Scuola Internazionale Superiore di Studi Avanzati - Trieste

String Field Theory: Time-Dependent Solutions and Other Aspects

Thesis submitted for the degree of
Doctor Philosophiæ

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Contents

Introduction

String Theory appeared in the late 60's as a tentative description to the strong interaction. This was so because string scattering amplitudes agreed with the ones found in meson scattering experiments at that time. As time passed, QCD had proved itself to be the right framework to study this problem, and the interest in String Theory faded. Moreover, it had some undesirable features for a theory of hadrons: more space-time dimensions were required and its spectrum presented other massless excitations than the spin-1 gluon. This last "drawback" has become later the reason for the theory's revival, since among these massless excitations there was a spin-2 particle and the only consistent interactions of massless spin-2 particles are gravitational interactions. String Theory thus contains in itself general relativity and, due to that, was proposed as an unified theory of the fundamental forces of nature.

Much progress had been achieved then. The inclusion of fermions in the theory led to what is called Superstring Theory, where supersymmetry is crucial for its consistency. Later, anomaly cancellation left us with only five consistent superstring theories in ten space-time dimensions, they were dubbed Type I, Type IIA, Type IIB, Heterotic SO(32) and Heterotic $E_8 \times E_8$. Using some of these theories plus the concept of compactification, we can construct models with features very similar to the ones of the standard model, albeit a complete agreement has not been found yet.

A major breakthrough occurred about a decade ago with the discovery of nonperturbative objects called Dp-branes. They are extended p-dimensional objects in space-time where open strings end, and its dynamics is given by the dynamics of the open strings attached to it. It was also shown that they are solitonic solutions to ten-dimension supergravity equations of motion, being charged under the RR $p+1$-forms present in the closed string spectrum, what makes them stable objects. For example, in Type IIA/IIB theories there are odd/even forms and consequently just even/odd dimensional branes are stable. On the contrary, the branes with wrong dimensionality do not have any RR form to couple to and so are unstable. The same occurs with the D-brane anti-D-brane system since it breaks supersymmetry completely.

Although D-branes were discovered in the context of superstring theory, they also exist in the bosonic one. This is reasonable, since they are hyperplanes where the ends of open strings are constrained to move. In this case, however, they are not stable, due to the nonexistence of RR forms in the spectrum. This instability (like the ones pointed out in the previous paragraph) is connected with the existence of the tachyon, thereby signalling that we are expanding the theory around the wrong vacuum. It is sensible to think that, once we compute the tachyon potential, it is going to present a minimum where no tachyons exist and, equivalently, no open string perturbative degrees of freedom are present. Quantitatively, it means that the difference between the value of the potential at the maximum and at the minimum should equal the tension of the brane. To compute the potential, and consequently the true vacuum, requires a second quantized version of the theory, which is known as String Field Theory. There are two kinds of string field theory (not considering supersymmetry): Open and Closed String Field Theory. Open String Field Theory (OSFT) is the tool to be used if we want to study the open string tachyon.

The basic component of string field theory is the string field, which is composed of an infinite number of space-time fields. Using the string field, we are able to construct an action for the theory, a very important ingredient for studying the tachyon potential. As usual, if we are dealing with static solutions, the action is nothing but the negative of the potential. Hence, by working with string fields whose components are space-time independent, we get the potential just by computing the action. As it is impossible to work with an infinite number of fields, we should truncate the string field at some level and then compute the potential. These computations showed with great accuracy the forementioned conjecture that the difference between the maximum and the minimum of the potential equals the brane tension. However, no solution to the equation of motion was known at that time[1], leading people to the construction of what is called Vacuum String Field Theory (VSFT).

The idea behind VSFT was to expand the theory around the minimum of the potential and, after doing field redefinitions, conjecture that the kinetic operator can be written in terms of only ghost operators. As a consequence, the equation of motion factorizes in ghost and matter parts. The matter part of the equation of motion becomes very simple, and its solutions are just projectors with respect to the product of string fields. Many static solutions to this equation have been found, for example, the sliver and the lump. The sliver solution represents the space filling $D25$-brane and the lump describes lower dimensional branes. One fact in support of this is that the ratio of their energies equals the ratio of the corresponding D-brane tensions.

As we have just said, static solutions to the equation of motion have been found and it would be very interesting if we could find time-dependent solutions in VSFT, as these would interpolate between the perturbative and the nonperturbative vacua, describing in this way the decay of the brane. In this thesis, we will focus on our results concerning these time-dependent solutions. We will also discuss other general results we obtained in the framework of VSFT.

This thesis is divided as follows. In chapter 1, we give a brief review of SFT, focusing on OSFT and the construction of VSFT. We also give some outline about the recently found solution to the OSFT equation of motion.

Chapter 2 deals with the construction of time-dependent solutions in VSFT. We start by reviewing how to construct static classical solutions and then we turn to the time-dependent ones. We construct these solutions both in a trivial and in a E-field background. We show that these solutions really interpolate between the two vacua and we point out some points of resemblance between our solution and Sen's rolling tachyon solution. In the case of an E-field background, after the decay we find a flat profile in the transverse direction to the brane and this ought to be interpreted as fundamental strings, which are polarized by the

[1] A solution to OSFT equation of motion was found at the end of last year by Schnabl [4]

eletric field and thereby hampered from decaying. We conclude the chapter by presenting some solutions in VSFT that are supposed to describe these fundamental strings.

In chapter 3, we exit from the main subject of this thesis and we show some results concerning integrability properties in Light-Cone Open String Field Theory. We show that the three strings vertex coefficients in light–cone open string field theory satisfy the Hirota equations for the dispersionless Toda lattice hierarchy. We also show that Hirota equations allow us to calculate the correlators of an associated quantum system where the Neumann coefficients represent the two–point functions. We consider next the three strings vertex coefficients of the light–cone string field theory on a maximally supersymmetric pp–wave background. Using the previous results we are able to show that these Neumann coefficients satisfy the Hirota equations for the full Toda lattice hierarchy at least up to second order in the string mass μ.

In chapter 4, we show that a family of 1/2–BPS states of $\mathcal{N} = 4$ SYM is in correspondence with a family of classical solutions of VSFT with a B–field playing the role of the inverse Planck constant. We establish this correspondence by relating the Wigner distributions of the N fermion systems representing such states to low energy space profiles of systems of VSFT D-branes. In this context the Pauli exclusion principle appears as a consequence of the VSFT projector equation. The family of 1/2–BPS states maps through coarse–graining to droplet LLM supergravity solutions. We also discuss the possible meaning of the corresponding coarse graining in the VSFT side.

Clearly, one should go into the realm of Superstring Field Theory in order to describe nature. Hence, in chapter 5, we make a brief review about Superstring Field Theory, describing some attempts to construct it and their respective failures. Firstly, we describe Witten's formulation that is an extension of the bosonic theory with insertions of picture changing operators, which introduce divergences in the theory already at the classical level. Moreover, the tachyon potential computed within this framework does not possess a minimum, contradicting Sen's conjectures. Secondly, we present an improved version of Witten's theory that uses a two-step picture changing operator, thereby solving the two forementioned problems. However, it presents other drawbacks like the possibility of different theories off-shell and non-physical states being also a solution to the equation of motion.

We finish with chapter 6, where we present some conclusions and open problems.

In the appendices, we give some technical details needed in the analysis we make within the body of the thesis.

Chapter 1

String Field Theory

The need to study nonperturbative aspects of String Theory has led to the formulation of a new framework called String Field Theory (SFT). One of these aspects is related to the existence of a tachyonic mode in the spectrum of the open string, indicating we are not expanding the theory around the true vacuum (minimum of the potential). It is necessary, therefore, to compute the effective tachyon potential and to find its minimum.

A tachyon potential is an off-shell concept, implying that the first-quantized version of String Theory is not adequate for studying this problem, since it is an on-shell theory. We must formulate a second-quantized theory of open strings if we want to analyze this question.

In this chapter, we describe the theory originally formulated by Witten [1], and analyze some aspects thereof. We focus on the points that will be important when we come to the discussion of our results.

1.1 Open String Field Theory (OSFT)

1.1.1 Some Basics

Witten proposed, as a candidate for string field theory, a formalism based on a non-commutative generalization of a gauge theory. He starts with a very simple action:

$$S = -\frac{1}{2}\int \Psi \star Q\Psi \;-\; \frac{g}{3}\int \Psi \star \Psi \star \Psi, \tag{1.1}$$

where g is interpreted as the open string coupling constant. The string field Ψ takes values in a graded algebra $\mathcal{A}$ with which is associated a star product:

$$\star : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}, \tag{1.2}$$

under which the degree is additive ($G_{\Psi\star\Phi} = G_\Psi + G_\Phi$). There is also a BRST operator of degree one

$$Q : \mathcal{A} \longrightarrow \mathcal{A} \qquad\qquad (G_{Q\Psi} = 1 + G_\Psi) \tag{1.3}$$

and an integration operation

$$\int : \mathcal{A} \longrightarrow \mathbf{C}. \tag{1.4}$$

As one can notice, this structure looks like the one of p-forms, where Q plays the role of the exterior derivative, $\star$ the role of the wedge product, and the integral the role of the integration of a p-form on a manifold.

String Field Theory is defined by these elements and the following group of axioms:

$$\begin{aligned}
Q^2\Psi &= 0, \quad \forall \Psi \in \mathcal{A}; \\
\int Q\Psi &= 0, \quad \forall \Psi \in \mathcal{A}; \\
Q(\Psi \star \Phi) &= (Q\Psi) \star \Phi + (-1)^{G_\Psi} \Psi \star (Q\Phi), \quad \forall \Psi, \Phi \in \mathcal{A}; \\
\int \Psi \star \Phi &= (-1)^{G_\Psi G_\Phi} \int \Phi \star \Psi, \quad \forall \Psi, \Phi \in \mathcal{A}; \\
(\Phi \star \Psi) \star \Xi &= \Phi \star (\Psi \star \Xi), \quad \forall \Psi, \Phi, \Xi \in \mathcal{A}.
\end{aligned} \tag{1.5}$$

Once these are satisfied, one can show the action (1.1) is invariant under the following gauge transformation:

$$\delta\Psi = Q\Lambda + g(\Psi \star \Lambda - \Lambda \star \Psi), \tag{1.6}$$

for any gauge parameter $\Lambda \in \mathcal{A}$ with degree 0.

If g is taken to vanish, the equation of motion and gauge transformations become $Q\Psi = 0$ and $\delta\Psi = Q\Lambda$, respectively. Hence, when $g = 0$ the SFT describes exactly the free bosonic string. The reason for introducing the extra g piece in the action was to find a simple interaction term which was consistent with the perturbative expansion of open bosonic string theory.

One question we could ask ourselves is: Why is this structure suitable for open string field theory? The most natural way to imagine the interaction of two oriented[1] open strings U and V is to say they interact by joining their end points, forming a new string U·V (which we might think as product between U and V). It is understood that U·V is obtained by gluing them together if the end of U is the beginning of V, and zero otherwise. This product is non-commutative because U·V is non-zero but V·U is (the final point of U coincides with the initial point of V, but the opposite is not true). This gives us a hint why the structure underlying open string field theory is supposedly non-commutative.

Witten argued that the above axioms are all satisfied if $\mathcal{A}$ is taken to be the space of string fields $\mathcal{A} = \Psi[x(\sigma); c(\sigma), b(\sigma)]$, which can be described as functionals of the matter, ghost and antighost fields describing the theory of an open string ($0 \leq \sigma \leq \pi$) in 26 dimensions. This string field can be formally written as a sum over open string Fock space states with coefficients given by an infinite family of space-time fields

$$\Psi[x(\sigma); c(\sigma), b(\sigma)] = \int d^{26}p \; [\phi(p) c_1|0;p\rangle + A_\mu(p)\alpha_{-1}^\mu c_1|0;p\rangle + \ldots]. \tag{1.7}$$

[1]The same can be done for unoriented open strings by just keeping states which are invariant under the twist operation.

However, there is a drawback in the previous definition for the interaction of open strings: it leads to non-associativity. If we work with parametrized strings, we must ask what is the parametrization of U·V after the gluing. The bad point is that there is no choice which preserves associativity. It is necessary, therefore, to modify this definition. One option is to single out the string midpoint, in this way the string will now have a left and a right part. We consider now a new product in which we join the right part of the first string with the left part of the second. This product factorizes into separate matter and ghost parts. In the matter sector (We will just show the matter sector, the ghost sector of the theory is defined in a similar fashion.), it is given by:

$$\begin{aligned}
(\Psi \star \Phi)[z(\sigma)] &\equiv \int \prod_{0\leq\tilde{\tau}\leq\frac{\pi}{2}} dy(\tilde{\tau})\, dx(\pi-\tilde{\tau}) \prod_{\frac{\pi}{2}\leq\tau\leq\pi} \delta[x(\tau)-y(\pi-\tau)]\Psi[x(\tau)]\Phi[y(\tau)],\\
x(\tau) &= z(\tau) \quad \text{for} \quad 0\leq\tau\leq\frac{\pi}{2},\\
y(\tau) &= z(\tau) \quad \text{for} \quad \frac{\pi}{2}\leq\tau\leq\pi.
\end{aligned} \tag{1.8}$$

Now, we need a definition of the integration operation. From the axioms, we see it ought to obey cyclicity. The only way to do so is to define the integration of a string field as the gluing of its left and right parts. More precisely:

$$\int \Psi = \int \prod_{0\leq\sigma\leq\pi} dx(\sigma) \prod_{0\leq\tau\leq\frac{\pi}{2}} \delta[x(\tau)-x(\pi-\tau)]\Psi[x(\tau)] \tag{1.9}$$

These definitions for the product and the integral, despite being very precise, lead to very awkward computations. Yet it is possible to cast the structure described above in a more generic framework, which can be used to write more general string field theory actions, including those whose interactions are not delta-function overlaps. In this language the action takes the form:

$$S(\Phi) = -\frac{1}{g^2}\left[\frac{1}{2}\langle\Phi, Q_B\Phi\rangle + \frac{1}{3}\langle\Phi, \Phi\star\Phi\rangle\right]. \tag{1.10}$$

The string field Φ is a state in the total matter plus ghost conformal field theory and $\langle .,. \rangle$ is a bilinear inner product on the state space of the conformal field theory. This action can be related to the action (1.1) by making $\Phi = g\Psi$ and $\langle A, B\rangle = \int A\star B$. The operator Q_B and the inner product satisfy the following identities:

$$\begin{aligned}
Q_B^2 A &= 0;\\
Q_B(A\star B) &= (Q_B A)\star B + (-1)^A A\star(Q_B B);\\
\langle Q_B A, B\rangle &= -(-1)^A\langle A, Q_B B\rangle;\\
\langle A, B\rangle &= (-1)^{AB}\langle B, A\rangle;\\
\langle A, B\star C\rangle &= \langle A\star B, C\rangle;\\
A\star(B\star C) &= (A\star B)\star C.
\end{aligned} \tag{1.11}$$

From these identities, we are able to determine the degree, which is identified with the ghost number in the case of bosonic open strings, of the string field. Using the above properties we get

$$\langle\Phi, Q_B\Phi\rangle = (-1)^{\Phi(1+\Phi)}\langle Q_B\Phi, \Phi\rangle = \langle Q_B\Phi, \Phi\rangle = -(-1)^{\Phi}\langle\Phi, Q_B\Phi\rangle, \tag{1.12}$$

telling us the ghost number of Φ must be odd if we want a non-zero kinetic term. We know that the zero-momentum tachyon state is $tc_1|0\rangle$. Since the tachyon is the first state in the expansion of the string field in the Fock basis and it has ghost number one, we deduce that Φ has also ghost number one.

Now, one can show that the action (1.10) is invariant under the following gauge transformation:

$$\delta\Phi = Q_B\Lambda + \Phi\star\Lambda - \Lambda\star\Phi, \tag{1.13}$$

with Λ any ghost number zero state.

Before proceeding to more practical issues, it is worth mentioning two additional facts about OSFT. The first is the existence of a twist operation, which reverses the parametrization of the open string. It is represented in OSFT by an operator Ω that satisfies the following properties:

$$\begin{aligned} \Omega(Q_BA) &= Q_B(\Omega A); \\ \langle\Omega A, \Omega B\rangle &= \langle A, B\rangle; \\ \Omega(A\star B) &= (-1)^{AB+1}\Omega(B)\star\Omega(A). \end{aligned} \tag{1.14}$$

These properties imply the twist invariance of the action: $S(\Omega\Phi) = S(\Phi)$.

The second fact is that the star algebra may have an identity $\mathcal{I}$, which is to satisfy

$$\mathcal{I}\star A = A\star\mathcal{I} = A, \tag{1.15}$$

for any state A. This shows $\mathcal{I}$ has ghost number zero and implies it is a twist odd string field. The last property can be seen from:

$$\Omega A = \Omega(\mathcal{I}\star A) = (-1)^{0.A+1}(\Omega A)\star(\Omega\mathcal{I}) = -(\Omega A)\star(\Omega\mathcal{I}). \tag{1.16}$$

The equality just holds if $\Omega\mathcal{I} = -\mathcal{I}$. Moreover, the identity string field must be annihilated by any derivation D of the star algebra:

$$DA = D(\mathcal{I}\star A) = (D\mathcal{I})\star A + \mathcal{I}\star DA = D\mathcal{I}\star A + DA, \tag{1.17}$$

imposing $D\mathcal{I} = 0$.

Now, it is time we define how to compute the kinetic and the interaction terms appearing in the action. Our approach to this issue will be the conformal one, which is based on the premise that conformal field theory computations are usually more economical.

We start by defining in a precise way how to compute the kinetic term appearing in the action (1.10):

$$\langle A, B \rangle \equiv \langle \mathrm{bpz}(A)|B\rangle,$$

where bpz is the BPZ conjugation[2].

The next step is to give a definion for the interaction term[3]:

$$\langle A, B, C\rangle \equiv \langle f_1 \circ \mathcal{O}_A(0), f_2 \circ \mathcal{O}_B(0), f_3 \circ \mathcal{O}_C(0)\rangle, \tag{1.18}$$

where $\mathcal{O}_A, \mathcal{O}_B$ and $\mathcal{O}_C$ are the vertex operators associated with the states A, B and C, and the correlator is computed on the disk or in the conformally equivalent upper-half plane. The conformal transforms are specified by the functions f_i as we explain in the following.

Consider three copies of the complex plane. The worldsheets of the three interacting strings are represented as unit half-disks in each one of these complex planes, with local coordinates ξ_i, $i = 1, 2, 3$. In these coordinates, the boundaries $|\xi_i| = 1$ in the respective upper half-disks are the strings, and the $\xi_i = i$ is the string midpoint (Fig. 1.1). We define the interaction of the three strings by gluing the three half-disks to form a single disk. This should be done in a way such that the midpoint of each string is mapped to a common interaction point in the disk, also the left half of the first string is to be glued with the right half of the second string, and so on. The functions f_i, therefore, must map each upper half-disk to a 120° wedge of the unit disk. The following functions

$$\begin{aligned}
f_1(\xi_1) &= e^{\frac{2\pi i}{3}}\left(\frac{1+i\xi_1}{1-i\xi_1}\right)^{\frac{2}{3}}, \\
f_2(\xi_2) &= \left(\frac{1+i\xi_2}{1-i\xi_2}\right)^{\frac{2}{3}}, \\
f_3(\xi_3) &= e^{-\frac{2\pi i}{3}}\left(\frac{1+i\xi_3}{1-i\xi_3}\right)^{\frac{2}{3}}
\end{aligned} \tag{1.19}$$

map the three upper half-disks to three wedges with punctures at $e^{\frac{2\pi i}{3}}$, 1, and $e^{-\frac{2\pi i}{3}}$ respectively (Fig. 1.2).

We could now map this interacting disk back to the upper half-plane if we want. This is done by composing the previous functions with the map $h^{-1}(z) = -i\frac{z-1}{z+1}$ that takes the unit disk to the upper half-plane with the three punctures on the real axis (Fig. 1.3), that is

$$f_i^{half-plane}(\xi_i) = h^{-1} \circ f_i(\xi_i) \tag{1.20}$$

[2]We recall that a primary field $\phi(z) = \sum_{n\neq 0} \frac{\phi_n}{z^{n+h}}$ of conformal dimension h transforms under BPZ conjugation as:

$$\mathrm{bpz}(\phi_n) = (-1)^{n+h}\phi_{-n}.$$

[3]This is just another way of writing the interacting term in the action:

$$\langle A, B, C\rangle \equiv \langle A, B \star C\rangle.$$

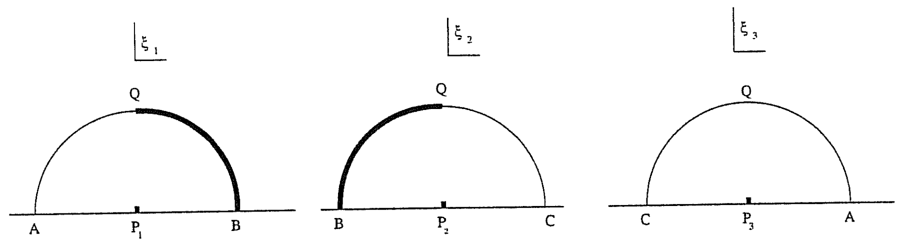

Figure 1.1: *The worldsheets of the three strings*

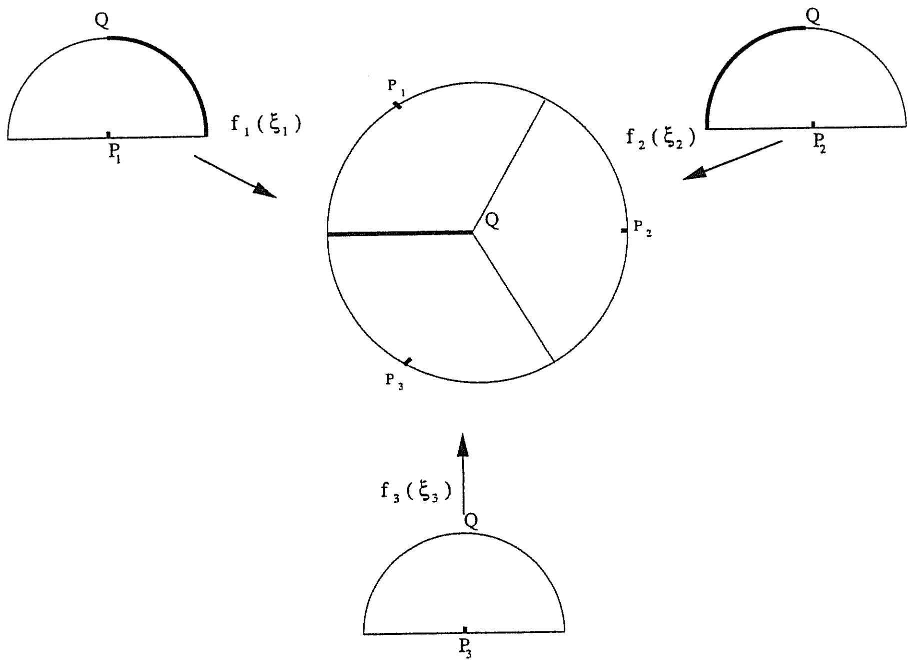

Figure 1.2: *The gluing of the three interacting strings*

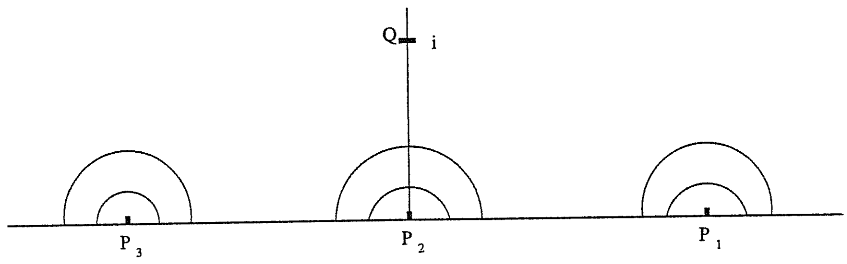

Figure 1.3: *Representation of the cubic vertex as the upper half-plane with 3 punctures on the real axis.*

1.1.2 An Analytic Solution

For a long time, an analytic solution to the equation of motion $Q_B\Psi+\Psi\star\Psi = 0$ coming from the action (1.1) has been searched without success. The main importance of finding this solution is to study tachyon condensation, especially by verifying whether Sen's conjectures are right. After identifying the open string tachyon with the instability of the D-brane where the open string ends, Sen made the following three conjectures[2, 3]:

- The difference between the values of the tachyon potential at the maximum and at the minimum should equal the tension of the D-brane.
- There are lump solutions with the correct tension so as to describe lower dimensional D-branes.
- There are no physical open string excitations around the minimum, as the brane decayed. Thus, the cohomology of the BRST-like kinetic operator is trivial there.

Due to the lack of analytic solutions in OSFT, these conjectures have been verified just numerically. But the situation changed at the end of last year when Schnabl finally found a solution. To find the solution, he used a class of states, called wedge states with insertion, which have a very simple multiplication rule under the $\star$-product. He also chose a more appropriate gauge that simplifies the process of obtaining the solution. Now, we briefly review how he obtained the solution and the proof of Sen's first and third conjectures. For a much more detailed exposition, see [4, 5, 6, 7, 8].

We first define a class of states called surface states. A surface state is defined as

$$\langle f|\phi\rangle \equiv \langle f\circ\phi(0)\rangle, \qquad \forall\phi \tag{1.21}$$

where $f(\xi)$ is a conformal map from the worldsheet coordinates to the upper-half plane. As we can see, there is a one-to-one correspondence between surface states and conformal maps $f(\xi)$. In the operator formalism, these states are written as

$$\langle f| = \langle 0|U_f, \qquad U_f = \exp\left[\sum_{n=0}^{\infty} v_n L_n\right] \tag{1.22}$$

where L_n are the usual virasoro generators. The operator U_f is such that

$$f\circ\phi(\xi) = [f'(\xi)]^d\phi(f(\xi)) \equiv U_f\phi(\xi)U_f^{-1} \tag{1.23}$$

Indeed, one can easily see that

$$\begin{aligned}\langle f|\phi\rangle &= \langle 0|U_f\phi(\xi=0)|0\rangle = \langle 0|U_f\phi(\xi=0)U_f^{-1}|0\rangle \\ &= \langle 0|f\circ\phi(\xi=0)|0\rangle = \langle f\circ\phi(0)\rangle.\end{aligned} \tag{1.24}$$

The problem now is how to find v_n given f, or the other way around. One can show that they are related by the Julia equation $v(\xi)\partial_\xi f(\xi) = v(f(\xi))$, where $v(\xi) = \sum v_n\xi^{n+1}$. If $v(\xi)$ is known, it is possible to find $f(\xi)$, but the inverse problem is very hard and just a few examples are known.

There are special surface states called wedge states, which have a great importance in SFT. They are denoted by $|n\rangle$, for a real $n \geq 1$, and are defined by the conformal map

$$f_n(\xi) = \frac{n}{2}\tan\left(\frac{2}{n}\arctan\xi\right) \tag{1.25}$$

They obey a very simple multiplication rule

$$|m\rangle \star |n\rangle = |m+n-1\rangle \tag{1.26}$$

and will play an extremely important role in Schnabl's solution. One can see that the SL(2,R) vacuum is the state $|2\rangle$ and the identity string field is the state $|1\rangle$.

Schnabl's strategy was to work in another coordinate than the ξ one. He did so because he realized that in the $z = f_\infty(\xi) = \arctan\xi$ coordinate the star product becomes quite simple. The upper half of the unit disk is mapped to a semi-infinite strip with a width of $\pi/2$, and the left and right halves of the open string are mapped to semi-infinite lines parallel to the imaginary axis (Fig. 1.4). The right half of one string and the left half of the other string are glued together in Witten's star product so that star products of states in the Fock space can be obtained simply by a translation in the z coordinate. For example, Fig. 1.5 represents the gluing of three string fields $|\Psi_1\rangle, |\Psi_2\rangle$ and $|\Psi_3\rangle$ ($|\Psi_i\rangle = \Psi_i(\xi_i = 0)|0\rangle$), where they are inserted at $P_1 = 0, P_2 = \frac{\pi}{2}$ and $P_3 = \pi$, respectively.

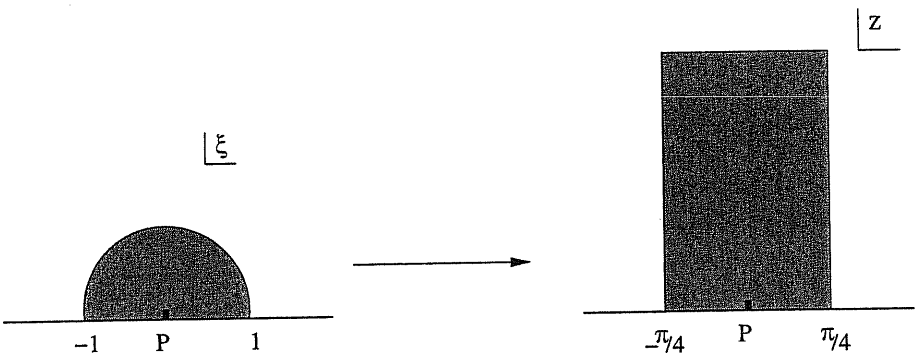

Figure 1.4: *The* $z = \arctan\xi$ *map.*

The next step was to try to find a solution that can be written in terms of wedge states, since they obey a very nice multiplication rule under the star product. If we use the Siegel gauge, the equation of motion we have to solve is

$$\Psi + \frac{b_0}{L_0}(\Psi \star \Psi) = 0. \tag{1.27}$$

The problem is that the action of $\frac{b_0}{L_0}$ on wedge states leaves the family of wedge states. The solution to this problem comes as follows. If we use the zero modes of the b field and of the energy momentum tensor in the z coordinate

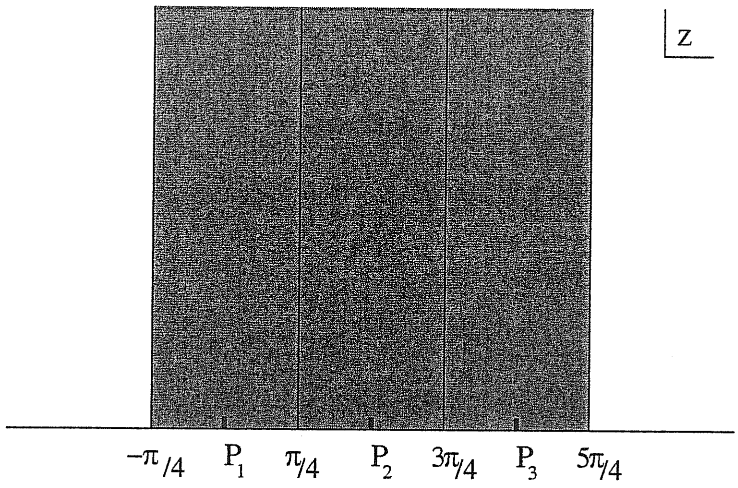

Figure 1.5: *The gluing of three strings in the z coordinate.*

$$\begin{aligned}\mathcal{B}_0 &= \oint \frac{dz}{2\pi i}\, z\, b(z) = \oint \frac{d\xi}{2\pi i}(1+\xi^2)\ \arctan\xi\ b(\xi) = b_0 + \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{4k^2-1} b_{2k} \qquad (1.28)\\ \mathcal{L}_0 &= \oint \frac{dz}{2\pi i}\, z\, T(z) = \oint \frac{d\xi}{2\pi i}(1+\xi^2)\ \arctan\xi\ T(\xi) = L_0 + \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{4k^2-1} L_{2k}\end{aligned}$$

and choose as a new gauge $\mathcal{B}_0\Psi = 0$, equation (1.27) can be rewritten as

$$\Psi + \frac{\mathcal{B}_0}{\mathcal{L}_0}(\Psi \star \Psi) = 0. \qquad (1.29)$$

The difference now is that $\frac{\mathcal{B}_0}{\mathcal{L}_0}$ has a very simple action on wedge states

$$\frac{\mathcal{B}_0}{\mathcal{L}_0}|n\rangle = -\mathcal{B}_0^\dagger \int_2^n \frac{dm}{m}|m\rangle, \qquad (1.30)$$

making it a bit easier to solve the equation (1.29). Using $\mathcal{L}_0$ level truncation, Schnabl could solve the equation recursively level by level and found that the solution can be written as

$$\Psi = \lim_{N\to\infty}\left[\sum_{n=0}^{N} \frac{d}{dn}\psi_n - \psi_N\right], \qquad (1.31)$$

where

$$\begin{aligned}\psi_n &= \frac{2}{\pi} c_1|0\rangle \star |n\rangle \star B_1^L c_1|0\rangle \qquad (1.32)\\ B_1^L &= \frac{1}{2}(b_1 + b_{-1}) + \frac{1}{\pi}(\mathcal{B}_0 + \mathcal{B}_0^\dagger). \qquad (1.33)\end{aligned}$$

To compute the derivative of ψ_n, we need the fact that the wedge state $|n\rangle$ can also be written as

$$\begin{aligned} |n\rangle &= e^{\frac{\pi(n-2)}{2}K_1^L}|0\rangle \\ K_1^L &= \frac{1}{2}(L_1+L_{-1})+\frac{1}{\pi}(\mathcal{L}_0+\mathcal{L}_0^\dagger). \end{aligned} \tag{1.34}$$

From this, it is easy to see that

$$\psi_n' \equiv \frac{d}{dn}\psi_n = c_1|0\rangle \star K_1^L|n\rangle \star B_1^L c_1|0\rangle. \tag{1.35}$$

Plugging (1.31) in the equation of motion, one can prove, after a very laborious computation, that it is really a solution.

To prove Sen's first conjecture, we should compute the string field theory action for the above solution. By doing so, Schnabl showed that

$$V(\Psi) = \frac{1}{g^2}\left[\frac{1}{2}\langle\Psi, Q_B\Psi\rangle + \frac{1}{3}\langle\Psi, \Psi\star\Psi\rangle\right] = -\frac{1}{2\pi^2 g^2}, \tag{1.36}$$

which is exactly minus the tension of the $D25$-brane.

The proof of Sen's third conjecture is not so direct. First, we expand the action around the solution Ψ and, due to that, the kinetic operator Q_B gets modified to Q, which can be written in terms of Q_B and Ψ as

$$Q\Lambda = Q_B\Lambda + \Psi\star\Lambda - \Lambda\star\Psi. \tag{1.37}$$

The task is then to prove that this kinetic operator around the nonperturbative vacuum has trivial cohomology, and proving so is equivalent to finding a state A such that $QA = \mathcal{I}$. This can be seen as follows:

- First, suppose that Q has trivial cohomology. Then take $Q\mathcal{I} = Q_B\mathcal{I} + \Psi\star\mathcal{I} - \mathcal{I}\star\Psi = Q_B\mathcal{I} = 0$. Since $\mathcal{I}$ is Q-closed and Q has no cohomology, it must be exact, that is, $\mathcal{I} = QA$.

- Now suppose that there exists a state A such that $QA = \mathcal{I}$. For any Q-closed state Λ we have

$$Q(A\star\Lambda) = (QA)\star\Lambda + (-1)^A A\star(Q\Lambda) = \mathcal{I}\star\Lambda = \Lambda. \tag{1.38}$$

  As any Q-closed state is also Q-exact, Q has trivial cohomology.

Thus, one must find a solution to the equation

$$QA = Q_B A + \Psi\star A + A\star\Psi = \mathcal{I}. \tag{1.39}$$

Schnabl, using some previous results concerning this equation in the Siegel gauge, found the state A to be

$$A = \frac{\mathcal{B}_0}{\mathcal{L}_0}\mathcal{I} = \frac{\pi}{2} B_1^L \int_1^2 dn|n\rangle, \tag{1.40}$$

thereby proving Sen's conjecture.

1.2 Vacuum String Field Theory (VSFT)

In principle, it is straightforward to analyze the structure of the tachyon vacuum. We just need to follow three basic steps:

- Find the classical solution Φ_0 that represents the nonperturbative vacuum;
- Expand the action around this solution, setting $\Phi = \Phi_0 + \Psi$;
- Analyze the spectrum of Ψ using the resulting kinetic term.

Nonetheless, this prescription was hard to carry out because there had been no known closed form for Φ_0 before Schnabl found it last year. This fact led Rastelli, Sen and Zwiebach to formulate, some years ago, what is called Vacuum String Field Theory (VSFT)[9]. Their reasoning in constructing this theory was the following: since shifting the string field by a classical solution does not change the interacting term, they guessed the form of the new kinetic term after the shift and then checked whether the new action satisfies some consistency conditions. First, this action should be invariant under a gauge transformation like the original SFT action. Second, the new kinetic operator must have vanishing cohomology, implying the absence of open string degrees of freedom. Third, the action must have classical solutions representing the original D-brane configuration, as well as lump solutions representing lower dimensional D-branes.

Let Φ_0 be a classical solution of the equation of motion

$$Q_B\Phi_0 + \Phi_0 \star \Phi_0 = 0, \tag{1.41}$$

representing the tachyon vacuum. If we shift the string field $\Phi = \Phi_0 + \tilde{\Phi}$, the action becomes

$$S(\Phi_0 + \tilde{\Phi}) = S(\Phi_0) - \frac{1}{g^2}\left[\frac{1}{2}\langle\tilde{\Phi}, Q\tilde{\Phi}\rangle + \frac{1}{3}\langle\tilde{\Phi}, \tilde{\Phi}\star\tilde{\Phi}\rangle\right], \tag{1.42}$$

where $S(\Phi_0)$ is a constant that is equal to the tension of the D-brane. The kinetic operator Q is given as:

$$Q\tilde{\Phi} = Q_B\tilde{\Phi} + \Phi_0 \star \tilde{\Phi} + \tilde{\Phi} \star \Phi_0. \tag{1.43}$$

Since neither the inner product nor the star product have changed, all the relations of OSFT and the properties of the kinetic operator are still valid. Again, this new action is invariant under the gauge transformation $\delta\tilde{\Phi} = Q\Lambda + \tilde{\Phi}\star\Lambda - \Lambda\star\tilde{\Phi}$.

Their argument follows by saying we can propose field redefinitions which leave the interaction term invariant and, at the same time, simplify the operator Q. They propose a transformation of the kind:

$$\tilde{\Phi} = e^{K}\Phi \tag{1.44}$$

where K is a ghost number zero operator. It is also required that

$$\begin{aligned} K(A\star B) &= (KA)\star B + A\star (KB); \\ \langle KA, B\rangle &= -\langle A, KB\rangle \end{aligned} \tag{1.45}$$

so that the form of the cubic term is unchanged. The action finally takes the form

$$S(\Psi) = -\frac{1}{g^2}\left[\frac{1}{2}\langle\Psi, \mathcal{Q}\Psi\rangle + \frac{1}{3}\langle\Psi, \Psi\star\Psi\rangle\right], \tag{1.46}$$

where

$$\mathcal{Q} = e^{-K}Qe^{K}. \tag{1.47}$$

Gauge invariance can be shown to hold once

$$\begin{aligned} \mathcal{Q}^2 &= 0; \\ \mathcal{Q}(A\star B) &= (\mathcal{Q}A)\star B + (-1)^{A}A\star(\mathcal{Q}B); \\ \langle \mathcal{Q}A, B\rangle &= -(-1)^{A}\langle A, \mathcal{Q}B\rangle, \end{aligned} \tag{1.48}$$

which are true due to the previous relations.

As a closed form for Φ_0 was not known, a $\mathcal{Q}$ operator had to be postulated. Nor only should $\mathcal{Q}$ satisfy relations (1.48), but also the following requisites:

- It must have ghost number one;
- It must have vanishing cohomology;
- It must be universal, that is, it must be written without any reference to the brane boundary conformal field theory.

These requirements are satisfied if $\mathcal{Q}$ is constructed purely from ghost operators, and Rastelli *et.al.* construct operators of the form

$$\begin{aligned} \mathcal{Q} &= \sum_{n=0}^{\infty} a_n \mathcal{C}_n; \\ \mathcal{C}_n &= c_n + (-1)^n c_{-n}, \end{aligned} \tag{1.49}$$

where the a_n's are constant coefficients. These operators are ghost number one and universal, as they are constructed just with ghosts oscillators. They also have trivial cohomology for the following reason: for each n, there is an operator $B_n = \frac{1}{2}(b_n + (-1)^n b_{-n})$ that satisfies $\{\mathcal{C}_n, B_n\} = 1$. Hence, when $\mathcal{C}_n\psi = 0$, we have

$$\psi = \{\mathcal{C}_n, B_n\}\psi = \mathcal{C}_n(B_n\psi), \tag{1.50}$$

proving that ψ is $\mathcal{C}_n$ trivial. This shows that each $\mathcal{C}_n$ has vanishing cohomology, and so does $\mathcal{Q}$.

Finally, in another paper [10], Rastelli *et.al.* propose a canonical choice for the ghost kinetic operator

$$\mathcal{Q} = \frac{1}{2i}(c(i) - c(-i)) = c_0 + \sum_{n=1}^{\infty}(-1)^n \mathcal{C}_{2n} \tag{1.51}$$

that is supported by level expansion computations in the Siegel gauge. We will not discuss the details here since they are rather technical and not needed in the forthcoming discussions.

Chapter 2

Time-Dependent Solutions in VSFT

As discussed previously, String Field Theory is the natural framework where to study some nonperturbative aspects of String Theory, for example, to compute the tachyon potential and to find its minimum. An interesting question, related to the previous one, is to find time-dependent solutions which interpolate between the perturbative and nonperturbative vacua, describing thereby the decay of the space-filling $D25$-brane.

In this chapter, we present our results concerning these time-dependent solutions and discuss some of their properties, showing they exhibit the correct behavior in time and making a parallel with Sen's rolling tachyon solution.

2.1 Classical Solutions to VSFT

2.1.1 The Three String Vertex

Before starting to analyze the classical solutions to VSFT equations of motion, we are going to describe one more tool needed: the three string vertex. It is defined in the following way:

$$\langle A, B, C\rangle \equiv {}_1\langle A| \otimes {}_2\langle B| \otimes {}_3\langle C||V_3\rangle_{123}, \tag{2.1}$$

where A, B and C are string fields expressed in the Fock basis and

$$\begin{aligned} |V_3\rangle_{123} &= \int d^{26}p_{(1)}d^{26}p_{(2)}d^{26}p_{(3)}\ \delta^{26}(p_{(1)}+p_{(2)}+p_{(3)})e^{-E}|0,p\rangle_{123} \\ E &= \sum_{r,s=1}^{3}\left(\frac{1}{2}\sum_{m,n\geq 1}\eta_{\mu\nu}a_m^{\dagger(r)\mu}V_{mn}^{rs}a_n^{\dagger(s)\nu} + \sum_{m\geq 1}\eta_{\mu\nu}p_{(r)}^{\mu}V_{0m}^{rs}a_m^{\dagger(s)\nu} + \right. \\ &\qquad \left. + \frac{1}{2}\eta_{\mu\nu}p_{(r)}^{\mu}V_{00}^{rs}p_{(s)}^{\nu}\right). \end{aligned} \tag{2.2}$$

Summation over the Lorentz indices $\mu, \nu = 0, ..., 25$ is understood with η the flat Lorentz metric. The operators $a_m^{(r)\mu}, a_m^{\dagger(r)\mu}$ denote the non-zero modes matter oscillators of the r-th string, normalized such that

$$[a_m^{(r)\mu}, a_n^{\dagger(s)\nu}] = \eta^{\mu\nu}\delta_{mn}\delta^{rs}, \qquad m,n \geq 1 \tag{2.3}$$

$p_{(r)}$ is the momentum of the r-th string and $|0,p\rangle_{123} \equiv |p_{(1)}\rangle \otimes |p_{(2)}\rangle \otimes |p_{(3)}\rangle$ is the tensor product of the Fock vacuum states relative to the three strings with definite center of mass momentum. $|p_{(r)}\rangle$ is annihilated by the operators $a_m^{(r)\mu}$ and it is an eigenstate of the momentum operator $\hat{p}^{\mu}_{(r)}$ with eigenvalue $p^{\mu}_{(r)}$. They are normalized as

$$\langle p_{(r)}|p'_{(s)}\rangle = \delta_{rs}\delta^{26}(p+p'). \tag{2.4}$$

To compute the coefficients V^{rs}_{mn}, V^{rs}_{0m} and V^{rs}_{00}, we take some standard correlator and evaluate it in two different ways. By comparing the results, we are able to get an expression for them. For example, take the correlator $\langle A, B, C\rangle$ with $|A\rangle = i\partial X^{(r)}(z)|0\rangle$, $|B\rangle = i\partial X^{(s)}(w)|0\rangle$ and $|C\rangle = |0\rangle$. Using

$$i\partial X(z) = \sum_n \frac{\alpha_n}{z^{n+1}}, \qquad [\alpha_n, \alpha_m] = n\delta_{n+m} \tag{2.5}$$

and (2.2), we get the following expression for the correlator:

$$\langle A|\langle B|\langle C|V_3\rangle_{123} = \frac{\delta^{rs}}{(z-w)^2} - \sum_{n,m=1}^{\infty} \frac{\sqrt{mn}}{z^{-m+1}w^{-n+1}} V^{rs}_{mn}, \tag{2.6}$$

the $\sqrt{mn}$ comes when we write $\alpha_n = \sqrt{n}a_n$. From the previous chapter (1.18), we know that this correlator can also be computed as

$$\begin{aligned} \langle f_r \circ i\partial X^{(r)}(z) f_s \circ i\partial X^{(s)}(w)\rangle &= f'_r(z) f'_s(w) \langle i\partial X^{(r)}(f_r(z)) i\partial X^{(s)}(f_s(w))\rangle \\ &= \frac{f'_r(z) f'_s(w)}{(f_r(z) - f_s(w))^2}, \end{aligned} \tag{2.7}$$

where we used $\langle i\partial X(z) i\partial X(w)\rangle = \frac{1}{(z-w)^2}$. As (2.6) and (2.7) must be equal, we obtain

$$V^{rs}_{mn} = -\frac{1}{\sqrt{mn}} \oint \frac{dz}{2\pi i} z^{-m} \oint \frac{dw}{2\pi i} w^{-n} \frac{f'_r(z) f'_s(w)}{(f_r(z) - f_s(w))^2}. \tag{2.8}$$

Proceeding in the same manner for $|A\rangle = e^{ip_{(r)}.X(0)}|0\rangle$, $|B\rangle = i\partial X^{(s)}(w)|0\rangle$ and $|A\rangle = e^{ip_{(r)}.X(0)}|0\rangle$, $|B\rangle = e^{ip_{(s)}.X(0)}|0\rangle$, we get

$$V^{rs}_{0m} = -\frac{1}{\sqrt{m}} \oint \frac{dw}{2\pi i} w^{-m} \frac{\log|f'_r(0)|^{\frac{1}{2}} f'_s(w)}{(f_r(0) - f_s(w))}, \tag{2.9}$$

$$V^{rs}_{00} = \begin{cases} \log|f'_r(0)| & \text{if } r = s; \\ \log|f_r(0) - f_s(0)| + \frac{1}{2}\log|f'_r(0) f'_s(0)| & \text{if } r \neq s. \end{cases}$$

Similarly, the same can be done for the ghost part of the vertex. Yet, we will not discuss it here since we do not need it in the forthcoming discussions.

2.1.2 Classical Solutions

Now we posses all the instruments to compute classical solutions for VSFT equation of motion

$$\mathcal{Q}\Psi + \Psi \star \Psi = 0. \tag{2.10}$$

Here, the fact that $\mathcal{Q}$ comprises just ghost degrees of freedom plays a crucial role. As the vertex factorizes in matter and ghost parts, we can say the solution also splits into two parts. Thus, we make the following ansatz for nonperturbative solutions

$$\Psi = \Psi_m \otimes \Psi_g \tag{2.11}$$

where Ψ_g and Ψ_m depend purely on ghost and matter degrees of freedom, respectively. Then eq.(2.10) splits into

$$\begin{aligned} \mathcal{Q}\Psi_g &= -\Psi_g \star_g \Psi_g & (2.12) \\ \Psi_m &= \Psi_m \star_m \Psi_m & (2.13) \end{aligned}$$

where $\star_g$ and $\star_m$ refer, respectively, to the star product involving only the ghost and matter part. The action for this type of solution becomes

$$\mathcal{S}(\Psi) = -\frac{1}{6g_0^2}\langle\Psi_g|\mathcal{Q}|\Psi_g\rangle\langle\Psi_m|\Psi_m\rangle. \tag{2.14}$$

It is well–known how to find solutions to (2.12), but since the ghost part will not play a role in our future discussions we will simply ignore it. We will instead focus on the matter part, eq.(2.13). The solutions are projectors of the $\star_m$ algebra, and the $\star_m$ product is defined as follows

$$|\Psi_1 \star_m \Psi_2\rangle_3 \;=\; {}_1\langle\Psi_1|\; {}_2\langle\Psi_2|V_3\rangle_{123}. \tag{2.15}$$

<u>The Sliver and The Inverse Sliver</u>

To represent the space-filling $D25$-brane, a solution to (2.13) must be translationally invariant and, as a consequence, all momenta can be set to zero in (2.2). In this way, the integration over the momenta can be dropped and the only surviving part in E will be the one involving V_{mn}^{rs}. The strategy we will follow is to make an ansatz for this solution

$$|\Xi\rangle = \mathcal{N}e^{-\frac{1}{2}a^\dagger \cdot S \cdot a^\dagger}|0\rangle, \qquad a^\dagger \cdot S \cdot a^\dagger = \sum_{m,n=1}^{\infty} a_m^{\mu\dagger} S_{mn} a_n^{\nu\dagger}\eta_{\mu\nu}, \tag{2.16}$$

plug it in (2.13) and find what conditions S should satisfy to fullfill the requirement the solution is a projector of the $\star_m$ algebra. This solution is called the *sliver*. This state satisfies eq.(2.13) provided the matrix S satisfies the equation

$$S = V^{11} + (V^{12}, V^{21})(1-\Sigma\mathcal{V})^{-1}\Sigma\begin{pmatrix} V^{21} \\ V^{12} \end{pmatrix} \tag{2.17}$$

where

$$\Sigma = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}, \qquad \mathcal{V} = \begin{pmatrix} V^{11} & V^{12} \\ V^{21} & V^{22} \end{pmatrix}, \tag{2.18}$$

The proof of this fact is well–known. First one expresses eq.(2.18) in terms of the twisted matrices $X = CV^{11}, X_+ = CV^{12}$ and $X_- = CV^{21}$, together with $T = CS = SC$, where $C_{nm} = (-1)^n \delta_{nm}$. The matrices X, X_+, X_- are mutually commuting, and requiring T to commute with them as well, one can show that eq.(2.17) reduces to the algebraic equation

$$XT^2 - (1+X)T + X = 0 \tag{2.19}$$

The sliver solution is

$$T = \frac{1}{2X}(1 + X - \sqrt{(1+3X)(1-X)}) \tag{2.20}$$

The normalization constant $\mathcal{N}$ is calculated to be

$$\mathcal{N} = (\mathrm{Det}(1 - \Sigma\mathcal{V}))^{\frac{D}{2}} \tag{2.21}$$

The contribution of the sliver to the matter part of the action (see (2.14)) is given by

$$\langle \Xi | \Xi \rangle = \frac{\mathcal{N}^2}{(\det(1 - S^2))^{\frac{D}{2}}} \tag{2.22}$$

Both eq.(2.21) and (2.22) are ill–defined and need to be regularized. However, we will not discuss this point here since it is a rather technical matter.

Now let us remark that there is another solution to (2.19), i.e. $1/T$. In fact (2.19) is invariant under the substitution $T \leftrightarrow 1/T$. $1/T$ is given by the RHS of eq.(2.20) with the $-$ sign replaced by the $+$ sign in front of the square root. We will call it the *inverse sliver*. This solution was previously discarded, [11], because of the bad asymptotic behavior of the $1/T$ eigenvalues. Nevertheless, it is exactly this behavior that will allow us to find interesting time–dependent solutions.

The Lump

In order to build our time-dependent solution, we will need another kind of solution, the *lump* [11]. The lump solution is engineered to represent a lower dimensional brane, therefore it will have transverse directions along which translational invariance is broken. Accordingly, we split the three string vertex into the tensor product of the perpendicular part and the parallel part

$$|V_3\rangle = |V_{3,\perp}\rangle \otimes |V_{3,\parallel}\rangle \tag{2.23}$$

and the exponent E as $E = E_{\parallel} + E_{\perp}$. The parallel part is the same as in the sliver case whereas the perpendicular part is modified as follows. Following [11], we denote by x^α, p^α, $\alpha = 1, ..., k$ the coordinates and momenta in the transverse directions and introduce the zero mode combinations

$$a_0^{(r)\alpha} = \frac{1}{2}\sqrt{b}\hat{p}^{(r)\alpha} - i\frac{1}{\sqrt{b}}\hat{x}^{(r)\alpha}, \qquad a_0^{(r)\alpha\dagger} = \frac{1}{2}\sqrt{b}\hat{p}^{(r)\alpha} + i\frac{1}{\sqrt{b}}\hat{x}^{(r)\alpha}, \tag{2.24}$$

where $\hat{p}^{(r)\alpha}, \hat{x}^{(r)\alpha}$ are the zero momentum and position operators of the r–th string, and we have introduced the parameter b as in [11]. It follows

$$[a_0^{(r)\alpha}, a_0^{(s)\beta\dagger}] = \eta^{\alpha\beta}\delta^{rs} \tag{2.25}$$

Denoting by $|\Omega_b\rangle$ the oscillator vacuum ($a_0^\alpha|\Omega_b\rangle = 0$), the relation between the momentum basis and the oscillator basis is defined by

$$|\{p^\alpha\}\rangle_{123} = \left(\frac{b}{2\pi}\right)^{\frac{3}{2}} \exp\left[\sum_{r=1}^{3}\left(-\frac{b}{4}p_\alpha^{(r)}\eta^{\alpha\beta}p_\beta^{(r)} + \sqrt{b}a_0^{(r)\alpha\dagger}p_\alpha^{(r)} - \frac{1}{2}a_0^{(r)\alpha\dagger}\eta_{\alpha\beta}a_0^{(r)\beta\dagger}\right)\right]|\Omega_b\rangle \tag{2.26}$$

Next we insert this equation inside $E_\perp$ and eliminate the momenta along the perpendicular directions by integrating them out. The overall result of this operation is that, while $|V_{3,||}\rangle$ is the same as in the ordinary case,

$$|V_{3,\perp}\rangle = K_2\, e^{-E'}|\Omega_b\rangle \tag{2.27}$$

with

$$K_2 = \frac{\sqrt{2\pi b^3}}{3(V_{00}+b/2)^2}, \tag{2.28}$$

$$E' = \frac{1}{2}\sum_{r,s=1}^{3}\sum_{M,N\geq 0} a_M^{(r)\alpha\dagger}V_{MN}^{'rs}a_N^{(s)\beta\dagger}\eta_{\alpha\beta} \tag{2.29}$$

where M, N denote the couple of indices $\{0, m\}$ and $\{0, n\}$, respectively. The coefficients $V_{MN}^{'rs}$ are given in Appendix B of [11]. The new Neumann coefficients matrices $V^{'rs}$ satisfy the same relations as the V^{rs} ones. In particular one can introduce the matrices $X^{'rs} = CV^{'rs}$, where $C_{NM} = (-1)^N\,\delta_{NM}$, which turn out to commute with one another. All the relations of Appendix A hold with primed quantities. We can therefore repeat word by word the derivation of the sliver from eq.(2.16) through eq.(2.22). The new solution will have the form (2.16) with S along the parallel directions and S replaced by S' along the perpendicular ones. In turn S' is obtained as a solution to eq.(2.17) where all the quantities are replaced by primed ones. This amounts to solving eq.(2.19) with primed quantities. Hence, in the transverse directions S is replaced by S', given by

$$S' = CT', \qquad T' = \frac{1}{2X'}(1 + X' - \sqrt{(1+3X')(1-X')}) \tag{2.30}$$

In a similar way we have to adapt the normalization and energy formulas (2.21,2.22). Exactly as in the sliver case, we can consider the solution with T' replaced by $1/T'$. The same considerations hold as in that case.

Before ending this section, let us make a brief comment. One could ask why the sliver and the lump solutions describe the space-filling $D25$-brane and the lower dimensional ones. First, we should expect it to be so because we are describing the theory around the nonperturbative vacuum, and a classical solution, therefore, is to represent the perturbative one. Second, when we compute the ratio of the energies of these solutions, the ghost contributions cancel as they are common to all of them, and the result is the same as the ratio of the tensions of two branes [12]. This is a further evidence that this interpretation is to be correct.

2.1.3 Spectroscopy and diagonal representation

In this section we summarize, for later use, some important results concerning the spectroscopy of the matrices previously defined. The diagonalization of the X matrix was carried out in [13], while the same analysis for X' was accomplished in [14] and [15]. The eigenvalues of $X = X^{11}, X_+ = X^{12}, X_- = X^{21}$ and T are given, respectively, by

$$\mu^{rs}(k) = \frac{1 - 2\,\delta_{r,s} + e^{\frac{\pi k}{2}}\,\delta_{r+1,s} + e^{-\frac{\pi k}{2}}\,\delta_{r,s+1}}{1 + 2\cosh\frac{\pi k}{2}} \tag{2.31}$$

$$t(k) = -e^{-\frac{\pi|k|}{2}} \tag{2.32}$$

where $-\infty < k < \infty$. The generating function for the eigenvectors is

$$f^{(k)}(z) = \sum_{n=1}^{\infty} v_n^{(k)} \frac{z^n}{\sqrt{n}} = \frac{1}{k}(1 - e^{-k \arctan z}) \tag{2.33}$$

The completeness and orthonormality equations for the eigenfunctions are as follows

$$\begin{aligned} \sum_{n=1}^{\infty} v_n^{(k)} v_n^{(k')} &= \mathcal{N}(k)\delta(k - k') \\ \int_{-\infty}^{\infty} dk\, \frac{v_n^{(k)} v_m^{(k)}}{\mathcal{N}(k)} &= \delta_{nm} \\ \mathcal{N}(k) &= \frac{2}{k}\sinh\frac{\pi k}{2} \end{aligned} \tag{2.34}$$

The spectrum of X is continuous and lies in the interval $[-1/3, 0)$. It is doubly degenerate except at $-\frac{1}{3}$. The continuous spectrum of X' lies in the same interval, but X' in addition has a discrete spectrum. To describe it we follow [15]. We consider the decomposition

$$X'^{rs} = \frac{1}{3}(1 + \alpha^{s-r} CU' + \alpha^{r-s} U'C) \tag{2.35}$$

where $\alpha = e^{\frac{2\pi i}{3}}$. It is convenient to express everything in terms of CU' eigenvalues and eigenvectors (see Appendix B). The discrete eigenvalues are denoted by ξ and $\bar{\xi}$. Since CU' is unitary they lie on the unit circle. They are more effectively represented via the parameter η, (B.1), which in turn is connected to the parameter b (B.3). To each value of b there corresponds a couple of values of η with opposite sign (except for $b = 0$ which implies $\eta = 0$).

The eigenvectors corresponding to the continuous spectrum are $V_N^{(k)}$ $(-\infty < k < \infty)$, while the eigenvectors of the discrete spectrum are denoted by $V_N^{(\xi)}$ and $V_N^{(\bar{\xi})}$. They form a complete basis. They will be normalized so that the completeness relation takes the form

$$\int_{-\infty}^{\infty} dk\, V_N^{(k)} V_M^{(k)} + V_N^{(\xi)} V_M^{(\xi)} + V_N^{(\bar{\xi})} V_M^{(\bar{\xi})} = \delta_{NM} \tag{2.36}$$

It has become familiar and very useful to expand all the relevant quantities in VSFT by means of this basis. To this end we define

$$a_k = \sum_{N=0}^{\infty} V_N^{(k)} a_N, \qquad a_\xi = \sum_{N=0}^{\infty} V_N^{(\xi)} a_N, \qquad a_{\bar\xi} = \sum_{N=0}^{\infty} V_N^{(\bar\xi)} a_N$$

$$a_N = \int_{-\infty}^{\infty} dk\, V_N^{(k)}\, a_k + V_N^{(\xi)} a_\xi + V_N^{(\bar\xi)} a_{\bar\xi} \tag{2.37}$$

and introduce the even and odd twist combinations

$$e_k = \frac{a_k + Ca_k}{\sqrt{2}}, \quad e_\eta = \frac{a_\xi + Ca_\xi}{\sqrt{2}}, \quad o_k = \frac{a_k - Ca_k}{i\sqrt{2}}, \quad o_\eta = \frac{a_\xi - Ca_\xi}{i\sqrt{2}}. \tag{2.38}$$

The commutation relations among them are

$$[e_k, e_{k'}^\dagger] = \delta(k-k'), \quad [e_\eta, e_\eta^\dagger] = 1, \quad [o_k, o_{k'}^\dagger] = \delta(k-k'), \quad [o_\eta, o_\eta^\dagger] = 1, \tag{2.39}$$

while all the other commutators vanish. The twist properties are defined by

$$Ca_k = a_{-k}, \qquad Ca_\xi = a_{\bar\xi}.$$

Using these combinations the three–strings vertex can be cast in diagonal form and, for instance, the exponent of the conventional lump state can be written

$$\begin{aligned} a^\dagger S' a^\dagger &= \int_{-\infty}^{\infty} dk\, t(k)\, (a_k^\dagger, Ca_k^\dagger) + 2\, t_\xi\, (a_\xi^\dagger, Ca_\xi^\dagger) \\ &= \frac{1}{2}\int_{-\infty}^{\infty} dk\, t(k)\, (e_k^\dagger e_k^\dagger + o_k^\dagger o_k^\dagger) + t_\eta\, (e_\eta^\dagger e_\eta^\dagger + o_\eta^\dagger o_\eta^\dagger) \end{aligned} \tag{2.40}$$

where $t_\eta \equiv t_\xi = e^{-|\eta|}$. The unconventional lump (inverse lump) is obtained by replacing t_η with its inverse $e^{|\eta|}$.

In the sequel we need the behavior of the eigenvectors when $b \to 0$ and when $b \to \infty$. Near $b = 0$ we have

$$\begin{aligned} &b \approx 0, \qquad \eta \approx 0, \qquad \xi \approx 1 \\ &V_0^{(\xi)} = \frac{1}{\sqrt{2}} + \mathcal{O}(\eta), \qquad V_n^{(\xi)} = \mathcal{O}(\eta^2) \end{aligned} \tag{2.41}$$

The same behavior holds for the $V^{(\bar\xi)}$ basis.

When $b \to \infty$ we have instead

$$\begin{aligned} &b \to \infty, \qquad b \approx 4\log\eta, \qquad \xi \approx -e^{\frac{\pi i}{3}} \\ &V_0^{(\xi)} \approx e^{-\frac{\eta}{2}}\sqrt{2\eta\log\eta}, \qquad V_n^{(\xi)} \sim e^{-\frac{\eta}{2}}\sqrt{\eta} \end{aligned} \tag{2.42}$$

and the same for $V^{(\bar\xi)}$.

These asymptotic behaviors will be used to evaluate matrix elements such as (2.50). In this regard they are completely reliable (and, in any case, backed up by numerical evidence).

If we consider instead the corresponding asymptotic expansions for the $V^{(k)}$ basis, we have to be more careful. The point is that the expression $(V_0^{(k)})^2$, see (B.7), would superficially seem to vanish in the limit $b \to \infty$, but it is in fact a representation of the Dirac delta function $\delta(k)$, see Appendix D. Therefore the result of taking the $b \to \infty$ limit in an integral containing $(V_0^{(k)})^2$ is to concentrate it at the point $k = 0$. This renders the generating function (B.6) very singular and, consequently, such integrals as $\int dk\, V_n^{(k)} V_m^{(k)} f(k)$ must be handled with care. As for the limit of the continuous basis when $b \to 0$, one can see that $V_0^{(k)} \to 0$, while the other eigenfunctions have a nonvanishing finite limit.

2.2 Time-Dependent Solutions

The search for time–dependent solutions has lately become one of the prominent research topics in string theory. Particularly interesting is the search for solutions describing the decay of D-branes. An archetype problem in open bosonic string theory is describing the evolution from the maximum of the tachyon potential to the (local) minimum. Such a solution known as rolling tachyon, if it exists, describes the decay of the space filling D25–brane corresponding to the unstable perturbative vacuum to the locally stable vacuum. That such a solution exists has been argued in many ways [16, 17, 18, 19] and, in this section, we discuss our results thereof.

2.2.1 Time dependent solutions: dead ends

In order to appreciate the very nature of the problem of finding time–localized VSFT solutions, let us examine first some obvious attempts and learn from their failure. The first thing that comes to one's mind is to start from a lump with one transverse space direction (therefore it represents a D24-brane) and inverse–Wick–rotate it. One such solution has been introduced above, see section 2.1.2 from eq.(2.23) through eq.(2.30). For simplicity we denote the transverse direction coordinate, momentum and oscillators simply by x, p and a_N. The solution is written as follows:

$$
\begin{aligned}
|\Psi'\rangle &= |\Xi\rangle_{25} \otimes |\Lambda'\rangle \\
|\Lambda'\rangle &= \mathcal{N}' \exp\left[-\frac{1}{2}\sum_{N,M\geq 0} a_N^\dagger S'_{NM} a_M^\dagger\right] |\Omega_b\rangle
\end{aligned}
\tag{2.43}
$$

where $|\Xi\rangle_{25}$ is the usual sliver along the longitudinal 25 directions and

$$
\mathcal{N}' = \sqrt{3}\frac{V_{00} + \frac{b}{2}}{(2\pi b^3)^{\frac{1}{4}}}\sqrt{\det(1 - X')\det(1 + T')}
\tag{2.44}
$$

In order to study the space profile of this solution in the transverse direction we contract it with the x_0–coordinate eigenstate

$$
|x_0\rangle = \sqrt{\frac{2}{b\pi}}\exp\left[-\frac{1}{b}x_0^2 - \frac{2}{\sqrt{b}}i a_0^\dagger x_0 + \frac{1}{2}(a_0^\dagger)^2\right] |\Omega_b\rangle
\tag{2.45}
$$

The result is

$$\langle x_0|\Lambda'\rangle = \sqrt{\frac{2}{b\pi}}\frac{\mathcal{N}}{\sqrt{1+s'}}\exp\left[\frac{1}{b}\frac{s'-1}{s'+1}x_0^2 - \frac{2i}{\sqrt{b}}\frac{x_0 f_0}{1+s'} - \frac{1}{2}a^\dagger W' a^\dagger\right] \tag{2.46}$$

where the condensed notation means

$$f_0 = \sum_{n=1} S'_{0n}a_n^\dagger, \quad a^\dagger W' a^\dagger = \sum_{n,m=1} a_n^\dagger W'_{nm} a_m^\dagger, \quad W'_{nm} = S'_{nm} - \frac{S'_{0n}S'_{0m}}{1+s'} \tag{2.47}$$

and

$$s' = S'_{00} \tag{2.48}$$

After an inverse Wick–rotation $x_0 \to i\mathrm{x}_0, a_n^\dagger \to i a_n^\dagger$ (2.46) becomes

$$\langle \mathrm{x}_0|\Lambda'\rangle = \sqrt{\frac{2}{b\pi}}\frac{\mathcal{N}}{\sqrt{1+s'}}\exp\left[\frac{1}{b}\frac{1-s'}{1+s'}\mathrm{x}_0^2 + \frac{2i}{\sqrt{b}}\frac{\mathrm{x}_0 f_0}{1+s'} + \frac{1}{2}a^\dagger W' a^\dagger\right] \tag{2.49}$$

We are interested in solutions localized in time. The second term in the exponent gives rise to time oscillations. Only the first term can guarantee time localization. Precisely this happens when $|s'| > 1$. However such a condition can never be achieved within the present scheme in which ordinary lump solutions are utilized. In fact it is possible to show that for such solutions $|s'| \leq 1$. Therefore with the simple–minded scheme considered so far it is impossible to achieve time localization (in this regard our negative conclusion is similar to [20]; as for the case $b \to 0$, see below).

Let us see this in more detail by showing that $|s'| \leq 1$. Using the basis of the previous section we can write

$$s' \equiv S'_{00} = \int_{-\infty}^{\infty} dk\, V_0^{(k)}(-e^{-\frac{\pi|k|}{2}})V_0^{(k)} + V_0^{(\xi)}e^{-|\eta|}V_0^{(\xi)} + V_0^{(\bar{\xi})}e^{-|\eta|}V_0^{(\bar{\xi})} \tag{2.50}$$

Using (B.7), one can see that the first term in the RHS does not contribute in the limit $b \to 0$ (i.e. $\eta \to 0$) and using the approximants (2.41) we immediately see that the remaining two terms add up to 1. Therefore when $b \to 0$, $s' \to 1$. Viceversa, in the limit $b \to \infty$, using (2.42) we see that the last two terms in the RHS of (2.50) do not contribute, while the first term contribute exactly –1. This can be also shown numerically or with the alternative analytical method of Appendix C. For generic values of b we cannot calculate s' analytically but it is easy to evaluate it numerically and to show that it is a monotonically decreasing function of b for $0 \leq b < \infty$. This in turn implies that the quantity $\frac{1-s'}{1+s'}$ is always *positive* (see figure 2.1).

Our conclusion is therefore that we cannot obtain a time–localized solution by inverse–Wick–rotating an ordinary lump solution. We will show elsewhere that also adding a constant B–field does not change this negative conclusion. Some drastic change has to be made in order to produce a localized time–dependent solution.

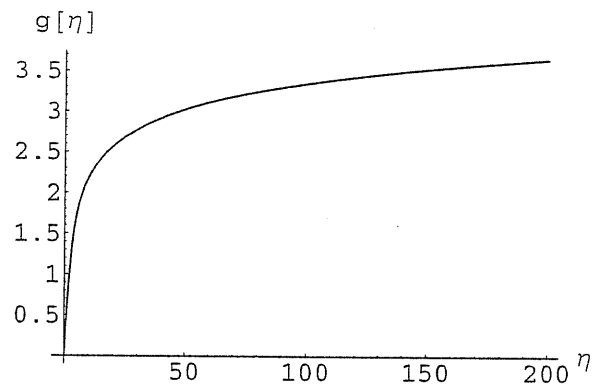

Figure 2.1: *The quantity* $g[\eta] = \frac{1-s'}{1+s'}$ *as a function of* η

2.2.2 A Rolling Tachyon–like Solution

It is not hard to realize that if we were to replace $e^{-|\eta|}$ with $e^{|\eta|}$ in eq.(2.50) we would reverse the conclusion at the end of the previous section. In fact, see below, we would have $|s'| \geq 1$. In this section we wish to exploit this possibility. In section 2.1.2 we have seen that if in the lump solution we replace T' by $1/T'$, formally, we still have a projector. Motivated by this fact we define an unconventional lump, by replacing $|\Lambda'\rangle$ in (2.43) with

$$|\check{\Lambda}'\rangle = \mathcal{N}\exp(a^\dagger C\check{T}'a^\dagger)|\Omega_b\rangle \tag{2.51}$$

where

$$\check{T}'_{NM} = -\int_{-\infty}^{\infty} dk\, V_N^{(k)}\, V_M^{(k)} \exp\left(-\frac{\pi|k|}{2}\right) + \left(V_N^{(\xi)}\, V_M^{(\xi)} + V_N^{(\bar{\xi})}\, V_M^{(\bar{\xi})}\right) \exp|\eta| \tag{2.52}$$

Due to the fact that the star product is split into eigenspaces of the Neumann coefficients X', X'_+, X'_-, the projector equation split accordingly into the continuous and discrete spectrum part. Therefore we are guaranteed that (2.51) is again a projector, as one can on the other hand easily verify by direct calculation. This is the solution we propose.

Before we proceed with our analysis we would like to clarify a basic question about the solution we have just put forward. Passing from a squeezed state solution with a matrix T' to another characterized by the inverse matrix $1/T'$ may lead in general to unacceptable features of the state, such as divergent terms in the oscillator basis. However in the case at hand, in which one inverts only the discrete spectrum, such unpleasant aspects disappear. First of all the matrix $\check{T}'$ is well defined both in the oscillator and in the diagonal basis. Second, such expression as $\sqrt{\det(1-\check{T}')}$ are well–defined. This is due to the fact that, if we are allowed to factorize the discrete and continuous spectrum contribution, the former can be written as $\det(1-\check{T}')^{(d)} = (1-\exp|\eta|)^2$, so that the possible dangerous – sign under the square root disappears due to the double multiplicity of the discrete eigenvalue. Third, the energy density of the (euclidean) solution (2.51) equals the energy density of the ordinary lump. In fact, using the formulas of [11], the ratio between the energy densities of the two solutions reduces to

$$\sqrt{\frac{\det(1+\check{T}')}{\det(1-\check{T}')}}\Big/\sqrt{\frac{\det(1+T')}{\det(1-T')}} = \sqrt{\frac{(1+e^{|\eta|})^2}{(1-e^{|\eta|})^2}}\Big/\sqrt{\frac{(1+e^{-|\eta|})^2}{(1-e^{-|\eta|})^2}} = 1 \tag{2.53}$$

after factorization of the discrete and continuous parts of the spectrum.

After these important remarks it remains for us to show that this solution has the appropriate features to represent a rolling tachyon solution. To see if this is true we have to represent it in a more explicit way. In particular we have to extract the explicit time dependence (better, the space dependence and then inverse–Wick–rotate it). To do so, we have to choose a (coordinate) basis on which to project (2.51). There seem to be two distinguished ways to make this choice. We will work them out explicitly and then discuss them.

To start with let us define the following coordinate and momentum operator, given by the twist even and twist odd parts of the discrete spectrum,

$$\hat{x}_\eta = \frac{i}{\sqrt{2}}(e_\eta - e_\eta^\dagger) \tag{2.54}$$

$$\hat{y}_\eta = \frac{i}{\sqrt{2}}(o_\eta - o_\eta^\dagger) \tag{2.55}$$

The eigenstates of the coordinate $\hat{x}_\eta$ are given by

$$\begin{gathered}|x\rangle = \frac{1}{\sqrt{\pi}}\exp\left(-\tfrac{1}{2}x^2 - \sqrt{2}ie_\eta^\dagger x + \tfrac{1}{2}e_\eta^\dagger e_\eta^\dagger\right)|\Omega_{\eta_e}\rangle, \\ e_\eta|\Omega_{\eta_e}\rangle = 0 \\ \hat{x}_\eta|x\rangle = x|x\rangle\end{gathered} \tag{2.56}$$

Correspondingly the eigenstates of the momentum $\hat{y}_\eta$ are

$$\begin{gathered}|y\rangle = \frac{1}{\sqrt{\pi}}\exp\left(-\tfrac{1}{2}y^2 - \sqrt{2}io_\eta^\dagger y + \tfrac{1}{2}o_\eta^\dagger o_\eta^\dagger\right)|\Omega_{\eta_o}\rangle, \\ o_\eta|\Omega_{\eta_o}\rangle = 0 \\ \hat{y}_\eta|y\rangle = y|y\rangle\end{gathered} \tag{2.57}$$

In order to make the x, y dependence explicit we project our solution (2.51) into the position/momentum eigenstates (2.56, 2.57). Using standard results[1] we get

$$\langle x, y|\check{\Lambda}'\rangle = \frac{1}{\pi(1+e^{|\eta|})}\exp\left(\frac{e^{|\eta|}-1}{e^{|\eta|}+1}(x^2+y^2)\right)|\check{\Lambda}'_c\rangle \tag{2.58}$$

The state $|\check{\Lambda}'_c\rangle$ is given by (2.51), but with only oscillators from the continuous spectrum, as the contribution of the discrete spectrum is now contained in the prefactor at the rhs of

[1]Here we are assuming that the vacuum factorizes into $|\Omega_{\eta_e}\rangle \otimes |\Omega_{\eta_o}\rangle \otimes |\Omega_c\rangle$ where the latter factor represents the vacuum with respect to the continuous oscillator component.

(2.58) . Now we perform the inverse Wick rotation $x \to i\mathrm{x}$, $y \to -i\mathrm{y}$ to recover the Lorentz signature, and obtain

$$|\check{\Lambda}'(\mathrm{x};\mathrm{y})\rangle = \frac{1}{\pi(1+e^{|\eta|})}\mathrm{exp}\left(-\frac{e^{|\eta|}-1}{e^{|\eta|}+1}(\mathrm{x}^2+\mathrm{y}^2)\right)|\check{\Lambda}'_c\rangle^{(Wick)} \tag{2.59}$$

It is evident that for every value of η the solution is localized in the x–time coordinate. The extra coordinate y is related to internal twist odd degrees of freedom and can be interpreted as a free parameter of the representation (2.59). This solution also contains the free parameter η which is nothing but a re-parametrization of b, through (B.3). Therefore it is characterized by two free parameters.

The 'time' x is not the ordinary time, i.e. the time coupled to the flat open string metric and related to the string center of mass. We will see later on a possible interpretation for x. Now, let us turn to the ordinary (open string) time, i.e. the time defined by the center of mass of the string and analyze the corresponding time profile. Despite the fact that this coordinate is not diagonal for the $\star$–product we can still have complete control on the profile along it. The center of mass position operator is given by

$$\hat{x}_0 = \frac{i}{\sqrt{b}}(a_0 - a_0^\dagger) \tag{2.60}$$

The center of mass position eigenstate is

$$|x_0\rangle = \sqrt{\frac{2}{b\pi}}\mathrm{exp}\left(-\frac{1}{b}x_0x_0 - \frac{2}{\sqrt{b}}ia_0^\dagger x_0 + \frac{1}{2}a_0^\dagger a_0^\dagger\right)|\Omega_b\rangle \tag{2.61}$$

Let us compute the center of mass time profile. After inverse–Wick–rotating it, it turns out to be

$$\begin{aligned} |\Lambda'(\mathrm{x}_0)\rangle &= \langle \mathrm{x}_0|\Lambda'\rangle = \sqrt{\frac{2}{b\pi}}\frac{\mathcal{N}}{\sqrt{1+\check{T}'_{00}}}\mathrm{exp}\left(\frac{1}{b}\frac{1-\check{T}'_{00}}{1+\check{T}'_{00}}\mathrm{x}_0^2+\right. \qquad (2.62)\\ &+ \left.\frac{2i}{\sqrt{b}(1+\check{T}'_{00})}\mathrm{x}_0\check{T}'_{0n}a_n^\dagger + \frac{1}{2}a_n^\dagger W'_{nm}a_m^\dagger\right)|\Omega_b\rangle \\ W'_{nm} &= \check{S}'_{nm} - \frac{1}{1+\check{T}'_{00}}\check{S}'_{0n}\check{S}'_{0m} \qquad (2.63) \end{aligned}$$

The quantities $\check{S}'_{0n}$ and $\check{S}'_{nm}$ can be computed in the diagonal basis

$$\begin{aligned} \check{S}'_{0n} &= \check{T}'_{0n} \qquad (2.64)\\ &= (1+(-1)^n)\left(-\int_0^\infty V_0^{(k)}V_n^{(k)}\mathrm{exp}\left(-\frac{\pi k}{2}\right) + V_0^{(\xi)}V_n^{(\xi)}\,\mathrm{exp}\,|\eta|\right) \\ \check{S}'_{nm} &= (-1)^n\check{T}'_{nm} = \qquad (2.65)\\ &= ((-1)^n+(-1)^m)\left(-\int_0^\infty V_n^{(k)}V_m^{(k)}\mathrm{exp}\left(-\frac{\pi k}{2}\right) + V_n^{(\xi)}V_m^{(\xi)}\,\mathrm{exp}\,|\eta|\right) \end{aligned}$$

It is evident that the leading time dependence in (2.62), for large x_0, is contained in $\exp\left(\frac{1}{b}\frac{1-\check{T}'_{00}}{1+\check{T}'_{00}}\mathrm{x}_0^2\right)$. The number $\check{T}'_{00}$ is $b(\eta)$–dependent and can be computed via

$$\check{T}'_{00}(\eta) = -2\int_0^\infty dk\left(V_0^{(k)}(b(\eta))\right)^2 \exp\left(-\frac{\pi k}{2}\right) + 2(V_0^{(\xi)})^2 \exp|\eta| \tag{2.66}$$

This is the crucial quantity as far as the time profile is concerned. An analytic evaluation of it is beyond our reach. However we will later show that

$$\lim_{\eta\to 0}\check{T}'_{00} = 1 \tag{2.67}$$

$$\lim_{\eta\to\infty}\check{T}'_{00} = \infty \tag{2.68}$$

A numerical analysis shows that this quantity is a function monotonically increasing with η within such limits. This means that the quantity $\frac{1-\check{T}'_{00}}{1+\check{T}'_{00}}$ is always *negative* (it lies in the interval $(-1, 0)$, see figure 2.2) and so the profile is always localized in the center of mass time, except in the extreme case $\eta \to 0$, which corresponds to the tensionless limit.

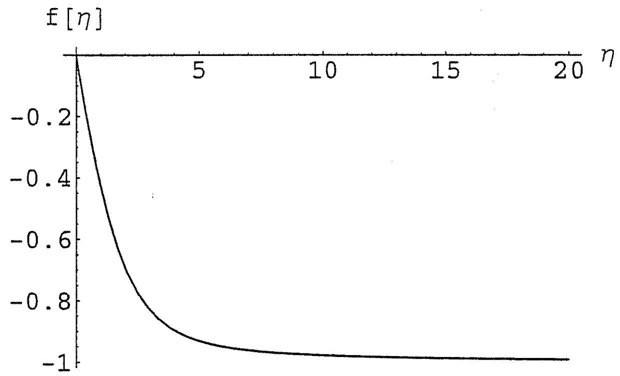

Figure 2.2: *The quantity* $f[\eta] = \frac{1-\check{T}'_{00}}{1+\check{T}'_{00}}$ *as a function of* η

This has to be compared with the usual lump solution (see previous section) for which the corresponding quantity is always positive and takes values in the interval $(0, \infty)$, allowing for localized *space* profiles but divergent along a timelike direction.

For reasons that will become clear in the next section, we extract also the free parameter y dependence, by projecting onto the corresponding twist–odd eigenstate (2.57). This operation can be done before or after the projection along the center of mass coordinate and does not interfere with it because $\hat{\mathrm{y}}$ does not contain the zero mode. We will therefore consider the following representation of our solution (inverse Wick rotation is again understood)

$$|\Lambda'(\mathrm{x}_0,y)\rangle = \langle \mathrm{x}_0,\mathrm{y}|\Xi_\eta\rangle = \sqrt{\frac{2}{b\pi}}\frac{\mathcal{N}}{\sqrt{2\pi(1+e^{|\eta|})}}\exp\left(\frac{1-e^{|\eta|}}{1+e^{|\eta|}}\mathrm{y}^2\right) \tag{2.69}$$
$$\times\frac{1}{\sqrt{1+\check{T}'_{00}}} \quad \exp\left(\frac{1}{b}\frac{1-\check{T}'_{00}}{1+\check{T}'_{00}}\mathrm{x}_0^2+\frac{2i}{\sqrt{b}(1+\check{T}'_{00})}\mathrm{x}_0\check{T}'_{0n}a_n^\dagger-\frac{1}{2}a_n^\dagger W''_{nm}a_m^\dagger\right)|0\rangle$$

The quantities $\check{T}'_{00}$ and $\check{T}'_{0n}$ are the same as in (2.66, 2.64) since the momentum $\hat{y}_\eta$ is twist–odd. Some changes occur in W''_{nm}

$$W''_{nm} = \check{S}''_{nm}-\frac{1}{1+\check{T}'_{00}}\check{S}'_{0n}\check{S}'_{0m} \tag{2.70}$$
$$\check{S}''_{nm} = ((-1)^n+(-1)^m)\left(-\int_0^\infty dk\, V_n^{(k)}\, V_m^{(k)}\exp\left(-\frac{\pi k}{2}\right)\right.$$
$$+ \left. V_n^{(\eta)}V_m^{(\eta)}\exp|\eta|\right) \qquad n,m \text{ even}$$
$$= -((-1)^n+(-1)^m)\int_0^\infty dk\, V_n^{(k)}\, V_m^{(k)}\exp\left(-\frac{\pi k}{2}\right) \qquad n,m \text{ odd}$$

Note that $\check{S}''_{nm}$ gets contribution only from the twist–even part of the discrete spectrum.

In conclusion (2.69) provides the solution we were looking for. It represents a solution localized in x_0, with the desired profile. It depends on two free parameters y and η (or b). These are all positive features. But let us start making a closer comparison with the rolling tachyon solution (such a comparison is made with the representation (2.69)). This can be done by considering the limit $b\to\infty$, which can be derived from the eqs.(2.42). $b\to\infty$ means $\eta\to\infty$ (for simplicity from now on we take η positive) and

$$\check{T}'_{00}\approx 2\,\eta\log\eta\,(1-\frac{\log(2\pi)}{\log\eta}+\ldots) \tag{2.71}$$

where dots denote higher order terms. Therefore we see that in this limit any time dependence in (2.69) disappears. Moreover, anticipating a result of the following section, we also have that $W''_{nm}\to S_{nm}$. In other words, in the limit $b\to\infty$ we obtain a static solution corresponding to the initial sliver. From this we understand that the parameter $1/b$, for large b, plays a role similar to Sen's parameter $\tilde{\lambda}$ near 0[2]. A second remark concerns the limit $y\to\infty$. In this case the first exponential factor in the RHS of (2.69) suppresses everything, so that the limit is the 0 state. In other words, we can consider this value of the parameter y as identifying the (relatively) stable vacuum state.

Although the rolling tachyon naturally compares with (2.69) rather than with (2.59), it is instructive to repeat something similar with the latter. Let us stress once more that both (2.69) and (2.59) represent the same solution, but in different bases, in particular with two different times: one, x_0, is the open string center of mass time, the other, x is related to the discrete spectrum. In the (2.59) case a parameter like b is missing. But this is something

[2]We recall that Sen's rolling tachyon solution depends on the parameter $\tilde{\lambda}$, which appears in the $\tilde{\lambda}\int_{\partial D} dt\cosh X^0(t)$–deformed BCFT.

that is simply not customary and can be easily remedied. We can in fact introduce a parameter b_e in (2.54,2.55), just replacing $\sqrt{2}$ with $\sqrt{b_e}$ in those equations. Then (2.59) would become

$$|\check{\Lambda}'(\mathrm{x};\mathrm{y})\rangle = \frac{1}{b_e\pi(1+e^\eta)}\exp\left(-\frac{1}{b_e}\frac{e^\eta-1}{e^\eta+1}(\mathrm{x}^2+\mathrm{y}^2)\right)|\check{\Lambda}'_c\rangle^{(Wick)} \tag{2.72}$$

and we could repeat the same argument as above and reach the same conclusion, except that in this case we have to take $b_e \to \infty$ as well as $\eta \to \infty$. The limit $y \to \infty$ plays the same role as in the (2.69) representation.

In the next section we will study the solution (2.51) in a regime we are more familiar with, the low energy regime $\alpha' \to 0$, and in the other extreme regime, $\alpha' \to \infty$, in which the solution considerably simplifies and an analytic treatment is possible. What we would like to see more closely is whether, for sufficiently small values of the parameters, the solution at time 0 is close enough to the sliver configuration (2.16), whose decay the solution is expected to describe.

2.2.3 Low energy and tensionless limits

As reviewed in appendix C, the low energy limit is obtained by performing an $\epsilon \to 0$ limit on the quantities that depend on the Neumann coefficients of the three strings vertex. ϵ is a dimensionless parameter that represents the smallness of α', [21]. As it happens, in all the expansions we consider, the parameters ϵ and b only appear through the ratio ϵ/b. Therefore, formally, the expansions for small ϵ/b are the same as the expansions for large b, i.e. $\eta \to \infty$. Therefore, in this section, when we consider the expansion in η near ∞ we really mean the expansion for ϵ/b small (i.e. ϵ small and b finite). A different attitude is required by the 'external' states like (2.45). There the rescaling of x_0 would lead to the replacement $b \to b\epsilon$. In this case we absorb ϵ into x_0 and keep b finite. In conclusion, throughout the analysis of the low energy limit, b should be considered as a finite free parameter.

Let us analyze in detail what is the limit of the various quantities appearing in (2.69). First of all we have

$$\lim_{\eta\to\infty}\frac{1-\check{T}'_{00}}{1+\check{T}'_{00}} = -1 \tag{2.73}$$

This follows from (2.42) and from the discussion at the end of section 2.1.3, in particular from the property of $(V_0^{(k)})^2$ of approximating $\delta(k)$ in the limit $b \to \infty$, which implies that $\check{T}'_{00} \to \infty$ in the same limit. For the oscillating term we have

$$\lim_{\eta\to\infty}\frac{\check{T}'_{0n}}{1+\check{T}'_{00}} = \lim_{\eta\to\infty}\frac{1}{\sqrt{2\log\eta}} = 0 \tag{2.74}$$

To evaluate this limit one must evaluate $\check{T}'_{0n}$. This in turn requires knowing the asymptotic expansion of the basis $V_n^{(k)}$ for $\eta \to \infty$. This is done in Appendix D. A numerical approximation confirm the above result.

Thus, in the limit, the oscillating part completely decouples from the time dependent part. It remains for us to consider the limit of the quadratic form W''_{nm}, (2.70). When n, m

are odd there are no contribution from the discrete spectrum, since the contraction with $\langle \mathrm{y}|$ has eliminated them.

$$W''_{2n-1,2m-1} = \check{S}'^{(c)}_{2n-1,2m-1} \tag{2.75}$$

When n, m are even we have, on the contrary, potentially dangerous terms because there are divergent contributions arising from the discrete spectrum. The latter have to be carefully evaluated.

$$\begin{aligned} W''^{(d)}_{2n,2m} &= 2V^{(\xi)}_{2n} V^{(\xi)}_{2m} \left(e^{\eta} - \frac{2(V^{(\xi)}_0)^2 e^{2\eta}}{2(V^{(\xi)}_0)^2 e^{\eta} + \mathcal{O}(\frac{e^{-\eta}}{\eta\,\log\eta})} \right) \\ &= 2V^{(\xi)}_{2n} V^{(\xi)}_{2m}\, \mathcal{O}(\frac{e^{-\eta}}{\eta\,\log\eta}) \approx \mathcal{O}(\frac{1}{\log\eta}) \end{aligned} \tag{2.76}$$

We see that the potentially divergent contributions arising from the discrete spectrum exactly cancel when $\eta \to \infty$. Therefore, as far as $W''_{2n,2m}$ is concerned, we are left only with the contribution from the continuous spectrum. Of the two pieces that contribute to $W''^{(c)}_{2n,2m}$, see eq.(2.70) only the first survives in the limit $\eta \to \infty$, the second vanishes for the usual reasons. Therefore we can conclude that

$$W''_{nm} = \check{S}'^{(c)}_{nm} + \dots$$

where dots denote subleading corrections of order at least $1/\log\eta$. At this stage we can do the calculation directly as in Appendix D, or we can resort to an indirect argument by noticing that $\check{S}'^{(c)}_{nm}$ approaches S'_{nm} in the same limit, because the discrete spectrum contribution to the latter vanishes, and then use the results of Appendix D. In both cases we conclude that

$$W''_{nm} = S_{nm} + \mathcal{O}(\epsilon/b) \tag{2.77}$$

Going back to equation (2.69) we see that, modulo a normalization factor, we obtain

$$\lim_{\alpha'\to 0} |\Lambda'(\mathrm{x}_0, \mathrm{y})\rangle = \mathcal{N}'(\mathrm{y})\, e^{-\frac{\mathrm{x}_0^2}{b}}\, |\Xi\rangle \tag{2.78}$$

where $|\Xi\rangle$ is the zero momentum sliver state. This result can be phrased as follows: in the low energy limit the solution takes the form of a time–Gaussian multiplying a sliver, the subleading terms being proportional to ϵ/b, eq.(2.77).

To end this section let us briefly consider the opposite limit, that is $\alpha' \to \infty$ (tensionless limit). As in the previous case this is formally achieved by taking the $\eta \to 0$ limit in all the quantities which are related to the Neumann coefficients, but leaving b as a free parameter. This limit is well defined. Using the results of appendix D we get

$$\lim_{\eta\to 0} \frac{1 - \check{T}'_{00}}{1 + \check{T}'_{00}} = 0 \tag{2.79}$$

The oscillating term in (2.69) vanishes as well. This result implies that the Gaussian representing time dependence in (2.69) is actually completely flat: time dependence has disappeared! We believe this to be related to the fact that all strings modes become massless in this limit [22], so there are no modes to decay into. It is easy to see that the only non vanishing term in the exponent of (2.69) is the quadratic part which gets contribution only from the continuous spectrum (on the contrary of the $\eta \to \infty$ limit the discrete eigenvector has only the 0–component, while the higher components disappear like positive powers of $1/\eta$). We remark that in the tensionless limit the center of mass time and the x time are identified.

2.2.4 Further Comments

In the last two sections we have shown that by inverting the discrete part of the spectrum we obtain a definite (unconventional) lump solution which, after inverse Wick–rotation, gives rise to a time–localized state with many properties characteristic of the rolling tachyon solution. We have left, however, some loose ends which we would like now to tie up or at least comment upon.

The first comment concerns normalization of the states we have come across. We have written down throughout normalization factors in quite a formal way. We have already recalled the fact that the sliver state and the lump state have a vanishing normalization, but we believe these problems have to be kept separated from the normalization of our time dependent solution. As a matter of fact a normalization problem appears only for the representation (2.69) and in the low energy limit, for the coefficient $\mathcal{N}'$ in (2.78) diverges exponentially for $\eta \to \infty$ once all the contributions are taken into account (this problem does not arise for the other representation (2.59)). We remark however that, as was noticed in the discussion after eq.(2.51), the energy density of the corresponding Euclidean solution is well–defined (once the conventional lump energy density is). Therefore the exploding normalization can only be an artifact of the representation. It means that we have to use the parameters of the state to regulate the normalization, although it is not clear a priori what is the right way to do it. A possibility is to use the factor $\exp\left(\frac{1-e^{|\eta|}}{1+e^{|\eta|}}\mathrm{y}^2\right)$ in (2.69). Since this vanishes for y large, we can view y as a suitable function of η as $\eta \to \infty$. This can settle the problem. Other possibilities are connected to dressing, [23, 24].

Another important question is the number of parameters. Our solution depends on two parameters y and b. One may wonder why we extracted the y dependence from (2.51). This is indeed not a choice but a constraint. Had we not done it, we would have found a different formula (2.70) in which also the n, m odd part of S''_{nm} would have taken a contribution from the discrete spectrum (exactly as the n, m even part). However in the odd–odd part no such cancellation (2.76) as in the even–even part occurs and we would find badly divergent coefficients in W'. We gather that y is a genuine free parameter of the time–dependent state. What about b? It was argued in [11] that this parameter represents a gauge degree of freedom. This need not be in contradiction with the meaning we have attributed to it in the previous sections. We recall that in ordinary gauge theory a singular gauge transformation may convey some physical information. Now, looking at (2.24), the values $b = 0$ and $b = \infty$ may well correspond to singular gauge transformations, and therefore contain physical information. More generally the gauge nature of b may mean that using a different formulation one may be able to write the solution in terms of a single physical parameter which contains the information carried by both b and y.

The third question we would like to address is the relation between the two representations (2.59) and (2.69). The latter is expressed in terms of the open string center of mass x_0 and its interpretation is obvious. The interpretation of the former is less clear since the 'time' x does not have a clear connection with the open string center of mass time. A rather bold speculation is that x be connected with the closed string time. The closed string time couples to the closed string metric, which, in correspondence with the D–brane, must develop a singularity (it must be a solution of the effective low energy field theory associated to the closed string). So the relation between the open and the closed string time should be something like $g_c(dt_c)^2 \sim g_o(dt_o)^2$ in the field theory limit, were $g_o = 1$ and g_c becomes larger and larger near the origin. Something similar indeed occurs between x_0 and x when $\eta \to \infty$. In fact the ratio between x_0 and the zero mode part of x decreases exponentially with η. We notice moreover that the normalization of the representation (2.59) does not need any regularization. In other words x seems to be a smoother choice of time, with respect x_0.

2.3 Time-Dependent Solutions with an E-field

We have just shown that it is possible to get time localized solutions of VSFT by taking an euclidean lump solution on a transverse direction (the euclidean time) and simply inverting the discrete eigenvalues of the lump Neumann matrix. This solution, preserving the same euclidean action as the conventional lump[3], has the remarkable feature of being localized in the time coordinate identified by the twist even discrete eigenvector of the Neumann matrices and, what is more important, of being localized in the center of mass time coordinate for every value of the free parameter b. Moreover, time dependence disappears when $b \to \infty$ (the solution becoming the zero momentum sliver state) and when $b \to 0$ (the solution becoming the 0 string field, i.e. the "stable" vacuum of VSFT). This lead us to propose that, at least in these singular limits, the b–parameter should be related to the $\tilde{\lambda}$ parameter of Sen's Rolling Tachyon BCFT.[4]

Now, we are going to study the corresponding time-dependent solution in the presence of a constant E–field background on a longitudinal or transverse direction. We obtain E–field physics by first going to an euclidean signature with *imaginary* B–field, $B = iE$, and then inverse Wick rotating, in the same way as [25] for what concerns the effective target space and BCFT analysis. One of the main differences with respect to the previous solution is that when the E–field reaches its critical value $E_c = \frac{1}{2\pi\alpha'}$, the (center of mass) time dependence is lost, regardless of the b–parameter, and we get a flat non zero time profile which, along the lines of [25], should be interpreted as a static background of fundamental strings, polarized by the E–field. This result persists when $b \to 0$ if we double scale appropriately $E \to E_c$ with $b \to 0$.

[3]We stress that in presence of time dependent backgrounds, one cannot anymore consider the value of the classical action mod volume as the space averaged energy

[4]Another approach in obtaining time dependent solutions in VSFT can be found in [20]

2.3.1 Longitudinal E–field

In this section we will analyze the case of switching the $E-$field along a tangential direction, i.e., along, say, the world volume of a $D25-$brane. As explained in [25], the presence of the E–field does not create non commutativity as the direction in which it is turned on is at zero momentum.

We use the double Wick rotation, that is we make space–time euclidean by sending $X^0(\sigma) \to iX^D(\sigma)$; then we construct an *unconventional* lump solution on the transverse spatial direction $X^D(\sigma)$ and inverse Wick rotate along it, $X^D(\sigma) \to -iX^0(\sigma)$. Let $\alpha, \beta = 1, D$ be the couple of directions on which the E–field is turned on. Then E–field physics is obtained by taking an imaginary B–field

$$B_{\alpha\beta} = B\epsilon_{\alpha\beta} = iE\epsilon_{\alpha\beta}, \quad E \in \Re \tag{2.80}$$

A localized time-dependent solution is easily given by straightforwardly changing the metric $\eta_{\alpha\beta}$ of the previous solution with the open string metric $G_{\alpha\beta}$

$$G_{\alpha\beta} = (1 - (2\pi\alpha' E)^2)\,\delta_{\alpha\beta} \tag{2.81}$$

$$G^{\alpha\beta} = \frac{1}{1-(2\pi\alpha' E)^2}\,\delta^{\alpha\beta} \tag{2.82}$$

Note that, contrary to the case of a real B–field, a critical value shows up for the imaginary analytic continuation[5]

$$E_c = \frac{1}{2\pi\alpha'} \tag{2.83}$$

From now on all indexes (α, β) are raised/lowered with the open string metric (2.81).

We have then the following commutators

$$[a_m^\alpha, a_n^{\beta\dagger}] = G^{\alpha\beta}\delta_{mn}, \qquad m, n \geq 1 \tag{2.84}$$

stating that the a^α's are canonically normalized with respect the open string metric (2.81)

We recall that, in case of a background $B_{\alpha\beta}$–field, the three string vertex is deformed to be, [26]

$$|V_3\rangle = |V_{3,\perp}\rangle \otimes |V_{3,\|}\rangle \tag{2.85}$$

The factor $|V_{3,\|}\rangle$ concerns the directions with no B–field and its expression is the usual one, [27, 28, 29, 30], on the other hand $|V_{3,\perp}\rangle$ deals with the directions on which the B field is turned on[6] .

$$|V_{3,\perp}\rangle = \int d^{26}p_{(1)} d^{26}p_{(2)} d^{26}p_{(3)} \delta^{26}(p_{(1)} + p_{(2)} + p_{(3)}) \exp(-E')\,|0, p\rangle_{123} \tag{2.86}$$

[5] In the rest of the paper we will set $\alpha' = 1$

[6] Note that in the case under consideration the symbols $\perp$ and $\|$ do not refer to perpendicular or transverse directions to the brane, but simply indicates directions with E–field turned on ($\perp$) or not ($\|$)

The operator in the exponent is given by, [26]

$$\begin{aligned}
E' &= \sum_{r,s=1}^{3}\left(\frac{1}{2}\sum_{m,n\geq 1} G_{\alpha\beta}a_m^{(r)\alpha\dagger}V_{mn}^{rs}a_n^{(s)\beta\dagger} + \sum_{n\geq 1} G_{\alpha\beta}p_{(r)}^{\alpha}V_{0n}^{rs}a_n^{(s)\beta\dagger}\right. \\
&\left. + \frac{1}{2}G_{\alpha\beta}p_{(r)}^{\alpha}V_{00}^{rs}p_{(s)}^{\beta} + \frac{i}{2}\sum_{r<s} p_{\alpha}^{(r)}\theta^{\alpha\beta}p_{\beta}^{(s)}\right) \qquad (2.87)
\end{aligned}$$

Note that the part giving rise to space–time non–commutativity, i.e. the last term, does not contribute due to the zero momentum condition in the 1 spatial direction.
Let's first consider the sliver solution at zero momentum along the 1 direction. The three string vertex in such a direction takes the form ($p^1 = p_1 = 0$)

$$\begin{aligned}
|V_3(E, p=0)\rangle &= |V_3(E=0, p=0)\rangle^{(\eta_{11}\to G(E)_{11})} \qquad (2.88) \\
&= \exp\left(\frac{1}{2}\sum_{r,s=1}^{3} G_{11}a^{(r)1\dagger}\cdot V^{rs}\cdot a^{(s)1\dagger}\right)|0\rangle \qquad (2.89)
\end{aligned}$$

This implies that the zero momentum sliver is in this case

$$\begin{aligned}
|S(E, p=0)\rangle &= |S(E=0, p=0)\rangle^{(\eta_{11}\to G(E)_{11})} \qquad (2.90) \\
&= \mathcal{N}\exp\left(-\frac{1}{2}G_{11}a^{1\dagger}\cdot S\cdot a^{1\dagger}\right)|0\rangle \qquad (2.91)
\end{aligned}$$

where the normalization $\mathcal{N}$ and the matrix S are given as usual, [11],

$$\begin{aligned}
T &= CS = \frac{1}{2X}(1 + X - \sqrt{(1+3X)(1-X)}) \qquad (2.92) \\
\mathcal{N} &= \sqrt{\det(1-X)(1+T)} \qquad (2.93)
\end{aligned}$$

On the euclidean time direction we need the full 3 string vertex in oscillator basis. This is given by

$$|V_{3,\perp}\rangle' = K\, e^{-E'}|\Omega_b\rangle \qquad (2.94)$$

with

$$K = \left(\frac{\sqrt{2\pi b^3}}{3(V_{00}+b/2)^2}(1-(2\pi E)^2)^{\frac{1}{2}}\right)^{\frac{1}{2}}, \qquad (2.95)$$

$$E' = \frac{1}{2}\sum_{r,s=1}^{3}\sum_{M,N\geq 0} a_M^{(r)D\dagger}V_{MN}^{\prime rs}a_N^{(s)D\dagger}G_{DD} \qquad (2.96)$$

where M, N denote the couple of indices $\{0, m\}$ and $\{0, n\}$, respectively, and D is the (euclidean) time direction. The coefficients $V_{MN}^{\prime rs}$ are given in Appendix B of [11]. In order to have localization in Minkowski time, we need an explosive profile in euclidean time (unconventional lump)

$$|\check{\Lambda}'\rangle = \mathcal{N}\exp\left(-\frac{1}{2}G_{DD}a^{\dagger D}C\check{T}'a^{\dagger D}\right)|\Omega_b\rangle \qquad (2.97)$$

where

$$\check{T}'_{NM} = -\int_{-\infty}^{\infty} dk\, V_N^{(k)}\, V_M^{(k)} \exp\left(-\frac{\pi|k|}{2}\right) + \left(V_N^{(\xi)}\, V_M^{(\xi)} + V_N^{(\bar{\xi})}\, V_M^{(\bar{\xi})}\right) \exp|\eta| \tag{2.98}$$

We get a localized time profile by projecting on the coordinates/momenta of the discrete spectrum

$$\hat{x}_\eta = \frac{i}{\sqrt{2}}\left(e_\eta - e_\eta^\dagger\right) \tag{2.99}$$

$$\hat{y}_\eta = \frac{i}{\sqrt{2}}\left(o_\eta - o_\eta^\dagger\right) \tag{2.100}$$

where e_η / o_η are oscillators constructed with the twist even/odd part of the discrete spectrum eigenvectors $V_N^{(\bar{\xi})}$, $V_N^{(\xi)}$

$$e_\eta = \sum_{N=0}^{\infty} \frac{1}{2}\left(1 + (-1)^N\right) V_N^{(\xi)} a_N \tag{2.101}$$

$$o_\eta = \sum_{N=0}^{\infty} \frac{1}{2i}\left(1 - (-1)^N\right) V_N^{(\xi)} a_N \tag{2.102}$$

The profile along these coordinates is given by (inverse Wick rotation, $(x, y) \to i(\mathrm{x}, -\mathrm{y})$ is assumed)

$$|\check{\Lambda}'(\mathrm{x},\mathrm{y})\rangle = \langle \mathrm{x},\mathrm{y}|\check{\Lambda}'\rangle = \frac{1}{\pi(1+e^{|\eta|})}\exp\left(-\frac{e^{|\eta|}-1}{e^{|\eta|}+1}(\mathrm{x}^2+\mathrm{y}^2)\right)|\check{\Lambda}'_c\rangle \tag{2.103}$$

where $|\check{\Lambda}'_c\rangle$ contains only continuous spectrum contributions. This profile is localized on the time coordinate x. Note however that there is no more reference to the E–field in the exponent. In order to see explicitly the presence of the E–field, we need to use the usual *open* string time, i.e. the center of mass.

Therefore we contract our solution with the center of mass euclidean time, x^D, and then inverse Wick rotate it, $x^D \to ix^0$, obtaining

$$\begin{aligned} |\Lambda'(\mathrm{x}_0, y)\rangle &= \langle \mathrm{x}_0, \mathrm{y}|\Xi_\eta\rangle = \sqrt{\frac{2}{b\pi}}\frac{\mathcal{N}}{\sqrt{2\pi(1+e^{|\eta|})}}\exp\left(\frac{1-e^{|\eta|}}{1+e^{|\eta|}}\mathrm{y}^2\right) \\ \times\frac{1}{\sqrt{1+\check{T}'_{00}}} &\exp\left(-\mathcal{A}(x^0)^2 + \frac{2i\sqrt{1-(2\pi E)^2}}{\sqrt{b}(1+\check{T}'_{00})}x^0\check{T}'_{0n}\tilde{a}_n^\dagger - \frac{1}{2}\tilde{a}_n^\dagger W''_{nm}\tilde{a}_m^\dagger\right)|0\rangle \end{aligned} \tag{2.104}$$

The extra coordinate y is given by the twist odd contribution of the discrete spectrum, we need to project along it in order to have a well defined $b \to \infty$ limit in the oscillator part W''_{nm} as before. The oscillators $\tilde{a}_n$ are canonically normalized with respect the η-metric and are given by

$$\tilde{a}_n = \sqrt{1-(2\pi E)^2}a_n \tag{2.105}$$

The quantity that give rise to time localization is then

$$\mathcal{A} = -\frac{1}{b}\frac{1-\check{T}'_{00}}{1+\check{T}'_{00}}(1-(2\pi E)^2) \tag{2.106}$$

This quantity depends on the free parameter b, as well as on the value of the E–field, through the open string metric, used to covariantize the quadratic form in time. The matrix element $\check{T}'_{00}$ is given by

$$\check{T}'_{00}(\eta) = -2\int_0^\infty dk\left(V_0^{(k)}(b(\eta))\right)^2 \exp\left(-\frac{\pi k}{2}\right) + 2(V_0^{(\xi)})^2 \exp|\eta|, \tag{2.107}$$

it is a monotonic increasing function of b, greater than 1: this is what ensures localization in time as opposed to the standard lump which is suited for space localization.

The life time of the brane is thus given by

$$\Delta T = \frac{1}{2}\sqrt{\frac{1}{2\mathcal{A}}} = \frac{1}{(1-(2\pi E)^2)^{\frac{1}{2}}}\,\Delta T^{(E=0)} \tag{2.108}$$

Note that for E going to the critical value $E_c = \frac{1}{2\pi}$, the lifetime becomes infinite. In particular we get a completely flat profile. This has to be traced back to the fact that open strings become effectively tensionless in this limit, [31], so we correctly recover the result that the D-brane is stable. This configuration should correspond to a background of fundamental strings stretched along the E–field direction, with closed strings completely decoupled.

2.3.2 Transverse E–field

In this section we study the time dynamics of a D–brane with transverse E–field. We will do this in two steps. First we will write down coordinates and momenta operators corresponding to the oscillators of the discrete diagonal basis and look at the profile of the lump solution with respect to them. Next we will determine the open string time profile of the lump solution by projecting it onto the center of mass coordinates. Since the solutions with E–field are equivalent to euclidean solutions with imaginary B–field, before proceeding further, we will first give a brief summary of the construction of lump solutions in VSFT with transverse B–field.

Lump solutions with B-field

The solitonic lump solutions in VSFT in the presence of a constant transverse B-field were determined in [32]. The $\star$ product is defined as follows

$$_{123}\langle V_3|\Psi_1\rangle_1|\Psi_2\rangle_2 =_3 \langle \Psi_1 \star_m \Psi_2| \tag{2.109}$$

where the 3-string vertex V_3, with a constant B-field turned on along the 24^{th} and 25^{th} directions (in view of the D-brane interpretation, these directions are referred to as transverse), is

$$|V_3\rangle = |V_{3,\perp}\rangle \otimes |V_{3,||}\rangle. \tag{2.110}$$

$|V_{3,||}\rangle$ corresponds to the tangential directions while $|V_{3,\perp}\rangle$ is obtained from [26] by passing to zero modes oscillator basis and integrating over transverse momenta, see [32]

$$|V_{3,\perp}\rangle = \frac{\sqrt{2\pi b^3 \Delta}}{A^2(4a^2+3)} \exp\left[\frac{1}{2}\sum_{r,s=1}^{3}\sum_{N,M\geq 0} a_M^{(r)\alpha\dagger} \mathcal{V}^{rs}_{\alpha\beta,MN} a_N^{(s)\beta\dagger}\right] |0\rangle \otimes |\Omega_{b,\theta}\rangle_{123}. \tag{2.111}$$

In the following we will set $\alpha, \beta = 1, 2$ for simplicity of notation. $|\Omega_{b,\theta}\rangle$ is the vacuum with respect to the zero mode oscillators

$$a_0^{(r)\alpha} = \frac{1}{2}\sqrt{b}\hat{p}^{(r)\alpha} - i\frac{1}{\sqrt{b}}\hat{x}^{(r)\alpha}, \qquad a_0^{(r)\alpha\dagger} = \frac{1}{2}\sqrt{b}\hat{p}^{(r)\alpha} + i\frac{1}{\sqrt{b}}\hat{x}^{(r)\alpha}. \tag{2.112}$$

$\mathcal{V}^{rs}_{\alpha\beta,MN}$ are the Neumann coefficients with zero modes in a constant B-field background, which are symmetric under simultaneous exchange of all the three pairs of indices and cyclic in the string label indices (r, s) where $r, s = 4$ is identified with $r, s = 1$. Moreover $\Delta = \sqrt{\mathrm{Det}G}$, $G_{\alpha\beta}$ being the open string metric along the transverse directions (2.81). We have also introduced the notations

$$A = V_{00} + \frac{b}{2}, \qquad a = -\frac{\pi^2}{A}|B|. \tag{2.113}$$

The lump solution is given by

$$|S\rangle = |S_{||}\rangle \otimes \mathcal{N}\exp\left(-\frac{1}{2}\sum_{M,N\geq 0} a_M^{\alpha\dagger} \mathcal{S}_{\alpha\beta,MN} a_N^{\beta\dagger}\right) |0\rangle \otimes |\Omega_{b,\theta}\rangle, \tag{2.114}$$

where

$$\mathcal{N} = \frac{A^2(3+4a^2)}{\sqrt{2\pi b^3}(\mathrm{Det}G)^{\frac{1}{4}}} \mathrm{Det}(\mathcal{I} - \mathcal{X})^{\frac{1}{2}} \mathrm{Det}(\mathcal{I} + \mathcal{T})^{\frac{1}{2}}, \tag{2.115}$$

and

$$\mathcal{X} = C'\mathcal{V}^{11}, \qquad \mathcal{T} = C'\mathcal{S}, \qquad C' = (-1)^N \delta_{NM} \tag{2.116}$$

In (2.114) $|S_{||}\rangle$ corresponds to the longitudinal part of the lump solution and it is a zero momentum sliver.

In order for (2.114) to satisfy the projector equation, $\mathcal{T}$ and $\mathcal{X}$ should satisfy the relation[7]

$$(\mathcal{T} - 1)(\mathcal{X}\mathcal{T}^2 - (\mathcal{I} + \mathcal{X})\mathcal{T} + \mathcal{X}) = 0. \tag{2.117}$$

In the above formulae the α, β, N, M indices are implicit. This equation is solved by $\mathcal{T}_0$, $1/\mathcal{T}_0$ and 1, where

$$\mathcal{T}_0 = \frac{1}{2\mathcal{X}}\left(1 + \mathcal{X} - \sqrt{(1+3\mathcal{X})(1-\mathcal{X})}\right) \tag{2.118}$$

$\mathcal{T} = 1$ gives the identity state, whereas the first and the second solutions give the lump and the inverse lump, respectively. As stated previously, although the inverse lump solution was discarded in earlier works [33, 11], because of the bad behavior of its eigenvalues in the

[7]We limit ourselves to twist invariant projectors, but our analysis can be straightforwardly generalized to projectors of the kind [34]

oscillator basis, it is possible to make sense out of it by considering (2.117) as a relation between eigenvalues relative to twist definite eigenvectors. In particular, in the diagonal basis, the projector equation factorizes into the continuous and discrete contributions, which separately satisfy equation (2.117). Therefore, one can just invert (for example) the discrete eigenvalues of $\mathcal{T}$: dangerous $-$ signs under the square root in the energy densities of the solution are indeed avoided by counting the double multiplicity of these eigenvalues, which is required by twist invariance. See Appendix E for the spectroscopy of $\mathcal{X}$, and hence of $\mathcal{T}$.

Diagonal Coordinates and Momenta

In Appendix F, τ–twist definite oscillators of the diagonal basis are introduced. Due to the structure of Neumann coefficients it is natural to define the twist matrix as τC, where $\tau = \sigma^3$ acts on space–time indices, see appendices E and F for details. In the following C–parity will be always understood as τC–parity. Now let's define the following coordinates and momenta operators in terms of the twist even and twist odd parts of the discrete spectrum, (F.6)

$$\hat{X}_{\xi_i} = \frac{i}{\sqrt{2}}(e_{\xi_i} - e^\dagger_{\xi_i}) \qquad \hat{Y}_{\xi_i} = \frac{i}{\sqrt{2}}(o_{\xi_i} - o^\dagger_{\xi_i}) \tag{2.119}$$

which are hermitian by definition and have the following eigenstates

$$|X_i\rangle = \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}X_i^2 - \sqrt{2}iX_i e^\dagger_{\xi_i} + \frac{1}{2}e^\dagger_{\xi_i} e^\dagger_{\xi_i}} |\Omega_{e_i}\rangle \tag{2.120}$$

$$|Y_i\rangle = \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}Y_i^2 - \sqrt{2}iY_i o^\dagger_{\xi_i} + \frac{1}{2}o^\dagger_{\xi_i} o^\dagger_{\xi_i}} |\Omega_{o_i}\rangle. \tag{2.121}$$

We made the assumption that the vacuum factorizes as

$$|0\rangle \otimes |\Omega_{b,\theta}\rangle = \prod_{i=1}^{2} \prod_k |\Omega_i(k)\rangle \otimes |\Omega_{e_i}\rangle \otimes |\Omega_{o_i}\rangle \tag{2.122}$$

where $|\Omega_i(k)\rangle$, $|\Omega_{e_i}\rangle$ and $|\Omega_{o_i}\rangle$ are vacua with respect to the continuous, the twist even discrete and twist odd discrete oscillators, respectively.

The explicit (X_i, Y_i) dependence of the lump state (2.30) can be obtained by projecting it onto the eigenstates $|X_i, Y_i\rangle$. After re-writing (2.30) in terms of the diagonal basis oscillators and performing the projection (see Appendix F), it follows

$$\langle X_i, Y_i | S\rangle = \frac{1}{\pi^2[1 + t_d(\eta_1)][1 + t_d(\eta_2)]} \exp\frac{1}{2}\left[\frac{t_d(\eta_1) - 1}{t_d(\eta_1) + 1}(X_1^2 + Y_1^2)\right.$$
$$\left. + \frac{t_d(\eta_2) - 1}{t_d(\eta_2) + 1}(X_2^2 + Y_2^2)\right] |S\rangle_c \otimes |S_{||}\rangle. \tag{2.123}$$

$|S\rangle_c$ is given by (F.11) with only continuous spectrum oscillators and $t_d(\eta_i) = e^{-|\eta_i|}$ are the discrete eigenvalues of $\mathcal{T}$ corresponding to the eigenvalue $\xi(\eta_i)$ of the operator $C'\mathcal{U}$.

In (2.123) the directions α, β are completely mixed. As a matter of fact, it is not apparent at this stage which of these variables (X_i, Y_i) contain the information about the center of mass time dependence of the lump. To make this clear let's recall the non-diagonal basis oscillators and write the coordinates and the momenta operators as

$$\hat{X}^{\alpha}_{N} = \frac{i}{\sqrt{2}}(a^{\alpha}_{N} - a^{\alpha\dagger}_{N}) \qquad \hat{P}^{\alpha}_{N} = \frac{1}{\sqrt{2}}(a^{\alpha}_{N} + a^{\alpha\dagger}_{N}). \tag{2.124}$$

In order to get the relation between these operators and the corresponding diagonal operators we have defined above, we need to re-write the diagonal basis oscillators in terms of the non-diagonal ones. In doing so, one has to be careful about taking the complex conjugate of the eigenstates, as we are dealing with hermitian rather then symmetric matrices. Taking this fact into account and using some results of Appendix F, we obtain

$$e_{\xi_i} = \frac{1}{\sqrt{2}}\sum_{N=0}^{\infty}(V_N^{(\xi_i)\alpha} + V_N^{(\bar{\xi}_i)\alpha})a_{N,\alpha} \qquad e^{\dagger}_{\xi_i} = \frac{1}{\sqrt{2}}\sum_{N=0}^{\infty}(\bar{V}_N^{(\xi_i)\alpha} + \bar{V}_N^{(\bar{\xi}_i)\alpha})a^{\dagger}_{N,\alpha} \tag{2.125}$$

$$o_{\xi_i} = \frac{-i}{\sqrt{2}}\sum_{N=0}^{\infty}(V_N^{(\xi_i)\alpha} - V_N^{(\bar{\xi}_i)\alpha})a_{N,\alpha} \qquad o^{\dagger}_{\xi_i} = \frac{i}{\sqrt{2}}\sum_{N=0}^{\infty}(\bar{V}_N^{(\xi_i)\alpha} - \bar{V}_N^{(\bar{\xi}_i)\alpha})a^{\dagger}_{N,\alpha} \tag{2.126}$$

and similar relations for the continuous spectrum oscillators. Hence, the diagonal coordinates and momenta can be written as

$$\hat{X}_{\xi_i} = \sqrt{2}\sum_{N=0}^{\infty} V_{2N}^{\xi_i,1}\hat{X}^1_{2N} + V_{2N+1}^{\xi_i,2}\hat{P}^2_{2N+1} \tag{2.127}$$

$$\hat{Y}_{\xi_i} = \sqrt{2}\sum_{N=0}^{\infty} V_{2N+1}^{\xi_i,1}\hat{P}^1_{2N+1} - iV_{2N}^{\xi_i,2}\hat{X}^2_{2N} \tag{2.128}$$

Now, to make the center of mass time dependence of the solution explicit, we need to extract the zero modes from these operators. Let's write the zero mode coordinate and momentum operators by introducing the b parameter as

$$\hat{X}^{\alpha}_{0} = \frac{i}{\sqrt{b}}(a^{\alpha}_{0} - a^{\alpha\dagger}_{0}) \qquad \hat{P}^{\alpha}_{0} = \frac{\sqrt{b}}{2}(a^{\alpha}_{0} + a^{\alpha\dagger}_{0}). \tag{2.129}$$

This gives

$$\hat{X}_{\xi_i} = \sqrt{2}\left[V_0^{\xi_i,1}\sqrt{\frac{2}{b}}X^1_0 + \sum_{n=1}^{\infty} V_{2n}^{\xi_i,1}\hat{X}^1_{2n} + V_{2n-1}^{\xi_i,2}\hat{P}^2_{2n-1}\right], \tag{2.130}$$

$$\hat{Y}_{\xi_i} = \sqrt{2}\left[V_0^{\xi_i,2}\sqrt{\frac{2}{b}}X^2_0 + \sum_{n=1}^{\infty} V_{2n-1}^{\xi_i,1}\hat{P}^1_{2n-1} - iV_{2n}^{\xi_i,2}\hat{X}^2_{2n}\right]. \tag{2.131}$$

Since our aim is to obtain the localization in time by making the inverse Wick rotation on direction 1, we see that it is X_{ξ_i} that contains the time coordinate, which we have to compare with the string center of mass time (see below).

Projection on the center of mass coordinates

In order to obtain the open string time profile of the lump solution, we need to project it onto the center of mass coordinates of the string. The center of mass position operator is given by

$$\hat{x}_{cm,\alpha} = \frac{i}{\sqrt{b}}(a_{0,\alpha} - a^\dagger_{0,\alpha}) \tag{2.132}$$

and its eigenstate is

$$|X_{CM}\rangle = \sqrt{\frac{2\Delta}{\pi b}}\, e^{-\frac{1}{b}x_\alpha x^\alpha - \frac{2}{\sqrt{b}}ix_\alpha a_0^{\alpha\,\dagger} + \frac{1}{2}a^\dagger_{0,\alpha}a_0^{\alpha\,\dagger}}|\Omega_{\theta,b}\rangle. \tag{2.133}$$

One can project the lump on this state to obtain the center of mass time profile. However, for reasons that will be clear later, we will first project on the Y_i momenta ($|\Lambda\rangle = \langle Y_1, Y_2|S\rangle$),

$$|\Lambda\rangle = \frac{\mathcal{N}}{\pi\sqrt{[1+t_d(\eta_1)][1+t_d(\eta_2)]}}\exp\frac{1}{2}\left[\frac{t_d(\eta_1)-1}{t_d(\eta_1)+1}Y_1^2 + \frac{t_d(\eta_2)-1}{t_d(\eta_2)+1}Y_2^2\right]$$
$$\times\exp-\frac{1}{2}\left[e^\dagger_{\xi_i}e^\dagger_{\xi_i}t_d(\eta_i) + \int_{-\infty}^{\infty}dka_i^\dagger(k)a^\dagger_{i+1}(-k)t_c(k)\right]|\Omega_e\rangle\otimes|\Omega_c\rangle\otimes|S_{||}\rangle. \tag{2.134}$$

Where we have used the notation

$$|\Omega_e\rangle = \prod_{i=1}^{2}|\Omega_{e_i}\rangle, \qquad |\Omega_c\rangle = \prod_{i=1}^{2}\prod_{k}|\Omega_i(k)\rangle. \tag{2.135}$$

Taking equation (2.126) and the corresponding relations for the continuous spectrum oscillators, equation (2.134) can be rewritten as

$$|\Lambda\rangle = \frac{\mathcal{N}}{\pi\sqrt{[1+t_d(\eta_1)][1+t_d(\eta_2)]}}\exp\frac{1}{2}\left[\frac{t_d(\eta_1)-1}{t_d(\eta_1)+1}Y_1^2 + \frac{t_d(\eta_2)-1}{t_d(\eta_2)+1}Y_2^2\right]$$
$$\times\exp\left[-\frac{1}{2}a^\dagger_{0,\alpha}\hat{S}^{\alpha\beta}_{00}a^\dagger_{0,\beta} - a^\dagger_{0,\alpha}S^\alpha_0 - \frac{1}{2}a^\dagger_{n,\alpha}\hat{S}^{\alpha\beta}_{nm}a^\dagger_{m,\beta}\right]|\hat{\Omega}_{b,\theta}\rangle\otimes|S_{||}\rangle, \tag{2.136}$$

where $|\hat{\Omega}_{b,\theta}\rangle = |\Omega_e\rangle\otimes|\Omega_c\rangle$ and

$$\hat{S}^{\alpha\beta}_{00} = \sum_{i=1}^{2}V_0^{(\xi_i^+)\alpha}\bar{V}_0^{(\xi_i^+)\beta}t_d(\eta_i) + \int_{-\infty}^{\infty}dkt(k)V_0^{i,\alpha}(k)\bar{V}_0^{i,\beta}(k) \tag{2.137}$$

$$S^\alpha_0 = \sum_{i=1}^{2}\sum_{n=1}\left[V_0^{(\xi_i^+)\alpha}\bar{V}_n^{(\xi_i^+)\beta}t_d(\eta_i) + \int_{-\infty}^{\infty}dkt(k)V_0^{i,\alpha}(k)\bar{V}_n^{i,\beta}(k)\right]a^\dagger_{n,\beta} = \hat{S}^{\alpha\beta}_{0n}a^\dagger_{n,\beta} \tag{2.138}$$

$$\hat{S}^{\alpha\beta}_{nm} = \sum_{i=1}^{2}(-1)^nV_n^{(\xi_i^+)\alpha}\bar{V}_m^{(\xi_i^+)\beta}t_d(\eta_i) + \int_{-\infty}^{\infty}dkt(k)(-1)^nV_n^{i,\alpha}(k)\bar{V}_m^{i,\beta}(k) \tag{2.139}$$

with $V_N^{(\xi_i^+)\alpha}$ being the twist even combination of the discrete eigenstates, see appendix F. Now let's project onto the center of mass coordinates

$$\langle X_{CM}|\Lambda\rangle = \frac{\mathcal{N}}{\pi\sqrt{[1+t_d(\eta_1)][1+t_d(\eta_2)]}}\exp\frac{1}{2}\left[\frac{t_d(\eta_1)-1}{t_d(\eta_1)+1}Y_1^2+\frac{t_d(\eta_2)-1}{t_d(\eta_2)+1}Y_2^2\right]$$

$$\times\frac{\sqrt{\frac{2\Delta}{\pi b}}}{\sqrt{[1+s_1][1+s_2]}}\exp\frac{1}{b}\left[\frac{s_1-1}{s_1+1}x_1x^1+\frac{s_2-1}{s_2+1}x_2x^2+2i\sqrt{b}\left(\frac{S_{0,1}x^1}{1+s_1}+\frac{S_{0,2}x^2}{1+s_2}\right)\right]$$

$$\times\exp\left[-\frac{1}{2}a^\dagger_{n,\alpha}\left(\hat{S}^{\alpha\beta}_{nm}-\frac{\hat{S}^{\alpha}_{n0,1}\hat{S}^{1\beta}_{0m}}{1+s_1}-\frac{\hat{S}^{\alpha}_{n0,2}\hat{S}^{2\beta}_{0m}}{1+s_2}\right)a^\dagger_{m,\beta}\right]|0\rangle\otimes|S_{||}\rangle \tag{2.140}$$

where

$$s_1 = 2\Delta[g_d^2(\eta_1,\eta_2)t_d(\eta_1)+g_d^2(\eta_2,\eta_1)t_d(\eta_2)]+\Delta\int_{-\infty}^{\infty}dk\ t_c(k)[g_c^2(k)+g_c^2(-k)),$$

$$s_2 = \Delta\int_{-\infty}^{\infty}dk\ t_c(k)[g_c^2(k)+g_c^2(-k)], \tag{2.141}$$

$t_c(k)=-e^{-\pi|k|/2}$ is the eigenvalue of $\mathcal{T}$ in the continuous spectrum, see Appendix E for the definition of the remaining terms which enter in the last two equations. The inverse Wick-rotation along direction 1 of (2.140) should give us a time-localized solution. It depends on two parameters, b and a, which can be expressed in terms of (η_1,η_2), through the eigenvalues equations (E.16) . Let's now take a look at every term in this solution and analyze it for different values of the such parameters.

In the Wick-rotated solution, to get time-localization, the term $-\frac{1}{b}\frac{s_1-1}{s_1+1}$ should be negative. We cannot achieve this using the conventional lump, since in this case $-1<s_1<1$. To correct this, as anticipated, we need to invert one or two discrete eigenvalues, ($t_d(\eta_1)$ or/and $t_d(\eta_2)$). In this case one can easily show that $1<s_1<\infty$ and we get the desired behavior. Given the possibility of inverting one or two eigenvalues, it might seem that there is some arbitrariness in our procedure. Actually there is none, since the cancellation of the potentially divergent terms when $b\to\infty$ (see below), requires the inversion of only one eigenvalue. In addition, time localization in small b regime requires the inversion of the eigenvalue of $\mathcal{T}$ corresponding to the greater between η_1 and η_2 (η_2 in our conventions). From now on we will then consider a solution in which $t_d(\eta_2)$ is inverted, i.e. $t_d(\eta_2)\to t_d^{-1}(\eta_2)$.

Next, look at the term $\frac{s_2-1}{s_2+1}x_2x^2$. Due to the $\langle Y_1,Y_2|$ projection, it gets a contribution only from the continuous spectrum, which is always negative and in the range $(-1,0)$. As a result, this second term is always negative and gives localization in the transverse space direction.

Now we would like to point out some facts about the two parameters on which our solution depends. We pointed out that the inverse of the parameter b, for large b, may play the role of Sen's $\tilde{\lambda}$ near zero. Here again we can repeat the same argument. Note however that in taking b to infinity we should keep a vanishing, see (2.113), since we cannot overcome the critical value $|E|_c=\frac{1}{2\pi}$. For this reason the result of taking $b\to\infty$ is insensible of the value of the E-field, making this limit completely commutative.

As it is justified in Appendix G, the proper way to send b to ∞ is to take $\eta_1 \approx \eta_2 \to \infty$ keeping $\eta_1 < \eta_2$. In this case one can easily see that

$$s_1 \approx \eta_2 \log \eta_1 \eta_2 + t_c(k_0 \approx 0), \qquad s_2 \approx t_c(k_0 \approx 0) \tag{2.142}$$

with k_0 as defined in Appendix G. Note that $t_c(k_0 \approx 0) = -1$. This is so because the E–field cannot scale to infinity due to existence of critical value. Then it follows

$$\lim_{\eta_1,\eta_2 \to \infty} \frac{s_1 - 1}{s_1 + 1} = 1, \qquad \lim_{\eta_1,\eta_2 \to \infty} \frac{s_2 - 1}{s_2 + 1} = -\infty \tag{2.143}$$

As justified in Appendix G.1.2, in this limit $\hat{S}_{n0}^{\alpha\beta(c)} = 0$ so that the oscillating term in (2.140) receives a contribution only from the discrete part. It is also pointed out that the discrete contribution vanishes except for $\alpha = \beta = 1$, which is the only non trivial contribution to the oscillating term. Moreover, we have

$$\lim_{\eta_1,\eta_2 \to \infty} \frac{\Delta \hat{S}_{n0}^{11}}{s_1 + 1} = (-1)^n \lim_{\eta_1,\eta_2 \to \infty} \frac{1}{2\sqrt{\log \eta_1 \eta_2}} = 0, \tag{2.144}$$

Therefore, the oscillating term in (2.140) vanishes when $b \to \infty$.

Now let's consider the non-zero mode terms, i.e, the last line in (2.140). In the $b \to \infty$ limit it is clear that $V_n^{(\xi_i^+)\alpha}$ vanishes for $\alpha = 2$ and $n \geq 1$. Therefore, the contribution of the discrete spectrum to $\hat{S}_{nm}^{\alpha\beta}$ is zero for α or $\beta = 2$ and $n, m \geq 1$. However, for $\alpha = \beta = 1$ this is not true and there are potentially divergent contributions from the discrete spectrum. We are now going to show that these divergences cancel and the expression

$$\check{S}_{nm}^{11} = \hat{S}_{nm}^{11(c)} + \hat{S}_{nm}^{11(d)} - \frac{\hat{S}_{n0,1}^{1(d)} \hat{S}_{0m}^{11(d)}}{1 + s_1}. \tag{2.145}$$

is finite when $b \to \infty$.

To this end we notice that, inverting only $t_d(\eta_2)$ but taking both η_1 and η_2 to infinity, the different terms which enter in the above expression have the following behaviors

$$\begin{aligned}
1 + s_1 &\approx \Delta t_d^{-1}(\eta_2) V_0^{(\xi_2^+),1} \bar{V}_0^{(\xi_2^+),1}, \\
\hat{S}_{n0,1}^{1(d)} &\approx \Delta (-1)^n t_d^{-1}(\eta_2) V_n^{(\xi_2^+),1} \bar{V}_0^{(\xi_2^+),1}, \\
\hat{S}_{n0}^{11(d)} &\approx (-1)^n t_d^{-1}(\eta_2) V_n^{(\xi_2^+),1} \bar{V}_0^{(\xi_2^+),1}, \\
\hat{S}_{nm}^{11(d)} &\approx (-1)^n t_d^{-1}(\eta_2) V_n^{(\xi_2^+),1} \bar{V}_m^{(\xi_2^+),1},
\end{aligned} \tag{2.146}$$

Note that $t_d^{-1}(\eta_2) = e^{|\eta_2|}$ gives a divergent contribution as $\eta_2 \to \infty$. However, using these results in eq.(2.145), it is easy to see that the divergent terms cancel and we are left with $\check{S}_{nm}^{11} = \hat{S}_{nm}^{11(c)}$. This, combined with the fact that for $\alpha = 2$ or $\beta = 2$ we have $\hat{S}_{NM}^{\alpha\beta(d)} = 0$, leads us to the conclusion that $\check{S}_{nm}^{\alpha\beta} = \hat{S}_{nm}^{\alpha\beta(c)} + O\left(\frac{1}{b}\right)$. It is also verified in Appendix G that $\hat{S}_{nm}^{11(c)} = \hat{S}_{nm}^{22(c)} = S_{nm}$.

This also shows that is not possible to invert both the discrete eigenvalues and obtain the same cancellation. Indeed, if we invert both, the term $\hat{S}^{1(d)}_{n0,1}\hat{S}^{11(d)}_{0m}$ contains mixed terms like $[t^{-1}(\eta_1)V_n^{(\xi_1^+),1}\bar{V}_0^{(\xi_1^+),1}][t^{-1}(\eta_2)V_m^{(\xi_2^+),1}\bar{V}_0^{(\xi_2^+),1}]$, for which we cannot find a counter term in $\hat{S}^{11(d)}_{nm}$ to cancel it. As a result we will not be able to get a regular time and space localized solution, since these terms diverge in the limit $\eta_1, \eta_2 \to \infty$.

After all these remarks, we can write the space-time localized solution in the $b \to \infty$ limit as

$$\lim_{b\to\infty} \langle X_{CM}|\Lambda\rangle_{Wick} = N(Y_1, Y_2) \lim_{b\to\infty} e^{-\frac{\Delta}{b}(x^0)^2} e^{-\epsilon(b)(x^2)^2}|S\rangle \tag{2.147}$$

where $|S\rangle$ is the space-time independent VSFT solution (the sliver). Note that time dependence completely disappears in this limit . A remark is in order for the quantity $\epsilon(b)$, this number is given by, see (2.140)

$$\epsilon(b) = \frac{\Delta}{b}\frac{s_2 - 1}{s_2 + 1} \tag{2.148}$$

a numerical analysis shows that this becomes vanishing as $b \to \infty$. One can indeed easily check (numerically) that the $\frac{1}{b}$ correction to $\frac{s_2(b)-1}{s_2(b)+1}$ diverges. This in turn implies that the loss of time dependence is accompanied by loss of transverse space dependence, giving a resulting zero momentum state (the D25–sliver). Therefore, taking b to infinity is like sitting at the original unstable vacuum (the D25–brane), which is the same situation as setting Sen's $\tilde{\lambda}$ to zero.

Another remark we would like to make is about small b limit, which we can get by taking $\eta_1 \to 0$ and keeping η_2 finite. Given that the large b limit corresponds to Sen's $\tilde{\lambda}$ near zero (i.e it represents the unstable vacuum), it is natural to think that the small b limit corresponds to $\tilde{\lambda}$ near $\frac{1}{2}$ (or the stable vacuum). As a matter of fact, taking this limit of b one gets the 0 state, which is also obtained in the $x_0 \to \infty$ limit and corresponds to the stable vacuum to which the D-brane decays. This can be seen by noting that, in this case, $V_0^\alpha(k) \to 0$, whereas $V_n^\alpha(k)$ for $n \geq 1$ have a finite nonvanishing limit. As a result s_1 do not get a contribution from the continuous spectrum and $s_2 = 0$. Then, it follows

$$-\frac{\Delta}{b}\frac{s_1 - 1}{s_1 + 1} \approx -\frac{\Delta}{\eta_1}\left(\left|\frac{s_1 - 1}{s_1 + 1}\right|\right), \qquad \frac{\Delta}{b}\frac{s_2 - 1}{s_2 + 1} \approx -\frac{\Delta}{\eta_1} \tag{2.149}$$

where we have used $(b \approx \eta_1)$, $(s_1 \approx 1 + O(\sqrt{\eta_1}))$ in the limit $\eta_1 \to 0$ and η_2 finite. These are results one can easily obtain from appendix G.2. For $\Delta \neq 0$ both of these terms gives a negative infinity in the exponent and suppress everything in front to give us the 0 state which corresponds to the stable vacuum. However, the case $\Delta = 0$ should be handled with care. In this case, one can send Δ and η_1 to zero, in such a way that the ratio $\frac{\Delta}{\eta_1}$ remains finite. As a result the time dependence will be lost while the solution is still space localized. One should compare this with the time-independent solution obtained when we send Sen's $\tilde{\lambda}$ to $\frac{1}{2}$ and, at the same time, tune the E–field to its critical value, [25], obtaining a static fundamental strings background.

2.4 Fundamental Strings in VSFT

As we argued above, it has been possible to find in VSFT an exact time-dependent solution with the characteristics of a rolling tachyon [16]. A rolling tachyon describes in various languages (effective field theory, BCFT, SFT) the decay of unstable D-branes. It is by now clear that the final product of a brane decay is formed by massive closed string states. However, it has been shown that, in the presence of a background electric field also (macroscopic) fundamental strings appear as final products of a brane decay. Now, since our aim is to be able to describe a brane decay in the framework of VSFT, we must show first of all that such fundamental strings exist as solutions of VSFT. Showing their existence will be the aim of this section.

2.4.1 Constructing new solutions

In this section we would like to show how qualitatively new solutions to (2.13) can be constructed by accretion of infinite many lumps. Let us start from a lump solution representing a D0–brane: it has a Gaussian profile in all space directions, the form of the string field – let us denote it $|\Xi_0'\rangle$ – will be the same as (2.20) with S replaced by S', while the $\star$–product will be determined by the primed three strings vertex (2.94). Let us pick one particular space direction, say the α–th. For simplicity in the following we will drop the corresponding label from the coordinate $\hat{x}^\alpha$, momenta $\hat{p}^\alpha$ and oscillators a^α along this direction. Next we need the same solution displaced by an amount s in the positive x direction (x being the eigenvalue of $\hat{x}$). The appropriate solution has been constructed by Rastelli, Sen and Zwiebach, [35]:

$$|\Xi_0'(s)\rangle = e^{-is\hat{p}}|\Xi_0'\rangle \tag{2.150}$$

It satisfies $|\Xi_0'(s)\rangle * |\Xi_0'(s)\rangle = |\Xi_0'(s)\rangle$. Eq.(2.150) can be written explicitly as

$$|\Xi_0'(s)\rangle = \mathcal{N}' e^{-\frac{s^2}{2b}(1-S_{00}')} \exp\left(-\frac{is}{\sqrt{b}}((1-S')\cdot a^\dagger)_0\right) \exp\left(-\frac{1}{2}a^\dagger\cdot S'\cdot a^\dagger\right)|\Omega_b\rangle \tag{2.151}$$

where $((1-S')\cdot a^\dagger)_0 = \sum_{N=0}^\infty((1-S')_{0N}a_N^\dagger)$ and $a^\dagger\cdot S'\cdot a^\dagger = \sum_{N,M=0}^\infty a_N^\dagger S_{NM}' a_M^\dagger$; $\mathcal{N}'$ is the $|\Xi_0'\rangle$ normalization constant. Moreover one can show that

$$\langle\Xi_0'(s)|\Xi_0'(s)\rangle = \langle\Xi_0'|\Xi_0'\rangle \tag{2.152}$$

The meaning of this solution is better understood if we make its space profile explicit by contracting it with the coordinate eigenfunction

$$|\hat{x}\rangle = \left(\frac{2}{\pi b}\right)^{\frac{1}{4}} \exp\left(-\frac{x^2}{b} - i\frac{2}{\sqrt{b}}a_0^\dagger x + \frac{1}{2}a_0^\dagger a_0^\dagger\right)|\Omega_b\rangle \tag{2.153}$$

The result is

$$\langle\hat{x}|\Xi_0'(s)\rangle = \left(\frac{2}{\pi b}\right)^{\frac{1}{4}} \frac{\mathcal{N}'}{\sqrt{1+S_{00}'}} e^{-\frac{1-S_{00}'}{1+S_{00}'}\frac{(x-s)^2}{b}} e^{-\frac{2i}{\sqrt{b}}\frac{x-s}{1+S_{00}'}S_{0m}'a_m^\dagger} e^{-\frac{1}{2}a_n^\dagger W_{nm} a_m^\dagger}|0\rangle \tag{2.154}$$

where $W_{nm} = S_{nm}' - \frac{S_{n0}'S_{0m}'}{1+S_{00}'}$. It is clear that (2.154) represents the same Gaussian profile as $|\Xi_0'\rangle = |\Xi_0'(0)\rangle$ shifted away from the origin by s.

It is important to remark now that two such states $|\Xi'_0(s)\rangle$ and $|\Xi'_0(s')\rangle$ are $\star$–orthogonal and bpz–orthogonal provided that $s \neq s'$. For we have

$$|\Xi'_0(s)\rangle \star |\Xi'_0(s')\rangle = e^{-\mathcal{C}(s,s')}|\Xi'_0(s,s')\rangle \tag{2.155}$$

where the state $|\Xi'_0(s,s')\rangle$ becomes proportional to $|\Xi'_0(s)\rangle$ when $s = s'$ and need not be explicitly written down otherwise; while

$$\mathcal{C}(s,s') = -\frac{1}{2b}\left[(s^2+s'^2)\left(\frac{T'(1-T')}{1+T'}\right)_{00} + ss'\left(\frac{(1-T')^2}{1+T'}\right)_{00}\right] \tag{2.156}$$

The quantity $\left(\frac{T'(1-T')}{1+T'}\right)_{00}$ can be evaluated by using the basis of eigenvectors of X' and T', [13, 15, 36]:

$$\begin{aligned}\left(\frac{T'(1-T')}{1+T'}\right)_{00} &= 2\int_0^\infty dk\,(V_0(k))^2\frac{t(k)(1-t(k))}{1+t(k)}\\ &+ \left(V_0^{(\xi)}V_0^{(\xi)} + V_0^{(\bar\xi)}V_0^{(\bar\xi)}\right)\frac{e^{-|\eta|}(1-e^{-|\eta|})}{1+e^{-|\eta|}}\end{aligned} \tag{2.157}$$

The discrete spectrum part of the RHS of (2.157) is just a number. Let us concentrate on the continuous spectrum contribution. We have $t(k) = -\exp(-\frac{\pi|k|}{2})$. Near $k=0$, $V_0(k) \sim \frac{1}{2}\sqrt{\frac{b}{2\pi}}$ and the integrand $\sim -\frac{b}{2\pi^2}\frac{1}{k}$, therefore the integral diverges logarithmically, a singularity we can regularize with an infrared cutoff ϵ. Taking the signs into account we find that the RHS of (2.157) goes like $\frac{b}{2\pi^2}\log\epsilon$ as a function of the cutoff. Similarly one can show that $\left(\frac{(1-T')^2}{1+T'}\right)_{00}$ goes like $-\frac{b}{\pi^2}\log\epsilon$. Since for $s \neq s'$, $s^2+s'^2 > 2ss'$, we can conclude that $\mathcal{C}(s,s') \sim -c\log\epsilon$, where c is a positive number. Therefore, when we remove the cutoff, the factor $e^{-\mathcal{C}(s,s')}$ vanishes, so that (2.155) becomes a $\star$–orthogonality relation. Notice that the above logarithmic singularities in the two pieces in the RHS of (2.157) neatly cancel each other when $s = s'$ and we get the finite number

$$\mathcal{C}(s,s) = -\frac{s^2}{2b}(1-S'_{00})$$

In conclusion we can write

$$|\Xi'_0(s)\rangle * |\Xi'_0(s')\rangle = \hat\delta(s,s')|\Xi'_0(s)\rangle \tag{2.158}$$

where $\hat\delta$ is the Kronecker (not the Dirac) delta function.

Similarly one can prove that

$$\langle\Xi'_0(s')|\Xi'_0(s)\rangle = \frac{\mathcal{N}^2}{\sqrt{\det(1-S'^2)}}\,e^{-\frac{s^2}{b}(1-S'_{00})}e^{\frac{1}{2b}\left[(s^2+s'^2)\left(\frac{S'(1-S')}{1+S'}\right)_{00}+2ss'\left(\frac{1-S'}{1+S'}\right)_{00}\right]} \tag{2.159}$$

We can repeat the same argument as above and conclude that

$$\langle\Xi'_0(s')|\Xi'_0(s)\rangle = \hat\delta(s,s')\,\langle\Xi'_0|\Xi'_0\rangle \tag{2.160}$$

After the above preliminaries, let us consider a sequence $s_1, s_2, \ldots$ of distinct real numbers and the corresponding sequence of displaced D0–branes $|\Xi'_0(s_n)\rangle$. Due to the property (2.158) also the string state

$$|\Lambda\rangle = \sum_{n=1}^{\infty} |\Xi'_0(s_n)\rangle \tag{2.161}$$

is a solution to (2.13): $|\Lambda\rangle * |\Lambda\rangle = |\Lambda\rangle$. To figure out what it represents let us study its space profile. To this end we must sum all the profiles like (2.154) and then proceed to a numerical evaluation. In order to get a one dimensional object, we render the sequence $s_1, s_2, \ldots$ dense, say, in the positive x–axis so that we can replace the summation with an integral. The relevant integral is

$$\int_0^{\infty} ds\, e^{-\alpha(x-s)^2 - i\beta(x-s)} = \frac{\sqrt{\pi}}{2\sqrt{\alpha}}\left(e^{-\frac{\beta^2}{4\alpha}}\left(1+\mathrm{Erf}\left(\frac{i\beta}{2\sqrt{\alpha}}+\sqrt{\alpha}x\right)\right)\right) \tag{2.162}$$

where Erf is the error function and

$$\alpha = \frac{1}{b}\frac{1-S'_{00}}{1+S'_{00}}, \qquad \beta = \frac{2}{\sqrt{b}}\frac{S'_{0m}a^{\dagger}_m}{1+S'_{00}}$$

Of course (2.162) is a purely formal expression, but it becomes meaningful in the $\alpha' \longrightarrow 0$ limit. As usual, [21], we parametrize this limit with a dimensionless parameter ϵ and take $\epsilon \longrightarrow 0$. Using the results of [36, 21], one can see that $\alpha \sim 1/\epsilon$, $\beta \sim 1/\sqrt{\epsilon}$, so that $\beta/\sqrt{\alpha}$ tends to a finite limit. Therefore, in this limit, we can disregard the first addend in the argument of Erf. Then, up to normalization, the space profile of $|\Lambda\rangle$ is determined by

$$\frac{1}{2}\left(1+\mathrm{Erf}(\sqrt{\alpha}x)\right) \tag{2.163}$$

In the limit $\epsilon \longrightarrow 0$ this factor tends to a step function valued 1 in the positive real x–axis and 0 in the negative one. Of course a similar result can be obtained numerically to any degree of accuracy by using a dense enough discrete $\{s_n\}$ sequence.

Another way of getting the same result is to use the recipe of [21] first on (2.154). In this way the middle exponential disappears, while the first exponential is regularized by hand (remember that $S'_{00} \longrightarrow -1$ as $\epsilon \longrightarrow 0$), so we replace S'_{00} by a parameter $\mathfrak{s}$ and keep it $\neq -1$. Now it is easy to sum over s_n. Again we replace the summation by an integration and see immediately that the space profile becomes the same as (2.163).

Let us stress that the derivation of the space profile in the low energy regime we have given above is far from rigorous. This is due to the very singular nature of the lump in this limit, first pointed out by [21]. A more satisfactory derivation will be provided in the next section after introducing a background B-field.

In summary, the state $|\Lambda\rangle$ is a solution to (2.13), which represents, in the low energy limit, a one–dimensional object with a constant profile that extends from the origin to infinity in the x–direction. Actually the initial point could be any finite point of the x–axis, and it is not hard to figure out how to construct a configuration that extend from $-\infty$ to $+\infty$. How should we interpret these condensate of D0–branes? In the absence of supersymmetry it is not easy to distinguish between D–strings and F–strings (see, for instance, [37] for a comparison), however in the last section we will provide some evidence that the one–dimensional solutions of the type $|\Lambda\rangle$ can be interpreted as fundamental strings. This kind of objects are very well–known in string theory as classical solutions, [25, 38, 39, 40, 41], see also [42, 43, 44]. For the time being let us notice that, due to (2.160),

$$\langle\Lambda|\Lambda\rangle = \sum_{n,m=1}^{\infty} \langle\Xi_0'(s_n)|\Xi_0'(s_m)\rangle = \sum_{n=1}^{\infty} \langle\Xi_0'|\Xi_0'\rangle \tag{2.164}$$

It follows that the energy of the solution is infinite. Such an (unnormalized) infinity is a typical property of fundamental string solutions, see [38].

2.4.2 An improved construction

In this section we would like to justify some of the passages utilized in discussing the space profile of the fundamental string solution in section 2.4.1. The problems in the last section are linked to the well–known singularity of the lump space profile, [21], which arises in the low energy limit ($\epsilon \to 0$) and renders some of the manipulations rather slippery. The origin of this singularity is the denominator $1 + S_{00}'$ that appears in many exponentials. Since, when $\epsilon \to 0$, $S_{00}' \to -1$ the exponentials are ill–defined because the series expansions in $1/\epsilon$ are. The best way to regularize them is to introduce a constant background B–field, [26, 45, 46]. The relevant formulas can be found in [32]. For the purpose of this paper we introduce a B-field along two space directions, say x and y (our aim is to regularize the solution in the x direction, but, of course, there is no way to avoid involving in the process another space direction).

Let us use the notation x^α with $\alpha = 1, 2$ to denote x, y and let us denote

$$G_{\alpha\beta} = \Delta\delta_{\alpha\beta}, \qquad \Delta = 1 + (2\pi B)^2 \tag{2.165}$$

the open string metric. As is well–known, as far as lump solutions are concerned, there is an isomorphism of formulas with the ordinary case by which X', S', T' are replaced, respectively, by $\mathcal{X}, \mathcal{S}, \mathcal{T}$, which explicitly depend on B. One should never forget that the latter matrices involve two space directions. We will denote by $|\hat{\Xi}_0\rangle$ the D0–brane solution in the presence of the B-field.

Without writing down all the details, let us see the significant changes. Let us replace formula (2.150) by

$$|\hat{\Xi}_0(\{s^\alpha\})\rangle = e^{-is^\alpha \hat{p}_\alpha}|\hat{\Xi}_0\rangle \tag{2.166}$$

It satisfies $|\hat{\Xi}_0(s)\rangle * |\hat{\Xi}_0(s)\rangle = |\hat{\Xi}_0(s)\rangle$ and $\langle\hat{\Xi}_0(s)|\hat{\Xi}_0(s)\rangle = \langle\hat{\Xi}_0|\hat{\Xi}_0\rangle$. Instead of (2.153) we have

$$|\{\hat{x}^\alpha\}\rangle = \left(\frac{2\Delta}{\pi b}\right)^{\frac{1}{2}} \exp\left[\left(-\frac{x^\alpha x^\beta}{b} - i\frac{2}{\sqrt{b}}a_0^{\alpha\dagger}x^\beta + \frac{1}{2}a_0^{\alpha\dagger}a_0^{\beta\dagger}\right)G_{\alpha\beta}\right]|\Omega_b\rangle \tag{2.167}$$

Next we have

$$\langle\{\hat{x}^\alpha\}|\hat{\Xi}_0(s)\rangle = \frac{\left(\frac{2\Delta}{\pi b}\right)^{\frac{1}{2}}\hat{\mathcal{N}}}{\sqrt{\det(1+\mathcal{S}_{00})}}\exp\left[-\frac{1}{b}(x^\alpha - s^\alpha)\left(\frac{1-\mathcal{S}_{00}}{1+\mathcal{S}_{00}}\right)_{\alpha\beta}(x^\beta - s^\beta)\right]$$

$$- \frac{2i}{\sqrt{b}}(x^\alpha - s^\alpha)(1+\mathcal{S}_{00})_{\alpha\beta}\mathcal{S}_{0m}{}^\beta{}_\gamma\, a_m^{\gamma\dagger}\Big] \exp\left[-\frac{1}{2}a_n^{\alpha\dagger}\mathcal{W}_{nm,\alpha\beta}a_m^{\beta\dagger}\right]|0\rangle \tag{2.168}$$

where $\det(1+\mathcal{S}_{00})$ means the determinant of the 2x2 matrix $(1+\mathcal{S}_{00})_{\alpha\beta}$ and

$$\mathcal{W}_{nm,\alpha\beta} = \mathcal{S}_{nm,\alpha\beta} - \mathcal{S}_{n0,\alpha}{}^\gamma \left(\frac{1}{1+\mathcal{S}}\right)_{00,\gamma\delta} \mathcal{S}_{0m}{}^\delta{}_\beta \tag{2.169}$$

The state we start from, i.e. $|\Xi_0'(s)\rangle$, and the relevant space profile, are obtained by setting $s^1 = s$ and $s^2 = 0$ in the previous formulas.

Next we have an analog of (2.155) with $\mathcal{C}(s,s')$ replaced by

$$\hat{\mathcal{C}}(s,s') = -\frac{1}{2b}(s^2+s'^2)\left(\frac{\mathcal{T}(1-\mathcal{T})}{1+\mathcal{T}}\right)_{00,11} - \frac{ss'}{2b}\left(\frac{(1-\mathcal{T})^2}{1+\mathcal{T}}\right)_{00,11} \tag{2.170}$$

Proceeding in the same way as in section 2.4.1, we can prove the analog of eq.(2.158). By using the spectral representation worked out in [47] one can show that $\hat{\mathcal{C}}$ picks up a logarithmic singularity unless $s = s'$. In a similar way one can prove the analog of (2.160).

Now let us discuss the properties of

$$|\hat{\Lambda}\rangle = \sum_{n=1}^{\infty} |\hat{\Xi}_0(s_n)\rangle$$

in the low energy limit. We refer to (2.168) with $s^1 = s$ and $s^2 = 0$. The fundamental difference between this formula and (2.154) is that in the low energy limit $\mathcal{S}_{00,\alpha\beta}$ becomes diagonal and takes on a value different from -1. More precisely

$$\mathcal{S}_{00,\alpha\beta} \to \frac{2|a|-1}{2|a|+1}G_{\alpha\beta}, \qquad a = -\frac{\pi^2}{V_{00}+\frac{b}{2}}B$$

see [32]. Therefore the $1+\mathcal{S}_{00}$ denominators in (2.168) are not dangerous any more. Similarly one can prove that in the same limit $\mathcal{S}_{0n} \to 0$. Moreover the ϵ–expansions about these values are well–defined. Therefore the space profile we are interested in is

$$\sim \exp[-\frac{\mu}{b}(x+s)^2 - \frac{\mu}{b}(y)^2]\exp[-\frac{1}{2}a_n^{\alpha\dagger}\mathcal{S}_{nm,\alpha\beta}a_m^{\beta\dagger}]|0\rangle \tag{2.171}$$

with a finite normalization factor and $\mu = \frac{2|a|-1}{2|a|+1}\Delta$. Now one can safely integrate s and obtain the result illustrated in section 2.4.1. This also shed light on how the resulting state couples to the $B_{\mu\nu}$ field. Indeed the length of this one dimensional objects is measured with the open string metric (2.165), in other words the B–field couples to the string by "stretching" it.

2.4.3 Fundamental strings

In this section we would like to discuss the properties of the Λ solutions we found in the previous sections. In order to justify the claim we made that they represent fundamental strings, we note that they are naturally attached to D–branes on both ends. These are the two extremal D0-branes in the s_n sequence (see also below). It easy to envisage systems in which such strings are attached to other D-branes as well. For instance let us pick $|\Lambda\rangle$ as given by (2.161) with $s_n > 0$ for all n's. Now let us consider a D24–brane with the only transverse direction coinciding with the x–axis and centered at $x = 0$. Let us call it $|\Xi'_{24}\rangle$. Due to the particular configuration chosen, it is easy to prove that $|\Xi'_{24}\rangle + |\Lambda\rangle$ is still a solution to (2.13). This is due to the fact that $|\Xi'_{24}\rangle$ is $\star$-orthogonal to the states $|\Xi'_0(s_n)\rangle$ for all n's. To be even more explicit we can study the space profile of $|\Xi'_{24}\rangle + |\Lambda\rangle$, assuming the sequence s_n to become dense in the positive x–axis. Using the previous results it is not hard to see that the overall configuration is a Gaussian centered at $x = 0$ in the x direction (the D24-brane) with an infinite prong attached to it and extending along the positive x–axis. The latter has a Gaussian profile in all space directions except x.

We remark that the condition $s_n > 0$ for all n's is important because $|\Xi'_{24}\rangle + |\Lambda\rangle$ is not anymore a projector if the $\{s_n\}$ sequence contains 0, since $|\Xi'_0(0)\rangle$ is not $\star$–orthogonal to $|\Xi'_{24}\rangle$. This remark tells us that it is not possible to have solutions representing configurations in which the string crosses the brane and/or is not attached to D-branes by the endpoints: the string has to stop at a brane[8].

Needless to say it is trivial to generalize the solution of the type $|\Xi'_{24}\rangle + |\Lambda\rangle$ to lower dimensional branes.

It is worth pointing out that it is also possible to construct string solutions of finite length. It is enough to choose the sequence $\{s_n\}$ to lie between two fixed values, say a and b in the x–axis, and then 'condense' the sequence between these two points. In the low energy limit the resulting solution shows precisely a flat profile for $a < x < b$ and a vanishing profile outside this interval (and of course a Gaussian profile along the other space direction). This solution is fit to represent a string stretched between two D–branes located at $x = a$ and $x = b$.

An important property for fundamental strings is the exchange property. Let us see if it holds for our solutions in a simple example. We consider first an extension of the solution (2.161) made of two pieces at right angles. Let us pick two space directions, x and y. We will denote by $\{s_n^x\}$ and $\{s_n^y\}$ a sequence of points along the positive x and y–axis. The string state

$$|\Lambda^{\pm\pm}\rangle = |\Xi'_0\rangle + \sum_{n=1}^{\infty} |\Xi'_0(\pm s_n^x)\rangle + \sum_{n=1}^{\infty} |\Xi'_0(\pm s_n^y)\rangle \tag{2.172}$$

is a solution to (2.13). The $\pm\pm$ label refers to the positive (negative) x and y–axis. This state represents an infinite string stretched along the positive (negative) x and y–axis including the origin. Now let us construct the string state

$$|\Xi'_0\rangle + \sum_{n=1}^{\infty} |\Xi'_0(s_n^x)\rangle + \sum_{n=1}^{\infty} |\Xi'_0(-s_n^x)\rangle + \sum_{n=1}^{\infty} |\Xi'_0(s_n^y)\rangle + \sum_{n=1}^{\infty} |\Xi'_0(-s_n^y)\rangle \tag{2.173}$$

This is still a solution to (2.13) and can be interpreted in two ways: either as $|\Lambda^{++}\rangle + |\Lambda^{--}\rangle$ or as $|\Lambda^{+-}\rangle + |\Lambda^{-+}\rangle$, up to addition to both of $|\Xi'_0\rangle$ (a bit removed from the origin). This

[8] It should be noted that, according to section 5 of [35], one could $\star$–rotate $|\Xi'_{24}\rangle \to |\check{\Xi}'_{24}\rangle$ so that $|\check{\Xi}'_{24}\rangle \star |\Xi'_0(0)\rangle = 0$; however this simply means that $|\Lambda\rangle$ has to be rotated too in a similar way into $|\check{\Lambda}\rangle$ in order for the corresponding string to end on $|\check{\Xi}'_{24}\rangle$. If we do not do so, we simply obtain a string that may cross the $|\check{\Xi}'_{24}\rangle$ brane, but still ends on the two terminal D0-branes.

addition costs the same amount of energy in the two cases, an amount that vanishes in the continuous limit. Therefore the solution (2.173) represents precisely the exchange property of fundamental strings.

So far we have considered only straight one–dimensional solutions (in terms of space profiles), or at most solutions represented by straight lines at right angles. However this is an unnecessary limitation. It is easy to generalize our construction to any curve in space. For instance, let us consider two directions in space and let us denote them again x and y ($\hat{p}^x$ and $\hat{p}^y$ being the relevant momentum operators). Let us construct the state

$$|\Xi_0'(s^x, s^y)\rangle = e^{-is^x\hat{p}^x} e^{-is^y\hat{p}^y} |\Xi_0'\rangle \tag{2.174}$$

It is evident that this represents a space–localized solution displaced from the origin by s^x in the positive x direction and s^y in the positive y direction. Using a suitable sequence $\{s_n^x\}$ and $\{s_n^y\}$, and rendering it dense, we can construct any curve in the $x-y$ plane, and, as a consequence, write down a solution to the equation of motion corresponding to this curve (even a closed one). The generalization to other space dimensions is straightforward.

We would like to remark that, by generalizing the above construction, one can also construct higher dimensional objects. For instance one could repeat the accretion construction by adding parallel D1–branes (that extend, say, in the y direction) along the x–axis. In this way we end up with a membrane-like configuration (with a flat profile in the x, y–plane), and continue in the same tune with higher dimensional configurations.

All the solutions we have considered so far are unstable. However the fundamental string solutions are endowed with a particular property. Since they end on a D–brane, their endpoints couple to the electromagnetic field on the brane, [38, 48, 49], and carry the corresponding charge. When the D–brane decays there is nothing that prevents the (fundamental) strings attached to it from decaying themselves. However in the presence of a background E–field, the latter are excited by the coupling with the E–field and persist (or, at least, persist longer than the other unstable objects). This phenomenon is described in [25] in effective field theory and BCFT language (see also [47]).

Chapter 3

Light-Cone SFT: Integrability Properties

In recent years it has become more and more evident that integrability plays an important role in string theory. This fact is conspicuous in the $AdS_5 \times S^5$ example of AdS/CFT duality, where integrability features on both sides of the correspondence. Integrability plays a major role in connection with topological strings. But there are also other less well–known cases. One of these is represented by the integrability properties of the Neumann coefficients for the three strings vertex of string field theory (SFT). In [50] it was shown that these Neumann coefficients in the case of Witten's covariant open bosonic string field theory (OSFT), [1], satisfy the Hirota equations, [51], of the *dispersionless* Toda lattice hierarchy [52, 53, 54]. This is to be traced back to the existence of a conformal mapping [28, 55] underlying the three strings vertex. The conclusions of [50] were limited to covariant OSFT. What we would like to show in this chapter is that the above integrability properties seem to be a general characteristic of the three strings vertex. Indeed, below, we will prove that it holds for the light–cone string field theory (LCSFT) in flat background. But, what is more important, we will present evidence that it may also hold for a nontrivial background. We will specifically examine the Neumann coefficients of the three strings vertex in a maximally supersymmetric pp–wave background and expand them in terms of the 'string mass' μ. We will show that, up to second order in this expansion, they satisfy the (correspondingly expanded) Hirota equations for the *dispersive* Toda lattice hierarchy (i.e. for the full Toda lattice hierarchy), [56]. This leads us to the conjecture that the Neumann coefficients in this background satisfy the Hirota equations of the full Toda lattice hierarchy.

Our results do not have an immediate practical impact. And perhaps they are not unexpected. Once the generating function of the Neumann coefficients are written in terms of conformal mappings (see below), they are expected to satisfy some kind of integrability condition, [55]. That this is true also for the three strings vertex in the pp–wave background may be seen as the consequence of the solvability of string theory on such background. However, the main value of our results is that they point towards the existence of an integrable model (of which the Hirota equations are a signal) underlying the SFT structure, very likely a matrix model, which, once uncovered, would greatly improve our knowledge of SFT. The fact that this is probably true also for a nontrivial background, such as the pp–wave one, may suggest a way to approach the problem of defining SFT on more general backgrounds.

3.1 The conformal maps for the light–cone SFT

In this section, we work out the conformal properties of the three strings vertex in light–cone SFT. We introduce first the relevant Neumann coefficients and then we show that, like in Witten's OSFT, they can be defined in terms of conformal mappings and determine them explicitly. We show how our formulas are related to the ones of Mandelstam, [57]. Finally we discuss the general features of these conformal mappings and describe an explicit example.

3.1.1 Neumann coefficients for bosonic three strings vertex

In LCSFT the three superstrings vertex is determined entirely in terms of the Neumann coefficients of the bosonic part. Therefore, we will concentrate on the latter, which can be written in the form

$$|V_3\rangle = \int \prod_{s=1}^{3} d\alpha_s \prod_{I=1}^{8} dp^I_{(r)} \delta(\sum_{r=1}^{3} \alpha_r) \delta(\sum_{r=1}^{3} p^I_{(r)}) \exp(-\Delta_B) \, |0,p\rangle_{123} \tag{3.1}$$

where $I = 1, \ldots, 8$ label the transverse directions, $\alpha_r = 2p^+_{(r)}$ and

$$\Delta_B = \sum_{r,s=1}^{3} \delta_{IJ} \left(\sum_{m,n\geq 1} a^{(r)I\dagger}_m V^{rs}_{mn} a^{(s)J\dagger}_n + \sum_{n\geq 1} p^I_{(r)} V^{rs}_{0n} a^{(s)J\dagger}_n + p^I_{(r)} V^{rs}_{00} p^J_{(s)} \right) \tag{3.2}$$

where summation over I and J is understood. The operators $a^{(s)J}_n, a^{(s)J\dagger}_n$ are the non–zero mode transverse oscillators of the s–th string. They satisfy

$$[a^{(r)I}_m, a^{(s)J\dagger}_n] = \delta_{mn} \delta^{IJ} \delta^{rs}$$

$p_{(r)}$ is the transverse momentum of the r–th string and $|0,p\rangle_{123} \equiv |p_{(1)}\rangle \otimes |p_{(2)}\rangle \otimes |p_{(3)}\rangle$ is the tensor product of the Fock vacua relative to the three strings. $|p_{(r)}\rangle$ is annihilated by $a^{(r)}_n$ and is the eigenstate of the operator $\hat{p}^I_{(r)}$ with eigenvalue $p^I_{(r)}$. The vertex coefficients V_{nm} are more conveniently expressed in terms of the Neumann coefficients N_{nm} as follows:

$$V^{rs}_{mn} = -\sqrt{mn} N^{rs}_{mn}, \qquad V^{rs}_{m0} = \frac{\sqrt{m}}{6} \epsilon^{stu} \frac{\alpha_t - \alpha_u}{\alpha_r} N^r_m \equiv \frac{1}{\sqrt{m}} N^{rs}_{m0} \tag{3.3}$$

$$V^{rr}_{00} = \frac{1}{3}(\alpha^2_{r-1} + \alpha^2_{r+1}) \frac{\tau_0}{\alpha}, \qquad V^{rs}_{00} = -\frac{2}{3} \alpha_r \alpha_s \frac{\tau_0}{\alpha}, \qquad r \neq s\,. \tag{3.4}$$

In these equations and in the sequel $r, s = 1,2,3$ modulo 3. In eq.(3.3) ϵ denotes the completely antisymmetric tensor with $\epsilon^{123} = 1$ and the indices t, u are summed over. In these equations the N's are related to the Mandelstam $\bar{N}$ by $N^{rs}_{mn} = \bar{N}^{rs}_{mn}$ and $N^r_m = \alpha_r \bar{N}^r_m$, with, see [57, 58],

$$N^r_m = \frac{\Gamma(-\frac{\alpha_{r+1}}{\alpha_r} m)}{m! \, \Gamma(-\frac{\alpha_{r+1}}{\alpha_r} m - m + 1)} e^{\frac{\tau_0}{\alpha_r} m}, \tag{3.5}$$

$$N^{rs}_{mn} = -\frac{\alpha \; m \; n}{\alpha_r \alpha_s (m \; \alpha_s + n \; \alpha_r)} N^r_m N^s_n \tag{3.6}$$

and

$$\alpha = \alpha_1\alpha_2\alpha_3, \qquad \tau_0 = \sum_{r=1}^{3} \alpha_r \ \ln\ |\alpha_r|, \qquad \sum_{r=1}^{3} \alpha_r = 0 \ . \tag{3.7}$$

The three strings vertex V^{rs}_{m0} and V^{rs}_{00} are not exactly the same as in [58]. We have symmetrized them using momentum conservation.

The Neumann coefficients (3.5–3.6) are homogeneous functions of the parameters α_r, so they actually depend on a single parameter β

$$\beta_r := \frac{\alpha_{r+1}}{\alpha_r}, \quad \beta_1\beta_2\beta_3 = 1, \quad \beta_r(\beta_{r+1}+1) = -1, \tag{3.8}$$

$$\beta_1 = \beta, \quad \beta_2 = -\frac{\beta+1}{\beta}, \quad \beta_3 = -\frac{1}{\beta+1}, \tag{3.9}$$

$$-\infty < \beta_r \le -1, \quad -1 \le \beta_{r+1} \le 0, \quad 0 \le \beta_{r+2} < \infty,$$

$$e^{\frac{\tau_0}{\alpha_1}} = \frac{|\beta|^\beta}{|\beta+1|^{\beta+1}}, \qquad e^{\frac{\tau_0}{\alpha_2}} = \frac{|\beta|}{|\beta+1|^{\frac{\beta+1}{\beta}}}, \qquad e^{\frac{\tau_0}{\alpha_3}} = \frac{|\beta+1|}{|\beta|^{\frac{\beta}{\beta+1}}} \ . \tag{3.10}$$

We can organize the Neumann coefficients by means of generating functions

$$N^r(z) := \sum_{n=1}^{\infty} \frac{1}{z^n} N^r_n, \qquad N^{rs}(z_1, z_2) := \sum_{n,m=1}^{\infty} \frac{1}{z_1^n}\frac{1}{z_2^m} N^{rs}_{nm} \ . \tag{3.11}$$

Our first purpose is to express these functions in terms of conformal mapping from the unit semidisk to the complex plane, in analogy the case of covariant OSFT, [50].

We write the conformal mappings for LCSFT as follows

$$f_r(z^{-1}) = f_{r+2}(0) - \frac{(f_{r+2}(0) - f_r(0))(f_{r+1}(0) - f_{r+2}(0))}{(f_{r+1}(0) - f_r(0))\varphi_r(z^{-1}) + f_r(0) - f_{r+2}(0)} \tag{3.12}$$

with $f_1(0) \ne f_2(0) \ne f_3(0)$. The functions $\varphi_r(z^{-1}) \equiv \varphi_r$ are solutions to the equations

$$\varphi_r^{\beta_r}(\varphi_r - 1) = \frac{1}{z} e^{\frac{\tau_0}{\alpha_r}} := x_r \ . \tag{3.13}$$

We remark that, as a consequence of (3.12–3.13), we have the following identities

$$f'_r(0) = e^{\frac{\tau_0}{\alpha_r}} \frac{(f_{r+1}(0) - f_r(0))(f_{r+2}(0) - f_r(0))}{f_{r+1}(0) - f_{r+2}(0)},$$

$$\frac{f'_r(0) f'_s(0)}{(f_r(0) - f_s(0))^2} = e^{\tau_0(\frac{1}{\alpha_r} + \frac{1}{\alpha_s})}, \quad r \ne s \ . \tag{3.14}$$

By means of the conformal mappings we now define the generating functions

$$N^r(z) = -\beta_{r+2}\ \ln\left(-\frac{f'_r(0)}{z}\left(\frac{1}{f_r(0) - f_r(z^{-1})} + \frac{1}{f_{r+1}(0) - f_r(0)}\right)\right), \tag{3.15}$$

$$N^{rr}(z_1, z_2) = \ln\left(\frac{f'_r(0)}{z_1 - z_2}\left(\frac{1}{f_r(0) - f_r(z_2^{-1})} - \frac{1}{f_r(0) - f_r(z_1^{-1})}\right)\right), \tag{3.16}$$

$$N^{rs}(z_1,z_2) = \ln\left(\frac{(f_r(z_1^{-1}) - f_s(z_2^{-1}))(f_r(0) - f_s(0))}{(f_r(0) - f_s(z_2^{-1}))(f_r(z_1^{-1}) - f_s(0))}\right), \quad r \neq s \,. \tag{3.17}$$

We notice that these definitions are very close to those in Witten's SFT, [50].

Let us analyze eq. (3.13). First of all we notice that any solution will have a branch that can be expanded around the value 1 as $1 + x_r + \ldots$

$$\begin{aligned}
\varphi_r &\equiv \varphi_r(z^{-1}) \equiv \varphi_r(x_r) = 1 + x_r + \sum_{k=2}^{\infty} a_k x_r^k \\
&= 1 + x_r - \beta_r x_r^2 + \frac{1}{2}\beta_r(1+3\beta_r)x_r^3 - \frac{1}{3}\beta_r(1+2\beta_r)(1+4\beta_r)x_r^4 \\
&+ \frac{1}{24}\beta_r(1+5\beta_r)(2+5\beta_r)(3+5\beta_r)x_r^5 + \ldots \quad .
\end{aligned} \tag{3.18}$$

Using these expansions inside the definitions (3.15–3.17) one can make a direct comparison and verify that they do generate the Neumann coefficients of Mandelstam (3.5–3.6).

It must be clarified that the three functions $\varphi_1, \varphi_2, \varphi_3$ are not distinct. With very simple manipulations one can see that the solutions to eqs.(3.13) with different r are related by the following transformations:

$$\varphi_{r+1}(x_{r+1}) = \frac{1}{1 - \varphi_r\left(-x_{r+1}^{\beta_r}\right)} = 1 - \frac{1}{\varphi_{r-1}\left((-x_{r+1})^{\frac{1}{\beta_{r+1}}}\right)}. \tag{3.19}$$

Therefore the three equations (3.13) give rise to a unique solution (which describes a Riemann surface, see below).

The importance of these transformations should not be underestimated. The three strings vertex represents the fusion of two strings that come together and give rise to a third one. In Witten's covariant OSFT this process is very symmetric: two strings evolving from $\tau = -\infty$ come together at $\tau = 0$ in such a way that the right half of one string overlap with the left half of the other; the three strings vertex describes this process which ends with the emergence of the third string. The three strings vertex in the present case (LCSFT), as we shall see, has a different stringy/geometric interpretation. It is however still characterized by precise overlapping conditions that are made possible by the above equations.

Let us analyze the gluing conditions for the mappings $f_r(z_r^{-1})$ (3.12), i.e. let us see how we can satisfy the conditions

$$f_{r+1}(z_{r+1}^{-1}) = f_r(z_r^{-1}) \,. \tag{3.20}$$

They lead to

$$\varphi_{r+1}(x_{r+1}) = \frac{1}{1 - \varphi_r(x_r)} \tag{3.21}$$

which can easily be solved if one compares eqs. (3.21) and (3.19)

$$x_r = -x_{r+1}^{\beta_r}. \tag{3.22}$$

Equivalently, we have the following relations for the string coordinates:

$$z_r = -z_{r+1}^{\beta_r} \,. \tag{3.23}$$

For later use we record that on the unit circle these conditions become

$$z_r = e^{i\theta_r}, \quad \theta_r = \beta_r \, \theta_{r+1} + \pi. \tag{3.24}$$

Starting from the definitions (3.15–3.17) and using (3.8), (3.12) and (3.13), one can derive the following explicit representations

$$\begin{aligned} N^r(z) &= \frac{1}{\beta_r} \ln\left(-\frac{f_r'(0)}{z}\left(\frac{1}{f_r(0)-f_r(z^{-1})} + \frac{1}{f_{r+2}(0)-f_r(0)}\right)\right) \\ &\equiv \ln \varphi_r(z^{-1}) \equiv -\beta_{r+2} \ln\left(-\frac{e^{\frac{\tau_0}{\alpha_r}}}{z}\left(\frac{1}{1-\varphi_r(z^{-1})} - 1\right)\right) \\ &\equiv \frac{1}{\beta_r} \ln\left(-\frac{e^{\frac{\tau_0}{\alpha_r}}}{z}\,\frac{1}{1-\varphi_r(z^{-1})}\right), \end{aligned} \tag{3.25}$$

$$\begin{aligned} N^{rr}(z_1,z_2) &= \ln\left(\frac{1}{z_1-z_2}\left(z_1\varphi_r^{\beta_r}(z_1^{-1}) - z_2\varphi_r^{\beta_r}(z_2^{-1})\right)\right) \\ &\equiv \ln\left(\frac{e^{\frac{\tau_0}{\alpha_r}}}{z_1-z_2}\left(\frac{1}{1-\varphi_r(z_2^{-1})} - \frac{1}{1-\varphi_r(z_1^{-1})}\right)\right), \end{aligned} \tag{3.26}$$

$$\begin{aligned} N^{rr+1}(z_1,z_2) &= \ln\left(\frac{1}{\varphi_r(z_1^{-1})} + \varphi_{r+1}(z_2^{-1}) - \frac{\varphi_{r+1}(z_2^{-1})}{\varphi_r(z_1^{-1})}\right) \\ &\equiv \ln\left(1 + \frac{(1-\varphi_r(z_1^{-1}))(1-\varphi_{r+1}(z_2^{-1}))}{\varphi_r(z_1^{-1})}\right), \\ & N^{rs}(z_1,z_2) = N^{sr}(z_2,z_1) \end{aligned} \tag{3.27}$$

which actually do not depend on the values of the conformal mappings (3.12) at the origin, $f_r(0)$, as long as they are distinct, i.e. $f_1(0) \neq f_2(0) \neq f_3(0)$. Therefore the parameters $f_r(0)$ in (3.12) are inessential gauge parameters.

3.1.2 A comparison with the Neumann function method

Eqs. (3.13) are related to the ones that appear in the derivation of open string tree amplitudes by means of the Neumann function method. At the tree level the interaction of open strings consists of the joining of two strings by the endpoints to form a unique string or the splitting of a string into two. These two processes can geometrically be described, [57, 59, 60], in terms of cutting and pasting string world–sheet strips. In turn such a geometry can be nicely represented in the complex plane by means of logarithmic conformal mappings. This leads to the explicit evaluation of the Neumann coefficients for the interaction of three strings that are precisely those introduced at the beginning of section 3.1. For instance, a suitable logarithmic map for the r–th three–string configuration is

$$\rho_r = \alpha_{r+1}\ln(z'-1) + \alpha_{r-1}\ln z' \tag{3.28}$$

where z' takes values in the upper half plane. This can be transformed into, see [59, 60],

$$y_r = -\beta_r \ln(1 + x_r e^{y_r}) \ . \tag{3.29}$$

This equation can easily be reduced to the form (3.13) if we set

$$\varphi_r = \exp(-\beta_{r+1}\beta_{r+2} y_r)$$

and make the identification

$$x_r \equiv \frac{e^{\frac{\tau_0}{\alpha_r}}}{z} = -z'^{\frac{1}{\beta_{r-1}}}(z'-1)^{\beta_r} \ . \tag{3.30}$$

This means that our variable z in (3.13) is a 'uniformizing' variable for the three equations. To clarify this issue, in the next subsection, we work out an explicit example.

3.1.3 Particular case $\beta = 1$

Let us study in detail a particular case, specified by values of α_r for which the parameter $\beta = 1$. This case turns out to be particularly simple and can be analyzed in full detail. In this case $\beta_1 = 1, \beta_2 = -2, \beta_3 = -\frac{1}{2}$ and eqs. (3.13) can easily be solved. For each φ_r we have two branches

$$\varphi_1^{\pm}(z^{-1}) = \frac{1 \pm \sqrt{1+\frac{1}{z}}}{2}, \tag{3.31}$$

$$\varphi_2^{\pm}(z^{-1}) = \frac{2}{1 \pm \sqrt{1-\frac{1}{z}}}, \tag{3.32}$$

$$\varphi_3^{\pm}(z^{-1}) = \left(\frac{1}{z} \pm \sqrt{1+\frac{1}{z^2}}\right)^2 . \tag{3.33}$$

The $^{(+)}$ branch is the one for which the expansion for small x_r is of the form $\varphi_r = 1 + x_r + \ldots$. We can now define the corresponding conformal mappings. To specify the latter we need to fix the values they take at the origin. We recall that these values can be arbitrary as long as they are distinct. Therefore we choose them simply as

$$f_1(0) = -1, \qquad f_2(0) = 0, \qquad f_3(0) = 1 \ .$$

With this choice we get

$$f_1^{(\pm)}(1/z) = -\frac{1 \pm \sqrt{1+1/z}}{3 \mp \sqrt{1+1/z}}, \tag{3.34}$$

$$f_2^{(\pm)}(1/z) = \frac{\sqrt{1-1/z} \mp 1}{\sqrt{1-1/z} \pm 3}, \tag{3.35}$$

$$f_3^{(\pm)}(1/z) = -\frac{1}{1 - 2(1/z \pm \sqrt{1+(1/z)^2})^2} \ . \tag{3.36}$$

Each couple of functions $f_r^{(\pm)}(1/z)$ defines a Riemann surface represented by two sheets joined through a cut. In the first case the cut runs from -1 to ∞, in the second it runs from 1 to ∞ and in the third between i and $-i$. These three Riemann surfaces however are related by the maps (3.19).

In the $\beta = 1$ case it is rather easy to see in detail the gluing conditions that characterize the three strings vertex. We consider three unit semidisks S_1, S_2 and S_3 cut out in the complex $\zeta_1 = 1/z_1$, $\zeta_2 = 1/z_2$ and $\zeta_3 = 1/z_3$ upper half planes. Each function $f_r^{(\pm)}$ maps the semidisk into a region $D_r^{(\pm)}$ in the image complex plane. $D_1^{(\pm)}$ and $D_2^{(\pm)}$ have the form of lobes, while $D_3^{(\pm)}$ are the outer part of a compact bi-lobed domain in the lower $(+)$ and upper $(-)$ half plane, respectively (see Figs. 3.1–3.3, where the $D_r^{(+)}$'s are shown, the $D_r^{(-)}$'s being the same regions reflected with respect to the real axis).

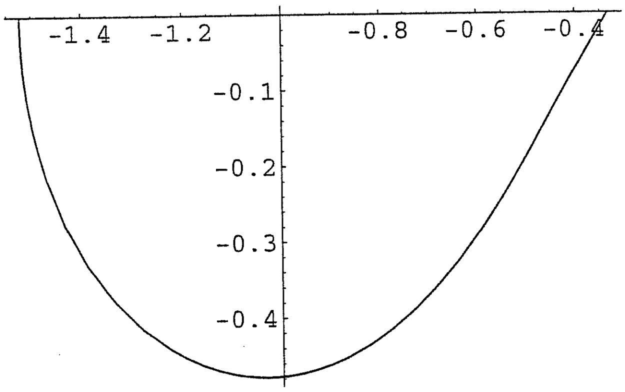

Figure 3.1: *The region $D_1^{(+)}$ is contained between the curve and the real axis.*

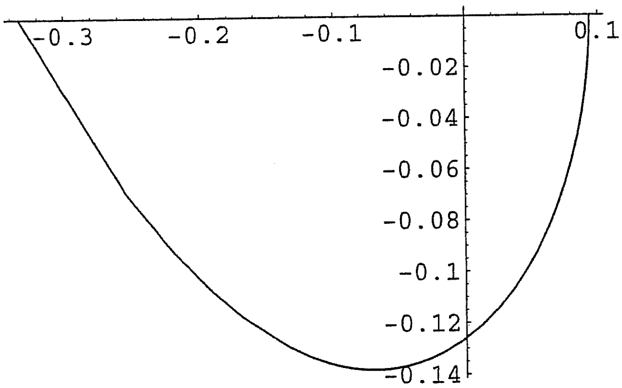

Figure 3.2: *The region $D_2^{(+)}$ is contained between the curve and the real axis.*

These domains are glued together along the borderlines in the following way. To start with, $f_1^{(+)}(\zeta_1) = f_2^{(-)}(\zeta_2)$ if $\zeta_1 = -\zeta_2$. This means that S_1 and S_2 are glued together along

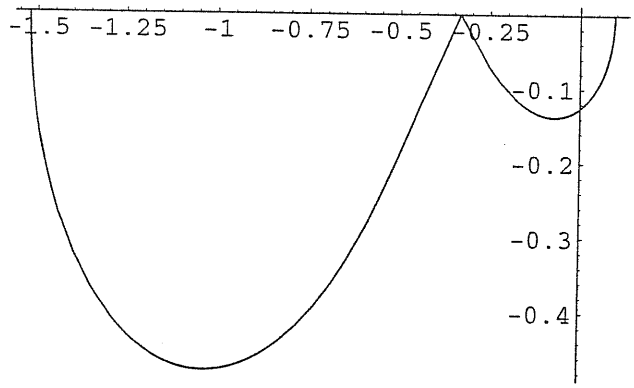

Figure 3.3: *The region $D_3^{(+)}$ is the portion of the lower half plane external to the curve.*

the real axis with opposite orientation. The image common boundary in $D_1^{(+)}$ and $D_2^{(-)}$ stretches also along the real axis. The second overlap condition we consider is between $D_1^{(+)}$ and $D_3^{(+)}$: $f_1^{(+)}(\zeta_1) = f_3^{(+)}(\zeta_3)$ if $\zeta_3 = -1/\sqrt{\zeta_1}$. This means the absolute values of ζ_1 and ζ_3 are 1, while their phases θ_1 and θ_3 are related by $\theta_3 = \pi - \frac{\theta_1}{2}$. So, while θ_1 runs between 0 and π, θ_3 spans the interval between π and $\pi/2$. These two angular intervals are mapped to the same curve in the target complex plane: the curved part of the lobe boundary in $D_1^{(+)}$ and a piece of the curved boundary of $D_3^{(+)}$. Finally $f_2^{(+)}(\zeta_2) = f_3^{(+)}(\zeta_3)$ if $\zeta_2 = -1/\zeta_3^2$. That means the moduli of ζ_2 and ζ_3 are 1, while their phases θ_2 and θ_3 are related by $\theta_2 = \pi - 2\theta_3$. So, while θ_2 runs between 0 and π, θ_3 spans the interval between $\pi/2$ and 0. Again these two angular intervals are mapped to the same curve in the target complex plane: the curved boundary of $D_2^{(+)}$ and the other piece of the curved boundary of $D_3^{(+)}$. In a similar way one can proceed with the remaining three correspondences.

So far the discussion has been purely mathematical, without any concern for the string process we want to describe. From the point of view of string theory the 'physical' branches are the ones where the expansion for small x_r is of the form $\varphi_r = 1 + x_r + \ldots$, i.e. the + branches in (3.31–3.33). So we have to glue together the images of the semidisks S_1, S_2, S_3 by $f_1^{(+)}, f_2^{(+)}, f_3^{(+)}$ respectively, according to the maps (3.23), identifying the common boundaries. This means that we have to glue the inner part of the lobe $D_1^{(+)}$ and $D_2^{(+)}$ with $D_3^{(+)}$, i.e. with the outer part of the bi–lobe. The result is the entire lower half plane. The process described is the joining of the strings 1 and 2 by two endpoints to form the third string at the interaction point $\zeta_1 = -1, \zeta_2 = 1$ and $\zeta_3 = i$. The picture obtained in this way allows us to make a comparison with the three string interaction in covariant SFT. As one can see there are remarkable differences. In particular the midpoints of the three strings never overlap (at variance with covariant OSFT); on the contrary, the LCSFT vertex preserves its characteristic perturbative geometry where the strings interact by the endpoints.

For completeness we write down the explicit expressions of the generating functions for the Neumann coefficients. In these definitions we always use the + branch of φ_r

$$
\begin{aligned}
N^1(z) &= \ln \frac{1+\sqrt{1+\frac{1}{z}}}{2}, \\
N^2(z) &= -N^1(-z), \\
N^3(z) &= 2\,\ln\left(\frac{1}{z}+\sqrt{1+\frac{1}{z^2}}\right)
\end{aligned}
\tag{3.37}
$$

and

$$
\begin{aligned}
N^{11}(z_1,z_2) &= \ln\left(\frac{1}{2(z_1-z_2)}\left(\frac{1}{1-\sqrt{1+\frac{1}{z_2}}}-\frac{1}{1-\sqrt{1+\frac{1}{z_1}}}\right)\right), \\
N^{22}(z_1,z_2) &= N^{11}(-z_1,-z_2), \\
N^{33}(z_1,z_2) &= \ln\left(\frac{1}{z_1-z_2}\left(\sqrt{1+z_1^2}-\sqrt{1+z_2^2}\right)\right)
\end{aligned}
\tag{3.38}
$$

as well as

$$
\begin{aligned}
N^{12}(z_1,z_2) &= -N^{11}(z_1,-z_2), \\
N^{23}(z_1,z_2) &= \ln\left(\frac{\sqrt{1-\frac{1}{z_1}}-\sqrt{1+z_2^2}}{1-\sqrt{1+z_2^2}}\right), \\
N^{13}(z_1,z_2) &= \ln\left(\frac{\sqrt{1+\frac{1}{z_1}}+\sqrt{1+z_2^2}}{1+\sqrt{1+z_2^2}}\right), \\
N^{rs}(z_1,z_2) &= N^{sr}(z_2,z_1).
\end{aligned}
\tag{3.39}
$$

Finally let us remark that, for $\beta \neq 1$, the conformal mappings f_r may be far more complicated than in the above example. With β rational they have a finite number of branches that describe a Riemann surface. Like in the $\beta = 1$ case, only one of them will define the correct Neumann coefficients. When β is irrational, as shown in (3.18), this branch can always be determined, but the Riemann surface interpretation is lost.

3.2 The dTL Hirota equations

Let us introduce the flow parameters $t_0^{(r)}, t_k^{(r)}, \bar{t}_k^{(r)}$ with $k = 1, 2, .., \infty$ and the differential operators

$$
D_r(z) = \sum_{k=1}^{\infty} \frac{1}{kz^k}\frac{\partial}{\partial t_k^{(r)}}, \quad \bar{D}_r(\bar{z}) = \sum_{k=1}^{\infty} \frac{1}{k\bar{z}^{\,k}}\frac{\partial}{\partial \bar{t}_k^{(r)}}, \quad r,s = 1,2,3 \; mod \; 3. \tag{3.40}
$$

The Hirota equations for the three decoupled copies of the dispersionless Toda lattice hierarchies fit for the present case are (for more details, see [54] and references therein)

$$(z_1 - z_2)e^{D_r(z_1)D_r(z_2)F} = z_1 e^{-\partial_{t_0^{(r)}} D_r(z_1)F} - z_2 e^{-\partial_{t_0^{(r)}} D_r(z_2)F}, \tag{3.41}$$

$$(\bar{z}_1 - \bar{z}_2)e^{\bar{D}_r(\bar{z}_1)\bar{D}_r(\bar{z}_2)F} = \bar{z}_1 e^{-\partial_{t_0^{(r)}} \bar{D}_r(\bar{z}_1)F} - \bar{z}_2 e^{-\partial_{t_0^{(r)}} \bar{D}_r(\bar{z}_2)F}, \tag{3.42}$$

$$-z_1\bar{z}_2\left(1 - e^{-D_r(z_1)\bar{D}_r(\bar{z}_2)F}\right) = e^{\partial_{t_0^{(r)}}(\partial_{t_0^{(r)}} + D_r(z_1) + \bar{D}_r(\bar{z}_2))F} \tag{3.43}$$

where $F \equiv F(\{t_0^{(r)}, t_k^{(r)}, \bar{t}_k^{(r)}\})$ is the τ–function (free energy) of the hierarchy. The minus sign on the l.h.s. of eq. (3.43) can be replaced by the standard plus sign via the transformations $\{t_n^r,\ \bar{t}_n^{\,r}\} \Rightarrow \{t_n^r,\ (-1)^n\bar{t}_n^{\,r}\}$ or $\{t_n^r,\ \bar{t}_n^{\,r}\} \Rightarrow \{(i)^n t_n^r,\ (i)^n\bar{t}_n^{\,r}\}$.

We remark that eqs. (3.41–3.43) form three distinct sets of equations, each one being formally the same as in Witten's OSFT, [50]. The fact is that in the latter case these sets of equations collapse to the same set.

Now, it is elementary to verify that the generating functions $N^r(z)$, $N^{rs}(z_1, z_2)$ (3.15–3.17) of the Neumann coefficients satisfy the following equations:

$$\begin{aligned}(z_1 - z_2)e^{N^{rr}(z_1,z_2)} &= z_1 e^{-\beta_{r+1}\beta_r N^r(z_1)} - z_2 e^{-\beta_{r+1}\beta_r N^r(z_2)} \\ &= z_1 e^{\beta_r N^r(z_1)} - z_2 e^{\beta_r N^r(z_2)}\end{aligned} \tag{3.44}$$

and

$$-z_1\bar{z}_2\left(1 - e^{N^{rr+1}(z_1,\bar{z}_2)}\right) = e^{\tau_0(\frac{1}{\alpha_r} + \frac{1}{\alpha_{r+1}}) + \beta_{r+1}\beta_r N^r(z_1) - \beta_{r+1}N^{r+1}(\bar{z}_2)}. \tag{3.45}$$

These are easily seen to reproduce the Hirota equations (3.41–3.43) provided we suitably identify the generating functions with the second derivatives of F as follows:

$$\begin{aligned}
&D_r(z_1)D_r(z_2)F = N^{rr}(z_1, z_2),\\
&\bar{D}_r(\bar{z}_1)\bar{D}_r(\bar{z}_2)F = N^{r+1r+1}(\bar{z}_1, \bar{z}_2),\\
&D_r(z_1)\bar{D}_r(\bar{z}_2)F = -N^{rr+1}(z_1, \bar{z}_2),\\
&\partial_{t_0^{(r)}} D_r(z)F = \beta_{r+1}\beta_r N^r(z),\\
&\partial_{t_0^{(r)}} \bar{D}_r(\bar{z})F = -\beta_{r+1}N^{r+1}(\bar{z}),\\
&\partial_{t_0^{(r)}}\partial_{t_0^{(r)}}F = \tau_0(\frac{1}{\alpha_r} + \frac{1}{\alpha_{r+1}})\ .
\end{aligned} \tag{3.46}$$

However we remark that we can also define a consistent reduction by:

$$\frac{\partial}{\partial \bar{t}_n^{(r)}}F = \pm\frac{\partial}{\partial t_n^{(r+1)}}F. \tag{3.47}$$

This allows us to define new identifications as follows

$$\begin{aligned}
&D_r(z_1)D_r(z_2)F = N^{rr}(z_1,z_2),\\
&D_r(z_1)D_{r+1}(z_2)F = \mp N^{rr+1}(z_1,z_2),\\
&\partial_{t_0^{(r)}} D_r(z)F = \beta_{r+1}\beta_r N^r(z),\\
&\partial_{t_0^{(r)}} D_{r+1}(z)F = \mp\beta_{r+1} N^{r+1}(z),\\
&\partial_{t_0^{(r)}}\partial_{t_0^{(r)}} F = \tau_0\Big(\frac{1}{\alpha_r}+\frac{1}{\alpha_{r+1}}\Big)\,.
\end{aligned}\tag{3.48}$$

The corresponding equations for the tau function are

$$\begin{aligned}
(z_1-z_2)e^{D_r(z_1)D_r(z_2)F} &= z_1 e^{-\partial_{t_0^{(r)}} D_r(z_1)F} - z_2 e^{-\partial_{t_0^{(r)}} D_r(z_2)F}\\
&= z_1 e^{\mp\partial_{t_0^{(r-1)}} D_r(z_1)F} - z_2 e^{\mp\partial_{t_0^{(r-1)}} D_r(z_2)F}\,,\\
-z_1z_2\left(1-e^{\mp D_r(z_1)D_{r+1}(z_2)F}\right) &= e^{\partial_{t_0^{(r)}}(\partial_{t_0^{(r)}}+D_r(z_1)\pm D_{r+1}(z_2))F}\,.
\end{aligned}\tag{3.49}$$

These Hirota equations refer to the dispersionless hierarchy produced by the cyclic (coupled) reduction (3.47) of the three copies of the dispersionless Toda lattice hierarchies. Our conjecture is that the resulting hierarchy can be related to the 3-punctures Whitham hierarchy [61] and the dispersionless limit of the 3-component KP hierarchy.

3.3 Hirota equations for more general correlators

As we will see in the next section, in order to verify the validity of the full Hirota equations we need to know correlators with more than two entries. The dispersionless Hirota equations for 'n–point functions' are obtained by differentiating $n-2$ times the equations (3.41,3.42) and (3.43). These derived Hirota equations, like the original ones, do not uniquely determine their solutions. In order to be able to write down the latter we have to provide some additional information. In fact what we are looking for are n–point correlators that are compatible with the Neumann coefficients of the previous section and satisfy the full Hirota equation (see next section). Although we don't have a proof of it, we believe these two requirements completely determine the full series of n–point functions.

In this section we would like to concentrate on three– and four–point functions. On the basis of what we have just said, in order to determine them we have to rely on plausibility arguments and verify the results *a posteriori*. We notice first that the Neumann coefficients for the three strings vertex can be interpreted in terms of two–point correlators of a 1D system. For instance

$$\langle V_3|a_m^{(r)I\dagger}a_n^{(s)J\dagger}|0\rangle = \delta^{IJ}\,V_{mn}^{rs} = -\sqrt{nm}\,N_{mn}^{rs}\,\delta^{IJ}\,.\tag{3.50}$$

Analogous relations hold for two–point functions involving zero modes. In the same way we can of course consider correlators with more insertions. It is obvious that all the correlators with an odd number of insertions identically vanish. We take these correlators as a model in order to calculate three– and four–point functions of a quantum system that satisfies the full Hirota equations. We will see below that the correct answer is not the three– and four–point correlators obtained by inserting one and two creation operators in (3.50), respectively, but combinations of them. We will refer to the underlying model, based on the two–point correlators (3.50) and satisfying the Hirota equation, as the *associated quantum system*. It must be clear that we do not know yet what this system is, we simply postulate its existence. In particular, as a working hypothesis, we assume that its three–point functions identically vanish.

The dispersionless Hirota equations for four–point functions are obtained by differentiating twice the equations (3.41,3.42) and (3.43). Of course we have the possibility of applying $\partial_{t_0^{(r)}}$ or $D_r(z)$. Therefore from (3.41), for instance, we obtain

$$
\begin{aligned}
(z_1 - z_2)e^{D_r(z_1)D_r(z_2)F}\prod_{i=1}^{4} D_r(z_i)F &= & (3.51)\\
z_2 e^{-\partial_{t_0^{(r)}}D_r(z_2)F}\partial_{t_0^{(r)}}D_r(z_2)\prod_{i=3}^{4} D_r(z_i)F &- z_1 e^{-\partial_{t_0^{(r)}}D_r(z_1)F}\partial_{t_0^{(r)}}D_r(z_1)\prod_{i=3}^{4} D_r(z_i)F,\\
(z_1 - z_2)e^{D_r(z_1)D_r(z_2)F}\partial_{t_0^{(r)}}\prod_{i=1}^{3} D_r(z_i)F &= & (3.52)\\
z_2 e^{-\partial_{t_0^{(r)}}D_r(z_2)F}\partial^2_{t_0^{(r)}}D_r(z_2)D_r(z_3)F &- z_1 e^{-\partial_{t_0^{(r)}}D_r(z_1)F}\partial^2_{t_0^{(r)}}D_r(z_1)D_r(z_3)F,\\
(z_1 - z_2)e^{D_r(z_1)D_r(z_2)F}\partial^2_{t_0^{(r)}}D_r(z_2)D_r(z_1)F &= & (3.53)\\
z_2 e^{-\partial_{t_0^{(r)}}D_r(z_2)F}\partial^3_{t_0^{(r)}}D_r(z_2)F &- z_1 e^{-\partial_{t_0^{(r)}}D_r(z_1)F}\partial^3_{t_0^{(r)}}D_r(z_1)F
\end{aligned}
$$

where we have assumed that three–point correlators vanish.

After expanding the above equations in powers of z_1 and z_2, and collecting terms of the form $z_1^m z_2^n$ we obtain equations involving two and four derivative of F. We already know how to identify the two–derivative terms in terms of the Neumann coefficients of LCSFT (see previous section). The task now is to try to make the corresponding identifications for the four–derivative ones. The solutions to eqs. (3.51–3.53) are not uniquely defined (see below). The solutions we are interested in are as follows:

$$
\begin{aligned}
F_{t_0^{(r)}t_0^{(r)}t_0^{(r)}t_m^{(r)}} &= -3(\alpha_r)^2\frac{1+\beta_r}{\beta_r}m(N^{rr+1}_{m0} - N^{rr}_{m0}),\\
F_{t_0^{(r)}t_0^{(r)}t_m^{(r)}t_n^{(r)}} &= -(\alpha_r)^2 mn\left[\frac{1+\beta_r}{\beta_r}(n+m)N^{rr}_{mn} + 2(N^{rr+1}_{m0} - N^{rr}_{m0})(N^{rr+1}_{n0} - N^{rr}_{n0})\right],\\
F_{t_0^{(r)}t_m^{(r)}t_n^{(r)}t_l^{(r)}} &= -(\alpha_r)^2 mnl[(n+l)(N^{rr+1}_{m0} - N^{rr}_{m0})N^{rr}_{nl} + (n+m)(N^{rr+1}_{l0} - N^{rr}_{l0})N^{rr}_{mn}\\
&\quad + (m+l)(N^{rr+1}_{n0} - N^{rr}_{n0})N^{rr}_{ml}], \qquad (3.54)\\
F_{t_m^{(r)}t_n^{(r)}t_l^{(r)}t_k^{(r)}} &= -(\alpha_r)^2 mnlk[(m+n)(l+k)N^{rr}_{mn}N^{rr}_{lk} + (m+l)(n+k)N^{rr}_{ml}N^{rr}_{nk}\\
&\quad + (m+k)(n+l)N^{rr}_{mk}N^{rr}_{nl}]
\end{aligned}
$$

where N^{rs}_{m0} are defined in eqs. (3.3). Multiplying these by the appropriate monomials of z variables and summing, leads to the following compact expressions (which will be used later on):

$$
\begin{aligned}
\partial^3_{t_0^{(r)}} D_r(z)F &= \frac{3\alpha_{r-1}^2}{\beta_r}\frac{\varphi_r(z^{-1})-1}{(1+\beta_r)\varphi_r(z^{-1})-\beta_r}, \\
\partial^2_{t_0^{(r)}} \prod_{i=1}^{2} D_r(z_i)F &= -3\alpha_{r-1}^2 \prod_{i=1}^{2}\frac{\varphi_r(z_i^{-1})-1}{(1+\beta_r)\varphi_r(z_i^{-1})-\beta_r}, \\
\partial_{t_0^{(r)}} \prod_{i=1}^{3} D_r(z_i)F &= 3\alpha_{r-1}^2\beta_r \prod_{i=1}^{3}\frac{\varphi_r(z_i^{-1})-1}{(1+\beta_r)\varphi_r(z_i^{-1})-\beta_r}, \\
\prod_{i=1}^{4} D_r(z_i)F &= -3\alpha_{r-1}^2\beta_r^2 \prod_{i=1}^{4}\frac{\varphi_r(z_i^{-1})-1}{(1+\beta_r)\varphi_r(z_i^{-1})-\beta_r}.
\end{aligned} \tag{3.55}
$$

Using these expressions one can easily verify that they satisfy equations (3.51), (3.52) and (3.53).

A comment is in order concerning the results we have just written down. We have already said that eqs. (3.55) are not unique solutions to the Hirota equations for four–point functions. We have single them out because they are compatible with the dispersive Hirota equations (see next section) and they will allow us to find a solution of the latter coherent with the dispersionless Neumann coefficients. We recall that they are also compatible with vanishing three–point functions. As was mentioned at the beginning of this section, at first sight it would seem that the two–point functions, three–point functions and four–point functions (i.e. the derivatives of F with respect to two, three and four t_n parameters) of our system are simply given by

$$
\langle V_3|a_{n_1}^{r_1\dagger}\dots a_{n_k}^{r_k\dagger}|0\rangle. \tag{3.56}
$$

While this is true for two– and three–point functions, it is not quite true for the four–point ones, as one can see by comparing (3.56) with (3.54). It is evident that the four–point functions given by (3.56) has the right form, but must be suitably combined in order to coincide with (3.54), which satisfy the Hirota equations. Therefore the correlators of the associated quantum system that underlies the Neumann coefficients of LCSFT are made of suitable combinations of the k–point functions (3.56). In conclusion: we have found a solution to the Hirota equations for the four–point functions, which is compatible with the Mandelstam's three strings vertex of the LCSFT (and with its deformations up to second order in the expansion parameter, see next section), however we do not have a proof that this solution is unique.

A discussion of the general solution to eqs. (3.51–3.53) can be found in Appendix H.

3.4 PP–wave SFT and the dispersive Hirota equations

String theory on a maximally supersymmetric pp–wave background (plane wave limit of $AdS_5 \times S^5$) [62, 63], is exactly solvable [64]. Building on this it has been recently possible to construct the exact three strings vertex for the LCSFT on this background, i.e. to completely specify the relevant Neumann coefficients. They depend on the 'string mass' μ

(determined by the five form flux of type IIB superstring theory). When $\mu \to 0$ one recovers Mandelstam's Neumann coefficients discussed in the previous sections. In other words the nontrivial string background deforms the Neumann coefficients. It is interesting to see whether the Hirota equations discussed in the previous sections get deformed accordingly in such a way as to preserve integrability. This is our guess and this is what we would like to provide evidence for in this section.

Let us start from the expression of the μ–deformed three strings vertex, [65, 66, 67, 68, 69, 70, 71, 72, 73, 74]. The bosonic part, to which we limit ourselves[1], is defined by

$$\Delta'_B = \frac{1}{2}\delta_{IJ}\left(\sum_{r,s=1}^{3}\sum_{m,n=-\infty}^{\infty} a_m^{(r)I\dagger}\mathcal{V}^{rs}_{mn}a_n^{(s)J\dagger}\right). \tag{3.57}$$

The commutation relations are as in section 3.1.1, but $a_m^{(r)I\dagger} \neq a_{-m}^{(r)I}$ and $a_m^{(r)I}|0\rangle = 0$ for $m \in \mathbb{Z}$. The coefficients with negative n, m labels can be obtained from those with positive ones[2]. In this paper we consider only the $\mathcal{V}^{rs}_{mn}$ with $m, n \geq 0$. We set

$$\mathcal{V}^{rs}_{mn} = -\sqrt{mn}\,\mathcal{N}^{rs}_{mn}. \tag{3.58}$$

The Neumann coefficients $\mathcal{N}^{rs}_{mn}$ with $m, n \geq 1$ have been calculated in an explicit way in [73, 74]:

$$\mathcal{N}^{rs}_{mn} = -\frac{mn\alpha}{\alpha_s\omega_{r,m} + \alpha_r\omega_{s,n}}\,\frac{\mathcal{N}^r_m\mathcal{N}^s_n}{\alpha_r\alpha_s} \tag{3.59}$$

where $\omega_{r,m} = \sqrt{m^2 + \alpha_r^2\mu^2}$ and

$$\mathcal{N}^r_m = \sqrt{\frac{\omega_{r,m}}{m}}\,\frac{\omega_{r,m} + \alpha_r\mu}{m}f_m^{(r)}. \tag{3.60}$$

In these formulas

$$f_m^{(r)} = -\frac{e^{\tau_0\omega_{\frac{m}{\alpha_r}}}}{m(\alpha_r + \alpha_{r+1})\omega_{\frac{m}{\alpha_r}}}\,\frac{\Gamma_\mu^{(r+1)}\left(-\frac{m}{\alpha_r}\right)}{\Gamma_\mu^{(r)}\left(\frac{m}{\alpha_r}\right)\Gamma_\mu^{(r-1)}\left(\frac{m}{\alpha_r}\right)} \tag{3.61}$$

and $\omega_z = \mathrm{sgn}(z)\sqrt{\mu^2 + z^2}$. The μ–deformed Γ functions are defined by

$$\Gamma_\mu^{(r)}(z) = \frac{e^{-\gamma\alpha_r\omega_z}}{\alpha_r z}\prod_{n=1}^{\infty}\left(\frac{n}{\omega_{r,n} + \alpha_r\omega_z}e^{\frac{\alpha_r\omega_z}{n}}\right) \tag{3.62}$$

---

[1] For the problems connected with the supersymmetric completion of the vertex and its prefactor see the reviews [75, 76, 77] and references therein.

[2] There is a subtle point here. The expressions of Δ'_B and Δ_B in eq.(3.2) for the three strings vertex refer to two different bases: Δ_B is expressed in terms of the transverse momenta p_r^I (momentum basis) while Δ'_B contains the zero mode oscillators $a_0^{(r)I}$ (oscillator basis). This means that starting from (2.86) we have passed from the former basis to the latter by explicitly integrating over the transverse momenta (see, for instance, [11]). This operation modifies in a well–known way the vertex coefficients. However, in the present case, it turns out that in the limit $\mu \to 0$ the coefficients $\mathcal{V}^{rs}_{mn}$ tend to the corresponding coefficients V^{rs}_{mn} in (3.2) at least for $m, n \geq 1$.

where γ is the Euler–Mascheroni constant. It must be remarked that the above coefficients are not written in the usual form, we have simplified them by dropping some intermediate inessential factors.

Here we present some results which are going to be needed in the next sections. Expanding (3.60) and (3.59) in powers of μ up to order μ^2 we obtain

$$\mathcal{N}_m^r = N_m^r \left[1 + \frac{\alpha_r}{m}\mu + \left(\frac{\alpha_r^2}{4m^2} + \frac{\alpha_r}{2m}\tau_0\right)\mu^2 + \ldots\right] \tag{3.63}$$

and

$$\mathcal{N}_{mn}^{rs} = N_{mn}^{rs} \left[1 + \left(\frac{\alpha_r}{m} + \frac{\alpha_s}{n}\right)\mu + \frac{1}{4}\left(\frac{\alpha_r}{m} + \frac{\alpha_s}{n}\right)\left(\frac{\alpha_r}{m} + \frac{\alpha_s}{n} + 2\tau_0\right)\mu^2 + \ldots\right] \tag{3.64}$$

where N_m^r and N_{mn}^{rs} are Mandelstam's Neumann coefficients. For reasons that will become clear in the next section, it is convenient to rewrite these expansions in powers of λ related to μ as: $\mu = \lambda - \frac{\tau_0}{2}\lambda^2 + \ldots$

$$\mathcal{N}_m^r = N_m^r \left[1 + \frac{\alpha_r}{m}\lambda + \left(\frac{\alpha_r^2}{4m^2}\right)\lambda^2 + \ldots\right] \equiv \sum_{j=0}^{\infty} \mathcal{N}_{j,m}^r \lambda^j, \tag{3.65}$$

$$\mathcal{N}_{mn}^{rs} = N_{mn}^{rs} \left[1 + \left(\frac{\alpha_r}{m} + \frac{\alpha_s}{n}\right)\lambda + \frac{1}{4}\left(\frac{\alpha_r}{m} + \frac{\alpha_s}{n}\right)^2\lambda^2 + \ldots\right] \equiv \sum_{j=0}^{\infty} \mathcal{N}_{j,mn}^{rs} \lambda^j \tag{3.66}$$

where the subscript j refers to the order of expansion in λ.

Again, for later use, it is convenient to organize these Neumann coefficients by means of associated generating functions

$$\mathcal{N}_j^r(z) := \sum_{m=1}^{\infty} \frac{1}{z^m}\, m^{2j}\, \mathcal{N}_{j,m}^r, \tag{3.67}$$

$$\mathcal{N}_j^{rs}(z_1, z_2) := \sum_{m,n=1}^{\infty} \frac{1}{z_1^m}\frac{1}{z_2^n}\,(mn)^j\, \mathcal{N}_{j,mn}^{rs}\,, \tag{3.68}$$

for $j = 0, 1, 2$. At each order the summation can easily be carried out to give

$$\mathcal{N}_1^r(z) = \alpha_r \frac{\varphi_r(z^{-1}) - 1}{(1+\beta_r)\varphi_r(z^{-1}) - \beta_r}, \tag{3.69}$$

$$\mathcal{N}_2^r(z) = \frac{\alpha_r^2}{4}\frac{\varphi_r(z^{-1})(\varphi_r(z^{-1}) - 1)}{((1+\beta_r)\varphi_r(z^{-1}) - \beta_r)^3}, \tag{3.70}$$

$$\mathcal{N}_1^{rs}(z_1, z_2) = -\frac{\alpha}{\alpha_r\alpha_s}\frac{\varphi_r(z_1^{-1}) - 1}{(1+\beta_r)\varphi_r(z_1^{-1}) - \beta_r}\frac{\varphi_s(z_2^{-1}) - 1}{(1+\beta_s)\varphi_s(z_2^{-1}) - \beta_s}, \tag{3.71}$$

$$\begin{aligned}\mathcal{N}_2^{rs}(z_1, z_2) = &-\frac{\alpha}{4\alpha_r\alpha_s}\frac{(\varphi_r(z_1^{-1}) - 1)(\varphi_s(z_2^{-1}) - 1)}{[(1+\beta_r)\varphi_r(z_1^{-1}) - \beta_r][(1+\beta_s)\varphi_s(z_2^{-1}) - \beta_s]}\\ &\times\left[\frac{\alpha_s\varphi_r(z_1^{-1})}{[(1+\beta_r)\varphi_r(z_1^{-1}) - \beta_r]^2} + \frac{\alpha_r\varphi_s(z_2^{-1})}{[(1+\beta_s)\varphi_s(z_2^{-1}) - \beta_s]^2}\right].\end{aligned} \tag{3.72}$$

We notice that in order to get these compact generating functions it is necessary to insert the m^{2j} and $(mn)^j$ factors in (3.67) and (3.68), respectively.

3.4.1 The dispersive Hirota equations

Here, we consider the full (dispersive) Hirota equations for the Toda lattice hierarchy (see, e.g. ref. [54] and references therein). For the sake of brevity we deal in the sequel with half of the story, namely only with those equations that involve unbarred z variables (corresponding to eq.(3.41))

$$\begin{aligned} & z_1 \left(e^{\lambda(\partial_{t_0} - D_r(z_1))} \tau_\lambda\right) \left(e^{-\lambda D_r(z_2)} \tau_\lambda\right) - z_2 \left(e^{\lambda(\partial_{t_0} - D_r(z_2))} \tau_\lambda\right) \left(e^{-\lambda D_r(z_1)} \tau_\lambda\right) \\ & = (z_1 - z_2) \left(e^{-\lambda(D_r(z_1) + D_r(z_2))} \tau_\lambda\right) \left(e^{\lambda \partial_{t_0}} \tau_\lambda\right) \end{aligned} \tag{3.73}$$

where λ is a deformation parameter and τ_λ is the full *tau function* of the Toda lattice (KP) hierarchy:

$$\tau_\lambda = \exp(\mathcal{F}_\lambda), \qquad \mathcal{F}_\lambda = \frac{1}{\lambda^2} F_0 + \frac{1}{\lambda} F_1 + F_2 + \dots \quad . \tag{3.74}$$

In order to find a solution to (3.73) it is useful to proceed in two steps, and solve first the Hirota equation that does not involve t_0 derivatives (corresponding to the KP hierarchy), that is

$$(z_1 - z_2) \left(e^{-\lambda(D_r(z_1) + D_r(z_2))} \tau_\lambda\right) \left(e^{-\lambda D(z_3)} \tau_\lambda\right) + \text{cycl. perms. of } 1,2,3 = 0. \tag{3.75}$$

Our conjecture is that these equations are obeyed by the Neumann coefficients $\mathcal{N}^{rs}_{mn}$ introduced above, provided we identify λ with a suitable function $f(\mu)$ of μ. At present we are not in the condition to prove this conjecture in a non–perturbative way. But we can expand in powers of λ and try to prove it order by order in λ. We will be able to do it up to second order in λ, with the identification $\lambda = \mu + \frac{\tau_0}{2}\mu^2 + \dots$.

Expanding (3.73) in powers of λ we obtain an infinite set of equations that constrain the correlators at the different orders of approximation. To order 0 we get the dispersionless Hirota equation. Our task is therefore to prove that the next order equations hold.

3.4.2 Order μ approximation

Expanding (3.75) to first order in λ and identifying λ with μ we find

$$(z_1 - z_2) e^{D_r(z_1) D_r(z_2) F_0} D_r(z_1) D_r(z_2) F_1 + \text{cycl. perms. of} \quad 1,2,3 = 0 \tag{3.76}$$

while, expanding (3.73), we get

$$\begin{aligned} -z_1 e^{-\partial_{t_0^{(r)}} D_r(z_1) F_0} \partial_{t_0^{(r)}} D_r(z_1) F_1 + z_2 e^{-\partial_{t_0^{(r)}} D_r(z_2) F_0} \partial_{t_0^{(r)}} D_r(z_2) F_1 \\ = (z_1 - z_2) e^{D_r(z_1) D_r(z_2) F_0} D_r(z_1) D_r(z_2) F_1 . \end{aligned} \tag{3.77}$$

In deriving these equations we have used the information that all third order derivatives of F_0 (three–point functions) vanish (see previous section). It is easy to see that if we make the following identifications:

$$D_r(z_1)D_r(z_2)F_1 = \mathcal{N}_1^{rr}(z_1,z_2) \tag{3.78}$$

with $D_r(z_1)D_r(z_2)F_0$ as in (3.46), eq.(3.76) is satisfied. If in addition we identify

$$\partial_{t_0^{(r)}} D_r(z)F_1 = \beta_{r+1}\beta_r \mathcal{N}_1^r(z), \tag{3.79}$$

with $\partial_{t_0^{(r)}} D_r(z)F_0$ as in (3.46), the more general eq.(3.77) is satisfied as well.

3.4.3 Order μ^2 approximation

Expanding (3.75) to order λ^2 and identifying λ with $\mu + \frac{\tau_0}{2}\mu^2 + \dots$ we find

$$\begin{aligned} &(z_1 - z_2)e^{D_r(z_1)D_r(z_2)F_0}\Big(D_r(z_1)D_r(z_2)F_2 + \frac{1}{2}\left(D_r(z_1)D_r(z_2)F_1\right)^2 \\ +\ &\frac{1}{6}D_r(z_1)D_r(z_2)^3F_0 + \frac{1}{4}D_r(z_1)^2D_r(z_2)^2F_0 + \frac{1}{6}D_r(z_1)^3D_r(z_2)F_0\Big) \\ +\ &\text{cycl. perms. of } 1,2,3 = 0 \end{aligned} \tag{3.80}$$

while expanding (3.73) to the same order we get

$$\begin{aligned} &z_1 e^{-\partial_{t_0^{(r)}} D_r(z_1)F_0}\Big(-\partial_{t_0^{(r)}} D_r(z_1)F_2 + \frac{1}{2}(\partial_{t_0^{(r)}} D_r(z_1)F_1)^2 + \frac{1}{4}\partial^2_{t_0^{(r)}} D_r(z_1)^2F_0 \\ &-\frac{1}{6}\partial_{t_0^{(r)}} D_r(z_1)^3F_0 - \frac{1}{6}\partial^3_{t_0^{(r)}} D_r(z_1)F_0\Big) - z_2 e^{-\partial_{t_0^{(r)}} D_r(z_2)F_0}\Big(-\partial_{t_0^{(r)}} D_r(z_2)F_2 \\ &+\frac{1}{2}(\partial_{t_0^{(r)}} D_r(z_2)F_1)^2 + \frac{1}{4}\partial^2_{t_0^{(r)}} D_r(z_2)^2F_0 - \frac{1}{6}\partial_{t_0^{(r)}} D_r(z_2)^3F_0 - \frac{1}{6}\partial^3_{t_0^{(r)}} D_r(z_2)F_0\Big) \\ &= (z_1 - z_2)e^{D_r(z_1)D_r(z_2)F_0}\Big(D_r(z_1)D_r(z_2)F_2 + \frac{1}{2}\left(D_r(z_1)D_r(z_2)F_1\right)^2 \\ &+\frac{1}{6}D_r(z_1)D_r(z_2)^3F_0 + \frac{1}{4}D_r(z_1)^2D_r(z_2)^2F_0 + \frac{1}{6}D_r(z_1)^3D_r(z_2)F_0\Big). \end{aligned} \tag{3.81}$$

In deriving this equation we have used once again the information that all odd order derivatives of F_0 and F_1 vanish. Eq.(3.81) is satisfied provided we make the following identifications:

$$D_r(z_1)D_r(z_2)F_2 = \mathcal{N}_2^{rr}(z_1,z_2) \tag{3.82}$$

with four-derivatives as defined in (3.55). If, in addition, we identify

$$\partial_{t_0^{(r)}} D_r(z)F_2 = \beta_{r+1}\beta_r \mathcal{N}_2^r(z) \tag{3.83}$$

the more general equation (3.81) is satisfied.

Taking into account eqs. (3.46), (3.78–3.79) and (3.82–3.83) it is plausible to expect that the following universal identification is valid in general:

$$D_r(z_1)D_r(z_2)F_j = \mathcal{N}_j^{rr}(z_1, z_2), \quad \partial_{t_0^{(r)}} D_r(z)F_j = \beta_{r+1}\beta_r \mathcal{N}_j^r(z) \tag{3.84}$$

for j=0,1,2,3,... Nevertheless, in this case one may have to modify the definitions (67-68) of the generating functions for $j > 2$.

As a final remark, we point out the four–point functions we have found are not the unique solutions to the Hirota equations for four-point functions. The one that we have found however are likely to be the unique solution that are consistent with the two–point functions of order 0, 1 and 2 in the μ expansion (although we have not been able to exclude other solutions). This question is intertwined with two related problems: on the one hand the question of defining in terms of matter oscillators $a_n^{(r)}$ the 1D associated quantum system whose two–point correlators are given by $\langle V_3 | a_m^{(r)\dagger} a_n^{(s)\dagger} |0\rangle$ and underlies the LCSFT three strings vertex; on the other hand defining an integrable system, very likely a matrix model, where all these correlators can explicitly be calculated. A successful search of such an integrable model is also likely to lead us to a natural explanation of why the somewhat mysterious factors m^{2j} and $(nm)^j$ need to be inserted in the generating functions (3.67–3.68) in order to square matters.

Chapter 4

1/2-BPS states in VSFT

The essence of this chapter is the observation that there is a remarkable correspondence between states one meets in the framework of the AdS/CFT duality, and solutions of vacuum string field theory (VSFT). The correspondence is simply sketched and is far from exhaustive (mainly because we do not know enough about supersymmetric VSFT), but it is very suggestive and, if confirmed, it could lead to very interesting consequences. Roughly speaking it goes as follows. In the framework of type IIB superstring theory, AdS/CFT duality establishes a correspondence between $\mathcal{N}=4$ superconformal $U(N)$ gauge field theory on the boundary of AdS_5 and supergravity on the background $AdS_5 \times S^5$. In the strongest formulation the correspondence is between the two theories as a whole. Here we will limit our consideration to a class of 1/2–BPS states which can be formulated as composite of the $U(N)$ gauge theory scalars. In general they can be cast in the form of Schur polynomials, and thus they are in one to one correspondence with Young diagrams, represented (in the case of giant gravitons [78], for instance) by columns of maximal size (number of boxes) N. On the supergravity side they correspond to 1/2 BPS states that are solutions to the supergravity equations of motion. The latter represent localized states in the AdS geometry that wrap around S^3 cycles of S^5; they are stabilized by their angular momentum J in S^5, with the condition $J \leq N$. There are other significant 1/2–BPS states with mass $\sim N^2$, the superstars [79]. They correspond to Young diagrams represented by approximate triangles. On the supergravity side these are singular 1/2–BPS states with a naked singularity, which are regarded as solutions on the verge of developing a black-hole horizon due to the quantum corrections. Giant gravitons and superstars are two examples of a zoo of new entities that can be constructed in similar ways.

An interesting question is the following one: what is the precise relation between a state in gauge field theory and the corresponding supergravity solution? More precisely, how does the geometry that characterize the latter arise from the former (which, at first sight, is a totally ungeometrical object)? The answer seems to be coarse-graining: geometry arises from averaging details of the quantum states in the gauge theory side.

The argument brought forth by [80, 81, 82] goes as follows. One remarkable aspect of the above gauge field theory states is that they can be represented also in terms of N fermionic oscillators in a harmonic potential. The correspondence can be once again established via Young diagrams: quantum systems with the same Young diagram describe the same quantum state. This lends itself to a very interesting development: to quantum systems of this type we can associate in a one–to–one way Wigner distribution functions. In general, to a point–like system in a (q, p) phase space we can associate a Wigner distribution function $W(q, p)$. This is nothing but the bosonization of the original fermion system, but $W(q, p)$ is also very close to a probability distribution in phase space. In this way we can associate a Wigner distribution to any state, such as the vacuum, "black rings", superstars, etc. Now, it so happens that these Wigner distributions are characterized in the large N limit by (coarse–grained) profiles that can be matched to the corresponding geometry (droplets) of the 1/2 BPS supergravity solutions.

With the above premise, the point we want to make is that the Wigner distributions for the above introduced states naturally appear in VSFT. More precisely, the same profiles appear as low energy space profiles of VSFT solutions. This correspondence leads us to a related subject: open–closed string duality as seen from the SFT point of view. A.Sen has recently conjectured, [83], that open string theory might be able to describe all the closed string physics, at least in a background where D–branes are present. In this sense VSFT should be a privileged vantage point: the tachyon condensation vacuum physics can only represent closed string theory and thus VSFT should be able to describe closed string theory in the sense of [83]. The existence of the D–brane solutions mentioned above is a confirmation of this. However these D–brane solutions are expressed as squeezed (or related) states. At most, in the presence of a background B-field, we can produce a space profile thereof. However it has not been known so far how to associate a corresponding geometry. The correspondence between space profiles and Wigner distributions may be the clue: by interpreting a space profile as a Wigner distribution, we can reconstruct a half–BPS state and as a consequence arrive at some definite geometry, which is the coarse–grained averaging over the corresponding fermion systems.

4.1 Half–BPS solutions

In the field theory side of the AdS/CFT correspondence, half–BPS multiplets of $\mathcal{N} = 4$ Yang–Mills theory fall into representations $(0, l, 0)$ of the $SO(6)$ R–symmetry group. Highest weight states can be constructed as gauge invariant polynomials of a complex scalar field X. The conformal dimension of the latter is $\Delta = 1$ and the $U(1)$ R–charge $J = 1$, where $U(1) \in SO(6)$. A highest weight therefore satisfies $\Delta = J$. Basically such states are constructed out of multiple traces of X. The most general state of this type of charge n takes the form

$$(\mathrm{tr}(X^{l_1}))^{k_1}(\mathrm{tr}(X^{l_2}))^{k_2} \dots (\mathrm{tr}(X^{l_p}))^{k_p} \tag{4.1}$$

where the integers l_i, k_i form a partition of n: $\sum_{i=1}^{p} l_i k_i = n$. A basis for these states is given by the degree n Schur polynomials of the group $U(N)$. These in turn correspond to Young tableaux of maximal column length N. Therefore we can classify these highest weight states (chiral primaries) by means of Young diagrams [84, 85, 86].

It can be shown that they can be represented in another useful way. In fact their correlators can be related, by using the canonical approach, by correlators of a suitable (time–dependent) matrix model with a quadratic potential [84, 87]. The matrix model can be solved also in another way, by diagonalizing it and producing in this way a Vandermonde determinant Jacobian factor [86]. The latter can be lifted to the exponential giving rise to a repulsive potential among the eigenvalues. The result is that we can interpret the eigenvalues λ_i as a system of N fermionic oscillators with Hamiltonian $H = \sum_i \lambda_i^\dagger \lambda_i + \frac{1}{2}$. The energy levels of this system are given by $E_i = n_i + \frac{1}{2}$, where n_i are nonnegative integers. The corresponding wavefunctions are given in terms of Slater determinants

$$\Psi(\lambda_1, \lambda_2, \dots, \lambda_n) \sim e^{-\sum_i \frac{\lambda_i^2}{2}} \operatorname{Det} \begin{pmatrix} H^{n_1}(\lambda_1) & H^{n_1}(\lambda_2) & \dots & H^{n_1}(\lambda_N) \\ H^{n_2}(\lambda_1) & H^{n_2}(\lambda_2) & \dots & H^{n_2}(\lambda_N) \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ H^{n_N}(\lambda_1) & H^{n_N}(\lambda_2) & \dots & H^{n_N}(\lambda_N) \end{pmatrix} \tag{4.2}$$

where H_n are the Hermite polynomials for a single harmonic oscillator. The ground state Ψ_0 corresponds to $n_1 = 0, n_2 = 1, \dots n_N = N-1$. Therefore the generic excited state can be represented by means of a Young diagram with rows $(r_1, r_2, \dots, r_N)$, with $r_i = n_i - i + 1$ not all vanishing natural numbers in decreasing order. The energy of the state above the Fermi sea is $E = J = \sum_i r_i$, which is the total number of boxes in the Young diagram (for the relation to Fermi systems, see [18, 86, 88, 89, 90, 91]).

Let us list a few states which will be considered in the sequel by means of their Young diagram representation. A giant graviton is represented by a single column Young diagram, whose maximum length is of course N. A giant graviton, [78, 92, 93] is a half–BPS state which can be described as a D3–brane wrapping around an S^3 cycles in the S^5 factor of $AdS_5 \times S^5$. Stability is guaranteed by the spinning of the brane around an axis in S^5. The angular momentum has an upper bound $J \le N$, which is a manifestation of the stringy exclusion principle. Since $\Delta = J$, the representation by means of a Young tableau incorporates in a simple way the exclusion principle.

A dual giant graviton, i.e. a D3–brane wrapping around an S^3 cycle in AdS_5, is represented by a one–row Young diagram of arbitrary length (no bound here). A black ring is represented by a large rectangular diagram of size N (see below). A superstar is represented by a large triangular diagram of size $\sim N$. It represents a stack of giant gravitons located at the origin of AdS_5. From the supergravity viewpoint, it is a singular solution in that it has a naked singularity. It is conjectured that due to string corrections it may actually be completely regular solution.

In the last two cases the energy of the states is proportional to the area of the Young tableau and therefore $\sim N^2$. Following in particular [81], these are the states we will be mostly interested in in the following and we will consider them in the large N limit. In [81] a limit shape was introduced for the corresponding Young tableaux in the continuous limit. This is a function $y(x)$, where x runs from left to right along the rows and y from bottom to top along the columns. The origin is set at the leftmost bottom box of the tableau. For instance, for the superstar ensemble we have $\Delta = NN_c/2$ and $y(x) = \frac{N_c}{N}x$, where N_c is the number of columns.

4.1.1 1/2–BPS states as supergravity solutions

In [94] a beautiful characterization of 1/2–BPS states in type IIB supergravity was found. Regular 1/2–BPS solutions with a geometry invariant under $SO(4) \times SO(4) \times R$ correspond to the following ansatz

$$\begin{aligned} ds^2 &= -h^{-2}(dt+V_i\,dx^i)^2 + h^2(dy^2+dx^i\,dx^i) + y\,e^G d\Omega_3^2 + y\,e^{-G} d\tilde{\Omega}_3^2 \\ h^{-2} &= 2y\,cosh\,G \\ y\,\partial_y V_i &= \epsilon_{ij}\,\partial_j z, \qquad y(\partial_i V_j - \partial_j V_i) = \epsilon_{ij}\,\partial_y z \\ z &= \frac{1}{2} tanh\,G \end{aligned} \tag{4.3}$$

where $i,j = 1,2$ and ϵ_{ij} is the antisymmetric symbol. There are also N units of 5–form flux, with

$$F_{(5)} = F_{\mu\nu} dx^\mu \wedge dx^\nu \wedge d\Omega_3 + \tilde{F}_{\mu\nu} dx^\mu \wedge dx^\nu \wedge d\tilde{\Omega}_3$$

where $\mu,\nu = 0,...,3$ refer to t, x^1, x^2, y. As for the ansatz for F and $\tilde{F}$, see [94]. The full solution is determined in terms of a single function z, which must satisfy the equation

$$\partial_i \partial_i\, z + y\partial_y \left(\frac{\partial_y z}{y}\right) = 0 \tag{4.4}$$

One can solve this equation by remarking its analogy with the Laplace equation for an electrostatic potential. Regular solutions can exist only if at the boundary $y = 0$ the function $z(0, x_1, x_2)$ takes the values $\pm\frac{1}{2}$. Therefore regular solutions correspond to boundary functions $z(0, x_1, x_2)$ that are locally constant in the x_1, x_2 plane. The region of this plane where $z = -1/2$ are called droplets and denoted by $\mathcal{D}$. Following [81] we reintroduce in the game $\hbar$ and make the identification $\hbar \leftrightarrow 2\pi\ell_p^4$, noticing that x_1, x_2 have the unusual dimension of a length square.

The area of the droplet must equal N:

$$N = \int_{\mathcal{D}} \frac{d^2x}{2\pi\hbar} \tag{4.5}$$

while the conformal dimension of the state corresponding to the droplet $\mathcal{D}$ is

$$\Delta = \int_{\mathcal{D}} \frac{d^2x}{2\pi\hbar} \frac{1}{2} \frac{x_1^2+x_2^2}{\hbar} - \frac{1}{2}\left(\int_{\mathcal{D}} \frac{d^2x}{2\pi\hbar}\right)^2 \tag{4.6}$$

In conclusion, the information about the solution is encoded in the droplet. For instance, if the droplet is a disk of radius r_0 we recover the $AdS_5 \times S^5$ solution; if the droplet is the upper half plane one gets the plane wave solution. In general if the droplet is compact the solution is asymptotically $AdS_5 \times S^5$. It is useful to introduce the new notation $u(0; x_1, x_2) = \frac{1}{2} - z(0; x_1, x_2)$; u is the characteristic function of the droplet, since it equals 1 inside the droplet and 0 outside. Solutions with such (sharp) characteristic functions are regular since the boundary conditions are satisfied. Solutions characterized by a function u which is not exactly 1 or 0, are singular [95, 96] (for a connection with quantum Hall effect, see [97, 98, 99]). This is the case of the superstar solution [79].

4.1.2 The Wigner distribution

It is clearly of utmost importance to establish a dictionary between the 1/2–BPS states introduced at the beginning of this section starting from $N = 4$ SYM and the droplet solutions. This is tantamount to finding a recipe to recognize the geometry emerging from a given gauge field theory state. The clue is the free fermion representation introduced above: any state represented by a Young diagram can be interpreted as a system of N fermions with energies above the Fermi sea. To this end it is useful to rewrite the formulas (4.5) and (4.6) in the more general form

$$\Delta = \int \frac{d^2x}{2\pi\hbar}\frac{1}{2}\frac{x_1^2+x_2^2}{\hbar}u(0;x_1,x_2) - \frac{1}{2}\left(\int \frac{d^2x}{2\pi\hbar}u(0;x_1,x_2)\right)^2 \tag{4.7}$$

$$N = \int \frac{d^2x}{2\pi\hbar}u(0;x_1,x_2), \tag{4.8}$$

where the integration extends over the whole x_1, x_2 plane. These formulas suggest that u be identified with the semiclassical limit of the quantum one-particle (q,p) phase–space distributions of the free dual fermions after the identification $(x_1,x_2) \leftrightarrow (q,p)$. A phase–space distribution may be understood as an attempt of assigning a probability distribution to the phase–space points. It is a heuristic concept and there is no unique prescription for it. The most well–known distribution is the Wigner one [100]:

$$W(q,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dy \langle q-y|\hat{\rho}|q+y\rangle e^{2ipy/\hbar} \tag{4.9}$$

where $\hat{\rho}$ is the density matrix. In the case of a pure state ψ, $\langle q'|\hat{\rho}|q''\rangle = \psi(q')\,\psi^*(q'')$, therefore

$$W(q,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dy\psi^*(q+y)\psi(q-y)e^{2ipy/\hbar} \tag{4.10}$$

In general $\hat{\rho}$ will take the form of

$$\hat{\rho}(q',q'') = \sum_{f\in\mathcal{F}} \psi_f(q')\psi_f^*(q'')$$

$\mathcal{F}$ being a given family of pure states. We will consider family of pure states representing excited states of N (fermionic) harmonic oscillators $f_n = r_n + n - 1$, with $n = 1,...,N$ (where we have dropped $\hbar$). In this case $\mathcal{F}$ will be a subset of the natural numbers and

$$\psi_{f_n} = A(f_n)H_{f_n}(q/\sqrt{\hbar})e^{-q^2/2\hbar}$$

where $A(n)$ is a normalization constant and H_n are, as above, the Hermite polynomials. Using a well–known integration formula for Hermite polynomials one gets, [100],

$$W(q,p) = \sum_{f_n\in\mathcal{F}} W_{f_n}(q,p) = \frac{1}{2\pi\hbar}e^{-(q^2+p^2)/\hbar}\sum_{f_n\in\mathcal{F}}(-1)^{f_n}L_{f_n}\left(2\frac{q^2+p^2}{\hbar}\right) \tag{4.11}$$

Thinking of W as a probability distribution is certainly a heuristic and approximate concept, because it may be negative (see the considerations in [81], where an improved always positive distribution is introduced, the Husimi distribution). However we will not need it in the following, because we will compare Wigner distributions with space profiles of VSFT (which are not probability distributions either).

Here we are interested in Wigner distributions because they represent a precise recipe to bosonize associated fermion systems: from the fermion system we easily get the Wigner distribution and from the latter we can reconstruct the former, [89, 101]. In the following we will use Wigner distributions in this sense, and will be concerned specifically with distributions relative to ensembles, in which N is supposed to be very large. The semiclassical limit will correspond to $\hbar \to 0$ keeping $\hbar N$ finite. We will use such distributions to make a comparison with the u droplet distributions, [81], and with space profiles in VSFT (for coarse–graining, see also [102, 103, 104]).

Let us consider a few significant cases. The first concerns the Fermi sea. The relevant distribution is

$$2\pi\hbar W_{FS} = 2\pi\hbar \sum_{n=0}^{N-1} W_n(q,p) \tag{4.12}$$

By using a well–known identity for Laguerre polynomials one formally obtains 1 when the summation extends to infinity with fixed $\hbar$. This would not correspond to $AdS_5 \times S^5$. However a numerical analysis shows that the limit $\hbar \to 0$ with $\hbar N$ fixed reproduces the finite disk characteristic of the latter solution (see, for instance, [105]).

The second example involves the Young diagram corresponding to a giant graviton. It has $r_n = 0$, $n < k$ and $r_n = 1$ for $k \leq n \leq N$. The distribution is

$$2\pi\hbar W_{GG} = 2\pi\hbar \left(\sum_{n=0}^{k-2} + \sum_{n=k}^{N} \right) W_n(q,p) \tag{4.13}$$

It is evident that in the large N limit with k fixed, this distribution will be indistinguishable from the Fermi sea one.

The third example is the case corresponding to a rectangular Young diagram of row length K. It represents N fermions all excited above the sea by the same amount K. There is no a priori relation between N and K, but we are interested in the limit of large N and K such that $\hbar K$ as well as $\hbar N$ are finite. The Wigner distribution is

$$2\pi\hbar W_{\text{rect}} = 2\pi\hbar \sum_{n=K}^{N+K-1} W_{K+n-1}(q,p) \tag{4.14}$$

Setting $u(0,x_1,x_2) = 2\pi\hbar W_{\text{rect}}$ this identifies a characteristic function which is (approximately) 1 in the ring $\hbar K \leq \frac{q^2+p^2}{2} \leq \hbar(N+K)$ and 0 outside, in the large N and K limit. This corresponds to the 1/2–BPS called "black ring" in [94]. It has conformal dimension $\Delta = NK \sim N^2$, since K must be some rational multiple of N.

The last example concerns Young diagrams which are approximately triangular with $\Delta = NN_c/2$ and so correspond to superstar ensembles. In this case we have $f_n = (n-1)\delta_n$, with δ_n an integer $\sim \frac{N_c}{N} + 1$. For illustrative purposes let us set $\delta_n = \delta = \frac{N_c}{N} + 1$. Then

$$2\pi\hbar W_{\text{triangle}} = 2\pi\hbar \sum_{n=0}^{N-1} W_{n\delta}(q,p) = 2e^{-\frac{2H}{\hbar}} \sum_{n=0} (-1)^{n\delta} L_{n\delta}(4H/\hbar) \tag{4.15}$$

where $H = (q^2 + p^2)/2$. The result of the analysis in [81] is that in the large N limit

$$2\pi\hbar W^{\infty}_{\text{triangle}} = \frac{1}{\delta} + \text{oscillations at scale}\, \Delta\mathrm{H} = \hbar$$

Therefore identifying once again $2\pi\hbar W^{\infty}_{\text{triangle}}$ with $u(0; x_1, x_2)$ we get approximately $u(0, x_1, x_2) = 1/\delta$ within a finite radius disk. This corresponds to a fractionally filled droplet and represents the superstar solution, which is singular. It is also an explicit example of the relations

$$u(0; r^2) = \frac{1}{1 + y'} = g(E) \tag{4.16}$$

which was conjectured and verified in various examples in [81]. The function $g(E)$ is called *grayscale distribution* and encodes the effective behavior of coarse–grained semiclassical observables in a given quantum state.

4.2 Building 1/2-BPS States in VSFT

We are looking for solutions that mimic the behavior of the half-BPS states discussed in the previous section. As it turns out they must be superpositions of matter projectors (stacks of VSFT D–branes). The latter have the following characteristics: they must cover the ordinary 4D Minkowski space (parallel directions), be, in the low energy limit ($\alpha' \to 0$), delta–function like in 16 directions and have some width in the remaining 6 directions (these 22 directions will be referred to as the transversal ones). Out of the latter two will have a special status, in that a constant B-field will be switched on along them. We can imagine that all the internal dimensions are compactified on tori, but this is not strictly necessary for our argument.

It is possible to construct a full family of such solutions which are $\star$– and *bpz*–orthonormal. This goes as follows, [30, 32]. First we introduce two 'vectors' $\xi = \{\xi_{N\alpha}\}$ and $\zeta = \{\zeta_{N\alpha}\}$, which are chosen to satisfy the conditions

$$\rho_1\xi = 0, \qquad \rho_2\xi = \xi, \qquad \text{and} \quad \rho_1\zeta = 0, \qquad \rho_2\zeta = \zeta, \tag{4.17}$$

where ρ_1, ρ_2 are the half–string projectors [35, 106]. Moreover we define the matrix τ as $\tau = \{\tau_\alpha{}^\beta\} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Next we set

$$\mathrm{x} = (a^\dagger \tau \xi)\,(a^\dagger C \zeta) = (a_N^{\alpha\dagger} \tau_\alpha{}^\beta \xi_{N\beta})(a_N^{\alpha\dagger} C_{NM} \zeta_{M\alpha}) \tag{4.18}$$

Finally we introduce the Laguerre polynomials $L_n(z)$, of the generic variable z, and define the sequence of states

$$|\Lambda_n\rangle = (-\kappa)^n L_n\Big(\frac{\mathrm{x}}{\kappa}\Big)|\mathrm{S}_{\perp\theta}\rangle \tag{4.19}$$

where, for simplicity, we have written down the tensorial factor involving the the y_1, y_2 directions only and understood the other directions. As part of the definition of $|\Lambda_n\rangle$ we require the two following conditions to be satisfied

$$\xi^T \tau \frac{1}{\mathbb{I} - \mathcal{T}^2}\zeta = 1, \qquad \xi^T \tau \frac{\mathcal{T}}{\mathbb{I} - \mathcal{T}^2}\zeta = \kappa \tag{4.20}$$

where κ is a real number. To guarantee Hermiticity for $|\Lambda_n\rangle$, we require

$$\zeta = \tau \xi^*. \tag{4.21}$$

The states $|\Lambda_n\rangle$ satisfy the remarkable properties

$$|\Lambda_n\rangle \star |\Lambda_m\rangle = \delta_{n,m}|\Lambda_n\rangle \tag{4.22}$$

$$\langle \Lambda_n|\Lambda_m\rangle = \delta_{n,m}\langle \Lambda_0|\Lambda_0\rangle \tag{4.23}$$

Therefore each Λ_n, as well as any combination of Λ_n with unit coefficients, are lump solution.

So far we have set $\alpha' = 1$. It is easy to insert back α'. In order to evaluate the low energy profile of $|\Lambda_n\rangle$ we first contract it with the eigenstate of the position operators with eigenvalues y^α: $\langle y|\Lambda_n\rangle$, and then take the limit $\alpha' \to 0$, [21, 32]. The leading term in the α' expansion turns out to be

$$\langle y|\Lambda_n\rangle = \frac{1}{\pi}(-1)^n L_n\left(\frac{2\rho^2}{\theta}\right) e^{-\frac{\rho^2}{\theta}}|\Xi\rangle + \mathcal{O}(\sqrt{\alpha'}) \tag{4.24}$$

where $\rho^2 = y^\alpha y^\beta \delta_{\alpha\beta}$ and $|\Xi\rangle$ is the sliver solution.

The projectors we need in the following have this $\alpha' \to 0$ limit in the y^α directions; as for the remaining directions, they have the form of the sliver in the parallel directions and, finally, they become delta–like functions multiplied by the sliver in the remaining transverse directions, i.e. they are localized at the origin of the latter. This can be easily seen by taking the limit $\theta \to 0$ in the case $n = 0$ in (4.24)[1].

4.3 A correspondence

Looking at eqs.(4.12,4.14,4.15) of subsection 4.1.2, one immediately notices that they can be seen (up to an overall normalization constant) as the low energy limit space profiles of combinations of the string states Λ_n introduced in the previous section, with unit coefficients. Since combinations of Λ_n with unit coefficients are solutions to the equation of motion of VSFT, we can see the above Wigner distributions as the low energy profile of VSFT solutions (up to the common $|\Xi\rangle$ factor). It is therefore tantalizing to make the following association

Wigner distribution for an N fermion system $\leftrightarrow$ VSFT solution

For this to work we must require the correspondence[2] $\hbar \leftrightarrow \theta$ and that the coordinates x_1, x_2 be identified with y_1, y_2. This is what we suggest and would like to motivate in this section. The previous correspondence can be read in two directions. First: one can say that to any 1/2–BPS state to which we can associate a Wigner distribution of the type (4.9), there corresponds a VSFT solution given by a combination

$$|W_{\mathcal{F}}\rangle = \sum_{f_n \in \mathcal{F}} |\Lambda_{f_n}\rangle, \qquad 2\hbar W(q,p)|\Xi\rangle = \langle y|W_{\mathcal{F}}\rangle \tag{4.25}$$

[1] One could easily construct projectors that are 'fat' also along other transverse directions, but we will not need them in the sequel.

[2] It should be recalled that on the SUGRA side we have three parameters, α', g_s and N. With the first two one forms the combination $\hbar = 2\pi g_s \alpha'^2$. On the VSFT side we have also three parameters α', θ and N.

where (p, q) is identified with (y_1, y_2) and the latter are the eigenvalues of $|y\rangle$. Vice versa: to any VSFT solution of the type (4.25) we can associate a Wigner distribution $W(q, p)$ according to (4.9). In this way we can associate to $|W\rangle$ a Young tableau and therefore a 1/2–BPS state in the $\mathcal{N} = 4$ superconformal field theory (before taking the large N limit) and we can associate a geometry (after taking it[3]). The latter point of view is probably the most interesting one. It implies that we may be able to associate a geometry to a given VSFT solution, therefore we are in the condition to answer some of the questions posed by open–closed string duality. Here we see how geometry emerges from a VSFT solution which is entirely expressed in terms of open string creation operators.

In the following we would like to list some arguments in support of our proposal.

1) With the above association we connect a microstate corresponding to a geometry, which is a supergravity solution, to a string state which is a solution of the VSFT equation of motion. The correspondence (4.25) is one–to–one[4] (before the large N limit).

2) The droplet geometry lives in a (x_1, x_2) plane which lies in the internal (compactified) dimensions. In the same way the plane (y_1, y_2) stays in the compactified part of the bosonic target space. As pointed out above, we identify the two planes. One could phrase it by saying that the two space coordinates x_1, x_2, which had been replaced by two phase–space coordinates q, p in the intermediate argument, have returned to their natural role via the identification with y_1, y_2.

3) The correspondence (4.25) tells us how the Pauli principle gets incorporated into a bosonic setting. The numbers f_n in the LHS of (4.25) correspond to the fermion energy levels in the original fermion system. Therefore, due to the Pauli exclusion principle, each f_n can appear only once in the family $\mathcal{F}$. Therefore in the summation each $|\Lambda_{f_n}\rangle$ appears only once. This guarantees that $|W\rangle$ is a VSFT solution[5]. On the other hand any VSFT solution that can be written in the form $\sum_{f_n \in \mathcal{F}} |\Lambda_{f_n}\rangle$ tells us that the numbers $f_n \in \mathcal{F}$ can be interpreted as energy levels of a fermionic harmonic oscillator system, since each appears only once. This is the way the D–brane solutions of VSFT manifest their fermionic nature.

4) The VSFT solution corresponding to the Fermi sea (4.12) is represented by a stack of N (unstable) VSFT D–branes. The giant graviton solution (4.13) is represented by a D–brane missing from the stack. The superstar solution (4.15) is represented by a stack of such missing (unstable) D–branes. This is in keeping with the interpretation of superstars as stack of giant gravitons, [79]. (It is worth remarking at this point that all the VSFT solutions we consider in this paper are composite of VSFT D–branes and there is no direct identification between single VSFT D–branes and single D3–branes in superstring theory.)

5) There is an algebra isomorphism between Wigner distributions of the type (4.9) and VSFT solutions like $|W\rangle$, an isomorphism that was pointed out in [32, 45, 46]. It is a well–known fact that any classical function $f(q, p)$ in a (q, p) phase space can be mapped to a quantum operator $\hat{O}_f$ via the Weyl transform, and that the product for quantum operators $\hat{O}_f \hat{O}_g$ is mapped into the Moyal product $f * g$ for functions. Under this correspondence the $(x_1, x_2) \leftrightarrow (q, p)$ plane becomes noncommutative. It is a well–known fact that, under this correspondence the classical Wigner distributions like (4.12,4.14,4.15) are mapped into

[3] In the process of taking the large N limit one smears out many details, so that multiple states are mapped to the same geometry

[4] See on this point the remark at the end of subsection 4.3.1

[5] If $|\Lambda\rangle$ is some $\star$–projector, $n|\Lambda\rangle$ is a $\star$–projector if and only if $n = 0, 1$.

projectors of the Moyal star algebra:

$$(2\pi\hbar W) * (2\pi\hbar W) = 2\pi\hbar W. \tag{4.26}$$

Actually these distributions turn out to coincide with families of the so–called GMS solitons, [107, 108]. Let us recall the relevant construction. Define the harmonic oscillator $a = (\hat{q} + i\hat{p})/\sqrt{(2\theta)}$ and its hermitian conjugate $a^\dagger$: $[a, a^\dagger] = 1$. The normalized harmonic oscillator eigenstates are: $|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$. Now, via the Weyl correspondence, we can map any rank one projector $|n\rangle\langle n|$ to a classical function of the coordinates x_1, x_2.

$$|n\rangle\langle n| \longleftrightarrow \psi_n(x_1, x_2) = 2\,(-1)^n L_n\left(\frac{2r^2}{\theta}\right) e^{-\frac{r^2}{\theta}} \tag{4.27}$$

where $r^2 = x_1^2 + x_2^2$. Each of these solutions, by construction, satisfy $\psi_n * \psi_n = \psi_n$. These are referred to as GMS solitons [107]. In the previous section we have shown that the low energy limit of $\langle y|\Lambda_n\rangle$ factorizes into the product of the sliver state and $\psi_n(y_1, y_2)$. This means that the VSFT star product factorizes into Witten's star product and the Moyal $*$ product, [45, 46]. More precisely we can formalize the following isomorphism

$$\begin{array}{ccccc} |\Lambda_n\rangle & \longleftrightarrow & P_n & \longleftrightarrow & \psi_n(y_1, y_2) \\ |\Lambda_n\rangle \star |\Lambda_{n'}\rangle & \longleftrightarrow & P_n P_{n'} & \longleftrightarrow & \psi_n * \psi_{n'} \end{array} \tag{4.28}$$

where $*$ denotes the Moyal product.

This remark should not be underestimated. Let us consider $2\pi\hbar W$ and suppose it is such that we can ignore its derivatives with respect to p and q. Then eq.(4.26) becomes $(2\pi\hbar W)^2 = 2\pi\hbar W$, which is the equation of a characteristic function (it can only be either 0 or 1). This is indeed what happens in the case of the vacuum and the black ring solutions, see [81, 82]. It is not the case of the superstar distribution because in that case we cannot ignore derivatives. But this remark suggests that the property of being Moyal projectors is basic for Wigner distributions to represent 1/2–BPS states. The string state $|W\rangle$ 'inherits' this property, it is the 'continuation' of the space profile to the whole string theory. In this sense it is natural that $|W\rangle$ be a string field theory solution.

6) Finally one should point out that there exists a solution generating technique that allows one to produce new solutions starting from a fixed one. As an example let us consider the partial isometries introduced in [109]. They are defined as follows

$$(\mathcal{P}_+)^k|\Lambda_n\rangle = |\Lambda_{n+k}\rangle \tag{4.29}$$

$$(\mathcal{P}_-)^k|\Lambda_n\rangle = |\Lambda_{n-k}\rangle \tag{4.30}$$

$$\tag{4.31}$$

where we define $\Lambda_n = 0$ for $n < 0$ (see [109] for definitions of $\mathcal{P}_\pm$)). Given a state represented by a certain Young diagram, the operator $(\mathcal{P}_+)^k$ adds k boxes in each of the N rows, while its inverse $(\mathcal{P}_-)^k$ removes them. Consider for instance the Fermi sea solution corresponding to eq.(4.12) and apply to it $(\mathcal{P}_+)^K$. According the above equations one gets the solution corresponding to the rectangular Young diagram (4.14). We have seen that (4.14), in the large N limit, leads to the black ring geometry. Note that the transformations that are generated by such partial isometries are area preserving on the droplet plane (the number of fermions is left unchanged). This points to the fact that partial isometries on the open string side are mapped to topology changing transformations on the closed string side.

4.3.1 Matching observables

In this section we deal with the identification of the quantities in VSFT that correspond to two basic observables in the superconformal field theory and in the supergravity side. The latter are given by the total five form flux, N, and by the energy, Δ, (4.5, 4.6). We would like to see how these two observables are encoded in the star algebra that characterizes the VSFT solutions.

The total flux is simply given by the bpz norm of the projector corresponding to the given Young diagram. We have indeed, (4.22),

$$N = \frac{\langle W_{\mathcal{F}} | W_{\mathcal{F}} \rangle}{\langle \Lambda_0 | \Lambda_0 \rangle} \tag{4.32}$$

This is perfectly expected as the total flux is determined by the number of boundary branes producing it. Note that, differently from the usual open string description given by the gauge theory, this observable *is not* part of the definition of the theory but is part of a classical solution (in much the same way as it happens in gravity).

In order to understand how the observable corresponding to Δ emerges from the star algebra, an extension of the (4.22) is necessary. Consider the following non twist invariant states, [30]

$$|\Lambda_{n,m}\rangle = \sqrt{\frac{n!}{m!}}(-\kappa)^n Y^{m-n} L_n^{m-n}\left(\frac{\mathrm{x}}{\kappa}\right)|\mathcal{S}_{\perp\theta}\rangle, \qquad n \leq m \tag{4.33}$$

$$|\Lambda_{n,m}\rangle = \sqrt{\frac{m!}{n!}}(-\kappa)^m X^{n-m} L_m^{n-m}\left(\frac{\mathrm{x}}{\kappa}\right)|\mathcal{S}_{\perp\theta}\rangle, \qquad n \geq m \tag{4.34}$$

where

$$X = a^\dagger \tau \xi \qquad Y = a^\dagger C \zeta \tag{4.35}$$

so that $\mathrm{x} = XY$, and $L_n^{m-n}(z) = \sum_{k=0}^m \binom{m}{n-k} (-z)^k/k!$ are the generalized Laguerre polynomials. Note that $\Lambda_n = \Lambda_{nn}$.

These states star–multiply among themselves in the following way

$$\Lambda_{nm} \star \Lambda_{pq} = \delta_{mp} \Lambda_{nq} \tag{4.36}$$

In VSFT they have been shown to be the vacuum states for perturbative strings stretched between the n-th and the m-th brane, [109]

Thanks to this extended algebra we can explicitly realize the fermionic system of our concern. To this end let's define the following *inner* operators acting on the string Hilbert space

$$\begin{aligned} A_+\phi &= \sum_{n=0}^{\infty} \sqrt{n+1}\, \Lambda_{n+1,n} \star \phi \star \Lambda_{n,n+1} \\ A_-\phi &= \sum_{n=0}^{\infty} \sqrt{n+1}\, \Lambda_{n,n+1} \star \phi \star \Lambda_{n+1,n} \end{aligned}$$

for any string state ϕ. These are the (adjoint representation of) the string field oscillators defined in [110] (at *fixed* half string vector) and behave as the raising/lowering operators of a harmonic oscillator

$$\begin{aligned} A_+ \Lambda_n &= \sqrt{n+1}\, \Lambda_{n+1} \\ A_- \Lambda_n &= \sqrt{n}\, \Lambda_{n-1} \end{aligned}$$

It is then natural to consider the operator

$$H = A_+ A_- \tag{4.37}$$

which, up to zero point energy, is the analog of the harmonic oscillator hamiltonian.

For single brane states we have

$$H\, \Lambda_n = n\, \Lambda_n$$

and, more important,

$$n = \frac{\langle \Lambda_n | H | \Lambda_n \rangle}{\langle \Lambda_0 | \Lambda_0 \rangle}$$

If we evaluate this operator on the stack $W_{\mathcal{F}}$ we get

$$\frac{\langle W_{\mathcal{F}} | H | W_{\mathcal{F}} \rangle}{\langle \Lambda_0 | \Lambda_0 \rangle} = \sum_{n=0}^{N-1} f_n \tag{4.38}$$

This is nothing but the energy of the corresponding fermion ensemble given by the Young diagram $\mathcal{F}$.

Now let us define the observable that corresponds to Δ, which we will denote with the same symbol. As in the gravity side Δ is defined as the difference in 'energy' between the state under consideration and the 'vacuum' (empty AdS), at *fixed* five form flux N, see (4.6).

$$\Delta = \frac{\langle W_{\mathcal{F}} | H | W_{\mathcal{F}} \rangle - \langle W_{\mathcal{F}_0} | H | W_{\mathcal{F}_0} \rangle}{\langle \Lambda_0 | \Lambda_0 \rangle} \tag{4.39}$$

which gives the number of boxes of the Young Tableau. $\mathcal{F}_0 = \{0, 1, ..., N-1\}$ is the Fermi sea at given N.

4.4 Discussion

The correspondence (4.25) in the previous section is based on a series of facts, which have been listed above. The coincidence might be accidental, but we tend to believe it has a deeper meaning. The suggestion that comes from the previous section is summarized in the following table:

$\mathcal{N}=4$ U(N) SYM chiral primaries, Young tableaux		
$\Updownarrow$		
N fermion systems of harmonic oscillators, Young tableaux		
$\Updownarrow$		
Wigner distributions Young tableaux	$\Longleftrightarrow$	VSFT solutions: sum of $*$–projectors, Young tableaux
$\downarrow$		$\downarrow$
Half BPS IIB SUGRA solutions	$\longleftrightarrow$	Singular gravity solutions (?)

where double–line arrows represent one–to–one correspondences, simple down arrows represent the large N limit and the question mark indicates the conjectural part of our proposed correspondence. Let us describe it in more detail.

The fact that $|W\rangle$ is a VSFT solution is the strongest support of our conjectured correspondence. The weak point is that we know it is a solution of bosonic VSFT but we do not know whether it is a solution of the supersymmetric vacuum string field theory. However we would like to notice that the 1/2–BPS states considered in [80, 81, 82] in the gauge theory side, are all (very heavy) bosonic states. It is not unconceivable that the bosonic part of 1/2–BPS states is well described by solutions of the bosonic string theory. Unfortunately the study of the tachyon condensation in superstring field theory has not progressed much, see [111]. From what we know nowadays it is possible that the bosonic parts of some solutions of supersymmetric VSFT take a form like $|W\rangle$, although a complete solutions has not yet been determined.

This raises a problem as to the interpretation of the lower right corner of the above table. Based on the above argument, they should represent the bosonic part of supergravity solutions. Now the Einstein equation for the latter is

$$R_{\mu\nu} \sim F_{\mu\lambda_1\lambda_2\lambda_3\lambda_4} F_\nu{}^{\lambda_1\lambda_2\lambda_3\lambda_4} \tag{4.40}$$

where F is the five–form field strength and where we have set the dilaton to 0. The contribution of the RHS is basic in the case of LLM solutions, as the latter do not satisfy the pure gravity equation $R_{\mu\nu} = 0$. As a consequence the solutions in the lower right corner above are not pure gravity solutions. This is the problem we alluded to above: from a purely bosonic theory we get, in the low energy limit, (the bosonic part of)supergravity solutions. The most likely explanation of this surprising result lies in the type of limit we have taken in VSFT: first $\alpha' \to 0$ with θ fixed, and then $\theta \to 0$ with θN fixed. This two–step limit selects the (bosonic part of the) droplet–like supergravity solutions. But, on the VSFT side, there are other possible limits. For instance, one could take the same limit, but in three steps, first $\alpha' \to 0$, then $\theta \to 0$ and finally $N \to \infty$. This shrinks the droplets to zero size and leads to singular (in the sense of delta–like) solutions, which we can identify with (singular) pure gravity solutions. However these solutions are too singular to base on them any serious discussion. So let us deal with this subject from another viewpoint.

From the above we see that the parameter θ, i.e. the B-field, plays a fundamental role in the VSFT side limit. There is no B-field in the IIB supergravity solutions side, there is instead a background five–form field, whose flux in suitable units equals N. Of course in the bosonic SFT side there cannot be any such background. However we have seen that $\theta \sim 1/B$ is identified with $\hbar$ and, in the large N limit, $\hbar N$ is kept finite. Therefore B and the five–form flux play a parallel role. The five–form flux supports the supergravity solution, while B supports the corresponding VSFT solution, which otherwise would collapse to a delta function. It looks like B is a surrogate of the five–form flux in the bosonic theory (see the considerations about noncommutativity in [112]). One may wonder whether this remark can be confirmed in some independent way. There actually exists a way in the gravity side, although it is very hard to verify it analytically.

The low energy equations of motion for bosonic string theory is

$$R_{\mu\nu} \sim H_{\mu\lambda\rho}H_{\nu}{}^{\lambda\rho} \tag{4.41}$$

where again we have set the dilaton to 0, and $H = dB$. Comparing this with (4.40) we see that the H^2 term might play the role of the five–form term there. There is however an obstacle. Our VSFT solutions contain a constant B–field and if we replace a constant B in (4.41) the H^2 term vanishes. Nevertheless there exists another interpretation. Let us consider for definiteness the vacuum solution, where the droplet is a disk of finite radius r_0 in the phase space. As we remarked above we can easily reproduce this solution on the VSFT side with a constant field $B = B_0$, by taking the limit $N \to \infty$ such that $\theta_0 N \sim r_0$. Now, the region $r > r_0$ corresponds to a vanishing distribution in the large N limit. This can be reproduced as well by taking first $\theta \to 0$ and *then* $N \to \infty$. In other words, in order to reproduce the vacuum solution it is not necessary to assume a constant B-field everywhere. Unfortunately we do not know how to deal with VSFT in the presence of a varying background B-field. But we believe it is reasonable to assume that, anyhow, well inside the droplet the VSFT solution will be described by the solution with $B = B_0$, far outside the droplet by a solution with very large B and in the intermediate region by some interpolating solution. If this is correct then our overall VSFT solution will correspond to a non constant B-field: after coarse–graining the profile will be such that the H^2 term in (4.41) mimics the five–form quadratic term in (4.40). Of course one should take care also of the other gravity equations of motion: in order to satisfy them all a nontrivial dilaton might be necessary and the solution is anyhow very likely to be singular. One may object that we are here in presence of two different VSFT solutions (with constant and non constant B-field) that correspond to the same space profile. This is true, but the profile we have been talking is the same only in the $\alpha' \to 0$ limit. We expect the α' corrections to remove the degeneracy between them.

In any case using bosonic string field theory to establish a correspondence with 1/2–BPS states of an $\mathcal{N} = 4$ YM theory can be taken, at this stage, only as a suggestion. However the elements we have listed above are striking. So let us suppose that our conjecture is correct at least for the class of states and solutions we are interested in. Then it is convenient to view it in the framework of open–closed string duality. One should not forget that VSFT is a version of *open* string field theory, i.e. its language is the language of open string theory. The correspondence we have established above is between 1/2–BPS states of SYM theory in 4D and *full* VSFT solutions. This suggests that the open string field theory we have been considering describes in fact the physics of open strings attached to the stack of D3–branes where the SYM is defined, that is it is the stringy completion of the latter theory. Once again, the appropriate treatment should make use of superstring field theory. But let us suppose that bosonic SFT is enough for the present purpose; then we must conclude that, using tachyon condensation, we have found a way to pass from open SFT solutions to space time geometry (via coarse–graining), which is one of the major problems in open–closed string duality, and this with the additional bonus of the α' corrections.

We would like to add two specifications. The first concerns Chan–Paton factors one is expected to introduce in order to represent a $U(N)$ theory, and we have not. This is in fact not necessary, since it was proven in [109] that VSFT contains solutions with all type of $U(N)$ CP factors without the need to introduce them by hand. The second concerns the string critical dimension, which, in the bosonic case, is D=26, while the physics of SYM theory lives in D=4. However we have seen that our VSFT solution spontaneously solve the problem, because we can choose them translationally invariant in 4D and of finite or zero size in the transverse dimensions, as need be.

We have just considered Wigner distributions. As pointed out in [81] there are other proposals, for instance the Husimi distribution, which is based on a convolution of the Wigner one. On the other hand there are many other solutions in VSFT, beside the family based on the sliver and the Laguerre polynomials we have considered so far, for instance the family of butterfly projectors. It would be interesting to see whether our correspondence extends to other phase–space distributions and to other VSFT solutions.

Concerning the future problems to be studied an interesting one relates to the possible utilization of the full VSFT to calculate α' corrections. It has been suggested that the superstar solution may develop a horizon due to the stringy corrections. Now, a string state like $|W\rangle$ contains the α' corrections to its low energy profile. It is therefore natural to ask whether this knowledge can be translated to the supergravity side. Related to this is the problem of counting the microstates in order to evaluate the entropy of the ensemble, [80, 81, 82]. To this end one should be able to count the distinct string fields corresponding to a given low energy profile. A problem nested into this is related to gauge equivalence. The states $|\Lambda_n\rangle$ are defined in terms of a vector ξ_n (likewise for the ghost part). These infinitely many numbers ξ_n are irrelevant in the low energy limit. This fact is understood in the sense that these numbers are likely to represent only gauge degrees of freedom (otherwise also our previous claim of one–to–one correspondence would seem to need a better phrasing). It would be interesting to find a real proof of this.

Chapter 5

Superstring Field Theory

Bosonic string theory is not the right one to describe nature, for the obvious reason that there are no fermions in it. One should then try to extend all the knowledge we have about SFT to the realm of superstrings. As Superstring Field Theory (SSFT) is a very interesting and important field of research, we would like to review in this chapter some developments made, although there has not been much progress so far.

5.1 Witten's Cubic Superstring Field Theory

Witten proposed an action for SSFT that is just an extension of the SFT one [113]. Nevertheless, the situation now is much more subtle due to the Ramond sector and the concept of picture. We start by analyzing the Never-Schwarz sector, assuming the string field A to have ghost number 1 and picture number -1. One can immediately see that, if we propose the same action as the bosonic one, the cubic term vanishes since it has picture number -3 instead of -2[1]. To fix this problem, we will need the picture changing operator X and its inverse Y

$$\begin{aligned} X(z) = \{Q_B, \xi(z)\} &= c\partial\xi + e^{\phi}G^m + e^{2\phi}b\partial\eta + \partial(e^{2\phi}b\eta) \\ Y(z) &= c\partial\xi e^{-2\phi} \\ \lim_{z\to w} X(z)Y(w) &= \lim_{z\to w} Y(z)X(w) = 1. \end{aligned} \tag{5.1}$$

We use them to define new star product and integration operations[2]

$$A \star B \equiv X(A * B), \qquad \oint A \equiv \int YA, \tag{5.2}$$

where X and Y are inserted at the string midpoint. The reason they are inserted at the midpoint is that we want to preserve the symmetry of the action under the following subalgebra of the Virasoro algebra

[1]We recall that in superstring theory, amplitudes on the disk must have picture charge -2 to be nonvanishing.

[2]In this chapter, we use another convention for the star-product than in the rest of the thesis. Now the usual star product is denoted by $*$.

$$K_n = L_n - (-1)^n L_{-n}. \tag{5.3}$$

This is so if $[K_n, X] = [K_n, Y] = 0$, implying that X and Y must have conformal weight 0 and are inserted at the string midpoint. We then propose the action for the Neveu-Schwarz sector as

$$S_{NS} = -\oint \left(\frac{1}{2} A \star Q_B A + \frac{g}{3} A \star A \star A \right). \tag{5.4}$$

We have written, at least formally, an action for the NS sector, and now we turn to the Ramond one. Since the product of two Ramond sector gauge parameters is supposed to be in the NS sector, we consider the combined R-NS string field $M = (A, \psi)$, where A is a NS state and ψ is a Ramond state with ghost number 1 and picture $-\frac{1}{2}$. We then define the product of two string fields M_1 and M_2 by

$$M_1 \hat{\star} M_2 = (A_1, \psi_1) \hat{\star} (A_2, \psi_2) = (A_1 \star A_2 + \psi_1 * \psi_2, A_1 \star \psi_2 + \psi_1 \star A_2), \tag{5.5}$$

where $*$ is the usual star-product, $\star$ is the product defined in (5.2) and we denote the new product by $\hat{\star}$. We also define an integration for the combined system by

$$\iint (A, \psi) \equiv \oint A, \tag{5.6}$$

since the integral of a Ramond string field must be zero by Lorentz invariance. Putting all of this together, we have finally the action for the combined R-NS system

$$S_{RNS} = -\iint \left(\frac{1}{2} M \hat{\star} Q_B M + \frac{g}{3} M \hat{\star} M \hat{\star} M \right). \tag{5.7}$$

Expanding the action in R-NS components, we have

$$S_{RNS} = -\int \left(\frac{1}{2} A * Q_B A + \frac{1}{2} Y \psi * Q_B \psi + \frac{g}{3} X A * A * A + g A * \psi * \psi \right), \tag{5.8}$$

where we can see that, if we set ψ to zero, it correctly reproduces the action (5.4) for the NS sector only. Also, the quadratic terms can be cast in the usual way once we choose the gauge $b_0 A = b_0 \psi = \beta_0 \psi = 0$, reproducing the correct propagators for the Neveu-Schwarz and Ramond sector fields.

Despite being a very natural extension, Witten's cubic SSFT presents serious problems. The first one has to do with gauge invariance. Formally, (5.7) possesses the following gauge invariance

$$\delta M = Q_B \Lambda + g(M \hat{\star} \Lambda - \Lambda \hat{\star} M), \tag{5.9}$$

but, when one computes it carefully, the variation of the cubic term involves a collision of two X's, producing a divergence since $X(z)X(w) \sim (z-w)^{-2}$. One could solve this problem by adding counterterms to the action, but these counterterms must be infinite even in the classical action, since the divergence exists at tree-level. This would render the action ill-defined unless some regularization was done. This collision problem also shows up when we compute some amplitudes, for example, the 4-boson amplitude.

The second problem concerns the computation of the tachyon potential. At level zero, the potential is negative and has no minimum. The reason for this lies in the fact that, at level zero, the string field becomes

$$A_+ = 0, \qquad A_- = t \cdot ce^{-\phi}(0)|0\rangle, \tag{5.10}$$

where the indices $+,-$ refer to GSO$(+,-)$ sectors. Due to the conservation of $e^{i\pi F}$ (F measures the world-sheet fermion number) at every string vertex, the action can include only

$$A_+ * Q_B A_+, \qquad A_- * Q_B A_-, \qquad A_+ * A_+ * A_+ \quad \text{and} \quad A_+ * A_- * A_-. \tag{5.11}$$

As $A_+ = 0$, the only term that survives is the second one, providing a purely quadratic potential. Going higher and higher in level will not solve the problem, producing steeper and even singular potentials. For example, at level $(\frac{1}{2}, 1)$ and $(1, 2)$, the other fields than the tachyon can be integrated out exactly giving the following effective potentials for the tachyon

$$\begin{aligned} V_{(\frac{1}{2},1)}(t) &= -\frac{t^2}{2} - \frac{81}{32}t^4 \\ V_{(1,2)}(t) &= -\frac{t^2}{2} - \frac{81}{32}\frac{t^4}{1-16t^2}. \end{aligned} \tag{5.12}$$

These problems show that Witten's original proposal for SSFT seems to be flawed, and some modifications might be needed. In the next section, we will describe one of these modifications.

5.2 Modifications to Witten's SSFT

In order to solve the problems described in the last section, a slightly different formalism was proposed [114, 115]. Basically, it consisted in using NS string fields in the 0-picture, not in the -1-picture. It is then clear that a kinetic term like $A * Q_B A$ and a cubic term like $A * A * A$ are not good, since they have picture number 0 instead of -2. We need them an operator that carries picture number -2, which we will call Y_{-2}. We can now write the action as

$$\begin{aligned} S_{RNS} &= -\frac{1}{g^2}\left(\frac{1}{2}\int Y_{-2}\mathcal{A} * Q_B\mathcal{A} + \frac{1}{3}\int Y_{-2}\mathcal{A} * \mathcal{A} * \mathcal{A} + \frac{1}{2}\int Y_{-2}X\psi * Q_B\psi \right. \\ &+ \left. \int Y_{-2}X\mathcal{A} * \psi * \psi\right), \end{aligned} \tag{5.13}$$

where $\mathcal{A}$ is the NS string field in the 0-picture, ψ is the Ramond string field in the natural $-\frac{1}{2}$-picture, and X is the picture changing operator of the previous section. As $\mathcal{A}$ and ψ both have ghost number 1, we see from the action that $Y_{-2}X$ must have ghost number 0 and picture number -1, the same characteristics of the inverse picture changing operator Y. Thus, we must have

$$\lim_{z\to w} Y_{-2}(z)X(w) = \lim_{z\to w} X(z)Y_{-2}(w) = Y(w). \tag{5.14}$$

By examining carefully the question of gauge invariance and amplitudes, one can see that just collisions of $Y_{-2}X$ and YX appear, showing that the theory now is free from collision divergences.

We now pass to the construction of Y_{-2}. We know that, besides (5.14), it is required to satisfy

$$\begin{aligned} L_0|Y_{-2}\rangle &= L_0 Y_{-2}(0)|0\rangle = 0; \\ Q_B|Y_{-2}\rangle &= 0; \\ |Y_{-2}\rangle &\neq Q_B|\Lambda\rangle. \end{aligned}$$

The first condition tells us that Y_{-2} is a Lorentz scalar and has conformal dimension 0. The second condition says it is BRST invariant, and the third one that it is not BRST trivial. Taking all these conditions into account, [114, 115] showed that there are only two possibilities

$$\begin{aligned} Y_{-2}(z) &= -4e^{-2\phi(z)} - \frac{16}{5}e^{-3\phi}c\partial\xi\psi_\mu\partial X^\mu(z), \\ Y_{-2}(z,\bar{z}) &= Y(z)Y(\bar{z}). \end{aligned} \tag{5.15}$$

Actually, these are the only two possibilities up to BRST equivalence, thus giving the same results for on-shell computations. However, once we go off-shell like in SSFT, they produce different results. For example, it was shown that two quadratic actions written in terms of component fields take different forms, and so equations of motion. It is still unknown if the theories arising from the various choices of Y_{-2} are equivalent or not.

We now briefly analyze the question of the tachyon potential, and for that we deal just with the NS sector and choose $Y_{-2} = Y(z)Y(\bar{z})$. The crucial point in this analysis is that Y has a non-trivial kernel, and this will change the result drastically. The lowest eigenstate of L_0^{tot} is now $c_1|0\rangle$ with eigenvalue -1, but this state is annihilated by Y showing why it was not present in the -1-picture version described in the previous section since $A^{(-1)} = Y\mathcal{A}^{(0)}$. The next state to be considered is $\gamma_{\frac{1}{2}}|0\rangle$ with L_0^{tot} eigenvalue $-\frac{1}{2}$, and since

$$\lim_{z\to 0} Y(z)\gamma(0) = \lim_{z\to 0} c\partial\xi e^{-2\phi}(z)\eta e^{\phi}(0) = -ce^{-\phi}(0), \tag{5.16}$$

it corresponds to the zero momentum tachyon in the -1-picture theory. As in the previous case we must consider both GSO(+) and GSO(−) sectors, for example, to analyze the potential at level $(\frac{1}{2}, 1)$, we take

$$\mathcal{A}_+(z) = uc(z), \qquad \mathcal{A}_-(z) = te^{\phi}\eta(z). \tag{5.17}$$

For the same reason as before, the only terms in the action that contribute are the ones with two $\mathcal{A}_+$'s, two $\mathcal{A}_-$'s, three $\mathcal{A}_+$'s, and one $\mathcal{A}_+$ and two $\mathcal{A}_-$'s. The result for the potential is

$$V_{(\frac{1}{2},1)}(t,u) = -u^2 + \frac{9}{16}ut^2 - \frac{1}{4}t^2, \tag{5.18}$$

and we can exactly integrate out the auxiliary tachyon u to get

$$V_{(\frac{1}{2},1)}(t) = \frac{81}{1024}t^4 - \frac{1}{4}t^2. \tag{5.19}$$

We can see from this potential that, since the t^4 term is positive, it has two minima, as expected. The effect of the auxiliary tachyon u is to make the tachyon potential to be of the double-well form. Going higher in level will not affect the form of the potential and will make the value of the potential at the minimum to approach -1, as it should according to Sen's conjecture.

We see that this modified version of Witten's SSFT solves the two problems presented in the previous section, that is, contact terms divergences and the absence of a minimum in the tachyon effective potential. However, new questions arise. For example, a non-physical field could solve the equation of motion $Y_{-2}(Q_B\mathcal{A} + \mathcal{A} * \mathcal{A}) = 0$ if it is in the kernel of Y_{-2}, and we should also deal with the question of different possible choices for Y_{-2}, as they lead to completely different theories off-shell when we use level truncation. Perhaps they are all equivalent once we consider the full theory.

Another approach to SSFT we would like to mention, although we will not discuss it here in detail, was developed by Berkovits ([116] and references therein). His idea was to embed a $\mathcal{N}$=1 critical RNS superstring into a critical $\mathcal{N}$=2 superstring, and further extend it to have a small $\mathcal{N}$=4 superconformal symmetry by introducing additional generators. Then, by generalizing this construction to go off-shell, he builds a Wess-Zumino-Witten-like action instead of a Chern-Simons one like in Witten's version. Berkovits' formulation is free from the problems mentioned previously, but no explicit solution has been found yet and it is not known how to include the Ramond sector in the theory.

Chapter 6

Conclusions

The desire to understand nonperturbative aspects of string theory has led to the development of String Field Theory, and we devoted most of this thesis to a particular version of it called Vacuum String Field Theory. As a result of our research, we obtained time-dependent solutions that interpolate between the perturbative and nonperturbative vacua, describing the process of tachyon condensation. We were also able to get solutions in VSFT which possibly correspond to some 1/2-BPS states appearing both in $\mathcal{N}=4$ SYM and type IIB supergravity, shedding some light into the open/closed string duality.

In this conclusion, we would like to discuss some points that merit further scrutiny. The first point we would like to comment concerns the relation between VSFT solutions and the Schnabl's one. VSFT has proved itself very useful since solutions to its equation of motion in the matter sector were merely projectors of the star algebra, making it easier to find solutions, the sliver and the identity being two of them. The fact that these two states are wedge states, corresponding to n=∞ and n=1 respectively, prompted Schnabl to try, as a solution to OSFT equation of motion, a state expanded in terms of wedge states with ghost insertions (in order to obtain a ghost number one state). A crucial ingredient in finding the solution was to work in the conformal frame of the sliver $z = \arctan\xi$, what, allied with a suitable choice of gauge, made the equation of motion much simpler to solve. Further developments [117] showed that an infinite class of coordinate maps $f(\xi)$ lead to solvable equations of motion and that all these functions define projectors. This points to the relevance of projectors in solving the equations of motion and hints some kind of correspondence between VSFT solutions (projectors) and OSFT solutions.

Another question of great interest is translating Schnabl's solution to the operator approach. Up to now, no proof of the solvability of the equation of motion has been given within it (all computations have been made in the CFT approach), and we believe this would be a very important check on the solution. However, things have proved to be not so easy. So far, we could not find the operator form of Schnabl's solution due to ambiguities in defining the ghost sector of the three string vertex properly. In this case the ghost vertex plays an important role, as the solution is made of wedge states with ghost insertions. Another hurdle we have been grappling with is how to deal with insertions, we do not know precisely how to star multiply states with insertions in the operator approach. We believe that obtaining the operator version of Schnabl's developments will help clarify many questions about VSFT and SFT in general.

Once the bosonic theory is well understood, we must proceed to the superstring case, and here not much progress has been obtained. The theories built so far present many problems related to divergences, ambiguities, or no consistent solutions have been found yet. Certainly, if we want to go on, we will have to solve these problems or construct a new theory free from these drawbacks. We ought to understand better how the concept of picture works in an off-shell theory, since using picture changing operators seems crucial in these theories. For example, two candidates for the double-step picture changing operator give the same theory on-shell but different ones if we are off-shell. Surely, a deeper understanding of this question is in order if we want to build a consistent Superstring Field Theory. Moreover, other attempts have been made like Berkovits' formulation for SSFT, where the problems mentioned previously have been solved. It also has the advantage that no picture changing operators are used, yet no explicit solution has been found and it is not known how to include the Ramond sector in the theory. As we can see, there is still a lot to be done concerning the development of SSFT.

Appendix A

Some Useful Relations

In this Appendix we collect some useful results and formulas involving the matrices of the three strings vertex coefficients.

To start with, we recall that

- (i) V^{rs}_{nm} are symmetric under simultaneous exchange of the two couples of indices;
- (ii) they are endowed with the property of cyclicity in the r, s indices, i.e. $V^{rs} = V^{r+1,s+1}$, where $r, s = 4$ is identified with $r, s = 1$.

Next, using the twist matrix C ($C_{mn} = (-1)^m \delta_{mn}$), we define

$$X^{rs} \equiv CV^{rs}, \qquad r, s = 1, 2, \tag{A.1}$$

These matrices are often rewritten in the following way $X^{11} = X$, $X^{12} = X_+$, $X^{21} = X_-$. They commute with one another

$$[X^{rs}, X^{r's'}] = 0, \tag{A.2}$$

moreover

$$CV^{rs} = V^{sr}C, \qquad CX^{rs} = X^{sr}C \tag{A.3}$$

Next we quote some useful identities:

$$\begin{aligned}
&X + X_+ + X_- = 1 \\
&X_+X_- = (X)^2 - X \\
&(X_+)^2 + (X_-)^2 = 1 - (X)^2 \\
&(X_+)^3 + (X_-)^3 = 2(X)^3 - 3(X)^2 + 1
\end{aligned} \tag{A.4}$$

The same relations hold if we replace X, X_+, X_-, T by X', X'_+, X'_-, T', respectively.

Appendix B

Diagonal representation of CU'

With reference to formula (2.35), we illustrate the spectroscopy and diagonal representation of CU'. The matrix CU' is hermitian, unitary and commutes with $U'C$. The discrete eigenvalues ξ and $\bar{\xi}$ are determined as follows, [15]. Let

$$\xi = -\frac{2 - \cosh\eta - i\sqrt{3}\sinh\eta}{1 - 2\cosh\eta} \tag{B.1}$$

and

$$F(\eta) = \psi(\frac{1}{2} + \frac{\eta}{2\pi i}) - \psi(\frac{1}{2}), \qquad \psi(z) = \frac{d\log\Gamma(z)}{dz} \tag{B.2}$$

Then the eigenvalues ξ and $\bar{\xi}$ are the solutions of

$$\Re F(\eta) = \frac{b}{4} \tag{B.3}$$

The eigenvectors $V_n^{(\xi)}$ are defined via the generating function

$$\begin{aligned} F^{(\xi)}(z) = \quad & \sum_{n=1}^{\infty} V_n^{(\xi)} \frac{z^n}{\sqrt{n}} = -\sqrt{\frac{2}{b}} V_0^{(\xi)} \left[\frac{b}{4} + \frac{\pi}{2\sqrt{3}} \frac{\xi - 1}{\xi + 1} + \log iz \right. \\ & \left. + \, e^{-2i(1+\frac{\eta}{\pi i})\arctan z} \Phi(e^{-4i\arctan z}, 1, \frac{1}{2} + \frac{\eta}{2\pi i}) \right] \end{aligned} \tag{B.4}$$

where $\Phi(x, 1, y) = 1/y \, {}_2F_1(1, y; y + 1; x)$, while

$$V_0^{(\xi)} = \left(\sinh\eta \frac{\partial}{\partial\eta} \left[\log\Re F(\eta)\right] \right)^{-\frac{1}{2}} \tag{B.5}$$

As for the continuous spectrum, it is spanned by the variable k, $-\infty < k < \infty$. The eigenvalues of CU' are given by

$$\nu(k) = -\frac{2 + \cosh\frac{\pi k}{2} + i\sqrt{3}\sinh\frac{\pi k}{2}}{1 + 2\cosh\frac{\pi k}{2}}$$

The generating function for the eigenvectors is

$$F_c^{(k)}(z) = \sum_{n=1}^{\infty} V_n^{(k)} \frac{z^n}{\sqrt{n}} = V_0^{(k)} \sqrt{\frac{2}{b}} \left[-\frac{b}{4} - \left(\Re F_c(k) - \frac{b}{4} \right) e^{-k \arctan z} - \log iz \right. \tag{B.6}$$
$$\left. - \left(\frac{\pi}{2\sqrt{3}} \frac{\nu(k)-1}{\nu(k)+1} + \frac{2i}{k} \right) + 2i\, f^{(k)}(z) - \Phi(e^{-4i \arctan z}, 1, 1 + \frac{k}{4i})\, e^{-4i \arctan z}\, e^{-k \arctan z} \right]$$

where

$$F_c(k) = \psi(1 + \frac{k}{4\pi i}) - \psi(\frac{1}{2}),$$

while

$$V_0^{(k)} = \sqrt{\frac{b}{2\mathcal{N}(k)}} \left[4 + k^2 \left(\Re F_c(k) - \frac{b}{4} \right)^2 \right]^{-\frac{1}{2}} \tag{B.7}$$

The continuous eigenvalues of X', X'_-, X'_- and T' (for the conventional lump) are given by same formulas as for the X, X_+, X_- and T case, eqs(2.31,2.32). As for the discrete eigenvalues, they are given by the formulas

$$\mu_\xi^{rs} = \frac{1 - 2\,\delta_{r,s} - e^{\eta}\,\delta_{r+1,s} - e^{-\eta}\,\delta_{r,s+1}}{1 - 2\cosh \eta}$$
$$t_\xi = e^{-|\eta|} \tag{B.8}$$

Appendix C

Limits of X' and T'

In this Appendix we briefly discuss the low energy and high energy limit of X' and T' in the oscillator basis. The Neumann coefficients $V'^{(rs)}_{NM}$ we use are given in Appendix B of [11]. They explicitly depend on the b parameter. In the low energy limit the three–strings vertex can be expanded by means of a parameter ϵ (a dimensionless parameter, in fact an alias of α') , see [21]. This translates into an expansion for $V'^{(rs)}_{NM}$ triggered by the following re-scalings

$$\begin{aligned} &V^{(rs)}_{mn} \to V^{(rs)}_{mn} \\ &V^{(rs)}_{m0} \to \sqrt{\epsilon} V^{(rs)}_{m0} \\ &V_{00} \to \epsilon V_{00} \end{aligned} \tag{C.1}$$

For instance X' is expanded as follows to the lowest orders of approximation

$$X' = \begin{pmatrix} -\frac{1}{3} + \frac{8}{3} V_{00} \frac{\epsilon}{b} & -\frac{4}{3}\sqrt{\frac{2\epsilon}{b}} \langle \mathbf{v}_e | \\ -\frac{4}{3}\sqrt{\frac{2\epsilon}{b}} | \mathbf{v}_e \rangle & X - \frac{8}{3}\frac{\epsilon}{b}(|\mathbf{v}_e\rangle\langle\mathbf{v}_e| - |\mathbf{v}_o\rangle\langle\mathbf{v}_o|) \end{pmatrix} \tag{C.2}$$

where

$$|\mathbf{v}_e\rangle_n = -\frac{3}{2\sqrt{2}} V^{(11)}_{0n}, \qquad |\mathbf{v}_o\rangle_n = \sqrt{\frac{3}{8}} (V^{(12)}_{0n} - V^{(21)}_{0n})$$

It is interesting to remark that the parameter ϵ appears always divided by b, so that one could just as well absorb ϵ into $1/b$ and say that the expansion is in the parameter $1/b$ for large b. However to avoid confusion it is useful to keep the two parameters distinct.

Now, it is immediate to see that

$$T' = \begin{pmatrix} -1 + \mathcal{O}(\frac{\epsilon}{b}) & \mathcal{O}(\sqrt{\frac{\epsilon}{b}}) \\ \mathcal{O}(\sqrt{\frac{\epsilon}{b}}) & T + \mathcal{O}(\frac{\epsilon}{b}) \end{pmatrix} \tag{C.3}$$

This is correct provided we can prove that the use of (C.2) to compute T' makes full sense, that is all the terms of the expansion in powers of $\sqrt{\frac{\epsilon}{b}}$ are well defined. One can actually see that a naive expansion leads to infinite coefficients. This is a well–known problem, pointed out for the first time in [21], which requires a regularization. A nice way to introduce a regulator is to switch on a constant background B field. We will not do it here, but we quote the result: in the presence of a B field the infinities disappear, and the expansion (C.3) makes full sense. From this we deduce in particular that

$$T'_{nm} = T_{nm} + \mathcal{O}(\frac{\epsilon}{b}) \tag{C.4}$$

Let us consider now another extreme expansion, that is the limit $\alpha' \to \infty$. In just the same way as above, we can introduce an alias, t ($t >> 1$) instead of ϵ. So, in particular,

$$\begin{aligned} &V^{(rs)}_{mn} \to V^{(rs)}_{mn} \\ &V^{(rs)}_{m0} \to \sqrt{t} V^{(rs)}_{m0} \\ &V_{00} \to t V_{00} \end{aligned} \tag{C.5}$$

In this case X' to the lowest orders of approximation becomes

$$X' = \begin{pmatrix} 1 + \frac{2}{3}\frac{1}{V_{00}}\frac{b}{t} & -\frac{2}{3}\sqrt{\frac{2b}{t}}\langle \mathbf{v}_e| \\ -\frac{2}{3}\sqrt{\frac{2b}{t}}|\mathbf{v}_e\rangle & X - \frac{4}{3}\frac{1}{V_{00}}(1 - \frac{1}{V_{00}}\frac{b}{2t})(|\mathbf{v}_e\rangle\langle \mathbf{v}_e| - |\mathbf{v}_o\rangle\langle \mathbf{v}_o|) \end{pmatrix} \tag{C.6}$$

The lowest order in this expansion is known as the tensionless limit [22]. Also here one must be careful about the use of this expansion in calculating T'. From eq.(C.6) one finds that

$$T'_{00} = 1 + \mathcal{O}(\frac{b}{t}) \tag{C.7}$$

Appendix D

The $\alpha' \to 0$ *limit of* $\check{S}'^{(c)}_{nm}$ *and* $\check{S}'^{(c)}_{0n}$

In this Appendix we discuss the limit of the unconventional lump matrix elements $\check{S}'^{(c)}_{nm}$ and $\check{S}'^{(c)}_{0n}$ by means of the diagonal basis. According to (2.42), we speak interchangeably of the $b \to \infty$ limit and the $\eta \to \infty$ one. When applying the results of this Appendix in the thesis, we understand that $1/b$ is replaced everywhere by ϵ/b with finite b.

As a preliminary step let us prove that

$$\lim_{b\to\infty} \left(V_0^{(k)}\right)^2 = \delta(k) \tag{D.1}$$

A rather informal way to see this is as follows. Looking at (B.7) it is easy to realize that the limit always vanishes provided $k \neq 0$. Therefore the support of the limiting distribution must be at $k = 0$. We can therefore expand all the functions involved in k around $k = 0$ and keep the leading terms. Since $\Re F_c(k) \approx 1.386...$ around this point, we can disregard $\Re F_c(k)$ compared to $b/4$ in the $b \to \infty$ limit. Therefore we easily find

$$\lim_{b\to\infty} (V_0^{(k)})^2 = \lim_{b\to\infty} = \frac{\bar{b}}{\pi}\frac{1}{1+\bar{b}^2k^2}$$

where $\bar{b} = b/8$. Now defining $\bar{\epsilon} = 1/\bar{b}$, the limit becomes

$$\lim_{\bar{\epsilon}\to 0} \frac{1}{\pi}\frac{\bar{\epsilon}}{k^2+\bar{\epsilon}^2} = \delta(k) \tag{D.2}$$

according to a well–known representation of the delta function. We can also show that

$$(V_0^{(k)})^2 = \delta(k) + \mathcal{O}(1/b)$$

From now on we suppose that, in the $\int dk$ integrals , we are allowed to replace the integrands with their $1/b$ expansions, and that the results we obtain are valid at least in an asymptotic sense. This attitude is always confirmed by numerical approximations.

D.1 Limit of $\check{S}'^{(c)}_{mn}$

Let us rewrite the generating function for $V_m^{(k)}$ as follows:

$$F^{(k)}(z) = A^{(k)} f^{(k)}(z) - \frac{(1-\nu(k))V_0^{(k)}}{\sqrt{b}} B(k,z) \tag{D.3}$$

where

$$A^{(k)} = V_0^{(k)}\sqrt{\frac{2}{b}}k\left(\Re F_c(k) - \frac{b}{4}\right) \tag{D.4}$$

and

$$\begin{aligned} B(k,z) &= \frac{2}{1-\nu(k)}\left[\Re F_c(k) + \frac{\pi}{2\sqrt{3}}\frac{\nu(k)-1}{\nu(k)+1} + \frac{2i}{k} + \log(iz) - 2if^{(k)}(z)\right. \\ &+ \left.\text{LerchPhi}(e^{-4i\arctan(z)}, 1, 1+\frac{k}{4i})e^{-4i\arctan(z)}e^{-k\arctan(z)}\right] \end{aligned} \tag{D.5}$$

From (D.3) we can derive a useful expression for $V_m^{(k)}$:

$$V_m^{(k)} = A^{(k)}\frac{\sqrt{m}}{2\pi i}\oint dz\frac{f^{(k)}(z)}{z^{m+1}} - \frac{(1-\nu(k))V_0^{(k)}}{\sqrt{b}}\frac{\sqrt{m}}{2\pi i}\oint dz\frac{B(k,z)}{z^{m+1}} \tag{D.6}$$

Since $v_m^{(k)} = \frac{\sqrt{m}}{2\pi i}\oint dz\frac{f^{(k)}(z)}{z^{m+1}}$ and $S'^{(c)}_{mn} = \int_{-\infty}^{\infty} dk\ t(k)V_m^{(k)}V_n^{(-k)}$ we get:

$$\begin{aligned} \check{S}'^{(c)}_{mn} &= \int_{-\infty}^{\infty} dk\ t(k)\left[A^{(k)}A^{(-k)}v_m^{(k)}v_n^{(-k)} - A^{(k)}V_0^{(k)}v_m^{(k)}(1-\bar{\nu}(k))\tilde{B}_n(-k)\frac{1}{\sqrt{b}}\right. \\ &- \left.A^{(-k)}V_0^{(k)}v_n^{(-k)}(1-\nu(k))\tilde{B}_m(k)\frac{1}{\sqrt{b}} + (V_0^{(k)})^2(1-\bar{\nu}(k))(1-\nu(k))\tilde{B}_m(k)\tilde{B}_n(-k)\frac{1}{b}\right] \end{aligned} \tag{D.7}$$

where

$$\tilde{B}_m(k) = \frac{\sqrt{m}}{2\pi i}\oint dz\frac{B(k,z)}{z^{m+1}}$$

Now we want to take the limit of (D.7) when $b \to \infty$. To this end we notice the following:

$$\begin{aligned} \lim_{b\to\infty} A^{(k)}A^{(-k)} &= \lim_{b\to\infty}(V_0^{(k)})^2\left(\frac{-2k^2}{b}\right)\left(\Re F_c(k) - \frac{b}{4}\right)^2 \\ &= \lim_{x\to-\infty}\left(\frac{-k^2}{N(k)}\right)\frac{x^2}{4+k^2x^2} = \left(\frac{-k^2}{N(k)}\right)\frac{1}{k^2} = -\frac{1}{N(k)} \end{aligned}$$

where $x = \left(\Re F_c(k) - \frac{b}{4}\right)$. When k is very large $\Re F_c(k)$ tends to (slowly) diverge, but the factor $t(k)$ in the integrand of (D.7) concentrates the integral in the small k region.

We also need:

$$\begin{aligned} \lim_{b\to\infty}\frac{A^{(k)}V_0^{(k)}}{\sqrt{b}} &= \sqrt{2}k\delta(k)\left(\frac{\Re F_c(k)}{b} - \frac{1}{4}\right) \\ \lim_{b\to\infty}\frac{A^{(-k)}V_0^{(k)}}{\sqrt{b}} &= -\sqrt{2}k\delta(k)\left(\frac{\Re F_c(k)}{b} - \frac{1}{4}\right) \end{aligned}$$

Finally using these limits

$$\begin{aligned} \lim_{b\to\infty}\check{S}'^{(c)}_{mn} &= -\int_{-\infty}^{\infty}\frac{dk}{N(k)}\ t(k)v_m^{(k)}v_n^{(-k)} \\ &+ \lim_{b\to\infty}\int_{-\infty}^{\infty} dk\ t(k)\delta(k)(1-\bar{\nu}(k))(1-\nu(k))\tilde{B}_m(k)\tilde{B}_n(-k)\frac{1}{b} \end{aligned}$$

while the other integrals vanish because they contain the factor $k\delta(k)$. Here we have used the fact that $\nu(0) = \bar{\nu}(0) = -1$ and $\tilde{B}_m(0)$ is finite, for a straightforward calculation gives

$$\tilde{B}_m(0) = \frac{\sqrt{m}}{2\pi i} \oint dz \frac{\log(1+z^2)}{z^{m+1}} = \begin{cases} 0 & \text{for m odd;} \\ \frac{\sqrt{2m}}{2}(-1)^{\frac{m}{2}+1}(\frac{m}{2}+1)! & \text{for m even.} \end{cases} \tag{D.8}$$

So we are left with:

$$\lim_{b\to\infty} \breve{S}'^{(c)}_{mn} = S_{mn} \tag{D.9}$$

This is the sliver. The corrections are of order $\frac{1}{b}$.

D.2 Limit of $\breve{S}'_{0m}$

In the rest of this appendix we would like to justify eq.(2.74). The limit of $\breve{S}'^{(c)}_{0m}$ can be computed the same way as before. We have:

$$\lim_{b\to\infty} \breve{S}'^{(c)}_{0m} = \lim_{b\to\infty} \int_{-\infty}^{\infty} dk\ t(k) V_0^{(k)} V_m^{(-k)} = \lim_{b\to\infty} \left(\int_{-\infty}^{\infty} dk\ t(k) V_0^{(k)} A^{(-k)} v_m^{(-k)} + \frac{2}{\sqrt{b}} \tilde{B}_m(0) \right) \tag{D.10}$$

The last term in the RHS of course vanishes in the limit $b \to \infty$, while the first limit diverges, but, recalling (2.74), what we are really need to know is the limit of $\frac{\breve{S}'_{0n}}{1+s'}$. Using the fact that $1+s' \approx 4\eta \log \eta$ when $b \to \infty$ ($b \approx 4\log \eta$) and that we can write $\breve{S}'_{0m} = \breve{S}'^{(c)}_{0m} + \breve{S}'^{(d)}_{0m}$ (factorization into continuous and discrete parts) we have:

$$\breve{S}'^{(d)}_{0m} \approx 2\eta\sqrt{2\log \eta}$$

$$\breve{S}'^{(c)}_{0m} \approx \int_{-\infty}^{\infty} dk\ t(k) v_m^{(-k)} (-\sqrt{2}k)(V_0^{(k)})^2 \left(\frac{\Re F_c(k)}{4\log \eta} - \frac{1}{4} \right) 2\sqrt{\log \eta}$$

Using these we get:

$$\frac{\breve{S}'^{(c)}_{0m}}{1+s'} \approx \int_{-\infty}^{\infty} dk\ t(k) v_m^{(-k)} (\sqrt{2}k)\delta(k) \left(\frac{\Re F_c(k)}{4\log \eta} - \frac{1}{4} \right) \frac{1}{2\eta} = 0$$

and

$$\frac{\breve{S}'^{(d)}_{0m}}{1+s'} \approx \frac{1}{\sqrt{2\log \eta}}$$

Hereby the conclusion (2.74) follows.

Appendix E

Spectroscopy of the Neumann matrices with B-field

In this appendix we present the computation of the eigenstates and eigenvalues of the Neumann matrix $\mathcal{X}^{\alpha}{}_{\beta}$ in the presence of B–field, along the line of [15]. A similar analysis was carried out in [118], but with no reference to the correct normalization of continuous and discrete eigenvectors; moreover the discrete eigenvectors presented in the first of [118] does not reproduce the known ones when $B \to 0$. Since the discrete spectrum is of crucial importance for our purposes we re–derive completely the whole spectroscopy taking care of the correct normalization of continuous and discrete eigenvalues, as in [15]. To avoid the degeneracy of the diagonal Neumann coefficient $\mathcal{X}^{\alpha}{}_{\beta}$, we consider the unitary matrices $C'\mathcal{U}^{\alpha}{}_{\beta}$ and $\mathcal{U}^{\alpha}{}_{\beta}C'$, which are related to $\mathcal{X}^{\alpha}{}_{\beta}$ as follows [32]

$$\left(\mathcal{X}^{\alpha}{}_{\beta}\right)_{NM} = \frac{1}{3}\left(\delta^{\alpha}{}_{\beta} + C'\mathcal{U}^{\alpha}{}_{\beta} + C'\bar{\mathcal{U}}^{\alpha}{}_{\beta}\right)_{NM}. \tag{E.1}$$

The matrix $(C'\mathcal{U}^{\alpha}{}_{\beta})_{NM}$ can be written explicitly as

$$C'\mathcal{U} \;=\; \begin{pmatrix} 1-3bK & 2\sqrt{3}bKa & 3\sqrt{b}K\langle W| & -2\sqrt{3b}Ka\langle W| \\ -2\sqrt{3}bKa & 1-3bK & 2\sqrt{3b}Ka\langle W| & 3\sqrt{b}K\langle W| \\ 3\sqrt{b}K|W\rangle & -2\sqrt{3b}Ka|W\rangle & CU-3K|W\rangle\langle W| & 2\sqrt{3}Ka|W\rangle\langle W| \\ 2\sqrt{3b}Ka|W\rangle & 3\sqrt{b}K|W\rangle & -2\sqrt{3}Ka|W\rangle\langle W| & CU-3K|W\rangle\langle W| \end{pmatrix}$$

where, see [36]

$$|W\rangle = -\sqrt{2}(|v_e\rangle + i|v_o\rangle), \qquad\qquad K = \frac{A^{-1}}{4a^2+3}. \tag{E.2}$$

CU is the non-zero mode analog of $C'\mathcal{U}$ without B field. We recall that, [32],

$$C'\bar{\mathcal{U}} = \tilde{\mathcal{U}}C', \tag{E.3}$$

where tilde means transposition with respect to α, β indices.

Our aim is to solve the eigenvalue equation

$$C'\mathcal{U}|\Psi\rangle = \xi|\Psi\rangle, \qquad |\Psi\rangle = \begin{pmatrix} g_1 \\ g_2 \\ |\Lambda_1\rangle \\ |\Lambda_2\rangle \end{pmatrix}, \tag{E.4}$$

which splits into

$$\langle W|\Lambda_1\rangle = \frac{A}{\sqrt{b}}[\xi - 1 + \frac{b}{A}]g_1 + \frac{2Aa}{\sqrt{3b}}(\xi - 1)g_2 \tag{E.5}$$

$$\langle W|\Lambda_2\rangle = \frac{A}{\sqrt{b}}[\xi - 1 + \frac{b}{A}]g_2 - \frac{2Aa}{\sqrt{3b}}(\xi - 1)g_1 \tag{E.6}$$

$$(CU - \xi)|\Lambda_1\rangle = \sqrt{\frac{1}{b}}g_1(\xi - 1)|W\rangle \tag{E.7}$$

$$(CU - \xi)|\Lambda_2\rangle = \sqrt{\frac{1}{b}}g_2(\xi - 1)|W\rangle \tag{E.8}$$

We know, [13], that CU has a continuous spectrum and the solution of (E.4) depends on whether the eigenvalue ξ is in the continuous spectrum of CU or not. So we will distinguish these two different cases and analyze each of them in detail.

E.1 Discrete spectrum

If ξ is not in the spectrum of CU, we can invert $(CU - \xi)$ in equations (E.7) and (E.8) to obtain

$$|\Lambda_1\rangle = \sqrt{\frac{1}{b}}g_1(\xi - 1)\frac{1}{(CU - \xi)}|W\rangle \tag{E.9}$$

$$|\Lambda_2\rangle = \sqrt{\frac{1}{b}}g_2(\xi - 1)\frac{1}{(CU - \xi)}|W\rangle. \tag{E.10}$$

As we can see the solutions get modified w.r.t. the $B = 0$ case, only via possible modifications of the eigenvalue ξ. Substitution of these solutions into equations (E.5) and (E.6) gives

$$\sqrt{\frac{1}{2b}}(\xi - 1)\langle W|\frac{1}{CU - \xi}|W\rangle g_1 - \frac{A}{\sqrt{2b}}\left(\xi - 1 + \frac{b}{A}\right)g_1 - \frac{2aA}{\sqrt{6b}}(\xi - 1)g_2 = 0 \tag{E.11}$$

$$\sqrt{\frac{1}{2b}}(\xi - 1)\langle W|\frac{1}{CU - \xi}|W\rangle g_2 - \frac{A}{\sqrt{2b}}\left(\xi - 1 + \frac{b}{A}\right)g_2 + \frac{2aA}{\sqrt{6b}}(\xi - 1)g_1 = 0 \tag{E.12}$$

The bracket which appears here is the same as the one in [15] and is given by

$$\langle W|\frac{1}{CU-\xi}|W\rangle = V_{00} + \frac{\xi+1}{\xi-1}2\Re F(\eta) \tag{E.13}$$

where

$$F(\eta) = \psi\left(\frac{1}{2}+\frac{\eta}{2\pi i}\right) - \psi\left(\frac{1}{2}\right), \qquad \xi = -\frac{1}{1-2\mathrm{cosh}\eta}[2-\mathrm{cosh}\eta - \mathrm{i}\sqrt{3}\mathrm{sinh}\eta]. \tag{E.14}$$

$\psi(x)$ is the logarithmic derivative of the Euler Γ-function.

Substitution of these in (E.11) and (E.12) gives us

$$\left(\Re F(\eta) - \frac{b}{4}\right) g_1 - \frac{aA}{\sqrt{3}}\frac{\xi-1}{\xi+1}g_2 = 0, \qquad \left(\Re F(\eta) - \frac{b}{4}\right) g_2 + \frac{aA}{\sqrt{3}}\frac{\xi-1}{\xi+1}g_1 = 0. \tag{E.15}$$

This system of equations will have non trivial solutions for g_2 and g_1 if the determinant of the coefficient matrix is zero, i.e.

$$\frac{b}{4} = \Re F(\eta) \pm aA\mathrm{tanh}\frac{\eta}{2}, \tag{E.16}$$

Using equations (E.15) we can show that $g_2 = \pm i g_1$. This is a constraint on g_1 and g_2 thus we cannot split the eigenstates in the two directions, choosing one of the constants to be zero. g_1 is now an overall constant, which can be chosen real and fixed by normalization completely.

The eigenstates are then

Case-1

$$\frac{b}{4} = \Re F(\eta) + aA\mathrm{tanh}\frac{\eta}{2}, \qquad g_2 = -ig_1 = -ig_d(\eta_1,\eta_2) \tag{E.17}$$

$$|V^{(\xi_1)}\rangle = g_d(\eta_1,\eta_2)\begin{pmatrix} 1 \\ -i \\ \sqrt{\frac{1}{b}}(\xi_1-1)\frac{1}{CU-\xi_1}|W\rangle \\ -i\sqrt{\frac{1}{b}}(\xi_1-1)\frac{1}{CU-\xi_1}|W\rangle \end{pmatrix} \tag{E.18}$$

$$|V^{(\bar{\xi}_2)}\rangle = g_d(\eta_2,\eta_1)\begin{pmatrix} 1 \\ -i \\ \sqrt{\frac{1}{b}}(\bar{\xi}_2-1)\frac{1}{CU-\bar{\xi}_2}|W\rangle \\ -i\sqrt{\frac{1}{b}}(\bar{\xi}_2-1)\frac{1}{CU-\bar{\xi}_2}|W\rangle \end{pmatrix} \tag{E.19}$$

Case-2

$$\frac{b}{4} = \Re F(\eta) - aA\mathrm{tanh}\frac{\eta}{2}, \qquad g_2 = ig_1 = ig_d(\eta_2,\eta_1) \tag{E.20}$$

$$|V^{(\xi_2)}\rangle = g_d(\eta_2,\eta_1)\begin{pmatrix} 1 \\ i \\ \sqrt{\frac{1}{b}}(\xi_2-1)\frac{1}{CU-\xi_2}|W\rangle \\ i\sqrt{\frac{1}{b}}(\xi_2-1)\frac{1}{CU-\xi_2}|W\rangle \end{pmatrix} \tag{E.21}$$

$$|V^{(\bar{\xi}_1)}\rangle = g_d(\eta_1,\eta_2)\begin{pmatrix} 1 \\ i \\ \sqrt{\frac{1}{b}}(\bar{\xi}_1 - 1)\frac{1}{CU-\bar{\xi}_1}|W\rangle \\ i\sqrt{\frac{1}{b}}(\bar{\xi}_1 - 1)\frac{1}{CU-\bar{\xi}_1}|W\rangle \end{pmatrix}. \tag{E.22}$$

Normalizing them in the following way[1]

$$\bar{V}^{\xi_i}_{\alpha} V^{\xi_j,\alpha} = \delta^{ij}$$

$$\bar{V}^{\bar{\xi}_i}_{\alpha} V^{\bar{\xi}_j,\alpha} = \delta^{ij}$$

$$\bar{V}^{\bar{\xi}}_{\alpha} V^{\xi,\alpha} = 0$$

we get, use the results of [15],

$$|g_d(\eta_1,\eta_2)|^2 = \frac{1}{2\Delta}\left[(1 - r(\eta_1,\eta_2)) + r(\eta_1,\eta_2)\mathrm{sinh}\eta_1\frac{\partial}{\partial\eta_1}[\mathrm{Log}\Re F(\eta_1)]\right)^{-1}, \tag{E.23}$$

where

$$r(\eta_1,\eta_2) = \Re F(\eta_1)\frac{\tanh(\frac{\eta_1}{2}) + \tanh(\frac{\eta_2}{2})}{\Re F(\eta_2)\tanh(\frac{\eta_1}{2}) + \Re F(\eta_1)\tanh(\frac{\eta_2}{2})}. \tag{E.24}$$

It is important to notice that $V^{(\xi_1)}$ and $V^{(\bar{\xi}_1)}$ are degenerate eigenstates of $\mathcal{X}$, and the same holds for $V^{(\xi_2)}$ and $V^{(\bar{\xi}_2)}$.

E.2 Continuous spectrum

If ξ is in the continuous spectrum of CU ($\xi = \nu(k)$, [15]), we cannot invert the operator $(CU - \xi)$. Thus, in this case, the solution of (E.7) and (E.8) is

$$|\Lambda_1\rangle = A_1(k)|k\rangle + \frac{1}{\sqrt{b}}g_1(\nu(k) - 1)\wp\frac{1}{(CU - \nu(k))}|W\rangle \tag{E.25}$$

$$|\Lambda_2\rangle = A_2(k)|k\rangle + \frac{1}{\sqrt{b}}g_2(\nu(k) - 1)\wp\frac{1}{CU - \nu(k)}|W\rangle. \tag{E.26}$$

Where $\wp$ is the principal value, [15]. Using these in (E.5) and (E.6), we get

$$A_1(k) = g_1\sqrt{\frac{2}{b}}k\left(\Re F_c(k) - \frac{b}{4}\right) - \frac{\sqrt{2}Aa}{\sqrt{3b}}k\left(\frac{\nu(k)-1}{\nu(k)+1}\right)g_2 \tag{E.27}$$

$$A_2(k) = g_2\sqrt{\frac{2}{b}}k\left(\Re F_c(k) - \frac{b}{4}\right) + \frac{\sqrt{2}Aa}{\sqrt{3b}}k\left(\frac{\nu(k)-1}{\nu(k)+1}\right)g_1 \tag{E.28}$$

[1] This is the standard way to normalized eigenvectors of hermitian matrices

Note that in this case g_1 and g_2 are completely free and we can choose them to construct two linearly independent orthogonal vectors as follows

Case-1 $\quad g_2 = ig_1 = ig_c(k)$

$$V^1(k) = g_c(k) \begin{pmatrix} 1 \\ i \\ P(k)|k\rangle + \frac{1}{\sqrt{b}}(\nu(k)-1)\wp\frac{1}{CU-\nu(k)}|W\rangle - iH(k,a)|k\rangle \\ iP(k)|k\rangle + i\frac{1}{\sqrt{b}}(\nu(k)-1)\wp\frac{1}{CU-\nu(k)}|W\rangle + H(k,a)|k\rangle \end{pmatrix} \tag{E.29}$$

Case-2 $\quad g_2 = -ig_1 = -ig_c(-k)$

$$V^2(k) = g_c(-k) \begin{pmatrix} 1 \\ -i \\ P(k)|k\rangle + \frac{1}{\sqrt{b}}(\nu(k)-1)\wp\frac{1}{CU-\nu(k)}|W\rangle + iH(k,a)|k\rangle \\ -iP(k)|k\rangle - i\frac{1}{\sqrt{b}}(\nu(k)-1)\wp\frac{1}{CU-\nu(k)}|W\rangle + H(k,a)|k\rangle \end{pmatrix}, \tag{E.30}$$

where

$$P(k) = \sqrt{\frac{2}{b}}k\left(\Re F_c(k) - \frac{b}{4}\right), \qquad H(k,a) = \frac{\sqrt{2}Aa}{\sqrt{3b}}k\left(\frac{\nu(k)-1}{\nu(k)+1}\right). \tag{E.31}$$

Imposing the continuous orthonormality condition

$$\bar{V}^{i,\alpha}(k)V^j_\alpha(k') = \delta^{ij}\delta(k-k') \tag{E.32}$$

we get

$$g_c(k) = \left[\frac{4\Delta}{b}N(k)\left(4 + k^2\left(\Re F_c(k) - \frac{b}{4} - \frac{Aa}{\tanh\frac{\pi k}{4}}\right)^2\right)\right]^{-1/2} \tag{E.33}$$

Sending $k \to -k$ we get the right degeneracy for $\mathcal{X}$.

Appendix F

Diagonalization of the 3-string vertex and the Lump state

We can express the oscillators $a^{(r)}_{N,\alpha}$, appearing in the 3-string vertex (2.111), in terms of the oscillators of the diagonal basis as

$$a^{(r)}_{N,\alpha} = \sum_{i=1}^{2}\left(a^{(r)}_{\xi_i}\bar{V}^{(\xi_i)}_{N,\alpha} + a^{(r)}_{\bar{\xi}_i}\bar{V}^{(\bar{\xi}_i)}_{N,\alpha} + \int_{-\infty}^{\infty} dk a^{(r)}_i(k)\bar{V}^{(i)}_{N,\alpha}(k)\right) \tag{F.1}$$

$$a^{(r)\dagger}_{N,\alpha} = \sum_{i=1}^{2}\left(a^{(r)\dagger}_{\xi_i}V^{(\xi_i)}_{N,\alpha} + a^{(r)\dagger}_{\bar{\xi}_i}V^{(\bar{\xi}_i)}_{N,\alpha} + \int_{-\infty}^{\infty} dk a^{(r)\dagger}_i(k)V^{(i)}_{N,\alpha}(k)\right). \tag{F.2}$$

Using these oscillators and the fact that $\tau\bar{V} = V$ ($\tau^\alpha_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$), we can rewrite the 3-string vertex as

$$|V^m_3\rangle = N_m \exp[-\frac{1}{2}\sum_{r,s}\sum_{i=1}^{2}\left(a^{(r)\dagger}_{\xi_i}\bar{V}^{(\xi_i)}_{N,\alpha} + a^{(r)\dagger}_{\bar{\xi}_i}\bar{V}^{(\bar{\xi}_i)}_{N,\alpha} + \int_{-\infty}^{\infty} dk a^{(r)\dagger}_i(k)\bar{V}^{(i)}_{N,\alpha}(k)\right)$$

$$\times(\tau C'\mathcal{X})^{\alpha(rs)}_{\beta,NM}\sum_{j=1}^{2}\left(a^{(s)\dagger}_{\xi_j}V^{(\xi_j)\beta}_M + a^{(s)\dagger}_{\bar{\xi}_j}V^{(\bar{\xi}_j)\beta}_M + \int_{-\infty}^{\infty} dk' a^{(s)\dagger}_j(k')V^{(j)\beta}_M(k')\right)]|\Omega_{b,\theta}\rangle \tag{F.3}$$

The twist operator $\tau C'$ acts on the eigenstates of the discrete and continuous spectra as follows

$$\tau C' V^{(\xi_i)} = V^{(\bar{\xi}_i)} \qquad \tau C' V^{(i)}(k) = V^{(i+1)}(-k), \tag{F.4}$$

where $V^3(k)$ is identified with $V^1(k)$. Then (F.3) becomes

$$|V^m_3\rangle = N_m \exp\left[-\frac{1}{2}\sum_{r,s}\sum_{i=1}^{2}\left(a^{(r)\dagger}_{\xi_i}\mu^{rs}(\bar{\xi}_i)a^{(s)\dagger}_{\bar{\xi}_i} + a^{(r)\dagger}_{\bar{\xi}_i}\mu^{rs}(\xi_i)a^{(s)\dagger}_{\xi_i} + \right.\right.$$

$$\left.\left. + \int_{-\infty}^{\infty} dk a^{(r)\dagger}_i(k)\mu^{rs}(-k)a^{(s)\dagger}_{i+1}(-k)\right)\right]|\Omega_b\rangle. \tag{F.5}$$

In order to write this in an exact diagonal form, we need to introduce oscillators with definite τ–twist parity

$$e^{\tau}_{\xi_i} = \frac{a^{\tau}_{\xi_i} + a^{\tau}_{\bar{\xi}_i}}{\sqrt{2}} = \frac{a^{\tau}_{\xi_i} + \tau C' a^{\tau}_{\xi_i}}{\sqrt{2}}, \qquad o^{\tau}_{\xi_i} = -i\frac{a^{\tau}_{\xi_i} - a^{\tau}_{\bar{\xi}_i}}{\sqrt{2}} = -i\frac{a^{\tau}_{\xi_i} - \tau C' a^{\tau}_{\xi_i}}{\sqrt{2}} \tag{F.6}$$

$$\begin{aligned} e^{\tau}_i(k) &= \frac{a^{\tau}_i(k) + a^{\tau}_{i+1}(-k)}{\sqrt{2}} = \frac{a^{\tau}_i(k) + \tau C' a^{\tau}_i(k)}{\sqrt{2}} \\ o^{\tau}_i(k) &= -i\frac{a^{\tau}_i(k) - a^{\tau}_{i+1}(-k)}{\sqrt{2}} = -i\frac{a^{\tau}_i(k) - \tau C' a^{\tau}_i(k)}{\sqrt{2}} \end{aligned} \tag{F.7}$$

These oscillators have the following BPZ conjugation property

$$\text{bpz } o_i = -o_i^{\dagger} \qquad \text{bpz } e_i = -e_i^{\dagger}\ , \tag{F.8}$$

and satisfy the commutation relations

$$\begin{aligned} [e_{\xi_i}, e^{\dagger}_{\xi_j}] &= \delta_{ij}, \qquad [o_{\xi_i}, o^{\dagger}_{\xi_j}] = \delta_{ij}, \\ [e_i(k), e^{\dagger}_j(k')] &= \delta_{ij}\delta(k-k'), \qquad [o_i(k), o^{\dagger}_j(k')] = \delta_{ij}\delta(k-k'), \end{aligned} \tag{F.9}$$

with all the other commutators vanishing. Using them into (F.5) we finally obtain

$$\begin{aligned} |V_3^m\rangle = N_m \text{exp}\Bigg[-\frac{1}{4}\sum_{r,s}\sum_{i=1}^{2} \Big(\left[\mu^{rs}(\xi_i) + \mu^{rs}(\bar{\xi}_i)\right] \left(e^{(r)\dagger}_{\xi_i} e^{(s)\dagger}_{\xi_i} + o^{(r)\dagger}_{\xi_i} o^{(s)\dagger}_{\xi_i} \right) \\ -i\left[\mu^{rs}(\xi_i) - \mu^{rs}(\bar{\xi}_i)\right] \left(o^{(r)\dagger}_{\xi_i} e^{(s)\dagger}_{\xi_i} - e^{(r)\dagger}_{\xi_i} o^{(s)\dagger}_{\xi_i} \right) \\ - \int_{-\infty}^{\infty} dk \mu^{rs}(k) \left(e^{(r)\dagger}_i(k) e^{(s)\dagger}_i(k) + o^{(r)\dagger}_i(k) o^{(s)\dagger}_i(k) \right) \\ -i\int_{-\infty}^{\infty} dk \mu^{rs}(k) \left(e^{(r)\dagger}_i(k) o^{(s)\dagger}_i(k) - o^{(r)\dagger}_i(k) e^{(s)\dagger}_i(k) \right) \Big) \Bigg] |\Omega_{b,\theta}\rangle \end{aligned} \tag{F.10}$$

This gives the diagonal representation of the 3-string interaction vertex. The same procedure gives the following diagonal representation of the transverse part of the Lump

$$\begin{aligned} |S_{\perp}\rangle = \frac{A^2(3+4a^2)}{\sqrt{2\pi b^3}(\text{Det}G)^{\frac{1}{4}}} \text{Det}(\mathcal{I}-\mathcal{X})^{\frac{1}{2}} \text{Det}(\mathcal{I}+\mathcal{T})^{\frac{1}{2}} \text{exp}\Bigg(-\frac{1}{2}\sum_{i=1}^{2} \Big[t_d(\eta_i) \\ \times \left(e^{\dagger}_{\xi_i} e^{\dagger}_{\xi_i} + o^{\dagger}_{\xi_i} o^{\dagger}_{\xi_i} \right) + \frac{1}{2}\int_{-\infty}^{\infty} dk t_c(k) \left(e^{\dagger}_i(k) e^{\dagger}_i(k) + o^{\dagger}_i(k) o^{\dagger}_i(k) \right) \Big] \Bigg) |\Omega_{b,\theta}\rangle \end{aligned} \tag{F.11}$$

Appendix G

Asymptotic behaviors

In subsection 2.3.2, we have analyzed our solution in the large and small limits of the parameter b. In this appendix we compute the relevant matrix elements in these asymptotic regimes.

G.1 The $b \to \infty$ Limit

From (E.16), we can write

$$|a| = \frac{\Re F(\eta_2) - \Re F(\eta_1)}{[V_{00} + 2\Re F(\eta_2)]\tanh(\frac{\eta_1}{2}) + [V_{00} + 2\Re F(\eta_1)]\tanh(\frac{\eta_2}{2})} \tag{G.1}$$

$$\frac{b}{4} = \frac{\Re F(\eta_2)\tanh(\frac{\eta_1}{2}) + \Re F(\eta_1)\tanh(\frac{\eta_2}{2})}{\tanh(\frac{\eta_1}{2}) + \tanh(\frac{\eta_2}{2})} \tag{G.2}$$

where we take, by definition, $\eta_2 > \eta_1 > 0$. There are two ways of taking $b \to \infty$

i) $\eta_2 \to \infty$; η_1 fixed

In this limit we can see that

$$\frac{b}{4} \approx \left(\frac{\tanh(\frac{\eta_1}{2})}{1+\tanh(\frac{\eta_1}{2})}\right)\log(\eta_2), \qquad a \approx \frac{1}{2\tanh(\frac{\eta_1}{2})} > \frac{1}{2}. \tag{G.3}$$

ii) $\eta_2 \to \infty$; $\eta_1 \to \infty$

We can parameterize $\eta_2 = \eta^y$, $\eta_1 = \eta^x$ and then take $\eta \to \infty$, while keeping $y > x$. We then obtain

$$\frac{b}{4} \approx \frac{1}{2}(y+x)\log(\eta), \qquad a \approx \frac{1}{2}\frac{y-x}{y+x} < \frac{1}{2} \tag{G.4}$$

We will be concerned with this second regime as it is the one connected to $a = 0$, which is a condition arising from the existence of the critical value for the E–field, when $b \to \infty$. In this second limit it can be easily seen that the discrete eigenvectors have the following behavior

$$V_0^{\xi_i,1} = V_0^{\bar{\xi}_i,1} \approx \frac{1}{\sqrt{2\Delta}}e^{-\eta_i/2}\sqrt{\eta_i \mathrm{Log}\eta_1\eta_2},$$

$$V_0^{\xi_i,2} = -V_0^{\bar{\xi}_i,2} \approx (-1)^i\frac{i}{\sqrt{2\Delta}}e^{-\eta_i/2}\sqrt{\eta_i \mathrm{Log}\eta_1\eta_2},$$

and

$$V_n^{\xi_i,\alpha} \approx -\frac{V_0^{\xi_i,\alpha}}{\sqrt{\mathrm{Log}\eta_1\eta_2}}, \qquad V_n^{\bar{\xi}_i,\alpha} \approx -\frac{V_0^{\bar{\xi}_i,\alpha}}{\sqrt{\mathrm{Log}\eta_1\eta_2}}. \tag{G.5}$$

For the continuous spectrum the situation is more complicated and getting this limits is not easy. However, it is possible to calculate the limit of $(V_0^{i,\alpha}(k))^2$, which is enough for our purposes. We have

$$(V_0^{1,1}(k))^2 = \left[\frac{4\Delta}{b}N(k)\left(4+k^2\left(\Re F_c(k)-\frac{b}{4}-\frac{Aa}{\tanh\frac{\pi k}{4}}\right)^2\right)\right]^{-1}. \tag{G.6}$$

When $b \to \infty$ this expression vanishes everywhere except at

$$k_0 \approx -\frac{4}{\pi}\mathrm{arctanh}(2a) \tag{G.7}$$

where it diverges. Expanding around k_0 one easily gets

$$(V_0^{1,1}(k))^2 \approx \Delta^{-1}\frac{4a}{k_0(1-4a^2)N(k_0)}\frac{\bar{b}}{\pi(1+(k-k_0)^2\bar{b}^2}$$

where

$$\bar{b} = \frac{k_0\pi(1-4a^2)}{64a}b.$$

Now taking the $b \to \infty$ limit one obtains

$$(V_0^{1,1}(k))^2 \approx \frac{1}{2\Delta}\delta(k-k_0). \tag{G.8}$$

Following the same procedure one can also show that

$$(V_0^{2,1}(k))^2 \approx \frac{1}{2\Delta}\delta(k+k_0) \tag{G.9}$$

remember that

$$|V_0^{1,2}(k)|^2 = (V_0^{1,1}(k))^2, \qquad |V_0^{2,2}(k)|^2 = (V_0^{2,1}(k))^2. \tag{G.10}$$

The non zero components, $V_m^{i,\alpha}(k)$, can be expressed in terms of a generating function. For instance, the generating function for $V_m^{1,1}(k)$ is given by

$$F^{(k)}(z) = A_1(k)f^{(k)}(z) - \frac{(1-\nu(k))V_0^{1,1}(k)}{\sqrt{b}}B(k,z) \tag{G.11}$$

where

$$A_1(k) = V_0^{1,1}(k)\sqrt{\frac{2}{b}}k\left(\Re F_c(k)-\frac{b}{4}-\frac{Aa}{\tanh(\frac{\pi k}{4})}\right),$$

$$B(k,z) = \frac{2}{1-\nu(k)}\left[\Re F_c(k)+\frac{\pi}{2\sqrt{3}}\frac{\nu(k)-1}{\nu(k)+1}+\frac{2i}{k}+\log(iz)-2if^{(k)}(z)\right]$$

$$+\frac{2}{1-\nu(k)}\left[\Phi(e^{-4i\arctan(z)},1,1+\frac{k}{4i})e^{-4i\arctan(z)}e^{-k\arctan(z)}\right] \tag{G.12}$$

where Φ is the LerchPhi function and $f^{(k)}$ is the generating function for the spectrum of the Neumann matrix without zero modes, [13]. Inverting this equation we can write $V_m^{1,1}(k)$ as

$$V_m^{1,1}(k) = A_1(k)\frac{\sqrt{m}}{2\pi i}\oint dz\frac{f^{(k)}(z)}{z^{m+1}} - \frac{(1-\nu(k))V_0^{1,1}(k)}{\sqrt{b}}\frac{\sqrt{m}}{2\pi i}\oint dz\frac{B(k,z)}{z^{m+1}} \tag{G.13}$$

With the same procedure one can also write

$$V_m^{2,1}(k) = A_1'(k)\frac{\sqrt{m}}{2\pi i}\oint dz\frac{f^{(k)}(z)}{z^{m+1}} - \frac{(1-\nu(k))V_0^{2,1}(k)}{\sqrt{b}}\frac{\sqrt{m}}{2\pi i}\oint dz\frac{B(k,z)}{z^{m+1}} \tag{G.14}$$

with

$$A_1'(k) = V_0^{2,1}(k)\sqrt{\frac{2}{b}}k\left(\Re F_c(k) - \frac{b}{4} + \frac{Aa}{\tanh(\frac{\pi k}{4})}\right). \tag{G.15}$$

The other vectors are related to these ones as

$$V_n^{1,2}(k) = iV_n^{1,1}(k), \qquad V_n^{2,2}(k) = -iV_n^{2,1}(k) \tag{G.16}$$

G.1.1 Limit of $\hat{S}_{mn}^{\alpha\beta(c)}$

With all these results at hand we can now calculate the continuous spectrum contribution to the non zero mode matrix elements in the limits under consideration. Recalling that spectrum of the Neumann matrix without zero modes is given by

$$v_m^{(k)} = \frac{\sqrt{m}}{2\pi i}\oint dz\frac{f^{(k)}(z)}{z^{m+1}} \tag{G.17}$$

we can write

$$\hat{S}_{nm}^{11(c)} = \int_{-\infty}^{\infty} dk\; t_c(k)(-1)^n\left[V_n^{1,1}(k)\bar{V}_m^{1,1}(k) + V_n^{2,1}(k)\bar{V}_m^{2,1}(k)\right] \tag{G.18}$$

as

$$\hat{S}_{mn}^{11(c)} = \int_{-\infty}^{\infty} dk\; t_c(k)(-1)^m[A_1(k)A_1(k)v_m^{(k)}v_n^{(k)} - A_1(k)V_0^{1,1}(k)v_m^{(k)}(1-\bar{\nu}(k))\tilde{B}_n(k)\frac{1}{\sqrt{b}}$$

$$-A_1(k)V_0^{1,1}(k)v_n^{(k)}(1-\nu(k))\tilde{B}_m(k)\frac{1}{\sqrt{b}} + (V_0^{1,1}(k))^2(1-\bar{\nu}(k))(1-\nu(k))\tilde{B}_m(k)\tilde{B}_n(k)\frac{1}{b}]$$

$$+\left[A_1(k)\to A_1'(k), V_0^{1,1}(k)\to V_0^{2,1}(k)\right] \tag{G.19}$$

where

$$\tilde{B}_m(k) = \frac{\sqrt{m}}{2\pi i}\oint dz\frac{B(k,z)}{z^{m+1}}. \tag{G.20}$$

Note that if the indices are separated by comma then the first index is the label of the vector and the second is the space-time index, otherwise both are space time indices. Now we want to calculate each term in the above expression in the limit when $b \to \infty$. To this end we notice the following

$$\lim_{b\to\infty} A_1(k)A_1(k) = \lim_{b\to\infty} (V_0^{1,1}(k))^2 \left(\frac{2k^2}{b}\right)\left(\Re F_c(k) - \frac{b}{4} - \frac{Aa}{\tanh(\frac{\pi k}{4})}\right)^2$$

$$= \lim_{x\to-\infty}\left(\frac{k^2}{2\Delta N(k)}\right)\frac{x^2}{4+k^2x^2} = \left(\frac{k^2}{2\Delta N(k)}\right)\frac{1}{k^2} = \frac{1}{2\Delta N(k)} \tag{G.21}$$

where $x = \left(\Re F_c(k) - \frac{b}{4} - \frac{Aa}{\tanh(\frac{\pi k}{4})}\right)$. The other terms are zero in the limit because, either they contain term like $(k-k_0)\delta(k-k_0)$ in the integral or they are of order $\frac{1}{b}$. Therefore, we are left with

$$\lim_{b\to\infty} \hat{S}_{mn}^{11(c)} = \lim_{b\to\infty} \hat{S}_{mn}^{22(c)} = \Delta^{-1}S_{mn}, \quad \text{where} \quad S_{nm} = -\int_{-\infty}^{\infty} \frac{dk\ t_c(k)}{N(k)} v_n^{(k)} v_m^{(-k)} \tag{G.22}$$

and

$$\lim_{b\to\infty} \hat{S}_{mn}^{21(c)} = \lim_{b\to\infty} \hat{S}_{mn}^{12(c)} = 0, \tag{G.23}$$

which is the sliver in each direction with corrections of order $\frac{1}{b}$.

G.1.2 Limit of $\hat{S}_{0m}^{\alpha\beta(c)}$

In this section we would like to justify that the contribution from the continuous spectrum to $\hat{S}_{0m}^{\alpha\beta}$ is zero in the limit. This can be computed the same way as before since we have

$$\lim_{b\to\infty} \hat{S}_{0m}^{\alpha\beta(c)} = \lim_{b\to\infty} \sum_{i=1}^{2} \int_{-\infty}^{\infty} dk\ t_c(k) V_0^{i,\alpha}(k) V_m^{i,\beta}(k). \tag{G.24}$$

For instance, lets calculate $\hat{S}_{0m}^{11(c)}$ which is given by

$$\hat{S}_{0m}^{11(c)} = \lim_{b\to\infty}\int_{-\infty}^{\infty} dk\ t_c(k) v_m^{(k)} \sqrt{\frac{2}{b}}k\left((V_0^{1,1}(k))^2\left[\Re F_c(k) - \frac{b}{4} - \frac{Aa}{\tanh(\frac{\pi k}{4})}\right]\right.$$

$$\left. + (V_0^{2,1}(k))^2\left[\Re F_c(k) - \frac{b}{4} + \frac{Aa}{\tanh(\frac{\pi k}{4})}\right]\right) + O(\frac{1}{\sqrt{b}}) \tag{G.25}$$

We have already verified that $\lim_{b\to\infty}(V_0^{i,\alpha}(k))^2 \approx \frac{1}{2}\delta(k \pm k_0)$. This will allow us to expand the terms in square brackets about the points $\pm k_0$ to get

$$\hat{S}_{0m}^{11(c)} = \frac{1}{\Delta}\lim_{b\to\infty}\int_{-\infty}^{\infty} dk\ t_c(k) v_m^{(k)}\left[\frac{1}{2}\delta(k-k_0)\sqrt{b}(k-k_0)k_0\pi\left(\frac{1-4a^2}{32a}\right)\right.$$

$$\left. + \frac{1}{2}\delta(k+k_0)\sqrt{b}(k+k_0)(-k_0)\pi\left(\frac{1-4a^2}{32a}\right)\right] + O(\frac{1}{\sqrt{b}}). \tag{G.26}$$

Due to the presence of the delta functions the terms $(k \pm k_0)\sqrt{b}$ are both finite in the $b \to \infty$ limit. As a matter of fact, we can safely do the integrals first and take the limits later. Since the integrals vanishes we note that

$$\hat{S}_{0m}^{11(c)} \approx 0. \tag{G.27}$$

Similar steps show that all the remaining terms of $\hat{S}_{0m}^{\alpha\beta(c)}$ are also zero.

G.2 The $b \to 0$ Limit

As it was mentioned before this limit can be obtained by taking $\eta_1 \to 0$. In this limit it is not hard to see that

$$b \approx 2\frac{\Re F(\eta_2)}{\tanh(\frac{\eta_2}{2})}\eta_1 \tag{G.28}$$

$$g_d(\eta_1, \eta_2) \approx \frac{1}{\sqrt{2\Delta}}\left(1 - \frac{\tanh(\frac{\eta_2}{2})}{2\Re F(\eta_2)}\eta_1\right) \tag{G.29}$$

$$g_d(\eta_2, \eta_1) \approx \frac{1}{\sqrt{2\Delta}}\left[2\tanh(\frac{\eta_2}{2})\left(\sinh\eta_2\frac{\partial}{\partial\eta_2}[\mathrm{Log}\Re F(\eta_2)] - 1\right)\right]^{-1/2}\sqrt{\eta_1}. \tag{G.30}$$

One can use these results and equations (E.18) through (E.22) to write down $V_0^{\xi_i,\alpha}$, $V_0^{\bar{\xi}_i,\alpha}$ $V_n^{\xi_i,\alpha}$ and $V_n^{\bar{\xi}_i,\alpha}$ as

$$V_0^{\xi_1,1} = V_0^{\bar{\xi}_1,1} \approx \frac{1}{\sqrt{2\Delta}}\left(1 - \frac{\tanh(\frac{\eta_2}{2})}{2\Re F(\eta_2)}\eta_1\right),$$

$$V_0^{\xi_1,2} = -V_0^{\bar{\xi}_1,2} \approx -i\frac{1}{\sqrt{2\Delta}}\left(1 - \frac{\tanh(\frac{\eta_2}{2})}{2\Re F(\eta_2)}\eta_1\right), \tag{G.31}$$

$$V_0^{\xi_2,1} = V_0^{\bar{\xi}_2,1} \approx \frac{1}{\sqrt{2\Delta}}\left[2\tanh(\frac{\eta_2}{2})\left(\sinh\eta_2\frac{\partial}{\partial\eta_2}[\mathrm{Log}\Re F(\eta_2)] - 1\right)\right]^{-1/2}\sqrt{\eta_1}$$

$$V_0^{\xi_2,2} = -V_0^{\bar{\xi}_2,2} \approx i\frac{1}{\sqrt{2\Delta}}\left[2\tanh(\frac{\eta_2}{2})\left(\sinh\eta_2\frac{\partial}{\partial\eta_2}[\mathrm{Log}\Re F(\eta_2)] - 1\right)\right]^{-1/2}\sqrt{\eta_1}, \tag{G.32}$$

and

$$V_n^{\xi_1,\alpha} = \pm V_n^{\bar{\xi}_1,\alpha} \approx \sqrt{\eta_1},$$

$$V_n^{\xi_2,\alpha} = \pm V_n^{\bar{\xi}_2,\alpha} \approx \frac{1}{\sqrt{2\Delta}}\left[2\tanh(\frac{\eta_2}{2})\left(\sinh\eta_2\frac{\partial}{\partial\eta_2}[\mathrm{Log}\Re F(\eta_2)] - 1\right)\right]^{-1/2} f(\eta_2). \tag{G.33}$$

The f is a regular function of η_2. On the other hand

$$g_c(k) \approx 0. \tag{G.34}$$

This shows all $V_0^{i,\alpha}(k)$ are zero, whereas $V_m^{i,\alpha}(k)$ are finite and b independent to the leading order. These results are extensively used in subsection 2.3.2 to calculate quantities like s_1, s_2 in the $b \to 0$ limit.

Appendix H

A discussion of four–point functions

Using Hirota equations (3.51–3.53) one can easily express the four–point functions $\prod_{i=1}^{4} D_r(z_i)F$, $\partial_{t_0^{(r)}} \prod_{i=1}^{3} D_r(z_i)F$ and $\partial^2_{t_0^{(r)}} \prod_{i=1}^{2} D_r(z_i)F$ in terms of the two–point functions $\prod_{i=1}^{2} D_r(z_i)F$ and $\partial_{t_0^{(r)}} D_r(z)F$ as well as the four–point function $\partial^3_{t_0^{(r)}} D_r(z)F$. The two–point functions were already identified with the corresponding generating functions of the Neumann coefficients (see, eqs. (3.46)), so one is left with the problem of first finding a suitable identification for the four–point function $\partial^3_{t_0^{(r)}} D_r(z)F$, which leads to the straightforward derivation of all other. It is interesting that if one additionally requires that the four–point functions admit a factorizable form as functions of z_i, then the identification of $\partial^3_{t_0^{(r)}} D_r(z)F$ is uniquely fixed modulo three arbitrary $SL(2)$-group parameters a_r, b_r, c_r and d_r ($b_r c_r - a_r d_r = 1$) as follows

$$\partial^3_{t_0^{(r)}} D_r(z)F = \frac{\varphi_r(z^{-1}) - 1}{\varphi_r(z^{-1})} \frac{a_r \varphi_r(z^{-1}) + b_r}{c_r \varphi_r(z^{-1}) + d_r} \,. \tag{H.1}$$

Using this and eqs. (3.51–3.53), one can derive the corresponding explicit expressions for the remaining four–point functions

$$\partial^2_{t_0^{(r)}} \prod_{i=1}^{2} D_r(z_i)F = -\prod_{i=1}^{2} \frac{\varphi_r(z_i^{-1}) - 1}{c_r \varphi_r(z_i^{-1}) + d_r} \,, \tag{H.2}$$

$$\partial_{t_0^{(r)}} \prod_{i=1}^{3} D_r(z_i)F = -d_r \prod_{i=1}^{3} \frac{\varphi_r(z_i^{-1}) - 1}{c_r \varphi_r(z_i^{-1}) + d_r} \,, \tag{H.3}$$

$$\prod_{i=1}^{4} D_r(z_i)F = -d_r^2 \prod_{i=1}^{4} \frac{\varphi_r(z_i^{-1}) - 1}{c_r \varphi_r(z_i^{-1}) + d_r} \,. \tag{H.4}$$

The four–point functions (3.55) are a particular case of the latter for

$$d_r = -\frac{1}{a_r} = -\frac{\beta_r}{\sqrt{3}\,\alpha_{r-1}}, \quad b_r = 0, \quad c_r = -\frac{1}{\sqrt{3}\,\alpha_r}. \tag{H.5}$$

Let us remark that if one rescales the parameters in eqs. (H.1–H.4) as follows

$$\{a_r,\ b_r,\ c_r,\ d_r\} \quad \Rightarrow \quad \frac{1}{\epsilon}\,\{a_r,\ b_r,\ c_r,\ d_r\} \tag{H.6}$$

and consider the limit $\epsilon \longrightarrow 0$, then the four–point function $\partial^3_{t_0^{(r)}} D_r(z)F$ preserves its form (H.1) with the parameters satisfying $b_r c_r - a_r d_r = 0$, so it becomes

$$\partial^3_{t_0^{(r)}} D_r(z)F = \frac{\varphi_r(z^{-1}) - 1}{\varphi_r(z^{-1})}\,\frac{a_r}{c_r}\ , \tag{H.7}$$

but all the other four–point functions degenerate and become equal to zero, $\partial^2_{t_0^{(r)}} \prod_{i=1}^{2} D_r(z_i)F = \partial_{t_0^{(r)}} \prod_{i=1}^{3} D_r(z_i)F = \prod_{i=1}^{4} D_r(z_i)F = 0$.

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