Renormalization Group Studies of Scalar-Tensor theories of Gravity

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Chapter 1

Introduction

The unification of Quantum Mechanics (QM) and General Relativity (GR) is one of the biggest challenges of contemporary physics. In spite of the fact that the two theories were discovered a century ago and a lot of effort has been made to combine the principles of the two theory, still we have not been a able to find a consistent quantum theory of gravitation, which describes the nature in a satisfactory manner.

One may ask what is the need for having a quantum theory of gravity? The replies to this are many, the important ones are the following:

1. Apart from gravitational interactions, the experimental results known so far tells that there are three other forces namely: electromagnetic, weak and strong, that are responsible for describing the physics of the known Universe. Gravity and electromagnetism are responsible for describing the physics of large scale structure in the Universe while the other two forces are used to the describe the physics at subatomic scale. The non-gravitational forces have been studied in past consistently using the mathematical framework of quantum field theory. For example electromagnetism at a macroscopic level can be very described using the classical field theory like gravity, but at the microscopic level it is consistently described using the quantum theory of electromagnetism, also known by name Quantum Electrodynamics (QED). One will naturally expect something similar to happen in the case of gravity. Furthermore the belief that there must exists a unified description of all the known fundamental forces demands to search for a quantum theory of gravity.

2. Classical GR produces solutions which contains spacetime singularities. These singularities are either masked by an event horizon (in case of black holes) or are naked (the big bang singularity). Classical GR breaks down near such singularities as the curvature of spacetime become infinite. This makes classical GR incomplete. It is hard to imagine that some classical extension of GR could avoid completely these pathologies. In fact black hole solutions and Freedman-Robertson- Walker (FRW) type solutions studied in the extension of GR, like scalar-tensor theories and higher-derivative theories of gravity like

$F(R)$ theories, show that singularities persists. It is expected that a quantum theory of gravity would resolve this.

3. Black holes are solutions of vacuum Einstein's equation and are uniquely characterized by just three parameters namely: mass, charge and angular momentum. It has been found that black holes are like thermodynamical objects that obey laws similar to laws of thermodynamics, with an entropy proportional to its area. Semi-classical quantization shows that the black holes radiate with temperature going as inverse of its mass. It was argued that as black holes radiate more and more, their mass decreases more when ultimately they vanish with an infinite temperature. This argument results in two problems: first the well known problem of information paradox and secondly the problem of temperature becoming infinite. Also, as mass decreases its size also decreases and goes to zero. It was argued that these problems are a consequence of semi-classical approximation being made, whose results cannot be trusted beyond a certain energy scale. As the black hole decreases and approaches the Planck's size and reaches the Planck's temperature, quantum gravity effects cannot be ruled out. This urges very strongly for the need of having a quantum theory of gravity. Besides as black holes are thermodynamical objects obeying laws similar to laws of thermodynamics and having an entropy, thus it becomes necessary to ask what is the microscopic structure underlying the macroscopic objects. Just as for a fluid which obey macroscopic thermodynamical laws and can be described using pressure, volume *etc.* while it also has a statistical description in terms of molecules; in the same way one expects to find "atoms" of spacetime.

The reasons and motivations given above are just a few important points which urges the need for having a quantum theory of gravity.

The traditional approach for doing QFT is through perturbative quantization of a classical field theory. Here one performs the loop computation by expanding the Green functions in to powers of $\hbar$ or in powers of coupling constants. If the divergences appearing in the computations can be reabsorbed by making suitable redefinition of a finite number of coupling constants, then the theory is termed perturbatively renormalizable. One is then left with a finite theory free of divergences at all loops depending on a finite number of parameters which are fixed through experiments. This methodology of perturbative quantization have been successfully used to study various quantum physical problems in four dimensions. Famous examples are: QED, non-abelian gauge theories, scalar theories etc. The success of these methods have led to the constructions of Standard Model. In general one can show that a theory is perturbatively renormalizable if the classical action contains couplings which have either zero or positive mass dimensions. Experimental verification of QED and Standard Model to a great accuracy has shown that perturbative quantum field theory methods are very reliable.

Motivated with the successes of these methods, physicists applied them to theories of gravity. The first action to which it was applied was the Einstein-Hilbert action given by,

$$S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{g}\, R \ , \tag{1.1}$$

where it was found that the theory is plagued by UV divergences [1]. The calculation were performed by breaking the quantum metric in to a background and a fluctuation. As the gravitational coupling has a negative mass dimension, thus it was expected that the theory will be non-renormalizable. Explicit calculations done in [1] showed that pure gravity eq. (1.1) is renormalizable at one-loop as the divergences vanish on-shell. Thus it was possible to absorb the divergences using a field redefinition. It was further noticed that EH-gravity (1.1) interacting with a scalar field is non-renormalizable even at one loop, where it was not possible to absorb the divergence and the counter term did not vanish on-shell. The results of non-renormalizability of EH-gravity interacting with matter were further confirmed in [2, 3], where it was concluded that gravity interacting with matter is non-renormalizable. Further more, a two loop computation showed that pure gravity eq. (1.1) is two loop non-renormalizable, where a term cubic in curvature could not be absorbed within the original Lagrangian even on-shell [4, 5]. This implied that as one goes further on the perturbation series more and more counter-terms need to be added to the original action to make the theory free from divergences, but then the theory will loose its predictability. However when perturbative methods of quantization were applied to higher-derivative theories of gravity, they were found to be renormalizable but violating unitarity [6].

The common thinking is that when a theory is not renormalizable perturbatively then it is not complete. Actually the reality is that such theories offers a low energy effective description of the more fundamental theory. This line of thought has been applied to the case of weak interactions where they were modeled using the Fermi's theory. It was later found that the theory of intermediate vector boson used to describe the weak interactions reduces to Fermi's theory in low energy limit. This makes one tempted to say that perhaps GR (or more generally a diffeomorphism invariant theory with metric as the dynamical variable) should not be considered as a fundamental theory, meaning that one should expect it to hold at only low energy scales, much below the fundamental scale, presumably the Planck scale. With this in mind one can treat these theories like Fermi's theory or the non-linear sigma model. Thus one can make precise calculations for the quantum effects, like for example the computation of the perturbative $\hbar$ correction to the Newtonian potential [7, 8, 9].

In the same spirit that led to the development of the Weinberg-Salam model from Fermi's theory, one can try to do the same to describe gravity in a quantum setting, by incorporating new features to the theory to make it consistent. The most prominent approach that has so far succeeded in extending QFT consistently is String Theory. It is not a QFT in the usual sense but in the low energy limit it has an effective description as a QFT. Current calculations rely on perturbative quantization around a fixed background metric, and are able to reproduce under some low energy limit the GR (and some higher order correction terms). String theory is beautiful and elegant but for the mathematical consistency requires large number of ingredients that so far have not been seen in nature. To be mathematically consistent it demands the existence of dimensions more than four. The nice feature of String theory is that they can be described by just one free parameter, thus they can describe a unified description of the four forces. The possible existence of extra-dimensions has a large phenomenological consequences. A decade ago it was shown [10] that the existence of a large extra-dimension in a model will bring down

Planck's scale to a range of energy accessible by the particle accelerators like LHC, thus allowing quantum gravity effects to be testable in accelerator labs. This model was also shown to solve the problem of large hierarchy present between the electroweak and the Planck scale.

Instead of extending the theory to make it consistent, one could more conservatively think that the problems do not lie with GR but with the method of quantization, the perturbative techniques. The non-renormalizability of GR does not imply that it is incompatible with Quantum Mechanics. The line of research that has gone farthest in this direction is Loop Quantum Gravity (LQG) or its covariant form, the Spin Foam Formalism. It is the canonical quantization of GR which originated from Ashtekar's reformulation of canonical GR which tries to impose diffeomorphism as a constraint on the Hilbert space of the quantum states. Unlike string theory this approach to quantum gravity is background independent. An important problem (one unlike in string theory) with this approach is that there is no proof that the low energy limit of this theory will be EH-gravity.

Quantum field theory has attained an iconic status ever since its success in formulating the Standard model of particle physics and its experimental verifications to a great accuracy. Not only these, it has been successfully used to study condensed matter systems, for example superconductivity. Therefore it becomes necessary before giving it up, that one should reconsider QFT from the nonperturbative perspective, and study the problem of Quantum Gravity in that framework. This is what I am going to discuss in this thesis. The issue that I am going to tackle in this work is whether GR (or a minimal modification of it *i.e.* a theory based on minimal symmetry principles) can be given the status of the fundamental theory, one from which predictions can be expected at all energy scales, without running in to problems of UV divergences. I will be as conservative as possible and see first if it is possible to build a consistent QFT with metric as the only dynamical field, without including any extra features like strings.

Simplical quantum gravity and causal dynamical trangulations are some of the discrete approaches to quantum gravity which can be used to calculate some of the numerical quantities of interests. These approaches are related to QG in the same way as lattice QFT are related to continuum QFT. Therefore the aim is to define the sensible continuum limit which by definition would be QG.

Among the approaches that maintain the ideology of continuous QFT, the most promising seems to be "Asymptotic Safety". It was a proposal made by Steven Weinberg in 1979 [11], and gained momentum in the last ten years. It is a generalized nonpertubative notion of renormalizability. It reduces to ordinary perturbative renormalizability under special circumstances. The general framework for the asymtotic safety scenario is the renormalization group. It is well known from the perturbative QFT that the coupling constants of theory runs *i.e.* the couplings are dependent on scale k and vary as the scale k is varied. The running of the couplings is described by the beta-function of the coupling, which in the case of perturbative QFT is obtained through a Callan-Symanzik equation. Beta functions are very important ingredient of the theory. For example in the case of non-abelian theories, the coupling approach zero as k goes to infinity, making the theory asymptotically free. At this point the beta function of the coupling vanishes, and for this reason this is called fixed point of the theory or more correctly the "Gaussian" Fixed point. Near this fixed point one can do perturbation theory as the couplings are small.

In some theories the Gaussian FP is reached by the couplings in the UV limit, while in others it is the FP in the IR limit. In the former case, theories have well defined UV limit and are perturbatively renormalizable, while in later case UV limit is questionable, even though they may be perturbatively renormalizable. The best and the simple example which describes the later case is the $\lambda\,\phi^4$ theory in four dimensions, which is perturbatively renormalizable but develops a Landau pole in the beta function and thus the UV limit cannot be taken using the perturbation theory. It is renormalizable near the Gaussian FP which lies in the in IR regime, but as energies become high the coupling strength increases and one is no longer within the realm of perturbative quantization.

Beside these there are theories which are perturbatively non-renormalizable, *e.g.* Einstein-Hilbert gravity. It has been found to be one-loop perturbatively renormalizable but at two loops the plague of UV divergences destroys the renromalizability. Putting matter fields in the system makes the situation even worse, when the one loop renormalizability is also sacrificed. Extending Einstein-Hilbert gravity by including higher-derivative terms brings new features, renormalizability is achieved at the price of Unitarity. Coupling these theories with large number of matter fields, the business of unitarity gets settled. It was in this setting that the idea of Asymptotic safety came to existence to settle the debatable issues by proposing a new line of sight to the problem.

The idea proposed that as gravity becomes strongly coupled in the UV regime thus should be treated nonperturbatively. The fact that it is perturbatively non-renormalizable was associated to the point of doing perturbation theory around a Gaussian FP and trying to extend the results beyond the realm of validity. It was suggested that gravity might possess a nontrivial FP in the UV regime where some of the couplings might be nonzero. If the theory possesses such a non Gaussian FP and has a finite number of UV attractive directions at the FP, then one can argue such a theory is free of uncontrollable UV divergences and is predictive. A QFT that possesses such properties is termed Asymptotically safe.

In four dimensions one cannot reach this non-Gaussian FP through perturbation theory, as the coupling are non-zero and thus one needs nonperturbative techniques to see this FP. Soon after the idea was proposed, physicists tried to find this FP. They started off by doing computations in $d = 2+\epsilon$ dimensions, where the canonical dimension of the Newton's constant is close to zero and perturbation theory can be trusted. On calculating the beta functions it was seen that it possesses a non-trivial UV attractive FP, and has the right properties to make gravity asymptotically safe in $d = 2 + \epsilon$. Using dimensional continuation one can argue that gravity is asymptotically safe in other near by dimensions too. However extending this to four dimensions was questionable as the calculations cannot be trusted in the limit $\epsilon \to 2$, because ϵ was taken as a small parameter. This was all done in eighties [12, 13, 14, 15, 16, 17, 18]. Research came to halt due to lack of technical tools to perform the computation in four dimensions.

In 1993 Wetterich discovered a Functional Renormalization Group Equation (FRGE) [20]. This equation was based on the Wilson idea [21] of integrating out degrees of freedom beyond scale k to obtain an effective averaged theory describing the physics at scale k [19]. This is done using functional integrals, where one integrates out modes with momenta above scale

k, to obtain the effective theory at scale k. From this one could extract the k-dependence of the couplings present and obtain their running, which are the beta-functions of the couplings. This was done in seventies and the approach was perturbative. However this idea can be implemented within a nonperturbative framework, thereby obtaining the flow equation for the effective action. This is a nonperturbative flow equation as there has not been made any assumptions about couplings being small. This equation is now known as FRGE and gives the beta-functions of the couplings in a theory in a nonperturbative way.

This equation although first constructed for scalar field theory to study phase transitions [22], was soon adapated to study gauge-invariant theories [23]. It was applied for the first time to gravity to obtain the beta functions of cosmological constant and Newton's constant in four dimensions [24, 25]. These beta functions were completely non-perturbative and were immediately used to search for nontrivial FP of Einstein-Hilbert gravity. A nontrivial FP was indeed found in [26] with two UV attractive directions thereby making the theory asymptotically safe.

FRGE contains an scale dependent cutoff in the equation, and thus one expects that the results will depend strongly on the cutoff. This was termed as "Scheme Dependence". However computations involving different types of cutoff showed that results depend very mildly on them. Only the quantitative properties are mildly affected while qualitative properties remain same, like the existence of FP etc [27, 28]. As FRGE for gravity was constructed using the euclidean functional integral, and thus involved gauge-fixing. So the gauge dependence of the results pose another threat. But it turned out that even gauge dependence was mild [27].

The strongest fear was that whether the results will continue to hold when the Einstein-Hilbert gravity is extended to include matter and/or higher-derivative terms. A study involving higher-derivative pure gravity was done for the first time in [29]. The system contained three operators $\sqrt{g}$, $\sqrt{g}R$ and $\sqrt{g}R^2$. It was found that nontrivial FP exists and has three UV attractive directions. This was both a good news and bad one. Good in the sense that results continue to hold *i.e* the theory is asymptotically safe. Bad in the sense that when we considered EH-gravity, there we had two couplings in system and at the nontrivial FP it was found that the theory possesses two UV attractive directions; and when we considered a system with three couplings, it has nontrivial FP and at FP it has three UV attractive directions. Thus it appeared that as one increases the number of couplings, the number of UV attractive directions would also increase. So it was expected that if for example one has ten couplings consisting of terms up to R^9, then there might perhaps be ten UV attractive directions.

To rule out such a possibility, a renormalization group study of higher-derivative gravity with four derivatives of metric, including all curvature invariants except total derivatives was considered in one-loop approximation [62]. This showed that the nontrivial FP was present except that the number of UV attractive directions was now found to be five (of which three were marginally attractive). This computation was in one-loop approximation thus demanded further study. A study of this system beyond one-loop (but omitting the Euler term) shows that there are no marginal couplings and the number of UV attractive directions is three [32]. This was further confirmed by a nonperturbative computation done in the context of $F(R)$-gravity with terms till R^8, where it was found that at the nontrivial FP there are three UV attractive directions [30, 31].

Inclusion of matter field was another issue which raised fear. In perturbative computations involving matter fields, it was found that renormalizability was sacrificed even at one loop. A first step within the asymptotic safety scenario was to consider minimally coupled field. The beta functions have been obtained in [25], while the study of fixed point and number of UV attractive directions was done in [33, 34]. It was found that under some circumstances when the number of matter fields is large, the existence of FP is questionable. This was used as a condition to put bound on the matter fields of various spins. This was extended to consider nonminimally coupled matter in [34]. It was found that gravity is asymptotically safe and possesses a nontrivial FP where matter couplings become asymptotically free, with four UV attractive directions. This was interesting to note that with the inclusion of nonminimally coupled matter in Einstein-Hilbert gravity, the number of UV attractive directions increases from two to four. This study was undertaken in a truncation involving a finite number of couplings, in which it was not clear if the results will remain the same when a better truncation is considered by including more number of couplings.

A study of minimally coupled scalar with an higher derivative R^2 (including all possible curvature invariants with four derivatives of metric) gravity showed that gravity is asymptotically safe with three UV attractive directions [35]. This was a good news that coupling with matter does not destroy the non-perturbative renormalizability of the theory.

In each of the work involving nonminimally coupled matter fields, it was not clear whether the picture of asymptotic safety will change under the inclusion of more nonminimal matter couplings. A numerical study conducted in four dimensions showed that a GMFP will exist in all gauges, with four UV attractive directions, but there was no rigorous argument for the same. In [36, 37] it has been mathematically proved using method of induction that at the GMFP the number of UV attractive directions remains finite in arbitrary dimensions. In the study involving nonminimal coupling of EH-gravity with scalar, it was found that in four dimensions the number of UV attractive directions will always remain four irrespective of the number of non-minimal couplings included [36]. A similar study involving higher-derivative truncations involving powers of R up to R^8, showed that in a non-minimal setting there will be two more UV attractive directions apart from the usual three, thereby making a total of five UV attractive directions [37]. These two works constitute the main part of my thesis and are described in detail in the later chapters.

Due to the fashionable proposal of extra-dimensions scenarios, there has been some work on the asymptotic safety of gravitational theories in extra-dimensions. A first step in this direction was taken in [38], where Einstein-Hilbert gravity was considered in arbitrary dimensions. It was found that a FP exists in extra dimensions with two UV attractive directions. Moreover studies also showed that the FP in four dimension is smoothly connected to the FP in $d = 2 + \epsilon$ (dimensional continuation) [31]. A study involving nonminimal coupling of scalar to EH-gravity showed similar results [36].

The outline of the thesis is as follows. In chapter 2, I will describe the construction of FRGE and discuss various issues related to it. Then I will describe the basic ides of asymptotic safety as set forth by Steven Weinberg. I will then use FRGE to extract the running of couplings in a $O(N)$-symmetric scalar field theory. This is followed by a discussion on scheme dependence in the beta functions derived using FRGE.

In chapter 3, I will show how to construct a FRGE for gravity. I will discuss the simplifications need to be made in order to do practical computations. Then I will use the FRGE for gravity to study the flow of the couplings in Einstein-Hilbert theory of gravity, thereby deriving the beta functions of the couplings in four dimensions and using them to show how the requirements of asymptotic safety are satisfied. I took this opportunity to also discuss various cutoff types that are used in computations and how the results depend on them.

In chapter 4 I will apply the FRGE constructed for gravity to a scalar coupled nonminimally to Einstein-Hilbert gravity. This is done in arbitrary dimensions. In chapter 5, I study the asymptotic safety of scalar coupled nonminimally to $F(R)$ gravity. Finally in chapter 6, I present the conclusion of the thesis.

Chapter 2

Functional Renormalization Group Equation

In this chapter I will describe how to derive the Functional Renormalization Group Equation (FRGE) for a simple scalar field. The main point in this derivation is the construction of the effective average action Γ_k, which is the coarse grained action describing effective field theory of the physics at scale k. The FRGE tells the running of this effective average action. After discussing the construction of FRGE and effective average action, I will discuss the notion of Asymptotic Safety as introduced by Steven Weinberg [11]. After demonstrating the process of construction of FRGE for the simple scalar case, I will show how to use this equation to extract the running of couplings in a theory. I will consider a simple example of an $O(N)$-symmetric scalar theory consisting of the usual kinetic term and a generic $O(N)$-symmetric potential. I will use FRGE to study its flow in arbitrary dimensions. The trivial Gaussian fixed point is found in all dimensions. By considering the linearized flow around this fixed point I will derive some general properties of critical exponents around this fixed point in arbitrary dimensions for arbitrary N. On studying the flow in three dimensions the Wilson-Fisher fixed point is found. I will then discuss the issue of scheme dependence in beta functions which are obtained using FRGE.

2.1. FRGE and its Approximations

2.1.1. Derivation of FRGE

In this section I would demonstrate how to derive the FRGE for the simple scalar system in d-dimensions and would introduce the concept of effective average action. The derivation given closely follows the one first given in [19, 20] (see for reviews [39, 40]). Consider the Euclidean path integral for the scalar field theory.

$$Z[J] = \int \mathcal{D}\varphi(x)\, \exp\Big\{ - S[\varphi] - \int d^d x J(x) \cdot \varphi(x) \Big\} \,, \tag{2.1}$$

where $S[\varphi]$ is a generic bare action. From this path integral one is able to get various n-point correlation functions. Using this one can construct the connected Green's functional W,

$$W[J] = -\ln Z[J] \,. \tag{2.2}$$

Using W we get the vacuum expectation value of the $\varphi(x)$.

$$\phi(x) = \langle \varphi(x) \rangle = -\frac{1}{Z}\frac{\delta Z}{\delta J(x)} = \frac{\delta W}{\delta J(x)} \ . \tag{2.3}$$

This is then used to construct the one–particle–irreducible (1PI) connected Green functional by Legendre transforming W:

$$\Gamma[\phi] = W - \int d^d x \, J(x) \cdot \phi(x) \ . \tag{2.4}$$

In these generating functional all the modes of the quantum field $\varphi(x)$ have been integrated out. But in order to study physical phenomenons at a given scale k, one need to compute the effective action at that scale k. The effective field theory Lagrangian describing the physics at that scale is obtained by integrating out all modes of quantum field with momenta $p > k$.

Wilson obtained this aim by sharply cutting off the functional integral given in eq. (2.1) at scale k and then considering its flow [21]. Thus in the Wilson's picture eq. (2.1) is written as,

$$Z_k[J] = \int_{|p|>k} \mathcal{D}\varphi(x) \, \exp\Big\{ - S[\varphi] - \int d^d x \, J(x) \cdot \varphi(x) \Big\} \ . \tag{2.5}$$

This job of sharply cutting off the functional integral can be done alternatively by introducing a cutoff action $\Delta S_k[\varphi]$ in the exponent of eq. (2.5). The purpose of $\Delta S_k[\varphi]$ is to sharply suppress the modes of quantum field with momenta below k. On introducing this cutoff in the path integral (2.1) one gets,

$$Z_k[J] = \int \mathcal{D}\varphi(x) \, \exp\Big\{ - S[\varphi] - \Delta S_k[\varphi] - \int d^d x \, J(x) \cdot \varphi(x) \Big\} \ . \tag{2.6}$$

The cutoff action $\Delta S_k[\varphi]$ has the following form,

$$\Delta S_k = \frac{1}{2} \int d^d x \, d^d y \, \varphi(x) \, \mathcal{R}_k(\Delta) \varphi(y) \ . \tag{2.7}$$

where $\mathcal{R}_k$ is the cutoff constructed from some differential operator Δ whose eigenfunctions can be used as basis for expanding the field. The purpose of cutoff $\mathcal{R}_k$ is to modify the propagator of the modes with momenta below k by adding a mass like term to it. An infinite mass term would result in a sharp cutoff and would be equivalent to eq. (2.5). However a sharply cut functional integral leads to technical problems. Replacing the sharp cutoff $\mathcal{R}_k$ with a smooth one overcomes such difficulties. The particular form of $\mathcal{R}_k(z)$ is arbitrary apart from the fact that it satisfies the following properties,

$$\mathcal{R}_k(z) \approx \begin{cases} k^2 & \text{for } \ z \ll k^2 \,, \\ 0 & \text{for } \ z \gg k^2 \,. \end{cases} \tag{2.8}$$

A cutoff satisfying the above conditions would give mass of order k^2 to modes of momenta $p < k$, thus causing suppression while giving almost no mass to modes with momenta $p > k$, thus allowing them propagate without suppression [20].

Having defined a well behaved path integral given by eq.(2.6), I define the connected Green functional W_k at scale k in the same way as in eq. (2.2),

$$\exp\{-W_k\} = Z_k \ . \tag{2.9}$$

Using which k-dependent expectation value of φ is obtained. This is denoted by $\phi(x)$.

$$\langle\varphi\rangle_k = \frac{\delta W_k}{\delta J(x)} = \phi(x) \ . \tag{2.10}$$

One can invert this relation to obtain source J as a function of ϕ. As ϕ is k-dependent, thus the source which depends on ϕ gets an implicit dependence on k. Using this k-dependent source I define the Legendre transform of the connected Green functional W_k which we call $\tilde{\Gamma}_k[\phi]$.

$$\tilde{\Gamma}_k[\phi] = W_k[J] - \int d^dx\, J(x)\cdot\phi(x) \ . \tag{2.11}$$

On taking the functional derivative of this Legendre transform with respect to the classical field we get,

$$\frac{\delta\tilde{\Gamma}_k}{\delta\phi(x)} = -J(x) \ . \tag{2.12}$$

Now we take derivative of this Legendre transform with respect to $t = \ln k$. This will give,

$$\begin{aligned}
(\partial_t\tilde{\Gamma}_k)[\phi] + \int d^dx\, \frac{\delta\tilde{\Gamma}_k}{\delta\phi(x)}\cdot\partial_t\phi(x) &= (\partial_t W_k)[J] + \int d^dx\, \frac{\delta W_k}{\delta J(x)}\cdot\partial_t J(x) \\
&- \int d^dx\, \partial_t J(x)\cdot\phi(x) - \int d^dx\, J(x)\cdot\partial_t\phi(x) \\
(\partial_t\tilde{\Gamma}_k)[\phi] - \int d^dx\, J(x)\cdot\partial_t\phi(x) &= (\partial_t W_k)[J] + \int d^dx\, \phi(x)\cdot\partial_t J(x) \\
&- \int d^dx\, \partial_t J(x)\cdot\phi(x) - \int d^dx\, J(x)\cdot\partial_t\phi(x) \ ,
\end{aligned} \tag{2.13}$$

where eq. (2.10) and (2.12) has been used to obtain the last line of eq. (2.13). Canceling the various terms we get,

$$(\partial_t\tilde{\Gamma}_k)[\phi] = (\partial_t W_k)[J] \ . \tag{2.14}$$

The RHS of the eq. (2.14) can be obtained in the following way,

$$\begin{aligned}
(\partial_t W_k)[J] &= -\frac{1}{Z_k[J]}\,(\partial_t Z_k)[J]\ , \\
&= \frac{1}{Z_k[J]}\int \mathcal{D}\varphi(x)\,(\partial_t \Delta S_k)\ \exp\Big\{-S[\varphi]-\Delta S_k[\varphi]-\int d^d x\, J(x)\cdot\varphi(x)\Big\}\ , \\
&= \frac{1}{2}\int d^d x\, d^d y\,\langle \varphi(x)\cdot\partial_t \mathcal{R}_k(\Delta)\cdot\varphi(y)\rangle\ , \\
&= \frac{1}{2}\int d^d x\, d^d y\left[-\frac{\delta^2 W_k}{\delta J(x)\,\delta J(y)}+\frac{\delta W_k}{\delta J(x)}\,\frac{\delta W_k}{\delta J(y)}\right]\cdot\partial_t \mathcal{R}_k(\Delta)\ , \\
&= \frac{1}{2}\int d^d x\, d^d y\left[-\frac{\delta^2 W_k}{\delta J(x)\,\delta J(y)}+\phi(x)\phi(y)\right]\cdot\partial_t \mathcal{R}_k(\Delta)\ ,
\end{aligned} \tag{2.15}$$

where I have used eq. (2.10) and (2.12) along with the identity,

$$\langle\varphi(x)\varphi(y)\rangle = \left[-\frac{\delta^2 W_k}{\delta J(x)\,\delta J(y)}+\frac{\delta W_k}{\delta J(x)}\,\frac{\delta W_k}{\delta J(y)}\right]\ . \tag{2.16}$$

The expression for the t-derivative of W_k derived in eq. (2.15) can then be plugged in eq. (2.14) to obtain an equation for the running of $\tilde{\Gamma}_k$. This is given by,

$$(\partial_t\tilde{\Gamma}_k)[\phi] = \frac{1}{2}\int d^d x\, d^d y\left[-\frac{\delta^2 W_k}{\delta J(x)\,\delta J(y)}+\phi(x)\phi(y)\right]\cdot\partial_t \mathcal{R}_k(\Delta) \tag{2.17}$$

At this point I eliminate W_k by making use of the following identity,

$$\begin{aligned}
\frac{\delta J(x)}{\delta J(y)} &= \delta(x-y) \\
\int d^d z\,\frac{\delta J(x)}{\delta\phi(z)}\cdot\frac{\delta\phi(z)}{\delta J(y)} &= \delta(x-y) \\
\int d^d z\,\frac{\delta^2\tilde{\Gamma}_k}{\delta\phi(x)\delta\phi(z)}\cdot\frac{\delta^2 W_k}{\delta J(z)\delta J(y)} &= -\delta(x-y)\ .
\end{aligned} \tag{2.18}$$

The last line of the above equation can be notationally written in the following way,

$$\frac{\delta^2 W_k}{\delta J\delta J} = -\left(\frac{\delta^2\tilde{\Gamma}_k}{\delta\phi\delta\phi}\right)^{-1} \tag{2.19}$$

At this particular stage I define the effective average action at scale k in the following way [19, 20],

$$\Gamma_k = \tilde{\Gamma}_k - \frac{1}{2}\int d^d x\,\phi(x)\mathcal{R}_k(\Delta)\phi(x)\ . \tag{2.20}$$

The running of the effective average action Γ_k can be obtained by making use of eq. (2.20) and (2.19) in eq. (2.17). This is done as follows,

$$\begin{aligned}(\partial_t \Gamma_k)[\phi] &= \frac{1}{2}\int d^dx\, d^dy \left[-\frac{\delta^2 W_k}{\delta J(x)\,\delta J(y)}\right]\cdot \partial_t \mathcal{R}_k(\Delta)\delta(x-y) \\ &= -\frac{1}{2}\mathrm{Tr}\left[\frac{\delta^2 W_k}{\delta J\,\delta J}\cdot \partial_t \mathcal{R}_k\right] \\ &= \frac{1}{2}\mathrm{Tr}\left[\left(\frac{\delta^2 \tilde{\Gamma}_k}{\delta\phi\delta\phi}\right)^{-1}\cdot \partial_t \mathcal{R}_k\right] \\ &= \frac{1}{2}\mathrm{Tr}\left[\left(\frac{\delta^2 \Gamma_k}{\delta\phi\delta\phi}+\mathcal{R}_k\right)^{-1}\cdot \partial_t \mathcal{R}_k\right] , \end{aligned} \tag{2.21}$$

where the last line gives the expression for the running of the effective average action Γ_k. This is the functional renormalization group equation, which we write separately,

$$(\partial_t \Gamma_k)[\phi] = \frac{1}{2}\mathrm{Tr}\left[\left(\frac{\delta^2 \Gamma_k}{\delta\phi\delta\phi}+\mathcal{R}_k\right)^{-1}\cdot \partial_t \mathcal{R}_k\right] , \tag{2.22}$$

where $\delta^2\Gamma_k/\delta\phi\delta\phi \equiv \Gamma_k^{(2)}$ is the Hessian of the effective average action.

2.1.2. Properties of the Flow Equation and Effective average action

Here I would describe the properties obeyed by the effective average action Γ_k given in eq. (2.20) and the FRGE eq. (2.22).

(1) The trace appearing in eq. (2.22) is the sum over all the eigenvalues λ of the operator Δ. This sum is both UV and IR regularized. This can be understood better in a simpler setting when the operator $\Delta = -\partial^2$. In this case the eigenvalues of Δ are p^2, where p is the momentum of the field. Thus the trace becomes a momentum integral with p running from zero to infinity. In this momentum integral there is no need to put a UV regulator. This is because of the properties of the cutoff $\mathcal{R}_k$ given in eq. (2.8). The trace contain a factor $\partial_t\mathcal{R}_k(p^2)$, which is significantly different from zero only in the region centered around $p^2 = k^2$. Hence the trace receives contributions from momenta $p^2 \lesssim k^2$ only and is therefore well convergent both in the UV and IR.

(2) In the computation of the one-loop effective action we get the following expression,

$$\Gamma^{(1)} = S[\phi] + \frac{1}{2}\mathrm{Tr}\ln\left[\frac{\delta^2 S}{\delta\phi\delta\phi}\right] , \tag{2.23}$$

where the second term contains the loop corrections. This is the full effective action where all the modes have been integrated out. If one were to compute the one loop effective action at scale k, then that can be obtained by just adding the cutoff action $\Delta S_k[\phi]$ to S appearing inside the trace and we get,

$$\begin{aligned}\Gamma_k^{(1)} &= S[\phi] + \frac{1}{2}\mathrm{Tr}\ln\left[\frac{\delta^2(S+\Delta S_k)}{\delta\phi\delta\phi}\right] \ , \\ &= S[\phi] + \frac{1}{2}\mathrm{Tr}\ln\left[\frac{\delta^2 S}{\delta\phi\delta\phi} + \mathcal{R}_k\right] \ , \end{aligned} \tag{2.24}$$

On taking the t-derivative of eq. (2.24) we get,

$$\begin{aligned}\partial_t\Gamma_k^{(1)} &= \frac{1}{2}\partial_t\left(\mathrm{Tr}\ln\left[\frac{\delta^2 S}{\delta\phi\delta\phi} + \mathcal{R}_k\right]\right) \ , \\ &= \frac{1}{2}\mathrm{Tr}\left[\left(\frac{\delta^2 S}{\delta\phi\delta\phi} + \mathcal{R}_k\right)\partial_t\mathcal{R}_k\right] \ . \end{aligned} \tag{2.25}$$

This equation has the same appearance as the one given in eq. (2.22) except that eq. (2.25) contains the bare action within the trace, while eq. (2.22) contains the effective average action at scale k. This is the renormalization group improved one-loop equation encoding the beta functions of our effective theory. The eq. (2.22) is non-perturbative in the sense that we make no assumptions about the couplings and is exact in the sense that solving it is equivalent to solving the complete theory. The effective average action Γ_k in general can be expanded as follows,

$$\Gamma_k = \sum_{n=0}^{\infty}\sum_i g_i^{(n)}(k)\,\mathcal{P}_i^{(n)} \ , \tag{2.26}$$

where $g_i^{(n)}$ are the couplings and $\mathcal{P}_i^{(n)}$ are all the operators of order n that can be constructed from the field and its derivatives in accordance with the symmetry requirements. With this effective average actions the LHS of FRGE would be,

$$\partial_t\Gamma_k = \sum_{n=0}^{\infty}\sum_i \beta_i^{(n)}(k)\,\mathcal{P}_i^{(n)} \ , \tag{2.27}$$

where $\beta_i^{(n)}(k) = \partial_t g_i^{(n)}(k)$ are the beta functions of the dimensionfull couplings. Thus one can think of the RHS of the FRGE as a beta functional.

(3) The effective average action Γ_k satisfies the following integro-differential equation,

$$\begin{aligned}\exp\{-\Gamma_k[\phi]\} &= \int\mathcal{D}\varphi\,\exp\left\{-S[\varphi] - \int d^dx\,(\varphi-\phi)\frac{\delta\Gamma_k[\phi]}{\delta\phi}\right\}\times \\ &\quad\times\exp\left\{-\int d^dx\,(\varphi-\phi)\mathcal{R}_k(\Delta)(\varphi-\phi)\right\} \ . \end{aligned} \tag{2.28}$$

This equation is easily obtained by using eq. (2.9), (2.11) and (2.20) combined with eq. (2.10) and (2.12).

(4) From the definition of the effective average action at scale k given in eq. (2.20) we note that it interpolates between the standard effective action $\Gamma = \Gamma_{k\to 0}$ and the bare action $S[\phi]$ for $k \to \infty$. The $k \to 0$ limit is obtained from the properties of cutoff $\mathcal{R}_k$ given in eq. (2.8). The $k \to \infty$ limit is obtained using the integro-differential equation given in (2.28). The argument goes as follows. In the limit $k \to \infty$, the cutoff $\mathcal{R}_k \approx k^2$. Thus the second exponential on the RHS of eq. (2.28) becomes $\exp\{-k^2 \int dx(\varphi - \phi)^2\}$, which, up to a normalization factor, approaches a delta-functional $\delta[\varphi - \phi]$. The φ integration can be performed trivially then and one ends up with $\lim_{k\to\infty} \Gamma_k[\phi] = S[\phi]$. In a more careful treatment [20] one shows that the saddle point approximation of the functional integral in eq. (2.28) about the point $\varphi = \phi$ becomes exact in the limit $k \to \infty$. As a result, $\lim_{k\to\infty} \Gamma_k$ and S differ at most by the infinite mass limit of a one-loop determinant, which we suppress here since it plays no role in typical applications (see [41] for a more detailed discussion).
(5) The FRGE (2.22) is independent of the bare action S which enters only via the initial condition $\Gamma_\infty = S$. In the FRGE approach, the calculation of the path integral for W_k is replaced by integrating the RG equation from $k = \infty$, where the initial condition $\Gamma_\infty = S$ is imposed, down to $k = 0$, where the effective average action equals the ordinary effective action Γ, the object which we actually would like to know.

2.1.3. Theory space

The arena in which the Wilsonian renormalization group dynamics takes place is known as "theory space". It is a very formal concept but it helps very much in visualizing the various notions related to functional renormalization group equations, see fig. 2.1.3. Here I will be very general while describing the notion. Lets consider the an arbitrary set of fields given by $\phi(x)$. Then the action which is a functional of $\phi(x)$ can be thought as a map from this set of fields to c-numbers. The space of all action functionals based on this set of field under the requirements of respective symmetries is called theory space. In formal notation one can write the map as $\Gamma : \phi \mapsto \Gamma[\phi]$. The theory space $\{\Gamma[\,\cdot\,]\}$ is fixed once the set of fields and symmetries are fixed. Let suppose it is possible to find the set of "basis functionals" $\{\mathcal{O}_\alpha[\phi]\}$ which can be used to expand each point of the theory space as follows,

$$\Gamma[\phi] = \sum_{\alpha=1}^{\infty} \bar{u}_\alpha \mathcal{O}_\alpha[\phi] \ . \tag{2.29}$$

The basis consists of both local field monomials and non-local invariants. The coefficients of these basis are "generalized couplings" $\{\bar{u}_\alpha, \alpha = 1, 2, \cdots\}$, which can be seen as the local coordinates of the action. More precisely, the theory space is coordinatized by the subset of essential couplings.

From the geometrical point of view the FRGE given in eq. (2.22) defines a vector field on the theory space. The integral curves along the vector field are the "RG trajectories" and are maps $k \mapsto \Gamma_k$ with Γ_k being parameterized by k. These trajectories start from $k = \infty$, at the bare action S and terminate at the ordinary effective action at $k = 0$. The orientation of the flow

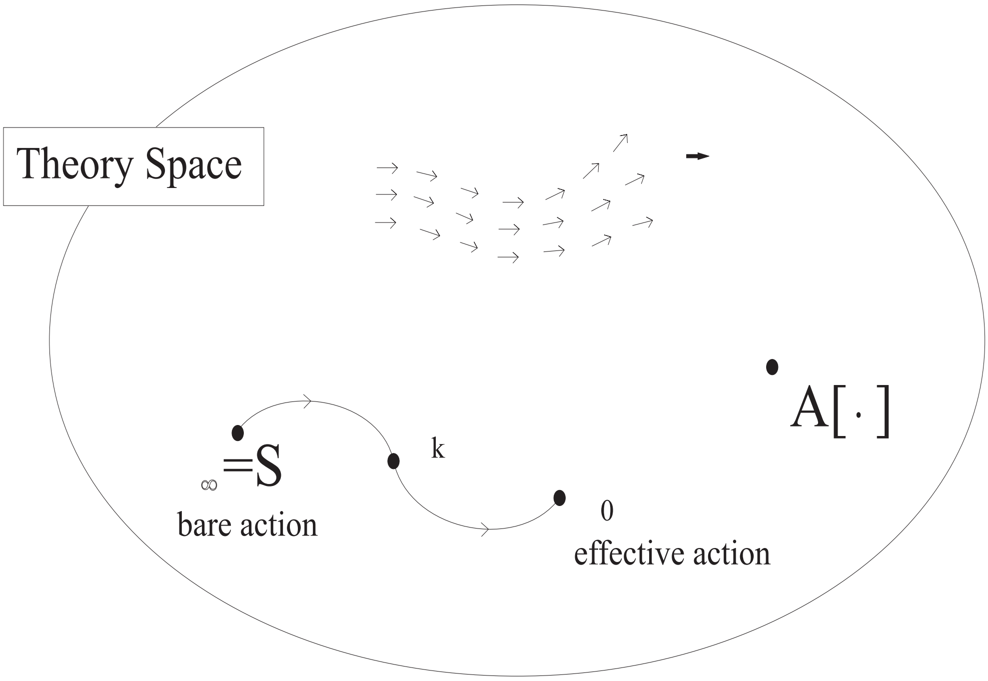

Figure 2.1.: The points of theory space are the action functionals $A[\,\cdot\,]$. The RG equation defines a vector field $\vec{\beta}$ on this space; its integral curves are the RG trajectories $k \mapsto \Gamma_k$. They start at the bare action S and end at the standard effective action Γ.

is the direction of decreasing k or increasing "coarse graining". One can expand the effective average action in terms of the basis,

$$\Gamma_k[\phi] = \sum_{\alpha=1}^{\infty} \bar{u}_\alpha(k)\, \mathcal{O}_\alpha[\phi] \ . \tag{2.30}$$

The trajectory would be described by infinitely many running couplings $\bar{u}_\alpha(k)$. On plugging the expansion of effective average action given by eq. (2.30) in the FRGE, one would obtain the infinitely many coupled differential equations for the couplings $\bar{u}_\alpha$.

$$k\partial_k\, \bar{u}_\alpha(k) = \overline{\beta}_\alpha(\bar{u}_1, \bar{u}_2, \cdots ; k) \ , \quad \alpha = 1, 2, \cdots \ . \tag{2.31}$$

Here the beta functions $\overline{\beta}_\alpha$ are the components of the vector field, and can be obtained from the FRGE by performing the trace on the RHS of FRGE and expanding it in the terms of $\mathcal{O}_\alpha[\,\cdot\,]$.

The presence of bar on $\bar{u}_\alpha$ and $\overline{\beta}_\alpha$ is to indicate that we are still dealing with the dimensionfull quantities. To search for fixed points and study the UV behavior of couplings at the fixed point, the flow equation is re-expressed in terms of dimensionless couplings $u_\alpha = k^{-d_\alpha}\bar{u}_\alpha$, where d_α is the mass dimension of the coupling $\bar{u}_\alpha$. Then these dimensionless couplings are used as coordinates of the theory space and the RG equation would be a coupled system of autonomous differential equations. The β_α would have no explicit k dependence and would define a "time independent" vector field in the theory space.

2.1.4. Effective average action and FRGE with UV cutoff

Till now I have been talking about the EAA and FRGE without a UV cutoff where the regularization in the trace of FRGE was achieved by the presence of the factor $\partial_t \mathcal{R}_k(q^2)$, which is nonzero only for momenta centered around $q^2 \approx k^2$ and goes to zero as $q^2 \to \infty$. But here in this section I would talk about the EAA and FRGE with a explicit UV cutoff Λ_c [42]. The reason why one should study this is because without a UV regularization the functional integral given in eq. (2.6) is not well defined. So it becomes important to introduce a UV cutoff to make the functional integral underlying the definition of EAA well defined.

As has been described in the section 2.1.1, on the construction of EAA and FRGE without a UV cutoff, we define the cutoff action and take its expression to be the one given in eq. (2.7). Then the UV-regulated analogue of eq. (2.9) is given by,

$$\exp\{W_{k,\Lambda_c}\} \equiv \int \mathcal{D}_{\Lambda_c}\varphi(x)\, \exp\Big\{ - S_{\Lambda_c}[\varphi] - \Delta S_k[\varphi] - \int d^dx\, J(x)\cdot\varphi(x)\Big\} \ . \tag{2.32}$$

Now following the same steps as outlined in section 2.1.1 on the construction of EAA and FRGE without UV cutoff, I would obtain the coarse grained expectation value of φ to be given by,

$$\phi_{\Lambda_c}(x) = \frac{\delta}{\delta J(x)} W_{k,\Lambda_c}[J] \ . \tag{2.33}$$

With which I define the Legendre transform of the W_{k,Λ_c} to be denoted by $\tilde{\Gamma}_{k,\Lambda_c}[\phi]$, where for notational simplicity we take $\phi(x) \equiv \phi_{\Lambda_c}(x)$. Then using this we finally define the UV regulated EAA as,

$$\Gamma_{k,\Lambda_c}[\phi] = \tilde{\Gamma}_{k,\Lambda_c}[\phi] - \frac{1}{2}\int d^dx\, \phi(x)\mathcal{R}_k(\Delta)\phi(x) \ . \tag{2.34}$$

This has the following UV-regulated FRGE,

$$\partial_t \Gamma_{k,\Lambda_c}[\phi] = \frac{1}{2}\mathrm{Tr}_{\Lambda_c}\left[\left(\Gamma^{(2)}_{k,\Lambda_c} + \mathcal{R}_k\right)^{-1}\cdot \partial_t \mathcal{R}_k\right] \ , \tag{2.35}$$

where the trace is now the sum over all the eigenvalues $\lambda \le \Lambda_c^2$ and $\Gamma^{(2)}_{k,\Lambda_c}$ denotes the Hessian of Γ_{k,Λ_c} Now for simplicity I would assume that $\Delta = -\partial^2$. Then the trace would be a momentum integration with the integration variable running from zero to Λ_c^2. The trace can be written in the following way,

$$\mathrm{Tr}_{\Lambda_c}[\cdots] = \mathrm{Tr}[\theta(\Lambda_c^2 - p^2)\{\cdots\}] \ , \tag{2.36}$$

where p^2 are the eigenvalues of $-\partial^2$. Since the cutoff $\mathcal{R}_k$ has the properties given in eq. (2.8), therefore due to the presence of term $\partial_t \mathcal{R}_k$ in FRGE, it is safe to take the limit $\Lambda_c \to \infty$. This would give the "Λ_c-free" FRGE without a UV cutoff, valid for all $k \geq 0$, and this FRGE is given by eq. (2.22). For simplicity I would write this as,

$$\partial_t \Gamma_k[\phi] = B_k\{\Gamma_k\}[\phi] \ , \tag{2.37}$$

where B_k denotes the beta functional,

$$B_k\{\Gamma_k\}[\phi] = \frac{1}{2}\text{Tr}\left[\left(\Gamma_k^{(2)}[\phi] + \mathcal{R}_k\right)^{-1} \cdot \partial_t \mathcal{R}_k\right] \ . \tag{2.38}$$

As has been discussed in the section on theory-space 2.1.3, B_k denotes a vector field.

Γ_k vs Γ_{k,Λ_c} in the limit $\Lambda_c \to \infty$

After having obtained the UV-regulated FRGE and described how to compute the Tr_{Λ_c} we now compare this with the Λ_c-free FRGE eq. (2.22), and see how the two things differ when Λ_c is made large [42].

The FRGE with UV cutoff eq. (2.35) contains the restricted trace Tr_{Λ_c} eq. (2.36). This can be written as,

$$\text{Tr}_{\Lambda_c}[\cdots] = \text{Tr}[\cdots] - \text{Tr}[\theta(p^2 - \Lambda_c^2)(\cdots)] \ . \tag{2.39}$$

Using this we have,

$$\partial_t \Gamma_{k,\Lambda_c} = B_k\{\Gamma_{k,\Lambda_c}\}[\phi] + \Delta B_{k,\Lambda_c}\{\Gamma_{k,\Lambda_c}\}[\phi] \ , \tag{2.40}$$

where the second term on RHS is given by,

$$\Delta B_{k,\Lambda_c}\{\Gamma_{k,\Lambda_c}\}[\phi] = -\frac{1}{2}\text{Tr}\left[\theta(p^2 - \Lambda_c^2)\left(\Gamma_k^{(2)}[\phi] + \mathcal{R}_k\right)^{-1} \cdot \partial_t \mathcal{R}_k\right] \ . \tag{2.41}$$

This term is a correction to the beta functional due to the presence of the UV cutoff. Its presence affects the Γ_{k,Λ_c} but not Γ_k. The corresponding RG flow are generated by the vector fields $B_k + \Delta B_{k,\Lambda_c}$ and B_k respectively. This correction term is small due to the presence of the step functions. It receives contributions only from modes with momenta $p^2 > \Lambda_c^2 \geq k^2$. However due to the properties of cutoff the term $\partial_t \mathcal{R}_k$ decays very quickly, when $p^2 \to \infty$. Thus this correction term receives substantial contributions from very small number of modes. It diminishes quickly as $\Lambda_c \to \infty$. This arguments tell that the flow equation for Γ_k and Γ_{k,Λ_c} are essentially same as long as $k \ll \Lambda_c$. But when k approaches Λ_c from below small deviations occur due to the correction term. Making Λ_c larger increases the range of k values in which Γ_k and Γ_{k,Λ_c} have the same beta functional, and finally in limit $\Lambda_c \to \infty$ both Γ_k and Γ_{k,Λ_c} have the same t-derivatives for any finite k.

This is the situation for the generic nonsingular cutoff $\mathcal{R}_k$. Certainly it will be good if there is a cutoff when the correction term vanishes. This indeed happens for the optimized cutoff [43]. The optimized cutoff is given by,

$$\mathcal{R}_k(p^2) = (k^2 - p^2)\theta(k^2 - p^2) \ , \qquad \partial_t \mathcal{R}_k(p^2) = 2k^2\theta(k^2 - p^2) \ . \tag{2.42}$$

Using eq. (2.42) it is easy to see that the correction term is zero for $k \leq \Lambda_c$, as it contains $\theta(k^2 - p^2)\theta(p^2 - \Lambda_c^2)$. This is a very strong argument in favor of optimized cutoff [43]. Thus from now onwards we will be using optimized cutoff in numerical computations given in this work.

With the optimized cutoff the solutions $\{\Gamma_{k,\Lambda_c}, 0 \leq k \leq \Lambda_c\}$ for the regularized FRGE becomes just the restriction of solutions $\{\Gamma_k, 0 \leq k \leq \infty\}$ of the regularized free FRGE in the interval $k < \Lambda_c$.

2.1.5. Approximation Schemes within FRGE framework

In most cases, when FRGE is used to study the renormalization group flow of a theory, one has to depend upon approximations of the full exact flow. This is due to the way the effective average actions are defined. As was mentioned in the section on theory space (2.1.3), they are functionals defined on an infinite dimensional theory space which is spanned by interaction monomials consistent with the symmetries, with co-ordinates as the couplings. The FRGE is an exact equation and tells the flow of the full theory. But in principal it is impossible to handle infinite number of couplings. Even if one starts from an EAA at a some scale k with only finite number of couplings, there is no reason to believe why the RG flow would not generate other couplings. In fact computations have shown the FRGE trace does generate other new couplings as one moves away from the initial condition. As it is generally impossible to follow the flow of the infinite number of couplings thus approximation schemes are needed in the framework of FRGE.

Perturbation Theory

One possible way to do the approximation is to use perturbation theory. This is achieved by performing the computations at one loop. In one loop computation of FRGE, one ignores the running of the couplings on the RHS of the FRGE. The t-derivative would not act on the couplings present in the cutoff $\mathcal{R}_k$. Thus the couplings gets k-independent on the RHS of FRGE. The t-derivative can be taken outside the trace and one get the FRGE at one loop to be given by,

$$\partial_t \Gamma_k^{1-\mathrm{loop}} = \frac{1}{2}\partial_t \left(\mathrm{Tr}\ln\left[\Gamma_{\bar{k}}^{(2)} + \mathcal{R}_k\right]\right) . \tag{2.43}$$

where $\bar{k}$ is some fixed scale and $\Gamma_{\bar{k}}$ can be thought of as the "bare action". This equation has the same form as the eq. (2.25), which was obtained when the one loop effective action is calculated by doing the computation perturbatively. It is for this reason this approximation of FRGE is called one-loop approximation.

Truncations

Using perturbative techniques and doing the one loop computation one can extract only perturbative properties of the flow. On the other hand one is interested in the nonperturbative properties of the flow, then the approximation scheme described in the previous section will not give us any nonperturbative information. In the context of FRGE the typical nonperturbative scheme that is employed is to truncate the renormalization group flow [44, 39, 40, 45], which is the procedure we will follow in this work. In this methodology one makes an ansatz for the EAA Γ_k, which consists of a finite number of interaction monomials which forms a subset of the infinite dimensional theory space spanned by interaction monomials. This ansatz is then plugged in FRGE. The FRGE trace when expanded would produce not only terms which lie within the subset of ansatz but also terms that lie outside this subset ansatz. In the truncation scheme the contributions coming from terms which lie outside the ansatz are put to zero, the coefficients of remaining operators on both side of equation are equated to get the beta functions of the couplings. These beta functions would be nonperturbative and would contain genuine nonperturbative information.

Although the information given by this approximation scheme is nonperturbative, there is a difficulty in checking its reliability of the ansatz made as there is no small expansion parameter which can guide. Moreover the renormalization group flow generates terms which do not lie in the ansatz subspace and discarding them might be potentially affecting the running of the couplings we are retaining. To start with a good ansatz requires some physical intuition and an educated guess as to what kind of interaction terms have to be a part of the ansatz subspace.

As far as the study of fixed points is concerned, it is well known that truncated flows can give rise to spurious fixed points. To check whether a fixed point is spurious or not, one has to extend the truncation subspace to full theory space. Then one should see if the fixed point persists on extending the truncation. This is well understood in the case of scalar field theory in the local potential approximation [44, 47], where it can be shown how physical solutions are recovered among all apparent fixed points of the flow.

In theories which are more involved where such procedures are absent, the best way to check if the results are reliable or not is to see if they remain stable in gradual extension of the truncation ansatz. Results are considered to be reliable if the relevant quantities like values of fixed point and dimensionless quantities computed in the previous truncation are not greatly affected by the presence of new terms in the enlarged truncation and are seen to converge. While studying theories of gravity and searching for non-trivial fixed point I will be employing this strategy. In subsequent chapters I would stress that encouraging results are not only that such fixed points have been found in all truncations studied, but also they share similar properties in all those truncations. Thus they constitute the physical solutions.

Another way to test the quality of the truncation is to consider the scheme dependence. This is done by considering different types of cutoff. One can do this job in two ways: first by considering different forms of shape function entering the cutoff and secondly by constructing the cutoff with different operators Δ. While the choice of cutoff introduces a scheme dependence in the results, physical quantities should be independent of such choices. Truncations introduce spurious cutoff dependence which can serve as an estimate on the quality of the approximation. Using this one can define an optimization criterion for the cutoff profile [43] with which such dependences are minimized and the truncated flow is rendered most stable.

2.2. General notion of Asymptotic safety

Having introduced the concept of effective average action Γ_k, and having described its flow given by FRGE, I now set to discuss the idea of "Asymptotic Safety" as introduced by Steven Weinberg [11] (see also [46] for reviews).

For simplicity I will denote the field content of the theory by ϕ_A. Then the effective action $\Gamma_k[\phi_A]$ used at the tree level gives the accurate description of the physical process taking place at scale k. In general one can expand the effective action as follows,

$$\Gamma_k[\phi_A, g_i] = \sum_i g_i(k)\mathcal{O}_i(\phi_A)\,, \tag{2.44}$$

where as mentioned before $g_i(k)$ are the running couplings and $\mathcal{O}_i(\phi_A)$ are all possible operators constructed from the field and its derivatives, which abide the symmetry requirements of the theory. These are functionals on $\mathcal{F} \times \mathcal{Q} \times R^+$; where $\mathcal{F}$ is the configuration space of all fields, $\mathcal{Q}$ is the space of all couplings and R^+ is the space parametrized by k. The FRGE (2.22) describes the dependence of Γ_k on k. We can write $\partial_t\Gamma_k(\phi_A, g_i) = \sum_i \beta_i\mathcal{O}_i(\phi_A)$, where $\beta_i(g_j, k) = \partial_t g_i$ are the beta functions and $t = \log(k/k_0)$.

Dimensional arguments tell us that when the fields, couplings and k are scaled by some arbitrary parameter b, then the effective action is invariant. This is expressed as,

$$\Gamma_k[\phi_A, g_i] = \Gamma_{bk}[b^{d_A}\phi_A, b^{d_i}g_i]\,, \tag{2.45}$$

where d_A and d_i are the mass dimensions of ϕ_A and g_i respectively, and $b \in R^+$ is a positive real scaling parameter. As Γ_k is a dimensionless quantity, it can be rewritten in terms of dimensionless fields $\tilde{\phi}_A = \phi_A k^{-d_A}$ and dimensionless couplings $\tilde{g}_i = g_i k^{-d_i}$. At this point if one performs the transformation given in eq. (2.45) with parameter $b = k^{-1}$, then one can define a quantity completely independent of the scale. It is a functional $\tilde{\Gamma}$ on the space $(\mathcal{F} \times \mathcal{Q} \times R^+)/R^+$,

$$\tilde{\Gamma}(\tilde{\phi}_A, \tilde{g}_i) := \Gamma_1(\tilde{\phi}_A, \tilde{g}_i) = \Gamma_k(\phi_A, g_i)\;. \tag{2.46}$$

Similarly the beta functions of the dimensionfull couplings can be written as $\beta_i(g_j, k) = k^{d_i}a_i(\tilde{g}_j,$ where $a_i(\tilde{g}_j) = \beta_i(\tilde{g}_j, 1)$. From this follows the dimensionless beta function of the dimensinless couplings:

$$\tilde{\beta}_i(\tilde{g}_j) \equiv \partial_t\tilde{g}_i = a_i(\tilde{g}_j) - d_i\tilde{g}_i\,, \tag{2.47}$$

which depends on k only through the dependence of k implicitily present in $\tilde{g}_i(t)$.

Changing the scale from k to $k-\delta k$ results in the corresponding change in the effective action Γ_k to $\Gamma_{k-\delta k}$. These two effective actions differ essentially by a functional integral over field modes with momenta between k and $k-\delta k$. This integration will not involve any divergences, so the beta functions obatined will also be finite. Once the beta function is known at a scale k, they are automatically determined at any other scale through dimensional analysis. Thus the scale k_0 and the initial action S act just as the initial condition: when the beta functions are known one can start from any arbitrary point on $\mathcal{Q}$ and use FRGE to obtain the RG flow in either direction. The effective action at a particular scale can be obtained by integrating the flow from the chosen initial point up to scale k. In particular, the UV behaviour can be studied by taking the limit $k \to \infty$.

It happens often that it is not possible to integrate the flow up to infinity, and can be integrated only to some limiting scale Λ, defining the point at which some "new physics" has to come into play. In such a case the theory is valid only for $k < \Lambda$ and is known as "effective" or "cutoff" QFT. However, it may happen that one can integrate the flow up to infinity, where the limit $t \to \infty$ can be taken; then in such cases we have a self-consistent description of physical phenomena which is valid up to arbitrarily high energy scales and does not need to refer to anything outside it. Then the theory is termed "fundamental".

The couplings appearing in the effective action can be related to physically measurable quantities such as cross sections and decay rates. Dimensional analysis implies that aside from an overall power of k, such quantities only depend on dimensionless kinematical variables X, like scattering angles and ratios of energies, and on the dimensionless couplings $\tilde{g}_i$ (recall that usually k is identified with one of the momentum variables).

$$R = k^D f\left(E/k, X, \tilde{g}_i(k)\right) , \tag{2.48}$$

where R is the reaction rate, D is the mass dimension of R and E is the energy at which the process is taking place. The important point to note is that the physicial quantity R cannot depend on any arbitrary choice of k at which the couplings are defined, so it is taken as the energy of the process, and thus we get

$$R = k^D f(1, X.\tilde{g}_i(k)) . \tag{2.49}$$

For example, a cross section can be expressed as $\sigma = k^{-2}\tilde{\sigma}(X, \tilde{g}_i)$. If some of the couplings $\tilde{g}_i$ go to infinity, when $t \to \infty$, also the reaction rate R can be expected to diverge. A sufficient condition to avoid this problem is to assume that in the limit $t \to \infty$ the RG trajectory tends to a FP of the RG, *i.e.* a point $\tilde{g}_*$ where $\tilde{\beta}_i(\tilde{g}_*) = 0$ for all i. The existence of such a FP is the first requirement for asymptotic safety. Before discussing the second requirement, we have to understand that one needs to impose this condition only on a subset of all couplings.

The fields ϕ_A are integration variables, and a redefinition of the fields does not change the physical content of the theory. This can be seen as invariance under a group $\mathcal{G}$ of coordinate transformations in $\mathcal{F}$. There is a similar arbitrariness in the choice of coordinates on $\mathcal{Q}$, due to the freedom of redefining the couplings g_i. Since, for given k, Γ_k is assumed to be the "most general" functional on $\mathcal{F} \times \mathcal{Q}$ (in some proper sense), given a field redefinition $\phi' = \phi'(\phi)$ one can find new couplings g_i' such that

$$\Gamma_k(\phi_B'(\phi_A), g_i) = \Gamma_k(\phi_A, g_i') . \tag{2.50}$$

At least locally, this defines an action of $\mathcal{G}$ on $\mathcal{Q}$. We are then free to choose a coordinate system which is adapted to these trasformations, in the sense that a subset $\{g_{\hat{\imath}}\}$ of couplings transform nontrivially and can be used as coordinates in the orbits of $\mathcal{G}$, while a subset $\{g_{\bar{\imath}}\}$ are invariant under the action of $\mathcal{G}$ and define coordinates on $\mathcal{Q}/\mathcal{G}$. The couplings $g_{\hat{\imath}}$ are called redundant or inessential, while the couplings $g_{\bar{\imath}}$ are called essential. In an adapted parametrization there exists, at least locally, a field redefinition $\bar{\phi}(\phi)$ such that using eq. (2.50) the couplings $g_{\hat{\imath}}$ can be given fixed values $(g_{\hat{\imath}})_0$. We can then define a new action $\bar{\Gamma}$ depending only on the essential couplings:

$$\bar{\Gamma}_k(\bar{\phi}_A, g_{\bar{\imath}}) := \Gamma_k(\bar{\phi}_A, g_{\bar{\imath}}, (g_{\hat{\imath}})_0) = \Gamma_k(\phi_A; g_{\bar{\imath}}, g_{\hat{\imath}}) \,. \tag{2.51}$$

Similarly, the values of the redundant couplings can be fixed also in the expressions for measurable quantities, so there is no need to constrain their RG flow in any way: they are not required to flow towards a FP.

For example, the action of a scalar field theory in a background $g_{\mu\nu}$,

$$\Gamma_k(\phi, g_{\mu\nu}; Z_\phi, \lambda_{2i}) = \int d^4x\sqrt{g}\left[\frac{Z_\phi}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \lambda_2\phi^2 + \lambda_4\phi^4 + \dots\right] \tag{2.52}$$

has the scaling invariance

$$\Gamma_k(c\phi, g_{\mu\nu}; c^{-2}Z_\phi, c^{-2i}\lambda_{2i}) = \Gamma_k(\phi, g_{\mu\nu}; Z_\phi, \lambda_{2i}) \,, \tag{2.53}$$

which is a special case of eq. (2.50). There exists an adapted coordinate system where Z is inessential and $\bar{\lambda}_{2i} = \lambda_{2i}Z_\phi^{-i}$ are the essential coordinates. A transformation with $c = \sqrt{Z_\phi}$ then leads to $Z_\phi = 1$, leaving the essential couplings unaffected.

A comparison of (1.2.4) and (1.2.7) shows that k behaves like a redundant coupling. In ordinary QFT's, it is generally the case that for each multiplet of fields ϕ_A there is a scaling invariance like (1.2.9) commuting with (1.2.3). One can use these invariances to eliminate simultaneously k and one other redundant coupling per field multiplet; the conventional choice is to eliminate the wave function renormalization Z_A. No conditions have to be imposed on the RG flow of the Z_A's, and the anomalous dimensions $\eta_A = \partial_t \log Z_A$, at a FP, can be determined by a calculation. More generally, (1.2.3) and (1.2.6) can be used to eliminate simultaneously the dependence of Γ_k on k and on the inessential couplings, and to define an effective action $\tilde{\Gamma}(\tilde{\phi}_A, \tilde{g}_{\bar{\imath}})$, depending only on the dimensionless essential couplings $\tilde{g}_{\bar{\imath}} = g_{\bar{\imath}}k^{-d_{\bar{\imath}}}$. It is only on these couplings that one has to impose the FP condition $\partial_t\tilde{g}_{\bar{\imath}} = 0$.

We can now state the second requirement for asymptotic safety. Denote $\tilde{\mathcal{Q}} = \mathcal{Q}/\mathcal{G}$ the space parametrized by the dimensionless essential couplings $\tilde{g}_{\bar{\imath}}$. The set $\mathcal{C}$ of all points in $\tilde{\mathcal{Q}}$ that flow towards the FP in the UV limit is called the UV critical surface. If one chooses an initial point lying on $\mathcal{C}$, the whole trajectory will remain on $\mathcal{C}$ and will ultimately flow towards the FP in the UV limit. Points that lie outside $\mathcal{C}$ will generally flow towards infinity (or other FP's). Thus, demanding that the theory lies on the UV critical surface ensures that it has a sensible UV limit. It also has the effect of reducing the arbitrariness in the choice of the coupling constants. In particular, if the UV critical surface is finite dimensional, the arbitariness is reduced to a finite number of parameters, which can be determined by a finite number of experiments. Thus, a theory with a FP and a finite dimensional UV critical surface has a controllable UV behaviour, and is predictive. Such a theory is called "asymptotically safe".

A perturbatively renormalizable, asymptotically free field theory such as QCD is a special case of an asymptotically safe theory. In this case the FP is the Gaußian FP, where all couplings vanish, and the critical surface is spanned, near the FP, by the couplings that are renormalizable in the perturbative sense (those with dimension $d_{\bar{\imath}} \geq 0$).

2.2.1. Dimension of Critical Surface

The dimension of the UV critical surface can be determined from the behavior of the running of dimensionless essential couplings near the fixed point. Lets denote the running of dimensionless essential couplings by $\tilde{\beta}_{\bar{\imath}}(\tilde{g})$. In the neighbourhood of the FP $\tilde{g}^*_{\bar{\imath}}$, the running of dimensionless essential couplings is given by,

$$\partial_t \tilde{g}_{\bar{\imath}} = \sum_j M_{ij} \left(\tilde{g}_{\bar{\jmath}} - \tilde{g}^*_{\bar{\jmath}} \right) , \tag{2.54}$$

where the sum is over the dimensionless essential couplings of the theory and M_{ij} is given by,

$$M_{ij} = \left. \frac{\partial \tilde{\beta}_{\bar{\imath}}(\tilde{g})}{\partial \tilde{g}_{\bar{\jmath}}} \right|_{\tilde{g}=\tilde{g}^*} . \tag{2.55}$$

The general solution of the eq. (2.54) is given by,

$$\tilde{g}_{\bar{\imath}}(k) = \sum_J C_J \, V^J_i k^{e_J} + \tilde{g}^*_{\bar{\imath}} , \tag{2.56}$$

where V^J are the eigenvectors of M_{ij} with eigenvalues e_J, and C_J are arbitrary coefficients. Clearly the condition that the couplings approach the FP is that C_J should vanish for all positive eigenvalues. Thus the dimensionality of the critical surface is the number of remaining C_J parameter *i.e.* the number of negative eigenvalues of M_{ij}.

2.3. FRGE applied to $O(N)$ scalar field theory

In the section 2.1.1 I showed how to derive a functional renormalization group equation and also defined the effective average action as given in [19, 20]. I then discussed the properties obeyed by FRGE and EAA, thereby giving the concept of theory space, which led to the discussion on approximation schemes [39, 40, 45, 44].

After having derived the FRGE, in this section I would demonstrate how this is used to obtain the flow of the couplings present in the truncated subspace of the theory. One strategy that is employed in making the guess for truncation is the derivative expansion of EAA. The effective average action is expanded and truncated at some order. Here I would consider the $O(N)$ scalar field theory and would consider the leading order terms in the truncation. This would be the local potential approximation, where I retain only the kinetic term for the field and its generic potential. The effective average action is then given by,

$$\Gamma_k = \int d^dx \left[\frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a + V(\phi_a\phi_a)\right] \ . \tag{2.57}$$

In all computations where FRGE is used, the first step to do is to obtain the Hessian $\Gamma_k^{(2)}$ of the theory. For the EAA given in eq. (2.57), the Hessian is obtained by doing the second variation around a constant background ϕ_a. This is given by

$$(\Gamma_k^{(2)})^{ab} = -\delta^{ab}\Box + 2V'\delta^{ab} + 4\phi^a\phi^b V'' \tag{2.58}$$

where prime (′) denotes derivative with respect to argument. In typical computation when the Hessian has been obtained, one decides which operator has to be used as Δ in the construction of the cutoff $\mathcal{R}_k$. Once Δ is identified, we first construct the modified inverse propagator of the theory denoted by $\mathcal{P}_k(\Delta)$. This is obtained by replacing Δ in $\Gamma^{(2)}$ with $P_k(\Delta)$, which is given by $\Delta + R_k(\Delta)$, and $R_k(\Delta)$ is the shape function. The cutoff is then given by $\mathcal{R}_k(\Delta) = \mathcal{P}_k(\Delta) - \Gamma_k^{(2)}(\Delta)$. This is summarized as follows,

$$\begin{gathered}\Gamma_k^{(2)}(\Delta) \ \Rightarrow \ \mathcal{P}_k(\Delta) = \Gamma_k^{(2)}(\,P_k(\Delta)\,) \\ \text{where } P_k(\Delta) = \Delta + R_k(\Delta) \ , \text{ and } R_k(\Delta) = \text{shape function} \ , \\ \mathcal{R}_k(\Delta) = \mathcal{P}_k(\Delta) - \Gamma_k^{(2)}(\Delta) \ .\end{gathered} \tag{2.59}$$

For the present case we note that one can take $\Delta = -\Box$. At this point I would like to stress that through out this work I will take $\Delta = -\Box$. Thus we have the modified inverse propagator and cutoff given by,

$$\begin{aligned}\mathcal{P}_k^{ab}(-\Box) &= P_k(-\Box)\,\delta^{ab} + 2V'\delta^{ab} + 4\phi^a\phi^b V'' \\ \mathcal{R}_k^{ab}(-\Box) &= \mathcal{P}_k^{ab}(-\Box) - (\Gamma_k^{(2)})^{ab}(-\Box) \\ &= R_k(-\Box)\,\delta^{ab} \ .\end{aligned} \tag{2.60}$$

To do the computation using the FRGE we would need the following projectors,

$$\alpha^{ab} = \frac{\phi^a\phi^b}{\phi^2} \ , \qquad \beta^{ab} = \delta^{ab} - \alpha^{ab} \ , \tag{2.61}$$

where $\phi^2 = \phi^a\phi^a$. These projectors have the following properties:

$$\begin{gathered}\alpha^{ab}\,\phi^a = \phi^b \ , \qquad \beta^{ab}\,\phi^a = 0 \\ \alpha^2 = \alpha \ , \qquad \beta^2 = \beta \ , \qquad \alpha\cdot\beta = 0 \\ \mathrm{Tr}\,\alpha = 1 \ , \qquad \mathrm{Tr}\,\beta = N-1 \ .\end{gathered} \tag{2.62}$$

Then using the projectors the modified inverse propagator and cutoff is written as,

$$\begin{aligned}\mathcal{P}(-\Box) &= (P_k(-\Box) + 2V')\,(\alpha+\beta) + 4\phi^2 V''\,\alpha \\ &= (P_k(-\Box) + 2V')\,\beta + (P_k(-\Box) + 2V' + 4\phi^2 V'')\,\alpha\ , \qquad (2.63)\\ \mathcal{R}(-\Box) &= R_k(-\Box)\,\alpha + R_k(-\Box)\,\beta\ . \qquad (2.64)\end{aligned}$$

With the help of projectors the modified inverse propagator can be easily inverted. This is given by,

$$(\mathcal{P}_k(-\Box))^{-1} = \frac{1}{P_k(-\Box)+2V'}\,\beta + \frac{1}{P_k(-\Box)+2V'+4\phi^2V''}\,\alpha\ . \qquad (2.65)$$

This information can be plugged in the RHS of FRGE which is written as,

$$\begin{aligned}(\partial_t\Gamma)[\phi] &= \frac{1}{2}\mathrm{Tr}\left[\frac{(\mathrm{Tr}\,\beta)\,\partial_t R_k(-\Box)}{P_k(-\Box)+2V'} + \frac{(\mathrm{Tr}\alpha)\,\partial_t R_k(-\Box)}{P_k(-\Box)+2V'+4\phi^2V''}\right]\ , \\ &= \frac{1}{2}\mathrm{Tr}\left[\frac{(N-1)\,\partial_t R_k(-\Box)}{P_k(-\Box)+2V'} + \frac{\partial_t R_k(-\Box)}{P_k(-\Box)+2V'+4\phi^2V''}\right]\ . \qquad (2.66)\end{aligned}$$

At this point after computing the RHS of the FRGE we note that there are two terms on the RHS. The term proportional to $(N-1)$ has a propagator coming from modes in "Goldstone directions" with mass $m^2 = 2V'$, which will be zero had the ϕ_a would have been the minima of the potential. While the other term has a propagator for the radial mode with mass $m^2 = 2V'+4\phi^2V''$. These two terms will always be present as long as $N > 1$. For $N = 1$, the projector $\alpha = 1$ while $\beta = 0$. Thus the system reduces to that of one scalar and we reproduces the results mentioned in [34].

The trace can be computed in two ways. This is done either by performing the trace in configuration space, in which case it is $\mathrm{Tr} \equiv \int d^dx \int d^dy \sum_a$, or in momentum space in which case it becomes $\mathrm{Tr} \equiv Vol\,(2\pi)^{-d} \int d^dq \sum_a$. Given that we have a constant background field ϕ_a, the FRGE trace in eq. (2.66) would give,

$$\partial_t\Gamma_k = \frac{1}{2}\frac{Vol}{(4\pi)^{d/2}}\left[(N-1)Q_{\frac{d}{2}}\left(\frac{\partial_t R_k}{P_k+2V'}\right) + Q_{\frac{d}{2}}\left(\frac{\partial_t R_k}{P_k+2V'+4\phi^2V''}\right)\right]\ , \qquad (2.67)$$

where $Vol = \int d^dx$ is the space-time volume and the Q function is defined as,

$$Q_n[W(z)] = \frac{1}{\Gamma(n)}\int_0^\infty dz z^{n-1} W(z)\ . \qquad (2.68)$$

At this point it is noted that while performing the trace the two contributions, one coming form Goldstone modes and the other coming from radial modes, do not mix with each other. When $N = 1$ only the radial mode survives, and we are left with the expression giving the flow of the effective average action for a single scalar [34, 48, 49]. Depending on the choice that one makes for the shape function, it possible to perform the integration in closed form or it has to be done numerically. As has been discussed in previous section on FRGE with UV cutoff, it was shown that an optimized cutoff is the one in which the dependence on UV cutoff disappears. Thus we will be using the optimized shape function [43, 48, 49], which is given by,

$$R_k(z) = (k^2 - z)\theta(k^2 - z)\,. \tag{2.69}$$

However computations have also been done in past using exponentials shape functions [20], which are given by,

$$R_k(z) = \frac{z \exp -z/k^2}{1 - \exp -z/k^2}\ . \tag{2.70}$$

With an optimized shape function (2.69), it is possible to do the Q-function integration in closed form, while with exponential shape function the integration has to be done in a numerical way. Since the background field ϕ_a is constant, one can extract the running of potential from eq. (2.67). For the optimized shape function this is given by,

$$\partial_t V = k^{d+2}\, c_d \cdot \left(\frac{N-1}{k^2 + 2V'} + \frac{1}{k^2 + 2V' + 4\phi^2 V''} \right)\ , \tag{2.71}$$

where $c_d = \frac{1}{(4\pi)^{d/2}\,\Gamma(\frac{d}{2}+1)}$.

As has been discussed while outlining the requirements of asymptotic safety in section 2.2, that one needs to work with dimensionless couplings in order to search for fixed points and compute the critical exponents. Thus with this in mind I define dimensionless field $\tilde{\phi}_a$ and dimensionless potential $\tilde{V}(\tilde{\phi}^a\tilde{\phi}^a)$ as,

$$\tilde{\phi}_a = k^{-(d-2)/2}\phi_a\ , \qquad \Rightarrow \qquad \tilde{\phi}^2 = k^{-(d-2)}\,\phi^2\ ,$$
$$\tilde{V} = k^{-d}\,V\ . \tag{2.72}$$

One can now derive the running of dimensionless potential.

$$\begin{aligned}
(\partial_t V)[\phi^2] &= \partial_t(k^d \tilde{V}[\tilde{\phi}^2])\ , \\
&= d\,k^d\,\tilde{V}[\tilde{\phi}^2]) + k^d\,\frac{\delta\tilde{V}}{\delta\tilde{\phi}^2}\,\partial_t\tilde{\phi}^2 + k^d(\partial_t\tilde{V})[\tilde{\phi}^2]\ , \\
&= d\,k^d\,\tilde{V}[\tilde{\phi}^2]) - k^d\,\tilde{V}'\,(d-2)\tilde{\phi}^2 + k^d(\partial_t\tilde{V})[\tilde{\phi}^2] \\
&= k^d\left\{ d\,\tilde{V} - (d-2)\,\tilde{\phi}^2\,\tilde{V}' + (\partial_t\tilde{V})[\tilde{\phi}^2] \right\}\ ,
\end{aligned} \tag{2.73}$$

where the t-derivative first acts on the factor k^d, then it acts on the k-dependence in the $\tilde{\phi}^2$ and then finally it acts on the k-dependent couplings present in the dimensionless potential. From the last line of eq. (2.73) one can obtain the expression for the running of $(\partial_t\tilde{V})[\tilde{\phi}^2]$ in terms of $(\partial_t V)[\phi^2]$. This is given by,

$$(\partial_t\tilde{V})[\tilde{\phi}^2] = -d\,\tilde{V} + (d-2)\,\tilde{\phi}^2\,\tilde{V}' + k^{-d}\,(\partial_t V)[\phi^2]\ , \tag{2.74}$$

where the last term in the eq. (2.74) comes from FRGE. At this point I use the expression of $\partial_t V$, computed for optimized cutoff in eq. (2.71). Plugging this in eq. (2.74) gives the expression for the running of dimensionless potential in the local potential approximation for the $O(N)$-symmetric scalar theory. This is given by,

$$(\partial_t \tilde{V})[\tilde{\phi}^2] = -d\,\tilde{V} + (d-2)\,\tilde{\phi}^2\,\tilde{V}' + c_d \left[\frac{N-1}{1+2\tilde{V}'} + \frac{1}{1+2\tilde{V}'+4\tilde{\phi}^2\tilde{V}''}\right] \ . \tag{2.75}$$

After having derived the beta-functional of the dimensionless potential, we note that wherever $\tilde{\phi}^2$ occurs explicitly, it occurs in combination $\tilde{\phi}^2\tilde{V}'$ and $\tilde{\phi}^2\tilde{V}''$. This would be crucial in the proof of the existence of "Gaussian" Fixed Point (FP). Equation (2.75) is a non-linear partial differential equation with second order in $\tilde{\phi}^2$ and first order in t. This equation cannot be solved analytically in general. Even in the case of large N limit, the equation remains very nonlinear although it become first order in $\tilde{\phi}^2$. The best thing one can do is to assume a polynomial form of the dimensionless potential $\tilde{V}$ and extract the beta functions of the dimensionless couplings. Over here we will assume the following polynomial form of the potential.

$$\tilde{V}(\tilde{\phi}^2) = \sum_{n=0}^{n_t} \tilde{\lambda}_{2n}\tilde{\phi}^{2n} \ , \tag{2.76}$$

where n_t is the maximum power of $\tilde{\phi}^2$ considered in the truncation. In this chapter we would put the first term of the expansion in eq. (2.76) to be zero. Thus the first non zero term would be $\tilde{\lambda}_2$ which is the dimensionless mass term. The next term of the series is the usual quartic coupling given by $\tilde{\lambda}_4$, whose corresponding dimensionfull coupling λ_4 is dimensionless in $d = 4$ and so on.

To extract the beta function of the couplings we simply take derivatives of eq. (2.75) with respect to $\tilde{\phi}^2$ and set $\tilde{\phi}^2 = 0$, as

$$\partial_t \tilde{\lambda}_{2n} = \frac{1}{n!}\frac{\delta^n \partial_t \tilde{V}}{\delta(\tilde{\phi}^2)^n}\bigg|_{\tilde{\phi}^2=0} \ . \tag{2.77}$$

Using the above equation we will give the beta functions of the couplings for the truncations

$n_t = 4$, that is for $\tilde{\lambda}_2$, $\tilde{\lambda}_4$, $\tilde{\lambda}_6$ and $\tilde{\lambda}_8$. They are as follows,

$$\partial_t\tilde{\lambda}_2 = -2\tilde{\lambda}_2 + c_d\left[-\frac{4(N-1)\tilde{\lambda}_4}{(1+2\tilde{\lambda}_2)^2} - \frac{12\tilde{\lambda}_4}{(1+2\tilde{\lambda}_2)^2}\right], \quad (2.78)$$

$$\partial_t\tilde{\lambda}_4 = (-d+2(d-2))\,\tilde{\lambda}_4 + c_d\Bigg[(N-1)\left(\frac{16\tilde{\lambda}_4^2}{(1+2\tilde{\lambda}_2)^3} - \frac{6\tilde{\lambda}_6}{(1+2\tilde{\lambda}_2)^2}\right) + \frac{144\tilde{\lambda}_4^2}{(1+2\tilde{\lambda}_2)^3} - \frac{30\tilde{\lambda}_6}{(1+2\tilde{\lambda}_2)^2}\Bigg], \quad (2.79)$$

$$\partial_t\tilde{\lambda}_6 = (-d+3(d-2))\,\tilde{\lambda}_6 + c_d\Bigg[(N-1)\left(-\frac{64\tilde{\lambda}_4^3}{(1+2\tilde{\lambda}_2)^4} + \frac{48\tilde{\lambda}_4\,\tilde{\lambda}_6}{(1+2\tilde{\lambda}_2)^3} - \frac{8\tilde{\lambda}_8}{(1+2\tilde{\lambda}_2)^2}\right) - \frac{1728\tilde{\lambda}_4^3}{(1+2\tilde{\lambda}_2)^4} + \frac{720\,\tilde{\lambda}_4\,\tilde{\lambda}_6}{(1+2\tilde{\lambda}_2)^3} - \frac{56\tilde{\lambda}_8}{(1+2\tilde{\lambda}_2)^2}\Bigg], \quad (2.80)$$

$$\partial_t\tilde{\lambda}_8 = (-d+4(d-2))\tilde{\lambda}_8 + c_d\Bigg[(N-1)\left(-\frac{256\tilde{\lambda}_4^4}{(1+2\tilde{\lambda}_2)^5} - \frac{72\,\tilde{\lambda}_4^2\,\tilde{\lambda}_6}{(1+2\tilde{\lambda}_2)^4} + \frac{36\tilde{\lambda}_6^2}{(1+2\tilde{\lambda}_2)^3} + \frac{64\,\tilde{\lambda}_4\,\tilde{\lambda}_8}{(1+2\tilde{\lambda}_2)^2}\right) - \frac{20736\tilde{\lambda}_4^4}{(1+2\tilde{\lambda}_2)^5} - \frac{1296\,\tilde{\lambda}_4^2\,\tilde{\lambda}_6}{(1+2\tilde{\lambda}_2)^4} + \frac{900\tilde{\lambda}_6^2}{(1+2\tilde{\lambda}_2)^3} + \frac{1344\,\tilde{\lambda}_4\,\tilde{\lambda}_8}{(1+2\tilde{\lambda}_2)^2}\Bigg]. \quad (2.81)$$

From these beta functions we note that if we had just one scalar ($N = 1$), then these beta functions in $d = 4$ would match with the ones given in [34] for the optimized cutoff. The first term in each beta function is coming from the tree level diagram, and consists of two factors: one being the canonical dimension of the coupling whose beta function is computed and second is the coupling itself. The terms in brackets contain quantum corrections and are coming from loop diagrams. For the typical ϕ^4, one can easily recognize the familiar terms in the beta function of mass λ_2 and λ_4, which are proportional to λ_4 and λ_4^2 respectively.

Now we look for the fixed points and study the critical surface.

2.3.1. Fixed Points of $O(N)$ scalar theory

$O(N)$ scalar theory possesses two kinds of Fixed Points (FPs) : one is the "Gaussian" fixed point which exists for all dimensions and in all truncations, while the other is the Wilson Fisher fixed point which exists for $2 < d < 4$. Since a FP is defined by requiring that the t-derivative of the dimensionless coupling is zero, thus the condition to search for FP in this truncation is the requirement that the t-derivative of the dimensionless potential *i.e.* $\partial_t\tilde{V} = 0$. Using this information in eq. (2.75), the fixed point equation for the dimensionless potential is given by,

$$0 = -d\,\tilde{V} + (d-2)\,\tilde{\phi}^2\,\tilde{V}' + c_d\left[\frac{N-1}{1+2\tilde{V}'} + \frac{1}{1+2\tilde{V}'+4\tilde{\phi}^2\tilde{V}''}\right]. \quad (2.82)$$

This equation is a second order nonlinear ordinary differential equation for $\tilde{V}$. This equation cannot be solved in closed form. In usual physical computation one assumes that the potential is analytic around $\tilde{\phi}^2 = 0$ and thus possess a polynomial expansion like the one given in eq. (2.76). Then in the truncation $n_t = 4$, eq. (2.82) is equivalent to setting the running of dimensionless couplings $\tilde{\lambda}_2$, $\tilde{\lambda}_4$, $\tilde{\lambda}_6$ and $\tilde{\lambda}_8$ given in equations (2.78), (2.79), (2.80) and (2.81) respectively, to zero.

Once the fixed points are found, one uses the dimensionless beta functions to obtain the critical exponents. Critical exponents are defined to be negative of eigenvalues of the "Stability Matrix", which is given for $O(N)$ scalar theory as,

$$M_{ij} = \left. \frac{\delta\left(\frac{1}{i!}\partial_t \tilde{V}^{(i)}(0)\right)}{\delta\left(\frac{1}{j!}\tilde{V}^{(j)}(0)\right)} \right|_{FP} = \left. \frac{\delta\beta^{\tilde{\lambda}}_{2i}}{\delta\tilde{\lambda}_{2j}} \right|_{FP} , \tag{2.83}$$

where i and j take values from 1 to n_t.

Gaussian Fixed Point

A Gaussian fixed point is a point in the theory space where all the couplings vanishes. The existence of this fixed point is important for the applicability of the perturbation theory. The typical field theory computation involving perturbation theory are possible due to presence of this fixed point.

In the case of $O(N)$-scalar theory, it is possible to prove its existence by assuming Taylor's series expansion of the dimensionless potential around $\tilde{\phi}^2 = 0$ like the one given in eq. (2.76) for $n_t = \infty$. Lets assume that the Gaussian FP exists, then that means $\tilde{V}^{(i)}(0) = 0$ for all $i \geq 1$. This is nothing but the condition that all the couplings present in the expansion are zero. By taking successive derivatives of eq. (2.82) and making use of this ansatz one can show easily show that at each order of derivative of eq. (2.82), the equation is identically satisfied. This can be more clearly seen in the case when $n_t = 4$. For this the dimensionless beta functions of couplings are given in equations (2.78), (2.79), (2.80) and (2.81). At FP LHS of these beta functions are zero. Then making use of the condition $\tilde{V}^{(i)}(0) = 0$ for all $i \geq 1$, which is equivalent to saying that $\tilde{\lambda}_2$, $\tilde{\lambda}_4$, $\tilde{\lambda}_6$ and $\tilde{\lambda}_8$ are all zero. This would imply that the RHS of equations (2.78), (2.79), (2.80) and (2.81) are also zero.

After showing the existence of the Gaussian FP, we study the linearized flow around it. To study linearized flow we compute the stability matrix given in eq. (2.83). From this definition, the numerical computations tell that the stability matrix has the following form,

$$\begin{pmatrix} M_{11} & M_{12} & 0 & 0 & \cdots \\ 0 & M_{22} & M_{23} & 0 & \cdots \\ 0 & 0 & M_{33} & M_{34} & \cdots \\ 0 & 0 & 0 & M_{44} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} , \tag{2.84}$$

The various non zero entries of M are related to each other by the following recursion relations (in d-dimensions):

$$M_{ii} = (d-2)\,(i-1) + M_{11} \ ,$$
$$M_{i,i+1} = \left(\frac{2i}{N}+1\right)\,(i+1)\,\frac{M_{12}}{2\left(\frac{2}{N}+1\right)} = (2i+N)\,(i+1)\frac{M_{12}}{2(N+2)} \ , \tag{2.85}$$

where M_{11} and M_{12} are given by,

$$M_{11} = -2 \ , \qquad M_{12} = -4c_d\,(N+2) \ . \tag{2.86}$$

Thus to study the linearized flow around Gaussian FP all we need to know is M_{11} and M_{12}. All the other nonzero entries can be obtained from them. Thus the smallest truncation of the dimensionless potential that one needs to consider is just containing the couplings for $\tilde{\phi}^2$ and $\tilde{\phi}^4$.

Now I will give a mathematical derivation of the relations mentioned in eq. (2.3.1). From the beta functional of dimensionless potential (for optimized cutoff) given in eq. (2.75) and the running of effective action (for generic cutoff) given in eq. (2.67), we note that the beta functional of the dimensionless potential for a generic cutoff can be written as follows,

$$\partial_t \tilde{V} = -d\tilde{V} + (d-2)\,\tilde{\phi}^2\,\tilde{V}' + (N-1)\,\tilde{H}\left(2\tilde{V}'\right) + \tilde{H}\left(2\,\tilde{V}' + 4\tilde{\phi}^2\tilde{V}''\right) \ , \tag{2.87}$$

where $\tilde{H}$ is a generic functional which one obtain after doing the integration in Q-functions of eq. (2.67). The exact form of the functional $\tilde{H}$ is not important, all that is required are the arguments of the functionals. One can compare eq. (2.87) with eq. (2.75) and notice that the last two terms of eq. (2.75) has the same functional form as the last two terms of eq. (2.87). The properties of the stability matrix mentioned in eq. (2.3.1) can be obtained by taking successive derivatives of eq. (2.87) with respect to $\tilde{\phi}^2$ at $\tilde{\phi}^2 = 0$.

For $i = 1$, we take one derivative of eq. (2.87). This gives,

$$\begin{aligned}\partial_t \tilde{V}' \ &= \ -d\,\tilde{V}' + (d-2)\,\tilde{V}' + (d-2)\,\tilde{\phi}^2\,\tilde{V}'' + (N-1)\,\frac{\delta\tilde{H}}{\delta(2\tilde{V}')}\,2\tilde{V}'' \\ &\quad + \frac{\delta\tilde{H}}{\delta(2\tilde{V}' + 4\tilde{\phi}^2\tilde{V}'')}(2\tilde{V}'' + 4\tilde{V}'' + 4\tilde{\phi}^2\tilde{V}''') \ .\end{aligned} \tag{2.88}$$

When we set $\tilde{\phi}^2 = 0$, we note from the above equation that $\partial_t\tilde{V}'(0)$ depends only on $\tilde{V}'(0)$ and $\tilde{V}''(0)$. We use this information in eq. (2.83) to calculate the $i = 1$ entries of the stability matrix. We note that $M_{1j} = 0$ for all $j \geq 3$. Now we find the remaining nonzero entries. For $j = 1$, we note that the dependence on $\tilde{V}'(0)$ is present (apart from the canonical terms) only in $\left.\frac{\delta\tilde{H}}{\delta(2\tilde{V}')}\right|_{\tilde{\phi}^2=0}$ and $\left.\frac{\delta\tilde{H}}{\delta(2\tilde{V}'+4\tilde{\phi}^2\tilde{V}'')}\right|_{\tilde{\phi}^2=0}$. But each of these non-canonical terms are multiplied with $\tilde{V}''(0)$. So when we compute the entries of stability matrix at Gaussian FP, these terms will not contribute. Thus for $j = 1$ we take the derivative of $\partial_t \tilde{V}'(0)$ with respect to $\tilde{V}'(0)$. This gives M_{11} at the Gaussian FP as,

$$M_{11} = -d + (d-2) = -2 \ . \tag{2.89}$$

While for $j = 2$ we take the derivative of $\partial_t \tilde{V}'(0)$ with respect to $\tilde{V}''/2$. This gives,

$$\begin{aligned} M_{12} &= 2(N-1) \cdot \left. \frac{\delta \tilde{H}}{\delta (2\tilde{V}')} \right|_{\tilde{\phi}^2=0} \cdot 2 \, + \, 2 \cdot \left. \frac{\delta \tilde{H}}{\delta (2\tilde{V}' + 4\tilde{\phi}^2 \tilde{V}'')} \right|_{\tilde{\phi}^2=0} \cdot 6 \\ &= 2(2(N-1)+6) \cdot \left. \frac{\delta \tilde{H}}{\delta (2\tilde{V}')} \right|_{\tilde{\phi}^2=0} = 4(N+2) \left. \frac{\delta \tilde{H}}{\delta (2\tilde{V}')} \right|_{\tilde{\phi}^2=0} \ . \end{aligned} \tag{2.90}$$

Thus we see that for $i = 1$ we have the following entries of the stability matrix,

$$M_{11} = -2 \ , \qquad M_{12} = 4(N+2) \left. \frac{\delta \tilde{H}}{\delta (2\tilde{V}')} \right|_{\tilde{\phi}^2=0} \ , \qquad M_{1j} = 0 \ \forall \ j \geq 3 \ . \tag{2.91}$$

Now we consider the $j = 2$ entries of the stability matrix. To do so we first compute the second derivative of eq. (2.87). This gives,

$$\begin{aligned} \partial_t \tilde{V}'' &= -d \, \tilde{V}'' + 2(d-2) \, \tilde{V}'' + (d-2) \, \tilde{\phi}^2 \, \tilde{V}''' + (N-1) \left(\frac{\delta^2 \tilde{H}}{\delta (2\tilde{V}')^2} (2\tilde{V}'')^2 + \frac{\delta \tilde{H}}{\delta (2\tilde{V}')} \, 2\tilde{V}''' \right) \\ &+ \frac{\delta^2 \tilde{H}}{\delta (2\tilde{V}' + 4\tilde{\phi}^2 \tilde{V}'')^2} (2\tilde{V}'' + 4\tilde{V}'' + 4\tilde{\phi}^2 \tilde{V}''')^2 + \frac{\delta \tilde{H}}{\delta (2\tilde{V}' + 4\tilde{\phi}^2 \tilde{V}'')} (10\tilde{V}''' + 4\tilde{\phi}^2 \tilde{V}'''') \end{aligned} \tag{2.92}$$

When we set $\tilde{\phi}^2 = 0$, we note from the above equation that $\partial_t \tilde{V}''(0)$ depends only on $\tilde{V}'(0)$, $\tilde{V}''(0)$ and $\tilde{V}'''(0)$. We use this information in eq. (2.83) to calculate the $i = 2$ entries of the stability matrix. Using the same arguments as before we conclude that $M_{2j} = 0$ for all $j \geq 4$. The other nonzero entries we compute one by one. For $j = 1$, note that the dependence on $\tilde{V}'(0)$ is present only in $\left. \frac{\delta \tilde{H}}{\delta (2\tilde{V}')} \right|_{\tilde{\phi}^2=0}$ and $\left. \frac{\delta \tilde{H}}{\delta (2\tilde{V}' + 4\tilde{\phi}^2 \tilde{V}'')} \right|_{\tilde{\phi}^2=0}$. But each of these non-canonical terms are multiplied with $\tilde{V}'''(0)$. So when we compute the entries of stability matrix at Gaussian FP, these terms like argued earlier will not contribute. Thus we conclude that $M_{21} = 0$.

For $j = 2$ we take the derivative of $\frac{1}{2}\partial_t \tilde{V}''(0)$ with respect to $\frac{1}{2}\tilde{V}''(0)$. Then at the Gaussian FP only the canonical terms contribute, which gives,

$$M_{22} = -d + 2(d-2) = -2 + (d-2) = (d-2) + M_{11} \ . \tag{2.93}$$

While for $j = 3$, we take the derivative of $\frac{1}{2}\partial_t \tilde{V}''(0)$ with respect to $\frac{1}{3!}\tilde{V}'''(0)$ at the Gaussian FP.

This gives,

$$\begin{aligned} M_{23} &= \frac{1}{2}\left(6(N-1)\cdot \frac{\delta\tilde{H}}{\delta(2\tilde{V}')}\bigg|_{\tilde{\phi}^2=0}\cdot 2 \,+\, 6\cdot \frac{\delta\tilde{H}}{\delta(2\tilde{V}'+4\tilde{\phi}^2\tilde{V}'')}\bigg|_{\tilde{\phi}^2=0}\cdot 10\right) \\ &= \frac{1}{2}\left(6(2(N-1)+10)\cdot \frac{\delta\tilde{H}}{\delta(2\tilde{V}')}\bigg|_{\tilde{\phi}^2=0}\right) = 6(N+4)\,\frac{\delta\tilde{H}}{\delta(2\tilde{V}')}\bigg|_{\tilde{\phi}^2=0} \\ &= \frac{3(N+4)}{2(N+2)}\,M_{12}\ . \end{aligned} \tag{2.94}$$

Thus for note that $i=2$ we have,

$$\begin{aligned} &M_{21}=0\ ,\quad M_{22}=(d-2)(2-1)+M_{11}\ , \\ &M_{23}=\frac{3(N+4)}{2(N+2)}M_{12}\ ,\quad M_{2j}=0\ \forall\ j\geq 4\ . \end{aligned} \tag{2.95}$$

In order to understand the structure of lines for $i\geq 3$ we proceed by method of induction. We assume that the i-th derivative has the following structure,

$$\begin{aligned} (\partial_t\tilde{V})^{(i)} &= -d\tilde{V}^{(i)}+(d-2)\left(\tilde{\phi}^2\,\tilde{V}^{(i+1)}+i\,\tilde{V}^{(i)}\right)+(N-1)\left\{\frac{\delta\tilde{H}}{\delta(2\tilde{V}')}(2\tilde{V})^{(i+1)}+\cdots\right\} \\ &\quad+\left\{\frac{\delta\tilde{H}}{\delta(2\,\tilde{V}'+4\tilde{\phi}^2\,\tilde{V}'')}\left(2\,\tilde{V}'+4\tilde{\phi}^2\,\tilde{V}''\right)^{(i)}+\cdots\right\}\ , \end{aligned} \tag{2.96}$$

where the $(\cdots)$ denote expressions having at least two factors of derivatives of potentials, which are irrelevant when calculating the entries of stability matrix. Clearly this property is true for $i=1$ and $i=2$. We show that if it holds for a given value of i, then it also holds for $i+1$. Thus we take one more derivative eq.(2.96) and we find

$$\begin{aligned} (\partial_t\tilde{V})^{(i+1)} &= -d\tilde{V}^{(i+1)}+(d-2)\left(\tilde{\phi}^2\,\tilde{V}^{(i+2)}+(i+1)\,\tilde{V}^{(i+1)}\right)+(N-1)\left[\frac{\delta\tilde{H}}{\delta(2\tilde{V}')}(2\tilde{V})^{(i+2)}\right. \\ &\quad\left.+\frac{\delta^2\tilde{H}}{\delta(2\tilde{V}')^2}(2\tilde{V})^{(i+1)}\,(2\tilde{V}')+\cdots\right]+\left[\frac{\delta\tilde{H}}{\delta(2\,\tilde{V}'+4\tilde{\phi}^2\,\tilde{V}'')}\left(2\,\tilde{V}'+4\tilde{\phi}^2\,\tilde{V}''\right)^{(i+1)}\right. \\ &\quad\left.+\frac{\delta^2\tilde{H}}{\delta(2\,\tilde{V}'+4\tilde{\phi}^2\,\tilde{V}'')^2}\left(2\,\tilde{V}'+4\tilde{\phi}^2\,\tilde{V}''\right)^{(i)}\left(2\,\tilde{V}'+4\tilde{\phi}^2\,\tilde{V}''\right)'+\cdots\right] \end{aligned} \tag{2.97}$$

Aside from the new terms containing two factors of derivatives of the potentials, which can be neglected for our purposes, the remaining terms have the same structure as eq. (2.96). Thus by induction eq. (2.96) holds for all i.

We use this result to compute the entries of the stability matrix in the i-th row. To do so we will need the following,

$$\left(2\,\tilde{V}' + 4\tilde{\phi}^2\,\tilde{V}''\right)^{(i)} = 2\,(2\,i+1)\,\tilde{V}^{(i+1)} + 4\tilde{\phi}^2\,\tilde{V}^{(i+2)} \ . \tag{2.98}$$

Thus to compute the i-th row of the stability matrix we put $\tilde{\phi}^2 = 0$ in eq. (2.96). Then at Gaussian FP $M_{ij} = 0$ for all $j \leq (i-1)$ and for all $j \geq (i+3)$. The only nonzero entries are M_{ii} and $M_{i,i+1}$, which are given by,

$$\begin{aligned}
M_{ii} &= \left.\frac{\delta\left(\frac{1}{i!}\partial_t\tilde{V}^{(i)}(0)\right)}{\delta\left(\frac{1}{(i)!}\tilde{V}^{(i)}(0)\right)}\right|_{FP} = -d + (d-2)i = (d-2)(i-1) + M_{11} \ , \\
M_{i,i+1} &= \left.\frac{\delta\left(\frac{1}{i!}\partial_t\tilde{V}^{(i)}(0)\right)}{\delta\left(\frac{1}{(i+1)!}\tilde{V}^{(i+1)}(0)\right)}\right|_{FP} = (i+1)\left.\frac{\delta\left(\partial_t\tilde{V}^{(i)}(0)\right)}{\delta\left(\tilde{V}^{(i+1)}(0)\right)}\right|_{FP} \\
&= (i+1)\ [2(N-1) + 2(2i+1)]\left.\frac{\delta\tilde{H}}{\delta(2\tilde{V}')}\right|_{\tilde{\phi}^2=0} = 2(i+1)(2i+N)\left.\frac{\delta\tilde{H}}{\delta(2\tilde{V}')}\right|_{\tilde{\phi}^2=0} \\
&= 2(i+1)(2i+N)\,\frac{M_{12}}{4(N+2)} = (i+1)(2i+N)\,\frac{M_{12}}{2(N+2)}
\end{aligned} \tag{2.99}$$

This completes the proof of our statement given in eq. (2.3.1).

Having established the properties of the stability matrix the usual question to ask what are the eigenvalues of such a stability matrix? A good feature of the matrix given in eq. (2.84) is that the eigenvalues are just the diagonal entries. We have already proven the relations among various nonzero entries given in eq. (2.3.1) of the stability matrix. Thus we conclude the if the eigenvalues of this system are named as e_i for i from 1 to nt, then we have,

$$e_i = (d-2)(i-1) + e_1 = -d + (d-2)i \ . \tag{2.100}$$

Thus if know M_{11} at the Gaussian FP, then we know the eigenvalues of the full stability matrix. Beside this if we also know M_{12} then the full stability matrix is known and eigenvectors can also be computed.

Due to the particular diagonal structure of stability matrix and the recursive relations followed by the nonzero entries, it is easy to demonstrate that eigenvectors have certain properties too. One can write the eigenvector as $v = (v_1, v_2, \cdots, v_q)^T$, where $q = n_t$. Then the vector $V_1 = (v_1, 0, 0, \cdots, 0)^T$ is an eigenvector if v_1 satisfies, $M_{11}\,v_1 = e_1\,v_1$ *i.e.* v_1 is an eigenvector of M_{11}. As M_{11} is one dimensional and is equal to e_1, therefore $v_1 = 1$.

Now consider a vector of the form $V_2 = (v_1', v_2, 0, 0, \cdots, 0)^T$. They are eigenvector of the stability matrix if it satisfies $M_{11}\,v_1' + M_{12}\,v_2 = e_2 v_1'$ and $M_{22}\,v_2 = e_2 v_2$. By making use of eq. (2.3.1 and 2.100) in the second equation I note that $v_2 = v_1 = 1$. Plugging this value in the former equation gives v_1'. In the same way one can determine the other eigenvectors. For example to compute the next eigenvector, consider $V_3 = (v_1'', v_2', v_3, 0 \cdots, 0)^T$. Then for it to be eigenvector, it must satisfy,

$$M_{11}\, v_1'' + M_{12}\, v_2' = e_3\, v_1'' \,, \qquad M_{22}\, v_2' + M_{23}\, v_3 = e_2\, v_2' \qquad M_{33}\,, v_3 = e_3\, v_3 \,. \tag{2.101}$$

Then using eq. (2.3.1 and 2.100) in the last equation one finds $v_3 = v_2 = v_1 = 1$. And so one can figure out the rest of the entries. This process can be continued to figure out all the eigenvectors.

Wilson Fisher Fixed Point

The Wilson Fisher FP correspond to the non-trivial solution of the fixed point equation given in (2.82). Over here I will discuss the solution in $d = 3$. The fixed point equation (2.82) is a non-linear second order ordinary differential equation. For $N \neq \infty$, one can obtain the analytical solution by doing a Taylor series expansion of the dimensionless potential around $\tilde{\phi}^2 = 0$ as in eq. (2.76) for $n_t = \infty$. By equating coefficients of various powers of $\tilde{\phi}^2$ in eq. (2.82) to zero we obtain the fixed point equations for various couplings. From these equations we will note that at the fixed point all the couplings $\tilde{\lambda}^*_{2n}$ for $n \geq 2$ can be expressed as a function of $\tilde{\lambda}^*_2$. This can be seen as follow. The beta function of $\tilde{\lambda}_2$ when put to zero, would give $\tilde{\lambda}^*_4$ in terms of $\tilde{\lambda}^*_2$. Then putting the beta function of $\tilde{\lambda}_4$ to zero would give $\tilde{\lambda}^*_6$ in terms of $\tilde{\lambda}^*_2$. This recursion goes on, and to determine $\tilde{\lambda}^*_{2n}$ in terms of $\tilde{\lambda}^*_2$, one has to put the beta function of $\tilde{\lambda}_{2(n-1)}$ to zero. Thus all the couplings becomes function of $\tilde{\lambda}^*_2$, and one gets the fixed point potential $\tilde{V}^*$ as a function of $\tilde{\lambda}^*_2$. All the fixed point values of the couplings starting from $\tilde{\lambda}^*_4$, have the following functional form,

$$\tilde{\lambda}^*_{2n} = \tilde{\lambda}^*_2 \cdot f(\tilde{\lambda}^*_2, N)\ \forall\, n \geq 2 \,, \tag{2.102}$$

where f is a function. The domain of validity of this expansion of potential is restricted to $0 \leq \tilde{\phi}^2 \leq \tilde{\phi}^2_c < \infty$. Such polynomial solutions given in eq. (2.76) for $n_t = \infty$ cannot be extended to large fields. The value $\tilde{\phi}^2_c$ defines the radius of convergence for the polynomial approximation. It is related to the gap associated to cutoff functions [50, 49]. Not all values of $\tilde{\lambda}^*_2$ lead to scaling solutions which remain finite and analytical for all $\tilde{\phi}^2 \leq \tilde{\phi}^2_c$. Only two values of $\tilde{\lambda}^*_2$ lead to well defined solutions of the fixed point equation. One is the Gaussian FP corresponding to $\tilde{\lambda}^*_2 = 0$ and has been discussed in the previous section, and other is the Wilson Fisher FP corresponding to $\tilde{\lambda}^*_2 = \tilde{\lambda}^c_2 \neq 0$. This value $\tilde{\lambda}^c_2$ is determined by fine tuning such that $\tilde{\phi}^2 = \tilde{\phi}^2_c$. In [49] the authors have calculated this value for $N = 1$.

Here I will not discuss that part. Rather as argued in [49], I would study the scaling solution using numerical methods. This is achieved by truncating the polynomial expansion in eq. (2.76) at some finite order n_t. By doing so we would note that the beta function of $\tilde{\lambda}_{2n_t}$ would not contain coupling $\tilde{\lambda}_{2(n_t+1)}$, thereby stopping the recursive process. Putting this beta function to zero would determine $\tilde{\lambda}^*_2$, and then all the other couplings in the truncation. By increasing the truncation the value of $\tilde{\lambda}^*_2$ starts to converge.

Now I would write the values of the couplings at the wilson-fisher FP and the critical exponents at this FP. Here I would consider the truncation up to $n_t = 10$ for $N = 1$. Other values of N has been considered in detail in [49].

n_t	$\tilde{\lambda}_2^*$	$\tilde{\lambda}_4^*$	$\tilde{\lambda}_6^*$	$\tilde{\lambda}_8^*$	$\tilde{\lambda}_{10}^*$	$\tilde{\lambda}_{12}^*$	$\tilde{\lambda}_{14}^*$	$\tilde{\lambda}_{16}^*$	$\tilde{\lambda}_{18}^*$	$\tilde{\lambda}_{20}^*$
2	-0.0385	0.3234								
3	-0.0714	0.5179	0.7511							
4	-0.0880	0.590	1.235	2.0460						
5	-0.0941	0.612	1.419	3.017	4.377					
6	-0.0947	0.614	1.436	3.112	4.839	2.263				
7	-0.0935	0.610	1.401	2.921	3.919	-2.212	-25.123			
8	-0.0928	0.608	1.380	2.804	3.366	-4.836	-39.53	-93.05		
9	-0.0928	0.6074	1.379	2.800	3.340	-4.961	-40.21	-97.40	-29.80	
10	-0.0930	0.6081	1.385	2.829	3.482	-4.290	-36.55	-73.94	131.3	1100.2

Table 2.1.: Position of Wilson Fisher FP for various truncation in $d = 3$ and $N = 1$.

n_t	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	θ_{10}
2	1.843	-1.176								
3	1.686	-1.133	-12.220							
4	1.586	-0.956	-8.824	-32.52						
5	1.537	-0.780	-6.522	-22.02	-60.30					
6	1.527	-0.6614	-4.844	-15.80	-40.15	-93.47				
7	1.534	-0.6191	-3.742	-11.76	-29.11	-63.19	-132.03			
8	1.540	-0.6331	-3.164	-9.170	-22.35	-47.21	-92.23	-177.97		
9	1.542	-0.6581	-2.986	-7.502	-17.91	-37.31	-70.87	-128.20	-232.55	
10	1.540	-0.6667	-3.053	-6.420	-14.78	-30.48	-57.08	-100.42	-171.24	-295.61

Table 2.2.: Critical exponents at the Wilson Fisher FP for various truncation in $d = 3$ and $N = 1$.

After having given the values of the couplings at the wilson fisher FP, I now give the critical exponents for various truncations considered. A critical exponent is defined as opposite the eigenvalue *i.e.* if e is the eigenvalue then the corresponding critical exponent denoted by $\theta = -e$.

This study of critical exponents at Wilson-Fisher FP is done without including any wave-function renormalization for the scalar field. However this computation can also be done by incorporating the wave-function renormalization for the scalar. In that case it will lead to improvements of the numerical results obtained above.

2.4. Cutoff Types and Scheme Dependence

After demonstrating the use of FRGE in deriving the beta functions of the couplings in a simple setup of an $O(N)$-symmetric scalar theory, I describe in this section the various ways of constructing the cutoff and how the results depend on them. By construction FRGE eq. (2.22) contains the cutoff $\mathcal{R}_k$, so the results obtained using it are expected to depend on the cutoff $\mathcal{R}_k$, *i.e.* depend on details of how one chooses to implement the cutoff. Scheme dependence is nothing new in quantum field theory. It is usually taken as a sign that the quantity one is calculating is not directly mesurable. In the computations done using perturbative methods in QFT, it has been observed that the beta functions generally depend on the way of the regularization of loop integral and the renormalization.

Implementing the cutoff $\mathcal{R}_k$ in different ways leads to different quantitative results but it is expected that qualitative properties of the renormalization group flow should remain same, for example the existence of fixed point and the dimension of critical surface. It is due to these reasons that in this section I will dicuss the different ways the cutoff can be contructed, thereby discussing how the beta functions and fixed point differ in each of way of implementing the cutoff [52].

The cutoff $\mathcal{R}_k$ are function of some differential operator Δ, which according to the form of the Hessian $\Gamma^{(2)}$ can be the full Hessian itself or a part of it. In computations done so far the cutoff $\mathcal{R}_k$ can be constructed in many different ways, which can be grouped in two classes. In the first method of construction the cutoff contains either some or all couplings of the theory, while in the second method of construction the cutoff does not contain any couplings at all and are called "pure" cutoffs. The first class of cutoffs are further categorized in three types based on way they are constructed using the Hessian. They are named as Type I, II and III cutoffs. the type III cutoffs are also known as "spectrally adjusted" cutoffs. Lets assume that the effective action Γ_k has the following derivative expansion:

$$\Gamma_k = \sum_i g_i \mathcal{O}_i \,, \tag{2.103}$$

where $\mathcal{O}_i$ are operators and g_i are numerical parameters depending on k. The operators $\mathcal{O}_i$ are integrals of the form

$$\mathcal{O}_i = \int d^d x \sqrt{g}\, \Omega_i, \tag{2.104}$$

where Ω_i are (possibly nonpolynomial) functions of the fields and their derivatives, respecting all the symmetries that the theory is supposed to possess. In gauge theories, Ω_i are constructed with covariant derivatives and curvatures. The number of derivatives increases with i, but the precise correspondence need not be spelled out here.

Some of the parameters appearing in the expansion may be eliminated by field redefinitions. This is the case, for example, for the wave function renormalization constants. Such parameters are said to be "redundant" or "inessential" [51, 11]. We assume that the theory has been parametrized in such a way that a certain subset of the g_i is redundant, while the remaining ones are "essential".

The fields do not have any scale dependence, so that

$$k\frac{d\Gamma_k}{dk} = \sum_i \beta_i \mathcal{O}_i, \tag{2.105}$$

where $\beta_i(g_j, k) = k\frac{dg_i}{dk}$ are the beta functions. In general they depend on all the g_i and also explicitly on k. Note that we call "beta functions" the derivatives of the parameters appearing in the action whether they are essential or not. One sometimes prefers to call "anomalous dimensions" the (logarithmic) derivatives of irrelevant parameters such as the wave function renormalization constants. We will not need to make this terminological distinction here. We will call β (without subscript i) the "beta functional" on the r.h.s. of the FRGE

$$\beta = \sum_i \beta_i \mathcal{O}_i \ .$$

If the operator Ω_i has dimension α_i, $\mathcal{O}_i$ has dimension $\alpha_i - d$ and g_i has dimension $d_i = d - \alpha_i$. One can now define dimensionless couplings $\tilde{g}_i$ and dimensionless operators $\tilde{\mathcal{O}}_i$ by $g_i = k^{d_i}\tilde{g}_i$ and $\mathcal{O}_i = k^{-d_i}\tilde{\mathcal{O}}_i$, so that eq. (2.103) can also be written as $\Gamma_k = \sum_i \tilde{g}_i\tilde{\mathcal{O}}_i$. The condition that has to be satisfied by a FP is

$$k\frac{d\tilde{g}_i}{dk} = 0 \ , \tag{2.106}$$

for all essential couplings g_i. We can rewrite this as follows. From the definition of $\tilde{g}_i$ we obtain $\partial_t g_i = d_i g_i + k^{d_i}\partial_t \tilde{g}_i$. Then we can rewrite eq. (2.105) as

$$k\frac{d\Gamma_k}{dk} = \sum_i d_i \tilde{g}_i \tilde{\mathcal{O}}_i + \sum_i \partial_t \tilde{g}_i \tilde{\mathcal{O}}_i \ .$$

Then the FP equation can be written compactly as

$$\left(-\sum_i d_i \tilde{g}_i \tilde{\mathcal{O}}_i + \beta\right)\Bigg|_{\text{essential}} = 0 \tag{2.107}$$

where the subscript "essential" means that the equation has to be projected on the subspace of essential couplings. The individual equations eq. (2.106) can be obtained from the functional equation by extracting the coefficient of the operator $\tilde{\mathcal{O}}_i$. We will now compare the functional form of this equation for two classes of cutoffs.

For definiteness we start by choosing a type III cutoff, defined as follows. The second variation of the action is a differential operator

$$\Delta(g_i) = \frac{\delta^2\Gamma_k}{\delta\phi\delta\phi} = \sum_i g_i \frac{\delta^2\mathcal{O}_i}{\delta\phi\delta\phi} \ . \tag{2.108}$$

By this notation we emphasize that the operator depends on all the parameters g_i. In the case of gauge theories the operator Δ is constructed with the covariant derivative ∇_μ. We choose the cutoff $\mathcal{R}$ to be a function of the full operator Δ: $\mathcal{R}_k = R_k(\Delta(g_i))$, where R_k is one of the functions that were discussed in section III. Then the modified inverse propagator is

$$\Delta(g_i) + R_k(\Delta(g_i)) = P_k(\Delta(g_i)) \ , \tag{2.109}$$

where P_k is defined as in eq. (2.3). This is a cutoff of "type III", in the terminology of [31]. Since the operator in the argument of the cutoff changes along the flow, this is called "spectrally adjusted".

The r.h.s. of the FRGE can now be written

$$\beta = \frac{1}{2}\mathrm{STr}\Big(\Delta(g_i) + R_k(\Delta(g_i))\Big)^{-1}\left(\frac{\partial R_k(\Delta(g_i))}{\partial t} + R_k'(\Delta(g_i))\frac{\partial \Delta}{\partial g_i}\beta_i\right) , \tag{2.110}$$

where a prime indicates the derivative of the function with respect to its argument. In the first term one derives only the explicit dependence of the cutoff on k and in the second the dependence that comes from the flow of the g_i. From here the beta functions β_i can be obtained in a two step procedure. First one has to extract from eq. (2.110) the coefficient of $\mathcal{O}_i$. Formally we can write

$$\beta_i = \frac{\delta\beta}{\delta\mathcal{O}_i}$$

This is usually the most labor-intensive part of the calculation, but still it does not immediately give the beta function, because the r.h.s. is itself a linear combination of the beta functions, of the form

$$\frac{\delta\beta}{\delta\mathcal{O}_i} = B_i + A_{ij}\beta_j \ .$$

where B_i are the one loop beta functions and A_{ij} are calculable coefficients. The beta functions can be obtained by solving this linear system:

$$\beta_i = (\mathbf{1} - A)^{-1}_{ij} B_j \ .$$

If one is only interested in the location of the FP, one can avoid this step by the following trick [33, 34]. Since at a FP $g_i = \tilde{g}_{i*}k^{d_i}$, for some constants $\tilde{g}_{i*}$, we obtain an equivalent set of FP equations if in the beta functional we replace β_i by $d_i g_i = d_i\tilde{g}_i k^{d_i}$. This modified beta functional is

$$\bar{\beta} = \frac{1}{2}\mathrm{STr}\Big(\Delta(g_i) + R_k(\Delta(g_i))\Big)^{-1}\left(\frac{\partial R_k(\Delta(g_i))}{\partial t} + R_k'(\Delta(g_i))\frac{\partial \Delta}{\partial g_i} d_i\tilde{g}_i k^{d_i}\right) \tag{2.111}$$

If we define

$$\bar{\beta}_i = \frac{\delta\bar{\beta}}{\delta\mathcal{O}_i}$$

these expressions do not contain the β functions anymore, and so they can be plugged directly in the FP equation. The FP equations obtained from the modified beta functions $\bar{\beta}_i$ have the same FP solutions as the ones obtained from the true beta functions β_i. We observe that the second term on the r.h.s. of eq. (2.111) is not just a function of Δ. In general it is a complicated operator that will not commute with Δ itself. We actually do not have the mathematical tools to extract beta functions from such complicated traces involving functions of several noncommuting operators. However, calculability is not required here, so we can proceed formally. We now simply assume that the FP equations determined in this way have a solution at $\tilde{g}_i = \tilde{g}_{i*}$.

The source of g_i-dependence in the cutoff definition given above is the operator Δ. We can turn the cutoff into a pure cutoff if we replace all the couplings appearing in Δ by arbitrary constants, multiplied by suitable powers of k to preserve the correct dimensionalities. The cutoff is then $R_k(\Delta(\gamma_i k^{d_i})))$. With this cutoff the r.h.s. of the FRGE reads

$$\beta = \frac{1}{2}\mathrm{STr}\left(\Delta(g_i) + R_k(\Delta(\gamma_i k^{d_i}))\right)^{-1}\left(\frac{\partial R_k(\Delta(\gamma_i k^{d_i}))}{\partial t} + R_k'(\Delta(\gamma_i k^{d_i}))\frac{\partial \Delta}{\partial g_i} d_i \gamma_i k^{d_i}\right) \tag{2.112}$$

From here one can extract beta functions

$$\beta_i(g_i, \gamma_i) = \frac{\delta\beta}{\delta\mathcal{O}_i}$$

that can be used to write FP equations. These FP equations depend parametrically on the arbitrary numbers γ_i. Recalling that $g_i = \tilde{g}_i k^{d_i}$ and comparing eq. (2.112) to eq. (2.111) we see that the only difference lies in the replacement of $\tilde{g}_i$ by γ_i in certain functional dependences.

It is clear that since the FP equation for the spectrally adjusted cutoff has a zero when we replace everywhere $\tilde{g}_i$ by the numbers $\tilde{g}_{i*}$, then the FP equation for the pure cutoff will also have a zero when we replace all the γ_i and all the $\tilde{g}_i$ by $\tilde{g}_{i*}$. Therefore with the particular choice of parameters $\gamma_i = \tilde{g}_{i*}$, the pure cutoff produces a FP in the same position as the spectrally adjusted type III cutoff.

This result has been derived using what we call a "type III" cutoff, because the argument is easier to make independently of the form of the action, but we believe that it holds more generally, also for other cutoffs. To illustrate this consider a generalization of what was called a "type I" cutoff in [31]. In a gauge theory the second variation defined in eq. (2.108) is a differential operator constructed with the covariant derivative ∇_μ. Let us assume that the truncation of the theory is such that Δ depends on ∇_μ only through the combination $-\Box = -\nabla_\mu\nabla^\mu$. To make this explicit let us write it as $\Delta(-\Box, g_i)$. A generalized type I cutoff can be defined by the requirement that the modified inverse propagator has the same form as the original one except for the replacement of $-\Box$ by $P_k(-\Box)$:

$$\mathcal{R}_k(-\Box, g_i) = \Delta(P_k(-\Box), g_i) - \Delta(-\Box, g_i)\ . \tag{2.113}$$

The beta functional that one obtains with this cutoff has the form

$$\beta = \frac{1}{2}\mathrm{STr}\left(\Delta(P_k(-\Box), g_i)\right)^{-1}\left(\frac{\partial\Delta}{\partial(-\Box)}\frac{\partial P_k(-\Box)}{\partial t} + \frac{\partial}{\partial g_i}\Big(\Delta(P_k(-\Box), g_i) - \Delta(-\Box, g_i)\Big)\beta_i\right) \tag{2.114}$$

which upon use of the trick explained above yields equivalent FP equations as

$$\bar{\beta} = \frac{1}{2}\mathrm{STr}\left(\Delta(P_k(-\Box), g_i)\right)^{-1}\left(\frac{\partial\Delta}{\partial(-\Box)}\frac{\partial P_k(-\Box)}{\partial t} + \frac{\partial}{\partial g_i}\Big(\Delta(P_k(-\Box), g_i) - \Delta(-\Box, g_i)\Big)d_i\tilde{g}_i k^{d_i}\right) \tag{2.115}$$

Again we can define a pure cutoff of generalized type I by replacing g_i by $\gamma_i k^{d_i}$ in $\mathcal{R}_k$

$$\mathcal{R}_k(-\Box, \gamma_i k^{d_i}) = \Delta(P_k(-\Box), \gamma_i k^{d_i}) - \Delta(-\Box, \gamma_i k^{d_i})\ . \tag{2.116}$$

The beta functional that one obtains with this cutoff has the form

$$\beta = \frac{1}{2}\mathrm{STr}\left(\Delta(P_k(-\Box),\gamma_i k^{d_i}) + \Delta(-\Box, g_i) - \Delta(-\Box, \gamma k^{d_i})\right)^{-1} \times \left(\frac{\partial\Delta}{\partial(-\Box)}\frac{\partial\mathcal{R}_k(-\Box)}{\partial t} + \left(\frac{\partial\Delta}{\partial g_i}(P_k(-\Box),\gamma_i k^{d_i}) - \frac{\partial\Delta}{\partial g_i}(-\Box,\gamma_i k^{d_i})\right) d_i\gamma_i k^{d_i}\right) \quad (2.117)$$

Again we see that the two beta functionals have the same form except for the replacement of $\tilde{g}_i$ by γ_i in certain functional dependences; additional terms in the first factor cancel for $\gamma_i = \tilde{g}_i$. Therefore the argument given above shows that if we set $\gamma_i = \tilde{g}_{i*}$ the pure cutoff will have a FP in the same position as the generalized type I cutoff.

The reason why this discussion is less general than the previous one is that this type of cutoff could only be defined if the inverse propagator has a specific form. It may be possible to generalize this argument, for example defining the cutoff by the rule

$$\nabla_\mu \mapsto \sqrt{\frac{P_k(-\Box)}{-\Box}}\,\nabla_\mu\ .$$

We will not pursue this further. The discussion of the type III cutoff is sufficient to make the point in generality. Furthermore, the type III cutoff is "ideologically" at the opposite extreme of a pure cutoff, being always fully dependent on all couplings. This is also supported by the numerical results, which show that type III cutoffs yield fixed points at at the extreme end of the range of variation [31]. So it is somewhat reassuring that one can reproduce at least the FP position of a spectrally adjusted, type III cutoff by a pure cutoff.

2.5. Summary

In this chapter I showed how to derive the functional renormalization group equation in the simple setting of a scalar system. Here I closely followed the methodology first introduced in [19, 20]. This FRGE was constructed where it was assumed that the UV cutoff of the theory is at infinity. Following that I have described the properties obeyed by the FRGE and the effective average action. As the cutoff $\mathcal{R}_k$ enters the FRGE as a t-derivative, thus due to its required properties it works as an UV-regulator thereby evading the necessity of introducing any regularization scheme. It was shown to be an one loop improved equation, whose r.h.s acts as a beta-functional of the theory. The effective average action entering the FRGE was shown to interpolate between the bare action in the UV limit and the usual effective action in the IR limit.

I then discussed the concept of theory space where the effective action is a point in theory space. The basis of such a space is the field monomials obeying the symmetries of the theories, while couplings act as co-ordinates in this space. In this notion, the FRGE is interpreted as a vector field, whose integral curves were interpreted as RG trajectories, interpolating between the bare action and the full effective action.

I then discussed the construction of FRGE in a theory with a UV cutoff. Here I followed the steps given in [42]. This is important as in physical situations one has UV cutoff. The construction of this FRGE with UV cutoff is same as the one without UV cutoff except now one does the FRGE trace over field modes with momenta only up to the UV cutoff, while in the FRGE without UV cutoff one performs the trace over all the modes (the high momentum modes are of course suppressed by the cutoff). I then discussed the difference that comes up when the computations are done using the two methods. It was further shown that this difference vanishes when the cutoff is chosen to be an optimized one [43].

As the theory space is an infinite one, thus in order to employ FRGE to practical computations, one needs approximation schemes. This constitute the discussion of section 2.1.5. The two approximations that are commonly made are the one-loop approximation where one ignores the running of couplings on the r.h.s of FRGE, while the second way of approximation is by truncating the theory space. Sometimes both are employed at the same time *i.e.* considering the one-loop flow of the couplings in the truncated theory space.

Then I introduced the concept of Asymptotic Safety, and obtained the properties that a theory is required to have in order to be asymptotically safe. These were that the dimensionless essential couplings parameters approach a nontrivial fixed point and the number of UV attractive directions at the FP be finite.

I then applied FRGE to a simple example of an $O(N)$-symmetric scalar field theory in a local potential approximation consisting of a kinetic term and a generic potential. I show how to compute the second variation of the action and using it how to construct the cutoff $\mathcal{R}_k$. Using the simple optimized cutoff the functional RG trace was computed in closed form, from which it the running of the potential is extracted by taking background of a constant scalar field. This expression was then used to do a lot of things. First the beta functions of the couplings were extracted by equating various powers of ϕ^2. Then using the existence of Gaussian FP I proved the properties obeyed by the various nonzero entries of the stability matrix. Wilson-Fisher FP was also obtained with the critical exponents showing that it has only one attractive direction at the FP.

Then in the final section 2.4 I discuss the two ways in which cutoffs are known to be constructed. One in which either some or all couplings are present are known as spectrally adjusted cutoff, while the other which does not contain any couplings at all are called pure cutoff. Discussion showed how the results computed using the two schemes differ.

Chapter 3

FRGE applied to Gravity

As mentioned in the Introduction, most of the progress towards asymptotic safety of the last ten years has come from applying functional renormalization group methods to gravity. In this chapter I will show how to construct the FRGE for gravity. After deriving the functional RG equation, I will apply it to the simplest truncation in the derivative expansion which will contain only two derivatives of the metric. This truncation is the Einstein-Hilbert (EH) truncation. By making use of FRGE I will derive the beta functions of couplings, look for fixed points and find the critical exponents at the fixed point. This will be done in different cutoff types keeping the spacetime dimension arbitrary, specializing to four dimensions wherever necessary.

3.1. Constructing the FRGE for gravity

3.1.1. Problems encountered in construction

I showed in the previous chapter that the FRGE of an effective action does not depend on the bare action S. There I discussed that in order to define a theory space one has to specify on which type of fields the functional Γ is supposed to depend, and what their symmetries are. Furthermore, in the theory space one choose coordinates g_i by writing the effective action Γ as,

$$\Gamma = \sum_i g_i \, \mathcal{O}_i \, . \tag{3.1}$$

This is the only input data needed for finding the renormalization group flow. Given a theory space, the form of FRGE and, as a result, the vector field β are completely fixed.

In the case of quantization of pure gravity, the theory space consists, by definition, of functionals Γ depending on a symmetric tensor field, the metric, in a diffeomorphism invariant way. Unfortunately it is not possible to straightforwardly apply the constructions of the previous chapter to this theory space. Diffeomorphism invariance leads to two types of complications one has to deal with [24].

The first complication is the need to gauge fix the functional integral. This problem already occurs during the functional integral quantization of any gauge or gravity theories, and so is quite a familiar one. Furthermore, fixing the gauge does not leave the effective action Γ constructed from it invariant, only the physical quantities constructed from Γ, for example S-matrix elements are gauge invariant.

The second problem is related to the fact that in gauge theory a "coarse graining" based on a naive Fourier decomposition of the gauge field is not gauge covariant, and hence not physical. In a non-gauge theory one decomposes the field in the eigenfunctions of the (positive) operator $-\partial^2$ and the coarse graining is achieved by declaring its eigenmodes long or short wavelength depending on whether the corresponding eigenvalue p^2 is smaller or larger than a given k^2. In a gauge theory, the best that can be done in incorporating a similar procedure is to decompose the field in to the eigenfunctions of the negative of *covariant* Laplacian (which is positive) or a similar operator, and then organize the modes based on the size of the eigenvalues. Although this approach is gauge covariant but it sacrifices to some extent the intuition of Fourier coarse graining in terms of slow and fast mode. Analogous remarks apply to theories of gravity covariant under general coordinate transformations.

The key idea which led to a solution of both problems was the use of the background field method. In fact, it is well known [53, 54] that the background gauge fixing method leads to an effective action which depends on its arguments in a gauge invariant way. As it turned out [23, 55] this technique also lends itself for implementing a covariant IR cutoff, and it is at the core of the effective average action for Yang-Mills theories [23, 55, 56] and for gravity [24].

3.1.2. Method of Construction

In this section I will introduce the effective average action for Euclidean quantum gravity in d-dimensions and will derive the flow equation governing its scale dependence.

Here I will closely follow the algorithm prescribed by Reuter [24]. The starting point of the computation is the diffeomorphism invariant Euclidean path-integral,

$$Z = \int \mathcal{D}\gamma_{\mu\nu} \, \exp(-S[\gamma_{\mu\nu}]) \ , \tag{3.2}$$

where $S[\gamma_{\mu\nu}]$ is the bare action for the gravity and $\gamma_{\mu\nu}$ is the quantum metric. The path-integral is invariant under the general co-ordinate transformation of $\gamma_{\mu\nu}$,

$$\delta\gamma_{\mu\nu} = \mathcal{L}_V \gamma_{\mu\nu} = \nabla_\mu V_\nu + \nabla_\nu V_\mu \ , \tag{3.3}$$

where $\mathcal{L}_V$ denotes the Lie-derivative with respect to the vector field V^μ and ∇_μ is the covariant derivative for the metric $\gamma_{\mu\nu}$. We will employ the background gauge fixing techniques. This means that we break the field $\gamma_{\mu\nu}$ as follows,

$$\gamma_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \ . \tag{3.4}$$

Here $\bar{g}_{\mu\nu}$ is the fixed background metric, and $h_{\mu\nu}$ is the fluctuation, which is not supposed to be small. This decomposition allows in the functional integral to replace the measure over $\gamma_{\mu\nu}$

by a measure over $h_{\mu\nu}$. Due to fixed background the gauge transformation acts only on the fluctuation field $h_{\mu\nu}$

$$\delta h_{\mu\nu} = \mathcal{L}_V \gamma_{\mu\nu} = \mathcal{L}_V(\bar{g}_{\mu\nu} + h_{\mu\nu}) \ , \qquad \delta\bar{g}_{\mu\nu} = 0 \ . \tag{3.5}$$

Due to background gauge invariance, the path-integral needs to be gauge fixed. Faddeev-Popov procedure is used to do this [53]. The gauge fixing action so obtained is given by,

$$S_{GF}[\bar{g}, h] = \frac{A}{2\alpha} \int d^d x \, \sqrt{\bar{g}} \bar{g}^{\mu\nu} F_\mu F_\nu \ , \tag{3.6}$$

where α is the gauge parameter, A is function of couplings, and $F_\mu = 0$ describes the gauge fixing condition. This procedure of gauge fixing introduces Faddeev-Popov ghost $\bar{C}_\mu$ and C^μ, whose action is given by,

$$S_{gh}[h, \bar{C}, C; \bar{g}] = -\int d^d x \, \sqrt{\bar{g}} \bar{C}_\mu \bar{g}^{\mu\nu} \frac{\partial F_\nu}{h_{\rho\sigma}} \mathcal{L}_C(\bar{g}_{\rho\sigma} + h_{\rho\sigma}) \ . \tag{3.7}$$

This procedure works in general for all arbitrary gauge fixing, but it is convenient to use F_μ which is linear in quantum field $h_{\mu\nu}$.

$$F_\mu = \mathcal{F}_\mu{}^{\alpha\beta} h_{\alpha\beta} \ , \tag{3.8}$$

where $\mathcal{F}_\mu{}^{\alpha\beta}$ is a differential operator. It is constructed from the background metric $\bar{g}_{\mu\nu}$. It is important to construct it from a background metric as otherwise the effective action obtained from it will not be diffeomorphism invariant. The most common gauge is given by,

$$\begin{aligned} F_\mu &= \left(\bar{\nabla}^\mu h_{\mu\nu} - \frac{\beta+1}{d} \bar{\nabla}_\nu h \right) \ , \\ &= \left(\delta^\sigma_\mu \bar{g}^{\rho\gamma} \, \bar{\nabla}_\gamma - \frac{\beta+1}{d} \bar{g}^{\rho\sigma} \bar{\nabla}_\mu \right) h_{\rho\sigma} \ , \end{aligned} \tag{3.9}$$

where α and β are the gauge parameters. For $\beta = d/2 - 1$ and in flat space, the condition $F_\mu = 0$ reduces to the usual harmonic gauge condition $\partial^\mu h_{\mu\nu} = \frac{1}{2}\partial_\nu h$. This gauge fixing would give the following ghost action,

$$S_{gh}[h, \bar{C}, C; \bar{g}] = -\int d^d x \, \sqrt{\bar{g}} \bar{C}_\mu \, \mathcal{M}^\mu{}_\nu \, C^\nu \ , \tag{3.10}$$

where $\mathcal{M}^\mu{}_\nu$ is given by,

$$\mathcal{M}^\mu{}_\nu = \left[\bar{g}^{\mu\rho} \bar{g}^{\sigma\lambda} \bar{\nabla}_\lambda (\gamma_{\sigma\nu} \nabla_\rho + \gamma_{\rho\nu} \nabla_\sigma) - \frac{2(\beta+1)}{d} \bar{g}^{\mu\lambda} \bar{g}^{\rho\sigma} \bar{\nabla}_\lambda \gamma_{\sigma\nu} \nabla_\rho \right] \ . \tag{3.11}$$

For every choice of background type gauge fixing F_μ, the ghost action is gauge invariant as the ghost field transforms as:

$$\delta C^\mu = \mathcal{L}_V C^\mu \, , \qquad \delta \bar{C}_\mu = \mathcal{L}_V \bar{C}_\mu \, . \tag{3.12}$$

With the gauge-fixing and ghost action one can write the connected Green functional at scale k by introducing the cutoff action ΔS_k in the same way as was done in the previous chapter. This is given as,

$$\begin{aligned} \exp\{-W_k[t^{\mu\nu},\sigma^\mu,\bar{\sigma}_\mu;\bar{g}_{\mu\nu}]\} &= \int \mathcal{D}h_{\mu\nu}\mathcal{D}\bar{C}_\mu\mathcal{D}C^\nu \exp\left[-S(\bar{g}+h)-\Delta S_k[h,\bar{C},C;\bar{g}]\right. \\ &\quad \left. -S_{GF}[h;\bar{g}]-S_{gh}[h,\bar{C},C;\bar{g}]-S_{sources}\right] \ , \\ S_{sources} &= \int d^dx\sqrt{\bar{g}}\left[t^{\mu\nu}\,h_{\mu\nu}+\sigma^\mu\,\bar{C}_\mu+\bar{\sigma}_\mu\,C^\mu\right] \ . \end{aligned} \tag{3.13}$$

The action for $S_{GF}[h;\bar{g}]$ and $S_{gh}[h,\bar{C},C;\bar{g}]$ are given by eq. (3.6) and (3.10) respectively. The action for the cutoff action is defined in the same way as it was defined in previous chapter, to be quadratic in fields. This is useful for obtaining a tractable evaolution equation later. For gravity it has the following form,

$$\Delta S_k[h,\bar{C},C;\bar{g}] = \frac{1}{2}\int d^dx\sqrt{\bar{g}}h_{\mu\nu}\,\mathcal{R}_k^{grav}[\bar{g}]^{\mu\nu\rho\sigma}h_{\rho\sigma} - \int d^dx\sqrt{\bar{g}}\bar{C}_\mu\mathcal{R}_k^{gh}[\bar{g}]^{\mu\nu}C_\nu \ , \tag{3.14}$$

where the cutoffs $\mathcal{R}_k^{grav}$ and $\mathcal{R}_k^{gh}$ are constructed from suitable differential operators using the background metric. This makes it invariant under background gauge transformation. The cutoffs have the properties defined in previous chapter. Given the functional W_k, we introduce the k-dependent classical fields

$$\bar{h}_{\mu\nu} = \frac{1}{\sqrt{\bar{g}}}\frac{\delta W_k}{\delta t^{\mu\nu}} \ , \qquad \bar{c}_\mu = \frac{1}{\sqrt{\bar{g}}}\frac{\delta W_k}{\delta \sigma^\mu} \ , \qquad c_\mu = \frac{1}{\sqrt{\bar{g}}}\frac{\delta W_k}{\delta \bar{\sigma}_\mu} \ . \tag{3.15}$$

Using the classical fields we define the Legendre transform of the W_k. This is given by,

$$\tilde{\Gamma}_k[\bar{h},\bar{c},c;\bar{g}] = W_k[t,\sigma,\bar{\sigma};\bar{g}] - \int d^dx\sqrt{\bar{g}}\left[t^{\mu\nu}\,\bar{h}_{\mu\nu}+\sigma^\mu\,\bar{c}_\mu+\bar{\sigma}_\mu\,c^\mu\right] \ . \tag{3.16}$$

Then the effective average action for gravity is obtained by subtracting the cutoff action from the $\tilde{\Gamma}_k$ as was done in previous chapter,

$$\Gamma'_k[\bar{h},\bar{c},c;\bar{g}] = \tilde{\Gamma}_k[\bar{h},\bar{c},c;\bar{g}] - \Delta S_k[h,\bar{c},c;\bar{g}] \ . \tag{3.17}$$

At this point it is useful to introduce the metric,

$$g_{\mu\nu} = \langle\gamma_{\mu\nu}\rangle = \bar{g}_{\mu\nu} + \bar{h}_{\mu\nu} \ . \tag{3.18}$$

Using this one can write the Γ_k as a functional of $g_{\mu\nu}$ rather than $\bar{h}_{\mu\nu}$ as follows,

$$\Gamma_k[g_{\mu\nu},\bar{g}_{\mu\nu},\bar{c},c] := \Gamma'_k[\bar{h}=g-\bar{g},\bar{g},\bar{c},c] \ . \tag{3.19}$$

The functional Γ_k obeys the following flow equation,

$$\begin{aligned}\partial_t \Gamma'_k[\bar{h}, \bar{g}, c, \bar{c}] &= \frac{1}{2}\mathrm{Tr}\left[\left(\Gamma'^{(2)}_{\mathrm{k}} + \mathcal{R}_{\mathrm{k}}\right)^{-1}_{\bar{\mathrm{h}}\bar{\mathrm{h}}} \cdot (\partial_{\mathrm{t}}\mathcal{R}_{\mathrm{k}})_{\bar{\mathrm{h}}\bar{\mathrm{h}}}\right] + \frac{1}{2}\mathrm{Tr}\left[\left(\Gamma'^{(2)}_{\mathrm{k}} + \mathcal{R}_{\mathrm{k}}\right)^{-1}_{\mathrm{c}\bar{\mathrm{c}}} \cdot (\partial_{\mathrm{t}}\mathcal{R}_{\mathrm{k}})_{\mathrm{c}\bar{\mathrm{c}}}\right] \\ &\quad -\frac{1}{2}\mathrm{Tr}\left[\left(\Gamma'^{(2)}_{\mathrm{k}} + \mathcal{R}_{\mathrm{k}}\right)^{-1}_{\bar{\mathrm{c}}\mathrm{c}} \cdot (\partial_{\mathrm{t}}\mathcal{R}_{\mathrm{k}})_{\bar{\mathrm{c}}\mathrm{c}}\right] \ , \end{aligned} \tag{3.20}$$

where the Hessian $\Gamma'^{(2)}_k$ for graviton and ghost is given by,

$$\begin{aligned}\left[\left(\Gamma^{(\prime 2)}_k\right)_{\bar{h}\bar{h}}\right]^{\mu\nu\alpha\beta} &= \frac{1}{\sqrt{\bar{g}}}\frac{\delta}{\delta \bar{h}_{\mu\nu}}\frac{1}{\sqrt{\bar{g}}}\frac{\delta}{\delta \bar{h}_{\alpha\beta}}\,\Gamma'_k[\bar{h}, \bar{g}, c, \bar{c}] \ , \\ \left[\left(\Gamma'^{(2)}_k\right)_{\bar{c}c}\right]^{\nu}_{\mu} &= \frac{1}{\sqrt{\bar{g}}}\frac{\delta}{\delta c^{\mu}}\frac{1}{\sqrt{\bar{g}}}\frac{\delta}{\delta \bar{c}_{\nu}}\,\Gamma'_k[\bar{h}, \bar{g}, c, \bar{c}] \ . \end{aligned} \tag{3.21}$$

This effective average action for gravity Γ_k has the following properties:
(1) The functional Γ_k is invariant under general co-ordinate transformation, where all arguments transform as tensors of corresponding rank:

$$\Gamma_k[\Phi + \mathcal{L}_V\Phi] = \Gamma_k[\Phi] \ , \qquad \Phi = \{g_{\mu\nu}, \bar{g}_{\mu\nu}, \bar{c}_\mu, c^\mu\} \ . \tag{3.22}$$

Note that in this transformation even the background metric also transforms as an ordinary tensor field. This is contrary to the "quantum gauge transformation" given in eq. (3.5) where the background metric was fixed. Equation (3.22) is a consequence of

$$W_k[\mathcal{J} + \mathcal{L}_V\mathcal{J}] = W_k[\mathcal{J}] \ , \qquad \mathcal{J} = \{t^{\mu\nu}, \sigma^\mu, \bar{\sigma}_\mu, \bar{g}_{\mu\nu}\} \ . \tag{3.23}$$

(2) Since the cutoff $\mathcal{R}_k$ vanishes for $k = 0$, the limit $k \to 0$ of $\Gamma_k[g_{\mu\nu}, \bar{g}_{\mu\nu}, \bar{c}, c]$ gives the standard effective action functional which still depends on two metrics. The effective action obtained by setting ghost to zero is,

$$\Gamma'[\bar{h}, \bar{g}] = \lim_{k\to 0}\Gamma'_k[\bar{h}, \bar{g}, \bar{c} = 0, c = 0] \tag{3.24}$$

This effective action is the generating functional of 1PI "Off-shell" Green functions. The Green functions depends on the background metric $\bar{g}_{\mu\nu}$. The "ordinary" effective action $\Gamma[g_{\mu\nu}]$ with one metric argument is obtained from this functional by setting $g_{\mu\nu} = \bar{g}_{\mu\nu}$:

$$\Gamma[g] = \lim_{k\to 0}\Gamma_k[g, \bar{g} = g, \bar{c} = 0, c = 0] = \lim_{k\to 0}\Gamma'_k[\bar{h} = 0, \bar{c} = 0, c = 0; g = \bar{g}] \ . \tag{3.25}$$

This functional is the generating functional of the 1PI "on-shell" Green function for the gravitons. They do not depend on the background metric. In this context "on-shell" means that the metric satisfies $\delta\Gamma[g]/\delta g_{\mu\nu} = 0$.

The k-dependent counterpart of eq. (3.25) is given by,

$$\bar{\Gamma}_k[g_{\mu\nu}] \equiv \Gamma_k[g_{\mu\nu}, g_{\mu\nu}, 0, 0] \ . \tag{3.26}$$

Due to eq. (3.22) , the functionals $\Gamma[g_{\mu\nu}]$ and $\bar{\Gamma}_k[g_{\mu\nu}]$ are invariant under general co-ordinate transformation $\delta g_{\mu\nu} = \mathcal{L}_V g_{\mu\nu}$. However there is a price to be paid that the functional $\bar{\Gamma}_k[g_{\mu\nu}]$ no longer satisfies the functional RG equation (3.20), because it contains insufficient information. The actual RG evolution has to be performed at the level of functional $\Gamma_k[\bar{h}, \bar{g}, c, \bar{c}]$. Only after the evolution one may set the $\bar{g} = g$, $c = 0$ and $\bar{c} = 0$.
(3) The functional RG trace in eq. (3.20) does not require any additional UV regularization. This is due to the properties that the cutoff $\mathcal{R}_k$ satisfies, and the way it enters the functional RG equation as a t-derivative. Due to this the trace gets contributions only from modes with eigenvalues close to k.
(4) The effective average action for gravity satisfies the following integro-differential equation,

$$\exp\{-\Gamma_k[\bar{h}, \bar{g}, c, \bar{c}]\} = \int \mathcal{D}h\mathcal{D}C\mathcal{D}\bar{C} \exp\Bigg[-\tilde{S}[h, C, \bar{C}; \bar{g}] - \int d^d x \Bigg\{(h_{\mu\nu} - \bar{h}_{\mu\nu}) \frac{\delta\Gamma_k}{\delta\bar{h}_{\mu\nu}} + (C^\mu - c^\mu) \frac{\delta\Gamma_k}{\delta c^\mu} + (\bar{C}_\mu - \bar{c}_\mu) \frac{\delta\Gamma_k}{\delta\bar{c}_\mu}\Bigg\}\Bigg] \cdot \exp\{-\Delta S_k[h - \bar{h}, C - c, \bar{C} - \bar{c}; \bar{g}]\} \ , \tag{3.27}$$

where,

$$\tilde{S} = S + S_{GF} + S_{gh} \ , \tag{3.28}$$

is expressed in terms of the "microscopic" field (h, C and $\bar{C}$). Eq. (3.27) can be obtained by inserting the definition of Γ_k in eq. (3.13) and using,

$$\frac{\tilde{\Gamma}_k}{\delta\bar{h}_{\mu\nu}} = \sqrt{\bar{g}}\, t^{\mu\nu} \ , \quad \frac{\tilde{\Gamma}_k}{\delta c^\mu} = -\sqrt{\bar{g}}\, \bar{\sigma}_\mu \ , \quad \frac{\tilde{\Gamma}_k}{\delta\bar{c}_\mu} = -\sqrt{\bar{g}}\, \sigma^\mu \ . \tag{3.29}$$

When one takes the limit $k \to \infty$, then the last term in eq. (3.27) becomes a delta function *i.e*

$$\exp\{-\Delta S_k\} \sim \delta[h - \bar{h}]\, \delta[C - c]\, \delta[\bar{C} - \bar{c}] \ . \tag{3.30}$$

As a consequence of which the EAA for gravity in UV limit is,

$$\Gamma'_{k\to\infty}[\bar{h}, c, \bar{c}; \bar{g}] = S[\bar{g} + \bar{h}] + S_{GF}[\bar{h}; \bar{g}] + S_{gh}[\bar{h}, c, \bar{c}; \bar{g}] \ . \tag{3.31}$$

The "initial value" $\Gamma'_{k\to\infty}$ includes the gauge fixing and ghost actions. At the level of functional $\bar{\Gamma}_k[g]$, eq. (3.31) boils down to $\bar{\Gamma}_{k\to\infty} = S[g]$. However as the second variation $\Gamma'^{(2)}_k$ involves derivatives with respect to $\bar{h}_{\mu\nu}$ (or equivalently $g_{\mu\nu}$) at fixed $\bar{g}_{\mu\nu}$, it is clear that the evolution can be formulated entirely in terms of $\bar{\Gamma}_k$ alone.

3.2. Truncated Flow Equation

Solving the FRGE eq. (3.20) subject to the initial condition eq. (3.31) is equivalent to (and in practice as difficult as) calculating the original functional integral over $\gamma_{\mu\nu}$. Both the task are severely difficult and complex. Thus it is important to devise efficient approximation methods to tackle the problem. The truncation of theory space is the one which makes maximum use of the FRGE reformulation of the quantum field theory problem at hand.

As for the flow on the theory space $\{\Gamma[g,\bar{g},c,\bar{c}]\}$, one way of simplifying things but still keeping the truncation very general is by neglecting the flow of the ghost by making the ansatz:

$$\Gamma_k[g,\bar{g},c,\bar{c}] = \bar{\Gamma}_k[g] + \widehat{\Gamma}_k[g,\bar{g}] + S_{GF}[g-\bar{g};\bar{g}] + S_{gh}[g-\bar{g},c,\bar{c};\bar{g}]\,, \tag{3.32}$$

where we extracted the classical S_{GF} and S_{gh} from Γ_k. The remaining functional depends on both $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$. It is further decomposed as $\bar{\Gamma}_k + \widehat{\Gamma}_k$ where $\bar{\Gamma}_k$ is defined as in eq. (3.26) and $\widehat{\Gamma}_k$ contains the deviations for $\bar{g} \neq g$. Hence, by definition, $\widehat{\Gamma}_k[g,g] = 0$, and $\widehat{\Gamma}_k$ contains in particular quantum corrections to the gauge fixing term which vanishes for $\bar{g} = g$, too. This ansatz satisfies the initial condition eq. (3.31) if

$$\bar{\Gamma}_{k\to\infty} = S \qquad \text{and} \qquad \widehat{\Gamma}_{k\to\infty} = 0\,. \tag{3.33}$$

Inserting eq. (3.32) into the exact FRGE eq. (3.20) one obtains an evolution equation on the truncated space $\{\Gamma[g,\bar{g}]\}$:

$$\begin{aligned}\partial_t\Gamma_k[g,\bar{g}] \;&=\; \frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)}[g,\bar{g}] + \mathcal{R}_k^{grav}[\bar{g}]\right)^{-1}\partial_t\mathcal{R}_k^{grav}[\bar{g}]\right] \\ &\quad -\mathrm{Tr}\left[\left(S_{gh}^{(2)}[g,\bar{g}] + \mathcal{R}_k^{gh}[\bar{g}]\right)^{-1}\partial_t\mathcal{R}_k^{gh}[\bar{g}]\right]\,,\end{aligned} \tag{3.34}$$

where $\Gamma_k^{(2)}[g,\bar{g}]$ and $S_{gh}^{(2)}[g,\bar{g}]$ are given by,

$$\begin{aligned}\left(\Gamma_k^{(2)}[g,\bar{g}]\right)^{\mu\nu\alpha\beta} &= \frac{1}{\sqrt{\bar{g}}}\frac{\delta}{\delta\bar{h}_{\mu\nu}}\frac{1}{\sqrt{\bar{g}}}\frac{\delta}{\delta\bar{h}_{\alpha\beta}}\,\Gamma_k[g,\bar{g}]\bigg|_{g=\bar{g}}\,, \\ \left(S_{gh}^{(2)}[g,\bar{g}]\right)^{\mu}{}_{\nu} &= \frac{1}{\sqrt{\bar{g}}}\frac{\delta}{\delta c^{\nu}}\frac{1}{\sqrt{\bar{g}}}\frac{\delta}{\delta\bar{c}_{\mu}}\,S[g,\bar{g}]\bigg|_{g=\bar{g}}\,.\end{aligned} \tag{3.35}$$

The eq. (3.34) is the one which is used in practical computations involving truncations. It evolves the functional

$$\Gamma_k[g,\bar{g}] \equiv \bar{\Gamma}_k[g] + S_{GF}[g-\bar{g};\bar{g}] + \widehat{\Gamma}_k[g,\bar{g}]\,. \tag{3.36}$$

The truncation ansatz given in eq. (3.32) is still too general for the praticial computations to be easily possible. The best way to simplify the computation is to do the derivative expansion of $\bar{\Gamma}_k[g]$ and consider terms up to some finite order. Under this strategy the first truncation that was considered was the Einstein-Hilbert action with a cosmological constant. In the next section I will describe how the FRGE was used to obtain the beta functions of the couplings in this truncation.

3.3. Einstein-Hilbert Truncation

In the Einstein-Hilbert truncation, the theory is parametrized by two couplings: cosmological constant Λ and Newtons' constant $G = 1/16\pi Z$. The functional $\bar{\Gamma}[g]$ consists of two operators $\int d^dx\,\sqrt{g}$ and $\int d^dx\,\sqrt{g}R$, and $\Gamma_k[g,\bar{g}]$ is given by,

$$\Gamma_k[g,\bar{g}] = \int d^dx\sqrt{g}\,(2\Lambda Z - Z\,R(g)) + S_{GF} + \widehat{\Gamma}_k[g,\bar{g}]\,, \tag{3.37}$$

where S_{GF} is the gauge fixing action given by,

$$S_{GF}(\bar{g},h) = \frac{Z}{2\alpha}\int d^dx\sqrt{\bar{g}}\left(\nabla^\rho h_{\rho\mu} - \frac{1+\beta}{d}\nabla_\mu h\right)\left(\nabla_\sigma h^{\sigma\mu} - \frac{1+\beta}{d}\nabla^\mu h\right)\,, \tag{3.38}$$

with α and β as gauge parameters. To extract the running of couplings from the FRGE one plugs the eq. (3.37) in to eq. (3.34) and sets $g = \bar{g}$. Setting of $g = \bar{g}$ implies $\bar{h} = 0$, which means that the gauge fixing term and $\widehat{\Gamma}_k[g,\bar{g}]$ drops out from the LHS of the FRGE. By expanding the functional RG trace to required order and comparing coefficients of respective operators gives the running of respective coupling.

This is done by first expanding to second order the action $\bar{\Gamma}_k[g]$ around the background $\bar{g}$, then adding to it the gauge fixing action S_{GF} and obtaining the Hessian for the gravitational field. From the Hessian the cutoff $\mathcal{R}_k$ is constructed and the information is plugged in FRGE to obtain the flow.

In this section I will fix the gauge parameters, $\alpha = 1$ and $\beta = d/2 - 1$ (de-Donder gauge). I will describe two procedures of implementing the cutoff and computing the flow. Then I will compare the results obtained from the two procedures.

The inverse propagator of $h_{\mu\nu}$, including the gauge fixing term, can be written in the form

$$\frac{1}{2}\int d^dx\,\sqrt{g}h_{\mu\nu}\Gamma_k^{(2)\mu\nu\rho\sigma}h_{\rho\sigma}$$

containing the minimal operator:

$$\Gamma^{(2)\mu\nu}_{k\ \rho\sigma} = Z\left[K^{\mu\nu}_{\rho\sigma}(-\Box - 2\Lambda) + U^{\mu\nu}_{\rho\sigma}\right]\,, \tag{3.39}$$

where $\Box = \nabla_\mu\nabla^\mu$ and

$$\begin{aligned} K^{\mu\nu}_{\rho\sigma} &= \frac{1}{2}\left(\delta^{\mu\nu}_{\rho\sigma} - \frac{d}{2}P^{\mu\nu}_{\rho\sigma}\right)\,; \qquad \delta^{\mu\nu}_{\rho\sigma} = \frac{1}{2}\left(\delta^\mu_\rho\delta^\nu_\sigma + \delta^\mu_\sigma\delta^\nu_\rho\right)\,; \qquad P^{\mu\nu}_{\rho\sigma} = \frac{1}{d}g^{\mu\nu}g_{\rho\sigma}\,; \\ U^{\mu\nu}_{\rho\sigma} &= R\,K^{\mu\nu}_{\rho\sigma} + \frac{1}{2}\left(g^{\mu\nu}R_{\rho\sigma} + R^{\mu\nu}g_{\rho\sigma}\right) - \delta^{(\mu}_{(\rho}R^{\nu)}_{\sigma)} - R^{(\mu}{}_{(\rho}{}^{\nu)}{}_{\sigma)}\,. \end{aligned} \tag{3.40}$$

In the following I will sometimes suppress indices for notational clarity; I will use boldface symbols to indicate linear operators on the space of symmetric tensors. For example, the objects defined above will be denoted $\mathbf{K}$, $\mathbf{1}$, $\mathbf{P}$, $\mathbf{U}$. Note that $\mathbf{P}$ and $\mathbf{1} - \mathbf{P}$ are projectors onto the trace and trace-free parts in the space of symmetric tensors: $h_{\mu\nu} = h^{(TF)}_{\mu\nu} + h^{(T)}_{\mu\nu}$ where $h^{(T)}_{\mu\nu} = P_{\mu\nu}^{\rho\sigma} h_{\rho\sigma} = \frac{1}{d} g_{\mu\nu} h$. Using that $\mathbf{K} = \frac{1}{2}\left((\mathbf{1}-\mathbf{P}) + \frac{2-d}{2}\mathbf{P}\right)$, if $d \neq 2$ eq. (3.39) can be rewritten in either of the following forms:

$$\begin{aligned} \mathbf{\Gamma}^{(2)}_k &= Z\,\mathbf{K}(-\Box - 2\Lambda\mathbf{1} + \mathbf{W}) \\ &= \frac{Z}{2}\left[(\mathbf{1}-\mathbf{P})\,(-\Box - 2\Lambda\mathbf{1} + 2\mathbf{U}) - \frac{d-2}{2}\mathbf{P}\left(-\Box - 2\Lambda\mathbf{1} - \frac{4}{d-2}\mathbf{U}\right)\right] \end{aligned} \tag{3.41}$$

where I have defined

$$W_{\rho\sigma}^{\mu\nu} = 2U_{\rho\sigma}^{\mu\nu} - \frac{(d-4)}{2(d-2)}(R_{\rho\sigma}g^{\mu\nu} + g_{\rho\sigma}R^{\mu\nu} - R g_{\rho\sigma} g^{\mu\nu}) \; .$$

Note that the overall sign of the second term in the second line of (3.41) is negative when $d > 2$. This is the famous problem of the unboundedness of the Euclidean Einstein–Hilbert action. I will show shortly how this is dealt with in the FRGE. Later on, I will need the traces:

$$\begin{aligned} \mathrm{tr}\,\mathbf{1} &= \frac{d(d+1)}{2}\;;\;\; \mathrm{tr}\,\mathbf{P} = 1\;;\;\; \mathrm{tr}\,(\mathbf{1}-\mathbf{P}) = \frac{d^2+d-2}{2}\;;\;\; \mathrm{tr}\,\mathbf{W} = \frac{d(d-1)}{2}R\;; \\ \mathrm{tr}\,\mathbf{W}^2 &= 3R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \frac{d^2-8d+4}{d-2}R_{\mu\nu}R^{\mu\nu} + \frac{d^3-5d^2+8d+4}{2(d-2)}R^2 \; . \end{aligned} \tag{3.42}$$

The ghost action is:

$$S_{\text{ghost}} = -\int \sqrt{g}\,\bar{C}_\mu\,(-\Box\delta^\mu_{\ \nu} - R^\mu_{\ \nu})\,C^\nu \; . \tag{3.43}$$

On the d-dimensional sphere the tensor $\mathbf{U}$ becomes,

$$\mathbf{U} = \frac{1}{2}\left[(\mathbf{1}-\mathbf{P})\,\frac{d^2-3d+4}{d(d-1)}R - \mathbf{P}\frac{d-2}{2}\frac{d-4}{d}R\right] \; .$$

Then, using the second line of (3.41), I have

$$\mathbf{\Gamma}^{(2)}_k = \frac{Z}{2}\left[(\mathbf{1}-\mathbf{P})\left(-\Box - 2\Lambda + \frac{d^2-3d+4}{d\,(d-1)}R\right) - \frac{d-2}{2}\mathbf{P}\left(-\Box - 2\Lambda + \frac{d-4}{d}R\right)\right] \; . \tag{3.44}$$

Now before I go on with the computations, I will give a short introduction t various cutoff types available and how the function RG trace is performed in each one of them.

3.3.1. Cutoff types and FRGE trace

In this section I will illustrate the method that is used to compute the trace in the r.h.s. of (3.34) in a gravitational setting and to evaluate the beta functions of the gravitational couplings. Quite generally, we will consider the contribution of fields whose inverse propagator $\Gamma^{(2)}$ is a differential operator of the form $\Gamma^{(2)} = -\Box + \mathbf{E}$ ($\Box = \nabla^2$), where ∇ is a covariant derivative, both with respect to the gravitational field and possibly also with respect to other gauge connections coupled to the internal degrees of freedom of the field, and $\mathbf{E}$ is a linear map acting on the quantum field. In general, $\mathbf{E}$ could contain mass terms or terms linear in curvature. For example, in the case of a nonminimally coupled scalar, $\mathbf{E} = \xi R$, where ξ is a coupling. A priori, nothing will be assumed about the gravitational action and also the spacetime dimension d can be left arbitrary at this stage.

In order to write the FRGE we have to define the cutoff. For the operator Δ to be used in the definition of (2.7), several possible choices suggest themselves. Let us split $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$, where $\mathbf{E}_1$ does not contain any couplings and $\mathbf{E}_2$ consists only of terms containing the couplings. We call a cutoff *of type I*, if R_k is a function of the "bare Laplacian" $\Delta = -\Box$, *of type II* if it is a function of $\Delta = -\Box + \mathbf{E}_1$ and *of type III* if it is a function of the full kinetic operator $\Delta = -\Box + \mathbf{E}$. The substantial difference between the first two types and the third is that in the latter case, due to the running of the couplings, the spectrum changes along the flow. For this reason these cutoffs are said to be "spectrally adjusted" [57]. [1]

Let us now restrict ourselves to the case when $\mathbf{E}_2 = 0$, i.e. the kinetic operator does not depend on the couplings; then there is only a choice between cutoffs of type I and II. The derivation of the beta functions is technically simpler with a type II cutoff. In this case we choose a real function R_k with the properties listed before and define a modified inverse propagator

$$P_k(\Delta) = \Delta + R_k(\Delta) \; . \tag{3.45}$$

If the operator $\mathbf{E}$ does not contain couplings, using (A.10) the trace in the r.h.s. of the FRGE reduces simply to:

$$\mathrm{Tr}\frac{\partial_t R_k(\Delta)}{P_k(\Delta)} = \frac{1}{(4\pi)^{d/2}} \sum_{i=0}^{\infty} Q_{\frac{d}{2}-i}\left(\frac{\partial_t R_k}{P_k}\right) B_{2i}(\Delta) \tag{3.46}$$

where $B_{2i}(\Delta)$ are the heat kernel coefficients of the operator Δ and the Q-functionals, defined in (A.14,A.15) are the analogs of momentum integrals in this curved spacetime setting. We have written $\partial_t R_k$ to denote the derivative with respect to the explicit dependence of R_k on k; when the argument of R_k does not contain couplings this coincides with the total derivative $\frac{d}{dt}R_k$.

With a type I cutoff we use the same profile function R_k but now with $-\Box$ as its argument. This implies the replacement of the inverse propagator Δ by

$$\Delta + R_k(-\Box) = P_k(-\Box) + \mathbf{E} \; . \tag{3.47}$$

Therefore the r.h.s. of the ERGE will now contain the trace $\mathrm{Tr}\frac{\partial_t R_k(-\nabla^2)}{P_k(-\nabla^2)+\mathbf{E}}$. Since $\mathbf{E}$ is linear in curvature, in the limit when the components of the curvature tensor are uniformly much

[1] In (2.7) it was assumed for simplicity that the operator Δ appearing in the argument of the cutoff function is also the operator whose eigenfunctions are used as a basis in the evaluation of the functional trace. It is worth stressing that this need not be the case, as discussed in Appendix A.1.

smaller than k^2, we can expand

$$\frac{\partial_t R_k}{P_k + \mathbf{E}} = \sum_{\ell=0}^{\infty} (-1)^\ell \mathbf{E}^\ell \frac{\partial_t R_k}{P_k^{\ell+1}} \; .$$

Each one of the terms on the r.h.s. can then be evaluated in a way analogous to (A.10), so in this case we get a double series:

$$\mathrm{Tr} \frac{\partial_t R_k(-\Box)}{P_k(-\Box) + \mathbf{E}} = \frac{1}{(4\pi)^{d/2}} \sum_{i=0}^{\infty} \sum_{\ell=0}^{\infty} Q_{\frac{d}{2}-i} \left(\frac{\partial_t R_k}{P_k^{\ell+1}} \right) \int dx \, \sqrt{g} (-1)^\ell \mathrm{tr} \mathbf{E}^\ell b_{2i}(-\Box) \; . \tag{3.48}$$

In order to extract the beta functions of the gravitational couplings one has to collect terms with the same monomials in curvature.

At this point it is interesting to consider the scheme–independent part of the trace. In general, on dimensional grounds, the functionals $Q_n\left(\frac{\partial_t R_k}{P_k^m}\right)$ appearing in (3.46) and (3.48) will be equal to $k^{2(n-m+1)}$ times a number depending on the profile function. As discussed in Appendix A, the integrals with $m = n + 1$ are independent of the shape of R_k. Thus, in even-dimensional spacetimes with a cutoff of type II, and using (A.19), the coefficient of the term in the sum (3.46) with $i = \frac{d}{2}$ is $Q_0\left(\frac{\partial_t R_k}{P_k}\right) B_d(\Delta) = 2B_d(\Delta)$. On the other hand with a type I cutoff, using (A.18), (A.19) and (A.5) the terms with $\ell = \frac{d}{2} - i$ add up to

$$\begin{aligned}
&\sum_{\ell=0}^{d/2} Q_\ell \left(\frac{\partial_t R_k}{P_k^{\ell+1}} \right) \int dx \, \sqrt{g} (-1)^\ell \mathrm{tr} \mathbf{E}^\ell b_{2i}(-\Box) \\
&= 2 \int dx \, \sqrt{g} \mathrm{tr} \left[b_d(-\Box) - \mathbf{E} b_{d-2}(-\Box) + \ldots + \frac{(-1)^{d/2}}{(d/2)!} \mathbf{E}^{d/2} b_0(-\Box) \right] \\
&= 2B_d(-\Box + \mathbf{E})
\end{aligned}$$

Therefore, in addition to being independent of the shape of the cutoff function, these coefficients are also the same using type I or type II cutoffs.

Now I will apply these cutoff schemes to the Einstein-Hilbert gravity case and perform the functional RG trace according to various schemes.

3.3.2. Cutoff of type Ia

This is the scheme that was used originally in [24]. It is defined by the cutoff term

$$\Delta S_k[h_{\mu\nu}] = \frac{1}{2} \int dx \, \sqrt{g} \, h_{\mu\nu} R_k(-\Box)^{\mu\nu\rho\sigma} \, h_{\rho\sigma} - \int dx \, \sqrt{g} \, \bar{C}_\mu R_k^{(gh)}(-\Box)^\mu{}_\nu C^\nu \; , \tag{3.49}$$

where

$$\begin{aligned}
\mathbf{R}_k(-\Box) &= Z \mathbf{K} R_k(-\Box) \\
R_k^{(gh)}(-\Box)^\mu{}_\nu &= \delta^\mu_\nu R_k(-\Box) \, .
\end{aligned} \tag{3.50}$$

for gravitons and ghosts respectively. Defining the anomalous dimension by

$$\eta = \frac{\partial_t Z}{Z} , \tag{3.51}$$

I then have

$$\partial_t \mathbf{R}_k = Z\mathbf{K}\left[\partial_t R_k(-\Box) + \eta R_k(-\Box)\right] . \tag{3.52}$$

The calculation in [24] proceeded as follows. The background metric is chosen to be that of Euclidean de Sitter space. The modified inverse propagator is obtained from (3.44) just replacing $-\Box$ by $P_k(-\Box)$, where $P_k(-\Box) = -\Box + R_k(-\Box)$. Using the properties of the projectors, its inversion is trivial:

$$\left(\mathbf{\Gamma}_k^{(2)} + \mathbf{R}_k\right)^{-1} = \frac{2}{Z}\left[(\mathbf{1}-\mathbf{P})\frac{1}{P_k - 2\Lambda + \frac{d^2-3d+4}{d(d-1)}R} - \frac{2}{d-2}\mathbf{P}\frac{1}{P_k - 2\Lambda + \frac{d-4}{d}R}\right] \tag{3.53}$$

Decomposing in the same way the term $\partial_t \mathbf{R}_k$, multiplying and tracing over spacetime indices one obtains

$$\partial_t \Gamma_k = \frac{1}{2}\mathrm{Tr}(\mathbf{1}-\mathbf{P})\frac{\partial_t R_k + \eta R_k}{P_k - 2\Lambda + \frac{d^2-3d+4}{d(d-1)}R} + \frac{1}{2}\mathrm{Tr}\mathbf{P}\frac{\partial_t R_k + \eta R_k}{P_k - 2\Lambda + \frac{d-4}{d}R} - \mathrm{Tr}\delta^\mu_\nu \frac{\partial_t R_k}{P_k - \frac{R}{d}} .$$

One can now expand to first order in R, use the traces (3.42) and formula (A.10) to obtain:

$$\begin{aligned}
\partial_t \Gamma_k \;=\; & \frac{1}{(4\pi)^{d/2}}\int d^dx\,\sqrt{g}\Bigg\{\frac{d\,(d+1)}{4}Q_{\frac{d}{2}}\left(\frac{\partial_t R_k + \eta R_k}{P_k - 2\Lambda}\right) - dQ_{\frac{d}{2}}\left(\frac{\partial_t R_k}{P_k}\right) \\
& + \left[\frac{d\,(d+1)}{24}Q_{\frac{d}{2}-1}\left(\frac{\partial_t R_k + \eta R_k}{P_k - 2\Lambda}\right) - \frac{d}{6}Q_{\frac{d}{2}-1}\left(\frac{\partial_t R_k}{P_k}\right)\right. \\
& \left. - \frac{d\,(d-1)}{4}Q_{\frac{d}{2}}\left(\frac{\partial_t R_k + \eta R_k}{(P_k - 2\Lambda)^2}\right) - Q_{\frac{d}{2}}\left(\frac{\partial_t R_k}{P_k^2}\right)\right] R + O(R^2)\Bigg\} .
\end{aligned} \tag{3.54}$$

This derivation highlights two noteworthy facts. The first is that the negative sign of the kinetic term for the trace part of h is immaterial. With the chosen form for the cutoff, any prefactor multiplying the kinetic operator in the inverse propagator cancels out between the two factors in the r.h.s. of the FRGE. The second fact, which I will exploit in the following, is that the singularity occurring in the kinetic operator for the trace part in $d = 2$ (see equation (3.41)) is actually made harmless by a hidden factor $d - 2$ occurring in $\mathbf{U}$. So, the final result (3.54) is perfectly well defined also in two dimensions.

The computation can be done also by not choosing any particular background. This is achieved by not decomposing the field $h_{\mu\nu}$ into tracefree and trace parts and using the form of the inverse propagtor given in the first line of (3.41). Then, the modified inverse propagator for gravitons is

$$\mathbf{\Gamma}_k^{(2)} + \mathbf{R}_k = Z\mathbf{K}\left(P_k(-\Box) - 2\Lambda\mathbf{1} + \mathbf{W}\right) . \tag{3.55}$$

On a general background it is impossible to invert $\mathbf{\Gamma}_k^{(2)} + \mathbf{R}_k$ exactly, but remembering that $\mathbf{W}$ is linear in curvature it can be expanded to first order:

$$\begin{aligned}\left(\mathbf{\Gamma}_k^{(2)} + \mathbf{R}_k\right)^{-1} &= \frac{\mathbf{K}^{-1}}{Z} \cdot \frac{1}{P_k - 2\Lambda}\left[\mathbf{1} - \frac{1}{P_k - 2\Lambda}\mathbf{W} + O(R^2)\right] \\ \left(\Gamma^{(2)}_{C\bar{C}}{}^{\mu}{}_{\nu} + R_k^{(gh)\mu}{}_{\nu}\right)^{-1} &= \frac{1}{P_k}\left[\delta^{\mu}_{\nu} + \frac{1}{P_k}R^{\mu}{}_{\nu} + O(R^2)\right] . \end{aligned} \tag{3.56}$$

Then the FRGE becomes, up to terms of higher order in curvature,

$$\partial_t \Gamma_k = \frac{1}{2}\mathrm{Tr}\frac{\partial_t R_k + \eta R_k}{P_k - 2\Lambda}\left[\mathbf{1} - \frac{1}{P_k - 2\Lambda}\mathbf{W}\right] - \mathrm{Tr}\frac{\partial_t R_k}{P_k}\left[\delta^{\mu}_{\nu} + \frac{1}{P_k}R^{\mu}{}_{\nu}\right] .$$

From here, using (A.10) one arrives again at (3.54). This alternative derivation explicitly highlights the background independence of the results.

One can now extract the beta functions. The first line of (3.54) gives the beta function of $2Z\Lambda$, while the other two lines give the beta function of $-Z$. Note the appearance of the beta function of Z in the η terms on the r.h.s. In a perturbative one–loop calculation such terms would be absent; they are a result of the "renormalization group improvement" implicit in the ERGE. The beta functions can be written in the form

$$\begin{aligned}\partial_t\left(\frac{2\Lambda}{16\pi G}\right) &= \frac{k^d}{16\pi}(A_1 + A_2\eta) \\ -\partial_t\left(\frac{1}{16\pi G}\right) &= \frac{k^{d-2}}{16\pi}(B_1 + B_2\eta) ,\end{aligned} \tag{3.57}$$

where A_1, A_2, B_1 and B_2 are dimensionless functions of Λ, k and of d which, by dimensional analysis, can also be written as functions of $\tilde{\Lambda} = \Lambda k^{-2}$ and d. One can solve these equations for $\partial_t\tilde{\Lambda}$ and $\partial_t\tilde{G}$, obtaining

$$\begin{aligned}\partial_t\tilde{\Lambda} &= -2\tilde{\Lambda} + \tilde{G}\,\frac{A_1 + 2B_1\tilde{\Lambda} + \tilde{G}(A_1B_2 - A_2B_1)}{2(1 + B_2\tilde{G})} , \\ \partial_t\tilde{G} &= (d-2)\tilde{G} + \frac{B_1\tilde{G}^2}{1 + B_2\tilde{G}} .\end{aligned} \tag{3.58}$$

The corresponding perturbative one loop beta functions are obtained by neglecting the η terms in (3.57), i.e. setting $A_2 = B_2 = 0$, and expanding A_1 and B_1 in $\tilde{\Lambda}$. The leading term is

$$\begin{aligned}\partial_t\tilde{\Lambda} &= -2\tilde{\Lambda} + \frac{1}{2}A_1(0)\tilde{G} + B_1(0)\tilde{G}\tilde{\Lambda} , \\ \partial_t\tilde{G} &= (d-2)\tilde{G} + B_1(0)\tilde{G}^2 ,\end{aligned} \tag{3.59}$$

where A_1 and B_1 are evaluated at $\tilde{\Lambda} = 0$. We will refer to it as the "perturbative Einstein–Hilbert flow".

The explicit form of the coefficients appearing in (3.57), with the optimized cutoff [43], is

$$\begin{aligned}
A_1 &= \frac{16\,\pi\,(d-3+8\,\tilde{\Lambda})}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})\,(1-2\,\tilde{\Lambda})} \\
A_2 &= \frac{16\pi\,(d+1)}{(4\pi)^{\frac{d}{2}}(d+2)\,\Gamma(\frac{d}{2})\,(1-2\,\tilde{\Lambda})} \\
B_1 &= \frac{-4\pi(-d^3+15d^2-12d+48+(2d^3-14d^2-192)\tilde{\Lambda}+(16d^2+192)\tilde{\Lambda}^2)}{3(4\pi)^{\frac{d}{2}}\,d\,\Gamma(\frac{d}{2})\,(1-2\,\tilde{\Lambda})^2} \\
B_2 &= \frac{4\pi\,(d^2-9\,d+14-2\,(d+1)\,(d+2)\,\tilde{\Lambda})}{3\,(4\pi)^{\frac{d}{2}}(d+2)\,\Gamma(\frac{d}{2})\,(1-2\,\tilde{\Lambda})^2}\,.
\end{aligned}$$

A similar form of the beta functions had been given in [28] in another gauge. For the sake of clarity I write here the beta functions in four dimensions:

$$\begin{aligned}
\beta_{\tilde{\Lambda}} &= -2\tilde{\Lambda}+\frac{\tilde{G}}{6\pi}\frac{3-4\tilde{\Lambda}-12\tilde{\Lambda}^2-56\tilde{\Lambda}^3+\frac{107-20\tilde{\Lambda}}{12\pi}\tilde{G}}{(1-2\tilde{\Lambda})^2-\frac{1+10\tilde{\Lambda}}{12\pi}\tilde{G}} \\
\beta_{\tilde{G}} &= 2\tilde{G}-\frac{\tilde{G}^2}{3\pi}\frac{11-18\tilde{\Lambda}+28\tilde{\Lambda}^2}{(1-2\tilde{\Lambda})^2-\frac{1+10\tilde{\Lambda}}{12\pi}\tilde{G}}\,. \qquad (3.60)
\end{aligned}$$

Note the nontrivial denominators, which in a series expansion could be seen as re-summations of infinitely many terms of perturbation theory. They are the result of the "RG improvement" in the FRGE.

3.3.3. Cutoff of type Ib

This type of cutoff was introduced in [25]. The fluctuation $h_{\mu\nu}$ and the ghosts are decomposed into their different spin components according to

$$h_{\mu\nu}=h^T_{\mu\nu}+\nabla_\mu\xi_\nu+\nabla_\nu\xi_\mu+\nabla_\mu\nabla_\nu\sigma-\frac{1}{d}g_{\mu\nu}\Box\sigma+\frac{1}{d}g_{\mu\nu}h. \qquad (3.61)$$

and

$$C^\mu=c^{T\mu}+\nabla^\mu c\ \ ,\bar{C}_\mu=\bar{c}^T_\mu+\nabla_\mu\bar{c}\,, \qquad (3.62)$$

where $h^T_{\mu\nu}$ is transverse and traceless, ξ is a transverse vector, σ and h are scalars, c^T and $\bar{c}^T$ are transverse vectors, and c and $\bar{c}$ are scalars. These fields are subject to the following differential constraints:

$$h^{T\mu}_\mu=0\,;\qquad \nabla^\nu h^T_{\mu\nu}=0\,;\qquad \nabla^\nu\xi_\nu=0\,;\nabla^\mu\bar{c}^T_\mu=0\,;\nabla_\mu c^{T\mu}=0\,.$$

Using this decomposition can be advantageous in some cases because it can lead to a partial diagonalization of the kinetic operator and it allows an exact inversion. This is the case for example when the background is a maximally symmetric metric. In this section we will therefore assume that the background is a sphere; this is enough to extract exactly and unambiguously the beta functions of the cosmological constant and Newton's constant. Then the FRGE (3.34) can be written down for arbitrary gauge α and β. We refer to [25] for more details of the calculation. In the gauge $\alpha = 1$, $\beta = d/2 - 1$ (de-Donder gauge) and without making any approximation, the inverse propagators of the individual components are

$$\begin{aligned}
\Gamma^{(2)}_{h^T_{\mu\nu}h^T_{\alpha\beta}} &= \frac{Z}{2}\left[-\Box + \frac{d^2-3d+4}{d(d-1)}R - 2\Lambda\right]\delta^{\mu\nu,\alpha\beta} \\
\Gamma^{(2)}_{\xi_\mu\xi_\nu} &= Z\left(-\Box - \frac{R}{d}\right)\left[-\Box + \frac{d-3}{d}R - 2\Lambda\right]g^{\mu\nu} \\
\Gamma^{(2)}_{hh} &= -Z\frac{d-2}{4d}\left[-\Box + \frac{d-4}{d}R - 2\Lambda\right] \\
\Gamma^{(2)}_{\sigma\sigma} &= Z\frac{d-1}{2d}(-\Box)\left(-\Box - \frac{R}{d-1}\right)\left[-\Box + \frac{d-4}{d}R - 2\Lambda\right] \\
\Gamma^{(2)}_{\bar{c}^T_\mu c^T_\nu} &= \left[\Box + \frac{R}{d}\right]g^{\mu\nu} \\
\Gamma^{(2)}_{\bar{c}c} &= -\Box\left[\Box + \frac{2}{d}R\right]
\end{aligned} \tag{3.63}$$

The change of variables (3.61) and (3.62) leads to Jacobian determinants involving the operators

$$J_V = -\Box - \frac{R}{d},\ J_S = -\Box\left(-\Box - \frac{R}{d-1}\right),\ J_c = -\Box$$

for the vector, scalar and ghost parts. The calculations of the Jacobians arising due to the field decomposition is obtained along the method described in [58, 59]. For the case of gravity when the field is decomposed as in eq. (3.61) the calculation of Jacobian is described in detail in [27, 64]. The inverse propagators (3.63) contain four derivative terms. In [25, 27] this was avoided by making the field redefinitions

$$\xi_\mu \ \rightarrow\ \sqrt{-\Box - \frac{R}{d}}\,\xi_\mu,\ \sigma \rightarrow \sqrt{-\Box}\sqrt{-\Box - \frac{R}{d-1}}\sigma. \tag{3.64}$$

At the same time, such redefinitions also eliminate the Jacobians. These field redefinitions work well for truncations containing up to two powers of curvature, but cause poles for higher truncations as the heat kernel expansion will involve derivatives of the trace arguments. Therefore while describing the flow equation for higher-derivative gravity I will not perform the field redefinitions, but treat the Jacobians instead as further contribution to the FRGE by exponentiating them, introducing appropriate auxiliary fields and a cutoff on these variables. Here I will describe the result of performing the field redefinitions.

For each of the Hessians for the various spin component of the graviton, it is noted that it has the following structure

$$\Gamma^{(2)}_k = aZ[-\Box + bR + c\Lambda]\,, \tag{3.65}$$

where a could be a constant tensor or scalar. The cutoff $\mathcal{R}_k$ is constructed for the Hessian of each of the spin component of graviton and for ghosts fields. It is constructed in the following way. First the modified inverse propagator is obtained for each of the field component by replacing $-\Box$ by $P_k(-\Box)$, where $P_k(-\Box) = -\Box + R_k(-\Box)$. Subtracting the Hessian from modified inverse propagator for each field gives the cutoff $\mathcal{R}_k$, having the form,

$$\mathcal{R}_k = a\, Z R_k(-\Box)\ . \tag{3.66}$$

The FRGE is

$$\begin{aligned}
\partial_t \Gamma_k \;=\;& \frac{1}{2}\mathrm{Tr}_{(2)} \frac{\partial_t R_k + \eta R_k}{P_k - 2\Lambda + \frac{d^2-3d+4}{d(d-1)}R} + \frac{1}{2}\mathrm{Tr}'_{(1)} \frac{\partial_t R_k + \eta R_k}{P_k - 2\Lambda + \frac{d-3}{d}R} \\
&+ \frac{1}{2}\mathrm{Tr}_{(0)} \frac{\partial_t R_k + \eta R_k}{P_k - 2\Lambda + \frac{d-4}{d}R} + \frac{1}{2}\mathrm{Tr}''_{(0)} \frac{\partial_t R_k + \eta R_k}{P_k - 2\Lambda + \frac{d-4}{d}R} \\
&- \mathrm{Tr}_{(1)} \frac{\partial_t R_k}{P_k - \frac{R}{d}} - \mathrm{Tr}'_{(0)} \frac{\partial_t R_k}{P_k - \frac{2R}{d}}\ .
\end{aligned} \tag{3.67}$$

The first term comes from the spin–2, transverse traceless components, the second from the spin–1 transverse vector, the third and fourth from the scalars h and σ. The last two contributions come from the transverse and longitudinal components of the ghosts. A prime or a double prime indicate that the first or the first and second eigenvalues have to be omitted from the trace. The reason for this is explained in Appendix A.2.

Expanding the denominators to first order in R, but keeping the exact dependence on Λ as in the case of a type Ia cutoff, and using the formula (A.10), one obtains

$$\begin{aligned}
\partial_t \Gamma_k \;=\;& \frac{1}{(4\pi)^{d/2}} \int dx\, \sqrt{g} \Bigg\{ \frac{d(d+1)}{4} Q_{\frac{d}{2}}\left(\frac{\partial_t R_k + \eta R_k}{P_k - 2\Lambda}\right) - d Q_{\frac{d}{2}}\left(\frac{\partial_t R_k}{P_k}\right) \\
&+ R\left[-\frac{d^4 - 2d^3 - d^2 - 4d + 2}{4d(d-1)} Q_{\frac{d}{2}}\left(\frac{\partial_t R_k + \eta R_k}{(P_k - 2\Lambda)^2}\right) - \frac{d+1}{d} Q_{\frac{d}{2}}\left(\frac{\partial_t R_k}{P_k^2}\right) \right. \\
&\left. + \frac{d^4 - 13d^2 - 24d + 12}{24d(d-1)} Q_{\frac{d}{2}-1}\left(\frac{\partial_t R_k + \eta R_k}{P_k - 2\Lambda}\right) - \frac{d^2 - 6}{6d} Q_{\frac{d}{2}-1}\left(\frac{\partial_t R_k}{P_k}\right) \right] + O(R^2) \Bigg\}.
\end{aligned} \tag{3.68}$$

In principle in two dimensions one has to subtract the contributions of some excluded modes. However, using the results in Appendix A.2, the contributions of these isolated modes turn out to cancel. Thus, the FRGE is continuous in the dimension also at $d = 2$.

In order to be able to perform the calculation in closed form we choose the optimized cutoff $R_k(z) = (k^2 - z)\theta(k^2 - z)$ [43]. The beta functions have again the form (3.58); the coefficients

A_1 and A_2 are the same as for the type Ia cutoff but now the coefficients B_1 and B_2 are

$$\begin{aligned} B_1 &= 4\pi\Big(d(d-1)(d^3-15d^2-36)+24-2(d^5-8d^4-5d^3-72d^2-36d+96)\tilde\Lambda \\ &\quad -16(d-1)(d^3+6d+12)\tilde\Lambda^2\Big)\Big/3(4\pi)^{\frac{d}{2}}d^2(d-1)\Gamma\left(\frac{d}{2}\right)(1-2\tilde\Lambda)^2 \\ B_2 &= 4\pi\frac{d(d^4-10d^3+11d^2-38d+12)-2(d+2)(d^4-13\,d^2-24d+12)\,\tilde\Lambda}{3(4\pi)^{\frac{d}{2}}\,(2+d)(d-1)d^2\,\Gamma(\frac{d}{2})(1-2\tilde\Lambda)^2} \end{aligned}$$

In four dimensions, the t-derivative of $\tilde Z$ and $\tilde Z\tilde\Lambda$ is given by,

$$\begin{aligned} \partial_t\tilde Z &= -2\tilde Z+\frac{373-654\tilde\Lambda+600\tilde\Lambda^2}{1152\pi^2(1-2\tilde\Lambda)^2}+\frac{\partial_t\tilde Z}{\tilde Z}\frac{29-9\tilde\Lambda}{1152\pi^2(1-2\tilde\Lambda)^2} \\ \partial_t(\tilde Z\tilde\Lambda) &= -4\tilde Z\tilde\Lambda+\frac{1+3\tilde\Lambda}{12\pi^2(1-2\tilde\Lambda)}+\frac{\partial_t\tilde Z}{\tilde Z}\frac{5}{192\pi^2(1-2\tilde\Lambda)}\,. \end{aligned} \tag{3.69}$$

From this the beta functions of $\tilde G$ and $\tilde\Lambda$ is extracted and is given by,

$$\begin{aligned} \beta_{\tilde\Lambda} &= -2\tilde\Lambda+\frac{1}{24\pi}\frac{(12-33\tilde\Lambda+20\tilde\Lambda^2-200\tilde\Lambda^3)\tilde G+\frac{467-572\tilde\Lambda}{12\pi}\tilde G^2}{(1-2\tilde\Lambda)^2-\frac{29-9\tilde\Lambda}{72\pi}\tilde G} \\ \beta_{\tilde G} &= 2\tilde G-\frac{1}{24\pi}\frac{(105-212\tilde\Lambda+200\tilde\Lambda^2)\tilde G^2}{(1-2\tilde\Lambda)^2-\frac{29-9\tilde\Lambda}{72\pi}\tilde G}\,. \end{aligned} \tag{3.70}$$

3.3.4. Cutoff of type II

Let us define the following operators acting on gravitons and on ghosts:

$$\Delta_2 = -\Box+\mathbf{W} \tag{3.71}$$

$$\Delta_{(gh)} = -\Box-\mathrm{Ricci}\,. \tag{3.72}$$

The traces of the $\mathbf{b}_2$–coefficients of the heat–kernel expansion for these operators are

$$\begin{aligned} \mathrm{tr}\mathbf{b}_2(\Delta_2) &= \mathrm{tr}\left(\frac{R}{6}\mathbf{1}-\mathbf{W}\right)=\frac{d(7-5d)}{12}R \\ \mathrm{tr}\mathbf{b}_2(\Delta_{gh}) &= \mathrm{tr}\left(\frac{R}{6}\mathbf{1}+\mathrm{Ricci}\right)=\frac{d+6}{6}R\,. \end{aligned}$$

The type II cutoff is defined by the choice

$$\begin{aligned} \mathbf{R}_k &= Z\mathbf{K}R_k(\Delta_2) \\ R_k^{(gh)\mu}{}_\nu &= \delta^\mu{}_\nu R_k(\Delta_{(gh)})\,, \end{aligned}$$

which results in

$$\begin{aligned}\mathbf{\Gamma}_k^{(2)} + \mathbf{R}_k &= Z\mathbf{K}\left(P_k(\Delta_2) - 2\Lambda\right) \\ \Gamma_{C\bar{C}}^{(2)} + R_k^{(gh)} &= P_k(\Delta_{(gh)})\end{aligned} \tag{3.73}$$

and

$$\frac{d\mathbf{R}_k}{dt} = Z\mathbf{K}\left(\partial_t R_k(\Delta_2) + \eta R_k(\Delta_2)\right) \ .$$

Collecting all terms and evaluating the traces leads to

$$\begin{aligned}\frac{d\Gamma_k}{dt} &= \frac{1}{2}\mathrm{Tr}\frac{\partial_t R_k(\Delta_2) + \eta R_k(\Delta_2)}{P_k(\Delta_2) - 2\Lambda} - \mathrm{Tr}\frac{\partial_t R_k(\Delta_{(gh)})}{P_k(\Delta_{(gh)})} \\ &= \frac{1}{(4\pi)^{d/2}}\int dx\,\sqrt{g}\left\{\frac{d(d+1)}{4}Q_{\frac{d}{2}}\left(\frac{\partial_t R_k + \eta R_k}{P_k - 2\Lambda}\right) - d\,Q_{\frac{d}{2}}\left(\frac{\partial_t R_k}{P_k}\right)\right. \\ &\left. + \left[\frac{d(7-5d)}{24}\,Q_{\frac{d}{2}-1}\left(\frac{\partial_t R_k + \eta R_k}{P_k - 2\Lambda}\right) - \frac{d+6}{6}Q_{\frac{d}{2}-1}\left(\frac{\partial_t R_k}{P_k}\right)\right] R + O(R^2)\right\} \ .\end{aligned} \tag{3.74}$$

The beta functions are again of the form (3.58), and the coefficients A_1 and A_2 are the same as in the case of the cutoffs of type I. The coefficients B_1 and B_2 are now

$$\begin{aligned}B_1 &= -\frac{4\pi(5d^2 - 3d + 24 - 8(d+6)\tilde{\Lambda})}{3(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})(1 - 2\,\tilde{\Lambda})} \\ B_2 &= -\frac{4\pi(5d-7)}{3(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})(1 - 2\tilde{\Lambda})}\end{aligned}$$

In four dimensions, the beta functions are

$$\begin{aligned}\beta_{\tilde{\Lambda}} &= -2\tilde{\Lambda} + \frac{1}{6\pi}\frac{(3 - 28\tilde{\Lambda} + 84\tilde{\Lambda}^2 - 80\tilde{\Lambda}^3)\tilde{G} + \frac{191 - 512\tilde{\Lambda}}{12\pi}\tilde{G}^2}{(1 - 2\tilde{\Lambda})(1 - 2\tilde{\Lambda} - \frac{13}{12\pi}\tilde{G})} \\ \beta_{\tilde{G}} &= 2\tilde{G} - \frac{1}{3\pi}\frac{(23 - 20\tilde{\Lambda})\tilde{G}^2}{(1 - 2\tilde{\Lambda}) - \frac{13}{12\pi}\tilde{G}} \ .\end{aligned} \tag{3.75}$$

3.3.5. Cutoff of type III

Finally we discuss the spectrally adjusted, or type III cutoff. This consists of defining the cutoff function as a function of the whole inverse propagator $\Gamma_k^{(2)}$, only stripped of the overall wave function renormalization constants. In the case of the graviton, $\mathbf{\Gamma}_k^{(2)} = Z\mathbf{K}(\Delta_2 - 2\Lambda\mathbf{1})$ while for the ghosts $\Gamma_{C\bar{C}}^{(2)} = \Delta_{gh}$, where Δ_2 and Δ_{gh} were defined in (3.71). Type III cutoff is defined by the choice

$$\mathbf{R}_k = Z\mathbf{K}R_k(\Delta_2 - 2\Lambda) \tag{3.76}$$

for gravitons, while for ghosts it is the same as in the case of type II cutoff. Since the operator in the graviton cutoff now contains the coupling Λ, the derivative of the graviton cutoff now involves an additional term:

$$\frac{d\mathbf{R}_k}{dt} = Z\mathbf{K}\left(\partial_t R_k(\Delta_2 - 2\Lambda) + \eta R_k(\Delta_2 - 2\Lambda) - 2R'_k(\Delta_2 - 2\Lambda)\partial_t\Lambda\right) \tag{3.77}$$

where R'_k denotes the partial derivative of $R_k(z)$ with respect to z. Note that the use of the chain rule in the last term is only legitimate if the t-derivative of the operator appearing as the argument of R_k commutes with the operator itself. This is the case for the operator $\Delta_2 - 2\Lambda$, since its t-derivative is proportional to the identity. The modified inverse propagator is then simply

$$\mathbf{\Gamma}_k^{(2)} + \mathbf{R}_k = Z\mathbf{K}P_k(\Delta_2 - 2\Lambda)$$

for gravitons, while for ghosts it is again given by equation (3.73). Collecting,

$$\frac{d\Gamma_k}{dt} = \frac{1}{2}\mathrm{Tr}\frac{\partial_t R_k(\Delta_2 - 2\Lambda) + \eta R_k(\Delta_2 - 2\Lambda) - 2R'_k(\Delta_2 - 2\Lambda)\partial_t\Lambda}{P_k(\Delta_2 - 2\Lambda)} - \mathrm{Tr}\frac{\partial_t R_k(\Delta_{(gh)})}{P_k(\Delta_{(gh)})} . \tag{3.78}$$

The traces over the ghosts are exactly as in the case of a cutoff of type II. As in previous cases, one should now proceed to evaluate the trace over the tensors using equation (A.10) and the heat kernel coefficients of the operator $\Delta_2 - 2\Lambda$. However, the situation is now more complicated because the heat kernel coefficients $B_{2k}(\Delta_2 - 2\Lambda)$ contain terms proportional to Λ^k and $\Lambda^{k-1}R$, all of which contribute to the beta functions of $2\Lambda Z$ and $-Z$. This is in contrast to the calculations with cutoffs of type I and II, where only the first two heat kernel coefficients contributed to the beta functions of $2\Lambda Z$ and $-Z$. In order to re-sum all these contributions, one can proceed as follows. We define the function $W(z) = \frac{\partial_t R_k(z) + \eta R_k(z) - 2R'_k(z)\partial_t\Lambda}{P_k(z)}$ and the function $\bar{W}(z) = W(z - 2\Lambda)$. It is shown explicitly in the end of Appendix A.1 (equation (A.33) and following) that $\mathrm{Tr}W = \mathrm{Tr}\bar{W}$. Then, the terms without R and the terms linear in R (which give the beta functions of $2\Lambda Z$ and $-Z$ respectively) correspond to the first two lines in (A.34). In this way we obtain

$$\begin{aligned}\frac{d\Gamma_k}{dt} &= \frac{1}{(4\pi)^{d/2}}\int dx\,\sqrt{g}\left\{\frac{d(d+1)}{4}\sum_{i=0}^{\infty}\frac{(2\Lambda)^i}{i!}Q_{\frac{d}{2}-i}\left(\frac{\partial_t R_k + \eta R_k - 2\partial_t\Lambda R'_k}{P_k}\right) - dQ_{\frac{d}{2}}\left(\frac{\partial_t R_k}{P_k}\right)\right.\\ &\left.+\frac{d(7-5d)}{24}R\sum_{i=0}^{\infty}\frac{(2\Lambda)^i}{i!}Q_{\frac{d}{2}-1-i}\left(\frac{\partial_t R_k + \eta R_k - 2\partial_t\Lambda R'_k}{P_k}\right) - \frac{d+6}{6}Q_{\frac{d}{2}-1}\left(\frac{\partial_t R_k}{P_k}\right)R\right\} .\end{aligned} \tag{3.79}$$

The remarkable property of the optimized cutoff is that in even dimensions the sums in those expressions contain only a finite number of terms; in odd dimensions the sum is infinite but can still be evaluated analytically. Using the results (A.22,A.23,A.24,A.29, A.30,A.31,A.32) the first sum in (3.79) gives

$$\frac{1}{(4\pi)^{d/2}}\frac{d+1}{2}\frac{(k^2+2\Lambda)^{d/2}}{\Gamma(d/2)}\left(2 + \frac{\eta}{\frac{d}{2}+1}\frac{k^2+2\Lambda}{k^2} + 2\frac{\partial_t\Lambda}{k^2}\right)\int dx\,\sqrt{g} \tag{3.80}$$

whereas the second sum gives

$$\frac{1}{(4\pi)^{d/2}}\frac{d(7-5d)}{24}\frac{(k^2+2\Lambda)^{\frac{d-2}{2}}}{\Gamma(d/2)}\left(2+\frac{\eta}{d/2}\frac{k^2+2\Lambda}{k^2}+2\frac{\partial_t\Lambda}{k^2}\right)\int dx\,\sqrt{g}R \tag{3.81}$$

This resummation can actually be done also with other cutoffs. An alternative derivation of these formulae, based on the proper time form of the ERGE is given in Appendix A.3.

The beta functions cannot be written in the form (3.58) anymore, because of the presence of the derivatives of Λ on the right hand side of the ERGE. Instead of (3.57) we have

$$\begin{aligned}
\frac{d}{dt}\left(\frac{2\Lambda}{16\pi G}\right) &= \frac{k^d}{16\pi}(A_1+A_2\eta+A_3\partial_t\tilde\Lambda)\,,\\
-\frac{d}{dt}\left(\frac{1}{16\pi G}\right) &= \frac{k^{d-2}}{16\pi}(B_1+B_2\eta+B_3\partial_t\tilde\Lambda)\,,
\end{aligned} \tag{3.82}$$

where

$$\begin{aligned}
A_1 &= \frac{16\pi(-4+(d+1)(1+2\tilde\Lambda)^{\frac{d}{2}+1})}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}\\
A_2 &= \frac{16\pi(d+1)(1+2\tilde\Lambda)^{\frac{d}{2}+1}}{(4\pi)^{\frac{d}{2}}(d+2)\Gamma(\frac{d}{2})}\\
A_3 &= \frac{16\pi(d+1)(1+2\tilde\Lambda)^{\frac{d}{2}}}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\\
B_1 &= \frac{4\pi(-4(d+6)+d(7-5d)(1+2\tilde\Lambda)^{\frac{d}{2}})}{3(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}\\
B_2 &= \frac{4\pi(7-5d)(1+2\tilde\Lambda)^{\frac{d}{2}}}{3(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}\\
B_3 &= \frac{4\pi d(7-5d)(1+2\tilde\Lambda)^{\frac{d}{2}-1}}{3(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}
\end{aligned}$$

Solving (3.82) for $d\tilde\Lambda/dt$ and $d\tilde G/dt$ gives

$$\begin{aligned}
\frac{d\tilde\Lambda}{dt} &= -2\tilde\Lambda+\frac{(A_1+2(B_1-A_3)\tilde\Lambda-4B_3\tilde\Lambda^2)\tilde G+(A_1B_2-A_2B_1+2(A_2B_3-A_3B_2)\tilde\Lambda)\tilde G^2}{2+(2B_2-A_3-2B_3\tilde\Lambda)\tilde G+(A_2B_3-A_3B_2)\tilde G^2}\\
\frac{d\tilde G}{dt} &= (d-2)\tilde G+\frac{2(B_1-2B_3\tilde\Lambda)\tilde G^2+(A_1B_3-A_3B_1)\tilde G^3}{2+(2B_2-A_3-2B_3\tilde\Lambda)\tilde G+(A_2B_3-A_3B_2)\tilde G^2}\,.
\end{aligned} \tag{3.83}$$

In four dimensions, the beta functions are

$$\begin{aligned}\beta_{\tilde\Lambda} &= -2\tilde\Lambda + \frac{1}{6\pi}\frac{(3+14\tilde\Lambda+8\tilde\Lambda^2)\tilde G + \frac{(1+2\tilde\Lambda)^2}{12\pi}(191-60\tilde\Lambda-260\tilde\Lambda^2)\tilde G^2}{1-\frac{1}{12\pi}(43+120\tilde\Lambda+68\tilde\Lambda^2)\tilde G+\frac{65}{72\pi^2}(1+2\tilde\Lambda)^4\tilde G^2}\\ \beta_{\tilde G} &= 2\tilde G - \frac{1}{3\pi}\frac{(23+26\tilde\Lambda)\tilde G^2 - \frac{51+152\tilde\Lambda+100\tilde\Lambda^2}{\pi}\tilde G^3}{1-\frac{1}{12\pi}(43+120\tilde\Lambda+68\tilde\Lambda^2)\tilde G+\frac{65}{72\pi^2}(1+2\tilde\Lambda)^4\tilde G^2}\ .\end{aligned} \tag{3.84}$$

3.3.6. Results in four dimensions

I will now consider Einstein-Hilbert gravity with cosmological constant in four dimensions.

The beta functions for $\tilde\Lambda$ and $\tilde G$ for the four cutoff types have been given in equations (3.60, 3.70, 3.75 and 3.84). All of these beta functions admit a trivial (Gaußian) FP at $\tilde\Lambda = 0$ and $\tilde G = 0$ and a nontrivial FP at positive values of $\tilde\Lambda$ and $\tilde G$. I will discuss the Gaußian FP first. As usual, the perturbative critical exponents are equal to 2 and -2, the canonical mass dimensions of Λ and G. However, the corresponding eigenvectors are not aligned with the $\tilde\Lambda$ and $\tilde G$ axes. It is instructive to trace the origin of this fact. Since it can be already clearly seen in perturbation theory, we consider the perturbative Einstein–Hilbert flow (3.59). The linearized flow is given by the stability matrix

$$M = \begin{pmatrix} \frac{\partial\beta_{\tilde\Lambda}}{\partial\tilde\Lambda} & \frac{\partial\beta_{\tilde\Lambda}}{\partial\tilde G} \\ \frac{\partial\beta_{\tilde G}}{\partial\tilde\Lambda} & \frac{\partial\beta_{\tilde G}}{\partial\tilde G}\end{pmatrix} = \begin{pmatrix} -2+B_1\tilde G+\frac{1}{2}\tilde G\frac{\partial A_1}{\partial\tilde\Lambda}+\tilde\Lambda\tilde G\frac{\partial B_1}{\partial\tilde\Lambda} & \frac{1}{2}A_1+B_1\tilde\Lambda \\ \tilde G^2\frac{\partial B_1}{\partial\tilde\Lambda} & 2+2B_1\tilde G\end{pmatrix}. \tag{3.85}$$

At the Gaussian FP this matrix becomes

$$M = \begin{pmatrix} -2 & \frac{1}{2}A_1(0) \\ 0 & 2\end{pmatrix}, \tag{3.86}$$

which has the canonical dimensions of Λ and G on the diagonal, as expected. However, the eigenvectors do not point along the Λ and G axes. At the Gaussian FP the “attractive” eigenvector is in the direction $(1,0)$ but the ‘repulsive” one is in the direction $(A_1(0)/4, 1)$. The slant is proportional to $A_1(0)$ and can therefore be seen as a direct consequence of the running of the vacuum energy. This fact has a direct physical consequence: it is not consistent to study the ultraviolet limit of gravity neglecting the cosmological constant. One can set $\tilde\Lambda = 0$ at some energy scale, but if $\tilde G \neq 0$, as soon as one moves away from that scale the renormalization group will generate a nontrivial cosmological constant. This fact persists when one considers the renormalization group improved flow.

I will now come to the nontrivial FP. We begin by making for a moment the drastic approximation of treating A_1 and B_1 as constants, independent of $\tilde\Lambda$ (this is the leading term in a series expansion in $\tilde\Lambda$). Thus we consider again the perturbative Einstein–Hilbert flow (3.59). In this

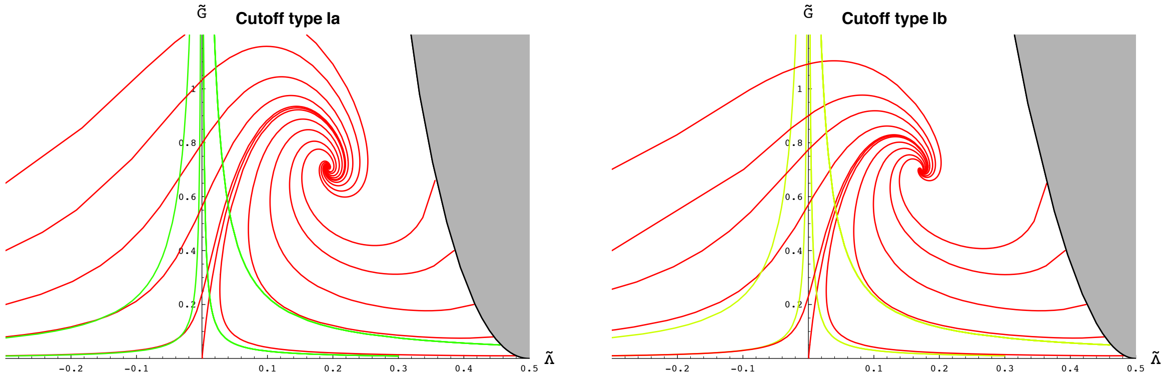

Figure 3.1.: The flow near the perturbative region with cutoffs of type Ia and Ib. The boundary of the shaded region is a singularity of the beta functions. The curves in light color are "classical" trajectories with constant $\tilde{\Lambda}\tilde{G}$.

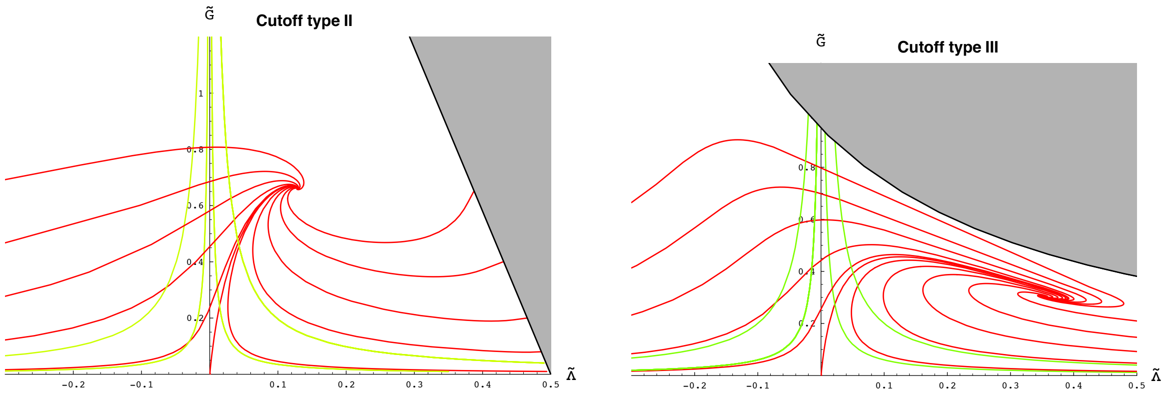

Figure 3.2.: The flow near the perturbative region with cutoffs of type II and III. The boundary of the shaded region is a singularity of the beta functions.

approximation the flow can be solved exactly:

$$\tilde{\Lambda}(t) = \frac{(2\tilde{\Lambda}_0 - \frac{1}{4}A_1\tilde{G}_0(1-e^{4t}))e^{-2t}}{2 + B_1\tilde{G}_0(1-e^{2t})} ,$$
$$\tilde{G}(t) = \frac{2\tilde{G}_0 e^{2t}}{2 + B_1\tilde{G}_0(1-e^{2t})} . \tag{3.87}$$

The FP would occur at $\tilde{\Lambda}_* = -A_1/4B_1$, $\tilde{G}_* = -2/B_1$, at which point the matrix (3.85) becomes

$$M = \begin{pmatrix} -4 & -\frac{1}{4}A_1 \\ 0 & -2 \end{pmatrix} . \tag{3.88}$$

It has real critical exponents 2 and 4, equal to the canonical dimensions of the constants $2Z\Lambda$ and $-Z$. This should not come as a surprise, since the linearized flow matrix for the couplings $g^{(0)}$ and $g^{(2)}$ is diagonal, with eigenvalues equal to their canonical dimensions, and the eigenvalues are invariant under regular coordinate transformations in the space of the couplings. So we see that a nontrivial UV–attractive FP in the $\tilde{\Lambda}$–$\tilde{G}$ plane appears already at the lowest level of perturbation theory.

All the differences between the perturbative Einstein–Hilbert flow and the exact flow are due to the dependence of the constants A_1 and B_1 on $\tilde{\Lambda}$, and in more accurate treatments to the RG improvements incorporated in the flow through the functions A_2, B_2 A_3, B_3. Such improvements are responsible for the non-polynomial form of the beta functions. In all these calculations the critical exponents at the nontrivial FP always turn out to be a complex conjugate pair, giving rise to a spiraling flow. The real part of these critical exponents is positive, corresponding to eigenvalues of the linearized flow matrix with negative real part. Therefore, the nontrivial FP is always UV-attractive in the $\tilde{\Lambda}$–$\tilde{G}$ plane. Conversely, an infinitesimal perturbation away from the FP will give rise to a renormalization group trajectory that flows towards lower energy scales away from the nontrivial FP. Among these trajectories there is a unique one that connects the nontrivial FP in the ultraviolet to the Gaußian FP in the infrared. This is called the "separatrix".

One noteworthy aspect of the flow equations in the Einstein–Hilbert truncation is the existence of a singularity of the beta functions. Looking at equations (3.60, 3.70, 3.75 and 3.84), we see that there are always choices of $\tilde{\Lambda}$ and $\tilde{G}$ for which the denominators vanish. The singularities are the boundaries of the shaded regions in figure 3.1. Of course the flow exists also beyond these singularities but those points cannot be joined continuously to the flow in the perturbative region near the Gaußian FP, which we know to be a good description of low energy gravity. When the trajectories emanating from the nontrivial FP approach these singularities, they reach it at finite values of t and the flow cannot be extended to $t \to -\infty$. The presence of these singularities can be interpreted as a failure of the Einstein–Hilbert truncation to capture all features of infrared physics and it is believed that they will be avoided by considering a more complete truncation. Let us note that for cutoffs of type I and II the singularities pass through the point $\tilde{\Lambda} = 1/2$, $\tilde{G} = 0$. Thus, there are no regular trajectories emanating from the nontrivial FP and

Scheme	$\tilde{\Lambda}_*$	$\tilde{G}_*$	$\tilde{\Lambda}_*\tilde{G}_*$	ϑ
Ia	0.1932	0.7073	0.1367	1.475±3.043i
Ia - 1 loop	0.1213	1.1718	0.1421	1.868±1.398i
Ib	0.1715	0.7012	0.1203	1.689± 2.486i
Ib - 1 loop	0.1012	1.1209	0.1134	1.903± 1.099 i
II	0.0924	0.5557	0.0513	2.425±1.270i
II - 1 loop	0.0467	0.7745	0.0362	2.310± 0.382 i
III	0.2742	0.3321	0.0910	1.752±2.069 i
III - 1 loop	0.0840	0.7484	0.0628	1.695± 0.504 i

Table 3.1.: The nontrivial fixed point for Einstein's theory in $d = 4$ with cosmological constant.

reaching the region $\tilde{\Lambda} > 1/2$. However, for type III cutoffs the shaded region is not attached to the $\tilde{\Lambda}$ axis and there are trajectories that avoid it, reaching smoothly the region $\tilde{\Lambda} > 1/2$.

In table 3.1 we collect the main features of the UV–attractive FP for the Einstein–Hilbert truncation with cosmological constant for the two different cutoff schemes described here.

3.3.7. Pure Cutoff

As anticipated, we would like to examine a different type of cutoff, not depending on any of the parameters that are present in the action [52]. The cutoff eq. (3.66) depends on the parameter Z, so to define a pure cutoff we replace Z by γk^2 where γ is an arbitrary number:

$$\mathcal{R}_k = a\gamma k^2 R_k(-\Box) \ . \tag{3.89}$$

The FRGE now reads

$$\begin{aligned}
\frac{d\Gamma_k}{dt} &= \frac{1}{2}\mathrm{Tr}_{(2)}\frac{\partial_t R_k + 2R_k}{\frac{Z}{\gamma k^2}\left(-\Box + \frac{2}{3}R - 2\Lambda\right) + R_k(-\Box)} + \frac{1}{2}\mathrm{Tr}'_{(1)}\frac{\partial_t R_k + 2R_k}{\frac{Z}{\gamma k^2}\left(-\Box + \frac{1}{4}R - 2\Lambda\right) + R_k(-\Box)} \\
&\quad + \frac{1}{2}\mathrm{Tr}_{(0)}\frac{\partial_t R_k + 2R_k}{\frac{Z}{\gamma k^2}\left(-\Box - 2\Lambda\right) + R_k(-\Box)} + \frac{1}{2}\mathrm{Tr}''_{(0)}\frac{\partial_t R_k + 2R_k}{\frac{Z}{\gamma k^2}\left(-\Box - 2\Lambda\right) + R_k(-\Box)} \\
&\quad - \mathrm{Tr}_{(1)}\frac{\partial_t R_k}{P_k - \frac{R}{4}} - \mathrm{Tr}'_{(0)}\frac{\partial_t R_k}{P_k - \frac{R}{2}} \ .
\end{aligned} \tag{3.90}$$

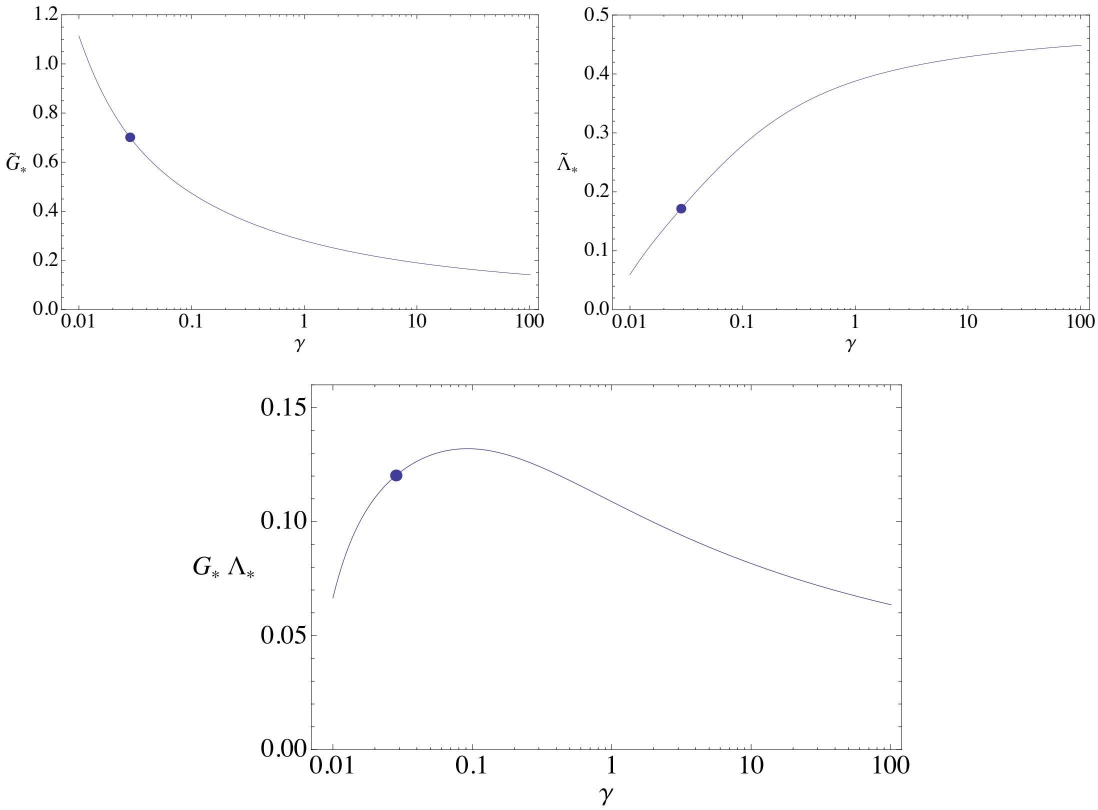

Figure 3.3.: Value of $\tilde{G}_*$ (left panel), $\tilde{\Lambda}_*$ (right panel) and $\Lambda_* G_*$ (lower panel) as functions of γ with a pure cutoff. The dot indicates the values for the type Ib cutoff.

This leads to the following beta functions

$$
\begin{aligned}
\partial_t \tilde{Z} &= -2\tilde{Z} + \frac{49\gamma(\gamma-\tilde{Z}) + (1-2\tilde{\Lambda})(25\tilde{Z}^2 - 151\tilde{Z}\gamma + 28\gamma^2)}{192\pi^2(\gamma-\tilde{Z})^2(1-2\tilde{\Lambda})} \\
&\qquad - \frac{\gamma\left[3(\gamma-\tilde{Z})^2 + \tilde{Z}(1-2\tilde{\Lambda})(101\tilde{Z}-3\gamma)\right]}{192\pi^2(\gamma-\tilde{Z})^3}\, X \\
\partial_t(\tilde{Z}\tilde{\Lambda}) &= -4\tilde{Z}\tilde{\Lambda} - \frac{9\gamma^2 + 4\tilde{Z}^2 - \gamma\tilde{Z}(23-20\tilde{\Lambda})}{32\pi^2(\gamma-\tilde{Z})^2} - \frac{5\gamma\left[\gamma^2 - 2\gamma\tilde{Z} + 4\tilde{Z}^2\tilde{\Lambda}(1-\tilde{\Lambda})\right]}{16\pi^2(\gamma-\tilde{Z})^3}\, X
\end{aligned}
\tag{3.91}
$$

where

$$
X = \ln\left(\frac{\tilde{Z}(1-2\tilde{\Lambda})}{\gamma - 2\tilde{Z}\tilde{\Lambda}}\right).
$$

The appearance of the logarithms is due to the mismatch between the coefficients of $-\Box$ and

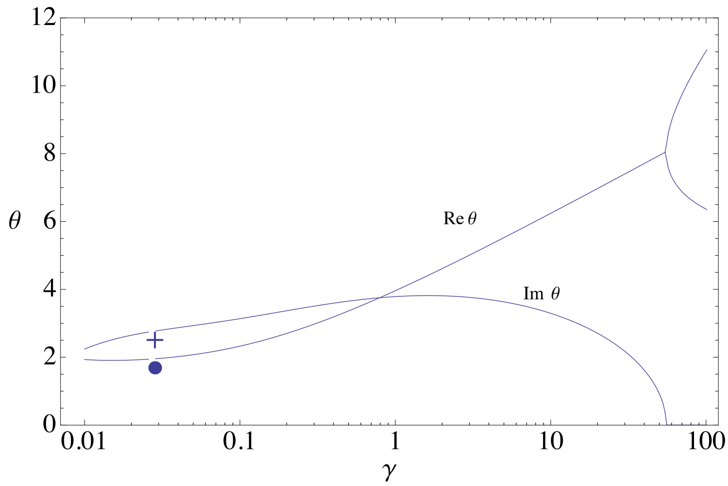

Figure 3.4.: Real and imaginary parts of the critical exponents as functions of γ. The dot and the cross indicate the real and imaginary part of the critical exponent for the type Ib cutoff.

$R_k(-\Box)$, which leaves some explicit terms with $z = -\Box$ to be integrated over. The FP now depends on the arbitrary parameter γ, which is part of the freedom in the definition of the cutoff. This reflects itself in the position of the fixed point, as shown in Fig. 3.3. We give separately the dependence of $\tilde{G}_*$, $\tilde{\Lambda}_*$ and of the dimensionless product $\Lambda_* G_*$. We see that as γ varies over four orders of magnitude, $\tilde{G}_*$, $\tilde{\Lambda}_*$ each vary by less than one order of magnitude, and $\Lambda_* G_*$ changes just by a factor smaller than 2. It had been observed before that the dimensionless product ΛG has a beta function that is gauge independent in lowest order in an expansion in $\tilde{\Lambda}$ [25]; also its value at the FP was found to be quite insensitive to the choice of gauge and cutoff. Our findings confirm this picture also for the dependence on the parameter γ. In figure 3.4 we also give the critical exponents as functions of γ. As with other cutoffs, they form a complex conjugate pair, but for large γ the imaginary part of the eigenvalue goes to zero and for $\gamma > 60$ we find two real eigenvalues. Clearly for very large or very small γ the properties of the FP are significantly affected, but there is a wide range of values for which the properties of the FP are quite stable.

We observe that the curves in Fig. 3.3 pass through the position of the fixed point in the type Ib cutoff examined previously, which is marked by a dot in the graphs. In other words, there is a value of γ for which the pure cutoff gives a FP in the same position as the type Ib cutoff. The corresponding value is precisely $\gamma = \tilde{Z}_* = 0.0284$. This is an example of the discussion presented in section 2.4.

It would seem from eq. (3.91) that the beta functions become singular when $\gamma = \tilde{Z}$ but if we put $\gamma = \tilde{Z} + \epsilon$ and expand in powers of ϵ, the coefficient of the negative powers of ϵ cancel out. Furthermore, one finds that the leading (ϵ-independent) terms in the expansion coincide with the first two terms on the r.h.s. of eq. (3.69).

This is just an example of the more general argument discussed in section 2.4, where it was showed that a pure cutoff obtained from any of the three type of cutoff schemes available, by replacing the couplings present in the cutoff by an arbitrary real dimensionless parameter times a scale dependent factor, will result in a FP which concides with the FP found in any of the three cutoff schemes if these arbitrary real parameters acquire special values which is the value of the FP of that coupling in the particular cutoff type adopted.

If for example one does the above analysis by obtaining the pure cutoff starting from the type III cutoff given in eq. (3.76), by replacing the couplings Z and Λ by $\gamma_1 k^2$ and $\gamma_2 k^2$ respectively. Then one will find that in the special case when $\gamma_1 = \tilde{Z}_*$ and $\gamma_2 = \tilde{\Lambda}_*$ (where $\tilde{Z}_*$ and $\tilde{\Lambda}_*$ are FP values of type III cutoff), the FP of the pure cutoff is same as FP obtained using the type III cutoff.

3.4. Summary

In this chapter I showed how to construct FRGE for gravity. This involved some complications and need to be fixed before one follows the steps outlined in chapter 2. Being a diffeomorphic invariant theory it involved gauge freedom in the functional integral which need to be fixed like in any other gauge theory. Apart from this the other complication was regarding the coarse graining. In a non gauge theory one could choose some positive operator like $-\partial^2$, in whose eigenfunction the field is decomposed, thereby classifying the modes as slow and fast according to the eigenvalue p^2 begin below and above the cutoff scale k. While in gauge theory the best one can do is to consider generalized positive operator like $-\Box = -\nabla_\mu \nabla^\mu$ or some other similar operator. Doing this is gauge covariant but it sacrifices the notion of Fourier coarse graining. Both these complications are settled by use of background gauge fixing technique.

Then after this, using the background field method I showed how to construct the cutoff. Fixing the gauge introduces Faddev-Popov ghost, whose contribution is also taken while constructing the FRGE. The FRGE so derived is quite complicated to use in practical computations. It depends on two metrics: classical metric g and background metric $\bar{g}$ and the ghost fields. In order to simplify, as a first step the running of the ghost action is ignored. Then after computing the Hessian for both graviton and ghost, the background metric is set equal to the classical metric. The other simplification is to then consider truncated effective actions.

In this chapter I demostrated how to do the task mentioned in the above paragraph in the simplest truncation for gravity: Einstein-Hilbert truncation. I took this opportunity to also demonstrate the scheme dependence in the results by doing the computation in different cutoff types, and comparing them in four dimensions.

Chapter 4

RG Flow of Scalar coupled to Einstein-Hilbert Gravity

In this chapter I will study the renormalization group flow in a class of scalar-tensor theories involving at most two derivatives of the fields *i.e.* a scalar field coupled to Einstein-Hilbert gravity. I will compute the running of potentials (arbitrary functions of scalar field). Using the expressions for the running of the potentials I will show that minimal coupling is self consistent, in the sense that when the scalar self couplings are switched off, their beta functions also vanish. I will then present complete, explicit beta functions in five parameter truncation of the theory in $d = 4$. I will then search for the nontrivial fixed point of the theory in dimensions greater than two. I will then study the linearized flow around the fixed points, thereby checking whether the theory satisfies the requirements of Asymptotic safety in various dimensions.

4.1. Truncation Ansatz and Motivation

Fundamental scalar fields have not yet been observed, but they play a crucial role in the standard model and in grand unified theories, as the order parameters whose VEV is used to distinguish between otherwise undifferentiated gauge interactions. Whether such scalar order parameters are elementary fields, as in the standard model, or composites, as in technicolor theories, is still an open question. Known examples of the Higgs phenomenon (superconductivity, the chiral condensate in QCD) point to the latter possibility, but even if this was the case it might still be possible to use scalar theory as an effective description (á la Landau-Ginzburg) at sufficiently low energy.

Scalar fields also play an important role in theories of gravity. Due to their simplicity they are very often used as models for matter. Also, because of the ease by which one can generate a nontrivial VEV, with an energy momentum tensor that resembles a cosmological constant, a scalar field is the most popular option as a driver of inflation. Furthermore, scalar fields easily mingle with the metric: by means of Weyl transformations it is possible to rewrite the dynamics in different ways [60], sometimes leading to new insight or to simplifications. Theories of gravity in which a scalar is present are often called scalar-tensor theories. In this chapter I will discuss the quantum properties of a class of theories of this type.

In the previous chapter I showed that pure Einstein-Hilbert gravity is Asymptotically safe. In the recent years a lot of progress has been made towards understanding the UV behavior of gravity. This included extending the Einstein-Hilbert gravity to include higher-derivative terms in the truncation. It seems that pure gravity possesses a Fixed Point (FP) with the right properties to make it asymptotically safe, or in other words nonperturbatively renormalizable [11, 25, 24, 26, 27, 38, 61, 62, 30, 31, 63, 29, 64, 65, 32, 35, 66, 67, 42, 68, 69, 70, 71, 72] (see also [46] for reviews). Let us assume for a moment that this ambitious goal can be achieved, and that pure gravity can be shown to be asymptotically safe. Still, from the point of view of phenomenology, one is not satisfied because the real world contains also dozens of matter fields that interact in other ways than gravitationally, and their presence affects also the quantum properties of the gravitational field, as is known since long [1]. Indeed, in a first investigation along these lines, it was shown in [33] that the presence of minimally coupled (*i.e.* non self interacting) matter fields shifts the position of the gravitational FP and the corresponding critical exponents. In some cases the FP ceases to exist, so it was suggested that this could be used to put bounds on the number of matter fields of each spin. More generally the asymptotic safety program requires that the fully interacting theory of gravity and matter has a FP with the right properties. Given the bewildering number of possibilities, in the search for such a theory one needs some guiding principle. One possibility that naturally suggests itself is that all matter self-interactions are asymptotically free [73]. Then, asymptotic safety requires the existence of a FP where the matter couplings approach zero in the UV, while the gravitational sector remains interacting. I will call such a FP a "Gaussian Matter FP" or GMFP. Following a time honored tradition, as a first step in this direction, scalar self interactions in four dimensions have been studied in [74, 34]. During my thesis studies I extended that work further in various ways, by studying these theories in arbitrary dimensions and in arbitrary gauge, which I will describe in this chapter.

The tool that was used in this study was the functional renormalization group equation, which was described in detail in chapter 2. There I discussed that any study involving FRGE necessarily requires making use of approximations, and if one is interested in nonperturbative studies, then one has to apply FRGE on a truncated theory space. A systematic study of truncations requires a derivative expansion of the effective action. In the case of a scalar theory the lowest order of this expansion is the local potential approximation (LPA), where one retains a standard kinetic term plus a generic potential [75, 20, 76, 77]. An exclusive example of this was shown in chapter 2, where I applied FRGE to $O(N)$-symmetric scalar theory. In the case of pure gravity with metric as the field, it involves operators that are powers of curvatures and derivatives thereof. The lowest truncation involves two derivative of metric along with a constant term, thereby constituting the Einstein-Hilbert truncation. A systematic study involving operators up to terms with four derivatives has been done in [62, 32, 35, 70], and for a limited class of operators (namely powers of the scalar curvature) up to sixteen derivatives of the metric has been done in [30, 31, 64].

In the case of scalar tensor theories of gravity, one will have to expand both in derivatives of the metric and of the scalar field. The lowest order in the expansion involving two derivatives of the fields (metric and scalar field) including a generic potential for the scalar can be written as,

$$\Gamma_k[g,\phi] = \int d^dx\,\sqrt{g}\left(V(\phi^2) - F(\phi^2)R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right) + S_{GF} + S_{gh}. \tag{4.1}$$

This can also be seen as a generalization of the LPA, where one also includes terms with two derivatives of the metric. Here the functions V and F are generic functions of ϕ^2 and for simplicity I call them potentials. Notice that the scalar kinetic term is fixed. Taking into account its running would yield information on the scalar anomalous dimension. We will not consider this effect here.

4.2. The beta functions

In this section I will obtain beta functionals for the functions V and F defined in (4.1). To achieve this, we use Wetterich's functional renormalization group equation (FRGE) [20]

$$\partial_t\Gamma_k = \frac{1}{2}\mathrm{STr}\left[\left(\frac{\delta^2\Gamma_k}{\delta\Phi\,\delta\Phi} + \mathcal{R}_k\right)^{-1}\partial_t\mathcal{R}_k\right], \tag{4.2}$$

where Φ are all the fields present in the theory and STr is the generalized functional trace including a minus sign for fermionic variables and a factor 2 for complex variables, and $\mathcal{R}_k$ is a suitable tensorial cutoff.

4.2.1. Second variations

In order to evaluate the r.h.s. of (4.2) we start from the second functional derivatives of the functional (4.1). These can be obtained by expanding the action to second order in the quantum fields around classical backgrounds: $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ and $\phi = \bar{\phi} + \delta\phi$, where $\bar{\phi}$ is constant. The gauge fixing action is given by

$$S_{GF} = \frac{1}{2\alpha}\int d^dx\,\sqrt{\bar{g}}\,F(\phi^2)\bar{g}^{\mu\nu}\,\chi_\mu\,\chi_\nu\ , \tag{4.3}$$

$$\chi^\mu = \left(\bar{\nabla}_\nu h^{\nu\mu} - \frac{\beta+1}{d}\bar{\nabla}^\mu h\right)\ .$$

and S_{gh} is the corresponding ghost action given by

$$S_{GH} = -\int d^dx\,\sqrt{\bar{g}}\,\bar{C}^\mu\,\left[\delta^\rho_\mu\,\bar{\Box} + \left(1 - \frac{2(1+\beta)}{d}\right)\bar{\nabla}_\mu\bar{\nabla}^\rho + \bar{R}^\rho_\mu\right]C_\rho\ . \tag{4.4}$$

These terms are already quadratic in the quantum fields. The second variation of eq. (4.1) is,

$$\begin{aligned}
\Gamma_k^{(2)} &= \frac{1}{2}\int d^dx\sqrt{g}\Bigg[\left(\frac{1}{4}h^2-\frac{1}{2}h_{\mu\nu}h^{\mu\nu}\right)\left(V(\phi^2)-F(\phi^2)\,R\right)+F(\phi^2)\,h\,h^{\mu\nu}\,R_{\mu\nu}+\frac{1}{2}F(\phi^2)\,h\,\Box\,h \\
&+ F(\phi^2)\,h^{\mu\nu}\nabla_\mu\nabla_\rho h^\rho{}_\nu - F(\phi^2)\,h_\alpha{}^\nu\,h^{\mu\alpha}\,R_{\mu\nu}-F(\phi^2)\,h^{\mu\nu}\,R_{\rho\mu\sigma\nu}\,h^{\rho\sigma}-\frac{1}{2}F(\phi^2)h^{\mu\nu}\,\Box\,h_{\mu\nu} \\
&- F(\phi^2)\,h\nabla_\mu\nabla_\nu h^{\mu\nu}\Bigg]+\int d^dx\sqrt{g}\Big[h\cdot\phi\left(V'(\phi^2)-F'(\phi^2)\,R\right)\,\delta\phi+2\,\phi F'(\phi^2)\,h^{\mu\nu}\,R_{\mu\nu}\,\delta\phi \\
&- 2\,\phi\,F'(\phi^2)\,\delta\phi\,(\nabla_\mu\nabla_\nu h^{\mu\nu}-\Box h)\Big]+\frac{1}{2}\int d^dx\sqrt{g}\,\delta\phi\Big[-\Box+2\,V'(\phi^2)+4\,\phi^2\,V''(\phi^2) \\
&- R\,\left(2\,F'(\phi^2)+4\,\phi^2\,F''(\phi^2)\right)\Big]+S_{GF}+S_{gh}\ . \qquad (4.5)
\end{aligned}$$

Since we will never have to deal with the original metric $g_{\mu\nu}$ and scalar field ϕ, in order to simplify the notation, in the preceding formula and everywhere else from now on we will remove the bars from the backgrounds. As explained in detail in [24], the functional that obeys the FRGE (4.2) depends separately on the background field $\bar{g}_{\mu\nu}$ and on a "classical field" $(g_{\rm cl})_{\mu\nu}=\bar{g}_{\mu\nu}+(h_{\rm cl})_{\mu\nu}$, where $(h_{\rm cl})_{\mu\nu}$ is Legendre conjugate to the sources that couple linearly to $h_{\mu\nu}$. The same applies to the scalar field. Here, like in most of the literature on the subject and as has been argued in the chapter 3, I will restrict myself to study the effective average action in the case when $(g_{\rm cl})_{\mu\nu}=\bar{g}_{\mu\nu}$ and $\phi_{\rm cl}=\bar{\phi}$. From now on the notation $g_{\mu\nu}$ and ϕ will be used to denote equivalently the "classical fields" or the background fields. For a discussion of the effective average action of pure gravity in the more general case when $(g_{\rm cl})_{\mu\nu}\neq\bar{g}_{\mu\nu}$ we refer to [68].

4.2.2. Decomposition

In order to partially diagonalize the kinetic operator, we use the decomposition of $h_{\mu\nu}$ into irreducible components

$$h_{\mu\nu}=h^T_{\mu\nu}+\nabla_\mu\xi_\nu+\nabla_\nu\xi_\mu+\nabla_\mu\nabla_\nu\sigma-\frac{1}{d}g_{\mu\nu}\Box\sigma+\frac{1}{d}g_{\mu\nu}h\ , \qquad (4.6)$$

where $h^T_{\mu\nu}$ is the (spin 2) transverse and traceless tensor, ξ_μ is the (spin 1) transverse vector component, σ and h are (spin 0) scalars. In some cases this decomposition allows an exact inversion of the propagator. This happens for example in the case of maximally symmetric background metric. Thus with that in mind we will work on a d-dimensional sphere. This change of variables in the functional integral gives rise to Jacobian determinants, which however can be absorbed by further field re-definitions $\hat{\xi}_\mu=\sqrt{-\Box-\frac{R}{d}}\,\xi_\mu$ and $\hat{\sigma}=\sqrt{-\Box}\sqrt{-\Box-\frac{R}{d-1}}\,\sigma$ [25, 27, 30, 31]. Then the inverse propagators for various components of the field are easily read from the second variation of the effective action. Thus for the spin-2 component $h^T_{\mu\nu}$ we get the following inverse propagator:

$$\frac{1}{2}F(\phi^2)\left(-\Box+\frac{d^2-3d+4}{d(d-1)}R\right)-\frac{1}{2}V(\phi^2)\,. \tag{4.7}$$

For the spin-1 component $\hat{\xi}$ we have the following inverse propagator:

$$\frac{1}{\alpha}F(\phi^2)\left(-\Box-\frac{R}{d}\right)-V(\phi^2)+\frac{d-2}{d}\,F(\phi^2)\,R\,. \tag{4.8}$$

The two spin-0 components of the metric, $\hat{\sigma}$ and h, mix with the fluctuation of ϕ resulting in an inverse propagator given by a symmetric 3×3 matrix S, with the following entries:

$$\begin{aligned}
S_{\sigma\sigma} &= \left(1-\frac{1}{d}\right)\left[\left\{\frac{1}{2}-\left(1-\frac{1}{\alpha}\right)\left(1-\frac{1}{d}\right)\right\}F(\phi^2)\,(-\Box)-\frac{1}{2}V(\phi^2)\right.\\
&\quad\left.+\frac{d-4}{2d}\,F(\phi^2)\,R+\left(1-\frac{1}{\alpha}\right)F(\phi^2)\,\frac{R}{d}\right]\;,\\
S_{\sigma h} &= S_{h\sigma}=\frac{1}{2}\left(1-\frac{1}{d}\right)\left[\frac{2}{d}\left(\frac{\beta}{\alpha}+1\right)-1\right]F(\phi^2)\sqrt{-\Box\left(-\Box-\frac{R}{d-1}\right)}\;,\\
S_{\sigma\phi} &= S_{\phi\sigma}=-2\,\phi\left(1-\frac{1}{d}\right)F'(\phi^2)\sqrt{-\Box\left(-\Box-\frac{R}{d-1}\right)}\,,\\
S_{hh} &= \frac{d-2}{4d}\left[\left\{-\left(1-\frac{1}{d}\right)+\frac{2\beta^2}{\alpha d(d-2)}\right\}2\,F(\phi^2)\,(-\Box)+V(\phi^2)-\frac{d-4}{d}F(\phi^2)\,R\right]\;,\\
S_{h\phi} &= S_{\phi h}=\left[-2\left(1-\frac{1}{d}\right)\phi\,F'(\phi^2)\,(-\Box)+\phi\,V'(\phi^2)-\left(1-\frac{2}{d}\right)\phi\,F'(\phi^2)\,R\right]\;,\\
S_{\phi\phi} &= -\Box+2V'(\phi^2)+4\phi^2V''(\phi^2)-(2F'(\phi^2)+4\phi^2F''(\phi^2))R\,.
\end{aligned} \tag{4.9}$$

In order to diagonalize the kinetic operator occuring in the ghost action eq. (4.4), we perform a similar decomposition of the ghost field into transverse and longitudinal parts in the following manner:

$$\bar{C}^\mu=\bar{C}^{\mu T}+\nabla^\mu\bar{C}\;,\qquad C_\mu=C^T_\mu+\nabla_\mu C\,. \tag{4.10}$$

where $\bar{C}^{\mu T}$ and C^T_μ satisfy the following constraints,

$$\nabla_\mu\bar{C}^{\mu T}=0\;,\qquad \nabla^\mu C^T_\mu=0. \tag{4.11}$$

Again this decomposition would give rise to a non trivial Jacobian in the path-integral, which is cancelled by the further redefinition $\hat{C}=\sqrt{-\Box}C$. For spin-1 component of the ghost, the inverse propagator is

$$-\Box-\frac{R}{d}\;, \tag{4.12}$$

while for spin-0 component we have the following inverse propagator

$$\left(2 - \frac{2(1+\beta)}{d}\right)(-\Box) - \frac{2R}{d} \ , \tag{4.13}$$

Now we have to specify the cutoff $\mathcal{R}_k$ occurring in FRGE eq. (4.2). We define $\mathcal{R}_k$ by the rule that $\Gamma_k^{(2)} + \mathcal{R}_k$ has the same form as $\Gamma_k^{(2)}$ except for the replacement of $-\Box$ by $P_k(-\Box)$, where $P_k(z) = z + R_k(z)$. $R_k(z)$ is a profile function which tends to k^2 for $z \to 0$ and it approaches zero rapidly for $z > k^2$. The quantity $\Gamma_k^{(2)} + \mathcal{R}_k$ is the "modified inverse propagator". This procedure applies both to the bosonic degrees of freedom and to the ghosts. The cutoff $\mathcal{R}_k$ occurring in the FRGE depends on k not only through the profile function $R_k(z)$, but also through k dependent couplings present in the function $F(\phi^2)$ and $F'(\phi^2)$. Thus the derivative $k\frac{d}{dk} = \frac{d}{dt}$ acts not only on the profile function $R_k(z)$, but also on the k-dependent couplings present in $F(\phi^2)$ and $F'(\phi^2)$. When this is neglected one recovers the one loop results. The presence of the beta functions on the RHS of the FRGE, produces a coupled system of linear equations, which has to be solved algebraically to yield the beta functions.

4.2.3. The β-functionals in $d = 4$

To read off the beta functions we have to compare the r.h.s. of the FRGE with the t-derivative of eq. (4.1), namely

$$\partial_t \Gamma[g,\phi] = \int d^d x \, \sqrt{g} \left(\partial_t V(\phi^2) - \partial_t F(\phi^2)\, R \right) \ . \tag{4.14}$$

(the gauge fixing and the kinetic term are not allowed to run in our approximations). Since the background R and ϕ are constant, the space-time integral produces just a volume factor, which eventually cancels with the same factor appearing on the RHS of the FRGE. Thus the running of V and F can be calculated using,

$$\partial_t V(\phi^2) = \frac{1}{Vol}\, \partial_t \Gamma_k \Big|_{R=0} \ , \qquad \partial_t F(\phi^2) = -\frac{1}{Vol} \frac{\partial(\partial_t \Gamma_k)}{\partial R}\Big|_{R=0} \ , \tag{4.15}$$

where Vol is the space-time volume. In order to exhibit the explicit form of these beta functionals we go to $d = 4$, where $Vol = \frac{384\pi^2}{R^2}$, and set $\alpha = 0$ and $\beta = 1$ (De-Donder gauge). Furthermore, we choose the optimized cutoff $R_k(z) = (k^2 - z)\theta(k^2 - z)$ [43], which allows to perform the integrations occurring in FRGE trace in closed form (see appendix A). From the FRGE we then get

$$\begin{aligned} \partial_t V \;&=\; \frac{k^4}{192\pi^2} \Bigg\{ 6 + \frac{30\, V}{\Psi} + \frac{6(k^2\,\Psi + 24\,\phi^2\, k^2\, F'\,\Psi' + k^2\, F\Sigma_1)}{\Delta} + \left(\frac{4}{F} + \frac{5\, k^2}{\Psi} + \frac{k^2\,\Sigma_1}{\Delta} \right) \partial_t F \\ &+ \frac{24\,\phi^2\, k^2\,\Psi'}{\Delta}\, \partial_t F' \Bigg\} \ , \end{aligned} \tag{4.16}$$

$$
\begin{aligned}
\partial_t F &= \frac{k^2}{2304\pi^2}\Bigg\{150 + \frac{120\,k^2\,F\,(3\,k^2 F - V)}{\Psi^2} - \frac{24}{\Delta}\left(24\,\phi^2\,k^2\,F'\,\Psi' + k^2\,\Psi + k^2\,F\Sigma_1\right) \\
&- \frac{36}{\Delta^2}\Big[-4\,\phi^2\,(6\,k^4\,F'^2 + \Psi'^2)\,\Delta + 4\,\phi^2\,\Psi\,\Psi'\,(7\,k^2\,F' - V')\,(\Sigma_1 - k^2) \\
&+ 4\,\phi^2\Sigma_1\,(7\,k^2\,F' - V')\,(2\,\Psi\,V' - V\,\Psi') + 2\,k^4\,\Psi^2\,\Sigma_2 + 48\,k^4\,F'\,\phi^2\,\Psi\,\Psi'\,\Sigma_2 \\
&- 24\,k^4 F\,\phi^2\,\Psi'^2\,\Sigma_2\Big] - \frac{\partial_t F}{F}\Bigg[30 - \frac{10\,k^2 F\,(7\,\Psi + 4\,V)}{\Psi^2} + \frac{6}{\Delta^2}\Big(k^2\,F\,\Sigma_1\,\Delta + 4\,\phi^2\,V'\,\Psi'\,\Delta \\
&- 24\,k^4\,F\,\phi^2\,\Psi'^2\,\Sigma_2 - 4\,\phi^2\,k^2\,F\,\Psi'\,\Sigma_1(7\,k^2\,F' - V')\Big)\Bigg] + \partial_t F'\,\frac{24\,k^2\,\phi^2}{\Delta^2}\Big[(k^2\,F' + 5\,V')\Delta \\
&- 12\,k^2\,\Psi\,\Psi'\,\Sigma_2 - 2\,(7\,k^2 F' - V')\,\Psi\,\Sigma_1\Big]\Bigg\}
\end{aligned}
\tag{4.17}
$$

where we have defined the shorthands:

$$
\Psi = k^2\,F - V\,; \quad \Sigma_1 = k^2 + 2\,V' + 4\,\phi^2\,V''\,; \quad \Sigma_2 = 2\,F' + 4\,\phi^2\,F''\,; \quad \Delta = \left(12\,\phi^2\,\Psi'^2 + \Psi\,\Sigma_1\right).
$$

Let us define the dimensionless fields $\tilde{\phi} = k^{\frac{2-d}{2}}\,\phi$, $\tilde{R} = k^{-2}\,R$ and the dimensionless functions $\tilde{V}(\tilde{\phi}^2) = k^{-d}\,V(\phi^2)$ and $\tilde{F}(\tilde{\phi}^2) = k^{2-d}\,F(\phi^2)$. The beta functionals of the dimensionless and dimensionful functions are related as follows:

$$
(\partial_t\tilde{V})[\tilde{\phi}^2] = -d\,\tilde{V}(\tilde{\phi}^2) + (d-2)\,\tilde{\phi}^2\,\tilde{V}'(\tilde{\phi}^2) + k^{-d}\,(\partial_t V)[\phi^2]\,, \tag{4.18}
$$

$$
(\partial_t\tilde{F})[\tilde{\phi}^2] = -(d-2)\,\tilde{F}(\tilde{\phi}^2) + (d-2)\,\tilde{\phi}^2\,\tilde{F}'(\tilde{\phi}^2) + k^{-(d-2)}\,(\partial_t F)[\phi^2]\,. \tag{4.19}
$$

Some comments are in order. From the expressions of $\partial_t V$ and $\partial_t F$ given in eq. (4.16) and (4.17) respectively, we note that where ever there is occurrence of ϕ^2, it occurs in combinations like $\phi^2 V'V'$, $\phi^2 V'F'$, $\phi^2 F'F'$, $\phi^2 V'\partial_t F'$, $\phi^2 F'\partial_t F'$, $\phi^2 V''$ and $\phi^2 F''$. Occurrence of such combinations are crucial, as they help us (as is demonstrated in [37]) in proving that minimal coupling is self consistent. Because of the occurrence of $\partial_t\tilde{F}'$ in the r.h.s. of (4.16), the system of equations cannot be solved algebraically for $\partial_t\tilde{F}$. It may be possible to solve it as a differential equation, but here we shall not pursue this. Rather, we observe that if $\tilde{F}$ and $\tilde{V}$ are assumed to be finite polynomials in $\tilde{\phi}^2$ of the form

$$
\tilde{V}(\tilde{\phi}^2) = \sum_{n=0}^{a}\tilde{\lambda}_{2n}\,\tilde{\phi}^{2n}\,; \qquad \tilde{F}(\tilde{\phi}^2) = \sum_{n=0}^{b}\tilde{\xi}_{2n}\,\tilde{\phi}^{2n}\,. \tag{4.20}
$$

with finite a and b, then $\partial_t\tilde{F}'$ is also a finite polynomial in the beta functions and it becomes possible to solve for the beta functions algebraically. As an explicit example, I present below these equations in the de-Donder gauge ($\alpha = 0$ and $\beta = 1$) in $d = 4$ with five couplings truncation ($a = 2$, $b = 1$).

4.2.4. Explicit beta functions

In the previous subsection I have presented equations which in principle determine the beta functionals for V and F in $d = 4$ and in the gauge $\alpha = 0$ and $\beta = 1$. At one loop the beta functionals can be immediately read off from there, but if one wants to get the "improved" beta functionals (meaning that no approximation is made beyond the truncation), then the system is too complicated to be solved. It can be solved if I assume that V and F are finite polynomials. I give here a set of ordinary nonlinear differential equations that determine the beta functions in a five coupling truncation, including λ_0, ξ_0, λ_2, ξ_2, λ_4. One can use algebraic methods to extract the beta functions of the couplings with the help of mathematical packages. This exercise will also enable us to compare with familiar one loop results. These beta functions had been written previously in [34], but since there we had left the cutoff generic, it was not possible to compute the integrals over momenta, which are contained in the expressions Q_2 and Q_1. [1] Here the integrals have already been performed, using an optimized cutoff [43], so the beta functions are in closed form and completely explicit. We use the notation $\eta = \frac{\partial_t \xi_0}{\xi_0} = \frac{\partial_t \tilde{\xi}_0}{\tilde{\xi}_0} + 2$.

$$\partial_t \tilde{\lambda}_0 = -4\tilde{\lambda}_0 + \frac{1}{32\pi^2}\left[2 + \frac{1}{1+2\tilde{\lambda}_2} + \frac{6\tilde{\lambda}_0}{\tilde{\xi}_0 - \tilde{\lambda}_0}\right] + \frac{\eta}{96\pi^2}\frac{5\tilde{\xi}_0 - 2\tilde{\lambda}_0}{\tilde{\xi}_0 - \tilde{\lambda}_0}\,, \tag{4.21}$$

$$\partial_t \tilde{\xi}_0 = -2\tilde{\xi}_0 + \frac{1}{384\pi^2}\left[25 - \frac{4}{1+2\tilde{\lambda}_2} - \frac{24\tilde{\xi}_2}{(1+2\tilde{\lambda}_2)^2} + \frac{8\tilde{\xi}_0(7\tilde{\xi}_0 - 2\tilde{\lambda}_0)}{(\tilde{\xi}_0 - \tilde{\lambda}_0)^2}\right] + \frac{\eta}{1152\pi^2}\frac{17\tilde{\xi}_0^2 + 18\tilde{\xi}_0\tilde{\lambda}_0 - 15\tilde{\lambda}_0^2}{(\tilde{\xi}_0 - \tilde{\lambda}_0)^2}\,, \tag{4.22}$$

$$\begin{aligned}\partial_t \tilde{\lambda}_2 = &-2\tilde{\lambda}_2 + \frac{1}{48\pi^2}\left[\frac{9\tilde{\lambda}_0(1+2\tilde{\xi}_2)}{2(\tilde{\xi}_0 - \tilde{\lambda}_0)^2} - \frac{9(2\tilde{\lambda}_0 - \tilde{\xi}_0)(1+2\tilde{\xi}_2)^2}{2(1+2\tilde{\lambda}_2)(\tilde{\xi}_0 - \tilde{\lambda}_0)^2} - \frac{9(1+2\tilde{\xi}_2)^2}{2(1+2\tilde{\lambda}_2)^2(\tilde{\xi}_0 - \tilde{\lambda}_0)} - \frac{18\tilde{\lambda}_4}{(1+2\tilde{\lambda}_2)^2}\right] \\ &+ \frac{\eta}{96\pi^2}\left[-\frac{2\tilde{\xi}_2}{\tilde{\xi}_0} + \frac{3\tilde{\xi}_0(1+2\tilde{\xi}_2)}{2(\tilde{\xi}_0 - \tilde{\lambda}_0)^2} - \frac{3\tilde{\xi}_0(1+2\tilde{\xi}_2)^2}{2(1+2\tilde{\lambda}_2)(\tilde{\xi}_0 - \tilde{\lambda}_0)^2}\right] \\ &+ \frac{1}{96\pi^2}\frac{\partial_t \tilde{\xi}_2}{\tilde{\xi}_0}\left[2 - \frac{3\tilde{\xi}_0}{\tilde{\xi}_0 - \tilde{\lambda}_0} + \frac{6\tilde{\xi}_0(1+2\tilde{\xi}_2)}{(1+2\tilde{\lambda}_2)(\tilde{\xi}_0 - \tilde{\lambda}_0)}\right]\,,\end{aligned} \tag{4.23}$$

[1] Note that the notation used in [34] is opposite to the one used here: parameters with a tilde are dimensionfull, those without tilde are dimensionless. The beta functions written in [34] contain a number of transcription errors, which however do not affect the subsequent results.

$$\partial_t \tilde\xi_2 = \frac{1}{576\pi^2}\Bigg[\frac{1+2\tilde\lambda_2}{\tilde\xi_0-\tilde\lambda_0}\left(9+\frac{39\tilde\xi_0}{\tilde\xi_0-\tilde\lambda_0}+\frac{60\tilde\xi_0^2}{(\tilde\xi_0-\tilde\lambda_0)^2}\right)+\frac{3(3+32\tilde\xi_2)}{\tilde\xi_0-\tilde\lambda_0}-\frac{6\tilde\xi_0(11+2\tilde\xi_2)}{(\tilde\xi_0-\tilde\lambda_0)^2}$$
$$-\frac{60\tilde\xi_0^2(1+2\tilde\xi_2)}{(\tilde\xi_0-\tilde\lambda_0)^3}+\frac{216\tilde\xi_2(1+2\tilde\xi_2)^2}{(1+2\tilde\lambda_2)^3(\tilde\xi_0-\tilde\lambda_0)}+\frac{9[\tilde\lambda_0(5-2\tilde\xi_2)-2\tilde\xi_0(1+2\tilde\xi_2)](1+2\tilde\xi_2)}{(1+2\tilde\lambda_2)(\tilde\xi_0-\tilde\lambda_0)^2}$$
$$+\frac{27(1+2\tilde\xi_2)(1-10\tilde\xi_2-16\tilde\xi_2^2)}{(1+2\tilde\lambda_2)^2(\tilde\xi_0-\tilde\lambda_0)}+\frac{108\tilde\xi_0\tilde\xi_2(1+2\tilde\xi_2)^2}{(1+2\tilde\lambda_2)^2(\tilde\xi_0-\tilde\lambda_0)^2}+\frac{72\tilde\lambda_4}{(1+2\tilde\lambda_2)^2}\frac{1+12\tilde\xi_2+2\tilde\lambda_2}{1+2\tilde\lambda_2}\Bigg]$$
$$+\frac{\eta}{1152\pi^2}\Bigg[\frac{1+2\tilde\lambda_2}{\tilde\xi_0-\tilde\lambda_0}\left(3+\frac{18\tilde\xi_0}{\tilde\xi_0-\tilde\lambda_0}+\frac{20\tilde\xi_0^2}{(\tilde\xi_0-\tilde\lambda_0)^2}\right)+\frac{15\tilde\xi_2}{\tilde\xi_0}-\frac{6(1+\tilde\xi_2)}{\tilde\xi_0-\tilde\lambda_0}-\frac{10\tilde\xi_0(3+4\tilde\xi_2)}{(\tilde\xi_0-\tilde\lambda_0)^2}$$
$$-\frac{20\tilde\xi_0^2(1+2\tilde\xi_2)}{(\tilde\xi_0-\tilde\lambda_0)^3}-\frac{3[\tilde\lambda_0-\tilde\xi_0(5-4\tilde\xi_2)](1+2\tilde\xi_2)}{(1+2\tilde\lambda_2)(\tilde\xi_0-\tilde\lambda_0)^2}+\frac{36\tilde\xi_0\tilde\xi_2(1+2\tilde\xi_2)^2}{(1+2\tilde\lambda_2)^2(\tilde\xi_0-\tilde\lambda_0)^2}\Bigg]$$
$$+\frac{1}{1152\pi^2}\frac{\partial_t\tilde\xi_2}{\tilde\xi_0}\left[-15+\frac{54\tilde\xi_0}{\tilde\xi_0-\tilde\lambda_0}+\frac{20\tilde\xi_0^2}{(\tilde\xi_0-\tilde\lambda_0)^2}-\frac{6\tilde\xi_0(7+2\tilde\xi_2)}{(1+2\tilde\lambda_2)(\tilde\xi_0-\tilde\lambda_0)}-\frac{144\tilde\xi_0\tilde\xi_2(1+2\tilde\xi_2)}{(1+2\tilde\lambda_2)^2(\tilde\xi_0-\tilde\lambda_0)}\right], \quad (4.24)$$

$$\partial_t\tilde\lambda_4 = \frac{1}{48\pi^2}\Bigg[\frac{9}{4(\tilde\xi_0-\tilde\lambda_0)^2}\Bigg(5(1+2\tilde\lambda_2)(1+4\tilde\xi_2)-(1+2\tilde\xi_2)(21+62\tilde\xi_2)+\frac{33(1+2\tilde\xi_2)^3}{1+2\tilde\lambda_2}$$
$$-\frac{(1+2\tilde\xi_2)^3(23+24\tilde\xi_2)}{(1+2\tilde\lambda_2)^2}+\frac{6(1+2\tilde\xi_2)^4}{(1+2\tilde\lambda_2)^3}\Bigg)+\frac{9\tilde\xi_0(\tilde\xi_2-\tilde\lambda_2)^2}{(\tilde\xi_0-\tilde\lambda_0)^3}\left(6\frac{(1+2\tilde\xi_2)^2}{(1+2\tilde\lambda_2)^2}-10\frac{1+2\tilde\xi_2}{1+2\tilde\lambda_2}+5\right)$$
$$-\frac{72\tilde\lambda_2\tilde\lambda_4(1+2\tilde\xi_2)(1-4\tilde\lambda_2+6\tilde\xi_2)}{(\tilde\xi_0-\tilde\lambda_0)(1+2\tilde\lambda_2)^3}+\frac{9\tilde\xi_0\tilde\lambda_4}{(\tilde\xi_0-\tilde\lambda_0)^2}\left(6\frac{(1+2\tilde\xi_2)^2}{(1+2\tilde\lambda_2)^2}-8\frac{1+2\tilde\xi_2}{1+2\tilde\lambda_2}+3\right)+\frac{216\tilde\lambda_4^2}{(1+2\tilde\lambda_2)^3}\Bigg]$$
$$+\frac{\eta}{96\pi^2}\Bigg[\frac{2\tilde\xi_2^2}{\tilde\xi_0^2}+\frac{3\tilde\xi_0(\tilde\xi_2-\tilde\lambda_2)^2}{(\tilde\xi_0-\tilde\lambda_0)^3}\left(6\frac{(1+2\tilde\xi_2)^2}{(1+2\tilde\lambda_2)^2}-10\frac{1+2\tilde\xi_2}{1+2\tilde\lambda_2}+5\right)$$
$$+\frac{3\tilde\xi_0\tilde\lambda_4}{(\tilde\xi_0-\tilde\lambda_0)^2}\left(6\frac{(1+2\tilde\xi_2)^2}{(1+2\tilde\lambda_2)^2}-8\frac{1+2\tilde\xi_2}{1+2\tilde\lambda_2}+3\right)\Bigg]$$
$$+\frac{1}{96\pi^2}\frac{\partial_t\tilde\xi_2}{\tilde\xi_0}\Bigg[-\frac{2\tilde\xi_2}{\tilde\xi_0}-\frac{3\tilde\xi_0(\tilde\xi_2-\tilde\lambda_2)}{(\tilde\xi_0-\tilde\lambda_0)^2}\left(12\frac{(1+2\tilde\xi_2)^2}{(1+2\tilde\lambda_2)^2}-21\frac{1+2\tilde\xi_2}{1+2\tilde\lambda_2}+10\right)$$
$$-\frac{24\tilde\xi_0\tilde\lambda_4(1-4\tilde\lambda_2+6\tilde\xi_2)}{(1+2\tilde\lambda_2)^2(\tilde\xi_0-\tilde\lambda_0)}\Bigg]. \quad (4.25)$$

If one neglects the terms involving η and $\partial_t\tilde\xi_2$ in the r.h.s., then the remaining terms are the one loop beta functions for the couplings. One can recognize among them some familiar terms. The term containing $-18\tilde\lambda_4$ in the first line of eq. (4.23) and the term containing $216\tilde\lambda_4^2$ in the third line of eq. (4.25) are the familiar beta functions of the mass and of the coupling in ϕ^4 theory in flat space. The term containing $72\lambda_4(1+12\xi_2)$ in the third line of eq. (4.24) is also known from earlier calculations [78, 79].

Notice the ubiquitous appearance of the factors $1/(1+2\tilde{\lambda}_2)$, which represent threshold effects for the contributions of scalar loops: for $k^2 \gg \lambda_2$, $\tilde{\lambda}_2 \ll 1$ and the denominator can be approximated by 1, whereas for $k^2 \ll \lambda_2$, $\tilde{\lambda}_2 \gg 1$ and the term is suppressed. The denominators $\tilde{\xi}_0 - \tilde{\lambda}_0$ have a somewhat similar effect. When written in terms of the more familiar variables Λ and G defined in eq. (4.66), they give rise to denominators $(1-2\tilde{\Lambda})$. These can be approximated by 1 when $\Lambda \ll k^2$ but they vanish when the dimensionless cosmological constant $\tilde{\Lambda}$ tends to $1/2$, corresponding to an infrared singularity in the RG trajectories. This is well documented in the literature.

The term proportional to $9\tilde{\xi}_0\tilde{\lambda}_4$ in the third line of eq. (4.25) is the leading gravitational correction (of order $\tilde{G}$) to the running of the scalar self coupling. Note that for small $\tilde{\lambda}_0$, $\tilde{\lambda}_2$ and $\tilde{\xi}_2$, the denominator and the bracket to its right can be expanded as $1+O(\tilde{\lambda}_0)+O(\tilde{\lambda}_2)+O(\tilde{\xi}_2)$. [2] The order of magnitude and sign of this term agree with the calculations done in [74], in the gauge $\alpha = 0$. One should not expect the results to agree exactly, because this term is gauge dependent and the calculation was done in a different gauge (namely $\beta = -1$). Notice that this term is proportional to $\tilde{\lambda}_4$, thus when we set $\tilde{\lambda}_4$ to zero, the beta function for $\tilde{\lambda}_4$ does not get any contribution from gravity, in agreement with the general statement that minimal coupling is self consistent. We observe that the same phenomenon happens in the case of the Yukawa coupling [81] and of the gauge coupling [82].

4.3. The Gaussian Matter Fixed Point

4.3.1. Minimal coupling is self consistent

We assume that V and F are real analytic so that they can be Taylor expanded around $\phi^2 = 0$. A given V and F define a FP if the corresponding dimensionless potentials satisfy $\partial_t\tilde{V} = 0$ and $\partial_t\tilde{F} = 0$. Because of analyticity, this is equivalent to requiring that all the derivatives of $\partial_t\tilde{V}$ and $\partial_t\tilde{F}$ with respect to $\tilde{\phi}^2$, evaluated at $\tilde{\phi}^2 = 0$ are zero. Taking n derivatives of eq. (4.18) and eq. (4.19) with respect to $\tilde{\phi}^2$ we get

$$0 = (\partial_t\tilde{V})^{(n)}(0) = ((d-2)n-d)\,\tilde{V}^{(n)}(0) + (k^{-d}\,\partial_t V)^{(n)}(0) \; ; \tag{4.26}$$
$$0 = (\partial_t\tilde{F})^{(n)}(0) = (n-1)(d-2)\,\tilde{F}^{(n)}(0) + (k^{-(d-2)}\,\partial_t F)^{(n)}(0) \; . \tag{4.27}$$

where in the last two terms the expressions in brackets can be thought of as functions of $\tilde{\phi}^2$. We can rewrite them as

$$\frac{\partial^n}{\partial(\tilde{\phi}^2)^n}(k^{-d}\partial_t V) = k^{(d-2)n-d}\frac{\partial^n}{\partial(\phi^2)^n}(\partial_t V) \; ; \qquad \frac{\partial^n}{\partial(\tilde{\phi}^2)^n}(k^{-(d-2)}\partial_t F) = k^{(d-2)(n-1)}\frac{\partial^n}{\partial(\phi^2)^n}(\partial_t F) \; .$$

We now make the following Ansatz:

$$V = k^d\,\tilde{\lambda}_0 \; , \qquad F = k^{d-2}\,\tilde{\xi}_0 \; , \tag{4.28}$$

[2] on a related note, we also observe that the first term in $\partial_t\tilde{\lambda}_4$, which is proportional to $\tilde{G}^2$, vanishes when we set $\tilde{\lambda}_2 = 0$ and $\tilde{\xi}_2 = 0$.

where $\tilde{\lambda}_0$ and $\tilde{\xi}_0$ are numbers to be determined. This corresponds to putting $a = b = 0$ in (4.20), or in other words to setting to zero all scalar self couplings. We are assuming here that all the derivatives of V and F at $\phi^2 = 0$ vanish, so that V and F are just constants. If a FP of this type exists, we call it a Gaussian Matter Fixed Point (GMFP). In order to check that this ansatz defines a FP we need to show that eq. (4.26) and (4.27) are identically satisfied for all $n \geq 1$, while for $n = 0$ they determine the numbers $\tilde{\lambda}_0$ and $\tilde{\xi}_0$.

For $n \geq 1$ the first term on the r.h.s. of eq. (4.26) and (4.27) vanishes because of the ansatz. There remains to show that $\frac{\partial^n}{\partial(\phi^2)^n}(\partial_t V)$ and $\frac{\partial^n}{\partial(\phi^2)^n}(\partial_t F)$ are zero at $\phi^2 = 0$. In $d = 4$ one can check this explicitly by inspecting eq. (4.16) and eq. (4.17). The crucial point to observe is that in $\partial_t V$ and $\partial_t F$, whenever ϕ^2 appears explicitly, it is multiplied by some derivative of V or F. So when the derivative removes ϕ^2, what remains is zero because of the ansatz, and otherwise it is zero because there remains some positive power of ϕ^2.

In other dimensions this crucial property remains valid, because it is true either for the second variations (in the case of the transverse traceless tensor and transverse vector components) or for the matrix trace of the second variations, in the case of the scalars. Since the beta functionals are obtained by taking functional traces of these expressions, this property will go through for them as well *i.e.* for $n \geq 1$ the eq. (4.26) and (4.27) are identically satisfied. For a detailed proof see [37].

Thus in any dimension the ansatz works for all $n \geq 1$. There remains to solve the equations for the constant terms in V and F, which are given by $\tilde{\lambda}_0$ and $\tilde{\xi}_0$. We are going to do this numerically in section 4.4. In the meanwhile we assume that such a solution exists, and we study the properties of the linearized flow around it.

4.3.2. Linearized Flow around GMFP

To study the linearized flow around GMFP it will be convenient to Taylor expand V and F as follows:

$$V(\phi^2) = \sum_{n=0}^{\infty} \lambda_{2n}\, \phi^{2n} \; ; \qquad F(\phi^2) = \sum_{n=0}^{\infty} \xi_{2n}\, \phi^{2n} \, . \tag{4.29}$$

We define dimensionless couplings $\tilde{\lambda}_{2n} = k^{-d+(d-2)n}\lambda_{2n}$ and $\tilde{\xi}_{2n} = k^{-(d-2)(1-n)}\xi_{2n}$, in such a way that the dimensionless potentials can be expanded as:

$$\tilde{V}(\tilde{\phi}^2) = \sum_{n=0}^{\infty} \tilde{\lambda}_{2n}\, \tilde{\phi}^{2n} \; ; \qquad \tilde{F}(\tilde{\phi}^2) = \sum_{n=0}^{\infty} \tilde{\xi}_{2n}\, \tilde{\phi}^{2n} \quad . \tag{4.30}$$

To obtain the running of dimensionless couplings we take derivatives of eq. (4.18) and eq. (4.19) with respect to $\tilde{\phi}^2$ and use eq. (4.15)

$$\partial_t \tilde{\lambda}_{2n} = \frac{1}{n!} \left. \frac{\delta^n \partial_t \tilde{V}}{\delta(\tilde{\phi}^2)^n} \right|_{\tilde{\phi}^2=0} \quad ; \qquad \partial_t \tilde{\xi}_{2n} = \frac{1}{n!} \left. \frac{\delta^n \partial_t \tilde{F}}{\delta(\tilde{\phi}^2)^n} \right|_{\tilde{\phi}^2=0} \quad . \tag{4.31}$$

Because of the presence of t-derivative on the RHS of FRGE, we do not obtain the beta functions of dimensionless couplings directly, rather we get algebraic equations for them, solving which one get the full beta functions.

Having defined the dimensionless couplings, we now define the stability matrix to be the matrix of derivatives of the dimensionless beta functions with respect to the dimensionless couplings at the FP. By definition it is a tensor quantity in the theory space. It will be convenient to write $V_0 = V$ and $V_1 = -F$. One can then define the corresponding dimensionless potentials as $\tilde{V}_a = k^{d-2a} V_a$, where a is either 0 or 1. Then the stability matrix is given by,

$$(M_{ij})_{ab} = \left. \frac{\delta\left(\frac{1}{i!}\partial_t \tilde{V}_a^{(i)}(0)\right)}{\delta\left(\frac{1}{j!}\tilde{V}_b^{(j)}(0)\right)} \right|_{\rm FP} \quad . \tag{4.32}$$

From the above definition of the stability matrix we note that the couplings get arranged in the following order: λ_0, ξ_0, λ_2, ξ_2, λ_4, $\xi_4 \ldots$. Then the matrix M at the GMFP has the following form:

$$\begin{pmatrix} M_{00} & M_{01} & 0 & 0 & \cdots \\ 0 & M_{11} & M_{12} & 0 & \cdots \\ 0 & 0 & M_{22} & M_{23} & \cdots \\ 0 & 0 & 0 & M_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} , \tag{4.33}$$

where each entry is a 2×2 matrix of the form

$$M_{ij} = \begin{pmatrix} \frac{\partial \beta^{\tilde{\lambda}}_{(2i)}}{\partial \tilde{\lambda}_{(2j)}} & \frac{\partial \beta^{\tilde{\lambda}}_{(2i)}}{\partial \tilde{\xi}_{(2j)}} \\ \frac{\partial \beta^{\tilde{\xi}}_{(2i)}}{\partial \tilde{\lambda}_{(2j)}} & \frac{\partial \beta^{\tilde{\xi}}_{(2i)}}{\partial \tilde{\xi}_{(2j)}} \end{pmatrix} . \tag{4.34}$$

Moreover the various non zero entries of M are related to each other by the following recursion relations (in d-dimensions):

$$M_{ii} = (d-2)\, i + M_{00} \; ; \qquad M_{i,i+1} = (i+1)\,(2i+1)\, M_{01} \; , \tag{4.35}$$

where

$$M_{00} = \begin{pmatrix} -d & 0 \\ 0 & -(d-2) \end{pmatrix} + \begin{pmatrix} \delta M_{\tilde{\lambda}_0 \tilde{\lambda}_0} & \delta M_{\tilde{\lambda}_0 \tilde{\xi}_0} \\ \delta M_{\tilde{\xi}_0 \tilde{\lambda}_0} & \delta M_{\tilde{\xi}_0 \tilde{\xi}_0} \end{pmatrix} \; ; \qquad M_{01} = \begin{pmatrix} \delta M_{\tilde{\lambda}_0 \tilde{\lambda}_2} & \delta M_{\tilde{\lambda}_0 \tilde{\xi}_2} \\ \delta M_{\tilde{\xi}_0 \tilde{\lambda}_2} & \delta M_{\tilde{\xi}_0 \tilde{\xi}_2} \end{pmatrix} . \tag{4.36}$$

We can prove these facts for the one loop beta functions, *i.e.* neglecting the t-derivatives of the couplings on the r.h.s. of FRGE. Using this we note that the running of dimensionless potentials can be written as follows:

$$\partial_t \tilde{V}_a = -(d-2a)\, \tilde{V}_a + (d-2)\, \tilde{\phi}^2\, \tilde{V}_a' + \tilde{H}_a \left(\tilde{V}_b,\, \tilde{\phi}^2\, \tilde{V}_b' \tilde{V}_c',\, 2\tilde{V}_b' + 4\tilde{\phi}^2\, \tilde{V}_b'' \right) \; . \tag{4.37}$$

We have indicated that the one loop beta functional depends on $\tilde{\phi}^2$ only through the three types of combinations indicated as the arguments for $\tilde{H}_a$. This can be verified in $d = 4$ by inspection of eq. (4.16) and eq. (4.17), when one drops the terms proportional to $\partial_t F$ and $\partial_t F'$ in the r.h.s. The properties of the stability matrix given above follow by taking successive derivatives of $\partial_t \tilde{V}_a$ with respect to $\tilde{\phi}^2$ at $\tilde{\phi}^2 = 0$.

The $i = 0$ entries of eq. (4.32) can be calculated by setting $\tilde{\phi}^2 = 0$ in eq. (4.37):

$$\partial_t \tilde{V}_a(0) = -(d - 2a)\, \tilde{V}_a(0) + \tilde{H}_a(\tilde{V}_a(0),\, 2\tilde{V}_a'(0)) \ . \tag{4.38}$$

Since $\partial_t \tilde{V}_a(0)$ depends only on $\tilde{V}_a(0)$ and $\tilde{V}_a'(0)$, in eq. (4.32) for $i = 0$, only $j = 0, 1$ will be non zero. Thus M_{00} and M_{01} are given by,

$$(M_{00})_{ab} = -(d - 2a)\, \delta_{ab} + \left.\frac{\delta \tilde{H}_a(\tilde{V}_c(0),\, 2\tilde{V}_c'(0))}{\delta\, \tilde{V}_b(0)}\right|_{\text{GMFP}} ; (M_{01})_{ab} = \left.\frac{\delta \tilde{H}_a(\tilde{V}_c(0),\, 2\tilde{V}_c'(0))}{\delta\, \tilde{V}_b'(0)}\right|_{\text{GMFP}} . \tag{4.39}$$

Now we take first derivative of $\partial_t \tilde{V}_a$ with respect to $\tilde{\phi}^2$. This gives,

$$\begin{aligned}\partial_t \tilde{V}_a' &= -(d - 2a)\tilde{V}_a' + (d-2)\tilde{V}_a' + (d-2)\tilde{\phi}^2 \tilde{V}_a'' + \frac{\delta \tilde{H}_a}{\delta \tilde{V}_c} \tilde{V}_c' \\ &+ \frac{\delta \tilde{H}_a}{\delta(\tilde{\phi}^2 \tilde{V}_c' \tilde{V}_d')}(\tilde{V}_c' \tilde{V}_d' + \tilde{\phi}^2 \tilde{V}_c'' \tilde{V}_d' + \tilde{\phi}^2 \tilde{V}_d'' \tilde{V}_c') + \frac{\delta \tilde{H}_a}{\delta(2\tilde{V}_c' + 4\tilde{\phi}^2 \tilde{V}_c'')}(2\tilde{V}_c'' + 4\tilde{V}_c'' + 4\tilde{\phi}^2 \tilde{V}_c''')\end{aligned} \tag{4.40}$$

When we set $\tilde{\phi}^2 = 0$, we note from the above equation that $\partial_t \tilde{V}_a'(0)$ depends only on $\tilde{V}_a(0)$, $\tilde{V}_a'(0)$ and $\tilde{V}_a''(0)$. We use this in eq. (4.32) to calculate the $i = 1$ entries of the stability matrix. We note that $M_{1j} = 0$ for all $j \geq 3$. Now we find the remaining possible non zero entries. For $j = 0$, we note that the dependence on $\tilde{V}_a(0)$ is present only in $\left.\frac{\delta \tilde{H}_a}{\delta \tilde{V}_c}\right|_{\tilde{\phi}^2=0}$, $\left.\frac{\delta \tilde{H}_a}{\delta(\tilde{\phi}^2 \tilde{V}_c' \tilde{V}_d')}\right|_{\tilde{\phi}^2=0}$ and $\left.\frac{\delta \tilde{H}_a}{\delta(2\tilde{V}_c' + 4\tilde{\phi}^2 \tilde{V}_c'')}\right|_{\tilde{\phi}^2=0}$. But each of these terms are multiplied either with $\tilde{V}_a'$ or $\tilde{V}_a''$, so when we calculate the stability matrix, these terms will not contribute due to GMFP conditions ($\tilde{V}_a^{(i)} = 0$ for all $i \geq 1$). Thus we conclude that $M_{10} = 0$.

For $j = 1$, we take the derivative of $\partial_t \tilde{V}_a'(0)$ with respect to $\tilde{V}_b'$. Thus using the condition of GMFP and eq. (4.39) we find,

$$(M_{11})_{ab} = -(d - 2a)\delta_{ab} + (d-2)\delta_{ab} + \left.\frac{\delta \tilde{H}_a}{\delta \tilde{V}_b}\right|_{\tilde{\phi}^2=0} = (d-2)\delta_{ab} + (M_{00})_{ab} \ , \tag{4.41}$$

while for $j = 2$ we take derivatve of $\partial_t \tilde{V}_a'(0)$ with respect to $\tilde{V}_b''/2$ and use eq. (4.39). Thus we get

$$(M_{12})_{ab} = 2 \left.\frac{\delta \tilde{H}_a}{\delta(2\tilde{V}_c')}\right|_{\tilde{\phi}^2=0} \cdot 6\delta_{bc} = 6\, (M_{01})_{ab} \ . \tag{4.42}$$

Thus we see that for $i = 1$ we have,

$$M_{10} = 0 \; ; \qquad M_{11} = (d-2)\cdot 1 + M_{00} \; ; \qquad M_{12} = 2\cdot 3\, M_{01} \; ; \qquad M_{1j} = 0 \;, \forall j \geq 3 \; . \tag{4.43}$$

In order to understand the structure of the lines $i \geq 2$ we will proceed by induction. We assume that the i-th derivative has the following structure,

$$\begin{aligned}(\partial_t \tilde{V}_a)^{(i)} &= -(d-2a)\tilde{V}_a^{(i)} + (d-2)\left(\tilde{\phi}^2\,\tilde{V}_a^{(i+1)} + i\,\tilde{V}_a^{(i)}\right) + \Bigg\{\cdots + \frac{\delta \tilde{H}_a}{\delta \tilde{V}_c}\tilde{V}_c^{(i)} \\ &+ \frac{\delta \tilde{H}_a}{\delta(\tilde{\phi}^2\,\tilde{V}_c'\,\tilde{V}_d')}\left(\tilde{\phi}^2\,\tilde{V}_c'\,\tilde{V}_d'\right)^{(i)} + \frac{\delta \tilde{H}_a}{\delta(2\,\tilde{V}_c' + 4\tilde{\phi}^2\,\tilde{V}_c'')}\left(2\,\tilde{V}_c' + 4\tilde{\phi}^2\,\tilde{V}_c''\right)^{(i)}\Bigg\} \; , \end{aligned} \tag{4.44}$$

where the $(\cdots)$ denote expressions having at least two factors of derivatives of potentials, which are irrelevant when calculating the entries of stability matrix. Clearly this property is true for $i = 1$. We show that if it holds for a given value of i, then it also holds for $i + 1$. Thus we take one more derivative eq.(4.44) and we find

$$\begin{aligned}(\partial_t \tilde{V}_a)^{(i+1)} &= -(d-2a)\tilde{V}_a^{(i+1)} + (d-2)\left(\tilde{\phi}^2\,\tilde{V}_a^{(i+2)} + (i+1)\,\tilde{V}_a^{(i+1)}\right) + \Bigg\{\cdots + \frac{\delta \tilde{H}_a}{\delta \tilde{V}_c}\tilde{V}_c^{(i+1)} \\ &+ \frac{\delta^2 \tilde{H}_a}{\delta \tilde{V}_c\,\delta \tilde{V}_d}\tilde{V}_c^{(i)}\tilde{V}_d' + \frac{\delta^2 \tilde{H}_a}{\delta \tilde{V}_e\,\delta(\tilde{\phi}^2\,\tilde{V}_c'\,\tilde{V}_d')}\tilde{V}_e^{(i)}\left(\tilde{\phi}^2\,\tilde{V}_c'\,\tilde{V}_d'\right)' \\ &+ \frac{\delta^2 \tilde{H}_a}{\delta \tilde{V}_e\,\delta(2\,\tilde{V}_c' + 4\tilde{\phi}^2\,\tilde{V}_c'')}\tilde{V}_e^{(i)}\left(2\,\tilde{V}_c' + 4\tilde{\phi}^2\,\tilde{V}_c''\right)' \\ &+ \frac{\delta \tilde{H}_a}{\delta(\tilde{\phi}^2\,\tilde{V}_c'\,\tilde{V}_d')}\left(\tilde{\phi}^2\,\tilde{V}_c'\,\tilde{V}_d'\right)^{(i+1)} + \frac{\delta^2 \tilde{H}_a}{\delta \tilde{V}_e\,\delta(\tilde{\phi}^2\,\tilde{V}_c'\,\tilde{V}_d')}\tilde{V}_e'\left(\tilde{\phi}^2\,\tilde{V}_c'\,\tilde{V}_d'\right)^{(i)} \\ &+ \frac{\delta^2 \tilde{H}_a}{\delta(\tilde{\phi}^2\,\tilde{V}_e'\,\tilde{V}_f')\,\delta(\tilde{\phi}^2\,\tilde{V}_c'\,\tilde{V}_d')}(\tilde{\phi}^2\,\tilde{V}_e'\,\tilde{V}_f')'\left(\tilde{\phi}^2\,\tilde{V}_c'\,\tilde{V}_d'\right)^{(i)} \\ &+ \frac{\delta^2 \tilde{H}_a}{\delta(2\,\tilde{V}_e' + 4\tilde{\phi}^2\,\tilde{V}_e'')\,\delta(\tilde{\phi}^2\,\tilde{V}_c'\,\tilde{V}_d')}(2\,\tilde{V}_e' + 4\tilde{\phi}^2\,\tilde{V}_e'')'\left(\tilde{\phi}^2\,\tilde{V}_c'\,\tilde{V}_d'\right)^{(i)} \\ &+ \frac{\delta \tilde{H}_a}{\delta(2\,\tilde{V}_c' + 4\tilde{\phi}^2\,\tilde{V}_c'')}\left(2\,\tilde{V}_c' + 4\tilde{\phi}^2\,\tilde{V}_c''\right)^{(i+1)} + \frac{\delta^2 \tilde{H}_a}{\delta \tilde{V}_e\,\delta(2\,\tilde{V}_c' + 4\tilde{\phi}^2\,\tilde{V}_c'')}\tilde{V}_e'\left(2\,\tilde{V}_c' + 4\tilde{\phi}^2\,\tilde{V}_c''\right)^{(i)} \\ &+ \frac{\delta^2 \tilde{H}_a}{\delta(2\,\tilde{V}_e' + 4\tilde{\phi}^2\,\tilde{V}_e'')\,\delta(\tilde{\phi}^2\,\tilde{V}_c'\,\tilde{V}_d')}(2\,\tilde{V}_e' + 4\tilde{\phi}^2\,\tilde{V}_e'')^{(i)}\left(\tilde{\phi}^2\,\tilde{V}_c'\,\tilde{V}_d'\right)' \\ &+ \frac{\delta^2 \tilde{H}_a}{\delta(2\,\tilde{V}_c' + 4\tilde{\phi}^2\,\tilde{V}_c'')\,\delta(2\,\tilde{V}_d' + 4\tilde{\phi}^2\,\tilde{V}_d'')}(2\,\tilde{V}_c' + 4\tilde{\phi}^2\,\tilde{V}_c'')^{(i)}\,(2\,\tilde{V}_d' + 4\tilde{\phi}^2\,\tilde{V}_d'')'\Bigg\} \; . \end{aligned} \tag{4.45}$$

Aside from the new terms containing two factors of derivatives of the potentials, which can be neglected for our purposes, the remaining terms have the same structure as eq. (4.44). Thus by induction eq. (4.44) holds for all i.

We can now use this result to calculate the entries of the stability matrix in the i-th row. Using

$$\left(2\,\tilde{V}_a' + 4\tilde{\phi}^2\,\tilde{V}_a''\right)^{(i)} = 2\,(2\,i+1)\,\tilde{V}_a^{(i+1)} + 4\tilde{\phi}^2\,\tilde{V}_a^{(i+2)} \ , \tag{4.46}$$

we note that at $\tilde{\phi}^2 = 0$ and using the condition of GMFP for calculating the stability matrix we have,

$$\begin{aligned} M_{ij} &= 0 \ ,\forall j \leq (i-1) \ ; \qquad M_{ii} = (d-2)\,i + M_{00} \ ; \\ M_{i,i+1} &= (i+1)\,(2\,i+1)M_{01} \ ; \qquad M_{ij} = 0 \ ,\forall j \geq (i+2) \ . \end{aligned} \tag{4.47}$$

This completes the proof of our statements in the one loop approximation. It is difficult to extend this proof to the exact equation, but we see in finite truncations that the previous properties of the stability matrix remain true.

Having established the properties of stability matrix we would like to compute its eigenvalues. The good feature of the block structure of stability matrix indicated in eq. (4.33) is that the eigenvalues are given just by the diagonal blocks. Since the consecutive diagonal blocks just differ by $d-2$, the eigenvalues of the consecutive diagonal blocks of M also differ by $d-2$. This is a very strong result, because it implies that, at a GMFP, the eigenvalues of M are all determined by the eigenvalues of M_{00}. Furthermore, the off diagonal blocks of M are all determined by M_{01}, so knowing M_{00} and M_{01} one can also determine all the eigenvectors. This is useful to understand the mixing among various operators at the FP. The smallest truncation that is required to calculate both M_{00} and M_{01} is when we retain terms up to ϕ^2 in each potential.

4.4. Numerical Results

4.4.1. The GMFP in $d = 4$.

We now look for GMFP in various dimensions and calculate the critical exponents of the system, which are defined to be the opposites of the eigenvalues of M, *i.e.* $\theta_i = -\lambda_i$, where λ_i is the eigenvalue. As explained in the previous section, it is enough to calculate the eigenvalues of M_{00}. We do this task first in $d = 4$.

In $d = 4$ for De-donder gauge we get the following FP equation,

$$24\,\tilde{\lambda}_0\left(16\pi^2 + \frac{1}{\tilde{\lambda}_0 - \tilde{\xi}_0}\right) = 19, \tag{4.48}$$

$$(265 - 2304\,\pi^2\,\tilde{\xi}_0)\,\tilde{\xi}_0 + (127 + 2304\,\pi^2\,\tilde{\xi}_0)\,\tilde{\lambda}_0 = \frac{160\,\tilde{\lambda}_0^2}{\tilde{\lambda}_0 - \tilde{\xi}_0} \ . \tag{4.49}$$

On solving these, the only real solution that we get is

$$\tilde{\lambda}_0^* = 0.00862 \ ; \qquad \tilde{\xi}_0^* = 0.02375 \ . \tag{4.50}$$

We now compute the critical exponents θ of the stability matrix in this gauge. The relations given in eq. (4.35) between the various nonzero entries of the stability matrix are independent of the gauge. However the entires of M_{00} and M_{01} are gauge dependent. For $d = 4$ eq. (4.35) reduces to,

$$M_{ii} = 2\,i + M_{00} \;; \qquad M_{i,i+1} = (i+1)\,(2\,i+1)\,M_{01} \;. \tag{4.51}$$

In De-Donder gauge for $d = 4$, the entries of M_{00} are

$$\begin{aligned} M_{\tilde{\lambda}_0\tilde{\lambda}_0} =& \frac{1}{32\pi^2}\frac{1}{\Theta}\tilde{\xi}_0\Bigg[\tilde{\xi}_0^2\left\{9667 + 3456\pi^2\tilde{\xi}_0(169 + 2304\pi^2\tilde{\xi}_0)\right\} + \tilde{\lambda}_0^2\left\{3279 + 1152\pi^2\tilde{\xi}_0(275 + 6912\pi^2\tilde{\xi}_0)\right\} \\ &- 18\tilde{\lambda}_0\tilde{\xi}_0\left\{551 + 128\pi^2\tilde{\xi}_0(331 + 6912\pi^2\tilde{\xi}_0)\right\}\Bigg] \;, \end{aligned} \tag{4.52}$$

$$\begin{aligned} M_{\tilde{\lambda}_0\tilde{\xi}_0} =& \frac{1}{32\pi^2}\frac{1}{\Theta}\Bigg[-48384\tilde{\lambda}_0^4 - 443520\pi^2\tilde{\xi}_0^4 + \tilde{\lambda}_0\tilde{\xi}_0^2\left\{-9667 + 2304\pi^2\tilde{\xi}_0(161 - 3456\pi^2\tilde{\xi}_0)\right\} \\ &- 3\tilde{\lambda}_0^3\left\{1093 + 4608\pi^2\tilde{\xi}_0(1 + 576\pi^2\tilde{\xi}_0)\right\} + 18\tilde{\lambda}_0^2\tilde{\xi}_0\left\{551 + 192\pi^2\tilde{\xi}_0(-1 + 4608\pi^2\tilde{\xi}_0)\right\}\Bigg] \;, \end{aligned} \tag{4.53}$$

$$M_{\tilde{\xi}_0\tilde{\lambda}_0} = \frac{12}{\Theta}\Bigg[\tilde{\xi}_0^2\left\{-252\tilde{\lambda}_0\tilde{\xi}_0(3 + 128\pi^2\tilde{\xi}_0) + 3\tilde{\lambda}_0^2(41 + 1536\pi^2\tilde{\xi}_0) + \tilde{\xi}_0^2(593 + 27648\pi^2\tilde{\xi}_0)\right\}\Bigg] \;, \tag{4.54}$$

$$\begin{aligned} M_{\tilde{\xi}_0\tilde{\xi}_0} =& \frac{-3}{\Theta}\Bigg[-135\tilde{\lambda}_0^4 + 1309\tilde{\xi}_0^4 + 12\tilde{\lambda}_0^3\tilde{\xi}_0(145 + 1536\pi^2\tilde{\xi}_0) + 36\tilde{\lambda}_0\tilde{\xi}_0^3(77 + 3072\pi^2\tilde{\xi}_0) \\ &- 6\tilde{\lambda}_0^2\tilde{\xi}_0^2(811 + 23504\pi^2\tilde{\xi}_0)\Bigg] \;. \end{aligned} \tag{4.55}$$

While the entries of M_{01} are

$$M_{\tilde{\lambda}_0\tilde{\lambda}_2} = \frac{1}{16\pi^2}\frac{1}{\Theta}\Bigg[\tilde{\xi}_0^2(37 - 1152\pi^2\tilde{\xi}_0) + 2\tilde{\lambda}_0\tilde{\xi}_0(-5 + 1152\pi^2\tilde{\xi}_0) - \tilde{\lambda}_0^2(7 + 1152\pi^2\tilde{\xi}_0)\Bigg] \;, \tag{4.56}$$

$$M_{\tilde{\lambda}_0\tilde{\xi}_2} = -\frac{3}{4\pi^2}\frac{(2\tilde{\lambda}_0 - 5\tilde{\xi}_0)(\tilde{\lambda}_0 - \tilde{\xi}_0)}{\Theta} \;, \tag{4.57}$$

$$M_{\tilde{\xi}_0\tilde{\lambda}_2} = \frac{24\tilde{\xi}_0(\tilde{\lambda}_0 - \tilde{\xi}_0)^2}{\Theta} \;, \tag{4.58}$$

$$M_{\tilde{\xi}_0\tilde{\xi}_2} = -\frac{72\tilde{\xi}_0(\tilde{\lambda}_0 - \tilde{\xi}_0)^2}{\Theta} \;, \tag{4.59}$$

where

$$\Theta = \left[-18\tilde{\lambda}_0\tilde{\xi}_0(1+128\pi^2\tilde{\xi}_0) + 3\tilde{\lambda}_0^2(5+384\pi^2\tilde{\xi}_0) + \tilde{\xi}_0^2(-17+1152\pi^2\tilde{\xi}_0)\right]^2 \ . \tag{4.60}$$

The relations eq. (4.51) tells that the critical exponents of consecutive diagonal blocks will differ by 2. In the truncation where we keep terms till ϕ^2 in each potential, the critical exponents are,

$$2.143 \pm 2.879i \ , \qquad 0.143 \pm 2.879i \tag{4.61}$$

The critical exponents $2.143 \pm 2.879i$ correspond to eigenvalues of M_{00}, while the critical exponents $0.143 \pm 2.879i$ which are shifted by 2 correspond to the eigenvalues of M_{11}. This justifies our claim. The eigenvectors in this truncation are

$$\begin{pmatrix} 0.3557 \pm 0.3776i \\ 0.8549 \\ 0 \\ 0 \end{pmatrix} , \qquad \begin{pmatrix} (-18.059 \pm 7.310i)\times 10^{-4} \\ (-30.723 \pm 10.763i)\times 10^{-4} \\ 0.3557 \pm 0.3776i \\ 0.8549 \end{pmatrix} , \tag{4.62}$$

where the first complex conjugate pair of eigenvector correspond to critical exponents $2.143 \pm 2.879i$, while the second pair correspond to critical exponents $0.143 \pm 2.879i$.

We then looked for GMFP in other gauges. In $d = 4$ we consider various values of the gauge parameters α and β. To study the gauge dependence we considered 50 different values of α in the range 0 to 1.225 at step of 0.025, and 25 different values of β in the range -1 to 1.4 at interval of 0.1. For each combination of α and β we solved the FP equation obtained for $\tilde{\lambda}_0$ and $\tilde{\xi}_0$. In general, this produces a set of FPs. In order to choose the correct GMFP from that set, we plot all the real FPs to see which one is continuously followed in other gauge values and which ones are spurious. For example one can take any value of β, and plot all the real FPs for various values of α. Some FPs don't exits for all values of α, and are assumed to be truncation artifacts. Only one GMFP exists for all values, and is continuous. This observation of continuity in α and β is useful to write a code for selecting the right GMFP for various gauges. After calculating the GMFP we calculate the critical exponents of M_{00}. We then plot the GMFP and critical exponents against the various gauge values and generate 3D graphs. In $d = 4$ we obtain the graphs shown in Fig. (4.1). We note that the existence of the FP has been actually verified in a much larger range of values of α and β.

4.4.2. The GMFP in other Dimensions

We now look for the GMFP in other dimensions. For any $d > 2$, in De-donder gauge, the FP equation for $\tilde{\lambda}_0$ and $\tilde{\xi}_0$ is given by,

$$2d\tilde{\lambda}_0 + \frac{(3d-2)\tilde{\lambda}_0 + (d-2)(d^2+d-1)\tilde{\xi}_0}{(4\pi)^{d/2}\Gamma\left(2+\frac{d}{2}\right)(\tilde{\lambda}_0 - \tilde{\xi}_0)} = 0 \ , \tag{4.63}$$

$$\frac{A\,\tilde{\lambda}_0^2 + B\,\tilde{\lambda}_0\tilde{\xi}_0 + C\,\tilde{\xi}_0^2}{(\tilde{\lambda}_0 - \tilde{\xi}_0)} + D\,(\tilde{\lambda}_0 - \tilde{\xi}_0) = 0 \ , \tag{4.64}$$

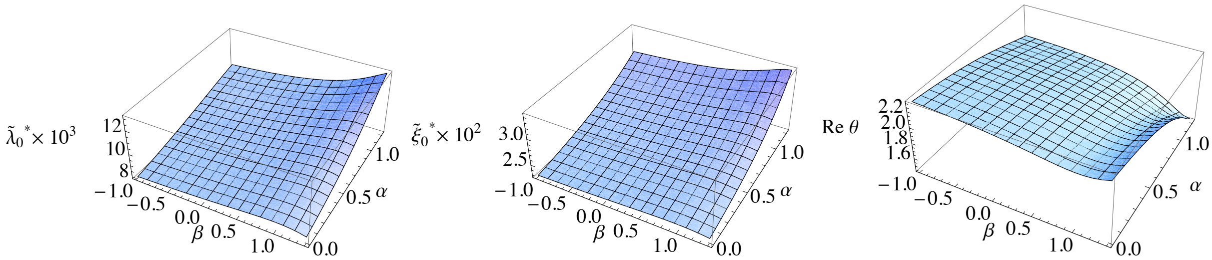

Figure 4.1.: The gauge dependence of GMFP and critical exponents in $d = 4$. For convenience of depicting the values in plot, we have rescaled $\tilde\lambda_0^*$ and $\tilde\xi_0^*$ by 1000 and 100 respectively.

where

$$\begin{aligned}
A &= -(d-1)(d^3+2d^2+36d+24) \ , \\
B &= -(d-1)(d^5-17d^3-38d^2-96d-48) \ , \\
C &= (d^6-13d^5+32d^4-104d^3+72d^2+36d+24) \ , \\
D &= 24d(d-1)(d-2)(4\pi)^{d/2}\Gamma\left(2+\frac{d}{2}\right) \ .
\end{aligned} \tag{4.65}$$

Solving these equations we find that in other dimensions, it is possible to have more than one real solution. But when we plot all the real the solutions against various d in a graph, we notice that not all solutions exist in all dimensions. Only one solution exists in all dimensions, and is continuous in d. Besides, the ones which don't exist in all dimensions, have large critical exponents and are probably unphysical. In Fig.2 we plot the position of the GMFP for $2 < d \leq 11$, both in terms of $\tilde\lambda_0$ and $\tilde\xi_0$ and of the more familiar dimensionless cosmological constant and Newton constant

$$\tilde\lambda_0 = \frac{2\tilde\Lambda}{16\pi\tilde G} \ ; \qquad \tilde\xi_0 = \frac{1}{16\pi\tilde G} \ . \tag{4.66}$$

After having found the GMFP in various dimensions, we set to calculate their critical exponents. In arbitrary dimensions, the various blocks of the stability matrix obey eq. (4.35). We plot the critical exponents of M_{00} for various dimensions. From the graph Fig. (4.3) we note that around $d = 2.8$ there is bifurcation. Below $d < 2.8$ the critical exponents are no more complex.

A summary of the properties of the GMFP in various dimensions is given in table (4.1). Notice that for all the dimensions considered, the real part of the critical exponents is greater than $d-2$ and less than $2(d-2)$. As a result, in all these cases there are exactly two pairs of complex conjugate critical exponents with positive real part, *i.e.* four relevant directions.

Finally we studied the gauge dependence in different dimensions in the same way as we did in $d = 4$, for example in $d = 6$ we obtain the graphs shown in Fig. (4.4).

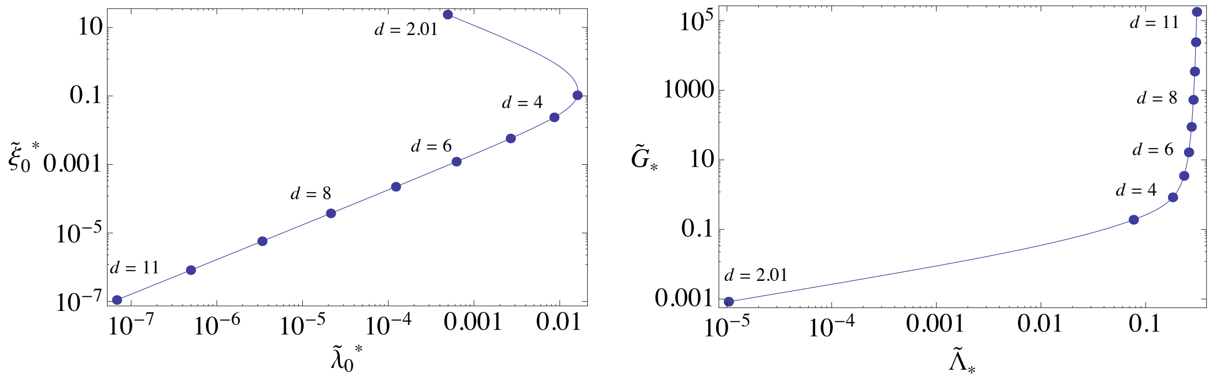

Figure 4.2.: In the first graph we plot the GMFP $\tilde{\lambda}_0^*$ and $\tilde{\xi}_0^*$ in various dimensions. In the second plot we calculate the corresponding FP values of the cosmological constant $\tilde{\Lambda}^*$ and Newton's constant $\tilde{G}^*$ in various dimensions.

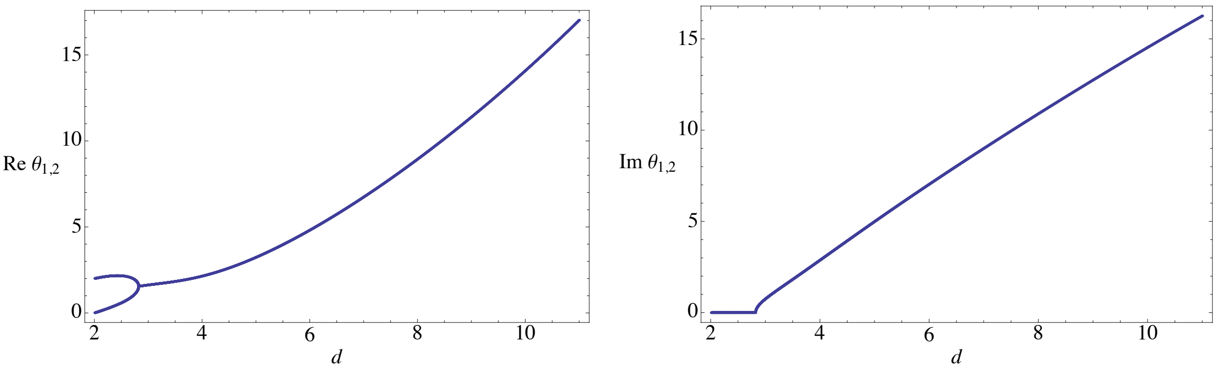

Figure 4.3.: Critical exponents at the GMFP in various dimensions. The left panel shows the real part of the critical exponents, the right panel shows the imaginary part of the critical exponents. We note that below $d = 2.8$ the critical exponents becomes real.

4.5. Other Non trivial Fixed Points

Having discussed the existence and properties of the GMFP, we can ask ourselves whether there exist other FP where the scalar field has nontrivial self-interactions. We look for (truncated) polynomial FP potentials of the form

$$\tilde{V}(\tilde{\phi}^2) = \sum_{n=0}^{a} \tilde{\lambda}_{2n}\, \tilde{\phi}^{2n}\, ; \qquad \tilde{F}(\tilde{\phi}^2) = \sum_{n=0}^{b} \tilde{\xi}_{2n}\, \tilde{\phi}^{2n}\, . \tag{4.67}$$

with finite $a \geq 1$, $b \geq 0$. Such potentials are known not to exist in a pure scalar theory in four dimensions [83], so we consider it unlikely that they exist in the presence of gravity, In fact

d	$\tilde{\lambda}_0^*$	$\tilde{\xi}_0^*$	$\tilde{\Lambda}^*$	$\tilde{G}^*$	θ_1	θ_2
2.001	4.968×10^{-5}	2.386×10^{2}	1.041×10^{-7}	8.339×10^{-5}	2.001	0.001
3	1.605×10^{-2}	1.047×10^{-1}	7.666×10^{-2}	1.900×10^{-1}	1.627 + 0.754 i	1.627 - 0.754 i
4	8.620×10^{-3}	2.375×10^{-2}	1.814×10^{-1}	8.375×10^{-1}	2.143 + 2.879 i	2.143 - 2.879 i
5	2.669×10^{-3}	5.744×10^{-3}	2.323×10^{-1}	3.463	3.236 + 4.996 i	3.236 - 4.996 i
6	6.230×10^{-4}	1.207×10^{-3}	2.581×10^{-1}	1.648×10	4.818 + 7.039 i	4.818 - 7.039 i
7	1.225×10^{-4}	2.235×10^{-4}	2.740×10^{-1}	8.900×10	6.744 + 9.004 i	6.744 - 9.004 i
8	2.133×10^{-5}	3.738×10^{-5}	2.853×10^{-1}	5.322×10^{2}	8.945 + 10.904 i	8.945 - 10.904 i
9	3.380×10^{-6}	5.747×10^{-6}	2.941×10^{-1}	3.462×10^{3}	11.396 + 12.748 i	11.396 - 12.748 i
10	4.960×10^{-7}	8.228×10^{-7}	3.014×10^{-1}	2.418×10^{4}	14.089 + 14.537 i	14.089 - 14.537 i
11	6.817×10^{-8}	1.107×10^{-7}	3.079×10^{-1}	1.797×10^{5}	17.025 + 16.261 i	17.025 - 16.261 i

Table 4.1.: Position of GMFP and critical exponents for various dimensions.

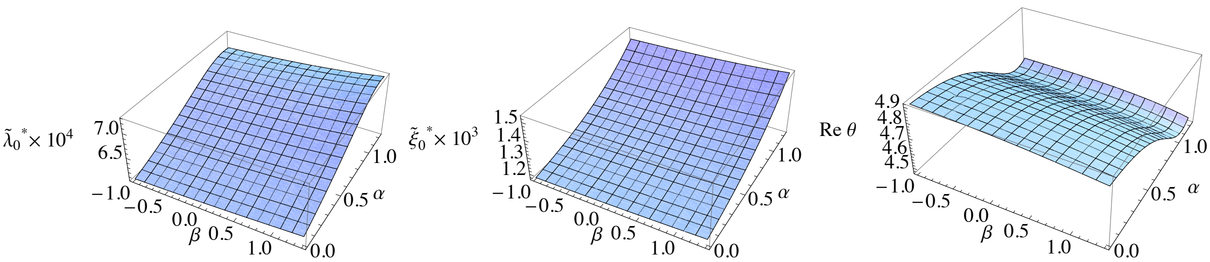

Figure 4.4.: The gauge dependence of GMFP and critical exponents in $d = 6$. For convenience of depicting the values in plot, we have rescaled $\tilde{\lambda}_0^*$ and $\tilde{\xi}_0^*$ by 10^4 and 10^3 respectively.

the outcome of our numerical searches is that no such FP's appear to exist in dimensions 4, 5 and 6. (Some FP do appear in certain truncations but not in others, so they are likely to be just truncation artifacts.)

The situation is somewhat different in three dimensions. We know that pure scalar theory in $d = 3$ has the Wilson-Fisher FP [84]. This FP can be seen in our calculations by taking the limit $\tilde{G} \to 0$ (where Newton's constant G is related to $\xi_0 = 1/16\pi G$) and $\tilde{\lambda}_0 \to 0$, in which case gravity decouples. Solving the FP equations of the scalar field in the LPA, truncated to order ϕ^4, one gets $\tilde{\lambda}_2^* = -0.0385$ and $\tilde{\lambda}_4^* = 0.3234$, with critical exponents $\theta_1 = 1.843$ and $\theta_2 = -1.176$. (These are not very good values, but we quote them here for the sake of comparison with what we find in the presence of gravity.) The FP persists when one goes to higher truncations.

One wonders whether there exists a "gravitationally dressed" Wilson-Fisher FP, with non-vanishing $\tilde{G}$, namely a FP where gravity and the scalar simultaneously have nontrivial interactions. Again in certain truncations one finds various FPs which turn out to be truncation artifacts. There seems however to exist one genuine FP: we find it in all truncations where $a \geq b$, and it has very similar properties in all truncations. To explore its properties we have looked in two directions: increasing simultaneously a and b, or keeping $b = 0$ and increasing a.

(a,b)	$\tilde\lambda_0^*$	$\tilde\lambda_2^*$	$\tilde\lambda_4^*$	$\tilde\lambda_6^*$	$\tilde\lambda_8^*$	$\tilde\xi_0^*$	$\tilde\xi_2^*$	$\tilde\xi_4^*$	$\tilde\xi_6^*$	$\tilde\xi_8^*$
(2,1)	0.0196	-0.1646	-0.1595			0.1088	-0.03108			
(3,1)	0.01994	-0.1758	-0.1958	-0.2796		0.1096	-0.03810			
(4,1)	0.02002	-0.1783	-0.2041	-0.3466	-0.5579	0.1098	-0.03969			
(2,2)	0.01894	-0.1408	-0.1241			0.1071	-0.01122	0.04297		
(3,2)	0.01971	-0.1680	-0.1848	-0.2879		0.1089	-0.03131	0.01731		
(4,2)	0.01988	-0.1735	-0.1975	-0.3544	-0.5687	0.1093	-0.03542	0.01121		
(3,3)	0.01911	-0.1469	-0.1469	-0.1935		0.1074	-0.01420	0.05017	0.1617	
(4,3)	0.01953	-0.1618	-0.1768	-0.3083	-0.6569	0.1084	-0.02571	0.03197	0.1102	
(4,4)	0.01923	-0.1512	-0.1572	-0.2496	-0.4911	0.1077	-0.01728	0.04732	0.1765	0.3868

Table 4.2.: Position of Nontrivial FP in $d = 3$ for various truncations.

(a,b)	θ_1'	θ_1''	θ_3	θ_4	θ_5	θ_6	θ_7			
(2,1)	1.648	0.592	-0.956	-3.902	-13.46					
(3,1)	1.650	0.554	-1.079	-3.776	-11.20	-29.397				
(4,1)	1.650	0.543	-1.105	-3.673	-10.02	-24.01	-49.31			
(a,b)	θ_1'	θ_1''	θ_2'	θ_2''	θ_5	θ_6	θ_7	θ_8		
(2,2)	1.649	0.656	-7.979	1.261	-0.559	-3.192				
(3,2)	1.652	0.589	-7.933	3.909	-0.835	-3.578	-27.67			
(4,2)	1.652	0.570	-7.635	4.083	-0.898	-3.626	-22.64	- 47.78		
(a,b)	θ_1'	θ_1''	θ_2'	θ_2''	θ_3'	θ_3''	θ_7	θ_8	θ_9	
(3,3)	1.649	0.641	-6.703	2.097	-14.12	8.990	-0.512	-2.991		
(4,3)	1.651	0.603	-6.448	3.343	-13.94	10.91	-0.657	-3.287	-42.28	
(a,b)	θ_1'	θ_1''	θ_2'	θ_2''	θ_3'	θ_3''	θ_3'	θ_3''	θ_9	θ_{10}
(4,4)	1.650	0.630	-5.958	2.008	-12.88	7.966	-20.07	19.03	-0.513	-2.977

Table 4.3.: Critical exponents at Non trivial FP in $d = 3$ for various truncations. When the critical exponents are complex we write them in the form $\theta_\ell' \pm i\theta_\ell''$

Tables (4.2) and (4.3) give the position and critical exponents of this FP for $a \geq b$ and $b \leq 4$. One notices that $\tilde\lambda_{2n}^* < 0$ for all $n > 0$ in the table. It is computationally demanding to continue in this direction, so to have some indication on the sign of $\tilde\lambda_{2n}^*$ for higher n we considered a simple truncation where $\tilde F$ is constant, *i.e.* $\tilde\xi_n = 0$ for $n > 0$. In this case we could push the truncation up to $a = 8$. The results are given in tables (4.4) and (4.5). One sees that the coefficients of the potential are indeed all negative. Furthermore, the coefficients grow in absolute value, so the series for V has a very small radius of convergence. This is similar to

(a,b)	$\tilde{\lambda}_0^*$	$\tilde{\lambda}_2^*$	$\tilde{\lambda}_4^*$	$\tilde{\lambda}_6^*$	$\tilde{\lambda}_8^*$	$\tilde{\lambda}_{10}^*$	$\tilde{\lambda}_{12}^*$	$\tilde{\lambda}_{14}^*$	$\tilde{\lambda}_{16}^*$	$\tilde{\xi}_0^*$
(1,0)	0.01813	-0.1088								0.1060
(2,0)	0.01880	-0.1343	-0.1561							0.1065
(3,0)	0.01894	-0.1395	-0.1942	-0.2633						0.1066
(4,0)	0.01898	-0.1407	-0.2032	-0.3284	-0.4998					0.1066
(5,0)	0.01899	-0.1410	-0.2053	-0.3437	-0.6182	-0.9604				0.1066
(6,0)	0.01899	-0.1411	-0.2058	-0.3472	-0.6452	-1.180	-1.826			0.1066
(7,0)	0.01899	-0.1411	-0.2059	-0.3479	-0.6511	-1.228	-2.229	-3.380		0.1066
(8,0)	0.01899	-0.1411	-0.2059	-0.3481	-0.6524	-1.238	-2.313	-4.091	-5.977	0.1066

Table 4.4.: Position of Nontrivial FP in $d = 3$ for other truncations.

(a,b)	θ_1'	θ_1''	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	θ_{10}
(1,0)	1.659	0.753	-1.699							
(2,0)	1.675	0.745	-1.594	-10.59						
(3,0)	1.679	0.742	-1.485	-8.384	-22.41					
(4,0)	1.68	0.741	-1.434	-7.341	-18.19	-36.96				
(5,0)	1.68	0.741	-1.414	-6.84	-15.93	-31.12	-54.10			
(6,0)	1.68	0.741	-1.407	-6.609	-14.70	-27.47	-47.25	-73.60		
(7,0)	1.68	0.740	-1.405	-6.509	-14.02	-25.287	-42.062	-66.663	-95.172	
(8,0)	1.68	0.740	-1.405	-6.469	-13.67	-23.94	-38.77	-59.72	-89.58	-118.4

Table 4.5.: Critical Exponents at Non trivial FP in $d = 3$ for other truncations.

the situation discussed in [83], making the FP unphysical. So we conclude that also in three dimensions there is probably no physically viable FP besides the GMFP.

4.6. Conclusions

The results given here confirm and extend the findings of [34]. The GMFP is found to exist also in other dimensions and in other gauges, and (with the possible exception of $d = 3$) there does not seem to be other FP's with nontrivial scalar self-interactions. In four dimensions this agrees with the findings of [83].

These results may be applied in various settings. The beta functions given in section 4.2.4 contain the full dependence on the dimensionless parameters $\tilde{\lambda}_0$, $\tilde{\xi}_0$, $\tilde{\lambda}_2$, $\tilde{\xi}_2$, $\tilde{\lambda}_4$, without making any assumption on the value of these couplings (which in the case of the first three means the ratio between the dimensionfull couplings λ_0, ξ_0, λ_2 and the RG scale k). In particular, threshold effects are taken into account by the denominators $1+2\tilde{\lambda}_2$ and $\tilde{\xi}_0 - \tilde{\lambda}_0$. One can easily recognize among various terms the ones that are obtained in perturbative approximations, but we emphasize that the derivation of these beta functions using the FRGE does not require that the couplings be small.

The most natural application of these results seems to be in the context of early cosmology, where a scalar field is used to drive inflation. In an asymptotic safety context, it would be attractive to obtain inflation as a result of FP behavior along the lines of [85, 86]. In fact the energy scale involved is sufficiently high that one may expect quantum gravity effects to play some role. Alternatively, it would also be of interest to apply the flow equations derived here to the scalar tensor theory, *e.g.* to improve the results of [79].

According to various speculations, quantum effects may play a role also on very large scales, and then again the RG flow of scalar-tensor theory could become relevant. In this connection I recall that scalar-tensor theories of a different type also arise in the conformal reduction of pure gravity, and have been studied from the FRGE point of view in [61, 66].

I have mentioned in the beginning of this chapter, that scalar-tensor theories can be reformulated classically also as pure gravity theories with $f(R)$ type actions, and one may wonder whether there is a relation also between their RG flows. In particular one could ask whether the FP that was found in [30, 31, 64] has a counterpart in the equivalent scalar-tensor theory. At first sight one would think that this is not the case, because the choice of cutoff breaks the classical equivalence between these theories. Still, this point deserves a more detailed investigation.

Another direction for research is the inclusion of other matter fields. As discussed in the introduction, if asymptotic safety is indeed the answer to the UV issues of quantum field theory, then it will not be enough to establish asymptotic safety of gravity: one will have to establish asymptotic safety for a theory including gravity as well as all the fields that occur in the standard model, and perhaps even other ones that have not yet been discovered. Ideally one would like to have a unified theory of all interactions including gravity, perhaps a GraviGUT along the lines of [80]. More humbly one could start by studying the effect of gravity on the interactions of the standard model or GUTs. Fortunately, for some important parts of the standard model it is already known that an UV Gaussian FP exists, so the question is whether the coupling to gravity, or some other mechanism, can cure the bad behavior of QED and of the Higgs sector. That this might happen had been speculated long ago [73]; see also [78] for some detailed calculations. It seem that the existence of a GMFP for all matter interactions would be the simplest solution to this issue. In this picture of asymptotic safety, gravity would be the only effective interaction at sufficiently high scale. The possibility of asymptotic safety in a nonlinearly realized scalar sector has been discussed in [87]. Aside from scalar tensor theories, the effect of gravity has been studied in [82, 88] for gauge couplings and [81] for Yukawa couplings.

Chapter 5

RG flow of Scalar Coupled to F(R) Gravity

In this chapter I will study renormalization group flow of a class of scalar-tensor theories where the scalar is coupled to a higher-derivative gravity which is polynomial in the scalar curvature R. I will start with an effective action in arbitrary dimensions. I will then derive the Hessian and construct the cutoff, which are then plugged in the FRGE eq. (4.2) to obtain the form of the flow of potentials. This information is then used to prove that the at the Gaussian Matter Fixed point, matter is asymptotically free as the matter couplings vanishes. I will then study the linearized flow around this FP and obtain some general properties of the flow, in particular the relations among the critical exponents. I will then specialize to four dimensions and give numerical results.

5.1. Truncation ansatz and motivation

In the previous chapter I studied the renormalization group flow of a scalar coupled non-minimalt to Einstein-Hilbert gravity in arbitrary dimensions. It was found that a GMFP exists in all the dimensions considered and the theory is asymptotically safe with a critical surface to be four dimensional for $3 \leq d \leq 11$.

In recent years calculations involving pure higher-derivative gravity showed that it is asymptotically safe in four dimensions. However results about the renormalizability of gravity can depend crucially on the inclusion of matter, as has been discussed in the previous chapter.

In this chapter I will consider the following truncation for the effective action,

$$\Gamma_k[g,\phi] = \int d^dx\sqrt{g}\left\{F(\phi^2,R) + \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\,\nabla_\nu\phi\right\} + S_{GF} + S_{gh}\ , \tag{5.1}$$

This effective action is the natural extension of the truncation that has been considered in the previous chapter and of the $F(R)$ gravity considered in [30, 31]. From the point of view of the former, the truncation here taking in to account higher-derivative of the metric in the form of powers of curvature, while from the later point of view it is the study of inclusion of matter to $F(R)$-gravity. Thus it is natural to study this truncation as a next step towards the understanding of asymptotic safety scenario.

5.2. The FRGE for $F(\phi^2, R)$

5.2.1. Second variations

Starting from the action given in eq. (5.1), we expand $F(\phi^2, R)$ in polynomial form in ϕ^2 and R as

$$F(\phi^2, R) = V_0(\phi^2) + V_1(\phi^2)\, R + V_2(\phi^2)\, R^2 + V_3(\phi^2)\, R^3 + \cdots + V_p(\phi^2)\, R^p = \sum_{a=0}^{p} V_a(\phi^2)\, R^a \ . \quad (5.2)$$

In order to evaluate the r.h.s. of eq. (4.2) we calculate the second functional derivatives of the functional given in eq. (5.1). These can be obtained by expanding the action to second order in the quantum fields around classical backgrounds $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ and $\phi = \bar{\phi} + \delta\phi$, where $\bar{\phi}$ is constant. The gauge fixing action quadratic in $h_{\mu\nu}$ is chosen to be

$$S_{GF} = \frac{1}{2} \int d^d x \sqrt{\bar{g}}\, \chi_\mu\, G^{\mu\nu}\, \chi_\nu \ , \quad (5.3)$$

where $\chi_\nu = \bar{\nabla}^\mu h_{\mu\nu} - \frac{1+\rho}{d} \bar{\nabla}_\nu h^\mu{}_\mu$, $G_{\mu\nu} = \bar{g}_{\mu\nu}\left(\alpha + \beta\bar{\Box}\right)$; α, β, and ρ are the gauge parameters, we denote $\bar{\Box} = \bar{\nabla}^\mu \bar{\nabla}_\mu$.

The gauge fixing action eq. (5.3) gives rise to a ghost action consisting of two parts, $S_{gh} = S_c + S_b$. The first part S_c arises from the usual Fadeev-Popov procedure leading to the complex ghost fields C_μ and $\bar{C}_\mu$. It is given by

$$S_c = \int d^d x \sqrt{\bar{g}} \bar{C}^\mu (\alpha + \beta\bar{\Box}) \left[\delta^\nu{}_\mu \bar{\Box} + \bar{R}^\nu{}_\mu + \frac{d-2-2\rho}{d} \bar{\nabla}_\mu \bar{\nabla}^\nu\right] C_\nu \ . \quad (5.4)$$

The second part S_b arises for $\beta \neq 0$ and comes from the exponentiation of a nontrivial determinant which requires the introduction of real anti-commuting fields b_μ which are usually referred to as the third ghost fields [78],

$$S_b = \frac{1}{2} \int d^d x \sqrt{\bar{g}}\, b_\mu\, G^{\mu\nu}\, b_\nu \,. \quad (5.5)$$

These terms are already quadratic in the quantum fields. Then the second variation of eq. (5.1) is given by

$$
\begin{aligned}
\Gamma_k^{(2)} &= \frac{1}{2}\int d^dx\sqrt{g}\Bigg[F(\phi^2,R)\left\{\frac{1}{4}h^2-\frac{1}{2}h_{\mu\nu}h^{\mu\nu}\right\}+\frac{\partial F(\phi^2,R)}{\partial R}\Bigg\{-hh^{\mu\nu}R_{\mu\nu}-\frac{1}{2}h\Box h \\
&+ \frac{1}{2}h^{\mu\nu}\Box h_{\mu\nu}+h^{\mu\alpha}h_{\alpha\beta}R^\beta_\mu+h_{\mu\nu}R^{\mu\rho\nu\lambda}h_{\rho\lambda}-h^\nu_\mu\nabla^\mu\nabla^\rho h_{\rho\nu}+h\nabla^\mu\nabla^\nu h_{\mu\nu}\Bigg\} \\
&+ \frac{\partial^2 F(\phi^2,R)}{\partial R^2}\Bigg\{h^{\mu\nu}R_{\mu\nu}\cdot h^{\alpha\beta}R_{\alpha\beta}-2h^{\mu\nu}R_{\mu\nu}\cdot\nabla^\rho\nabla^\sigma h_{\rho\sigma}+2h^{\mu\nu}R_{\mu\nu}\cdot\Box h \\
&+ \nabla^\alpha\nabla^\beta h_{\alpha\beta}\cdot\nabla^\mu\nabla^\nu h_{\mu\nu}-2\Box h\cdot\nabla^\mu\nabla^\nu h_{\mu\nu}+\Box h\cdot\Box h\Bigg\}\Bigg] \\
&+ \int d^dx\sqrt{g}\left[h\cdot\phi\frac{\partial F(\phi^2,R)}{\partial\phi^2}\delta\phi+2\phi\,\delta\phi\frac{\partial^2F(\phi^2,R)}{\partial R\,\partial\phi^2}\left\{\nabla^\mu\nabla^\nu h_{\mu\nu}-\Box h-h^{\mu\nu}R_{\mu\nu}\right\}\right] \\
&+ \frac{1}{2}\int d^dx\sqrt{g}\,\delta\phi\left[-\Box+2\frac{\partial F(\phi^2,R)}{\partial\phi^2}+4\phi^2\frac{\partial^2F(\phi^2,R)}{\partial(\phi^2)^2}\right]\delta\phi+S_{GF}+S_{gh}\ , \qquad (5.6)
\end{aligned}
$$

where $\Box = \nabla^\mu\nabla_\mu$ and $h = h^\mu_\mu$. Since we will never have to deal with the original metric $g_{\mu\nu}$ and scalar field ϕ, in order to simplify the notation, in the preceding formula and everywhere else from now on we will remove the bars from the backgrounds. As explained in [24], the functional that obeys the FRGE (4.2) has a separate dependence on the background field $\bar{g}_{\mu\nu}$ and on a "classical field" $(g_{\rm cl})_{\mu\nu} = \bar{g}_{\mu\nu} + (h_{\rm cl})_{\mu\nu}$, where $(h_{\rm cl})_{\mu\nu}$ is the Legendre conjugate of the sources coupling linearly to $(h_{\rm cl})_{\mu\nu}$. The same applies to the scalar field. In this paper, like in most of the literature on the subject, we will restrict ourselves to the case when $(g_{\rm cl})_{\mu\nu} = \bar{g}_{\mu\nu}$ and $\phi_{\rm cl} = \bar{\phi}$. From now on the notation $g_{\mu\nu}$ and ϕ will be used to denote equivalently the "classical fields" or the background fields.

5.2.2. Decomposition

In order to simplify the terms and partially diagonalize the kinetic operator, we perform a decomposition of $h_{\mu\nu}$ in tensor, vector, and scalar parts as in [30, 31],

$$h_{\mu\nu} = h^T_{\mu\nu} + \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu + \nabla_\mu\nabla_\nu\sigma - \frac{1}{d}g_{\mu\nu}\Box\sigma + \frac{1}{d}g_{\mu\nu}h \qquad (5.7)$$

where $h^T_{\mu\nu}$ is the (spin 2) transverse and traceless part, ξ_μ is the (spin 1) transverse vector component, σ and h are (spin 0) scalars. This decomposition allows an exact inversion of the second variation under the restriction to a spherical background. With that in mind, we work on a d-dimensional sphere. For the spin-2 part, the inverse propagator is

$$\frac{\delta^2\Gamma_k}{\delta h^T_{\mu\nu}\delta h^T_{\rho\sigma}} = \left[\frac{1}{2}\frac{\partial F(\phi^2,R)}{\partial R}\left\{\Box+\frac{2(d-2)}{d(d-1)}R\right\}-\frac{1}{2}F(\phi^2,R)\right]\delta^{\mu\nu,\rho\sigma}\ , \qquad (5.8)$$

where $\delta^{\mu\nu,\rho\sigma} = \frac{1}{2}(g^{\mu\rho}\, g^{\nu\sigma} + g^{\mu\sigma}\, g^{\nu\rho})$. For the spin-1 part it is

$$\frac{\delta^2\Gamma_k}{\delta\xi_\mu\delta\xi_\nu} = \left(\Box + \frac{R}{d}\right)\left[(\alpha+\beta\Box)\left(\Box+\frac{R}{d}\right) - 2\frac{\partial F(\phi^2,R)}{\partial R}\,\frac{R}{d} + F(\phi^2,R)\right] g^{\mu\nu}\,. \tag{5.9}$$

The two spin-0 components of the metric, σ and h, mix with $\delta\phi$ resulting in an inverse propagator given by a symmetric 3×3 matrix S with the entries

$$\begin{aligned}
S_{\sigma\sigma} &= \left(1-\frac{1}{d}\right)(-\Box)\left(-\Box-\frac{R}{d-1}\right)\left[\left(1-\frac{1}{d}\right)\left(-\Box-\frac{R}{d-1}\right)\left\{\alpha+\beta\left(\Box+\frac{R}{d}\right)\right\} - \frac{1}{2}F(\phi^2,R)\right.\\
&- \left.\left(\frac{2-d}{2d}\right)\left(-\Box-\frac{2R}{2-d}\right)\frac{\partial F(\phi^2,R)}{\partial R} + \left(1-\frac{1}{d}\right)(-\Box)\left(-\Box-\frac{R}{d-1}\right)\frac{\partial^2 F(\phi^2,R)}{\partial R^2}\right]\,,\\
S_{\sigma h} &= S_{h\sigma} = \frac{1}{2}\left(1-\frac{1}{d}\right)(-\Box)\left(-\Box-\frac{R}{d-1}\right)\left[\frac{2\rho}{d}\left\{\alpha+\beta\left(\Box+\frac{R}{d}\right)\right\}\right.\\
&+ \left.\left(1-\frac{2}{d}\right)\frac{\partial F(\phi^2,R)}{\partial R} + 2\left(1-\frac{1}{d}\right)\left(-\Box-\frac{R}{d-1}\right)\frac{\partial^2 F(\phi^2,R)}{\partial R^2}\right]\,,\\
S_{\sigma\phi} &= S_{\phi\sigma} = 2\phi\left(1-\frac{1}{d}\right)(-\Box)\left(-\Box-\frac{R}{d-1}\right)\frac{\partial^2 F(\phi^2,R)}{\partial R\,\partial\phi^2}\,,\\
S_{hh} &= \left[-\Box\left\{\alpha+\beta\left(\Box+\frac{R}{d}\right)\right\}\left(\frac{\rho}{d}\right)^2 + \frac{d-2}{4d}F(\phi^2,R) + \left(1-\frac{1}{d}\right)\left(\frac{1}{2}-\frac{1}{d}\right)\right.\\
&\times \left.\left(-\Box-\frac{2R}{d-1}\right)\frac{\partial F(\phi^2,R)}{\partial R} + \left(1-\frac{1}{d}\right)^2\left(-\Box-\frac{R}{d-1}\right)^2\frac{\partial^2 F(\phi^2,R)}{\partial R^2}\right]\,,\\
S_{h\phi} &= S_{\phi h} = \phi\frac{\partial F(\phi^2,R)}{\partial\phi^2} + 2\phi\left(1-\frac{1}{d}\right)\left(-\Box-\frac{R}{d-1}\right)\frac{\partial^2 F(\phi^2,R)}{\partial R\,\partial\phi^2}\,,\\
S_{\phi\phi} &= -\Box + 2\frac{\partial F(\phi^2,R)}{\partial\phi^2} + 4\phi^2\frac{\partial^2 F(\phi^2,R)}{\partial(\phi^2)^2}\,.
\end{aligned} \tag{5.10}$$

As discussed in more detail in [31], to match the trace-spectra of the Laplace-operator acting on $h_{\mu\nu}$ with those obtained for the constrained fields after the decomposition, the first eigenmode of the operator trace over the vector contribution and the first two eigenmodes of the operator trace over the σ contribution have to be omitted. The trace over the h and $\delta\phi$ components should be taken over the whole operator spectrum instead. To handle the mixing of the scalar components in an easy way, we subtract first the two first eigenmodes from the complete scalar contribution from the matrix S and then add the first two trace modes which should have been retained for h and $\delta\phi$. This requires to take into account a further scalar matrix B formed by the components of h, ϕ and their mixing term. It is given by

$$B = \begin{pmatrix} S_{hh} & S_{h\phi} \\ S_{h\phi} & S_{\phi\phi} \end{pmatrix}\,, \tag{5.11}$$

whose trace contribution to the FRGE will be calculated on the first two eigenmodes of the spectrum of the Laplacian.

Again, in order to diagonalize the kinetic operators occurring in the ghost actions eqs. (5.4) and (5.5), we perform a decomposition of the ghost fields C_μ, $\bar{C}_\mu$ and b_μ into transverse and longitudinal parts,

$$\bar{C}^\mu = \bar{C}^{\mu T} + \nabla^\mu \bar{C}, \qquad C_\mu = C^T_\mu + \nabla_\mu C, \qquad b_\mu = b^T_\mu + \nabla_\mu b, \tag{5.12}$$

with $\nabla_\mu \bar{C}^{\mu T} = 0$, $\nabla^\mu C^T_\mu = 0$ and $\nabla^\mu b^T_\mu = 0$.

After this decomposition, the inverse propagators for the vector and scalar components of the ghost and third ghost fields are

$$\frac{\delta^2\Gamma_k}{\delta\bar{C}^T_\mu\,\delta C^T_\nu} = (\alpha + \beta\Box)\left(\Box + \frac{R}{d}\right) g^{\mu\nu} \ , \tag{5.13}$$

$$\frac{\delta^2\Gamma_k}{\delta\bar{C}\,\delta C} = \frac{2(d-1-\rho)}{d}(-\Box)\left[\alpha + \beta\left(\Box + \frac{R}{d}\right)\right]\left[\Box + \frac{R}{d-1-\rho}\right] \ , \tag{5.14}$$

$$\frac{\delta^2\Gamma_k}{\delta b^T_\mu\,\delta b^T_\nu} = (\alpha + \beta\Box)\, g^{\mu\nu} \ , \tag{5.15}$$

$$\frac{\delta^2\Gamma_k}{\delta b\,\delta b} = -\Box\left[\alpha + \beta\left(\Box + \frac{R}{d}\right)\right] . \tag{5.16}$$

5.2.3. Contributions by Jacobians

The decomposition of $h_{\mu\nu}$, $\bar{C}_\mu$, C_μ, and b_μ gives rise to nontrivial Jacobians in the path integral, given by

$$J_\xi = \left[det'\left(-\Box - \frac{R}{d}\right)\right]^{1/2} \ , \ J_\sigma = \left[det''\left\{\Box\left(\Box + \frac{R}{d-1}\right)\right\}\right]^{1/2} \ ,$$
$$J_c = [det'(-\Box)]^{-1} \ , \ J_b = [det'(-\Box)]^{-1} \ . \tag{5.17}$$

These Jacobians can be absorbed by field redefinitions which however introduce terms which involve non-integer powers of the Laplacian. To avoid technical difficulties, we therefore prefer to exponentiate these Jacobians by the introduction of auxiliary anti-commuting and commuting fields according to the sign of the exponent of the determinant, see also [64, 30, 31]. One has to take their contribution into account while writing the FRGE.

5.3. The Gaussian Matter fixed point

The running of $V_a(\phi^2)$ is calculated from the FRGE as

$$(\partial_t V_a)[\phi^2] = \frac{1}{\mathrm{Vol}}\,\frac{1}{a!}\,\frac{\partial^a(\partial_t\Gamma_k)[\phi^2, R]}{\partial R^a} \tag{5.18}$$

where $(\partial_t \Gamma_k)[\phi^2, R]$ is obtained for various fields in an analogous way as in [30, 31]. Rescaling all fields with respect to the cutoff scale k, we obtain the dimensionless quantities $\tilde{\phi} = k^{\frac{2-d}{2}}\phi$, $\tilde{R} = k^{-2}R$ and $\tilde{V}_a(\tilde{\phi}^2) = k^{-(d-2a)}V_a(\phi^2)$. These dimensionless quantities we can use to analyze the RG flow and its FP structure. From the running of $V_a(\phi^2)$ one can calculate the running of $\tilde{V}_a(\tilde{\phi}^2)$ using

$$(\partial_t \tilde{V}_a)[\tilde{\phi}^2] \quad = \quad -(d-2a)\tilde{V}_a(\tilde{\phi}^2) + (d-2)\tilde{\phi}^2\, \tilde{V}_a'(\tilde{\phi}^2) + k^{-(d-2a)}\,(\partial_t V_a)[\phi^2] \tag{5.19}$$

where the last term is calculated using eq. (5.18). A FP is a solution of the infinite set of functional equations $\partial_t \tilde{V}_a = 0$ for $a = 0, \ldots, \infty$. This means that, at the FP, for each a the function $\tilde{V}_a(\tilde{\phi}^2)$ is k-independent, or equivalently that each coefficient of its Taylor expansion is k-independent. Since we assume that each $\tilde{V}_a$ is analytic it can be Taylor expanded around $\tilde{\phi}^2 = 0$, and therefore

$$\partial_t \tilde{V}_a^{(i)}(0) = 0 \tag{5.20}$$

for $i = 0, \ldots, \infty$, where the superscript i denotes the i-th derivative with respect to $\tilde{\phi}^2$.

5.3.1. Minimal matter coupling of gravity at the GMFP

The existence of a Gaussian Matter Fixed Point (GMFP), where all the matter couplings approach zero for $k \to \infty$ and only the purely gravitational couplings have nontrivial values, was observed for finite polynomial truncations in [34]. In [36], its existence was proven for effective average actions of the form

$$\Gamma_k[g,\phi] = \int d^dx\, \sqrt{g}\left(V_0(\phi^2) + V_1(\phi^2)\,R + \frac{1}{2}\,g^{\mu\nu}\,\partial_\mu\phi\partial_\nu\phi\right) + S_{GF} + S_{gh}. \tag{5.21}$$

The existence of a GMFP can be shown to hold for the more general class of effective average actions considered in this paper. By definition, a GMFP is a point where $\tilde{V}_a$ are $\tilde{\phi}^2$-independent, *i.e.*

$$\tilde{V}_a^{(i)}(0) = 0 \tag{5.22}$$

for $i = 1, \ldots, \infty$. In this subsection we will prove that with the ansatz in eq. (5.22) all the equations in (5.20) with $i = 1, \ldots, \infty$ are identically satisfied, thus leaving only the equations with $i = 0$ to be solved. We will give numerical solutions of these remaining equations for $a = 0, 1 \ldots, 8$ in section 5.4.

Now we explicitly analyze the structure of $\partial_t F$ related to the second variation of the effective average action given in eq. (5.1) for the various field components. The second variation for $h^T_{\mu\nu}$ and ξ_μ has the form

$$\Gamma_k^{(2)}\Big|_{\mathrm{T,V}} = f(z,\,R) + f_a(z,\,R)\,V_a\ , \tag{5.23}$$

where we denote $z := -\Box$. The functional form for $\Gamma_k^{(2)}\Big|_{T,V}$ is motivated by eqs. (5.8) and (5.9) from which we notice that it depends on V_a at most linearly, with coefficients being functions of z and R, which are denoted here by $f(z,\,R)$ and $f_a(z,\,R)$.

For the scalar part, the second variation has the form

$$\Gamma_k^{(2)}\Big|_{\mathrm{s}} = \begin{pmatrix} l^{11}(z,R) + f_a^{11}(z,R)\,V_a & l^{12}(z,R) + f_a^{12}(z,R)\,V_a & g_a^1(z,R)\,\phi\,V_a' \\ l^{12}(z,R) + f_a^{12}(z,R)\,V_a & l^{22}(z,R) + f_a^{22}(z,R)\,V_a & g_a^2(z,R)\,\phi\,V_a' \\ g_a^1(z,R)\,\phi\,V_a' & g_a^2(z,R)\,\phi\,V_a' & z + R^a(2\,V_a' + 4\,\phi^2\,V_a'') \end{pmatrix} , \tag{5.24}$$

where a prime denotes derivative with respect to ϕ^2. Again the functional form for $\Gamma_k^{(2)}\Big|_s$ is motivated by eq. (5.10) which clearly tells that entries $S_{\sigma\sigma}$, $S_{\sigma h}$, and S_{hh} depend at most linearly on V_a, while the entries $S_{\phi\sigma}$ and $S_{\phi h}$ are linear combinations of $\phi\,V_a'$. The coefficients in these linear combinations are functions of z and R denoted here by $l^{ij}(z,R)$ and $g_a^i(z,R)$.

For the ghost part the second variation has the form

$$\Gamma_k^{(2)}\Big|_{\mathrm{gh}} = D(z,\,R)\ . \tag{5.25}$$

This can be verified from eqs. (5.13, 5.14, 5.15 and 5.16). We first consider the contributions from $h_{\mu\nu}^T$ and ξ_μ. Since for them the second variation has the form given by eq. (5.23), the modified inverse propagator $\mathcal{P}_k := \Gamma_k^{(2)} + \mathcal{R}_k$ and the cutoff $\mathcal{R}_k$ will have the functional form

$$\mathcal{P}_k = f(P_k,R) + f_a(P_k,R)V_a\ , \qquad \mathcal{R}_k = f(P_k,R) - f(z,R) + \{f_a(P_k,R) - f_a(z,R)\}V_a\ , \tag{5.26}$$

where we have simply replaced z by $P_k(z) := z + R_k(z)$ to obtain the modified inverse propagator. $R_k(z)$ is a profile function which tends to k^2 for $z \to 0$ and approaches zero rapidly for $z > k^2$. The RG-time derivative of the cutoff $\mathcal{R}_k$ in eq. (5.26) is

$$\partial_t\mathcal{R}_k = \partial_t f(P_k,R) + \partial_t f_a(P_k,R)\,V_a + \{f_a(P_k,R) - f_a(z,R)\}\,\partial_t V_a\ . \tag{5.27}$$

Using eq. (5.27) in the FRGE one finds that the contributions from $h_{\mu\nu}^T$ and ξ_μ have the form

$$\partial_t V_a = H_a(V_c) + H_{ab}(V_c)\partial_t V_b\ . \tag{5.28}$$

This can be justified by noticing that $\partial_t\mathcal{R}_k$ given by eq. (5.27) depends at most linearly on $\partial_t V_b$. On the r.h.s. of the FRGE, $\partial_t\mathcal{R}_k$ occurs in the numerator, while the denominator contains the modified inverse propagator given in eq. (5.26) which depends at most linearly on V_a. So we find that the r.h.s of the FRGE depends at most linearly on $\partial_t V_a$. The coefficients in front of $\partial_t V_a$ are functionals of V_a and are denoted by $H_a(V_c)$ and $H_{ab}(V_c)$.

The contributions from the ghost parts will be simpler. Since they do not depend on the potentials, they will only give a constant contribution to H_a. The contributions from the scalars are more involved due to the matrix structure. The modified inverse scalar propagator is obtained by replacing all z with P_k in eq. (5.24).

The cutoff is constructed in the usual way by subtracting the inverse propagator from the modified inverse propagator. This cutoff can be written as

$$\begin{aligned}\mathcal{R}^s_k &= \begin{pmatrix} l^{11}(P_k,R)-l^{11}(z,R) & l^{12}(P_k,R)-l^{12}(z,R) & 0\\ l^{12}(P_k,R)-l^{12}(z,R) & l^{22}(P_k,R)-l^{22}(z,R) & 0\\ 0 & 0 & P_k-z\end{pmatrix}\\ &+\begin{pmatrix} f^{11}_a(P_k,R)-f^{11}_a(z,R) & f^{12}_a(P_k,R)-f^{12}_a(z,R) & 0\\ f^{12}_a(P_k,R)-f^{12}_a(z,R) & f^{22}_a(P_k,R)-f^{22}_a(z,R) & 0\\ 0 & 0 & 0\end{pmatrix} V_a\\ &+\begin{pmatrix} 0 & 0 & g^1_a(P_k,R)-g^1_a(z,R)\\ 0 & 0 & g^2_a(P_k,R)-g^2_a(z,R)\\ g^1_a(P_k,R)-g^1_a(z,R) & g^2_a(P_k,R)-g^2_a(z,R) & 0\end{pmatrix}\phi V'_a \quad (5.29)\end{aligned}$$

Then the t derivative of the cutoff given in eq. (5.29) is

$$\begin{aligned}\partial_t\mathcal{R}^s_k &= \begin{pmatrix} \partial_t l^{11}(P_k,R) & \partial_t l^{12}(P_k,R) & 0\\ \partial_t l^{12}(P_k,R) & \partial_t l^{22}(P_k,R) & 0\\ 0 & 0 & \partial_t P_k\end{pmatrix}+\begin{pmatrix} \partial_t f^{11}_a(P_k,R) & \partial_t f^{12}_a(P_k,R) & 0\\ \partial_t f^{12}_a(P_k,R) & \partial_t f^{22}_a(P_k,R) & 0\\ 0 & 0 & 0\end{pmatrix} V_a\\ &+\begin{pmatrix} f^{11}_a(P_k,R)-f^{11}_a(z,R) & f^{12}_a(P_k,R)-f^{12}_a(z,R) & 0\\ f^{12}_a(P_k,R)-f^{12}_a(z,R) & f^{22}_a(P_k,R)-f^{22}_a(z,R) & 0\\ 0 & 0 & 0\end{pmatrix} \partial_t V_a\\ &+\begin{pmatrix} 0 & 0 & \partial_t g^1_a(P_k,R)\\ 0 & 0 & \partial_t g^2_a(P_k,R)\\ \partial_t g^1_a(P_k,R) & \partial_t g^2_a(P_k,R) & 0\end{pmatrix}\phi V'_a\\ &+\begin{pmatrix} 0 & 0 & g^1_a(P_k,R)-g^1_a(z,R)\\ 0 & 0 & g^2_a(P_k,R)-g^2_a(z,R)\\ g^1_a(P_k,R)-g^1_a(z,R) & g^2_a(P_k,R)-g^2_a(z,R) & 0\end{pmatrix}\phi\partial_t V'_a \quad (5.30)\end{aligned}$$

The modified propagator for scalars is the matrix inverse of eq. (5.24) with z replaced by P_k. It is given by

$$(\mathcal{P}^s_k)^{-1} = \frac{1}{\mathrm{Det}\,\mathcal{P}^s_k}\,\mathrm{Adj}\,(\mathcal{P}^s_k) \ . \quad (5.31)$$

where Adj $(\mathcal{P}^s_k)$ denotes the adjoint of the matrix $(\mathcal{P}^s_k)$ (the matrix of cofactors). The determinant is a functional depending only on V_a, $\phi^2\, V'_a V'_b$, and $2\,V'_a + 4\phi^2\, V''_a$. This can be easily derived from the modified inverse propagator obtained from eq. (5.24).

All entries of the adjoint of $\mathcal{P}^s_k$ consist of cofactors, thus it has the form

$$\mathrm{Adj}\,(\mathcal{P}^s_k) = \begin{pmatrix} A^{11}\left(V_a,\phi^2\, V'_a V'_b, 2\,V'_a+4\phi^2\, V''_a\right) & A^{12}\left(V_a,\phi^2\, V'_a V'_b, 2\,V'_a+4\phi^2\, V''_a\right) & A^{13}(V_a)\phi\, V'_a\\ A^{21}\left(V_a,\phi^2\, V'_a V'_b, 2\,V'_a+4\phi^2\, V''_a\right) & A^{22}\left(V_a,\phi^2\, V'_a V'_b, 2\,V'_a+4\phi^2\, V''_a\right) & A^{23}(V_a)\phi\, V'_a\\ A^{31}(V_a)\phi\, V'_a & A^{32}(V_a)\phi\, V'_a & A^{33}(V_a)\end{pmatrix} , \quad (5.32)$$

where each entry depends additionally on P_k and R. In order to calculate the RG trace, we multiply $(\mathcal{P}_k^s)^{-1}$ with $\partial_t \mathcal{R}_k^s$ and then take the matrix trace. Doing this we note that $\phi V_a'$ is either multiplied with another $\phi V_a'$ or it is multiplied with $\phi \partial_t V_a'$. So the scalar contribution to the FRGE has the form

$$
\begin{aligned}
\partial_t V_a\big|_{\mathrm{s}} &= H_a\left(V_a, \phi^2\, V_a' V_b', 2\,V_a' + 4\phi^2\, V_a''\right) + H_{ab}\left(V_a, \phi^2\, V_a' V_b', 2\,V_a' + 4\phi^2\, V_a''\right)\,\partial_t V_b \\
&+ H_{abc}\left(V_a, \phi^2\, V_a' V_b', 2\,V_a' + 4\phi^2\, V_a''\right)\,\phi^2\, V_b'\,\partial_t V_c'\,.
\end{aligned} \tag{5.33}
$$

The contributions from the transverse traceless tensor and transverse vector can also be combined in the above expression to write the full FRGE contribution in the same way as above. Then $\partial_t F = R^a\,\partial_t V_a$.

After having calculated the structural form for the running of $V_a(\phi^2)$, we use it to calculate the dimensionless beta functional using eq. (5.19), which gives

$$
\begin{aligned}
(\partial_t \tilde{V}_a)[\tilde{\phi}^2] &= -(d-2a)\tilde{V}_a + (d-2)\tilde{\phi}^2\tilde{V}_a' + \tilde{H}_a\left(\tilde{V}_a,\, \tilde{\phi}^2\,\tilde{V}_a'\,\tilde{V}_b',\, 2\tilde{V}_a' + 4\tilde{\phi}^2\,\tilde{V}_a''\right) \\
&+ \tilde{H}_{ab}\left(\tilde{V}_a,\, \tilde{\phi}^2\,\tilde{V}_a'\,\tilde{V}_b',\, 2\tilde{V}_a' + 4\tilde{\phi}^2\,\tilde{V}_a''\right)\left\{(d-2b)\tilde{V}_b - (d-2)\tilde{\phi}^2\tilde{V}_b' + (\partial_t\tilde{V}_b)[\tilde{\phi}^2]\right\} \\
&+ \tilde{H}_{abc}\left(\tilde{V}_a,\, \tilde{\phi}^2\,\tilde{V}_a'\,\tilde{V}_b',\, 2\tilde{V}_a' + 4\tilde{\phi}^2\,\tilde{V}_a''\right)\tilde{\phi}^2\tilde{V}_b'\left\{(d-2c)\tilde{V}_c'\right. \\
&- \left.(d-2)\left(\tilde{\phi}^2\tilde{V}_c'' + \tilde{V}_c'\right) + (\partial_t\tilde{V}_c)'[\tilde{\phi}^2]\right\}\,.
\end{aligned} \tag{5.34}
$$

Inserting eq. (5.20) in eq. (5.34) we get the fixed point equation

$$
\begin{aligned}
0 = &-(d-2a)\tilde{V}_a + (d-2)\tilde{\phi}^2\tilde{V}_a' + \tilde{H}_a\left(\tilde{V}_a,\, \tilde{\phi}^2\,\tilde{V}_a'\,\tilde{V}_b',\, 2\tilde{V}_a' + 4\tilde{\phi}^2\,\tilde{V}_a''\right) \\
&+\tilde{H}_{ab}\left(\tilde{V}_a,\, \tilde{\phi}^2\,\tilde{V}_a'\,\tilde{V}_b',\, 2\tilde{V}_a' + 4\tilde{\phi}^2\,\tilde{V}_a''\right)\left\{(d-2b)\tilde{V}_b - (d-2)\tilde{\phi}^2\tilde{V}_b'\right\} \\
&+\tilde{H}_{abc}\left(\tilde{V}_a,\, \tilde{\phi}^2\,\tilde{V}_a'\,\tilde{V}_b',\, 2\tilde{V}_a' + 4\tilde{\phi}^2\,\tilde{V}_a''\right)\tilde{\phi}^2\tilde{V}_b'\left\{(d-2c)\tilde{V}_c' - (d-2)\left(\tilde{\phi}^2\tilde{V}_c'' + \tilde{V}_c'\right)\right\}
\end{aligned} \tag{5.35}
$$

The above equation is identically satisfied when we take its Taylor expansion around $\tilde{\phi}^2 = 0$

and use eq. (5.22). For example, taking one derivative with respect to $\tilde{\phi}^2$ gives

$$\begin{aligned} 0 \;=\;& -(d-2a)\tilde{V}_a' + (d-2)\left\{\tilde{\phi}^2\tilde{V}_a'' + \tilde{V}_a'\right\} + \frac{\delta\tilde{H}_a}{\delta\tilde{V}_c}\tilde{V}_c' + \frac{\delta\tilde{H}_a}{\delta(\tilde{\phi}^2\tilde{V}_c'\tilde{V}_d')}(\tilde{V}_c'\tilde{V}_d' + \tilde{\phi}^2\tilde{V}_c''\tilde{V}_d' + \tilde{\phi}^2\tilde{V}_d''\tilde{V}_c') \\ & + \frac{\delta\tilde{H}_a}{\delta(2\tilde{V}_c' + 4\tilde{\phi}^2\tilde{V}_c'')}(2\tilde{V}_c^{(2)} + 4\tilde{V}_c^{(2)} + 4\tilde{\phi}^2\tilde{V}_c^{(3)}) + \left\{(d-2b)\tilde{V}_b' - (d-2)\tilde{\phi}^2\tilde{V}_b'' - (d-2)\tilde{V}_b'\right\}\tilde{H}_{ab} \\ & + \left\{(d-2b)\tilde{V}_b - (d-2)\tilde{\phi}^2\tilde{V}_b'\right\}\left(\frac{\delta\tilde{H}_{ab}}{\delta\tilde{V}_c}\tilde{V}_c' + \frac{\delta\tilde{H}_{ab}}{\delta(\tilde{\phi}^2\tilde{V}_c'\tilde{V}_d')}(\tilde{V}_c'\tilde{V}_d' + \tilde{\phi}^2\tilde{V}_c''\tilde{V}_d' + \tilde{\phi}^2\tilde{V}_d''\tilde{V}_c')\right. \\ & \left. + \frac{\delta\tilde{H}_{ab}}{\delta(2\tilde{V}_c' + 4\tilde{\phi}^2\tilde{V}_c'')}(2\tilde{V}_c^{(2)} + 4\tilde{V}_c^{(2)} + 4\tilde{\phi}^2\tilde{V}_c^{(3)})\right) + \tilde{V}_b'\left\{(d-2c)\tilde{V}_c - (d-2)\tilde{\phi}^2\tilde{V}_c'\right\}\tilde{H}_{abc} \\ & + \tilde{\phi}^2\tilde{V}_b''\left\{(d-2c)\tilde{V}_c - (d-2)\tilde{\phi}^2\tilde{V}_c'\right\}\tilde{H}_{abc} + \tilde{\phi}^2\tilde{V}_b'\left\{(d-2c)\tilde{V}_c' - (d-2)\tilde{\phi}^2\tilde{V}_c'' - (d-2)\tilde{V}_c'\right\}\tilde{H}_{abc} \\ & + \tilde{\phi}^2\tilde{V}_b'\left\{(d-2c)\tilde{V}_c - (d-2)\tilde{\phi}^2\tilde{V}_c'\right\}\left(\frac{\delta\tilde{H}_{abc}}{\delta\tilde{V}_d}\tilde{V}_d' + \frac{\delta\tilde{H}_{abc}}{\delta(\tilde{\phi}^2\tilde{V}_d'\tilde{V}_e')}(\tilde{V}_d'\tilde{V}_e' + \tilde{\phi}^2\tilde{V}_d''\tilde{V}_e' + \tilde{\phi}^2\tilde{V}_e''\tilde{V}_d')\right. \\ & \left. + \frac{\delta\tilde{H}_{abc}}{\delta(2\tilde{V}_d' + 4\tilde{\phi}^2\tilde{V}_d'')}(2\tilde{V}_d^{(2)} + 4\tilde{V}_d^{(2)} + 4\tilde{\phi}^2\tilde{V}_d^{(3)})\right). \end{aligned} \tag{5.36}$$

Setting $\tilde{\phi}^2 = 0$ and using the GMFP conditions, we see that the right hand side will be zero. One can take successive derivatives to verify that this property indeed holds when higher derivatives are taken. The only equation which is not automatically solved in this way is the one where we evaluate eq. (5.35) at $\tilde{\phi}^2 = 0$ and use eq. (5.22). This is just the FP equation for an $f(R)$ theory with a single minimally coupled scalar. We will solve these equations in section 5.4.

5.3.2. Linearized Flow around the GMFP

The attractivity properties of a FP are determined by the signs of the critical exponents defined to be minus the eigenvalues of the linearized flow matrix, the so-called stability matrix, at the FP. The eigenvectors corresponding to negative eigenvalues (positive critical exponent) span the UV critical surface. At the Gaussian FP the critical exponents are equal to the mass dimension of each coupling, so the relevant couplings are the ones that are power–counting renormalizable (or marginally renormalizable). In a perturbatively renormalizable theory they are usually finite in number.

At the GMFP, the situation is more complicated as the eigenvalues being negative or positive do not correspond to couplings being relevant or irrelevant. In principle, at the GMFP the eigenvectors corresponding to negative eigenvalues get contributions from all the couplings present in the truncation, thus making it more difficult to find the fixed point action. Thus understanding the properties of the stability matrix around the GMFP becomes crucial.

Therefore we now discuss the structure of the linearized flow around the GMFP. It is convenient to Taylor expand the potentials $V_a(\phi^2)$ as

$$V_a(\phi^2) = \sum_{i=0}^{q} \lambda_{2i}^{(a)}(k)\phi^{2i}, \tag{5.37}$$

where $\lambda_{2i}^{(a)}$ are the corresponding couplings with mass dimension $d - 2a - i(d-2)$. We are assuming a finite truncation with up to p powers of R, *i.e.* a going from 0 to p, and q powers of ϕ^2. In practice it has been possible to deal with $p \leq 8$; as we shall see, it is possible to understand the structure of the theory for any polynomial in ϕ^2, so one could also let $q \to \infty$. Rescaling these couplings with respect to the RG scale defines dimensionless couplings $\tilde{\lambda}_{2i}^{(a)} = k^{d-2a-i(d-2)}\lambda_{2i}^{(i)}$ and the corresponding beta functions $\beta_{2i}^{(a)} = \partial_t \lambda_{2i}^{(a)}$.

The stability matrix is defined as

$$\left(M_{ij}\right)_{ab} = \left. \frac{\delta\left(\frac{1}{i!}\partial_t \tilde{V}_a^{(i)}(0)\right)}{\delta\left(\frac{1}{j!}\tilde{V}_b^{(j)}(0)\right)} \right|_{\mathrm{FP}} = \left. \frac{\partial \beta_{2i}^{a}}{\partial \tilde{\lambda}_{2j}^{(b)}} \right|_{\mathrm{FP}} \tag{5.38}$$

Using the above definitions, numerical results tell that the stability matrix M has the form

$$\begin{pmatrix} M_{00} & M_{01} & 0 & 0 & \cdots \\ 0 & M_{11} & M_{12} & 0 & \ddots \\ 0 & 0 & M_{22} & M_{23} & \ddots \\ 0 & 0 & 0 & M_{33} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{5.39}$$

where each entry is a $(p+1) \times (p+1)$ matrix of the form

$$M_{ij} = \begin{pmatrix} \frac{\partial \beta_{2i}^{(0)}}{\partial \tilde{\lambda}_{2j}^{(0)}} & \cdots & \frac{\partial \beta_{2i}^{(0)}}{\partial \tilde{\lambda}_{2j}^{(p)}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \beta_{2i}^{(p)}}{\partial \tilde{\lambda}_{2j}^{(0)}} & \cdots & \frac{\partial \beta_{2i}^{(p)}}{\partial \tilde{\lambda}_{2j}^{(p)}} \end{pmatrix}, \tag{5.40}$$

while p is the highest power of scalar curvature included in the action. It turns out that,

$$M_{ij} = 0 \;\forall\, i \geq 1,\, \forall\, j < i \;; \qquad M_{ij} = 0 \;\forall\, i,\, \forall\, j > (i+1)\;. \tag{5.41}$$

The various nonzero entries follow the same relations that were observed in [36]. In d dimensions they are

$$M_{ii} = (d-2)i\,\mathbf{1} + M_{00} \;; \qquad M_{i,i+1} = (i+1)(2i+1)M_{01} \;; \tag{5.42}$$

where

$$M_{00} = \begin{pmatrix} \delta M_{\tilde\lambda_0^{(0)},\tilde\lambda_0^{(0)}} & \cdots & \cdots & \delta M_{\tilde\lambda_0^{(0)},\tilde\lambda_0^{(p)}} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \delta M_{\tilde\lambda_0^{(p)},\tilde\lambda_0^{(0)}} & \cdots & \cdots & \delta M_{\tilde\lambda_0^{(p)},\tilde\lambda_0^{(p)}} \end{pmatrix} + \begin{pmatrix} -d & 0 & 0 & \dots & 0 \\ 0 & -(d-2) & 0 & \dots & 0 \\ 0 & 0 & -(d-4) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & -(d-2p) \end{pmatrix} ; \tag{5.43}$$

$$M_{01} = \begin{pmatrix} \delta M_{\tilde\lambda_0^{(0)},\tilde\lambda_2^{(0)}} & \cdots & \cdots & \delta M_{\tilde\lambda_0^{(0)},\tilde\lambda_2^{(p)}} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \delta M_{\tilde\lambda_0^{(p)},\tilde\lambda_2^{(0)}} & \cdots & \cdots & \delta M_{\tilde\lambda_0^{(p)},\tilde\lambda_2^{(p)}} \end{pmatrix} . \tag{5.44}$$

Using the same arguments as in [36], one can prove the above properties starting from eq. (5.33) neglecting $\partial_t V_a$ and $\partial_t V'_a$ on the right hand side (corresponding to a one-loop approximation). Solving eq. (5.33) beyond that level would require solving a functional differential equation and would be beyond the scope of this paper. However, the results presented in the next section suggest that these relations should hold exactly. They are relations independent of the gauge choice, however the entries of M_{00} and M_{01} are gauge dependent.

The physical nature of the relations among the eigenvalues can be understood from the difference between the GMFP and the Gaussian fixed point where also the gravitational couplings would vanish. At a Gaussian fixed point, the critical exponents are determined by the mass dimension of the couplings, and therefore are all spaced by $d-2$. At the GMFP, the gravitational couplings lead to some corrections to the critical exponents, but the correction to all exponents is the same, such that the spacing remains equal to $d-2$.

These relations have important consequences. Because the stability matrix at the GMFP has the block diagonal structure given by eq. (5.39), its eigenvalues are just the eigenvalues of the diagonal blocks. Since the diagonal blocks are related by eqs. (5.42), the eigenvalues of the various blocks differ only by multiples of $d-2$. That means if $\rho_0^{(0)}, \dots, \rho_0^{(p)}$ are the eigenvalues of M_{00}, then all the eigenvalues of M are of the form

$$\rho_{2i}^{(a)} = \rho_0^{(a)} + (d-2)\, i \; . \tag{5.45}$$

As M_{00} depends only on the couplings $\lambda_0^{(a)}$, it is enough to include only these couplings into the action to find all the eigenvalues of the stability matrix. Therefore, the results for minimally coupled scalar-tensor theory determine the eigenvalues of the nonminimally coupled scalar-tensor theory. In particular, if one has calculated the dimension of the UV critical surface of the minimally coupled theory, one can also predict the dimension of the UV critical surface of the nonminimally coupled theory.

To find all the eigenvectors of the stability matrix it is necessary to know also M_{01}. One can write the eigenvectors as $v = (v_0, v_1, \ldots, v_q)^T$ where each v_i is itself a $p+1$ dimensional vector. Then the vector $V_0 = (v_0, 0, 0, \ldots, 0)^T$ is an eigenvector if v_0 is an eigenvector of M_{00} which can be seen immediately by multiplying it with M. The eigenvectors of M with the above form are eigenvectors for the eigenvalues of M_{00} and can therefore be completely determined by just using M_{00}. Thus we note at this point that these eigenvectors are mixtures of gravitational couplings only, they do not contain any contribution from matter couplings.

Now consider a vector of the form $V_1 = (v_0', v_1, 0, 0, \ldots, 0)^T$. Acting on it with M, and demanding V_1 to be an eigenvector of M corresponding to some eigenvalue $\rho_2^{(a)}$, we obtain two relations,

$$M_{00}\, v_0' + M_{01}\, v_1 = \rho_2^{(a)}\, v_0' \ , \qquad M_{11}\, v_1 = \rho_2^{(a)}\, v_1 \ . \tag{5.46}$$

The second equation in (5.46) tells that v_1 is an eigenvector of M_{11}. Now due to equations given in (5.42) and (5.45), we note that $v_1 = v_0$. Determining v_1 will then determine also v_0'. In the same way one can go on to determine the next eigenvector. Consider $V_2 = (v_0'', v_1', v_2, 0, \ldots, 0)^T$. We then demand it to be a eigenvector of M. That means it should satisfy

$$M_{00}\, v_0'' + M_{01}\, v_1' = \rho_4^{(a)}\, v_0'' \ , \qquad M_{11}\, v_1' + M_{12}\, v_2 = \rho_4^{(a)}\, v_1' \ , \qquad M_{22}\, v_2 = \rho_4^{(a)}\, v_2 \ . \tag{5.47}$$

One notices immediately that v_2 is the eigenvector of M_{22}, and using equations in (5.42) and (5.45) we conclude that $v_2 = v_0$. Other equations would determine v_0'' and v_1'. This process can be continued to find all the eigenvectors.

We will now illustrate the validity of these results in various truncations with scalar fields coupled minimally and nonminimally to gravity.

5.4. Numerical results

5.4.1. Nonminimally coupled scalar field

From here on we proceed as in [30, 31]. We choose the gauge $\alpha = 0$, $\beta \to \infty$, and $\rho = 0$. This simplifies the calculation considerably because with that choice several arguments in the FRGE cancel with each other. The cutoff operators are chosen so that the modified inverse propagator is identical to the inverse propagator except for the replacement of $z = -\nabla^2$ by $P_k(z) = z + R_k(z)$; we use exclusively the optimized cutoff functions $R_k(z) = (k^2 - z)\theta(k^2 - z)$ [43]. Then knowledge of the heat kernel coefficients which contain at most R^4 taken from [89] is sufficient to calculate all the beta functions. A further benefit of this choice of cutoff is that the trace arguments will be polynomial in z. This simplifies the integrations in the trace evaluation and is done in closed form.

Inserting everything into the FRGE and comparing the terms with equal powers of R and ϕ^2 on each side of the equation will give a system of algebraic equations for the beta functions of the couplings $\tilde{\lambda}^{(a)}_{2i}$. The fixed points of the flow equations are evaluated and the corresponding critical exponents $\vartheta^{(a)}_{2i}$ are determined.

We carried out the calculation for effective average actions including up to R^4 and up to ϕ^2 in each potential V_a. Such truncations include at most ten couplings. We find that a GMFP does indeed exist for all these truncations.

p	$\tilde{\lambda}^{(0)}_{0*}$	$\tilde{\lambda}^{(1)}_{0*}$	$\tilde{\lambda}^{(2)}_{0*}$	$\tilde{\lambda}^{(3)}_{0*}$	$\tilde{\lambda}^{(4)}_{0*}$
1	6.495	-21.579			
2	5.224	-16.197	1.834		
3	6.454	-20.756	1.071	-6.474	
4	6.354	-21.342	0.792	-6.807	-3.865

Table 5.1.: Nonvanishing couplings at the GMFP. The index p is the highest power of R included in the truncation. All values are multiplied by a factor 1000.

p	ϑ'_0	ϑ''_0	$\vartheta^{(2)}_0$	$\vartheta^{(3)}_0$	$\vartheta^{(4)}_0$	ϑ'_2	ϑ''_2	$\vartheta^{(2)}_2$	$\vartheta^{(3)}_2$	$\vartheta^{(4)}_2$
1	2.493	2.368				0.493	2.368			
2	1.847	2.397	21.031			-0.153	2.397	19.031		
3	3.077	2.524	2.033	-3.852		1.077	2.524	0.033	-5.852	
4	3.261	2.772	1.670	-3.593	-5.182	1.261	2.772	-0.330	-5.593	-7.182

Table 5.2.: Critical exponents at the GMFP. The index p is the highest power of R included in the truncation. Critical exponents are labeled $\vartheta^{(a)}_{2i}$, like the couplings, but the corresponding eigenvectors involve strong mixing, as discussed in the text. For each i, the first two critical exponents form a complex conjugate pair given by $\vartheta'_0 \pm \vartheta''_0 i$ and $\vartheta'_2 \pm \vartheta''_2 i$.

The nonvanishing fixed point values for various truncations are given in table 5.1, the corresponding critical exponents (the negative of the eigenvalues of the stability matrix) in table 5.2.

From the critical exponents one realizes at once several features. Though we carry out the full FRGE calculation we find that in general the real parts of the critical exponents $\vartheta^{(a)}_2$ differ from $\vartheta^{(a)}_0$ exactly by two as proven in the one-loop case while the imaginary parts of the critical exponents are unchanged. This suggests strongly that the relations among the eigenvalues will also hold at the exact level. The qualitative and quantitative properties turn out to be very similar to those of the purely gravitational theory.

The inclusion of only four couplings with $a = 0, 1$ and $i = 0, 1$ leads to four attractive directions. The complex critical exponents $\vartheta'_0 \pm \vartheta''_0 i$ are expected from the experience with the Einstein-Hilbert truncation. The existence of a second pair of complex critical exponents

$\vartheta_2' \pm \vartheta_2'' i$ follows from the relation between the eigenvalues given in eq. (5.45). These complex conjugate pairs occur also when higher scalar curvature terms are included.

When one includes also R^2 couplings, one encounters large positive critical exponents as known from the calculations in pure gravity [29, 30, 31, 64]. Using eq. (5.45) one concludes that one has to go up to power ϕ^{20} before encountering a negative critical exponent, so the critical surface would be twelve dimensional. But this is a fluke of the R^2 truncations due to the anomalously large positive critical exponent. The situation quickly normalizes when one adds further powers of R.

Including R^3 couplings, classically one would expect only three positive critical exponents as the classical mass dimensions of $\lambda_0^{(0)}$, $\lambda_0^{(1)}$, $\lambda_0^{(2)}$, $\lambda_0^{(3)}$, $\lambda_2^{(0)}$, $\lambda_2^{(1)}$, $\lambda_2^{(2)}$, and $\lambda_1^{(3)}$, are 4, 2, 0, -2, 2, 0, -2, and -4 respectively. Apparently, the FRGE calculation, which includes quantum corrections with large mixing between the various couplings, produces instead six positive critical exponents in the R^3 truncation. The critical exponent $\vartheta_2^{(2)}$ is however very close to zero in consistency with the eigenvalue shift in eq. (5.45). This tells us that the truncation with $p = 3$ has a six-dimensional UV critical surface for any $i \geq 1$.

With the inclusion of the coupling for the R^4 operator whose classical mass dimension is -4, one notices that $0 < \vartheta_0^{(2)} < 2$. Thus one would expect that including the coupling for the operator $\phi^2 R^4$ with classical mass dimension -6, in consistency with eq. (5.45), the critical exponent $\vartheta_2^{(2)}$ would be negative, and the critical surface would be five dimensional. Indeed, the inclusion of those couplings does make $\vartheta_2^{(2)}$ negative, leading to five negative and five positive critical exponents. One can then say, using eq. (5.45) in the truncation $p = 4$, that for any $i \geq 1$, the critical surface would be five dimensional.

To illustrate our results we display here the stability matrix for the R^4 truncation. The entries in the upper left 5×5 block and in the lower right 5×5 block are the same except the ones on the diagonals which differ by two. The upper right block is M_{01}, the lower left one contains only zero entries:

$$
M|_{\mathrm{GMFP}} = \left(\begin{array}{cccccccccc}
-0.81 & 1.87 & 0.40 & -1.24 & 0.41 & -0.0057 & 0.0021 & 0.0011 & -0.00039 & -0.000051 \\
-8.01 & -6.05 & 2.95 & 2.78 & -1.80 & -0.0031 & -0.0093 & 0.00083 & 0.0024 & 0.00024 \\
2.16 & 0.27 & -4.57 & 1.64 & -0.041 & 0.00021 & -0.00018 & -0.0032 & -0.00038 & -5.5510^{-}6 \\
2.95 & -0.61 & -7.46 & 4.13 & 0.44 & -0.00026 & -0.0032 & -0.0098 & -0.0019 & -0.000091 \\
5.12 & 4.95 & 3.34 & -10.52 & 7.79 & 0.00065 & 0.0021 & -0.0010 & -0.0071 & -0.00075 \\
0 & 0 & 0 & 0 & 0 & 1.19 & 1.87 & 0.40 & -1.24 & 0.41 \\
0 & 0 & 0 & 0 & 0 & -8.01 & -4.05 & 2.95 & 2.78 & -1.80 \\
0 & 0 & 0 & 0 & 0 & 2.16 & 0.27 & -2.57 & 1.64 & -0.041 \\
0 & 0 & 0 & 0 & 0 & 2.95 & -0.61 & -7.46 & 6.13 & 0.44 \\
0 & 0 & 0 & 0 & 0 & 5.12 & 4.95 & 3.34 & -10.52 & 9.79
\end{array} \right) \tag{5.48}
$$

The eigenvectors corresponding to the five positive critical exponents in the R^4 truncation are given by

$$\begin{pmatrix} -0.2774 \pm 0.2693i \\ 0.8574 \\ -0.1206 \pm 0.0634i \\ 0.0473 \pm 0.1254i \\ -0.2202 \pm 0.1746i \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} (15.381 \pm 5.409i) \times 10^{-4} \\ (-33.008 \pm 13.931i) \times 10^{-4} \\ (4.894 \pm 1.980i) \times 10^{-4} \\ (-2.535 \pm 1.083i) \times 10^{-4} \\ (5.437 \pm 8.333i) \times 10^{-4} \\ -0.2774 \pm 0.2692i \\ 0.8574 \\ -0.1205 \pm 0.0634i \\ 0.0473 \pm 0.1254i \\ -0.2202 \pm 0.1746i \end{pmatrix}, \quad \begin{pmatrix} -0.3845 \\ -0.07586 \\ -0.7103 \\ -0.5667 \\ -0.1437 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5.49}$$

The first complex conjugate pair of eigenvectors corresponds to the complex conjugate pair of critical exponents $\vartheta_0' \pm \vartheta_0''$ with values $3.2608 \pm 2.7722i$, while the second pair of complex conjugate eigenvectors corresponds to the complex conjugate pair of critical exponents $\vartheta'_2 \pm \vartheta''_2$ with values $1.2608 \pm 2.7722i$. The last eigenvector corresponds to the critical exponent $\vartheta_0^{(2)} = 1.6698$. We note that the eigenvectors corresponding to the eigenvalues of M_{00}, namely the first complex conjugate pair of eigenvectors and the last one, have the same structure as was described in the previous section, *i.e.* $(v_0, 0, 0, \ldots, 0)^T$, where v_0 is determined by just using M_{00}. We note that these eigenvectors do not get mixing from the matter couplings, but only from the purely gravitational couplings. Further more, if we look at the eigenvectors corresponding to the eigenvalues of M_{11}, namely the second complex conjugate pair of eigenvectors in eq. (5.49), which has the form $(v_0', v_1, 0, \ldots, 0)^T$, we clearly notice that $v_1 = v_0$, as described in the previous section.

5.4.2. Minimally coupled scalar field

Having verified that the properties of the stability matrix proved at one-loop level do also hold in the exact calculation, we now analyze higher order curvature terms retaining only the couplings $\tilde{\lambda}_0^{(a)}$ corresponding to a truncation with a minimally coupled scalar field. Then one obtains the non-Gaussian fixed points and critical exponents given in tables 5.3 and 5.4. We analyze these results and use them to make predictions for the nonminimal truncation. One observes that the addition of the scalar fields alters the results for pure gravity in [30, 31] only by a small amount. Just as there, the UV critical surface becomes at most three-dimensional, and fixed point values for the cosmological and the Newton constant remain very stable. It has to be remarked that for those two couplings the oscillation in the fixed point value after the introduction of the R^2-term is not as strong as in pure gravity. Also the critical exponent obtained after the introduction of the R^2-coupling becomes large, but not as large as in pure gravity. So the addition of the scalar field seems to have already a little stabilizing effect on the R^2-truncation. The introduction of the R^4 and R^5-couplings leads to a second complex conjugate pair of critical exponents as soon as both couplings are included.

p	$\tilde{\Lambda}_*$	$\tilde{G}_*$	$\Lambda_* G_*$	$10^3\times$								
				$\tilde{\lambda}_{0*}^{(0)}$	$\tilde{\lambda}_{0*}^{(1)}$	$\tilde{\lambda}_{0*}^{(2)}$	$\tilde{\lambda}_{0*}^{(3)}$	$\tilde{\lambda}_{0*}^{(4)}$	$\tilde{\lambda}_{0*}^{(5)}$	$\tilde{\lambda}_{0*}^{(6)}$	$\tilde{\lambda}_{0*}^{(7)}$	$\tilde{\lambda}_{0*}^{(8)}$
1	0.150	0.923	0.139	6.495	-21.579							
2	0.161	1.228	0.198	5.224	-16.197	1.834						
3	0.155	0.958	0.149	6.454	-20.756	1.071	-6.474					
4	0.149	0.932	0.139	6.354	-21.342	0.792	-6.807	-3.865				
5	0.149	0.932	0.139	6.355	-21.339	0.793	-6.793	-3.854	-0.024			
6	0.146	0.918	0.134	6.312	-21.669	0.586	-7.169	-5.576	-0.537	2.702		
7	0.146	0.917	0.133	6.318	-21.702	0.534	-6.469	-5.530	-1.979	2.761	2.565	
8	0.148	0.926	0.137	6.344	-21.489	0.678	-5.922	-4.574	-2.074	1.863	2.393	0.829

Table 5.3.: Position of the FP for increasing number p of couplings included. The first three columns give the FP values in the form of cosmological and Newton constant and their dimensionless product. The values $\tilde{\lambda}_{0*}^{(a)}$ (and only them) have been rescaled by a factor 1000.

Now it is easy to analyze how the dimension of the UV critical surface changes under the introduction of nonminimal matter couplings. In general, if a critical exponent $\vartheta_0^{(a)}$ is negative then $\vartheta_{2i}^{(a)}$ will also be negative for all $i > 0$. From table 5.4 we see that $\vartheta_0^{(a)} < 0$ for all $a \geq 3$, thus all $\vartheta_{2i}^{(a)} < 0$ for all $a \geq 3$ and $i > 0$. However, since $4 > \vartheta_0' > 2$, using eq. (5.45) we conclude that $2 > \vartheta_2' > 0$. This means that there are two more attractive directions. From table 5.4 one sees however that $0 < \vartheta_0^{(2)} < 2$ as soon as R^4 is included, thus we do not obtain any other attractive directions. So compared to [30, 31] where a three-dimensional UV critical surface was obtained for pure gravity, interactions with scalar matter lead to a five-dimensional UV critical surface.

5.5. Summary

We have shown that a Gaussian matter fixed point does exist also under the inclusion of higher order curvature terms and their coupling to scalar fields. We verified that the properties of the stability matrix proven only at one-loop level hold also in the exact calculations. We exploited these properties to show the relations between minimally and nonminimally coupled scalar-tensor theory. In particular, we were able to calculate the critical exponents for the nonminimal scalar tensor theory from those of the minimal one. The introduction of minimally coupled scalar matter fields gives only slight quantitative corrections to the fixed point properties of the purely gravitational theory. The critical exponents again seem to converge with the inclusion of more curvature terms. The minimally coupled theory produces three positive critical exponents. We derived that the additional critical exponents in the nonminimally coupled theory will be the ones of the minimal theory shifted by constant values. This produces two more positive critical exponents. From that we can conclude that, in four dimensions, the scalar-tensor

p	ϑ_0'	ϑ_0''	$\vartheta_0^{(2)}$	$\vartheta_0^{(3)}$	$\vartheta_0^{(4)}$	$\vartheta_0^{(5)}$	$\vartheta_0^{(6)}$	$\vartheta_0^{(7)}$	$\vartheta_0^{(8)}$
1	2.493	2.368							
2	1.847	2.397	21.031						
3	3.077	2.524	2.033	-3.852					
4	3.261	2.772	1.670	-3.593	-5.182				
5	2.777	2.908	1.795	-4.176	-4.196	-6.764			
6	2.841	2.813	1.386	-4.000	-3.798	-5.947	-8.538		
7	2.930	2.964	1.312	-4.009	-2.760	-4.623	-7.459	-11.166	
8	2.331	2.902	1.570	-4.063	-0.673	-7.120	-7.323	-9.854	-11.611

Table 5.4.: Critical exponents for increasing number p of couplings included. The first two critical exponents are a complex conjugate pair of the form $\vartheta' \pm \vartheta'' i$. The same is the case for the fourth and fifth critical exponent $\vartheta_0^{(4)} \pm \vartheta_0^{(5)} i$.

theory based on an action polynomial in scalar curvature and in even powers of scalar field gives rise to a five-dimensional UV critical surface.

Chapter 6

Conclusion

Pure GR has been shown to be the best classical theory of gravity to date which is very successful in explain a variety of cosmological and astrophysical phenomenons in a wide range of energy. However this success is masked by a gloomy fact that it is inconsistent with the quantum theory of matter, which has very well described the world at small scales. Gravity is the only force among the four known forces which has not been successfully combined with the quantum ideas. Different methods of incorporating the quantum notions in the classical gravity picture, leads to several approaches to quantum gravity. These are broadly classified in two categories: top-bottom picture and bottom-up picture. In the former one starts with a theory which is drastically different from the well known physical theories describing the present universe, and one tries to find the correct low energy limit, while in later one starts with the effective theory describing the present universe and tries to extend it to higher energy scales.

In this thesis I have taken the bottom-up approach. It is the renormalization group approach to gravity. I started by discussing the new methods to study renormalization group of a theory, by describing the construction of Functional Renormalization Group Equation (FRGE) and discussing its properties. I then discussed the generalized nonperturbative notion of renormalizability as first described by Weinberg [11], and showed how gravity can very well satisfy these notions. It was called Asymptotic Safety Scenario. I demonstrated how the new tool FRGE can be effectively used to study this nonperturbative notion of renormalizabiltiy by computing the beta functions of the theory in a nonperturbative way, in the sense that the assumption of coupling being small is not made.

FRGE has been very useful to study asymptotic safety scenario in the context of quantum field theory of gravity, which has been formulated with metric as the field variable. Past studies have shown that pure gravity is asymptotically safe as there exists a nontrivial UV attractive FP satisfying the requirements of asymptotic safety outlined in section 2.2 [11, 25, 24, 26, 27, 38, 61, 62, 30, 31, 63, 29, 64, 65, 32, 66, 67, 42, 68, 69, 70, 71, 72] (see also [46] for reviews). Studies involving inclusion of matter minimally interacting with gravity puts bounds on the number of matter fields one can have in a system if the requirement of asymptotic safety needs to be met [33], while considerations of a nonminimally coupled matter with gravity shows that the theory is asymptotically safe [34, 36]. Higher-derivative gravity interacting with matter have also been studied in [35, 37], where they were found to be asymptotically safe with an increased dimensionality of UV critical surface.

Apart from the possible application of the work [36, 37] to cosmology as has been discussed in the end of chapter 4, the research conducted in this thesis can be extended along various directions.

First way of extending the work is to formulate a quantum field theory of gravity where both metric and connection are dynamical, and investigate whether the requirements of asymptotic safety are met or not. So far any asymptotically safe theory of gravity that has been constructed is done with only the metric as the dynamical field. It would be interesting to see how the situation changes when both metric and connection are dynamical. This is also called first order formalism while the theory constructed with only metric is the second order formalism.

The second way to generalize the computations done in this thesis is by considering the effect of higher-derivative gravity. To start with one can do the truncations with four derivatives of metric like the one considered in [62, 32, 35] and include a nonminimally coupled scalar field. This can be then gradually extended to include more derivative terms of the metric. Along this line a special class of such theories have been considered in [37], where the effective average action was a scalar coupled nonminimally to $F(R)$ theories of gravity. There it was found that the UV critical surface is five dimensional while it has been noted that in pure $F(R)$ theories the UV critical surface is three dimensional [30, 31]. Thus it would be interesting to see how the results gets affected when a non-minimally coupled scalar is introduced in the system of higher-derivative gravity. Already it has been noted that in a theory involving maximum four derivatives of metric, the UV critical surface is three dimensional [32, 35], irrespective of whether one considers pure gravity or gravity minimally coupled to a scalar. The proposal stated in above lines will thus be an improved computation. The only problem in pursuing such studies are regarding the background metric that one should choose so as to extract the beta functions of all the couplings present in the truncation. Choosing a very nontrivial background leads to computational difficulties while choosing a simple background may not give all the information required. Thus in regard to this perhaps one needs to develop some new tools extract the information from the FRGE. Development of new tools will also open the door for studying the theories of gravity which contains the notorious two loop counter-term of perturbative gravity, which made EH-gravity two loop non-renormalizable. This will further shed light on the asymptotic safety scenario.

The third line of research could be to include the effect of anomalous dimension of the scalar. In a scalar theory in flat space background not interacting with gravity it has been noted that in three dimensions, the presence of scalar wave-function renormalization leads to improved results for the Wilson-Fisher FP. Thus one expects that a similar thing might happen in the case when the scalar is interacting with gravity.

The fourth line of research could be to study the asymptotic safety of higher-derivative gravity theories in dimensions other than four, in particular focussing on extra dimensions. It has been well established that EH-gravity whether interacting with matter or not is asymptotically safe in dimensions other than four also [38, 36]. This was important to study after the proposal of extra-dimensions scenarios in [10], which has gained immense attention since its birth. Within the community of asymptotic safety it is well known that at the nontrivial fixed point of gravity all possible operators allowed by the symmetries of the theory will become equally important. Thus it is important to study the asymptotic safeness of higher-derivative theories of gravity both with and without matter in dimensions other than four.

The fifth line of research could be to study the effect of inclusion of other matter fields. In a more realistic world apart from scalar fields there are dozens of other types of matter field. Thus it is justified to consider a model of gravity interacting with various types of matter field. A particular model which will be of great interest is the standard model interacting with gravity. A first step in this directions was taken in [25, 33], where minimally coupled matter fields were considered. It was found that demanding the existence of a fixed point required putting constraints on the number of matter fields allowed in the theory. Thus considering a realistic model consisting of various matter fields interacting nonminimally with gravity will be of crucial importance to the field of aysmptotic safety. Such a study will nomatter be very complicated to do but will shed light on the various important problems like the triviality and hierarchy problem.

Apart from the proposals mentioned above there are several issues related to quantum behavior of gravity from a more general perspective which needs to be addressed. Particular examples concering them that needs to be investigated are what is the structure of spacetime in the UV limit within our approach to QG and what are the quantum observables. While such issues have been started to be considered within the framework of FRGE [94, 95], they have also been the primary focus of other nonperturbative approaches to QG. It would thus be interesting to explore the connection between our approach and other approaches. This can be done both at the level of reconstructing quantities such as bare action and regularized path integral from RG trajectories using FRGE [42] and of relating the theoretical predictions between such different approaches. In the later context a possible point of contact is the phenomenon of spontaneous dimensional reduction that has been reported across various approaches to QG (see [96] for a review) and which also is an immediate consequence of asymptotic safety, though whether it is a mere coincidence or not remains to be established. Although nonperturbative approaches are few but still establishing this connection is not any straightforward task, as the method of quantization in various approaches is very different from each other. However understanding these connections will be important as this will give crucial insights in to gravitational physics in the UV.

Ultimately, however we would like to arrive at a testable theory of quantum gravity. In the approach of asymptotic safety it requires what are the observational consequesce of our putative asymptotically safe theory of gravity. This is important as apart from the theoretical evidence that have been gathered for asymptotic safety scenario and its relation with other approaches to quantum gravity, the question of ultimate nature of quantum gravity is something that must be determined by the experimental data. However it is perhaps too optimistic to expect any direct experimental evidence or observational tests of quantum gravity in a near future, but one can rely on indirect test of such theories in the light of astrophysical and cosmological observations [85]. Mapping the renormalization grop flow of gravity could be an important step in that directions.

Chapter A

Appendices

A.1. Trace technology

The r.h.s. of the FRGE is the trace of a function of a differential operator. To illustrate the methods employed to evaluate such traces, we begin by considering the covariant Laplacian in a metric g, $-\Box$ (where $\Box = \nabla_\mu \nabla^\mu$). If the fields carry a representation of a gauge group G and are coupled to gauge fields for G, the covariant derivative ∇ contains also these fields. We will denote $\Delta = -\Box \mathbf{1} + \mathbf{E}$ a second order differential operator. $\mathbf{E}$ is a linear map acting on the spacetime and internal indices of the fields. In our applications to de Sitter space it will have the form $\mathbf{E} = qR\,\mathbf{1}$ where $\mathbf{1}$ is the identity in the space of the fields and q is a real number.

The trace of a function W of the operator Δ can be written as

$$\mathrm{Tr} W(\Delta) = \sum_i W(\lambda_i) \tag{A.1}$$

where λ_i are the eigenvalues of Δ. Introducing the Laplace anti-transform $\tilde{W}(s)$

$$W(z) = \int_0^\infty ds\, e^{-zs} \tilde{W}(s) \tag{A.2}$$

we can rewrite eq. (A.1) as

$$\mathrm{Tr} W(\Delta) = \int_0^\infty ds \mathrm{Tr} K(s) \tilde{W}(s) \tag{A.3}$$

where $\mathrm{Tr} K(s) = \sum_i e^{-s\lambda_i}$ is the trace of the heat kernel of Δ. We assume that there are no negative and zero eigenvalues; if present, these will have to be dealt with separately. The trace of the heat kernel of Δ has the well-known asymptotic expansion for $s \to 0$:

$$\mathrm{Tr}\left(e^{-s\Delta}\right) = \frac{1}{(4\pi)^{\frac{d}{2}}} \left[B_0(\Delta)\, s^{-\frac{d}{2}} + B_2(\Delta)\, s^{-\frac{d}{2}+1} + \ldots + B_d(\Delta) + B_{d+2}(\Delta)\, s + \ldots \right] \tag{A.4}$$

where $B_n = \int d^d x \sqrt{g} \mathrm{tr} \mathbf{b}_n$ and $\mathbf{b}_n$ are linear combinations of curvature tensors and their covariant derivatives containing $2n$ derivatives of the metric.

Assuming that $[\Delta, \mathbf{E}] = 0$, the heat kernel coefficients of Δ are related to those of $-\Box$ by

$$\mathrm{Tr} e^{-s(-\Box+\mathbf{E})} = \frac{1}{(4\pi)^{\frac{d}{2}}} \sum_{k,\ell=0}^{\infty} \frac{(-1)^{\ell}}{\ell!} \int d^d x \sqrt{g}\, \mathrm{tr}\, \mathbf{b}_k(\Delta) \mathbf{E}^{\ell} s^{k+\ell-2}. \tag{A.5}$$

The first six coefficients have the following form [90]:

$$\mathbf{b}_0 = \mathbf{1} \tag{A.6}$$

$$\mathbf{b}_2 = \frac{R}{6}\mathbf{1} - \mathbf{E} \tag{A.7}$$

$$\begin{aligned}\mathbf{b}_4 = {} & \frac{1}{180}\left(R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} - R^{\mu\nu}R_{\mu\nu} + \frac{5}{2}R^2 + 6\Box R\right)\mathbf{1} \\ & + \frac{1}{12}\mathbf{\Omega}_{\mu\nu}\mathbf{\Omega}^{\mu\nu} - \frac{1}{6}R\mathbf{E} + \frac{1}{2}\mathbf{E}^2 - \frac{1}{6}\Box\mathbf{E}\end{aligned} \tag{A.8}$$

$$\begin{aligned}\mathbf{b}_6 = {} & \frac{1}{180}R\mathbf{1}\left(R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} - R^{\mu\nu}R_{\mu\nu} + \frac{5}{6}R^2 + \frac{7}{2}\Box R\right) \\ & + \frac{R}{2}\mathbf{E}^2 + \mathbf{E}^3 + \frac{1}{30}\mathbf{E}\left(R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} - R^{\mu\nu}R_{\mu\nu} + \frac{5}{2}R^2 + 6\Box R\right) \\ & + \frac{R}{12}\mathbf{\Omega}_{\mu\nu}\mathbf{\Omega}^{\mu\nu} + \frac{1}{2}\mathbf{E}\mathbf{\Omega}_{\mu\nu}\mathbf{\Omega}^{\mu\nu} + \frac{1}{2}\mathbf{E}\Box\mathbf{E} - \frac{1}{2}\mathbf{J}_{\mu}\mathbf{J}^{\mu} \\ & + \frac{1}{30}\left(2\mathbf{\Omega}^{\mu}{}_{\nu}\mathbf{\Omega}^{\nu}{}_{\alpha}\mathbf{\Omega}^{\alpha}{}_{\mu} - 2R^{\mu}{}_{\nu}\mathbf{\Omega}_{\mu\alpha}\mathbf{\Omega}^{\alpha\nu} + R^{\mu\nu\alpha\beta}\mathbf{\Omega}_{\mu\nu}\mathbf{\Omega}_{\alpha\beta}\right) \\ & + \mathbf{1}\left[-\frac{1}{630}R\Box R + \frac{1}{140}R_{\mu\nu}\Box R^{\mu\nu} + \frac{1}{7560}\left(-64R^{\mu}{}_{\nu}R^{\nu}{}_{\alpha}R^{\alpha}{}_{\mu} + 48R^{\mu\nu}R_{\alpha\beta}R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}\right.\right. \\ & \left.\left. + 6R_{\mu\nu}R^{\mu}{}_{\rho\alpha\beta}R^{\nu\rho\alpha\beta} + 17R_{\mu\nu}{}^{\alpha\beta}R_{\alpha\beta}{}^{\rho\sigma}R_{\rho\sigma}{}^{\mu\nu} - 28R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}R^{\alpha}{}_{\rho}{}^{\beta}{}_{\sigma}R^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu}\right)\right]\end{aligned} \tag{A.9}$$

where $\mathbf{\Omega}^{\mu\nu} = [\nabla^{\mu}, \nabla^{\nu}]$ is the curvature of the connection acting on a set of fields in a particular representation of the Lorentz and internal gauge group and $\mathbf{J}_{\mu} = \nabla_{\alpha}\mathbf{\Omega}^{\alpha}{}_{\mu}$. We neglect total derivative terms. The coefficient $\mathbf{b}_8$, which is also used in this work, is much too long to write here, and can be found in [89]. These coefficients are for unconstrained fields. The ones for fields satisfying differential constraints such as $h^T_{\mu\nu}$ and ξ_{μ} in the field decompositions (3.61) are given in the appendix A.2.

Let us return to equation (A.3). If we are interested in the local behavior of the theory (*i.e.* the behavior at length scales much smaller than the typical curvature radius) we can use the asymptotic expansion (A.4) and then evaluate each integral separately. Then we get

$$\begin{aligned}\mathrm{Tr} W(\Delta) = \frac{1}{(4\pi)^{\frac{d}{2}}} & \left[Q_{\frac{d}{2}}(W)B_0(\Delta) + Q_{\frac{d}{2}-1}(W)B_2(\Delta) + \ldots \right. \\ & \left. + Q_0(W)B_d(\Delta) + Q_{-1}(W)B_{d+2}(\Delta) + \ldots\right],\end{aligned} \tag{A.10}$$

where

$$Q_n(W) = \int_0^{\infty} ds\, s^{-n}\tilde{W}(s)\,. \tag{A.11}$$

In the case of four dimensional field theories, it is enough to consider integer values of n. However, in odd dimensions half-integer values of n are needed and we are also interested in the analytic continuation of results to arbitrary real dimensions. We will therefore need expressions for (A.11) that hold for any real n.

If we denote $W^{(i)}$ the i-th derivative of W, we have from (A.2)

$$W^{(i)}(z) = (-1)^i \int_0^\infty ds\, s^i e^{-zs} \tilde{W}(s) \; . \tag{A.12}$$

This formula can be extended to the case when i is a real number to define a notion of "noninteger derivative". From this it follows that for any real i

$$Q_n(W^{(i)}) = (-1)^i Q_{n-i}(W) \; . \tag{A.13}$$

For n a positive integer one can use the definition of the Gamma function to rewrite (A.11) as a Mellin transform:

$$Q_n(W) = \frac{1}{\Gamma(n)} \int_0^\infty dz\, z^{n-1} W(z) \tag{A.14}$$

while for m a positive integer or $m = 0$

$$Q_{-m}(W) = (-1)^m W^{(m)}(0) \; . \tag{A.15}$$

More generally, for n a positive real number we can define $Q_n(W)$ by equation (A.14), while for n real and negative we can choose a positive integer k such that $n + k > 0$; then we can write the general formula

$$Q_n(W) = \frac{(-1)^k}{\Gamma(n+k)} \int_0^\infty dz\, z^{n+k-1} W^{(k)}(z) \; . \tag{A.16}$$

This reduces to the two cases mentioned above when n is integer. In the case when n is a negative half integer $n = -\frac{2m+1}{2}$ we will set $k = m + 1$ so that we have

$$Q_{-\frac{2m+1}{2}}(W) = \frac{(-1)^{m+1}}{\sqrt{\pi}} \int_0^\infty dz\, z^{-1/2} f^{(m+1)}(z) \tag{A.17}$$

Let us now consider some particular integrals that are needed in this paper. As discussed in section 3.3.1, there are two natural choices of cutoff function: type I cutoff is a function $R_k(-\Box)$ such that the modified inverse propagator is $P_k(-\Box) = -\Box + R_k(-\Box)$; type II cutoff is the same function but its argument is now the entire inverse propagator: $R_k(\Delta)$, such that the modified inverse propagator is $P_k(\Delta) = \Delta + R_k(\Delta)$.

We now restrict ourselves to the case when $\mathbf{E} = q\mathbf{1}$, so that we can write $\Delta = -\Box + q\mathbf{1}$. The evaluation of the r.h.s. of the FRGE reduces to knowledge of the heat kernel coefficients and calculation of integrals of the form $Q_n\left(\frac{\partial_t R_k}{(P_k+q)^\ell}\right)$. It is convenient to measure everything in units of k^2. Let us define the dimensionless variable y by $z = k^2 y$; then $R_k(z) = k^2 r(y)$ for some dimensionless function r, $P_k(z) = k^2(y + r(y))$ and $\partial_t R_k(z) = 2k^2(r(y) - yr'(y))$.

In general the coefficients $Q_n(W)$ will depend on the details of the cutoff function. However, if $q = 0$ and $\ell = n + 1$ they turn out to be independent of the shape of the function. Note that they are all dimensionless. For $n > 0$, as long as $r(0) \neq 0$:

$$Q_n\left(\frac{\partial_t R_k}{P_k^{n+1}}\right) = \frac{2}{\Gamma(n)}\int_0^\infty dy \frac{d}{dy}\left[\frac{1}{n}\frac{y^n}{(y+r)^n}\right] = \frac{2}{n!} \,. \tag{A.18}$$

Similarly, if $r(0) \neq 0$ and $r'(0)$ is finite,

$$Q_0\left(\frac{\partial_t R_k}{P_k}\right) = 2 \,. \tag{A.19}$$

Finally, for $n = -m < 0$

$$Q_n\left(\frac{\partial_t R_k}{P_k^{1-m}}\right)\Big|_{y=0} = (-1)^m \left(\frac{\partial_t R_k}{P_k^{1-m}}\right)^{(m)}\Big|_{y=0} = \sum_{n=0}^{m}\binom{m}{n}\left(r - y\,r'\right)^{(n)}(y+r)^{(m-1)}\Big|_{y=0} = 0 \tag{A.20}$$

as $(r - y\,r')^{(n)} = r^{(n)} - y\,r^{(n+1)} - r^{(n)} = -y\,r^{(n+1)}$ which vanishes at $y = 0$. This concludes the proof that $Q_n\left(\frac{\partial_t R_k}{P_k^{n+1}}\right)$ are scheme–independent.

Regarding the other coefficients $Q_n\left(\frac{\partial_t R_k}{(P_k+q)^\ell}\right)$ whenever explicit evaluations are necessary, we will use the so-called "optimized cutoff function" [43]

$$R_k(z) = (k^2 - z)\theta(k^2 - z) \tag{A.21}$$

With this cutoff $\partial_t R_k = 2k^2\theta(k^2 - z)$. Since the integrals are all cut off at $z = k^2$ by the theta function in the numerator, we can simply use $P_k(z) = k^2$ in the integrals. For $n > 0$ we have

$$Q_n\left(\frac{\partial_t R_k}{(P_k+q)^\ell}\right) = \frac{2}{n!}\frac{1}{(1+\tilde{q})^\ell}k^{2(n-\ell+1)} \tag{A.22}$$

where $\tilde{q} = qk^{-2}$. For $n = 0$ we have

$$Q_0\left(\frac{\partial_t R_k}{(P_k+q)^\ell}\right) = \frac{\partial_t R_k}{(P_k+q)^\ell}\Bigg|_{z=0} = \frac{2}{(1+\tilde{q})^\ell}k^{2(-\ell+1)} \,. \tag{A.23}$$

Finally, owing to the fact that the function $\frac{\partial_t R_k(z)}{(P_k(z)+q)^\ell}$ is constant in an open neighborhood of $z = 0$, we have

$$Q_n\left(\frac{\partial_t R_k}{(P_k+q)^\ell}\right) = 0 \;\text{ for } n < 0 \,. \tag{A.24}$$

This has the remarkable consequence that with the optimized cutoff the trace in the FRGE consists of finitely many terms.

For noninteger n let us calculate

$$Q_{-\frac{2n+1}{2}}\left(\frac{\partial_t R_k}{P_k}\right) = \frac{(-1)^{n+1}}{\sqrt{\pi}}\int_0^\infty dz\, z^{-1/2}\frac{d^{n+1}}{dx^{n+1}}\frac{\partial_t R_k(z)}{P_k(z)} \tag{A.25}$$

where $P_k(z) = z + (k^2 - z)\theta(k^2 - z)$. We change the variable to $x = z/k^2$ so we have

$$Q_{-\frac{2n+1}{2}}\left(\frac{\partial_t R_k}{P_k}\right) = \frac{(-1)^{n+1}k^{-(2n+1)}}{\sqrt{\pi}}\int_0^\infty dx\, x^{-1/2}\frac{d^{n+1}}{dx^{n+1}}\frac{2x\theta(1-x)}{x+(1-x)\theta(1-x)} \tag{A.26}$$

We find

$$\begin{aligned}\int_0^\infty dx\, x^{-1/2}\frac{d}{dx}f(x) &= 2 \\ \int_0^\infty dx\, x^{-1/2}\frac{d^2}{dx^2}f(x) &= -5\end{aligned} \tag{A.27}$$

so that

$$\begin{aligned}Q_{-1/2}\left(\frac{\partial_t R_k}{P_k}\right) &= -\frac{2}{\sqrt{\pi}k} \\ Q_{-3/2}\left(\frac{\partial_t R_k}{P_k}\right) &= -\frac{5}{\sqrt{\pi}k^3}\end{aligned} \tag{A.28}$$

We also need some Q-functionals of $\frac{R_k}{(P_k+q)^\ell}$. For $n > 0$ we have

$$Q_n\left(\frac{R_k}{(P_k+q)^\ell}\right) = \frac{1}{(n+1)!}\frac{1}{(1+\tilde{q})^\ell}k^{2(n-\ell+1)}\ . \tag{A.29}$$

The function $\frac{R_k(z)}{(P_k(z)+q)^\ell}$ is equal to $\frac{k^2-z}{(k^2+q)^\ell}$ in an open neighborhood of $z = 0$; therefore

$$Q_0\left(\frac{R_k}{(P_k+q)^\ell}\right) = \left.\frac{R_k}{(P_k+q)^\ell}\right|_{z=0} = \frac{1}{(1+\tilde{q})^\ell}k^{2(-\ell+1)} \tag{A.30}$$

$$Q_{-1}\left(\frac{R_k}{(P_k+q)^\ell}\right) = \frac{1}{(1+\tilde{q})^\ell}k^{-2\ell}\ ,\quad Q_n\left(\frac{R_k}{(P_k+q)^\ell}\right) = 0\ \text{ for } n < -1\ . \tag{A.31}$$

Finally, for the type III cutoff one also needs the following

$$Q_n\left(\frac{1}{(P_k+q)^\ell}\right) = \frac{1}{n!}\frac{k^{2(n-\ell)}}{(1+\tilde{q})^\ell}\ \text{ for } n \geq 0\ ;\ \ Q_n\left(\frac{1}{(P_k+q)^\ell}\right) = 0\ \text{ for } n < 0\ . \tag{A.32}$$

In conclusion let us address a general problem concerning the choice of the operator $\mathcal{O}$, whose eigenfunctions are taken as a basis in the functional space. In some calculations the r.h.s.

of the FRGE takes the form $\frac{1}{2}\mathrm{Tr}\frac{\partial_t R_k(\Delta+q\mathbf{1})}{P_k(\Delta+q\mathbf{1})}$ where Δ is an operator and q is a constant. Equation (A.10) tells us how to compute the trace of this function, regarded as a function of the operator $\Delta+q\mathbf{1}$. In the derivation of this result it was implicitly assumed that $\mathcal{O}=\Delta+q\mathbf{1}$. However, the trace must be independent of the choice of basis in the functional space. It is instructing to see this explicitly, namely to evaluate the trace regarding $\frac{1}{2}\mathrm{Tr}\frac{\partial_t R_k(\Delta+q\mathbf{1})}{P_k(\Delta+q\mathbf{1})}$ as a function of Δ. Given any function $W(z)$ we can define $\bar{W}(z)=W(z+q)$; in general, expanding in q we then have

$$\begin{aligned}
Q_n(\bar{W}) &= \frac{1}{\Gamma(n)}\int_0^\infty dz\, z^{n-1}W(z+q) \\
&= \frac{1}{\Gamma(n)}\int_0^\infty dz\, z^{n-1}(W(z)+qW'(z)+\frac{1}{2!}q^2W''(z)+\frac{1}{3!}q^3W'''(z)\dots) \\
&= Q_n(W)+qQ_n(W')+\frac{1}{2!}q^2Q_n(W'')+\frac{1}{3!}q^3Q_n(W''')+\dots \\
&= Q_n(W)-qQ_{n-1}(W)+\frac{1}{2!}q^2Q_{n-2}(W)-\frac{1}{3!}q^3Q_{n-3}(W)\dots
\end{aligned} \tag{A.33}$$

where in the last step we have used equation (A.13). Using (A.10) for the function $\bar{W}$ we then have

$$\begin{aligned}
\mathrm{Tr}\bar{W}[\Delta] &= \frac{1}{(4\pi)^{\frac{d}{2}}}\left[Q_{\frac{d}{2}}(\bar{W})B_0(\Delta)+Q_{\frac{d}{2}-1}(\bar{W})B_2(\Delta)+\dots+Q_0(\bar{W})B_{2d}(\Delta)+\dots\right] \\
&= \frac{1}{(4\pi)^{\frac{d}{2}}}\Bigg[\left(Q_{\frac{d}{2}}(W)-qQ_{\frac{d}{2}-1}(W)+\frac{1}{2!}q^2Q_{\frac{d}{2}-2}(W)-\frac{1}{3!}q^3Q_{\frac{d}{2}-3}(W)+\dots\right)B_0(\Delta) \\
&\quad+\left(Q_{\frac{d}{2}-1}(W)-qQ_{\frac{d}{2}-2}(W)+\frac{1}{2!}q^2Q_{\frac{d}{2}-3}(W)-\frac{1}{3!}q^3Q_{\frac{d}{2}-4}(W)+\dots\right)B_2(\Delta) \\
&\quad+\qquad\dots \\
&\quad+\left(Q_0(W)-qQ_{-1}(W)+\frac{1}{2!}q^2Q_{-2}(W)-\frac{1}{3!}q^3Q_{-3}(W)+\dots\right)B_{2d}(\Delta) \\
&\quad+\qquad\dots\Bigg]
\end{aligned} \tag{A.34}$$

We can now collect the terms that have the same Q-functions. They correspond to the anti-diagonal lines in (A.34). Using equation (A.5) one recognizes that the coefficient of $Q_{\frac{d}{2}-k}$ is $B_{2k}(\Delta+q\mathbf{1})$. Therefore

$$\begin{aligned}
\mathrm{Tr}\bar{W}[\Delta] = \quad & \frac{1}{(4\pi)^{\frac{d}{2}}}\Big[Q_{\frac{d}{2}}(\bar{W})B_0(\Delta+q\mathbf{1})+Q_{\frac{d}{2}+1}(\bar{W})B_2(\Delta+q\mathbf{1}) \\
&\qquad+\dots+Q_0(\bar{W})B_{2d}(\Delta+q\mathbf{1})+\dots\Big]
\end{aligned} \tag{A.35}$$

which coincides term by term with the expansion of $\mathrm{Tr}W[\Delta+q]$ using the basis of eigenfunctions of the operator $\mathcal{O}=\Delta+q\mathbf{1}$. This provides an explicit check, at least in this particular example, that the trace of this function is independent of the basis in the functional space.

A.2. Spectral geometry of differentially constrained fields

In this appendix we work on a sphere. Consider the decomposition of a vector field A_μ into its transverse and longitudinal parts:

$$A_\mu \rightarrow A^T_\mu + \nabla_\mu \Phi$$

The spectrum of $-\Box$ on vectors is the disjoint union of the spectrum on transverse and longitudinal vectors. The latter can be related to the spectrum of $-\Box - \frac{R}{d}$ on scalars using the formula

$$-\Box \nabla_\mu \Phi = -\nabla_\mu \left(\Box + \frac{R}{d}\right) \Phi. \tag{A.36}$$

Therefore one can write for the heat kernel

$$\mathrm{Tr}\, \mathrm{e}^{-s(-\Box)} \,|_{A_\mu} = \mathrm{Tr}\, \mathrm{e}^{-s(-\Box)} \,|_{A^T_\mu} + \mathrm{Tr}\, \mathrm{e}^{-s\left(-\Box - \frac{R}{d}\right)} \,|_{\Phi} - \mathrm{e}^{\left(s\frac{R}{d}\right)}. \tag{A.37}$$

The last term has to be subtracted because a constant scalar is an eigenfunction of $-\Box - \frac{R}{d}$ with negative eigenvalue, but does not correspond to an eigenfunction of $-\Box$ on vectors. The spectrum of $-\Box$ on scalars and transverse vectors is obtained from the representation theory of $SO(d+1)$ and is reported in table A.1.

A similar argument works for symmetric tensors, when using the decomposition (3.61). One can use equation

$$-\Box \left(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu\right) = \nabla_\mu \left(-\Box - \frac{d+1}{d\,(d-1)} R\right) \xi_\nu + \nabla_\nu \left(-\Box - \frac{d+1}{d\,(d-1)} R\right) \xi_\mu \tag{A.38}$$

and equation

$$-\Box \left(\nabla_\mu \nabla_\nu - \frac{1}{d} g_{\mu\nu} \Box\right) \sigma = \left(\nabla_\mu \nabla_\nu - \frac{1}{d} g_{\mu\nu} \Box\right) \left(-\Box - \frac{2}{d-1} R\right) \sigma \tag{A.39}$$

to relate the spectrum of various operators on vectors and scalars to the spectrum of $-\Box$ on tensors. One has to observe that the $d(d+1)/2$ Killing vectors are eigenvectors of $-\Box - \frac{d+1}{d(d-1)} R$ on vectors but give a vanishing tensor $h_{\mu\nu}$, so they do not contribute to the spectrum of $-\Box$ on tensors. Likewise, a constant scalar and the $d+1$ scalars proportional to the Cartesian coordinates of the embedding $\mathbf{R}^n$, which correspond to two the lowest eigenvalues of $-\Box - \frac{2}{d-1} R$, also do not contribute to the spectrum of tensors. So one has for the heat kernel on tensors

$$\begin{aligned} \mathrm{Tr}\, \mathrm{e}^{(-s(-\Box))}\Big|_{h_{\mu\nu}} &= \mathrm{Tr}\, \mathrm{e}^{(-s(-\Box))}\Big|_{h^T_{\mu\nu}} + \mathrm{Tr}\, \mathrm{e}^{\left(-s\left(-\Box - \frac{(d+1)R}{d(d-1)}\right)\right)}\Big|_{\xi} + \mathrm{Tr}\, \mathrm{e}^{(-s(-\Box))}\Big|_{h} \\ &+ \mathrm{Tr}\, \mathrm{e}^{\left(-s\left(-\Box - \frac{2}{d-1} R\right)\right)} \,|_{\sigma} - \mathrm{e}^{\left(\frac{2}{d-1} sR\right)} - (d+1)\, \mathrm{e}^{\left(\frac{1}{d-1} sR\right)} - \frac{d\,(d+1)}{2} \mathrm{e}^{\left(\frac{2}{d(d-1)} sR\right)}. \end{aligned} \tag{A.40}$$

The last exponentials can be expanded in Taylor series as $\sum_{m=0}^{\infty} c_m R^m$ and these terms can be viewed as modifications of the heat kernel coefficients of $-\Box$ acting on the differentially

Spin s	Eigenvalue $\lambda_l(d,s)$	Multiplicity $D_l(d,s)$
0	$\frac{l(l+d-1)}{d(d-1)}R;\, l=0,1\dots$	$\frac{(2l+d-1)(l+d-2)!}{l!(d-1)!}$
1	$\frac{l(l+d-1)-1}{d(d-1)}R;\, l=1,2\dots$	$\frac{l(l+d-1)(2l+d-1)(l+d-3)!}{(d-2)!(l+1)!}$
2	$\frac{l(l+d-1)-2}{d(d-1)}R;\, l=2,3\dots$	$\frac{(d+1)(d-2)(l+d)(l-1)(2l+d-1)(l+d-3)!}{2(d-1)!(l+1)!}$

Table A.1.: Eigenvalues and their multiplicities of the Laplacian on the d-sphere

constrained fields. To see where these modifications enter, recall that the volume of the sphere is

$$V_{\mathrm{dS}} = (4\pi)^{\frac{d}{2}} \left(\frac{d\,(d-1)}{R}\right)^{\frac{d}{2}} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\,(d)} \tag{A.41}$$

so that

$$\int \mathrm{d}^d x \sqrt{g}\, \mathrm{tr}\, \mathbf{b}_n \propto R^{\frac{n-d}{2}}\ . \tag{A.42}$$

This means a coefficent c_m from the Taylor series will contribute to a heat kernel coefficient for which $2m = n - d$. So there are contributions to $\mathbf{b}_n$ only for $n \geq d$.

We have discussed how the negative and zero modes from constrained scalar and vector fields affect the heat kernel coefficients of the decomposed vector and tensor fields. These modes have to be excluded also from the traces over the constrained fields; this is denoted by one or two primes, depending on the number of excluded modes. This can be done by calculating the trace and subtracting the contributions to the operator trace from the excluded modes. Thus the trace with m primes is

$$\mathrm{Tr}^{'\cdots'}\left[W(-\Box)\right] = \mathrm{Tr}\left[W(-\Box)\right] - \sum_{l=1}^{m} D_l\,(d,s)\, W\left(\lambda_l\,(d,s)\right) \tag{A.43}$$

where $\lambda_l(d,s)$ are the eigenvalues, $D_l(d,s)$ their multiplicities, both depending on the dimension d and on the spin of the field, s. The eigenvalues and multiplicities for the m-th mode of the Laplacian on the sphere are given in table A.1.

The expressions that we will need are those for the cases where one mode is excluded from the transverse vector trace ($s = 1$, $m = 1$), or one or two modes from the scalar trace ($s = 0$, $m = 1,2$), each one in two and four dimensions. The results obtained by calculating the corresponding multiplicity and eigenvalue from table A.1 are given in table A.2. To see what is the relevant contribution to one of the heat-kernel coefficients, one can expand the obtained expression in R. For the case $s = 0$, $d = 4$, $m = 2$ one has for example

$$\begin{aligned}\sum_{l=1}^{2} D_l\,(4,0)\, W\left(\lambda_l\,(4,0)\right) &= W\,(0) + 5W\left(\frac{R}{3}\right)\\ &= \frac{R^2}{4\,(4\pi)^2}\int \mathrm{d}x\sqrt{g}\left(W\,(0) + \frac{5R}{18}W'\,(0) + \frac{5}{108}R^2 W''\,(0) + \frac{5}{36\cdot 27}R^3 W'''\,(0) + \dots\right)\end{aligned} \tag{A.44}$$

	s=1	s=0
$m=1, d=2$	$3W\left(\frac{R}{2}\right)$	$W(0)$
$m=1, d=4$	$10W\left(\frac{R}{4}\right)$	$W(0)$
$m=2, d=2$		$W(0)+3W(R)$
$m=2, d=4$		$W(0)+5W\left(\frac{R}{3}\right)$

Table A.2.: $\sum_{l=1}^{m} D_l(d,s)\,W(\lambda_l(d,s))$ for $s=0,1$, $d=2,4$, $m=1,2$

	S	V	VT	T	TT	TTT
$\mathrm{tr}\mathbf{b}_0$	1	d	$d-1$	$\frac{d(d+1)}{2}$	$\frac{(d+2)(d-1)}{2}$	$\frac{(d-2)(d+1)}{2}$
$\mathrm{tr}\mathbf{b}_2$	$\frac{R}{6}$	$d\,\frac{R}{6}$	$\frac{(d+2)(d-3)\,R}{6d}+\frac{1}{2}\,R\,\delta_{d,2}$	$\frac{d(d+1)R}{12}$	$\frac{(d+2)(d-1)\,R}{12}$	$\frac{(d+1)(d+2)(d-5)\,R}{12(d-1)}+\frac{7}{2}\,R\,\delta_{d,2}$

Table A.3.: Heat Kernel coefficients for S^d. The columns are for scalar (S), vector (V), transverse vector (TV), tensor (T), traceless-tensor (TT) and transverse traceless tensor (TTT). The contribution proportional to $\delta_{d,2}$ comes from excluded modes. The entries are calculated using (A.6) and (A.7), and (A.37) and (A.40).

From this one sees that, in this case, the $\mathbf{b}_{2n}$ receive a correction for $n \geq 2$. In two dimensions, that would be already the case for $n \geq 1$. The full list of heat kernel coefficients of $-\Box$ in 4d is given in table A.4.

A.3. Proper time ERGE

Let us start from the ERGE for gravity in the Einstein–Hilbert truncation with a type III cutoff, written in equation (3.78). Define the functions:

$$A_k(z) = \frac{\partial_t R_k(z)}{z+R_k(z)} \qquad B_k(z) = \frac{R_k(z)}{z+P_k(z)} \qquad C_k(z) = \frac{\partial_z R_k(z)}{z+R_k(z)}\,. \tag{A.45}$$

The term in equation (3.78) containing C is nontrivial. To rewrite it in a managable form we take the Laplace transform:

$$C_k(z) = \int_0^\infty ds\,\tilde{C}_k(s)\,e^{-sz}\,. \tag{A.46}$$

	S	V	VT	T	TT	TTT
$\mathrm{tr}\mathbf{b}_0$	1	4	3	10	9	5
$\mathrm{tr}\mathbf{b}_2$	$\frac{R}{6}$	$\frac{2R}{3}$	$\frac{R}{4}$	$\frac{5R}{3}$	$\frac{3R}{2}$	$-\frac{5R}{6}$
$\mathrm{tr}\mathbf{b}_4$	$\frac{29R^2}{2160}$	$\frac{43R^2}{1080}$	$-\frac{7R^2}{1440}$	$\frac{11R^2}{216}$	$\frac{81R^2}{2160}$	$-\frac{R^2}{432}$
$\mathrm{tr}\mathbf{b}_6$	$\frac{37R^3}{54432}$	$-\frac{R^3}{17010}$	$-\frac{541R^3}{362880}$	$-\frac{1343R^3}{136080}$	$\frac{-319R^3}{30240}$	$\frac{311R^3}{54432}$
$\mathrm{tr}\mathbf{b}_8$	$\frac{149R^4}{6531840}$	$-\frac{2039R^4}{13063680}$	$-\frac{157R^4}{2488320}$	$-\frac{2999R^4}{3265920}$	$\frac{683R^4}{725760}$	$\frac{109R^4}{1306368}$

Table A.4.: Heat kernel coefficients for S^4. The columns for the transverse vector (VT) and transverse traceless tensor (TTT) are obtained from equations (A.37) and (A.40) in $d = 4$. Note that the excluded modes contribute to $\mathrm{tr}\mathbf{b}_n$ only for $n \geq 4$.

Since the operator $\partial_t(\Delta_2 - 2\Lambda)$ commutes with $\Delta_2 - 2\Lambda$, we can write

$$\begin{aligned} C_k\,(\Delta_2 - 2\Lambda)\,\partial_t(\Delta_2 - 2\Lambda) &= \int_0^\infty ds\,\tilde{C}_k(s)\,\partial_t(\Delta_2 - 2\Lambda)\,e^{-s(\Delta_2-2\Lambda)} \\ &= -\int_0^\infty \frac{ds}{s}\,\tilde{C}_k(s)\,\partial_t e^{-s(\Delta_2-2\Lambda)}\,. \end{aligned} \tag{A.47}$$

Laplace transforming also A_k and B_k, the first term in equation (3.78) becomes

$$\frac{1}{2}\int_0^\infty ds\left[\tilde{A}_k(s) + \tilde{B}_k(s)\,\eta - \frac{1}{s}\,\tilde{C}_k(s)\,\partial_t\right]\mathrm{Tr}\,e^{-s(\Delta_2-2\Lambda)}\,. \tag{A.48}$$

This is the functional RG equation in "proper time" form [91, 92, 93]. Note that the first term corresponds precisely to the one loop approximation. The trace of the heat kernel can be expanded

$$\begin{aligned} \mathrm{Tr}\,e^{-s(\Delta_2-2\Lambda)} &= e^{-s(-2\Lambda)}\frac{1}{(4\pi)^{d/2}}\int dx\sqrt{g}\,\mathrm{tr}\left[\mathbf{1}s^{-\frac{d}{2}} + \left(\mathbf{1}\frac{R}{6} - \mathbf{W}\right)s^{-\frac{d}{2}+1} + O(R^2)\right] \\ &= e^{-s(-2\Lambda)}\frac{1}{(4\pi)^{d/2}}\int dx\sqrt{g}\left[\frac{d(d+1)}{2}s^{-\frac{d}{2}} + \frac{d(7-5d)}{12}R\,s^{-\frac{d}{2}+1} + O(R^2)\right]\,, \end{aligned}$$

whereas for the ghosts

$$\begin{aligned} \mathrm{Tr}\,e^{-s(\delta^\mu_\nu\Delta - R^\mu_\nu)} &= \frac{1}{(4\pi)^{d/2}}\int dx\sqrt{g}\,\mathrm{tr}\left[\delta^\mu_\nu s^{-\frac{d}{2}} + \left(\delta^\mu_\nu\frac{R}{6} + R^\mu_\nu\right)s^{-\frac{d}{2}+1} + O(R^2)\right] \\ &= \frac{1}{(4\pi)^{d/2}}\int dx\sqrt{g}\left[ds^{-\frac{d}{2}} + \frac{d+6}{6}R\,s^{-\frac{d}{2}+1} + O(R^2)\right]\,. \end{aligned}$$

The ERGE then takes the form:

$$\begin{aligned}\partial_t\Gamma_k &= \frac{1}{(4\pi)^{d/2}}\int dx\sqrt{g}\Big\{\frac{d(d+1)}{4}Q_{\frac{d}{2}}\left(\bar{A}_k+\eta\bar{B}_k-2\partial_t\Lambda\,\bar{C}_k\right)-dQ_{\frac{d}{2}}\left(A_k\right)\\ &+\left[\frac{d(7-5d)}{12}\,Q_{\frac{d}{2}-1}\left(\bar{A}_k+\eta\bar{B}_k-2\partial_t\Lambda\,\bar{C}_k\right)-\frac{d+6}{d}\,Q_{\frac{d}{2}-1}\left(A_k\right)\right]\,R+O(R^2)\Big\}\,.\end{aligned}$$

where $\bar{W}(z)=W(z-2\Lambda)$. Using an optimized cutoff one can now reproduce equations (3.80) and (3.81). However, in this way the sums in equation (3.79) can be resummed for any type of cutoff shape.

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Curriculum Vitae

Gaurav Narain was born on 26 of September, 1984 in the capital city Lucknow of the state Uttar Pradesh, India. He completed his secondary education from City Montessori School, Mahanagar branch located in the city Lucknow, India in the year 2001. In the same year he passed the Joint Entrance Examination for admission in Indian Institute of Technology (IIT), and got enrolled in IIT, Kanpur for his undergraduate studies in Physics. He graduated from there in 2006 with a degree in Master of Science (Integrated) in Physics. During the undergraduate studies he visited Indian Institute of Astrophysics (IIA), Banglore, India as a summer student in 2004, and visited Institute of Theoretical Physics, Goethe University, Frankfurt, Germany for summer Internship of three months in 2005. After finishing the undergraduate studies in 2006, he joined SISSA/ISAS in the same year, where he was awarded the esteemed graduate fellowship for four years. There he joined the group of Roberto Percacci and started his doctoral research in the Astroparticle physics in the area of Qauntum Gravity. The results of that research are presented in this thesis.