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ADE superpotentials, Seiberg Duality and Matrix Models

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Abstract

In this thesis we study some issues of the nonperturbative dynamics of $\mathcal{N}=1$ supersymmetric gauge theories. We consider SQCD with two chiral superfields in the adjoint representation and superpotential deformations, whose flows fall into Arnold's ADE classification of simple singularities. We study in detail the confining phase deformation of the A_n SQCD and its Seiberg dual in the classical and quantum chiral ring and find the duality map by means of the DV method. Then we analyze the deformation of the D_{n+2} SQCD and describe its three classical branches and its cubic curve. In all the cases we can continuously interpolate between the classical vacua by following a path in the moduli space. We are led to the proposal that, for an $\mathcal{N}=1$ supersymmetric gauge theory with a mass gap, the degree of its algebraic curve corresponds to the number of semiclassical branches.

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1. INTRODUCTION

Nonabelian gauge theories describe successfully the fundamental interactions, with the possible exception of gravity. The second example is the Standard Model of electroweak interactions, which is described by a gauge theory at weak coupling, that is under the control of perturbation theory. The other example is the strong interaction described by QCD. This gauge theory is strongly coupled at large distances, its degrees of freedom in the infrared are not the ones appearing in the lagrangian, henceforth we cannot study its dynamics with the usual tools of perturbation theory and we need some different techniques.

Supersymmetry is the key ingredient that allows us to delve deep into the nonperturbative nature of strongly coupled gauge theory. The closest cousins of real world QCD are the $\mathcal{N} = 1$ supersymmetric gauge theories, for which we have many kits at our disposal. Most of them where developed during the last decade mainly due to Seiberg [1]: by virtue of holomorphy and symmetries we can tell exact results about the low energy dynamics of these theories, in particular we can study their vacuum structure and uncover a vast zoo of different behaviours. In the paradigmatic case of supersymmetric QCD, Seiberg [2] was able to completely classify all the possible phases we encounter by varying the number of flavors N_f. On the way, he discovered a generalization of the usual electric–magnetic duality to the case of nonabelian gauge groups, which is known as *Seiberg duality.*

This kind of duality is not exact at all scales, but it holds at large distances only. Consider an asymptotically free supersymmetric gauge theory, that we will call *electric*, whose renormalization group flow has a fixed point at a long distance scale, where the physics is described by a superconformal field theory. For some range of flavors N_f inside the so called *conformal window*, the theory is in a nonabelian Coulomb phase, where the infrared degrees of freedom are interacting quarks and gluons and the potential between two external charges scales as $\frac{1}{R}$. The *magnetic* dual is another theory which flows to the same fixed point. In other words, the physics at the infrared point is described equivalently by both theories and there is no experimental way to tell whether the Coulomb potential is mediated by the interacting electric or the interacting magnetic variables. The two sets of degrees of freedom of the dual pair might be in general very different at the level of the microscopic lagrangian, as duality holds only for the two low energy effective field theories. A particular feature of Seiberg duality is that it works not only inside the conformal window, where it was firstly discovered, but for all ranges of flavors. Indeed, it may even happen that a very strongly coupled electric theory has an infrared dual formulation in terms of a free magnetic theory (thus not asymptotically free) and viceversa.

1.1. ADE Classification and the "a" Theorem

An arena where we can study the infrared physics and Seiberg duality in a controlled way is a generalization of $SU(N_c)$ SQCD, that we obtain by adding two chiral superfields X and Y that transform in the adjoint representation of the gauge group. This is in a sense the most general theory with adjoints and fundamentals, since if we had three adjoints the theory would loose its asymptotic freedom and the infrared dynamics would be just a free electric phase. Recently, Intriligator and Wecht [3] studied the possible RG flows of this theory deformed by relevant operators made out of the adjoints. In particular, the relevant superpotential deformations involving adjoints are found to be

$$\begin{array}{cc} \widehat{O} & W_{\widehat{O}} = 0 \\ \widehat{A} & W_{\widehat{A}} = \mathrm{Tr}Y^2 \\ \widehat{D} & W_{\widehat{D}} = \mathrm{Tr}XY^2 \\ \widehat{E} & W_{\widehat{E}} = \mathrm{Tr}Y^3 \\ A_n & W_{A_n} = \mathrm{Tr}(X^{n+1} + Y^2) \\ D_{n+2} & W_{D_{n+2}} = \mathrm{Tr}(X^{n+1} + XY^2) \\ E_6 & W_{E_6} = \mathrm{Tr}(Y^3 + X^4) \\ E_7 & W_{E_7} = \mathrm{Tr}(Y^3 + YX^3) \\ E_8 & W_{E_8} = \mathrm{Tr}(Y^3 + X^5). \end{array} \tag{1.1}$$

Each of the superpotentials in (1.1) describes a different infrared fixed point of SQCD with two adjoints. These fixed points fall into Arnold's ADE classification of simple singularities [4].

Once we have the relevant superpotential deformations (1.1), we can study the possible flows between them to have a picture of the phases of SQCD with two adjoints. Intriligator and Wecht were interested in particular in verifying the predictions of the conjectured a theorem. This is the four dimensional analogue of Zamolodchikov's two dimensional c theorem [5]: there exists a "central charge," which roughly speaking counts the number of degrees of freedom of a quantum field theory and monotonically decreases along RG flows to the IR, as degrees of freedom are integrated out. It is further conjectured [6], that an appropriate such central charge at the RG fixed points is the coefficient "a" of Euler density, a certain curvature–squared term of the conformal anomaly on a curved space–time background. The conjectured a theorem is that all RG flows satisfy $a_{IR} < a_{UV}$. This central charge a exhibits a number of special features. We can compute its exact value at the fixed points as a combination of the 't Hooft anomalies for the superconformal R–symmetry $a = \mathrm{Tr}R^3 - \mathrm{Tr}R$, once we know the exact charges of the fields at the fixed point [7]. This in general would be a very difficult task, due to the fact that the gauge theory at the fixed point is usually strongly coupled, however the exact superconformal R–symmetry has the property of maximizing a among all the possible assignments of R–charges [8]. Therefore, we can compute the superconformal R–charges by maximizing a and then use the 't Hooft anomalies to check the a theorem along the flows between the various fixed points.

Since the theories (1.1) are generalizations of ordinary SQCD, one would like to extend to them the complete analysis of the phases that Seiberg obtained for the latter and in particular one can look for different examples of Seiberg duality. Unfortunately, for the theory $\widehat{O}$ without superpotential and also for $\widehat{A}, \widehat{D}, \widehat{E}$ no hint of a Seiberg dual has been found. However, we can further simplify the dynamics by turning on additional deformations that restrict the number of nontrivial operators, or in other words that further truncate the chiral ring, which is the ring of gauge invariant chiral operators. It turns out that the A_n and D_{n+2} theories have a phase structure very similar to that of SQCD and a Seiberg dual description has been proposed for both fixed points. These two theories will be the main focus of our investigation. The E_n theories remain somewhat misterious.

If we want to have a better understanding of the low energy dynamics of these ADE SQCDs, we have to study their effective descriptions in the vicinity of the IR fixed points.

1.2. The Dijkgraaf–Vafa Approach

We are interested in studying the physics of the ADE SQCDs in (1.1) not in the superconformal but rather in the confining phase. To this aim we have to add to (1.1) some relevant superpotential deformations such that they drive the theory to confinement. The usual strategy then would be to use holomorphy and symmetry to constrain the form of the low energy superpotential along the lines of ordinary SQCD. However, due to the large number of couplings, these methods are difficult to implement in this case. Fortunately, the Dijkgraaf–Vafa method provides a sistematic way to address this problem [9]. In gauge theories that confine and have a mass gap, the lightest elementary degree of freedom is generally believed to be the glueball field. Thus it makes sense to consider the low energy description in terms of an effective action for the glueballs, even if in general it would be difficult to obtain such a quantity, which is nonperturbative. In the specific example of $\mathcal{N}=1$ supersymmetric gauge theories, however, DV gives a recipe to compute the effective glueball superpotential.

Relying initially on a large N_c topological string duality, embedded into the superstring, the original DV conjecture states that the effective glueball superpotential of the $\mathcal{N}=1$ supersymmetric $U(N_c)$ gauge theory with an adjoint chiral superfield X and tree level superpotential[1]

$$\begin{aligned} W_{el} =& \mathrm{Tr} V(X), \\ V(z) =& \sum_{k=1}^{n} \frac{t_k}{k+1} z^{k+1}, \end{aligned} \tag{1.2}$$

at finite N_c is computed by the planar limit of an auxiliary matrix model, whose action is the gauge theory tree level superpotential (1.2). This is a deformation of the A_n theory in (1.1). Subsequently, purely field theoretical arguments were presented in favor of the conjecture: a diagrammatic explanation regarding the glueball as a background field was given in [10], while in the $\mathcal{N}=2$ theory softly broken to $\mathcal{N}=1$ an explanation in terms of the factorization of the Seiberg–Witten curve was discussed in [11]. Then, Cachazo, Douglas, Seiberg and Witten

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[1] These operators are usually referred to as dangerously irrelevant, meaning that they are irrelevant at the UV fixed point when $n \geq 3$, but they become relevant as we flow to the infrared near the A_n fixed point in (1.1).

[12] proved the conjecture by using a generalization of the Konishi anomaly [13]. They showed that certain anomalous Ward identities in the field theory, obtained by considering holomorphic variations of the fields such as $\delta X = f(X, W_\alpha)$, are the Schwinger–Dyson equations for the generators of the chiral ring as functions of the glueball superfield $S \sim W_\alpha W^\alpha$. This equations are identified with the loop equations in the matrix model. By the method of anomaly equations we can compute the terms in the effective superpotential that depend on the couplings. To obtain the complete effective superpotential we have to add also the Veneziano–Yankielowicz term, which is responsible for the gaugino condensation in the low energy theory.

In our investigation we will make an extensive use of the Konishi anomaly approach to analyze the details of the effective theory just above the mass gap. First, we will compare the confining phase deformation of the A_n adjoint SQCD in (1.1) to its magnetic dual, to find the duality map at the level of the glueball effective theory. Then, we will consider the confining phase deformation of the D_{n+2} two–adjoint SQCD and study its curve and its phases, and describe its classical magnetic dual.

1.3. Seiberg duality in the A_n Theory

Consider a supersymmetric gauge theory with gauge group $U(N_c)$ and N_f flavors of quarks Q^f and antiquarks $\tilde{Q}_{\tilde{f}}$ and a chiral superfield X in the adjoint representation of the gauge group. As anticipated above, the magnetic dual of the theory without a superpotential (which corresponds to the $\widehat{A}$ fixed point in (1.1)) is not known. But we can study deformations by relevant superpotential couplings like the A_n fixed point, for which a dual theory was proposed by Kutasov, Schwimmer and Seiberg [14][15][16]. A way to simplify the dynamics, which was studied by KSS, is to add a generic polynomial superpotential for the adjoint (1.2), that drives the theory to a confining phase in the infrared, leaving at low energy no dynamics but rather just a discrete set of vacua.

The magnetic dual of the theory (1.2) is a supersymmetric gauge theory with gauge group $U(\bar{N}_c)$, where $\bar{N}_c = nN_f - N_c$, and N_f flavors of dual quarks $q_{\tilde{f}}$ and antiquarks $\tilde{q}^f$, an adjoint chiral superfield Y and nN_f^2 gauge singlets $(P_j)^{f}_{\tilde{f}}$,

$j = 1, \ldots, n$, that represent the electric mesons $P_j = \tilde{Q}X^{j-1}Q$. The magnetic theory is defined by the tree level superpotential

$$W_{mag} = -\mathrm{Tr}V(Y) + \tilde{q}\tilde{m}(P, Y)q, \tag{1.3}$$

where $\tilde{m}(P, z)$ is a certain degree $n-1$ polynomial, whose coefficients depend on the gauge singlets P_j. This magnetic polynomial will be the crucial quantity to evaluate in the quantum theory. Even if classically the chiral rings and the vacua of the two theories are very different, quantum mechanically they coincide.

The purpose of our investigation on the A_n theory is to generalize the analysis of KSS by considering the most generic electric superpotential, obtained by adding to (1.2) a meson deformation

$$\mathrm{Tr}V(X) + \tilde{Q}_{\tilde{f}}\, m(X)_f^{\tilde{f}} Q^f, \tag{1.4}$$

where in the classical chiral ring the degree of the meson polynomial $m(z)$ is at most $n-1$. At the classical level, this electric theory presents two different kinds of vacua. In the first vacuum, that we denote as *pseudoconfining*, the fundamentals vanish and the adjoint acquires a vacuum expectation values equal to the roots of $V'(x)$, that drives the theory to a product of low energy $U(N_i)$ SQCD blocks such that $\sum_{i=1}^{n} N_i = N_c$. In the *higgs* vacuum, also Q and $\widetilde{Q}$ acquire an expectation value and the adjoint is equal to the roots of $m(x)$.[2] In this case the rank of the gauge group decreases.

Our first analysis of the duality will focus on the map between the electric and magnetic classical vacua in both the pseudoconfining and the higgs phase. The magnetic dual of the theory (1.4) contains, in addition to the superpotential (1.3), a source term for the gauge singlets $\sum_{k=1}^{\deg m+1} m_k P_k$, where m_k are the coefficients of $m(z)$. The magnetic vacua will depend then on the details of the electric meson polynomial: each flavor appearing in $m(z)_f^{\widetilde{f}}$ turns on a higgsed block in the magnetic adjoint $\langle Y \rangle$. In particular, we will study the magnetic vacuum corresponding to the electric higgs phase, characterized by a nonzero classical vev for the magnetic singlets P_j. In our classical solution, as we increase

[2] In presence of matter in the fundamental representation of the gauge group there is no phase transition between higgs and confining regimes and in the quantum theory one can continuously interpolate between them [17].

the higgsed directions in the electric theory, thus driving it to weaker coupling, the higgsed block in the magnetic theory decreases its rank, driving the dual theory to stronger coupling.

We will consider then the map between the chiral rings of the two quantum theories. Due to the presence of a large number of couplings in the tree level action (1.4), the study of the effective superpotentials by the conventional methods of holomorphy and symmetries is more involved in this case. Therefore, we found more convenient to analyze the quantum theory with the DV method. In particular, we will concentrate on the operators that generate the chiral ring

$$M(z) = \langle \tilde{Q} \frac{1}{z-X} Q \rangle, \qquad T(z) = \langle \mathrm{Tr} \frac{1}{z-X} \rangle. \tag{1.5}$$

A generalized version of the Konishi anomaly allows us to solve explicitly for these operators as functions of the glueball superfield S and the couplings [18][19] and we can integrate them to obtain the glueball effective superpotentials. By matching first the electric mesons with the magnetic singlets and then the two effective superpotentials, we will derive the map between the electric and magnetic chiral operators. The low energy electric and magnetic theories will be both described by the same hyperelliptic Riemann surface

$$y^2 = V'(z)^2 + \hbar f(z), \tag{1.6}$$

a double–sheeted cover of the plane, where the quantum deformation $f(z)$ is a degree $n-1$ polynomial. The pseudoconfining and higgs duality map will turn out to be rather different, though. In particular, in the electric pseudoconfining phase the magnetic anomaly equations are solved by the simple condition

$$m(a_i)\tilde{m}(a_i) = f(a_i), \tag{1.7}$$

for $i = 1, \ldots, n$, where a_i are the roots of $V'(z)$. This condition will ensure also the match of the electric and magnetic chiral rings and will reproduce the Konishi anomaly in each low energy SQCD block.

The DV method allows us to study also the rich analytic structure of the low energy effective theory. Even if the electric and magnetic theory have the same curve, the meromorphic functions $M(z)$ and $T(z)$ living on the curve have very different analytic structures on the two sides. We will picture their analytic behavior as follows. According to [19], an higgs eigenvalue in the electric theory is seen as a pole of $M(z)$ on the first (semiclassical) sheet of the curve. As we will see, in the magnetic theory the corresponding $\tilde{M}(z)$ will have $n-1$ poles on the first sheet. We can higgs twice the electric theory by bringing a second pole of $M(z)$ from the second (invisible) sheet into the first one. The magnetic theory behaves in two different ways depending on whether we higgs different electric flavors or several times the same flavor.[3] We will see that, in this latter case, the second electric higgsing corresponds in the magnetic theory to moving one of the $n-1$ poles away from the first into the second sheet.

1.4. The D_{n+2} Theory and its Three Phases

Once we understood the quantum theory corresponding to the A_n series, we can pass to the next infrared fixed point D_{n+2}, whose deformation presents some interesting and unexpected features. Consider then a supersymmetric $SU(N_c)$ gauge theory with N_f flavors and two adjoint chiral superfields X and Y and superpotential

$$W = \mathrm{Tr}V(X) + \lambda \mathrm{Tr} XY^2 + \tilde{Q}\, m(X) Q. \tag{1.8}$$

We will find that this theory has three kind of vacua, the pseudoconfining phase, the usual "abelian" higgs phase and a new branch that we will denote "nonabelian higgs phase". The pseudoconfining vacua are the irreps of the equations of motion with vanishing fundamentals. In the one adjoint case we discussed above, we have just one dimensional vacua. In this case, a part from the usual one dimensional vacua $X = a_i \mathbb{1}$ and $Y = b_i \mathbb{1}$, that we will call *abelian* vacua, we have also two dimensional irreps, that we will call *nonabelian* vacua, in which the adjoints are proportional to the Pauli matrices $X = \hat{a}_i \sigma_3$ and $Y = d_i \sigma_3 + c_i \sigma_1$.[4] The higgs vacua are the ones in which also the fundamentals acquire an expectation value. First of all, there are the usual one dimensional higgs vacua, where X and Y are proportional to the identity, as in the usual abelian vacua. For this reason, we will denote this vacuum the *abelian higgs* branch. But there is also a new

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[3] At most we can higgs $n-1$ color directions on the same flavor, corresponding to the degree of the meson polynomial $m(z)$.

[4] This phenomenon was first noted in [20] and then discussed in [21] in the case of a supersymmetric gauge theory with adjoint fields and no fundamentals.

kind of higgs vacuum, the *nonabelian higgs* branch, in which the adjoints are two dimensional $X = x_h\sigma_3$ and $Y = y_h\sigma_3 + y_1\sigma_1$ and the fundamentals are nonvanishing.

Due to the presence of fundamentals, we expect no phase transition and in the full quantum theory the three branches will be connected by continuously varying the couplings. We will study then the chiral ring in the quantum theory, by means of the DV method. In order to compute the curve of the gauge theory, we will use the matrix model loop equations discussed by Ferrari [21], that in the gauge theory are reproduced by a set of generalized Konishi anomaly equations. Our analysis confirms that the DV method works for theories with two adjoint chiral superfields as well as for one adjoint theories. The gauge theory curve is *cubic*

$$y^3 + a(x^2)y^2 + b(x^2)y + c(x^2) = 0, \tag{1.9}$$

where the coefficients are even polynomials depending on the couplings and the quantum deformations. This curve is the same as the curve of Ferrari's two matrix model with action (1.8) in the planar limit [21].

To have a clear picture of the phase structure of the quantum theory, we will consider again the chiral operators $M(x)$ and $T(x)$ defined in (1.5). One can solve for these operators by the method of anomaly equations and find that they are meromorphic functions on the cubic curve (1.9), whose only singularities are simple poles. In particular, the poles of $T(x)$ have integer residue as in the one adjoint case [19]. We will show that, by moving poles between the three sheets, it is possible to connect continuously all the three branches. Moreover, a natural correspondence arises between the branches and the sheets: we can characterize each of the three branches by specifying the sheet on which $M(z)$ is regular, or by some combination of poles and residues of $T(x)$.

Finally, we consider the magnetic dual of the D_{n+2} fixed point. This was proposed by Brodie [22] in the superconformal case, but we are interested in finding its generalization when we deform it by the confining phase superpotential (1.8) allowing for abelian as well as and nonabelian vacua (the latter are not present in the SCF case). The dual theory is an $SU(\bar{N}_c)$ SQCD, where $\bar{N}_c = 3nN_f - N_c$, N_f magnetic flavors, two magnetic adjoints $\widetilde{X}$ and $\widetilde{Y}$ and $3nN_f^2$ gauge singlets $(P_{lj})_{\tilde{f}}^{f}$, where $j = 1, \ldots, n$ and $l = 1, 2, 3$, that represent the electric mesons $P_{lj} = \tilde{Q}X^{j-1}Y^{l-1}Q$. One finds that the magnetic tree level superpotential is not just the analogue of (1.8), but contains an extra term

$$W_{mag} = \mathrm{Tr}\bar{V}(\widetilde{X}) + \bar{\lambda}\mathrm{Tr}\widetilde{X}\widetilde{Y}^2 + \bar{s}\mathrm{Tr}\widetilde{Y}^2 + \widetilde{q}\widetilde{m}(X,Y)q, \tag{1.10}$$

corresponding to the extra coupling $\bar{s}$. We will find the classical duality map and see that $\bar{s}$ is a function of the other couplings. Unfortunately, one cannot solve the anomaly equations in the magnetic theory on a closed set of resolvents, due to this extra $\bar{s}$ coupling, so we have to stick to the classical duality map.

1.5. The Meaning of the "Classically Invisible Sheets"

The chiral ring of the one adjoint theory (1.4) is encoded in the quantum theory by the hyperelliptic Riemann surface (1.6). This surface is a double–cover of the x plane, which describes the expectation values of the adjoint $\langle X \rangle$. The first sheet is visible classically, while the second one is not accessible semiclassically. In the quantum theory, the two sheets are connected by n branch cuts and, at first, the meaning of this "invisible sheet" was not clear. Only when coupling the theory to the chiral superfields in the fundamental representation it was possible to understand the nature of the second sheet [19]. Let us see briefly why. As we discussed above, this theory has two kinds of vacuum. In the pseudoconfining vacuum the fundamentals vanish and the adjoint has diagonal expectation values equal to the roots of the adjoint polynomial $V'(x)$. In the higgs vacuum, also Q and $\widetilde{Q}$ acquire an expectation value. The gauge group is generically broken to $\prod_{i=1}^{k} U(N_i)$ with $\sum_i N_i = N_c - L$ and $k \leq n$, where L is the number of higgsed colors, and at low energy the nonabelian factors confine, leaving a $U(1)^k$ theory. In theories with fundamentals, once we fix the number of $U(1)$ factors, there is no order parameter to distinguish the pseudoconfining and higgs phases in an invariant way [17]. Thus one expects that in the full quantum theory the different classical vacua with the same number of low energy photons can be connected to each other. So we would use the word *branch* rather than phase to label the pseudoconfining and higgs vacua.

The concept of branches only makes sense in the semiclassical limit of large expectation values. As we discusse, the classical limit of the operators $M(x)$ and $T(x)$ characterizes the different classical vacua. In the pseudoconfining branch, $M(x)$ and $T(x)$ are regular on the first semiclassical sheet, while the higgs branch these generators have poles on the first sheet at the higgs eigenvalues of the adjoint $\langle X \rangle$. We can continuously interpolate between the two branches by moving the poles between the two sheets through the branch cuts. Therefore, in this case the first sheet corresponds to the pseudoconfining branch and the second sheet to the higgs branch and the connection between classical phases, or branches, and degree of the curve is clear.

However, more general supersymmetric gauge theories have algebraic curves of higher degree, which give rise to branched coverings of the plane with a larger number of sheets. It is not clear what the meaning of the "invisible sheets" is in general.

In the following we will suggest that this correspondence between the degree of the curve and the number of branches is a generic feature of $\mathcal{N} = 1$ theories. Consider a supersymmetric gauge theory with a matter content such that, once we fix the number of low energy $U(1)$s, there is no order parameter to distinguish between the various classical branches in an invariant way. This is the case of a theory with fundamentals, for instance. Under these assumption, we propose that[5]

An $\mathcal{N} = 1$ supersymmetric gauge theory with a mass gap is described by a degree k algebraic curve, where k is the number of different branches of the theory. The curve is a k−sheeted covering of the plane, where each sheet corresponds to a different branch.

Our proposal is trivially verified in the case of ordinary SQCD and is true also in the one adjoint SQCD (1.4) due to the work of Cachazo, Seiberg and Witten [19]. If we take into account the analysis of the curve of the two adjoint SQCD (1.8), we see that the conjectre is verified also in this new example.

[5] Note that we exclude the case in which the theory has a Coulomb branch, as it happens in the theory (1.4) for $n = N$. In this case in fact there is no mass gap.

2. SEIBERG DUALITY

The dynamics of supersymmetric QCD is extremely rich and it has led to the discovery of Seiberg duality, a generalization of the usual electric–magnetic duality to the case of nonabelian gauge group. In this Chapter we will summarize the basic facts about the phases of $\mathcal{N}=1$ SQCD and show that, for some range of flavors, it has an IR fixed point and the superconformal field theory at this fixed point can be described by a completely different theory, that we will refer to as the magnetic theory. In this Chapter we refer mainly to [1] and [2].

2.1. Classical SQCD

Consider an $\mathcal{N}=1$ supersymmetric gauge theory with gauge group $SU(N_c)$ coupled to N_f flavors. The field content is given by N_c^2-1 gauge fields and gauginos transforming in the adjoint representation of the gauge group and $2N_fN_c$ matter fields, in this case scalars and fermions transforming in the fundamental and antifundamental representation. We will always work in the $\mathcal{N}=1$ superspace, let us set the notations and introduce the basic objects of the supersymmetric action. The superspace field strength $W_\alpha = \bar{D}^2 e^{-V} D_\alpha e^V$, where V is the vector superfield, is a chiral superfield that transforms in the adjoint, its lowest component is the gaugino $W_\alpha|_{\theta=0}=\lambda_\alpha$. The matter fields are described by two chiral superfields Q^f and $\widetilde{Q}_{\widetilde{f}}$, for $f,\widetilde{f}=1,\ldots,N_f$, that transform in the fundamental. Their superspace expansion is $Q^f=\varphi^f+\theta^\alpha\psi^f_\alpha+\ldots$, where ψ^f_α are the quarks and φ^f their scalar superpartners. We omitted the gauge indices. The superspace Lagrangian of SQCD is given by

$$\mathcal{L}=\int d^4\theta\left(Q^\dagger_f e^V Q^f+\widetilde{Q}^{\dagger f}e^{-V}\widetilde{Q}_f\right)+\int d^2\theta\left(\frac{\tau}{16\pi i}\mathrm{Tr}W_\alpha W^\alpha+W_{el}(\widetilde{Q},Q)\right)+h.c.,$$

where $\tau=\frac{\theta}{2\pi}+\frac{4\pi i}{g^2}$ is the complexified gauge coupling and the last term W_{el}, called the superpotential, is a holomorphic function of the chiral superfields $\widetilde{Q}$ and Q. Consider first the theory without superpotential. The global anomaly free symmetries are the flavor symmetries and an R–symmetry

$$\begin{array}{ccccc} & SU(N_f) & SU(N_f) & U(1)_B & U(1)_R \\ Q^f & N_f & 1 & 1 & \frac{N_f-N_c}{N_f} \\ \widetilde{Q}_{\widetilde{f}} & 1 & \overline{N}_f & -1 & \frac{N_f-N_c}{N_f}. \end{array} \qquad (2.1)$$

If we have no superpotential, the scalar potential is just the D–term for the squarks and the condition for a supersymmetric vacuum is

$$\sum_{A=1}^{N_c^2-1} (T^A)^a_b \left(Q^\dagger_{fa} Q^{fb} - \widetilde{Q}^{\dagger f}_a \widetilde{Q}^b_f \right) = 0, \tag{2.2}$$

where $a, b = 1 \ldots, N_c$ are the gauge indices and $(T^A)^a_b$ are the gauge group generators in the fundamental representation. We used the fact that the generators of the antifundamental are $(T^A_{\bar{N}_c})^a_b = -(T^A_{N_c})^a_b$. Since T^A are generators of hermitian traceless matrices, we write this as $Q^\dagger_{fa} Q^{fb} - \widetilde{Q}^{\dagger f}_a \widetilde{Q}^b_f = \frac{\delta^b_a}{N_c} \left(Q^\dagger_{fc} Q^{fc} - \widetilde{Q}^{\dagger f}_c \widetilde{Q}^c_f \right)$. This equation have different solutions depending on the number of flavors and we will briefly discuss it in the following.

The vacuum expectation values of the squarks $\langle Q_f \rangle = \langle \varphi_f \rangle$ subject to the condition (2.2) and up to gauge equivalence cooks up a manifold which is called the classical moduli space of vacua. It can always be given a gauge invariant description in terms of the space of expectation values of gauge invariant polynomials in the fields subject to additional classical relations. This is because setting the scalar potential to zero and modding out by the gauge group is equivalent to modding out by the complexified gauge group. In our case the solution to (2.2) is

$$Q = \widetilde{Q} = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \cdot & \\ & & & a_{N_f} \end{pmatrix} \tag{2.3}$$

for $N_f < N_c$, with a_i arbitrary, and by

$$Q = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \cdot & \\ & & & a_{N_c} \\ & & & \end{pmatrix}, \quad \widetilde{Q} = \begin{pmatrix} \widetilde{a}_1 & & & \\ & \widetilde{a}_2 & & \\ & & \cdot & \\ & & & \widetilde{a}_{N_c} \\ & & & \end{pmatrix}, \tag{2.4}$$

$$|a_i|^2 - |\widetilde{a}_i|^2 = \text{independent of } i,$$

for $N_f \geq N_c$.

If $N_f < N_c$, the gauge invariant description of the moduli space is in terms of arbitrary expectation values of the meson superfields $M^f_{\tilde{f}} = \widetilde{Q}_{\tilde{f}} Q^f$. When $N_f \geq N_c$, it is also possible to form baryons superfields $B^{f_1 \cdots f_{N_c}} = Q^{f_1} \ldots Q^{f_{N_c}}$ and $\widetilde{B}_{\tilde{f}_1 \cdots \tilde{f}_{N_c}} = \widetilde{Q}_{\tilde{f}_1} \ldots \widetilde{Q}_{\tilde{f}_{N_c}}$ where we contracted the color indices with the anti-symmetric tensor. The gauge invariant description of the classical moduli space for $N_f \geq N_c$ is given in terms of the expectation values of M, B and $\widetilde{B}$, subject to the following classical constraints. Up to global symmetry transformations, the expectation values are

$$
\begin{aligned}
M &= \begin{pmatrix} a_1\widetilde{a}_1 & & & & \\ & a_2\widetilde{a}_2 & & & \\ & & \cdot & & \\ & & & a_{N_c}\widetilde{a}_{N_c} & \\ & & & & \\ & & & & \end{pmatrix} \\
B^{1,\ldots,N_c} &= a_1 a_2 \ldots a_{N_c} \\
\widetilde{B}_{1,\ldots,N_c} &= \widetilde{a}_1 \widetilde{a}_2 \ldots \widetilde{a}_{N_c}
\end{aligned}
\qquad (2.5)
$$

with all other components of M, B and $\widetilde{B}$ vanishing. Therefore, the rank of M is at most N_c. If it is less than N_c, either $B = 0$ with $\widetilde{B}$ having rank at most one or $\widetilde{B} = 0$ with B having rank at most one. If the rank of M is equal to N_c, both B and $\widetilde{B}$ have rank one and the product of their eigenvalues is the same as the product of non-zero eigenvalues of M. The physical interpretation of the flat directions is that the gauge group is higgsed. If $B = \widetilde{B} = 0$ and M has rank k, $SU(N_c)$ is broken to $SU(N_c - k)$ with $N_f - k$ massless flavors.

2.2. The Quantum Theory

Let us see what happens to the classical vacua of SQCD when we go to the quantum theory. The behavior depends crucially on the number of flavors.

When $N_f < N_c$, the global symmetries (2.1) allow for the presence of a unique nonperturbative superpotential term

$$
W_{ADS} = (N_f - N_c) \left(\frac{\Lambda^{3N_c - N_f}}{\det \widetilde{Q}Q} \right)^{1/(N_c - N_f)}, \qquad (2.6)
$$

where Λ is the dynamical scale of the SQCD. This is the Affleck–Dine–Seiberg superpotential [23]. If $N_f < N_c - 1$, this is generated dynamically by the condensation of gauginos, while if $N_f = N_c - 1$ it is generated by an instanton. The scalar potential that we obtain by (2.6) is the well known runaway potential that slopes to zero as $\langle M \rangle$ goes to infinity. Therefore, the classical moduli space of vacua of SQCD with $N_f < N_c$ is completely lifted and actually the quantum theory has no supersymmetric vacuum.

For $N_f \geq N_c$, the ADS superpotential (2.6) does not exist, because either the exponential diverges, when $N_f = N_c$, or the determinant vanishes, when $N_f > N_c$, and the vacuum degeneracy can not be lifted. The theory in this range of flavors was solved by Seiberg in [24].

If $N_f = N_c$, we see from (2.5) that the classical moduli space is described by mesons and baryons subject to the constraint $\det M - \widetilde{B}B = 0$. This manifold is singular at the origin where we expect an enhancement of the gauge symmetry and the appearance of massless gluons. In the quantum theory, however, this singularity is smoothed out and the manifold becomes $\det M - \widetilde{B}B = \Lambda^{2N_c}$. Therefore, in the quantum theory there is no enhancement of the gauge symmetry at the origin of moduli space. The only massless particles are the moduli, the fluctuations of M, B, and $\widetilde{B}$ under the constraint. In the semi-classical region of large expectation values it is appropriate to think of the theory as higgsed. Near the origin, it is appropriate to think of the theory as being confining. There is a smooth transition from the region where a Higgs description is more appropriate to the region where a confining description is more appropriate, due to the presence of matter fields in the fundamental representation of the gauge group. Since the origin is not on the moduli space, in the quantum theory we find a spontaneous breaking of the chiral symmetries (2.1).

When $N_f = N_c + 1$, we find that the expectation values of the mesons and the baryon (2.5) are subject to the constraints

$$\det M \left(\frac{1}{M}\right)_f^{\tilde{f}} - B_f \widetilde{B}^{\tilde{f}} = 0, \qquad M^f_{\tilde{f}} B_f = M^f_{\tilde{f}} \widetilde{B}^{\tilde{f}} = 0.$$

In this case, the quantum dynamics does not modify the moduli space, that remains the same. However, the singularity at the origin of the moduli space is interpreted differently. While in the classical theory the singularity is associated to the enhancement of the gauge symmetry and the appearance of massless gluons, in the quantum theory it is given by the presence of massless mesons and baryons, which are considered as the elementary fields of the IR description. As a check, we can see that they satisfy the 't Hooft anomaly matching. The dynamics of the theory is described by the effective superpotential

$$W_{eff} = \frac{1}{\Lambda^{2N_c-1}}(M_{\tilde{f}}^{f} B_{\tilde{f}} \widetilde{B}^{\tilde{f}} - \det M) \tag{2.7}$$

and, since at the origin the global symmetries are unbroken, we have confinement without chiral symmetry breaking.

If we want to understand the quantum moduli space of theory for $N_f \geq N_c + 2$, we have to introduce its Seiberg dual.

2.3. The Conformal Window

Consider SQCD in the range of flavors $\frac{3}{2}N_c < N_f < 3N_c$, in which the theory is still asymptotically free. The exact beta function [25] is

$$\begin{aligned} \beta &= -\frac{g^3}{16\pi^2}\frac{3N_c - N_f + N_f\gamma(g^2)}{1 - N_c\frac{g^2}{8\pi^2}} \\ \gamma &= -\frac{g^2}{8\pi^2}\frac{N_c^2 - 1}{N_c} + \mathcal{O}(g^4), \end{aligned} \tag{2.8}$$

where $\gamma(g^2)$ is the anomalous dimension of the fundamentals. Since there are values of N_f and N_c where the one loop beta function is negative but the two loop contribution is positive, Banks and Zaks [26] showed that, at least for large N_c and when the theory is just barely asymptotically free, there is a non-trivial fixed point of the RG flow in the IR. It was then argued by Seiberg [2] that such a fixed point exists for every $\frac{3}{2}N_c < N_f < 3N_c$.

Therefore, for this range of N_f, the infrared theory is a non-trivial four dimensional superconformal field theory. The elementary quarks and gluons are not confined but appear as interacting massless particles. The potential between external electric sources behaves as a Coulomb potential $V \sim \frac{1}{R}$ and for this reason Seiberg called this phase of SQCD the nonabelian Coulomb phase.

The R–symmetry of the superconformal fixed point is not anomalous and commutes with the flavor $SU(N_f) \times SU(N_f) \times U(1)_B$ symmetry. By the superconformal algebra, the R–symmetry determines the dimension D of the field as $D = \frac{3}{2}R$ and we can fix the anomalous dimensions of our gauge invariant operators $M = \widetilde{Q}Q$ to $D(M) = 3\frac{N_f - N_c}{N_f}$ and similarly for the baryons $D(B) = D(\widetilde{B}) = \frac{3N_c(N_f - N_c)}{2N_f}$. All of the gauge invariant operators at the infrared fixed point should be in unitary representations of the superconformal algebra. One of the constraints on the representations is that spinless operators have $D \geq 1$ and the bound is saturated for free fields. For $D < 1$ a highest weight representation includes a negative norm state which cannot exist in a unitary theory. We will use these facts about SCFTs to argue the existence of phases described by a free field theory.

The fixed point coupling g_* gets larger as the number of flavors is reduced. For $N_f < \frac{3}{2}N_c$ the value of $D(\widetilde{Q}Q)$ flows below the unitarity bound $D \geq 1$. The theory is very strongly coupled and goes over to a new phase, different from the interacting nonabelian Coulomb phase, that we will see after describing the Seiberg dual theory. Since the dimension of the meson becomes one for $N_f = \frac{3}{2}N_c$, at this value M becomes a free field. This suggests that in the the new phase at $N_f = \frac{3}{2}N_c$ the field M, and perhaps the whole IR theory, becomes free.

2.4. Seiberg Duality

When $N_f \geq N_c + 2$, the IR description in terms of mesons and baryons (2.5) does not work any more. The only term that is invariant under the global symmetries, a generalization of the superpotential (2.7) for $N_f = N_c + 1$, in this case does not have the correct dimension. Moreover, if we regard the mesons and baryons as the infrared degrees of freedom of the theory, then the 't Hooft anomaly matching does not hold.

The baryons in (2.5) have $\bar{N}_c \equiv N_f - N_c$ indices. We can see these composite operators as bound states of $\bar{N}_c$ constituents and associate to them two new chiral superfields q and $\widetilde{q}$ and actually introduce a new SQCD with gauge group $SU(\bar{N}_c)$ such that q and $\widetilde{q}$ transform in the fundamental and antifundamental. In this way we can give a dual description of the baryons as $B_{\widetilde{f}_1 \ldots \widetilde{f}_{\bar{N}_c}} = \epsilon_{a_1 \ldots a_{\bar{N}_c}} q^{a_1}_{\widetilde{f}_1} \cdots q^{a_{\bar{N}_c}}_{\widetilde{f}_{\bar{N}_c}}$.

This observation can be put on firm grounds and lead to the Seiberg dual of SQCD we are describing it in the following.

For $N_f \geq N_c + 2$, the IR description of SQCD is captured by another $\mathcal{N} = 1$ supersymmetric gauge theory (which we will refer to as the *magnetic* thery, as opposed to the original SQCD which is the *electric* theory) with gauge group $SU(\bar{N}_c)$, where $\bar{N}_c = N_f - N_c$. It is coupled to N_f chiral superfields of magnetic quarks $q_{\tilde{f}}$ and antiquarks $\tilde{q}^f$ that transform in fundamental and antifundamental of the magnetic gauge group. In addition, there is a gauge singlet $M^f_{\tilde{f}}$ with N_f^2 components which represents the electric mesons of the original description and couples to the fundamentals through the superpotential

$$W_{mag} = \frac{1}{\mu} M^f_{\tilde{f}} q_f \tilde{q}^{\tilde{f}}. \tag{2.9}$$

Without the superpotential (2.9), the magnetic theory also flows to a fixed point because $\frac{3}{2}(N_f - N_c) < N_f < 3(N_f - N_c)$ for the above range of N_f. This point is in a nonabelian Coulomb phase. At this fixed point M is a free field of dimension one and would have an additional $U(1)$ global symmetry. By adding the superpotential (2.9), which has dimension $D = 1 + 3N_c/N_f < 3$ at the fixed point of the magnetic gauge theory and is thus a relevant perturbation, we drive the theory to a new fixed point and break this unwanted $U(1)$ symmetry. The statement of Seiberg duality [2] is that this new fixed point is identical to that of the original electric SQCD.

The two theories have different gauge groups and different numbers of interacting particles. Nevertheless, they describe the same fixed point. In other words, there is no experimental way to determine whether the Coulomb potential between external sources is mediated by the interacting electric or the interacting magnetic variables. The scale μ in (2.9) is needed for the following reason. In the electric description $M^f_{\tilde{f}} = \tilde{Q}_{\tilde{f}} Q^f$ has dimension two at the UV fixed point and acquires anomalous dimension $D(M) = 3\frac{N_f - N_c}{N_f}$ at the IR fixed point. In the magnetic description, M_{mag} is an elementary field of dimension one at the UV fixed point which flows to the same operator with dimension $D(M_{mag}) = 3\frac{N_f - N_c}{N_f}$ at the IR fixed point. In order to relate M_{mag} to M of the electric description in the UV, a coupling μ must be introduced with the relation $M = \mu M_{mag}$. Below we will write all the expressions in terms of M and μ rather than in terms of

M_{mag}. However, since duality strictly speaking is defined at the IR fixed point, where the deformation (2.9) is marginal, this is an auxiliary scale.

The magnetic theory has a dynamical scale $\widetilde{\Lambda}$ which is related to the scale Λ of the electric theory by

$$\Lambda^{3N_c-N_f}\widetilde{\Lambda}^{3(N_f-N_c)-N_f} = (-1)^{N_f-N_c}\mu^{N_f}, \tag{2.10}$$

where μ is the dimensionful scale explained above. The factor $(-1)^{N_f-N_c}$ is fixed by consistency of the dual of the dual theory. We will see very often this relation, and its generalization to theories with adjoint chiral superfields. This relation of the scales has several consequences. First of all, it is consistent with all possible flows we can trigger by turning on relevant operators or expectation values, as we will briefly discuss below. Moreover, it ensures that Seiberg duality is a strong–weak coupling duality, since as the dynamical electric scale Λ increases, the magnetic scale $\widetilde{\Lambda}$ decreases, and viceversa. Differentiating the action with respect to $\log\Lambda$ relates the field strengths of the electric and the magnetic theories as $W_\alpha^2 = -\widetilde{W}_\alpha^2$, which we will refer to as the glueballs in the following. We will encounter very often this crucial relation is the course of our study of more complicated Seiberg dual pairs. The minus sign in this expression is common in electric magnetic duality, which maps $E^2 - B^2 = -(\widetilde{E}^2 - \widetilde{B}^2)$. In our case it shows that the gaugino bilinear in the electric and the magnetic theories are related by $\lambda\lambda = -\tilde{\lambda}\tilde{\lambda}$.

The electric and magnetic theories have different gauge symmetries. This is possible because gauge symmetries really have to do with a redundant description of the physics rather than with symmetry. On the other hand, global symmetries are physical and should be the same in the electric and magnetic theories. Indeed, the magnetic theory has the same anomaly free global $SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_R$ as the electric theory, with the singlet $M^f_{\tilde{f}}$ transforming as $Q^f\widetilde{Q}_{\tilde{f}}$ and the magnetic quarks transforming as

$$\begin{array}{ccccc} & SU(N_f) & SU(N_f) & U(1)_B & U(1)_R \\ q_{\tilde{f}} & \overline{N}_f & 1 & \frac{N_c}{N_f-N_c} & \frac{N_c}{N_f} \\ \tilde{q}^f & 1 & N_f & -\frac{N_c}{N_f-N_c} & \frac{N_c}{N_f} \end{array} \tag{2.11}$$

This symmetry is anomaly free in the magnetic theory and it is preserved by the superpotential (2.9). Furthermore, the magnetic spectrum with these charges satisfies the 'tHooft anomaly matching conditions, which is the first nontrivial check of the duality.

In order for the dual theory to describe the same physics as the electric theory, there must be a mapping of all gauge invariant primary operators of the electric theory to those of the dual theory. For example, the electric mesons $M^f_{\bar{f}} = Q^f \widetilde{Q}_{\bar{f}}$ and the singlets $M^f_{\bar{f}}$ of the magnetic theory are identified in the infrared. All such mappings must be compatible with the global symmetry charges discussed above. Another set of gauge invariant operators of the electric theory are the baryons $B^{f_1 \cdots f_{N_c}} = Q^{f_1} \cdots Q^{f_{N_c}}$ and $\widetilde{B}_{\bar{f}_1 \cdots \bar{f}_{N_c}} = \widetilde{Q}_{\bar{f}_1} \cdots \widetilde{Q}_{\bar{f}_{N_c}}$. As we discussed above, in the magnetic theory we can similarly form the baryons $b_{f_1 \cdots f_{\tilde{N}_c}} = q_{f_1} \cdots q_{f_{\tilde{N}_c}}$ and $\widetilde{b}^{\bar{f}_1 \cdots \bar{f}_{\tilde{N}_c}} = \widetilde{q}^{\bar{f}_1} \cdots \widetilde{q}^{\bar{f}_{\tilde{N}_c}}$, where $\tilde{N}_c = N_f - N_c$. At the fixed point, these operators are in one to one correspondence via the mapping

$$\begin{aligned} B^{i_1 \dots i_{N_c}} &= c\, \epsilon^{i_1 \dots i_{N_c} j_1 \dots j_{\tilde{N}_c}} b_{j_1 \dots j_{\tilde{N}_c}}, \\ \widetilde{B}_{\bar{i}_1 \dots \bar{i}_{N_c}} &= c\, \epsilon_{\bar{i}_1 \dots \bar{i}_{N_c} \bar{j}_1 \dots \bar{j}_{\tilde{N}_c}} \widetilde{b}^{\bar{j}_1 \dots \bar{j}_{\tilde{N}_c}}, \end{aligned} \tag{2.12}$$

with $c = \sqrt{-(-\mu)^{N_c - N_f} \Lambda^{3N_c - N_f}}$. Note that these mappings respect the global symmetries discussed above. The normalization constant can be fixed by the symmetries and by flowing to the cases $N_f \leq N_c + 1$.

2.5. Matching the Deformations

This duality is defined at the IR fixed point we have discussed. However, it must continue to hold along whatever flat directions or if we deform the dual theories by corresponding relevant operators. In other words, duality must hold along the flows from this fixed point. We will consider two kinds of flow of the electric theory.

Let us higgs the electric theory by giving an expectation value to the first l colors of electric quarks. Correspondingly, the electric meson $M = \widetilde{Q}Q$ acquires an expectation value such that $\operatorname{rank} M = l$. The electric theory flows to an $SU(N_c - l)$ SQCD with $N_f - l$ massless flavors. On the magnetic side, when the gauge singlet, which corresponds to the electric meson, gets such an expectation value, the corresponding l magnetic quarks in (2.9) a mass. Thus we can integrate them out and see that the magnetic theory flows to a theory with $SU(N_c - N_f)$ gauge group, but just $N_f - l$ light flavors, as expected from duality. Note that by higgsing the electric theory we go to weaker coupling, while in the magnetic theory, as we integrate out massive fundamentals, we go towards stronger coupling. This is therefore in agreement with our intuition about strong–weak coupling duality and actually one can check this on the scale matching relation (2.10), which is preserved along this flow.

Suppose instead we deform the electric theory by turning on a mass term $W_{el} = m\widetilde{Q}_1 Q^1$ for one flavor. Then, the magnetic superpotential is deformed to

$$W_{mag} = \frac{1}{\mu} M\widetilde{q}q + mM_1^1. \tag{2.13}$$

The F–term equations for the singlet then fixes the expectation value of the magnetic quarks to $\langle \widetilde{q}^1 q_1 \rangle = -\mu m$. But this means that one flavor is higgsed and the magnetic theory flows to an $SU(N_f - N_c - 1)$ gauge theory with $N_f - 1$ flavors, which is exactly what we expect from duality. In fact, one can check that the scale matching relation (2.10) is again preserved along this flow.

Let us add a brief comment about the meaning of Seiberg duality. We defined it as the infrared equivalence of two different UV theories. That is to say that the electric and magnetic theories are in the same universality class. This fact has been shown by Argyres, Plesser and Seiberg [27] by considering $\mathcal{N} = 2$ SQCD and showing that by turning on a mass term for the adjoint chiral superfield, which breaks supersymmetry to $\mathcal{N} = 1$, and tuning the mass parameter, one can flow either to the electric or to the magnetic UV fixed point theories. Since there can be no phase transition as one continuously adjusts a relevant parameter in a supersymmetric theory, one can then argue that the dual pairs are in the same universality class. This argument, in principle, holds both for the F–terms, that we discussed above, as well as for the D–term, i.e. the Kahler potential.

2.6. Free Magnetic Phase

Finally, by using duality we can describe what happens to SQCD in the case $N_c + 2 \leq N_f \leq \frac{3}{2}N_c$. We saw above that the mesons and baryons, that for $N_f = N_c + 1$ were the correct IR degrees of freedom, when we increase the number of flavors do not satisfy the 't Hooft anomaly matching any more. Now we can solve this puzzle by considering the Seiberg dual theory. In the magnetic description, the situation is clear: since $3(N_f - N_c) \leq N_f$, the magnetic $SU(N_f - N_c)$ gauge theory is not asymptotically free when $N_f \leq \frac{3}{2}N_c$ and thus weakly coupled at large distances, while the magnetic superpotential (2.9) is irrelevant. Therefore, the low energy spectrum of the theory consists of the $SU(N_f - N_c)$ gauge fields and the fields M, q, and $\widetilde{q}$ in the dual magnetic Lagrangian. These magnetic massless states are composites of the elementary electric degrees of freedom, which are strongly coupled. The massless composite gauge fields exhibit a gauge invariance which is not visible in the underlying electric description. Because there are massless magnetically charged fields, the theory is in a nonabelian free magnetic phase. On the other hand, when $N_f \geq 3N_c$ the electric theory looses asymptotic freedom and the theory is in a nonabelian free electric phase. So, we completed the list of the possible phases of SQCD:

$$\begin{array}{cc} N_f < N_c & \text{no vacuum} \\ N_c \leq N_f < N_c + 2 & \text{free nlsm of } M, B, \widetilde{B} \\ N_c + 2 \leq N_f < \frac{3}{2}N_c & \text{free magnetic} \\ \frac{3}{2}N_c \leq N_f \leq 3N_c & \text{nonabelian Coulomb phase} \\ N_f > 3N_c & \text{free electric.} \end{array} \tag{2.14}$$

3. *ADE* SUPERPOTENTIALS

After the discovery by Seiberg [2] in '94, there have been many generalizations of the $\mathcal{N}=1$ nonabelian duality, extending the proposal to theories with different gauge groups [1] as well as with different matter content. Among the tons of dual pairs that appeared in the literature, we will concentrate to the case of $SU(N_c)$ SQCD with chiral superfields in the adjoint representation of the gauge group. The reason is that in the case of supersymmetric gauge theories with adjoints, there are some special techniques available to study a subsector of the quantum theory, the chiral ring, which go under the name of the Dijkgraaf–Vafa method. In this Chapter we summarize the classification of the RG flows in SQCD with two adjoints (and some of their Seiberg duals, the ones we know). We will follow the detailed classification by Intriligator and Wecht [3]. In this way we give a picture of the different phases these theories undergo and we will be able to list them in tables such as the one for SQCD (2.14). We do not consider more than two adjoints because, already with three adjoints, the theory looses its asymptotic freedom and thus we do not expect any nice fixed point to be there in the IR. This exhausts the possible flows of an $\mathcal{N}=1$ supersymmetric gauge theory with fundamental and adjoint matter and adjoint deformations.[6]

Let us consider an $SU(N_c)$ supersymmetric gauge theory with N_f flavors $\widetilde{Q}_{\widetilde{f}}$ and Q^f and two chiral superfields X and Y transforming in the adjoint representation of the gauge group. For all N_f in the asymptotically free range $N_f \leq N_c$ the theory flows to an IR fixed point $\widehat{O}$, which is described by an interacting $\mathcal{N}=1$ superconformal field theory. By means of the a–maximization [8], we can compute the exact R–charges of the fields at the fixed point and find their anomalous dimensions $D=\frac{3}{2}R$. In this way, it is possible to classify all the relevant deformations of the $\widehat{O}$ fixed point and the corresponding flows. In particular, we focus on the operators which involve only the two adjoints X and Y. It turns out that the relevant deformations of the $\widehat{O}$ fixed point are just the following three

$$\begin{array}{cc} \widehat{O} & W_{\widehat{O}}=0 \\ \widehat{A} & W_{\widehat{A}}=\mathrm{Tr}Y^2 \\ \widehat{D} & W_{\widehat{D}}=\mathrm{Tr}XY^2 \\ \widehat{E} & W_{\widehat{E}}=\mathrm{Tr}Y^3. \end{array} \tag{3.1}$$

[6] For a classification of the meson deformations see [28].

Let us take a closer look at these new fixed points.

$\widehat{A}$ **theory**. The $\widehat{A}$ fixed point is reached via integrating out one adjoint chiral superfield Y and flowing to an $SU(N_c)$ theory with one adjoint X and N_f light flavors. This theory has received a lot of attention in the past. However, until very recently there was no way to solve for the exact R–charges and anomalous dimensions of the theory without superpotential and therefore to classify the relevant deformations that involve only the adjoint field. Kutasov and collaborators [29], by means of the a–maximization technique [8], where able to do the job. Let us define the parameter

$$x = \frac{N_c}{N_f}.$$

For any given n there exists an x_{A_n} such that

$$R(X^{n+1}) \leq 2, \qquad \text{for} \quad x \geq x_{A_n}.$$

In the range of flavors $x \geq x_n$ then the operator $\mathrm{Tr}X^{n+1}$ is a relevant deformation of the $\widehat{A}$ infrared fixed point and it triggers the flow to another fixed point, that we call $\widehat{A}_n$ and analyze below.

$\widehat{D}$ **theory**. The $\widehat{D}$ SCFT is the endpoint of the flow from $\widehat{O}$ with the superpotential $W = \mathrm{Tr}XY^2$. This and the $\widehat{E}$ fixed point, that we discuss below, where studied by [3]. The only relevant operators are of the form $\mathrm{Tr}X^{n+1}$. By means of the a–maximization, they found that for any given n there exists an $x_{D_{n+2}}$ such that

$$R(X^{n+1}) \leq 2, \qquad \text{for} \quad x \geq x_{D_{n+2}}.$$

In the range of flavors $x \geq x_{D_{n+2}}$ then the operator $\mathrm{Tr}X^{n+1}$ is a relevant deformation of the $\widehat{D}$ infrared fixed point and it triggers the flow to another fixed point, that we call $\widehat{D}_{n+2}$ and analyze below.

$\widehat{E}$ **theory**. The $\widehat{E}$ SCFT is the endpoint of the flow from $\widehat{O}$ with the superpotential $W = \mathrm{Tr}Y^3$. We would like to classify all the possible flows from this fixed point by adding relevant deformations to the superpotential. First, any quadratic ΔW superpotential is relevant. The possibilities, and where they drive the $\widehat{E}$ SCFTs, are

$$\begin{aligned} \Delta W &= \mathrm{Tr}Y^2 : \ \widehat{E} \to \widehat{A}, \\ \Delta W &= \mathrm{Tr}X^2 : \ \widehat{E} \to \widehat{A}_1, \\ \Delta W &= \mathrm{Tr}XY : \ \widehat{E} \to \mathrm{SQCD}. \end{aligned} \tag{3.2}$$

At the level of cubic ΔW, the only independent, relevant possibility is

$$\Delta W = \mathrm{Tr}X^2Y : \widehat{E} \to \widehat{D}_4, \tag{3.3}$$

Deforming $\mathrm{Tr}Y^3$ by $\Delta W = \mathrm{Tr}X^3$ is equivalent to (3.3) via a change of variables, and $\mathrm{Tr}XY^2$ is eliminated by the $\widehat{E}$ chiral ring relation. The operator $\mathrm{Tr}X^2Y$ is always relevant, however, we do not expect that it ever wins out over the original $W_{\widehat{E}} = \mathrm{Tr}Y^3$ term; both are important in determining the eventual RG fixed point. Finally, we have the higher powers ΔW. These are only relevant if x is sufficiently large, and the independent possibilities for $W = W_{\widehat{E}} + \Delta W$ are:

$$\begin{aligned}
\widehat{E} \to E_6 :\ W_{E_6} &= \mathrm{Tr}(Y^3 + X^4) \qquad \text{if} \quad x \geq x_{E_6} \approx 2.55, \\
\widehat{E} \to E_7 :\ W_{E_7} &= \mathrm{Tr}(Y^3 + YX^3) \qquad \text{if} \quad x \geq x_{E_7} \approx 4.12, \\
\widehat{E} \to E_8 :\ W_{E_8} &= \mathrm{Tr}(Y^3 + X^5) \qquad \text{if} \quad x \geq x_{E_8} \approx 7.28.
\end{aligned} \tag{3.4}$$

The values of x_{E_6}, x_{E_7}, and x_{E_8} are obtained by studying with the a maximization the R-charge of the corresponding deformation, $\Delta W_{E_6} = \mathrm{Tr}X^4$, $\Delta W_{E_7} = \mathrm{Tr}YX^3$, and $\Delta W_{E_8} = \mathrm{Tr}X^5$, at the $\widehat{E}$ RG fixed point, and seeing when $R(\Delta W)$ just drops below $R = 2$. If $x > x_{E_{6,7,8}}$, these ΔW drive $\widehat{E}$ to new SCFTs, which we call E_6, E_7, and E_8 and will analyze below.

3.1. A_n: SQCD with One Adjoint and its Dual

Let us generalize the Seiberg duality of Chapter 2 by adding an adjoint chiral superfield. Consider a supersymmetric gauge theory with gauge group $U(N_c)$ and N_f flavors of quarks Q^f and antiquarks $\tilde{Q}_{\tilde{f}}$ and a chiral superfield X in the adjoint representation of the gauge group, following Kutasov, Schwimmer and Seiberg (KSS) [14][15][16]. The magnetic dual of the theory $\widehat{A}$ without a superpotential is not known. But we can study deformations by relevant superpotential couplings, for which we know the dual theory. In particular, we know the Seiberg dual of the A_n fixed point. A way to simplify the dynamics, which was studied by KSS, is to add a generic polynomial superpotential for the adjoint[7]

$$\begin{aligned}
W_{el} =& \mathrm{Tr}V(X), \\
V(z) =& \sum_{k=1}^{n} \frac{t_k}{k+1} z^{k+1},
\end{aligned} \tag{3.5}$$

[7] These operators are usually referred to as dangerously irrelevant, meaning that they are irrelevant at the UV fixed point when $n \geq 3$, but they become relevant as we flow to the infrared.

that drives the theory from the fixed point A_n to a confining phase in the infrared, leaving at low energy no dynamics but rather just a discrete set of vacua. The classical vacua of this theory satisfy the equations of motion $V'(X) = 0$. Using the transformation in the complexified gauge group we can diagonalize the adjoint field X, then the eigenvalues of the adjoint are the roots a_i of the degree n polynomial $V'(z)$. The vacuum of this theory is then given by $X = \mathrm{diag}(a_1, \dots, a_n)$ where each a_i appears with multiplicity N_i. The low energy gauge group is thus $\prod_i U(N_i)$. At low energy we can integrate out the massive adjoint X and the theory flows into a product of decoupled $U(N_i)$ SQCD theories, which we will refer to as *Seiberg blocks* and we have just discussed.

The magnetic dual of the theory (3.5) is a supersymmetric gauge theory with gauge group $U(\bar{N}_c)$, where $\bar{N}_c = nN_f - N_c$, and N_f flavors of dual quarks $q_{\tilde{f}}$ and antiquarks $\tilde{q}^f$, an adjoint chiral superfield $\widetilde{X}$ and n gauge singlets $(P_j)^f_{\tilde{f}}$, $j = 1, \dots, n$, that transform in the $(\bar{N}_f, N_f)$ of the flavor symmetry group and represent the electric mesons

$$P_j = \tilde{Q}X^{j-1}Q.$$

The n singlets P_j are the straightforward generalization of the one magnetic singlet we found in the dual of SQCD. The way duality is to be understood is the following. First, we let the theory without superpotential flow to its IR fixed point. Then, we turn on the following magnetic superpotential

$$W_{mag} = -\mathrm{Tr}V(\widetilde{X}) + \tilde{q}\tilde{m}(P, \widetilde{X})q, \tag{3.6}$$

where $\tilde{m}(P, z)$ is a certain degree $n-1$ polynomial, whose coefficients depend on the gauge singlets P_j. Relying on the a–maximization, Kutasov and collaborators [29] found that the leading term in the magnetic superpotential $\mathrm{Tr}\widetilde{X}^{n+1}$, that corresponds to the A_n electric IR fixed point, is such that its R–charge

$$R(\widetilde{X}^{n+1}) \leq 2, \qquad \text{for} \quad \widetilde{x} \geq \widetilde{x}_{A_n},$$

where $\widetilde{x} = \frac{\bar{N}_c}{N_f}$ is the magnetic version of $x = N_c/N_f$.

At this point, we can completely classify the phases of SQCD with one adjoint at the A_n fixed point. The phases of the electric and magnetic theories can be summarized as

$$\begin{array}{cc} x \leq 1 & \text{free electric} \\ 1 < x < x_{A_n} & \widehat{A} \text{ electric} \\ x_{A_n} < x < n - \widetilde{x}_{A_n} & A_n \text{ conformal window} \\ n - \widetilde{x}_{A_n} < x < n-1 & \widehat{A} \text{ magnetic} \\ n-1 \leq x & \text{free magnetic.} \end{array} \tag{3.7}$$

For $x \leq 1$ the electric theory is not asymptotically free, so it flows to a free theory in the IR. In this case, we should use the free-electric description. To see the analogous free-magnetic phase of the magnetic dual, we can use the dual variable $\widetilde{x}$. The magnetic theory is asymptotically free if $\widetilde{x} > 1$. When the magnetic theory is not asymptotically free, i.e. $\widetilde{x} \leq 1$ and thus $x \geq n-1$, the magnetic theory becomes free in the IR. In this case, we should definitely use the magnetic description. Within the range $1 < x < n-1$, where both electric and magnetic theories are asymptotically free, we still have three possibilities. If $1 < x < x_{A_n}$ the $\mathrm{Tr}X^{n+1}$ on the electric side is irrelevant, and the electric theory flows back to the $\widehat{A}$ SCFT. In this case the electric description is again definitely better, since it is easier to see the enhanced symmetries associated with the fact that $\mathrm{Tr}X^{n+1}$ is irrelevant. Likewise, in the magnetic theory, the $\mathrm{Tr}\widetilde{X}^{n+1}$ superpotential is irrelevant if $\widetilde{x} < \widetilde{x}_{A_n}$ and the magnetic theory then flows to a magnetic version of the $\widehat{A}$ SCFT. In this case, the magnetic description is definitely better, since it's easier there to see the enhanced symmetries associated with the fact that part of W_{mag} is irrelevant. Finally, there is a "conformal window," where $\mathrm{Tr}X^{n+1}$ is relevant on the electric side, $x > x_{A_n}$, and $\mathrm{Tr}\widetilde{X}^{n+1}$ is relevant on the magnetic side, $\widetilde{x} > \widetilde{x}_{A_n}$. In the conformal window, both the electric and the magnetic theories flow to the same A_n SCFT. Either the electric or the magnetic description is a useful description in the conformal window.

Let us give a few details on the duality map. Even if classically the vacua of the two theories are very different, quantum mechanically they coincide and we will consider them below. KSS proposed that the dynamically generated scales Λ of the electric theory and $\tilde{\Lambda}$ of the magnetic theory obey the matching relation

$$\Lambda^{2N_c - N_f} \tilde{\Lambda}^{2\bar{N}_c - N_f} = \mu^{2N_f} t_n^{-2N_f}, \tag{3.8}$$

which is very similar to the corresponding scale matching of SQCD (2.10) we discussed above, and they checked the matching against various higgs and massive flows.

3.2. D_{n+2} SQCD with Two Adjoints and its Dual

We can carry on an analysis of the D_{n+2} fixed point analogously to the one for the A_n theory. Consider a supersymmetric gauge theory with gauge group $U(N_c)$ and N_f flavors of quarks Q^f and antiquarks $\tilde{Q}_{\tilde{f}}$ and two chiral superfields X and Y in the adjoint representation of the gauge group. The magnetic dual of the theory $\widehat{D}$ without a superpotential is not known. But we can study deformations by relevant superpotential couplings, for which we know the dual theory. In particular, we know the Seiberg dual of the D_{n+2} fixed point. Consider then the SCFT with superpotential

$$W_{el} = t_n \mathrm{Tr} X^{n+1} + \lambda \mathrm{Tr} X Y^2, \tag{3.9}$$

The magnetic dual of (3.9) was proposed by Brodie [22]. It is a supersymmetric gauge theory with gauge group $U(\bar{N}_c)$, where $\bar{N}_c = 3nN_f - N_c$, and N_f flavors of dual quarks $q_{\tilde{f}}$ and antiquarks $\tilde{q}^f$, two adjoint chiral superfields $\widetilde{X}$ and $\widetilde{Y}$ and $3n$ gauge singlets $(P_{lj})^f_{\tilde{f}}$, $j = 1, \ldots, n$, $l = 1, 2, 3$, transforming in the $(\bar{N}_f, N_f)$ of the flavor symmetry group, that represent the electric mesons

$$P_{lj} = \tilde{Q} X^{j-1} Y^{l-1} Q,$$

The way duality is to be understood is the following. First, we let the theory without superpotential flow to its IR fixed point. Then, we turn on the following magnetic superpotential

$$W_{mag} = -t_n \mathrm{Tr} \widetilde{X}^{n+1} - \lambda \mathrm{Tr} \widetilde{X} \widetilde{Y}^2 + \tilde{q} \tilde{m}(P, \widetilde{X}, \widetilde{Y}) q, \tag{3.10}$$

where $\tilde{m}(P, x, y)$ is a certain degree $n-1$ polynomial in x and quadratic in y, whose coefficients depend on the gauge singlets P_{lj}. Relying on the a–maximization, Intriligator and Wecht [3] found that the operator $\mathrm{Tr}\widetilde{X}^{n+1}$, that corresponds to the D_{n+2} electric IR fixed point, is such that its R–charge

$$R(\widetilde{X}^{n+1}) \leq 2, \qquad \text{for} \quad \widetilde{x} \geq \widetilde{x}_{D_{n+2}},$$

where $\widetilde{x} = \frac{\bar{N}_c}{N_f}$.

At this point, we can completely classify the phases of SQCD with two adjoints at the D_{n+2} fixed point. The phases of the electric and magnetic theories can be summarized as

$$\begin{array}{cc} x \leq 1 & \text{free electric} \\ 1 < x < x_{D_{n+2}} & \widehat{D} \text{ electric} \\ x_{D_{n+2}} < x < 3n - \widetilde{x}_{D_{n+2}} & D_{n+2} \text{ conformal window} \\ 3n - \widetilde{x}_{D_{n+2}} < x < 3n-1 & \widehat{D} \text{ magnetic} \\ 3n-1 \leq x & \text{free magnetic.} \end{array} \tag{3.11}$$

For $x \leq 1$ the electric theory is not asymptotically free, so it flows to a free theory in the IR. In this case, we should definitely use the free-electric description. To see the analogous free-magnetic phase of the magnetic dual, it's useful to introduce a dual variable to x, $\widetilde{x} \equiv \frac{\widetilde{N}_c}{N_f} = 3n - x$. The magnetic theory is asymptotically free if $\widetilde{x} > 1$. When the magnetic theory is not asymptotically free, i.e. $\widetilde{x} \leq 1$ and thus $x \geq 3n-1$, the magnetic theory becomes free in the IR. In this case, we should definitely use the magnetic descrption. Within the range $1 < x < 3n-1$, where both electric and magnetic theories are asymptotically free, we still have three possibilities. If $1 < x < x^{min}_{D_{n+2}}$ the $\text{Tr}X^{n+1}$ on the electric side is irrelevant, and the electric theory flows back to the $\widehat{D}$ SCFT. In this case the electric description is again definitely better, since it is easier to see the enhanced symmetries associated with the fact that $\text{Tr}X^{n+1}$ is irrelevant. Likewise, in the magnetic theory, the $\text{Tr}\widetilde{X}^{n+1}$ superpotential is irrelevant if $\widetilde{x} < \widetilde{x}_{D_{n+2}}$ and the magnetic theory then flows to a magnetic version of the $\widehat{D}$ SCFT. In this case, the magnetic description is definitely better, since it's easier there to see the enhanced symmetries associated with the fact that part of W_{mag} is irrelevant. Finally, there is a "conformal window," where $\text{Tr}X^{n+1}$ is relevant on the electric side, $x > x_{D_{n+2}}$, and $\text{Tr}\widetilde{X}^{n+1}$ is relevant on the magnetic side, $\widetilde{x} > \widetilde{x}_{D_{n+2}}$. In the conformal window, both the electric and the magnetic theories flow to the same D_{n+2} SCFT. Either the electric or the magnetic description is a useful description in the conformal window.

Brodie [22] proposed that the dynamically generated scales Λ of the electric theory and $\tilde{\Lambda}$ of the magnetic theory obey the matching relation

$$\Lambda^{N_c-N_f}\tilde{\Lambda}^{\tilde{N}_c-N_f} = t_n^{-3N_f}\lambda^{-3kN_f}\mu^{4N_f} \tag{3.12}$$

where μ is an auxiliary scale appearing in $\widetilde{m}$. This is very similar to the corresponding scale matching of the KSS theory (3.8) we discussed above.

3.3. E_n Theories: No Seiberg Dual!

The flows of the E_6, E_7 and E_8 SCFTs (3.4) have been analyzed in [3] in connection with the a theorem. Unfortunately, the magnetic dual of neither of these theories is known. Since the chiral ring is not truncated, it contains an infinite number of operators and therefore it is impossible to identify which field content and deformations to include in a magnetic description of these fixed points. Nobody even knows if the dual theory exists. However, by studying the flows from the E_n fixed points, it has been argued that, in the quantum theory, some mechanism should be responsible for the truncation of the chiral ring [3]. Therefore, even if the classical theory does not seem to be a suitable scenario to find the Seiberg dual, it might be that in the quantum theory a Seiberg dual might actually exists.

4. A_n THEORY: THE CHIRAL RING AND THE GENERALIZED KONISHI ANOMALY

A powerful tool to study the nonperturbative low energy physics of $\mathcal{N} = 1$ supersymmetric gauge theories is the Dijkgraaf–Vafa method [9]. It applies to theories that confine and exhibit a mass gap. Assuming that the elementary degree of freedom just above the mass gap is the glueball superfield, this technique computes the effective superpotential as a function of the glueball and the couplings, by solving an auxiliary zero dimensional matrix model. After integrating out the glueball, then, we recover the usual low energy superpotential that describes the vacuum structure of the gauge theory. Although the original derivation of the DV conjecture relies on string theory arguments, a purely field theoretical explanation has been proposed subsequently by Cachazo, Douglas, Seiberg and Witten [12]. This field theory approach is based on the Konishi anomaly and its generalization and in this Chapter we will outline its basic ingredients.

We will describe this method focussing on the adjoint SQCD, which is the electric side of the A_n fixed point we discussed in Section *3.1*. We will closely follow the original setup of [12][19] and the review [30]. We will describe the classical vacua and the chiral ring of the theory, explaining in particular a subtle issue in the nilpotence of the perturbative and exact glueball operator [31]. We will discuss the meaning of the nonperturbative glueball superpotential and then explain the generalization of the Konishi anomaly that leads to the Schwinger–Dyson equations for the generators of the chiral ring. We will see how these anomaly equations give rise to a hyperelliptic Riemann surface which encodes the vacuum structure of the theory. In the one adjoint case the curve is hyperelliptic, but this is not a generic feature. In the two adjoint case, for instance, we will see that the curve is cubic. We will show then how the different classical phases are all connected in the quantum theory [19]. This is not unexpected since there is no phase transition between semiclassical confining and higgs phases in a theory with fundamentals.

4.1. The Classical Vacua

We will consider an $\mathcal{N}=1$ supersymmetric gauge theory with $U(N_c)$ gauge group. The matter content consists of N_f flavors of quarks Q^f and antiquarks $\tilde{Q}_{\tilde{f}}$ and a chiral superfield X in the adjoint representation of the gauge group. We will at first let the theory flow to its infrared superconformal fixed point $\widehat{A}$. Then we will turn on the generic tree level superpotential

$$\begin{aligned} W_{el} =& \mathrm{Tr} V(X) + \tilde{Q}_{\tilde{f}} m(X)^{\tilde{f}}_{f} Q^f, \\ V'(z) =& \sum_{i=1}^{n} t_i z^i, \\ m(z)^{\tilde{f}}_{f} =& \sum_{k=1}^{l+1} (m_k)^{\tilde{f}}_{f} z^{k-1}, \end{aligned} \tag{4.1}$$

which corresponds to a deformation of the A_n fixed point. It is convenient to parameterize the adjoint polynomial as $V'(z) = t_n \prod_{i=1}^{n}(z-a_i)$ in terms of its roots. We denote the roots of the meson polynomial $m(z)$ as x_k, for $k=1,\ldots,l$. We are assuming here for simplicity that the meson matrix is diagonal in the flavor indices. The degree of $m(z)$ is $l \leq n-1$, since higher mesons are trivial in the classical chiral ring.

This theory exhibits two kinds of classically distinct vacua, that we will call pseudoconfining and higgs vacua. The pseudoconfining vacua are characterized by vanishing expectation values for the fundamentals

$$\begin{aligned} X =& \begin{pmatrix} a_1 & & \\ & \cdot & \\ & & a_n \end{pmatrix} \\ \tilde{Q}_{\tilde{f}} =& 0, \quad Q^f = 0, \end{aligned} \tag{4.2}$$

where each a_i has multiplicity N_i such that $\sum_i N_i = N_c$. The reason why these are called "pseudoconfining" rather than "confining" vacua is that, due to the presence of fields in the fundamental representation of the gauge group, there is no phase transition between these vacua and the higgs ones and in the quantum theory they are continuously connected. At low energy the theory consists of a set of decoupled $U(N_i)$ SQCD with N_f flavors, while the adjoint has been integrated out.[8] The rank of the gauge group does not decrease along this flow.

The Higgs vacua are characterized by a nonvanishing expectation value for the fundamentals. We consider the simple case of $m(z) = m_1 + m_2 z$, whose only root is $x_1 = -m_1/m_2$ and we have

$$\begin{aligned} X &= \begin{pmatrix} x_1 & & & \\ & a_1 & & \\ & & \cdot & \\ & & & a_n \end{pmatrix} \\ \tilde{Q}_{N_f} &= (\tilde{h}_1, 0, \ldots, 0), \quad Q^{N_f} = (h_1, 0, \ldots, 0), \end{aligned} \tag{4.3}$$

where each a_i is a root of $V'(z)$ and has multiplicity N_i such that $\sum_{i=1}^{n} N_i = N_c - 1$. One can work out the most general higgs phase solution of the equations of motion [19], but we just need this simple case. The adjoint equations of motion set $\tilde{h}_1 h_1 = -V'(x_1)/m_2$.

4.2. The Chiral Ring

We introduce here the chiral ring of the theory, following [12]. Chiral operators are operators that are annihilated by the supersymmetries $\bar{Q}_{\dot{\alpha}}$ of one chirality. The product of two chiral operators is also chiral. Chiral operators are usually considered modulo operators of the form $\{\bar{Q}_{\dot{\alpha}}, \ldots\}$. The equivalence classes can be multiplied, and form a ring called the chiral ring. A superfield whose lowest component is a chiral operator is called a chiral superfield.

Chiral operators are independent of the position, up to $\bar{Q}^{\dot{\alpha}}$-commutators. If $\{\bar{Q}_{\dot{\alpha}}, \mathcal{O}(x)\} = 0$, then

$$\frac{\partial}{\partial x^\mu}\mathcal{O}(x) = [P^\mu, \mathcal{O}(x)] = \{\bar{Q}^{\dot{\alpha}}, [Q^\alpha, \mathcal{O}(x)]\}. \tag{4.4}$$

This implies that the expectation value of a product of chiral operators is independent of each of their positions: $\partial_\mu \langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n)\rangle = 0$. Thus we can write $\langle \prod_I \mathcal{O}^I(x)\rangle = \langle \prod_I \mathcal{O}^I\rangle$ without specifying the positions x. Using this invariance,

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[8] If we allow for double roots in $V'(z)$ we end up with adjoint SQCD with a cubic tree level superpotential for the low energy adjoint superfield. For simplicity we will consider superpotential with only single roots, though.

we can take a correlation function of chiral operators at distinct points, and separate the points by an arbitrarily large distance. Cluster decomposition then implies that the correlation function factorizes

$$\langle \mathcal{O}^{I_1}(x_1)\mathcal{O}^{I_2}(x_2)\ldots\mathcal{O}^{I_n}(x_n)\rangle = \langle\mathcal{O}^{I_1}\rangle\langle\mathcal{O}^{I_2}\rangle\ldots\langle\mathcal{O}^{I_n}\rangle. \tag{4.5}$$

There are no contact terms in the expectation value of a product of chiral fields, because as we have just seen a correlation function such as $\langle\mathcal{O}^{I_1}(x_1)\mathcal{O}^{I_2}(x_2)\ldots\rangle$ is entirely independent of the positions x_i and so in particular does not have delta functions. A correlation function of chiral operators together with the upper component of a chiral superfield can have contact terms. In the theory considered here, with an adjoint superfield X, we can form gauge-invariant chiral superfields $\mathrm{Tr}X^k$ for positive integer k. These are all non-trivial chiral fields. The gauge field strength W_α is likewise chiral, and though it is not gauge-invariant, it can be used to form gauge-invariant chiral superfields such as $\mathrm{Tr}\,X^kW_\alpha$, $\mathrm{Tr}X^kW_\alpha X^lW_\beta$, etc. Setting $k=l=0$, we get, in particular, chiral superfields constructed from vector multiplets only.

There is, however, a very simple fact that drastically simplifies the classification of chiral operators. If $\mathcal{O}$ is any adjoint-valued chiral superfield, we have[9]

$$\left[\bar{Q}^{\dot\alpha}, D_{\alpha\dot\alpha}\mathcal{O}\right\} = \left[W_\alpha, O\right\}, \tag{4.6}$$

using the Jacobi identity and definition of W_α plus the assumption that $\mathcal{O}$ (anti)commutes with $\bar{Q}^{\dot\alpha}$. Taking $\mathcal{O}=X$, it suffices to consider only operators $\mathrm{Tr}X^nW_\alpha W_\beta$. Moreover, taking $\mathcal{O}=W_\beta$ in the same identity, we learn that

$$\{\bar{Q}^{\dot\alpha}, [D_{\alpha\dot\alpha}, W_\beta]\} = \{W_\alpha, W_\beta\}, \tag{4.7}$$

so in the chiral ring we can make the substitution $W_\alpha W_\beta \to -W_\beta W_\alpha$. So in any string of W's, say $W_{\alpha_1}\ldots W_{\alpha_s}$, we can assume antisymmetry in $\alpha_1,\ldots,\alpha_s$. As the α_i only take two values, we can assume $s\le 2$.

If we consider the chiral ring of just pure super Yang Mills, the only nontrivial invariant is the glueball superfield

$$S = -\frac{1}{32\pi^2}\mathrm{Tr}W_\alpha W^\alpha. \tag{4.8}$$

[9] $D_{\alpha\dot\alpha} = \frac{D}{Dx^{\alpha\dot\alpha}}$ is the bosonic covariant derivative.

In spite of the fact that $S \propto \bar{D}^2 \mathrm{tr}(W^\alpha e^{-V}(D_\alpha e^V))$, the glueball superfield is not chirally exact because $\bar{D}_{\dot\alpha}\mathrm{Tr}(W^\alpha e^{-V}(D_\alpha e^V))$ is not gauge invariant.

If we have also chiral superfields in the fundamental and antifundamental representations of the gauge group Q and $\widetilde{Q}$, as in our SQCD case, then we have additional operators in the chiral ring. Since $W_\alpha Q = \bar{\nabla}^2 \nabla_\alpha Q$, inserting this operator inside gauge invariant quantities gives a vanishing chiral operator. So, the operators $W_\alpha Q$ and $W_\alpha \widetilde{Q}$ are not in the chiral ring. A complete list of single-trace chiral operators in adjoint SQCD is

$$\mathrm{Tr}\, X^k, \quad \mathrm{Tr}\, X^k W_\alpha, \quad \mathrm{Tr}\, X^k W_\alpha W^\alpha, \quad \widetilde{Q} X^k Q. \tag{4.9}$$

Our main attention will be on computing the vacuum expectation value of the following chiral operators, which are the generators of the chiral ring

$$\begin{aligned} R(z) &= -\frac{1}{32\pi^2}\left\langle \mathrm{Tr}\frac{W_\alpha W^\alpha}{z-X}\right\rangle, \\ M_{\tilde f}^{f}(z) &= \left\langle \tilde{Q}_{\tilde f}\frac{1}{z-X}Q^f\right\rangle, \\ T(z) &= \left\langle \mathrm{Tr}\frac{1}{z-X}\right\rangle, \\ w_\alpha(z) &= \frac{1}{4\pi}\left\langle \mathrm{Tr}\frac{W_\alpha}{z-X}\right\rangle. \end{aligned} \tag{4.10}$$

Since we consider just supersymmetric vacua, we set to zero w_α and do not consider it in the following. We will see that some anomaly equations constrain these expectation values and fix them exactly in the chiral ring of the quantum theory.

The operators (4.9) are not completely independent, though, and are subject to relations. The first kind of relation stems from the fact that X is an $N_c \times N_c$ matrix. Therefore, $\mathrm{Tr}X^k$ with $k > N_c$ can be expressed as a polynomial in $u_l = \mathrm{Tr}X^l$ with $l \leq N_c$

$$\mathrm{Tr}X^k = P_k(u_1, ..., u_{N_c}) \tag{4.11}$$

The general story is that to every classical relation corresponds a quantum relation, but the quantum relations may be different.[10] A second kind of relations

[10] The classical relations (4.11) are modified by instantons for $k \geq 2N_c$.

follows from the presence of a tree level superpotential $W(X)$, as in our A_n SQCD. The equation of motion of $\partial_X W(X) = D^2 X^\dagger$ shows that in the chiral ring $\partial_X W(X) = 0$ in the chiral ring. In a gauge theory, we want to consider gauge-invariant chiral operators; classically, for any k, $\mathrm{Tr}X^k \partial_X W(X)$ vanishes in the chiral ring. This is a nontrivial relation among the generators. We will discuss in detail how this classical relation is modified by the Konishi anomaly and its generalizations.

4.3. Nilpotence of the Glueball

We now turn to discuss interesting relations which are satisfied by the glueball operator $S = -\frac{1}{32\pi^2}\mathrm{Tr}W_\alpha^2$. We discuss the pure gauge $\mathcal{N} = 1$ theory with gauge group $SU(N_c)$. The operator S is subtle because it is a bosonic operator which is constructed out of fermionic operators. Since the gauge group has $N_c^2 - 1$ generators, the Lorentz index α in W_α take two different values, and S is bilinear in W_α, it follows from Fermi statistics that in perturbation theory we find

$$(S^{N_c^2})_{pert} = 0, \tag{4.12}$$

so in particular S is nilpotent. Soon we will argue that this relation receives quantum corrections. It is important that (4.12) is true for any S which is constructed out of fermionic W_α. The latter does not have to satisfy the equations of motion.

If we are interested in the chiral ring, we can derive a more powerful result. Consider an $SU(2)$ gauge theory. For any $SU(2)$ generators A, B, C, D we have the identity

$$\mathrm{Tr}\, ABCD = \frac{1}{2}\left(\mathrm{Tr}AB\mathrm{Tr}CD + \mathrm{Tr}DA\mathrm{Tr}BC - \mathrm{Tr}AC\mathrm{Tr}BD\right) \tag{4.13}$$

Hence, allowing for Fermi statistics (which imply $\mathrm{Tr}W_1W_1 = \mathrm{Tr}W_2W_2 = 0$), we have

$$\mathrm{Tr}W_1W_1W_2W_2 = \left(\mathrm{Tr}W_1W_2\right)^2. \tag{4.14}$$

The left hand side is non-chiral, as we have seen above, and the right hand side is a multiple of S^2, so we found that $S^2 = 0$ perturbatively in the chiral ring of $SU(2)$. One can generalize this to $SU(N_c)$ [12] and find

$$(S^{N_c})_{pert} = \{\bar{Q}_{\dot\alpha}, X^{\dot\alpha}\}. \tag{4.15}$$

for some $X^{\dot{\alpha}}$, and therefore in the chiral ring $S^{N_c} = 0$ at all orders in perturbation theory.

If the relation $S^{N_c} = \{\bar{Q}_{\dot{\alpha}}, X^{\dot{\alpha}}\}$ were an exact quantum statement, it would follow that in any supersymmetric vacuum, $\langle S^{N_c}\rangle = 0$, and hence by factorization and cluster decomposition, also $\langle S\rangle = 0$. But maybe it can receive some quantum corrections. In perturbation theory, because of R-symmetry and dimensional analysis, there are no possible quantum corrections to this relation. Nonperturbatively, the instanton factor Λ^{3N_c} has the same chiral properties as S^{N_c}, in fact instantons lead to an expectation value

$$\langle S^{N_c}\rangle = \Lambda^{3N_c}, \tag{4.16}$$

and therefore, they do indeed modify the classical operator relation to

$$S^{N_c} = \Lambda^{3N_c} + \{\bar{Q}_{\dot{\alpha}}, X^{\dot{\alpha}}\} \tag{4.17}$$

where in the chiral ring we can set the last term to zero. Equation (4.17) is an exact operator relation in the theory. It is true in all correlation functions with all operators. Also, since it is an operator equation, it is satisfied in all the vacua of the theory. The relation $S^{N_c{}^2} = 0$, which we recall is an exact relation, not just a statement in the chiral ring, must also receive instanton corrections so as to be compatible with (4.17). To be consistent with the existence of a supersymmetric vacuum in which $\langle S^{N_c}\rangle = \Lambda^{3N_c}$, as well as with the classical limit in which $S^{N_c{}^2} = 0$, the corrected equation must be of the form $(S^{N_c} - \Lambda^{3N_c})P(S^{N_c}, \Lambda^{3N_c}) = 0$, where P is a homogeneous polynomial of degree $N_c - 1$ with a non-zero coefficient of $(S^{N_c})^{N_c-1}$. We will argue that the exact quantum relation is [31]

$$\left(S^{N_c} - \Lambda^{3N_c}\right)^{N_c} = 0. \tag{4.18}$$

There are two ways to calculate the gluino condensate $\langle S^{N_c}\rangle = \Lambda^{3N_c}$. One is based on the weak–coupling instanton calculations, giving the above result, whereas with the strong–coupling instanton calculations the right hand side of (4.16) is replaced by $2[(N_c - 1)!(3N_c - 1)]^{-1/N_c}\Lambda^{3N_c}$. However, it turns out that cluster decomposition does not hold in the strong coupling computations [32]. Furthermore, it has been observed that on $\mathbf{R}^3 \times S^1$ the results do not depend on the radius of S^1, so that one ends up with $\mathbf{R}^4$ in the infinite radius limit, or more precisely $\mathbf{R}^3 \times \widehat{\mathbf{R}}$, where $\widehat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ is the one point compactification of $\mathbf{R}$. Consider then nonperturbative configurations of $\mathcal{N} = 1$ supersymmetric gauge theory on $\mathbf{R}^3 \times \widehat{\mathbf{R}}$ rather than $\mathbf{R}^4$. This essentially avoids possible infrared divergences due to the fact that the calculations are performed by choosing a perturbative vacuum which is different from the true one [33]. This one–point compactification can be seen as a way to impose boundary conditions on nonperturbative configurations rather than a change of topology of the space where the gauge theory lives. This is of interest for the definition of the gluino condensate. We also note that the point splitting is a quantum operation which leads to a modification of the classical operator definition. Let us go back to the analysis of the chiral ring. The above discussion shows that we can use instanton results in order to define

$$\mathcal{O}_\Lambda \equiv S^{N_c} - \lim_{\substack{x_i \to x_j \\ \forall ij}} \langle S(x_1) \dots S(x_{N_c}) \rangle = S^{N_c} - \Lambda^{3N_c}. \tag{4.19}$$

Note that the $X^{\dot\alpha}$ in (4.15) and (4.17) can differ only by a chiral operator: dimensional analysis and R–symmetry forbid the existence of terms $\{\overline{Q}_{\dot\alpha}, \delta X^{\dot\alpha}\}$ that vanish as $\Lambda \to 0$. The correction from $S_{cl}^{N_c}$ to S^{N_c} concerns a redefinition of the glueball superfield and not $\{\overline{Q}_{\dot\alpha}, X^{\dot\alpha}\}$, that is

$$S_{cl}^{N_c} = S^{N_c} - \Lambda^{3N_c}. \tag{4.20}$$

Therefore, the basic observation is that it is the N_c–th power of the glueball superfield that gets quantum corrections. For these reasons we used the notation $\mathcal{O}_\Lambda$ in (4.19) instead of $S_{cl}^{N_c}$. However, since $S_{cl}^{{N_c}^2} = 0$ was derived as an identity, and since, as we said, $\{\overline{Q}_{\dot\alpha}, X^{\dot\alpha}\}$ does not receive quantum corrections, it follows by (4.12) and (4.15) that $\{\overline{Q}_{\dot\alpha}, X^{\dot\alpha}\}^{N_c} = 0$. On the other hand, being $\mathcal{O}_\Lambda = \{\overline{Q}_{\dot\alpha}, X^{\dot\alpha}\}$ we have $\mathcal{O}_\Lambda^{N_c} = 0$, that is Eq.(4.18).

The emerging structure is reminiscent of the property of forms in a $(N_c - 1)$–dimensional space. To realize the similarity let us define $\omega \doteq \{\overline{Q}_{\dot\alpha}, X^{\dot\alpha}\}$, where ω is a one–form on a $(N_c - 1)$–dimensional space. Then

$$\{\overline{Q}_{\dot\alpha}, X^{\dot\alpha}\}^{N_c} = \wedge_{k=1}^{N_c} \omega = 0, \tag{4.21}$$

leading to an interesting unexpected structure.

4.4. The Effective Glueball Superpotential

We will be interested in computing the effective superpotential of our theory in the confining phase. Let us consider then the Wilsonian action. If S_{μ_0} denotes the (bare) action at a scale μ_0, then S_μ (for $\mu < \mu_0$) denotes the action describing the same physics but where the degrees of freedom with momenta between μ and μ_0 have been integrated out. We can be more precise if we restrict our attention to the Wilsonian superpotential because any non-holomorphic dependence on the couplings must be regarded as a (now properly IR regularized) D-term. The Wilsonian superpotential W_{eff} is thus a holomorphic quantity in both fields and coupling constants and it is not perturbatively renormalized. Thus we write:

$$W_{eff} = W_{tree} + W_{np}, \tag{4.22}$$

where W_{tree} is the tree level superpotential and W_{np} a nonperturbative contribution. In the case where interacting massless fields are absent, the Wilsonian and the 1PI effective superpotentials coincide. Furthermore, the perturbative non-renormalization of the superpotential can also be proven using holomorphy and the symmetries of the theory. Notice that one can always write $W_{tree} = \sum g_r Z_r(\Phi_i)$ for the gauge invariant quantities $Z_r(\Phi_i)$. The question now is what is the form of W_{np}. One would expect that it depends the scale Λ, the couplings g_r and the invariants $Z_r(\Phi_i)$. But, in fact, W_{np} is independent of the couplings g_r. This fact is referred to as the "linearity principle" because it implies that the full superpotential is linear in the couplings. Having established that $W_{np} = W_{np}(Z_r, \Lambda)$ it is sometimes possible to completely fix the functional dependence up to some numerical constant by dimensional analysis and symmetry considerations, as in the case of the ADS superpotential of SQCD. Clearly, we can integrate out the field Z_1 at low energy (where the superpotential piece dominates) by solving

$$\frac{\partial}{\partial Z_1} W_{eff} = 0,$$

which is just a Legendre transform

$$g_1 = -\frac{\partial}{\partial Z_1} W_{np}.$$

Solving this equation for Z_1 in terms of g_1 and the other variables and substituting back one obtains W_{eff} where now the dependence on g_1 will be complicated because we have integrated out its partner. Thus we can think of g_1 and Z_1 as forming a canonical pair. One could integrate out all the composite fields Z_r to give W_{eff} solely in terms of the couplings g_r and the scale Λ. This is the low energy superpotential.

The Legendre transform is clearly invertible and as long as we are only interested in F–terms we do not lose any information integrating out a field and, in fact, we can integrate it back in by reversing the procedure:

$$\langle Z_r \rangle = \frac{\partial}{\partial g_r} W_{eff}. \tag{4.23}$$

This equation will be relevant later in a slightly different way: if we happen to know the vacuum expectation value of Z_r in some other way, (4.23) can be used to determine W_{eff}. This is the way we will compute our effective glueball superpotential.

We can also introduce a canonical conjugate for Λ. We can use the perturbatively exact definition of the holomorphic scale to rewrite the gauge kinetic term as a tree level plus one loop contribution to the Wilsonian superpotential:

$$\frac{\tau(\mu)}{16\pi i} \mathrm{Tr} W^\alpha W_\alpha \equiv \beta \log(\Lambda/\mu) S,$$

where S is the glueball superfield (4.8) and β is the coefficient of the one–loop beta function, in the case of our adjoint SQCD $\beta = 2N_c$. The linearity principle still applies in this case. While S does not appear in the original superpotential one can "integrate it in" by solving for Λ:

$$\langle S \rangle = \frac{1}{\beta} \Lambda \frac{d}{d\Lambda} W_{eff}. \tag{4.24}$$

Even without matter fields there is a superpotential for S: the so–called Veneziano-Yankielowicz (VY) superpotential [34]. Consider, as an example, pure $SU(N)$ SYM theory for which $\beta = 3N$. By the assumption of confinement and a mass gap, we expect that all degrees of freedom are massive and thus the effective superpotential at low energy can only be a constant. Dimensional analysis shows that $W_{eff} = a\Lambda^3$ for some numerical constant a. Using (4.24) yields $\langle S \rangle = (a/N)\Lambda^3$. From an explicit instanton computation we know that

$$\langle S \rangle = \Lambda^3, \tag{4.25}$$

The computation of the exact numerical coefficient is a subtle issue. To derive the correct result (4.25) one needs to perform a computation at weak coupling [32]. We can now express the nonperturbative superpotential as a function of S as

$$W_{np} = -NS \log \frac{S}{\mu^3} + NS.$$

Sometimes one also writes in a mixed notation

$$W_{eff} = W_{np} + 3NS \log \frac{\Lambda}{\mu} = NS \left(1 - \log \frac{S}{\Lambda^3}\right) \equiv W_{VY}, \tag{4.26}$$

usually referred to as the VY superpotential. The advantage of this notation is that upon minimizing W_{VY} with respect to S one recovers (4.25). Thus in the following we will consider W_{eff} as dependent on S although its "natural" variable should be Λ. Generically, all glueball fields S will be massive. However, the glueball mass scale Λ is understood as the lowest scale in the theory, assuming the existence of a mass gap. Therefore, it makes sense to use as effective superpotential W_{eff} as a function of the glueballs. We can always recover the low energy nonperturbative superpotential, that describes the physics of the discrete vacua, by just integrating out the glueball.

4.5. The Generalized Konishi Anomaly

The tool we need for computing the effective glueball superpotential, and in general for analyzing the theory just above the mass gap, is the Konishi anomaly. We will describe this method in the case of adjoint SQCD with superpotential (4.1). The Konishi anomaly [13] is an anomaly for the current

$$J = \mathrm{Tr} X^\dagger e^{ad\, V} X, \tag{4.27}$$

which generates the infinitesimal rescaling of the chiral field, $\delta X = \epsilon X$. Here $ad\, V$ means the adjoint representation, $(ad\, V\ X)^i{}_j = [V, X]^i_j$. It can be computed by any of the standard techniques: point splitting, Pauli-Villars regularization (since our model is non-chiral), anomalous variation of the functional measure (at one loop) or simply by computing Feynman diagrams. The result is a superfield generalization of the familiar $U(1) \times SU(N)^2$ mixed chiral anomaly for the fermionic component of X; in the theory with zero superpotential,

$$\bar{D}^2 J = \frac{\hbar}{32\pi^2} \mathrm{Tr}_{adj}\ (ad\, W_\alpha)(ad\, \dot{W}^\alpha) \tag{4.28}$$

where the trace is taken in the adjoint representation. When a tree level superpotential is present, the transformation $\delta X = \epsilon X$ is not a symmetry of the theory, so we have to add the classical variation. Evaluating this trace and adding the classical term present in the theory with superpotential, we obtain

$$\bar{D}^2 J = \mathrm{Tr} X \frac{\partial W(X)}{\partial X} + \hbar \frac{N}{16\pi^2} \mathrm{Tr} W_\alpha W^\alpha - \frac{\hbar}{16\pi^2} \mathrm{Tr} W_\alpha \mathrm{Tr} W^\alpha . \tag{4.29}$$

One way to see why this combination of traces appears here is to check that the diagonal $U(1)$ subgroup of the gauge group decouples.

We now take the expectation value of this equation. Since the divergence $\bar{D}^2 J$ is a $\bar{Q}^{\dot{\alpha}}$-commutator, it must have zero expectation value in a supersymmetric vacuum. Furthermore, we can use (4.5) and $\langle \mathrm{Tr} W_\alpha \rangle = 0$ to see that the last term is zero. Thus, we infer that

$$\left\langle \mathrm{Tr} X \frac{\partial W(X)}{\partial X} \right\rangle = -2\hbar \frac{N}{32\pi^2} \left\langle \mathrm{Tr} W_\alpha W^\alpha \right\rangle .$$

In general, this anomaly receives higher loop contributions, which are renormalization scheme dependent and somewhat complicated. However, one can see without detailed computation that these contributions can all be written as non-chiral local functionals [12]. Furthermore the correction must vanish for $g_k = 0$, so inverse powers of the couplings cannot appear. One might expect nonperturbative corrections, involving the dynamical scale Λ. However, such corrections can be excluded by considering the algebra of chiral transformations $\delta X = \sum_n \epsilon_n X^{n+1}$, a partial Virasoro algebra, showing that it can undergo no quantum corrections, and using this algebra to constrain the anomalies, along the lines of the Wess-Zumino consistency condition for anomalies [35]. This implies that, for purposes of computing the chiral ring and effective superpotential, the result (4.29) is exact.

Readers familiar with matrix models will notice the close similarity between the Konishi anomaly and the L_0 Virasoro constraint of the one bosonic matrix model. Indeed, the similarity is not a coincidence as the matrix model L_0 constraint has a very parallel origin: it is a Ward identity for the matrix variation $\delta M = \epsilon M$. In the matrix model, one derives further useful constraints from the variation $\delta M = \epsilon M^{n+1}$. This similarity suggests that we consider the most general variation in the chiral ring

$$\delta X = f(\Phi) \tag{4.30}$$

for a general holomorphic function f of all chiral superfields $\{\Phi\} = \{X, Q, \widetilde{Q}, W_\alpha\}$. Let us compute

$$\bar{D}^2 \langle J_f \rangle = \bar{D}^2 \langle \mathrm{Tr} X^\dagger e^{ad\, V} f(\Phi) \rangle. \tag{4.31}$$

Let us first do this at zeroth order in the couplings g_k (except that we assume a mass term); we will then argue that holomorphy precludes corrections depending on these couplings.

At zero coupling, the one loop contributions to (4.31) come from graphs involving $X^\dagger$ and a single X in $f(\Phi)$. In any one of these graphs, the other appearances of X and W_α in f are simply spectators. In other words, given

$$A_{ij,kl} \equiv \bar{D}^2 \langle X^\dagger_{ij} e^{adV} X_{kl} \rangle,$$

the generalized anomaly at one loop is

$$\bar{D}^2 \langle \mathrm{Tr} X^\dagger e^{ad\, V} f(\Phi) \rangle = \sum_{ijkl} A_{ij,kl} \frac{\partial f(\Phi)_{ji}}{\partial X_{kl}}. \tag{4.32}$$

The computation of $A_{ij,kl}$ is the same as the computation of (4.28), with the only difference being that we do not take the trace. Thus

$$A_{ij,kl} \equiv \frac{\hbar}{32\pi^2} \left[W_\alpha, [W^\alpha, e_{lk}]\right]_{ij}$$

where e_{lk} is the basis matrix with the single non-zero entry $(e_{lk})_{ij} = \delta_{il}\delta_{jk}$.

Substituting this in (4.32) and adding the classical variation, we obtain the final result

$$\bar{D}^2 J_f = \mathrm{Tr} f(\Phi) \frac{\partial W(X)}{\partial X} + \frac{\hbar}{32\pi^2} \sum_{i,j} \left[W_\alpha, \left[W^\alpha, \frac{\partial f}{\partial X_{ij}} \right] \right]_{ji}. \tag{4.33}$$

Finally, one can show that this result cannot receive other perturbative (or nonperturbative) corrections in the coupling, by holomorphy and symmetry, just as for the standard Konishi anomaly [13]. Taking expectation values, we obtain the Ward identities

$$\left\langle \mathrm{Tr} f(\Phi) \frac{\partial W}{\partial X} \right\rangle = -\frac{\hbar}{32\pi^2} \left\langle \sum_{i,j} \left(\left[W_\alpha, \left[W^\alpha, \frac{\partial f(\Phi)}{\partial X_{ij}} \right] \right] \right)_{ij} \right\rangle . \tag{4.34}$$

4.6. The Hyperelliptic Curve

By making use of the Konishi anomaly (4.34) we now compute the exact vacuum expectation values of the operators in (4.10). By solving for the first operator $R(z)$ we find the $\mathcal{N}=1$ curve of the gauge theory. The variation

$$\delta X_{ij} = f_{ij}(X, W_\alpha) = -\frac{1}{32\pi^2} \left(\frac{W_\alpha W^\alpha}{z - X} \right)_{ij} . \tag{4.35}$$

gives the anomaly equation

$$\hbar \left\langle R(z)\ R(z) \right\rangle = \left\langle \mathrm{Tr}\left(V'(X) R(z) \right) \right\rangle .$$

Since expectation values of products of gauge invariant chiral operators factorize, as expressed in (4.5), we get $\hbar \langle R(z) \rangle^2 = \langle \mathrm{Tr}\,(V'(X)R(z)) \rangle$. Thus, both sides have been expressed purely in terms of the vacuum expectation values $\langle \mathrm{Tr} W_\alpha W^\alpha X^k \rangle$, allowing us to write a closed equation for $R(z)$.

If we introduce

$$f(z) = -4 \left\langle \mathrm{Tr} W_\alpha W^\alpha \frac{V'(z) - V'(X)}{z - X} \right\rangle ,$$

which is a degree $n-1$ polynomial $f(z) = f_0 + ... + f_{n-1} z^{n-1}$, then we rewrite our equation in the form

$$\hbar R(z)^2 = V'(z) R(z) + \frac{1}{4} f(z) \tag{4.36}$$

We can use a Laurent series notation to rexpress (4.36) by noting that

$$-\frac{1}{32\pi^2} \mathrm{Tr} \frac{V'(X) W_\alpha W^\alpha}{z - \Phi} = [V'(z) R(z)]_-$$

where the notation $[F(z)]_-$ means to drop the non-negative powers in a Laurent expansion in z, *i.e.*

$$[z^k]_- = \begin{cases} z^k & \text{for } k < 0 \\ 0 & \text{for } k \geq 0 \end{cases}.$$

Using this notation, we can write the Ward identity as

$$\hbar R(z)^2 = [V'(z)R(z)]_-, \tag{4.37}$$

and $f(z) = -4\,[V'(z)R(z)]_+$. In other words, the role of $f(z)$ in (4.36) is just to cancel the polynomial part of the right hand side.[11]

The solution to (4.36) is

$$2\hbar R(z) = V'(z) - \sqrt{V'(z)^2 + \hbar f(z)}. \tag{4.38}$$

This solution is parameterized by the n coefficients in $f(z)$. We see that $R(z)$ has cuts in the complex z plane. In the semiclassical approximation of small f, each cut A_i is naturally associated with a zero a_i of $V'(z)$. If we set $\hbar = 0$ in the anomaly equation (4.36) and keep the expectation values of $\mathrm{Tr}W_\alpha W^\alpha X^k$ as nonvanishing parameters, we get the semiclassical expression for the resolvent

$$R(z)|_{cl} = -\frac{f(z)}{4V'(z)}. \tag{4.39}$$

Therefore, classically the resolvent lives on the z plane and has poles at the vacua a_i in (4.2).

Quantum mechanically, it is natural to analytically continue z through the cuts to a second sheet. The double–sheeted complex plane is a hyperelliptic

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[11] Note that (4.36) and (4.37) are the standard loop equations for the one bosonic matrix model. If the resolvent of the matrix field M is defined by $R_M(z) = \frac{g_m}{\widehat{N}}\left\langle \mathrm{Tr}\frac{1}{z-M}\right\rangle$, then it obeys precisely the above Ward identities. One can actually prove a precise relation between gauge theory expectation values $\langle\ldots\rangle_{g.t.}$ and matrix model expectation values $\langle\ldots\rangle_{m.m.}$,

$$-\frac{1}{32\pi^2}\left\langle \mathrm{Tr}\frac{W_\alpha W^\alpha}{z-\Phi}\right\rangle_{g.t.} = \frac{g_m}{\widehat{N}}\left\langle \mathrm{Tr}\frac{1}{z-M}\right\rangle_{m.m.}.$$

Riemann surface Σ. Define $y(z) = V'(z) - 2R(z)$ and write (4.38) as an equation for a Riemann surface Σ

$$y^2 = V'(z)^2 + \hbar f(z). \tag{4.40}$$

This genus $n-1$ Riemann surface was introduced in [36]. We can naturally understand Σ as a double cover of the complex z–plane branched at the roots of $W'(z)^2 + \hbar f(z) = t_n^2 \prod_{i=1}^n (z - a_i^+)(z - a_i^-)$. There are n branch cuts; as above we denote as A_i a contour that circles around the i^{th} cut. In the weak coupling limit, each pair of branch points $a_i^{\pm}$ comes from the splitting of the classical pole at a_i. Most of the functions we will consider have singularities at infinity; if the points on Σ with $z = \infty$ are removed, the n cycles A_i with $i = 1, ..., n$ are independent. Instead of parameterizing Σ by the coefficients in f, we alternatively parameterize it by the variables

$$S_i = \oint_{A_i} R(z) dz \qquad i = 1, ..., n. \tag{4.41}$$

The convention for the contour integrals is that we always understand a factor of $\frac{1}{2\pi i}$ in the measure dz.

The branch of the square root in (4.38) is chosen such that for large z in the first sheet we recover the semiclassical answer

$$R(z) \approx -\frac{f(z)}{4V'(z)} = \frac{S}{z} + \mathcal{O}(1/z^2). \tag{4.42}$$

A simple contour deformation argument shows that $S = \sum_i S_i$. It is easy to see that the asymptotic limit in the second sheet is

$$R(z) = V'(z) + \mathcal{O}(1/z) = t_n z^n + \mathcal{O}(z^{n-1}). \tag{4.43}$$

Let us identify the variables S_i in (4.41) with the glueballs in each low energy SQCD block in (4.2). First of all, in the semiclassical limit, the definition (4.41) gives what we want. In the classical limit, to evaluate the integral, we set X to its vacuum value (4.2). X is a diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_N$ (which are equal to the a_i with multiplicity N_i). If M is any matrix and M_{xy}, $x, y = 1, \ldots, N$ are the matrix elements of M in this basis, then

$$\mathrm{Tr} M \frac{1}{z - X} = \sum_x \frac{M_{xx}}{z - \lambda_x}, \tag{4.44}$$

so

$$\frac{1}{2\pi i}\oint_{A_i} dz \mathrm{Tr} M \frac{1}{z-X} = \sum_x \frac{1}{2\pi i}\oint_{A_i} dz \frac{M_{xx}}{z-\lambda_x} = \sum_{\lambda_x \in C_i} M_{xx} = \mathrm{Tr}\, M P_i. \quad (4.45)$$

Here $\lambda_x \in A_i$ means that λ_x is inside the contour A_i, and P_i is the projector onto eigenspaces of X corresponding to eigenvalues that are inside this contour. In the classical limit, P_i is just the projector onto the subspace in which $X = a_i$. Hence the above definitions amount in the classical limit to

$$S_i = -\frac{1}{32\pi^2}\mathrm{Tr}\, W_\alpha W^\alpha P_i. \quad (4.46)$$

The formula (4.45) is still valid in perturbation theory around the classical limit, except that the projection matrix P_i might undergo perturbative quantum fluctuations. Although the projection matrix P_i can fluctuate in perturbation theory, the dimension of the space onto which it projects cannot fluctuate, since perturbation theory only moves eigenvalues by a bounded amount. Since the S_i have no quantum corrections, they are functions only of the low energy gauge fields and not of the t_k. In the following, we will differentiate W_{eff} with respect to the couplings at "fixed gauge field background"; this can be done simply by keeping S_i fixed. We can describe this intuitively by saying that modulo $\{\bar{Q}_{\dot{\alpha}},\ \}$, the fluctuations in P_i are pure gauge fluctuations, roughly since there are no invariant data associated with the choice of an N_i-dimensional subspace in $U(N)$.

4.7. The Complete Solution

We solve now for the other generators $M(z)$ and $T(z)$ in (4.10). We consider the theory with superpotential (4.1) with a meson deformation $m(z)$ diagonal in the flavors and of degree l. The variation

$$\delta Q^f = \frac{1}{z-X} Q^f,$$

gives the anomaly equation

$$\left[(M(z)m(z))_f^{f'}\right]_- = \hbar R(z)\delta_f^{f'} \quad (4.47)$$

and the polynomials $[Mm]_+$ depend on the classical vacuum we choose.

In the pseudoconfining vacuum (4.2), the fundamentals vanish and classically $M(z)_{cl} = 0$. So we choose $[Mm]_+$ such that M is regular on the first sheet also in the quantum theory

$$M(z) = \frac{\hbar R(z)}{m(z)} - \sum_{i=1}^{l} \frac{1}{m'(x_i)} \frac{\hbar R(x_i)}{z - x_i}, \tag{4.48}$$

where x_i are the roots of $m(z)$. With such a solution we find on the first sheet $\oint_{x_i} M(z) = 0$.

In the Higgs vacua (4.3) we have in general nonvanishing fundamentals. Let us introduce the notation that $r_i = 1$ if the corresponding x_i appears $\langle X \rangle$, while $r_i = 0$ otherwise. A set of $\{r_i\}$ for $i = 1 \ldots, l$, where deg $m(z) = l$, completely specifies the classical phase of the gauge theory. If we have L higgs eigenvalues $x_1, \ldots, x_L$ in the $\langle X \rangle$, then $\sum_{i=1}^{l} r_i = L$ and $M(z)_{cl}$ is given by

$$M(z)_{cl} = -\sum_{i=1}^{L} \frac{V'(x_i)}{m'(x_i)}. \tag{4.49}$$

and the gauge symmetry breaking is $U(N) \to \prod_{i=1}^{n} U(N_i)$ such that $\sum_{i=1}^{n} N_i + L = N$. Then we have to chose $[Mm]_+$ such that (4.49) is reproduced in the classical limit. The solution corresponding to this generic case is

$$M(z) = \hbar \frac{R(z)}{m(z)} - \sum_{i=1}^{L} \frac{r_i V'(x_i) + (1 - 2r_i)\hbar R(x_i)}{z - x_i} \frac{1}{m'(x_i)}, \tag{4.50}$$

where $r_i = 1$ if the corresponding x_i appears in the classical $\langle X \rangle$ while $r_i = 0$ if the corresponding x_i is not present. Note that $M(z)$ has a singularity with residue $r_i(2\hbar R(x_i) - V'(x_i))\frac{1}{m'(x_i)}$ at x_i on the first sheet and a singularity with residue $(r_i - 1)(2\hbar R(x_i) - V'(x_i))\frac{1}{m'(x_i)}$ at x_i on the second sheet. In particular, the singularities are on the second sheet if $r_i = 0$ and on the first sheet if $r_I = 1$. In the classical limit of zero $R(z)$, this expression coincides with our classical answer (4.49) on the first sheet if $r_i = 1$.

Let us solve now for $T(x)$ in (4.10). The variation

$$\delta X = \frac{1}{z - X}, \tag{4.51}$$

gives the anomalous Ward identity

$$[V'(z)T(z)]_- + \mathrm{tr}\,[m'(z)M(z)]_- = 2\hbar R(z)T(z), \tag{4.52}$$

where the trace is over flavor indices. Note that in this case we have a contribution from the fundamentals by $m'(z)$. In particular, the solution for $T(z)$ depends on (4.50). If we take the generic phase in which L colors are higgsed out of the total l roots of $m(z)$, just as in (4.50), then we find the generic solution as

$$T(z) = \sum_{i=1}^{l} \frac{1}{2(z-x_i)} - \sum_{i=1}^{L} \frac{y(x_i)}{2y(z)(z-x_i)} + \frac{c(z)}{y(z)} \tag{4.53}$$

where

$$c(z) = \left\langle \mathrm{Tr}\frac{V'(z)-V'(X)}{z-X} \right\rangle - \frac{1}{2}\sum_{i=1}^{L} \frac{V'(z)-V'(x_i)}{z-x_i} \tag{4.54}$$

is a polynomial of degree $n-1$.

For large z, the first term in (4.54) behaves as $Nt_n z^{n-1}$ since $\langle \mathrm{Tr}\mathbf{1}\rangle = N$. The second term behaves as $-\frac{1}{2}Lt_n z^{n-1}$. This is enough to determine the large z behavior of $T(z)$,

$$T(z) = \begin{cases} \frac{N}{z} + \mathcal{O}(\frac{1}{z^2}) & \text{in the first sheet} \\ \frac{l-N}{z} + \mathcal{O}(\frac{1}{z^2}) & \text{in the second sheet .} \end{cases} \tag{4.55}$$

For the pseudo-confining vacua, we expect $T(z)$ to be regular at the points x_i on the first sheet. This indeed follows from (4.53), as $y(x_i)/y(z) \to 1$ for $z \to x_i$. On the second sheet, since $y(x_i)/y(z) \to -1$ for $z \to x_i$, $T(z)$ has a pole, $T(z) \sim 1/(z-x_i)$ for $z \to x_i$. Finally, $c(z)$ can be determined by the requirements

$$\oint_{A_i} T(z)dz = N_i\ . \tag{4.56}$$

The N_i is the rank of the i–th low energy $U(N_i)$ SQCD, $N_i = \mathrm{Tr}P_i$ in the notations of (4.45). By analogous considerations as the ones for the glueballs S_i, this integer corresponds to its classical value, since the rank of the projector P_i cannot change.

4.8. Interpolating Between the Pseucodonfining and the Higgs Phase

Consider varying the parameters in $m(z)$ with fixed $V(X)$ and fixed S_i. The Riemann surface Σ is unchanged but the zeros x_i of $m(z)$ change. Let us start with M and T being regular on the first sheet, corresponding to the pseudoconfining vacuum (4.2). Then M and T have poles at x_i on the second sheet. Let us move then one of the higgs poles, say x_1, through one of the cuts, say A_1 from the second to the first sheet. After this, our solutions for M and T have poles at x_1, but now on the first sheet. This process has a simple physical interpretation. We started semiclassically with $\langle X \rangle$ whose eigenvalues, as shown in (4.2), are approximately equal to the roots a_i of V'; this corresponds to having all $r_i = 0$. In a semiclassical limit, the cut A_1 is near a_1. When x_1 passes through the cut, it is near a_1 and the solution (4.2) is near the solution (4.3) that describes the Higgs branch. At this stage, the strong quantum dynamics are important and a semiclassical treatment is not precise enough. Passing x_1 through the cut and taking it to be again large (or at least far away from all cuts), we may find ourselves in a Higgs branch with $r_1 = 1$. On this branch, M and T are expected to have poles on the first sheet. Thus, in a process in which x_1 moves through one of the cuts, a branch with $r_1 = 0$ is continuously transformed into a branch with $r_1 = 1$.

5. A_n AND ITS DUAL: THE QUANTUM THEORY

In this Chapter we will generalize the analysis of KSS [16] about Seiberg duality in one–adjoint SQCD with a deformation of the A_n superpotential. We summarized this duality in Chapter 4. This Chapter is based on [37]. Here we consider the most generic electric superpotential, obtained by adding to (3.5) a meson deformation

$$\mathrm{Tr}V(X) + \tilde{Q}_{\tilde{f}} m(X)^{\tilde{f}}_{f} Q^{f}, \tag{5.1}$$

where in the classical chiral ring the degree of the meson polynomial $m(z)$ is at most $n-1$. Let us recall the classical vacua that we analyzed in Section *4.1*. This electric theory presents two different kinds of vacua. In the first vacuum, that we denote as *pseudoconfining*, the fundamentals vanish and the adjoint acquires a vacuum expectation that drives the theory to a product of low energy $U(N_i)$ SQCD blocks such that $\sum_{i=1}^{n} N_i = N_c$. The other vacuum is called the *higgs* vacuum and is characterized by a nonvanishing classical vev for the fundamentals.[12] In this case the rank of the gauge group decreases. If we higgs L colors, then the low energy theory is still a product of $U(N_i)$ SQCD blocks, but now $\sum_{i=1}^{n} N_i = N_c - L$.

Our first analysis of the duality will focus on the map between the electric and magnetic classical vacua in both the pseudoconfining and the higgs phase. The magnetic dual of the theory (5.1) contains, in addition to the superpotential (3.10), a source term for the gauge singlets $\sum_{k=1}^{\deg m+1} m_k P_k$, where m_k are the coefficients of $m(z)$. The magnetic vacua will depend then on the details of the electric meson polynomial: each flavor appearing in $m(z)^{\tilde{f}}_{f}$ turns on a higgsed block in the magnetic adjoint $\langle Y \rangle$. In particular, we will study the magnetic vacuum corresponding to the electric higgs phase, characterized by a nonzero classical vev for the magnetic singlets P_j. In our classical solution, as we increase the higgsed directions in the electric theory, thus driving it to weaker coupling, the higgsed block in the magnetic theory decreases its rank, driving the dual theory to stronger coupling.

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[12] In presence of matter in the fundamental representation of the gauge group there is no phase transition between higgs and confining regimes and in the quantum theory one can continuously interpolate between them.

We will consider then the map between the chiral rings of the two quantum theories. Due to the presence of a large number of couplings in the tree level action (5.1), the study of the effective superpotentials by the conventional methods of holomorphy and symmetries is more involved in this case. Therefore, we found more convenient to analyze the quantum theory with the DV method, along the lines discussed above for SQCD. In particular, we will concentrate on the operators that generate the chiral ring

$$M(z) = \langle \tilde{Q} \frac{1}{z-X} Q \rangle, \qquad T(z) = \langle \mathrm{Tr} \frac{1}{z-X} \rangle.$$

We will follow the method of the Konishi anomaly, discussed in Section *4.5*, to solve explicitly for these operators as functions of the glueball superfield S and the couplings [18][19] and then we will integrate them to obtain the glueball effective superpotentials. By matching first the electric mesons with the magnetic singlets and then the two effective superpotentials, we will derive the relation between the two gauge groups $\bar{N}_c = nN_f - N_c$ as well as the scale matching (3.8) and the map between the electric and magnetic chiral ring operators. The low energy electric and magnetic theories will be both described by the same hyperelliptic Riemann surface $y^2 = V'(z)^2 + f(z)$, a double–sheeted cover of the plane, where the quantum deformation $f(z)$ is a degree $n-1$ polynomial. The pseudoconfining and higgs duality map will turn out to be rather different, though. In particular, in the electric pseudoconfining phase the magnetic anomaly equations are solved by the simple condition

$$m(a_i)\tilde{m}(a_i) = f(a_i), \tag{5.2}$$

for $i = 1, \ldots, n$, where a_i are the roots of $V'(z)$ and $\widetilde{m}(z)$ is the magnetic polynomial (3.10). This condition will ensure also the match of the electric and magnetic chiral rings and will reproduce the Konishi anomaly in each low energy SQCD block.

The DV method allows us to study also the rich analytic structure of the low energy effective theory. Even if the electric and magnetic theory have the same curve, the meromorphic functions $M(z)$ and $T(z)$ living on the curve have very different analytic structures on the two sides. We will picture their analytic behavior as follows. As we discussed in Chapter 4, according to [19], an higgs eigenvalue in the electric theory is seen as a pole of $M(z)$ on the first (semiclassical) sheet of the curve. As we will see, in the magnetic theory the corresponding

$\tilde{M}(z)$ will have $n-1$ poles on the first sheet. We can higgs twice the electric theory by bringing a second pole of $M(z)$ from the second (invisible) sheet into the first one. The magnetic theory behaves in two different ways depending on whether we higgs different electric flavors or several times the same flavor.[13] We will see that, in this latter case, the second electric higgsing corresponds in the magnetic theory to moving one of the $n-1$ poles away from the first into the second sheet.

Our main concern will be to compare electric and magnetic results at every stage of the computation. For this reason, we will tackle separately the two electric pseudoconfining and higgs vacua and in each of them we will match first the classical and then the quantum theories.

In section 1 we will sketch the idea of how to use the Konishi anomaly to study Seiberg duality. We do this in the example of SQCD.

In section 2 we will consider the electric pseudoconfining vacuum in presence of the generic deformation (5.1). We will find the corresponding classical magnetic solution and see that this is only valid for a small number of massive electric flavors, due to the presence of instanton effects in the broken magnetic gauge group. We will then study the quantum chiral rings of the dual pair by the DV method. First we will consider the case in which the electric meson superpotential is just a mass term, and we will show that in this case duality works exactly offshell. For a generic meson polynomial, instead, the solution (5.2) that we found is not exact offshell, but still it reproduces the usual Konishi anomaly in the low energy SQCD blocks and we believe it to hold onshell.

We will consider then the electric higgs phase in section 3 and follow the same steps of the previous section, first the classical and then the quantum analysis, gaining in this way a complete picture of duality in the different vacua. Even if the solution of the quantum theory in this case will be implicit, we will be able to sketch the analytic behavior of the magnetic resolvents when moving the poles between the two sheets in the electric theory.

In section 4 we will consider the case of cubic tree level superpotential to see how duality works in a specific example and finally, in section 5, we will speculate about some questions raised by our analysis.

[13] At most we can higgs $n-1$ color directions on the same flavor, corresponding to the degree of the meson polynomial $m(z)$.

In the appendices we postponed some details of our computation of the effective superpotential of section 2, which is different from the one in [19]. In the last appendix we show a classical magnetic solution that generalizes the ones in section 3 for the higgs phase.

5.1. Seiberg Duality in SQCD Revisited

Before addressing Seiberg duality in our favourite A_n theory, let us see how these Konishi anomaly equations work in the prototypical example of electric–magnetic duality, namely Seiberg duality in SQCD with $U(N_c)$ gauge group and N_f flavors of quarks Q^f and antiquarks $\tilde{Q}_{\tilde{f}}$ [2], that we discussed at length in Chapter 2. If we want to apply DV, the theory has to be massive and we need a tree level mass term

$$W_{tree} = m\tilde{Q}_f Q^f. \tag{5.3}$$

Even if classically the mesons vanish, at the quantum level their expectation value is set by the Konishi anomaly to $\langle \tilde{Q}_f Q^f \rangle = N_f S/m$, where S is the glueball superfield [13]. The effective glueball superpotential is recovered by integrating this exact expectation value with respect to the corresponding coupling. We have to add also possible coupling independent terms, that in this case are the Veneziano–Yankielowicz superpotential $N_c S(1-\log S)$ and the one–loop exact renormalization of the gauge field $(3N_c - N_f)S\log\Lambda$, obtaining the glueball effective superpotential

$$W_{eff} = S\left(\log\frac{m^{N_f}\Lambda^{3N_c-N_f}}{S^{N_c}} + N_c\right). \tag{5.4}$$

The magnetic dual of this theory is a supersymmetric gauge theory with $U(\bar{N}_c)$ gauge group, N_f flavors of magnetic quarks q_f and antiquarks $\widetilde{\tilde{q}^f}$ and $N_{\tilde{f}}^2$ gauge singlets $P^f_{\tilde{f}}$, that represent the electric mesons. The classical magnetic superpotential corresponding to (5.3) is $\overline{W}_{tree} = \frac{1}{\mu}P\tilde{q}q + \overline{m}\mathrm{tr}P$. In the magnetic theory we have two basic equations to solve. The first is the singlet equation of motion, which completely fix the magnetic mesons to $\langle \tilde{q}q \rangle = -\mu\overline{m}$. Since the singlets are not coupled to the gauge fields, their equations of motion are exact in the chiral ring of the quantum theory. Then we have the Konishi anomaly, that sets

$\langle P\rangle = -\bar{S}/\overline{m}$. The effective glueball superpotential is then computed as in the electric case and we get

$$\overline{W}_{eff} = \bar{S}\left(\log\frac{\bar{S}^{N_f-\bar{N}_c}\tilde{\Lambda}^{3\bar{N}_c-N_f}}{(-\overline{m}\mu)^{N_f}} + (\bar{N}_c - N_f)\right). \tag{5.5}$$

To find the duality map, we first match the electric mesons with the magnetic singlets, since they are directly related by a Legendre transform, and we see that $S = -\bar{S}$ and $m = \overline{m}$. Then we match the effective glueball superpotential and find the relation between the gauge groups $\bar{N}_c = N_f - N_c$ as well as the scale matching relation $\Lambda^{3N_c-N_f}\tilde{\Lambda}^{3\bar{N}_c-N_f} = (-)^{N_f-N_c}\mu^{N_f}$.

However, the DV method is not really necessary in this case.[14] We can easily obtain the onshell expectation values of chiral operators by studying the nonperturbative low energy superpotentials of electric and magnetic theories, without ever introducing the glueball superfield. On the other hand, we can also integrate in the glueball superfield to obtain directly the glueball effective superpotential. On the electric side, the low energy theory is just pure $U(N_c)$ SYM, whose nonperturbative superpotential is $W_{low} = N_c(\Lambda_{low}^{3N_c})^{\frac{1}{N_c}}$, which is responsible for gaugino condensation. One first matches the low energy scale $\Lambda_{low}^{3N_c} = m^{N_f}\Lambda^{3N_c-N_f}$ and then just integrate in the glueball to obtain directly (5.4). On the other side, in the magnetic theory the singlet equations of motion force all the flavors to be higgsed, thus the low energy theory is pure SYM with gauge group $U(\bar{N}_c - N_f)$, whose low energy superpotential is the same as the electric one but with the appropriate magnetic quantities instead. By matching the magnetic scales $\tilde{\Lambda}_{low}^{3(\bar{N}_c-N_f)} = \tilde{\Lambda}^{3\bar{N}_c-N_f}/\langle\tilde{q}q\rangle^{N_f}$ and again integrating in the glueball we obtain (5.5).

5.2. The Electric Pseudoconfining Phase

In this section we will see how electric–magnetic duality works in the electric pseudoconfining phase. Our notations will be as follows. In the classical analysis, we will always use the electric couplings to describe the magnetic theory, assuming that we know the duality map. In the analysis of the quantum theory, we

[14] The matrix model approach to Seiberg duality has been first used in [38].

will overline the magnetic couplings to avoid possible confusion and then derive the duality map.

The Classical Vacua

The Electric Theory

Let us set the stage for our calculations. We will consider an $\mathcal{N}=1$ supersymmetric gauge theory with $U(N_c)$ gauge group, that we will call *electric.* The matter content consists of N_f flavors of quarks Q^f and antiquarks $\tilde{Q}_{\tilde{f}}$ and a chiral superfield X in the adjoint representation of the gauge group. We will at first let the theory flow to its infrared superconformal fixed point. Then we will turn on the generic tree level superpotential

$$\begin{aligned} W_{el} =& \mathrm{Tr} V(X) + \tilde{Q}_{\tilde{f}} m(X)^{\tilde{f}}_{f} Q^f, \\ V'(z) =& \sum_{i=1}^{n} t_i z^i, \\ m(z)^{\tilde{f}}_{f} =& \sum_{k=1}^{l+1} (m_k)^{\widetilde{f}}_{f} z^{k-1}, \end{aligned} \tag{5.6}$$

which is irrelevant in the UV but becomes relevant in the infrared. It will be useful to parameterize the adjoint polynomial as $V'(z) = t_n \prod_{i=1}^{n}(z-a_i)$ in terms of its roots. We denote the roots of the meson polynomial $m(z)$ as x_k, for $k=1,\ldots,l$. The degree of $m(z)$ is $l \leq n-1$, since higher mesons are trivial in the classical chiral ring, that contains the following operators

$$\mathrm{Tr}X^j, \qquad \tilde{Q}X^{j-1}Q, \tag{5.7}$$

for $j=1,\ldots,n$, as well as operators of the kind $\mathrm{Tr}W_\alpha X^j$ and $\mathrm{Tr}W_\alpha W^\alpha X^j$. However, $W_\alpha Q^f$ and $\tilde{Q}_{\tilde{f}} W_\alpha$ are not in the chiral ring. Also, since the gauge group is $U(N)$ rather than $SU(N)$ we do not include "baryonic operators". Our main attention will be focused on the following chiral operators, that generate the chiral ring

$$\begin{aligned} R(z) &= -\frac{1}{32\pi^2}\mathrm{Tr}\frac{W_\alpha W^\alpha}{z-X}, \\ M^f_{\tilde{f}}(z) &= \tilde{Q}_{\tilde{f}}\frac{1}{z-X}Q^f, \\ T(z) &= \mathrm{Tr}\frac{1}{z-X}, \\ w_\alpha(z) &= \frac{1}{4\pi}\mathrm{Tr}\frac{W_\alpha}{z-X}. \end{aligned} \tag{5.8}$$

We will set to zero in the following $w_\alpha(z)$ since its duality properties are automatic and does not constrain the other results.

This theory exhibits two kinds of classically distinct vacua, that we will call pseudoconfining and higgs vacua. In this section we will be concerned only with the former and leave the analysis of the higgs vacuum to the section 5.3. The pseudoconfining vacua are characterized by vanishing expectation values for the fundamentals

$$\begin{aligned} X = & \begin{pmatrix} a_1 & & \\ & \cdot & \\ & & a_n \end{pmatrix} \\ \tilde{Q}_{\tilde{f}} = & 0, \quad Q^f = 0, \end{aligned} \tag{5.9}$$

where each a_i has multiplicity N_i such that $\sum_i N_i = N_c$. The reason why these are called "pseudoconfining" rather than "confining" vacua is that, due to the presence of fields in the fundamental representation of the gauge group, there is no phase transition between these vacua and the higgs ones and in the quantum theory they are continuously connected. At low energy the theory consists of a set of decoupled $U(N_i)$ SQCD with N_f flavors, while the adjoint has been integrated out.[15] The rank of the gauge group does not decrease along this flow.

The Magnetic Theory

The magnetic theory corresponding to (5.6) is again an $\mathcal{N}=1$ supersymmetric gauge theory with gauge group $U(\bar{N}_c)$ and N_f flavors of dual quarks q_f and antiquarks $\tilde{q}^{\tilde{f}}$.[16] We also add a chiral superfield Y in the adjoint and N_f^2 gauge singlets $(P_j)^{f}_{\tilde{f}}$, for $j = 1, \ldots, n$. We first let this theory flow to its interacting superconformal fixed point, then we add the following superpotential

$$\begin{aligned} W_{mag} = & - \mathrm{Tr} V(Y) + \tilde{q} \tilde{m}(P, Y)\, q + \sum_{j=1}^{l+1} m_j P_j, \\ \tilde{m}(z) = & \frac{1}{\mu^2} \sum_{k=1}^{n} t_k \sum_{j=1}^{k} P_j z^{k-j}, \end{aligned} \tag{5.10}$$

[15] If we allow for double roots in $V'(z)$ we end up with adjoint SQCD with a cubic tree level superpotential for the low energy adjoint superfield. For simplicity we will consider superpotential with only single roots, though.

[16] Note that $\tilde{q}^{\tilde{f}}$ is in the fundamental representation of the flavor symmetry group, while q_f is in the antifundamental.

where we suppressed the flavor indices and $V(z)$ and the m_k's are the electric ones in (5.6).

We introduced the degree $n-1$ polynomial $\tilde{m}(P,z)$, which can be conveniently cast in the form[17]

$$\tilde{m}(z) = \frac{1}{\mu^2}\oint_A d\zeta \frac{V'(\zeta)-V'(z)}{\zeta - z}P(\zeta), \tag{5.11}$$

where A is a contour that sorrounds all the roots of $V'(z)$. We introduced also a meromorphic function that collects for the gauge singlets

$$P(z) = P_1 z^{-1} + \ldots + P_n z^{-n}, \tag{5.12}$$

and note that the last term in the superpotential (5.10) can be rewritten as

$$\oint_A m(z)P(z). \tag{5.13}$$

Moreover, by inverting (5.11) we find that the general expression for the singlets is fixed by $\tilde{m}(z)$ to

$$P(z) = \mu^2 \left[\frac{\tilde{m}(z)}{V'(z)}\right]_{-n}, \tag{5.14}$$

meaning that we take the Laurent expansion up to $\mathcal{O}(z^{-n})$. The equations of motion for the singlets are

$$\begin{aligned} &\sum_{i=j}^{n} t_i \tilde{q} Y^{i-j} q = -\mu^2 m_j, \qquad j=1,\ldots,l, \\ &\sum_{i=j}^{n} t_i \tilde{q} Y^{i-j} q = 0, \qquad j=l+1,\ldots,n. \end{aligned} \tag{5.15}$$

Therefore, the classical chiral ring of this theory does not contain the mesons $\tilde{q}Y^{j-1}q$, which are in fact replaced by the n singlets P_j. We already see here

[17] We will always understand a factor $\frac{1}{2\pi i}$ in the measure of the contour integrals.

that the analysis of the chiral ring in this theory will be slightly different than the usual electric one. We will be still interested in the following chiral operators

$$\begin{aligned}
\tilde{R}(z) &= -\frac{1}{32\pi^2}\mathrm{Tr}\frac{W_\alpha W^\alpha}{z-Y}, \\
\widetilde{M_f^f}(z) &= \tilde{q}^{\tilde{f}}\frac{1}{z-Y}q_f, \\
\tilde{T}(z) &= \mathrm{Tr}\frac{1}{z-Y}.
\end{aligned} \tag{5.16}$$

We already set to zero the magnetic w_α generator analogous to the one in (5.8).

Now we want to look at the magnetic vacuum corresponding to the pseudo-confining electric one in (5.9). This phase is characterized by a vanishing classical expectation value for the gauge singlets P_j, since they represent to the electric mesons. We have to satisfy the singlet equations of motion (5.15), as well as the adjoint ones $V'(Y) = 0$. Consider at first the simple case in which only the last flavor appears in the electric meson superpotential (5.6), i.e. $m(z)_f^{\tilde{f}} = m(z)_{N_f}^{N_f}$. Correspondingly, the right hand side of the singlet equations of motion (5.15) has nonvanishing entries only along these flavor directions. Let us denote

$$b_1 = \left(-\frac{m_1\mu^2}{t_n}\right)^{\frac{1}{n+1}}, \tag{5.17}$$

which has the dimension of a mass, and introduce the following bra–ket notation

$$|i\rangle \leftrightarrow i^\alpha = \delta_i^\alpha,$$

where a ket corresponds to a field in the fundamental representation of the gauge group and a bra to a field in the antifundamental. We introduce also the shift operator acting on the first n entries

$$R_n|i\rangle = \begin{cases} |i-1\rangle & i = 2,\dots,n, \\ 0 & \text{otherwise.} \end{cases} \tag{5.18}$$

In this notations, the classical expectation value for the adjoint can be represented in block diagonal form

$$Y = \mathrm{diag}(Y_{N_f}, Y_{a_1}, \dots, Y_{a_n}), \tag{5.19}$$

where

$$\begin{aligned} Y_{N_f} =& |1\rangle\langle v_{N_f}| + b_1 R_n, \\ |v_{N_f}\rangle =& -\sum_{k=1}^{n-1} \frac{t_{n-k}}{t_n} b_1^{1-k} |k\rangle. \end{aligned} \tag{5.20}$$

The first $n \times n$ block (5.20) reads

$$Y_{N_f} = \begin{pmatrix} -\frac{t_{n-1}}{t_n} & b_1 & 0 & & & \\ -\frac{t_{n-2}}{t_n}\frac{1}{b_1} & 0 & b_1 & . & & \\ . & & 0 & . & . & \\ . & & & . & b_1 & 0 \\ -\frac{t_1}{t_n}\frac{1}{b_1^{n-2}} & 0 & . & . & 0 & b_1 \\ 0 & & & & . & 0 \end{pmatrix} \tag{5.21}$$

The blocks $Y_{a_1}, \ldots, Y_{a_n}$ in (5.19) correspond to the electric pseudoconfining eigenvalues and are given by

$$Y_{a_i} = \text{diag}(a_i, \ldots, a_i). \tag{5.22}$$

Each Y_{a_i} block has rank $\tilde{N}_i = N_f - N_i - 1$. Note that this is different from the usual relation $\overline{N}_i = N_f - N_c$ that we have in each low energy Seiberg block when the meson polynomial $m(z)$ is switched off. We can check that with this solution the correct magnetic rank is reproduced. Since $\sum_i N_i = N_c$, we have

$$\sum_{i=1}^{n}(N_f - N_i - 1) + n = nN_f - N_c = \overline{N}_c. \tag{5.23}$$

The magnetic quarks are all vanishing except the last flavor

$$\begin{aligned} |\tilde{q}^{N_f}\rangle =& b_1|1\rangle, \\ |q_{N_f}\rangle =& \sum_{i=1}^{l+1} b_1^i \frac{m_i}{m_1} |n+1-i\rangle, \end{aligned} \tag{5.24}$$

whose vevs are along the first n color directions, in order to sandwich the first block Y_{N_f} in the adjoint and satisfy the singlet equations of motion. Note that, in the simplest case in which $V'(z) = t_n z^n$ and $m(z)_{\tilde{f}}^{f} = m_{N_f}^{N_f}$, this solution reduces to the usual KSS solution [15][16]. We can also write down the classical expressions for the generators (5.16) for this particular solution. The resolvent $\tilde{R}(z)$ vanishes while

$$\begin{aligned}\tilde{M}_{cl}(z)_{N_f}^{N_f} &= -\,\mu^2\frac{m(z)}{V'(z)},\\ \tilde{T}_{cl}(z) &= \frac{d}{dz}\ln\frac{V'(z)}{z}+\frac{1}{z}+\sum_{i=1}^{n}\frac{\bar{N}_i}{z-a_i},\end{aligned}\tag{5.25}$$

where $m(z)$ is the electric meson polynomial. Moreover, at large z we find $\tilde{T}_{cl}\sim\bar{N}_c/z$ since $\sum_i\bar{N}_i=\bar{N}_c-n$. In the electric pseudoconfining phase, the magnetic singlets vanish classically, thus we find that in the classical chiral ring $\tilde{m}(z)=0$.

The low energy theory described by (5.20) can be studied in two steps, following the KSS procedure. First, the $n\times n$ block (5.21) higgses the theory down to

$$U(\bar{N}_c)\to U(\bar{N}_c-n),$$

and note that $\bar{N}_c-n=n(N_f-1)-N_c$ as expected from the electric theory, where we integrated out the last massive flavor. At this stage, $\tilde{q}_\alpha^{N_f}$, $q^\alpha_{N_f}$, Y^α_m, Y^s_α for $\alpha=n+1,\dots,\bar{N}_c$ and $m=2,\dots,n$, $s=1,\dots,n-1$ conspire to join n massive vector superfields in the fundamental representation of the low energy gauge group $U(\bar{N}_c-n)$ with mass squared b_1^2. But then as we decompose the adjoint we find that the higgsed flavor gets replaced by a new flavor Y_1^α, Y_α^n for $\alpha=n+1,\dots,\bar{N}_c$, so the number of flavors does not decrease here. Secondly, the superpotential for the adjoint generates a mass term for this new flavor. Only the leading term $\mathrm{Tr}Y^{n+1}$ contributes

$$\frac{t_n}{n+1}\mathrm{Tr}Y^{n+1}=t_nY_\alpha^\gamma\langle Y^{n-1}\rangle^\alpha_\beta Y^\beta_\gamma=t_nb_1^{n-1}Y_1^\gamma Y_\gamma^n.\tag{5.26}$$

The number of flavors effectively decreases by one unit also in the magnetic theory. The singlets $(P_j)^i_{N_f}$ and $(P_j)^{N_f}_i$, $i=1,\dots,N_f$ become also massive. Now we can set the massive fields to the solution of their equations of motion and integrate them out. The effective superpotential at a scale below b_1 is $\mathrm{Tr}V(\widehat{Y})+\widehat{\tilde{q}}\tilde{m}(\widehat{Y})\widehat{q}$, where the hatted fields transform in the representation of the low energy gauge group $U(\bar{N}_c-n)$ and we are left with N_f-1 flavors. The matching of the scales goes as follows

$$\tilde{\Lambda}^{2\bar{N}_c-N_f}_{\bar{N}_c,N_f}=\frac{m_1\mu^2}{t_n^2}\tilde{\Lambda}^{2(\bar{N}_c-n)-(N_f-1)}_{\bar{N}_c-n,N_f-1}.\tag{5.27}$$

We can use the relation between the scales (3.8) and the electric scale matching and find

$$\Lambda_{N_c,N_f-1}^{2N_c-(N_f-1)}\tilde{\Lambda}_{\bar{N}_c-n,N_f-1}^{2(\bar{N}_c-n)-(N_f-1)} = \left(\frac{\mu^2}{t_n^2}\right)^{N_f-1}. \tag{5.28}$$

If we keep flowing to energies below the a_i of (5.22) we will find the usual product of the magnetic theories dual to each electric SQCD block.

This solution can be generalized to the case in which the electric meson polynomial has nonvanishing entries on different flavors. If also the one but last flavor appears in the electric superpotential, i.e. $m(z)_{\tilde{f}}^{f} = m(z)_{N_f}^{N_f} + p(z)_{N_f-1}^{N_f-1}$ with $\deg m(z)_{N_f}^{N_f} = l$ and $\deg p(z)_{N_f-1}^{N_f-1} = l'$, then the new flavor contributes an additional $n \times n$ higgsed block

$$Y = \mathrm{diag}(Y_{N_f}, Y_{N_f-1}, Y_{a_1}, \ldots, Y_{a_n}), \tag{5.29}$$

where the first block is always (5.20) and the second block is similar but with the substitution $b_1 \to b_1'$. For what concerns the magnetic quarks, in addition to (5.24) also the one but last flavor is higgsed as follows

$$\begin{aligned} |\tilde{q}^{N_f-1}\rangle =& b_1'|n+1\rangle, \\ |q_{N_f-1}\rangle =& \sum_{i=1}^{l} \frac{p_i}{p_1}(b_1')^i|2n+1-i\rangle. \end{aligned} \tag{5.30}$$

The magnetic gauge group is now higgsed down to $U(\bar{N}_c - 2n)$ and in each low energy Seiberg block we have the correspondence $\bar{N}_i = N_f - N_i - 2$.

This classical analysis can be pushed further until we hit the following bound on the number of massless quarks

$$N_f \geq \frac{N_c}{n}. \tag{5.31}$$

Suppose in fact that in the meson polynomial $m(z)$ there appear $N_f - N_c/n$ flavors so that we saturate the bound (5.31). Then the magnetic gauge group would be completely higgsed and we will see no low energy SQCD blocks. The solution to this problem is that as the magnetic gauge group is completely higgsed, a new superpotential is triggered by instantons in the broken gauge group and the singlet equations of motion get modified.

The Electric Stability Bound and Magnetic Instantons

Let us briefly describe the stability bound on the electric theory [15]. We have seen that a superpotential (5.6) drives the theory to a product of low energy decoupled SQCD, breaking the gauge group down to $\prod_{i=1}^{n} U(N_i)$. Consider each $U(N_i)$ SQCD block separately: it is well known that this gauge theory admits a stable vacuum iff the number of flavors is larger than the number of colors [23], i.e. $N_f \geq N_i \; \forall i$. Therefore the original theory admits a stable vacuum iff the bound (5.31) is satisfied.[18]

When we completely break the magnetic gauge group, the weak coupling analysis we carried out is no longer valid due to the presence of instantons. A well known example is $SU(N_c)$ SQCD with $N_f = N_c + 2$ flavors and its magnetic dual with gauge group $SU(N_f - N_c) = SU(2)$ [2]. If we add a mass term for the last electric flavor, the magnetic gauge group gets completely higgsed, so that instantons in the broken $SU(2)$ generate a superpotential term. By passing to the electric variables, one can see that the sum of the magnetic tree level and instanton superpotentials reproduces the usual nonperturbative superpotential of SQCD with $N_f = N_c + 1$.

We would like to generalize this issue to our case of adjoint SQCD and check whether we can generate an instanton term in the magnetic superpotential when approaching the stability bound. We consider the case in which $\bar{N}_c = n + 1$, i.e. we have $N_f = N_c/n + 1 + 1/n$ flavors. At this point, we are just above the bound (5.31) and our classical analysis still makes sense. We further specialize to $n = 2$ and take the electric deformation to be just $t_2 \text{Tr} X^3$.[19] Note that we do not include a mass term for the adjoint. We further add a mass term for the last flavor. Our electric superpotential reads

$$W_{el} = \frac{t_2}{3} \text{Tr} X^3 + m \tilde{Q}_{N_f} Q^{N_f}. \tag{5.32}$$

[18] The theory is always stable if all the flavors are massive, which is the case we will consider when solving the quantum theory.

[19] We consider the $\bar{N}_c = n+1$ rather than the $\bar{N}_c = n$ case because in the latter the magnetic deformation $\text{Tr} Y^{n+1}$ is trivial in the classical chiral ring and the analysis of the instantons is more involved due the presence of additional flat directions.

The magnetic theory is a $U(3)$ gauge theory defined by the superpotential

$$W_{mag} = -\frac{t_2}{3}\mathrm{Tr}Y^3 + \frac{t_2}{\mu^2}(M_1\tilde{q}Yq + M_2\tilde{q}q) + m(M_1)^{N_f}_{N_f}. \tag{5.33}$$

The classical solution (5.20) still applies and the magnetic gauge group is higgsed down to $U(1)$. Since the low energy dynamics is abelian, we may expect instanton effects in the broken gauge group. One can perform a standard analysis of the zero modes in the instanton 't Hooft vertex $\tilde{\Lambda}^{2\tilde{N}_c-N_f}\lambda^{2\tilde{N}_c}\psi_Y^{2\tilde{N}_c}\psi_{\tilde{q}}^{N_f}\psi_q^{N_f}$, where ψ_Φ denotes the second component of the chiral superfield Φ. By using the interactions in the tree level action, such as the scalar–fermion–gaugino D–term vertex as well as the superpotential couplings in (5.33), one can read out from this vertex the following contribution to the superpotential

$$W_{inst} = \frac{t_2^{2N_f+3}}{m^2\mu^{4N_f}}(\Lambda^{6-N_f})^2 \det\widehat{M}_2(\widehat{M}_1 \mathrm{cof}\widehat{M}_2), \tag{5.34}$$

where $\mathrm{cof}M \equiv M^{-1}\det M$ and the hatted fields transform in the $SU(N_f-1)$ low energy flavor symmetry group [39]. Note that this is the contribution by a two–instanton. As explained in [40], this is due to the absence of the mass coupling for the adjoint, that would have been an overall factor in the one–instanton term.

We see that, when hitting the bound (5.31), the classical solution (5.20) is no longer valid, due to the presence of the instanton term that couples the singlets. We can also translate this superpotential to the electric variables by using the scale matching relation (3.8) and the electric low energy scale $\Lambda_{low}^{2N_c-(N_f-1)} = m\Lambda^{2N_c-N_f}$, obtaining the electric superpotential

$$W_{nonpert} = \frac{\det M_2(M_1\mathrm{cof}M_2)}{t_2^{2N_f}(\Lambda^{2N_c-N_f})^2}, \tag{5.35}$$

In this expression we dropped the hats and the subscript on the scale. It is to be understood as the superpotential of a theory with N_c colors and $N_f = (N_c+1)/2$ flavors. The magnetic instanton superpotential is seen on the electric side as a nonperturbative superpotential arising from strong coupling effects [39], in a very similar way to ordinary SQCD with $N_f = N_c + 1$ flavors.

The Chiral Ring

In this section we will find the chiral ring of the quantum theory by solving the generalized Konishi anomaly equations [12][19]. In appendix A we quote the results we need about DV to set the notations, while for a basic review and a guide to the vast literature we refer to [30]. As a first step, we will consider the case in which the meson superpotential is just a mass term for all the flavors, with no Yukawa–type interactions between quarks and adjoint. While the solution of the electric theory is standard, the anomaly equations in the magnetic theory are somewhat different, due to the presence of the gauge singlets. This massive case is useful to illustrate the general procedure without worrying about the rich analytic structure of the generators of the chiral ring, that we will encounter later.

The Electric Theory

We will focus on the case in which the electric meson superpotential is just a mass term

$$W_{el} = \mathrm{Tr}\, V(X) + \tilde{Q}_{\tilde{f}} m_f^{\tilde{f}} Q^f, \tag{5.36}$$

where m is a diagonal matrix. If the second derivatives of $V(z)$ at the saddle points are nonvanishing, all the fields will be massive and it makes sense to use the effective action as a function of the glueball superfield S. We will be interested in the chiral operators (5.8).

The solution of the anomaly equation for the resolvent $R(z)$ gives

$$2R(z) = V'(z) - \sqrt{V'(z)^2 + f(z)}, \tag{5.37}$$

where $f(z)$ is a $n-1$ degree polynomial $f(z) = f_1 + ... + f_n z^{n-1}$. This defines the curve of the electric theory to be the hyperelliptic Riemann surface $y^2 = V'(z)^2 + f(z)$.

Since the meson polynomial $m(z)$ is just constant, the anomaly equation for the matrix $M(z)$ reduces to the following simple form $[M(z)m]_- = R(z)$, where we suppressed flavor indices. The solution is

$$M(z) = R(z)m^{-1}, \tag{5.38}$$

M being a diagonal matrix.

The anomaly equation for $T(z)$ is

$$[y(z)T(z)]_- + [\operatorname{tr} m'(z)M(z)]_- = 0, \tag{5.39}$$

but since $m(z) = \text{const}$ the last term drops. The solution is

$$T(z) = \frac{c(z)}{\sqrt{V'(z)^2 + f(z)}}, \tag{5.40}$$

where $c(z)$ is another $n-1$ degree polynomial $c(z) = c_1 + ... + c_n z^{n-1}$. Since m is not z-dependent, in the electric theory the fundamentals do not influence the solution for T .

The parameters f_j, c_j are related to the glueballs S_i of the low energy SQCD blocks and the ranks N_i of their gauge groups as follows

$$\begin{aligned} S_i &= \oint_{A_i} R(z)dz, \\ N_i &= \oint_{A_i} T(z)dz, \end{aligned} \tag{5.41}$$

where A_i is classically a contour around a_i. At the quantum level, each stationary point a_i opens up into a branch cut for $R(z)$ and the contour A_i actually encircles the two branch points. One can get exact formulae for the total glueball $S = \sum_i S_i$ and the rank of the high energy gauge group $N_c = \sum_i N_i$ by looking at the $1/z$ terms in (5.37) and (5.40), since choosing a contour A around all the branch points is equivalent to closing it around ∞. In this way we can fix the first coefficient of the polynomials $c(z)$ and $f(z)$

$$S = -\frac{f_n}{4t_n}, \qquad N_c = \frac{c_n}{t_n}. \tag{5.42}$$

We calculate now the relevant relations in the chiral ring. We can extract from (5.38) the mesons operators by

$$\tilde{Q}X^{j-1}Q = \oint_A z^{j-1}M(z), \tag{5.43}$$

where the contour A encircles all the branch points of the resolvent $R(z)$, obtaining

$$\tilde{Q}X^{j-1}Q = -\sum_{i=1}^{n} \frac{a_i^{j-1} f(a_i)}{4mV''(a_i)}, \tag{5.44}$$

for $j = 1, \ldots, n$, coming from the negative power expansion of the first term in the semiclassical expansion of the resolvent

$$R(z) = -\frac{f(z)}{4V'(z)} + \mathcal{O}\left(\frac{f(z)^2}{V'(z)^3}\right), \tag{5.45}$$

In particular we find the usual Konishi anomaly

$$\tilde{Q}_f Q^f = \frac{N_f S}{m}, \tag{5.46}$$

where we used (5.42). Higher meson operators receive additional contributions from the semiclassical expansion. The single trace of the adjoint X can be obtained as the coefficients of inverse powers of z in the expansion of $T(z)$ at large z

$$\mathrm{Tr} X^j = \oint_A z^j T(z) dz, \tag{5.47}$$

where A circles all the the branch points of the resolvent $R(z)$. Expanding $T(z)$ we get

$$T(z) = \frac{c(z)}{V'(z)} - \frac{c_n f_n}{2t_n^3 z^{n+2}} + \ldots \tag{5.48}$$

and we can extract the chiral operators

$$\mathrm{Tr} X^j = \sum_{i=1}^{n} \frac{c(a_i) a_i^j}{V''(a_i)} + \delta_{n+1}^j \frac{2N_c S}{t_n}, \tag{5.49}$$

for $j = 1, \ldots, n+1$. Clearly the equation $V'(X) = 0$ is obeyed in the chiral ring, but relations obtained by multiplying it with X get quantum corrections.

The Magnetic Theory

The magnetic theory corresponding to (5.36) has a tree level superpotential

$$W_{mag} = \mathrm{Tr} \bar{V}(Y) + \tilde{q}^{\bar{f}} \tilde{m}_{\bar{f}}^{f}(Y) q_f + \tilde{m} \mathrm{tr}(P_1). \tag{5.50}$$

Note that the quantities appearing in (5.50) are the magnetic ones, as explained at the beginning of this section. In particular we have that

$$\tilde{m}(z) = -\frac{1}{\mu^2} \oint_{\tilde{A}} \frac{\bar{V}'(\zeta) - \bar{V}'(z)}{\zeta - z} P(\zeta), \tag{5.51}$$

and, inverting this, we find the gauge singlets

$$P(z) = -\mu^2 \left[\frac{\tilde{m}(z)}{\bar{V}(z)}\right]_{-n}, \tag{5.52}$$

We are ready to use now the anomaly equations. The form of $\tilde{R}(z)$, which is independent on the fundamentals, will be the same as for the electric theory

$$2\tilde{R}(z) = \bar{V}'(z) - \sqrt{\bar{V}(z)^2 + \bar{f}(z)}, \tag{5.53}$$

where the quantum deformation $\bar{f}(z)$ is a degree $n-1$ polynomial. Since we will see that the quantum deformations on both sides are equivalent under the offshell duality map, we will conclude that the magnetic theory has the same curve of the electric one.

In addition to the usual anomaly equations, that we encountered in the electric theory, there are new ones following from variations of the gauge singlets P's. Since P is not coupled to the gauge fields, these are just its equations of motion. For the special case we are studying, after rearranging the equations, (5.15) reduce to

$$\begin{aligned} \bar{t}_n \tilde{q} Y^{n-1} q &= \tilde{m}\mu^2, \\ \tilde{q} Y^{j-1} q &= 0, \qquad j = 1, ... n-1. \end{aligned} \tag{5.54}$$

On the other hand, the role of the electric meson polynomial is played now by $\tilde{m}(z)$. The anomaly equation for the meson generator is then

$$[\tilde{M}(z)\tilde{m}(z)]_- = \tilde{R}(z). \tag{5.55}$$

Its generic solution is

$$\tilde{M}(z) = \tilde{R}(z)\tilde{m}^{-1}(z) + r(z)\tilde{m}^{-1}(z), \tag{5.56}$$

in our case all the matrices being diagonal. The crucial piece of information about the magnetic theory is the quantum expression of $\tilde{m}(z)$, which contains the gauge singlets and fixes the analytic properties of the meson generator. The way in which (5.56) is supposed to be used is the following

1. We fix the polynomial $r(z)$ such that there are no additional singularities in (5.56) arriving from the zeroes of $\tilde{m}(z)$.

2. We fix the polynomial $\tilde{m}(z)$ imposing that the mesons $\tilde{q}X^{j-1}q$ extracted from (5.56) fulfill the singlet equations of motion (5.54). In this way we fix also $P(z)$.

The unique solution to these requirements is

$$r(z) = 0, \tag{5.57}$$

and

$$\tilde{m}(z) = -\frac{\overline{f}(z)}{4\bar{m}\mu^2}. \tag{5.58}$$

Since $\tilde{m}(z)$ is proportional to $\overline{f}(z)$ and the the resolvent $\tilde{R}(z)$ vanishes at the zeroes of $\overline{f}(z)$, we see that $\tilde{m}^{-1}(z)$ does not give additional singularities in (5.56). The analytic structure of $\tilde{M}(z)$ in this case turns out to be very simple, while the singlets are

$$P(z) = \frac{1}{4\bar{m}}\left[\frac{\overline{f}(z)}{\bar{V}'(z)}\right]_{-n}, \tag{5.59}$$

where the expansion in inverse powers of z is understood to stop at z^{-n}. Comparing with (5.44) we see that the matching

$$P_j = \tilde{Q}X^{j-1}Q, \tag{5.60}$$

for $j = 1, \dots, n$, is implied for a sign choice which will be discussed later. Of course we could go backwards and requiring (5.60) prove the form of the Kutasov kernel $\frac{\bar{V}'(\zeta)-\bar{V}'(z)}{\zeta - z}$ which determines the form of the fundamental magnetic superpotential. We can extract the expectation values of the magnetic singlets out of (5.59)

$$P_j = \frac{1}{4\bar{m}}\sum_{i=1}^{n}\frac{\bar{f}(\bar{a}_i)\bar{a}_i^{j-1}}{\bar{V}''(\bar{a}_i)} = -\sum_{i=1}^{n}\frac{\bar{a}_i^{j-1}\bar{S}_i}{\bar{m}}, \tag{5.61}$$

where we used the definition of the glueballs in (5.41).

We can now calculate $\tilde{T}(z)$. Its anomaly equation is

$$[\tilde{y}(z)\tilde{T}(z)]_- + \mathrm{tr}[\tilde{m}'(z)\tilde{M}(z)]_- = 0. \tag{5.62}$$

The solution here, as opposed to (5.40), depends also on the fundamentals

$$\tilde{T}(z) = \frac{1}{\tilde{y}(z)}[-\tilde{m}'(z)\tilde{M}(z) + \tilde{c}(z)] \tag{5.63}$$

with $\tilde{m}'(z)$, $\tilde{M}(z)$ given by (5.59) and (5.56). Since $\bar{c}(z)$ is a polynomial of degree $n-1$, while $\tilde{m}'(z)\tilde{M}(z)$ starts with z^{-2}, the contribution of the fundamentals will start only from the power z^{-n-2}. Recalling that $\tilde{m}$ is a diagonal matrix, we expand (5.63) at large z

$$\tilde{T}(z) = \frac{\bar{c}(z)}{\bar{V}'(z)} - \frac{\bar{c}_n \bar{f}_n}{2\bar{t}_n^2 z^{n+2}} + N_f \frac{\bar{f}_n}{4\bar{t}_n^2} \frac{1}{z^{n+1}} \frac{\bar{f}'(z)}{\bar{f}(z)}, \tag{5.64}$$

and the chiral ring is

$$\mathrm{Tr} Y^j = \sum_{i=1}^{n} \frac{\bar{a}_i^j \bar{c}(\bar{a}_i)}{\bar{V}''(\bar{a}_i)}, \tag{5.65}$$

for $j = 0, \ldots, n$. The first operator which will receive a contribution from the last two terms in (5.64) will be $\mathrm{Tr}Y^{n+1}$ which in the magnetic theory becomes[20]

$$\mathrm{Tr} Y^{n+1} = \sum_{i=1}^{n} \frac{\bar{c}(\bar{a}_i)\bar{a}_i^{n+1}}{\bar{V}''(\bar{a}_i)} + \frac{2\bar{N}_c \bar{S}}{\bar{t}_n} - N_f \frac{\bar{S}(n-1)}{\bar{t}_n}. \tag{5.66}$$

We would like to stress again a basic property of the solution (5.58). Since it is proportional to $\bar{f}(z)$, the meson generator $\tilde{M}(z)$ and also $\tilde{T}(z)$ have a very simple analytic structure in both sheets, as opposed to the generic cases we will solve below. Because of this fact, we will be able to see that the electric–magnetic duality map here works exactly offshell.

The Effective Actions

Once that we have solved the chiral ring, we can determine the superpotential part of the low energy effective action by integrating the derivatives with respect to the parameters appearing in the lagrangian, which are the expectation values of the chiral operators we just computed above. This offshell effective action will be valid at energies above the glueball mass, that sets the scale of mass gap.

The Electric Theory

20 We used the fact that $\oint f'/f = \#$ zeroes of f.

The electric couplings are m and $t_1, \ldots, t_n$. It is covenient to use as independent parameters t_n and $\widehat{t}_j = \frac{t_j}{t_n}$ for $j = 1, ..., n-1$. The parameters $\widehat{t}_j$ are homogenous polynomials in a_i. The derivatives of the effective action are

$$\frac{\partial W_{eff}}{\partial \widehat{t}_j} = t_n \frac{1}{j+1} \langle \mathrm{Tr} X^{j+1} \rangle = -\frac{1}{j+1} \sum_{i=1}^{n} \frac{t_n c(a_i) a_i^{j+1}}{4V''(a_i)}, \tag{5.67}$$

for $j = 1, ..., n-1$ and

$$\frac{\partial W_{eff}}{\partial t_n} = \frac{1}{n+1} \langle \mathrm{Tr} X^{n+1} \rangle = -\frac{1}{n+1} \sum_{i=1}^{n} \frac{c(a_i) a_i^{n+1}}{4V''(a_i)} + \frac{1}{n+1} \frac{2N_c S}{t_n}, \tag{5.68}$$

$$\langle \tilde{Q}_f Q^f \rangle = \frac{\partial W_{eff}}{\partial m} = \frac{N_f S}{m}. \tag{5.69}$$

Since we are looking for the offshell effective action, these equations are supposed to be integrated at fixed S_i, N_i. Now observe that (5.69) and the second term in (5.68) satisfy the integrability condition by themselves. Therefore we can integrate them separately and there is a solution $\mathcal{W}_{eff}$ without them. The general effective action we obtain by (5.67), (5.68) and (5.69) is

$$W_{eff} = \mathcal{W}_{eff} + \frac{2N_c S}{n+1} \log t_n + N_f S \log m + [t_j, m - \text{independent terms}] \tag{5.70}$$

Let us consider the coupling independent terms. There are two contributions, the first is the one–loop exact renormalization of the gauge field kinetic term $(2N_c - N_f) S \log \Lambda$, that contains the dynamically generated scale Λ through the running gauge coupling constant. Then we have a Veneziano–Yankielowicz type superpotential $bS(\log S - 1)$. One can fix the numerical coefficient b by requiring that the effective action is $U(1)_R$ invariant. Since the R–current and the dilatation current lie in the same $\mathcal{N} = 1$ supermultiplet, this is the same as fixing them by dimensional analysis. By the usual localization trick, we promote the couplings to background chiral superfields so that we can assign them a charge. The dimensions Δ of the various fields are

$$\begin{array}{cc} & \Delta \\ S & 3 \\ t_j & 2-j \\ m & 1 \\ \Lambda^{2N_c - N_f} & 2N_c - N_f \end{array} \tag{5.71}$$

so that we find $b = -2N_c/(n+1)$. Since $\mathcal{W}_{eff}$ is invariant by itself we get the effective superpotential

$$W_{eff} = \mathcal{W}_{eff} + S \log \frac{\Lambda^{2N_c - N_f} t_n^{\frac{2N_c}{n+1}} m^{N_f}}{S^{\frac{2N_c}{n+1}}} + \frac{2N_c}{n+1} S. \tag{5.72}$$

We will now turn to the evaluation of the term $\mathcal{W}_{eff}$. It is most convenient to parameterize the degree $n-1$ polynomial $c(z)$ in the following way

$$c(z) = V'(z) \sum_{i=1}^{n} \frac{h_i}{z - a_i}. \tag{5.73}$$

where $N_c = \sum_{i=1}^{n} h_i$. The n coefficients h_i are fixed by the contour integral

$$h_i = \oint_{A_i} \frac{c(z)}{V'(z)}, \tag{5.74}$$

so that classically we have just $h_i = N_i$. Using this parametrization we can rewrite the relevant part of (5.67) and (5.68) as

$$\frac{\partial \mathcal{W}_{eff}}{\partial t_j} = \sum_{i=1}^{n} h_i a_i^j. \tag{5.75}$$

In particular, we see that $\mathrm{Tr} X^j = \sum_i h_i a_i^j$ for $j = 1, \ldots, n$, while $\mathrm{Tr} X^{n+1}$ contains in addition the last term in (5.49). The coefficients h_i depend on t_j, S_i and N_k, as we can see from (5.74). It is convenient to use in place of the glueballs S_i the new variable

$$y = \sum_{i=1}^{n} \log S_i, \tag{5.76}$$

and $n-1$ independent ratios of glueballs, e.g. $\frac{S_1}{S_n}, \ldots, \frac{S_{n-1}}{S_n}$. Introduce now the following functions

$$d_i = h_i - e^y \int_{-\infty}^{y} dy' e^{-y'} \frac{\partial h_i}{\partial y'}. \tag{5.77}$$

We claim that integrating (5.75) we find

$$\mathcal{W}_{eff} = \sum_{i=1}^{n} d_i V(a_i). \tag{5.78}$$

In the Appendix B we prove, up to an assumption of integrability, that indeed differentiating (5.78) one recovers (5.75).

Putting everything together, the effective superpotential of the electric theory is

$$W_{eff} = \sum_{i=1}^{n} d_i V(a_i) + S \log \frac{\Lambda^{2N_c - N_f} t_n^{\frac{2N_c}{n+1}} m^{N_f}}{S^{\frac{2N_c}{n+1}}} + \frac{2N_c}{n+1} S. \tag{5.79}$$

The Magnetic Theory

We can follow again the same procedure of integrating the expectation values with respect to the parameters, but now we have a new coupling μ and the derivative with respect to $\bar{t}_j$ gets a contribution also from the magnetic fundamentals and singlets

$$\frac{\partial \overline{W}_{eff}}{\partial \bar{t}_j} = \frac{1}{j+1} \langle \mathrm{Tr} Y^{j+1} \rangle + \frac{1}{\mu^2} \sum_{i=1}^{j} \langle P_i \tilde{q} Y^{j-i} q \rangle. \tag{5.80}$$

for $j = 1, \ldots, n$. Then we have

$$\mu^2 \frac{\partial \overline{W}_{eff}}{\partial \mu^2} = \bar{m} \langle \mathrm{tr} P_1 \rangle, \qquad \frac{\partial \overline{W}_{eff}}{\partial \bar{m}} = \langle \mathrm{tr} P_1 \rangle, \tag{5.81}$$

where we used the fact that the expectation values of gauge invariant chiral operators factorize and the singlet equations of motion (5.15), which are exact in the quantum theory. We can substitute the expectation values (5.61), (5.65), and (5.66) into (5.80), obtaining

$$\frac{\partial \overline{W}_{eff}}{\partial \bar{t}_j} = \frac{1}{j+1} \sum_{i=1}^{n} \frac{\bar{c}(\bar{a}_i) \bar{a}_i^{j+1}}{\bar{V}''(\bar{a}_i)} + \frac{\delta_n^j}{n+1} \frac{2(\bar{N}_c + N_f)\bar{S}}{\bar{t}_n}, \tag{5.82}$$

for $j = 1, \ldots, n$ and into (5.81)

$$\mu^2 \frac{\partial \overline{W}_{eff}}{\partial \mu^2} = -N_f S, \qquad \frac{\partial \overline{W}_{eff}}{\partial \bar{m}} = -N_f \frac{S}{\bar{m}}. \tag{5.83}$$

The first term in (5.82) is analogous to the corresponding electric one in (5.67). We assume, as in that case, that it satisfies the integrability condition by itself and integrate it to obtain $\overline{\mathcal{W}}_{eff}$. This is formally equal to (5.78) but with magnetic quantities instead. On the other hand, also (5.83) and the second term in (5.82) satisfy the integrability condition, so that we find

$$\overline{W}_{eff} = \overline{\mathcal{W}}_{eff} + \frac{2(\bar{N}_c + N_f)}{n+1}\bar{S}\log\bar{t}_n - 2N_f\bar{S}\log\mu - N_f\bar{S}\log\bar{m} + [\bar{t}_j, \bar{m}, \mu\text{−indep. terms}] \tag{5.84}$$

Then we need to add the magnetic one–loop renormalization of the gauge fields $(2\bar{N}_c - N_f)\bar{S}\log\tilde{\Lambda}$ and the Veneziano–Yankielowicz type superpotential $\bar{b}\bar{S}(\log\bar{S} - 1)$. Again we fix the coefficient $\bar{b}$ requiring $U(1)_R$ invariance, as we did for the electric case, and get $\bar{b} = 2(nN_f - \bar{N}_c)/(n+1)$. Putting everything together we obtain the magnetic effective action

$$\overline{W}_{eff} = \sum_{i=1}^{n} \bar{d}_i \bar{V}(\bar{a}_i) + \bar{S}\log\frac{\tilde{\Lambda}^{2\bar{N}_c - N_f}\bar{t}_n^{\frac{2(\bar{N}_c+N_f)}{n+1}}\bar{S}^{\frac{2(nN_f-\bar{N}_c)}{n+1}}}{\bar{m}^{N_f}\mu^{2N_f}} - \frac{2(nN_f - \bar{N}_c)}{n+1}\bar{S}. \tag{5.85}$$

The Offshell Duality Map

At this point we will look for the duality map between the electric and magnetic operators in the chiral ring. As we discussed in the introduction, in this case the duality holds exactly offshell. First we will consider the match of the meson operators and then the effective actions.

The gauge singlets equations of motion (5.15) are exact in the chiral ring of the magnetic quantum theory. They tell us that the magnetic meson operators $\tilde{q}Y^{j-1}q$ are trivial. They are replaced by the gauge singlets, which represent the electric mesons through a Legendre transform, as it is clear from the expression of $\bar{m}(z)$ in the magnetic tree level superpotential (5.10). Therefore we should match directly the electric mesons with the corresponding magnetic gauge singlets through the relation

$$P_j = \tilde{Q}X^{j-1}Q, \tag{5.86}$$

independently on the other relations between the gauge groups. Comparing the two expressions (5.61) and (5.44)

$$\tilde{Q}X^{j-1}Q = \sum_{i=1}^{n}\frac{a_i^{j-1}S_i}{m}, \qquad P_j = -\sum_{i=1}^{n}\frac{\bar{a}_i{}^{j-1}\bar{S}_i}{\bar{m}},$$

for $j = 1, \ldots, n$, we get the relations

$$S_i = -\bar{S}_i, \qquad m = \bar{m}, \tag{5.87}$$

while the roots of the electric and magnetic polynomials for the adjoint coincide $a_i = \bar{a}_i$, i.e. the electric polynomial $V'(z)$ and the magnetic one $\overline{V}'(z)$ are identified up to a minus sign. Let us recall the definition (5.41) of the glueballs in terms of the resolvent

$$S_i = -\frac{f(a_i)}{4V''(a_i)},$$

which holds both for the electric and magnetic theories with the respective quantities. The relation (5.87) then fixes the the duality map as

$$f(z) = \bar{f}(z), \qquad V'(z) = -\overline{V}'(z). \tag{5.88}$$

This last relation, in particular, tells us that electric and magnetic theories have the same curve

$$y^2 = V'(z)^2 + f(z). \tag{5.89}$$

Now let us consider the electric and the magnetic effective actions (5.79) and (5.85). By comparing their second and third terms we get again the match between the glueballs and the mass terms (5.87) and $t_n = -\bar{t}_n$, which fixes the ambiguity in the sign choice of (5.88), together with the scale matching relation

$$\Lambda^{2N_c - N_f} \tilde{\Lambda}^{2\bar{N}_c - N_f} = t_n^{-2N_f} \mu^{2N_f}, \tag{5.90}$$

and the usual relation between the electric and magnetic gauge groups $\bar{N}_c = nN_f - N_c$. The scale matching (5.90) is consistent with the fact that $\log \Lambda^{2N_c - N_f}$ and $\log \tilde{\Lambda}^{2\bar{N}_c - N_f}$ are the sources for the respective electric and magnetic total glueballs and that we found $\bar{S} = -S$. Let us analyze in more detail the relation between the gauge groups. The rank of the electric and magnetic gauge groups fixes the pole at infinity of $T(z)$

$$N_c = \oint_A T(z), \tag{5.91}$$

where A is the large contour, and analogously for the magnetic theory. The matching $\bar{N}_c = nN_f - N_c$ translates into the following relation

$$\oint_{\tilde{A}} \tilde{T}(z) = N_f \oint_A \frac{V'(z)}{V''(z)} - \oint_A T(z). \tag{5.92}$$

We evaluate the contour integrals by expanding (5.40) and (5.63) at large z and get

$$\bar{c}(z) = c(z) - N_f V''(z). \tag{5.93}$$

By the definition (5.74) of the coefficients h_i we find that $\bar{h}_i = N_f - h_i$ or equivalently $\bar{d}_i = N_f - d_i$, which fixes the map between the operators

$$\mathrm{Tr}Y^j = -\mathrm{Tr}X^j + N_f \sum_{i=1}^{n} a_i^j, \tag{5.94}$$

for $j = 1, \ldots, n$, in agreement with the KSS results [16]. The match between the electric and magnetic $\mathcal{W}_{eff}$ using the relation $\bar{d}_i = N_f - d_i$ shows that the magnetic effective superpotential contains an additional Y–independent term, which in this case is just $N_f \sum_i V(a_i)$.

The classical limit of the coefficients h_i is N_i, the rank of each low energy SQCD block. Thus we recover the usual matching relation $\bar{N}_i = N_f - N_i$, which is somewhat different from the one we found in our classical analysis of (5.23), which anyway was only valid in the case where the number of massive electric quarks is less than $N_f - N_c/n$, because of the stability bound. The higgsed blocks in the magnetic adjoint we found in (5.19) and (5.29), that were responsible for the singularities in (5.25), are an artifact of the classical theory. When we pass to the full quantum theory, in this pseudoconfining case all the classical singularities of $\tilde{M}(z)$ are smoothed out, and this is the reason why we get back the usual Seiberg duality map $\bar{N}_i = N_f - N_i$ for the rank of the gauge groups of the low energy SQCD blocks.

The Generic Pseudoconfining Case

In Section *5.2* we saw that, in the case of massive quarks without Yukawa couplings, duality works offshell, that is at the level of the dynamical effective actions. We will consider in this section the most generic pseudoconfining case, where in addition to the mass terms for the quarks we allow for a generic z–dependent meson polynomial. As a consequence, the analytic properties of the various resolvents in the quantum chiral ring get more involved and in the end the match between electric and magnetic quantities will not hold anymore exactly offshell, but we expect it to hold only onshell. We will not compute the effective action, as we did above, but we will match the electric mesons with the magnetic singlets and find a map that reproduces the Konishi anomaly in each low energy SQCD block as a classical equation in the magnetic theory.

The Electric Theory

Let us consider the electric theory with a generic yet diagonal meson polynomial

$$\begin{aligned} W_{el} =& \mathrm{Tr} V(X) + \tilde{Q}^{\tilde{f}} m(X)_{\tilde{f}}^{f} Q_f, \\ m(z)_{f}^{\tilde{f}} =& \sum_{i=1}^{l+1} m_i z^{i-1} \delta_{f}^{\tilde{f}} \end{aligned} \tag{5.95}$$

We denote the roots of $m(z)$ as x_k, for $k = 1, \ldots, l$. The degree of the polynomial $m(z)$ is at most $n-1$ and its constant term m_1 must be nonzero for all the flavors in order for the theory to be massive. The classical pseudoconfining vacuum is (5.9), while the generators (5.8) of the classical chiral ring all vanish except

$$T_{cl}(z) = \sum_{i=1}^{n} \frac{N_i}{z - a_i}. \tag{5.96}$$

This phase is characterized by a vanishing classical expection value for the fundamentals.

Let us consider the generalized Konishi anomaly equations. The resolvent $R(z)$ is still given by (5.37). The story is different for $M(z)$, the generator of the mesons. When solving its anomaly equation, we have to cancel the additional singularities coming from the zeroes of $m(z)$. We have to specify the boundary conditions coming from our choice of the vacuum. In this pseudoconfining case, $M(z)$ is regular in the first sheet (up to the residue at infinity). Implementing these boundary conditions we find

$$M(z) = \frac{R(z)}{m(z)} - \sum_{k=1}^{l} \frac{R(q_k)}{z - x_k} \frac{1}{m'(x_k)}. \tag{5.97}$$

Let us extract the expectation value of the mesons. We can evaluate (5.43) by expanding semiclassically the resolvent in powers of $f(z)/V'(z)^2$ as in (5.45) and find

$$\tilde{Q}X^{j-1}Q = -\sum_{i=1}^{n} \frac{a_i^{j-1} f(a_i)}{4m(a_i)V''(a_i)} + \dots, \tag{5.98}$$

where we showed only the leading approximation. Here we see the crucial difference between the purely massive case (5.44) and this general case. There, we took the semiclassical expansion and then we opened up the contour A to the big circle, throwing away all the higher terms in the expansion. Here, we cannot open up the contour A after taking the semiclassical expansion, because in this process we would hit the additional poles at the zeroes of $m(z)$ for each term in the expansion. Due to the richer analytic structure, we are forced to keep in (5.98) all the semiclassical expansion. We will see that a duality map exists at the first order in this expansion.

A similar story carries on to the last anomaly equation (5.39), whose solution with the classical limit (5.96) is

$$T(z) = \sum_{k=1}^{l} \frac{1}{2(z-x_k)} - \sum_{k=1}^{l} \frac{y(q_k)}{2y(z)(z-x_k)} + \frac{c(z)}{y(z)}, \tag{5.99}$$

where

$$c(z) = V'(z)\sum_{i=1}^{n} \frac{h_i}{z-a_i} - \frac{1}{2}\sum_{k=1}^{l} \frac{V'(z)-V'(x_k)}{z-x_k}, \tag{5.100}$$

is a degree $n-1$ polynomial whose leading coefficient is $c_n/t_n = N - l/2$. Note that in this case the fundamentals do contribute to $T(z)$. We have considered a convenient parametrization of (5.100) similar to the one in (5.73) but now slightly modified to take into account the more complicated analytic structure. We still have $\sum_i h_i = N_c$. Since the roots x_k of the meson polynomial $m(z)$ are supposed to be very large in the semiclassical limit, we see that the definition of the coefficients h_i is still (5.74), the last term in (5.100) not contributing to the contour integral. Now we can integrate the generator on the contour A to obtain the expectation values. The first term in (5.99) does not contribute because the x_k's lie outside the contour and we obtain

$$\mathrm{Tr}X^j = \sum_{i=1}^{n} h_i a_i^j + \sum_{k=1}^{l} R(x_k) \sum_{i=1}^{n} \frac{a_i^j}{(a_i - x_k)V''(a_i)} + \delta_{n+1}^j \frac{2S(N-l/2)}{t_n} + \dots, \tag{5.101}$$

for $j = 1, \ldots, n+1$. By $\ldots$ we denote higher terms in the semiclassical expansion (5.45).

The Magnetic Theory and the Match

The magnetic theory corresponding to (5.95) has the tree level superpotential

$$W_{mag} = \mathrm{Tr}\bar{V}(Y) + \tilde{q}\bar{m}(P,Y)q + \oint \bar{m}(z)P(z), \tag{5.102}$$

where we use the same notations as in (5.51) and $\bar{m}(z)$ corresponds to the electric meson polynomial. This phase is characterized by a vanishing classical value of the singlets and thus also of $\tilde{m}(z)$.

Let us solve the anomaly equations. The resolvent $\tilde{R}(z)$ is still given by (5.53). The anomaly equation for the generator of the magnetic mesons is always (5.55), whose general solution is

$$\tilde{M}(z) = \tilde{R}(z)\tilde{m}^{-1}(z) + r(z)\tilde{m}^{-1}(z). \tag{5.103}$$

We recall that the polynomial $r(z)$ is fixed in order to cancel the additional singularities coming from the zeroes of $\tilde{m}(z)$. Then $\tilde{m}(z)$ is fixed by imposing that the magnetic singlet equations of motion are satisfied. Denote the roots of the degree $n-1$ polynomial $\tilde{m}(z)$ as e_k, for $k = 1, \ldots, n-1$. In this case our boundary conditions are such that $\tilde{M}(z)$ is regular in the first sheet at the zeroes e_k

$$\frac{r(z)}{\tilde{m}(z)} = -\sum_{k=1}^{n-1} \frac{\tilde{R}(e_k)}{z-e_k} \frac{1}{\tilde{m}'(e_k)}. \tag{5.104}$$

Note that in the previous case (5.59) there was no need to keep the polynomial $r(z)$, since $\tilde{m}(z)$ was proportional to the quantum deformation $\bar{f}(z)$ of the resolvent. In that case, no additional singularity was present. Now the story is quite different and to find the result we should first rewrite the singlet equations of motion in a more convenient way. First note that, just as we can usually trade the glueballs S_i for the coefficients of the quantum deformation $f(z)$ [41], we can also trade the n singlets P_l for the n coefficients of the polynomial $\tilde{m}(z) = \sum_{l=1}^{n} \tilde{m}_l z^{l-1}$, that are a linear combination thereof

$$\tilde{m}_l = -\frac{1}{\mu^2} \sum_{k=l}^{n} \bar{t}_k P_{k-l+1}. \tag{5.105}$$

Now we cast the superpotential in a suitable form to replace the $P(z)$ with the $\tilde{m}(z)$. Recall that the singlets are fixed by $\tilde{m}(z)$ as in (5.52). By using (5.102), the relevant part of the superpotential we need is

$$\tilde{q}\tilde{m}(Y)q - \mu^2 \oint_{\tilde{A}} \frac{\bar{m}(z)\tilde{m}(z)}{\bar{V}'(z)}. \tag{5.106}$$

Differentiating w.r.t. $\tilde{m}_l$ we get

$$\tilde{q}Y^{l-1}q - \mu^2 \oint_{\tilde{A}} z^{l-1}\frac{\bar{m}(z)}{\bar{V}'(z)} = 0, \tag{5.107}$$

for $l = 1, \ldots, n$, that we can also write as

$$\oint_{\tilde{A}} z^{l-1}\left[\tilde{M}(z) - \mu^2 \frac{\bar{m}(z)}{\bar{V}'(z)}\right] = 0, \tag{5.108}$$

Note that while in the electric case the zeroes of $m(z)$ are very large in the semiclassical regime, in the magnetic case it turns out that the zeroes of $\tilde{m}(z)$ do lie inside the $\tilde{A}$ contour, as we will see explicitly in section 6.5 for the cubic superpotential. We can expand the resolvent semiclassically as in (5.45) and only will the residue at the zeroes of $V'(z)$ contribute. Remember that the singlet equations of motion (5.108) are supposed to fix the unknown polynomial $\tilde{m}(z)$. Indeed the solution of (5.108) at the first order in the semiclassical expansion is

$$4\mu^2\bar{m}(\bar{a}_i)\tilde{m}(\bar{a}_i) = -\bar{f}(\bar{a}_i), \tag{5.109}$$

for $i = 1, \ldots, n$, where $\bar{a}_i$ are the roots of $\bar{V}'(z)$ and the flavor indices are suppressed (note that they are not summed over). Eq. (5.109) consists of n conditions that account for the n unknown coefficients $\tilde{m}_l$.

Some comments are in order. The classical limit of (5.109) is well defined, since both sides vanish (remember that classically the singlets vanish in this phase). Now look at the meson generator $\tilde{M}(z)$ in (5.103) with boundary conditions (5.104). At the quantum level it is regular in the first semiclassical sheet, while it has $n-1$ poles on the second sheet. Nevertheless, when taking the classical limit, both $\tilde{R}(z)$ and $\tilde{m}(z)$ vanish, but the result is a nonvanishing classical value for $\tilde{M}(z)$, that reproduces our classical understanding of the theory being higgsed, as explained in Section *5.2*. Here we see again the same issue discussed thereof. The singularities of $\tilde{M}(z)$ on the first sheet are an artifact of the classical theory in the pseudoconfining case: they are smoothed out in the full quantum theory.

We can complete the analysis of the magnetic chiral ring by solving the anomaly equation (5.62) for $\tilde{T}(z)$ where $\tilde{m}(z)$ and $\tilde{M}(z)$ are given by (5.103) and (5.109) and get

$$\tilde{T}(z) = \sum_{k=1}^{n-1} \frac{1}{2(z-e_k)} - \sum_{k=1}^{n-1} \frac{\tilde{y}(e_k)}{2\tilde{y}(z)(z-e_k)} + \frac{\bar{c}(z)}{\tilde{y}(z)}, \tag{5.110}$$

where we can choose the following parametrization for the degree $n-1$ polynomial $\bar{c}(z)$

$$\bar{c}(z) = \bar{V}'(z) \sum_{i=1}^{n} \frac{\bar{h}_i}{z-\bar{a}_i} - \frac{1}{2}\sum_{k=1}^{n-1} \frac{\bar{V}'(z) - \bar{V}'(e_k)}{z-e_k}, \tag{5.111}$$

where we can fix $\bar{c}_n/\bar{t}_n = \bar{N}_c - n + 1$ and $\sum_i \bar{h}_i = \bar{N}_c$. By following the same procedure as in the electric case (5.101), we can extract again the corresponding magnetic expectation values

$$\mathrm{Tr} Y^j = \sum_{i=1}^{n} \bar{h}_i \bar{a}_i^j + \sum_{k=1}^{n-1} \bar{R}(e_k) \sum_{i=1}^{n} \frac{\bar{a}_i^j}{(\bar{a}_i - e_k)\bar{V}''(\bar{a}_i)} + \delta^j_{n+1} \frac{\bar{S}(2\bar{N}_c - n + 1)}{\bar{t}_n} + \dots, \tag{5.112}$$

for $j = 1, \dots, n+1$. By ... we denote the higher terms in the semiclassical expansion.

Now we can check that our singlets $P(z)$ in (5.14) match the electric mesons (5.98)

$$\tilde{Q}X^{j-1}Q = -\sum_{i=1}^{n} \frac{a_i^{j-1} f(a_i)}{4m(a_i)V''(a_i)} + \dots, \qquad P_l = -\sum_{i=1}^{n} \frac{\bar{a}_i^{l-1} \bar{m}(\bar{a}_i)\mu^2}{\bar{V}''(\bar{a}_i)}. \tag{5.113}$$

At the leading approximation in the electric semiclassical expansion (5.45), the match is ensured by the condition (5.109) that solves the singlet equations of motion, provided that the relation between electric and magnetic polynomials and the quantum deformations is again

$$\overline{V}'(z) = -V'(z), \qquad \overline{m}(z) = m(z), \qquad \overline{f}(z) = f(z), \tag{5.114}$$

just the same we found in the massive case (5.88). Therefore electric and magnetic theory still have the same curve (5.89). Note that this is equivalent to the simple relation between the glueballs

$$\bar{S}_i = -S_i.$$

At this point we can rewrite the solution (5.109) of the magnetic theory is terms of the electric quantities, recalling that $\tilde{m}(z)$ in (5.51) reverses its sign

$$4\mu^2 m(a_i)\tilde{m}(a_i) = f(a_i). \qquad (5.115)$$

The Konishi Anomaly

Consider the low energy theory described by the vacuum (5.9). It is a product of decoupled SQCDs with N_f flavors. We can look at the physics of each separate $U(N_i)$ SQCD by integrating the resolvents around the contour A_i, that encircles the branch points of the resolvent appeared by the splitting of the a_i root. In particular, by (5.98) the mesons, even if classically vanishing, at the quantum level satisfy the Konishi anomaly

$$\langle\tilde{Q}Q\rangle_i = \frac{S_i}{m(a_i)} = -\frac{f(a_i)}{4m(a_i)V''(a_i)}, \qquad (5.116)$$

where we dropped the higher terms in the semiclassical expansion (5.45).

We can perform a similar analysis in the magnetic theory: the electric mesons in (5.116) correspond to the gauge singlets P_1. The general expression for the singlets is given in (5.52). The low energy magnetic theory is a product of decoupled n Seiberg blocks, each one dual to a corresponding electric SQCD. The relation corresponding to the Konishi anomaly (5.116) in each low energy block is

$$\langle P_1\rangle_i = -\mu^2\frac{\tilde{m}(\bar{a}_i)}{\bar{V}''(\bar{a}_i)}, \qquad (5.117)$$

and it matches the electric one due to the relations (5.109) and (5.114).

5.3. The Electric Higgs Phase

In this section we will find the classical magnetic solution when the electric theory is in the higgs vacuum and check its properties. Then we will solve the chiral rings and look for the duality map. In the end we will consider the analytic structure of the solution we found as well as its behaviour when moving poles between the sheets. Our notations will be as follows: we will describe the classical magnetic theory with electric couplings, while in the quantum theory we will distinguish explicitly electric and magnetic couplings.

The Classical Vacua

The Electric Theory

To be definite we will consider the case in which only one flavor, e.g. the last one, is higgsed. We will begin by considering the classical theory with the simple KSS perturbation, namely the special case of (5.6) with

$$W_{el} = \frac{t_n}{n+1}\mathrm{Tr}X^{n+1} + m_2\tilde{Q}_{N_f}XQ^{N_f} + m_1\tilde{Q}_{N_f}Q^{N_f}. \tag{5.118}$$

This theory does not confine in the IR. Instead, the superpotential (5.118) drives the flow to an interacting SCFT. The higgs vacuum in the electric theory is obtained by giving a classical expectation value to the last flavor of fundamentals

$$\tilde{Q}_{N_f\alpha} = (\tilde{h}, 0, \ldots, 0), \quad Q_\alpha^{N_f} = (h, 0, \ldots, 0), \tag{5.119}$$

then the quark equations of motion fix the value of the adjoint to $X = \mathrm{diag}(x_1, 0, \ldots, 0)$ where $x_1 = -m_1/m_2$. The expectation value of the quarks is fixed by the adjoint equations of motion to $h\tilde{h} = -t_n x_1^n/m_2$.

As usual, we can think of the low energy theory in two stages. First, by higgsing the quarks we decrease the number of colors from N_c to $N_c - 1$. The quarks $Q^\alpha_{N_f}$, $\tilde{Q}^{N_f}_\alpha$ for $\alpha = 2, \ldots, N_c$ become the transverse component of a massive vector superfield of mass squared $\tilde{h}h$, while the components X^1_α, X^α_1 for $\alpha = 2, \ldots, N_c$ of the adjoint replace the last flavor, so the total number of flavors does not decrease. Secondly, this latter new flavor acquires a mass $t_n x_1^{n-1}$ by expanding the adjoint superpotential. The low energy theory is a $U(N_c - 1)$ gauge theory with $N_f - 1$ flavors. The matching of the electric scales is

$$\Lambda_{N_c,N_f}^{2N_c-N_f} = \frac{(-x_1)}{m_2}\Lambda_{N_c-1,N_f-1}^{2(N_c-1)-(N_f-1)}. \tag{5.120}$$

The Magnetic Theory

The magnetic theory corresponding to (5.118) is defined by

$$W_{mag} = -\frac{t_n}{n+1}\mathrm{Tr}Y^{n+1} + \frac{t_n}{\mu^2}\sum_{j=1}^{n} P_j \tilde{q} Y^{n-j} q + m_2 (P_2)_{N_f}^{N_f} + m_1 (P_1)_{N_f}^{N_f}. \quad (5.121)$$

Unlike the previous pseudoconfining case, this vacuum is characterized by a non-vanishing classical expectation value for the singlets, corresponding to the electric higgsed quarks, that we classically match as

$$(P_j)_{N_f}^{N_f} = \tilde{Q}_{N_f} X^{j-1} Q^{N_f} = -\frac{t_n x_1^n}{m_2} x_1^{j-1}, \quad (5.122)$$

for $j = 1\ldots, n$. In this case the classical chiral ring is more complicated, due to the nonvanishing singlets. In addition to the usual singlet equations of motion (5.15), we have also the quark

$$\sum_{j=1}^{n} P_j (Y^{n-j} q) = 0, \quad \sum_{j=1}^{n} P_j (\tilde{q} Y^{n-j}) = 0, \quad (5.123)$$

as well as the adjoint equations of motion. Nevertheless one can check that, in the convenient notation of (5.19) and (5.20), the adjoint is in block diagonal form $Y = \mathrm{diag}(Y_{higgs}, 0, \ldots, 0)$ and the nonvanishing part of the solution is

$$\begin{aligned} Y_{higgs} =& |1\rangle\langle v| + b_2 R_{n-1}, \\ |v\rangle =& -\sum_{j=1}^{n-1} x_1 \left(\frac{x_1}{b_2}\right)^{j-1} |j\rangle, \\ |\tilde{q}^{N_f}\rangle =& b_2|1\rangle, \qquad |q_{N_f}\rangle = b_2|n-1\rangle, \end{aligned} \quad (5.124)$$

where b_2 is given by (5.17). The first $(n-1)\times(n-1)$ block of the adjoint reads

$$Y_{higgs} = \begin{pmatrix} -x_1 & b_2 & 0 & & \\ -x_1(\frac{x_1}{b_2}) & 0 & b_2 & . & \\ . & & 0 & . & . \\ . & & & . & b_2 \\ -x_1(\frac{x_1}{b_2})^{n-2} & 0 & . & . & 0 \end{pmatrix} \quad (5.125)$$

Let us figure the low energy theory. First, by higgsing the theory we break the gauge symmetry as $U(\bar{N}_c) \to U(\bar{N}_c - n + 1)$ and note that $\bar{N}_c - n + 1 = n(N_f - 1) - (N_c - 1)$, as expected from the electric theory. Accordingly, $\tilde{q}_\alpha^{N_f}$, $q_{N_f}^\alpha$, Y_m^α, Y_α^s for $\alpha = n, \ldots, \bar{N}_c$, $m = 2, \ldots, n-1$ and $s = 1, \ldots, n-2$ conspire to join $n-1$ massive vector superfields in the fundamental of $U(\bar{N}_c - n + 1)$ with mass b_2. The flavor that disappears is replaced by a new flavor Y_1^α, Y_α^{n-1} for $\alpha = n, \ldots, \bar{N}_c$, so the number of flavors does not decrease. Secondly, we look for a mass term for the new flavor coming from the superpotential

$$\frac{t_n}{n+1}\mathrm{Tr}Y^{n+1} \simeq t_n(-x_1)b_2^{n-2}Y_1^\alpha Y_\alpha^1, \quad \alpha = n, \ldots, \bar{N}_c. \tag{5.126}$$

The number of flavors thus decreases by one unit also in the magnetic theory. The matching of the magnetic scale goes as follows

$$\tilde{\Lambda}_{\bar{N}_c, N_f}^{2\bar{N}_c - N_f} = \frac{b_2^n}{t_n(-x_1)}\tilde{\Lambda}_{\bar{N}_c - n+1, N_f - 1}^{2(\bar{N}_c - n+1)-(N_f-1)}. \tag{5.127}$$

We can use the relation (3.8) between the scales and find that this solution is consistent with the flows

$$\Lambda_{N_c-1, N_f-1}^{2(N_c-1)-(N_f-1)}\tilde{\Lambda}_{\bar{N}_c-n+1, N_f-1}^{2(\bar{N}_c-n+1)-(N_f-1)} = \left(\frac{\mu^2}{t_n^2}\right)^{N_f-1}. \tag{5.128}$$

Generic Polynomial Deformation

We can generalize this to an arbitrary polynomial deformation

$$W_{el} = \mathrm{Tr}V(X) + m_2\tilde{Q}_{N_f}XQ^{N_f} + m_1\tilde{Q}_{N_f}Q^{N_f}. \tag{5.129}$$

The classical solution is the same as in (5.119) the only difference being that now $\tilde{h}h = -V'(x_1)/m_2$.

In the magnetic theory, the corresponding solution is as in (5.124) but now the vector $|v\rangle$ is replaced by

$$|v'\rangle = -\sum_{j=1}^{n-1}\left(\frac{x_1}{b_2}\right)^{j-1}\left(x_1 + \sum_{l=0}^{j-1} x_1^{-l}\frac{t_{n-l-1}}{t_n}\right)|j\rangle. \tag{5.130}$$

Now that we have the generic adjoint polynomial $V(z)$, we can keep on flowing by further breaking the gauge group down to the low energy SQCD blocks. The electric adjoint is then $X = \mathrm{diag}(x_1, a_1, \ldots, a_n)$, where a_i are the roots of $V'(z)$ that appear with multiplicity N_i such that $\sum_i N_i = N_c - 1$. In the magnetic theory we have correspondingly a bunch of diagonal blocks $Y = \mathrm{diag}(Y_{higgs}, Y_{a_1}, \ldots, Y_{a_n})$, where the first one is (5.130) and the others are as in (5.22). In this vacuum the relation between the low energy electric and magnetic gauge groups is

$$\bar{N}_i = N_f - N_c - 1, \tag{5.131}$$

since in the higgsed electric theory we have $\sum_{i=1}^{n} N_i = N_c - 1$. We can also compute the classical expression of the generators of the chiral ring operators in this vacuum. The resolvent $\tilde{R}(z)$ vanishes, while

$$\begin{aligned} \tilde{M}_{cl}(z) =& -\mu^2 \frac{m(z)}{V'(z) - V'(x_1)}, \\ \tilde{T}_{cl}(z) =& \sum_{i=1}^{n} \frac{\bar{N}_i}{z - a_1} + \frac{d}{dz} \ln \frac{V'(z) - V'(x_1)}{z - x_1}. \end{aligned} \tag{5.132}$$

Note that (5.132) gives the correct behaviour at infinity $T_{cl} \sim \bar{N}_c/z$ since $\sum_{i=1}^{n} \bar{N}_i = \bar{N}_c - n + 1$.

Let us mention that this description is agreement with the expectations from electric magnetic duality. If we compare this solution to the pseudoconfining one (5.20), we see that while the electric theory, being higgsed, becomes more weakly coupled, in the magnetic theory the rank of the higgsed block in the adjoint decreases from n to $n-1$, thus making the theory more strongly coupled.

One can find a small generalization of the solution (5.124) by turning on higher meson perturbations in the electric theory, always along the last electric flavor direction. In Appendix C we will give more details about the solutions with several higgsed electric colors, but now let us add just few comments. In this way we can have more higgsed entries in the same flavor $Q^{N_f} = (h_1, \ldots, h_l, 0, \ldots, 0)$ and correspondingly $X = \mathrm{diag}(x_1, \ldots, x_l, 0, \ldots, 0)$, the electric theory being at weaker coupling. The general structure of the magnetic expectation values is that the first Y_{higgs} block decreases its rank down to $n - l$. Hence, the magnetic side looks more strongly coupled. The rank of the generic Seiberg blocks is still $\bar{N}_i = N_f - N_i - 1$ and one can check that still

$$\sum_{i=1}^{n} (N_f - N_i - 1) + n - l = nN_f - N_c = \bar{N}_c, \tag{5.133}$$

since now $\sum_{i=1}^{n} N_i = N_c - l$. We can carry on this procedure until $l = n-1$: one further higgsing would get the rank of Y_{higgs} vanished. In fact $l = n-1$ is also the maximal number of Higgs eigenvalues we can turn on on the same electric flavor, i.e. the largest value the degree of the meson polynomial $m(z)$ can reach, higher mesons being trivial in the electric chiral ring. Finally, if we allow for different electric flavors to get higgsed then in the magnetic theory we have to add a new block analogous to Y_{higgs} for each higgsed flavor. We can not go on higgsing forever, issues similar to the one that led our discussion of (5.34) arise also in this phase.

The Chiral Ring

The Electric Theory

Let us consider the minimal case in which the electric theory admits a higgs vacuum and we can safely apply the DV method: all flavors are massive and a Yukawa interaction is turned on only for the last flavor. The tree level superpotential is

$$W_{tree} = \mathrm{Tr}V(X) + m_1 \tilde{Q}_f Q^f + m_2 \tilde{Q}_{N_f} X Q^{N_f}, \tag{5.134}$$

so that the meson polynomial reads $m(z)^{\tilde{f}}_{f} = m_1 \delta^{\tilde{f}}_{f} + z\, m_2 \delta^{N_f}_{f} \delta^{\tilde{f}}_{N_f}$ and has only one root $x_1 = -m_1/m_2$. We give a classical expectation value to the last flavor of quarks and consider the following solution to the equations of motion

$$\begin{aligned} X =& \mathrm{diag}(x_1, a_1, \ldots, a_n) \\ \tilde{Q}_{N_f} =& (\tilde{h}_1, 0, \ldots, 0), \quad Q^{N_f} = (h_1, 0, \ldots, 0), \end{aligned} \tag{5.135}$$

where each a_i is a root of $V'(z)$ and has multiplicity N_i such that $\sum_{i=1}^{n} N_i = N_c - 1$. The adjoint equations of motion set $\tilde{h}_1 h_1 = -V'(x_1)/m_2$. In the classical chiral ring the resolvent $R(z)$ vanishes, while the nonvanishing generators are

$$\begin{aligned} T(z)|_{cl} =& \frac{1}{z - x_1} + \sum_{i=1}^{n} \frac{N_i}{z - a_i}, \\ M^{N_f}_{N_f}(z)|_{cl} =& -\frac{V'(x_1)}{z - x_1} \oint_{x_1} \frac{dx}{m^{N_f}_{N_f}(x)} = -\frac{1}{m_2} \frac{V'(x_1)}{z - x_1}, \end{aligned} \tag{5.136}$$

Let us solve the anomaly equations. The resolvent $R(z)$ is always (5.37). The story is different now for the generator of the mesons $M(z)$. The boundary conditions in the higgs vacuum require a pole on the first sheet along the last flavor direction. The solution along the pseudoconfining flavor directions is the usual one

$$M(z)_f^{\widetilde{f}} = R(z)m_1^{-1}\delta_f^{\widetilde{f}}, \qquad (f,\widetilde{f}) \neq (N_f, N_f), \tag{5.137}$$

while the solution along the last flavor direction is

$$M(z)_{N_f}^{N_f} = \frac{R(z)}{m_1 + z m_2} - \frac{V'(x_1) - R(x_1)}{z - x_1} m_2^{-1}. \tag{5.138}$$

We can integrate (5.138) on the contour A that encircles all the branch points of the resolvent and obtain the quantum expressions for the mesons. There are two types of mesons, the ones in the $(f,\widetilde{f}) \neq (N_f, N_f)$ flavor directions that are exactly given by (5.44), and the ones in the last flavor direction that are

$$\tilde{Q}_{N_f} X^{j-1} Q^{N_f} = -\sum_{i=1}^{n} \frac{a_i^{j-1} f(a_i)}{(m_1 + a_i m_2)V''(a_i)} + \dots, \tag{5.139}$$

where the dots stand for higher terms in the semiclassical expansion of the resolvent (5.45).

The Magnetic Theory and its Analytic Structure

The magnetic theory corresponding to (5.134) is defined by the following tree level superpotential

$$W_{mag} = \mathrm{Tr}\bar{V}(Y) + \tilde{q}^{\tilde{f}} \bar{m}(Y)_{\tilde{f}}^{f} q_f + \bar{m}_1 \mathrm{tr} P_1 + \bar{m}_2 (P_2)_{N_f}^{N_f}, \tag{5.140}$$

The anomaly equation for the resolvent $\tilde{R}(z)$ gives the usual solution (5.53). The equations for $\tilde{M}(z)$ and the singlet equations of motion now have different boundary conditions depending on the flavor directions. The first $(f,\widetilde{f}) \neq (N_f, N_f)$ flavors have the same solution (5.58) and (5.59) as in the first massive case we considered, in which $\tilde{m}(z)$ is proportional to the quantum deformation

$$\begin{aligned} \tilde{M}(z)_f^{\widetilde{f}} &= \tilde{R}(z)\tilde{m}(z)^{-1}{}_f^{\widetilde{f}}, \\ \tilde{m}(z)_{\widetilde{f}}^{f} &= -\frac{\bar{f}(z)}{4\mu^2}(\bar{m}^{-1})_{\widetilde{f}}^{f}. \end{aligned} \tag{5.141}$$

The remaining flavor direction $(f, \tilde{f}) = (N_f, N_f)$ corresponds to the higgsed electric meson. The new boundary conditions for $\tilde{M}(z)$ are $n-1$ poles on the first sheet and no pole on the second sheet, as opposed to the previous pseudoconfining case (5.103) in which no pole was there on the first sheet and $n-1$ poles appeared on the second sheet

$$\tilde{M}(z)_{N_f}^{N_f} = \frac{\tilde{R}(z)}{\tilde{m}(z)_{N_f}^{N_f}} - \sum_{i=1}^{n-1} \frac{\tilde{V}'(e_k) - \tilde{R}(e_k)}{z - e_k} \frac{1}{\tilde{m}'(e_k)_{N_f}^{N_f}}, \tag{5.142}$$

where e_k for $k = 1, \ldots, n-1$ are the roots of $\tilde{m}(z)$.

The picture of the analytic structure of $\tilde{M}(z)$ is the following. We saw that in the pseudoconfining electric case (5.9), the magnetic solution (5.103) does not have poles on the first sheet, but it has $n-1$ poles on the second sheet. In the electric higgs phase (5.119), the magnetic solution (5.142) gets $n-1$ poles appearing on the first sheet and no pole on the second sheet. The classical limit of this last solution has still $n-1$ poles, coming from the second term in (5.142) and the fact that classically $\tilde{m}$ is nonvanishing. In the classical limit, these poles are very large, but in the quantum theory they move to the region near the branch cuts, as we will check explicitly in section 5.4. Now let us move back to the electric theory and higgs two color direction on the same electric flavor, replacing (5.135) with

$$\tilde{Q}_{N_f} = (\tilde{h}_1, \tilde{h}_2, 0, \ldots, 0), \qquad Q^{N_f} = (h_1, h_2, 0, \ldots, 0). \tag{5.143}$$

and $X = \mathrm{diag}(x_1, x_2, 0, \ldots, 0)$. The gauge group is higgsed down to $U(N_c - 2)$ and the electric theory becomes more weakly coupled. Classically we saw in (5.133) that the rank of the corresponding magnetic higgs block decreases by one. Quantum mechanically this corresponds to moving one of the $n-1$ poles in (5.142) from the first to the second sheet

$$M(z)_{N_f}^{N_f} = \frac{R(z)}{\tilde{m}(z)} - \sum_{i=2}^{n-1} \frac{V'(e_k) - R(e_k)}{z - e_k} \frac{1}{\tilde{m}'(e_k)} - \frac{R(e_1)}{(z - e_1)} \frac{1}{\tilde{m}'(e_1)}. \tag{5.144}$$

In this way the magnetic theory becomes more strongly coupled. In the classical limit $\tilde{m}(z)$ is nonvanishing so we are left with just $n-2$ poles. Note that we can higgs at most $n-1$ electric color directions on the same flavor, $Q^{N_f} =$

$(h_1, \dots, h_{n-1}, 0, \dots, 0)$, corresponding to the largest degree the electric meson polynomial $m(z)$ can have. On the magnetic side, there are at most $n-1$ poles to be moved all the way to the second sheet. When we pass them all, the corresponding meson generator looks much like (5.103), but actually it is different. While the classical limit of (5.103) is nonzero due to the fact that $\tilde{m}(z)$ vanishes classically, in this case $\tilde{m}(z)$ is always nonvanishing and therefore the meson generator vanishes classically.

In the previous pseudoconfining case, we noted that, even if classically $\tilde{M}(z)$ has some singularities, in the quantum theory these singularities are smoothed out and we end up with a regular expression in the first semiclassical sheet. In the higgs case, instead, the singularities we might expect in the classical generator do not disappear at the quantum level but are genuine poles in the quantum expressions (5.142).

We still have to fix $\tilde{m}(z)$ by requiring that the singlet equations of motion (5.108) are satisfied. The contour $\tilde{A}$ in (5.108) encircles all the branch points of the resolvent, but now it encircles also the $n-1$ poles at e_k. The evaluation of this contour integral is much more complicated than in the pseudoconfining case (5.108), since we get additional residues at e_k. Dropping higher terms in the semiclassical expansion of the resolvent (5.45) and showing just the leading approximation we get

$$\sum_{i=1}^{n}\left[-\frac{\bar{a}_i^{l-1}\bar{f}(\bar{a}_i)}{4\tilde{m}(\bar{a}_i)\bar{V}''(\bar{a}_i)} - \mu^2\frac{\bar{a}_i^{l-1}\tilde{m}(\bar{a}_i)}{\bar{V}''(\bar{a}_i)}\right] + \sum_{k=1}^{n-1}\frac{2\tilde{R}(e_k) - \bar{V}'(e_k)}{\tilde{m}'(e_k)}e_k^{l-1} = 0, \quad (5.145)$$

for $l = 1, \dots, n$. Again we see that (5.145) amounts to n conditions that implicitely fix the unknown polynomial $\tilde{m}(z)$. However, in this case it is hard to solve these equations explicitly since the roots e_k appear inside the resolvent.

Now consider the matching (5.86) between the gauge singlets and the electric mesons. The mesons in the directions $(f, \tilde{f}) \neq (N_f, N_f)$ match as in the first massive case, reobtaining the map

$$\bar{V}'(z) = -V'(z), \qquad \bar{f}(z) = f(z), \qquad m_{\tilde{f}}^{f} = \tilde{m}_{\tilde{f}}^{f}. \quad (5.146)$$

The last direction $(f, \tilde{f}) = (N_f, N_f)$ gives a new condition, that we can write as

$$\oint_{A'} z^{l-1}\left[M_{el}(z)_{N_f}^{N_f} + \mu^2\frac{\tilde{m}(z)_{N_F}^{N_f}}{V'(z)}\right] = 0, \quad (5.147)$$

for $l = 1, \ldots, n$. The electric meson generator is given by (5.138) and we replaced the magnetic adjoint polynomial with the electric one by (5.146). The contour A' now is a very large contour that encircles the branch points of the resolvent as well as the electric pole at the point x_1 in the first sheet. Evaluating the contour integral at first order in the semiclassical expansion and dropping the higher terms we find

$$\sum_{i=1}^{n} \frac{a_i^{l-1} f(a_i)}{4m(a_i)V''(a_i)} - \frac{2R(x_1) - V'(x_1)}{m_2} x_1^{l-1} + \sum_{i=1}^{n} \mu^2 \frac{a_i^{l-1}\tilde{m}(a_i)}{V''(a_i)} = 0, \qquad (5.148)$$

for $l = 1, \ldots, n$.

Had we not allowed the contour to encircle the pole at x_1, this expression would have had an inconsistent classical limit. Let us consider in fact the classical limit of the conditions we have found so far. This is achieved by setting to zero the quantum deformation $f(z)$ so that the resolvent vanishes in the first sheet. It is more transparent to write the two classical conditions as contour integrals. We fix the classical polynomial $\tilde{m}_{cl}(z)$ by the singlet equations

$$\oint_{A_{cl}} z^{l-1} \left[\mu^2 \frac{m(z)}{V'(z)} + \frac{V'(z)}{\tilde{m}(z)} \right] = 0, \qquad (5.149)$$

for $l = 1, \ldots, n$, where the contour encircles all the poles of the two meromorphic functions. By picking up the residues we get

$$\mu^2 \sum_{i=1}^{n} \frac{a_i^{l-1} m(a_i)}{V''(a_i)} + \sum_{k=1}^{n} \hat{e}_k^{l-1} \frac{V'(\hat{e}_k)}{\tilde{m}'_{cl}(\hat{e}_k)} = 0, \qquad (5.150)$$

where we hatted the classical roots $\hat{e}_k$. This condition is much easier to solve than (5.145) due to the disappearance of the resolvent. Once we fix $\tilde{m}_{cl}(z)$, the classical limit of the matching condition (5.147) is satisfied

$$\oint_{A'_{cl}} z^{l-1} \left[\mu^2 \frac{\tilde{m}(z)}{V'(z)} + \frac{V'(z)}{m(z)} \right] = 0, \qquad (5.151)$$

for $l = 1, \ldots, n$, whose evaluation yields

$$\mu^2 \sum_{i=1}^{n} \frac{a_i^{l-1} \tilde{m}_{cl}(a_i)}{V''(a_i)} + x_1^{l-1} V'(x_1) m_2^{-1} = 0. \qquad (5.152)$$

5.4. The Cubic Superpotential

In this section we will illustrate the pseudoconfining and higgs phase computations, worked out in the previous sections, in the simplest example that allows for a higgs phase, namely a cubic interaction for the adjoint.

The Pseudoconfining Case

Let us consider an electric tree level superpotential as in (5.134) and let us specialize to $n = 2$. We take the following adjoint polynomial

$$V'(z) = t_1 z + t_2 z^2,$$

whose roots are $a_1 = 0$ and $a_2 = -t_1/t_2$. We also have a meson polynomial $m(z)_f^{\tilde{f}} = m_1 \delta_f^{\tilde{f}} + z\, m_2 \delta_f^{N_f} \delta_{N_f}^{\tilde{f}}$. The resolvent is $2R(z) = V'(z) - \sqrt{V'(z)^2 + f(z)}$ and its quantum deformation is $f(z) = f_0 + f_1 z$.

In the magnetic theory, all the flavor directions $(f, \tilde{f}) \neq (N_f, N_f)$ correspond to the massive case solved in Section *5.2*. In the following we will focus instead on the last direction $(f, \tilde{f}) = (N_f, N_f)$ only and suppress the flavor indices. We will see an explicit example of the computations in Section *5.2*. Let us consider $\tilde{m}(z) = \tilde{m}_1 + \tilde{m}_2 z$, whose one root we denote as $e_1 = -\tilde{m}_1/\tilde{m}_2$. The solution (5.115) of the magnetic theory is given by the condition $4\mu^2 m(a_i)\tilde{m}(a_i) = f(a_i)$ for $i = 1, 2$, from which we get the $\tilde{m}(z)$ coefficients in terms of $m(z)$ and $f(z)$

$$\begin{aligned} \tilde{m}_1 =& \frac{f_0}{4\mu^2 m_1}, \\ \tilde{m}_2 =& -\frac{t_2}{4\mu^2 t_1}\left(\frac{t_2 f_0 - t_1 f_1}{t_2 m_1 - t_1 m_2} - \frac{f_0}{m_1}\right), \end{aligned} \tag{5.153}$$

so that the singlet equations of motion (5.15) are satisfied. This condition also ensures that the singlets P_j extracted from (5.14) match the electric mesons $\tilde{Q}X^{j-1}Q$.

We would like to check that the root e_1 lies inside the contour $\tilde{A}$ that encircles the branch points of the resolvent. Consider the classical limit of this setup. In this limit both $f(z)$ and $\tilde{m}(z)$ vanish, but we still have to satisfy the singlet equations of motion. We first want to obtain the dependence of $f(z)$ on the total glueball and then perform the limit by sending the glueball to zero. For this purpose we have to choose a vacuum for the electric theory and solve the factorization of gauge theory curve. Let us consider the phase in which the gauge group is unbroken, which corresponds to the one–cut case, namely the electric adjoint is $X = \mathrm{diag}(a_i, \ldots, a_i)$. Then the curve factorizes as

$$V'(z)^2 + f(z) = t_2^2(z-k)^2(z-a+b)(z-a-b), \tag{5.154}$$

with one double root and two branch points. We already know from (5.42) that $f_1 = -4t_2S$ and we can find [42]

$$\begin{aligned} k &= -\frac{t_1}{t_2} + a, \\ a &= \frac{t_2}{t_1}S + \mathcal{O}(S^2), \\ b &= \sqrt{\frac{S}{2m}}\left(2 + \mathcal{O}(S)\right). \end{aligned} \tag{5.155}$$

We don't need the full result, but just the leading terms in the glueball, from which we find $f_0 = -2t_1S + \mathcal{O}(S^2)$. Then in the classical limit $S \to 0$ we have

$$\frac{f_0}{f_1} \sim \frac{t_1}{t_2} + \mathcal{O}(S). \tag{5.156}$$

The root e_1 of $\tilde{m}(z)$ in the classical limit is

$$\widehat{e}_1 = \frac{t_1}{t_2}\frac{t_2m_1 - t_1m_2}{t_1m_2 - 2t_2m_1}. \tag{5.157}$$

In the limit of large mass m_1 for the electric quarks, we find $\widehat{e}_1 \sim -\frac{t_1}{2t_2}$, which is not large but lie inside the contour $\tilde{A}$ that encircles the branch points of the resolvent, as we claimed below Eq.(5.108). In particular, this classical pole is halfway between the two roots of $V'(z)$.

The Higgs Phase

We keep the same superpotential, but consider now the electric higgs vacuum $X = \mathrm{diag}(x_1, 0, \ldots, 0)$ and

$$\tilde{Q}_{N_f} = (\tilde{h}_1, 0, \ldots, 0), \qquad Q^{N_f} = (h_1, 0, \ldots, 0), \tag{5.158}$$

where the gauge group is higgsed down to $U(N_c - 1)$ and the electric equations of motion set $\tilde{h}_1 h_1 = -V'(x_1)/m_2$. This vacuum is characterized by a nonvanishing classical expectation value for the electric mesons

$$\tilde{Q}_{N_f} X^{j-1} Q^{N_f} = (P_j)^{N_f}_{N_f} = -\frac{x_1^{j-1} V'(x_1)}{m_2}. \tag{5.159}$$

We want to check the prescription we outlined in Section *5.3* in the magnetic theory. In the higgs phase the singlets $P(z)$ as well as the magnetic polynomial $\tilde{m}(z)$ acquire a classical expectation value. From (5.159) we can read out their classical expressions

$$\begin{aligned} P(z)_{cl} =& \frac{\bar{V}'(\bar{x}_1)}{\bar{m}_2 z^2}(z + \bar{x}_1), \\ \tilde{m}(z)_{cl} =& \frac{\bar{V}'(\bar{x}_1)}{\mu^2 \bar{m}_2}(t_1 + t_2 \bar{x}_1 + z t_2). \end{aligned} \tag{5.160}$$

Now we would like to solve the quantum theory at first order in the semi-classical expansion. If we look at the flavor directions $(f, \widetilde{f}) \neq (N_f, N_f)$ we find the duality map (5.146), that we can use in the following computation. In the higgsed direction, first we have to solve the singlet equations of motion (5.145) and then check that the matching relation (5.148) is satisfied. But this is kind of hard, due to the presence of the resolvent in the last term of (5.145) that makes the equations pretty much involved. However, since the solutions of (5.148) must be solutions of (5.145) too, the best we can do is we solve the matching condition (5.148) and then try to check that this solution satisfies the singlet equations of motion (5.145), thus getting it the other way around.

The matching condition at first order is

$$\sum_{i=1}^{2} \left[-\frac{a_i^{l-1} f(a_i)}{4\bar{m}(a_i) V''(a_i)} - \mu^2 \frac{a_i^{l-1} \tilde{m}(a_i)}{V''(a_i)} \right] + \frac{2R(x_1) - V'(x_1)}{\bar{m}_2} x_1^{l-1} = 0, \tag{5.161}$$

for $l = 1, 2$. This can be solved easily with the result

$$\begin{aligned} \tilde{m}_1 =& -\frac{f_0 - 4V'(\bar{x}_1)[V'(\bar{x}_1) - 2R(\bar{x}_1)]}{4\mu^2 \bar{m}_1}, \\ \tilde{m}_2 =& \frac{t_2}{4\mu^2 \bar{m}_2 V'(\bar{x}_1)} \left[f(\bar{x}_1) - 4V'(\bar{x}_1)[V'(\bar{x}_1) - 2R(\bar{x}_1)] \right]. \end{aligned} \tag{5.162}$$

Quantum mechanically, the one root of $\tilde{m}(z)$ is $e_1 = -\tilde{m}_1/\tilde{m}_2$. If we take the classical limit of (5.162) we obtain the expected expression (5.160) and its classical root $\widehat{e}_1 = -x_1 - t_1/t_2$, by identifying $\tilde{m}(z) = m(z)$. In the semiclassical electric picture in which the higgs vev x_1 is large, this root gets very large, too. This phase is very different from (5.157), where for large electric quark masses we got small $\widehat{e}_1$.

It would be very hard to check that (5.162) satisfies the singlet equations of motion (5.145) at first order in the semiclassical expansion, due to the fact that the resolvent should be evaluated at the root of $\tilde{m}(z)$. But one can still easily check that indeed the classical limit of the singlet equations

$$\mu^2 \sum_{i=1}^{2} \frac{a_i^{l-1} m(a_i)}{V''(a_i)} + \widehat{e}_1^{l-1} \frac{V'(\widehat{e}_1)}{\tilde{m}'_{cl}(\widehat{e}_1)} = 0, \qquad l = 1, 2, \tag{5.163}$$

is satisfied by $\widehat{e}_1$ and the classical limit of (5.162).

5.5. Discussion

Let us summarize our results and suggest some further speculations. At the classical level, we generalized the KSS solution to the case of polynomial superpotentials, allowing for generic meson deformations, and we found the solutions of the magnetic theory corresponding to the electric pseudoconfining and higgs vacua. We considered then duality in the quantum theory and we used the DV approach to solve for the chiral rings just above the mass gap: we studied the effective glueball superpotential. We analyzed the following three cases:

1. The electric meson superpotential is a mass term for all the flavors. We saw that electric–magnetic duality holds exactly offshell in this case.
2. The generic pseudoconfining phase, where we allow for a generic meson deformation, has a way richer analytic structure. We matched the electric mesons with the magnetic singlets at first order in the semiclassical expansion of the resolvent. In this way we found a condition that reproduces the Konishi anomaly equation in the low energy SQCD blocks and their magnetic dual. In this case duality does not hold exactly offshell.
3. In the electric higgs phase, we found the solution to the magnetic theory, at first order in the semiclassical expansion, and showed that it is consistent

with the classical limit. Neither in this case does duality work exactly off-shell. Moreover, while in the pseudoconfining case the classical singularities in $\tilde{M}(z)$ are just an artifact of the classical solution and in the quantum theory they disappear, in the higgs case the classical singularities are preserved in the quantum theory.

We could draw a picture of the analytic properties of the magnetic theory as we continuously interpolate between different higgs vacua in the electric theory (when we move poles from the second to the first electric sheet). An interesting extension of our analysis would be to show what happens on the magnetic side when we smoothly pass from the pseudoconfining to the higgs phase in the electric theory. In this way, one might shed some light on the onshell process that takes place when a branch cut of the resolvent closes up, as recently investigated in [43]. On the electric side this is a strong coupling phenomenon, but one should describe it easily in the dual regime.

On the other hand, it would be interesting to use our quantum duality map to gain insight on the meaning of the electric parameter L introduced in [19] as the degree of the determinant of the meson polynomial $B(z) = \det m(z)$. This parameter plays the role of an effective number of flavors and is related to the appearence of instanton corrections to the classical chiral ring. In particular, if the electric superpotential $V(z)$ has degree $N_c + 1$, when $L \geq N_c$ the strong coupling analysis shows that the classical Casimirs $\mathrm{Tr}X^j$ for $j = 1, \ldots, N_c$ are modified in the quantum chiral ring by terms proportional to the instanton factor. It would be interesting to understand the corresponding phenomenon in the magnetic theory. In our setup, $L \leq N_f(n-1)$, so the condition for the appearence of instanton corrections is related to $N_f \leq \bar{N}_c$ on the magnetic side.

A natural generalization of our analysis would be to consider $SO(N_c)$ and $Sp(2N_c)$ gauge groups. In particular, one could translate into a magnetic language the map between $Sp(2N_c)$ theory with an antisymmetric tensor and $U(2N_c + 2n)$ with an adjoint, recently proposed in [44].

We would like to make one last remark on the theory without superpotential, whose magnetic dual is not known. In [16] it was suggested that one might try to obtain this theory as a certain limit of the KSS theory with superpotential $t_n \mathrm{Tr}X^{n+1}$. Since the limit of vanishing t_n is singular, it was suggested to study the $k \to \infty$ limit instead, so that the magnetic dual might look like an $U(\infty)$ gauge theory, which is expected to behave like a string theory. The story might be simpler, though. Due to the recent work of Intriligator and Wecht [3], we know that an analogue of the conformal window of SQCD exists also for the KSS theory: it is the region in the range of N_f in which both the electric and the magnetic deformations $\mathrm{Tr}X^{n+1}$ and $\mathrm{Tr}Y^{n+1}$ are relevant [29]. Now, if we take a sufficiently large number of flavors we can make the deformation $\mathrm{Tr}X^{n+1}$ irrelevant, but still keeping the electric theory asymptotically free. Therefore, the electric theory at the fixed point will be the theory without superpotential. But on the magnetic side, the corresponding superpotential keeps being relevant and we have the usual full magnetic theory. So we might not really need to take k very large to remove the electric superpotential, hence the magnetic dual of the theory without superpotential need not be a kind of string theory. This point might deserve further study.

Appendix A. Some Properties of the Effective Glueball Superpotential

In this Appendix we will consider some properties of the coefficients h_i introduced in (5.74) and, by using these expressions, we will prove (5.75) up to an assumption of integrability.

A.1. Properties of the h_i

Consider the function $V(a_i)$ of the couplings t_j defined as

$$V(a_i) = \sum_{j=1}^{n} \frac{t_j}{j+1} a_i^{j+1}, \tag{5.164}$$

where a_i is solution of $V'(a_i) = 0$. Note that we are considering the $a_i = a_i(t_j)$ as functions of the couplings. Taking a derivative of $V(a_i)$ with respect to t_k we obtain

$$\frac{\partial V(a_i)}{\partial t_k} = \frac{a_i^{k+1}}{k+1}, \tag{5.165}$$

the second term in taking the derivative vanishing since it is multiplied by $V'(a_i)$. Since (5.165) is a derivative, it fulfills the condition

$$\frac{\partial}{\partial t_l} \frac{a_i^{k+1}}{k+1} = \frac{\partial}{\partial t_k} \frac{a_i^{l+1}}{l+1}, \tag{5.166}$$

and therefore

$$\frac{\partial}{\partial t_{n-l}} \frac{a_i^{j+l}}{j+l} = \frac{\partial}{\partial t_{n-k}} \frac{a_i^{j+k}}{j+k}. \tag{5.167}$$

which is our classical integrability condition.

Now we will assume that also the effective superpotential (5.75) satisfies the integrability condition

$$\frac{\partial^2 \mathcal{W}_{eff}}{\partial t_l \partial t_j} = \frac{\partial^2 \mathcal{W}_{eff}}{\partial t_j \partial t_l}. \tag{5.168}$$

By using the classical integrability (5.167), we find the relation

$$\sum_{i=1}^{n} \frac{\partial h_i}{\partial t_l} \frac{a_i^{j+1}}{j+1} = \sum_{i=1}^{n} \frac{\partial h_i}{\partial t_j} \frac{a_i^{l+1}}{l+1}. \tag{5.169}$$

Note that this relation will hold also for the d_i defined in (5.77)

$$\sum_{i=1}^{n} \frac{\partial d_i}{\partial t_l} \frac{a_i^{j+1}}{j+1} = \sum_{i=1}^{n} \frac{\partial d_i}{\partial t_j} \frac{a_i^{l+1}}{l+1}. \tag{5.170}$$

Finally, let us consider a scaling argument on the coefficients $h_i = h_i(t_k, N_l, S_j)$. Since

$$N_c = \sum_{i=1}^{n} h_i = \oint_A \frac{c(z)}{\sqrt{V'(z)^2 + f(z)}}, \tag{5.171}$$

if we rescale the glueballs $S_i \longrightarrow \lambda S_i$ and the couplings $t_k \longrightarrow \lambda t_k$, we have correspondingly that $V'(z) \longrightarrow \lambda V'(z)$ and $f(z) \longrightarrow \lambda^2 f(z)$, while the N_i are unchanged in (5.41). But by (5.171) also the h_i are invariant under the scaling, meaning that they are homogeneous functions of the couplings and the glueballs

$$\sum_{i=1}^{n} \left(t_i \frac{\partial}{\partial t_i} + S_i \frac{\partial}{\partial S_i} \right) h_l = 0, \tag{5.172}$$

and this property carries on to the d_i.

A.2. Evaluation of $\mathcal{W}_{eff}$

We will prove, up to the assumption (5.168), that, in the notations of Section *5.2*, if we define

$$\mathcal{W}_{eff} = \sum_{i=1}^{n} d_i V(a_i), \tag{5.173}$$

then we have

$$\frac{\partial \mathcal{W}_{eff}}{\partial t_j} = \frac{1}{j+1} \sum_{i=1}^{n} h_i a_i^{j+1}. \tag{5.174}$$

Let us differentiate (5.173)

$$\frac{\partial \mathcal{W}_{eff}}{\partial t_j} = \sum_{i=1}^{n} \frac{d_i}{j+1} a_i^{j+1} + \sum_{i,k=1}^{n} t_k \frac{\partial d_i}{\partial t_j} \frac{a_i^{k+1}}{k+1} + \sum_{i=1}^{n} d_i \frac{\partial a_i}{\partial t_j} \sum_{k=1}^{n} t_k a_i^k, \tag{5.175}$$

but the last term vanishes since $V'(a_i) = 0$. Now we need to evaluate $\partial d_i / \partial t_j$. First note that $\partial_y d_i = d_i - h_i$ where $y = \sum_{i=1}^{n} \log S_i$. Then by using the homogeneity (5.172) of the d_i we have that

$$\sum_{k=1}^{n} t_k \frac{\partial d_i}{\partial t_k} = -\partial_y d_i. \tag{5.176}$$

Then we can use the integrability condition (5.170) for the second term in (5.175) and get

$$\begin{aligned} \frac{\partial \mathcal{W}_{eff}}{\partial t_j} &= \sum_{i=1}^{n} d_i \frac{a_i^{j+1}}{j+1} - \sum_{i=1}^{n} \frac{\partial d_i}{\partial y} \frac{a_i^{j+1}}{j+1} \\ &= \sum_{i=1}^{n} d_i \frac{a_i^{j+1}}{j+1} - \sum_{i=1}^{n} (d_i - h_i) \frac{a_i^{j+1}}{j+1} \\ &= \sum_{i=1}^{n} h_i \frac{a_i^{j+1}}{j+1}. \end{aligned} \tag{5.177}$$

Appendix B. Several Higgs Solution

In this Appendix we will generalize the higgs solution (5.124) to the case in which more than one electric color direction is higgsed on the same electric flavor.

B.1. Two–Higgs case

The Electric Theory

Consider the electric theory with superpotential

$$W_{el} = \frac{t_n}{n+1}\mathrm{Tr}X^{n+1} + m_3\tilde{Q}_{N_f}X^2Q^{N_f} + m_2\tilde{Q}_{N_f}XQ^{N_f} + m_1Q_{N_f}Q^{N_f}. \quad (5.178)$$

We can get a classical vacuum in which the gauge group is higgsed as $U(N_c) \to U(N_c - 2)$ by considering the following expectation values

$$\begin{aligned} X =& \mathrm{diag}(x_1, x_2, 0, \dots, 0) \\ \tilde{Q}_{N_f} =& (\tilde{h}_1, \tilde{h}_2, 0, \dots, 0), \quad Q^{N_f} = (h_1, h_2, 0, \dots, 0), \end{aligned} \quad (5.179)$$

where $\tilde{h}_i h_i = -V'(x_i)/m'(x_i)$. We denoted by $x_{1,2}$ the two roots of the meson polynomial $m_3z^2+m_2z+m_1$. As is well known, the roots of a quadratic algebraic equation satisfy

$$-(x_1 + x_2) = \frac{m_2}{m_3}, \qquad x_1x_2 = \frac{m_1}{m_3}. \quad (5.180)$$

The Magnetic Theory

The superpotential for the magnetic theory is

$$W_{mag} = -\frac{t_n}{n+1}\mathrm{Tr}X^{n+1} + \tilde{q}\tilde{m}(P,Y)q + m_1(P_1)^{N_f}_{N_f} + m_2(P_2)^{N_f}_{N_f} + m_3(P_3)^{N_f}_{N_f}. \quad (5.181)$$

By (5.128), the singlets acquire the classical expectation value $P_j = \tilde{h}_1h_1x_1^{j-1} + \tilde{h}_2h_2x_2^{j-1}$ corresponding to the electric mesons.

We expected the magnetic gauge group to break down to $U(\bar{N}_c) \to U(\bar{N}_c - n+2)$, so that the vev for the adjoint will be a nonvanishing block of rank $n-2$. By using the property (5.180) of the roots of $m(x)$ we can find the solution to the singlet, fundamental and adjoint equations of motion. In the notations of (5.20), the only nonvanishing entries in the adjoint are

$$\begin{aligned} Y =& |1\rangle\langle v_2| + b_3R_{n-2}, \\ |v_2\rangle =& -\sum_{j=1}^{n-2}\frac{1}{b_3}^{j-1}\Bigg[\left(-\frac{m_2}{m_3}\right)^j + \sum_{k=1}^{[j/2]}(j-k)\left(-\frac{m_2}{m_3}\right)^{j-2k}\left(-\frac{m_1}{m_3}\right)^k \\ &+ \delta^j_{even}\binom{j}{2}\left(-\frac{m_1}{m_3}\right)^{[j/2]}\Bigg]|j\rangle, \end{aligned} \quad (5.182)$$

while the fundamentals are $|q_{N_f}\rangle = b_3|n-2\rangle$ and $|\tilde{q}^{N_f}\rangle = b_3|1\rangle$ and b_3 is defined in (5.17).

B.2. Several Higgs

The Electric Theory

Consider the electric theory with a generic meson polynomial on the last flavor

$$\begin{aligned} W_{el} =& \frac{t_n}{n+1}\mathrm{Tr}X^{n+1} + \tilde{Q}_{N_f} m(X) Q^{N_f}, \\ m(x) =& \sum_{k=1}^{l+1} m_k x^{k-1}. \end{aligned} \tag{5.183}$$

The polynomial $m(x)$ has l roots that we denote $x_1, \ldots, x_l$. The following property between the coefficients of the polynomial and its roots holds

$$\frac{m_{l+1-i}}{m_{l+1}} = (-)^i \sum_{k_1<k_2<\ldots<k_i} x_{k_1} x_{k_2} \ldots x_{k_i}. \tag{5.184}$$

We can consider the following vacuum

$$\begin{aligned} X =& \mathrm{diag}(x_1, x_2, \ldots, x_l, 0, \ldots, 0) \\ \tilde{Q}_{N_f} =& (\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_l, 0, \ldots, 0), \quad Q^{N_f} = (h_1, h_2, \ldots, h_l, 0, \ldots, 0), \end{aligned} \tag{5.185}$$

where $\tilde{h}_i h_i = -V'(x_i)/m'(x_i)$. Note that each root x_k can appear just once in the adjoint expectation value. In this way we break the gauge symmetry as $U(N_c) \to U(N_c - l)$. We can higgs at most $n-1$ colors on the same flavor, corresponding to the largest degree the meson polynomial $m(z)$ can have.

The Magnetic Theory

According to the above discussion, in the magnetic theory we will have to solve the singlet equations of motion

$$\begin{aligned} \tilde{q}^{N_f} Y^{n-i} q_{N_f} =& \frac{m_i}{m_{l+1}} b_{l+1}^{n-l+1}, \qquad i = 1, \ldots, l+1, \\ \tilde{q}^{N_f} Y^j q_{N_f} =& 0, \quad j = 0, \ldots, n-l-2, \end{aligned} \tag{5.186}$$

where we used (5.184) to ease the notation and b_{l+1} is defined in (5.17). The solution for the adjoint, which generalizes (5.182), can be sketched as the nonvanishing block of rank $n-l$

$$\begin{aligned} Y =& |1\rangle\langle v_l| + b_{l+1} R_{n-l}, \\ |v_l\rangle =& -\sum_{j=1}^{n-l} \frac{1}{b_{l+1}}^{j-1} \left[\left(-\frac{m_l}{m_{l+1}} \right)^j + \dots \right] |j\rangle, \\ |q_{N_f}\rangle =& b_{l+1}|n-l\rangle, \qquad |\tilde{q}^{N_f}\rangle = b_{l+1}|1\rangle, \end{aligned} \tag{5.187}$$

where the dots stand for an expression analogous to the one in (5.182) but more involved. In this way we break the magnetic gauge group down to $U(\bar{N}_c - n + l) = U(n(N_f - 1) - (N_c - l))$. Note that this solution holds only for $l \leq n-1$, as we saw on the electric side.

6. D_{n+2}: A NEW PHASE AND THE CUBIC CURVE

In the previous Chapter we have given a detailed analysis of the effective description of the A_n theory and its dual in the confining phase. The next fixed point in the ADE classification of (3.1) is the D_{n+2} theory. Here we will present a detailed analysis of the chiral ring of this theory in the confining phase, classically as well as quantum mechanically. This discussion follows [45]. We discussed in Section *3.2* the SCFT living at the infrared fixed point, its phase diagram (3.11) and Brodie's magnetic dual, following [3][22]. Here we will consider an $SU(N_c)$ rather than $U(N_c)$ gauge group and we will do this by adding two Lagrange multipliers that enforce the tracelessness condition on the adjoints X and Y. The tree level superpotential of the electric theory will be (1.8). The reason why we drop the overall abelian factor is the following. If we consider the low energy theory we flow to in the generic confining vacuum, we find a product of semiclassically decoupled SQCDs. In the $U(N_c)$ theory the low energy spectrum of these SQCDs contains a massless degree of freedom consisting in the overall $U(1)$ part of the second adjoint $\mathrm{Tr}Y$. This field, albeit being neutral under the gauge group, nevertheless couples to the light flavors through Yukawa type interactions. When a massless particle is present in the low energy dynamics, we cannot make use of the DV method any more, since it requires the presence of a mass gap. In general, as we will see, the duality map in the $SU(N_c)$ case is way more complicated than the $U(N_c)$ case, since we need to impose the tracelessness condition on the magnetic adjoints.

In this Chapter, we will at first introduce the classical theory and show that classically there are three different phases: *pseudoconfining*, *abelian higgs* and *nonabelian higgs* phases. The one–adjoint theory we discussed in the previous Chapter has only the first two phases instead. The pseudoconfining vacua are interesting because of the presence of one as well as two dimensional irreps of the equations of motion, giving rise to what we will call abelian and nonabelian pseudoconfining vacua.

In Section 6.2 we will then use the generalized Konishi anomalies to compute some anomalous Ward identities, following [21]. At the end we will obtain the curve of the gauge theory, that is a cubic algebraic equation, giving rise to a three sheeted covering of the plane. This is a new feature, since the usual one–adjoint SQCD is described by an hyperelliptic Riemann surface. We will analyze in detail the semiclassical expansion of the resolvents and identify their analytic structure in terms of the branch points and the holomorphic differentials. There are some subtleties in the definition of the glueballs for the nonabelian cuts, due to an automorphism of the cubic curve.

We will begin the study of the phase structure of the quantum theory in Section 6.3, where we study the meson operators. We will see that each of the three phases is characterized by the meson generator $M(x)$ being regular on one of the three sheets. In this way we can associate each sheet to a different phase and we will confirm the general idea that, if an $\mathcal{N} = 1$ SQCD has n different classical phases, then its curve is a degree n algebraic equation, giving rise to an n–sheeted covering of the plane.

The three phases are continuously connected in the full quantum theory, and we will show this in Section 6.4 and 6.5 by studying the analytic behaviour of the resolvent $T(x)$ as we vary the parameters in the bare lagrangian. In Section 6.6 we will consider the D_3 case in detail.

In Section 6.7 we will confirm a prediction of the a theorem [3], that the chiral ring of $D_{n'+2}$ for n' even, classically untruncated, gets truncated along the flow from an odd n, $n > n'$. In Section 6.8 then we discuss our proposal that the number of sheets of the algebraic curve is equal to the number of branches of the semiclassical theory and check it to hold in the case of SQCD with different extra matter.

Finally, in Section 6.9 we will address the magnetic theory. We would like to extend the duality map of the Brodie's SCFT [22] to the generic confining phase superpotential, by borrowing some tricks of singularity theory, along the lines of KSS [16]. The classical duality map can be worked out, with some unusual features. Unfortunately, the anomaly equations do not close on a finite set of resolvents and so we cannot use the DV method to study the quantum theory as in [37]. Only in the D_3 case we can completely solve the quantum theory and find the map between the quantum deformations.

In Section 6.10 we present some speculations and further directions.

6.1. The Classical Theory

In this section we study the classical vacua of the theory. We have the usual pseudoconfining vacua, with vanishing fundamentals, and we distinguish them in abelian ones, that is one dimensional irreps of the algebra of the equations of motion, and nonabelian ones, denoting two dimensional irreps. Then we have the abelian higgs vacua and a new classical phase that we will call nonabelian higgs vacuum. The theory is slightly different depending on whether n is odd or even. In the following we will consider in detail the former case. In the latter, as we will show, the pseudoconfining vacua are still one and two dimensional only, but the chiral ring is not truncated. The analysis of the quantum theory goes through for both cases with analogous treatments.

Consider an $\mathcal{N}=1$ supersymmetric $SU(N_c)$ gauge theory with matter content consisting in two chiral superfields X and Y in the adjoint representation, N_f fundamentals Q^f and N_f anti-fundamentals $\widetilde{Q}_{\widetilde{f}}$ (f and $\widetilde{f}$ are the flavor indices). We let this theory flow to its IR fixed point and then we turn on the following tree level superpotential

$$W = \mathrm{Tr}V(X) + \lambda\mathrm{Tr}XY^2 + \alpha\mathrm{Tr}Y + \beta\mathrm{Tr}X + Q\,m(X)Q. \tag{6.1}$$

where we suppressed flavor indices and we introduced the adjoint polynomial

$$V(z) = \sum_{k=1}^{n} \frac{t_k}{k+1} z^{k+1} \tag{6.2}$$

and the meson deformation $m(x) = m_1 + m_2\,x$ is diagonal in the flavor indices, while α and β are two Lagrange multipliers enforcing the tracelessness condition.[21] It will be convenient in the following to separate the odd and even part of the adjoint polynomial as $V'(x) = -v_+(x^2) - xv_-(x^2)$. The equations of motion are

$$\begin{aligned} &V'(X) + \lambda Y^2 + m_2\widetilde{Q}Q = 0, \\ &\lambda\{X,Y\} + \alpha = 0, \end{aligned} \tag{6.3}$$

[21] The superpotential (6.2) would be irrelevant in the UV for $n > 2$, however there always exists a range of flavors N_f such that it is a relevant deformation of the IR fixed point [3][28].

$$\widetilde{Q}m(X) = 0, \quad m(X)Q = 0. \tag{6.4}$$

The (6.3) are the X and Y equations of motion, while (6.4) are the equations for the fundamentals. In addition, by varying (6.1) with respect to the Lagrange multipliers we get the tracelessness condition $\operatorname{Tr} X = \operatorname{Tr} Y = 0$.

Pseudoconfining Vacua

We consider at first the pseudoconfining vacua, in which the fundamentals vanish. We want to study the irreducible representations of the algebra defined by the adjoint equations of motion (6.3) for $\langle Q\rangle = \langle\tilde{Q}\rangle = 0$. The Casimirs are $X^2 = x^2\mathbb{1}$, $Y^2 = y^2\mathbb{1}$. Then the first equation reads $\lambda y^2 = v_+(x^2) + Xv_-(x^2)$ and we can outline two different cases.

i) *abelian vacua*

The one–dimensional representation are the solutions to

$$\begin{cases} y = -\frac{\alpha}{2\lambda x}, \\ \lambda y^2 + V'(x) + \beta = 0. \end{cases} \tag{6.5}$$

Thus we have $n + 2$ vacua

$$\langle X\rangle = \begin{pmatrix} a_1 & & & \\ & \cdot & & \\ & & a_2 & \\ & & & \cdot \\ & & & & \cdot \end{pmatrix}, \quad \langle Y\rangle = \begin{pmatrix} b_1 & & & \\ & \cdot & & \\ & & b_2 & \\ & & & \cdot \\ & & & & \cdot \end{pmatrix} \tag{6.6}$$

where the X expectation values a_i are the roots of the degree $n + 2$ abelian polynomial

$$p(x) = x^2[V'(x) + \beta] + \frac{\alpha^2}{4\lambda} = 0, \tag{6.7}$$

and $b_i = -\frac{\alpha}{2\lambda a_i}$. Each a_i, b_i has multiplicity N_i such that $\sum_{i=1}^{n+2} N_i = N_c$. The symmetry breaking pattern is $SU(N_c) \to U(1)^{n+1} \times \prod_{i=1}^{n+2} SU(N_i)$.

ii) *nonabelian vacua*

The only higher dimensional irreps are two dimensional ones, that we parameterize in terms of the Pauli matrices $X = \widehat{a}_i\sigma_3$ and $Y = c_i\sigma_1 + d_i\sigma_3$. To satisfy the X equation of motion, the odd part of the adjoint polynomial must vanish, so we have $\frac{n-1}{2}$ nonabelian vacua $\widehat{a}_i$ which are the roots of $v_-(x^2) = 0$. The Y expectation values are $d_i = -\frac{\alpha}{2\lambda}\frac{1}{\widehat{a}_i}$ and $c_i = \sqrt{\lambda^{-1}(v_+(\widehat{a}_i^2) - \beta) - d_i^2}$. The nonabelian vacua display a $\mathbb{Z}_2$ symmetry that acts by reflection of the eigenvalues around the origin. Note also that $x = 0$ is not a solution. Consider the gauge symmetry breaking in the nonabelian vacua, for simplicity consider unbroken gauge group $SU(N_c)$ with N_c even. The generic nonabelian vacuum is given by

$$\langle X \rangle = \begin{pmatrix} \widehat{a}_1\sigma_3 & & & \\ & \cdot & & \\ & & \widehat{a}_2\sigma_3 & \\ & & & \cdot \\ & & & & \cdot \end{pmatrix}, \qquad \langle Y \rangle = \begin{pmatrix} c_1\sigma_1 + d_1\sigma_3 & & & \\ & \cdot & & \\ & & c_2\sigma_1 + d_2\sigma_3 & \\ & & & \cdot \\ & & & & \cdot \end{pmatrix}, \tag{6.8}$$

where each $\widehat{a}_i$ has multiplicity $\widehat{N}_i$ such that $2\sum_{i=1}^{\frac{n-1}{2}} \widehat{N}_i = N_c$. In this case, unlike the usual one–dimensional one, the vacuum decreases the rank of the gauge group. The gauge symmetry is broken as $SU(N_c) \to U(1)^{\frac{n-1}{2}} \times \prod_{i=1}^{\frac{n-1}{2}} SU(\widehat{N}_i)$.

One can easily show that there are no higher dimensional irreps of the equations of motion (6.3), following [20]. One can shift $X \to X + aY$ and $Y \to Y + bX$ and get to a new algebra with $X^2 = Y^2 = 0$ and $\{X, Y\} + c = 0$. This algebra has just one irreducible representation, which is two dimensional and corresponds to the Fock space of a single fermionic creation–annihilation algebra.[22] The generic gauge symmetry breaking pattern, in the pseudoconfining case, is the following

$$SU(N_c) \longrightarrow U(1)^{\frac{3}{2}(n+1)-1} \times \prod_{i=1}^{n+2} SU(N_i) \times \prod_{i=1}^{\frac{n-1}{2}} SU(\widehat{N}_i), \tag{6.9}$$

where $N_c = \sum_{i=1}^{n+2} N_i + 2\sum_{i=1}^{\frac{n-1}{2}} \widehat{N}_i$. At energies below the vevs but above the dynamical scale of the theory, we flow to a bunch of $\frac{3}{2}(n+1)$ low energy SQCDs with massive fundamentals, whose number is N_f in the $n+2$ abelian sectors and $2N_f$ in the $(n-1)/2$ nonabelian sectors. At low energies, the nonabelian factors confine and we are left with a $U(1)^{\frac{3}{2}(n+1)-1}$ theory for odd n.

The Higgs Vacua

The equations of motion (6.3) allow also for higgs solutions, in which the fundamentals acquire a vacuum expectation value. The Yukawa coupling contains just terms in the dressed X–mesons. There are two different kinds of higgs

[22] This argument holds irrespectively of n, so we have a finite number of vacua both if n is odd and even.

solutions. The first one is the usual one dimensional vacuum, that we will denote abelian higgs, but it turns out that there are also new two dimensional solutions, similar to (6.8), that we will denote nonabelian higgs. We consider for simplicity the higgsing of just the last flavor.

i) *abelian higgs*

The usual one–dimensional higgs vacua are given by

$$\begin{aligned} \widetilde{Q}_{N_f} &= (\widetilde{h}, 0, \ldots, 0), \quad Q^{N_f} = (h, 0 \ldots, 0), \\ X &= \mathrm{diag}(x_h, 0, \ldots, 0), \quad Y = \mathrm{diag}(y_h, 0 \ldots, 0), \end{aligned} \tag{6.10}$$

where x_h is a root of the meson deformation $m(x)$, i.e. $x_h = -m_1/m_2$, and $y_h = -\frac{\alpha}{2\lambda x_h}$ and the squark expectation values are fixed by the X equations of motion to $\widetilde{h}h = -\frac{1}{m_2}[\lambda y_h^2 + V'(x_h)]$. This solution higgses the gauge group $SU(N_c)$ down to $SU(N_c - 1)$.

ii) *nonabelian higgs*

The equations of motion (6.3) admit also two–dimensional representations with nonvanishing fundamentals

$$\begin{aligned} \langle X \rangle &= \mathrm{diag}(x_h \sigma_3, 0, \ldots 0), \qquad \langle Y \rangle = \mathrm{diag}(y_1 \sigma_1 + y_h \sigma_3, 0, \ldots, 0) \\ \widetilde{Q}_{N_f} &= (\widehat{\widetilde{h}}, 0, \ldots, 0), \qquad Q^{N_f} = (\widehat{h}, 0 \ldots, 0), \end{aligned} \tag{6.11}$$

where x_h is always a root of the meson deformation $m(x)$, while

$$y_h = -\frac{\alpha}{2\lambda x_h}, \qquad \lambda \widehat{y}_1^2 + V'(-x_h) + \frac{\alpha^2}{4\lambda x_h^2} = 0,$$

and the quark expectation values are $\widehat{\widetilde{h}}\widehat{h} = -\frac{1}{m_2}[V'(x_h) - V'(-x_h)]$. This is a new classical phase of SQCD and it higgses the gauge groups $SU(N_c)$ down to $SU(N_c - 2)$.

D–terms

Consider the kinetic term for the adjoints

$$\int d^2\theta \, d^2\bar{\theta} \left(X^\dagger e^{adV} X + Y^\dagger e^{adV} Y \right),$$

the D–term equations of motion are $[X, X^\dagger] + [Y, Y^\dagger] = 0$. The abelian vacua (6.6), satisfy the D–term equation as usual. For the nonabelian vacua (6.8) and (6.11), however, due to the nonvanishing commutator of the Pauli matrices, we get the additional condition

$$\begin{array}{cc} \text{pseudoconf.} & \text{nonabelian higgs} \\ \text{Im } cd^* = 0, & \text{Im } \widehat{y}_1 y_h^* = 0. \end{array} \tag{6.12}$$

Note also that, if we set to zero the Lagrange multiplier α, then the term proportional to σ_3 in $\langle Y \rangle$ vanishes, so that the nonabelian vacuum automatically satisfies the D–term. This would amount to consider Y transforming in the adjoint of $U(N_c)$, rather than $SU(N_c)$. In this way we would get rid of this additional D–term condition, since the vev that is subject to the constraint is proportional to α. However, if we compute the low energy matter content in the nonabelian vacua (6.8), we find that the TrY, which is the $U(1)$ part of the adjoint, becomes massless in this case. Albeit being neutral under the gauge interactions, the TrY field interacts with the other massive low energy degrees of freedom through superpotential terms. On the other hand, we need a mass gap in order to make sense of the glueball superpotential, so we are forced to keep the Lagrange multiplier α and the additional constraint (6.12).

The Classical Chiral Ring

Consider the superpotential (6.1) and for simplicity drop all the lower relevant operators, keeping just the leading deformations

$$W = \frac{t_n}{n+1}\text{Tr}X^{n+1} + \lambda\text{Tr}XY^2 + \beta\text{Tr}X + \alpha\text{Tr}Y.$$

In this case the theory is superconformal and its flows have been studied in [3]. Using the equations of motion we get

$$t_n\left((-)^n + 1\right)X^kY = -2\lambda Y^3, \tag{6.13}$$

so that in the n odd case the chiral ring is truncated to $Y^3 = Y$ and is generated by the products Tr$X^{k-1}Y^{j-1}$, for $k = 1, \ldots, n$ and $j = 1, 2, 3$, regardless of the ordering. Due to $\{X, Y\} = -\frac{\alpha}{\lambda}$ and the cyclicity of the trace, the only nonvanishing chiral ring operators are actually

$$\begin{aligned} &\mathrm{Tr}X^{k-1}, \quad k = 3, \ldots, n, \qquad \mathrm{Tr}Y^2, \\ &\mathrm{Tr}X^{2k}Y^2, \quad k = 1, \ldots, \frac{1}{2}(n-1), \end{aligned} \tag{6.14}$$

and also the dressed mesons

$$M_{kj} = \widetilde{Q}X^{k-1}Y^{j-1}Q, \quad k = 1, \ldots, n; \quad j = 1, 2, 3. \tag{6.15}$$

In the n even case (6.13) does not do the job, so that, apparently, the chiral ring would not be truncated. We will consider just the n odd case in the following and get back to this issue in Section [], where we will show that indeed, by considering the flow from n odd to n' even, with $n' < n$, the chiral ring is truncated also in the even case.

We will be interested in solving for the expectation values of the operators of the chiral ring. We can collect them in four generating functions

$$\begin{aligned} Z(x,y) &= -\frac{1}{32\pi^2}\left\langle \mathrm{Tr}\frac{W_\alpha W^\alpha}{x-X}\frac{1}{y-Y}\right\rangle, \\ u_\alpha(x,y) &= \frac{1}{4\pi}\left\langle \mathrm{Tr}\frac{W_\alpha}{x-X}\frac{1}{y-Y}\right\rangle, \\ U(x,y) &= \left\langle \mathrm{Tr}\frac{1}{x-X}\frac{1}{y-Y}\right\rangle, \\ M_{\tilde{f}}^{f}(x,y) &= \left\langle \widetilde{Q}_{\tilde{f}}\frac{1}{x-X}\frac{1}{y-Y}Q^f\right\rangle. \end{aligned} \tag{6.16}$$

In a supersymmetric vacuum u_α must be vanishing, therefore we set it to zero. These loop functions (6.16) can be expanded in Laurent series of x or y, for instance the first one is

$$Z(x,y) = \sum_{k=0}^{\infty} x^{-1-k}R_k^Y(y) = \sum_{k=0}^{\infty} y^{-1-k}R_k^X(x), \tag{6.17}$$

where we introduced the *generalized resolvents*

$$R_k^X(x) = -\frac{1}{32\pi^2}\left\langle \mathrm{Tr}\frac{W_\alpha W^\alpha}{x-X}Y^k\right\rangle, \tag{6.18}$$

and analogously for $R_k^Y(y)$. The leading term in the expansion (6.17) is the usual resolvent of the one–adjoint theory. It will be useful to introduce also a *generalized glueball* $\widetilde{S} = -\frac{1}{32\pi^2}\langle \mathrm{Tr} W_\alpha W^\alpha Y\rangle$. Since all the single trace operators of two adjoints can be extracted from $R_k^X(x)$, we can just solve for this operators and do not consider $R_k^Y(y)$. Analogous expressions to (6.18) hold for the other generalized resolvents

$$M_k(x) = \left\langle \widetilde{Q}\frac{1}{x-X}Y^k Q\right\rangle, \qquad T_k(x) = \left\langle \mathrm{Tr}\frac{1}{x-X}Y^k\right\rangle. \tag{6.19}$$

Let us consider the semiclassical expressions for the generators that we obtain by plugging into (6.16) the solutions for $\langle X\rangle$ and $\langle Y\rangle$ and for the fundamentals.

Classically, the glueball vanishes, however we can keep it as a fixed parameter to study $Z(x,y)$: in fact it has poles at the eigenvalues of the adjoints, whose residues give the glueballs S_i in the corresponding i–th low energy SQCD block. By inserting vacua (6.6) and (6.8) and evaluating the trace, we get the classical expression of the loop function

$$Z(x,y) = \sum_{i=1}^{n+2} \frac{S_i}{x-a_i}\frac{1}{y-b_i} + \sum_{i=1}^{\frac{n-1}{2}} \frac{\widehat{S}_i}{x^2-\widehat{a}_i^2}\frac{1}{y^2-(c_i^2+d_i^2)} 2(xy+\widehat{a}_i c_i), \tag{6.20}$$

where S_i, for $i = 1,\dots,n+2$ are the glueballs for the SQCD we flow to in the abelian vacua (6.6), while $\widehat{S}_i$, for $i = 1,\dots,(n-1)/2$ are the glueballs for the SQCD we flow to in the nonabelian vacua (6.8). The leading term in the x Laurent expansion gives the resolvent

$$R(x) = \sum_{i=1}^{n+2} \frac{S_i}{x-a_i} + 2x\sum_{i=1}^{\frac{n-1}{2}} \frac{\widehat{S}_i}{x^2-\widehat{a}_i^2}. \tag{6.21}$$

The glueballs are the residues of the resolvent at the corresponding poles.

The meson generator depends on the phase we consider. In the pseudoconfining phase $M(x,y)$ vanish, since the fundamentals vanish. In the abelian higgs

(6.10) and nonabelian higgs phases (6.11), however, it is nonvanishing and we can get the expression for just $M_0(x) \equiv M(x)$

$$\begin{array}{cc} \text{abelianhiggs} & \text{nonabelianhiggs} \\ M(x) = -\frac{1}{m'(x_h)}\frac{V'(x_h)+\frac{\alpha^2}{4\lambda x_h^2}}{x-x_h}, & M(x) = -\frac{1}{m'(x_h)}\frac{V'(x_h)-V'(-x_h)}{x-x_h}, \end{array} \tag{6.22}$$

where we considered just the last flavor direction, i.e. $M(x) = M(x)^{N_f}_{N_f}$, according to the classical solutions (6.10) and (6.11). The meson generator has poles at the higgs eigenvalues, whose residue depends on the couplings and the phase.

The classical expressions for the generator $U(x, y)$ depends on the phase, too. In particular we can extract the expression for $T_0(x) \equiv T(x)$

$$T(x) = \sum_{i=1}^{n+2} \frac{N_i}{x-a_i} + 2x \sum_{i=1}^{\frac{n-1}{2}} \frac{\widehat{N}_i}{x^2-\widehat{a}_i^2} + r_{ab}\frac{1}{x-x_h} + r_{nab}\frac{2x}{x^2-x_h^2}, \tag{6.23}$$

where r_{ab} vanishes except in the abelian higgs phase (6.10), when is equal to 1, and r_{nab} vanishes unless in the nonabelian higgs phase (6.11), when is equal to 1.

6.2. The Three–sheeted Curve

In this Section we will study the chiral ring in the quantum theory by making use of the Konishi anomaly equations. We will first make a brief summary of the generalized anomaly equations to set the notations. At first, one would hope to find some closed algebraic equations for the chiral operators Z, M, U that we introduced in (6.16). However, it is not possible to solve directly for those generators. What one can do, instead, is to derive some equations involving those generators and then, by considering the Laurent expansion of these equations, derive some recursion relations that magically close on the resolvents $R(x), M(x), T(x)$. In this Section we will concentrate on the resolvent $R(x)$, whose algebraic equation defines the curve of the gauge theory. On the other hand, due to the DV correspondence, this curve provides the solution to the planar limit of the two–matrix model, whose action is given by the gauge theory superpotential (6.1).

We will first study the semiclassical expansion of the resolvent and describe its analytic structure, i.e. the branch points. Then we will work out the holomorphic differentials on the curve. At the end of the Section we will consider the algebraic equations for the effective description in terms of the resolvent $R(y)$.

It turns out that, unlikely the one–adjoint theory, for which the curve is an hyperelliptic Riemann surface, in this two adjoint case the algebraic curve is a cubic. The quantum theory is therefore described by a three–sheeted covering of the complex plane. In the one–adjoint case, each of the two sheets corresponds to a classical phase of the gauge theory: the pseudoconfining and the higgs phase. The interpolation between the two phases is possible by continuously move the poles of the resolvents $M(x)$ and $T(x)$ through the two sheets. In our two adjoint theory, we will see that again each of the three sheets corresponds to a different classical phase: the pseudoconfining, the abelian higgs and the nonabelian higgs phase. This leads to the suggestion that the degree of the $\mathcal{N} = 1$ curve corresponds to the number of semiclassical phases of its gauge theory. In the quantum theory we can interpolate between all the phases by moving poles around the curve. It is somewhat surprising though that the curve of the gauge theory "knows" about the three phases even if we do not have fundamentals, in fact the algebraic equation does not depend on the couplings to the fundamentals.

The generalized Konishi Anomaly

We flash a summary of the generalized Konishi anomaly in order to set the notation. We will display explicitly $\hbar$ to identify the semiclassical expansion.

Given the variation $\delta\Phi = \epsilon f(W_\alpha, X, Y)$, where $Phi = X, Y$, we denote the corresponding Ward identity as

$$\hbar\langle J(W_\alpha, X, Y)\rangle = \langle K(W_\alpha, X, Y)\rangle. \tag{6.24}$$

The l.h.s. is the anomalous variation of the measure

$$J(W_\alpha, X, Y) = -\frac{1}{32\pi^2}\left\langle \sum_{i,j}\left(\left[W_\alpha, \left[W^\alpha, \frac{\partial f(W_\alpha, X, Y)}{\partial \Phi_{ij}}\right\}\right\}\right)_{ij}\right\rangle.$$

The r.h.s. is the classical variation of the superpotential

$$K(W_\alpha, X, Y) = \left\langle \mathrm{Tr} f(W_\alpha, X, Y)\frac{\partial W_{tree}}{\partial \Phi}\right\rangle.$$

Note that, if we set $\hbar$ to zero in (6.24), we obtain the classical Ward identity, that gives the classical expression for the resolvent.

Cubic Equation for the Resolvent

There are many variations that one can try, but only very few of them are useful. In particular, Ferrari [21] has shown that the following three variations can be combined to obtain the curve

$$
\begin{aligned}
1st: \quad & \delta X = 0, \qquad \delta Y = -\frac{1}{32\pi^2}\frac{W_\alpha W^\alpha}{x - X}, \\
2nd: \quad & \delta X = -\frac{1}{32\pi^2}\frac{W_\alpha W^\alpha}{x - X}\frac{1}{y - Y}, \qquad \delta Y = 0, \\
3rd: \quad & \delta X = 0, \qquad \delta Y = -\frac{1}{32\pi^2} W_\alpha W^\alpha \frac{1}{x - X}\frac{1}{y - Y}\frac{1}{-x - X}.
\end{aligned}
\tag{6.25}
$$

The three variations (6.25) give the following anomalous Ward identity

$$
\begin{aligned}
1st: \quad & \lambda R_1(x) = \lambda \frac{\widetilde{S}}{x} - \frac{\alpha}{2x} R(x), \\
2nd: \quad & \left[V'(x) + \beta - \hbar R(x) + \lambda y^2\right] Z(x,y) = \lambda y R(x) + \lambda R_1(x) \\
& \qquad - \frac{1}{32\pi^2}\left\langle \mathrm{Tr} W_\alpha W^\alpha \frac{V'(x) - V'(X)}{x - X}\frac{1}{y - Y}\right\rangle, \\
3rd: \quad & Z(x,y) Z(-x,y) = \lambda \left[R(x) + R(-x)\right] - \left(\lambda y + \frac{\alpha}{2x}\right) Z(x,y) \\
& \qquad - \left(\lambda y - \frac{\alpha}{2x}\right) Z(-x,y),
\end{aligned}
\tag{6.26}
$$

where $R_k(x)$ are the generalized resolvent in (6.18), $Z(x,y)$ is the chiral operator in (6.16), and $\widetilde{S} = -\frac{1}{32\pi^2}\langle W_\alpha W^\alpha Y\rangle$. We want to find some recursion relations for the resolvents R_k by expanding the loop equations (6.26) in powers of y.[23] Let us introduce the degree $n-1$ polynomials $F_k(x)$

$$
F_k(x) \equiv -\frac{1}{32\pi^2}\left\langle \mathrm{Tr} W_\alpha W^\alpha \frac{V'(x) - V'(X)}{x - X} Y^k \right\rangle. \tag{6.27}
$$

[23] In the following we will use the convention to reabsorb the Lagrange multiplier β, associated with the tracelessness of X, as a constant term in the adjoint polynomial $V'(x) + \beta \to V'(x)$. In particular we have $v_+(x^2) + \beta \to v_+(x^2)$ and $v_-(x^2)$ unchanged. This is unambigous, since the original $V'(x)$ does not have a linear term.

These are the generalization of the usual quantum deformation $f(x)$ in the one–adjoint hyperelliptic Riemann surface $w^2 = V'(x)^2 + f(x)$ and their coefficients are proportional to the glueballs, hence they vanish classically.[24] By the last two anomaly equations in (6.26) we get

$$\lambda R_{k+2}(x) = [\hbar R(x) - V'(x)]\, R_k(x) + F_k(x), \tag{6.28}$$

$$\lambda\,[R_{q+2}(x) + R_{q+2}(-x)] + \frac{\alpha}{2x}\,[R_{q+1}(x) - R_{q+1}(-x)] + \hbar \sum_{k+k'=q} R_k(x) R_k(-x) = 0, \tag{6.29}$$

for $k \geq 0$. The strategy is to plug (6.28) into (6.29) and get at $k = 0$ an equation for $R(-x)$, then at $k = 2$ use it to obtain the closed equation in $R(x)$. By introducing $\widetilde{w} = \hbar R(x) - V'(x)$ we get the following cubic equation

$$\widetilde{w}^3 + \widetilde{a}(x^2)\widetilde{w}^2 + \widetilde{b}(x^2)\widetilde{w} + \widetilde{c}(x^2) = 0, \tag{6.30}$$

where the coefficients are

$$\begin{cases} \widetilde{a}(x) = V'(x) + V'(-x) - \dfrac{\alpha^2}{4\lambda}, \\ \widetilde{b}(x) = V'(x)V'(-x) - \dfrac{\alpha^2}{4\lambda}[V'(x) + V'(-x)] + \hbar[F_0(x) + F_0(-x) + \dfrac{\alpha \widetilde{S}}{x^2}], \\ \widetilde{c}(x) = -\dfrac{\alpha^2}{4\lambda} V'(x)V'(-x) + \hbar\Big(F_0(-x)V'(x) + F_0(x)V'(-x) \\ \quad + \lambda[F_2(x) + F_2(-x) - \hbar\dfrac{\widetilde{S}^2}{x^2}] + \dfrac{\alpha}{2x}\left[\dfrac{\widetilde{S}}{x}[V'(x) + V'(-x)] + F_1(x) - F_1(-x)\right]\Big). \end{cases} \tag{6.31}$$

Since in the coefficients there appears negative powers of x, we have to rescale the equation by multiplying by x^2. Setting $w = x^2 \widetilde{w}$ we find our cubic equation

$$w^3 + a(x^2)w^2 + b(x^2)w + c(x^2) = 0, \tag{6.32}$$

[24] Our notations are slightly different from those of CDSW [12], i.e. $f(x) = -4F_0(x)$.

where the coefficients are given by

$$
\begin{cases}
a(x^2) = x^2[V'(x)+V'(-x)] - \dfrac{\alpha^2}{4\lambda}, \\
b(x^2) = x^4 V'(x)V'(-x) - \dfrac{\alpha^2 x^2}{4\lambda}[V'(x)+V'(-x)] \\
\qquad + \hbar x^2\left[x^2[F_0(x)+F_0(-x)] + \alpha\widetilde{S}\right], \\
c(x^2) = x^4\Big\{ -\dfrac{\alpha^2}{4\lambda}V'(x)V'(-x) + \hbar\Big[x^2\big[F_0(-x)V'(x) + F_0(x)V'(-x) \\
\qquad + 2\lambda\widetilde{F}_2(x^2) + \alpha\widetilde{F}_1(x^2)\big] + \dfrac{\alpha}{2}\widetilde{S}[V'(x)+V'(-x)]\Big] - \hbar^2\lambda\widetilde{S}^2\Big\}.
\end{cases}
\tag{6.33}
$$

We denoted by $2x^2\widetilde{F}_1(x^2) = x[F_1(x) - F_1(-x)]$ and $2\widetilde{F}_2(x^2) = F_2(x) + F_2(-x)$. Note that the coefficient of the leading term of the first polynomial $F_0(x)$ gives the glueball $S = F_{0(n-1)}/t_n$.

Once we solved for the resolvent $R(x)$, then by (6.26) we get

$$
\begin{aligned}
Z(x,y) =& \frac{1}{\lambda y^2 + V'(x) - \hbar R(x)}\Big[\Big(\lambda y - \frac{\alpha}{2x}\Big)R(x) + \lambda\frac{\widetilde{S}}{x} \\
&- \frac{1}{32\pi^2}\Big\langle \mathrm{Tr}\frac{W_\alpha W^\alpha}{y-Y}\frac{V'(x)-V'(X)}{x-X}\Big\rangle\Big]
\end{aligned}
\tag{6.34}
$$

Note that the coefficients a, b, c are even functions of x. The curve in fact is invariant under the automorphism $x \to -x$, that will appear explicitely in the study of its analytic structure.

There are some standard techniques to study the cubic equations. It is convenient to shift $w \to w + a(x)/3$ to get rid of the subleading term and cast (6.32) to its normal form

$$
f(w,x) = w^3 + 3\gamma(x^2)w + 2\delta(x^2) = 0, \tag{6.35}
$$

where $\gamma(x^2)$ and $\delta(x^2)$ are the combinations $3\gamma = (b - \frac{a^2}{3})$ and $2\delta = (c - \frac{ab}{3} + \frac{2a^3}{27})$. Let us introduce the discriminant of the cubic equation $\Delta(x^2) = \gamma^3 + \delta^2$ and the auxiliary function $u(x) = (-\beta + \sqrt{\Delta})^{\frac{1}{3}}$. Then, as explained in Appendix A, the general solutions to (6.32) can be expressed in the form

$$
\begin{aligned}
w^{(I)} =& e^{i\frac{2}{3}\pi}u - e^{-i\frac{2}{3}\pi}\frac{\gamma}{u} - \frac{a}{3}, \\
w^{(II)} =& u - \frac{\gamma}{u} - \frac{a}{3}, \\
w^{(III)} =& e^{-i\frac{2}{3}\pi}u - e^{i\frac{2}{3}\pi}\frac{\gamma}{u} - \frac{a}{3},
\end{aligned}
\tag{6.36}
$$

and recalling the definition $w = x^2[R(x) - V'(x)]$ we find the three expressions for the resolvents

$$\hbar R^{(i)}(x) = V'(x) + w^{(i)}(x)/x^2, \qquad i = I, II, III.$$

To identify the physical resolvent we need to study the asymptotic behavior of the three solutions at large x. In particular, the physical sheet will be identified as usual with the asymptotics $R(x) \sim S/x$. We can rearrange the asymptotic expansion as a semiclassical expansion in powers of $\hbar$ and find that the solution $w^{(I)}(x)$ has the correct physical behavior

$$R(x) = \frac{x^2 F_0(x) + \frac{\alpha\widetilde{S}}{2}}{x^2 V'(x) + \frac{\alpha^2}{4\lambda}} + \lambda x \frac{2\widetilde{F}_2(x^2) + \alpha\widetilde{F}_1(x^2)}{2v_-(x^2)[x^2V'(x) + \frac{\alpha^2}{4\lambda}]} + \mathcal{O}(\hbar), \tag{6.37}$$

where $-2xv_-(x^2) = V'(x) - V'(-x)$ is the odd part of the adjoint polynomial. We will denote this sheet as the first sheet. By looking at the leading term of (6.37) we can identify the glueball as $S = F_{0(n-1)}/t_n$. The other two solutions in (6.36) describe the second and the third sheet, which are not visible classically. The semiclassical expansion of the resolvent $R(x)$ on the other two sheets is

$$\begin{aligned}
\hbar R^{(II)}(x) =& V'(x) + \frac{\alpha^2}{4\lambda x^2} - \hbar\frac{x^2F_0(x) + \alpha\widetilde{S}/2}{p(x)} - \hbar\frac{x^2F_0(-x) + \alpha\widetilde{S}/2}{p(-x)} \\
& - \hbar\lambda x^2 \frac{2\widetilde{F}_2(x^2) - \alpha\widetilde{F}_1(x^2)}{p(x)p(-x)} + \mathcal{O}(\hbar^2), \\
\hbar R^{(III)}(x) =& V'(x) - V'(-x) + \hbar\frac{x^2F_0(-x) + \frac{\alpha\widetilde{S}}{2}}{p(-x)} \\
& - \hbar\lambda x \frac{2\widetilde{F}_2(x^2) + \alpha\widetilde{F}_1(x^2)}{2v_-(x^2)p(-x)} + \mathcal{O}(\hbar^2),
\end{aligned} \tag{6.38}$$

where $p(x)$ is the polynomial (6.7), whose roots are the abelian vacua (6.6), and $v_-(x^2)$, whose roots are the nonabelian vacua (6.8), is the odd part of the adjoint polynomial $V'(x)$. By looking at (6.38) we get a quick preview of the structure of the branch points of the gauge theory curve (6.32). Indeed, each pole in the semiclassical resolvent splits up into two branch points in the full quantum theory. If the resolvent has a pole on the same value of x on two different sheets, in the quantum theory a branch cut will appear, connecting those same sheets.

Therefore, the branch cuts coming from the splitting of the abelian vacua at the roots of $p(x)$ will connect the first and the second sheet. The branch cuts symmetric of the former with respect to the origin, i.e. coming from the roots of $p(-x)$, connect the second and the third sheets. The branch cuts from the nonabelian vacua, i.e. the roots of $v_-(x^2)$, connect the first and the third sheets.

If we put $\hbar = 0$ in the anomaly equations, but still keeping the glueballs as parameters, $F_k(x) \neq 0$, we find as classical expressions for the resolvent precisely the physical solution (6.37).

The Branch Points

Let us look at the analytic structure of the curve (6.35). The branch points are the singular points of the curve, that is the points at which $f = df = 0$. Since $\partial_w f = 3(w + \gamma)$, one can easily find that the singular points are given by the zeros of the discriminant $\Delta(x^2) = \gamma^3 + \delta^2$, that we encountered above in building the solution to the cubic. The ramification index r_i of each of these branch points is such that $f(w, x)$ together with its $r_i - 1$ derivatives vanish at the point. This index tells how many sheets we can reach by winding around the branching point. The number of branch points would be $\deg \Delta = 6(n + 2)$ where n is the degree of $V'(x)$ in the gauge theory. However, we can collect out an overall x^6 factor in front of Δ. Therefore, the number of branch points is $6(n + 1)$. All of these branching points have ramification index $r_i = 2$ since $\partial_w^2 f(w, x) = 6w$ never vanishes at these points.

As a first to understand the analytic structure, we can write down the semi-classical expansion of the discriminant

$$-27\,\Delta = x^6 v_-(x^2)^2 p(x)^2 p(-x)^2 + x^6 \mathcal{O}(\hbar). \tag{6.39}$$

Classically, we have $6(n+1)$ double zeros, which come in pairs symmetric under $x \longrightarrow -x$. The first $n - 1$ of them come from the zeros of $v_-(x^2)$, the odd part of the adjoint polynomial, and correspond to the nonabelian vacua (6.8). Other $n + 2$ double zeros are given by the roots of $p(x)$ in (6.7)and correspond to the abelian vacua (6.6). The last $n + 2$ double zeros are given by the roots of $p(-x)$ and are the reflection of the abelian vacua (6.7) under $x \longrightarrow -x$. These last zeros are not classical vacua and we will shortly see how they arise. Consider now the image of $x = 0$. Even if Δ has an overall factor x^6, it turns out that $\partial_w f$

vanishes at $x = 0$ on the first and third sheet, but it is nonvanishing on the second sheet, so this is actually a cusp and not a branch point. Each double zero of the discriminant corresponds to a pole in the semiclassical expansion of the resolvents (6.37) and (6.38).

In the quantum theory, each of these double zeros split up into two branch point. Since they all have branching number two, each of the branch points connect two different sheets. Now, each branch point belongs to two sheets of the algebraic surface. Therefore, to tell which sheets are connected by which branch point, we just need to consider the three sheets (6.36), solve the conditions

$$w^{(i)}(a) = w^{(j)}(a), \qquad i, j = I, II, III, \tag{6.40}$$

and identify the branch points. For instance, the points $x = a$ that lie on the first and second sheet satisfy $w^{(I)}(a) = w^{(II)}(a)$, which gives the condition $(-\delta(a))^{\frac{2}{3}} = e^{i\frac{\pi}{3}}\gamma(a)$. However, the expressions of $\gamma(a)$ and $\delta(a)$ are very complicated and involve the quantum deformations $F_k(x)$, so we would like to find an easier way.

There is indeed a simple way to study the monodromy of the curve and identify the branch points, by making use of the fact that in the limit $\hbar = 0$ the cubic (6.32) factorizes into three disconnected sheets. The classical limit of the curve (6.32) is

$$\left[w - \frac{\alpha^2}{4\lambda}\right] [w^2 + x^2[V'(x) + V'(-x)]w + x^4 V'(x)V'(-x)] = 0, \tag{6.41}$$

whose solutions we can identify as the semiclassical limits of the resolvents in (6.37) and (6.38)

$$w_{cl}^{(I)}(x) = -x^2 V'(x), \quad w_{cl}^{(II)}(x) = \frac{\alpha^2}{4\lambda}, \quad w_{cl}^{(III)}(x) = -x^2 V'(-x). \tag{6.42}$$

The branch points are coming from the splitting of each classical pole of the resolvent and in the exact solutions (6.36) they satisfy (6.40), which means that they are special points lying on two different sheets. In the classical limit, each couple of branch points degenerates into a pole located at the corresponding vacuum. We can solve the conditions (6.40)on the classical curve and identify which classical pole connects which sheets: on the curve (6.41) the vacua are represented by marked points on each disconnected sheets, such that the above conditions are satisfied. We find then

$$w_{cl}^{(I)}(a) = w_{cl}^{(II)}(a) \quad iff \quad a^2 V'(a) + \frac{\alpha^2}{4\lambda} = 0, \tag{6.43}$$

which is the condition (6.7). The branch points connecting the 1st with the 2nd sheet come from the splitting of the abelian vacua $x = a_i$, for $i = 1, \ldots, n+2$, that we showed in (6.6). We call A_i the $n+2$ branch cuts coming from the splitting of the abelian vacua at a_i. These branch cuts connect the 1st and 2nd sheet. Then we have

$$w_{cl}^{I}(a) = w_{cl}^{III}(a) \quad iff \quad v_-(a^2) = 0, \tag{6.44}$$

so the 1st and the 3rd sheet are connected by the branch points coming from the splitting of the nonabelian vacua $x = \pm\widehat{a}_i$, for $i = 1, \ldots, (n-1)/2$, that we showed in (6.8). Note that for each nonabelian vacuum corresponding to a value of $\widehat{a}_i^2$, we have on the 1st sheet a pole in $\widehat{a}_i$ and another one in $-\widehat{a}_i$ and each pole splits into two branch points that connect to the 3rd sheet. We denote $\widehat{A}_i$ the branch cuts around $\widehat{a}_i$ and $\widehat{A}'_i$ the branch cuts around $-\widehat{a}_i$. These branch cuts connect the 1st and 3rd sheets. Then the last condition

$$w_{cl}^{(II)}(a) = w_{cl}^{(III)}(a) \quad iff \quad a^2 V'(-a) + \frac{\alpha^2}{4\lambda} = 0, \tag{6.45}$$

namely the 2nd and 3rd sheet have classical poles at $x = -a_i$, where a_i are again the abelian vacua. In the quantum theory each pole splits into two branch points connecting the two sheets. We will denote A'_i the branch cut around the pole at $-a_i$. These cuts connect the 2nd and 3rd sheet. Note that these cuts do not correspond to a gauge theory vacuum and, indeed, we cannot see them on the physical sheet.

In this way we have accounted for all the $6(n+1)$ branch points. We can summarize the monodromy structure of the curve as follows

$$\begin{array}{cc} \text{Branch Cuts} & \text{Sheets} \\ I_i & I \leftrightarrow II \\ I'_i & II \leftrightarrow III \\ \widehat{I}_i, \widehat{I}'_i & I \leftrightarrow III \end{array} \quad . \tag{6.46}$$

point. We could have found the same results by looking at the classical expression for the resolvents (6.37) and (6.38) and their semiclassical expansion in Appendix A. In fact, the classical poles in the resolvents correspond precisely to the marked points on the three disconnected sheets. In the quantum theory, each marked point splits into two branch points. The classical discriminant (6.39) has double zeroes corresponding to the classical marked points in (6.46). Finally, let us point out that the automorphism $x \to -x$ of the curve (6.32) exchanges the 1st and the 3rd sheets, while leaving the second sheet invariant. We do not discuss the noncompact B cycles. For our purpose, in fact, we will always consider the gauge theory in the weak coupling expansion, so the periods on compact and noncompact cycle will never mix.

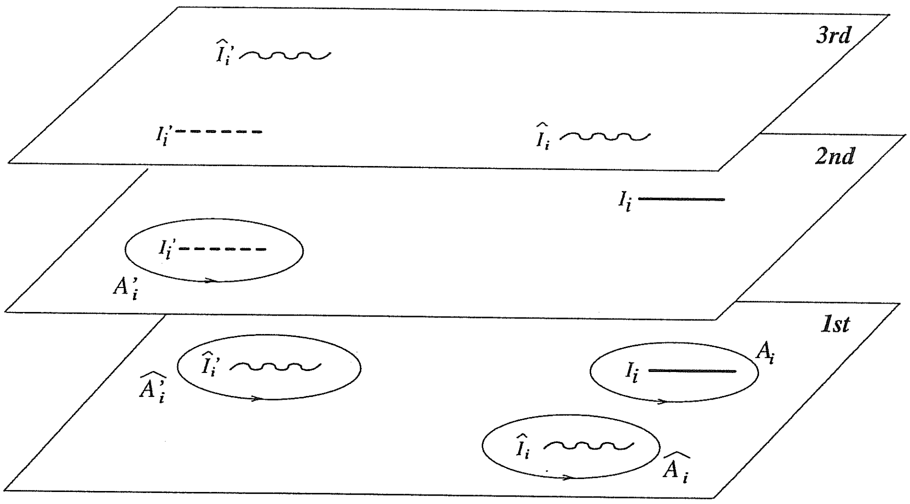

Fig. 1: The three sheeted curve. The cut A_i comes from the splitting of the abelian vacua at $x = a_i$. The cuts $\widehat{A}_i$ and $\widehat{A}'_i$ come from the splitting of the nonabelian vacuum, at $x = \pm\widehat{a}_i$. The cut A'_i is not visible from the physical sheet. It comes from the splitting of the pole at $x = -a_i$.

Now we can use the Hurwitz formula to compute the genus g of the curve

$$2g - 2 = -2p + \sum_i (r_i - 1),$$

where p is the number of sheets and the sum runs over all the branch points, $r_i = 2$ being the branching number of each. We find

$$g = 3(n+1) - 2. \qquad (6.47)$$

Let us compute a basis for the holomorphic differentials on the curve (6.32). We use the method of divisors in the notations of [46]. Let us denote by

$$[f] = \frac{P_1^{\alpha_1} \dots P_n^{\alpha_n}}{Q_1^{\beta_1} \dots Q_m^{\beta_m}},$$

the divisor of f, where P_i is a zero of degree α_i and Q_j is a pole of degree β_j. The degree of the divisor is given by $deg[f] = \sum_i \alpha_i - \sum_j \beta_j$. The Riemann–Roch theorem states the degree of a meromorphic function f is $deg[f] = 0$, while for a meromorphic differential ω the degree is $deg[\omega] = 2g - 2$. We need to compute the divisors of dx, x, $\partial_w F = 3(w(x)^2 + \gamma(x^2))$ and $w(x)$.

The differential dx vanishes at the branch points and has a double pole at ∞ on each of the three sheets. If we denote by Δ_B the divisor corresponding to the $6(n+1)$ branch points we find

$$[dx] = \frac{\Delta_B}{P_{\infty_I}^2 P_{\infty_{II}}^2 P_{\infty_{III}}^2},$$

so that $deg[dx] = 6n = 2g - 2$. The function x has the following divisor

$$[x] = \frac{\mathcal{O}_I \mathcal{O}_{II} \mathcal{O}_{III}}{P_{\infty_I} P_{\infty_{II}} P_{\infty_{III}}},$$

where $\mathcal{O}_i$ represents the origin on each sheet.

Now let us consider $\partial_w F$. It vanishes at the branch point locus Δ_B, while $\partial_w F(x, w) \sim_{x \sim \infty} x^{2(n+2)}$ on each sheet. Thus, by Riemann–Roch, we are missing six zeroes. If we study the asymptotics at small x we find that $\partial_w F_{x,w(x)} \sim x^3$ on the 1st and 3rd sheets while $\partial_w F_{x,w(x)} \sim const$ on the 2nd sheet, so that

$$[\partial_w F] = \frac{\Delta_B \mathcal{O}_I^3 \mathcal{O}_{III}^3}{P_{\infty_I}^{2(n+2)} P_{\infty_{II}}^{2(n+2)} P_{\infty_{III}}^{2(n+2)}}. \tag{6.48}$$

Consider now $w(x)$. We need to study its zeroes for small x, in order to cancel the poles coming from (6.48). For small w we can approximate the curve (6.32) by $b(x)w + c(x) = 0$ so that w vanishes at the roots of $c(x)$ which are not roots of $b(x)$. So we expect a double zero at $x = 0$ and a bunch of $2n$ nonvanishing other zeroes, whose corresponding divisor we denote by C_{2n}. The asymptotic expansion of the solutions $w(x)$ is $w(x) \sim_{x\sim 0} x^2$ on the 1st and 3rd sheets and $w(x) \sim_{x\sim 0} const$ on the 2nd sheet so that its divisor is

$$[w] = \frac{\mathcal{O}_I^2 \mathcal{O}_{III}^2 C_{2n}}{P_{\infty_I}^{n+2} P_{\infty_{III}}^{n+2}}.$$

To build the holomorphic differentials we have to take care of the poles coming from (6.48) at the points $\mathcal{O}_{I,III}$, so that

$$\frac{dx}{\partial_w F}, \quad \frac{xdx}{\partial_w F}, \tag{6.49}$$

have triple and double poles, respectively, while

$$\frac{x^2 dx}{\partial_w F}, \quad \frac{wdx}{\partial_w F}, \tag{6.50}$$

have just single poles at $\mathcal{O}_I, \mathcal{O}_{III}$. Therefore we have to eliminate (6.49) but we can take a linear combination of (6.50) with vanishing residue. Therefore, a basis for the holomorphic differentials is given by

$$\begin{aligned} &\frac{(c_1 x^2 + c_2 w)dx}{w^2 + \gamma(x)}, \\ &\frac{x^j dx}{w^2 + \gamma(x)}, \qquad j = 3, \ldots, 2n+2, \\ &\frac{x^k w dx}{w^2 + \gamma(x)}, \qquad k = 1, \ldots, n. \end{aligned} \tag{6.51}$$

We have in total $3n+1 = 3(n+1) - 2 = g$ holomorphic differential as expected.

The Glueballs

In the one–adjoint theory [12], the usual way one defines the glueball S_i in the i–th low energy SQCD is by computing the period of the resolvent around the i–th cut on the physical sheet. In our case, the generic low energy SQCDs (6.9) come from abelian as well as nonabelian vacua and they require different definitions. In the case of the abelian vacua (6.6), we define the glueballs as usual

$$S_i = \oint_{A_i} R(x), \tag{6.52}$$

where the A_i contour sorrounds the corresponding I_i abelian cut, see Fig.1. This definition reproduces the semiclassical result (6.21) and is the same prescription as in [12]. On the other hand, the SQCD we flow to in the nonabelian vacuum is described on the physical sheet by the two cuts $\widehat{I}_i$ and $\widehat{I}'_i$, which are symmetric with respect to the origin. This phenomenon has been called "eigenvalue entanglement" in the related two–matrix model [21], where it was shown that the eigenvalue density $\rho(x)$ for such representations is symmetric, $\rho(x) = \rho(-x)$ for $x \in \widehat{I}_i \cup \widehat{I}'_i$. Since the gauge theory glueball corresponds to the matrix model filling fraction of the eigenvalues, the periods of the resolvent $R(x)$ around the cuts $\widehat{I}_i$ and $\widehat{I}'_i$ is the same. We define therefore the glueball as either period

$$\widehat{S}_i = \oint_{\widehat{A}_i} R(x) = \oint_{\widehat{A}'_i} R(x). \tag{6.53}$$

This definition is consistent with the semiclassical resolvent (6.21), in fact we have that the total glueball is $S = \sum_{i=1}^{n+2} S_i + 2\sum_{i=1}^{\frac{n-1}{2}} \widehat{S}_i$ and is the residue of the resolvent at the pole at infinity. We will see below that this definition reproduces also the Konishi anomalies in these low energy SQCDs.

We would like to find that the number of glueballs corresponds to the number of parameters in the equation for the resolvent (6.32), which in turn is related to the genus of the curve. Recall first what happens in the one–adjoint theory with gauge group $U(N)$ [12]. There, a degree n adjoint polynomial $V'(x)$ gives n low energy SQCD blocks with gauge group $U(N_i)$, each of which defines a glueball S_i [12]. The n glueballs S_i are in one to one correspondence to the n coefficients of the quantum deformation $f_{n-1}(x)$ of the $\mathcal{N} = 1$ hyperelliptic curve $y^2 = V'(x)^2 + f_{n-1}(x)$. Finally, we can fix the coefficient of the leading term of $f(x)$ by the residue of the resolvent at infinity, due to the overall relation $\sum_i S_i = S$. The number of moduli of the curve is just the genus $g = n - 1$, and the free parameters in $f(x)$ actually parameterize the moduli of the curve.

Now let us look at the cubic curve (6.32) and its coefficients (6.33) and identify the independent parameters. We have: the generalized glueball $\widetilde{S}$; the degree $n-1$ polynomial $F_0(x)$, with n coefficients; the polynomial $\widetilde{F}_2(x^2)$, which has $(n+1)/2$ coefficients; $\widetilde{F}_1(x^2)$ which has $(n-1)/2$ coefficients. However, by making use of the first anomaly equation in Appendix A, one can show that the coefficients of $\widetilde{F}_1(x^2)$ can be recast as combinations of coefficients of $F_0(x)$, so they are not free parameters. We are left with a total of $1+n+(n+1)/2 = \frac{3}{2}(n+1)$ parameters, which is precisely the number of vacua, i.e. the low energy SQCDs in the generic vacuum (6.9). However, it might seem this is not in agreement with the number of independent deformations of the curve, which has genus $g = 3(n+1) - 2$. But recall that the coefficients (6.33) of the curve are even functions, namely the curve has the automorphism $x \to -x$ that halves the number of moduli: this means that the periods of $R(x)$ around A_i and $\widehat{A}_i$ are respectively the same as those around A'_i and $\widehat{A}'_i$. Finally, the coefficient of the leading term of the quantum deformation $F_0(x)$ is fixed as the sum of all the glueballs, just as in the one–adjoint case we discussed above.

6.3. The mesons

We will solve now for the generator of the X–dressed mesons. Its classical expression depends on which of the three classical phases we are considering: it vanishes in the pseudoconfining phase, while it is given by (6.22) in the abelian and nonabelian higgs phase. Our strategy is again to consider a variation of the fundamentals and get an anomalous Ward identity.

Consider the following variation for the quarks

$$\delta Q^i = f^i(X,Y,Q).$$

The corresponding anomalous Ward identity in the chiral ring is

$$\left\langle \widetilde{Q}_{\tilde{i}} m(X,Y)^{\tilde{i}}_{i} f^i(X,Y,Q) \right\rangle = -\frac{\hbar}{32\pi^2} \left\langle (W_\alpha W^\alpha)_{ab} \delta^i_f \frac{\partial f^f_a(X,Y,Q)}{\partial Q^i_b} \right\rangle,$$

where $m(X,Y)$ is the most generic meson deformation containing both X– and Y–dressed Yukawa couplings.

In our case (6.1) the meson deformation is just X–dependent, $m(x) = m_1 + m_2 x$. Let us focus then on the X–dependent variation

$$\delta Q^f = Q^f \frac{1}{x - X},$$

which gives the usual anomaly equation $[m(x)M(x)]_- = \hbar R(x)$, where we suppressed flavor indices. These considerations still hold if we consider the generalized meson generators $M_k(x) = \widetilde{Q}\frac{1}{x-X}Y^k Q$, whose anomaly equation are

$[m(x)M_k(x)]_- = \hbar R_k(x)$. The explicit solution depends on the vacuum we are considering. We have three cases

1. *pseudoconfining phase.* In this case the classical meson generator vanishes on the first sheet $M(x)|_{cl} = 0$, so we require that the spurious poles coming from the zeroes of $m(x)$ be cancelled

$$M(x) = \hbar\left(\frac{R(x)}{m(x)} - \frac{R^I(x_h)}{m'(x_h)}\frac{1}{x-x_h}\right). \tag{6.54}$$

2. *abelian higgs phase.* This is characterized by the meson generator having a pole at the higgs eigenvalue x_h on the first sheet, whose residue is computed according to the first expression in (6.22). If we recall the expression of the resolvent on the second sheet in (6.38), we see that the boundary conditions are that the meson generator be regular on the second sheet

$$M(x) = \hbar\left(\frac{R(x)}{m(x)} - \frac{R^{II}(x_h)}{m'(x_h)}\frac{1}{x-x_h}\right). \tag{6.55}$$

   This means that we can connect the phases (6.54) and (6.55) by moving the pole at x_h from the first to the second sheet by passing through one of the abelian cuts A_i.

3. *nonabelian higgs phase.* This phase is characterized by the meson generator having a pole at x_h on the first sheet, whose residue is computed according to the second expression in (6.22). If we recall the expression of the resolvent on the third sheet in (6.38), we see that the boundary conditions in this case are that the meson generator be regular on the third sheet

$$M(x) = \hbar\left(\frac{R(x)}{m(x)} - \frac{R^{III}(x_h)}{m'(x_h)}\frac{1}{x-x_h}\right). \tag{6.56}$$

   This means that we can connect the phases (6.55) and (6.56) by moving the pole at x_h from the second to the third sheet by passing through one of the cuts A'_i around $x = -a_i$, where a_i is one of the abelian pseudoconfining vacua. We can connect the nonabelian higgs solution to the pseudoconfining one by passing the pole from the third to the first sheet through one of the nonabelian cuts $\widehat{A}_i$.

We can summarize the three phases as follows

$$\begin{array}{cc} \textit{phase} & M(x) \\ \text{pseudoconfining} & \text{regular on 1st sheet} \\ \text{abelian higgs} & \text{regular on 2nd sheet} \\ \text{nonabelian higgs} & \text{regular on 3rd sheet} \end{array} \tag{6.57}$$

If the meson deformation only depends on the X adjoint, as in (6.1), then we can solve for the meson generator $M(x)$. However, we can't find a closed equation for the generator of the Y–dressed mesons $M(y)$. In fact, consider

$$\delta Q^f = Q^f \frac{1}{x-X} \frac{1}{y-Y},$$

we get the anomaly equation $[m(x)M(x,y)]_- = \hbar Z(x,y)$, where $Z(x,y)$ is the chiral operator defined in (6.16), whose expression is given in Appendix A. In the pseudoconfining phase, the mesons vanish classically, so we expect that the residues of $M(x,y)$ around the poles of $m(x)$ be vanishing in the classical limit

$$\oint_{x_k} dx\, M(x,y) = 0,$$

where x_k are the roots of $m(x)$. This gives the following solution to the anomaly equation

$$M(x,y) = \frac{\hbar Z(x,y)}{m(x)} - \hbar \sum_k \frac{Z(x_k,y)}{m'(x_k)} \frac{1}{x-x_k}.$$

If we expand for large x we find

$$M(y) = -\hbar \sum_k \frac{Z(x_k,y)}{m'(x_k)}.$$

The same reasoning applies in the case the meson deformation only depends on Y instead, i.e. we have $\widetilde{Q}m(Y)Q$. Here, we can solve for the generator $M(y)$ but we can't get a nice expression for the generator $M(x)$. Eventually, if the meson deformation depends on both X and Y, then there is no easy way to study either meson generators.

Konishi Anomaly

To get the expectation values of the meson operators in each low energy SQCD block, we just integrate the meson generator $M(x)$ around the corresponding cut. The i–th SQCD coming from the abelian vacuum (6.6) has N_f flavors and

$$\left\langle \widetilde{Q} X^j Q \right\rangle |_i = \oint_{A_i} x^j M(x),$$

where we suppressed the flavor indices. The first meson gives the usual Konishi anomaly equation $\widetilde{Q}_f Q^f = \hbar N_f S_i / m(a_i) + \mathcal{O}(\hbar^2)$, where $m(a_i)$ is the effective quarks mass in the i–th SQCD.

The j–th SQCD that comes from the nonabelian vacuum (6.8) gets twice as many flavors and requires some additional considerations. We can parameterize the $2N_f$ fundamentals as $(\widetilde{Q}_\alpha)^{\pm}_{\widetilde{f}}$ and $(Q^\alpha)^{\pm,f}$ where $f = 1, \ldots, N_f$ is the flavor index and $\alpha = 1, \ldots, \widehat{N}_j$ is the color index and the additional index $\pm$ is another flavor index comes from the splitting of the color indices in the nonabelian vacuum and the fact that the rank of the gauge group is halved in this vacuum. The effective quark mass is different for the two type of fundamentals Q^+ and Q^- and we can easily compute it

$$\langle \widetilde{Q}_f m(X) Q^f \rangle |_i = m(\widehat{a}_j) \widetilde{Q}^+_{\widetilde{f}} Q^{+,f} + m(-\widehat{a}_j) \widetilde{Q}^-_{\widetilde{f}} Q^{-,f}.$$

We have two different kind of mesons in this SQCD, they are decoupled and their expectation values are different. The $+$ mesons are given by the contour integral around the $\widehat{A}_j$ cut, the $-$ mesons around the $\widehat{A}'_j$ cut

$$\left\langle \widetilde{Q}^+_{\widetilde{f}} X^l Q^{+f} \right\rangle |_j = \oint_{\widehat{A}_j} x^l M^{f}_{\widetilde{f}}(x), \qquad \left\langle \widetilde{Q}^-_{\widetilde{f}} X^l Q^{-f} \right\rangle |_j = \oint_{\widehat{A}'_j} x^l M^{f}_{\widetilde{f}}(x). \tag{6.58}$$

If we recall the definition of the glueball in this vacuum (6.53), we find that the Konishi anomaly is again satisfied in the following form

$$\widetilde{Q}^+_{\widetilde{f}} Q^{+f} = N_f \widehat{S}_j / m(\widehat{a}_j), \qquad \widetilde{Q}^-_{\widetilde{f}} Q^{-f} = N_f \widehat{S}_j / m(-\widehat{a}_j). \tag{6.59}$$

6.4. The resolvent $T(x)$

In this section we will solve for the last resolvent $T(x) = \mathrm{Tr}\frac{1}{x-X}$ and study its analytic behavior. We will see that the three different classical phases of the gauge theory, pseudoconfining, abelian higgs and nonabelian higgs, are described by three different configurations of the simple poles of $T(x)$. This phenomenon is analogous to what happens in the one–adjoint theory, in which the poles of $T(x)$ characterize either of the two phases of the theory [19]. In that case, a pole of $T(x)$ on the physical sheet signals a semiclassical higgs phase, while when $T(x)$ is regular on the physical sheet we are in a semiclassical pseudoconfining phase. One can interpolate continuously between the two phases by moving the pole of $T(x)$ between the first and the second sheet through the branch cuts. In the present case, we will see that, again, a regular $T(x)$ on the physical sheet means that the theory is in the semiclassical pseudoconfining phase, while more complicated configurations describe the two higgs phases. Once again, we can reach all three phases by moving poles around the three sheets of the Riemann surface.

We solve for the last generator $T(x)$ by using the same strategy of used for the resolvent $R(x)$. The variations we will use are analogous to the ones in (6.25) but without the gluino insertions

$$\begin{aligned} 1st: &\quad \delta X = 0, \qquad \delta Y = \frac{1}{x-X}, \\ 2nd: &\quad \delta X = \frac{1}{x-X}\frac{1}{y-Y}, \qquad \delta Y = 0, \\ 3rd: &\quad \delta X = 0, \qquad \delta Y = \frac{1}{x-X}\frac{1}{y-Y}\frac{1}{-x-X}. \end{aligned} \tag{6.60}$$

The variations (6.60) give the following three anomaly equations

$$\begin{aligned} 1st: &\quad T_1(x) = -\frac{\alpha}{2\lambda x}T(x), \\ 2nd: &\quad \left(V'(x) + \lambda y^2 - \hbar R(x)\right) U(x,y) + m_2 M(x,y) = \hbar T(x) Z(x,y) + \\ &\qquad + \lambda y T(x) + \lambda T_1(x) + \left\langle \mathrm{Tr}\frac{V'(x)-V'(X)}{x-X}\frac{1}{y-Y}\right\rangle, \\ 3rd: &\quad \hbar\left[Z(x,y)U(-x,y) + Z(-x,y)U(x,y)\right] = -\lambda y\left[U(x,y)+U(-x,y)\right] \\ &\qquad - \frac{\alpha}{2x}\left[U(x,y) - U(-x,y)\right] + \lambda\left[T(x)+T(-x)\right], \end{aligned} \tag{6.61}$$

where the chiral operators $Z(x,y)$, $U(x,y)$ and $M(x,y)$ are given in (6.16) and we set to zero the terms proportional to u^α, w^α in the supersymmetric vacuum. $T_k(x)$ are the generalized resolvents (6.19). The mesons contribute only through the second anomaly equation with the term proportional to m_2. Let us introduce the degree $n-1$ polynomials

$$C_k(x) = \left\langle \mathrm{Tr} \frac{V'(x) - V'(X)}{x - X} Y^k \right\rangle, \tag{6.62}$$

which are analogous to (6.27), but do not vanish classically. In particular, the leading term of the first polynomial $C_0(x)$ is the rank of the unbroken gauge group $N = C_{0(n-1)}/t_n$. We consider the Laurent expansion in powers of y of the second and third equations in (6.61) and find the recursion relations

$$\lambda T_{k+2}(x) = [\hbar R(x) - V'(x)] T_k(x) - m_2 M_k(x) + \hbar R_k(x) T(x) + C_k(x), \tag{6.63}$$

$$\begin{aligned} &\lambda \left[T_{k+2}(x) + T_{k+2}(-x)\right] + \frac{\alpha}{2x} \left[T_{k+1}(x) - T_{k+1}(-x)\right] + \\ &+\hbar \sum_{q+q'=k} \left[R_q(x) T_{q'}(-x) + R_q(-x) T_{q'}(x)\right] = 0, \end{aligned} \tag{6.64}$$

for $k \geq 0$, where $M_k(x)$ are the generalized meson generators in (6.19). Notice that these recursion relations are linear in T_k, whereas the recursion relations for R_k in (6.28) and (6.29) are bilinear. Plugging (6.63) into (6.64) at $k = 0$ we solve for $T(-x)$, then at $k = 2$ we find a linear equation for $T(x)$, whose solution is precisely

$$T(x) = \frac{N(x) + \delta N(x)}{D(x)}, \tag{6.65}$$

It turns out that the only case in which one can solve the anomaly equations explicitly is when $m(x)$ is just linear in x as in (6.1). If higher Yukawa couplings are present this procedure does not work any more. The notation is the following

$$\begin{aligned} N(x) =& x^2 C_0(x) \left[V'(x) - V'(-x)\right] - \hbar R(x) \left[C_0(x) + C_0(-x)\right] \\ &- x^2 \left[2\lambda \widetilde{C}_2(x^2) + \alpha \widetilde{C}_1(x^2)\right], \\ D(x) =& \left[x^2 V(x) + \frac{\alpha^2}{4\lambda} - 2x^2 \hbar R(x)\right] \left[V'(x) - V'(-x) - 2\hbar R(x)\right] - \hbar^2 x^2 R(x)^2 \\ &+ \hbar x^2 \left[F_0(x) + F_0(-x)\right] + \hbar \alpha \widetilde{S}. \end{aligned} \tag{6.66}$$

We have used the combinations $2x^2\widetilde{C}_1(x^2) = x[C_1(x) - C_1(-x)]$ and $2\widetilde{C}_2(x^2) = C_2(x) + C_2(-x)$. The term $\delta N(x)$ depends on the meson generators

$$\begin{aligned}\delta N(x) =& m_2\Big(-M(x)[V'(x) - V'(-x)] + \lambda[M_2(x) + M_2(-x)] \\ &+ \frac{\alpha}{2x}[M_1(x) - M_1(-x)] + \hbar R(x)[M(x) + M(-x)]\Big),\end{aligned} \tag{6.67}$$

where $M_k(x)$ are the generalized meson generators in (6.19).[25] The explicit expressions for $M_k(x)$ depend on the vacuum we choose and are discussed in Section 4. We can summarize the different phases as

$$M_k(x) = \frac{\hbar}{m(x)}\left(R_k(x) - R_k^{(i)}(x_1)\right), \tag{6.68}$$

where the index $i = 1, 2, 3$ labels respectively the pseudoconfining, abelian and nonabelian higgs phases and gives the resolvent on the different sheets as in (6.37) and (6.38). The explicit expressions for $R_1(x)$ and $R_2(x)$ are given in Appendix A by the recursion relations in term of $R(x)$. If we plug (6.68) in (6.67) we find

$$\begin{aligned}\delta N(x) =& -\hbar\frac{R(x) - R^{(i)}(x_1)}{x - x_1}[V'(x) - V'(-x)] + \hbar\lambda\frac{R_2(x) - R_2^{(i)}(x_1)}{x - x_1} \\ &- \hbar\lambda\frac{R_2(-x) - R_2^{(i)}(x_1)}{x + x_1} + \frac{\alpha}{2x}\hbar\frac{R_1(x) - R_1^{(i)}(x_1)}{x - x_1} \\ &+ \frac{\alpha}{2x}\hbar\frac{R_1(-x) - R_1^{(i)}(x_1)}{x + x_1} + \hbar R(x)\Big(\hbar\frac{R(x) - R^{(i)}(x_1)}{x - x_1} \\ &- \hbar\frac{R(-x) - R^{(i)}(x_1)}{x + x_1}\Big).\end{aligned} \tag{6.69}$$

Analytics of $T(x)$

Once we have the explicit solution (6.65) we can study its semiclassical expansion, its asymptotics and the analytic structure. Consider first the case in

[25] The term (6.67) vanishes if the superpotential does not have Yukawa couplings $\widetilde{Q}m(X)Q$ between the fundamentals and the adjoints. In our case (6.1), this coupling is given by $m_2\widetilde{Q}XQ$ and if we set m_2 to zero $\delta N(x)$ vanishes.

which there is no meson deformation in (6.1) and set the term $\delta N(x)$ to zero. The semiclassical expansion of $T(x) = N(x)/D(x)$ on the physical sheet is

$$\frac{N(x)}{D(x)} = \frac{x^2 C_0(x)}{x^2 V'(x) + \frac{\alpha^2}{4\lambda}} + \lambda x \frac{2\widetilde{C}_2(x^2) + \alpha \widetilde{C}_1(x^2)}{2v_-(x^2)\left[x^2 V'(x) + \frac{\alpha^2}{4\lambda}\right]} + \hbar \frac{N_c S}{t_n x^{n+2}} + \mathcal{O}(\hbar^2). \quad (6.70)$$

Note that the terms of $\mathcal{O}(\hbar)$ have the asymptotics x^{-n-2}, so that they will contribute to $\langle \mathrm{Tr} X^{n+1} \rangle$. The higher quantum corrections $\mathcal{O}(\hbar^2)$ begin at x^{2n+3}, so they do not contribute to the expectation values of the nontrivial operators in the chiral ring (6.14) and we can safely drop them. If we compare (6.70) to the semiclassical expansion of the resolvent $R(x)$ in (6.37), we see that they have the same poles and the quantum deformations F_k are replaced by the polynomials C_k. The large x behaviour of $T(x)$ is dominated by the first classical term, which yields $\frac{N}{D} \sim \frac{N}{x}$. The behaviour of $T(x)$ on the other two sheets is just inherited from $R(x)$, so that we just have to continue analytically the resolvent through the other sheets and read off the expression for $T(x)$. The classical part of $T(x)$ has poles at the same location of the semiclassical expansion of $R(x)$, discussed in Appendix A. If we take into account also the contribution of $\delta N(x)$, we find that the resolvent $T(x)$ has a pole at infinity on the first and third sheets with residue $-N$ and $N-2$ respectively,[26] while it is regular at infinity on the second sheet.

Let us find out the other poles of $T(x)$ on the various sheets in the three different branches. The only singularities of $T(x)$ are the ones at infinity and the simple poles at the images of the point x_h. The branches enter in the expression (6.65) of $T(x)$ only through the meson generators in (6.67). There is a nice pictorial way to see the three branches. Let us denote by a cross "×" a simple pole with residue -1 and with a dot "•" a simple pole with residue 1. We can

[26] The -2 comes from $\delta N(x)$.

summarize the singularities in the three branches as[27]

	pseudoconf.		abel. higgs		nonab. higgs	
	$-x_h$	x_h	$-x_h$	x_h	$-x_h$	x_h
III sheet	$\bullet$	$\bullet$	0	$\bullet$	0	0
II sheet	$\times$	$\bullet$	0	0	$\times$	$\bullet$
I sheet	0	0	0	$\bullet$	$\bullet$	$\bullet$

(6.71)

For each branch, in the first column we collect the residues at the images of $-x_h$ on the three sheets and in the second column the residues at the images of x_h. Note that, just as the meson generator in (6.57), each branch is characterized by the generator being regular on one of the three sheets. Therefore, we can label the second sheet the *abelian higgs sheet* and the third one the *nonabelian higgs sheet*. When the resolvents are regular on the physical sheet we are of course in the pseudoconfining phase, as shown in Fig. 2.

Let us consider the A–periods of T, recalling the definition of the glueballs in Section 3.3. In the one–adjoint theory [12], the A–periods of T define the ranks of the i–th low energy SQCD as $N_i = \oint_{A_i} T(x)$, but in this case we have two different kinds of SQCDs, by the flows in the abelian and nonabelian vacua. The A_i periods define the ranks of the SQCD in the abelian vacua (6.6) as usual

$$N_i = \oint_{A_i} T(x),$$

while the ranks of the SQCD in the nonabelian blocks (6.8) is computed by either periods around the nonabelian cuts

$$\widehat{N}_i = \oint_{\widehat{A}_i} T(x) = \oint_{\widehat{A}'_i} T(x). \qquad (6.72)$$

[27] The appearance of a pole with residue -1 for $T(x)$ might seem unexpected. Consider the semiclassical expansion of $T^{II}(x)$ in (6.65) on the 2nd sheet: $D(x)$, $N(x)$ and $\delta N(x)$ are all even functions of x as $T^{II}(x)$, which is regular at infinity. The contour integral of $T^{II}(x)$ on a large contour henceforth vanish. We can close the contour around the finite singularities, which are the poles at x_h and $-x_h$ and the periods around A_i and A'_i. We have $\sum_{i=1}^{n+2}\left(\oint_{A_i} + \oint_{A'_i}\right) T^{II}(x) = 0$ and $\left(\oint_{x_h} + \oint_{-x_h}\right) T^{II}(x) = 0$. The residue around $-x_h$ is thus the opposite of the residue around x_h.

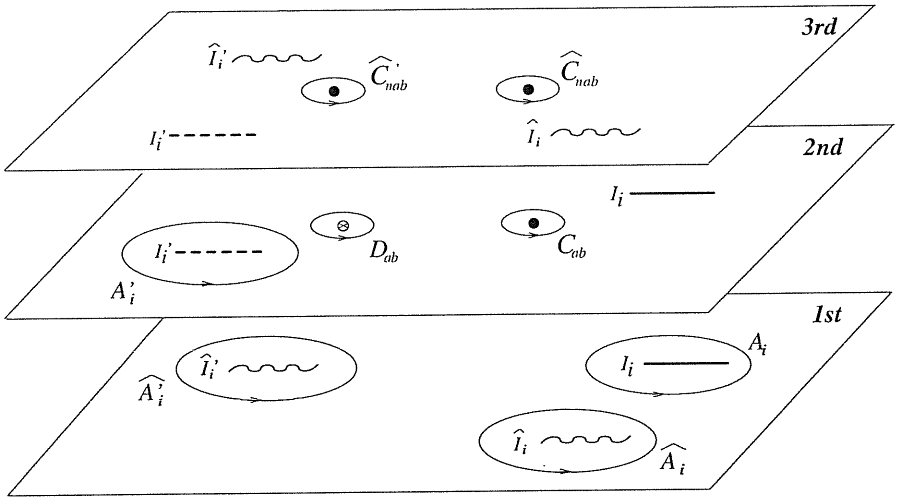

Fig. 2: The pseudoconfining phase. The black and white dots represent poles for $T(x)$ with residue respectively one and minus one. The contours $\widehat{C}'_{nab}$ and D_{ab} enclose the images of the higgs eigenvalue $-x_h$, while $\widehat{C}_{nab}$ and C_{ab} enclose the images of x_h.

With these definitions we recover the residue of $T(x)$ at infinity in the physical sheet as the sum of the the ranks plus the higgs poles $N_c = \sum_{i=1}^{n+2} N_i + 2\sum_{i=1}^{\frac{n-1}{2}} + r_{ab} + 2r_{nab}$, where r_{ab} is 1 in the abelian higgs branch and vanishes otherwise, while r_{nab} is 1 in the nonabelian higgs branch and vanishes otherwise.

6.5. Interpolating Between the Three Phases

Looking at the table (6.71), we can check that the sum of all residues of $T(x)$ on the curve vanishes. Moreover, when a cross meets a dot, they annihilate and, viceversa, from a vanishing residue we can create a pair cross–dot: $\times + \bullet = 0$. Now we can picture the way we interpolate between the three different branches as follows.

i) *pseudoconfining* $\leftrightarrow$ *abelian higgs*: Start with the pseudoconfining phase and move the dot $\bullet$ from the 2nd sheet to the 1st through the cut I_i. Due to the automorphism $x \to -x$, the other cross $\times$ in the second sheet moves through the symmetric cut I'_i from the 2nd to the 3rd sheet. Once on the 3rd sheet, the cross $\times$ annihilates with the $\bullet$, being both residues of a pole at $-x_h$, and we are left with the abelian higgs phase.

	pseudo conf.		$\longrightarrow$	abel.	higgs
	$-x_h$	x_h		$-x_h$	x_h
III sheet	$\bullet$	$\bullet$		$\times + \bullet = 0$	$\bullet$
II sheet	$\times \uparrow I'_i$	$\bullet \downarrow I_i$		0	0
I sheet	0	0		0	$\bullet$

When passing the pole through the i–th abelian cut I_i, the rank N_i of the corresponding i–th SQCD decreases by one. This is depicted in Fig.3. The new contour in fact is $A_i|_{new} = A_i - C_{ab}$ and we find

$$N'_i = \oint_{A_i} T(x) - \oint_{C_{ab}} T(x) = N_i - 1. \qquad (6.73)$$

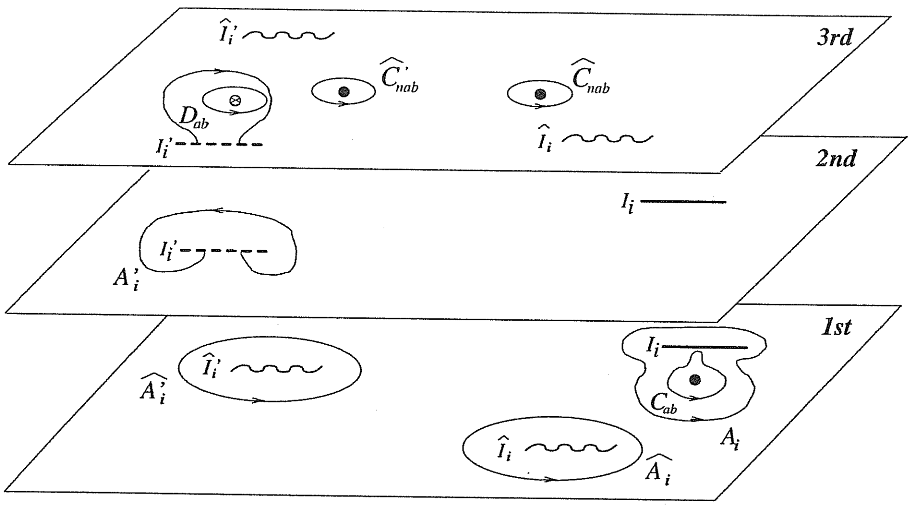

Fig. 3: Fig. 3: Interpolating between the pseudoconfining and the abelian higgs phase. Start in the pseudoconfining phase in fig. 2. Then move the pole D_{ab} to the 3rd sheet through the cut I'_i and the pole C_{ab} to the 1st sheet through the cut I_i. On the 3rd sheet, the contours D_{ab} and $\widehat{C}'_{nab}$ combine giving vanishing residue for $T(x)$ at $-x_h$ on the 3rd sheet. The new period of $T(x)$ on the first sheet is around the contour $A_i|_{new} = A_i - C_{ab}$ and we find (6.73).

ii) *pseudoconfining* $\leftrightarrow$ *nonabelian higgs*: Start with the pseudoconfining phase and move the two dots $\bullet$ from the 3rd sheet to the 1st through the nonabelian cuts: the pole at $-x_h$ moves through $\widehat{I}'_i$ and the pole at x_h moves through $\widehat{I}_i$.

	pseudo conf.		$\longrightarrow$	nonab. higgs	
	$-x_h$	x_h		$-x_h$	x_h
III sheet	$\bullet \uparrow \widehat{I}'_i$	$\bullet \uparrow \widehat{I}_i$		0	0
II sheet	$\times$	$\bullet$		$\times$	$\bullet$
I sheet	0	0		$\bullet$	$\bullet$

When passing the pole through the i–th nonabelian cut $\widehat{I}_i$, the rank $\widehat{N}_i$ of the corresponding i–th SQCD decreases by one. The new cycles are in fact $\widehat{A}_i|_{new} = \widehat{A}_i - \widehat{C}_{nab}$ and $\widehat{A}'_i|_{new} = \widehat{A}'_i - \widehat{C}'_{nab}$ and we find

$$\widehat{N}'_i = \oint_{\widehat{A}_i} T(x) - \oint_{\widehat{C}_{nab}} T(x) = \widehat{N}_i - 1,$$

or equivalently for the other period around $\widehat{A}'_i$.

iii) *nonabelian higgs* $\leftrightarrow$ *abelian higgs*: Start with the nonabelian higgs phase, pass to the pseudoconfining phase by moving the poles from the physical sheet to the third one and then move to the abelian higgs phase.

6.6. The Resolvent for Y

Up to now we have considered the effective description of the gauge theory when integrating out the adjoint superfield X. However, we can as well integrate out the other adjoint Y and study the effective theory encoded in the resolvent

$$R(y) = -\frac{1}{32\pi^2}\left\langle \mathrm{Tr}\frac{W_\alpha W^\alpha}{y-Y}\right\rangle.$$

To find an algebraic equations for $R(y)$ there are two methods [21], but we will use the most intuitive one. Consider the anomaly equations (6.26). To solve for $R(x)$ we used their Laurent expansion in y, but now we can use their Laurent expansion in powers of x and find the following recursion relations

$$\sum_{i=0}^{n} t_i R_i(y) + \lambda y^2 R(y) - \lambda y S - \lambda\widetilde{S} = 0, \tag{6.74}$$

$$\sum_{i=0}^{n} t_i R_{k+i+1}(y) - \hbar\sum_{i=0}^{k}\widehat{S}_i R_{k-i}(y) + \lambda y^2 R_{k+1}(y) - \lambda y\widehat{S}_{k+1} + \frac{\alpha}{2}\widehat{S}_k = 0, \tag{6.75}$$

$$2\lambda y R_{2k+1}(y) = 2\lambda \widehat{S}_{2k+1} + \hbar \sum_{i=0}^{2k} (-1)^i R_i(y) R_{2k-i}(y) - \alpha R_{2k}(y), \qquad (6.76)$$

for $k \geq 0$, where $R_k(y) = -\frac{1}{32\pi^2}\mathrm{Tr}\frac{W_\alpha W^\alpha}{y-Y}X^k$ are the generalized resolvents and $\widehat{S}_k = -\frac{1}{32\pi^2}\mathrm{Tr}W_\alpha W^\alpha X^k$. If we combine these three equations we get a closed degree 2^n algebraic equation for $R(y)$.

By solving the recursion relations one finds that $R(y)$ satisfies a degree 2^n algebraic equation. The curve in the Y–effective description is thus a 2^n sheeted covering of the plane. This might seem weird at first, since we conjectured that each sheet of the gauge theory curve is related to a different semiclassical phase. However, the three phases of our gauge theory are just the ones that we obtain by coupling the fundamentals with the adjoint X, and are the ones we can study in the X–effective description of the theory. It is likely that, when adding the most generic meson deformation $\delta W = \widetilde{Q}m(X,Y)Q$ to the superpotential, new vacua appear corresponding to new higgs phases, that we can just tell from each other in the effective Y description. On the X side they would be undistinguishable from the three phases we already considered.

The massive case D_3

We will not study the generic 2^n degree curve, but we show here what happens in the case $n = 1$. We will compare this case to its magnetic dual in Section 10. and recover the usual KSS duality discussed in [37].

Consider the tree level superpotential

$$W_{tree}(X,Y) = \mathrm{Tr}\left(\frac{t_1}{2}X^2 + \beta X + \lambda X Y^2 + \alpha Y\right) + m\widetilde{Q}Q, \qquad (6.77)$$

which corresponds to the case $n = 1$ and $V'(x) = t_1 x$. In this case we just have the $n + 2 = 3$ abelian vacua (6.6) but no nonabelian vacua (6.8), which are only present if $n \geq 3$. Here X is massive and we can integrate it out upon its equations of motion, obtaining an effective superpotential $U_{eff}(Y) = -\frac{1}{2t_1}\mathrm{Tr}(\beta + \lambda Y^2)^2 + \alpha \mathrm{Tr}Y$ at energies below the mass scale t_1, whose derivative is

$$U'(y) = -\frac{2\lambda}{t_1}(\beta y + \lambda y^3) + \alpha. \qquad (6.78)$$

In the quantum theory, we can use the anomaly equations derived in Appendix A to find a degree $2^n = 2$ algebraic equation for the resolvent $R^Y(y)$, which in this case is the hyperelliptic curve

$$\hbar^2 R(y)^2 - U'(y)R(y) - \frac{1}{4}f(y) = 0. \tag{6.79}$$

This is the usual anomaly equation for the one matrix model [12], where $U'(y)$ is the effective superpotential (6.78) and the quantum deformation

$$f(y) = \frac{8\lambda}{t_1^2}(\beta F_0 + \lambda F_2 + \lambda y t_1 \widetilde{S} + \lambda y^2 F_0), \tag{6.80}$$

that we expressed in terms of the parameters that we used in the solution for $R(x)$ in (6.33).[28] The solution of the anomaly equation (6.79) is

$$2\hbar R(y) = U'(y) - \sqrt{U'(y)^2 + \hbar f(y)}. \tag{6.81}$$

The physical picture in this case is the following. Classically, the resolvent $R(y)$ has three poles located at the classical vacua $y = b_i$, where the b_i's are given in (6.6). These are the roots of the cubic effective polynomial $U'(y)$. In the quantum theory, each pole splits into two branch points, that connect the first and the second sheet of the hyperelliptic curve. In this case we just have two classical phases, the pseudoconfining and the abelian higgs phase. The nonabelian higgs phase is only present if $n \geq 3$. So we just have two sheets as in the one–adjoint theory.

We can get the expectation values of the dressed mesons in the quantum theory by computing the contour integrals of the meson generator $M(y) = \widetilde{Q}\frac{1}{y-Y}Q = m^{-1}R(y)$ and find

$$\widetilde{Q}Q = \frac{S}{m}, \qquad \widetilde{Q}YQ = \frac{\widetilde{S}}{m}, \qquad \widetilde{Q}Y^2Q = \frac{F_2}{t_1}. \tag{6.82}$$

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[28] From the algebraic equation in Appendix A we get $f(y) = \frac{8\lambda}{t_1}(\lambda y^2 S + \lambda y \widetilde{S} - t_1 \widehat{S}_1) = 0$, where $\widehat{S}_1 = -\frac{1}{32\pi^2}\langle \mathrm{Tr} W_\alpha W^\alpha X \rangle$. By using the recursion relations in Appendix A we can also express the parameter $\widehat{S}_1$ in terms of the parameters $F_0 = t_n S, F_2$ that we used in the solution for $R(x)$ in (6.33) as $t_1 \widehat{S}_1 = -\frac{2\lambda}{t_1}F_2 - \beta S$.

The classical chiral ring of the gauge theory (6.1) is very different depending on whether $n = \deg V'(x)$ is odd or even. In the case $n =$ odd, we showed for $V'(x) = x^n$ that the independent operators are the ones in (6.14). Below we will show that this still holds for the generic superpotential (6.1) with n odd. If n is even, on the other hand, the trick (6.13) does not work any more and the chiral ring is not truncated. This would mean that there are an infinite number of independent operators in the classical chiral ring.

n odd

Let us first look at the chiral ring for odd n. A basis for the chiral operators in the $SU(N_c)$ theory is given by

$$\begin{aligned} &\mathrm{Tr}X^j, \qquad && j = 2, \ldots, n, \\ &\mathrm{Tr}X^{2l}Y^2, \qquad && l = 1, \ldots, \frac{n-1}{2}. \end{aligned} \tag{6.83}$$

We show that all the other chiral operators of the kind $\mathrm{Tr}X^jY^l$ can be expressed in terms of the basis (6.83). The classical chiral ring is characterized by the following relations, which are the classical limit of the anomaly equations in Appendix A we used to solve for $T(x)$. We have

$$\mathrm{Tr}X^{k+1}Y = -\frac{\alpha}{2\lambda}\mathrm{Tr}X^k, \tag{6.84}$$

$$\mathrm{Tr}X^l \left(V'(X) + \lambda Y^2\right) Y^k = 0, \tag{6.85}$$

$$\mathrm{Tr}X^{2l+1}Y^{k+2} = -\frac{\alpha}{2\lambda}\mathrm{Tr}X^{2l}Y^{k+1}, \tag{6.86}$$

for $k, l \geq 0$. The chiral operators we will consider are $\mathrm{Tr}X^{2l+1}Y^2$, higher powers of $\mathrm{Tr}X^{2l\geq n}Y^2$, $\mathrm{Tr}Y^{j\geq 3}$ and $\mathrm{Tr}X^{j\geq n+1}$.

Consider the operators containing odd powers of X, we have $\mathrm{Tr}X^{2l+1}Y^2 = -\frac{\alpha}{2\lambda}\mathrm{Tr}X^{2l}Y = \left(\frac{\alpha}{2\lambda}\right)^2 \mathrm{Tr}X^{2l-1}$, so we relate these operators to the truncation of the $\mathrm{Tr}X^j$ operators and discussed it below. The crucial point that distinguish the n odd from the n even case is that, in the former, we can use (6.86) to eliminate $\mathrm{Tr}X^nY^2$, while in the latter we can not, leaving the chiral ring untruncated.

Consider the operators $\mathrm{Tr}X^{2l}Y^2$, with $l \geq n$. The first one is

$$t_n \mathrm{Tr}X^{n+1}Y^2 = -\left(t_{n-1}\mathrm{Tr}X^nY^2 + \ldots + \beta\mathrm{Tr}XY^2 + \lambda\mathrm{Tr}XY^4\right), \tag{6.87}$$

and since $\mathrm{Tr}XY^4 = -\frac{\alpha}{2\lambda}\mathrm{Tr}Y^3$, we relate (6.87) to the truncation of higher powers of Y, discussed below. Then we proceed analogously until

$$t_n \mathrm{Tr}X^{2n}Y^2 = -\left(t_{n-1}\mathrm{Tr}X^{2n-1}Y^2 + \ldots + \beta\mathrm{Tr}X^nY^2 + \lambda\mathrm{Tr}X^nY^4\right),$$

and we relate the last term to $\mathrm{Tr}X^{n-1}Y^3$ by (6.86), and then to higher powers of X, to discuss below.

Consider higher powers of Y

$$\begin{aligned}\lambda\mathrm{Tr}Y^3 &= -\left(t_n\mathrm{Tr}X^nY + \ldots + t_1\mathrm{Tr}XY\right) = \mathrm{Tr}Xv_-(X^2)Y + \mathrm{Tr}\,v_+(X^2)Y \\ &= -\frac{\alpha}{2\lambda}\left(\mathrm{Tr}\,v_-(X^2) + t_{n-1}\mathrm{Tr}X^{n-2} + t_{n-3}\mathrm{Tr}X^{n-4} + \ldots\right).\end{aligned} \tag{6.88}$$

Then,

$$\begin{aligned}\lambda\mathrm{Tr}Y^4 =& \mathrm{Tr}Xv_-(X^2)Y^2 + \mathrm{Tr}\,v_+(X^2)Y^2 = -\frac{\alpha}{2\lambda}\mathrm{Tr}\,v_-(X^2)Y + \mathrm{Tr}\,v_+(X^2)Y^2 \\ =& \left(\frac{\alpha}{2\lambda}\right)^2\left(t_n\mathrm{Tr}X^{n-2} + t_{n-2}\mathrm{Tr}X^{n-4} + \ldots\right) + \mathrm{Tr}\,v_+(X^2)Y^2,\end{aligned} \tag{6.89}$$

and the last term is in the basis (6.83). Also $\lambda\mathrm{Tr}Y^5 = \mathrm{Tr}Xv_-(X^2)Y^3 + \mathrm{Tr}\,v_+(X^2)Y^3$, and the first one is related to the basis by (6.86). The last term instead $\lambda\mathrm{Tr}X^{n-1}Y^3 = -\mathrm{Tr}X^{n-1}V'(X)Y = \frac{\alpha}{2\lambda}\mathrm{Tr}X^{n-2}V'(X)$ is related to $\mathrm{Tr}X^{2n-2}$.

Consider eventually the higher powers of X

$$\begin{aligned}t_n\mathrm{Tr}X^{n+1} &= -\left(t_{n-1}\mathrm{Tr}X^n + \ldots + t_1\mathrm{Tr}X^2 + \lambda\mathrm{Tr}XY^2\right), \\ &= -\left(t_{n-1}\mathrm{Tr}X^n + \ldots + t_1\mathrm{Tr}X^2\right) + \frac{\alpha}{2}\mathrm{Tr}Y,\end{aligned} \tag{6.90}$$

Then we have $t_n\mathrm{Tr}X^{n+2} = -\left(t_{n-1}\mathrm{Tr}X^{n+1} + \ldots + \beta\mathrm{Tr}X^2\right) - \lambda\mathrm{Tr}X^2Y^2$ and so on. Then $t_n\mathrm{Tr}X^{2n+1} = -\left(t_{n-1}\mathrm{Tr}X^{2n} + \ldots + \beta\mathrm{Tr}X^{n+1}\right) - \lambda\mathrm{Tr}X^{n+1}Y^2$ and the last one is trivial by (6.87) and (6.88).

So we can conclude that the chiral ring is truncated to (6.83) in the case of n odd.

The n' even case

We want to address the n' even case by the point of view of the RG flows. We start at the IR fixed point $\widehat{D}$ of the theory with superpotential $\mathrm{Tr}XY^2$. Then we have two possibilities. If we consider the flow triggered by the relevant deformation $V'(X) = t_{n'}X^{n'}$ with $n' = 2m$, we can easily see that the chiral ring is not truncated. But we can consider the different flow, in the bottom line

$$\begin{array}{llllll} \widehat{\mathbf{D}} & \longrightarrow & \longrightarrow & \longrightarrow & \mathbf{n}' = even & \text{not truncated} \\ & \searrow & & & & \\ & & \mathbf{n} = odd \quad \text{truncated} & \longrightarrow & \text{new } \mathbf{n}' = even & \text{truncated} \end{array} \tag{6.91}$$

by first turning on a deformation such that $V'(X) = t_nX^n$ with $n = 2m+1$ and flow to the fixed point where the chiral ring is truncated. Then, we can switch on another relevant deformation such that $V'(X) = t_nX^n + t_{n'}X^{n'}$ with $n' = 2m < n$, that triggers a flow to another fixed point corresponding to the even case, but this fixed point is different from the untruncated one, namely here the chiral ring, inherited by the odd case, is still truncated. We can see this by reconsidering the computation above and use the superpotential $V'(X) = t_nX^n + t_{n'}X^{n'}$, i.e. $v_-(X^2) = t_nX^{n-1}$ and $v_+(X^2) = t_{n'}X^{n'}$. At the second fixed point, the first coupling t_n can be set to one, while the coupling $t_{n'}$ becomes marginal, so that actually $V'(X) = X^n + t_{n'}X^{n'}$. The crucial point here is that if we started directly with the even coupling $t_{n'}X^{n'}$, we could not use (6.86) to eliminate $\mathrm{Tr}X^{n'}Y^2$. If we start with the odd coupling, on the contrary, we can do the job and then, when flowing to the even case, this equation is still valid, by just setting $t_n = 1$ and keeping the marginal coupling $t_{n'}$. Therefore, the chiral ring for n' even that we get by flowing down from $n > n'$ contains the following operators

$$\begin{aligned} &\mathrm{Tr}X^j, \qquad j = 2, \dots, n'+1, \\ &\mathrm{Tr}X^{2i}Y^2, \qquad j = 1, \dots, \frac{n'}{2}. \end{aligned} \tag{6.92}$$

The analogous computation in the chiral ring, for the generalized glueballs $\mathrm{Tr}W_\alpha^2X^kY^j$, gives the following nontrivial operators

$$\begin{aligned} &\mathrm{Tr}W_\alpha^2X^j, \qquad j = 0, \dots, n'+1, \\ &\mathrm{Tr}W_\alpha^2Y,\ \mathrm{Tr}W_\alpha^2X^{2i}Y^2, \qquad j = 1, \dots, \frac{n'}{2}, \end{aligned} \tag{6.93}$$

which are actually $\frac{3}{2}n+3$. The chiral ring is truncated and we have two operators exceeding the number of vacua we would expect by sitting at the fixed point on the first line in (6.91). This is the mechanism by which, flowing to the $n' =$ even theory, the chiral ring gets truncated, thus confirming the a–theorem computations of [3].

6.8. The Classically Invisible Sheets and the Branches

We would like to draw some general lessons on the curve Σ of the $\mathcal{N} = 1$ supersymmetric gauge theory from this analysis of the different branches of SQCD. Consider a supersymmetric gauge theory with a matter content such that, once we fix the number of $U(1)$s in the low energy theory, there is no order parameter to distinguish between the various classical vacua in an invariant way, so we would use the word *branches* rather than phases. This means that, even if the classical theory has different kinds of solution to the equations of motion, in the quantum theory we can reach all the different semiclassical behaviors, with the same low energy photons, by continuously moving the couplings along their moduli space, as in the case of a theory with matter in the fundamental representation.[29] It is clear that the different branches can only make sense in the limit of

i) large expectation values, which is the semiclassical approximation;
ii) well separated branch cuts, that is far from singular points in the moduli space.

The observables that characterize the different branches are in the chiral ring of the onshell theory. If we have matter in the fundamental and adjoint representation they are the resolvents $M(x)$ and $T(x)$, that we defined in (6.19). The semiclassical branches are characterized by the analytic properties of these

[29] Vacua with a different number of low energy photons, however, describe two different phases of the theory. In fact, as suggested in [19] and discussed in [43] for the one adjoint theory, if the i–th low energy SQCD has $N_i = 1$, it is not possible to further pass any pole through the corresponding cut in the onshell theory.

resolvents on the curve Σ, that is by their poles and the respective residues. Under these assumptions, we can formulate the following proposal:[30]

An $\mathcal{N}=1$ supersymmetric gauge theory with a mass gap is described by a degree k algebraic curve, where k is the number of different branches of the theory. The curve is a k–sheeted covering of the plane, where each sheet corresponds to a different branch.

In this way we can explain the appearance, in the quantum theory, of the "classically invisible sheets", to quote [19]. Let us see how this works and focus the attention on the meson operator $M(x)$. In our SQCD, the mesons are dressed by the adjoints, but with a more general matter content they would be dressed in some other ways and the general picture would not change. Each branch is characterized by a set of classical expectation values for the matter fields, which are set to the solutions to the equations of motion. In our case (6.1) for instance we have the pseudoconfining, abelian higgs and nonabelian higgs branches. Each branch is defined by the poles and residues of $M(x)$ on the first sheet at large expectation values (semiclassical regime). By the generalized Konishi anomaly equations, it follows that the generic form of $M(x)$ in the branch A is given by

$$M^A(x) = \hbar g(x) R(x) + q^A(x), \tag{6.94}$$

where $g(x)$ is a rational function of the couplings, $R(x)$ is the gauge theory resolvent (which defines the curve) and $q^A(x)$ is another rational function that sets the boundary conditions on the meson operator, its poles and residues. Only in the classical limit do the branches make sense, thus we are interested in just the last term $q^A(x)$. In general, this depends on the resolvent $R(p_i)$ evaluated at the poles p_i, which are the images of the classical higgs expectation values. Since we assumed that there is no invariant way to distinguish the branches, we can connect all of them by changing continuously the boundary conditions $q^A(x)$, that is by moving the poles p_i between the sheets through the branch cuts. Now,

[30] We are excluding the case in which there is also a Coulomb branch, as it happens in the one adjoint theory when $n = N$. In this case, for instance, the gauge group is broken to its Cartan subalgebra, there is no mass gap and, in the limit of vanishing superpotential, we recover the $\mathcal{N}=2$ SQCD.

since the resolvent $R(p_i)$ on the curve Σ gets as many different classical limits as the number of sheets (this is the way we identify the sheets), it turns out that, when taking the classical limit of large poles in the first sheet, we obtain as many different expressions as the number of sheets. Each one of them is a solution to the equations of motion and therefore a different branch. Suppose that we have k branches but $k+1$ sheets. Then, we can continue the pole p_i to that extra sheet and compute the classical limit for $M(x)$ on the first sheet, but this corresponds to a new solution to the equations of motion and so we have found a new branch.

For a generic $\mathcal{N}=1$ gauge theory, this holds with the following two caveats:

- The number of sheets corresponds to the number of branches that we can distinguish in the effective description we are using. In our case of the deformed D_{n+2} theory (6.1), in the X effective description we can see only three branches, but we will argue below that more branches could be identified in the Y effective description.
- If there is an order parameter that characterizes one phase in an invariant way, then it seems plausible that the corresponding sheet be disconnected.

Let us see how our proposal works in the paradigmatic case of SQCD, where we have now a complete picture of all the possible branches. Depending on the extra matter content we can test our conjecture in different situations.

Ordinary SQCD: One Sheet

Consider SQCD with gauge group $U(N_c)$. We can describe the offshell curve of this theory in a confining vacuum by adding a massive adjoint superfield X and integrating it out. The tree level superpotential is

$$W_{SQCD} = \frac{t_1}{2}\mathrm{Tr}X^2 + m\widetilde{Q}Q,$$

with $t_1 \gg m$. This theory classically has only one branch, the pseudoconfining one, in which both X and the fundamentals vanish. The corresponding curve is

$$y^2 = t_1^2 x^2 + 4\hbar t_1 S, \qquad (6.95)$$

where S is the glueball. This looks like a double cover of the x plane with two branch points at $a^{\pm} = \pm 2\sqrt{S/t_1}$. But it is just a fake covering and the curve (6.95) actually describes the Riemann sphere. We have just one sheet corresponding to the one classical branch.

SQCD with One Adjoint: Two Sheets

Consider $U(N_c)$ SQCD with one adjoint X and a confining phase superpotential

$$W = \mathrm{Tr}V(X) + \widetilde{Q}m(X)Q,$$

where $V'(x)$ has degree n and $m(x)$ degree $n-1$. This theory has received a lot of attention. For $n < N$, it has two branches, the pseudoconfining and the (abelian) higgs one. The curve is the well known hyperelliptic Riemann surface $y^2 = V'(x)^2 + \hbar f(x)$, where the degree $n-1$ polynomial $f(x)$ is the quantum deformation. For $n > 1$, this is a genuine double–sheeted covering of the x plane. As explained in [19], we can continuously interpolate between the pseudoconfining and higgs branch by moving the poles of $M(x)$ and $T(x)$ from the second to the first sheet. The first sheet corresponds to the pseudoconfining branch and the second to the higgs branch.

SQCD with Two Adjoints: Three Sheets

This is the theory (6.1) that we have discussed at length. Classically, it has three branches: pseudoconfining (6.6)–(6.8), abelian higgs (6.10) and nonabelian higgs branch (6.11). The curve (6.32) is a three–sheeted covering of the plane and each sheet corresponds to a different branch, as explained in (6.71).

6.9. The Magnetic Dual

In this Section we will consider an equivalent description of the theory (6.1) in terms of magnetic degrees of freedom. We will first

A Seiberg dual description of the D_{n+2} SCFT theory with gauge group $SU(N_c)$ and superpotential[31]

$$W_{el} = t_n \mathrm{Tr}X^{n+1} + \lambda \mathrm{Tr}XY^2 + \beta \mathrm{Tr}X, \qquad (6.96)$$

has been proposed for $n =$ odd by Brodie in [22]. The magnetic theory is an $\mathcal{N} = 1$ $SU(3nN_f - N_c)$ gauge theory with two adjoint chiral superfields $\widetilde{X}$ and $\widetilde{Y}$, N_f magnetic fundamentals $\widetilde{q}^f$ and q_f and $3n$ gauge singlets $(P_{l,j})^f_{\widetilde{f}}$ for $l = 1, \ldots, n$

[31] This theory has no Lagrange multiplier for Y. Setting $\alpha = 0$ means that the adjoint Y gains an overall nonvanishing $U(1)$ component $\mathrm{Tr}Y$.

and $j = 1,2,3$, each of which transforms in the $(N_f, \bar{N}_f)$ of the flavor symmetry group. A magnetic superpotential was proposed in the case $\alpha = 0$

$$W_m = \bar{t}_n \mathrm{Tr}\widetilde{X}^{n+1} + \bar{\lambda}\mathrm{Tr}\widetilde{X}\widetilde{Y}^2 + \widetilde{q}\widetilde{m}(X,Y)q + \widetilde{\beta}\mathrm{Tr}\widetilde{X}, \tag{6.97}$$

where $\bar{t}_n = -t_n$ and $\bar{\lambda} = -\lambda$ and $\widetilde{m}(X,Y)$ is a polynomial that couples the magnetic fundamentals to the gauge singlets and the adjoints. This represents a Legendre transform between the electric and magnetic mesons. As opposed to our theory with confining phase superpotential (6.1), the theory (6.96) with $\alpha = 0$, that Brodie considered, has just $n+2$ one–dimensional vacua and no two–dimensional vacua is present in this case. The magnetic polynomial was proposed in the case $\alpha = \beta = 0$

$$\widetilde{m}(X,Y) = \frac{\bar{t}_n}{\mu^4}\sum_{l=1}^{n}\sum_{j=1}^{3} P_{lj} X^{n-l} Y^{3-j}. \tag{6.98}$$

This duality was checked to hold against different flows. Now, we would like to find the Seiberg dual of the theory (6.1), that is when generic deformations and Lagrange multipliers are present, so that we have both abelian (6.6) and nonabelian vacua (6.8). Let us state the results first. If the electric theory has superpotential

$$W_{el} = \mathrm{Tr}V(X) + \lambda\mathrm{Tr}XY^2 + \beta\mathrm{Tr}X + \alpha\mathrm{Tr}Y, \tag{6.99}$$

then the magnetic tree level superpotential corresponding to (6.99) is

$$W_{mag} = \mathrm{Tr}\bar{V}(\widetilde{X}) + \bar{\lambda}\mathrm{Tr}\widetilde{X}\widetilde{Y}^2 + \bar{s}\mathrm{Tr}\widetilde{Y}^2 + \widetilde{\beta}\mathrm{Tr}\widetilde{X} + \bar{\alpha}\mathrm{Tr}\widetilde{Y} + \widetilde{q}\widetilde{m}(X,Y)q + f(t_l,\lambda), \tag{6.100}$$

where $f(t_l\, l)$ depends only the couplings t_i, λ and the magnetic polynomial $\widetilde{m}(X,Y)$ will be discussed below. Note that, even if on the electric side there is no term such as $\mathrm{Tr}Y^2$, on the magnetic side we need the additional coupling $\bar{s}$ which will be fixed by duality to

$$\bar{s} = \lambda\frac{t_{n-1}}{t_n}\frac{N_f}{\bar{N}_c}.$$

Let us make a brief digression about the magnetic polynomial (6.98). The rationale behind this Legendre transform term is that it must decouple the electric and magnetic mesons in different low energy SQCD blocks. Namely, let us set the multiplier α to zero and consider the following coupling

$$\tilde{q}\tilde{m}(X,Y)q = \tilde{Q}\,\tilde{q}\frac{V'(X)-V'(\tilde{X})}{X-\tilde{X}}\frac{Y(\lambda Y^2+\beta)-\tilde{Y}(\lambda\tilde{Y}^2+\beta)}{Y-\tilde{Y}}q\,Q, \qquad (6.101)$$

Let us denote by M_i and m_i the N_f^2 electric and magnetic mesons in the i–th abelian vacuum and by $M_i^{\pm\pm}=\tilde{Q}^{\pm}Q^{\pm}$ and $M_i^{\pm\pm}=\tilde{Q}^{\pm}Q^{\pm}$ the $4N_f^2$ mesons in the i–th nonabelian vacuum, that we discussed in (6.59). By evaluating (6.101) on the classical expectation values for the adjoints in the abelian and nonabelian vacua, we see that

$$\begin{aligned}\tilde{q}\tilde{m}(X,Y)q = \beta\left(\sum_{i=1}^{n} m_i M_i V''(a_i) - t_1(M_{n+1}m_{n+1}+M_{n+2}m_{n+2})\right)\\ -2\beta\sum_{i=1}^{\frac{n-1}{2}} V''(\hat{a}_i)\left(\widehat{M}_i^{++}\hat{m}_i^{++}+\widehat{M}_i^{++}\hat{m}_i^{++}\right),\end{aligned}$$

the magnetic polynomial completely decouples the mesons in the different dual SQCD blocks. This superpotential term can be conveniently parameterized by a kernel that generalizes the one–adjoint theory [16][37],

$$\tilde{m}(X,Y) = \frac{1}{\mu^4}\oint dz\,dw\,\frac{V'(z)-V'(X)}{z-X}\frac{w(\lambda w^2+\beta)-Y(\lambda Y^2+\beta)}{w-Y}P(z,w), \qquad (6.102)$$

where we collected the $3n$ gauge singlets $(P_j)_{\tilde{f}}^{f}$ into a single meromorphic function

$$P(z,w) = \frac{P^{(1)}(z)}{w}+\frac{P^{(2)}(z)}{w^2}+\frac{P^{(3)}(z)}{w^3}, \qquad P^{(j)}(z)=\sum_{l=1}^{n}\frac{P_l^{(j)}}{z^l}. \qquad (6.103)$$

Note however that (6.102) is only valid if we set the Lagrange multiplier $\alpha = 0$.

Consider the vacua of the magnetic theory (6.100), they are very similar to the electric ones (6.6) and (6.8). The equations of motion are

$$\tilde{V}'(\tilde{X})+\tilde{\beta}+\tilde{\lambda}Y^2=0, \qquad \{\tilde{\lambda}\tilde{X}+\tilde{s},\tilde{Y}\}+\tilde{\alpha}=0, \qquad (6.104)$$

and their irreps are still one–dimensional vacua and two–dimensional vacua. The $n+2$ abelian vacua are analogous to (6.6) but with magnetic eigenvalues and multiplicities instead, such that (6.7) is replaced by $\bar{p}(x) = \left(x^2 + \frac{\bar{s}}{\bar{\lambda}}\right) [\bar{V}'(x) + \widetilde{\beta}] + \frac{\widetilde{\alpha}^2}{4\bar{\lambda}}$ and $\bar{b}_i = -\frac{\widetilde{\alpha}}{2\bar{\lambda}(\bar{a}_i + \frac{\bar{s}}{\bar{\lambda}})}$. The $(n-1)/2$ nonabelian vacua are analogous to (6.8), but each block of the adjoint is replaced by $\widetilde{X} = \widehat{a}_i\sigma_3 - \frac{\bar{s}}{\bar{\lambda}}\mathbb{1}_2$ and $\widetilde{Y} = \bar{d}_i\sigma_3 + \bar{c}_i\sigma_1$ and $\widehat{a}_i, \bar{d}_i, \bar{c}_i$ are fixed by (6.104).

We can use the SQCD duality relation $\bar{N}_i = \#\,\text{flavors} - N_i$ in each low energy SQCD block to check that the ranks of the gauge groups match. The electric low energy theory we flow to in the $n+2$ abelian vacua is SQCD with N_f flavors, while in the $\frac{n-1}{2}$ nonabelian vacua it is SQCD with $2N_f$ flavors. Therefore we have

$$\bar{N}_c = \sum_{i=1}^{n+2} \bar{N}_i + 2\sum_{i=1}^{\frac{n-1}{2}} \widehat{\bar{N}}_i = \sum_{i=1}^{n+2}(N_f - N_i) + 2\sum_{i=1}^{\frac{n-1}{2}}(2N_f - \widehat{N}_i) = 3nN_f - N_c. \quad (6.105)$$

Now we want to find the classical duality map. We will use the same strategy as KSS [16]. Let us first try a naive map between electric and magnetic eigenvalues $\bar{a}_i = a_i$ and $\bar{b}_i = b_i$. Since we will stick to the electric pseudo-confining phase, we can forget about the magnetic polynomial $\widetilde{m}(X, Y)$ for the moment. We have to impose the tracelessness condition on both the electric and magnetic adjoint vacua. On the electric side, the nonabelian vacua (6.8) are already traceless, since they are proportional to the Pauli matrices. We are left with the abelian vacua only $\mathrm{Tr}X = \sum_{i=1}^{n+2} N_i a_i = 0$ and $\mathrm{Tr}Y = \sum_{i=1}^{n+2} N_i b_i = 0$, which fix the Lagrange multipliers β and α. On the magnetic side, consider $\mathrm{Tr}\widetilde{Y} = \sum_{i=1}^{n+2} \bar{N}_i b_i = N_f \sum_{i=1}^{n+2} b_i$. Since $b_i = -\alpha/(2\lambda a_i)$ we get the condition $\sum_{i=1}^{n+2} \frac{1}{a_i} = 0$, but luckily this is automatically satisfied because the linear term in the abelian polynomial (6.7) vanishes. Then we have to impose also $\mathrm{Tr}\widetilde{X} = \sum_{i=1}^{n+2} \bar{N}_i \bar{a}_i = N_f \sum_{i=1}^{n+2} a_i - 2\frac{\bar{s}}{\bar{\lambda}}(N_f(n-1) - \sum_{i=1}^{\frac{n-1}{2}} \widehat{N}_i) = 0$, where also a contribution from the nonabelian magnetic vacua appear. Since a_i are the roots of (6.7) we have $\sum_{i=1}^{n+2} a_i = -\frac{t_{n-1}}{t_n}$ and the tracelessness condition can not be satisfied, unless $t_{n-1} = \bar{s} = 0$. This is very similar to what happens in the one–adjoint SQCD, that was discussed in [16]. We will use the strategy outlined there to solve for the duality map.

The Shift of the Electric and Magnetic Theory

To find the map we follow the usual trick in singularity theory and shift both electric and magnetic adjoints X and $\widetilde{X}$. The new feature is that we need to add the new coupling $\mathrm{Tr}\widetilde{Y}^2$ to the magnetic side. Then we impose the tracelessness conditions on the shifted adjoints and find that the naive map works in the shifted variables.

Consider the electric theory. Following the KSS reasoning, we shift X as[32]

$$X = X_s - B\mathbb{1}, \tag{6.106}$$

and the electric superpotential reads

$$W_{el} = \mathrm{Tr}V_s(X_s) + \beta_s\left(\mathrm{Tr}X_s - BN_c\right) + \lambda\mathrm{Tr}X_sY^2 + \alpha\mathrm{Tr}Y - \lambda B\mathrm{Tr}Y^2 + \phi N_c, \tag{6.107}$$

where

$$\begin{aligned} V_s(X_s) &= \sum_{i=1}^{n} \frac{g_i}{i+1} X_s^{i+1}, \\ g_l &= \sum_{i=l}^{n} \binom{i}{l} t_i(-B)^{i-l}, \\ \beta_s &= \beta + \sum_{i=1}^{n} t_i(-B)^i, \\ \phi &= \sum_{i=1}^{n} \frac{i+2}{i+1} t_i(-B)^{i+1}. \end{aligned} \tag{6.108}$$

The shifted equations of motion are $V_s'(X_s)+\beta_s+\lambda Y^2 = 0$ and $\lambda\{X_s-B, Y\}+\alpha = 0$. We still have abelian and nonabelian vacua. If we introduce the shifted eigenvalues $X_s = a_{s,i}$, keeping Y unshifted, the abelian vacua are the roots of

$$p_s(a_{s,i}) \equiv \left(V_s'(a_{s,i}) + \beta_s\right)(a_{s,i} - B)^2 + \frac{\alpha^2}{4\lambda} = 0, \tag{6.109}$$

for $i = 1, \ldots, n+2$ and $b_i = -\frac{\alpha}{2\lambda(a_{s,i}-B)}$. For the nonabelian vacua we find instead $X_s = \widehat{a}_{s,i}\sigma_3 + B\mathbb{1}_2$ and $Y = c_i\sigma_1 + d_i\sigma_3$ such that $d_i = -\frac{\alpha}{2\lambda\widehat{a}_{s,i}}$. If we plug X_s and Y into the X equation of motion we get two different terms, one proportional to σ_3 and one to the identity. We set them to zero separately

32 We do not shift Y since one would see that duality fixes the Y shift to zero.

and find two equations, the one proportional to σ_3 has $(n-1)/2$ roots $\widehat{a}_{s,i}$, the one proportional to $\mathbb{1}_2$ fixes c_i, we skip the details. So, we have a nonabelian solution also in the shifted variables. Once we found the vacua we can impose the tracelessness condition. On the electric side this fixes the Lagrange multipliers. $\mathrm{Tr}X_s = \sum_{i=1}^{n+2} N_i a_{s,i} + 2B\widehat{N} = 0$ where $\widehat{N} = \sum_{i=1}^{\frac{n-1}{2}} \widehat{N}_i$, which fixes β_s, while $\mathrm{Tr}Y = \sum_{i=1}^{n+2} N_i b_{s,i} = 0$, which fixes α.

On the magnetic side we have to switch on the deformation $\delta\bar{W} = \bar{s}\mathrm{Tr}\widetilde{Y}^2$ and consider the magnetic superpotential (6.100). At the end this new coupling will be fixed by duality as a function of the other couplings. Let us shift the magnetic theory as $\widetilde{X} = \widetilde{X}_s - \bar{B}\mathbb{1}$, the superpotential reads

$$
\begin{aligned}
\bar{W}_m =& \mathrm{Tr}\bar{V}_s(\widetilde{X}_s) + \widetilde{\beta}_s\left(\mathrm{Tr}\widetilde{X}_s - \bar{B}\bar{N}_c\right) + \bar{\lambda}\mathrm{Tr}\widetilde{X}_s\widetilde{Y}^2 + \widetilde{\alpha}\mathrm{Tr}\widetilde{Y} \\
&+ (\bar{s} - \bar{\lambda}\bar{B})\mathrm{Tr}\widetilde{Y}^2 + \bar{\phi}\bar{N}_c + f_s(t_l, \lambda, B),
\end{aligned}
\tag{6.110}
$$

where the notation is as in (6.108) but with magnetic quantities instead and $f_s(t_l, \lambda, B$ is the shifted coupling dependent function. If we solve the equations of motion we still find the abelian and nonabelian vacua. In the shifted magnetic eigenvalues, the abelian vacua are $\widetilde{X}_s = \bar{a}_{s,i}$ and $\widetilde{Y} = \bar{b}_i$, where

$$
\begin{aligned}
\bar{b}_i &= -\frac{\widetilde{\alpha}}{2\bar{\lambda}(\bar{a}_{s,i} + \frac{\bar{s}}{\bar{\lambda}} - \bar{B})}, \\
\bar{p}_s(\bar{a}_{s,i}) &\equiv \left(\bar{V}_s'(\bar{a}_{s,i}) + \widetilde{\beta}_s\right)(\bar{a}_{s,i} + \frac{\bar{s}}{\bar{\lambda}} - \bar{B})^2 + \frac{\widetilde{\alpha}^2}{4\bar{\lambda}} = 0,
\end{aligned}
\tag{6.111}
$$

for $i = 1, \ldots, n+2$. The shifted nonabelian vacua are $\widetilde{X}_s = \widehat{\bar{a}}_{s,i}\sigma_3 + \left(\bar{B} - \frac{\bar{s}}{\bar{\lambda}}\right)\mathbb{1}_2$ and $\widetilde{Y} = \bar{c}_i\sigma_1 + \bar{d}_i\sigma_3$ Now we impose the tracelessness condition on the magnetic adjoints

$$
\mathrm{Tr}\widetilde{X}_s = \sum_{i=1}^{n+2}(N_f - N_i)(\bar{a}_{s,i}) + 2\left(\bar{B} - \frac{\bar{s}}{\bar{\lambda}}\right)\sum_{i=1}^{\frac{n-1}{2}}(2N_f - \widehat{N}_i) = 0. \tag{6.112}
$$

Once we impose both electric and magnetic tracelessness conditions, we can postulate the naive match of the shifted eigenvalues

$$
\bar{a}_{s,i} = a_{s,i}, \tag{6.113}
$$

and we drop the dependence upon the vacua inside (6.112) by fixing the shifts

$$\begin{aligned} B &= \frac{g_{n-1}}{2(n-1)g_n}, \\ \bar{B} &= B + \frac{\bar{s}}{\bar{\lambda}}. \end{aligned} \tag{6.114}$$

By comparing the two abelian polynomials (6.109) and (6.111) we get the map between the shifted couplings and Lagrange multipliers

$$\begin{aligned} \bar{g}_l &= -g_l, \qquad & \bar{\lambda} &= -\lambda, \\ \bar{\beta}_s &= -\beta_s, \qquad & \bar{\alpha} &= -\alpha. \end{aligned} \tag{6.115}$$

The map between the operators is independent of the vacua if the electric and magnetic superpotential match and the coupling dependent function $f_s(t_l, \lambda, B)$ does not depend on the vacuum. We can check this last requirement by differentiating the effective action with respect to the shifted couplings

$$\mathrm{Tr} X_s^{l+1} = -\mathrm{Tr} \tilde{X}_s^{l+1} + (l+1)\frac{\partial f_s}{\partial g_l},$$

that gives two relations, depending on whether l is even or odd:

$$\begin{aligned} \frac{\partial f_s}{\partial g_{2l}} &= \frac{N_f}{2l+1}\left(\sum_{i=1}^{n+2} a_{s,i}^{2l+1} + 4B^{2l+1}\sum_{i=1}^{\frac{n-1}{2}}\sum_{k=0}^{l}\left(\frac{\widehat{a}_i}{B}\right)^{2k}\right), \\ \frac{\partial f_s}{\partial g_{2l+1}} &= \frac{N_f}{2l+2}\left(\sum_{i=1}^{n+2} a_{s,i}^{2l+2} + 4B^{2l+2}\sum_{i=1}^{\frac{n-1}{2}}\sum_{k=0}^{l+1}\left(\frac{\widehat{a}_i}{B}\right)^{2k}\right), \end{aligned} \tag{6.116}$$

while $\partial f_s/\partial\lambda = 0$ so that f_s does not depend on λ. By (6.116) we see that f_s does not depend on the vacua, so the operator map is independent of the vacua, too.

The Map in the Original Couplings

By using (6.113), (6.114) and the map (6.115) we can reobtain the relation between the eigenvalues and the couplings in the original parametrization of the theory

$$\begin{aligned} \bar{a}_i &= a_i - \frac{\bar{s}}{\bar{\lambda}}, \\ \bar{t}_l &= -\sum_{i=l}^{n}\binom{i}{l} t_l \left(\frac{\bar{s}}{\bar{\lambda}}\right)^{i-l}. \end{aligned} \tag{6.117}$$

While the abelian X eigenvalues are shifted by the duality, the nonabelian eigenvalues match exactly, as well as the Y expectation values

$$\begin{aligned} &\widehat{\bar{a}}_i = \widehat{a}_i, \\ &\bar{b}_i = b_i, \qquad \bar{c}_i = c_i, \qquad \bar{d}_i = d_i. \end{aligned} \tag{6.118}$$

By imposing the tracelessness condition in the unshifted magnetic adjoint we fix $\bar{s}$ in terms of the other couplings of the theory

$$\bar{s} = \lambda \frac{t_{n-1}}{t_n} \frac{N_f}{\overline{N}_c}. \tag{6.119}$$

In other words, if we introduce the shift $A = -\bar{s}/\bar{\lambda}$ such that $\bar{a}_i = a_i + A$, we can write down the electric superpotential as (6.99), while the magnetic one (6.100) reads in electric variables

$$W_m = -\mathrm{Tr}V(\tilde{X} - A) - \lambda \mathrm{Tr}(\tilde{X} - A)\widetilde{Y}^2 - \alpha \mathrm{Tr}\widetilde{Y} - \beta \mathrm{Tr}\widetilde{X} + f(coupl.). \tag{6.120}$$

Duality for D_3: the Quantum Theory

We would like to solve for the chiral ring operator (6.16) in both the electric and magnetic side in the quantum theory and find the map between the dual quantum deformations $F_k(x)$. For generic superpotentials (6.99) and (6.100) this seems impossible, since on the magnetic side the anomaly equations do not close any more on an algebraic equation for the magnetic resolvent $\widetilde{R}(x)$, due to the extra coupling $\mathrm{Tr}\widetilde{Y}^2$. The only case in which we can solve both electric and magnetic theory is when the adjoint polynomial is just a mass term

$$W_{el} = \frac{t_1}{2}\mathrm{Tr}X^2 + \lambda \mathrm{Tr}XY^2 + \beta \mathrm{Tr}X + \alpha \mathrm{Tr}Y + m\widetilde{Q}Q,$$

We will denote this the D_3 superpotential. In this case, X can be integrated out and the effective theory is described by the resolvent $R(y)$, that we worked out in (6.81). So we can compare the effective Y quantum theories on both sides of the duality and luckily get a map. This will reproduce exactly the KSS duality in the case of a one–adjoint SQCD with superpotential (6.78) in [37], that is we will see that $D_3 \sim A_4$, as expected in singularity theory.

The dual D_3 magnetic theory is

$$W_{mag} = \frac{\bar{t}_1}{2}\mathrm{Tr}\widetilde{X}^2 + \bar{\lambda}\mathrm{Tr}\widetilde{X}\widetilde{Y}^2 + \widetilde{\beta}\mathrm{Tr}\widetilde{X} + \widetilde{\alpha}\mathrm{Tr}\widetilde{Y} + \widetilde{q}\widetilde{m}(X,Y)q + \widetilde{m}\mathrm{tr}P_1^{(1)},$$

where the last term corresponds to the electric mass term for the fundamentals and the trace is over flavor indices. The magnetic polynomial in this case is very simple and we can actually use (6.102) even if we keep a nonvanishing α multiplier. Actually, the two adjoint polynomial $\widetilde{m}(X,Y)$ is equivalent to what we get by using the effective quartic one–adjoint polynomial $\bar{U}(\widetilde{Y})$ in (6.78). In fact, the $\widetilde{X}$ dependence drops and we are left with

$$\widetilde{m}(X,Y) = \frac{\bar{t}_1}{\mu^4}\oint dw\, \frac{\bar{U}'(w) - \bar{U}'(\widetilde{Y})}{w - \widetilde{Y}}, \tag{6.121}$$

The singlet equations of motion are

$$\widetilde{q}Y^j q = -\delta_{j,2}\frac{\widetilde{m}\mu^4}{\bar{\lambda}\bar{t}_1}. \tag{6.122}$$

The anomaly equations for the resolvents $\widetilde{R}^{\widetilde{X}}(x)$ and $\widetilde{R}^{\widetilde{Y}}(y)$ are the same as in the electric theory, (6.32) and (6.79). We want to solve for the singlets in the magnetic theory, following the method in [37]. The magnetic meson generator $\widetilde{M}(y) = \widetilde{q}\frac{1}{y-\widetilde{Y}}q$ satisfies the anomaly equation $[\widetilde{m}(y)\widetilde{M}(y)]_- = \widetilde{R}^{\widetilde{Y}}(y)$. The generic solution is

$$\widetilde{M}(y) = \frac{\widetilde{R}^{\widetilde{Y}}(y)}{\widetilde{m}(y)} + \frac{\widetilde{r}(y)}{\widetilde{m}(y)}. \tag{6.123}$$

The way we solve it is by first fixing $\widetilde{r}(y)$ to cancel spurious singularities from the zeros of $\widetilde{m}(y)$ and then solving for $\widetilde{m}(y)$ such that the singlet equations of motion (6.122) are satisfied. The solution in the pseudoconfining phase is $\widetilde{r}(y) = 0$ and

$$\widetilde{m}(y) = -\frac{\bar{t}_1^2}{\bar{\lambda}}\frac{\widetilde{f}(y)}{8\widetilde{m}\mu^4}, \tag{6.124}$$

where $\widetilde{f}(y) = \frac{8\bar{\lambda}}{\bar{t}_1^2}(\widetilde{\beta}\widetilde{F}_0 + \bar{\lambda}\widetilde{F}_2 + \bar{\lambda}y\bar{t}_1\widetilde{S} + \bar{\lambda}y^2\widetilde{F}_0)$ is the quantum deformation of $\widetilde{R}^{\widetilde{Y}}(y)$ and $\widetilde{F}_k$ are the quantum deformations of $\widetilde{R}^{\widetilde{X}}(x)$, the magnetic version of

(6.33). Formally it is the same expression as in (6.80), but replacing electric with magnetic quantities.

The magnetic polynomial in terms of the singlets reads $\widetilde{m}(y) = \frac{bart_1}{\mu^4}(\bar{\lambda}P_1^{(3)} + \bar{\lambda}yP_1^{(2)} + (\widetilde{\beta} + \bar{\lambda}y^2)P_1^{(1)})$. We can read off the expression of the quantum expectation value of the gauge singlets in terms of the quantum deformations

$$P_1^{(1)} = -\frac{\widetilde{F}_0}{\bar{m}\bar{t}_1}, \qquad P_1^{(2)} = -\frac{\bar{\widetilde{S}}}{\bar{m}}, \qquad P^{(3)} = -\frac{\widetilde{F}_2}{\bar{m}\bar{t}_1}. \tag{6.125}$$

If we match them directly to the electric mesons (6.82) we find the map duality map in the quantum theory between the quantum deformations $\widetilde{F}_0 = F_0$ and $\widetilde{F}_2 = F_2$ and then between the couplings, the Lagrange multipliers and the glueballs

$$\begin{aligned} \bar{t}_1 &= -\, t_1, \qquad \bar{\lambda} = -\lambda, \qquad \widetilde{\beta} = -\beta, \\ \bar{S} &= -\, S, \qquad \bar{\widetilde{S}} = -\widetilde{S}. \end{aligned} \tag{6.126}$$

6.10. Further Directions

In this Chapter we have proposed a general explanation of the presence of the classically "invisible" sheets in the curves of $\mathcal{N} = 1$ supersymmetric gauge theories. In general, the gauge theory curve is realized as a k–sheeted covering of the plane. One of these sheets is visible in the classical theory, while the remaining sheets are not accessible semiclassically bu only in the full quantum theory. A convenient method to compute this curve is by the DV prescription, that relies on the planar limit of a related matrix model or, correspondingly, on solving a set of anomaly equations in the gauge theory. We considered theories with matter content such that, once we fix the number of low energy photons, there is no order parameter to distinguish the various classical vacua, hence we denoted the different kinds of classical solutions as branches. Our proposal is that, under these circumstances, there is a one to one correspondence between the number of branches and the degree of the curve.

This proposal holds trivially in the case of ordinary SQCD and has been verified also for SQCD with one adjoint chiral superfield in [19]. In this paper, we have worked out the classical and quantum theory of SQCD with two adjoints and superpotential (6.1) and we have verified that the proposal works also in this case. In particular, we have shown that this theory has three classical vacua, namely the pseudoconfining, the abelian higgs and the nonabelian higgs ones. We have proven that in the quantum theory we can associate each sheet of the cubic curve to each of these three branches by looking at the singularities of some meromorphic functions on the curve. Moreover, we have argued that one can interpolate continuously between all the classical vacua with the same number of low energy $U(1)$ factors. It would be interesting to verify our conjecture for other gauge theories with a higher degree DV curve, in particular one can address the following cases.

Consider a $U(N_c)$ gauge theory with one adjoint and an additional chiral superfield in the symmetric (or antisymmetric) representation. Its DV curve is a cubic, as in our two adjoint SQCD, and has been computed in [47][48]. One could couple this theory to matter in the fundamental representation and find out the classical branches. According to our proposal, we expect to see, in addition to the pseudoconfining vacua of [47][48], two different higgs vacua and, in the quantum theory, we expect the three branches (with the same low energy photons) to be connected continuously.

The second theory is a quiver $SU(N_c) \times SU(N_c)$ gauge theory with matter in the bifundamental representation. The curve of this theory is again a cubic [48], but it has a weird feature, namely each node of the quiver sees a particular sheet as its own physical sheet and the leftover sheet seems mysterious. It would be interesting to add fundamental matter to this theory and classify its classical branches, then study the quantum theory and see how we can connect the different branches by moving the poles between the sheets. In this way one could clarify our proposal in the case of a quiver theory. Moreover, a Seiberg dual theory to this quiver with fundamentals has been discussed in [49]. It would be nice to see the dual description of the electric branches on the DV curve, which is the same for both dual pairs, along the lines of [37].

The E–type SQCD

Finally, an extremely interesting theory where to test our proposal is SQCD with two adjoints and E_n type (according to the ADE classification of [3]) tree level superpotential. For instance, one can consider the E_6 theory with superpotential $W = \mathrm{Tr}Y^3 + \mathrm{Tr}X^4$ deformed by lower dimensional operators. The classical vacua of this theories are not known. However, by studying their flows in connection with the a theorem, [3] argued that there are an infinite number of irreps of the equations of motion with vanishing fundamentals (which we called the pseudoconfining branch). First of all, it would be nice to see explicitly whether the number of pseudoconfining vacua is actually infinite. One could find also the higgs vacua and classify all the branches of the theory, then compute the $\mathcal{N} = 1$ curve and verify if the degree of the curve agrees with the number of branches.

As a byproduct of this analysis, one would shed light on the following mystery. The analytic structure of an $\mathcal{N} = 1$ curve is such that, on the physical sheet, the number of branch cuts are in correspondence with the classical pseudoconfining vacua and, in the classical limit, each branch cut shrinks to a point corresponding to a pseudoconfining vacuum. In this case, if the number of pseudoconfining vacua is infinite, it is not at all clear what the curve would look like, since we would expect an infinite number of branch cuts on the physical sheet. Moreover, one could consider the geometric engineering of this theory theory as a type IIB superstring theory on a certain local Calabi Yau threefold, in the framework of [20][21]. The classical theory is described by the resolved geometry of a P^1 bundle over a particular ALE space (for a review see [50]). The classical pseudoconfining vacua of the gauge theory should be seen in the geometry as the compact holomorphic curves of the threefold. According to the geometric transition conjecture, in the quantum theory these holomorphic curves are replaced by three spheres, whose volume is proportional to the gauge theory glueballs. But if we have an infinite number of pseudoconfining vacua, as argued by [3], it is not at all clear how to make sense of the classical geometric picture in the first place, whether there are an infinite number of holomorphic curves in the resolved geometry and, finally, how to perform the blow down map, if any, and compute the deformed Calabi Yau.

Appendix A. Solution of the Cubic Equation

Consider the cubic equation

$$w^3 + aw^2 + bw + c = 0. \tag{6.127}$$

First get rid of the subleading term by the shift $w = z - a/3$, obtaining $z^3 + 3\gamma z + 2\delta = 0$, where we introduced $3\gamma = b - \frac{a^2}{3}$ and $2\delta = c - \frac{ab}{3} + \frac{2}{27}a^3$. Now the trick is to replace z with two variables under a useful constraint. Set $z = u + v$ and get $u^3 + v^3 + 3(u+v)(uv+\gamma) + 2\delta = 0$. If we just choose $uv + \gamma = 0$ then we get

$$\begin{cases} u^3 + v^3 + 2\delta = 0, \\ uv + \gamma = 0. \end{cases}$$

We solve the quadratic equation $u^6 + 2\delta u^3 - \gamma^3 = 0$, obtaining $u^3 = -\delta + \sqrt{\delta^2 + \gamma^3}$. The solutions for u picks up the three cubic roots of unity, 1, $e^{i\frac{2\pi}{3}}$ and $e^{-i\frac{2\pi}{3}}$, obtaining

$$\begin{aligned} u^{(I)} &= (-\delta + \sqrt{\delta^2+\gamma^3})^{\frac{1}{3}}, \\ u^{(II)} &= e^{i\frac{2\pi}{3}} u^{(I)}, \\ u^{(III)} &= e^{-i\frac{2\pi}{3}} u^{(I)}, \end{aligned}$$

and analogous solutions for $v = -\frac{\gamma}{u}$. The solutions to (6.127) are therefore

$$w^{(i)} = u^{(i)} - \frac{\gamma}{u^{(i)}} - \frac{a}{3},$$

that we can list

$$\begin{aligned} w^{(I)} &= (-\delta + \sqrt{\delta^2+\gamma^3})^{\frac{1}{3}} - \frac{\gamma}{(-\delta + \sqrt{\delta^2+\gamma^3})^{-\frac{1}{3}}} - \frac{a}{3}, \\ w^{(II)} &= e^{i\frac{2}{3}\pi}(-\delta + \sqrt{\delta^2+\gamma^3})^{\frac{1}{3}} - e^{-i\frac{2}{3}\pi}\frac{\gamma}{(-\beta + \sqrt{\beta^2+\gamma^3})^{-\frac{1}{3}}} - \frac{a}{3}, \\ w^{(III)} &= e^{-i\frac{2}{3}\pi}(-\beta + \sqrt{\beta^2+\gamma^3})^{\frac{1}{3}} - e^{i\frac{2}{3}\pi}\frac{\gamma}{(-\beta + \sqrt{\beta^2+\gamma^3})^{-\frac{1}{3}}} - \frac{a}{3}. \end{aligned} \tag{6.128}$$

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