Moduli Spaces of Semistable Sheaves
on Projective Deligne-Mumford Stacks

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Introduction

The notion of stack was introduced at the beginning of the sixties, during the writing of what is called nowadays SGA4. It is a "natural" refinement of the concept of fibered category: a fibered category where local objects and morphisms can be glued in a "sheaf-like" way. This kind of categorical construction is particularly suitable to deal with moduli problems in an abstract way. First examples of stacks were the Picard stack introduced by Deligne, that is a generalization of the Picard scheme, and Gerbes introduced by Giraud in his PhD thesis.

In 1969 Deligne and Mumford provided an amplification of the definition of stack. They introduced a kind of stack, now called by their names, that not only is an abstract categorical construction, but also can be interpreted as a true geometric object. A Picard stack was conceived as a generalization of a group variety, a gerbe as a generalization of a principal bundle but a Deligne-Mumford stack was a generalization of the notion of algebraic scheme. In 1974 Artin introduced a generalization of the notion of Deligne-Mumford stack. The objects he defined still preserved a geometric meaning, despite of the increased generality, and they are known as Artin stacks or algebraic stacks.

In the last three decades many people developed various geometrical aspects of algebraic stacks in perfect analogy with the scheme theoretic setup. There is a well established sheaf theory for stacks, including the notion of coherent sheaf and a full cohomological machinery. These features are obtained with almost no effort because stacks are founded on the notion of topos so that they already carry a "natural" sheaf theory. Intersection theory for stacks was founded by Vistoli in 1989 and generalized by Kresch in 1999. Deformation theory descends from the seminal work of Illusie about deformation theory for topoi with some modifications to the cotangent complex introduced by Olsson. In 2005 Borisov Chen and Smith extended toric geometry to Deligne-Mumford stacks, and papers of Olsson, Starr and Kresch led to a good notion of projective stack. In the last few years there has been an increasing interest in moduli problems for objects defined on stacks or stack related. In this mainstream we can mention the proof of the representability of the Quot functor on a Deligne-Mumford stack by Olsson and Starr in 2003 and the foundation of Gromov-Witten theory for Deligne-Mumford stacks by Abramovich Graber and Vistoli in 2006. In the same spirit we extend to projective stacks the construction of the moduli space (moduli stack) of semistable sheaves as performed in the scheme theoretic setup by Simpson, Maruyama, Le Potier, Gieseker, Seshadri, Narasimhan, Mukai and many others from the early seventies to the nineties. The second aim of this dissertation is a generalization to projective stacks of Grothendieck/Serre duality for quasicoherent sheaves.

Overview

We define a notion of stability for coherent sheaves on stacks, and construct a moduli stack of semistable sheaves. The class of stacks that is suitable to approach this problem is the class of projective stacks: tame stacks with projective moduli-scheme and a "very ample" locally free sheaf (a *generating sheaf* in the sense of [OS03]). The hypothesis of tameness let us reproduce useful scheme-theoretic results such as a *cohomology and base change* theorem, *semicontinuity* for cohomology and Ext functors and other results related to flatness. The class of projective stacks includes for instance every DM toric stack with projective moduli scheme and more generally every smooth DM stack proper over an algebraically closed field with projective moduli scheme. We also introduce a notion of *family of projective stacks* parameterized by a noetherian finite-type scheme: it is a separated tame global quotient whose geometric fibers are projective stacks. These objects will play the role of projective morphisms.

In the first chapter we recall the notion of tame stack and projective stack and some results about their geometry taken from [AOV08], [Kre06] and [OS03]. Moreover we collect all the results about flatness that we are going to use in the following. The second chapter is essentially a proof that the stack of coherent sheaves on a projective stack is algebraic. This is already stated in more generality (no projectivity is required) in [Lie06]; however, we have decided to include this proof since it is elementary and provides an explicit smooth atlas for the stack of sheaves; it is also a first example of a practical usage of generating sheaves on a stack.

Let $\mathcal{X} \to S$ be a family of projective Deligne-Mumford stacks with moduli scheme X, a chosen polarization $\mathcal{O}_X(1)$ and $\mathcal{E}$ a generating sheaf of $\mathcal{X}$. We denote with $\mathcal{Q}_{N,m} := \mathrm{Quot}_{\mathcal{X}/S}(\mathcal{E}^{\oplus N} \otimes \pi^* \mathcal{O}_X(-m))$ the functor of flat quotient sheaves on $\mathcal{X}$ of the locally free sheaf $\mathcal{E}^{\oplus N} \otimes \pi^* \mathcal{O}_X(-m))$ (N, m are integers). It is proven in [OS03] that this functor is representable and a disjoint union of projective schemes on S.

Theorem (2.25). *Let $\mathcal{U}_{N,m}$ be the universal quotient sheaf of $\mathcal{Q}_{N,m}$. For every couple of integers N, m there is an open subscheme $\mathcal{Q}_{N,m}^0 \subseteq \mathcal{Q}_{N,m}$ (possibly empty) such that:*

$$\coprod_{N,m} \mathcal{Q}_{N,m}^0 \subseteq \coprod_{N,m} \mathcal{Q}_{N,m} \xrightarrow{\coprod H_{N,m}} \mathfrak{Coh}_{\mathcal{X}/S}$$

is a smooth atlas and the map $H_{N,m}$ is given by $\mathcal{U}_{N,m} \otimes \mathcal{End}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E})$.

The stack $\mathfrak{Coh}_{\mathcal{X}/S}$ is a locally finite-type Artin stack.

The second part of the work is devoted to the study of Grothendieck/Serre duality for projective Deligne-Mumford stacks; to be more specific we prove Grothendieck duality for morphisms from a projective Deligne-Mumford stack to a scheme, and Grothendieck duality for proper representable morphisms. This result of duality is used in the last part of the work to handle the definition of dual sheaf in the case of sheaves of non maximal dimension. Given such a sheaf $\mathcal{F}$ on a projective Cohen-Macaulay stack $p \colon \mathcal{X} \to \mathrm{Spec}\, k$ over k an algebraically closed field, the dual $\mathcal{F}^D$ is defined to be $R\,\mathcal{Hom}_{\mathcal{X}}(\mathcal{F}, p^! k)$. If the sheaf is torsion free on a smooth stack this is just the usual definition twisted by $\omega_{\mathcal{X}}$. Using Grothendieck duality we will be able to prove that there is a natural morphism $\mathcal{F} \to \mathcal{F}^{DD}$ and it is injective if and only if the sheaf is pure. We will use this basic result in the GIT study of the moduli scheme of semistable pure sheaves.

The first chapter is foundational, it deals with the existence of the dualizing complex through the abstract machinery developed by Deligne in [Har66] and refined by Neeman in [Nee96]. Having studied the property of flat base change of the dualizing sheaf we are able to prove Serre Duality for smooth projective stacks and duality for finite morphisms. We obtain that the dualizing sheaf for a smooth projective stack is the canonical bundle shifted by the dimension of the stack. For a closed embedding $i\colon \mathcal{X} \to \mathcal{Y}$ in a smooth projective stack $\mathcal{Y}$ the dualizing complex of $\mathcal{X}$ is $\mathcal{E}xt_{\mathcal{Y}}^{\bullet}(\mathcal{O}_{\mathcal{X}}, \omega_{\mathcal{Y}})$ where $\omega_{\mathcal{Y}}$ is the canonical bundle. This is a coherent sheaf if $\mathcal{X}$ is Cohen-Macaulay, an invertible sheaf if it is Gorenstein.

In the second chapter we use this abstract machinery to compute the dualizing sheaf of a projective nodal curve. We prove that the dualizing sheaf of a curve without smooth orbifold points is just the pullback of the dualizing sheaf of its moduli space. Smooth orbifold points give a non trivial contribution that can be computed in a second time using the root construction (Cadman [Cad07], Abramovich-Graber-Vistoli [AGV06]; see also Section). We compute also the dualizing sheaf a local complete intersection reproducing the result already well known in the scheme theoretic setup.

The last part contains the definition of stability and the study of the moduli space of semistable sheaves. The motivation for the kind of stability we propose comes from studies of stability in two well known examples of *decorated sheaves* on projective schemes that can be interpreted as sheaves on algebraic stacks: *twisted sheaves* and *parabolic bundles*. In the case of twisted sheaves it is possible to associate to a projective scheme X and a chosen twisting cocycle $\alpha \in H^2_{\text{ét}}(X, \mathbb{G}_m)$, an abelian $\mathbb{G}_m$-gerbe $\mathcal{G}$ on X such that the category of coherent α-twisted sheaves on X is equivalent to the category of coherent sheaves on $\mathcal{G}$ (Donagi-Pantev [DP03], Căldăraru [Căl00], Lieblich [Lie07]). In the case of a parabolic bundle on X, with parabolic structure defined by an effective Cartier divisor D and some rational weights, it is possible to construct an algebraic stack whose moduli scheme is X by a root construction. It was proven (Biswas [Bis97], Borne [Bor06], [Bor07]) that the category of parabolic bundles on X with parabolic structure on D and fixed rational weights is equivalent to the category of vector bundles on the associated root stack.

Since intersection theory on algebraic stacks was established in [Vis89] and [Kre99] it is possible to define μ-stability for a stack in the usual way. It is proven in [Bor06] that the degree of a sheaf on a root stack is the same as the parabolic-degree defined in [MY92] and used there (and also in [MS80]) to study stability. It is also well known that the degree of a sheaf on a gerbe, banded by a cyclic group, can be used to study stability [Lie07], or equivalently it can be defined a modified degree for the corresponding twisted sheaf on the moduli scheme of the gerbe [Yos06]. From these examples it looks reasonable that the degree of a sheaf on a stack could be a good tool to study stability, and we are lead to think that the naïve definition of μ-stability should work in a broad generality.

However it is already well known that a Gieseker stability defined in the naïve way doesn't work. Let $\mathcal{X}$ be a projective Deligne-Mumford stack with moduli scheme $\pi\colon \mathcal{X} \to X$ and $\mathcal{O}_X(1)$ a polarization of the moduli scheme and $\mathcal{F}$ a coherent sheaf on $\mathcal{X}$. Since π_* is exact and preserves coherence of sheaves and cohomology groups, we can define a Hilbert polynomial:

$$P(\mathcal{F}, m) = \chi(\mathcal{X}, \mathcal{F} \otimes \pi^*\mathcal{O}_X(m)) = \chi(X, \pi_*\mathcal{F}(m))$$

We could use this polynomial to define Gieseker stability in the usual way. We observe immediately that in the case of gerbes banded by a cyclic group this definition is not reasonable at all. A quasicoherent sheaf on such a gerbe splits in a direct sum of eigensheaves of the characters of the cyclic group, however every eigensheaf with non trivial character does not contribute to the Hilbert polynomial and eventually semistable sheaves on the gerbe, according to this definition, are the same as semistable sheaves on the moduli scheme of the gerbe. In the case of root stacks there is a definition of a parabolic Hilbert polynomial and a parabolic Gieseker stability (see [MY92]) which is not the naïve Hilbert polynomial or equivalent to a naïve Gieseker stability; moreover it is proven in [MY92] and in [Bor06] that the parabolic degree can be retrieved from the parabolic Hilbert polynomial, while it is quite unrelated to the naïve Hilbert polynomial. We introduce a new notion of Hilbert polynomial and Gieseker stability which depends not only on the polarization of the moduli scheme, but also on a chosen generating sheaf on the stack (see Def 2.2). If $\mathcal{E}$ is a generating sheaf on $\mathcal{X}$ we define a functor from $\mathfrak{Coh}_{\mathcal{X}/S}$ to $\mathfrak{Coh}_{X/S}$:

$$F_{\mathcal{E}} \colon \mathcal{F} \mapsto F_{\mathcal{E}}(\mathcal{F}) = \pi_* \mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}, \mathcal{F})$$

and the *modified Hilbert polynomial*:

$$P_{\mathcal{E}}(\mathcal{F}, m) = \chi(\mathcal{X}, \mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}, \mathcal{F}) \otimes \pi^* \mathcal{O}_X(m)) = \chi(X, F_{\mathcal{E}}(\mathcal{F})(m))$$

which is a polynomial if $\mathcal{X}$ is tame and the moduli space of $\mathcal{X}$ is a projective scheme. Using this polynomial we can define a Gieseker stability in the usual way. It is also easy to prove that given $\mathcal{X}$ with orbifold structure along an effective Cartier divisor, there is a choice of $\mathcal{E}$ such that this is the parabolic stability, and if $\mathcal{X}$ is a gerbe banded by a cyclic group this is the same stability condition defined in [Lie07] and [Yos06] (the twisted case is developed with some detail in the appendix). There is also a wider class of examples where the degree of a sheaf can be retrieved from this modified Hilbert polynomial (see proposition 5.18).

In order to prove that semistable sheaves form an algebraic stack we prove that Gieseker stability is an open condition. To prove that the moduli stack of semistable sheaves is a finite type global quotient we need to prove that semistable sheaves form a bounded family. To achieve this result we first prove a version of the well known Kleiman criterion, suitable for sheaves on stacks, that is theorem 6.13. In particular we prove that $\mathfrak{F}$ a set-theoretic family of sheaves on a projective stack $\mathcal{X}$ is bounded if and only if the family $F_{\mathcal{E}}(\mathfrak{F})$ on the moduli scheme X is bounded. We are then left with the task of proving that the family of semistable sheaves is mapped by the functor $F_{\mathcal{E}}$ to a bounded family.

First we prove that the functor $F_{\mathcal{E}}$ maps pure dimensional sheaves to pure dimensional sheaves of the same dimension (Proposition 5.7): it preserves the torsion filtration. However it doesn't map semistable sheaves on $\mathcal{X}$ to semistable sheaves on X: he doesn't preserver neither the Harder-Narasimhan filtration nor the Jordan-Hölder filtration. For this reason the boundedness of the family $F_{\mathcal{E}}(\mathfrak{F})$ is not granted for free.

Given $\mathcal{F}$ a semistable sheaf on $\mathcal{X}$ with chosen modified Hilbert polynomial, we study the maximal destabilizing subsheaf of $F_{\mathcal{E}}(\mathcal{F})$ and prove that its slope has an upper bound which doesn't depend on the sheaf $\mathcal{F}$. This numerical estimate, together with the Kleiman criterion for stacks and results of Langer [Lan04b] and [Lan04a] (applied on the moduli

scheme), is enough to prove that semistable sheaves on a projective stack with fixed modified Hilbert polynomial form a bounded family. The theorem of Langer we use here, replaces the traditional Le Potier-Simpson's result [HL97, Thm 3.3.1] in characteristic zero.

The result of boundedness leads to an explicit construction of the moduli stack of semistable sheaves as a global quotient of a quasiprojective scheme by the action of a reductive group.

Let $\mathcal{X}$ be a projective Deligne-Mumford stack over an algebraically closed field k with a chosen polarization, that is a couple $\mathcal{E}, \mathcal{O}_X(1)$ where $\mathcal{E}$ is a generating sheaf and $\mathcal{O}_X(1)$ is a very ample line bundle on the moduli scheme X. Fix an integer m, such that semistable sheaves on $\mathcal{X}$ with chosen modified Hilbert polynomial P are m-regular. Denote with V the linear space $k^{\oplus N} \cong H^0(X, F_{\mathcal{E}}(\mathcal{F})(m))$ where $N = h^0(X, F_{\mathcal{E}}(\mathcal{F})(m)) = P(m)$ for every semistable sheaf $\mathcal{F}$.

Theorem (7.1). *There is an open subscheme $\mathcal{Q}$ in $\mathrm{Quot}_{\mathcal{X}/k}(V \otimes \mathcal{E} \otimes \pi^*\mathcal{O}_X(-m), P)$, such that the algebraic stack of pure dimensional semistable sheaves on $\mathcal{X}$ with modified Hilbert polynomial P is the global quotient:*

$$[\mathcal{Q}/GL_{N,k}] \subseteq [\mathrm{Quot}_{\mathcal{X}/k}(V \otimes \mathcal{E} \otimes \pi^*\mathcal{O}_X(-m), P)/GL_{N,k}]$$

where the group $GL_{N,k}$ acts in the evident way on V.

Using GIT techniques we study in the last chapter the quotient $\mathcal{Q}/GL_{N,k}$. We prove that the open substack of pure stable sheaves has a moduli scheme which is a quasiprojective scheme, while the whole GIT quotient provides a natural compactification of this moduli scheme, and parameterizes classes of S-equivalent semistable sheaves. As in the case of sheaves on a projective scheme the GIT quotient is a moduli scheme of semistable sheaves if and only if there are no strictly semistable sheaves.

Our results on the moduli space of semistable sheaves depend both on the choice of $\mathcal{O}_X(1)$ and of generating sheaf $\mathcal{E}$. As in the case of schemes a change of polarization modifies the geometry of the moduli space of sheaves, we expect also in the case of stacks a change of generating sheaf to produce modifications to the moduli space. For the moment we have not investigated this kind of problem but we will probably approach it in a near future.

We strongly believe that our moduli space of semistable sheaves on a root stack is isomorphic to the moduli space constructed by Maruyama and Yokogawa, however for the moment we only know that the notion of stability we are using is equivalent, under certain assumption, to the parabolic stability they use. To prove that the two moduli spaces are isomorphic we need some deeper investigation of the quasi-isomorphism between the category of coherent sheaves on a root stack and the category of parabolic sheaves. In particular we need to know how the quasi-isomorphism behaves with respect to the pureness of sheaves and if it preserves flat families.

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Conventions and notations

Every scheme is assumed to be noetherian and also every tame stack (Def 1.1) is assumed noetherian if not differently stated. Unless differently stated every scheme, stack is defined over an algebraically closed field. With S we will denote a generic base scheme of finite type over the base field; occasionally it could be an algebraic space but in that case it will be explicitly stated. We will just say *moduli space* for the *coarse moduli space* of an algebraic stack and we will call it *moduli scheme* if it is known to be a scheme. We will always denote with $\pi\colon \mathcal{X} \to X$ the map from an S-stack $\mathcal{X}$ to its moduli space X, with $p\colon \mathcal{X} \to S$ the structure morphism of $\mathcal{X}$. With the name *orbifold* we will always mean a smooth Deligne-Mumford stack of finite type over a field and with generically trivial stabilizers.

We will call a *root stack* an orbifold whose only orbifold structure is along a simple normal crossing divisor. To be more specific let X be a scheme over a field k of characteristic zero. Let $\mathcal{D} = \sum_{i=1}^{n} \mathcal{D}_i$ be a simple normal crossing divisor. Let $\mathbf{a} = (a_1, \ldots, a_n)$ a collection of positive integers . We associate to this collection of data a stack:

$$\sqrt[\mathbf{a}]{D/X} := \sqrt[a_1]{\mathcal{D}_1/X} \times_X \ldots \times_X \sqrt[a_n]{\mathcal{D}_n/X}$$

that we will call a root stack. See [Cad07] and [AGV06] for a comprehensive treatment of the subject.

A projective morphism of schemes $f : X \to Y$ will be projective in the sense of Grothendieck, that is f is projective if there exists a coherent sheaf E on Y such that f factorizes as a closed immersion of X in $\mathbb{P}(E)$ followed by the structure morphism $\mathbb{P}(E) \to Y$.

Part I

Coherent sheaves on projective stacks

Chapter 1

Cohomology and base change

The natural generality to state a *Cohomology and base change* result for algebraic stacks is provided by the concept of *tame stack*. We recall the definition of tame stack from [AOV08]. Let S be a scheme and $\mathcal{X} \to S$ an algebraic stack locally of finite type over S. Assume that the stack has finite stabilizer, that is the natural morphism $I_\mathcal{X} \to \mathcal{X}$ is finite. Under this hypothesis $\mathcal{X}$ has a moduli space $\pi : \mathcal{X} \to X$ and the morphism π is proper [KM97].

Definition 1.1. Let $\mathcal{X}$ be an algebraic stack with finite stabilizer as above and moduli space $\pi : \mathcal{X} \to X$. The stack $\mathcal{X}$ is *tame* if the functor $\pi_* : \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(X)$ is exact where QCoh is the category of quasicoherent sheaves.

We recall also the main result in [AOV08, Thm 3.2]:

Theorem 1.2. *The following conditions are equivalent:*

1. *$\mathcal{X}$ is tame.*

2. *For every k algebraically closed field with a morphism $\mathrm{Spec}\, k \to S$ and every $\xi \in \mathcal{X}(\mathrm{Spec}\, k)$ an object, the stabilizer at the point ξ (which is the group scheme $\mathcal{A}ut_k(\xi) \to \mathrm{Spec}\, k$) is linearly reductive.*

3. *There exists an fppf cover $X' \to X$, a linearly reductive group scheme $G \to X'$ acting on a finite and finitely presented scheme $U \to X'$, together with an isomorphism*

$$\mathcal{X} \times_X X' \cong [U/G]$$

   of algebraic stacks over X'.

4. *The same as the previous statement but $X' \to X$ is an étale cover.*

For the definition of a *linearly reductive* group scheme see in the same paper the second section and in particular definition 2.4.

We recall also the results in [AOV08, Cor 3.3]

Corollary 1.3. *Let $\mathcal{X}$ be a tame stack over a scheme S and let $\mathcal{X} \to X$ be its moduli space:*

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1. If $X' \to X$ is a morphism of algebraic spaces, then X' is the moduli space of $X' \times_X \mathcal{X}$.

2. If $\mathcal{X}$ is flat over S then X is flat over S.

3. Let $\mathcal{F} \in QCoh(\mathcal{X})$ be a flat sheaf over S, then $\pi_* \mathcal{F}$ is quasicoherent and if we assume $\mathcal{X}$ to be Deligne-Mumford $\pi_* \mathcal{F}$ is flat over S.

Remark 1.4. For the convenience of the reader we recall also the following properties:

1. the functor π_* maps coherent sheaves to coherent sheaves. A proof can be found in [AV02, Lem 2.3.4]

2. the natural map $\mathcal{O}_X \to \pi_* \mathcal{O}_\mathcal{X}$ is an isomorphism

3. since π_* is an exact functor on $QCoh(\mathcal{X})$ and maps injective sheaves to flasque sheaves (Lem 1.10), we have that $H^\bullet(\mathcal{X}, \mathcal{F}) \cong H^\bullet(X, \pi_* \mathcal{F})$ for every quasicoherent sheaf $\mathcal{F}$.

In order to reproduce the *Cohomology and base change* theorem as in [Har77] or [Mum70] for an algebraic stack we need the following statement about tame Deligne-Mumford stacks:

Proposition 1.5. *Let $\mathcal{X}$ be a tame Deligne-Mumford stack with moduli space $\pi : \mathcal{X} \to X$ and $\rho : X' \to X$ a morphism of algebraic spaces. Consider the 2-cartesian diagram:*

$$
\begin{array}{ccc}
\mathcal{X} \times_X X' & \xrightarrow{\sigma} & \mathcal{X} \\
\downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\
X' & \xrightarrow{\rho} & X
\end{array}
$$

For every quasicoherent sheaf $\mathcal{F}$ on $\mathcal{X}$ the natural morphism $\rho^* \pi_* \mathcal{F} \to \pi'_* \sigma^* \mathcal{F}$ is an isomorphism.

Proof. Since the problem is local in both X and X' we can assume that $X = \operatorname{Spec} A$ and $X' = \operatorname{Spec} A'$ are affine schemes and the base scheme S is X. Applying theorem 1.2.3 we may assume that $\mathcal{X} = [\operatorname{Spec} B/G]$ where G is a finite linearly reductive group on $\operatorname{Spec} k$ (the base field) acting on $\operatorname{Spec} B$, the map $\operatorname{Spec} B \to \operatorname{Spec} A$ is finite and of finite presentation and $A = B^G$. By the same theorem we obtain that the fibered product $\mathcal{X}' \times_X X'$ is $[Spec(B \otimes_A A')/G]$ where the action of G is induced by the action of G on B and $A' = (B \otimes_A A')^G$. In this setup a coherent sheaf $\mathcal{F}$ is a finitely generated B-module M which is equivariant for the groupoid $G \times_{\operatorname{Spec} A} \operatorname{Spec} B \overset{p}{\underset{a}{\rightrightarrows}} \operatorname{Spec} B$ where the two arrows p, a are respectively the projection and the action. We have also an induced G-equivariant structure on the A-module $_A M$ where G acts trivially on A. To prove the proposition is now the same as proving that the natural morphism:

$$
A' \otimes_A (_A M)^G \overset{\psi_M}{\longrightarrow} (M \otimes_A A')^G \tag{0.1}
$$

is an isomorphism. The equivariant structure of the $B \otimes_A A'$-module $M \otimes_A A'$ is the obvious one; the G-invariant part of a module can be computed as follows: take the

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coaction $_AM \xrightarrow{\alpha} M \otimes \mathcal{O}_G$ and the trivial coaction $_AM \xrightarrow{\iota} M \otimes_A \mathcal{O}_G$ mapping $m \mapsto m \otimes 1$; the G-invariant part $_AM^G$ is $\ker \alpha - \iota$. Since B is finitely generated as an A-module, the A-module $_AM$ is finitely generated (the push forward of a coherent sheaf to the moduli space is coherent). Moreover $_AM$ admits a finite free presentation $P_2 \to P_1 \to {}_AM \to 0$. Since the tensor product $P_i \otimes \mathcal{O}_G$ is a flat resolution of $M \otimes \mathcal{O}_G$ the resolution P_i inherits an equivariant structure from $_AM$.

First we prove the statement for P a projective A-module. To construct the natural map ψ_P we start from the following exact diagram of A-modules:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & P^G & \longrightarrow & P & \xrightarrow{\alpha - \iota} & P \otimes \mathcal{O}_G \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & (P \otimes_A A')^G & \longrightarrow & P \otimes_A A' & \xrightarrow{(\alpha - \iota) \otimes \mathrm{id}} & P \otimes \mathcal{O}_G \otimes_A A'
\end{array}
$$

where the vertical map $P \to P \otimes_A A'$ is induced by $A \to A'$. We apply now the functor $\otimes_A A'$ and obtain:

$$
\begin{array}{ccccc}
A' \otimes_A P^G & \longrightarrow & A' \otimes_A P & \longrightarrow & A' \otimes_A P \otimes \mathcal{O}_G \\
\downarrow{\scriptstyle \psi_P} & & \downarrow{\scriptstyle \wr} & & \downarrow{\scriptstyle \wr} \\
0 \longrightarrow (P \otimes_A A')^G & \longrightarrow & P \otimes_A A' & \longrightarrow & P \otimes \mathcal{O}_G \otimes_A A'
\end{array}
$$

Since G is linearly reductive on a field and it acts on P with a finite representation the A-module P^G is a direct summand of P and the A'-module $(P \otimes_A A')^G$ is also a direct summand of $P \otimes_A A'$ of the same rank. Since the morphism ψ_P is a surjective morphism between two free A'-modules of the same rank it is an isomorphism.

Since the formation of ψ_M is functorial and the free resolution of M is compatible with the coaction we obtain:

$$
\begin{array}{ccccccc}
A' \otimes_A P_2^G & \longrightarrow & A' \otimes_A P_1^G & \longrightarrow & A' \otimes_A ({}_AM)^G & \longrightarrow & 0 \\
{\scriptstyle \wr} \downarrow {\scriptstyle \psi_2} & & {\scriptstyle \wr} \downarrow {\scriptstyle \psi_1} & & \downarrow {\scriptstyle \psi_M} & & \\
(A' \otimes_A P_2)^G & \longrightarrow & (A' \otimes_A P_1)^G & \longrightarrow & (A' \otimes_A M)^G & \longrightarrow & 0
\end{array}
$$

We have exactness on the right since G if a finite group and in particular linearly reductive. Eventually ψ_M is an isomorphism since the other two columns are isomorphisms. To extend the proof to quasicoherent sheaves we first observe that a quasi coherent sheaf is just a B-module N with a coaction. Quasi coherent sheaves on stacks are filtered limits of coherent sheaves, so we can assume that N and the coaction are a filtered limit of coherent equivariant B-modules M_λ. We first observe that the tensor product commutes with filtered limits because it has a right adjoint. The functor $()^G$ commutes with filtered limits because it involves a tensor product and a kernel (which is a finite limit). The result follows now from the statement in the coherent case. $\qquad \square$

Remark 1.6. We don't know if the previous statement is true if we drop the hypothesis "Deligne-Mumford". In that generality $G \to \operatorname{Spec} A$ would be a linearly reductive flat group scheme, and given P a free A-module we don't know if P^G is again a direct summand of P. Apart from this dubious point, the rest of the proof holds true for G linearly reductive. We notice also that we can drop the hypothesis "Deligne-Mumford" in Corollary 1.33 if P^G is a direct summand of P.

Theorem 1.7 (Cohomology and base change)**.** *Let $p : \mathcal{X} \to S$ be a tame Deligne-Mumford stack over S with moduli scheme $\pi : \mathcal{X} \to X$ and such that $q : X \to S$ is projective (proper is actually enough). Let $\operatorname{Spec} k(y) \to S$ be a point. Let $\mathcal{F}$ be a quasicoherent sheaf on $\mathcal{X}$ flat over S. Then:*

1. *if the natural map*

$$\phi^i(y) : R^i p_* \mathcal{F} \otimes k(y) \to H^i(\mathcal{X}_y, \mathcal{F}_y)$$

   is surjective, then it is an isomorphism, and the same is true for all y' in a suitable neighborhood of y;

2. *Assume that $\phi^i(y)$ is surjective. Then the following conditions are equivalent:*

   (a) *$\phi^{i-1}(y)$ is also surjective;*

   (b) *$R^i p_* \mathcal{F}$ is locally free in a neighborhood of y.*

Proof. It follows from 1.3.3 that $\pi_* \mathcal{F}$ is flat over S and according to [Har77, Thm 12.11] the statement is true for the quasicoherent sheaf $\pi_* \mathcal{F}$ and the natural map $\psi^i(y) :$ $R^i q_*(\pi_*(\mathcal{F})) \otimes k(y) \to H^i(X_y, (\pi_* \mathcal{F})_y)$. Since π_* is exact we have $R^i q_* \circ \pi_* \cong R^i(q_* \circ \pi_*)$. Applying 1.5 we deduce that $(\pi_* \mathcal{F})_y$ is isomorphic to $\pi_{y_*}(\mathcal{F}_y)$. According to 1.3.1 the morphism $\pi_y : \mathcal{X}_y \to X_y$ is the moduli scheme of $\mathcal{X}_y$ so that π_y is exact and we can conclude that $H^i(X_y, \pi_{y_*}(\mathcal{F}_y)) \cong H^i(\mathcal{X}_y, \mathcal{F}_y)$. $\square$

Repeating exactly the same proof we can reproduce the *Semicontinuity* theorem and a standard result of *Flat base change*.

Theorem 1.8 (Semicontinuity)**.** *Let $p : \mathcal{X} \to S$ be a tame Deligne-Mumford stack over S with moduli scheme $\pi : \mathcal{X} \to X$ and $q : X \to S$ is projective. Let $\mathcal{F}$ be a quasicoherent sheaf on $\mathcal{X}$ flat over S. Denote with y a point of S. For every $i \geq 0$ the function $y \mapsto h^i(\mathcal{X}_y, \mathcal{F}_y)$ is upper semicontinuous on S.*

Theorem 1.9. *Let $p : \mathcal{X} \to S$ be a separated tame Deligne-Mumford stack over S; let $u : S' \to S$ be a flat morphism and $\mathcal{F}$ a quasicoherent sheaf on $\mathcal{X}$.*

$$
\begin{array}{ccc}
\mathcal{X} \times_S S' & \xrightarrow{\ v\ } & \mathcal{X} \\
\downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\
S' & \xrightarrow{\ u\ } & S
\end{array}
$$

For all $i \geq 0$ the natural morphisms $u^* R^i p_* \mathcal{F} \to R^i p'_*(v^* \mathcal{F})$ are isomorphisms.

We conclude the chapter with the following lemma, proving that π_* maps injectives to flasque sheaves (as anticipated in Remark 1.4). We guess it is well known to experts since many years, nevertheless we prefer to write a proof for lack of references.

Lemma 1.10. *Assume $\pi \colon \mathcal{X} \to X$ is a tame stack and $\mathcal{F}$ is an abelian sheaf on $\mathcal{X}$. If $\mathcal{I}$ is an injective sheaf on $\mathcal{X}$, the pushforward $\pi_* \mathcal{I}$ is flasque[1] on X.*

---

[1] A sheaf on a site is flasque if it is acyclic on every object of the site (in agreement with Milne)

Proof. We choose a smooth presentation $X_0 \to \mathcal{X}$ and we associate to it the simplicial nerve $X^\bullet$. Let $f^i \colon X_i \to X$ be the obvious composition. For every sheaf $\mathcal{I}$ on $\mathcal{X}$ represented by $\mathcal{I}^\bullet$ on $X^\bullet$ we have a resolution (see [Ols07, Lem 2.5]):

$$0 \to \pi_* \mathcal{I} \to f_*^0 \mathcal{I}_0 \to f_*^1 \mathcal{I}_1 \to \dots \tag{0.2}$$

Assume now that $\mathcal{I}$ is injective, according to [Ols07, Cor 2.5] the sheaves $\mathcal{I}_i$ are injective for every i so that $H^p(X_q, \mathcal{I}_q)$ is zero for every $p > 0$ and every q and $H^p(\mathcal{X}, \mathcal{I})$ is zero for every $p > 0$. Using [Ols07, Cor 2.7] and [Ols07, Th 4.7] we have a spectral sequence $E_1^{p,q} = H^p(X_q, \mathcal{I}_q)$ abutting to $H^{p+q}(\mathcal{X}, \mathcal{I})$; for our previous observation this sequence reduces to the complex:

$$H^0(X_0, \mathcal{I}_0) \to H^0(X_1, \mathcal{I}_1) \to H^0(X_2, \mathcal{I}_2) \to \dots \tag{0.3}$$

Now we observe that, being $\mathcal{I}_q$ injective, $R^p f_*^q \mathcal{I}_q = 0$ for every $p > 0$ [Mil80, III 1.14]. Using the Leray spectral sequence [Mil80, III 1.18] we have that $H^0(X_i, \mathcal{I}_q) = H^0(X, f_*^q \mathcal{I}_q)$. The resolution (0.2) is actually a flasque resolution of $\pi_* \mathcal{I}$ (apply Lemma [Mil80, III 1.19]) and applying the functor $\Gamma(X, \cdot)$ it becomes resolution (0.3). This proves that $H^i(X, \pi_* \mathcal{I}) = H^i(\mathcal{X}, \mathcal{I})$ and eventually zero for $i > 0$. With the same argument (and actually applying Proposition 1.5) we can prove that $\pi_* \mathcal{I}$ is acyclic on every open of the étale site of X and conclude that $\pi_* \mathcal{I}$ is flasque using [Mil80, III 2.12.c]. $\qquad\square$

Remark 1.11 (psychological). We don't know if π_* maps injectives to injectives. If π is flat (gerbes and root stacks) the answer is trivially yes, but in non flat cases we guess it could be false.

1   More results related to flatness

In this part of the work we collect a few technical results taken from EGA which are related to flatness. We put these in a separate section since they are technically necessary but not so interesting in their own right. First of all we can reproduce for algebraic stacks a result of generic flatness [EGAIV.2, 6.9.1].

Lemma 1.12 (a result of hard algebra [EGAIV.2, 6.9.2])**.** *Let A be a noetherian and integral ring and B a finite type A-algebra; M a finitely generated B-module. There is a principal open subscheme $\operatorname{Spec} A_f$ such that M_f is a free A_f-module.*

Proposition 1.13. *Let $\mathcal{X} \to S$ be a finite type noetherian algebraic stack. Let $\mathcal{F}$ be a coherent $\mathcal{O}_\mathcal{X}$-module. There is a finite stratification $\coprod S_i \to S$ where S_i are locally closed in S such that, denoted with $\mathcal{X}_{S_i}$ the fibered product $\mathcal{X} \times_S S_i$ the $\mathcal{O}_{\mathcal{X}_i}$-module $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{O}_{S_i}$ is flat on $\mathcal{O}_{S_i}$.*

Proof. The proof works more or less as in the case of finite type noetherian schemes. First of all S is noetherian and has finitely many irreducible components. We can work on each irreducible component taking the open subscheme (with the reduced structure) that doesn't intersect other components. We are now in the case where S is integral, we can also assume that it is an affine $\operatorname{Spec} A$ taking a finite cover. Now let $X_0 \to \mathcal{X}$ be an atlas. Since $X_0 \to S$ is of finite type $X_0 = \bigcup_i \operatorname{Spec} B_i$ is a finite union of affine schemes

where B_i is a finitely generated A-algebra. A cartesian coherent sheaf is now a collection of finitely generated B_i-modules M_i plus additional structures, and it is A-flat if and only if each M_i is A-flat. We can now apply the lemma for every i and find open subschemes $\operatorname{Spec} A_i \subset \operatorname{Spec} A$ such that $M_i \otimes_A A_i$ is A_i-flat for every i. If we take the intersection of every $\operatorname{Spec} A_i$ we have an open of $\operatorname{Spec} A$ with the desired property. We complete the proof by noetherian induction on S. $\qquad\square$

Remark 1.14. The previous result is obviously weaker then a flattening-stratification result. In the case of a projective scheme it is possible to prove the existence of the flattening-stratification using generic-flatness with some cohomology and base change and some extra feature coming from the projective structure. In [OS03] Olsson and Starr proved a deeper result for stacks, that is the existence of *the* flattening stratification[2]; with no assumption of noetherianity they can produce a surjective quasi-affine morphism to S (which seems to be the optimal result in such a generality). They conjectured also that the flattening stratification is labeled by "generalized" Hilbert polynomial (as defined in the same paper).

We state a stack theoretic version of [EGAIII.2, 6.9.9.2] which is similar to 1.9 but it can be used in the case of an arbitrary base change.

Proposition 1.15. *Let $p : \mathcal{X} \to S$ be a separated tame Deligne-Mumford stack over S with S-projective moduli scheme $\pi : \mathcal{X} \to X$; let $u : S' \to S$ be a morphism of schemes and $\mathcal{F}$ a coherent sheaf on $\mathcal{X}$ which is flat on $\mathcal{O}_S$ and such that $R^i p_* \mathcal{F}$ are locally free for every $i \geq 0$, then for all $i \geq 0$ the natural morphisms $u^* R^i p_* \mathcal{F} \to R^i p'_*(v^* \mathcal{F})$ are isomorphisms.*

Proof. It follows from [EGAIII.2, 6.9.9.2] applying proposition 1.5. For a direct proof consider a point s in S' and take the fiber $\mathcal{X}_s$ with moduli scheme X_s. The cohomology on the fiber $H^i(\mathcal{X}_s, \mathcal{F}_s)$ can be computed as $H^i(X_s, \pi_{s*}\mathcal{F}_s)$ which in turn can be computed in Zariski topology since $\pi_{s*}\mathcal{F}_s$ is coherent. By Grothendieck vanishing there is a positive integer q such that $H^i(\mathcal{X}_s, \mathcal{F}_s) = 0$ for every $i \geq q$. Using the result of generic flatness 1.13 we deduce that there exists a finite stratification of S' such that the fiber X_s is a flat family embedded in $\mathbb{P}^N_{S'}$ for some N. In particular the number q can be chosen so that it doesn't depend on the point s. Applying recursively 1.7 starting from $i = q$ we obtain that $R^i p_* \mathcal{F} \otimes_k k(s) \cong H^i(\mathcal{X}_s, \mathcal{F}_s)$ for every $i \geq 0$. Since this is true for every point in S' the statement follows. $\qquad\square$

We put here also a classical criterion about flatness of fibers which is theorem [EGAIV.3, 11.3.10]. This will be used to fix a detail in the proof of the Kleiman criterion 6.13. This kind of result cannot be deduced from the analogous result for the moduli space, since flatness of $\mathcal{X}$ on S implies flatness of X on S, but the contrary is not true. First we recall the statement in the affine case:

Lemma 1.16 (Lemme [EGAIV.3, 11.3.10.1])**.** *Let $A \to B$ be a local homomorphism of noetherian local rings. Let k be the residue field of A and M be a finitely generated non zero B-module. The following two conditions are equivalent:*

1. M is A flat and $M \otimes_A k$ is a flat $B \otimes_A k$-module.

---

[2]The flattening stratification is the one coming with a nice universal property.

2. B is a flat A-module and M is B-flat.

Proposition 1.17 (flatness for fibers)**.** *Let $p\colon \mathcal{X} \to S$ be a tame stack locally of finite type with moduli space $\pi\colon \mathcal{X} \to X$. Let $\mathcal{F}$ be a coherent $\mathcal{O}_\mathcal{X}$-module flat on $\mathcal{O}_S$. Let x be a point of $\mathcal{X}$ and $s = p(x).$ The following statements are equivalent:*

1. *$\mathcal{F}$ is flat at the point x and the fiber $\mathcal{F}_s$ is flat at x.*

2. *The morphism π is flat at the point x and $\mathcal{F}$ is flat at x.*

If one of the two conditions is satisfied for a point x then there is an open substack of $\mathcal{X}$ such that for every point in it, the condition is satisfied.

Proof. We can reduce to the affine case using Theorem 1.2 as we have done in the previous proofs. Let X_0 be an atlas of $\mathcal{X}$, we can assume that $X_0 \to \mathcal{X} \to S$ is a finite type morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ where $\operatorname{Spec} B$ is a smooth chart of the atlas of $\mathcal{X}$ containing the point x (it exists according to the smooth neighborhood theorem [LMB00, thm. 6.3]) and $\operatorname{Spec} A$ is an open affine in S. The coherent module $\mathcal{F}$ is an equivariant B-module and we can apply lemma 1.16. The last part of the proposition follows from the result about the open nature of flatness in the affine case ([EGAIV.3, 11.1.1.1]) applied to the atlas $\operatorname{Spec} B$; there is an open subscheme of $\operatorname{Spec} B$ with the desired property and it is mapped to an open substack since the morphism is representable and flat. $\square$

Chapter 2

The algebraic stack of coherent sheaves on a projective stack

In this chapter we prove that $\mathfrak{Coh}_{\mathcal{X}/S}$, the stack of coherent sheaves over an algebraic stack $\mathcal{X} \to S$, is algebraic if the stack $\mathcal{X}$ is tame Deligne-Mumford and satisfies some additional conditions. For every S-scheme U, the objects in $\mathfrak{Coh}_{\mathcal{X}/S}$ are all the coherent sheaves on $\mathcal{X}_U = \mathcal{X} \times_S U$ which are $\mathcal{O}_U$-flat. Morphisms are isomorphisms of $\mathcal{O}_{\mathcal{X}_U}$-modules. We can also define a functor of flat quotients of a given coherent sheaf $\mathcal{F}$, and we will denote it by $\mathrm{Quot}_{\mathcal{X}/S}(\mathcal{F})$ in the usual way. We have seen in the previous chapter that, if $\mathcal{X}$ is tame, we have the same results of cohomology and base change and semicontinuity we have on schemes. To prove that $\mathfrak{Coh}_{\mathcal{X}/S}$ is algebraic we need some more structure. We need a polarization on the moduli scheme of $\mathcal{X}$ and a *very ample* sheaf on $\mathcal{X}$. It is known that there are no very ample invertible sheaves on a stack unless it is an algebraic space, however it was proven in [OS03] that, under certain hypothesis, there exist locally free sheaves, called *generating sheaves*, which behave like "very ample sheaves". Moreover in [EHKV01] is introduced another class of locally free sheaves that could be interpreted as "ample" sheaves on stacks. Relations between these two classes of sheaves and the ordinary concept of ampleness are explained with some details in [Kre06]. We briefly recall these notions. Let $\pi\colon \mathcal{X} \to X$ be a Deligne-Mumford S-stack with moduli space X:

Definition 2.1. A locally free sheaf $\mathcal{V}$ on $\mathcal{X}$ is π-*ample* if for every geometric point of $\mathcal{X}$ the representation of the stabilizer group at that point on the fiber is faithful.

Definition 2.2. A locally free sheaf $\mathcal{E}$ on $\mathcal{X}$ is π-*very ample* if for every geometric point of $\mathcal{X}$ the representation of the stabilizer group at that point contains every irreducible representation.

The following proposition is the reason why we have decided to use the word "ample" for the first class of sheaves.

Proposition 2.3 ([Kre06, 5.2]). *Let $\mathcal{V}$ be a π-ample sheaf on $\mathcal{X}$, there is a positive integer r such that the locally free sheaf $\bigoplus_{i=0}^{r} \mathcal{V}^{\otimes i}$ is π-very ample.*

We recall here the notion of generating sheaf together with the existence result in [OS03]. Let $\mathcal{X}$ be a tame S-stack.

Definition 2.4. Let $\mathcal{E}$ be a locally free sheaf on $\mathcal{X}$. We define a functor $F_{\mathcal{E}} : \mathfrak{QCoh}_{\mathcal{X}/S} \to \mathfrak{QCoh}_{X/S}$ mapping $\mathcal{F} \mapsto \pi_* \mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}, \mathcal{F})$ and a second functor $G_{\mathcal{E}} : \mathfrak{QCoh}_{X/S} \to \mathfrak{QCoh}_{\mathcal{X}/S}$ mapping $\mathcal{F} \mapsto \pi^* \mathcal{F} \otimes \mathcal{E}$.

Remark 2.5.   1. The functor $F_{\mathcal{E}}$ is exact since the dual $\mathcal{E}^{\vee}$ is locally free and the push-forward π_* is exact. The functor $G_{\mathcal{E}}$ is not exact unless the morphism π is flat. This happens for instance if the stack is a flat gerbe over a scheme or in the case of root stacks.

2. (Warning) The notation $F_{\mathcal{E}}$ is the same as in [OS03] but $G_{\mathcal{E}}$ is not. What they called $G_{\mathcal{E}}$ there, is actually our $G_{\mathcal{E}} \circ F_{\mathcal{E}}$.

Definition 2.6. A locally free sheaf $\mathcal{E}$ is said to be a *generator* for the quasi coherent sheaf $\mathcal{F}$ if the adjunction morphism (left adjoint of the identity of the identity $\pi_* F \otimes \mathcal{E}^{\vee} \xrightarrow{\mathrm{id}} \pi_* F \otimes \mathcal{E}^{\vee}$):

$$\theta_{\mathcal{E}}(\mathcal{F}) : \pi^* \pi_* \mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \to \mathcal{F} \tag{0.1}$$

is surjective. It is a *generating* sheaf of $\mathcal{X}$ if it is a generator for every quasicoherent sheaf on $\mathcal{X}$.

Proposition 2.7. *[OS03, 5.2] A locally free sheaf on a tame Deligne-Mumford stack $\mathcal{X}$ is a generating sheaf if and only if it is π-very ample.*

   In the following we will use the word generating sheaf or π-very ample (or just very ample) sheaf interchangeably. The property expressed by (0.1) suggests that a generating sheaf should be considered as a very ample sheaf relatively to the morphism $\pi \colon \mathcal{X} \to X$. Indeed the fundamental theorem of Serre [EGAIII.1, Thm 2.2.1] says that: if $f \colon Y \to Z$ is a proper morphism and $\mathcal{O}_Y(1)$ is a very ample invertible sheaf on Y with respect to f, then there is a positive integer n such that the adjunction morphism $f^* f_* \mathcal{H}om(\mathcal{O}_Y(-n), \mathcal{F}) \otimes \mathcal{O}_Y(-n) \to \mathcal{F}$ is surjective for every coherent sheaf $\mathcal{F}$ on Y.
   As we have defined $\theta_{\mathcal{E}}$ as the left adjoint of the identity we can define $\varphi_{\mathcal{E}}$ the right adjoint of the identity. In order to do this we recall the following lemma from [OS03]:

Lemma 2.8. *Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_{\mathcal{X}}$-module and G a coherent O_X-module. A projection formula holds:*

$$\pi_*(\pi^* G \otimes \mathcal{F}) = G \otimes \pi_* \mathcal{F}$$

Moreover it is functorial in the sense that if $\alpha \colon \mathcal{F} \to \mathcal{F}'$ is a morphism of quasicoherent sheaves and $b \colon G \to G'$ is a morphism of coherent sheaves we have

$$\pi_*(\pi^* b \otimes \alpha) = b \otimes \pi_* \alpha$$

Proof. We can prove the statement working locally. If we assume that G is coherent it has a finite free presentation and we conclude using exactness of π_*, right exactness of π^* and $\otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{F}$ and the projection formula in the free case. Functoriality follows with a similar argument. We can extend the result to quasicoherent sheaves with a standard limit argument. $\square$

Let F be a quasicoherent $\mathcal{O}_X$-module:

$$F \xrightarrow{\varphi_{\mathcal{E}}(F)} \pi_* \mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}, \pi^* F \otimes \mathcal{E}) = F_{\mathcal{E}}(G_{\mathcal{E}}(F))$$

According to lemma 2.8 it can be rewritten as:

$$F \xrightarrow{\varphi_{\mathcal{E}}(F)} F \otimes \pi_* \mathcal{E}nd_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}) \tag{0.2}$$

and it is the map given by tensoring a section with the identity endomorphism and in particular it is injective.

Lemma 2.9. *Let $\mathcal{F}$ be a quasicoherent sheaf on $\mathcal{X}$. The following composition is the identity:*

$$F_{\mathcal{E}}(\mathcal{F}) \xrightarrow{\varphi_{\mathcal{E}}(F_{\mathcal{E}}(\mathcal{F}))} F_{\mathcal{E}} \circ G_{\mathcal{E}} \circ F_{\mathcal{E}}(\mathcal{F}) \xrightarrow{F_{\mathcal{E}}(\theta_{\mathcal{E}}(\mathcal{F}))} F_{\mathcal{E}}(\mathcal{F})$$

Let H be a coherent sheaf on X then the following is the identity

$$G_{\mathcal{E}}(H) \xrightarrow{G_{\mathcal{E}}(\varphi_{\mathcal{E}}(H))} G_{\mathcal{E}} \circ F_{\mathcal{E}} \circ G_{\mathcal{E}}(H) \xrightarrow{(\theta_{\mathcal{E}}(G_{\mathcal{E}}(H)))} G_{\mathcal{E}}(H)$$

Proof. This statement is precisely [ML98, IV Thm 1]. It's also easy to explicitly compute the composition because in the first statement the second map is the composition of $\mathcal{E}nd_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E})$ with $\mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}, \mathcal{F})$ while the first one is tensoring with the identity; in the second the first map is tensoring with the identity endomorphism of $\mathcal{E}$ while the second is $\mathrm{id}_{\pi^* H} \otimes \theta_{\mathcal{E}}(\mathcal{E})$. $\square$

As we have said before there are no very ample invertible sheaves on a stack which is not an algebraic space, however there can be ample invertible sheaves.

Example 2.10. Let $\mathcal{X}$ be a global quotient $[U/G]$ where U is a scheme and G a finite group. We have a natural morphism $\iota \colon [U/G] \to BG$. Let V be the sheaf on BG given by the left regular representation; the sheaf $\iota^* V$ is a generating sheaf of $\mathcal{X}$.

Example 2.11. A root stack $\mathcal{X} := \sqrt[r]{\mathcal{D}/X}$ over a scheme X has an obvious ample invertible sheaf which is the tautological bundle $\mathcal{O}_{\mathcal{X}}(\mathcal{D}^{\frac{1}{r}})$ associated to the orbifold divisor. If the orbifold divisor has order r the locally free sheaf $\bigoplus_{i=0}^{r-1} \mathcal{O}_{\mathcal{X}}(\mathcal{D}^{\frac{i}{r}})$ is obviously very ample and it has minimal rank.

Example 2.12. A gerbe over a scheme banded by a cyclic group μ_r has an obvious class of ample locally free sheaves which are the twisted bundles, and there is an ample invertible sheaf if and only if the gerbe is essentially trivial (see [Lie07, Lem 2.3.4.2]). As in the previous example if $\mathcal{T}$ is a twisted locally free sheaf, $\bigoplus_{i=0}^{r-1} \mathcal{T}^{\otimes i}$ is very ample.

Example 2.13. Let $\mathcal{X}$ be a weighted projective space, the invertible sheaf $\mathcal{O}_{\mathcal{X}}(1)$ is ample, and denoted with m the least common multiple of the weights, $\bigoplus_{i=1}^{m} \mathcal{O}_{\mathcal{X}}(i)$ is very ample. Usually it is not of minimal rank.

Example 2.14. If $\mathcal{X}$ is a toric orbifold with $\mathcal{D}_i$, $1 \leq i \leq n$ the T-divisors associated to the coordinate hyperplanes, the locally free sheaf $\bigoplus_{i=1}^n \mathcal{O}_{\mathcal{X}}(\mathcal{D}_i)$ is ample. Indeed if $\mathcal{X} = [Z/G]$ where Z is quasi affine in $\mathbb{A}^n$ and G is a diagonalizable group scheme and the action of G on Z is given by irreducible representations χ_i for $i = 1, \ldots, n$ then the map

$$ 1 \longrightarrow G \overset{\chi}{\longrightarrow} (\mathbb{C}^*)^n $$

is injective. To complete the argument we just notice that $\mathcal{O}_{\mathcal{X}}(\mathcal{D}_i)$ is the invertible sheaf given by the character χ_i and the structure sheaf of Z.

With the following theorem Olsson and Starr proved the existence of generating sheaves, and proved also that the notion of generating sheaf is stable for arbitrary base change on the moduli space.

Definition 2.15 ([EHKV01, Def 2.9]). An S-stack $\mathcal{X}$ is a global quotient if it is isomorphic to a stack $[Z/G]$ where Z is an algebraic space of finite type over S and $G \to S$ is a flat group scheme which is a subgroup scheme (a locally closed subscheme which is a subgroup) of $\mathrm{GL}_{N,S}$ for some integer N.

Theorem 2.16 ([OS03, Thm. 5.7]).   *1. Let $\mathcal{X}$ be a Deligne-Mumford tame stack which is a separated global quotient over S, then there is a locally free sheaf $\mathcal{E}$ over $\mathcal{X}$ which is a generating sheaf for $\mathcal{X}$.*

2. Let $\pi : \mathcal{X} \to X$ be the moduli space of $\mathcal{X}$ and $f : X' \to X$ a morphism of algebraic spaces. Moreover let $p : \mathcal{X}' := \mathcal{X} \times_X X' \to \mathcal{X}$ be the natural projection from the fibered product, then $p^*\mathcal{E}$ is a generating sheaf for $\mathcal{X}'$.

In order to produce a smooth atlas of $\mathfrak{Coh}_{\mathcal{X}/S}$ we need to study the representability of the Quot functor. Fortunately this kind of study[1] can be found in [OS03].

Theorem 2.17 ([OS03, Thm. 4.4]). *Let S be a noetherian scheme of finite type over a field. Let $p \colon \mathcal{X} \to S$ be a Deligne-Mumford tame stack which is a separated global quotient and $\pi : \mathcal{X} \to X$ the moduli space which is a scheme with a projective morphism to $\rho \colon X \to S$ with $p = \rho \circ \pi$. Suppose $\mathcal{F}$ is a coherent sheaf on $\mathcal{X}$ and P a generalized Hilbert polynomial in the sense of Olsson and Starr [OS03, Def 4.1] then the functor $\mathrm{Quot}_{\mathcal{X}/S}(\mathcal{F}, P)$ is represented by a projective S-scheme.*

The theorem we have stated here is slightly different from the theorem in the paper of Olsson and Starr. They have no noetherian assumption but they ask the scheme S to be affine. Actually the proof doesn't change.

First they prove this statement:

Proposition 2.18. *[OS03, Prop 6.2] Let S be an algebraic space and $\mathcal{X}$ a tame Deligne-Mumford stack over S which is a separated global quotient. Let $\mathcal{E}$ be a generating sheaf on $\mathcal{X}$ and P a generalized Hilbert polynomial: the natural transformation $F_{\mathcal{E}} : \mathrm{Quot}_{\mathcal{X}/S}(\mathcal{F}, P) \to \mathrm{Quot}_{X/S}(F_{\mathcal{E}}(\mathcal{F}), P_V)$ is relatively representable by schemes and a closed immersion (see the original paper for the definition of P_V; we are not going to use it).*

---

[1]We are not interested in quasicoherent sheaves so we don't state [OS03, Thm 4.4] in its full generality

We obtain theorem 2.17 from this proposition and using the classical result of Grothendieck about the representability of $\mathrm{Quot}_{X/S}(F_{\mathcal{E}}(\mathcal{F}))$ when S is a noetherian scheme.

Remark 2.19. As in the case of schemes the functor $\mathrm{Quot}_{\mathcal{X}/S}(\mathcal{F})$ is the disjoint union of projective schemes $\mathrm{Quot}_{\mathcal{X}/S}(\mathcal{F}, P)$ where P ranges through all generalized Hilbert polynomial.

Assume now that $\mathcal{X}$ is defined over a field; it is known that $\mathcal{X}$ has a generating sheaf and projective moduli scheme if and only if $\mathcal{X}$ is a global quotient and has a projective moduli scheme. In characteristic zero this is also equivalent to the stack $\mathcal{X}$ to be a closed embedding in a smooth proper Deligne-Mumford stack with projective moduli scheme [Kre06, Thm 5.3]; in general this third property implies the first twos. This motivates the definition of projective stack:

Definition 2.20. Let k be a field. We will say $\mathcal{X} \to \mathrm{Spec}\,k$ is a *projective stack* (quasi projective) over k if it is a tame separated global quotient with moduli space which is a projective scheme (quasi projective).

For the reader convenience we summarize here equivalent definitions in characteristic zero:

Theorem 2.21 ([Kre06, Thm 5.3]). *Let $\mathcal{X} \to \mathrm{Spec}\,k$ be a Deligne-Mumford stack over a field k of characteristic zero. The following are equivalent:*

1. *the stack $\mathcal{X}$ is projective (quasi projective)*

2. *the stack $\mathcal{X}$ is a global quotient and the moduli scheme is projective (quasi projective)*

3. *the stack $\mathcal{X}$ has a closed embedding (locally closed) in a smooth Deligne-Mumford stack over k which is proper over k and has projective moduli scheme.*

Remark 2.22. It could seem more natural to define a projective stack via the third statement in the previous theorem; however we prefer to use a definition that is well behaved in families, also in mixed characteristic.

We give a relative version of the definition of projective stack. We first observe that if $\mathcal{X} = [Z/G]$ is a global quotient over a scheme S, for every geometric point s of S the fiber $\mathcal{X}_s$ is the global quotient $[Z_s/G_s]$ where Z_s and G_s are the fibers of Z and G. Moreover if $X \to S$ is a projective morphism the fibers X_s are projective schemes and according to Corollary 1.3 1 they are the moduli schemes of $\mathcal{X}_s$. This consideration leads us to the definition:

Definition 2.23. Let $p \colon \mathcal{X} \to S$ be a tame stack on S which is a separated global quotient with moduli scheme X such that p factorizes as $\pi \colon \mathcal{X} \to X$ followed by $\rho \colon X \to S$ which is a projective morphism. We will call $p \colon \mathcal{X} \to S$ a *family of projective stacks*.

Remark 2.24. 1. we don't say it is a *projective morphism* from $\mathcal{X}$ to S since this is already defined and means something else. There are no projective morphisms in the sense of [LMB00, 14.3.4] from $\mathcal{X}$ to S, indeed such a morphism cannot be representable unless $\mathcal{X}$ is a scheme.

2. Each fiber over a geometric point of S is actually a projective stack, which motivates the definition.

3. A family of projective stacks $\mathcal{X} \to S$ has a generating sheaf $\mathcal{E}$ according to Theorem 2.16, and according to the same theorem the fibers of $\mathcal{E}$ over geometric points of S are generating sheaves for the fibers of $\mathcal{X}$.

1 A smooth atlas for the stack of coherent sheaves

Let $\pi : \mathcal{X} \to X$ be a family of projective Deligne-Mumford stacks. Choose a polarization $\mathcal{O}_X(1)$ and a generating sheaf $\mathcal{E}$ on $\mathcal{X}$. Consider the disjoint union of projective schemes $Q_{N,m} := \mathrm{Quot}_{\mathcal{X}/S}(\mathcal{E}^{\oplus N} \otimes \pi^*\mathcal{O}_X(-m))$ where N is a non negative integer and m is an integer and let $\mathcal{E}_Q^{\oplus N} \otimes \pi_Q^*\mathcal{O}_{X_Q}(-m)) \xrightarrow{u_{N,m}} \mathcal{U}_{N,m}$ be the universal quotient sheaf. We can define the morphism:

$$\mathcal{U}_{N,m} \colon Q_{N,m} \longrightarrow \mathfrak{Coh}_{\mathcal{X}/S}$$

Denote with $\rho \colon X \to S$, with p the composition $\rho \circ \pi$ and with π_U, ρ_U, p_U every map obtained by base change from a scheme U with a map to S. We assume that for every base change $U \to S$ it is satisfied $\rho_{U*}\mathcal{O}_{X_U} = \mathcal{O}_U$ so that we have also $\rho_{U*}\rho_U^* = \mathrm{id}$. We define an open subscheme $Q_{N,m}^0 \to Q_{N,m}$. Let U be an S scheme with a map to $Q_{N,m}$ given by a quotient $\mathcal{E}_U^{\oplus N} \otimes \pi_U^*\mathcal{O}_{X_U}(-m) \xrightarrow{\mu} \mathcal{M}$. In order for the map to factor through $Q_{N,m}^0$ it must satisfy the following conditions:

1. The higher derived functors $R^i\rho_{U*}(F_{\mathcal{E}_U}(\mathcal{M})(m))$ vanish for every positive i, and for $i = 0$ it is a free sheaf. This condition is open because of proposition 1.7.

2. The $\mathcal{O}_U$-module $\rho_{U*}(F_{\mathcal{E}_U}(\mathcal{M})(m))$ is free and has constant rank N. This is an open condition because of 1.8.

3. Consider the morphism:

$$E_{N,m}(\mu) \colon \mathcal{O}_{X_U}^{\oplus N} \xrightarrow{\varphi_{\mathcal{E}}(\mathcal{O}_{X_U}^{\oplus N})} F_{\mathcal{E}_U} \circ G_{\mathcal{E}_U}(\mathcal{O}_{X_U}^{\oplus N}) \xrightarrow{F_{\mathcal{E}}(\mu)} F_{\mathcal{E}_U}(\mathcal{M})(m)$$

The pushforward $\rho_{U*}E_{N,m}$ is a morphism of free $\mathcal{O}_U$-modules of the same rank because of the previous point. We ask this map to be an isomorphism which is an open condition since it is a map of free modules.

Proposition 2.25. *The following composite morphism:*

$$Q_{N,m}^0 \subseteq Q_{N,m} \xrightarrow{\mathcal{U}_{N,m}} \mathfrak{Coh}_{\mathcal{X}/S}$$

denoted with $\mathcal{U}_{N,m}^0$ is representable locally of finite type and smooth for every couple of integers m, N.

Proof. This proof follows the analogous one for schemes in [LMB00, Thm 4.6.2.1] with quite a number of necessary modifications.

Let V be an S-scheme with a map $\mathcal{N}$ to $\mathfrak{Coh}_{\mathcal{X}/S}$. In order to study the representability and smoothness of $\mathcal{U}_{N,m}^0$ we compute the fibered product $Q_{N,m}^0 \underset{\mathfrak{Coh}_{\mathcal{X}/S}, \mathcal{N}}{\times} V$. Denote with QV

the fibered product $Q^0_{N,m} \times_S V$, with σ_Q, σ_V its two projections and with $\tau_Q \colon X_{QV} \to X_{Q^0_{N,m}}$, $\tau_V \colon X_{QV} \to X_V$ the two projections induced by base change and with η_Q and η_V the two analogous projections from $\mathcal{X}_{QV}$. It follows almost from the definition that the fibered product is given by:

$$\mathcal{I}so_{\mathcal{X}_{QV}}\left(\eta^*_Q \mathcal{U}^0_{N,m}, \eta^*_V \mathcal{N}\right) \xrightarrow{p_1} V$$

As in [LMB00] we observe that there is a maximal open subscheme $V_{N,m} \subseteq V$ such that the following conditions are satisfied (here and in the following we write V instead of $V_{N,m}$ since it is open in V and in particular smooth):

1. The higher derived functors $R^i \rho_{V*}(F_{\mathcal{E}_V}(\mathcal{N})(m))$ vanish for all $i > 0$.

2. The coherent sheaf $\rho_{V*}(F_{\mathcal{E}_V}(\mathcal{N})(m))$ is locally free of rank N (not free as we have assumed before).

3. The following adjunction morphism is surjective:

$$\rho^*_V \rho_{V*} F_{\mathcal{E}_V}(\mathcal{N})(m) \xrightarrow{\psi_V} F_{\mathcal{E}_V}(\mathcal{N})(m)$$

The last condition is a consequence of Serre's fundamental theorem about projective morphisms [EGAIII.1, 2.2.1] applied to the moduli scheme. Keeping in mind the conditions we have written we can define a natural transformation:

$$\mathcal{I}so_{\mathcal{X}_{QV}}\left(\eta^*_Q \mathcal{U}^0_{N,m}, \eta^*_V \mathcal{N}\right) \xrightarrow{I_{N,m}} \mathcal{I}so_V(\mathcal{O}^{\oplus N}_V, \rho_{V*}(F_{\mathcal{E}_V}(\mathcal{N})(m)))$$

factorizing the projection p_1 to V. An object of the first functor over a scheme W is a morphism $f \colon W \to V$, a morphism $g \colon W \to Q^0_{N,m}$ and an isomorphism:

$$\alpha \colon g^* \mathcal{U}^0_{N,m} \longrightarrow f^* \mathcal{N}$$

The transformation $I_{N,m}(W)$ associates to these data the morphism f and the isomorphism:

$$\rho_{W*}(F_{\mathcal{E}_W}(\alpha) \circ (g^* E_{N,m}(u^0_{N,m}))) \colon \mathcal{O}^{\oplus N}_W \to \rho_{W*}(F_{\mathcal{E}_W}(f^* \mathcal{N})(m))$$

This transformation is *relatively representable*[2], moreover we can prove that it is an isomorphism of sets for every $f \colon W \to V$. To do this we construct an explicit inverse of $I_{N,m}$, call it $L_{N,m}$. The map $L_{N,m}$ is defined in this way: to an isomorphism $\beta \colon \mathcal{O}^{\oplus N}_W \to \rho_{W*}(F_{\mathcal{E}_W}(f^* \mathcal{N})(m))$ associate the following surjective map:

$$\tilde{g} := \theta_{\mathcal{E}}(f^* \mathcal{N}) \circ G_{\mathcal{E}_W}(\psi_W \circ \rho^*_W \beta) \colon \mathcal{E}^{\oplus N}_W \longrightarrow f^*(\mathcal{N} \otimes \pi^*_V \mathcal{O}_{X_V}(m))$$

To give an object in $\mathcal{I}so_{\mathcal{X}_{QV}}$ we need to verify that this quotient is a map to $Q^0_{N,m}$: we have to check that $\rho_{W*} E_{N,m}(\tilde{g})$ is an isomorphism. To achieve this we analyze the morphism

---

[2]This notion appears in some notes of Grothendieck, it just means that for every natural transformation from a scheme to the second functor, the fibered product is a scheme.

with the following diagram:

$$
\begin{array}{c}
\mathcal{O}_{X_W}^{\oplus N} \\
\end{array}
$$

The diagram contains:

- Top vertex: $\mathcal{O}_{X_W}^{\oplus N}$
- Right vertical arrow labeled $\varphi_{\mathcal{E}_W}(\mathcal{O}_{X_W}^{\oplus N})$ down to $F_{\mathcal{E}_W} \circ G_{\mathcal{E}_W}(\mathcal{O}_{X_W}^{\oplus N})$
- Diagonal arrow labeled $\psi_W \circ \rho_W^* \beta$ down to $F_{\mathcal{E}_W}(f^*(\mathcal{N} \otimes \pi_V^* \mathcal{O}_{X_V}(m)))$
- Arrow labeled $F_{\mathcal{E}} \circ G_{\mathcal{E}}(\psi_W \circ \rho_W^* \beta)$ down to $F_{\mathcal{E}_W} \circ G_{\mathcal{E}_W} \circ F_{\mathcal{E}_W}(f^*(\mathcal{N} \otimes \pi_V^* \mathcal{O}_{X_V}(m)))$
- Horizontal arrow labeled $\varphi_{\mathcal{E}_W}$ from $F_{\mathcal{E}_W}(f^*(\mathcal{N} \otimes \pi_V^* \mathcal{O}_{X_V}(m)))$ to $F_{\mathcal{E}_W} \circ G_{\mathcal{E}_W} \circ F_{\mathcal{E}_W}(f^*(\mathcal{N} \otimes \pi_V^* \mathcal{O}_{X_V}(m)))$
- Arrow labeled $F_{\mathcal{E}_W}(\theta_{\mathcal{E}_W}(f^*(\mathcal{N} \otimes \pi_V^* \mathcal{O}_{X_V}(m))))$ down to $F_{\mathcal{E}_W}(f^*(\mathcal{N} \otimes \pi_V^* \mathcal{O}_{X_V}(m)))$

where the upper triangle is commutative because $\varphi_{\mathcal{E}}$ is a natural transformation, the lower triangle is commutative according to Lemma 2.9. We can conclude that $E_{N,m}(\tilde{g}) = \psi_W \circ \rho_W^* \beta$; then we have to apply ρ_{W*} and we obtain exactly β (recall that $\rho_{W*}\rho_W^* = \mathrm{id}$). It is now immediate to verify that $\tilde{g}^* \mathcal{U}_{N,m}^0$ is isomorphic to $f^* \mathcal{N}$, to explicitly obtain the isomorphism we must compare the universal quotient $\tilde{g}^* u_{N,m}^0$ and $\theta_{\mathcal{E}}(f^* \mathcal{N}) \circ G_{\mathcal{E}_W}(\psi_W \circ \rho_W^* \beta)$. The identity $I_{N,m}(W) \circ L_{N,m}(W)(\beta) = \beta$ is implicit in the construction. To prove that $L_{N,m}(W) \circ I_{N,m}(W)(\alpha) = \alpha$ we use the following diagram:

The diagram contains:

- Top vertex: $G_{\mathcal{E}_W}(\mathcal{O}_{X_W}^{\oplus N})$
- Right vertical arrow labeled $G_{\mathcal{E}_W}(\varphi_{\mathcal{E}_W}(\mathcal{O}_{X_W}^{\oplus N}))$ to $G_{\mathcal{E}_W} \circ F_{\mathcal{E}_W} \circ G_{\mathcal{E}_W}(\mathcal{O}_{X_W}^{\oplus N})$
- Diagonal double line to $G_{\mathcal{E}_W}(\mathcal{O}_{X_W}^{\oplus N})$
- Horizontal arrow labeled $\theta_{\mathcal{E}_W}$ from $G_{\mathcal{E}_W} \circ F_{\mathcal{E}_W} \circ G_{\mathcal{E}_W}(\mathcal{O}_{X_W}^{\oplus N})$ to $G_{\mathcal{E}_W}(\mathcal{O}_{X_W}^{\oplus N})$
- Left vertical arrow labeled $g^* u_{N,m}^0$ to $g^*(\mathcal{U}_{N,m}^0 \otimes \pi_Q^* \mathcal{O}_{X_Q}(m))$
- Right vertical arrow labeled $G_{\mathcal{E}_W} \circ F_{\mathcal{E}_W}(g^* u_{N,m}^0)$ to $G_{\mathcal{E}_W} \circ F_{\mathcal{E}_W}(g^*(\mathcal{U}_{N,m}^0 \otimes \pi_Q^* \mathcal{O}_{X_Q}(m)))$
- Horizontal arrow labeled $\theta_{\mathcal{E}_W}$
- Left vertical arrow labeled α to $f^*(\mathcal{N} \otimes \pi_V^* \mathcal{O}_{X_V}(m))$
- Right vertical arrow labeled $G_{\mathcal{E}_W} \circ F_{\mathcal{E}_W}(\alpha)$ to $G_{\mathcal{E}_W} \circ F_{\mathcal{E}_W}(f^*(\mathcal{N} \otimes \pi_V^* \mathcal{O}_{X_V}(m)))$
- Bottom horizontal arrow labeled $\theta_{\mathcal{E}_W}$

where the first triangle is commutative because of Lemma 2.9 and the two squares are commutative because $\theta_{\mathcal{E}}$ is natural.

Since the functor $\mathcal{I}so_V(\mathcal{O}_V^{\oplus N}, \rho_{V*}F_{\mathcal{E}_V}(\mathcal{N})(m))$ is represented by a scheme of finite type according to [LMB00, Thm. 4.6.2.1] and $I_{N,m}$ is a relatively representable isomorphism we deduce that the functor $\mathcal{I}so_{\mathcal{X}_{QV}}\left(\eta_Q^* \mathcal{U}_{N,m}^0, \eta_V^* \mathcal{N}\right)$ is represented by a scheme of finite type and it is a $\mathrm{GL}(N, \mathcal{O}_V)$-torsor over V so that it is represented by a scheme[3] smooth over V. $\qquad \square$

Proposition 2.26. *The morphism:*

$$\coprod_{N,m} Q_{N,m}^0 \subseteq \coprod_{N,m} Q_{N,m} \xrightarrow{\;\amalg \mathcal{U}_{N,m}^0\;} \mathfrak{Coh}_{\mathcal{X}/S}$$

is surjective.

Proof. To prove surjectivity of the map $\coprod \mathcal{U}_{N,m}^0$ we observe that given an S-scheme U and an object $\mathcal{N} \in \mathfrak{Coh}_{\mathcal{X}/S}(U)$, we can construct the coherent $\mathcal{O}_{X_U}$-module $F_{\mathcal{E}_U}(\mathcal{N})$, and according to Serre [EGAIII.1, Thm 2.2.1] there is m big enough such that the adjunction morphism is surjective:

$$H^0(F_{\mathcal{E}_U}(\mathcal{N})(m)) \otimes \mathcal{O}_{X_U}(-m) \longrightarrow F_{\mathcal{E}_U}(\mathcal{N})$$

Now we apply the functor $G_{\mathcal{E}_U}$ and the adjunction morphism $\theta_{\mathcal{E}_u}$ and we obtain the surjection:

$$H^0(F_{\mathcal{E}_U}(\mathcal{N})(m)) \otimes \mathcal{E}_U \otimes \pi_U^* \mathcal{O}_{X_U}(-m) \longrightarrow \mathcal{N}$$

We will denote this composition with $\widetilde{\mathrm{ev}}(\mathcal{N}, m)$. Now let N be the dimension of $H^0(F_{\mathcal{E}_U}(\mathcal{N})(m))$; the point $\mathcal{N}$ in the stack of coherent sheaves is represented on the chart $Q_{N,m}^0$. $\qquad \square$

Corollary 2.27. *The stack $\mathfrak{Coh}_{\mathcal{X}/S}$ is an Artin stack locally of finite type with atlas* $\coprod_{N,m} Q_{N,m}^0$.

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[3]It is an application of [EGAIII.2, 7.7.8-9] as explained in detail in the proof of [LMB00, Thm 4.6.2.1]

Part II

Grothendieck duality

Chapter 3

Foundation of duality for stacks

1 History

Part of what we are going to prove about the moduli space of semistable sheaves relies on Serre duality for projective Deligne-Mumford stacks. Serre duality can be easily proven with some ad hoc argument in specific examples, such as orbifold curves, gerbes, toric stacks and others; however a general enough proof requires some abstract machineries. Hartshorne's approach in Residues and Duality [Har66] is not suitable to be generalized to algebraic stacks (not in an easy way at least). In the appendix of the same book Deligne proves (in few pages) the following statement:

Lemma 3.1. *Let X be a quasi-compact scheme (non necessarily noetherian) and $QCoh_X$ the category of quasi coherent sheaves on X. Let $F\colon QCoh_X^{\circ} \to \mathfrak{Set}$ be a left exact contravariant functor sending filtered colimits to filtered limits, then the functor F is representable.*

Using this statement it's easy to prove the following:

Theorem 3.2. *Let $p\colon X \to Y$ be a morphism of separated noetherian schemes, F a sheaf on X and $C^{\bullet}(F)$ a functorial resolution of F acyclic with respect to p_*. Moreover let G be a quasi coherent sheaf on Y and $I^{\bullet}$ an injective resolution of G:*

1. *the functor $\mathrm{Hom}_Y(p_* C^q(F), I^p)$ is representable for every q, p and represented by a quasi coherent injective $p_q^! I^p$.*

2. *the injective quasicoherent double-complex $p_q^! I^p$ defines a functor $p^!\colon D_{qc}(Y) \to D_{qc}(X)$ which is right adjoint of Rp_*:*

$$R\,\mathrm{Hom}_Y(Rp_* F, G) \cong R\,\mathrm{Hom}_X(F, p^! G)$$

The proof of the first lemma relies on the fact that every finite presentation sheaf on X can be obtained as a colimit of sheaves of the kind $j_! \mathcal{O}_U$ where $j\colon U \to X$ is an open immersion, and every quasi coherent sheaf is a filtered colimit of finite presentation sheaves. In the case of noetherian algebraic stacks every quasi coherent sheaf is again a filtered colimit of coherent sheaves, however the first statement is trivially false.

We will prove Grothendieck duality for stacks using a further generalization of the previous technique. In [Nee96] Neeman proved Grothendieck duality using Brown representability theorem (adapted to triangulated category) and Bousfield localization.

Theorem 3.3. *Let $\mathcal{T}$ be a triangulated category which is compactly generated and $H : \mathcal{T}^\circ \to$ $\mathfrak{Ab}$ be a homological functor. If the natural map:*

$$H\Big(\coprod_{\lambda \in \Lambda} x_\lambda\Big) \to \prod_{\lambda \in \Lambda} H(x_\lambda)$$

is an isomorphism for every small coproduct in $\mathcal{T}$, then H is representable.

If a scheme has an ample line bundle then it's easy to prove that $D_{qc}(X)$ is compactly generated. Every scheme admits locally an ample line bundle (take an affine cover then the structure sheaf is ample); to verify that local implies global Bousfield localization is used.

In the case of stacks we will prove that if $\mathcal{X}$ has a generating sheaf and X an ample invertible sheaf then $D_{qc}(\mathcal{X})$ is compactly generated, so that we can use Brown representability. It's true again that every Deligne-Mumford stack $\mathcal{X}$ has *étale* locally a generating sheaf [OS03, Prop 5.2], however the argument used by Neeman to prove that local implies global heavily relies on Zariski topology and cannot be generalized to stacks in an evident way.

Once existence and uniqueness are proved, we will be able to determine the shape of the dualizing functor in many examples computing it on injective sheaves. In many examples we will be able to compute locally a functor that behaves like a dualizing functor on injective sheaves; since sheaves can be glued we will assemble a global functor and we will be able to argue it is *the* dualizing complex by uniqueness (examples of this procedure are Prop 3.12, Cor 3.22, Lem 3.25, Thm 3.37).

2 Existence

Assumption. In this section every stack and every scheme is separated and quasi-compact. Noetherianity will be explicitly stated if needed.

In this section we will prove Grothendieck duality for three classes of morphisms: morphisms from a scheme to an algebraic stack, morphisms from a projective stack to an algebraic stack and representable proper morphisms of algebraic stacks. We will derive also some properties tightly related to existence and not depending on the geometric properties of the objects involved. We will denote with $D(\mathcal{X})$ the derived category of quasicoherent sheaves on $\mathcal{X}$, we drop the notation $D_{qc}(\mathcal{X})$ of the old literature for the derived category of sheaves with quasicoherent cohomology since it is proven to be equivalent to $D(\mathcal{X})$ in [BN93, Cor 5.5].

Lemma 3.4. *Let $\pi : \mathcal{X} \to X$ be an algebraic stack with moduli space X. The functor $\pi_* : D(\mathcal{X}) \to D(X)$ respects small coproducts, that is the natural morphism:*

$$\coprod_{\lambda \in \Lambda} \pi_* x_\lambda \to \pi_* \coprod_{\lambda \in \Lambda} x_\lambda \tag{2.1}$$

is an isomorphism for every small Λ.

Proof. We recall that the category of sheaves of modules is an abelian category that satisfies the axiom $(AB4)$; in particular the coproduct is left exact, and as a matter of fact exact. We recall also that the coproduct in a derived category is just the coproduct of complexes. We choose a smooth presentation $X_0 \to \mathcal{X}$ and we associate to it the simplicial nerve $X^\bullet$. Let $f^i \colon X_i \to X$ be the obvious composition. For every quasicoherent sheaf $\mathcal{F}$ on $\mathcal{X}$ represented by $\mathcal{F}^\bullet$ on $X^\bullet$ we have a resolution (see [Ols07, Lem 2.5]):

$$0 \to \pi_*\mathcal{F} \to f^0_*\mathcal{F}_0 \to f^1_*\mathcal{F}_1 \to \ldots \tag{2.2}$$

We just need to keep the first three terms in this sequence and the result follows from left exactness of the coproduct, the analogous result for schemes [Nee96, Lem 1.4] and the existence of the natural arrow (2.1). $\qquad\square$

Corollary 3.5. *Let $\pi \colon \mathcal{X} \to X$ be as in the previous statement and $f \colon \mathcal{X} \to Y$ be a separated morphism to a scheme Y. The functor $Rf_* \colon D(\mathcal{X}) \to D(Y)$ respects small coproducts.*

Proof. This is an immediate consequence of the universal property of the moduli scheme X and the previous lemma. $\qquad\square$

Corollary 3.6. *Let $\pi \colon \mathcal{X} \to X$ be as in the previous statement and $f \colon \mathcal{X} \to \mathcal{Y}$ is a separated morphism to an algebraic stack $\mathcal{Y}$. The functor $Rf_* \colon D(\mathcal{X}) \to D(\mathcal{Y})$ respects small coproducts.*

Proof. To prove this we compute the push-forward as in [LMB00, Lem 12.6.2] and use the previous corollary. $\qquad\square$

Let $\pi \colon \mathcal{X} \to X$ be an algebraic stack with moduli space X. Let $\mathcal{E}$ be a generating sheaf of $\mathcal{X}$ and $\mathcal{O}_X(1)$ an ample invertible sheaf of X. In the following we will indicate this set of hypothesis with the sign $(*)$

Lemma 3.7. *Let the stack $\pi \colon \mathcal{X} \to X$, the sheaves $\mathcal{E}$ and $\mathcal{O}_X(1)$ satisfy $(*)$. The derived category $D(\mathcal{X})$ is compactly generated and the set $T = \{\mathcal{E} \otimes \pi^*\mathcal{O}_X(n)[m] \mid m, n \in \mathbb{Z}\}$ is a generating set.*

Proof. Same proof as in [Nee96, Ex 1.10], but using that every quasi coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ can be written as a quotient of $\mathcal{E} \otimes \pi^*\mathcal{O}_X(-t)$ for some integer number t. $\qquad\square$

Remark 3.8. The most important class of algebraic stacks $\mathcal{X}$ satisfying conditions in the previous lemma is composed by projective stacks and more generally families of projective stacks; the second class we have in mind is given by stacks of the kind $[\operatorname{Spec} B/G] \to \operatorname{Spec} A$ where G is a linearly reductive group scheme on $\operatorname{Spec} A$, which is the structure of a tame stack étale locally on its moduli space.

Proposition 3.9. *Let $\pi \colon \mathcal{X} \to X$, $\mathcal{E}$ and $\mathcal{O}_X(1)$ satisfy $(*)$, let the morphism $f \colon \mathcal{X} \to \mathcal{Y}$ be separated and $\mathcal{Y}$ an algebraic stack. The functor $Rf_* \colon D(\mathcal{X}) \to D(\mathcal{Y})$ has a right adjoint $f^! \colon D(\mathcal{Y}) \to D(\mathcal{X})$.*

Proof. This is a formal consequence of Brown representability [Nee96, Thm 4.1], Corollary 3.6 and Lemma 3.7. $\qquad\square$

Proposition 3.10. *Let $\mathcal{X}$ be an algebraic stack and $f: Z \to \mathcal{X}$ a separated morphism from a scheme Z. The functor $Rf_*: D(Z) \to D(\mathcal{X})$ has a right adjoint $f^!: D(Z) \to D(\mathcal{X})$.*

Proof. We only need to prove that the functor Rf_* respects coproducts and then use [Nee96, Thm 4.1] again. Let $X^\bullet$ be an étale presentation and $Z^\bullet$ the pullback presentation of Z and F a quasi coherent sheaf on Z. Denote with f_i the morphism $Z_i \to X_i$. The sheaf $(f_*F)|_{X_0}$ is just $f_0^*(F|_{Z_0})$. Coproducts commute with pullback because it has a right adjoint, so the result follows. $\square$

Remark 3.11. Let $f: \mathcal{X} \to \mathcal{Y}$ be a morphisms of stacks such that $f^!$ exists. It is clear that the existence of a right adjoint is enough to guarantee uniqueness. Moreover assume we have a composition $g \circ f: \mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ such that both $g^!$ and $f^!$ exist. The two functors Rg_*Rf_* and $R(g \circ f)_*$ are canonically isomorphic. Duality gives us a canonical isomorphism:

$$f^!g^! \xrightarrow{\eta_{f,g}} (g \circ f)^! \tag{2.3}$$

If we want to explicitly compute Serre duality for a smooth proper Deligne-Mumford stack we need to prove existence of duality in one last case, which is duality for proper representable morphisms of noetherian algebraic stacks in general (no projectivity is assumed). Let $f: \mathcal{Y} \to \mathcal{X}$ be such a morphism and choose a smooth presentation $X_1 \overset{s}{\underset{t}{\rightrightarrows}} X_0 \xrightarrow{p_0} \mathcal{X}$ and produce the pullback presentation $Y_1 \overset{u}{\underset{v}{\rightrightarrows}} Y_0 \xrightarrow{p_0} \mathcal{Y}$. Let I be an injective sheaf on $\mathcal{X}$ and $\alpha: s^*I_0 \to t^*I_0$ the isomorphism defining the sheaf I. Using flat base change theorem for schemes [Ver69, Thm 2] we can produce the following chain of isomorphisms:

$$u^*f_0^!I_0 \xrightarrow{c_s} f_1^!s^*I_0 \xrightarrow{f_1^!\alpha} f_1^!t^*I_0 \xleftarrow{c_t} v^*f_0^!I_0$$

Call this isomorphism β. It satisfies the cocycle condition because α does and the isomorphisms c_t, c_s defined in [Ver69, pg 401] satisfy it according to the same reference (see next section for a recall of the construction of c_s). The data $\beta, f_0^!I_0$ define a complex of injective sheaves on $\mathcal{Y}$ and we will denote it with $f^\natural I$.

Proposition 3.12. *Let $f: \mathcal{Y} \to \mathcal{X}$ be a morphism as above, the functor $Rf_*: D^+(\mathcal{Y}) \to D^+(\mathcal{X})$ admits a right adjoint $f^!: D^+(\mathcal{X}) \to D^+(\mathcal{Y})$. Let $\mathcal{F}^\bullet \in D^+(\mathcal{X})$ and $I^\bullet$ an injective complex quasi-isomorphic to it. The derived functor $f^!\mathcal{F}^\bullet$ is computed by $f^\natural I^\bullet$.*

Proof. Let J be an injective sheaf on $\mathcal{Y}$ and I an injective sheaf on $\mathcal{X}$. Keeping notations introduced above we can write the following exact sequence:

$$0 \longrightarrow \mathcal{H}om_{\mathcal{X}}(f_*J, I) \longrightarrow p_{0*}\mathcal{H}om_{X_0}(p_0^*f_*J, p_0^*I) \longrightarrow p_{0*}s_*\mathcal{H}om_{X_1}(s^*p_0^*f_*J, s^*p_0^*I)$$

Using duality for proper morphisms of schemes and flat base change for the twisted inverse

immage we have the following commutative square:

$$p_{0*}\,\mathcal{H}om_{X_0}(p_0^*f_*J, p_0^*I) \longrightarrow p_{0*}s_*\,\mathcal{H}om_{X_1}(s^*p_0^*f_*J, s^*p_0^*I)$$

$$\downarrow\wr \qquad\qquad\qquad\qquad \downarrow\wr$$

$$p_{0*}\,\mathcal{H}om_{X_0}(f_{0*}q_0^*J, p_0^*I) \qquad\qquad p_{0*}s_*\,\mathcal{H}om_{X_1}(f_{1*}u^*q_0^*J, s^*p_0^*I)$$

$$\downarrow\wr \qquad\qquad\qquad\qquad \downarrow\wr$$

$$p_{0*}f_{0*}\,\mathcal{H}om_{X_0}(q_0^*J, f_0^!p_0^*I) \qquad\qquad p_{0*}s_*f_{1*}\,\mathcal{H}om_{X_1}(u^*q_0^*J, f_1^!s^*p_0^*I)$$

$$\downarrow\wr \qquad\qquad\qquad\qquad \downarrow\wr$$

$$f_*q_{0*}\,\mathcal{H}om_{X_0}(q_0^*J, f_0^!p_0^*I) \longrightarrow f_*q_{0*}u_*\,\mathcal{H}om_{X_1}(u^*q_0^*J, u^*f_0^!p_0^*I)$$

In the picture we have applied duality for f_0, f_1 but there are no higher derived push-forwards for the two morphisms because both $\mathcal{H}om_{X_0}(q_0^*J, f_0^!p_0^*I)$ and $\mathcal{H}om_{X_1}(u^*q_0^*J, f_1^!s^*p_0^*I)$ are injective. The morphism $f_*q_{0*}\,\mathcal{H}om_{X_0}(q_0^*J, f_0^!p_0^*I) \to f_*q_{0*}u_*\,\mathcal{H}om_{X_1}(u^*q_0^*J, u^*f_0^!p_0^*I)$ in the picture is clearly induced by β defined above so that its kernel is $f_*\,\mathcal{H}om_{\mathcal{Y}}(J, f^\sharp I)$. This gives us a duality isomorphism:

$$\mathcal{H}om_{\mathcal{X}}(f_*J, I) \to f_*\,\mathcal{H}om_{\mathcal{Y}}(J, f^\sharp I)$$

The result follows. □

Remark 3.13. Actually we have proven something stronger then bare duality, we have a sheaf version of the result. We will obtain an analogous sheaf version of the duality for the morphism from a stack to a scheme in the next section.

To conclude we study the behavior of the twisted inverse image with respect to the tensor product

Proposition 3.14. *Let* $f\colon \mathcal{X} \to Y$ *be a morphism from an algebraic stack to a scheme. Suppose* Rf_* *has a right adjoint* $f^!$*, then for every* $F, G \in D^b(Y)$ *there is a natural morphism:*

$$Lf^*F \overset{L}{\otimes} f^!G \to f^!(F \otimes G)$$

moreover if G *is compact it is an isomorphism. In particular we have the following natural isomorphism:*

$$Lf^*F \overset{L}{\otimes} f^!\mathcal{O}_{\mathcal{Y}} \to f^!F$$

If $f^!G$ *is of finite Tor-dimension we can have* $F \in D^+(Y)$.

Proof. The proof mostly relies on the existence of $f^!$ and a general enough projection formula. Putting together [OS03, Cor 5.3] and [Nee96, Prop 5.3] we actually have a general enough projection formula that is a natural isomorphism $F \overset{L}{\otimes} Rf_*\mathcal{G} \to Rf_*(Lf^*F \overset{L}{\otimes} \mathcal{G})$ for $\mathcal{G} \in D(\mathcal{X})$. Since we are working on a site we cannot use the fancy stuff of [Nee96, Thm 5.4] to define derived functors in non bounded derived categories. For this reason we have more restrictive conditions on F, G. Once everything is well defined the proof goes just like in [Nee96, Thm 5.4]. □

Definition 3.15. Let $f\colon \mathcal{X} \to S$ be an S-stack, f the structure morphism. Suppose $f^!$ exists, we will call $f^!\mathcal{O}_S$ the dualizing complex of $\mathcal{X}$.

This definition agrees with literature.

3 Duality and flat base change

Now that we have proven existence our aim is to explicitly write $f^!$ (*twisted inverse image*) in some interesting case. At the end of the next section we will obtain Serre duality for smooth projective stacks and Grothendieck duality for finite morphisms. To achieve this, we will use existence and a result of flat base-change in the same spirit as [Ver69, Thm 2]. In this section every scheme and stack is again noetherian and derived categories are bounded below.

   We start proving base change for open immersions. We anticipate a technical lemma which is a variation of [Ver69, Lem 2]

Lemma 3.16. *Let $\mathcal{X}$ be an algebraic stack and $i\colon \mathcal{U} \to \mathcal{X}$ an open substack. Let $\mathcal{I}$ be an ideal sheaf defining the complementary of $\mathcal{U}$. For any $\mathcal{F} \in D_{qc}^+(\mathcal{X})$ the canonical morphisms:*

$$\lim_{n\to\infty} \mathrm{Ext}_{\mathcal{X}}^p(\mathcal{I}^n, \mathcal{F}) \to H^p(\mathcal{U}, i^*\mathcal{F}) \tag{3.1}$$

$$\lim_{n\to\infty} \mathcal{E}xt_{\mathcal{X}}^p(\mathcal{I}^n, \mathcal{F}) \to R^p i_* i^*\mathcal{F} \tag{3.2}$$

are isomorphisms for every p. Moreover if $\mathcal{G}$ is a bounded above complex over $\mathcal{X}$ with coherent cohomology we can generalize the first isomorphism to the following:

$$\lim_{n\to\infty} \mathrm{Ext}_{\mathcal{X}}^p(\mathcal{G} \overset{L}{\otimes} \mathcal{I}^n, \mathcal{F}) \to \mathrm{Ext}_{\mathcal{U}}^p(i^*\mathcal{G}, i^*\mathcal{F}) \tag{3.3}$$

Proof. The statement is the derived version of [Har66, App Prop 4]. This last proposition holds also for stacks. To see this we can use the usual trick of writing $\mathrm{Hom}_{\mathcal{X}}$ as the kernel of an opportune morphism between Hom_{X_0} and Hom_{X_1} for some given presentation $X^\bullet$. $\qquad\square$

   Consider the following cartesian square:

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\ i\ } & \mathcal{X} \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ U & \xrightarrow{\ j\ } & Y \end{array}$$

where $\pi\colon \mathcal{X} \to X$ is an algebraic stack satisfying the set of hypothesis $(*)$, the morphisms f, g are proper, i, j are flat morphisms and Y, U are quasi-compact separated schemes. We can define a canonical morphism $c_j\colon i^*f^! \to g^!j^*$. We can actually define it in two equivalent ways according to [Ver69, pg 401]. We recall here the two construction of this morphism for completeness, and to include a small modification that occurs in the case of stacks.

   1. Since j is flat we have that j^* is the left adjoint of Rj_* so that we have a unit and a counit:

$$\phi_j\colon \mathrm{id} \to Rj_*j^* \qquad \psi_j\colon j^*Rj_* \to \mathrm{id}$$

We can also apply theorem 1.9 and obtain an isomorphism: $\sigma\colon j^*Rf_* \to Rg_*i^*$. The right adjoint of this gives as $\widetilde{\sigma}\colon Ri_*g^! \to f^!Rj_*$. The canonical morphism we want is now the composition:

$$i^*f^! \xrightarrow{i^*f^!\circ\phi_j} i^*f^!Rj_*j^* \xrightarrow{i^*\circ\widetilde{\sigma}\circ j^*} i^*Ri_*g^!j^* \xrightarrow{\psi_i\circ g^!j^*} g^!i^*$$

2. Since Rf_* has a right adjoint $f^!$ we have a unit and a counit:

$$\mathrm{cotr}_f\colon \mathrm{id} \to f^!Rf_* \qquad \mathrm{tr}_f\colon Rf_*f^! \to \mathrm{id}$$

We can now consider the following composition:

$$i^*f^! \xrightarrow{\mathrm{cotr}_g\circ i^*f^!} g^!Rg_*i^*f^! \xrightarrow{g^!\circ\sigma\circ f^!} g^!j^*Rf_*f^! \xrightarrow{g^!j^*\,\mathrm{tr}_f} g^!j^*$$

According to [Ver69, pg 401] both these two compositions define c_j; moreover it satisfies a cocycle condition when composing two base changes.

Proposition 3.17. *In the above setup, assume also that j is an open immersion, the canonical morphism $c_j\colon i^*f^! \to g^!j^*$ is an isomorphism.*

Proof. Same proof as in [Ver69, Thm 2, case 1] but using Lemma 3.16. $\square$

Definition 3.18. We recall that a morphism of schemes (or stacks also) $f\colon X \to Y$ is *compactifiable* if it can be written as an open immersion i followed by a proper morphism p.

$$
\begin{array}{ccc}
X & \stackrel{i}{\hookrightarrow} & \overline{X} \\
{\scriptstyle f}\downarrow & \swarrow{\scriptstyle p} & \\
Y & &
\end{array}
$$

Deligne defined in [Har66, App] a notion of duality for compactifiable morphisms (duality with compact support) of separated noetherian schemes. First of all we observe that given an open immersion i or more generally an object in a site, the functor i^* has a left adjoint $i_!\colon \mathrm{QCoh}_X \to \mathrm{QCoh}_{\overline{X}}$ which is an exact functor (see for instance [Mil80, II Rem 3.18] for a general enough construction). Given f compactifiable we can define the derived functor $Rf_! = R(p_*i_!) = (Rp_*)i_!$. It is clear that this last functor has a right adjoint in derived category that is $i^*p^!$ and we will denote it as $f^!$. Deligne proved that this definition of $f^!$ is independent from the chosen compactification and well behaved with respect to composition of morphisms.

The functor $i_!$ is actually everything left of all the local cohomology mess in Residues and Duality. To prove Serre duality for stacks we will use $i_!$ for both open immersions and étale maps, being confident that they are compatible in the following sense:

Proposition 3.19. *[Mil80, VI Thm 3.2.b] Let $f\colon X \to Y$ be an étale morphism of noetherian separated schemes. Split it in an open immersion i followed by a finite morphism g then we have $f_! = g_*i_!$.*

Let $\mathcal{X}$ be a Deligne-Mumford stack satisfying $(*)$ and $f_0\colon X_0 \to \mathcal{X}$ an étale atlas. The morphism $X_0 \to X$ is quasi finite and a fortiori compactifiable. We can choose a compactification using Zariski main theorem and split the morphism as $X_0 \xrightarrow{k} \overline{X} \xrightarrow{h} X$ where h is finite and k is an open immersion. We can also apply Zariski main theorem for stacks [LMB00, Thm 16.5] to the morphism f_0 and obtain a different compactification $X_0 \xrightarrow{l} \overline{\mathcal{X}} \xrightarrow{\rho} \mathcal{X} \xrightarrow{\pi} X$ where l is open, ρ is finite and $\overline{\mathcal{X}}$ is a Deligne-Mumford stack. Now we are ready to prove that these two different compactifications are equivalent from the point of view of duality with compact support.

Proposition 3.20. *Consider the commutative square:*

$$
\begin{array}{ccc}
 & X_0 & \\
{}^{k}\swarrow & & \searrow{}^{l} \\
\overline{\mathcal{X}} & & \overline{X} \\
{}_{\pi\circ\rho}\searrow & & \swarrow{}_{h} \\
 & X &
\end{array}
$$

Let $F \in D^+(X)$, there is a canonical isomorphism $k^*(\pi \circ \rho)^! F \cong l^* h^! F$

Proof. Same proof as in [Ver69, Cor 1] but using base change result in 3.17. $\qquad\square$

We can now generalize to stacks Proposition 3.19:

Corollary 3.21. *In the setup of the previous Proposition, for every $F \in D^+(X)$ there is a canonical isomorphism $f_0^* \pi^! F \cong l^* h^! F$*

Proof. We use the previous proposition, the representability of f_0 and the analogous result for schemes. $\qquad\square$

As a further consequence we can prove Grothendieck duality in its sheaf version:

Corollary 3.22. *Let $f\colon \mathcal{X} \to Y$ be a proper morphism from an algebraic stack satisfying $(*)$, Y be a scheme and $\mathcal{F} \in D^+(\mathcal{X})$, $G \in D^+(Y)$. The natural morphism:*

$$
Rf_* R\mathcal{H}om_{\mathcal{X}}(\mathcal{F}, f^! G) \longrightarrow R\mathcal{H}om_Y(Rf_*\mathcal{F}, Rf_* f^! G) \xrightarrow{\mathrm{tr}_f} R\mathcal{H}om_Y(Rf_*\mathcal{F}, G)
$$

is an isomorphism.

Proof. Take $I^\bullet, J^\bullet$ injective complexes quasi isomorphic to $\mathcal{F}, G$. Let $j\colon U \to Y$ be an étale morphism $X^\bullet$ an étale presentation of $\mathcal{X}$. We can construct the following:

$$
\begin{array}{ccccccc}
U_1 & \underset{u}{\overset{v}{\rightrightarrows}} & U_0 & \xrightarrow{l} & \mathcal{U} & \xrightarrow{g} & U \\
{\scriptstyle m}\downarrow & & {\scriptstyle k}\downarrow & & {\scriptstyle j}\downarrow & & {\scriptstyle i}\downarrow \\
X_1 & \underset{s}{\overset{t}{\rightrightarrows}} & X_0 & \xrightarrow{h} & \mathcal{X} & \xrightarrow{f} & Y
\end{array}
$$

As usual we have the exact sequence:

$$
0 \to f_* \mathcal{H}om_{\mathcal{X}}(I^p, f^! J^q) \to f_* h_* \mathcal{H}om_{X_0}(h^* I^p, h^* f^! J^q) \to f_* h_* s_* \mathcal{H}om_{X_1}(s^* h^* I^p, s^* h^* f^! J^q)
$$

for every p, q. We first use flat base change to obtain $i^* f_* h_* \mathcal{H}om_{X_0}(h^* I^p, h^* f^! J^q) \cong g_* l_* k^* \mathcal{H}om_{X_0}(h^* I^p, h^* f^! J^q) \cong g_* l_* \mathcal{H}om_{U_0}(k^* h^* I^p, k^* h^* f^! J^q)$. Now we observe that $i^* = i^!$ if we consider an étale morphism as a compactifiable morphism [Mil80, V Prop 1.13], and using 3.20 we obtain $k^* h^* f^! = l^* g^! i^*$. Eventually we have:

$$i^* f_* h_* \mathcal{H}om_{X_0}(h^* I^p, h^* f^! J^q) \cong g_* l_* \mathcal{H}om_{U_0}(l^* j^* I^p, l^* g^! i^* J^q)$$

With the same argument we have also:

$$i^* f_* h_* s_* \mathcal{H}om_{X_1}(s^* h^* I^p, s^* h^* f^! J^q) \cong g_* l_* u_* \mathcal{H}om_{U_1}(u^* l^* j^* I^p, u^* l^* g^! i^* J^q)$$

and eventually:

$$i^* f_* \mathcal{H}om_{\mathcal{X}}(I^p, f^! J^q) \cong g_* \mathcal{H}om_{\mathcal{U}}(j^* I^p, g^! i^* J^q)$$

Now we just take global sections of this and use the non sheaf version of Grothendieck duality to complete the proof. $\square$

We have now all the ingredients to prove the flat base change result:

Theorem 3.23. *Consider the following cartesian square:*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{i} & \mathcal{X} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{j} & Y \end{array}$$

where $\mathcal{X}$ is an algebraic stack satisfying $(*)$, the morphisms f, g are proper and i, j are flat. The canonical morphism $c_j \colon i^* f^! \to g^! j^*$ is an isomorphism.

Proof. Same proof as in [Ver69, Thm 2, case 2] but using the stacky Corollary 3.22. $\square$

4 Duality for smooth morphisms

In this section $\mathcal{X} \to \operatorname{Spec} k$ is a smooth projective Deligne-Mumford stack over a field if not differently specified. It clearly satisfies $(*)$.

We start with two local results that don't rely on smoothness.

Lemma 3.24. *Let $f \colon \mathcal{Y} \to \mathcal{Z}$ be a representable finite étale morphism of noetherian algebraic stacks (non necessarily smooth), then the functor $f^!$ is the same as f^*.*

Proof. If $\mathcal{Y}$ and $\mathcal{Z}$ are two schemes the result is true, then the result for stacks follows by Proposition 3.12. $\square$

Let $\pi \colon \mathcal{X} \to X$ be a non necessarily smooth Deligne-Mumford stack with moduli scheme; using the previous lemma we can study the étale local structure of $\pi^!$. The morphism π étale locally is the same as $\rho \colon [\operatorname{Spec} B/G] \to \operatorname{Spec} A$ where G is a finite group and $\operatorname{Spec} A$ is the moduli scheme. Let $\operatorname{Spec} B \xrightarrow{p} [\operatorname{Spec} B/G]$ be the obvious étale (and finite) atlas and s, t source and target in the presentation (both étale and finite). Let I be an injective A-module. Consider the following chain of isomorphisms:

$$s^* (\rho \circ p)^! I \xrightarrow[\sim]{\eta_{s, \rho \circ p}} (\rho \circ p \circ s)^! I \xleftarrow[\sim]{\eta_{t, \rho \circ p}} t^* (\rho \circ p)^! I$$

31

where we have replaced $s^!, t^!$ with s^*, t^* using the lemma, and every isomorphism is given by equation (2.3). Call this isomorphism γ. The data of γ and $(\rho \circ p)^! I$ define a complex of injective sheaves on $[\operatorname{Spec} B/G]$ and we will denote it $\rho^\nabla I$. Let $F \in D^+(\operatorname{Spec} A)$ quasi isomorphic to a complex of injectives $I^\bullet$; the injective complex $\rho^\nabla I^\bullet$ on $[\operatorname{Spec} B/G]$ defines a functor $\rho^\nabla \colon D^+(\operatorname{Spec} A) \to D^+([\operatorname{Spec} B/G])$

Lemma 3.25. *The functor ρ^∇ above is actually $\rho^!$.*

Proof. We start observing that the twisted inverse image $(\rho \circ p)^! I$ is just the B-module $\operatorname{Hom}_A(B, I)$, the twisted inverse image $(\rho \circ p \circ s)^! I$ is $\operatorname{Hom}_A(B \otimes_A \mathcal{O}_G, I)$. The natural isomorphism for the composition of twisted inverse images $s^*(\rho \circ p)^! I \cong (\rho \circ p \circ s)^! I$ is just the canonical isomorphism $\operatorname{Hom}_A(B, I) \otimes_A \mathcal{O}_G \xrightarrow{\delta} \operatorname{Hom}_A(B \otimes_A \mathcal{O}_G, I)$. Let M be a B-module with α a coaction of $\mathcal{O}_G$. From the exact sequence in (2.2) and using duality we obtain the following exact diagram:

$$
\begin{array}{ccc}
\operatorname{Hom}_A(M \otimes_A \mathcal{O}_G, I) & \xrightarrow{\operatorname{Hom}(\alpha - \iota, \mathrm{id})} & \operatorname{Hom}_A(M, I) \\
\Big\downarrow{\wr} & & \Big\downarrow{\wr} \\
\operatorname{Hom}_{B \otimes_A \mathcal{O}_G}(M \otimes_A \mathcal{O}_G, \operatorname{Hom}_A(B \otimes_A \mathcal{O}_G, I)) & & \\
{\scriptstyle \operatorname{Hom}(\mathrm{id}, \delta^{-1})} \Big\downarrow{\wr} & & \\
\operatorname{Hom}_{B \otimes_A \mathcal{O}_G}(M \otimes_A \mathcal{O}_G, \operatorname{Hom}_A(B, I) \otimes \mathcal{O}_G) & & \\
\Big\| & & \\
\operatorname{Hom}_B(M, \operatorname{Hom}_A(B, I)) \otimes_A \mathcal{O}_G & \longrightarrow & \operatorname{Hom}_B(M, \operatorname{Hom}_A(B, I))
\end{array}
$$

The cokernel of the first horizontal arrow is $\operatorname{Hom}_A(M^G, I)$ while the cokernel of the last horizontal arrow is just $\operatorname{Hom}_B^G(M, \operatorname{Hom}_A(B, I))$ where the coaction of $\mathcal{O}_G$ on M is α and the coaction on $\operatorname{Hom}_A(B, I)$ is the one of $\rho^\nabla I$. The diagram induces an isomorphism[1]:

$$
\operatorname{Hom}_A(M^G, I) \to \operatorname{Hom}_B^G(M, \operatorname{Hom}_A(B, I))
$$

By uniqueness of the adjoint we conclude that ρ^∇ is exactly $\rho^!$. $\qquad\square$

Remark 3.26. Let F be a quasicoherent sheaf on $\operatorname{Spec} A$; if we know for some reason that $\rho^! F$ is a quasi coherent sheaf itself, then we can conclude that it glues as $\rho^!$ of an injective sheaf in the previous lemma.

We can now start using smoothness hypothesis on $\mathcal{X}$. We recall a result of Verdier in [Ver69, Thm 3]:

Theorem 3.27. *Let $f \colon X \to Y$ a proper morphism of Noetherian schemes and $j \colon U \to X$ an open immersion such that $f \circ j$ is smooth of relative dimension n. There exists a canonical isomorphism:*

$$
j^* f^! \mathcal{O}_Y \xrightarrow{can} \omega_{U/Y}[n] \tag{4.1}
$$

where $\omega_{U/Y}$ is the canonical sheaf.

---

[1] Despite of the unhappy notation Hom_B^G is not a B-module but a $B^G = A$-module.

In order to use Lemma 3.25 we need to explicitly know the isomorphism $f^! \circ g^! \cong (g \circ f)^!$ when f, g are compactifiable smooth morphisms of schemes. For this purpose we state a compactified version of a statement of Hartshorne [Har66, III Prop 2.2]:

Lemma 3.28. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be smooth compactifiable morphisms of noetherian schemes of relative dimensions n, m respectively. There is a natural isomorphism $\zeta\colon \omega_{X/Z} \to \omega_{x/Y} \otimes f^* \omega_{Y/Z}$. Called $\eta_{f,g}\colon f^! \circ g^! \cong (g \circ f)^!$ the natural isomorphism obtained for adjunction from $R(g \circ f)_! \cong Rf_! \circ Rg_!$ (same as equation (2.3)) we have the following commutative diagram:*

$$
\begin{array}{ccc}
\omega_{X/Z} & \xrightarrow{\quad\zeta\quad} & \omega_{X/Y} \otimes f^* \omega_{Y/Z} \\
{\scriptstyle can}\big\uparrow & & {\scriptstyle can}\big\uparrow \\
(g \circ f)^! \mathcal{O}_Z & \xleftarrow{\quad\eta_{f,g}\quad} & f^!(g^! \mathcal{O}_Z) = f^! \mathcal{O}_Y \otimes f^* g^! \mathcal{O}_Z
\end{array}
$$

With this machinery we can solve the local situation:

Proposition 3.29. *Let $\rho\colon \mathcal{X} = [\operatorname{Spec} B/G] \to \operatorname{Spec} A$ be an n-dimensional smooth Deligne-Mumford stack over $\operatorname{Spec} k$ with structure map $\sigma\colon \operatorname{Spec} A \to \operatorname{Spec} k$ a compactifiable morphism; $p\colon X_0 \to \mathcal{X}$ an étale atlas. The dualizing complex $\rho^! \sigma^! k$ is canonically isomorphic to $\omega_{\mathcal{X}}[n]$*

Proof. According to Theorem 3.27 we have that $(\sigma \circ \rho \circ p)^! k = \omega_B[n]$ (it's important to remember that duality along σ is duality with compact support). By Lemma 3.28 and using that s, t are finite we have:

$$
s^*(\sigma \circ \rho \circ p)^! k = s^* \omega_B[n] \xrightarrow{\text{can}} \omega_{B \times G}[n] = (\sigma \circ \rho \circ p \circ s)^! k
$$

and the same for t^*. Since the canonical isomorphism is the one described in 3.28 and using Lemma 3.25 we obtain that $(\sigma \circ \rho)^! k$ is canonically isomorphic to $\omega_{\mathcal{X}}[n]$. $\square$

Remark 3.30. Suppose now to change the atlas in the previous proposition to some scheme W_0 étale over $\mathcal{X}$ but not necessarily finite. We have a new étale presentation $W_1 \xrightarrow[\substack{v \\ u}]{} W_0 \xrightarrow{\tau} \mathcal{X}$. The two morphisms $\sigma \circ \rho \circ \tau\colon W_0 \to \operatorname{Spec} k$ and $\sigma \circ \rho \circ \tau \circ u\colon W_1 \to \operatorname{Spec} k$ are both compactifiable. Using duality with compact support we have canonical isomorphisms $u^*(\sigma \circ \rho \circ \tau)^! k \cong (\sigma \circ \rho \circ \tau \circ u)^! k$ and the same with v^*. Recall now that for an étale morphism the twisted inverse image (duality with compact support) is the same as the pullback; according to 3.28 and using the previous proposition the two isomorphisms are the two canonical isomorphisms $u^* \omega_{W_0} \cong \omega_{W_1}$ and $v^* \omega_{W_0} \cong \omega_{W_1}$.

To deal with the global case we have to study more the two natural isomorphisms we have: $\eta_{f,g}$ for the composition of twisted inverse images $f^!, g^!$ and c_j for the flat base change by a map j of a twisted inverse image. They are compatible according to a pentagram relation.

Lemma 3.31. *Consider the following diagram of noetherian schemes where horizontal arrows are compactifiable morphisms, vertical arrows are flat and the two squares cartesian:*

$$
\begin{array}{ccc}
X' & \xrightarrow{g} Y' \xrightarrow{\rho} & Z' \\
{\scriptstyle i}\big\downarrow & {\scriptstyle h}\big\downarrow & {\scriptstyle \sigma}\big\downarrow \\
X & \xrightarrow{f} Y \xrightarrow{\pi} & Z
\end{array}
$$

Let $F \in D^+(Z)$, the following pentagram relation holds:

$$(\rho \circ g)^! \sigma^* F \qquad (4.2)$$

$$i^*(\pi \circ f)^! F \xrightarrow{c_\sigma} \qquad \xleftarrow{\eta_{g,\rho}} g^! \rho^! \sigma^* F$$

$$i^* \eta_{f,\pi} \qquad g^! c_\sigma$$

$$i^* f^! \pi^! F \xrightarrow{\quad c_h \quad} g^! h^* \pi^! F$$

We are ready for the global case:

Theorem 3.32 (Smooth Serre duality). *Let $\sigma \colon \mathcal{X} \to \operatorname{Spec} k$ be a smooth projective Deligne-Mumford stack of dimension n. The complex $\sigma^! k$ is canonically isomorphic to the complex $\omega_{\mathcal{X}}[n]$.*

Proof. We start with a picture that reproduces the local setup and summarizes all the morphisms we are going to use:

$$
\begin{array}{ccccccc}
Y_1 \overset{v}{\underset{u}{\rightrightarrows}} Y_0 & \xrightarrow{g} & \coprod_i [\operatorname{Spec} B_i/G_i] & \xrightarrow{\rho} & \coprod \operatorname{Spec} A_i & & (4.3) \\
\downarrow{\scriptstyle h_1} \quad \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \sigma} & & \\
X_1 \overset{t}{\underset{s}{\rightrightarrows}} X_0 & \xrightarrow{f} & \mathcal{X} & \xrightarrow{\pi} & X & \longrightarrow & \operatorname{Spec} k
\end{array}
$$

We denote with k_X the dualizing complex of the scheme X. First we observe that $\pi^! k_X$ is a sheaf. Indeed we have an isomorphism $c_\sigma \colon \rho^! \sigma^* k_X \to h^* \pi^! k_X$ for Theorem 3.23; according to Proposition 3.29 the complex $\rho^! \sigma^* k_X$ is a sheaf and since h is faithfully flat $\pi^! k_X$ must be a sheaf itself. Denote with ξ the isomorphism on the double intersection X_1 defining the sheaf $\pi^! k_X$. Using again Proposition 3.29 and the following remark we have a commutative diagram expressing the isomorphism ξ in relation with the base change isomorphism c_σ:

$$u^* h_0^* (\pi \circ f)^! k_X \xrightarrow{u^* c_\sigma} u^* (\rho \circ g)^! \sigma^* k_X$$

$$\downarrow{\scriptstyle u^* h_0^* \eta_{f,\pi}}$$

$$u^* h_0^* f^* \pi^! k_X \qquad \qquad \qquad \searrow{\scriptstyle \eta_{u,\rho g}}$$

$$\downarrow{\scriptstyle h_1^* \xi} \qquad \qquad (\rho \circ g \circ u)^! \sigma^* k_X$$

$$v^* h_0^* f^* \pi^! k_X \qquad \qquad \nearrow{\scriptstyle \eta_{v,\rho g}}$$

$$\uparrow{\scriptstyle v^* h_0^* \eta_{f,\pi}}$$

$$v^* h_0^* (\pi \circ f)^! k_X \xrightarrow{v^* c_\sigma} v^* (\rho \circ g)^! \sigma^* k_X$$

Now we can use two times the pentagram relation for compactifiable morphisms in (4.2) (remember that $u^!, v^!, t^!, s^! = u^*, v^*, t^*, s^*$) and obtain the following commutative diagram:

$$\begin{array}{c} (4.4) \end{array}$$

$$u^* h_0^* (\pi \circ f)^! k_X \xrightarrow{\ u^* c_\sigma\ } u^* (\rho \circ g)^! \sigma^* k_X$$

$$\|$$

$$h_1^* s^* (\pi \circ f)^! k_X$$

$$\xrightarrow{h_1^* \eta_{s,\pi \circ f}}$$

$$h_1^* (\pi \circ f \circ s)^! k_X \xrightarrow{\ c_\sigma\ } (\rho \circ g \circ u)^! \sigma^* k_X$$

$$\eta_{u,\rho \circ g}$$

$$h_1^* \eta_{t,\pi \circ f}$$

$$h_1^* t^* (\pi \circ f)^! k_X$$

$$\|$$

$$\eta_{v,\rho \circ g}$$

$$v^* h_0^* (\pi \circ f)^! k_X \xrightarrow{\ v^* c_\sigma\ } v^* (\rho \circ g)^! \sigma^* k_X$$

Comparing the two commutative diagrams we have the following commutative square:

$$\begin{array}{ccc} h_1^* s^* f^* \pi^! k_X & \xrightarrow{h_1^* s^* \eta_{f,\pi}} & h_1^* s^* (\pi \circ f)^! k_X \\ \downarrow{\scriptstyle h_1^* \xi} & & \downarrow{\scriptstyle h_1^* (\eta_{t,\pi \circ f}^{-1} \circ \eta_{s,\pi \circ f})} \\ h_1^* t^* f^* \pi^! k_X & \xrightarrow{h_1^* t^* \eta_{f,\pi}} & h_1^* t^* (\pi \circ f)^! k_X \end{array} \qquad (4.5)$$

First of all we observe that the sheaf $(\pi \circ f)^! k_X$ glued by $\eta_{t,\pi \circ f}^{-1} \circ \eta_{s,\pi \circ f}$ is exactly $\omega_{\mathcal{X}}$ according to Lemma 3.28. The commutative square tells us that the isomorphism $\eta_{f,\pi}$ is an isomorphism between the dualizing sheaf and $\omega_{\mathcal{X}}$ once it is restricted to Y_0; unfortunately Y_0 is finer then the atlas we are using (X_0) and we actually don't know if this isomorphism descends. To make it descend we produce a finer presentation of $\mathcal{X}$, that is we use Y_0 as an atlas and we complete the presentation to the groupoid $\overline{X}_2 \rightrightarrows \overline{X}_1 \xrightarrow[u']{v'} Y_0$. This gives us an arrow λ from $\overline{X}_1$ to Y_1 and an arrow from $\overline{X}_2$ to Y_2. We can take the square in 4.5 and pull it back with λ^* to $\overline{X}_1$. Now the isomorphism $h_0^* \eta_{f,\pi}$ descends to an isomorphism of sheaves on $\mathcal{X}$. The main problem now is that we don't know if $\lambda^* h_1^* \xi$ and $\lambda^* h_1^* (\eta_{t,\pi \circ f}^{-1} \circ \eta_{s,\pi \circ f})$ are still the gluing isomorphisms of respectively the dualizing sheaf and $\omega_{\mathcal{X}}$. For what concerns the second we have the following commutative square:

$$\begin{array}{ccc} \lambda^* h_1^* s^* (\pi \circ f)^! k_X & \xrightarrow{u'^* \eta_{h_0, \pi \circ f}} & u'^* (\pi \circ f \circ h_0)^! k_X \\ \downarrow{\scriptstyle \lambda^* h_1^* \eta_{s,\pi \circ f}} & & \downarrow{\scriptstyle \eta_{u',\pi \circ f \circ h_0}} \\ \lambda^* h_1^* (\pi \circ f \circ s)^! k_X & \xrightarrow{v'^* \eta_{h_0, \pi \circ f}} & v'^* (\pi \circ f \circ h_0)^! k_X \end{array}$$

and an analogous one for t, v'. This square implies that the sheaf $h_0^* (\pi \circ f)^! k_X$ with the gluing isomorphism $\lambda^* h_1^* \eta_{t,\pi \circ f}^{-1} \circ \eta_{s,\pi \circ f}$ is canonically isomorphic via $h_0^* \eta_{h_0, \pi \circ f}$ to the sheaf given by $(\pi \circ f \circ h_0)^! k_X$ and the gluing isomorphism $\eta_{v',\pi \circ f \circ h_0}^{-1} \circ \eta_{u',\pi \circ f \circ h_0}$ which is actually

$\omega_{\mathcal{X}}$. For what concerns the dualizing sheaf we start considering the following picture:

$$
\begin{array}{ccccc}
\overline{X}_1 & \xrightarrow{\lambda} & Y_1 & \xrightarrow{v} & Y_0 \\
{\scriptstyle \lambda}\downarrow & & {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle h_0} \\
Y_1 & \xrightarrow{h_1} & X_1 & \xrightarrow{t} & X_0 \\
{\scriptstyle u}\downarrow & & {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle f} \\
Y_0 & \xrightarrow{h_0} & X_0 & \xrightarrow{f} & \mathcal{X}
\end{array}
\tag{4.6}
$$

where every single square is 2-cartesian. Only the square at the bottom right has a non trivial canonical two-arrow, let's call it γ. If we think of $\pi^! k_X$ as a sheaf on the étale site of $\mathcal{X}$, the gluing isomorphism ξ on the presentation $X_1 \rightrightarrows X_0$ is induced by the two-arrow γ. If we change the presentation to $\overline{X}_1 \rightrightarrows Y_0$ the gluing isomorphism is induced by $\gamma * \mathrm{id}_{h_1 \circ \lambda}$ according to the picture 4.6; this implies that the induced isomorphism is exactly $\lambda^* h_1^* \xi$. We can conclude that the sheaf $h_0^* f^* \pi^! k_X$ with gluing data $\lambda^* h_1^* \xi$ is again the dualizing sheaf. $\qquad \square$

Keeping all the notations of the previous theorem we have the following non-smooth result:

Theorem 3.33. *Let $\sigma \colon \mathcal{X} \to \operatorname{Spec} k$ be a projective Deligne-Mumford stack. Let F be a quasicoherent sheaf on the moduli scheme X, assume that $\pi^! F$ is a quasicoherent sheaf on $\mathcal{X}$ then its equivariant structure is given by the isomorphism $\eta_{t,\pi\circ f}^{-1} \circ \eta_{s,\pi\circ f}$.*

Proof. First of all we observe that $\pi^! F$ being a sheaf can be checked étale locally as in the previous theorem, to be more specific it is enough to know that $\rho^! \sigma^* F$ is a sheaf. We achieve the result of the theorem repeating the same proof as in the smooth case and keeping in mind Remark 3.26. $\qquad \square$

5 Duality for finite morphisms

We are going to prove that given $f \colon \mathcal{X} \to \mathcal{Y}$ a representable finite morphism of Deligne-Mumford stacks the functor $f^!$ is perfectly analogous to the already familiar one in the case of schemes.

To start with the proof we first need to state a couple of results, well known in the scheme-theoretic set up ([EGAII, Prop 1.3.1] and [EGAII, Prop 1.4.1]), in the stack-theoretic set-up. The first one is taken from [LMB00, Prop 14.2.4].

Lemma 3.34. *Let $\mathcal{X}$ be an algebraic stack over a scheme S. There is an equivalence of categories between the category of algebraic stacks $\mathcal{Y}$ together with a finite schematically representable S-morphism $f \colon \mathcal{X} \to \mathcal{Y}$ and quasicoherent $\mathcal{O}_{\mathcal{X}}$-algebras. This equivalence associates to the stack $\mathcal{Y}$ and the morphism f the sheaf of algebras $f_* \mathcal{O}_{\mathcal{Y}}$; to a sheaf of algebras $\mathcal{A}$ the affine morphism $f_{\mathcal{A}} \colon \operatorname{Spec} \mathcal{A} \to \mathcal{X}$.*

From this lemma we can deduce the following result on quasicoherent sheaves:

Lemma 3.35. *Let $\mathcal{X}$ be as above and $\mathcal{A}$ an $\mathcal{O}_\mathcal{X}$-algebra. There is an equivalence of categories between the category of quasicoherent $\mathcal{A}$-modules and the category of quasicoherent sheaves on $\operatorname{Spec}\mathcal{A}$. Denoted with f the affine morphism $\operatorname{Spec}A \to \mathcal{X}$, and given $\mathcal{F}$ a quasicoherent sheaf on $\operatorname{Spec}A$, the equivalence associates to $\mathcal{F}$ the sheaf $\overline{f}_*\mathcal{F}$ that is the sheaf $f_*\mathcal{F}$ with its natural structure of $\mathcal{A}$-algebra. The inverted equivalence is the left-adjoint of $\overline{f}_*$, we will denote it with $\overline{f}^*$ and it maps the category $QCoh_{f_*\mathcal{O}_\mathcal{Y}}$ to $QCoh_{\mathcal{O}_\mathcal{Y}}$.*

We need also a couple of properties of the functor $\overline{f}^*$:

Lemma 3.36.   *1. The functor $\overline{f}^*$ is exact.*

2. Let $f \colon \mathcal{X} \to \mathcal{Y}$ be an affine morphism of algebraic stacks and consider a base change p:

$$
\begin{array}{ccc}
\mathcal{X}_0 & \xrightarrow{\ q\ } & \mathcal{X} \\
{\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\
\mathcal{Y}_0 & \xrightarrow{\ p\ } & \mathcal{Y}
\end{array}
$$

The following base change rule holds:

$$\overline{f}^*p^* \cong q^*\overline{f}_0^* \tag{5.1}$$

and the isomorphism is canonical.

Proof. See [Har66, III.6] for some more detail. $\qquad\square$

Given a finite representable morphism of algebraic stacks $f \colon \mathcal{Y} \to \mathcal{X}$ we are now able to define the following functor:

$$f^\flat\mathcal{F} = \overline{f}^* R\,\mathcal{H}om_\mathcal{X}(f_*\mathcal{O}_\mathcal{Y}, \mathcal{F}) = R(\overline{f}^*\,\mathcal{H}om_\mathcal{X}(f_*\mathcal{O}_\mathcal{Y}, \mathcal{F})); \quad \mathcal{F} \in D^+(\mathcal{X}) \tag{5.2}$$

where the complex $R\,\mathcal{H}om_\mathcal{X}(f_*\mathcal{O}_\mathcal{Y}, \mathcal{F})$ must be considered as a complex of $f_*\mathcal{O}_\mathcal{Y}$-modules.

Theorem 3.37 (finite duality). *Let $f \colon \mathcal{Y} \to \mathcal{X}$ be a finite representable morphism of algebraic stacks. The twisted inverse image $f^!$ is the functor $f^\flat$.*

Proof. Let I be an injective quasicoherent sheaf on $\mathcal{X}$ defined by the couple (I_0, α) on a presentation of $\mathcal{X}$ (We keep notations in Proposition 3.12). We start observing that the sheaf $\mathcal{H}om_\mathcal{X}(f_*\mathcal{O}_\mathcal{Y}, I)$ is determined, on the same presentation by the following isomorphism:

$$s^*\,\mathcal{H}om_{X_0}(f_{0*}\mathcal{O}_{Y_0}, I_0) \xrightarrow[\sim]{b_s} \mathcal{H}om_{X_1}(f_{1*}\mathcal{O}_{Y_1}, s^*I_0) \xrightarrow[\sim]{\widetilde{\alpha}} \mathcal{H}om_{X_1}(f_{1*}\mathcal{O}_{Y_1}, t^*I_0)\dots$$

$$\dots \xleftarrow[\sim]{b_t} t^*\,\mathcal{H}om_{X_0}(f_{0*}\mathcal{O}_{Y_0}, I_0)$$

where b_s, b_t are the two natural isomorphisms and $\widetilde{\alpha}$ is induced by α. Applying $\overline{f}_1^*$ to this isomorphism we obtain the one defining $f^\flat I$. According to equation (5.1) we have $\overline{f}_1^*s^* \cong u^*\overline{f}_0^*$ and $\overline{f}_1^*t^* \cong v^*\overline{f}_0^*$. The composed isomorphism:

$$u^*\overline{f}_0^*\,\mathcal{H}om_{X_0}(f_{0*}\mathcal{O}_{Y_0}, I_0) \xrightarrow{\text{can}} \overline{f}_1^*s^*\,\mathcal{H}om_{X_0}(f_{0*}\mathcal{O}_{Y_0}, I_0) \xrightarrow{\overline{f}_1^*b_s} \overline{f}_1^*\,\mathcal{H}om_{X_1}(f_{1*}\mathcal{O}_{Y_1}, s^*I_0)$$

is the same as c_s and we can repeat the argument for t. Comparing with the isomorphism called β in Proposition 3.12 we prove the claim. $\qquad\square$

To be more explicit we need some vanishing result for $R\mathcal{H}om_{\mathcal{X}}$ like in Hartshorne [Har77, Lem 7.3]. First a technical lemma:

Lemma 3.38. *Let $\mathcal{X}$ be a projective Deligne-Mumford stack, $\mathcal{O}_X(1)$ and $\mathcal{E}$ as in $(*)$. Let $\mathcal{F}, \mathcal{G}$ be coherent sheaves on $\mathcal{X}$. For every integer i There is an integer $q_0 > 0$ such that for every $q \geq q_0$:*

$$\text{Ext}^i_{\mathcal{X}}(\mathcal{F}, \mathcal{G} \otimes \mathcal{E}^{\vee} \otimes \pi^* \mathcal{O}_X(q)) \cong \Gamma(\mathcal{X}, \mathcal{E}xt^i_{\mathcal{X}}(\mathcal{F}, \mathcal{G} \otimes \mathcal{E}^{\vee} \otimes \pi^* \mathcal{O}_X(q)))$$

Proof. Same proof as in [Har77, Prop 6.9] with obvious modifications. $\square$

Lemma 3.39. *Let $\mathcal{Y}$ be a codimension r closed substack in an n-dimensional smooth projective stack $\mathcal{X}$. Then $\mathcal{E}xt^i_{\mathcal{X}}(\mathcal{O}_{\mathcal{Y}}, \omega_{\mathcal{X}}) = 0$ for all $i < r$.*

Proof. The proof goes more or less like in [Har77, Lem 7.3]. Denote with F^i the coherent sheaf $\mathcal{E}xt^i_{\mathcal{X}}(\mathcal{O}_{\mathcal{Y}}, \omega_{\mathcal{X}})$. For q large enough the coherent sheaf $F_{\mathcal{E}}(F^i)(q)$ is generated by the global sections; if we can prove that $\Gamma(X, F_{\mathcal{E}}(F^i)(q)) = 0$ for $q >> 0$ we have also that $F_{\mathcal{E}}(F^i) = 0$. In Lemma 5.5 we will prove that $\pi \operatorname{supp} F^i = \operatorname{supp} F_{\mathcal{E}}(F^i)$, using this result we conclude that if $F_{\mathcal{E}}(F^i)(q)$ has no global sections for q big enough the sheaf F^i is the zero sheaf. We can now study the vanishing of $\Gamma(X, F_{\mathcal{E}}(F^i)(q))$:

$$\Gamma(X, F_{\mathcal{E}}(F^i)(q)) = \Gamma(\mathcal{X}, \mathcal{E}xt^i_{\mathcal{X}}(\mathcal{O}_{\mathcal{Y}}, \omega_{\mathcal{X}} \otimes \mathcal{E}^{\vee} \otimes \pi^* \mathcal{O}_X(q))) = \text{Ext}^i_{\mathcal{X}}(\mathcal{O}_{\mathcal{Y}}, \omega_{\mathcal{X}} \otimes \mathcal{E}^{\vee} \otimes \pi^* \mathcal{O}_X(q))$$

The last equality holds for a possibly bigger q according to Lemma 3.38. Applying smooth Serre duality we have the following isomorphism:

$$\text{Ext}^i_{\mathcal{X}}(\mathcal{O}_{\mathcal{Y}}, \omega_{\mathcal{X}} \otimes \mathcal{E}^{\vee} \otimes \pi^* \mathcal{O}_X(q)) \cong H^{n-i}(\mathcal{X}, \mathcal{O}_{\mathcal{Y}} \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-q))$$

This last cohomology group is the same as $H^{n-i}(X, F_{\mathcal{E}}(\mathcal{O}_{\mathcal{Y}})(-q))$; using again Lemma 5.5 we have that the dimension of $F_{\mathcal{E}}(\mathcal{O}_{\mathcal{Y}})(-q)$ is $n - r$ so that the cohomology group vanishes for $i < r$. $\square$

Proposition 3.40. *Let $f \colon \mathcal{Y} \to \mathcal{X}$ be a codimension r closed immersion of an equidimensional Cohen-Macaulay algebraic stack $\mathcal{Y}$ in a smooth projective stack $\mathcal{X} \to \operatorname{Spec} k$ of dimension n. The quasicoherent $\mathcal{O}_{\mathcal{Y}}$-module $\mathcal{E}xt^r_{\mathcal{X}}(\mathcal{O}_{\mathcal{Y}}, \omega_{\mathcal{X}})[n-r]$ is the dualizing complex of $\mathcal{Y}$.*

Proof. This is an immediate consequence of smooth Serre duality, Theorem 3.37 and Lemma 3.39. $\square$

Corollary 3.41 (Serre Duality). *Let $\mathcal{X}$ be a projective stack of pure dimension n and $i \colon \mathcal{X} \to \mathcal{P}$ a codimension r closed embedding in a smooth proper Deligne-Mumford stack $f \colon \mathcal{P} \to \operatorname{Spec} k$. The complex $\mathcal{E}xt^{\bullet}_{\mathcal{P}}(\mathcal{O}_{\mathcal{X}}, \omega_{\mathcal{P}})[n]$ is the dualizing complex of $\mathcal{X}$:*

$$i^! f^! k = \mathcal{E}xt^{\bullet}_{\mathcal{P}}(\mathcal{O}_{\mathcal{X}}, \omega_{\mathcal{P}})[n]$$

if $\mathcal{X}$ is also Cohen-Macaulay the dualizing complex is just the dualizing sheaf $\mathcal{E}xt^r_{\mathcal{P}}(\mathcal{O}_{\mathcal{X}}, \omega_{\mathcal{P}})$.

Proof. We have to prove that $\mathcal{E}\kern-1pt xt_{\mathcal{P}}^{j}(\mathcal{O}_{\mathcal{X}}, \omega_{\mathcal{P}}) = 0$ for $j > r$. We prove that for every point of $\mathcal{X}$ the stalk of $\mathcal{E}\kern-1pt xt_{\mathcal{P}}^{q}(\mathcal{O}_{\mathcal{X}}, \omega_{\mathcal{P}})$ vanishes for $q > r$. We use that for x a point in $\mathcal{X}$ we have $\mathcal{E}\kern-1pt xt_{\mathcal{P}}^{j}(\mathcal{O}_{\mathcal{X}}, \omega_{\mathcal{P}})_{x} = \mathrm{Ext}_{\mathcal{O}_{\mathcal{P},i(x)}}^{j}(\mathcal{O}_{\mathcal{X},x}, \omega_{\mathcal{P},i(x)})$. The stack $\mathcal{P}$ étale locally is $[\mathrm{Spec}\,C/G]$ where C is regular and G a finite group, since a closed embedding is given by a sheaf of ideals we can assume that $i\colon \mathcal{X} \to \mathcal{P}$ is given locally by:

$$[\mathrm{Spec}\,B/G] \xrightarrow{\ i\ } [\mathrm{Spec}\,C/G]$$

where B is local and Cohen-Macaulay. We denote with ω_C the canonical sheaf of $[\mathrm{Spec}\,C/G]$. As usual we have the long exact sequence:

$$0 \longrightarrow \mathrm{Hom}_C^G(B, \omega_C) \longrightarrow \mathrm{Hom}_C(B, \omega_C) \longrightarrow \mathrm{Hom}_C(B, \omega_C) \otimes G \longrightarrow$$
$$\longrightarrow \mathrm{Hom}_C(B, \omega_C) \otimes G \otimes G \longrightarrow \cdots$$

where the arrows are induced by the coactions of ω_C and of the structure sheaf of $[\mathrm{Spec}\,B/G]$. If we replace ω_C with an injective (equivariant) resolution $I^{\bullet}$ we obtain a double complex spectral sequence $E_1^{p,q} = \mathrm{Ext}_C^p(B, \omega_C) \otimes G^{\otimes q}$ abouting to the equivariant $R\,\mathrm{Hom}_C^G(B, \omega_C)$ (the sheaves $\mathrm{Hom}_C^G(B, I^{\bullet})$ are considered as B^G-modules) that is the stalk of the curly Ext. Since B is Cohen-Macaulay of the same dimension as $\mathcal{X}$ we have $\mathrm{Ext}_C^p(B, \omega_C) = 0$ for $p > r$ and the desired result follows. $\square$

It is important to stress that the previous Corollary 3.41 holds for every projective stack in characteristic zero; indeed according to [Kre06] such a stack can be embedded in a smooth proper Deligne-Mumford stack. To conclude we prove that π_* maps the dualizing sheaf of a projective Deligne-Mumford stack to the dualizing sheaf of its moduli scheme.

Proposition 3.42. *Let $\mathcal{X}$ be a projective Cohen-Macaulay Deligne-Mumford stack with moduli scheme $\pi\colon \mathcal{X} \to X$. Denote with $\omega_{\mathcal{X}}$ the dualizing sheaf of $\mathcal{X}$. The quasi coherent sheaf $\pi_*\omega_{\mathcal{X}} =: \omega_X$ is the dualizing sheaf of X.*

Proof. First of all we observe that the moduli scheme X is projective and Cohen-Macaulay so that we already know that its dualizing complex is actually a sheaf ω_X. We also know that $\pi^!\omega_X = \omega_{\mathcal{X}}$. We have just to prove that $\pi_*\pi^!\omega_X = \omega_X$. Let F be a quasi coherent sheaf on X, by duality we know that $R\,\mathrm{Hom}_X(F, \pi_*\pi^!\omega_X) = R\,\mathrm{Hom}_{\mathcal{X}}(L\pi^*F, \pi^!\omega_X) = R\,\mathrm{Hom}_X(\pi_*L\pi^*F, \omega_X)$. However we already know that $\pi_*\pi^* = \mathrm{id}$ and taking a locally free resolution (X is projective) we have also $\pi_*L\pi^* = \mathrm{id}$. Using duality on X and uniqueness of the dualizing sheaf we obtain $\pi_*\pi^!\omega_X = \omega_X$. $\square$

Corollary 3.43. *Let $X \to \mathrm{Spec}\,k$ be a variety with finite quotient singularities. Denote with $\pi\colon \mathcal{X}^{can} \to X$ the canonical stack associated to X as in [FMN07, Rem 4.9] and with ω_{can} its canonical bundle. The coherent sheaf $\pi_*\omega_{can}$ is the dualizing sheaf of X.*

Proof. We just observe that $\mathcal{X}^{\mathrm{can}}$ is smooth so that its dualizing sheaf is canonical bundle and apply the previous corollary. $\square$

Chapter 4

Applications and computations

1   Duality for nodal curves

Despite being probably already known, it is a good exercise to compute the dualizing sheaf for a nodal curve using the machinery developed so far. First of all we specify that by nodal curve we mean a non necessarily balanced nodal curve. We can assume from the beginning that the curve has generically trivial stabilizer. If it is not the case, we can always rigidify the curve and treat the gerbe separately. We assume also that if the node is reducible none of the two components has a non trivial generic stabilizer. With this assumption a stacky node étale locally looks like $\left[\operatorname{Spec}\frac{k[x,y]}{(xy)}/\mu_{a,k}\right]$. The action of $\mu_{a,k}$ is given by:

$$\frac{k[k,y]}{(xy)} \longrightarrow \frac{k[x,y]}{(xy)} \otimes \mu_{a,k}$$
$$x, y \longmapsto \lambda^i x, \lambda^j y$$

where $(i,j) = 1$; $i, j \neq 0 \mod a$ and a is coprime with the characteristic of k so that the stack is tame. The result of this section is the following theorem:

Theorem 4.1. *Let $\mathcal{C}$ be a proper tame nodal curve as specified above. Let $\pi\colon \mathcal{C} \to C$ be its moduli space. Let D be the effective Cartier divisor of C marking the orbifold points, and denote with $\mathcal{D} = \pi^{-1}(D)_{red}$. Denote also with ω_C the dualizing sheaf of C and with $\omega_{\mathcal{C}} = \pi^!\omega_C$ the dualizing sheaf of $\mathcal{C}$. The following relation holds:*

$$\omega_{\mathcal{C}}(\mathcal{D}) = \pi^*\omega_C(D)$$

We start proving this theorem with the following local computation:

Lemma 4.2. *Consider the orbifold node $\mathcal{Y} := \left[\operatorname{Spec}\frac{k[x,y]}{(xy)}/\mu_{a,k}\right]$ described above. Let $\rho\colon \mathcal{Y} \to Y := \operatorname{Spec}\frac{k[u,v]}{(uv)}$ be the moduli scheme, then we have $\omega_{\mathcal{Y}} = \rho^!\mathcal{O}_Y = \mathcal{O}_{\mathcal{Y}}$.*

Proof. We will denote the ring $\frac{k[x,y]}{(xy)}$ with B and $\frac{k[u,v]}{(uv)}$ with A. We also choose $\operatorname{Spec} B$ as atlas for the stack. Let α, β be the smallest positive integers such that $i\alpha = 0 \mod a$, $j\beta = 0 \mod a$, then the morphism from the atlas to the moduli scheme is the following:

$$A \xrightarrow{\ p_0\ } B$$
$$u, v \longmapsto x^\alpha, y^\beta$$

The dualizing sheaf for $\operatorname{Spec} A$ is isomorphic to the structure sheaf, so it's enough to compute duality for the structure sheaf denoted as the free A-module $\langle e \rangle$. According to 3.25 we first need to compute the B-module $R \operatorname{\mathcal{H}\!\mathit{om}}_A(B, \langle e \rangle)$. We take the infinite projective resolution of B as an A-module:

$$\cdots \longrightarrow A^{\oplus(\alpha+\beta-2)} \longrightarrow A^{\oplus(\alpha+\beta-2)} \longrightarrow A^{\oplus(\alpha+\beta-1)} \longrightarrow B \longrightarrow 0$$
$$e_0 \longmapsto 1$$
$$\cdots \qquad e_l \longmapsto u e_l, \; e_l \longmapsto v e_l, \; e_l \longmapsto x^l$$
$$\cdots \qquad f_m \longmapsto v f_m, \; f_m \longmapsto u f_m, \; f_m \longmapsto y^m$$

where $1 \leq l \leq \alpha - 1$, $1 \leq m \leq \beta - 1$. We apply the functor $\operatorname{Hom}_A(\cdot, \langle e \rangle)$ and compute cohomology. The complex is obviously acyclic as expected, and h^0 is the A-module $\bigoplus_l (u) e_l^\vee \oplus \bigoplus_m (v) f_m^\vee \oplus e_0^\vee$. The A-module h^0 is naturally a sub-module of $\operatorname{Hom}_A(B, \langle e \rangle)$ and its B-module structure is induced by the natural B-module structure of this last one. Let $g_l^\vee \in \operatorname{Hom}_A(B, \langle e \rangle)$ the morphism such that $g_l^\vee(x^l) = e$ and zero otherwise, $h_m^\vee$ the morphism such that $h_m^\vee(y^m) = e$ and zero otherwise, and $g_0^\vee$ such that $g_0^\vee(1) = e$. The B-module $\operatorname{Hom}_A(B, \langle e \rangle)$ can be written as $\frac{\langle g_{\alpha-1}^\vee, h_{\beta-1}^\vee \rangle}{x^{\alpha-1} g_{\alpha-1}^\vee - y^{\beta-1} h_{\beta-1}^\vee}$. The B module structure of h^0 is then given by:

$$\langle e_0^\vee \rangle \rightarrowtail \frac{\langle g_{\alpha-1}^\vee, h_{\beta-1}^\vee \rangle}{x^{\alpha-1} g_{\alpha-1}^\vee - y^{\beta-1} h_{\beta-1}^\vee}$$
$$e_0^\vee \longmapsto x^{\alpha-1} g_{\alpha-1}^\vee$$

Eventually we have $\overline{p}_0^* R \operatorname{Hom}_A(B, \langle e \rangle) = \langle e_0^\vee \rangle$. To compute the equivariant structure of $\langle e_0^\vee \rangle$ we follow the recipe in Lemma 3.25. We find out that the coaction on $e_0^\vee, e_l^\vee, f_m^\vee$ is as follows:

$$e_0^\vee \longrightarrow e_0^\vee$$
$$e_l^\vee \longrightarrow \lambda^{-il} e_l^\vee$$
$$f_m^\vee \longrightarrow \lambda^{-jm} e_m^\vee$$

so that the equivariant structure of $\langle e_0^\vee \rangle$ is the trivial one and $\rho^! \omega_Y$ is then canonically isomorphic to the structure sheaf. $\qquad \square$

Remark 4.3. We can notice that the assumptions on the two integers i, j have never been used in the previous proof, however they are going to be necessary in what follows.

With the following proposition we take care of smooth orbifold points.

Proposition 4.4. *Let $\mathcal{X} \to \operatorname{Spec} k$ be a projective Deligne-Mumford stack that is generically a scheme, let $D = \sum_{i=1}^d D_i$ be a simple normal crossing divisor whose support does not contain any orbifold structure. Let $\mathbf{a} = (a_1, \ldots, a_d)$ positive integers. Denote with $\mathcal{X}_{\mathbf{a},D} = \sqrt[\mathbf{a}]{D/\mathcal{X}} \xrightarrow{\tau} \mathcal{X}$ and with $\mathcal{D}_i = (\tau^{-1} D_i)_{red}$. For every $\mathcal{F}$ quasicoherent sheaf on $\mathcal{X}$ the object $\tau^! \mathcal{F} \in D(\mathcal{X}_{\mathbf{a},D})$ is the quasicoherent sheaf $\tau^* \mathcal{F}(\sum_{i=1}^d (a_i - 1) \mathcal{D}_i)$.*

Proof. Since τ is a flat morphism we already know that $\tau^!$ maps quasicoherent sheaves to quasicoherent sheaves. The precise statement can be retrieved using some of the computations in [AGV06, Thm 7.2.1] and the machinery in Theorem 3.33 (details left to the reader). $\qquad \square$

proof of Theorem 4.1. For the moment we can assume that the curve has no other orbifold points then the nodes and without loss of generality we can assume that there is only one node. First we prove that $\pi^!\omega_C = \pi^*\omega_C$, then we can add smooth orbifold points in a second time applying the root construction; the formula claimed in the theorem follows then from Proposition 4.4. First of all we take an étale cover $\mathcal{Y}$ of $\mathcal{C}$ in this way: we choose an étale chart of the node that is an orbifold node like the one in Lemma 4.2 and we complete the cover with a chart that is the curve $\mathcal{C}$ minus the node, denoted with C_0. The setup is summarized by the following cartesian square:

$$
\begin{array}{ccc}
C_0 \coprod [\operatorname{Spec} B/\mu_a] & \xrightarrow{i \coprod \overline{\sigma}} & \mathcal{C} \\
\downarrow{\scriptstyle \operatorname{id} \coprod p_0} & & \downarrow{\scriptstyle \pi} \\
C_0 \coprod \operatorname{Spec} A & \xrightarrow{i \coprod \sigma} & C
\end{array}
$$

where $B = \frac{k[x,y]}{(xy)}$, $A = \frac{k[u,v]}{(uv)}$, the map p_0 sends u, v to x^α, y^β, the map i is the inclusion of C_0 in C, and σ is étale. We use as an atlas $C_0 \coprod \operatorname{Spec} B$ with the obvious map to $\mathcal{C}$. Completing the presentation we obtain the following groupoid:

$$
C_0 \coprod \operatorname{Spec} B \times \mu_a \coprod \operatorname{Spec} B \times_{\sigma \circ p_0, C} C_0 \underset{t}{\overset{s}{\rightrightarrows}} C_0 \coprod \operatorname{Spec} B
$$

We can divide it in three pieces: one is the trivial groupoid over C_0, the second is $\operatorname{Spec} B \times \mu_a \rightrightarrows \operatorname{Spec} B$ where the arrows are action and projection; the last one is $\operatorname{Spec} B \times_{\sigma \circ p_0, C} C_0 \overset{p_0 \circ q_1}{\underset{q_2 \overline{p_0}}{\rightrightarrows}} C_0 \coprod \operatorname{Spec} B$, where q_1 is projection to $\operatorname{Spec} B$, q_2 is projection to C_0 and $\overline{p}_0$ fits inside the following cartesian square:

$$
\begin{array}{ccccc}
\operatorname{Spec} B \times_{\sigma \circ p_0, C} C_0 & \xrightarrow{\overline{p}_0} & \operatorname{Spec} A \times_{\sigma, C} C_0 & \xrightarrow{q_2} & C_0 \\
\downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle i} \\
\operatorname{Spec} B & \xrightarrow{p_0} & \operatorname{Spec} A & \xrightarrow{\sigma} & C
\end{array} \qquad (1.1)
$$

Now we check if the dualizing sheaf glues like $\pi^*\omega_C$ on this presentation and we achieve this using Theorem 3.33. The result is trivially true for the first piece of the presentation. For the second piece of the presentation it is implied immediately by Lemma 4.2. For what concerns the last piece of presentation we have that $\pi^!\omega_C$ glues with the canonical isomorphism $\overline{p}_0^! q_2^* i^* \omega_C \cong q_1^* p_0^! \sigma^* \omega_C$ where the canonical isomorphism comes from the cartesian square in picture (1.1). However we have $q_1^* p_0^! \sigma^* \omega_C = q_1^* p_0^! \mathcal{O}_A \otimes q_1^* p_0^* \sigma^* \omega_C$ and $\overline{p}_0^! \mathcal{O}_{\operatorname{Spec} A \times_{\sigma, C} C_0} \otimes \overline{p}_0^* q_2^* i^* \omega_C p_0^! \sigma^* \omega_C$ where the canonical isomorphism comes from the cartesian square in picture (1.1). However we have $q_1^* p_0^! \sigma^* \omega_C = q_1^* p_0^! \mathcal{O}_A \otimes q_1^* p_0^* \sigma^* \omega_C$ and $\overline{p}_0^! q_2^* i^* \omega_C = \overline{p}_0^! \mathcal{O}_{\operatorname{Spec} A \times_{\sigma, C} C_0} \otimes \overline{p}_0^* q_2^* i^* \omega_C$. According to Lemma 4.2 the sheaves $p_0^! \mathcal{O}_A$ and $\overline{p}_0^! \mathcal{O}_{\operatorname{Spec} A \times_{\sigma, C} C_0}$ are respectively equal to $\mathcal{O}_B$ and $\mathcal{O}_{\operatorname{Spec} B \times_{\sigma \circ p_0, C} C_0}$. Eventually the gluing isomorphism is just the identity and we can conclude that the dualizing sheaf is $\pi^*\omega_C$. $\qquad \square$

2 Other examples of singular curves

In the previous section we have seen that nodal curves, balanced or not, have a dualizing sheaf that is an invertible sheaf and carry a trivial representation on the fiber on the

node. It is not difficult to find examples of singularities where the representation on the fiber of the singularity is non trivial. What follows is a collection of computations of duality with compact support, performed with the same technique used in Lemma 4.2. These examples are mere applications of Lemma 3.25 and we will be able to retrieve these results with a better technique in the next section.

Example 4.5. **Cusp over a line** Let B be the cusp $k[x,y]/(y^2 - x^3)$ with an action of $\mu_{2,k}$ given by $y \mapsto \lambda y$ and $x \mapsto x$. The moduli scheme of the quotient stack is the affine line $k[x] =: A$ with the morphism:

$$A \xrightarrow{\ f\ } B$$
$$x \longmapsto x$$

The morphism f is flat and the dualizing sheaf restricted to the atlas is the B-module $\mathrm{Hom}_A(B, A)$. As a B-module this is just $\langle e_1^\vee \rangle$ where $e_1^\vee(y) = 1$ and zero otherwise. The coaction is given by $e_1^\vee \mapsto \lambda^{-1} e_1^\vee$.

Example 4.6. **Tac-node over a node** Let B be the tac-node $k[x,y]/(y^2 - x^4)$ with an action of $\mu_{2,k}$ given by $y \mapsto y$ and $x \mapsto \lambda x$. The moduli scheme of the quotient stack is the node $k[u,y]/(y^2 - u^2) =: A$ with the morphism:

$$A \xrightarrow{\ f\ } B$$
$$u, y \longmapsto x^2, y$$

The stack is reducible. The morphism f is flat again and the dualizing sheaf is the B-module $\langle e_1^\vee \rangle$ ($e_1^\vee(x) = 1$ and otherwise zero) with the coaction $e_1^\vee \mapsto \lambda^{-1} e_1^\vee$.

These two examples look pretty similar but they are actually of a quite different nature. With a simple computation we obtain that the tac-node is actually a root construction over the node $\sqrt[2]{\mathcal{O}_A, 0/\operatorname{Spec} A}$. For a root construction we expected that kind of dualizing sheaf from, the already studied, smooth case (Lemma 4.4). The cusp is not a root construction, however it is flat on the moduli scheme anyway, and the dualizing sheaf is the same we have for the root construction.

With the following example we see that the dualizing sheaf can be the structure sheaf for nodes other than $xy = 0$.

Example 4.7. **Tac-node over a cusp (an irreducible node)** Let B be the tac-node $k[x,y]/(y^2 - x^4)$ with an action of $\mu_{2,k}$ given by $y \mapsto \lambda y$ and $x \mapsto \lambda x$. The moduli scheme of the quotient stack is the cusp $k[u,t]/(t^2 - u^3) =: A$ with the morphism:

$$A \xrightarrow{\ f\ } B$$
$$u, t \longmapsto x^2, xy$$

This one stack is irreducible, none of $y - x^2$ and $y + x^2$ can be a closed substack. With a computation very similar to the one in Lemma 4.2 we obtain that the dualizing sheaf is $\langle e_0^\vee \rangle$ ($e_0^\vee(1) = 1$ and otherwise zero) with the trivial coaction.

3   Local complete intersections

According to Corollary 3.41 every Cohen-Macaulay projective Deligne-Mumford stack (in characteristic zero) admits a dualizing coherent sheaf; using Corollary 3.21 it's immediate

to prove that if the stack also have a Gorenstein atlas then the dualizing sheaf is an invertible sheaf. The aim of this section is to reconstruct the classic duality result for local complete intersections as in [Har77, III.7.11].

Theorem 4.8. *Let $\mathcal{X}$ be a projective Deligne-Mumford stack that has a regular codimension r closed embedding in a smooth projective Deligne-Mumford stack $\mathcal{P}$. Denote with $\mathcal{I}$ the ideal sheaf defining the closed-embedding and with $\omega_{\mathcal{P}}$ the canonical sheaf of $\mathcal{P}$. The dualizing sheaf of $\mathcal{X}$ is $\omega_{\mathcal{P}} \otimes \wedge^r(\mathcal{I}/\mathcal{I}^2)^{\vee}$.*

Proof. Our task is to compute $\mathcal{E}xt^r_{\mathcal{P}}(\mathcal{O}_{\mathcal{X}}, \omega_{\mathcal{P}})$. We can produce an étale cover of $\mathcal{P}$ so that it is locally $[\operatorname{Spec} C/G]$ where C is a regular ring and G a finite group. We can assume that the regular closed embedding is locally:

$$[\operatorname{Spec} B/G] \xrightarrow{i} [\operatorname{Spec} C/G]$$

where B is defined by $(f_1, \ldots, f_r)$ a regular sequence in C. Once we have fixed a basis for the ideal sheaf $\mathcal{I}$, the coaction of G is also determined on that basis. We denote with β_f the coaction on the basis $(f_1, \ldots, f_r)$, and it is an r-dimensional representation. We denote with ω_C the canonical sheaf on $\operatorname{Spec} C$, and it also comes with a coaction that is a one-dimensional representation ρ_C (we are assuming that ω_C is free). We now take the Koszul resolution $K^{\bullet}$ of B. The coactions of G on ω_C and $\mathcal{I}$ induce a coaction of G on $K^{\bullet}$ and a coaction on the complex $\operatorname{Hom}_C(K^{\bullet}, \omega_C)$. In particular the coaction on $\operatorname{Hom}_C(\wedge^r C^{\oplus r}, \omega_C)$ is the representation $\rho_C \otimes \det \beta_f^{\vee}$. Both the induced coaction (denoted with $\gamma^{\bullet}$) and the trivial coaction are morphisms of complexes:

$$0 \longrightarrow \operatorname{Hom}^G_C(K^{\bullet}, \omega_C) \longrightarrow \operatorname{Hom}_C(K^{\bullet}, \omega_C) \xrightarrow{\gamma^{\bullet}} \operatorname{Hom}_C K^{\bullet}, \omega_C \otimes \mathcal{O}_G$$

and the equalizer is the C^G-module of equivariant morphisms. The cohomology of this first complex computes the global sections of $\mathcal{E}xt^{\bullet}_{\mathcal{P}}(\mathcal{O}_{\mathcal{X}}, \omega_C)$ restricted to $[\operatorname{Spec} B/G]$. We can also compute cohomology of the second and third complex and we have arrows between cohomologies induced by both the coaction $\gamma^{\bullet}$ and the trivial coaction:

$$h^{\bullet}(\operatorname{Hom}_C(K^{\bullet}, \omega_C)) \xrightarrow{h^{\bullet}(\gamma^{\bullet})} h^{\bullet}(\operatorname{Hom}_C K^{\bullet}, \omega_C) \otimes \mathcal{O}_G$$

This gives the sheaves $h^{\bullet}(\operatorname{Hom}_C(K^{\bullet}, \omega_C))$ an equivariant structure; eventually these sheaves with the equivariant structure are $\mathcal{E}xt^{\bullet}_{\mathcal{P}}(\mathcal{O}_{\mathcal{X}}, \omega_C)$ restricted to $[\operatorname{Spec} B/G]$. However, we already know that $h^r(\operatorname{Hom}_C(K^{\bullet}, \omega_C)) = \frac{\omega_C}{(f_1, \ldots, f_r)\omega_C}$ and the others are zero. It is easy to check that the equivariant structure of the non vanishing one is the representation $\rho_C \otimes \det \beta_f^{\vee}$. To summarize, the dualizing sheaf restricted to $[\operatorname{Spec} B/G]$ is isomorphic to $\omega_C \otimes_C B$ where B has the non necessarily trivial coaction $\det \beta_f^{\vee}$. As in the case of schemes this isomorphism is not canonical. If we change the basis $(f_1, \ldots, f_r)$ to a new one $(g_1, \ldots, g_r)$ where $f_i = \delta_{i,j} g_j$ we produce an automorphism on the Koszul complex $K^{\bullet}$; in particular the last term of the complex $\wedge^r C^{\oplus r}$ carries an automorphism given by $\det \delta$. In the equivariant setup, also the representation β_f is affected by a change of basis. In particular the new basis carries a representation β_g such that $(a^*\delta) \circ \beta_g = \beta_f \circ \delta$, where a denotes the action of G on $\operatorname{Spec} C$.

As in the case of schemes the sheaf $\wedge^r(\mathcal{I}/\mathcal{I}^2)^\vee$ is trivial on $[\operatorname{Spec} B/G]$ and the change of basis $(f_1, \ldots, f_r) \mapsto (g_1, \ldots, g_r)$ induces an automorphism on the sheaf that is multiplication by $\det \delta^{-1}$. Moreover it is straightforward to check that the equivariant structure of the sheaf is given by the representation $\det \beta_f^\vee$. It is also obvious that the representation changes, after a change of basis, according to the formula $(a^*\delta) \circ \beta' = \beta \circ \delta$. Eventually we can conclude that there is an isomorphism between $\mathcal{E}\!xt_\mathcal{P}^r(\mathcal{O}_\mathcal{X}, \omega_C)|_{[\operatorname{Spec} B/G]}$ and $\omega_\mathcal{P} \otimes \wedge^r(\mathcal{I}/\mathcal{I}^2)^\vee|_{[\operatorname{Spec} B/G]}$ that doesn't depend on the choice of the basis of $\mathcal{I}$, so to speak a canonical isomorphism. This implies that we can glue these local isomorphisms to obtain a global one. $\qquad\square$

In the proof of this theorem we have seen how to compute the sheaf $\mathcal{E}\!xt_\mathcal{P}^r(\mathcal{O}_\mathcal{X}, \omega_\mathcal{P})$ locally on a stack that is $[\operatorname{Spec} B/G]$ using the Koszul resolution. Even if the stack is not locally complete intersection but Cohen-Macaulay we can use the same technique to compute $\mathcal{E}\!xt^r$, replacing the Koszul complex with some other equivariant resolution. This approach is obviously a much faster and reliable technique then the one used in section 3.7.

Example 4.9. **A non Gorenstein example** Let B be the triple point $k[u, v, t]/(uv - t^2, ut - v^2, vt - u^2)$ with an action of $\mu_{a,k}$ given by $u, v, t \mapsto \lambda u, \lambda v, \lambda t$ and we study duality of the quotient stack. This is non Gorenstein since the reducible ideal (uvt) is a system of parameters and B is Cohen-Macaulay (we apply the Ubiquity Theorem in [Bas63]). We can close it in $k[u, v, t]$ with the same action of $\mu_{a,k}$. The canonical sheaf of this quotient stack is the free module $\langle du \wedge dv \wedge dt \rangle$ with the coaction $du \wedge dv \wedge dt \mapsto \lambda^{-3} du \wedge dv \wedge dt$. It is completely straightforward to check that the dualizing sheaf of B is the B-module:

$$\omega_B := \frac{\langle e_1, e_2 \rangle}{(te_1 + ue_2, ve_1 + te_2, ue_1 + ve_2)}$$

It is also easy to check that the equivariant resolution of $(uv - t^2, ut - v^2, vt - u^2)$ induces a coaction $e_i \mapsto \lambda^3 e_i$ for $i = 1, 2$ that compensates the coaction carried by $\langle du \wedge dv \wedge dt \rangle$. Eventually the dualizing sheaf of $[\operatorname{Spec} B/G]$ is ω_B with the trivial coaction.

Part III

Moduli of semistable sheaves

Chapter 5

Gieseker stability

In the second section we have fixed the setup we use here to define a good notion of stability for coherent sheaves. We define a concept of *Gieseker stability* that relies on a modified Hilbert polynomial.

Assumption. In this section $p\colon \mathcal{X} \to \operatorname{Spec} k$ is a projective stack over a field k with moduli scheme $\pi\colon \mathcal{X} \to X$; a very ample invertible sheaf $\mathcal{O}_X(1)$ and a generating sheaf $\mathcal{E}$ are chosen. We will call this couple of sheaves a *polarization* of $\mathcal{X}$

1 Pure sheaves

As in the case of sheaves on schemes we can define the support of a sheaf in the following way [HL97, 1.1.1]

Definition 5.1. Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$, we define the support of $\mathcal{F}$ to be the closed substack associated to the ideal:

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathcal{X}} \to \mathcal{E}\!nd_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F})$$

Since the stack $\mathcal{X}$ has finite stabilizers we can deal with the dimension of the support of a sheaf as we do with schemes and define [HL97, 1.1.2]:

Definition 5.2. A pure sheaf of dimension d is a coherent sheaf $\mathcal{F}$ such that for every non zero subsheaf $\mathcal{G}$ the dimension of the support of $\mathcal{G}$ is d.

Remark 5.3. 1. Assume now that $\mathcal{X}$ is Deligne-Mumford and let $X_1 \xrightarrow[t]{\;s\;} X_0 \xrightarrow{\;\phi\;} \mathcal{X}$ be an étale presentation and $\mathcal{F}$ a pure sheaf on $\mathcal{X}$ of dimension d. If the ideal sheaf $\mathcal{I}$ defines the support of $\mathcal{F}$, the ideal $\phi^*\mathcal{I}$ defines the support of $\phi^*\mathcal{F}$ in X_0. This follows from the flatness of ϕ. For the same reason we can produce an étale presentation of $\operatorname{supp}(\mathcal{F})$ as $\operatorname{supp}(t^*\phi^*\mathcal{F}) \xrightarrow[t]{\;s\;} \operatorname{supp}(\phi^*\mathcal{F})$. It is then clear that $\dim(\mathcal{F}) = 2\dim(\phi^*\mathcal{F}) - \dim(t^*\phi^*\mathcal{F}) = \dim \phi^*\mathcal{F}$ in every point of $\operatorname{supp}(\mathcal{F})$.

2. The notion of associated point in the case of a Deligne-Mumford stack is the usual one [Lie07, 2.2.6.5]. A geometric point x of $\mathcal{X}$ is associated for the coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ if x is associated for the stalk $\mathcal{F}_x$ as on $\mathcal{O}_{\mathcal{X},x}$-module. A sheaf is pure

if and only if every associated prime has the same dimension (*i.e.*, the support has pure dimension and there are no embedded primes in the sheaf). Moreover $\phi(\text{Ass}\,(\phi^*\mathcal{F})) = \text{Ass}\,(\mathcal{F})$ [Lie07, 2.2.6.6]. It is now clear that $\mathcal{F}$ is pure of dimension d if and only if $\phi^*\mathcal{F}$ is pure of the same dimension.

As in [HL97, 1.1.4] we have the torsion filtration:

$$0 \subset T_0(\mathcal{F}) \subset \ldots \subset T_d(\mathcal{F}) = \mathcal{F}$$

where every factor $T_i(\mathcal{F})/T_{i-1}(\mathcal{F})$ is pure of dimension i or zero.

In the case $\mathcal{X}$ is an Artin stack we don't know what is the meaning of the dimension of the support, but we can use the notion of associated point as defined in [Lie07, 2.2.6.4] and of torsion subsheaf [Lie07, 2.2.6.10]. Lieblich proves that the sum of torsion subsheaves is torsion so that there is a maximal torsion subsheaf [Lie07, 2.2.6.11]. We will denote it with $T(\mathcal{F})$. Maybe there is a more general notion of a torsion filtration for sheaves on Artin stacks but for the moment we prefer to limit the study to the case of torsion free sheaves.

In the following we prove that the functor $F_{\mathcal{E}}$ maps torsion free sheaves to torsion free sheaves and more generally it preserves the pureness and the dimension of a sheaf. This will be of great help to prove Corollary 3. The proof goes in two parts. First we observe that the morphism π is an omeomorphism so that it preserves the dimension of points and we prove that $\pi\,\text{Ass}\,\mathcal{F} = \text{Ass}(\pi_*\mathcal{F})$ for every coherent sheaf $\mathcal{F}$, unless the push-forward $\pi_*\mathcal{F}$ vanishes. Second, we prove that $F_{\mathcal{E}}(\mathcal{F})$ is non zero unless $\mathcal{F}$ itself is zero. To clarify the situation we show the following example. Let $\pi\colon \mathcal{X} \to X$ be an abelian G-gerbe over a scheme. Every sheaf $\mathcal{F}$ on $\mathcal{X}$ decomposes into a direct sum on the characters of the banding group $\mathcal{F} = \bigoplus_{\chi \in C(G)} \mathcal{F}_\chi$. The push-forward $\pi_*\mathcal{F}$ is just $\pi_*\mathcal{F}_0$ where $\mathcal{F}_0$ corresponds to the trivial character. This example explains that a sheaf supported on a gerbe can be sent to zero, however we shall prove that this is the only pathology that occurs, and tensoring with the generating sheaf removes this pathology.

The following lemma is a simplified version of the general result 5.7 and its Corollary 5.8; the proof is much easier but it holds only for torsion free sheaves on integral projective stacks (but even Artin stacks).

Lemma 5.4. *Let $\mathcal{F}$ be a coherent torsion free sheaf on $\mathcal{X}$, a projective integral stack over a field, then $F_{\mathcal{E}}(\mathcal{F})$ (see 2.4 for its definition) is torsion free on the moduli scheme X. If $T(\mathcal{F})$ is the torsion subsheaf of $\mathcal{F}$ then $F_{\mathcal{E}}(T(\mathcal{F}))$ is the torsion subsheaf of $F_{\mathcal{E}}(\mathcal{F})$.*

Proof. Since $\mathcal{E}$ is locally free, $\mathcal{F}$ is torsion free if and only if $\mathcal{F} \otimes \mathcal{E}^\vee$ is torsion free. The torsion of a sheaf can be checked on the stalks so we can work locally and assume that X is an affine local scheme $\text{Spec}\,A$, the stack $\mathcal{X}$ is a quotient $[\text{Spec}\,B/G]$ of an affine scheme by a linearly reductive group scheme $p\colon G \to \text{Spec}\,A$ such that $A = B^G$, and the sheaf is a coherent B-module with a coaction α of G. We can assume that M is a torsion free B-module by [Lie07, Lem 2.2.6.17]. The push forward of the sheaf is just $(_A M)^G$ and we want to prove it is a torsion free A-module. Let $A \xrightarrow{f} B$ be the map of rings and for every

$a \in A$ consider the following exact diagram of A-modules:

$$
\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 \longrightarrow ({}_A M)^G \longrightarrow & & ({}_A M) & \xrightarrow{\ \alpha - \iota\ } & M \otimes_A p_* \mathcal{O}_G & & \\
& & \downarrow{\scriptstyle \cdot f(a)} & & \downarrow{\scriptstyle \cdot f(a)} & & \\
0 \longrightarrow ({}_A M)^G \longrightarrow & & ({}_A M) & \xrightarrow{\ \alpha - \iota\ } & M \otimes_A p_* \mathcal{O}_G & &
\end{array}
$$

where ι is the trivial coaction. The two squares are commutative since multiplying by $f(a)$ is equivariant, the central column is injective since M is torsion free (here we use that $\operatorname{Spec} B$ is integral and $\operatorname{Spec} A$ is integral too according to [MFK94, (2) pag.5]), the right column is injective because the group scheme is flat and the first one can be checked to be injective too. This argument still doesn't imply that the right column is not zero. We give an argument for the non vanishing of $({}_A M)^G$ in a more general setup in the next lemma. The second statement follows from the exactness of $F_{\mathcal{E}}$. $\qquad\square$

Here begins the proof for pure sheaves.

Lemma 5.5. *Let $\mathcal{X} \to \operatorname{Spec} k$ be a projective stack and $\pi\colon \mathcal{X} \to X$ the moduli scheme. Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$, then we have:*

$$
\pi \operatorname{supp} \mathcal{F} = \pi \operatorname{supp} \mathcal{F} \otimes \mathcal{E}^{\vee} = \operatorname{supp} F_{\mathcal{E}}(\mathcal{F}) \tag{1.1}
$$

Proof. Since the sheaf $\mathcal{E}$ is locally free and supported everywhere we have the first claimed equality (the tensor product intersects the supports). This is made evident by the following diagram:

$$
\begin{array}{ccccc}
0 \longrightarrow \mathcal{I} & \longrightarrow & \mathcal{O}_{\mathcal{X}} & \longrightarrow & \mathcal{E}nd_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}) \\
\downarrow{\scriptstyle \wr} & & \| & & \downarrow{\scriptstyle \otimes \operatorname{id}_{\mathcal{E}}} \\
0 \longrightarrow \mathcal{I}_{\mathcal{E}} & \longrightarrow & \mathcal{O}_{\mathcal{X}} & \longrightarrow & \mathcal{E}nd_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F} \otimes \mathcal{E}^{\vee})
\end{array}
$$

where $\mathcal{I}$ defines $\operatorname{supp}(\mathcal{F})$ and $\mathcal{I}_{\mathcal{E}}$ the support of $\mathcal{F} \otimes \mathcal{E}^{\vee}$. The second equality is less trivial. We start from the following commutative and exact diagram:

$$
\begin{array}{ccccc}
0 \longrightarrow \pi_* \mathcal{I}_{\mathcal{E}} & \longrightarrow & \mathcal{O}_X & \longrightarrow & \pi_* \mathcal{E}nd_{\mathcal{X}}(\mathcal{F} \otimes \mathcal{E}^{\vee}) \\
\downarrow & & \| & & \downarrow \\
0 \longrightarrow \mathcal{I}_{F_{\mathcal{E}}(\mathcal{F})} & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{E}nd_X(F_{\mathcal{E}}(\mathcal{F}))
\end{array}
$$

The vertical arrow on the left is injective for free, the vertical arrow on the right is the natural one. Our target is to prove that the vertical arrow on the left is an isomorphism. To achieve this it's enough to prove that the right vertical arrow is injective. This would be false if we removed the generating sheaf from the diagram, consider for instance our previous example of the gerbe; in that case the arrow would be surjective but not injective.

To prove injectivity we reduce the problem to an étale local computation using the cartesian square:

$$
\begin{array}{ccc}
\coprod_i [\operatorname{Spec} B_i / G_i] & \xrightarrow{\ \coprod f_i\ } & \mathcal{X} \\
\downarrow{\scriptstyle \coprod \rho_i} & & \downarrow{\scriptstyle \pi} \\
\coprod_i \operatorname{Spec} A_i & \xrightarrow{\ \coprod g_i\ } & X
\end{array}
$$

where G_i's are linearly reductive, f, g are étale and $A_i = B_i^{G_i}$ for every i. Clearly we have the following commutative diagram:

$$\prod_i \rho_{i*} \mathcal{E}nd_{[\operatorname{Spec} B_i/G_i]}(f_i^* \mathcal{F} \otimes \mathcal{E}^\vee) \xrightarrow{\ \sim\ } \prod_i g_i^* \pi_* \mathcal{E}nd_{\mathcal{X}}(\mathcal{F} \otimes \mathcal{E}^\vee)$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\prod_i \mathcal{E}nd_{\operatorname{Spec} A_i}(\rho_{i*} f_i^* \mathcal{F} \otimes \mathcal{E}^\vee) \xrightarrow{\ \sim\ } \prod_i g_i^* \mathcal{E}nd_X(\pi_*(\mathcal{F} \otimes \mathcal{E}^\vee))$$

If we can prove that the left column is injective then the right column is as well, and we are done because $\prod_i g_i$ is étale and surjective. In the local picture $f^* \mathcal{F}$ is a B-module with a coaction (we drop the i index from now on), this implies that the morphism can fail to be injective if and only if $\rho_* f^*(\mathcal{F} \otimes \mathcal{E}^\vee)$ vanishes. We prove that this can't occur.

We denote with M the B-module $f^* \mathcal{F}$; we assume that the generating sheaf is a sum $\bigoplus_{\chi \in C(G)} \mathcal{E}_\chi$ over the characters of G, each of them appearing exactly once (to simplify notations). The sheaf $\mathcal{E}_\chi$ is the free B-module $B^{\oplus r_\chi}$ where r_χ is the dimension of χ and it carries χ as a coaction. Denote with r the sum $\sum_{\chi \in C(G)} r_\chi$ the rank of $\mathcal{E}$. We can produce an equivariant presentation of M using powers of $\mathcal{E}$, and use it to compute $\rho_* f^*(\mathcal{F} \otimes \mathcal{E}^\vee)$:

$$
\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
(B^{\oplus r^2 c})^G & \longrightarrow & (B^{\oplus r^2 b})^G & \longrightarrow & (M \otimes B^{\oplus r})^G \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
B^{\oplus r^2 c} & \longrightarrow & B^{\oplus r^2 b} & \longrightarrow & M \otimes B^{\oplus r} \longrightarrow 0
\end{array}
$$

For every χ the sheaf $\mathcal{E}_\chi \otimes \mathcal{E}_\chi^\vee$ carries a representation that to the standard basis $e_i \otimes e_j^\vee$ associates 0 if $i \neq j$ and $e_i \otimes e_i^\vee$ otherwise; this implies that its pushforward is the free module A^{r_χ}. As a consequence the A-module $(B^{\oplus r^2 c})^G$ contains as a summand A^{rc} and the same for $(B^{\oplus r^2 b})^G$. Moreover the map between them $A^{\oplus rc} \to A^{\oplus rb}$ is induced by the map in the resolution $B^{\oplus rc} \to B^{\oplus rb}$. Since this last one is not surjective so is not the first one and this concludes the proof. $\qquad \square$

We still don't know if the morphism π "respects" associated points of $\mathcal{F}$.

Lemma 5.6. *Let* $f \colon \operatorname{Spec} B \to \operatorname{Spec} A$ *be a surjective flat morphism of noetherian schemes, E an A-module. We have the following:*

$$f \operatorname{Ass}(E \otimes_A B) = \operatorname{Ass}(E) \tag{1.2}$$

Proof. It is a special case of [Mat80, Thm 12]. $\qquad \square$

Proposition 5.7. *Let* $\mathcal{X} \to \operatorname{Spec} k$ *be a projective Deligne-Mumford stack and* $\pi \colon \mathcal{X} \to X$ *the moduli scheme. Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$. If the sheaf $\mathcal{F}$ is pure of dimension d the sheaf $F_{\mathcal{E}}(\mathcal{F})$ is pure of the same dimension.*

Proof. We can use Theorem 1.2 4 to produce the usual local picture of a tame stack with a presentation:

$$
\begin{array}{ccccc}
\coprod_i \operatorname{Spec} B_i & \xrightarrow{\ \ \chi\ \ } & \mathcal{X} & \xrightarrow{\ \ \pi\ \ } & X \\
& & \phi \uparrow & & \psi \uparrow \\
& & \coprod_i [\operatorname{Spec} B_i/G_i] & \xrightarrow{\ \rho\ } & \coprod_i \operatorname{Spec} A_i
\end{array}
$$

where vertical arrows are étale, the obvious map $\coprod_i \operatorname{Spec} B_i \to [\operatorname{Spec} B_i/G_i]$ composed with ρ gives a finite morphism $h\colon \coprod \operatorname{Spec} B_i \to \coprod_i \operatorname{Spec} A_i$ and the square in the picture is cartesian. The sheaf $\chi^*(\mathcal{F} \otimes \mathcal{E}^\vee)$ is given by finitely generated B_i-modules M_i. It is clear that:

$$\operatorname{Ass}\left(_{A_i} M_i\right)^{G_i} \subseteq \operatorname{Ass}\left(_{A_i} M_i\right) = h \operatorname{Ass} M_i$$

We can rewrite this as:

$$\operatorname{Ass} \rho_* \phi^*(\mathcal{F} \otimes \mathcal{E}^\vee) \subseteq \operatorname{Ass} h_* \chi^*(\mathcal{F} \otimes \mathcal{E}^\vee) = h \operatorname{Ass} \chi^*(\mathcal{F} \otimes \mathcal{E}^\vee) = \rho \operatorname{Ass} \phi^*(\mathcal{F} \otimes \mathcal{E}^\vee) \quad (1.3)$$

where the second equality follows from 5.3 2. For the same reason if $\mathcal{F}$ is pure of dimension d the module M_i is pure of the same dimension, moreover h is finite and preserves the dimension of points so that $(_{A_i} M_i)^G$ is pure of the same dimension. Since there are no embedded primes in M_i and using Lemma 5.5 we deduce that the inclusion in (1.3) is actually an equality. Using 1.5 we have $\rho_* \phi^*(\mathcal{F} \otimes \mathcal{E}^\vee) = \psi^* \pi_*(\mathcal{F} \otimes \mathcal{E}^\vee)$ that implies:

$$\operatorname{Ass} \psi^* \pi_*(\mathcal{F} \otimes \mathcal{E}^\vee) = \rho \operatorname{Ass} \phi^*(\mathcal{F} \otimes \mathcal{E}^\vee)$$

Using Lemma 5.6 we obtain:

$$\operatorname{Ass} \pi_*(\mathcal{F} \otimes \mathcal{E}^\vee) = \psi \operatorname{Ass} \psi^* \pi_*(\mathcal{F} \otimes \mathcal{E}^\vee) = \psi \circ \rho \operatorname{Ass} \phi^*(\mathcal{F} \otimes \mathcal{E}^\vee) = \pi \operatorname{Ass} \mathcal{F} \otimes \mathcal{E}^\vee$$

Since ϕ and ψ are étale and preserve the dimension of points [Mil80, I Prop 3.14], h is finite and preserves the dimension of points, we obtain that $F_\mathcal{E}(\mathcal{F})$ is pure of dimension d. $\qquad\square$

With this result the following corollary is immediate:

Corollary 5.8. *Let $\mathcal{X} \to \operatorname{Spec} k$ be a projective DM stack and $\mathcal{F}$ a coherent sheaf on $\mathcal{X}$ of dimension d. Consider the torsion filtration $0 \subset T_0(\mathcal{F}) \subset \ldots \subset T_d(\mathcal{F}) = \mathcal{F}$. The functor $F_\mathcal{E}$ sends the torsion filtration to the torsion filtration of $F_\mathcal{E}(\mathcal{F})$ that is $F_\mathcal{E}(T_i(\mathcal{F})) = T_i(F_\mathcal{E}(\mathcal{F}))$.*

Remark 5.9. 1. The statement of Lemma 5.4 can be generalized in the evident way to non integral stacks using the same techniques used to prove the analogous statement for pure sheaves on Deligne-Mumford stacks.

2. In order to classify coherent sheaves on a scheme or a stack we consider three filtrations which let us split the problem in simpler pieces. The first one is the torsion filtration that reduces the problem to the study of pure sheaves, the second is the Harder-Narasimhan filtration that reduces the problem to the study of pure dimensional semistable sheaves, and the last one is the Jordan-Hölder filtration that reduces the problem to the study of stable sheaves. We have these three filtrations both on a projective stack $\mathcal{X}$ and on its projective moduli scheme X; while the torsion filtration on $\mathcal{X}$ is sent to the torsion filtration on X, the functor $F_\mathcal{E}$ doesn't respect the other two filtrations as it will be clear in the following of this work.

2 Stability condition

Assumption. From now on when we write *pure sheaf* on a projective stack $\mathcal{X}$ (or family of) we will mean a pure sheaf of arbitrary dimension if the stack is Deligne-Mumford, a torsion free sheaf if it is not.

Definition 5.10. Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$, we define the following *modified Hilbert polynomial*:

$$P_{\mathcal{E}}(\mathcal{F}, m) = \chi(\mathcal{X}, \mathcal{F} \otimes \mathcal{E}^{\vee} \otimes \pi^* \mathcal{O}_X(m)) = P(F_{\mathcal{E}}(\mathcal{F})(m)) = \chi(X, F_{\mathcal{E}}(\mathcal{F})(m))$$

Remark 5.11.   1. If the sheaf $\mathcal{F}$ is pure of dimension d, the function $m \mapsto P_{\mathcal{E}}(\mathcal{F}, m)$ is a polynomial and we will denote it with:

$$P_{\mathcal{E}}(\mathcal{F}, m) = \sum_{i=0}^{d} \alpha_{\mathcal{E},i}(\mathcal{F}) \frac{m^i}{i!} \tag{2.1}$$

This is true since the functor $F_{\mathcal{E}}$ preserves the pureness and dimension of sheaves 5.7 and 5.4, so that we can conclude as in the case of schemes using Grothendieck-Riemann-Roch.

2. The modified Hilbert polynomial is additive on short exact sequences since the functor $F_{\mathcal{E}}$ is exact 2.5 and the Euler characteristic is additive on short exact sequences.

3. The modified Hilbert polynomial is not a generalized Hilbert polynomial in the sense of Olsson and Starr [OS03, Def 4.1].

Definition 5.12. As usual we define the *reduced Hilbert polynomial* for pure sheaves, and we will denote it with $p_{\mathcal{E}}(\mathcal{F})$, the monic polynomial with rational coefficients $\frac{P_{\mathcal{E}}(\mathcal{F})}{\alpha_{\mathcal{E},d}(\mathcal{F})}$.

Definition 5.13. We define also the *slope* of a sheaf of dimension d (not necessarily pure):

$$\hat{\mu}_{\mathcal{E}}(\mathcal{F}) = \frac{\alpha_{\mathcal{E},d-1}(\mathcal{F})}{\alpha_{\mathcal{E},d}(\mathcal{F})}$$

We will also use the ordinary slope of a sheaf F on a scheme, and we will denote it with $\hat{\mu}(F)$ as usual (see [HL97, Def 1.6.8]).

And here the definition of stability:

Definition 5.14. Let $\mathcal{F}$ be a pure coherent sheaf, it is *semistable* if for every proper subsheaf $\mathcal{F}' \subset \mathcal{F}$ it is verified $p_{\mathcal{E}}(\mathcal{F}') \leq p_{\mathcal{E}}(\mathcal{F})$ and it is stable if the same is true with a strict inequality.

The notion of μ-stability and semistability for torsion free sheaves are defined in the evident way and they are related to Gieseker stability as they are in the case of schemes.

Remark 5.15. The functor $F_{\mathcal{E}}$ doesn't map semistable sheaves on $\mathcal{X}$ to semistable sheaves on X. Indeed it induces a closed immersion of the Quot-scheme of $\mathcal{X}$ in the Quot-scheme of X; this means that in general we have "more quotients" on X then on $\mathcal{X}$.

The *multiplicity* or rank of the sheaf $F_{\mathcal{E}}(\mathcal{F})$ is the usual thing: if the Hilbert polynomial of $\mathcal{O}_X$ has coefficients $a_d(\mathcal{O}_X), \ldots, a_0(\mathcal{O}_X)$ it is given by:

$$\operatorname{rk} F_{\mathcal{E}}(\mathcal{F}) = \frac{\alpha_{\mathcal{E},d}(\mathcal{F})}{a_d(\mathcal{O}_X)}$$

We can also try to relate the rank of the sheaf $\mathcal{F}$ to the Hilbert polynomial. Let $P(\mathcal{F}, m) = \chi(\mathcal{F} \otimes \pi^* \mathcal{O}_X(m)) = \sum_{i=0}^d \alpha_i(\mathcal{F}) \frac{m^i}{i!}$ be the Hilbert polynomial of $\mathcal{F}$ with respect to the polarization $\pi^* \mathcal{O}_X(1)$ alone. We could be tempted to define the rank of $\mathcal{F}$ using this polynomial. Assume that $\mathcal{X}$ is an orbifold, we can put $\operatorname{rk} \mathcal{F} := \frac{\alpha_d(\mathcal{F})}{\alpha_d(\mathcal{O}_X)}$. This is a reasonable definition. Indeed if $\mathcal{F}$ is locally free this is the rank of the free module. This is because the only contribution to the rank from the Toën-Riemann-Roch formula is from the piece $\int_{\mathcal{X}} \operatorname{ch}(\mathcal{F} \otimes \pi^* \mathcal{O}_X(m)) \operatorname{Td}(T_{\mathcal{X}})$ (see the next subsection for some recall about the Toën-Riemann-Roch formula). Assume that $\mathcal{X}$ is a smooth Deligne-Mumford stack with non generically trivial stabilizer and $\mathcal{F}$ is locally free. In this case this is not the rank of the locally free sheaf but the rank of a direct summand[1] of $\mathcal{F}$.

To conclude the section we write a technical lemma. It states that given a flat family of sheaves the modified Hilbert polynomial is locally constant on the fibers. It replaces the analogous one for generalized Hilbert polynomials [OS03, Lem 4.3].

Lemma 5.16. *Let $\mathcal{X} \to S$ be a family of projective stacks with chosen $\mathcal{E}, \mathcal{O}_X(1)$ and $\mathcal{F}$ an $\mathcal{O}_S$-flat sheaf on $\mathcal{X}$. Assume S is connected. There is an integral polynomial P such that for every point $\operatorname{Spec} K \xrightarrow{s} S$ the modified Hilbert polynomial of the fiber $\chi(\mathcal{X}_s, \mathcal{F} \otimes \mathcal{E}^{\vee} \otimes \pi^* \mathcal{O}_X(m)|_{\mathcal{X}_s}) = P(m)$.*

Proof. Since π preserves flatness and using 1.5 together with 2.16 2 we reduce the problem to the moduli scheme X. We have to prove that the integral polynomial $\chi(X_s, F_{\mathcal{E}}(\mathcal{F})(m)|_{X_s})$ doesn't depend from s, but this is the statement of [EGAIII.2, Thm 7.9.4]. $\qquad\square$

Remark 5.17. Using this lemma and generic-flatness (Prop 1.13) we can produce a stratification $\coprod S_i \to S$ such that on each S_i the sheaf $\mathcal{F}$ is flat and its modified Hilbert polynomial is constant. Again this is not the same as a flattening stratification since the universal property of a flattening stratification described by Mumford in [Mum66] is missing. Let $p \colon X \to S$ be a projective morphism of schemes, Mumford constructed the flattening stratification for such a morphism relying on the couple of functors $\Gamma_*, \widetilde{}$ where $\Gamma_*(F) = \bigoplus_{m \geq 0} p_* F(m)$ and $\widetilde{}$ is its inverse. In particular he was able to prove that F is S-flat if and only if for all sufficiently large m the sheaves $p_* F(m)$ are locally free. In the case of projective stacks $\mathcal{X} \xrightarrow{\pi} X \xrightarrow{p} S$ we would need an analogous couple of functors. We could think of using $\Gamma_* \circ F_{\mathcal{E}}$ and its left inverse $\eta \circ \widetilde{}$; however it's evident that the statement $\mathcal{F}$ is S-flat if and only if for all sufficiently large m the sheaves $p_* F_{\mathcal{E}}(\mathcal{F})(m)$ are locally free is quite false.

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[1] A quasicoherent sheaf on a G-gerbe, where G is a diagonalizable group scheme decomposes according to the irreducible representations of the group (This is written in many papers). The direct summand is the one corresponding to the trivial representation.

3 Toën-Riemann-Roch and geometric motivations

It is natural to ask if the degree of the sheaf $\mathcal{F}$, computed with respect to $\pi^*\mathcal{O}_X(1)$ is related to the slope $\widehat{\mu}_{\mathcal{E}}$. It is, in a wide class of examples, but in general it is not. To explain this kind of relation we need some machinery from the paper of Toën [Toe99]. We recall a couple of ideas from the Riemann-Roch theorem for smooth tame Deligne-Mumford stacks. Let $\mathcal{X}$ be a smooth tame Deligne-Mumford stack over an algebraically closed field k which is a global quotient; denote with $\sigma\colon I_{\mathcal{X}} \to \mathcal{X}$ the inertia stack. Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$ and consider the sheaf $\sigma^*\mathcal{F}$. The inertia stack can be written as a disjoint union of closed substacks $\mathcal{X}_g$ of $\mathcal{X}$ where (g) is a conjugacy class of the stabilizer of $\mathcal{X}$ at some geometric point. The coherent sheaf $\sigma^*\mathcal{F}$ is a disjoint union of sheaves on the $\mathcal{X}_g$'s. Each of these components carry an action of the cyclic group $\langle g \rangle$, whose order is prime to the characteristic of k by the tameness assumption. This implies that we can choose an isomorphism between $\langle g \rangle$ and $\mu_{a,k}$, a the order of $\langle g \rangle$, that sends g to ξ, a generator of $\mu_{a,k}$. In particular we can decompose the sheaf $\sigma^*\mathcal{F}$ according to the irreducible representations of $\langle g \rangle$ in a direct sum of eigensheaves $\mathcal{F}^{(z)}$ where $z \in \mu_{\infty}$. On each $\mathcal{F}^{(z)}$ the element g acts by multiplication by z. We will denote with $\rho_{\mathcal{X}}([\sigma^*\mathcal{F}])$ the element $\sum_{z \in \mu_{\infty}} z[\mathcal{F}^{(z)}]$ in $K_0(I_{\mathcal{X}}) \otimes_{\mathbb{Z}} \mathbb{Q}(\mu_{\infty})$. Denote also by $I_{\mathcal{X}}^1$ the substack of the inertia stack made of connected components of codimension 1 and be $\sigma_1\colon I_{\mathcal{X}}^1 \to \mathcal{X}$ the composition of the inclusion of $I_{\mathcal{X}}^1$ in $I_{\mathcal{X}}$ and the morphism σ.

Proposition 5.18. *Assume that $\mathcal{X}$ is a projective (connected) orbifold (indeed a global quotient $[Z/G]$). The generating sheaf $\mathcal{E}$ is chosen so that $\rho_{\mathcal{X}}([\sigma_1^*\mathcal{E}])$ is a sum of locally free sheaves of the same rank on each connected component of $I_{\mathcal{X}}^1$ (the rank can change from a component to the other); let $\mathcal{F}$ be a locally free sheaf of rank r and e be the rank of $\mathcal{E}$ then we have:*

$$\frac{\deg(\mathcal{F})}{r} = \frac{\alpha_{\mathcal{E},d-1}(\mathcal{F})}{re} - \frac{\alpha_{\mathcal{E},d-1}(\mathcal{O}_{\mathcal{X}})}{e} \tag{3.1}$$

Proof. This is a computation with the Toën-Riemann-Roch formula. The degree receives contributions only from pieces of codimension zero and codimension one in the inertia stack. Since we have assumed that $\mathcal{X}$ is an orbifold the only piece in codimension zero contributing to $P_{\mathcal{E}}(\mathcal{F}, m)$ is:

$$\int_{\mathcal{X}} \operatorname{ch}(\mathcal{F} \otimes \mathcal{E}^{\vee}) \operatorname{ch}(\pi^*\mathcal{O}_X(m)) \operatorname{Td}(T_{\mathcal{X}})$$

Let $\sigma_g\colon \mathcal{X}_g \to \mathcal{X}$ be the connected components of $I_{\mathcal{X}}^1$ and $\langle g \rangle \cong \mu_{n_g,k}$. Each component contributes with:

$$\sum_{i,j=1}^{n_g} \xi^{i-j} \int_{\mathcal{X}_g} \frac{\operatorname{ch}(\mathcal{F}_i \otimes \mathcal{E}_j^{\vee}) \operatorname{ch}(\sigma_g^*\pi^*\mathcal{O}_X(m)) \operatorname{Td}(T_{\mathcal{X}_g})}{Q(\xi, c(N_{\mathcal{X}_g|\mathcal{X}}))}$$

The coherent sheaves $\mathcal{F}_i$ and $\mathcal{E}_j$ are summands in the decomposition in eigensheaves of $\sigma_g^*\mathcal{F}$ and $\sigma_g^*\mathcal{E}$ with respect to the group $\mu_{n_g,k}$. The function $Q(\xi, c(N_{\mathcal{X}_g|\mathcal{X}}))$ is an opportune integral polynomial in ξ and the Chern classes of the normal bundle $N_{\mathcal{X}_g|\mathcal{X}}$; the complex number $\xi \in \mu_{n_g}$ is different from 1. The sum $\sum_{j=1}^{n_g} \xi^{-j}$ vanishes, if we can also assume that $\operatorname{rk} \mathcal{E}_j$ does not depend from j we retrieve the claimed identity with a little algebra. $\square$

Remark 5.19.   1. If the projective orbifold is a global quotient by a finite group $[Z/G]$ we can choose the generating sheaf as follow. Take the natural map $\iota\colon [Z/G] \to BG$, let E be the locally free sheaf on BG carrying the left regular representation of G; the locally free sheaf $\iota^* E$ is a generating sheaf according to [OS03, Prop 5.2]. It is an easy exercise in representation theory to check that it satisfies the condition in Proposition 5.18. It is not evident to us if it is possible to switch from this local situation to the general case of an orbifold and prove that it is always possible to find a generating sheaf satisfying the condition in Proposition 5.18. Passing from local to global is done in [OS03] through Corollary (5.5) and Lemma (5.6); both of them are quite non constructive.

2. If the stack has a non generically trivial stabilizer we have to take care of contributions to the Toën-Riemann-Roch formula coming from twisted sectors of codimension zero. In order to retrieve formula (3.1) we have to take care of the vanishing of expressions like:

$$\sum_{j=1}^{n_g} z^{-j}(\operatorname{rk}(F_i)c_1(\mathcal{E}_j^\vee) + c_1(F)\operatorname{rk}(\mathcal{E}_j^\vee))$$

where $z \neq 1$ and n_g is again the order of the cyclic group generated by g. We can achieve this requiring that $\operatorname{rk}\mathcal{E}_j^\vee$ does not depend from j and that the determinant of $\mathcal{E}^\vee$ is some fixed invertible sheaf. This second request doesn't sound very reasonable in general.

3. There are cases where $\mathcal{E}$ can be chosen so that $\operatorname{rk}\mathcal{E}_j^\vee$ in Proposition 5.18 and in the previous point are equal to one. The component $\mathcal{X}_g$ of the inertia stack can be considered as a $Z(g)$-gerbe over an orbifold (where $Z(g)$ is the centralizer of (g) inside G; a fortiori it is a $\langle g\rangle$-gerbe over its rigidification $\mathcal{X}_g /\!\!/ \langle g\rangle$ which is a Deligne-Mumford stack in general. The existence of an invertible sheaf over a $\langle g\rangle$-gerbe which is an eigensheaf with respect to the representation associated to a generator of the group (an invertible twisted sheaf) is not a trivial fact. If $\mathcal{X}_g$ is a $\langle g\rangle$-gerbe over a scheme, such an invertible sheaf exists if and only if the gerbe is essentially trivial [Lie07, Lem 2.3.4.2]. Despite this being a stringent assumption there are significant cases where this is satisfied. If $\mathcal{X}$ is a toric stack the inertia stack is again toric and every toric gerbe is abelian and essentially trivial. In the toric case it is always possible to find a generating sheaf $\mathcal{E}$ satisfying the condition in 5.18. A Deligne-Mumford gerbe over a point is always trivial and a Deligne-Mumford abelian gerbe over a curve is always essentially trivial so that the condition in 5.18 can be always satisfied for a Deligne-Mumford curve.

4   Harder-Narasimhan and Jordan-Hölder filtrations

The last part of this section is devoted to the definition of the Harder-Narasimhan filtration and the Jordan-Hölder filtrations. The construction of these two filtrations doesn't differ from the case of sheaves on schemes which can be found in great detail in [HL97, 1.3] and [HL97, 1.5]; their existence in the case of stacks is granted by the fact that the

functor $F_{\mathcal{E}}$ is exact (Remark 2.5) and that the modified Hilbert polynomial $P_{\mathcal{E}}$ is additive for short exact sequences (Remark 5.11).

Definition 5.20. Let $\mathcal{F}$ be a pure sheaf on $\mathcal{X}$. A strictly ascending filtration:

$$0 = HN_0(\mathcal{F}) \subset HN_1(\mathcal{F}) \ldots \subset HN_l(\mathcal{F}) = \mathcal{F}$$

is a Harder-Narasimhan filtration if it satisfies the following:

1. the i-th graded piece $gr_i^{HN}(\mathcal{F}) := \frac{HN_i(\mathcal{F})}{HN_{i-1}(\mathcal{F})}$ is a semistable sheaf for every $i = 1, \ldots, l$.

2. denoted with $p_i = p_{\mathcal{E}}(gr_i^{HN}(\mathcal{F}))$, the reduced Hilbert polynomial are ordered in a strictly decreasing way:

$$p_{\max}(\mathcal{F}) := p_1 > \ldots > p_l =: p_{\min}(\mathcal{F})$$

Definition 5.21. Let $\mathcal{F}$ be a semistable sheaf on $\mathcal{X}$ with reduced Hilbert polynomial $p_{\mathcal{E}}(\mathcal{F})$. A strictly ascending filtration:

$$0 = JH_0(\mathcal{F}) \subset JH_1(\mathcal{F}) \ldots \subset JH_l(\mathcal{F}) = \mathcal{F}$$

is a Jordan-Hölder filtration if $gr_i^{JH}(\mathcal{F}) = \frac{JH_i(\mathcal{F})}{JH_{i-1}(\mathcal{F})}$ is stable with reduced Hilbert polynomial $p_{\mathcal{E}}(\mathcal{F})$ for every $i = 1, \ldots, l$.

Theorem 5.22 ([HL97, Thm 1.3.4]). *For every pure sheaf $\mathcal{F}$ on $\mathcal{X}$ there is a unique Harder-Narasimhan filtration.*

Theorem 5.23 ([HL97, Prop 1.5.2]). *For every semistable sheaf $\mathcal{F}$ on $\mathcal{X}$ there is a Jordan-Hölder filtration and the sheaf $gr^{JH}(\mathcal{F}) := \bigoplus_i gr_i^{JH}(\mathcal{F})$ doesn't depend on the particular chosen filtration.*

Remark 5.24.   1. All the summands $gr_i^{JH}(\mathcal{F})$ of the Jordan-Hölder filtration are semistable with reduced Hilbert polynomial $p_{\mathcal{E}}(\mathcal{F})$; also the graded object $\bigoplus_i gr_i^{JH}(\mathcal{F})$ is semistable with polynomial $p_{\mathcal{E}}(\mathcal{F})$ [HL97, 1.5.1].

2. If $\mathcal{F}$ is pure with Harder-Narasimhan filtration $HN_i(\mathcal{F})$ the sheaf $F_{\mathcal{E}}(\mathcal{F})$ is again pure, $F_{\mathcal{E}}(HN_i(\mathcal{F}))$ is again a filtration but it is not the Harder-Narasimhan filtration in general. This is clear in the trivial case where the sheaf $\mathcal{F}$ is already semistable, and the sheaf $F_{\mathcal{E}}(\mathcal{F})$ is not semistable and has a non trivial filtration. To fix the ideas we can think of the structure sheaf on $\mathcal{X}$ which is semistable (stable) and $\pi_* \mathcal{E}^\vee$ which is not semistable in most situations.

3. If $\mathcal{F}$ is semistable the sheaf $F_{\mathcal{E}}(\mathcal{F})$ is not; in general there is no hope to send a Jordan-Hölder filtration to a Jordan-Hölder filtration using the functor $F_{\mathcal{E}}$. Again consider the simple case of an invertible sheaf L on X. The pullback $\pi^* L$ is always stable on $\mathcal{X}$ (we have a trivial filtration), however $F_{\mathcal{E}}(\pi^* L) = L \otimes \pi_* \mathcal{E}^\vee$ is not stable in general since $\pi_* \mathcal{E}^\vee$ is not, and usually it is not even semistable.

Definition 5.25. As usual [HL97, 1.5.3-1.5.4] two semistable sheaves $\mathcal{F}_1, \mathcal{F}_2$ with the same reduced modified Hilbert polynomial are called *S-equivalent* if they satisfy $gr^{JH}(\mathcal{F}_1) \cong gr^{JH}(\mathcal{F}_2)$. A semistable sheaf $\mathcal{F}$ is *polystable* if it is the direct sum of stable sheaves or equivalently $\mathcal{F} \cong gr^{JH}(\mathcal{F})$.

Chapter 6

Boundedness

In order to construct the stack of semistable sheaves as a finite type stack and a global quotient we first need to know if the family of semistable sheaves is bounded. In the previous section we have defined the Mumford regularity of a sheaf $\mathcal{F}$ on a projective stack to be the Mumford regularity of $F_{\mathcal{E}}(\mathcal{F})$, however it is not of great help to know that the family $F_{\mathcal{E}}(\mathcal{F})$ is bounded by a family of sheaves on the moduli scheme, since this family cannot be "lifted" to a bounding family on the stack. To obtain a boundedness result we need to study a *Kleiman criterion* on the stack; the fact that we are using Mumford regularity of $F_{\mathcal{E}}(\mathcal{F})$ means that we are just going to consider regular sequences of hyperplane sections of the polarization $\mathcal{O}_X(1)$ pulled back to the stack. A priori we could decide to study a more general class of sections, for instance the global sections of the generating sheaf $\mathcal{E}$, however the generating sheaf is not suitable to produce the standard induction arguments that are commonly used with Mumford regularity.

1 Kleiman criterion for stacks

Assumption. In this section the morphism $p \colon \mathcal{X} \to S$ will denote a family of projective stacks (Def 2.23) on S with a fixed polarization $\mathcal{O}_X(1), \mathcal{E}$.

We prove here that general enough sequences of global sections of $\mathcal{O}_X(1)$ are enough to establish a result of boundedness for semistable sheaves. We recall a couple of results from Kleiman's *exposé* ([MR071, XIII]) about Mumford regularity and the definition and properties of (b)-sheaves. Let k be an algebraically closed field and X a projective k-scheme with a very ample line bundle $\mathcal{O}_X(1)$.

Definition 6.1 ([HL97, 1.7.1]). Let F be a coherent sheaf on X. The sheaf F is m-regular (Mumford-Castelnuovo regular) if for every $i > 0$ we have $H^i(X, F(m - i)) = 0$. The regularity of F denoted with $\mathrm{reg}(F)$ is the least integer m such that F is m-regular.

Definition 6.2. We define the Mumford regularity of a coherent sheaf on $\mathcal{X}$ to be the Mumford regularity of $F_{\mathcal{E}}(\mathcal{F})$ on X and we will denote it by $\mathrm{reg}_{\mathcal{E}}(\mathcal{F})$.

Proposition 6.3. *[MR071, XIII-1.2] Let F be a coherent m-regular sheaf on X. For $n \geq m$ the following results hold:*

    1. F is n-regular

2. $H^0(F(n)) \otimes H^0(\mathcal{O}_X(1)) \to H^0(F(n+1))$ is surjective

3. $F(n)$ is generated by its global sections.

Definition 6.4. Let F be a coherent sheaf on X and r an integer $\geq \dim(\operatorname{supp} F)$. Let $(b) = (b_0, \ldots, b_r)$ a collection of $r+1$ non negative integers. The sheaf F is a (b)-sheaf if there is an F-regular sequence $\sigma_1, \ldots, \sigma_r$ of r global sections of $\mathcal{O}_X(1)$ such that, denoted with F_i the restriction of F to the intersection $\cap_{j \leq i} Z(\sigma_j)$ of the zero schemes of the sections $(i = 0, \ldots, r)$, the dimension of the global sections of F_i is estimated by $h^0(F_i) \leq b_i$.

Now let $\mathcal{X}$ be a family of projective stacks on S with moduli scheme X and generating sheaf $\mathcal{E}$.

Definition 6.5. Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$; it is defined to be a (b)-sheaf if $F_\mathcal{E}(\mathcal{F})$ is a (b)-sheaf on X.

Remark 6.6. Let $\sigma_1, \ldots, \sigma_r$ an $F_\mathcal{E}(\mathcal{F})$-regular sequence of sections of $\mathcal{O}_X(1)$ making $\mathcal{F}$ into a (b)-sheaf; let $Z(\sigma_i)$ the associated zero-scheme. The closed substack $\pi^{-1}Z(\sigma_i) = Z(\pi^*\sigma_i)$ is the zero-stack of $\pi^*\sigma_i \in H^0(\mathcal{X}, \pi^*\mathcal{O}_X(1))$. Denote by $\mathcal{F}_i$ the restriction to $\cap_{j \leq i} Z(\pi^*\sigma_j)$. An obvious application of 1.5 and exactness of π_* imply that the following holds: $h^0(\mathcal{F}_i \otimes \mathcal{E}^\vee) \leq b_i$ for $i = 0, \ldots, r$.

As in [MR071, XIII-1.9] we define inductively a class of polynomials with rational coefficients $P_i \in \mathbb{Q}[x_0, \ldots, x_i]$:

$$\begin{cases} P_{-1} = 0 \\ P_i(x_0, \ldots, x_i) = P_{i-1}(x_1, \ldots, x_i) + \sum_{j=0}^i x_j \binom{P_{i-1}(x_1, \ldots, x_i) - 1 + j}{j} \end{cases}$$

Proposition 6.7. *Let $\mathcal{F}$ be a coherent (b)-sheaf on $\mathcal{X}$. Every $\mathcal{F}'$ subsheaf of $\mathcal{F}$ is a (b)-sheaf.*

Proof. It follows from [MR071, XIII-1.6] that every subsheaf $\mathcal{F}''$ of $F_\mathcal{E}(\mathcal{F})$ is a (b)-sheaf, and in particular $F_\mathcal{E}(\mathcal{F}')$ is such a subsheaf. $\square$

Proposition 6.8. *Let $\mathcal{F}$ be a coherent (b)-sheaf on $\mathcal{X}$ with Hilbert polynomial $P_\mathcal{E}(\mathcal{F}, m) = \sum_{i=0}^r a_i \binom{m+i}{i}$ and $(b) = (b_0, \ldots, b_r)$; let $(c) = (c_0, \ldots, c_r)$ integer numbers such that $c_i \geq b_i - a_i$ for $i = 0, \ldots, r$ then $n := P_r(c_0, \ldots, c_r)$ is non negative and $\mathcal{F}$ is n-regular.*

Proof. This has nothing to do with the stack $\mathcal{X}$ so that the proof in [MR071, XIII-1.11] is enough. $\square$

Lemma 6.9. *Let $X \to S$ be a projective morphism of schemes and F a coherent sheaf on X which is flat on $\mathcal{O}_S$. Assume that for every point s of S the line bundle $\mathcal{O}_{X_s}(1) := \mathcal{O}_X(1) \otimes_k k(s)$ is generated by the global sections. Let r be the degree of the Hilbert polynomial $P(F)$. There is a finite stratification $\coprod_i S_i \to S$ such that for every i the module $F \otimes_{\mathcal{O}_S} \mathcal{O}_{S_i}$ has a regular sequence $\sigma_1^i, \ldots, \sigma_r^i$ with $\sigma_j^i \in H^0(X_i, \mathcal{O}_{X_i}(1))$ where $X_i := X \times_S S_i$.*

Proof. This is an application of cohomology and base change. For every geometric point s the fiber X_s is a projective scheme so that it follows from [HL97, Lem 1.1.12] that F_s-regular sections are dense in $H^0(X_s, \mathcal{O}_{X_s}(1))$. Applying 1.7 we know that there is an open neighborhood U of s such that $H^0(X, \mathcal{O}_X(1)) \otimes_k k(t) \cong H^0(X_t, \mathcal{O}_{X_t}(1))$ for every $t \in U$. In particular we can take σ_s an F_s-regular section and extend it to a section σ_U in $\mathcal{O}_X(1)(U)$. The zero locus $Z(\sigma_s)$ is closed in $Z(\sigma_U)$ and doesn't contain associated points of F; in particular $Z(\sigma_U)$ is not contained in $\text{Ass}(F)$. Since F is coherent $\text{Ass}(F)$ is finite and we can assume that U doesn't contain any associated point of F, so that σ_U is again regular for $F|_{X_U}$. By noetherian induction we obtain the stratification. $\square$

To state the Kleiman criterion we first need to recall the notion of family of sheaves and bounded family.

Definition 6.10. By *family of sheaves* on $\mathcal{X} \to S$ we will mean a flat family, that is a coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ flat on $\mathcal{O}_S$.

Given s a point of S with residue field $k(s)$ and given also K a field extension of $k(s)$ a sheaf on a fiber is a coherent sheaf $\mathcal{F}_K$ on $\mathcal{X} \times_S \text{Spec}\, K$. If we are given two field extensions K, K' and two sheaves, respectively $\mathcal{F}_K, \mathcal{F}_{K'}$, they are equivalent if there are $k(s)$-homomorphisms of K, K' to a third extension K'' of $k(s)$ such that $\mathcal{F}_K \otimes_{k(s)} K''$ and $\mathcal{F}_{K'} \otimes_{k(s)} K''$ are isomorphic.

Definition 6.11. A *set-theoretic family* of sheaves on $p\colon \mathcal{X} \to S$ is a set of sheaves defined on the fibers of p.

Definition 6.12. A set theoretic family $\mathfrak{F}$ of sheaves on $\mathcal{X}$ is *bounded* if there is an S-scheme T of finite type and a family $\mathcal{G}$ on $\mathcal{X}_T := \mathcal{X} \times_S T$ such that every sheaf in $\mathfrak{F}$ is contained in the fibers of $\mathcal{G}$.

The following theorem is the stacky version of [MR071, XIII-1.13].

Theorem 6.13 (Kleiman criterion for stacks)**.** *Let $p\colon \mathcal{X} \to S$ be a family of projective stacks with moduli scheme $\pi : \mathcal{X} \to X$. Assume $\mathcal{O}_X(1)$ is chosen so that for every point s of S the line bundle restricted to the fiber $\mathcal{O}_{X_s}(1)$ is generated by the global sections (for instance $\mathcal{O}_X(1)$ is very ample relatively to $X \to S$). Let $\mathfrak{F}$ be a set-theoretic family of coherent sheaves on the fibers of $\mathcal{X} \to S$. The following statements are equivalent:*

1. *The family $\mathfrak{F}$ is bounded by a coherent sheaf $\mathcal{G}$ on $\mathcal{X}_T := \mathcal{X} \times_S T$. Moreover if every $\mathcal{F}_K \in \mathfrak{F}$ is locally free of rank r a bounding family can be chosen locally free of rank r ($\mathcal{F}_K$ is a sheaf on a K-fiber of $\mathcal{X} \to S$).*

2. *The set of the Hilbert polynomials $P_{\mathcal{E}_K}(\mathcal{F}_K)$ for $\mathcal{F}_K \in \mathfrak{F}$ is finite and there is a sequence of integers (b) such that every $\mathcal{F}_K$ is a (b)-sheaf ($\mathcal{E}_K$ is the K-fiber of the generating sheaf $\mathcal{E}$).*

3. *The set of Hilbert polynomials $P_{\mathcal{E}_K}(\mathcal{F}_K)$ for $\mathcal{F}_K \in \mathfrak{F}$ is finite and there is a non negative integer m such that every $\mathcal{F}_K$ is m-regular.*

4. *The set of Hilbert polynomials $P_{\mathcal{E}_K}(\mathcal{F}_K)$, $\mathcal{F}_K \in \mathfrak{F}$ is finite and there is a coherent sheaf $\mathcal{H}$ on $\mathcal{X}_T$ such that every $\mathcal{F}_K$ is a quotient of $\mathcal{H}_K$ for some point K-point in T. We can assume that $T = S$ and $\mathcal{H} = \mathcal{E}^{\oplus N} \otimes \pi^* \mathcal{O}_X(-m)$ for some integers N, m.*

5. There are two coherent sheaves $\mathcal{H}, \mathcal{H}'$ on $\mathcal{X}_T$ such that every $\mathcal{F}_K$ is the cokernel of a morphism $\mathcal{H}'_K \to \mathcal{H}_K$ for some K-point of T. Moreover we can assume that $T = S$ and $\mathcal{H} = \mathcal{E}^{\oplus N} \otimes \pi^* \mathcal{O}_X(-m), \mathcal{H}' = \mathcal{E}^{\oplus N'} \otimes \pi^* \mathcal{O}_X(-m')$.

Proof. Part of the proof, that is $1 \Rightarrow 2$, $2 \Rightarrow 3$ is just the proof in the exposé of Kleiman. We recall it just for completeness.

$(1) \Rightarrow (2)$ If $\mathfrak{F}$ is bounded by $\mathcal{G}$ we can take $F_{\mathcal{E}_T}(\mathcal{G})$ on X_T and produce a finite flat stratification of T (using [EGAIV.3, 6.9.1]) such that the sheaf $F_{\mathcal{E}_T}(\mathcal{G})$ is flat on the stratification. We can assume that $F_{\mathcal{E}_T}(\mathcal{G})$ is flat on $\mathcal{O}_T$. Moreover the number of Hilbert polynomials $P_{\mathcal{E}_K}(\mathcal{F}_K)$ for $\mathcal{F}_K \in \mathfrak{F}$ is bounded by the number of connected components of T ([EGAIII.2, 7.9.5]). If the sheaf $\mathcal{F}_K$ is locally free, according to proposition 1.17 there is an open substack $\mathcal{X}_{T^K}$ of $\mathcal{X}_T$ such that for every point y in $\mathcal{X}_{T^K}$ the fiber $\mathcal{G}_y$ is a locally free sheaf of the same rank; moreover we can assume that $\mathcal{X}_{T^K}$ is flat on T^K and T^K is open. If every sheaf $\mathcal{F}_K$ is locally free of the same rank we can assume that $\mathcal{G}$ is locally free of the same rank. Taking a finer stratification and using 6.9 we can assume that there is a $F_{\mathcal{E}_T}(\mathcal{G})$-regular sequence $\sigma_1, \ldots, \sigma_r$ in $H^0(\mathcal{O}_X(1))$. Let Z_j be the intersection $\cap_{l \leq j} Z(\sigma_l)$. Using semicontinuity for schemes on the function $t \mapsto h^0(F_{\mathcal{E}_t}(\mathcal{G}_t)|_{Z_j})$ for t in T and $0 \leq j \leq r$ we obtain the sequence of integers (b) such that every $\mathcal{F}_K \in \mathfrak{F}$ is a (b)-sheaf.

$(2) \Rightarrow (3)$ It follows from 6.8.

$(3) \Rightarrow (4)$ Take m such that every $\mathcal{F}_K$ is m-regular. Let N be the maximal $P_{\mathcal{E}_K}(\mathcal{F}_K, m) = h^0(F_{\mathcal{E}_K}(\mathcal{F}_K)(m))$. According to Proposition 6.3 3 there is a surjective map $\mathcal{O}_{X_T}^{\oplus N} \otimes \mathcal{O}_{X_T}(-m) \to F_{\mathcal{E}_T}(\mathcal{G})$. Composing with $\theta_{\mathcal{E}_T}(\mathcal{G})$ we obtain the surjection we wanted.

$(4) \Rightarrow (5)$ Assume that there is a coherent sheaf $\mathcal{H}$ on $\mathcal{X}_T$ satisfying (4). For every $\mathcal{F}_K$ there is a point t such that:

$$0 \to F_{\mathcal{E}_T}(\mathcal{F}'_t) \to F_{\mathcal{E}_T}(\mathcal{H}_t) \to F_{\mathcal{E}_K}(\mathcal{F}_K) \to 0$$

Since $\mathcal{H}$ is bounded, the number of Hilbert polynomials $P_{\mathcal{E}_T}(\mathcal{H}_t)$ is finite and there is (b) such that every $\mathcal{H}_t$ is a (b)-sheaf. The number of Hilbert Polynomials of $\mathcal{F}_K$ is finite by hypothesis so that the number of Hilbert polynomials of $\mathcal{F}'_t$ is finite too. Moreover according to 6.7 $\mathcal{F}'_t$ are (b)-sheaves. We can apply $(2) \Rightarrow (4)$ to $\mathcal{F}'_t$ and deduce (5).

$(5) \Rightarrow (1)$ First we prove that $\mathcal{H}$ and $\mathcal{H}'$ can be chosen of the kind $\mathcal{E}_T^{\oplus N} \otimes \pi^* \mathcal{O}_{X_T}(-m)$. Given a point $t \in T$ consider a cokernel $\mathcal{H}'_t \xrightarrow{\beta} \mathcal{H}_t \to \operatorname{coker} \beta \to 0$. We apply the functor $F_{\mathcal{E}_T}$ and observe that it commutes with $\otimes k(t)$ so that the family $F_{\mathcal{E}_t}(\operatorname{coker} \beta)$ belongs to the cokernels of the fibers of the two sheaves $F_{\mathcal{E}_T}(\mathcal{H}')$ and $F_{\mathcal{E}_T}(\mathcal{H})$. According to the Kleiman criterion for coherent sheaves on a scheme the family $F_{\mathcal{E}_t}(\operatorname{coker} \beta)$ is bounded and in particular the number of Hilbert polynomials $P_{\mathcal{E}_t}(\operatorname{coker} \beta)$ is finite. Since $\mathcal{H}$ is bounded we can assume that there is a surjection $\mathcal{L} := \mathcal{E}_T^{\oplus N} \otimes \pi^* \mathcal{O}_{X_T}(-m) \to \mathcal{H}$ so that we have an exact sequence:

$$0 \to \mathcal{C}_\beta \to \mathcal{L}_t \to \operatorname{coker} \beta \to 0$$

The family $\mathcal{C}_\beta$ has just a finite number of different Hilbert polynomials since $\mathcal{L}$ is bounded and $\operatorname{coker} \beta$ has a finite number of polynomials. Moreover there is (b) such that every sheaf $\mathcal{L}_t$ is a (b)-sheaf and since $\mathcal{C}_\beta$ are all subsheaves of $\mathcal{L}_t$ they are (b)-sheaves according

to 6.7. Using (2) $\Rightarrow$ (3) $\Rightarrow$ (4) we deduce that there is a sheaf $\mathcal{E}_T^{\oplus N'} \otimes \pi^* \mathcal{O}_{X_T}(-m_1)$ such that the family C_β is contained in the quotients of its fibers. This completes the first part. To complete the proof we take a finite stratification of T so that the coherent sheaf $\mathcal{H}om_{\mathcal{O}_{\mathcal{X}_T}}(\mathcal{H}', \mathcal{H})$ is flat on T and $R^i p_{T*} \mathcal{H}om_{\mathcal{O}_{\mathcal{X}_T}}(\mathcal{H}', \mathcal{H})$ are locally free for every $i \geq 0$ (they are in finite number according to proof of 1.15). By Proposition 1.15 we obtain that p_{T*} commutes with an arbitrary base-change. This implies that the representable functor $\mathbb{V}(p_{T*}\mathcal{H}om_{\mathcal{O}_{\mathcal{X}_T}}(\mathcal{H}', \mathcal{H})) = \mathbb{V}$ is the same as a functor associating to a map $f\colon U \to T$ the group $\Gamma(U, p_{U*}\widetilde{f}^* \mathcal{H}om_{\mathcal{O}_{\mathcal{X}_T}}(\mathcal{H}', \mathcal{H}))$ where $\widetilde{f}\colon \mathcal{X}_U \to \mathcal{X}_T$. To conclude we observe that $\mathbb{V}$ is a vector bundle and the map $\mathbb{V} \to T$ is smooth so that the universal section is an object $\mathcal{U} \in \Gamma(\mathbb{V}, \pi_{\mathbb{V}*} \mathcal{H}om_{\mathcal{O}_{\mathcal{X}_{\mathbb{V}}}}(\mathcal{H}'|_{\mathbb{V}}, \mathcal{H}|_{\mathbb{V}}))$. Eventually we obtain a universal quotient:

$$\mathcal{H}'|_{\mathbb{V}} \xrightarrow{\mathcal{U}} \mathcal{H}|_{\mathbb{V}} \to \mathcal{G} \to 0$$

where $\mathcal{G}$ bounds the family $\mathfrak{F}$. $\qquad\square$

We state here a useful lemma of Grothendieck about the boundedness of family of sheaves. The version of this lemma for schemes [HL97, Lem 1.7.9] does not require the Kleiman criterion, however in the case of stacks there is an easy way to pull-back the result from the moduli scheme using the Kleiman criterion for stacks.

Lemma 6.14 (Grothendieck)**.** *Let $\mathcal{X}$ be a projective stack over a field k with moduli scheme $\pi\colon \mathcal{X} \to X$. Let P be an integral polynomial of degree $d = \dim(X)$ $(0 \leq d \leq \dim(X))$ and ρ an integer. There is a constant $C = C(P, \rho)$ such that if $\mathcal{F}$ is coherent sheaf of dimension d on $\mathcal{X}$ with $P_{\mathcal{E}}(\mathcal{F}) = P$ and $\mathrm{reg}_{\mathcal{E}}(\mathcal{F}) \leq \rho$, then for every $\mathcal{F}'$ purely d-dimensional quotient $\hat{\mu}_{\mathcal{E}}(\mathcal{F}') \geq C$. Moreover, the family of purely d-dimensional quotients $\mathcal{F}'_i$, $i \in I$ (for some set of indices I) with $\hat{\mu}_{\mathcal{E}}(\mathcal{F}'_i)$ bounded from above is bounded.*

Proof. The first part of the lemma is just the original Grothendieck lemma applied to the moduli scheme, we have just to observe that if $\mathcal{F}$ has dimension d the sheaf $F_{\mathcal{E}}(\mathcal{F})$ has the same dimension. To prove the second part we observe that the lemma in the case of schemes provides us a coherent sheaf $\mathcal{G}$ on $X \times R$ for some finite type scheme $R \to \mathrm{Spec}\, k$ bounding the family of quotients $\pi_*(\mathcal{F}'_i \otimes \mathcal{E}^\vee)$. A bounded family of sheaves on a projective scheme has a finite number of Hilbert polynomial, so in particular the number of polynomials $P_{\mathcal{E}}(\mathcal{F}'_i)$ is finite. We can pull back the problem to the stack using the functor $G_{\mathcal{E}}$, and obtain that the family $\mathcal{F}'_i$ is contained in the quotients of $\pi_R^* \mathcal{G} \otimes \mathcal{E}_R$ and we write it as:

$$\pi^* \mathcal{G}_i \otimes \mathcal{E}_i \longrightarrow\mathrel{\mkern-14mu}\rightarrow \mathcal{F}'_i$$

The family of sheaves $\pi_R^* \mathcal{G} \otimes_{\mathcal{O}_{\mathcal{X} \times R}} \mathcal{E}_R$ is bounded and in particular it is a quotient of a sheaf $\mathcal{E}_R^{\oplus N} \otimes \pi_R^* \mathcal{O}_{X \times R}(-m)$ for some m and N; applying 6.13 we deduce that $\mathcal{F}'_i$ is a bounded family. $\qquad\square$

Remark 6.15. The same statement is obviously true if $\mathcal{F}'$ is a subsheaf (a family of subsheaves) and the inequalities are all inverted.

With this machinery we can prove that semistability and stability are open conditions.

Proposition 6.16. *Let F be a flat family of d-dimensional coherent sheaves on $p\colon \mathcal{X} \to S$ (a family of projective stacks again) and fixed modified Hilbert Polynomial P of degree d. The set $\{\, s \in S \mid F_s \text{ is pure and semistable}\}$ is open in S. The same is true for stable sheaves and geometrically stable sheaves.*

Proof. The same proof as in [HL97, 2.3.1] but using the Grothendieck lemma for stacks and projectivity of the Quot-scheme for sheaves on stacks proved in [OS03]. □

Corollary 6.17. *The stack of semistable sheaves on $\mathcal{X}$ is an algebraic open substack of* $\mathfrak{Coh}_{\mathcal{X}/S}$.

Proof. It follows from the previous one and Corollary 2.27. □

Corollary 6.18. *Let $\mathcal{X} \to X \to S$ be a polarized stack satisfying hypothesis of Theorem 6.13, and $\mathfrak{F}$ is a set-theoretic family of coherent sheaves on its fibers. The family $\mathfrak{F}$ is bounded if and only if $F_{\mathcal{E}}(\mathfrak{F})$ is bounded.*

Proof. If $\mathfrak{F}$ is bounded then there is (b) such that $F_{\mathcal{E}}(\mathfrak{F})$ are (b)-sheaves or equivalently there is an integer m such that $F_{\mathcal{E}}(\mathfrak{F})$ are m-regular. From the Kleiman criterion for schemes it follows that $F_{\mathcal{E}}(\mathfrak{F})$ is a bounded family. If $F_{\mathcal{E}}(\mathfrak{F})$ is a bounded family from the Kleiman criterion for schemes it follows that there is (b) or equivalently there is m such that $F_{\mathcal{E}}(\mathfrak{F})$ are (b)-sheaves or m-regular; from the Kleiman criterion for stacks this implies that $\mathfrak{F}$ is bounded. □

2 A numerical criterion for boundedness

With the last corollary we have reduced the problem of boundedness to a study of boundedness on the moduli scheme X of the family of projective stacks. Working on the moduli scheme we have at disposal very fine results to establish whether a family of sheaves is bounded or not: in characteristic zero we can use the well known theorem of Le Potier and Simpson [HL97, 3.3.1] relying on the Grauert Mülich theorem, in positive and mixed characteristic we can use a finer result of Langer in [Lan04b, Thm 4.2].

Let $\mathfrak{F}$ be the family of pure semistable sheaves of dimension d on the fibers of $\mathcal{X} \to S$ with fixed modified Hilbert polynomial P; as we have noticed before it is not true that $F_{\mathcal{E}}(\mathfrak{F})$ are semistable, however we can study how much this family is destabilized and try to bound this loss of stability. Given $\mathcal{F}$ in the family we can consider the Harder-Narasimhan filtration $0 \subset F_n \subset \ldots \subset F_1 \subset F_{\mathcal{E}}(\mathcal{F})$ and look for some estimate of the maximal slope $(\hat{\mu}(F_n))$ depending only on the fixed Hilbert polynomial and possibly the sheaf $\mathcal{E}$ and the geometry of $\mathcal{X}$. The rest of this section is devoted to this problem. First of all we show a simple result of boundedness for smooth projective curves whose proof is analogous to the one for schemes.

Proposition 6.19. *Let $\mathcal{X} \to \operatorname{Spec} k$ be a smooth projective stack of dimension 1 and k is an algebraically closed field. The family of torsion-free semistable sheaves on $\mathcal{X}$ is bounded*

Proof. This is an application of Serre duality for stacky curves and Kleiman criterion for stacks as in [HL97, Cor 1.7.7]. □

There is also a very standard result of Maruyama and Yokogawa about the boundedness of parabolic bundles:

Proposition 6.20. *Let $X \to \operatorname{Spec} k$ be a smooth projective scheme, and consider a root construction on it $\pi \colon \mathcal{X} \to X$. The family of semistable locally free sheaves on $\mathcal{X}$ is bounded.*

Proof. This is a direct computation that can be found in the original paper [MY92]. $\square$

What follows is devoted to the study of the problem in a greater generality. Let $\mathcal{F}$ a coherent sheaf on $\mathcal{X}$ and P a polynomial with integral coefficients. We will denote with $\mathrm{Quot}_{\mathcal{X}/S}(\mathcal{F}, P)$ the functor of quotients of $\mathcal{F}$ with modified Hilbert polynomial P. The natural transformation $F_{\mathcal{E}}$ maps $\mathrm{Quot}_{\mathcal{X}/S}(\mathcal{F}, P)$ to the ordinary Quot-scheme $\mathrm{Quot}_{X/S}(F_{\mathcal{E}}(\mathcal{F}), P)$ of quotient sheaves on X with ordinary Hilbert polynomial P.

Proposition 6.21. *The natural transformation $F_{\mathcal{E}}$ is relatively representable with schemes and actually a closed immersion. In particular $\mathrm{Quot}_{\mathcal{X}/S}(\mathcal{F}, P)$ is a projective scheme.*

Proof. It is the same proof as in [OS03, Lem 6.1] and [OS03, Prop 6.2] but using Lemma 5.16 instead of [OS03, Lem 4.3]. $\square$

Consider now T an S-scheme, $\mathcal{F}$ a coherent sheaf on $\mathcal{X}_T$ and P as before. We recall the definition of the natural transformation $\eta_T \colon \mathrm{Quot}_{X/S}(F_{\mathcal{E}}(\mathcal{F}), P)(T) \to \mathrm{Quot}_{\mathcal{X}/S}(\mathcal{F})(T)$ from [OS03]. Let $F_{\mathcal{E}}(\mathcal{F}) \xrightarrow{\rho} Q$ be a quotient sheaf in $\mathrm{Quot}_{X/S}(F_{\mathcal{E}}(\mathcal{F}), P)(T)$. First consider the kernel $K \xrightarrow{\sigma} F_{\mathcal{E}}(\mathcal{F})$, apply $G_{\mathcal{E}}$ and compose with the natural morphism $\theta_{\mathcal{E}}$:

$$G_{\mathcal{E}}(K) \xrightarrow{G_{\mathcal{E}}(\sigma)} G_{\mathcal{E}}F_{\mathcal{E}}(\mathcal{F}) \xrightarrow{\theta_{\mathcal{E}}(\mathcal{F})} \mathcal{F}$$

Let $\mathcal{Q}$ be the cokernel of this composition so that we have defined a quotient $\mathcal{F} \to \mathcal{Q}$ which is $\eta_T(\rho)$.

Lemma 6.22. *[OS03, Lem 6.1] Let $\mathcal{Q}$ be a coherent sheaf in $\mathrm{Quot}_{\mathcal{X}/S}(\mathcal{F})(T)$, the composition of natural transformations $\eta_T(F_{\mathcal{E}}(\mathcal{Q}))$ is the same sheaf $\mathcal{Q}$ moreover the association $T \mapsto \eta_T$ is functorial.*

Remark 6.23. In this context there is no reason why $\eta_T(Q)$ should have modified Hilbert polynomial P, unless it is in the image of $F_{\mathcal{E}}$.

Lemma 6.24. *Let $\mathcal{F}$ be a quasicoherent sheaf on $\mathcal{X}$. Let Q be the following quotient:*

$$0 \longrightarrow K \xrightarrow{\alpha} F_{\mathcal{E}}(\mathcal{F}) \xrightarrow{\beta} Q \longrightarrow 0$$

Let $\mathcal{F} \to \mathcal{Q}$ be the quotient associated to Q by the transformation η_S and let $\mathcal{K}$ be its kernel, then we have the following 9-roman:

Moreover the map γ factorizes in the following way:

$$K \rightarrowtail^{\varphi_{\mathcal{E}}(K)} F_{\mathcal{E}} \circ G_{\mathcal{E}}(K) \xrightarrow{\ \widetilde{\gamma}\ } \twoheadrightarrow F_{\mathcal{E}}(\mathcal{K})$$

Proof. First of all we produce the sheaf $\mathcal{Q}$ using the following diagram:

where $\mathcal{Q} = \operatorname{coker} \theta_{\mathcal{E}}(\mathcal{F}) \circ G_{\mathcal{E}}(\alpha)$ and $\mathcal{K} = \ker \beta$. Now we apply the exact functor $F_{\mathcal{E}}$ and use the transformation $\varphi_{\mathcal{E}}$ and formula (0.2) to obtain the following diagram:

The middle column is the identity according to lemma 2.9 so that the left column is injective and the right column is surjective. It is immediate to produce the 9-roman. $\square$

Proposition 6.25. *Let $\mathcal{X}$ be a projective polarized stack over an algebraically closed field k. Let $\mathcal{F}$ be a pure μ-semistable sheaf on $\mathcal{X}$. Let $\overline{F}$ be the maximal destabilizing sheaf of $F_{\mathcal{E}}(\mathcal{F})$. Take $\widetilde{m}$ an integer such that $\pi_* \mathcal{E}nd_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E})(\widetilde{m})$ is generated by the global sections and denote with $N = h^0(X, \pi_* \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})(\widetilde{m}))$. The following inequality holds:*

$$\hat{\mu}(\overline{F}) \leq \hat{\mu}_{\mathcal{E}}(\mathcal{F}) + \widetilde{m} \deg(\mathcal{O}_X(1)) \tag{2.1}$$

Proof. The coherent sheaf $\pi_* \mathcal{E}nd_{\mathcal{O}_{\mathcal{X}}} \mathcal{E}$ is unstable in almost every example, however we have a surjection

$$\mathcal{O}_X(-\widetilde{m})^{\oplus N} \longrightarrow \twoheadrightarrow \pi_* \mathcal{E}nd_{\mathcal{O}_{\mathcal{X}}} \mathcal{E}$$

with $\widetilde{m}, N$ as in the hypothesis, which is given by the evaluation map. Since the sheaf $\overline{F}$ is semistable the sheaf $\overline{F}(-\widetilde{m})$ is again semistable. This is an immediate consequence of Riemann-Roch for projective schemes. Moreover we observe that $\hat{\mu}(\overline{F}(-\widetilde{m})^{\oplus N}) = \hat{\mu}(\overline{F}(-\widetilde{m}))$ and in particular it doesn't depend on N. Therefor the sheaf $\overline{F}(-\widetilde{m})^{\oplus N}$ is semistable (for a proof see [OSS80, Lem 1.2.4.ii]). We have a surjection $\overline{F} \otimes_X \mathcal{O}_X(-\widetilde{m})^{\oplus N} \to F_{\mathcal{E}}(\overline{\mathcal{F}})$ where $\overline{\mathcal{F}}$ is the sheaf associated to $\overline{F}$ by the transformation η_k. Since it is a subsheaf of $\mathcal{F}$ it is pure. Using that $\overline{F}(-\widetilde{m})^{\oplus N}$ is μ-semistable we obtain:

$$\hat{\mu}(\overline{F} \otimes_X \mathcal{O}_X(-\widetilde{m})^{\oplus N}) \leq \hat{\mu}_{\mathcal{E}}(\overline{\mathcal{F}}) \leq \hat{\mu}_{\mathcal{E}}(\mathcal{F})$$

where the second inequality comes from the fact that $\mathcal{F}$ is μ-semistable and $\overline{\mathcal{F}}$ is a subsheaf. The desired inequality follows from this one with a simple computation. $\square$

3 A couple of results of Langer

To complete the proof of boundedness for semistable sheaves we have just to use the result of Langer about the boundedness of sheaves on projective schemes together with 6.25. We first recall the precise statement of [Lan04b, Th 4.2]

Theorem 6.26 (Langer). *Let $q: X \to S$ be a projective morphism of schemes of finite type over an algebraically closed field, let $\mathcal{O}_X(1)$ be a q-very ample locally free sheaf on X. Let P be an integral polynomial of degree d and μ_0 is a real number. The set-theoretic family of pure sheaves of dimension d on the geometric fibers of q with fixed Hilbert polynomial P and maximal slope bounded by μ_0 is bounded.*

In order to use 6.25 for a family of projective stacks $p: \mathcal{X} \to S$ we need a homogeneous bound for $\widetilde{m}$ in the theorem for every fiber of $\mathcal{E}$ and a bound for $\deg(\mathcal{O}_{X_s}(1))$ for every geometric point s of S.

Lemma 6.27. *Let $p: \mathcal{X} \to S$ be family of projective stacks polarized by $\mathcal{E}, \mathcal{O}_X(1)$. There is an integer $\widetilde{m}$ and a geometric point $\overline{s}$ of S such that for every sheaf $\mathcal{F}$ in the family of purely d-dimensional semistable sheaves on the fibers of p with fixed modified Hilbert polynomial P we have:*

$$\hat{\mu}_{\max}(F_{\mathcal{E}_s}(\mathcal{F})) \leq \hat{\mu}_{\mathcal{E}}(\mathcal{F}) + \widetilde{m} \deg(\mathcal{O}_{X_{\overline{s}}}(1)) \tag{3.1}$$

where s is the point of S on which $\mathcal{F}$ is defined.

Proof. Let $\widetilde{m}$ be the integer such that $\pi_* \mathcal{E}nd_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E})(\widetilde{m})$ is generated by the global sections. Since $k(s)$ is right exact for every point s and it commutes with $F_{\mathcal{E}}$ this $\widetilde{m}$ has the desired property on each fiber. Choose a finite flat stratification of S for $\mathcal{O}_X(1)$. Since the Euler characteristic $\chi(X_s, \mathcal{O}_{X_s}(1))$ is locally constant the function $s \mapsto \deg(\mathcal{O}_{X_s}(1))$ assume only a finite number of values, in particular we can choose $\overline{s}$ such that $\deg(\mathcal{O}_{X_{\overline{s}}}(1))$ is maximal. $\square$

Theorem 6.28. *Let $p: \mathcal{X} \to S$ be a family of projective stacks over an algebraically closed field, polarized by $\mathcal{E}, \mathcal{O}_X(1)$. Let P be an integral polynomial of degree d and μ_0 a real number.*

1. *Every set-theoretic family $\mathcal{F}_i$, $i \in I$ (I a set) of purely d-dimensional sheaves on the fibers of p with fixed modified Hilbert polynomial and such that $\hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F}_i)) \leq \mu_0$ is bounded.*

2. *The family of semistable purely d-dimensional sheaves on the fibers of q with fixed modified Hilbert polynomial P is bounded.*

Proof. (1) It is an immediate consequence of [Lan04b, Thm 4.2] that $F_{\mathcal{E}}(\mathcal{F}_i)$ form a bounded family, so according to Corollary 6.18 the family $\mathcal{F}_i$ is bounded too.

(2) We choose $\widetilde{m}$ and $\overline{s}$ as in Lemma 6.27; it follows from Corollary 6.25 that we have:

$$\hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F}_i) \leq \hat{\mu}_{\mathcal{E}}(\mathcal{F}_i) + \widetilde{m} \deg(\mathcal{O}_{X_{\overline{s}}}(1)) =: \mu_0$$

From the previous point we have that $\mathcal{F}_i$ is a bounded family. $\square$

Remark 6.29. This result improves boundedness for sheaves on curves in 6.19 since we have no normality assumption on $\mathcal{X}$. In particular we can study sheaves on nodal curves as in [AGV06]. This result improves the boundedness result for parabolic bundles. Indeed the equivalence between parabolic sheaves and sheaves on stacks of roots is proven only for locally free-sheaves [Bor06] and [Bor07], and we cannot use the result in [MY92] to prove boundedness of semistable sheaves. This generalizes also the result in [Lie07] about gerbes since we have no assumptions on the banding of the gerbe, we just need the gerbe to be a projective stack.

The second result of Langer that we need is estimate in [Lan04a, Cor 3.4]. In characteristic zero it is possible to bound the number of global sections of a family of semistable sheaves with fixed Hilbert polynomial restricted to a general enough hyperplane or intersection of hyperplanes. This is known as Le Potier Simpson theorem [HL97, 3.3.1]. In positive characteristic it is known that it is not possible to reproduce such a result (for a counterexample [Lan04a, Ex. 3.1]). However Langer was able to prove that it is possible to produce a bound for the number of global sections.

Theorem 6.30 ([Lan04a, Cor 3.4]). *Let X be a projective scheme with a very ample invertible sheaf $\mathcal{O}_X(1)$. For any pure sheaf F of dimension d we have:*

$$h^0(X,F) \leq \begin{cases} r\binom{\hat{\mu}_{\max}(F)+r^2+f(r)+\frac{d-1}{2}}{d} & \text{if } \hat{\mu}_{\max}(F) \geq \frac{d+1}{2} - r^2 \\ 0 & \text{if } \hat{\mu}_{\max}(F) < \frac{d+1}{2} - r^2 \end{cases}$$

where r is the multiplicity of F and $f(r) = -1 + \sum_{i=1}^{r} \frac{1}{i}$ is an approximation of $\ln r$.

From this we deduce a stacky version for semistable sheaves:

Corollary 6.31. *Let $\mathcal{X}$ be a projective stack with polarization $\mathcal{E}, \mathcal{O}_X(1)$. For any pure semistable sheaf $\mathcal{F}$ on $\mathcal{X}$ of dimension d we have:*

$$h^0(\mathcal{X},\mathcal{F}\otimes\mathcal{E}^\vee) \leq \begin{cases} r\binom{\hat{\mu}_{\mathcal{E}}(\mathcal{F})+\widetilde{m}\deg(\mathcal{O}_X(1))+r^2+f(r)+\frac{d-1}{2}}{d} & \text{if } \hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F})) \geq \frac{d+1}{2} - r^2 \\ 0 & \text{if } \hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F})) < \frac{d+1}{2} - r^2 \end{cases} \quad (3.2)$$

where r is the multiplicity of $F_{\mathcal{E}}(\mathcal{F})$, the integer $\widetilde{m}$ is like in Proposition 6.25.

This corollary is a replacement for [HL97, Cor 3.3.8] and it will play a fundamental role in the study of the GIT quotient producing the moduli scheme of semistable sheaves.

Chapter 7

The stack of semistable sheaves

In the previous section we have collected enough machinery to write the algebraic stack of semistable-sheaves on a projective Deligne-Mumford stack $\mathcal{X}$ as a global quotient. Let $\pi\colon \mathcal{X} \to X$ be a projective stack with moduli scheme X over an algebraically closed field k. Fix a polarization $\mathcal{E}, \mathcal{O}_X(1)$ and a polynomial P with integer coefficients and degree $d \leq \dim X$. Fix an integer m such that every semistable sheaf on $\mathcal{X}$ is m-regular, which exists since semistable sheaves are bounded and according to Kleiman criterion there is m such that every sheaf in a bounded family is m-regular. Let N be the positive integer $h^0(X, F_\mathcal{E}(\mathcal{F})(m)) = P_\mathcal{E}(\mathcal{F}, m) = P(m)$ and denote with V the linear space $k^{\oplus N}$.

Proposition 7.1. *There is an open subscheme $\mathcal{Q}$ in $\mathrm{Quot}_{\mathcal{X}/k}(V \otimes \mathcal{E} \otimes \pi^*\mathcal{O}_X(-m), P)$ such that $\mathcal{S}(\mathcal{E}, \mathcal{O}_X(1), P)$ the algebraic stack of semistable sheaves on $\mathcal{X}$ with Hilbert polynomial P is the global quotient:*

$$\mathcal{S}(\mathcal{E}, \mathcal{O}_X(1), P) = [\mathcal{Q}/GL_{N,k}] \subseteq [\mathrm{Quot}_{\mathcal{X}/k}(V \otimes \mathcal{E} \otimes \pi^*\mathcal{O}_X(-m), P)/GL_{N,k}]$$

Proof. First of all we consider the set of pairs $\mathcal{F}, \rho$ where $\mathcal{F}$ is a semistable sheaf and ρ is an isomorphism $\rho\colon V \to H^0(X, F_\mathcal{E}(\mathcal{F})(m))$. Every sheaf $F_\mathcal{E}(\mathcal{F})$ with isomorphism ρ can be written in a unique way as a quotient $\mathcal{O}_X(-m)^{\oplus N} \xrightarrow{\widetilde{\rho}} F_\mathcal{E}(\mathcal{F})$ such that the map induced by $\widetilde{\rho}$ between V and $H^0(X, F_\mathcal{E}(\mathcal{F})(m))$ is ρ. In particular the quotient $\widetilde{\rho}$ is unique and it is the following composition:

$$V(-m) \xrightarrow{\rho \otimes \mathrm{id}} H^0(X, F_\mathcal{E}(\mathcal{F})(m)) \otimes \mathcal{O}_X(-m) \xrightarrow{\mathrm{ev}} F_\mathcal{E}(\mathcal{F})$$

where the second map is the evaluation and we have denoted with $V(-m)$ the tensor product $V \otimes \mathcal{O}_X(-m)$. Now consider a quotient $G_\mathcal{E}(V(-m)) \xrightarrow{\widetilde{\sigma}} \mathcal{F}$; apply the functor $F_\mathcal{E}$ and compose on the left with the transformation $\varphi_\mathcal{E}(V(-m))$:

$$V(-m) \xrightarrow{\varphi_\mathcal{E}(V(-m))} F_\mathcal{E} \circ G_\mathcal{E}(V(-m)) \xrightarrow{F_\mathcal{E}(\widetilde{\sigma})} F_\mathcal{E}(\mathcal{F})$$

This induces a morphism in cohomology $V \xrightarrow{\sigma} H^0(X, F_\mathcal{E}(\mathcal{F})(m))$. The subset of points of $\mathrm{Quot}_{\mathcal{X}/k}(V \otimes \mathcal{E} \otimes \pi^*\mathcal{O}_X(-m), P)$ such that this procedure induces an isomorphism is an open (see the proof of 2.25) and we denote it with $\mathcal{Q}$. We claim that there is a bijection between points of $\mathcal{Q}$ and couples $\mathcal{F}, \rho$. Given a couple $\mathcal{F}, \rho$ we first associate to it a quotient $V(-m) \xrightarrow{\widetilde{\rho}} F_\mathcal{E}(\mathcal{F})$ where $\widetilde{\rho} = \mathrm{ev} \circ (\rho \otimes \mathrm{id})$, then we produce the quotient

$\theta_{\mathcal{E}}(\mathcal{F}) \circ G_{\mathcal{E}}(\widetilde{\rho})\colon G_{\mathcal{E}}(V(-m)) \to \mathcal{F}$. This quotient is clearly the same as $\widetilde{\mathrm{ev}}(N, m) \circ (\rho \otimes \mathrm{id})$. Given a quotient $\widetilde{\sigma}$ we associate to it the isomorphism induced in cohomology by $F_{\mathcal{E}}(\widetilde{\sigma}) \circ \varphi_{\mathcal{E}}(V(-m))$. We show that these two maps of sets are inverse to each other. First we consider the following composition:

$$G_{\mathcal{E}}(V(-m)) \xrightarrow{G_{\mathcal{E}}(\varphi_{\mathcal{E}}(V(-m)))} G_{\mathcal{E}} \circ F_{\mathcal{E}} \circ G_{\mathcal{E}}(V(-m)) \xrightarrow{G_{\mathcal{E}} \circ F_{\mathcal{E}}(\widetilde{\sigma})} G_{\mathcal{E}} \circ F_{\mathcal{E}}(\mathcal{F}) \xrightarrow{\theta_{\mathcal{E}}(\mathcal{F})} \mathcal{F}$$

where the first triangle commutes because of lemma 2.9 and the second triangle commutes because $\theta_{\mathcal{E}}$ is a natural transformation. Now we consider the composition the other way round:

$$V(-m) \xrightarrow{\varphi_{\mathcal{E}}(V(-m))} F_{\mathcal{E}} \circ G_{\mathcal{E}}(V(-m)) \xrightarrow{F_{\mathcal{E}} \circ G_{\mathcal{E}}(\rho)} F_{\mathcal{E}} \circ G_{\mathcal{E}} \circ F_{\mathcal{E}}(\mathcal{F}) \xrightarrow{F_{\mathcal{E}}(\theta_{\mathcal{E}}(\mathcal{F}))} F_{\mathcal{E}}(\mathcal{F})$$

The first triangle commutes because $\phi_{\mathcal{E}}$ is a natural transformation and the second because of 2.9. $\qquad\square$

It is clear that given a quotient $G_{\mathcal{E}}(V(-m)) = V \otimes \mathcal{E} \otimes \pi^*\mathcal{O}_X(-m) \xrightarrow{\widetilde{\sigma}} \mathcal{F}$ there is an action of the group $\operatorname{Aut}\mathcal{E}$ that is composing on the left with an automorphism α of $\mathcal{E}$. A priori it is not clear if this is an action of $\operatorname{Aut}\mathcal{E}$ on $\mathcal{Q}$. Eventually it is.

Lemma 7.2. *The open subscheme $\mathcal{Q}$ is invariant by the action of $\operatorname{Aut}\mathcal{E}$ on $\operatorname{Quot}_{\mathcal{X}/k}(V \otimes \mathcal{E} \otimes \pi^*\mathcal{O}_X(-m), P)$.*

Proof. We have just to prove that given a quotient $\widetilde{\sigma}$ inducing an isomorphism in cohomology and given $\alpha \in \operatorname{Aut}\mathcal{E}$ the composition $\widetilde{\sigma} \circ \alpha$ induces an isomorphism in cohomology σ_α. The morphism σ_α is induced by the following composition:

$$V(-m) \xrightarrow{\varphi_{\mathcal{E}}(V(-m))} F_{\mathcal{E}} \circ G_{\mathcal{E}}(V(-m)) \xrightarrow{F_{\mathcal{E}}(\alpha \otimes \mathrm{id})} F_{\mathcal{E}} \circ G_{\mathcal{E}}(V(-m)) \xrightarrow{F_{\mathcal{E}}(\widetilde{\sigma})} F_{\mathcal{E}}(F)$$

The composition of the first two arrows on the left is the same as a map $V(-m) \to F_{\mathcal{E}} \circ G_{\mathcal{E}}(V(-m))$ selecting the automorphism $\pi_*\alpha$ in $\pi_*\mathcal{E}nd_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E})$. A simple computation in local coordinates shows that σ_α is the same as the composition of σ and the endomorphism of $H^0(X, F_{\mathcal{E}}(F)(m))$ given by the action of α on $\mathcal{E}^\vee$. Since this is actually an automorphism we obtain that σ_α is an isomorphism. $\qquad\square$

Remark 7.3 (Psychological remark)*.* Despite the action of $\operatorname{Aut}\mathcal{E}$ on $\mathcal{Q}$ being well defined, there is no reason to quotient it in order to obtain the moduli stack of semistable sheaves, even if it could seem natural at a first sight. Actually we don't know what kind of moduli problem could represent a quotient by this group action, for sure a much harder one from the GIT viewpoint.

Remark 7.4*.* As in the case of sheaves on schemes we can consider the subscheme $\mathcal{Q}^s \subset \mathcal{Q}$ of stable sheaves which is an open subscheme 6.16.

Remark 7.5. The multiplicative group $\mathbb{G}_{m,k}$ is contained in the stabilizer of every point of $[\mathcal{Q}/\operatorname{GL}_{N,k}]$ so that it is natural to consider the rigidification $[\mathcal{Q}/\operatorname{GL}_{N,k}]\!\!\fatslash\mathbb{G}_m$ which is the stack $[\mathcal{Q}/\operatorname{PGL}_{N,k}]$ where the action is induced by the exact sequence:

$$1 \to \mathbb{G}_{m,k} \to \operatorname{GL}_{N,k} \to \operatorname{PGL}_{N,k} \to 1$$

In particular $[\mathcal{Q}/\operatorname{GL}_{N,k}]$ is a $\mathbb{G}_{m,k}$-gerbe on $[\mathcal{Q}/\operatorname{PGL}_{N,k}]$. In the same way we could consider the global quotient $[\mathcal{Q}/\operatorname{SL}_{N,k}]$ where the action is induced by the inclusion $\operatorname{SL}_{N,k} \to \operatorname{GL}_{N,k}$; again we have that $[\mathcal{Q}/SL_{N,k}]\!\!\fatslash\mu_{N,k}$ is isomorphic to $[\mathcal{Q}/\operatorname{PGL}_{N,k}]$ and $[\mathcal{Q}/\operatorname{SL}_{N,k}]$ is a $\mu_{N,k}$-gerbe on it. For this reason in the GIT study of this global quotient it is equivalent to consider one these three quotients.

Proposition 7.6. *The algebraic stack $[\mathcal{Q}/GL_{N,k}]$ is an Artin stack of finite type on k.*

Proof. Since the group scheme $GL_{N,k}$ is smooth separated and of finite presentation on k, the global quotient is an Artin stack and $\mathcal{Q}$ is a smooth atlas [LMB00, 4.6.1]. Since $\mathcal{Q} \to \operatorname{Spec} k$ is a finite type morphism the stack itself is of finite type on k. $\qquad\square$

Chapter 8

The moduli scheme of semistable sheaves

The aim of this section is to prove that the global quotient $\mathcal{S}(\mathcal{E}, \mathcal{O}_X(1), P)$ of semistable sheaves exists as a GIT quotient and it is a projective scheme. As in the case of sheaves on schemes the GIT quotient is the moduli scheme for the stack $\mathcal{S}(\mathcal{E}, \mathcal{O}_X(1), P)$ only if there are no strictly semistable sheaves, otherwise it just parametrizes classes of S-equivalent sheaves. To prove this result we use the standard machinery of Simpson [Sim94] to compare semistability of sheaves to semistability for the GIT quotient $\mathcal{Q}/SL_{N,k}$. The first section is devoted to the definition of the GIT problem, while the second contains the results.

1 Closed embedding of $\mathrm{Quot}_{\mathcal{X}/S}$ in the projective space

Let $\rho \colon \mathcal{X} \to S$ a family of projective Deligne-Mumford stacks with moduli scheme $\mathcal{X} \xrightarrow{\pi} X \xrightarrow{p} S$ and $\mathcal{H}$ a locally free sheaf on $\mathcal{X}$ and P a polynomial with integral coefficients. In this subsection we write explicitly a class of very ample line bundles on $\mathrm{Quot}_{\mathcal{X}/S}(\mathcal{H}, P)$.

Lemma 8.1. *Let G and H be two representable functors, represented by $\overline{G}$ and $\overline{H}$ respectively. Let $\mathcal{U}_G$ and $\mathcal{U}_H$ the two universal objects. Given $\iota \colon G \to H$ a natural transformation relatively represented by $\overline{\iota}$ the following square is cartesian:*

$$
\begin{array}{ccc}
G & \xrightarrow{\iota} & H \\
\scriptstyle u_G \uparrow & & \uparrow \scriptstyle u_H \\
\overline{G} & \xrightarrow{\overline{\iota}} & \overline{H}
\end{array}
$$

and in particular $\iota(\mathcal{U}_G) = \overline{\iota}^* \mathcal{U}_H$.

Proof. Almost the definition of representable functor. $\qquad \square$

Let $\widetilde{\mathcal{U}}$ be the universal quotient sheaf of $\widetilde{\mathcal{Q}} := \mathrm{Quot}_{\mathcal{X}/S}(\mathcal{H}, P)$. Let $\mathcal{U}$ be the universal quotient sheaf of $\mathcal{Q} := \mathrm{Quot}_{X/S}(F_{\mathcal{E}}(\mathcal{H}), P)$. First we recall that there is a class of closed embeddings $j_l \colon \mathrm{Quot}_{X/S}(F_{\mathcal{E}}(\mathcal{H}), P) \to \mathrm{Grass}_S(p_* F_{\mathcal{E}}(\mathcal{H})(l), P(l))$ where l is a big enough integer, and it is given by the class of very ample line bundles $\det(p_{\mathcal{Q}*}\mathcal{U}(l))$ where

$p_\mathcal{Q} \colon X_\mathcal{Q} \to \mathcal{Q}$ is the pull back of p through the Plücker embedding k_l in $\mathbb{P} := \operatorname{Proj}(\Lambda^{P(l)}W)$ of the grassmanian and the closed embedding j_l. The locally free sheaf W is the universal quotient of $\operatorname{Grass}_S(p_* F_\mathcal{E}(\mathcal{H})(l), P(l))$. According to [OS03, Prop 6.2] in its modified version 6.21 there is a closed embedding $\iota \colon \widetilde{\mathcal{Q}} \to \mathcal{Q}$ representing the natural transformation $F_\mathcal{E}$. We will denote with $\widetilde{\iota}$ the pull-back morphism $X_{\widetilde{\mathcal{Q}}} \to X_\mathcal{Q}$.

Proposition 8.2. *Let $\pi_{\widetilde{\mathcal{Q}}}, p_{\widetilde{\mathcal{Q}}}$ be the pull-back of π, p through the Plücker embedding the closed embedding j_l and ι. The class of invertible sheaves $L_l := \det\left(p_{\widetilde{\mathcal{Q}}*}(F_\mathcal{E}(\widetilde{\mathcal{U}})(l))\right)$ is very ample for l big enough as before.*

Proof. According to the previous recall about the Quot scheme of sheaves on a projective scheme we have that:

$$j_l^* k_l^* \mathcal{O}(1) = \det\left(p_{\mathcal{Q}*}\mathcal{U}(l)\right)$$

Moreover lemma 8.1 implies that $F_\mathcal{E}(\widetilde{\mathcal{U}}) = \widetilde{\iota}^*\mathcal{U}$. We observe that $\iota^* p_{\mathcal{Q}*}\mathcal{U}(l) \cong p_{\widetilde{\mathcal{Q}}*}\widetilde{\iota}^*\mathcal{U}(l)$. This is an application of cohomology and base-change; in general we have this isomorphism for an arbitrary base change and a flat family $\mathcal{G}$ on $X_\mathcal{Q}$ whenever $H^1(\mathcal{G}_q) = 0$ for every closed point q in $\mathcal{Q}$ (this can be easily derived from cohomology and base change and for an explicit reference see [MFK94, 0.5]). In this particular case the fiber $\mathcal{U}_q(l)$ is just a sheaf of the Quot-functor and l in the hypothesis is big enough so that every quotient sheaf is l-regular. This implies $H^1(\mathcal{U}_q(\overline{l} - 1)) = 0$ for every q and $\overline{l} \geq l$ and eventually the desired relation:

$$\iota^* j_l^* k_l^* \mathcal{O}(1) = \det\left(p_{\widetilde{\mathcal{Q}}*}(F_\mathcal{E}(\widetilde{\mathcal{U}})(l))\right)$$

$\square$

Lemma 8.3. *The class of very ample invertible sheaves L_l of proposition 8.2 carries a natural $\operatorname{GL}_{N,k}$-linearization.*

Proof. The universal sheaf $\widetilde{\mathcal{U}}$ carries a natural $\operatorname{GL}_{N,k}$-linearization coming from the universal automorphism of $\operatorname{GL}_{N,k}$ (see [HL97, pg. 90] for the details). This linearization induces a linearization of L_l because the formation of L_l commutes with arbitrary base changes. This follows from 1.5, the compatibility of $\mathcal{E}$ with base change and the criterion in 8.2 . $\square$

Remark 8.4. The invertible sheaf L_l together with the $GL_{N,k}$-linearization of the previous lemma defines an invertible sheaf on the global quotient $[\mathcal{Q}/\operatorname{GL}_{N,k}]$. We will denote this sheaf with $\mathcal{L}_l$.

With lemma 8.3 we can define a notion of GIT semistable (stable) points in the projective scheme $\overline{\mathcal{Q}}$, the closure of $\mathcal{Q}$, with respect to the invertible sheaf L_l and the action of $\operatorname{GL}_{N,k}$. Since a $\operatorname{GL}_{N,k}$-linearization induces an $\operatorname{SL}_{N,k}$-linearization we can consider the GIT problem with respect to the action of $\operatorname{SL}_{N,k}$. We will denote the subscheme of GIT semistable points with $\overline{\mathcal{Q}}^{ss}(L_l)$ and the subscheme of GIT stable points with $\overline{\mathcal{Q}}^{s}(L_l)$.

2 A couple of technical lemmas of Le Potier

We collect here two technical results of Le Potier [Pot92] which are useful to compare the semistability of sheaves on $\mathcal{X}$ and the GIT stability that we will study in the next section. The first statement is a tool to relate the notion of semistability to the number of global sections of subsheaves or quotient sheaves.

Theorem 8.5. *Let $\mathcal{F}$ be a pure dimensional coherent sheaf on a projective stack $\mathcal{X}$ with modified Hilbert polynomial $P_\mathcal{E}(\mathcal{F}) = P$, multiplicity r and reduced Hilbert polynomial p and let m be a sufficiently large integer. The following conditions are equivalent:*

1. *$\mathcal{F}$ is semistable (stable)*

2. *$r \cdot p(m) \leq h^0(F_\mathcal{E}(\mathcal{F})(m))$ and for every subsheaf $\mathcal{F}' \subset \mathcal{F}$ with multiplicity r', $0 < r' < r$ we have $h^0(F_\mathcal{E}(\mathcal{F}')(m)) \leq r' \cdot p(m)$;* $\quad$ *($<$)*

3. *for every quotient sheaf $\mathcal{F} \to \mathcal{F}''$ of multiplicity r'', $0 < r'' < r$ we have $r'' \cdot p(m) \leq h^0(F_\mathcal{E}(\mathcal{F}'')(m))$;* $\quad$ *($<$)*

moreover equality holds in 2 and 3 if and only if $\mathcal{F}'$ and $\mathcal{F}''$ respectively are semistable and it holds for every m.

Proof. The proof of this is just the same as in [HL97, 4.4.1]. This is true since the proof relies only on the Grothendieck lemma 6.14, the Kleiman criterion 6.13 and above all Langer's inequality (3.2) (replacing [HL97, Cor 3.3.8]). $\qquad\square$

Remark 8.6. If m is chosen such that every semistable sheaf on $\mathcal{X}$ is m-regular then it is an m big enough in the sense of the previous theorem.

Before stating the second lemma we recall the definition of dual sheaf of a pure dimensional sheaf.

Definition 8.7. Let $\mathcal{X}$ be a smooth projective Deligne-Mumford stack and $\mathcal{F}$ a coherent sheaf of codimension c on $\mathcal{X}$. We define the dual of $\mathcal{F}$ to be the coherent sheaf $\mathcal{F}^D = \mathcal{E}xt^c_\mathcal{X}(\mathcal{F}, \omega_\mathcal{X})$.

If the stack $\mathcal{X}$ is non smooth we could think of studying the dual choosing an embedding $i \colon \mathcal{X} \to \mathcal{P}$ in a smooth projective ambient $\mathcal{P}$ and using $(i_*\mathcal{F})^D = \mathcal{E}xt^e_\mathcal{P}(i_*\mathcal{F}, \omega_\mathcal{P})$ where e is now the codimension of $\mathcal{F}$ in $\mathcal{P}$. This is reasonable because of the following remark in [HL97]:

Lemma 8.8. *Let $\mathcal{X}$ be a smooth projective stack and $i \colon \mathcal{X} \to \mathcal{P}$ a closed embedding in a smooth projective stack $\mathcal{P}$. Let c be the codimension of $\mathcal{F}$ in $\mathcal{X}$ and e the codimension of $\mathcal{F}$ in $\mathcal{P}$. We have the following isomorphism:*

$$i_* \mathcal{E}xt^c_\mathcal{X}(\mathcal{F}, \omega_\mathcal{X}) \cong \mathcal{E}xt^e_\mathcal{P}(i_*\mathcal{F}, \omega_\mathcal{P})$$

Proof. It follows from the fact that i_* is exact and an application of Grothendieck duality to the closed immersion i (Corollary 3.41). $\qquad\square$

This lemma implies that our new definition of duality doesn't depend on the smooth ambient space.

Lemma 8.9. *Let $\mathcal{F}$ be a coherent sheaf on a projective stack $\mathcal{X}$. There is a natural morphism:*

$$\rho_{\mathcal{F}} \colon \mathcal{F} \to \mathcal{F}^{DD}$$

and it is injective if and only if $\mathcal{F}$ is pure dimensional.

Proof. Using Serre duality 3.41 we can rewrite [HL97, Prop 1.1.6], [HL97, Lem 1.1.8] and [HL97, Prop 1.1.10] tensoring occasionally with a generating sheaf to achieve some vanishing. □

With this machinery we can write the stacky version of a lemma of Le Potier [HL97, 4.4.2]. Using this lemma we can deal with possibly non pure dimensional sheaves that can be found in the closure $\overline{\mathcal{Q}}$.

Lemma 8.10. *Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$ that can be deformed to a pure sheaf of the same dimension d. There is a pure d-dimensional sheaf $\mathcal{G}$ on $\mathcal{X}$ with a map $\mathcal{F} \to \mathcal{G}$ such that the kernel is $T_{d-1}(\mathcal{F})$ and $P_{\mathcal{E}}(\mathcal{F}) = P_{\mathcal{E}}(\mathcal{G})$.*

Proof. Since $\mathcal{F}$ can be deformed to a pure sheaf there is a smooth connected curve $C \to \operatorname{Spec} k$ and a family $\mathfrak{F}$ of sheaves on $\mathcal{X}_C$ flat on C such that there is a point $0 \in C$ satisfying $\mathfrak{F}_0 \cong \mathcal{F}$ and for every point $t \neq 0$ the fibers $\mathfrak{F}_t$ are pure dimensional. Using the technique in [HL97, 4.4.2] together with Lemma 8.9 we can find a flat family $\mathfrak{G}$ on $\mathcal{X}_C$ and a map $0 \to \mathfrak{F} \to \mathfrak{G}$ which induces isomorphisms between the fibers for every $t \neq 0$, and for the special fiber $t = 0$ the map $\mathfrak{F}_0 \to \mathfrak{G}_0$ has kernel the torsion of $T_{d-1}(\mathcal{F})$. The two sheaves have the same Hilbert polynomial $P = P(\mathcal{F}) = P(\mathfrak{G}_0)$ since the family $\mathcal{G}$ is flat on a connected scheme. We observe that the two families $F_{\mathcal{E}_C}(\mathfrak{F})$ and $F_{\mathcal{E}_C}(\mathfrak{G})$ are again flat over C since $\mathcal{E}$ is locally free and using corollary 1.3 3. Moreover picking a fiber of a family and the functor $F_{\mathcal{E}}$ commute because of proposition 1.5 and the compatibility of $\mathcal{E}$ with base change. We have a morphism $F_{\mathcal{E}}(\mathcal{F}) \to F_{\mathcal{E}}(\mathcal{G}_0)$ and the kernel is $F_{\mathcal{E}}(T_{d-1}(\mathcal{F}))$ which is the same as $T_{d-1}(F_{\mathcal{E}}(\mathcal{F}))$ because of Corollary 5.8. Eventually $P(F_{\mathcal{E}}(\mathcal{F})) = P(F_{\mathcal{E}}(\mathfrak{G}_0))$ since $F_{\mathcal{E}}$ preserves flatness. □

3   GIT computations

The GIT problem is well posed and we can compute the Hilbert Mumford criterion for points in $\overline{\mathcal{Q}}$ and establish a numerical criterion for stability, which is actually the standard condition we have for stability of points in a grassmanian.

Suppose we are given $\rho \colon V \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m) \to \mathcal{F}$ a closed point in $\overline{\mathcal{Q}}$. Let $\lambda \colon \mathbb{G}_{m,k} \to \mathrm{SL}_{N,k}$ be a group morphism. This representation splits V into weight subspaces V_n such that $\lambda(t) \cdot v = t^n \cdot v$ for every n, every $t \in \mathbb{G}_{m,k}$ and every $v \in V_n$. We construct an ascending filtration $V_{\leq n} = \oplus_{i \leq n} V_n$ of V. In general, given a subspace $W \subseteq V$, it induces a subsheaf of $\mathcal{F}$ which is the image sheaf $\rho(W \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m))$. In this case we can produce a filtration of $\mathcal{F}$ with the subsheaves $\mathcal{F}_{\leq n} = \rho(V_{\leq n} \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m))$. We have an induced surjection $\rho_n \colon V_n \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m) \to \mathcal{F}_{\leq n}/F_{\leq n-1} =: \mathcal{F}_n$. Taking the sum over all the weights we obtain a new quotient sheaf:

$$\overline{\rho} := V \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m) \to \bigoplus_n \mathcal{F}_n =: \overline{\mathcal{F}}$$

It is a very standard result that:

Lemma 8.11. *The quotient $[\overline{\rho}]$ is the limit $\lim_{t\to 0} \lambda(t) \cdot [\rho]$ in the sense of the Hilbert Mumford criterion.*

Proof. It is just the same proof as in the case of sheaves on a projective scheme. See for instance [HL97, 4.4.3]. □

Lemma 8.12. *The action of $\mathbb{G}_{m,k}$ via the representation λ on the fiber of the invertible sheaf L_l at the point $[\overline{\rho}]$ has weight*

$$\sum_n n \cdot P_{\mathcal{E}}(\mathcal{F}_n, l)$$

Proof. The action of $\mathbb{G}_{m,k}$ on V_n induces an action on $\mathcal{F}_n$ which is again multiplication by t^n on the sections. The k-linear space $H^0(\mathcal{F}_n \otimes \mathcal{E}^\vee \otimes \pi^*\mathcal{O}_X(-l))$ inherits the same action. We recall that l is chosen big enough so that the Quot-scheme is embedded in the grassmanian; in particular this means that every quotient of $V \otimes \mathcal{E} \otimes \pi^*\mathcal{O}_X(-m)$ with Hilbert polynomial $P = P_{\mathcal{E}}(\mathcal{F})$ is l-regular. This means that $\overline{\mathcal{F}}$ is l-regular and $P(l) = P_{\mathcal{E}}(\overline{\mathcal{F}}, l) = h^0(X, F_{\mathcal{E}}(\overline{\mathcal{F}})(l))$. Now we take the fiber $[\overline{\rho}]$ in L_l:

$$L_l|_{[\overline{\rho}]} = \det\left(H^0(X, F_{\mathcal{E}}(\overline{\mathcal{F}})(l))\right) = \bigotimes_n \det\left(H^0(X, F_{\mathcal{E}}(\mathcal{F}_n)(l))\right) = \bigotimes_n \det\left(H^\bullet(X, F_{\mathcal{E}}(\mathcal{F}_n)(l))\right)$$

The last equality follows from the fact that $H^i(X, F_{\mathcal{E}}(\overline{\mathcal{F}})(l)) = \bigoplus_n H^i(X, F_{\mathcal{E}}(\mathcal{F}_n)(l))$ for every $i \geq 0$ and it vanishes for $i > 0$. This proves also that $h^0(X, F_{\mathcal{E}}(\mathcal{F}_n)(l)) = P_{\mathcal{E}}(\mathcal{F}_n, l)$ so that the weight of the action of $\mathbb{G}_{m,k}$ on $L_l|_{[\overline{\rho}]}$ is the one we have stated. □

An application of the Hilbert Mumford criterion translates in the following very standard lemma:

Lemma 8.13. *A closed point $\rho\colon V \otimes \mathcal{E} \otimes \pi^*\mathcal{O}_X(-m) \to \mathcal{F}$ is semistable (stable) if and only if for every non trivial subspace $V' \subset V$ the induced subsheaf $\mathcal{F}' \subset \mathcal{F}$ satisfies:*

$$\dim(V) \cdot P_{\mathcal{E}}(\mathcal{F}', l) \geq \dim(V') \cdot P_{\mathcal{E}}(\mathcal{F}, l); \quad (>) \tag{3.1}$$

Remark 8.14. As we have already seen, given a quotient $\rho\colon V \otimes \mathcal{E} \otimes \pi^*\mathcal{O}_X(-m) \to \mathcal{F}$ and a linear subspace $V' \subset V$ we can associate to it a subsheaf $\mathcal{F}' \subset \mathcal{F}$ which is $\rho(V' \otimes \mathcal{E} \otimes \pi^*\mathcal{O}_X(-m))$. Given a subsheaf $\mathcal{F}' \subset \mathcal{F}$ we can associate to it a subspace V' of V. Consider the following cartesian square:

$$
\begin{array}{ccc}
V & \longrightarrow V \otimes H^0(X, F_{\mathcal{E}}(\mathcal{E})(m)) \longrightarrow & H^0(X, F_{\mathcal{E}}(\mathcal{F})(m)) \\
\uparrow & & \uparrow \\
V \cap H^0(X, F_{\mathcal{E}}(\mathcal{F}')(m)) & \longrightarrow & H^0(X, F_{\mathcal{E}}(\mathcal{F}')(m))
\end{array}
$$

where the first map on the top is induced by $\varphi_{\mathcal{E}}(V(-m))$ and the second by $F_{\mathcal{E}}(\rho)$. The linear space we associate to $\mathcal{F}'$ is $V \cap H^0(X, F_{\mathcal{E}}(\mathcal{F}')(m))$. If we take a linear subspace V' and associate to it a subsheaf $\mathcal{F}'$ and then we associate to $\mathcal{F}'$ a linear space V'' with this procedure we obtain an inclusion $V'' \rightarrowtail V'$. On the contrary if we start from a subsheaf $\mathcal{F}'$, associate to it a linear space $V' = V \cap H^0(X, F_{\mathcal{E}}(\mathcal{F}')(m))$ and we use V' to generate a subsheaf $\mathcal{F}''$ we obtain again a natural injection of sheaves $\mathcal{F}'' \rightarrowtail \mathcal{F}'$.

From this observation follows the lemma:

Lemma 8.15. *Let* $\rho\colon V \otimes \mathcal{E} \otimes \pi^*\mathcal{O}_X(-m) \to \mathcal{F}$ *be a closed point in* $\overline{\mathcal{Q}}$. *It is GIT semistable (stable) if and only if for any coherent subsheaf* $\mathcal{F}'$ *of* $\mathcal{F}$ *and denoted* $V' = V \cap H^0(X, F_{\mathcal{E}}(\mathcal{F}')(l))$ *we have the following inequality:*

$$\dim(V) \cdot P_{\mathcal{E}}(\mathcal{F}') \geq \dim(V') \cdot P_{\mathcal{E}}(\mathcal{F}) \qquad (3.2)$$

Proof. We first observe that, fixed the point $[\rho]$, the family of subsheaves generated by a linear subspace of V is bounded because exact sequences of linear spaces split so that every subsheaf generated by a subspace has the same regularity as $\mathcal{F}$. This implies also that this family has a finite number of Hilbert polynomials. If the number of polynomials is finite the inequality (3.2) is equivalent to (3.1). The rest follows from the previous remark. $\square$

Lemma 8.16. *Let* $\rho\colon V \otimes \mathcal{E} \otimes \pi^*\mathcal{O}_X(-m) \to \mathcal{F}$ *be a closed point in* $\overline{\mathcal{Q}}$ *which is GIT semistable then the induced morphism* $V \to H^0(F_{\mathcal{E}}(\mathcal{F})(m))$ *is injective.*

Proof. Take the kernel K of the morphism $V \to H^0(F_{\mathcal{E}}(\mathcal{F})(m))$. It generates the null subsheaf in $\mathcal{F}$. Even if the null sheaf has no global sections we have $0 \cap V = K$ which is not null. Inequality (3.2) reads $0 \geq \dim(K) \cdot P_{\mathcal{E}}(F)$ which is impossible unless $\dim(K) = 0$. $\square$

Proposition 8.17. *Let* m *be a large enough integer (possibly larger of the one we have used so far) and* l *large enough in the usual sense. The scheme* $\mathcal{Q}$ *of semistable sheaves is equal to* $\overline{\mathcal{Q}}^{ss}(L_l)$ *the scheme of GIT semistable points in* $\overline{\mathcal{Q}}$ *with respect to* L_l*; moreover the stable points coincides* $\mathcal{Q}^s = \overline{\mathcal{Q}}^s(L_l)$.

Like the one in [HL97, 4.3.3] with possibly a correction). Denote with r the multiplicity deduced from the chosen polynomial P. Choose m so that semistable sheaves with polynomial P are m-regular and also semistable sheaf with reduced Hilbert polynomial p and multiplicity $r' < r$ are m-regular. Let $\rho\colon V \otimes \mathcal{E} \otimes \pi^*\mathcal{O}_X(-m) \to \mathcal{F}$ be a closed point in $\mathcal{Q}$. Let $\mathcal{F}'$ be a subsheaf of $\mathcal{F}$ with multiplicity $0 < r' < r$. Denote with V' the linear space $V \cap H^0(F_{\mathcal{E}}(\mathcal{F}')(m))$. It has the same dimension as $H^0(F_{\mathcal{E}}(\mathcal{F}')(m))$ since the map $V \to H^0(F_{\mathcal{E}}(\mathcal{F})(m))$ is an isomorphism. According to theorem 8.5 2 we have $\frac{1}{r'}h^0(F_{\mathcal{E}}(\mathcal{F}')(m)) \leq p_{\mathcal{E}}(\mathcal{F})$. If equality holds $\mathcal{F}'$ is semistable and has the same reduced Hilbert polynomial as $\mathcal{F}$ and it is m-regular for our previous assumption on m. This implies that $\dim(V') = h^0(F_{\mathcal{E}}(\mathcal{F}')(m)) = r' \cdot p(m)$ and eventually $\dim(V') \cdot P_{\mathcal{E}}(\mathcal{F}) = (r' \cdot p(m)) \cdot (r \cdot p(m)) = \dim(V) \cdot P_{\mathcal{E}}(\mathcal{F}')$. If the strict inequality holds we have: $\dim(V) \cdot r' = r \cdot p(m) \cdot r' > h^0(F_{\mathcal{E}}(\mathcal{F}')(m)) \cdot r = \dim(V') \cdot$. This implies that inequality (3.2) is satisfied and more precisely that $[\rho] \in \mathcal{Q}^s$ implies $[\rho] \in \overline{\mathcal{Q}}^s(L_l)$ and if $[\rho]$ is strictly semistable in $\mathcal{Q}$ then it is strictly semistable in $\overline{\mathcal{Q}}^{ss}(L_l)$. Now we prove that a point $[\rho]$ in $\overline{\mathcal{Q}}^{ss}(L_l)$ belongs to $\mathcal{Q}$. Indeed this is enough to complete the proof since we have already proven that a stable points in $\mathcal{Q}$ is GIT stable and a strictly semistable in $\mathcal{Q}$ is GIT strictly semistable. Since $[\rho]$ is in the closure of $\mathcal{Q}$ we can deform the sheaf $\mathcal{F}$ to a semistable sheaf and we can use lemma 8.10 to obtain a sheaf $\mathcal{G}$ and a morphism of sheaves $\mathcal{F} \to \mathcal{G}$ such that $\mathcal{G}$ is pure dimensional, $P_{\mathcal{E}}(\mathcal{F}) = P_{\mathcal{E}}(\mathcal{G})$ and the kernel of the morphism is $T_{d-1}(\mathcal{F})$. Since $\mathcal{F}$ is GIT semistable the morphism $V \to H^0(F_{\mathcal{E}}(\mathcal{F})(m))$ is injective according to

lemma 8.16. Consider the composition $V \to H^0(F_{\mathcal{E}}(\mathcal{F})(m)) \to H^0(F_{\mathcal{E}}(\mathcal{G}))$ and assume its kernel is K. It has an injection in $H^0(F_{\mathcal{E}}(T_{d-1}(\mathcal{F}))(m))$ compatible with all the other maps, this implies that there is also an injection $K \to V \cap H^0(F_{\mathcal{E}}(T_{d-1}(\mathcal{F}))(m))$. If K is non zero then $V \cap H^0(F_{\mathcal{E}}(T_{d-1}(\mathcal{F}))(m))$ is non zero but the sheaf $T_{d-1}(\mathcal{F})$ is of dimension $d-1$ and this is not compatible with inequality (3.2). We have proven that the map $V \to H^0(F_{\mathcal{E}}(\mathcal{G})(m))$ is injective. Consider now a quotient $\mathcal{G}''$ of $\mathcal{G}$ with multiplicity $0 < r'' < r$ and let $\mathcal{F}'$ be the kernel of the composition $\mathcal{F} \to \mathcal{G} \to \mathcal{G}''$. Denote with $V' = V \cap H^0(F_{\mathcal{E}}(\mathcal{F}')(m))$. We have the following cartesian square:

$$
\begin{array}{ccc}
V & \rightarrowtail & H^0(F_{\mathcal{E}}(\mathcal{F})(m)) \\
\uparrow & & \uparrow \\
V' & \rightarrowtail & H^0(F_{\mathcal{E}}(\mathcal{F}')(m))
\end{array}
$$

This implies that $h^0(F_{\mathcal{E}}(\mathcal{F})(m)) - h^0(F_{\mathcal{E}}(\mathcal{F}')(m)) \geq N - dim(V')$; moreover inequality (3.2) applied to $\mathcal{F}$ and $\mathcal{F}'$ translates in $r' \cdot \dim(V) \geq r \cdot \dim(V')$ and recall also that $\dim(V) = r \cdot p(m)$. Putting it all together:

$$
\begin{aligned}
h^0(F_{\mathcal{E}}(\mathcal{G}'')(m)) &\geq h^0(F_{\mathcal{E}}(\mathcal{F})(m)) - h^0(F_{\mathcal{E}}(\mathcal{F}')(m)) \\
&\geq N - \dim(V') \geq r \cdot p(m) - r' \cdot p(m) = r'' \cdot p(m)
\end{aligned}
$$

By theorem 8.5 3 the sheaf $\mathcal{G}$ is semistable and since $V \to H^0(F_{\mathcal{E}}(\mathcal{G})(m))$ is injective it is actually an isomorphism. We can use this isomorphism to produce a quotient $V \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m) \to \mathcal{G}$ that factorizes through $\mathcal{F}$. This implies that the morphism $\mathcal{F} \to \mathcal{G}$ is surjective and since they have the same Hilbert polynomial it is an isomorphism. $\square$

To prove next theorem, which completes the GIT study, we need a result of semicontinuity for the Hom functor on a family of projective stacks. Since we were not able to retrieve this result from an analogous one on the moduli scheme of $\mathcal{X}$, we prove it here and not in the first chapter.

Lemma 8.18. *Let $p \colon \mathcal{X} \to S$ be a family of projective stacks, let $\mathcal{G}$ be a coherent sheaf on $\mathcal{X}$ flat on S and $\mathcal{F}$ a coherent sheaf on $\mathcal{X}$. Let s be a point of S, the function $s \mapsto \hom_{\mathcal{X}_s}(\mathcal{F}_s, \mathcal{G}_s)$ is upper semicontinuous.*

Proof. We prove this using Grothendieck original approach to the problem and we keep most of the notations in section $III.12$ of [Har77]. Since the problem is local in the target we can assume that $S = \operatorname{Spec} A$ is affine. Let T^0 be the functor mapping an A-module M to $\operatorname{Hom}_{\mathcal{X}}(\mathcal{F}, \mathcal{G} \otimes_A M)$. Since $\mathcal{X}$ is projective we can take a locally free resolution $\mathcal{E}^{\oplus N_1} \otimes \pi^* \mathcal{O}_X(-m_1) \to \mathcal{E}^{\oplus N_2} \otimes \pi^* \mathcal{O}_X(-m_2) \to \mathcal{F} \to 0$ where m_1, m_2 are positive and big enough integers. We produce the exact sequence:

$$
0 \longrightarrow \operatorname{Hom}_{\mathcal{X}}(\mathcal{F}, \mathcal{G} \otimes_A M) \longrightarrow H^0(X, F_{\mathcal{E}}(\mathcal{G})^{\oplus N_2}(m_2)) \otimes_A M \longrightarrow \cdots
$$

$$
\cdots \longrightarrow H^0(X, F_{\mathcal{E}}(\mathcal{G})^{\oplus N_1}(m_1)) \otimes_A M
$$

The coherent sheaf $F_{\mathcal{E}}(\mathcal{G})^{\oplus N_i}(m_i)$ is A-flat for $i = 1, 2$ and choosing m_1, m_2 even larger we can assume (Serre vanishing plus semicontinuity for cohomology) that $H^1(X_y, F_{\mathcal{E}}(\mathcal{G})^{\oplus N_i}(m_i) \otimes_A k(y)) = 0$ for every point y in S. Denote with $q \colon X \to S$ the morphism from the moduli scheme to S. Using [MFK94, pag 19] we can conclude that $p_* F_{\mathcal{E}}(\mathcal{G})^{\oplus N_i}(m_i)$ is locally free, so that the module $L_i := H^0(X, F_{\mathcal{E}}(\mathcal{G})^{N_i}(m_i))$ is A-flat; it is also finitely generated since the morphism $X \to S$ is projective. The A-module L_i is flat and finitely generated so that it is a free A-module. Denote now with W^1 the cokernel of $L_0 \to L_1$. The module W_1 is finitely generated and according to [Har77, Ex 12.7.2] the function $y \mapsto \dim_{k(y)} W_1 \otimes_A k(y)$ is upper semicontinuous; moreover since L_i is a free module we can conclude that the function $y \mapsto T^0(k(y))$ is upper semicontinuous as in the proof of [Har77, Prop 12.8]. We are left to prove that $T^0(k(y)) = \mathrm{Hom}_{\mathcal{X}_y}(\mathcal{F}_y, \mathcal{G}_y)$. Since $\mathcal{E}^{\oplus N_i} \otimes \pi^* \mathcal{O}_X(-m_i) \otimes_A k(y)$ is still locally free we have the following exact diagram:

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{X}}(\mathcal{F}, \mathcal{G}_y) & \dashrightarrow & \mathrm{Hom}_{\mathcal{X}_y}(\mathcal{F}_y, \mathcal{G}_y) \\
\downarrow & & \downarrow \\
H^0(X, F_{\mathcal{E}}(\mathcal{G})^{\oplus N_2}(m_2)) \otimes_A k(y) & \xrightarrow{\sim} & H^0(X_y, F_{\mathcal{E}}(\mathcal{G})^{\oplus N_2}(m_2) \otimes_A k(y)) \\
\downarrow & & \downarrow \\
H^0(X, F_{\mathcal{E}}(\mathcal{G})^{\oplus N_1}(m_1)) \otimes_A k(y) & \xrightarrow{\sim} & H^0(X_y, F_{\mathcal{E}}(\mathcal{G})^{\oplus N_1}(m_1) \otimes_A k(y))
\end{array}
$$

The first two horizontal arrows from the bottom are isomorphisms because of [Har77, Cor 9.4] and Proposition 1.5 so that the first horizontal arrow is also an isomorphism. $\qquad\square$

Remark 8.19.   1. The argument of the previous proof is very ad-hoc, even if we believe that a good result of semicontinuity for Ext functors should hold, we don't know about a general proof.

2. It is suggested by the proof of this lemma that the original approach of Grothendieck to cohomology and base change still holds for stacks, however it relies on Proposition 1.5.

Theorem 8.20. *In the setup of the previous theorem, let $\rho \colon V \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m) \to \mathcal{F}$ be a semistable sheaf in $\mathcal{Q}$.*

1. *The polystable sheaf $gr^{JH}(\mathcal{F})$ S-equivalent to $[\rho]$ belongs to the closure of the orbit of $[\rho]$.*

2. *The orbit of $[\rho]$ is closed if and only if it is polystable.*

3. *Given two semistable sheaves $[\rho]$ and $[\rho']$, their orbits intersect if and only if they are S-equivalent.*

Proof. 1) We can explicitly construct $gr^{JH}(\mathcal{F})$ as an element of the closure of the orbit. Let $JH_\bullet \mathcal{F}$ be a Jordan Hölder filtration of $\mathcal{F}$. Define the linear space $V_{\leq n}$ as the intersection $V \cap H^0(F_{\mathcal{E}}(JH_n(\mathcal{F}))(m))$ and consider the morphism:

$$
V_{\leq n} \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m) \xrightarrow{\sim} H^0(F_{\mathcal{E}}(JH_n(\mathcal{F})(m))) \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m) \xrightarrow{\widetilde{\mathrm{ev}}(N,m)} JH_n(\mathcal{F})
$$

The sheaf $JH_n(\mathcal{F})$ is semistable with reduced Hilbert polynomial p and multiplicity $r_n \leq r$ since it is a subsheaf of $\mathcal{F}$. Our assumptions on m implies that it is m-regular so that the morphism $\widetilde{\mathrm{ev}}(N, m)$ is surjective and the composition defines a quotient $\rho_{\leq n}\colon V_{\leq n} \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m) \to JH_n(\mathcal{F})$. Denote with $V_n := V_{\leq n}/V_{\leq n-1}$ so that we have a quotient $\rho_n\colon V_n \otimes \mathcal{E} \otimes \mathcal{O}_X(-m) \to gr_n^{JH}(\mathcal{F})$. Define now a one parameter subgroup λ in $\mathrm{SL}_{N,k}$ such that $\lambda(t) \cdot v = t^n \cdot v$ for every $t \in \mathbb{G}_{m,k}$ and $v \in V_n$. It follows from the construction in [HL97] that $\lim_{t \to 0}[\rho] \cdot \lambda(t) = \oplus_n[\rho_n] =: [\overline{\rho}]$ where $[\overline{\rho}]$ is precisely $\overline{\rho}\colon V \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m) \to gr^{JH}(\mathcal{F})$ and this proves the firs statement.

2) It is almost the same as in [HL97, 4.3.3]. We have just to use semicontinuity for Hom proved in Lemma 8.18.

3) It follows from the first two points and the fact that a good quotient separates closed invariant subschemes. □

Theorem 8.21. *The stack of stable sheaves $\mathcal{S}(\mathcal{E}, \mathcal{O}_X(1), P)^s = [\mathcal{Q}^s / \mathrm{GL}_{N,k}]$ is a $\mathbb{G}_{m,k}$-gerbe over its moduli space $M^s(\mathcal{O}_X(1), \mathcal{E}) := \mathcal{Q}^s/GL_{N,k}$ which is a quasi projective scheme.*

Proof. Since the orbits of stable sheaves are closed it follows from the previous theorem and [MFK94, Thm 1.4.1.10]. □

Theorem 8.22. *Denote with $M^{ss} := M^{ss}(\mathcal{O}_X(1), \mathcal{E})$ the GIT quotient $\mathcal{Q}/GL_{N,k}$. Let ψ be the natural morphism $\psi\colon \mathcal{S}(\mathcal{E}, \mathcal{O}_X(1), P) = [\mathcal{Q}/GL_{N,k}] \to M^{ss}(\mathcal{O}_X(1), \mathcal{E})$.*

1. *It has the following universal property: let Z be an algebraic space and $\phi\colon \mathcal{S}(\mathcal{E}, \mathcal{O}_X(1), P) \to Z$ a morphism, then there is only one morphism $\theta\colon M^{ss} \to Z$ making the following diagram commute:*

$$\mathcal{S}(\mathcal{E}, \mathcal{O}_X(1), P) \xrightarrow{\psi} M^{ss}$$
$$\phi \searrow \quad \swarrow \exists!\theta$$
$$Z$$

2. *The natural morphism $\mathcal{O}_{M^{ss}} \to \psi_* \mathcal{O}_{\mathcal{S}(\mathcal{E}, \mathcal{O}_X(1), P)}$ is an isomorphism and the functor ψ_* is exact; in different words M^{ss} is a good moduli space in the sense of Alper [Alp08, Def 4.1]. Moreover there is an invertible sheaf $\mathcal{M}$ on M^{ss} and an integer m such that denoted $\sigma\colon [\mathcal{Q}/\mathrm{SL}_{N,k}] \to M^{ss}$ we have:*

$$\sigma^* \mathcal{M} \cong \mathcal{L}_l^{\otimes m}$$

3. *The algebraic stack $\mathcal{S}(\mathcal{E}, \mathcal{O}_X(1), P)$ has no moduli space or no tame moduli space in the sense of Alper [Alp08, Def 7.1].*

Proof. (1) According to [MFK94, Thm 1.4.1.10] the map $\overline{\psi}\colon \mathcal{Q} \to M^{ss}$ factorizing through the stack $[\mathcal{Q}/\mathrm{SL}_{N,k}]$ and the morphism $[\mathcal{Q}/\mathrm{SL}_{N,k}] \to [\mathcal{Q}/\mathrm{PGL}_{N,k}]$ is a categorical quotient and this implies the universal property in the first point for the stack $[\mathcal{Q}/\mathrm{SL}_{N,k}]$ and actually also for $[\mathcal{Q}/\mathrm{PGL}_{N,k}]$. The map $\mathcal{S}(\mathcal{E}, \mathcal{O}_X(1), P) = [\mathcal{Q}/\mathrm{GL}_{N,k}] \to [\mathcal{Q}/\mathrm{PGL}_{N,k}]$ is a gerbe so that it has the universal property in [Gir71, IV Prop 2.3.18.ii]; this implies that if $[\mathcal{Q}/\mathrm{PGL}_{N,k}]$ has the universal property in the statement then also $\mathcal{S}(\mathcal{E}, \mathcal{O}_X(1), P)$ has the same universal property.

(2) It is just the stacky interpretation of [MFK94, 1.4.1.10.ii].

(3) Theorem 8.20 implies that M^{ss} is not a moduli scheme since its points are in bijection with S-equivalence classes, and semistable sheaves can be S-equivalent even if not isomorphic. So to speak the scheme M^{ss} has not enough points to be a moduli scheme for $\mathcal{S}(\mathcal{E}, \mathcal{O}_X(1), P)$. However it satisfies the universal property in the first point (condition (C) in [KM97]) and this implies that if a moduli space exists it is isomorphic to M^{ss}. $\qquad\square$

Appendix A

Moduli of twisted sheaves.

In this appendix we want to make a more precise comparison between our result on semistable sheaves on gerbes and analogous results in [Lie07] and [Yos06]. In this section $\pi\colon \mathcal{X} \to X$ is a G-gerbe over X where X is a projective scheme over an algebraically closed field k and G is a diagonalizable group scheme over X (its Cartier dual is constant). The stack $\mathcal{X}$ can have non finite inertia; the most interesting case, that is $G = \mathbb{G}_m$, has non finite inertia so that $\mathcal{X}$ is not tame according to our definition. However π_* is exact and X is the moduli space of $\mathcal{X}$ so that all the construction of the moduli space of semistable sheaves still holds (we have just asked for finite inertia in order to grant the existence of the moduli space of $\mathcal{X}$).

Let $\chi \in C(G)$ be the characters of G; in the following we will prove that the moduli space of semistable sheaves on $\mathcal{X}$ is made of connected of components labelled by characters and each of these is the moduli space of χ-twisted sheaves on X.

Lemma A.1. *Let $\pi\colon \mathcal{X} \to X$ be a G-gerbe and $q\colon X \to S$ a projective morphism of finite type schemes over a field, and S is connected. Fix $\mathcal{E}, \mathcal{O}_X(1)$ a polarization. Let $\mathcal{F}$ be a coherent torsion-free sheaf on $\mathcal{X}$ flat on S and $\mathcal{F} = \bigoplus_{\chi \in C(G)} \mathcal{F}_\chi$ its decomposition in eigensheaves. Let $P = P_{\mathcal{E}}(\mathcal{F}_s)$ be the modified Hilbert polynomial of the geometric fiber over s a point of S. The polynomial P splits as $P = \sum_{\chi \in C(G)} P_\chi$ where $P_\chi(m) = \chi(X_s, \pi_*(\mathcal{F}_\chi \otimes \mathcal{E}_\chi^\vee)(m)|_{X_s})$ and it doesn't depend on the point s.*

Proof. Since $\mathcal{F}$ is S-flat each summand $\mathcal{F}_\chi$ is S-flat, moreover we observe that $\mathcal{X}_s$ is again a G-gerbe and that the decomposition in eigensheaves is compatible with the restriction to the fiber. Applying Toën-Riemann-Roch we have:

$$P_{\mathcal{E}}(\mathcal{F}|_{\mathcal{X}_s}, m) = \sum_{\chi \in C(G)} \chi(X_s, \pi_*(\mathcal{F}_\chi \otimes \mathcal{E}_\chi^\vee)(m)|_{X_s}) = \sum_{\chi \in C(G)} P_\chi(m)$$

Each P_χ doesn't depend on the fiber because of [EGAIII.2, 7.9.4]. $\qquad\square$

Let $\mathcal{G}$ be a locally free sheaf. Thanks to this lemma it makes sense to define the functor $\mathrm{Quot}_{\mathcal{X}/k}(\mathcal{G}_\chi, P_\chi)$ of quotients $\mathcal{F}_\chi$ with modified Hilbert polynomial $P_\chi(m) = \chi(X, \pi_*(\mathcal{F}_\chi \otimes \mathcal{E}_\chi^\vee)(m))$. Every quotient must be a χ-twisted sheaf or zero because the only morphism between sheaves twisted by different characters is the zero morphism. We have also natural transformations:

$$\mathrm{Quot}_{\mathcal{X}/k}(\mathcal{G}_\chi, P_\chi) \xrightarrow{\iota_\chi} \mathrm{Quot}_{\mathcal{X}/k}(\mathcal{G}, P)$$

and they are monomorphisms (of sets) since there are evident sections.

Lemma A.2. *Let $\mathcal{G}$ be a locally free sheaf on $\mathcal{X}$ and $\mathcal{G} = \bigoplus_{\chi \in C(G)} \mathcal{G}_\chi$ its decomposition in eigensheaves. Fix P an integral polynomial of degree $d = \dim X$. The natural transformation:*

$$\coprod_{\chi \in C(G)} \mathrm{Quot}_{\mathcal{X}/k}(\mathcal{G}_\chi, P_\chi) \xrightarrow{\amalg \iota_\chi} \mathrm{Quot}_{\mathcal{X}/k}(\mathcal{G}, P)$$

is relatively representable, surjective and a closed immersion.

Proof. Since there are no morphisms between sheaves twisted by different characters it is clear that the image of each natural transformation is disjoint from the others and the coproduct of all of them covers the target. We have just to prove that each ι_χ is relatively representable and a closed immersion. To prove this we first observe that, while $\mathcal{E}_\chi$ is not a generating sheaf, it is a generating sheaf for every χ-twisted sheaf. Having this in mind the result follows with the same proof as in [OS03, Prop 6.2] but using Lemma A.1 instead of [OS03, Lem 4.3]. $\square$

Let N, V, m be as in Proposition 7.1. Let $\mathcal{Q}$ the open subscheme of $\mathrm{Quot}_{\mathcal{X}/k}(V \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m), P)$ defined in Proposition 7.1 and denote with $\mathcal{Q}_\chi$ its intersection with $\mathrm{Quot}_{\mathcal{X}/k}(\mathcal{G}_\chi, P_\chi)$, then we have the following result:

Proposition A.3. *The moduli stack of torsion-free semistable sheaves on $\mathcal{X}$ with fixed modified Hilbert polynomial P is made of the following connected components:*

$$[\mathcal{Q}/GL_{N,k}] \cong \coprod_{\chi \in C(G)} [\mathcal{Q}_\chi/GL_{N,k}]$$

In the same way the good moduli scheme of $[\mathcal{Q}/GL_{N,k}]$ decomposes in connected components:

$$\mathcal{Q}/GL_{N,k} \cong \coprod_{\chi \in C(G)} \mathcal{Q}_\chi/GL_{N,k}$$

and each of them is the good moduli scheme of $[\mathcal{Q}_\chi/GL_{N,k}]$

Proof. The first statement is an immediate consequence of Lemma A.2. Since each $\mathcal{E}_\chi$ is a generating sheaf for the subcategory of quasicoherent χ-twisted sheaves all the results in section 7.3 can be reproduced for each quotient $\mathcal{Q}/GL_{N,k}$ and this implies the second statement. $\square$

If the group scheme G is $\mathbb{G}_m$ or μ_a for some integer a each $\mathcal{Q}_\chi/GL_{N,k}$ is the moduli scheme of χ-twisted sheaves constructed by Yoshioka for an evident choice of the generating sheaf $\mathcal{E}$. To obtain exactly the same moduli scheme produced by Lieblich it's enough to choose the generating sheaf $\mathcal{E}$ such that each summand $\mathcal{E}_\chi$ has trivial Chern classes $c_i(\mathcal{E}_\chi)$ for $i = 1, \ldots, \dim \mathcal{X}$.

Bibliography

[AGV06] Dan Abramovich, Tom Graber, and Angelo Vistoli, *Gromov–Witten theory of Deligne–Mumford stacks*, arXiv.math.AG/0603151 (2006), 57. vii, xiii, 42, 68

[AOV08] Dan Abramovich, Martin Olsson, and Angelo Vistoli, *Tame stacks in positive characteristic*, Ann. Inst. Fourier (Grenoble) **58** (2008), no. 4, 1057–1091. MR MR2427954 vi, 3

[AV02] Dan Abramovich and Angelo Vistoli, *Compactifying the space of stable maps*, J. Amer. Math. Soc. **15** (2002), no. 1, 27–75 (electronic). MR MR1862797 (2002i:14030) 4

[Alp08] Jarod Alper, *Good moduli spaces for Artin stacks*, arXiv:0804.2242v2 (2008). 81

[Bas63] Hyman Bass, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8–28. MR MR0153708 (27 #3669) 46

[Bis97] Indranil Biswas, *Parabolic bundles as orbifold bundles*, Duke Math. J. **88** (1997), no. 2, 305–325. MR MR1455522 (98m:14045) vii

[BN93] Marcel Bökstedt and Amnon Neeman, *Homotopy limits in triangulated categories*, Compositio Math. **86** (1993), no. 2, 209–234. MR MR1214458 (94f:18008) 24

[Bor06] Niels Borne, *Fibrés paraboliques et champ des racines*, arXiv:math/0604458v1 (2006). vii, viii, 68

[Bor07] _____, *Sur les représentations du groupe fondamental d'une variété privée d'un diviseur à croisements normaux simples*, arXiv:0704.1236v2 (2007). vii, 68

[Cad07] Charles Cadman, *Using stacks to impose tangency conditions on curves*, Amer. J. Math. **129** (2007), no. 2, 405–427. vii, xiii

[Căl00] Andrei Căldăraru, *Derived categories of twisted sheaves on Calabi-Yau manifolds*, 2000, Ph.D. Thesis, Cornell University. vii

[DP03] Ron Donagi and Tony Pantev, *Torus fibrations, gerbes, and duality*, arXiv:math/0306213v2 (2003). vii

[EHKV01] Dan Edidin, Brendan Hassett, Andrew Kresch, and Angelo Vistoli, *Brauer groups and quotient stacks*, Amer. J. Math. **123** (2001), no. 4, 761–777. MR MR1844577 (2002f:14002) 11, 14

[FMN07] Barbara Fantechi, Etienne Mann, and Fabio Nironi, *Smooth toric DM stacks*, arXiv:0708.1254v1 (2007). 39

[Ful98] William Fulton, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998. MR MR1644323 (99d:14003)

[Gir71] Jean Giraud, *Cohomologie non abélienne*, Springer-Verlag, Berlin, 1971, Die Grundlehren der mathematischen Wissenschaften, Band 179. 81

[EGAII] A. Grothendieck, *Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes*, Inst. Hautes Études Sci. Publ. Math. (1961), no. 8, 222. MR MR0163909 (29 #1208) 36

[EGAIII.1] ———, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I*, Inst. Hautes Études Sci. Publ. Math. (1961), no. 11, 167. MR MR0217085 (36 #177c) 12, 17, 19

[EGAIII.2] ———, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. II*, Inst. Hautes Études Sci. Publ. Math. (1963), no. 17, 91. MR MR0163911 (29 #1210) 8, 19, 55, 62, 83

[EGAIV.2] ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II*, Inst. Hautes Études Sci. Publ. Math. (1965), no. 24, 231. MR MR0199181 (33 #7330) 7

[EGAIV.3] ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III*, Inst. Hautes Études Sci. Publ. Math. (1966), no. 28, 255. MR MR0217086 (36 #178) 8, 9, 62

[Har66] Robin Hartshorne, *Residues and duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin, 1966. MR MR0222093 (36 #5145) vii, 23, 28, 29, 33, 37

[Har77] ———, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR 57 #3116 4, 6, 38, 45, 79, 80

[HL97] Daniel Huybrechts and Manfred Lehn, *The geometry of moduli spaces of sheaves*, Aspects of Mathematics, E31, Friedr. Vieweg & Sohn, Braunschweig, 1997. MR MR1450870 (98g:14012) ix, 49, 50, 54, 57, 58, 59, 61, 63, 64, 68, 74, 75, 76, 77, 78, 81

[KM97] Seán Keel and Shigefumi Mori, *Quotients by groupoids*, Ann. of Math. (2) **145** (1997), no. 1, 193–213. MR MR1432041 (97m:14014) 3, 82

[Kre99] Andrew Kresch, *Cycle groups for artin stacks*, Invent. Math. **138** (1999), no. 3, 495–536. MR MR1719823 (2001a:14003) vii

[Kre06] ———, *On the geometry of Deligne-Mumford stacks*, 2006, to appear on the Seattle proceedings. vi, 11, 15, 39

[Lan04a] Adrian Langer, *Moduli spaces of sheaves in mixed characteristic*, Duke Math. J. **124** (2004), no. 3, 571–586. MR MR2085175 (2005g:14082) viii, 68

[Lan04b] _____, *Semistable sheaves in positive characteristic*, Ann. of Math. (2) **159** (2004), no. 1, 251–276. MR MR2051393 (2005c:14021) viii, 64, 67

[Lie06] Max Lieblich, *Remarks on the stack of coherent algebras*, Int. Math. Res. Not. (2006), Art. ID 75273, 12. MR MR2233719 vi

[Lie07] _____, *Moduli of twisted sheaves*, Duke Math. J. **138** (2007), no. 1, 23–118. MR MR2309155 vii, viii, 13, 49, 50, 57, 68, 83

[LMB00] Gérard Laumon and Laurent Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 39, Springer-Verlag, Berlin, 2000. 9, 15, 16, 17, 19, 25, 30, 36, 71

[Mat80] Hideyuki Matsumura, *Commutative algebra*, second ed., Mathematics Lecture Note Series, vol. 56, Benjamin/Cummings Publishing Co., Inc., Reading, Mass., 1980. MR MR575344 (82i:13003) 52

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, third ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994. MR MR1304906 (95m:14012) 51, 74, 80, 81, 82

[Mil80] James S. Milne, *Étale cohomology*, Princeton Mathematical Series, vol. 33, Princeton University Press, Princeton, N.J., 1980. MR MR559531 (81j:14002) 7, 29, 31, 53

[ML98] Saunders Mac Lane, *Categories for the working mathematician*, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998. MR MR1712872 (2001j:18001) 13

[MR071] *Théorie des intersections et théorème de Riemann-Roch*, Springer-Verlag, Berlin, 1971, Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6), Dirigé par P. Berthelot, A. Grothendieck et L. Illusie. Avec la collaboration de D. Ferrand, J. P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud et J. P. Serre, Lecture Notes in Mathematics, Vol. 225. MR MR0354655 (50 #7133) 59, 60, 61

[MS80] V. B. Mehta and C. S. Seshadri, *Moduli of vector bundles on curves with parabolic structures*, Math. Ann. **248** (1980), no. 3, 205–239. MR MR575939 (81i:14010) vii

[Mum66] David Mumford, *Lectures on curves on an algebraic surface*, With a section by G. M. Bergman. Annals of Mathematics Studies, No. 59, Princeton University Press, Princeton, N.J., 1966. MR MR0209285 (35 #187) 55

[Mum70] _____, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, No. 5, Published for the Tata Institute of Fundamental Research, Bombay, 1970. MR MR0282985 (44 #219) 4

[MY92] M. Maruyama and K. Yokogawa, *Moduli of parabolic stable sheaves*, Math. Ann. **293** (1992), no. 1, 77–99. MR MR1162674 (93d:14022) vii, viii, 65, 68

[Nee96] Amnon Neeman, *The Grothendieck duality theorem via Bousfield's techniques and Brown representability*, J. Amer. Math. Soc. **9** (1996), no. 1, 205–236. MR MR1308405 (96c:18006) vii, 23, 25, 26, 27

[Ols07] Martin Olsson, *Sheaves on Artin stacks*, J. Reine Angew. Math. **603** (2007), 55–112. 7, 25

[OS03] Martin Olsson and Jason Starr, *Quot functors for Deligne-Mumford stacks*, Comm. Algebra **31** (2003), no. 8, 4069–4096, Special issue in honor of Steven L. Kleiman. MR MR2007396 (2004i:14002) vi, 8, 11, 12, 14, 24, 27, 54, 55, 57, 64, 65, 74, 84

[OSS80] Christian Okonek, Michael Schneider, and Heinz Spindler, *Vector bundles on complex projective spaces*, Progress in Mathematics, vol. 3, Birkhäuser Boston, Mass., 1980. MR MR561910 (81b:14001) 66

[Pot92] J. Le Potier, *L'espace de modules de Simpson*, Séminaire de géométrie algébrique, Jussieu (1992). 75

[Sim94] Carlos T. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety. i*, Inst. Hautes Études Sci. Publ. Math. (1994), no. 79, 47–129. MR MR1307297 (96e:14012) 73

[Toe99] B. Toën, *Théorèmes de Riemann-Roch pour les champs de Deligne-Mumford*, K-Theory **18** (1999), no. 1, 33–76. MR MR1710187 (2000h:14010) 56

[Ver69] Jean-Louis Verdier, *Base change for twisted inverse image of coherent sheaves*, Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968), Oxford Univ. Press, London, 1969, pp. 393–408. MR MR0274464 (43 #227) 26, 28, 29, 30, 31, 32

[Vis89] Angelo Vistoli, *Intersection theory on algebraic stacks and on their moduli spaces*, Invent. Math. **97** (1989), no. 3, 613–670. vii

[Yos06] Kōta Yoshioka, *Moduli spaces of twisted sheaves on a projective variety*, Moduli spaces and arithmetic geometry, Adv. Stud. Pure Math., vol. 45, Math. Soc. Japan, Tokyo, 2006, pp. 1–30. MR MR2306170 vii, viii, 83