an ample line bundle on it. Let (E, ϕ) be a μ-semistable pair where $\phi: E \to E \otimes M$ with M μ-semistable with respect to H and* $\deg M = 0$*, then E is μ-semistable.*

Proof. Let us assume E not μ-semistable and consider its Harder-Narasimhan filtration :

$$0 \subset E_1 \subset \cdots \subset E_t \subset E.$$

Then $\mu(E_1) > \mu(E)$ and so it is not ϕ-invariant. Let j the smallest integer such that $\phi(E_1) \subset E_j \otimes M$. The homomorphism $\phi: E_1 \to E_j/E_{j-1} \otimes M$ is not zero. Hence since $\mu(M) = 0$ and they are μ-semistable, we obtain

$$\mu(E_1) \leq \mu(E_j/E_{j-1})$$

but since they are elements of the Harder-Narasimhan filtration we should have the inequality

$$\mu(E_1) > \mu(E_j/E_{j-1})$$

and this is impossible. □

So if the canonical bundle K_X has degree zero and the tangent bundle is μ-semistable, a μ-semistable Higgs bundle (E, ϕ) over X is μ-semistable in the classical way. This geometric condition is strictly connected to the existence of a Kähler-Einstein metric on X. Aubin and Yau ([2] and [58]) show the existence of a Kähler-Einstein metric whenever the canonical line bundle K_X is ample or trivial. Kobayashi [25] and Lübke [23] show that the existence of a Kähler-Einstein metric implies the μ-polystability of the tangent bundle.

Definition 7.15. We say that a projective variety X is a Higgs variety if $\dim(X) = 1$, or, in the case $\dim(X) > 1$, the following property holds : if $\mathcal{E} = (E, \phi)$ is a Higgs vector bundle on X such that for any morphism $f\colon C \to S$ from a smooth projective curve C the pull-back $f^*(\mathcal{E})$ is semistable as a Higgs bundle, then $\Delta(E) = 0$.

We will prove that rationally connected varieties and quasi-abelian varieties are Higgs varieties. Moreover, we shall see that finite étale quotients of Higgs varieties, and rationally connected fibration on Higgs varieties are Higgs varieties. In particular this facts allow us to prove that any projective variety with nef tangent bundle is a Higgs variety.
Finally, using the Lefschetz Hyperplane section theorem, we shall show that if a variety X of dimension $n \geq 5$ admits an ample divisor D which is a Higgs variety then X is a Higgs variety as well.

7.2.1 Vector bundles over rationally connected varieties

7.2.2 Rationally connected varieties

Here we recall some general facts about rationally connected varieties, the reader can see [28] for more details.

Definition 7.16. A variety X is called rationally connected if any two general points in X are connected by a chain of rational curves.

Proposition 7.17. *Let $D \subset X$ be a smooth ample divisor on a smooth projective variety X. If D is rationally connected then X is also rationally connected.*

Theorem 7.18. *[26] Let X be a smooth projective variety, if $K_X < 0$ then X is rationally connected.*

In particular Fano varieties are rationally connected.

7.2.3 Higgs bundles on Rationally connected varieties

By Theorem 7.14 the negativity of the canonical bundle is an obstruction to the existence of semistable Higgs bundle which are not semistable. For vector bundles on rationally connected varieties we have the following results [6].

Proposition 7.19. *Let X be a rationally connected variety. Let $E \to X$ be a vector bundle such that for every morphism $f\colon \mathbb{P}^1 \to X$, the pullback f^*E is trivial. Then E itself is trivial.*

Corollary 7.20. Let $E \to X$ be a vector bundle over a rationally connected variety, such that for any morphism $f\colon \mathbb{P}^1 \to X$ the pull back is semistable then $E \cong \oplus_{i=1}^r L$ where L is a line bundle on X.

Proof. Let us consider a morphism $f\colon \mathbb{P}^1 \to X$. Since $f^*(E)$ is semistable we get $f^*(E) \cong \mathcal{O}(a)^r$ for some $a \in \mathbb{Z}$. In particular $\mathrm{End}(f^*(E)) = f^*(\mathrm{End}(E))$ is trivial for any such f so by the previous proposition $\mathrm{End}(E)$ is trivial.
This implies that, for any $x \in X$, the evaluation map

$$\mathrm{H}^0(X, \mathrm{End}(E)) \to \mathrm{End}(E_x)$$

is an isomorphism; let $A\colon E \to E$ be an isomorphism such that all the eigenvalues $\lambda_1, \ldots, \lambda_r$ of A_x are distinct. Hence E is isomorphic to the direct sum of the line subbundles $L_i\colon = \ker(\lambda_i - A) \subset E$, $1 \le i \le r$. Since the evaluation map is an isomorphism, we have

$$\dim \mathrm{H}^0(X, L_i \otimes L_j^\vee) \le 1$$

for all $i \neq j$. Note that if $\mathrm{H}^0(X, L_i \otimes L_j^\vee) = 0$ for some i and j then

$$\dim \mathrm{H}^0(X, \mathrm{End}(E)) < r^2,$$

which contradicts the fact that $\mathrm{End}(E)$ is trivial and so $E \cong \oplus_{i=1}^r L$ for some line bundle L. □

Theorem 7.21. *Let $\mathcal{E} = (E, \phi)$ be a μ-semistable Higgs vector bundle on a rationally connected variety X. If for any morphism $f\colon C \to X$, where C is a smooth projective curve, the Higgs bundle $f^*(\mathcal{E})$ is semistable, then $\Delta(E) = 0$.*

Proof. Let us consider a morphism $f\colon C \to S$ where C is a rational projective curve. By hypothesis the pull-back $f^*(\mathcal{E})$ is semistable as a Higgs bundle, applying Theorem 7.14 we get that $f^*(E)$ is semistable as a vector bundle. By the previous corollary we get that $E \cong \oplus L$ and this implies that $\Delta(E) = 0$. □

In particular the previous Theorem tells us that rationally connected varieties are Higgs varieties.

7.2.4 Vector bundles on quasi-abelian varieties

Let X be an abelian variety. Since the tangent bundle of X is trivial, its pull-back via any morphism remains trivial, hence semistable. The same facts holds true also under the weaker assumption that X is quasi-abelian "i.e.", X admits an étale cover by an abelian variety. Indeed, if $q\colon A \to X$ is an étale cover with

A an abelian variety and $f\colon C \to X$ is a morphism form a smooth projective curve in X, the pull back of f^*T_X to the fiber product $C \times_X A$ is trivial, hence f^*T_X is semistable. A natural question is whether there are other varieties with this property. It was proved by P. Jahnke and I. Radloff [24] that quasi-abelian varieties are the only ones which satisfy the condition that the pull-back of tangent bundle is semistable for any morphism from a smooth projective curve. This condition has many geometrically implications; for example, we have the following result:

Proposition 7.22. *Let X be a smooth projective variety such that the pull-back of the tangent bundle f^*T_X is semistable for any morphism $f\colon C \to X$, where C is a smooth projective curve. Any morphism $g\colon \mathbb{P}^1 \to X$ is constant.*

Proof. Assume $g\colon \mathbb{P}^1 \to X$ is not constant. Then g^*T_X is semistable and so $g^*T_X \simeq \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus n}$ for some $a \in \mathbb{Z}$. From $T_{\mathbb{P}^1} \hookrightarrow g^*T_X$ we infer $a > 0$. Then g^*T_X is ample for any morphism hence $X \simeq \mathbb{P}^n$ (see [28]). It is well known that $T_{\mathbb{P}^n}$ is $\mathcal{O}_{\mathbb{P}^n}(1)$-stable. However, the restriction of $T_{\mathbb{P}^n}$ to a line l splits as

$$T_{\mathbb{P}^n}|_l \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus n-1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$$

and is not semistable. This contradicts our hypothesis, $\square$

So in some sense quasi-abelian varieties are in the opposite direction of rationally connected varieties.

7.2.5 Higgs bundles over quasi-abelian varieties

Let (E, ϕ) be a μ-semistable Higgs bundle over a quasi-abelian variety X and fix an ample line bundle H. Since $c_1(T_X) = 0$, by Theorem 7.14 E is μ-semistable as a vector bundle. Now we want to show that if for any morphism $f\colon C \to X$ the vector bundle $(f^*E, f^*(\phi))$ is semistable as a Higgs bundle then $\Delta(E) = 0$, "i.e.", we want to show that X is a Higgs variety. For any such morphism f one can consider the pair (f^*E, ϕ') where

$$\phi'\colon f^*E \to f^*E \otimes f^*\Omega^1_X.$$

Clearly a ϕ'-invariant subbundle $F \subset E$ is also $f^*(\phi)$-invariant, where $f^*(\phi)\colon f^*E \to f^*E \otimes K_C$ is the induced Higgs field. In particular we have

Lemma 7.23. *If the Higgs bundle $(f^*E, f^*(\phi))$ is semistable then is so the pair (f^*E, ϕ').*

Proof. Let F be a ϕ'-invariant subbundle of f^*E, then since it is $f^*(\phi)$-invariant and the Higgs bundle $(f^*E, f^*(\phi))$ is semistable, we have

$$\mu(F) \le \mu(f^*E),$$

and so (f^*E, ϕ') is semistable as a pair. □

Corollary 7.24. A quasi-abelian variety is a Higgs variety.

Proof. Let $\mathcal{E} = (E, \phi)$ be a Higgs bundle and $f\colon C \to X$ be a morphism from a smooth projective curve. Assume that the Higgs bundle $(f^*E, f^*(\phi))$ is semistable, then the pair (f^*E, ϕ') is semistable. Since $f^*\Omega^1_X$ is semistable of degree zero, one can apply Theorem 7.14 and conclude that f^*E is semistable. Hence by Theorem 7.13 one gets $\Delta(E) = 0$.

□

7.2.6 More Higgs varieties

We showed that rationally connected varieties and abelian varieties are examples of Higgs varieties. We want to produce new examples of Higgs varieties. The first technique is to use the Lefschetz hyperplane theorem.

Theorem 7.25. *Let X be a smooth complex projective variety of dimension n, and let D be an effective ample divisor on X. The restriction map*

$$H^i(X, \mathbb{C}) \to H^i(D, \mathbb{C})$$

is an isomorphism for $i \le n-2$, and injective for $i = n-1$.

Corollary 7.26. Let (X, H) be a polarized variety with $\dim X \ge 5$ and let D be a smooth effective ample divisor on X. If D is a Higgs variety then X is a Higgs variety as well.

Proof. Let E be a Higgs vector bundle on X such that for any $f\colon C \to X$, $f^*(E)$ is semistable as Higgs bundle. Replacing E by $\mathrm{End}(E)$ we can assume $c_1(E) = 0$. Let us consider the vector bundle $E|_D$. For any morphism $g\colon C \to D$, the pull-back of $E|_D$ is semistable. Since D is a Higgs variety we have $\Delta(E|_D) = c_2(E|_D) = 0$. As by the Lefschetz theorem the morphism

$$H^i(X, \mathbb{C}) \to H^i(D, \mathbb{C})$$

is injective, we have $\Delta(E) = c_2(E) = 0$, and so X is a Higgs variety. □

Remark 7.27. Corollary 7.17 tells us that if X has a rationally connected ample divisor then its self is rationally connected so in this case the previous result does not give anything new.

Lemma 7.28. *Let $f\colon Y \to X$ be a surjective morphism and E be a vector bundle on Y such that $E|_{Y_x}$ is trivial for any $x \in X$, where Y_x is the fibre over x. There exists a vector bundle F over X such that*

$$E = f^*(F).$$

Corollary 7.29. Let X be a Higgs variety, and assume that there is a surjective morphism $g\colon Y \to X$ where each Y_x is rationally connected. Then Y is a Higgs variety.

Proof. Let E be a vector bundle on Y such that $f'^*(E)$ is Higgs semistable for any $f'\colon C' \to Y$. In particular by Proposition 7.19, E is trivial on the fibres Y_x, so that by the previous Lemma $E = g^*(F)$ for some vector bundle F on X. Let $f\colon C \to X$ be a morphism with C a smooth projective curve, then we have the following commutative diagram

$$\begin{tikzcd} C \times_X Y \arrow[r, "\bar{f}"] \arrow[d, "\bar{g}"'] & Y \arrow[d, "g"] \\ C \arrow[r, "f"] & X \end{tikzcd} \tag{7.1}$$

By hypothesis the vector bundle $\bar{f}^*(E) = (g \circ \bar{f})^*(F)$ is semistable over $C \times_X Y$, but since $g \circ \bar{f} = f \circ \bar{g}$ we obtain that the $(f \circ \bar{g})^*(F)$ is semistable as a Higgs bundle, hence also $f^*(F)$ is so. Since X is a Higgs variety we get $\Delta(F) = 0$ which clearly implies $\Delta(f^*(F)) = \Delta(E) = 0$ and Y is a Higgs variety. □

In particular the previous Corollary implies that ruled surfaces are Higgs variety.

Proposition 7.30. *Let $g\colon Y \to X$ be a finite étale cover of a projective variety X. If Y is a Higgs variety then also X is so.*

Proof. Let $\mathcal{E} = (E, \phi)$ be a Higgs vector bundle on X such that for any morphism $f\colon C \to X$ from a smooth projective curve C the pullback $f^*\mathcal{E}$ is semistable as a Higgs bundle. Let $h\colon C' \to Y$ be any morphism, we can consider the composition $g \circ h\colon C' \to X$ and so we get $h^*(g^*\mathcal{E}) = (g \circ h)^*\mathcal{E}$ is semistable as Higgs bundle, hence, since Y is a Higgs variety, $\Delta(g^*E) = 0$. Our hypothesis on g tells us that the morphism g^* is injective in cohomology in particular $\Delta(E) = 0$ and we are done. □

Proposition 7.31. *Let X and Y smooth surfaces and $g\colon X \to Y$ be a birational map which is an isomorphism between big open subset of X and Y. Then X is a Higgs variety if and only if Y is so.*

Proof. Let $\mathcal{E} = (E, \phi)$ be a vector bundle on Y, $g^*(E)$ is a vector bundle on a big open subset of X hence it can be extended uniquely to a vector bundle on all X. Moreover g induces a 1-to-1 correspondence between $Mor(C, Y)$ and $Mor(C, X)$, where C is a smooth curve, given by

$$(f\colon C \to Y) \longmapsto (g^{-1} \circ f\colon C \to X),$$

since birational maps between curves are actually isomorphisms we get that the map $g^{-1} \circ f$ is actually a morphism. The Higgs bundle $f^*\mathcal{E}$ is semistable if and only if $(g^{-1} \circ f)^*(g^*(\mathcal{E}))$ is semistable. Since X is a Higgs variety we obtain $\Delta(\phi^*(E)) = 0$ hence $\Delta(E) = 0$ and E is a Higgs variety. $\square$

7.3 Varieties with nef tangent bundle

Let us observe that all varieties we described in previous sections have nef tangent bundle. Now we give a converse of this fact, "i.e.", we show that varieties with nef tangent bundle are a Higgs varieties.
The main theorem we use is the follwing result given by Demailly, Peternell and Schneider ([13])

Theorem 7.32. *Let X be a compact Kähler manifold with nef tangent bundle T_X. Let $\widetilde{X}$ be a finite étale cover of X of maximum irregularity $q = q(\widetilde{X}) = h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$. Then*

1. $\pi_1(\widetilde{X}) \cong \mathbb{Z}^{2q}$.
2. *The Albanese map $\alpha\colon \widetilde{X} \to A(\widetilde{X})$ is a smooth fibration over a q-dimensional torus with nef relative tangent bundle.*
3. *The fibres F of α are Fano manifolds with nef tangent bundles.*

Corollary 7.33. Any projective variety with nef tangent bundle is a Higgs variety.

Proof. Let us observe that thanks to Proposition 7.30 we can study Higgs varieties up to finite étale cover. So we can assume that X satisfies the conditions in the previous Theorem. In particular since Fano varieties are rationally connected varieties and abelian varieties are Higgs varieties, thesis follows from Corollary 7.29. $\square$

Examples

For the dimension 2 and 3 we have the following classification theorem (Loc. cit.)

Theorem 7.34. *Let X be a surface such that T_X is nef then X is one of the following :*

1 $X = \mathbb{P}^2$*;*

2 X *is a* $\mathbb{P}^1 \times \mathbb{P}^1$*;*

3 X *is an abelian variety.*

4 X *is hyperelliptic;*

5 $X = \mathbb{P}(E)$*, where* E *is a rank 2-vector bundle on a elliptic curve* C *with either*

a) $E = \mathcal{O}_C \oplus L$ *with* $L \in Pic^0(C)$ *or*

b) E *is given by a non split extension*

$$0 \to \mathcal{O}_C \to E \to L \to 0,$$

with $L = \mathcal{O}_C$ *or* $\deg L = 1$*.*

Theorem 7.35. *Let X be a projective 3-fold with nef tangent bundle. Then X is up to finite étale cover one of the manifold in the following list :*

1 $X = \mathbb{P}^3$*;*

2 $X = Q_3$*, the 3-dimensional quadric;*

3 $X = \mathbb{P}^1 \times \mathbb{P}^2$*;*

4 $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$*;*

5 $X = \mathbb{P}(T_{\mathbb{P}^2})$*;*

6 $\mathbb{P}(E)$*, whit* E *a numerically flat rank 3 bundle over an elliptic curve* C*;*

7 $\mathbb{P}(E) \times \mathbb{P}(F)$*, with* E *and* F *numerically flat rank 2 bundles over an elliptic curve* C*;*

8 $\mathbb{P}(E)$*, whit* E *a numerically flat rank 2 bundle over an abelian surface*

9 X *is an abelian variety.*

In particular all previous varieties are examples of Higgs varieties.

7.4 Principal Higgs bundles

7.4.1 Semistable principal bundles and the adjoint bundle

Let G be a complex reductive linear algebraic group, and $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ the adjoint representation of G into its Lie algebra $\mathfrak{g}$. Given a principal G-bundle P the representation Ad leads us to construct a vector bundle,

$$\mathrm{Ad}(P) \colon = P \times \mathfrak{g}$$

Since G is reductive, there exists a nondegenerate G-invariant bilinear form on $\mathfrak{g}$. Via this form, $\mathrm{Ad}(P)$ is dual to itself and hence we have that $c_1(\mathrm{Ad}(P)) = 0$. The definition of semistable principal bundles over a polarized projective variety (X, H) is analogue to the classical definition of semistability given by Ramanathan.

Definition 7.36. A principal G-bundle P is (semi)stable if and only if for any proper parabolic subgroup $K \subset G$, any open dense subset $U \subset X$ such that $\mathrm{codim}(X - U) \geq 2$, and any reduction $\sigma\colon U \to P/K_{|U}$ of G to K on U, one has $\deg \sigma^*(T_{P/K,X})(\geq)0$.

We have the following result

Proposition 7.37. *A holomorphic principal G-bundle P over X is semistable if and only if the associated vector bundle Ad(P) is a μ-semistable vector bundle.*

So in order to study semistable principal G-bundles one can restrict the attention only on μ-semistable vector bundles. The same statement for stability does not hold in general, however one has the follwing result

Proposition 7.38. *Let P be a principal G-bundle, and let us assume that the vector bundle Ad(P) is stable then P is stable as principal G-bundle.*

Proof. Suppose that P is an Ad-stable G-bundle, ("i.e." $\mathrm{Ad}(P)$ is μ-stable). Let Q be a parabolic subgroup of G and $\mathfrak{q}$ be its Lie algebra. Also, let $\sigma\colon X \to P/Q$ be a reduction of structure group to Q. We have the exact sequence of Q-modules

$$0 \to \mathfrak{q} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{q} \to 0,$$

which induces the exact sequence of vector bundles

$$0 \to \sigma^*(P(\mathfrak{q})) \to \sigma^*(P(\mathfrak{g})) \to \sigma^*(P(\mathfrak{g}/\mathfrak{q})) \to 0$$

By construction we have $\sigma^*(P(\mathfrak{g})) \cong \mathrm{Ad}(P)$ and $\sigma^*(P(\mathfrak{g}/\mathfrak{q})) \cong \sigma^*(T_{P/Q,X})$. Since $\mathrm{Ad}(E)$ is a μ-stable vector bundle of degree zero, $\deg \sigma^*(T_{P/Q,X}) > 0$ and we are done. $\square$

$\mathrm{Ad}(P)$ can be strictly semistable even if P is stable in the sense of Ramanathan. For example, for any $n > 1$, no $GL(n, \mathbb{C})$-bundle is Ad-stable, since $\mathrm{Ad}(P) = End(P(\mathbb{C}^n))$ has the trivial line subbundle generated by the identity section.

The previous discussion holds also for Higgs G-principal bundle, with the only difference that we have to consider Higgs reduction (see Definition 2.15) when we give the notion of semistability.
Given a principal Higgs G-bundle $\mathcal{P} = (P, \phi)$ over X we have the following result ([11])

Theorem 7.39. *Let $\mathcal{P} = (P, \phi)$ be a principal Higgs G-bundle on X. If $\mathcal{P}$ is semistable and $c_2(Ad(P)) = 0$ in $H^4(X, \mathbb{R})$, then for every morphism $f\colon C \to X$, where C is a smooth projective curve, the pullback $f^*(\mathcal{P})$ is semistable as principal Higgs G-bundle.*

Now we can give a partial converse of this fact on Higgs varieties.

Proposition 7.40. *Let (X, H) a polarized Higgs variety. If for any morphism $f\colon C \to X$, where C is a smooth projective curve, the principal Higgs bundle $f^*\mathcal{P}$ is semistable, then $\mathcal{P}$ is semistable for any polarization on X, and $c_2(Ad(P)) = 0$.*

7.5 Vector bundles on Calabi-Yau manifold

7.5.1 Calabi-Yau manifolds

Definition 7.41. A (compact) Calabi-Yau manifold of (complex) dimension n is a compact n-dimensional Kähler manifold X with trivial canonical bundle.

The study of stable sheaves on Calabi-Yau is a very important problem with many applications; for instance, stable holomorphic bundles and sheaves on Calabi-Yau manifolds give informations about Donaldson-Thomas invariants.

In dimension one, the classification of vector bundles on an elliptic curve is due to Atiyah [1]. The set of isomorphism classes of indecomposable bundles of a fixed rank and degree is isomorphic to the elliptic curve itself. In dimension two, Mukai [32, 33] studied the moduli space M of Gieseker-semistable sheaves E on a smooth projective polarized K^3 surface. He showed that in this case if the moduli space M is smooth, it is symplectic. Nakashima [37] showed the non-emptiness of this moduli space for suitable choices of the topological invariants $(r, c_1(E), c_2(E))$.

The most important theorem we need here is the following.

Theorem 7.42. *The tangent bundle to a smooth projective variety with trivial canonical bundle is μ-semistable with respect to any polarization.*

Proof. Since $\mathrm{K}_X \equiv 0$ then X carries a Kähler-Einstein metric g, therefore (T_X, g) is a Hermitian-Einstein vector bundle and thus it is μ-semistable with respect to ϕ_g [25]. The fact that T_X is actually μ-semistable with respect to any polarization see [15]. □

Theorem 7.43. *Let (X, H) be a Calabi-Yau manifold of dimension n and $\mathcal{E} = (E, \phi)$ a μ-semistable Higgs bundle, then E is μ-semistable as a vector bundle.*

Proof. It follows from the previous discussion and Theorem 7.14 □

Let us observe that the same result for Abelian varieties was proved in [7]

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