Chen–Ruan cohomology of Moduli of Curves

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To my father Andrea

Contents

Chapter 1

Introduction

1.1 Historical background

This dissertation, broadly speaking, is devoted to the task of investigating the geometry of the moduli spaces of curves. It is well known after Poincaré that, from a topological or differentiable point of view, a closed surface is identified by one discrete invariant: the genus. On the contrary, algebraic, complex or conformal structures on a closed surface give rise to moduli, *i.e.* the set of all the equivalence classes of such structures on a genus g curve can naturally be given a topology and a structure of an algebraic variety. The task of understanding the geometry of all the spaces thus constructed has been an outstanding problem, of interest to geometers even before their actual precise mathematical construction. In 1857 Riemann computed the dimension of these spaces as a function of g, and since then moduli spaces have been widely used by the German and Italian schools. These moduli spaces, called $\mathcal{M}_g$, were defined as topological spaces by topologists in the first half of the XX century by means of Teichmüller theory, and were given a structure of algebraic variety only in 1965 via GIT by Mumford [MFK]. They were later described as smooth irreducible Deligne–Mumford stacks in the seminal paper [DM] by Deligne and Mumford (1969), where they were also compactified. Since then, the geometry of such spaces has become a matter of interest for geometers and topologists, as well as theoretical physicists (*e.g.* in topological string theory) and mathematical physicists (*e.g.* in integrable systems). In the following, we will always refer to smooth Deligne–Mumford stacks as *orbifolds*. The moduli of curves with marked points, $\overline{\mathcal{M}}_{g,n}$ ([K83]), have also been constructed, and, according to an insight of Grothendieck, they should be studied altogether for all g and n.

Moduli spaces have many applications. One way they appear in computations is via enumerative geometry: imposing geometric conditions usually corresponds to cutting appropriate subspaces in the moduli space. Thus, enumerative geometry is reduced to intersection theory on moduli spaces. The prototype of this program is what we could call the linear case: the Grassmannian, that is the moduli space of linear subspaces of a given vector space. Here, the development of the theory of Schubert calculus reduces the computation of the cohomology ring of the Grassmannian to a solvable combinatorial problem. Moduli spaces of curves can be seen as the easiest nonlinear analogue to the Grassmannian, and the analogue to Schubert calculus is called Gromov–Witten theory. The first step to mimic such a program is to have a smooth and compact space, and the moduli spaces of stable curves with marked points $\overline{\mathcal{M}}_{g,n}$ were constructed as compact, smooth geometric objects (though not varieties!) in [DM] and [K83]. In the beginning of the eighties Harer and Mumford started to tackle the fundamental question of determining the cohomology ring of these moduli spaces. Meanwhile, in the development of a theory of quantum gravity, the counting of curves in the target space of the theory represents the quantum corrections in string compactification. This, together with the birth of Gromov–Witten theory and mirror symmetry in [CDGP], led physicists to make predictions on the intersection theory of moduli spaces of curves, and to answer conjecturally some long–standing mathematical problems, such as the celebrated count of curves passing through n points in $\mathbb{P}^2$ of given degree (Witten's conjecture, now Kontsevich theorem) and the prediction for the number of rational curves in the quintic threefold in $\mathbb{P}^4$. So, in the nineties this problem was investigated by many researchers using techniques coming from topology, algebraic geometry, number theory and mathematical physics, and this led to a drastic increase in the knowledge of this topic.

Although some results have been obtained for specific values of g and n, it seems out of our present reach to give an – even conjectural – description of the cohomology for all g and n. Traditionally, two successful approaches have been followed: to tackle the problem starting in low genus, and to try to give results on the $k-$th cohomology group for all g and n. In the present work we proceed in the same spirit, giving general results and construction that hold for all g and n, and then trying to produce more explicit results in low genus.

An easier object of study, which was first introduced by Mumford ([Mu83]), is a subring of the cohomology of $\mathcal{M}_{g,n}$ (and $\overline{\mathcal{M}}_{g,n}$), called the *Tautological Ring* $R^*(\mathcal{M}_{g,n})$ (and $R^*(\overline{\mathcal{M}}_{g,n})$). It contains all the classes that can easily be constructed geometrically, which happen to be the ones of interest in Gromov–Witten theory and in mathematical physics. A very neat definition of these classes was given for $\overline{\mathcal{M}}_{g,n}$ in [FP05]. Tautological Rings arising from partial compactifications of $\mathcal{M}_{g,n}$ are defined in [Pa99].

In 2001, a new cohomology theory for orbifolds was developed; the so called Chen–Ruan cohomology [CR04]. Analogously a stringy Chow ring in the algebraic context was defined one year later by Abramovich, Graber and Vistoli [AGV02]. This cohomology theory is defined exactly as the degree zero part of the small quantum cohomology of orbifolds, which is developed in the same articles. This cohomology theory gives an object, H^*_{CR}, which is an H^*-algebra and takes into account orbifold phenomena. It seems interesting to determine the Chen–Ruan cohomology ring, which should be thought of as a refined version of the cohomology. Generally speaking our task is twofold. On one hand, we want to better understand the geometry and topology of the spaces $\overline{\mathcal{M}}_{g,n}$. Chen–Ruan cohomology contains important informations on the topological and orbifold structure of these moduli spaces, information that is not contained in the ordinary cohomology, not even with integral coefficients. On the other hand, it is a matter of interests on its own to provide new examples of explicit computations of orbifold Quantum Cohomology. This could be of interests in other fields when reformulated in the language of Frobenius Manifolds and Integrable Systems, or Topological String Theory.

1.2 Introduction to the problem

We want to substitute the question:

Question 1.1. *What is the cohomology ring $H^*(\mathcal{M}_{g,n}, \mathbb{Q})$? What is the cohomology ring $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$?*

with the new one:

Question 1.2. *What is the Chen–Ruan cohomology ring $H^*_{CR}(\mathcal{M}_{g,n}, \mathbb{Q})$? What is the cohomology ring $H^*_{CR}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$?*

The idea behind Chen–Ruan cohomology, that has analogous precursors in K-theory (see for instance [T99]), is that in computing the cohomology of a global quotient X/G, the equivariant cohomology $H^*(X)^G$ is somehow too small. One has to take into account the cohomology of the so called "twisted sectors". If g is an element of G, a twisted sector Y_g is, loosely speaking, the locus of points y such that g is in the stabilizer group of y. The equivariant cohomology then appears in the picture as the cohomology of the "untwisted sector": the locus labelled by the identical automorphism that coincides with the whole space X/G. The *Inertia Stack* is obtained by taking both the untwisted and twisted sectors, and the new cohomology theory is, additively, the ordinary cohomology of this new space. The Chen–Ruan product is then the usual product when two classes that belong to the untwisted sector are intersected. The intersection of two "new" classes is defined in such a way that the product of a class in Y_g with a class in Y_h is a class in Y_{gh}. To define it, an auxiliary space is introduced, that parametrizes loci having both g and h in the stabilizer group: it is called the *second Inertia Stack*. The Chen–Ruan intersection product is then computed via an intersection of the two classes restricted to this common locus, taking into account an orbifold excess intersection, whose information is encoded in a vector bundle on the second Inertia Stack. The Chen–Ruan cohomology ring turns out to be a Poincaré Duality ring if the grading of the cohomologies of each twisted sector is suitably shifted by a rational number, called the *age* or *fermionic shift*, which depends on the action of the stabilizer group on the normal bundle of the twisted sectors in the original space. We underline that part of the computation of the Chen–Ruan cohomology, namely the determination of the Inertia Stack and of the second Inertia Stack, is strongly related with an old research topic: the study of automorphism groups of curves.

Since the explicit computation is too difficult to address even on the ordinary cohomology, one expects the same from Chen–Ruan cohomology. Therefore, our study of the Chen–Ruan cohomology will be made *assuming knowledge of the ordinary cohomology ring*. Moreover, one wishes to define, in an analogous way, an *orbifold Tautological Ring*, and to state and prove conjectures on its structure.

1.3 Description of the approach to the problem

We sketch here the research program that we have developed to investigate the Chen–Ruan cohomology ring of moduli of curves with marked points:

Program 1.3. *We want to compute the ring $H^*_{CR}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ as an algebra over $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$. To do this, we follow the steps:*

1. *We give an explicit description of the Inertia Stack $I(\overline{\mathcal{M}}_{g,n})$. An intermediate step is usually to compute the Inertia Stack $I(\mathcal{M}_{g',n'})$ for all (g', n') with $g' < g$;*

2. *We compute the ordinary cohomology of this Inertia Stack;*

3. *We compute the age or degree shifting number for all the twisted sectors of the Inertia Stack;*

4. *We study the second Inertia Stack $I_2(X)$;*

5. *We compute the excess intersection bundle,* i.e. *a bundle on each connected component of $I_2(X)$, and then we compute all the Euler classes (top Chern classes);*

6. *We study the pull–back in cohomology of the natural map $f : I(X) \to X$: when it happens to be surjective H^*_{CR} is generated as an H^*–algebra by the fundamental classes of the twisted sectors, so one seeks for relations;*

7. *We study the push–forwards in cohomology of the natural maps $g : I_2(X) \to I(X)$ to determine all such relations;*

8. *Finally, we study the pull–back f^* when restricted to the Tautological Ring, and the push–forward f_*. For every twisted sector Y, we look for a natural candidate for a subring of its cohomology $R^*(Y)$ such that $f_{*|R^*(Y)}$ is included in $R^*(X)$ and $f^*_{|R^*(X)}$ is (possibly) still surjective;*

9. *We give a possible definition of a subring of the Chen–Ruan cohomology ring $H^*_{CR}(\overline{\mathcal{M}}_{g,n})$ that we call orbifold Tautological Ring ($R^*_{CR}(\overline{\mathcal{M}}_{g,n})$). We study it as an algebra over the usual Tautological Ring $R^*(\overline{\mathcal{M}}_{g,n})$.*

1.4 The results obtained

The case of genus 0 is trivial, since the moduli spaces $\overline{\mathcal{M}}_{0,n}$ are smooth algebraic varieties (rigid), and hence their Chen–Ruan cohomology coincides with their ordinary cohomology (which is explicitly determined combinatorially in [Ke92]). We ran the program in the first non trivial case, with $g = 1$, in the paper [P08]. We found that, in this case, simplifications occur at each step of 1.3, as follows:

1. The map $I(X) \to X$ is a closed embedding when restricted to each twisted sector;

2. The rational cohomology of the twisted sectors is simply the cohomology of products of moduli of genus 0 stable curves;

3. (No particular simplification appears here);

4. For every double twisted sector $(Z, g, h) \subset I_2(X)$ at least one of the three maps $I_2(X) \to I(X)$, when restricted to (Z, g, h), induces an isomorphism with a twisted sector of $I(X)$;

5. There are "few" excess intersection bundles whose top Chern class is not 0 nor 1, and these are all ψ classes on moduli of genus 0 stable curves;

6. The pull–back is surjective;

7. The push–forward is easily computed thanks to point 4;

8. The pull–back is still surjective when restricted to the Tautological Ring. The Push–Forward of the twisted sectors has image in the Tautological Ring.

9. We succeed in defining an orbifold Tautological Ring in genus 1. Unfortunately it seems to us that the specificity of this case does not allow us to foresee what a general definition in all genera could be.

We recall here the two main theorems that we found running Program 1.3 in genus 1:

Theorem 1.4. *(Theorem 3.17, Theorem 3.26) Each twisted sector of $\overline{\mathcal{M}}_{1,n}$ is isomorphic to a product:*

$$A \times \overline{\mathcal{M}}_{0,n_1} \times \overline{\mathcal{M}}_{0,n_2} \times \overline{\mathcal{M}}_{0,n_3} \times \overline{\mathcal{M}}_{0,n_4}$$

where $n_1, \ldots, n_4 \geq 3$ are integers and A is in the set:

$$\{B\mu_3, B\mu_4, B\mu_6, \mathbb{P}(4,6), \mathbb{P}(2,4), \mathbb{P}(2,2)\}$$

Theorem 1.5. *(Theorem 8.2) The Chen–Ruan cohomology ring of* $\overline{\mathcal{M}}_{1,n}$ *is generated as* $H^*(\overline{\mathcal{M}}_{1,n}, \mathbb{Q})$*-algebra by the fundamental classes of the twisted sectors with explicit relations.*

In the present thesis, we want to describe all the results in a new framework, which is introduced in [F09]. We outline the results of that paper in the beginning of the thesis, and then we take advantage of the easy combinatorial description that it gives of the Inertia Stack of $\mathcal{M}_{g,n}$. We generalize this framework to $\overline{\mathcal{M}}_{g,n}$, and then we use it to follow the steps of Program 1.3 in higher genus. A big part of the thesis, contained in Chapter 4, is devoted to giving an easy description of the Inertia Stacks of $\overline{\mathcal{M}}_{g,n}$. We think that the first point of the Program 1.3 (the study of the Inertia Stacks) is of interest on its own, due to the following (vague) principle that a true statement in the category of schemes, continues to be true in the 2−category of Deligne–Mumford stacks once some of the occurrences of the stack X are replaced with its Inertia Stack $I(X)$.

The first point of the program is solved for all g and n in:

Proposition 1.6. *(Proposition 4.60) The twisted sectors of* $I(\overline{\mathcal{M}}_{g,n})$ *are all constructed following the recipe given in the constructions 4.53 and 4.58*

In genus 2 we almost complete the program, giving similar results to the ones obtained in genus 1. In genus 3 we only cover the first and the third step. Our results partially agree with the ones of [S04] (he deals with the case of genus 2 without marked points). An important result that we obtain, is the Chen–Ruan Poincaré polynomial for $\overline{\mathcal{M}}_2$ (this condensates the results obtained in the first three points of Program 1.3):

Theorem 1.7. *(Theorem 7.52) The Chen–Ruan Poincaré polynomial of* $\overline{\mathcal{M}}_2$ *is:*

$$P_2^{CR}(t) = 2+4t^{\frac{1}{2}}+2t^{\frac{3}{4}}+16t^1+2t^{\frac{6}{5}}+8t^{\frac{5}{4}}+2t^{\frac{4}{3}}+2t^{\frac{7}{5}}+21t^{\frac{3}{2}}+2t^{\frac{8}{5}}+2t^{\frac{5}{3}}+8t^{\frac{7}{4}}+2t^{\frac{9}{5}}+16t^2+2t^{\frac{9}{4}}+4t^{\frac{5}{2}}+2t^3$$

We have developed two computer programs that make the solution more explicit: [MP1] and [MP2]. The first of the two programs computes all the stable graphs of genus g with n marked points, and we hope that it will be useful for further purposes. The second one computes all the discrete data associated with the twisted sectors of $\mathcal{M}_{g,n}$ for all g and n.

The results obtained in points 1 and 2 of Program 1.3 relates the orbifold Euler characteristic of the moduli spaces of curves with their ordinary Euler characteristic. According to a formula in [B04, p. 21], we have:

$$\chi(I(\overline{\mathcal{M}}_{g,n})) = e(\overline{\mathcal{M}}_{g,n}), \quad \chi(I(\mathcal{M}_{g,n})) = e(\mathcal{M}_{g,n})$$

Note that on the left hand sides the contributions from the untwisted sectors are simply the orbifold Euler characteristics of the moduli spaces. Both the latter and the ordinary Euler characteristics have been widely studied in recent years, starting from the seminal paper of Harer and Zagier [HZ86]. Our results for points 1 and 2 explicitly compute the difference between the two known terms, and so to have a consistency check on our results on the determination of the Inertia Stacks. In the genus 1 case, we have expressed all this in terms of a compact power series.

1.5 Contents of the chapters

In **Chapter** 2, we introduce the moduli spaces of curves with marked points as Deligne–Mumford stacks. We study their Deligne–Mumford compactifications. Then we focus our attention on two partial compactifications: the moduli of curves with rational tails and the moduli of curves of compact type. We define a map that forgets the rational tails π^{rt}, and another map that forgets the rational bridges π^{rb} [1]. We give some basic results on the stabilizers of the points of these moduli stacks in the section devoted to automorphism groups of curves. Then we study the duality between stable graphs and stable curves that lie in the boundary of the Deligne–Mumford compactification. We introduce the natural maps among moduli spaces of curves and we construct two special gluing maps that reconstruct the rational tails and the rational bridges (in a sense, they are inverses of the maps π^{rt} and π^{rb}). Finally, we devote a section to the study of the deformation theory of the smooth and stable curves with marked points.

In **Chapter** 3, we give a very brief survey of the known results on the rational cohomology of the moduli spaces of curves, focusing on what we will need in order to state and prove our own results. We introduce the Tautological Ring, following the definition by Faber and Pandharipande ([FP05]), and then state the Faber Conjectures concerning the Gorenstein property of the Tautological Ring of the moduli spaces of curves.

Chapter 4 contains our first results. After recalling the definition of the Inertia Stack and its elementary properties (in the first section), we study it for moduli of curves. We review the construction of Fantechi ([F09]), which introduces a convenient notation that holds for all the twisted sectors of $\mathcal{M}_{g,n}$ in Definition 4.22. The central result of this section, which Fantechi proves in its full generality in the paper [F09], is stated in Theorem 4.33. A proof of this theorem is given in the cases that are used in the subsequent chapters. We call the twisted sectors of $\mathcal{M}_{g,n}$ *base twisted sectors* (Definition 4.42). In Definition 4.35 we then generalize the Definition 4.22, to include the computations of the twisted sectors of the Inertia Stacks of all the quotients $\left[\mathcal{M}_{g,n}/S\right]$ where $S \subset S_n$ are subgroups of the symmetric group generated by products of disjoint cycles. With this, we can give a description of the Inertia Stack of $\overline{\mathcal{M}}_{g,n}$. Some twisted sectors of these stacks are obtained by simply compactifying the twisted sectors of $\mathcal{M}_{g,n}$. When g is fixed, there is only a finite number of such compactified twisted sectors [2]. So the biggest part of the twisted sectors of $\overline{\mathcal{M}}_{g,n}$ "comes from the boundary" (Definition 4.49). We construct all the twisted sectors that come from the boundary of the moduli spaces in a combinatorial construction 4.53. Some of these sectors are actually the boundary of the (few) twisted sectors of the Inertia Stack of the open part, $\mathcal{M}_{g,n}$. These are exactly the sectors in the boundary whose general element corresponds to a nodal curve with an automorphism (C, α), such that this couple is *smoothable* (Definition 2.53, Theorem 4.51) *i.e.* it is possible to find a deformation of the curve that smooths the nodes, which preserves the automorphism α.

In the last section of the chapter, we study the behaviour of the Inertia Stack under the forgetful maps of moduli spaces of curves with marked points. The idea of this section is to try to construct all the twisted sectors of $\overline{\mathcal{M}}_{g,n}$ assuming knowledge of the twisted sectors of $\overline{\mathcal{M}}_{g}$. This section is based on the simple observation that if C is a stable curve and α is an automorphism of it, it is possible to add marked points without "breaking the symmetry" of the automorphism in three ways: adding marked points on the irreducible components of the curves at the points that are stabilized by α, adding marked points "on former marked points" (Definition 2.35), and adding marked points "on the nodes" (Definition 2.39). The way the twisted sectors have to be modified when the marked points are added, is explained in Theorem 4.61, Lemma 4.69 and Proposition 4.73. These three results together prove the reconstruction Theorem 4.74 for the twisted sectors of $\overline{\mathcal{M}}_{g,n}$.

In **Chapter** 5 we use the techniques developed in Chapter 4 to study the Inertia Stacks of moduli of genus 1, 2 and 3 curves. We start by showing the strategy used in paper [P08] to construct $I(\mathcal{M}_{1,n})$ and $I(\overline{\mathcal{M}}_{1,n})$. In this case, several combinatorial complications simplify drastically. The results obtainable within the framework of [F09] are checked to be equal to the results obtained in [P08]. We then study the Inertia Stack of $\overline{\mathcal{M}}_{2,n}$ and sketch the construction

[1] these maps are only maps of sets –or of category fibered in groupoids– since they do not respect the topologies

[2] this is basically due to the fact that $\mathcal{M}_{g,n}$ is a scheme if $n \geq 2g + 2$

of the Inertia Stack of $\overline{\mathcal{M}}_3$. To represent the twisted sectors in the boundary, we have developed a notation (see Example 4.57), which consists of giving a stable graph with an automorphism of it, and assigning a base twisted sector (in the sense of Definition 4.42) to each vertex of the graph.

Chapter 6 is devoted to the computation of the Chen–Ruan cohomology group [3] of $\overline{\mathcal{M}}_{1,n}$ and $\overline{\mathcal{M}}_{2,n}$, again with some sketches on $\overline{\mathcal{M}}_3$. The Chen–Ruan cohomology of a stack is defined as vector space as the cohomology of its Inertia Stack. So in this chapter, we have to study the cohomologies of the twisted sectors. The simplest case is when the twisted sector is reduced to a point (in this case its rational cohomology is trivial). We see that the next easiest case of twisted sectors are those constructed as moduli stacks of cyclic coverings of genus 0 curves. The general results in Chapter 5 culminate in Remark 6.7, where we observe that we can explicitly compute for all g and n, the cohomologies of those twisted sectors that are constructed as moduli stacks of cyclic coverings of genus 0 curves. Then, we compute the dimensions of $H^*_{CR}(\overline{\mathcal{M}}_{1,n})$ and construct a generating series from all these numbers, and relate it to the generating series of the dimensions of the ordinary cohomologies of $\overline{\mathcal{M}}_{1,n}$ and $\overline{\mathcal{M}}_{0,n}$ (Theorem 6.16). In the genus 2 case, we could not describe explicitly the analogous power series, so we have only given the results for the dimension of $\overline{\mathcal{M}}_2$ and $\overline{\mathcal{M}}_{2,1}$. The section on the Chen–Ruan cohomology of $\overline{\mathcal{M}}_3$ contains only partial results.

We end the chapter with a consistency check (that we have written down explicitly only in the genus 1 cases). The determination of the twisted sectors of the Inertia Stack $I(X)$ of a stack X, leads in particular to the computation of the orbifold Euler characteristic (or virtual Euler characteristic) of the twisted sectors of $I(X)$. This orbifold Euler characteristic can be computed in another way as the difference of the ordinary Euler characteristic $e(X)$ and the orbifold Euler characteristic of X. These last two numbers have been studied widely in the XX century, starting from the seminal paper of Harer and Zagier [HZ86], and we can therefore check that our results are consistent with the existing literature.

In **Chapter** 7 we introduce the age grading, a particular shifting in the degree of the cohomologies of the twisted sectors. We compute it for the twisted sectors of $I(\mathcal{M}_{g,n})$ following [F09] (Proposition 7.7 and Corollary 7.10), and then we compute it for all the twisted sectors of $I(\overline{\mathcal{M}}_{g,n})$ for all g and n. For this last result, we first study the age for the twisted sectors of $\overline{\mathcal{M}}_g$ (Proposition 7.14), and then we study how age changes when adding marked points, in the spirit of Section 4.3. Proposition 7.17 and Lemma 7.19 conclude the computation of the age for all g and n.

Then we introduce the second Inertia Stack and the excess intersection bundle, two tools that appear in the definition of the Chen–Ruan cup product. We study everything in the genus 1 case. Thus we provide explicit formulas for the age in Lemma 7.33, and all the Chen–Ruan Poincaré polynomials in Theorem 7.39. Then we study the second Inertia Stack for $\overline{\mathcal{M}}_{1,n}$ in Proposition 7.41. The excess intersection bundles for $\overline{\mathcal{M}}_{1,n}$ and their first Chern classes (that appear in the definition for the Chen–Ruan cup product) are first computed and then described in Theorem 7.51. In the genus 2 case, we give all the degree shifting numbers and then we give the Chen–Ruan Poincaré polynomial in Theorem 7.52.

In **Chapter** 8, we deal only with the case of $\overline{\mathcal{M}}_{1,n}$, and we follow closely the paper [P08]. The main result is Theorem 8.2. It is proved in Section 8, assuming some results that are proved in the subsequent sections. Theorem 8.2, gives generators for the ring $H^*_{CR}(\overline{\mathcal{M}}_{1,n}, \mathbb{Q})$ as an algebra on $H^*(\overline{\mathcal{M}}_{1,n}, \mathbb{Q})$, [4] and the relations among all these multiplicative generators are described in Section 8.4.

In the last section we speculate on the orbifold Tautological Ring.

[3] in fact, a vector space

[4] these generators are simply the fundamental classes of the twisted sectors!

Finally, in **Chapter** 9, we give the tables of the product for the Chen–Ruan cohomology of $\overline{\mathcal{M}}_{1,n}$, $n \leq 4$, making completely explicit the results of Chapter 8 in the first few cases.

1.6 Further directions

The computation of the Chen–Ruan cup product for $\overline{\mathcal{M}}_2$ will be the content of a second paper. We also expect some partial results on the Chen–Ruan cup product for the spaces $\overline{\mathcal{M}}_{2,n}$. We expect to express all the Chern classes of the excess intersection bundles as linear combinations of pull–backs of ψ classes from $\overline{\mathcal{M}}_{2,n}$, $\overline{\mathcal{M}}_{1,n}$ and $\overline{\mathcal{M}}_{0,n}$.

As for genus 3, we think that it will be possible, using the Leray spectral sequence, to complete the computation of the Chen–Ruan cohomology group of $\overline{\mathcal{M}}_3$. We also expect the computation of the Chen–Ruan cohomology ring of $\overline{\mathcal{M}}_3$ to be possible. We hope that this may give better insight into a possible definition of an orbifold Tautological Ring for all g and n.

Note that the completion of the entire program 1.3 from point 4 onwards, requires a stratification by automorphism group of the moduli space $\mathcal{M}_g$ (this makes it possible to explicitly describe the second Inertia Stack). This is a non trivial result, which up to now is available in the literature only for genus ≤ 3 ([BV04, Section 5], [MSSV, Section 7.1]). We expect a computation of the Chen–Ruan Poincaré polynomials to be possible for $\overline{\mathcal{M}}_{g,n}$ with $g \geq 4$, but the product structure seems to be quite inaccessible at present.

It would also be fascinating to compute the whole Gromov–Witten theory for the moduli spaces of curves themselves.

1.7 Notation

The generality we adopt for the category of schemes is the schemes of finite type over $\mathbb{C}$. Although we treat only this case, our results can easily be extended to the case of an arbitrary field of characteristic 0.

In the thesis, algebraic stack means Deligne–Mumford stack. The intersection theory on schemes is defined in [Fu84], and on Deligne–Mumford stack it is defined in [Vi89]. We refer to these texts for definitions and basic properties of the Chow groups A_*. In particular, since all spaces we consider are smooth, there is a standard identification (which could be taken as a definition) of A^* with the dual of A_*. For the sake of simplicity, here we present the case of cohomology and Chow ring with rational coefficients.

The discrete group subscheme of $\mathbb{C}^*$ of the N-th roots of 1 is μ_N. The generators of μ_2, μ_4 and μ_6 are conventionally chosen to be respectively $-1, i$ and ϵ.

If G is a finite abelian group, $G^\vee = \mathrm{Hom}(G, \mathbb{C}^*)$ is the group of characters of G. BG is the trivial gerbe over a point. Instead, if G is a semigroup we indicate with G^* the group of invertible elements of it.

Being over the field of complex numbers, we can fix an isomorphism of μ_N with $\mathbb{Z}/N\mathbb{Z}$ that allows us, by a little abuse of notation, to identify the two groups. If $G = \mu_N$ the generator is canonically chosen to be $\zeta_N = e^{\frac{2\pi i}{N}}$. Since μ_N is identified with its dual $\mu_N^\vee$, ζ_N is a generator for $\mu_N^\vee$ too.

We call S_n the group of permutations on the set of the first n natural numbers: $[n] := \{1, 2, \ldots, n\}$.

Definition 1.8. Let G be a finite abelian group. Then we define $\mathrm{Pic}(BG)$ as $\mathrm{Hom}(G, \mathbb{C}^*)$.

Notation 1.9. As a consequence, if X is a scheme, the datum of a line bundle over $X \times BG$, is a pair (L, χ) where $L \in \mathrm{Pic}(X)$ and $\chi \in G^\vee$.

The following proposition allows us to identify the cohomology rings of a stack and that of its coarse moduli space, provided that we work with rational coefficients.

Proposition 1.10. *([B04, Proposition 36]) Let X be a Deligne–Mumford stack with coarse moduli space $\tilde{X}$. Then the canonical morphism $X \to \tilde{X}$ induces isomorphisms on $\mathbb{Q}$-valued cohomologies:*

$$H^k(\tilde{X}, \mathbb{Q}) \to H^k(X, \mathbb{Q})$$

Summary of notation

- $\mathcal{M}_{g,n},\ \mathcal{M}_{g,n}^{rt},\ \mathcal{M}_{g,n}^{ct},\ \overline{\mathcal{M}}_{g,n},$ — moduli spaces of smooth curves, of curves with rational tails, of curves of compact type, of nodal stable curves;
- $\mathcal{C}_{g,n},\ \overline{\mathcal{C}}_{g,n}$ — universal curves;
- $\mathcal{G}_{g,n},\ \mathcal{A}_{g,n}$ — the set of stable graphs of genus g and n marked points, the set of couples (graph, automorphism of the graph) (see Definition 4.52);
- $j_{g,k},\ j_g^m, j,\ j_{irr}$ — gluing maps of moduli spaces of curves (definitions 2.35, 2.39, 2.32);
- $\pi,\ \pi_I$ — the forgetful map forgetting the last marked point, the forgetful map forgetting all but the marked points in the set I (Definition 2.32);
- $R^*,\ RH^*$ — Tautological Ring, the image of the Tautological Ring in the cohomology ring;
- $I(X),\ I_r(X),\ \bar{I}(X)$ — various Inertia Stacks of X (see the specific Chapter);
- $(g', N, d_1, \ldots, d_{N-1})$ — set of admissible data (Definition 4.44);
- $(e_1, \ldots, e_N),\ (a_1, \ldots, a_d)$ — other notation in use in the literature for the data $d_1, \ldots, d_{N-1}$ (4.24);
- $\mathcal{M}_N(g', d_1, \ldots, d_{N-1}),\ \mathcal{M}'_N(g', d_1, \ldots, d_{N-1}),\ \mathcal{M}_N(g', d_1, \ldots, d_{N-1}, \alpha),\ \mathcal{M}'_N(g', d_1, \ldots, d_{N-1}, \alpha)$ — moduli spaces introduced in 4.22, 4.32, 4.35, 4.39;
- $\overline{\mathcal{M}}_N(g', d_1, \ldots, d_{N-1}),\ \overline{\mathcal{M}}_N(g', d_1, \ldots, d_{N-1}, \alpha)$ — compactifications of the former moduli spaces (4.47);
- $M_{g,n}(e_1, \ldots, e_N; B\mu_N),\ \overline{M}_{g,n}(e_1, \ldots, e_N; B\mu_N)$ — moduli spaces of twisted smooth and stable maps (6.3);
- $\mathrm{Adm}_{g,n}(G)$ — moduli space of admissible G-covers;
- $C_4,\ C_4',\ C_6,\ C_6',\ C_6'',\ A_i,\ \overline{A}_i$ — closed substacks of $\mathcal{M}_{1,n}$ and $\overline{\mathcal{M}}_{1,n}$ see 5.1;
- $(C_4, i),\ (C_4, -i), \ldots$ — base twisted sectors (4.42) of $\mathcal{M}_{1,n}$ and $\overline{\mathcal{M}}_{1,n}$, see 5.1;
- $II,\ II_1,\ II_{11},\ \overline{II},\ \overline{II}_1,\ \overline{II}_{11}$ — base twisted sectors of $\mathcal{M}_{2,n}$ and $\overline{\mathcal{M}}_{2,n}$ constructed as coverings of genus 1 curves, see 5.2;
- $Z^{I_1, \ldots, I_k}$ — twisted sectors of $\overline{\mathcal{M}}_{1,n}$, see 5.16;
- $\mathbb{P}(a_1, \ldots, a_{n+1})$ — weighted projective spaces;
- We use several times a notation for the twisted sectors. We use stable graphs and a label on each vertex, this is explained in Example 4.57.

1.8 Acknowledgments

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Chapter 2

Moduli Stacks of Curves with Marked Points

In this chapter we study the first definitions and properties of the moduli spaces of curves with marked points. We define them as Deligne–Mumford stack. We focus on the topic of automorphism groups of smooth and stable curves. Then we introduce the notion of stable graph and of duality between stable graphs and stable curves. We define a way to reconstruct rational tails and rational bridges that will be important in the study of the Inertia Stack of moduli of curves. Finally, in the last section, we develop the aspects of deformation theory that will be needed in the rest of the thesis.

Introduced to rigidify the smooth genus g curves, the *marked points*, helped to give a better insight into the study of the moduli spaces of curves. Following Grothendieck's viewpoint, the geometry of the moduli spaces of curves is better investigated when they are considered *alltogether*, including all the moduli spaces of curves with marked points. An application of these latter spaces is easily recognizable for instance in Gromov–Witten theory.

The moduli space of smooth $n-$pointed genus g curves, denoted $\mathcal{M}_{g,n}$, parametrizes isomorphism classes of objects of the form $(C, p_1, ..., p_n)$ where C is a smooth genus g curve, and $p_1, ..., p_n$ are distinct points of C, provided that $2g - 2 + n > 0$. The points of $\overline{\mathcal{M}}_{g,n}$ correspond to isomorphism classes of stable $n-$pointed genus g curves.

Definition 2.1. Let g, n be two natural numbers with $2g - 2 + n > 0$. We define $\mathcal{M}_{g,n}$, the *moduli space of smooth genus g curves with n marked points*, as the category fibered in groupoids:

$$\mathcal{M}_{g,n}(S) := \big\{\text{Smooth families of genus } g \text{ curves with } n \text{ sections}\big\} =$$

$$\left\{ \begin{tikzcd} \mathcal{C} \arrow[d, bend right, "\pi"'] \\ S \arrow[u, "x_1"'] \arrow[u, bend right=20, "x_2"'] \arrow[u, bend right=40, "\cdots"'] \arrow[u, bend right=60, "x_n"'] \end{tikzcd} \right\}$$

where π is a smooth proper morphism, and for every geometric point $s \in S$, $\mathcal{C}_s := \pi^{-1}(s)$ is a (smooth) genus g curve. The maps x_i are sections of π with disjoint image. A map between two families is a fibered diagram that respects the sections.

One can prove that this category fibered in groupoids is in fact a stack in the (e.g.) étale topology, and that this stack is actually Deligne–Mumford (cfr. Deligne–Mumford [DM] and Knudsen [K83]). In this framework, one can prove the following:

Theorem 2.2. *[DM] The moduli stacks $\mathcal{M}_{g,n}$ are smooth, separated Deligne–Mumford stacks.*

Nevertheless, these spaces are not compact. One seeks for a compactification of them, which carries a modular interpretation like the one of definition 2.1. Many compactifications, that for this reason are called *modular* ([Sm09, Def 1.1]), are available.

The most successful one is probably the Deligne–Mumford compactification:

Definition 2.3. Let $(C, p_1, \ldots, p_n)$ be a nodal curve with marked points. It will said to be *stable* if every genus 0 component has at least 3 special points on it, where special points include marked points and nodes. (A non separating node has to be accounted for twice)

Definition 2.4. Let g, n be two natural numbers with $2g - 2 + n > 0$. We define $\overline{\mathcal{M}}_{g,n}$, the *moduli space of stable genus g curves with n marked points* as a category fibered in groupoids:

$$\overline{\mathcal{M}}_{g,n}(S) := \big\{\text{Flat families of arithmetic genus } g \text{ nodal curves with } n \text{ sections}\big\} =$$

$$\left\{ \begin{tikzcd} \mathcal{C} \arrow[d, bend right, "\pi"'] \\ S \arrow[u, "x_1"'] \arrow[u, bend right=20, "x_2"'] \arrow[u, bend right=50, "x_n"', "\cdots"] \end{tikzcd} \right\}$$

where π is a proper flat morphism, and for every geometric point $s \in S$, $\mathcal{C}_s := \pi^{-1}(s)$ is a nodal, arithmetic genus g curve. The maps x_i are sections of π with disjoint image, inside the smooth locus of the image. Finally, the stability condition (2.3) holds for every irreducible component of $\mathcal{C}_s$. Again the maps between two families is a fibered diagram that respects the sections.

A full treatment on compactifications, as well as an attempt to classify all of the modular ones, is given in [Sm09].

Theorem 2.5. *([ACG2, Chapter 11, Section 2], [K83]) The moduli stack $\overline{\mathcal{M}}_{g,n}$ is compact. Moreover the natural inclusion:*

$$\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$$

is an open embedding.

One of the reasons of the great success of the Deligne–Mumford compactification is the result below, which we are going to use in the following chapters:

Proposition 2.6. *[K83] Let I be a subset of $\{1, \ldots, n\}$. The forgetful morphism $\pi_{[n-1]} : \overline{\mathcal{M}}_{1,n} \to \overline{\mathcal{M}}_{1,n-1}$ is the universal curve. The forgetful morphisms $\pi_I : \mathcal{M}_{1,n} \to \mathcal{M}_{1,I}$ and $\pi_I : \overline{\mathcal{M}}_{1,n} \to \overline{\mathcal{M}}_{1,I}$ are representable.*

We will make use of partial compactifications in what follows. These partial compactifications fit in a topological stratification:

$$\mathcal{M}_{g,n} \subset \mathcal{M}^{rt}_{g,n} \subset \mathcal{M}^{ct}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$$

The definition of the two central stacks is given taking special classes of curves inside the definition 2.4. Namely we have that:

Definition 2.7. A stable genus g curve (Definition 2.4) with n marked points, is said to be *of rational tail* if it has a smooth irreducible component of genus g.

and

Definition 2.8. A stable genus g curve (Definition 2.4) with n marked points, is said to be *of compact type* if its arithmetic genus is exactly the sum of the geometric genera of all the irreducible components (that are smooth *a posteriori*).

Remark 2.9. If $g \geq 2$ the following 2-cartesian diagram defines the curves of rational tail:

$$\begin{tikzcd} \mathcal{M}^{rt}_{g,n} \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{g,n} \arrow[d] \\ \mathcal{M}_g \arrow[r] & \overline{\mathcal{M}}_g \end{tikzcd}$$

where the vertical arrows are the forgetful maps and the orizontal arrows are inclusions.

The curves of compact type are exactly the curves in the complement of the closure (in $\overline{\mathcal{M}}_{g,n}$) of the locus of curves that have one irreducible, singular component. The name comes from the fact that these curves are exactly the ones whose Jacobian is compact (see, for instance, [ACG2])

We will see in the next section how to distinguish curves of rational tail and of compact type by simply looking at their dual graphs. In the subsequent chapters we deal with the smoothness of all these moduli spaces.

We will need also a new class of curves:

Definition 2.10. If $(C, x_1, \dots, x_n)$ is a stable curve, a *rational tail* is a proper genus 0 subcurve, which meets the closure of the complement in exactly 1 point. A *maximal rational tail* is a rational tail that is maximal with respect to inclusion. A stable curve will be said to be *without rational tails* if it does not contain any rational tail. We will call the moduli space of stable curves without rational tails $\overline{\mathcal{M}}^R_{g,n}$.

Definition 2.11. We define π^{rt} as the map of categories fibered in groupoids:

$$\pi^{rt} : \overline{\mathcal{M}}_{g,n} \to \coprod_{k=1}^{n} \overline{\mathcal{M}}_{g,k}$$

that forgets every maximal rational tail (Definition 2.10) and puts a marked point in the former gluing point among the two curves.

To be more precise, the marked point is chosen as the x_i such that if x_j was on the same maximal rational tail, then $j \geq i$. Then afterwards, as usual, the map reorders the marked points in such a way that they belong to $\{1, \dots, k\}$ (according to 2.31). This process is explained in Figures 2.1, 2.2 and 2.3.

Remark 2.12. The map π^{rt} is manifestly not a morphism of stacks, as one can easily see since the left hand side is irreducible while the right hand side is disconnected. The map factorizes in a surjection plus an inclusion:

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n} \arrow[r, two heads, "\pi^{rt}"] & \coprod_{k=1}^{n} \overline{\mathcal{M}}^R_{g,k} \arrow[r, hook, "i"] & \coprod_{k=1}^{n} \overline{\mathcal{M}}_{g,k} \end{tikzcd}$$

We stress the fact that the map forgetting rational tails does not change the automorphism group of the curve.

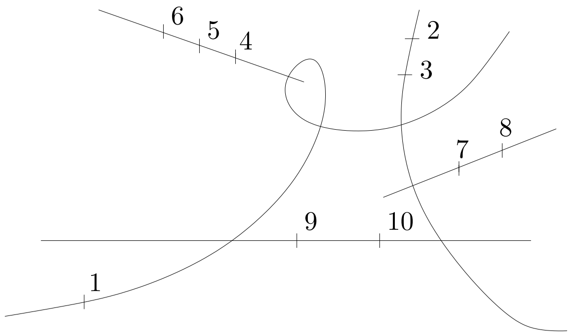


Figure 2.1: A curve, before forgetting the rational tails

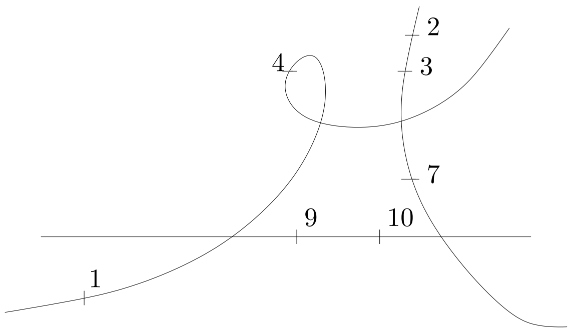


Figure 2.2: The same curve, after forgetting the rational tails

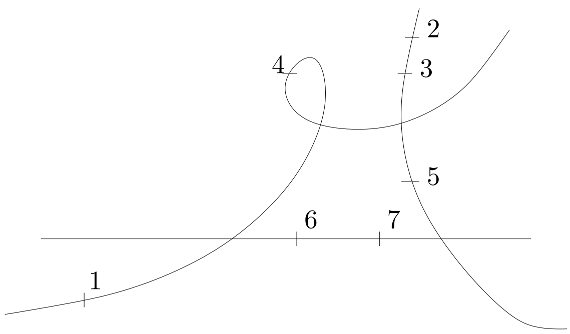


Figure 2.3: The same curve, after reordering the marked points

Definition 2.13. If $(C, x_1, \ldots, x_n)$ is a stable curve, a *rational bridge* is a proper genus 0 subcurve, which meets the closure of the complement in exactly 2 points. A *maximal rational bridge* is a rational bridge which is maximal with respect to inclusion. A stable curve will be said to be *without rational bridges and tails* if it does not contain any rational tail and any rational bridge. We will call the moduli space of stable curves without rational bridges and tails $\overline{\mathcal{M}}_{g,n}^{RB}$.

Definition 2.14. We define π^{rb} as the map:

$$\pi^{rb} : \overline{\mathcal{M}}_{g,n}^{R} \to \coprod_{I \subset [n]} \overline{\mathcal{M}}_{g,I}^{RB}$$

that forgets all the marked points lying on maximal rational bridges, and stabilizes. It also forgets all the marked points but the first one, when the whole stable curve is a geometric genus 0, irreducible curve with one node.

The process of forgetting maximal rational bridges clearly terminates and gives isomorphic results if the order of forgetting is exchanged.

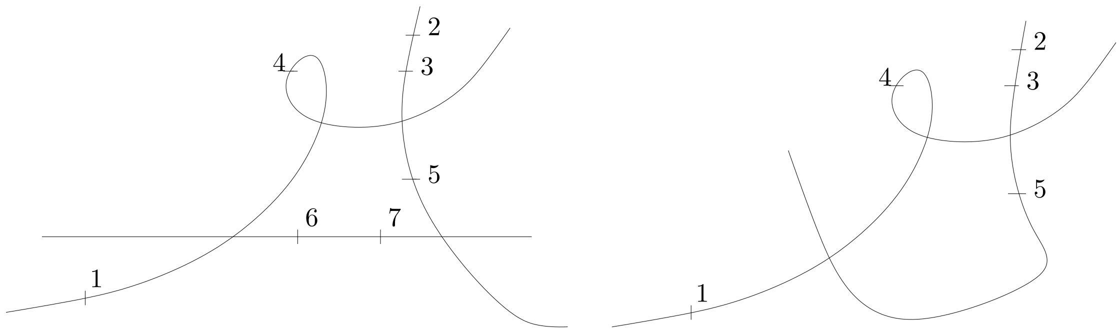


Figure 2.4: A curve, before and after forgetting the rational bridges and reordering the points

2.1 Automorphism groups of curves

There are several reasons to deal with automorphisms of genus g smooth curves (possibly with marked points).

Definition 2.15. Let $(C, p_1, \ldots, p_n)$ be a stable curve. An automorphism of it is an automorphism α of C, such that $\alpha(p_i) = p_i$.

We recall the following general result:

Proposition 2.16. *For a nodal marked curve $(C, p_1, \ldots, p_n)$ it is equivalent to be stable (2.3) and to admit a finite automorphism group.*

Definition 2.17. We define $\overline{\mathcal{M}}^o_{g,n}$ as the subcategory of $\overline{\mathcal{M}}_{g,n}$ which is made of curves whose automorphism group is trivial.

Theorem 2.18. *[HMo98] The category fibered in groupoids $\overline{\mathcal{M}}^o_{g,n}$ is equivalent to a scheme.*

Corollary 2.19. *The moduli stacks $\mathcal{M}_{0,n}$ and $\overline{\mathcal{M}}_{0,n}$ are representable by schemes. The first is open and dense in the second, which is compact.*

After these first results, we see that if we fix genus, adding a finite number of marked points makes it rigid.

Proposition 2.20. *If $n > 2g + 2$, then the moduli stack $\mathcal{M}_{g,n}$ is equivalent to a scheme.*

Proof. We use Riemann-Hurwitz formula. If C is a smooth genus g curve and ϕ is an automorphism of order N, then $C \to C/\langle\phi\rangle$ is a finite ramified cyclic covering. A point of C is stabilized under ϕ if and only if it is a point of total ramification. Let g' be the genus of the quotient curve $C/\langle\phi\rangle$. The Riemann-Hurwitz formula gives a bound on the number k of points of total ramification:

$$(N-1)k \leq 2g - 2 - N(2g' - 2) \leq 2g - 2 + 2N$$

This gives a way to estimate uniformly in g an upper bound for the maximum number of marked points that can be fixed by an automorphism of a genus g curve. Then the locus where this maximum is obtained is the hyperelliptic locus ($N = 2, g' = 0$). So one can conclude using Theorem 2.18. $\square$

Proposition 2.21. *([AV02] Lemma 4.4.3) Let $g : \mathcal{G} \to \mathcal{F}$ be a morphism of Deligne–Mumford stacks. The following conditions are equivalent:*

- *The morphism $g : \mathcal{G} \to \mathcal{F}$ is representable.*
- *For any $\xi \in \mathcal{G}(k)$, the natural group homomorphism $Aut(\xi) \to Aut(g(\xi))$ is injective.*

Proposition 2.22. *Let $(C, x_1, \ldots, x_n)$ be a stable curve. Then its automorphism group injects into the automorphism group of $(\tilde{C}, x_1, \ldots, x_{n-1})$, where $\tilde{C}$ is the stabilization of C after forgetting x_{n-1}.*

Proof. If the curve $\tilde{C}$ equals C, then the result is obvious and follows from the definition of an automorphism of a smooth pointed curve. If not, it follows from the fact that $\pi_{[n-1]}$ is the universal curve (Theorem 2.6), hence representable, and from Proposition 2.21. $\square$

For a brief historical summary on the study of some aspects of automorphism groups of smooth curves, see [MSSV, Chapter 1]. In particular, we will be interested in bounds on the cardinality of such groups. The first classical result:

$$|\operatorname{Aut}(G)| \leq 84(g-1) \tag{2.23}$$

follows as an easy consequence of the Riemann–Hurwitz formula (1893). This bound is attained in infinitely many genera. In the literature, such automorphisms are known as *Hurwitz groups*, and the curves as *Hurwitz curves*. Independently, Accola and Maclachlan found a sharp lower bound for the maximum cardinality of the automorphism group of smooth genus g curves. If we call the corresponding number $N(g)$, they found:

$$N(g) \geq 8(g+1) \tag{2.24}$$

Proposition 2.25. *Let C be a smooth curve, and G an automorphism subgroup of $\operatorname{Aut}(C)$, such that $|\operatorname{Aut}(G)| > 4(g-1)$. Then the quotient C/G is a genus 0 curve and the projection on the quotient has 3 or 4 ramification points.*

Definition 2.26. Such an automorphism group G is called a *large automorphism group.*

We now deal with the problem of bounding the cardinality of the largest cyclic automorphism group of a smooth genus g curve. Let t be the order of an automorphism of C. Hurwitz showed that:

$$t \leq 10(g-1). \tag{2.27}$$

In 1895, Wiman improved this bound to be:

$$t \leq 2(2g+1) \tag{2.28}$$

and moreover showed that this is the best possible. If t is a prime then:

$$t \leq 2g+1 \tag{2.29}$$

Homma [Ho80] shows that this bound is achieved if and only if the curve is birationally equivalent to:

$$y^{m-s}(y-1)^s = x^q, \quad 1 \leq s < m \leq g_x + 1 \tag{2.30}$$

2.2 Modular operadic aspects of moduli of curves

If I is a finite set, we can define the moduli spaces $\mathcal{M}_{g,I}$ (and the rational tail, compact type, stable analogous) in analogy with Definition 2.1, forcing the labels of the sections to belong to I instead of $\{1,\dots,n\}$.

Remark 2.31. Any bijection between I and $\{1,\dots,n\}$ determines an isomorphism of $\mathcal{M}_{g,I}$ with $\mathcal{M}_{g,|I|}$. In all the cases that we consider, I is a disjoint union of a subset J of $\{1,\dots,n'\}$ and a set of ordered symbols $\{\bullet_1,\dots,\bullet_k\}$. A canonical ordering of I is then the one induced on it by $1\le\dots\le n'\le\bullet_1\le\dots\le\bullet_k$. We will abuse the notation and assume implicitly this canonical isomorphism, thus dealing only with moduli spaces of curves with marked points in $\{1,\dots,n=|I|\}$.

Definition 2.32. (See [K83]) The following maps, are called *natural maps among the moduli spaces of curves.*

$$\pi_i:\overline{\mathcal{M}}_{0,n+1}\to\overline{\mathcal{M}}_{0,n}$$

by definition is the map that forgets the i-th marked points and, if necessary, stabilizes the resulting curve.

$$j_{(g_1,n_1),(g_2,n_2)}:\overline{\mathcal{M}}_{g_1,n_1\sqcup\bullet_1}\times\overline{\mathcal{M}}_{g_2,n_2\sqcup\bullet_2}\to\overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$$

by definition is the map that glues two curves C and C' in the points $\bullet_1$ and $\bullet_2$.

$$j_{irr}:\overline{\mathcal{M}}_{g,n+2}\to\overline{\mathcal{M}}_{g+1,n}$$

by definition is the map that glues the last two marked points, to produce a singular nodal curve of arithmetic genus increased by one.

If $I=\{i,\bullet_1\}$, then we conventionally define $\overline{\mathcal{M}}_{0,I}$ as a point labelled by i. The second natural map, if the first space is $\overline{\mathcal{M}}_{0,I}$, is just the identity.

This definition is taken from [GP03]:

Definition 2.33. A *stable graph* is the datum of:

$$G=(V,H,L,g:V\to\mathbb{Z}_{\ge0},a:H\to V,i:H\to H)$$

satisfying the following properties:

1. V is a vertex set with a genus function g,
2. H is a half–edge set equipped with a vertex assignment a and a fixed point free involution i,
3. E, the edge set, is defined by the orbits of i in H (self–edges at vertices are permitted),
4. (V,E) define a connected graph,
5. L is a set of numbered legs attached to the vertices,
6. For each vertex v, the stability condition holds:

$$2g(v)-2+n(v)>0$$

where $n(v)$ is the valence of A at v including both half–edges and legs.

The genus of G is defined to be:

$$g(G) := \sum_{v \in V} g(v) + h^1(G)$$

Let $n := |L|$. We define $\mathcal{G}_{g,n}$ as the set of stable genus g, n–marked graphs.

Construction 2.34. Let C be a stable curve. Then its associated stable graph G_C is constructed taking V to be the set of all the irreducible components of C. The set of half–edges H is the set of points in the normalization of C that lie over a node. If $h \in H$, $a(h)$ is defined to be the corresponding irreducible component. The set of legs L consists, for each vertex, of the marked points on the irreducible component corresponding to that vertex. We call $\mathcal{M}_G$ the locus of points in $\overline{\mathcal{M}}_{g,L}$ whose dual graph is G, and we call $\overline{\mathcal{M}}_G$ its closure. Let now:

$$\mathcal{M}'_G := \prod_{v \in V} \mathcal{M}_{g(v),n(v)} \quad \overline{\mathcal{M}}'_G := \prod_{v \in V} \overline{\mathcal{M}}_{g(v),n(v)}$$

Analogously to the second point of Definition 2.32 we can define a clutching morphism, (see [K83, Theorem 3.4]) ξ_G. Let x be a point of $\overline{\mathcal{M}}'_G$, consisting of a $H(v) \sqcup L(v)$-marked curve C_v for each vertex v. Then $\xi_G(x)$ is obtained from the disjoint union of the C_v by identifying points labelled by l and l' for any edge $i(l) = l'$. By construction, the image of ξ_G is contained in $\overline{\mathcal{M}}_G$, and $\mathrm{Aut}(G)$ acts on $\overline{\mathcal{M}}'_G$ in the obvious way. Again by construction, the map ξ_G induces a morphism of stacks:

$$\xi_G : \left[\overline{\mathcal{M}}'_G / \mathrm{Aut}(G)\right] \to \overline{\mathcal{M}}_G$$

that induces an isomorphism onto its image if restricted to the open substack $\left[\mathcal{M}'_G / \mathrm{Aut}(G)\right]$.

Using stable graphs, it is possible to restate the definition of curves with rational tails and of compact type (Definition 2.7, Definition 2.8). A stable pointed curve of arithmetic genus g is of rational tail iff its dual graph contains a vertex of geometric genus g. A stable pointed curve is of compact type iff its dual graph is a tree.

We will be needing two maps $j_{g,k}$ and j_g^m, that glue rational curves on a stable curve with marked points. The first map is common:

Definition 2.35. Let $k > 0$ and $X \subset \overline{\mathcal{M}}_{g,\coprod_{i=1}^k \bullet_i}$ be a substack, and let $(I_1, \ldots, I_k)$ be a partition of $[n]$. We define $j_{g,k}$ as the morphism gluing the marked points labelled with the same symbol:

$$j_{g,k} : X \times \overline{\mathcal{M}}_{g_1, I_1 \sqcup \bullet_1} \times \ldots \times \overline{\mathcal{M}}_{g_n, I_k \sqcup \bullet_k} \to \overline{\mathcal{M}}_{g + \sum g_i, n}$$

(cfr. Definition 2.32, Construction 2.34)

To define the second map, we need the following stratification:

Definition 2.36. Let $\overline{\mathcal{M}}^{(k)}_{g,n}$ be the closed locus of curves having at least k nodes.

$$\overline{\mathcal{M}}^{(3g-3+n)}_{g,n} \subset \ldots \subset \overline{\mathcal{M}}^{(k)}_{g,n} \subset \ldots \subset \overline{\mathcal{M}}^{(0)}_{g,n} = \overline{\mathcal{M}}_{g,n}$$

We define $\mathcal{M}^{(k)}_{g,n}$ as the locally closed locus of $\overline{\mathcal{M}}_{g,n}$ given by the locus of curves with exactly k nodes, *i.e.*:

$$\mathcal{M}^{(k)}_{g,n} := \overline{\mathcal{M}}^{(k)}_{g,n} \setminus \overline{\mathcal{M}}^{(k+1)}_{g,n}$$

Definition 2.37. Let $I \subset [n]$ and let G be a genus g stable graph with marked points in I, with m edges $e(1), \dots, e(m)$, t vertices and no rational tails nor bridges. To an $m-$partition of $[n]$:

$$[n] = J_1 \sqcup \dots \sqcup J_m$$

we associate a gluing map of moduli spaces j_g^m. We construct G' as the graph obtained by gluing a vertex of genus 0 with marked points in J_i in the middle of every edge $e(i)$.

Let:

$$\mathrm{Aut}(G)(J_1, \dots, J_m) := \{\beta \in \mathrm{Aut}(G) |\text{ for all } i \text{ such that } J_i \neq \emptyset,\ \beta \text{ fixes } e(i)\}$$

with this definition we have that $\mathrm{Aut}(G') = \mathrm{Aut}(G)(J_1, \dots, J_m)$.

Let now $\mathfrak{S}$ be the set of all the partitions $J_1 \sqcup \dots \sqcup J_m = [n]$. We consider the equivalence relation on $\mathfrak{S}$ induced by $\mathrm{Aut}(G)$. The partition $\{J_1, \dots, J_m\}$ and $\{J_1', \dots, J_m'\}$ are equivalent if there exists $\beta \in \mathrm{Aut}(G)$ such that for all i, $|J_{\tilde{\beta}(i)}| = |J_i'|$. Here $\tilde{\beta}$ indicates the permutation of $[m]$ induced by the permutation of the edges that β induces on the graph G. We denote the quotient set of $\mathfrak{S}$ via this equivalence relation $\mathfrak{S}_G$

Definition 2.38. Let G be a stable graph and β an automorphism of it. An edge e stabilized by β is said to be stabilized *as a directed graph* if β acts on it switching the two vertices that it links. Otherwise it is said to be stabilized *as an undirected graph.* If C is a nodal curve whose dual graph is G, a node p of C stabilized by β is stabilized as an undirected graph if β acts locally on p as:

$$\frac{\mathbb{C}[[x,y]]}{(xy)} \to \frac{\mathbb{C}[[x,y]]}{(xy)}$$
$$x \to y$$
$$y \to x$$

and as the identity in the other case.

Definition 2.39. With the notation introduced above, if G is a stable graph without rational tails and bridges with m edges, and $J_1, \dots, J_m$ is a partition of $[n]$, we define:

$$\mathcal{M}_G(J_1, \dots, J_m) := \left[\left(\prod_{i=1}^{t} \mathcal{M}_{g_i, n_i} \right) \times \left(\prod_{k=1}^{m} \overline{\mathcal{M}}_{0, J_k + 2} \right) / \, \mathrm{Aut}(G)(J_1, \dots, J_m) \right]$$

where $\beta \in \mathrm{Aut}(G)(J_1, \dots, J_m)$ acts on $\overline{\mathcal{M}}_{0,J_k+2}$ as the identity if the edge $e(k)$ is stabilized as a directed edge, and exchanging the last two marked points if the edge $e(k)$ is stabilized as an undirected edge.

Moreover, we define the gluing map:

$$j_G : \coprod_{\{J_1, \dots, J_m\} \in \mathfrak{S}_G} \mathcal{M}_G(J_1, \dots, J_m) \to \overline{\mathcal{M}}_{g,n}^{(m)}$$

which corresponds to the map defined in Definition 2.37. The map glues the rational curves E_l "on the nodes". By this, we mean: blow up the node and put the marked points on the exceptional component as in E_l, blowing up again the rational component if necessary. The components of the image have codimension between m and $2m$, according to how many among the J_i are empty.

This gluing map is explained in the picture (we indicate the genus of every irreducible component by a number close to one of the two extreme points of it. Rational curves are simply pictured as straight lines):

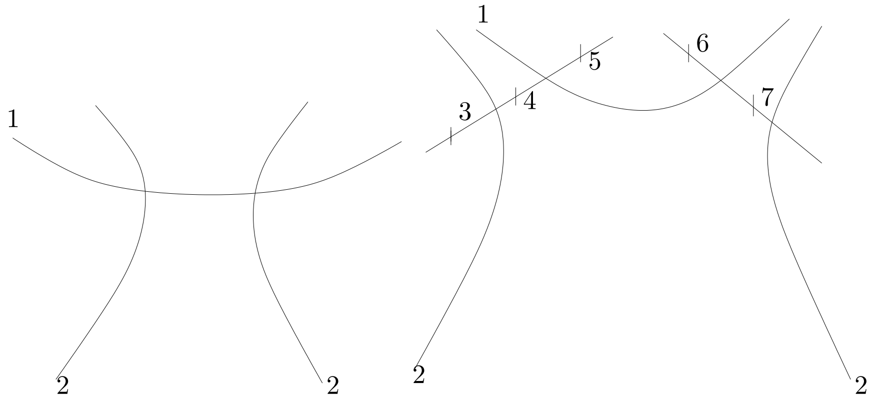


Figure 2.5: A curve before and after adding rational bridges.

Remark 2.40. There exists no map globally defined on the closed strata of curves with m nodes, which restricted to the locally closed strata equals the one defined above. However, such a map can be extended to the closure of each irreducible component of this closed strata of codimension m. In general, none of these extensions agree on the intersections.

Proposition 2.41. *There exists a map ϕ^{RT} of categories fibered in groupoids, which induces an isomorphism on the automorphism groups of the objects, and expresses $\overline{\mathcal{M}}_{g,n}$ as a partition of substacks.*

$$\phi^{RT} : \coprod_{I\subset[n],|I|=n-k} \; \coprod_{\{I_1,\ldots,I_k\}|\;[n]=\sqcup I_j} \left(\overline{\mathcal{M}}^R_{g,k} \times \overline{\mathcal{M}}_{0,I_1+1} \times \ldots \times \overline{\mathcal{M}}_{0,I_k+1}\right) \to \overline{\mathcal{M}}_{g,n}$$

where all the subsets that form the partition of $[n]$ are non empty. The composition $\pi^{rt} \circ \phi^{RT}$ is the projection onto the first factor.

Proof. Rational pointed curves are rigid, as we have already seen in the former section. Therefore the map ϕ^{RT} induces an isomorphism on the automorphism group of all the objects.

If an element $(C, x_1, \ldots, x_k)$ is taken in $\coprod_{k=1}^{n} \overline{\mathcal{M}}_{g,k}$, its fiber via π^{rt} is exactly the disjoint union over all the possible partitions $I_1 \sqcup \ldots \sqcup I_k = [n]$ of

$$j_{g,k}(C \times \overline{\mathcal{M}}_{0,I_1+1} \times \ldots \times \overline{\mathcal{M}}_{0,I_k+1})$$

Therefore there exists a functor ϕ^{RT}, that makes the diagram commutative:

$$\begin{tikzcd}
\coprod_{I\subset[n]} \coprod_{\{I_1,\ldots,I_k\}|\;[n]=\sqcup I_j} \left(\overline{\mathcal{M}}_{g,k} \times \overline{\mathcal{M}}_{0,I_1+1} \times \ldots \times \overline{\mathcal{M}}_{0,I_k+1}\right) \arrow[d, "\sqcup_{k=1}^{n} j_{g,k}"] \arrow[dr, "\phi^{RT}"] \arrow[dd, bend right=60, "pr_1"'] & \\
\coprod_{I\subset[n]} \coprod_{\{I_1,\ldots,I_k\}|\;[n]=\sqcup I_j} j_{g,k}\left(\overline{\mathcal{M}}_{g,k} \times \overline{\mathcal{M}}_{0,I_1+1} \times \ldots \times \overline{\mathcal{M}}_{0,I_k+1}\right) \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{g,n} \arrow[dl, "\pi^{rt}"] \\
\coprod_{k=1}^{n} \overline{\mathcal{M}}_{g,k} &
\end{tikzcd}$$

where the map $j_{g,k}$ is the gluing map defined in Definition 2.35 and pr_1 is the projection onto the first factor of the product in each term of the disjoint union. $\square$

In a similar fashion, one can also reconstruct curves without rational tails in terms of curves without rational bridges and tails (cfr. Definition 2.10 and Definition 2.13).

Lemma 2.42. *There exists a map induced on the quotient by the projection map:*

$$\tilde{pr}_1 : \mathcal{M}_G(J_1, \ldots, J_m) \to \mathcal{M}_G \subset \overline{\mathcal{M}}_{g,n}^{(m)}$$

which makes the following diagram commutative:

$$\begin{tikzcd}
\prod_{i=1}^t \mathcal{M}_{g_i,n_i} \times \prod_{k=1}^m \overline{\mathcal{M}}_{0,J_k+2} \arrow[r] \arrow[d, "pr_1"] & \mathcal{M}_G(J_1, \ldots, J_m) \arrow[d, "\tilde{pr}_1"] \\
\prod_{i=1}^t \mathcal{M}_{g_i,n_i} \arrow[r] & \mathcal{M}_G
\end{tikzcd}$$

Proposition 2.43. *There exists a map ϕ^{RB} of categories fibered in groupoids, which expresses $\overline{\mathcal{M}}_{g,n}^R$ as a partition of substacks:*

$$\phi^{RB} : \coprod_{I \subset [n]} \coprod_{m=0}^{3g-3+|I|} \coprod_{G \in \mathcal{G}_{g,I}^{(m)}} \coprod_{\{J_1,\ldots,J_m\} \in \mathfrak{S}_G} (\mathcal{M}_G(J_1, \ldots, J_m)) \to \overline{\mathcal{M}}_{g,n}^R$$

where $\mathcal{G}_{g,I}^{(m)}$ is the set of all graphs without rational bridges and tails of genus g and marked points in I, and with m edges (cfr. Definition 2.33). The composition $\pi^{rb} \circ \phi^{RB}$ equals the map $\tilde{pr}_1$ defined in Lemma 2.42.

This allows us to write the big cartesian diagram:

$$\begin{tikzcd}
\coprod_{\ldots} \overline{\mathcal{M}}_G^{RB}(J_1, \ldots J_m) \times \overline{\mathcal{M}}_{0,I_1} \times \ldots \times \overline{\mathcal{M}}_{0,I_k} \arrow[r, "\phi"] \arrow[dd, "\tilde{pr}_1 \circ pr_1"] & \overline{\mathcal{M}}_{g,n} \arrow[d, "\pi^{rt}"] \\
& \coprod_{k=1}^n \overline{\mathcal{M}}_{g,n}^R \arrow[d, "\pi^{rb}"] \\
\coprod_{\ldots} \overline{\mathcal{M}}_G^{RB} \arrow[r, "="] & \coprod_{\ldots} \overline{\mathcal{M}}_G^{RB}
\end{tikzcd}$$

We will use the principles and the results introduced in this section in studying the Inertia Stack of moduli of stable curves with marked points.

2.3 The deformation theory of genus g stable marked curves

For the basic notions on Deformation Theory we will refer to [FGIKNV, Chapter 6]

In this Section, $(C, x_1, \ldots, x_n)$ is a stable marked curve.

Definition 2.44. Let A be an artinian local ring, and 0 its closed point. We define the deformation functor $\mathrm{Def}_{(C,x_1,\ldots,x_n)}(A)$ as the set of proper flat families $\phi : \mathcal{C} \to A$, with a fixed isomorphism $\beta : C \to \phi^{-1}(0)$, and n sections $\sigma_1, \ldots, \sigma_n$ such that $\sigma_i(0) = \beta(x_i)$.

The following is a very classical theorem concerning the tangent and obstruction to the deformation of a scheme:

Theorem 2.45. *([Se06], [ACG2]) The tangent space to the deformation functor* $\mathrm{Def}_{(C,x_1,\ldots,x_n)}$ *is* $\mathrm{Ext}^1(\Omega_C(\sum x_i), \mathcal{O}_C)$*. An obstruction space for it is* $\mathrm{Ext}^2(\Omega_C(\sum x_i), \mathcal{O}_C)$*.*

Proposition 2.46. *([DM, Lemma 1.3]) If* $(C, x_1, \ldots, x_n)$ *is a marked nodal curve, the vector space* $\mathrm{Ext}^2(\Omega_C(\sum x_i), \mathcal{O}_C)$ *is zero.*

Corollary 2.47. *The moduli spaces of genus g smooth, rational tail, compact type, stable curves with n marked point are smooth.*

In low degrees, the local-to-global spectral sequence ([ACG2, Chapter 12, Section 2]) of Ext implies the local-to-global exact sequence:

$$0 \to H^1(X, \mathcal{H}om(\mathcal{F}, \mathcal{G})) \to \mathrm{Ext}^1(\mathcal{F}, \mathcal{G}) \to H^0(\mathcal{E}xt^1(\mathcal{F}, \mathcal{G})) \to H^2(X, \mathcal{H}om(\mathcal{F}, \mathcal{G})) \tag{2.48}$$

Now we consider such an exact sequence in the case where X is a stable curve C, $\mathcal{F} = \Omega_C(\sum x_i)$ and $\mathcal{G} = \mathcal{O}_C$. By the Grothendieck vanishing theorem the H^2 term vanishes on a curve. We end with a short exact sequence:

$$0 \to H^1(X, \mathcal{H}om(\Omega_C(\Sigma x_i), \mathcal{O}_C)) \to \mathrm{Ext}^1(\Omega_C(\Sigma x_i), \mathcal{O}_C) \to H^0(\mathcal{E}xt^1(\Omega_C(\Sigma x_i), \mathcal{O}_C)) \to 0 \tag{2.49}$$

Therefore, the tangent space to the deformation functor of curves with marked points, splits into two parts. These two parts can be interpreted as the tangent space to the deformations that smooth the nodes and the deformations which fix the node ([ACG2]).

We will be interested in deformations that preserve a certain automorphism group of the curve. So let $(C, x_1, \ldots, x_n, G)$ be the couple of a nodal marked curve and its automorphism group G. We can define:

Definition 2.50. Let A be an artinian local ring, and 0 its closed point. We define the deformation functor $\mathrm{Def}^G_{(C,x_1,\ldots,x_n)}(A)$ as the set of proper flat families $\phi : \mathcal{C} \to A$, with a fixed isomorphism $\beta : C \to \phi^{-1}(0)$, and n sections $\sigma_1, \ldots, \sigma_n$ such that $\sigma_i(0) = \beta(x_i)$, and such that the automorphism group G of $(C, x_1, \ldots, x_n)$ extends to an automorphism group $\mathcal{G}$ of $(\mathcal{C}, \sigma_1, \ldots, \sigma_n)$.

In analogy with Theorem 2.45, one can prove:

Theorem 2.51. *([Se06], [ACG2]) The tangent space to the deformation functor* $\mathrm{Def}^G_{(C,x_1,\ldots,x_n)}$ *is* $\mathrm{Ext}^1(\Omega_C(\sum x_i), \mathcal{O}_C)^G$*. An obstruction space for it is* $\mathrm{Ext}^2(\Omega_C(\sum x_i), \mathcal{O}_C)^G$*.*

The Deformation functor is unobstructed as a consequence of 2.46. Now the Sequence 2.49 remains exact taking G-invariant parts:

$$0 \to H^1(X, \mathcal{H}om(\Omega_C(\Sigma x_i), \mathcal{O}_C))^G \to \mathrm{Ext}^1(\Omega_C(\Sigma x_i), \mathcal{O}_C)^G \to H^0(\mathcal{E}xt^1(\Omega_C(\Sigma x_i), \mathcal{O}_C))^G \to 0 \tag{2.52}$$

Note the fact that the sheaf $\mathcal{E}xt^1(\Omega_C(\sum x_i), \mathcal{O}_C)$ is supported on the nodes of C.

Definition 2.53. Let $(C, x_1, \ldots, x_n, G)$ be a nodal marked curve with an automorphism G of it. This couple will be said to be *smoothable* if the G-invariant part of $H^0(\mathcal{E}xt^1(\Omega_C(\sum x_i), \mathcal{O}_C))$ is non trivial.

Example 2.54. Let C be a curve with exactly one node q. Then étale locally an automorphism of C of order N acts on $\mathrm{Spec}(\mathbb{K}[x,y]/(xy))$, via $x \to \zeta_n^a x, y \to \zeta_n^b y$ or via $x \to \zeta_n^a y, y \to \zeta_n^b x$ for certain integers a and b. Then the couple (G, C) is smoothable if and only if N divides $a+b$. An action of this kind on a node is usually called *balanced*.

Example 2.55. Let C be a nodal curve made of isomorphic irreducible components and G an automorphism of it that simply permutes the irreducible components. Then the couple (C, G) is always smoothable.

Corollary 2.56. *If $(C, x_1, \dots, x_n)$ is a curve of rational tail, and G is an automorphism of it, there are no deformations of it that smooth the nodes.*

Proof. The automorphism group G acts as the identity on each rational component. Therefore the action on the nodes can be balanced if and only if it is the identity. $\square$

Chapter 3

The Rational Cohomology and Chow Ring of Moduli of Curves

In this chapter, we will be studying the cohomologies of the moduli spaces of curves with marked points. We deal only with rational coefficients and for this case we will take advantage of Proposition 1.10. We will be concerned with different levels of the general task of determining the ordinary cohomologies of moduli of marked curves. A first step is to know the dimension of the cohomology groups, *i.e.* the Betti numbers, or the more refined Hodge numbers. Since there is a natural action of the symmetric group S_n on the moduli of $n-$marked curves, the cohomology groups are representations of these symmetric groups, and it will be useful for us to know the decomposition of these representations in irreducible representations. This is the information contained in the equivariant Poincaré polynomials. In the subsequent chapters, we will also be dealing with the ring structure. a common way to present the ring structure, consists of producing some natural classes in the cohomology that generate multiplicatively the whole ring, and then to produce the set of all the relations among these (thus describing the ring as a polynomial algebra).

3.1 The Tautological Rings

Since the task of studying the whole cohomology ring of moduli of curves with marked point has appeared to be particularly hard, a notion of a subring of it, was introduced. This subring contains all the classes that come from geometry. A very neat definition of the Tautological Rings was given in [FP05]:

Definition 3.1. The *tautological system of rings* $\{R^*(\overline{\mathcal{M}}_{g,n})\}_{(g,n)}$ is defined to be the set of smallest $\mathbb{Q}$-subalgebras of the Chow rings:

$$R^*(\overline{\mathcal{M}}_{g,n}) \subset A^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

that is stable under push–forward via the natural maps introduced in Definition 2.32.

Note that being a subalgebra, each Tautological Ring contains the fundamental class (the identity element) of $\overline{\mathcal{M}}_{g,n}$. Many other geometric classes are actually contained in it too:

Definition 3.2. A *boundary stratum* is the closure of the locus of curves in $\overline{\mathcal{M}}_{g,n}$ that share the same dual graph G (Defintion 2.33, Construction 2.34). A *boundary strata class* is the class of such a locus in the Chow ring.

We will introduce a special notation for the boundary divisors of $\overline{\mathcal{M}}_{1,n}$.

Notation 3.3. We will call D_I the closure of the substack of $\overline{\mathcal{M}}_{1,n}$ of reducible nodal curves with two smooth components. The marked points in the set I are on the genus 0 component and the marked points on the genus 1 curve are in the complementary, $[n] \setminus I$. Consequently, we call D_{irr} the closure of the substack of $\overline{\mathcal{M}}_{1,n}$ of irreducible curves of geometric genus 0. We will sometimes indicate with D_I also the class $[D_I] \in H^2(\overline{\mathcal{M}}_{1,n})$ represented by the closed divisor D_I.

We then define some classes we will be using several times in the last part of the thesis. Let $\pi : \mathcal{C}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ be the universal curve, let ω_π be the dualizing sheaf and let s_i be the i-th section.

Definition 3.4. We define $\mathbb{L}_i$ to be the line bundle $s_i^*(\omega_\pi)$. We define $\psi_i = c_1(\mathbb{L}_i)$.

Remark 3.5. [FP05] The boundary strata classes (3.2), and the ψ classes (3.4) belong to the Tautological Ring. The Tautological Ring is closed under pull–back via the natural maps. Moreover, it contains other geometric classes such as κ classes and λ classes (see [AC96] and [Bi05]).

The pull–back and push–forward via the natural maps 2.32, gives rise to maps among the cohomologies or the Chow rings. For instance:

$$j_{(g_1,n_1),(g_2,n_2)*} : A^*_{\mathbb{Q}}(\overline{\mathcal{M}}_{g_1,n_1 \sqcup \bullet_1}) \otimes A^*_{\mathbb{Q}}(\overline{\mathcal{M}}_{g_2,n_2 \sqcup \bullet_2}) \to A^*_{\mathbb{Q}}(\overline{\mathcal{M}}_{g_1+g_2,n_1+n_2})$$

Remember that if X is a smooth stack, Y is a smooth closed substack of codimension l, and $U := X \setminus Y$, there is a localization sequence:

$$A^k(Y) \to A^{k+l}(X) \to A^{k+l}(U) \to 0 \tag{3.6}$$

Definition 3.7. We define Tautological Rings $R^*(\mathcal{M}^{ct}_{g,n})$, $R^*(\mathcal{M}^{rt}_{g,n})$ and $R^*(\mathcal{M}_{g,n})$ via the localization sequence 3.6 as the quotients 3.1 in the respective open subsets.

Note that it is not known whether the localization sequences in the cases at hand split in the middle.

If X is a Deligne–Mumford stack, there is a cycle map (see [Vi89]):

$$cyc : A^*_{\mathbb{Q}}(X) \to H^*(X, \mathbb{Q})$$

so we can define:

Definition 3.8. If $X \in \{\mathcal{M}_{g,n}, \mathcal{M}^{rt}_{g,n}, \mathcal{M}^{rt}_{g,n}, \overline{\mathcal{M}}_{g,n}\}$, the Tautological Ring in cohomology $RH^*(X)$ is defined as the image in cohomology of the cycle map on the Tautological Ring R^*.

3.2 Faber conjectures

The following conjectures are usually referred to as *Faber Conjectures*:

Conjecture 3.9. *(Faber Conjectures) (cfr. [Fa08])*

1. *The Ring $R^*(\mathcal{M}^{rt}_{g,n})$ is Gorenstein with socle in degree $g - 2 + n - \delta_{0g}$.*
2. *The Ring $R^*(\mathcal{M}^{ct}_{g,n})$ is Gorenstein with socle in degree $2g - 3 + n$.*
3. *The Ring $R^*(\overline{\mathcal{M}}_{g,n})$ is Gorenstein with socle in degree $3g - 3 + n$.*

Faber conjectures have been verified for $\mathcal{M}_g$ for $g \leq 23$.

3.3 A survey of known results

In low genus and marked points, the Tautological Ring coincides with the Chow ring and with the cohomology ring. We review here some of the results that we shall need in the sequel.

Proposition 3.10. *The Tautological Ring $R^*(\overline{\mathcal{M}}_{1,n})$ is spanned (additively generated) by boundary strata classes 3.2.*

Proof. This follows as a consequence of Theorem $*$ [GV05]. ☐

Proposition 3.11. *([Be98] Theorem 3.1.1) For $n \leq 10$ the Chow group $A^*(\overline{\mathcal{M}}_{1,n})$ is spanned by boundary strata classes.*

From this, it follows that the Tautological Ring is the whole Chow ring for $\overline{\mathcal{M}}_{1,n}$, $n \leq 10$. Moreover Getzler has claimed the following results about the cohomology and its relation with the Chow ring:

Claim 3.12. ([Ge97] second paragraph) The boundary strata classes of $\overline{\mathcal{M}}_{1,n}$ span the even cohomology of $\overline{\mathcal{M}}_{1,n}$.

Claim 3.13. ([Ge97], second paragraph) The ideal of relations among the boundary cycles is generated by the genus 0 relations together with pull–backs to $\overline{\mathcal{M}}_{1,n}$ of the relation in $H^4(\overline{\mathcal{M}}_{1,4}, \mathbb{Q})$, which is stated in Lemma 1.1, [Ge97].

In [Pa99, Theorem 1] the relation in $H^4(\overline{\mathcal{M}}_{1,4}, \mathbb{Q})$ is shown to be algebraic. The claims therefore split in the two ring isomorphisms:

$$R^*(\overline{\mathcal{M}}_{1,n}) \cong RH^*(\overline{\mathcal{M}}_{1,n}) \tag{3.14}$$

$$RH^*(\overline{\mathcal{M}}_{1,n}) = H^{2*}(\overline{\mathcal{M}}_{1,n}) \tag{3.15}$$

for $n \geq 1$.
For a part of this work, namely in the section of pull–back of the Tautological Ring to the twisted sectors, we assume Getzler's claim.

Corollary 3.16. *Faber conjectures hold for $\overline{\mathcal{M}}_{1,n}$.*

We observe that in higher genus it is still true that the cycle map restricted to the Tautological Ring induces an isomorphism of it with the cohomology ring.

The following Theorem condenses results that have been obtained by Belorousski–Pandharipande, Faber, Faber–Izadi, Getzler, Getzler–Looijenga, Tommasi.

Theorem 3.17. *The map $R^*(\overline{\mathcal{M}}_{g,n}) \to H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ is an isomorphism if (g, n) belongs to the set: {(2,0),(2,1),(2,2),(2,3),(3,0),(3,1),(3,2),(4,0),(5,0)}*

We will not use this result, but we stress that some of the results that we will obtain in genus 1 could be obtained in the same way in the range where Theorem 3.17 holds.

Theorem 3.18. *([KL02, Theorem 2.9]) The S_n-equivariant Poincaré polynomial of $\mathcal{M}_{0,n}$ is known.*

We will require in what follows the S_n-equivariant Poincaré-Serre polynomial of $\mathcal{M}_{0,5}$:

$$P_{0,5} = t^2 q^2 s[5] + tqs[3, 2] + s[3, 1, 1]$$

Theorem 3.19. *([GK]) The S_n-equivariant Poincaré polynomial of $\overline{\mathcal{M}}_{0,n}$ is known.*

The equivariant Poincaré polynomial of $\overline{\mathcal{M}}_{0,n}$ for $n \leq 8$ will also be necessary, and is given by:

$\overline{\mathcal{M}}_{0,3}$	s_3
$\overline{\mathcal{M}}_{0,4}$	$(t+1)s[4]$
$\overline{\mathcal{M}}_{0,5}$	$s[4,1]t + (t^2+t+1)s[5]$
$\overline{\mathcal{M}}_{0,6}$	$(t^3+2t^2+2t+1)s[6] + (t^2+t)s[5,1] + (t^2+t)s[4,2]$
$\overline{\mathcal{M}}_{0,7}$	$s[4,2,1]t^2 + (t^4+2t^3+4t^2+2t+1)s[7] + (2t^3+3t^2+2t)s[6,1]+$ $(t^3+3t^2+t)s[5,2] + (t^3+2t^2+t)s[4,3]$
$\overline{\mathcal{M}}_{0,8}$	$(t^5+3t^4+6t^3+6t^2+3t+1)s[8] + (2t^4+6t^3+6t^2+2t)s[7,1]+$ $(2t^4+7t^3+7t^2+2t)s[6,2] + (t^3+t^2)s[6,1,1]+$ $(t^4+5t^3+5t^2+t)s[5,3] + (2t^3+2t^2)s[5,2,1] + (t^4+3t^3+3t^2+t)s[4,4]+$ $(2t^3+2t^2)s[4,3,1] + (t^3+t^2)s[4,2,2]$

We shall also assume the knowledge of the Poincaré polynomials of $\overline{\mathcal{M}}_{1,n}$ ([Ge98, p.8]) and of the Poincaré polynomials of $\overline{\mathcal{M}}_{2,n}$ for $n = 0,1,2$ ([Get98, p.18]).

Chapter 4

The Inertia Stack of Moduli of Curves

In this chapter, we introduce what is probably the most important object of the present thesis: the Inertia Stack. In the first section, we review the definition and the basic properties, and develop some notation. In the second section, we study $I(\mathcal{M}_{g,n})$ and $I(\overline{\mathcal{M}}_{g,n})$. Then we propose a way to construct the Inertia Stacks of $\overline{\mathcal{M}}_{g,n}$ inductively on n. In fact, we study the behaviour of the Inertia Stacks under the forgetful maps among moduli spaces of stable marked curves.

4.1 The Inertia Stack

The following is a natural stack associated to a stack X, which points to where X fails to be a space.

Definition 4.1. ([Vi89], Definition 1.12) Let X be an algebraic stack. The *Inertia Stack* $I(X)$ of X is defined as the fiber product $X \times_{X \times X} X$ where both morphisms $X \to X \times X$ are the diagonal ones.

In other words, the Inertia Stack fits in the following 2-cartesian diagram (for this assertion see [Vi89]).

$$\begin{tikzcd} I(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "\Delta_X"] \\ X \arrow[r, "\Delta_X"'] & X \times X \end{tikzcd}$$

Proposition 4.2. *([Vi89, Lemma 1.13]) The map $I(X) \to X$ is finite.*

Example 4.3. The Inertia Stack of the global quotient stack $[X/G]$. Let X be a scheme and G a group acting on it with finite, reduced stabilizers. Then the quotient $[X/G]$ is a Deligne–Mumford stack ([Vi89, Example 7.17]). Its Inertia Stack is isomorphic to:

$$I([X/G]) = \coprod_{g \in T} [X^g / C(g)]$$

where T is a set of choices of one representative for each conjugacy class of G, X^g is the g-invariant part of X and $C(g)$ is the centralizer subgroup of g in G.

Corollary 4.4. *Let Y be a connected component of $I(X)$. Then the map $Y \to X$ can be written as a composition of an étale map and a closed embedding.*

Proof. The last Proposition proves that the map $Y \to X$ is finite. To see that the map is étale on a closed subscheme of X it is enough to check it étale locally. Étale locally the stack X is isomorphic to a global quotient $[U/G]$ ([LMB, Theorem 6.2]) where U is an affine scheme and G is a finite group, and for that one can use the above Example. $\square$

From now on, we fix an isomorphism of μ_r with $\mathbb{Z}/r\mathbb{Z}$.

Definition 4.5. (For the complete definition, see: [AGV06, Definition 3.3.1]) Define the *rigidified Inertia Stack* as a $2-$category:

$$\overline{I}(X) := \coprod_r \overline{I}_r(X)$$

where each component $\overline{I}_r(X)(S)$ has objects:

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\phi"] \arrow[d] & X \\ S & \end{tikzcd}$$

where:

- $\mathcal{G} \to s$ is a μ_r-banded gerbe and,
- $\phi : \mathcal{G} \to X$ is representable.

Lemma 4.6. *[AGV06, Lemma 3.3.2] The $2-$category $\overline{I}(X)$ is equivalent to a category. Moreover this category is a Deligne–Mumford stack.*

If X is an algebraic stack, the objects of its Inertia Stack are:

$$I(X) = \{(x, g) |\ x \in Ob(X), g \in \mathrm{Aut}(x)\} = \{(x, \phi) |\ x \in Ob(X), \phi : \mathbb{Z}/r\mathbb{Z} \to \mathrm{Aut}(x)\}$$

where $Ob(X)$ are the objects of the category X.

Remark 4.7. In [AGV02] and [AGV06] the authors define a stringy Chow ring using the rigidified cyclotomic Inertia Stack. The comparison between the Gromov–Witten theories that one obtains with the rigidified Inertia Stack and the usual one, is explained in [AGV02, 4.4,4.5]. Since we work over $\mathbb{C}$, there is a canonical isomorphism of the Inertia Stack to the cyclotomic Inertia Stack. Moreover, since we deal with cohomologies with rational coefficients, the cohomology of the cyclotomic Inertia Stack is isomorphic to the cohomology of the rigidified cyclotomic Inertia Stack. In fact, the two share the same coarse moduli space (cfr Proposition 1.10).

Lemma 4.8. *([AGV06, Proposition 3.4.1]) The r-th component of the rigidified Inertia Stack $\overline{I}_r(X)$ is the rigidification (in the sense of [AGV06, Appendix C]) of $I_r(X)$. Therefore $\overline{I}_r(X)$ is a μ_r-banded gerbe on $I_r(X)$.*

$$\overline{I}_r(X) = I_r(X) /\!\!/ \mu_r$$

Remark 4.9. There is a natural involution $\iota : I(X) \to I(X)$ given by:

$$(x, \xi) \to (x, \xi^{-1})$$

This involution descends to an involution $\bar{I}(X) \to \bar{I}(X)$, which we shall call with the same name.

Definition 4.10. If X is an algebraic stack, the connected component associated with the identity automorphism of the Inertia Stack is called the *untwisted sector* of the Inertia Stack. All the remaining connected components are called the *twisted sectors* of $I(X)$.

The twisted sectors are embeddable as smooth closed substacks of the original stack thanks to Cartan's lemma.

Proposition 4.11. *[AGV06, Corollary 3.1.4] Let X be a smooth Deligne–Mumford stack. Then $I(X)$ is smooth.*

Definition 4.12. Let X be a smooth algebraic stack and denote T a set of indices in bijection with the twisted sectors of $I(X)$. We refer to the equality:

$$I(X) = X \sqcup \coprod_{i \in T} (X_i, g_i)$$

as a *decomposition of the Inertia Stack of X into twisted sectors* if each (X_i, g_i) is a twisted sector.

Notation 4.13. We will find some special cases when several isomorphic twisted sectors are distinguished only by the choice of the automorphism acting on their general element. If this is the case, in order to simplify the notation, we put together all pairs (A, g), (A, g') writing every possible disjoint union:

$$(A, g/g') := (A, g) \coprod (A, g')$$

When we simply write A we refer to the image of the closed embedding of the twisted sector inside the original stack X.

Now we study the behaviour of the Inertia Stack under arbitrary morphism of stacks.

Definition 4.14. Let $f : X \to Y$ be a morphism of stack. We define $f^*(I(Y))$ as the stack that makes the following diagram is 2−cartesian:

$$\begin{tikzcd} f^*(I(Y)) \arrow[r, "I(f)"] \arrow[d] & I(Y) \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

and $I(f)$ as the map that lifts f in the diagram.

Obviously, there is a map induced $I(X) \to f^*(I(Y))$. There is a necessary and sufficient condition for $I(X)$ to coincide with $f^*(I(Y))$:

Proposition 4.15. *Let $f : X \to Y$ be a morphism of stacks. Then $I(X)$ coincides with $f^*(I(Y))$ if and only if the map f induces an isomorphism on the automorphism groups of objects.*

Proof. Suppose that the morphism induces an isomorphism on the automorphism groups of the objects. To prove the statement, we must prove that the left face of the pictured below cube is 2−cartesian:

$$\begin{tikzcd}
 & I(Y) \arrow[rr] \arrow[dd] & & Y \arrow[dd] \\
I(X) \arrow[ur] \arrow[rr] \arrow[dd] & & X \arrow[ur] \arrow[dd] & \\
 & Y \arrow[rr] & & Y \times Y \\
X \arrow[ur] \arrow[rr] & & X \times X \arrow[ur] &
\end{tikzcd}$$

We prove that the top face, the bottom face, and the right face are 2−cartesian. Therefore the left face (which equals the top face) must be 2− cartesian too (by standard arguments on gluing 2− cartesian diagrams). The top and bottom faces of the two cubes are 2− cartesian by definition of Inertia Stack. The right face of the cube is 2−cartesian since the map restricted to the points gives an isomorphisms between the automorphism groups. If there is a point $x \to X$ such that $f_{|x}$ does not induce an isomorphism on the stabilizer groups, the converse is easily estabilished by taking the fiber product of the left face to the two points $x \to X$ and $f(x) \to Y$ □

4.2 General results

Here we outline the construction due to Fantechi [F09] of the twisted sectors of the Inertia Stack of $\mathcal{M}_{g,n}$. In this paragraph we assume that two integers g and n with $2g - 2 + n$ are fixed once and for all.

Let C be a smooth genus g curve and $\phi : C \to C$ be an automorphism of finite order $N \geq 2$. Let $G := \langle \phi \rangle$ be the subgroup of $\mathrm{Aut}(C)$ generated by ϕ. Let $C' := C/G$ be the quotient as algebraic schemes (the group G is finite), and call $\psi : C \to C'$ the quotient map. We take advantege of Pardini's results [Pa91, Theorem 2.1, Proposition 2.1] to describe C and f in terms of C' and the so called ([Pa91, Definition 2.3]) *reduced building data of the cover.*

Remark 4.16. In the special case at hand, G is cyclic and canonically isomorphic to $\mathbb{Z}/N\mathbb{Z}$ since we have fixed ϕ as a generator. The character group of G is then canonically isomorphic to μ_N (via $\chi \to \chi(1)$). In this special case, moreover, there is a canonical bijection:

$$\mu : \{1, \dots N-1\} \to \{(H, \psi) |\ H \text{ non trivial subgroup of } G,\ \psi \in H^* \text{ an injective character }\} \tag{4.17}$$

given by $\mu(m) = (\langle m \rangle < \mathbb{Z}/N\mathbb{Z}, m \to e^{\frac{2\pi i N}{\gcd(m,N)}})$.

The quotient C' is a smooth curve, and we call its genus g'. We call D the branch divisor inside C':

$$D := \{p \in C' |\ \exists q \in \psi^{-1}(p) \text{ s.t. } \mathrm{Stab}(q) \neq \{e\}\}$$

where we indicate with $\mathrm{Stab}(q)$ the stabilizer group of q w.r.t. the action of G. Now if $H = \mathrm{Stab}(q)$ for some point $q \in \psi^{-1}(p)$, the group H acts on the tangent space T_qC by a non trivial character $\psi : H \to \mathbb{C}^*$. Hence there is a map:

$$\gamma : D \to \{(H, \psi) |\ H \subset G,\ H \neq e,\ \psi \in H^* \text{ a non trivial character}\}$$

We define:

$$D_i := \gamma^{-1}(\mu(i))$$

where the map μ was defined in 4.17. According to this, the branch divisor splits:

$$D = \coprod_{i=1}^{N-1} D_i$$

Observe that $\psi_* \mathcal{O}_C$ is a representation of the group G and hence splits as a direct sum of irreducible (one dimensional) representations (see [Pa91, 1.1]):

$$\psi_* \mathcal{O}_C = \bigoplus_{\chi \in G^\vee} L_\chi^{-1}$$

Having fixed a basis $\langle \phi \rangle$ of G, by Remark 4.16, a basis of $G^\vee$ is also fixed. We call L the line bundle L_χ that corresponds to the representation ζ_n (in other words, the line bundle L_1 in the chosen basis).

Definition 4.18. ([Pa91, Definition 2.3]) $(D_1, \ldots, D_{N-1}, L)$ will be said to be *reduced building data* of the covering ψ.

Now Pardini's equations for the reduced building data ([Pa91, Proposition 2.1]) simplify to the unique equation:

$$NL = \sum_{i=1}^{n-1} iD_i \tag{4.19}$$

It is now convenient to rephrase Proposition 2.1 of [Pa91] in this simplified context:

Proposition 4.20. *([Pa91, Proposition 2.1]) Let $C \to C/\langle \phi \rangle$ be a cyclic ramified covering between smooth curves. Then the relation 4.19 holds for the reduced building data of the covering. Conversely, given an invertible sheaf L and divisors D_i such that 4.19 holds, then it is possible to construct a $G-$cover $\tilde{\psi} : \tilde{C} \to C'$ uniquely up to isomorphisms of G-covers.*

4.2.1 The Inertia Stack of moduli of smooth curves

We first give a description of the Inertia Stack of $\mathcal{M}_g$. We fix $0 \leq g' \leq g$ and $N \geq 2$, and we look for the moduli space of all finite cyclic (possibly) ramified coverings of degree N from genus g curves onto genus g' curves. Let $\psi : C \to C' = C/\mu_N$ be the induced map on the quotient, and let $D \subset C$ be the reduced branch divisor. We define $D_m \subset D$ as the sublocus of points whose fiber under ψ consists of $\gcd(N, m)$ points, and the action of $\mathrm{Stab}(\mu_N)$ on the tangent spaces at the fibers is given by multiplication by $e^{\frac{2\pi i N}{\gcd(m,N)}}$ (see the previous section). We call $d_i := \deg(D_i)$.

Equations 4.21. *([F09]) A set of admissible data will be a tuple $(g', N, d_1, \ldots, d_{N-1})$ satisfying:*

1. *The Riemann–Hurwiz equation:*

$$(2g-2) = N(2g'-2) + \sum_{i=1}^{N-1} d_i \gcd(m,i) \left(\frac{n}{\gcd(m,i)} - 1 \right)$$

2. *Pardini's relation 4.19:*

$$N \text{ divides } \sum_{i=1}^{N-1} id_i$$

Definition 4.22. Whenever $(g', N, d_1, \dots, d_{N-1})$ satisfy the two equations 4.21, we define the category fibered in groupoids $\mathcal{M}_N(g', d_1, \dots, d_{N-1})$, whose objects over S are tuples:

$$\{(C, L, D_1, \dots, D_{N-1}, \phi)\}$$

such that:

1. the map $C \to S$ is a family of genus g' smooth curves,
2. the map D_i assigns to every closed point of $s \in S$ a reduced, effective divisor on the fiber C_s,
3. L is a line bundle on C,
4. the map ϕ is an isomorphism of line bundles:

$$\phi : L^{\otimes N} \to \mathcal{O}_C \left(\sum iD_i\right) \tag{4.23}$$

A morphism between two such objects $(C, L, D_1, \dots D_{N-1}, \phi)$ and $(C', L', D'_1, \dots D'_{N-1}, \phi')$ consists of a couple of maps (f, g) where $f : C \to C'$ is a morphism of schemes such that $f^*(D'_j) = D_j$, and g is a morphism of line bundles on C such that the following diagram commutes:

$$\begin{tikzcd} L^{\otimes N} \arrow[r, "\phi"] \arrow[d, "g"] & \mathcal{O}_C\left(\sum iD_i\right) \arrow[d, "\cong"] \\ f^*L'^{\otimes N} \arrow[r, "f^*(\phi')"] & f^*\mathcal{O}_{C'}\left(\sum iD'_i\right) \end{tikzcd}$$

The objects of this stack are (possibly disconnected) cyclic $N-$coverings of genus g' curves.

Remark 4.24. Defining a_i in such a way that:

$$a_1 = \dots = a_{d_1} = 1,\ a_{d_1+1} = \dots = a_{d_1+d_2} = 2,\ \dots,\ a_{d-d_{N-1}+1} = \dots = a_d = N-1$$

where $d := \sum d_i$, the data of the d_i's is equivalent to the set of data $1 \leq a_1 \leq a_2 \leq \dots \leq a_d \leq N-1$. Conversely, one can obtain the d_i's from the a_i's by simply posing:

$$d_i := |\{a_j \mid a_j = i\}|$$

The data a_i are used to describe moduli spaces of cyclic coverings of prime order over smooth genus g curves in [Co87]. Another notation used in the literature (for example in [BC07] is to take the exponentials:

$$e_i := e^{\frac{2\pi i a_i}{N}}$$

Let $\{y_1, \dots, y_d\}$ be a set of symbols (where again $d := \sum d_i$). We call C_1 the set of the first d_1 elements $\{y_1, \dots y_{d_1}\}$, C_2 the set of the second d_2 elements $\{y_{d_1+1}, \dots, y_{d_1+d_2}\}$, and so on.

Definition 4.25. We define $G(N, d_1, \dots d_{N-1})$ as the subgroup of the symmetric group S_d consisting of:

$$\left\{f \text{ bijection on } \{y_1, \dots, y_d\} \text{ s.t. } f(C_i) = C_i\right\}$$

Remark 4.26. In [BC07, 2.2], Bayer and Cadman describe the moduli space $\overline{M}_{0,n}(e_1, \dots, e_n; B\mu_r)$ of balanced twisted stable maps from genus 0 stable marked curves to $B\mu_r$ in the sense of [AV02]. Their definition can be extended to higher genus as a stack $M_{g,n}(e_1, \dots, e_n; B\mu_r)$. Let $M_{g,n}(e_1, \dots, e_n; B\mu_N)$ be the open substack of $\overline{M}_{g,n}(e_1, \dots, e_n; B\mu_N)$ whose objects are twisted smooth maps in $B\mu_N$ (twisted on the image of the n sections corresponding to $e_i \neq 1$). Suppose that no one among the e_i is equal to one, and let (see Remark 4.24)

$$d_k := |\left\{e_j \; |e_j = e^{\frac{2\pi i k}{N}}\right\}|$$

The stack $\mathcal{M}_N(g', d_1, \dots, d_{N-1})$ that we defined in 4.22 is a quotient of the stack $M_{g',n}(e_1, \dots, e_n; B\mu_N)$ by the action of the group $G(N, d_1, \dots, d_{N-1})$ defined above:

$$\mathcal{M}_N(g', d_1, \dots, d_{N-1}) \cong \left[M_{g',d}(e_1, \dots, e_d; B\mu_N)/G(N, d_1, \dots d_{N-1})\right]$$

Namely, it is the quotient of the moduli space of smooth maps into $B\mu_N$, modulo the action that symmetrizes the sections corresponding to the same divisor D_i.

Remark 4.27. The spaces $\mathcal{M}_N(g', d_1, \dots, d_{N-1})$ have generic stabilizers that contain the group μ_N. They can be rigidified (see [AGV06, Appendix C]):

$$\mathcal{M}_N(g', d_1, \dots, d_{N-1}) \to \mathcal{M}_N(g', d_1, \dots, d_{N-1}) \;/\!\!/\mu_N$$

This last space is equivalent to the moduli space of data:

$$\{(C, L, D_1, \dots, D_{N-1})\}$$

(the same objects of the above definition without the map ϕ), satisfying (1), ..., (3) above, and substituting condition (4) with: *there exists an isomorphism of line bundles* ϕ such that:

$$\phi : L^{\otimes N} \to \mathcal{O}_C\left(\sum iD_i\right) \tag{4.28}$$

The morphisms are also modified in the obvious way.

Remark 4.29. To the discrete data $(g', N, d_1, \dots, d_{N-1})$ we can associate another moduli space:

$$\left[\mathcal{M}_{g',d}/G(N, d_1, \dots d_{N-1})\right]$$

Now there is the following commutative diagram of forgetful functors:

$$\begin{tikzcd}
\left[\{M_{g',d}(e_1, \dots, e_d; B\mu_N)/G(N, d_1, \dots d_{N-1})\right] \arrow[r] \arrow[ddr, "\mathfrak{G}"] & \mathcal{M}_N(g', d_1, \dots, d_{N-1}) \arrow[d, "\mathfrak{F}"] \arrow[dd, bend left=80] \\
& \mathcal{M}_N(g', d_1, \dots, d_{N-1}) \;/\!\!/\mu_N \arrow[d, "\mathfrak{L}"] \\
& \left[\mathcal{M}_{g',d}/G(N, d_1, \dots d_{N-1})\right]
\end{tikzcd}$$

where $\mathfrak{F}$ is the functor that forgets the isomorphism of line bundles explained in Remark 4.27, and $\mathfrak{L}$ is the functor that forgets the line bundle, while $\mathfrak{G}$ forgets the twisted smooth map. The top morphism is the isomorphism described in Remark 4.26.

Proposition 4.30. *The map $\mathfrak{L}$ is a finite étale map of degree $N^{2g'}$*

Proof. The stacks $\mathcal{M}_N(g', d_1, \ldots, d_{N-1}) /\!\!/ \mu_N$ and $\left[\mathcal{M}_{g',d}/G(N, d_1, \ldots d_{N-1})\right]$ are smooth, so it is enough to prove that the map has constant, finite fiber over the points. Let C be a point of $\left[\mathcal{M}_{g',d}/G(N, d_1, \ldots d_{N-1})\right]$. If L_1 and L_2 are two line bundles in $\mathrm{Pic}(C)$ in the fiber $\mathfrak{L}^{-1}(C)$, then

$$L_1^{\otimes N} \otimes \left(L_2^{\otimes N}\right)^{\vee} = \mathcal{O}_C$$

and so there is a bijection:

$$\mathfrak{L}^{-1}(C) \to \{L \in \mathrm{Pic}^0 \mid L^{\otimes N} = \mathcal{O}\}$$

and the latter is isomorphic (though not canonically) to $\mu_N^{2g'}$. $\square$

Corollary 4.31. *If $g' = 0$, then $\mathcal{M}_N(0, d_1, \ldots, d_{N-1})$ is isomorphic to a μ_N gerbe over the stack quotient $\left[\mathcal{M}_{0,\sum d_i}/G(N, d_1, \ldots d_{N-1})\right]$.*

We shall see that the stack description of the compactification of $\mathcal{M}_N(0, d_1, \ldots, d_{N-1})$ is more complicated (Theorem 6.3).

Recall that the total genus of a disconnected curve whose two components have genera g_1 and g_2 is defined to be $g_1 + g_2 - 1$. This definition makes the Riemann–Hurwiz formula work in the disconnected case. If $\gcd(N, a_1, \ldots, a_d) \neq 1$ then it is possible to describe the covering as a composition of a disconnected covering of degree $\gcd(N, a_1, \ldots, a_d) \neq 1$ and a connected covering of degree $N/\gcd(N, a_1, \ldots, a_d) \neq 1$. The objects of $\mathcal{M}_N(g', d_1, \ldots, d_{N-1})$ may correspond to disconnected coverings.

Definition 4.32. Let $(g', N, d_1, \ldots, d_{N-1})$ satisfy Equations 4.21. Then we define $\mathcal{M}'_N(g', d_1, \ldots, d_{N-1})$ as the open and closed substack of $\mathcal{M}_N(g', d_1, \ldots, d_{N-1})$ whose objects correspond to connected coverings.

Note that, by definition of Inertia Stack, there is a map $\mathcal{M}'_N(g', d_1, \ldots, d_{N-1}) \to I(\mathcal{M}_{g,n})$.

Theorem 4.33. *[F09]The stack $\mathcal{M}'_N(g', d_1, \ldots, d_{N-1})$ (cfr. 4.32), whose points correspond to connected ramified cyclic coverings of genus g' curves, is connected. The canonical map $\mathcal{M}'_N(g', d_1, \ldots, d_{N-1}) \to I(\mathcal{M}_{g,n})$ induces an isomorphism on its image. Conversely, if Y is a twisted sector of the Inertia Stack of $\mathcal{M}_{g,n}$, then there are discrete data $(g', N, d_1, \ldots, d_{N-1})$ satisfying 4.21, such that the canonical map $\mathcal{M}'_N(g', d_1, \ldots, d_{N-1}) \to I(\mathcal{M}_{g,n})$ induces an isomorphism of the domain with Y.*

Proof. We prove the proposition only in the cases that we are going to need in the present thesis, that is for the cases of genus $g = 1, 2$. As we shall see, in these cases, the admissible data of 4.21 satisfy one of the following conditions:

1. the genus of the covered curve g' equals 0,
2. the degree of the cyclic covering N, is a prime number.

In Case 1, as we have seen in Corollary 4.31, the stack $\mathcal{M}_N(0, d_1, \ldots, d_{N-1})$ is connected. In Case 2, the connectedness has been proved by Cornalba in [Co87, pag.3]. $\square$

4.2.2 A Generalization

This construction can be generalized to the case with marked points (cfr. [F09]). We want to deal here with a slightly more general case, namely the computation of the twisted sectors for the quotient stack:

$$[\mathcal{M}_{g,n}/S]$$

where $S \subset S_n$ is a subgroup generated by a product of disjoint cycles. Suppose that the natural action of the subgroup S on $\{1, \ldots, n\}$ is made of t orbits. Up to conjugacy in S_n, the subgroup S is given once a partition $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_t$ of n is given. We call $\lambda := \mathrm{lcm}(\lambda_i)$. We now add a discrete datum to: $(g', N, d_1, \ldots, d_{N-1})$. This is a map:

$$\alpha : \{1, \ldots, t\} \to \{0, \ldots, N-1\}$$

where $\alpha(s)$ is one among the i such that $\lambda_s = \gcd(i, N)$. (Note that $N = \gcd(0, N)$). We add three more equations relating all the data:

Equations 4.34. *If g, n, S are fixed as above, a set of data $N, d_1, \ldots, d_{N-1}, \alpha$ is said to be a set of generalized admissible data if they satisfy:*

1. $(2g-2) = N(2g'-2) + \sum_{i=1}^{N-1} d_i \gcd(N, i) \left(\frac{N}{\gcd(N,i)} - 1\right)$
2. *N divides $\sum_{i=1}^{N-1} i d_i$*
3. *λ divides N*
4. *$|\alpha^{-1}(i)| \leq d_i$ for all $i > 0$*
5. $\sum_{s=1}^{t} \gcd(\alpha(s), N) = n$.

We can now modify Definition 4.22, to obtain our description of the twisted sectors of $[\mathcal{M}_{g,n}/S]$.

Definition 4.35. Whenever $(g', N, d_1, \ldots, d_{N-1}, \alpha)$ is a set of generalized admissible data (4.34), we define the category fibered in groupoids $\mathcal{M}_N(g', d_1, \ldots, d_{N-1}, \alpha)$, whose objects over S are tuples:

$$\{(C, L, D_1, \ldots, D_{N-1}, x_1, \ldots, x_t, \phi)\}$$

such that:

1. the map $C \to S$ is a family of genus g' smooth curves,
2. the map $x_i : S \to C$ is a section of the previous map,
3. the map D_i assigns to every geometric point $s \in S$ a reduced effective divisor in the fiber C_s,
4. L is a line bundle on C,
5. the map ϕ is an isomorphism of line bundles:

$$\phi : L^{\otimes N} \to \mathcal{O}_C\left(\sum i D_i\right) \tag{4.36}$$

Note that D_i gives a map from the points s of S to divisors of the fiber C_s of degree d_i. We define $D_0(s) := C \setminus \coprod_i D_i(s)$. These data satisfy the condition that for every geometric point s in the base S:

$$x_i(s) \in D_{\alpha(i)}(s)$$

A morphism between two such objects $(C, L, D_1, \ldots D_{N-1}, x_1, \ldots x_t, \phi)$ and $(C', L', D'_1, \ldots D'_{N-1}, x'_1, \ldots x'_t, \phi')$ is a couple of maps (f, g) where $f : C \to C'$ is a morphism of schemes such that $f^*(x'_i) = x_i$,

$f^*(D'_j) = D_j$, and g is a morphism of line bundles on C such that the following diagram commutes:

$$\begin{tikzcd} L^{\otimes N} \arrow[r, "\phi"] \arrow[d, "g"] & \mathcal{O}_C\left(\sum iD_i\right) \arrow[d, "\cong"] \\ f^*L'^{\otimes N} \arrow[r, "f^*(\phi')"] & f^*\mathcal{O}_{C'}\left(\sum iD'_i\right) \end{tikzcd}$$

As before, let $\{y_1, \dots, y_d\}$ be a set of symbols (where $d := \sum d_i$). We call C_1 the set of the first d_1 elements $\{y_1, \dots y_{d_1}\}$, C_2 the set of the second d_2 elements $\{y_{d_1+1}, \dots, y_{d_1+d_2}\}$, and so on.

Definition 4.37. We define $G(N, d_1, \dots d_{N-1}, \alpha)$ as the subgroup of S_d consisting of:

$$\Big\{ f \text{ bijection on } \{y_1, \dots, y_d\} \text{ s.t. } f(C_i) = C_i,\ \forall i \in \{1, \dots, N\} \text{ the first } |\alpha^{-1}(i)|$$

$$\text{points in } C_i \text{ are fixed}\Big\}$$

Remark 4.38. Let $a := |\alpha^{-1}(0)|$. We can mimic Remark 4.26 and construct the stack $M_{g',d+a}(e_1, \dots, e_d, e_{d+1}, \dots, e_{d+a}; B\mu_N)$([BC07]) where $e_{d+1} = \dots = e_{d+a} = 1$ and the relation between the d_i and the e_j for $j \leq d$ is as in 4.24 or 4.26. Then we have an isomorphism that generalizes the one described in 4.26:

$$\left[M_{g',d+a}(e_1, \dots, e_d, e_{d+1}, \dots, e_{d+a}; B\mu_N)/G(N, d_1, \dots d_{N-1}, \alpha)\right] \cong \mathcal{M}_N(g', d_1, \dots, d_{N-1}, \alpha)$$

where the group acts on the points d_i and fixes the last a points.

As in Remark 4.29, the functor that forgets the map, describes the left hand side as a finite étale covering on:

$$\left[\mathcal{M}_{g',d+a}/G(N, d_1, \dots d_{N-1}, \alpha)\right]$$

With the notation introduced in this section, we are ready to state and prove the following generalization of Theorem 4.33:

Definition 4.39. Let $(g', N, d_1, \dots, d_{N-1}, \alpha)$ satisfy Equations 4.34. Then we define $\mathcal{M}'_N(g', d_1, \dots, d_{N-1}, \alpha)$ as the open and closed substack of $\mathcal{M}_N(g', d_1, \dots, d_{N-1}, \alpha)$ whose objects correspond to connected coverings.

Again, by definition of Inertia Stack, there is a map $\mathcal{M}'_N(g', d_1, \dots, d_{N-1}, \alpha) \to I([\mathcal{M}_{g,n}/S])$.

Theorem 4.40. *(cfr. 4.33) The stack $\mathcal{M}'_N(g', d_1, \dots, d_{N-1}, \alpha)$ (cfr. 4.32) whose points correspond to connected ramified cyclic coverings of genus g' curves, is connected. The canonical map $\mathcal{M}'_N(g', d_1, \dots, d_{N-1}, \alpha) \to I\left(\left[\mathcal{M}_{g,n}/S\right]\right)$ induces an isomorphism onto its image. Conversely, if Y is a twisted sector of the Inertia Stack of $\left[\mathcal{M}_{g,n}/S\right]$, then there are discrete data $(g', N, d_1, \dots, d_{N-1}, \alpha)$, satisfying 4.34, such that the canonical map $\mathcal{M}'_N(g', d_1, \dots, d_{N-1}, \alpha) \to I\left(\left[\mathcal{M}_{g,n}/S\right]\right)$ induces an isomorphism of the domain with Y.*

Proof. If $(g', N, d_1, \dots, d_{N-1})$ satisfy 4.21, then the choice of a product of cyclic disjoint permutations $S \subset S_n$ and of a map α such that $(g', N, d_1, \dots, d_{N-1}, \alpha)$ satisfying 4.34 give a map: $\mathcal{M}'_N(g', d_1, \dots, d_{N-1},) \to \mathcal{M}'_N(g', d_1, \dots, d_{N-1}, \alpha)$. Since the domain is connected (4.33), the image must be connected too. The converse is obvious from the definition of the Inertia Stack and by Construction (4.20). ∎

Remark 4.41. With the notation introduced in this section, the twisted sectors of $\mathcal{M}_{g,n}$ are isomorphic to $\mathcal{M}_N(g', d_1, \dots, d_{N-1}, \alpha)$, where:

$$\alpha : \{1, \dots, n\} \to \mathbb{Z}/N\mathbb{Z}^*$$

Definition 4.42. Let g, n be fixed integers, with $g > 0, n \geq 1$. We shall call *k-base twisted sectors of rational type for g and n* (or simply base twisted sector) all the twisted sectors of the Inertia Stacks of $\mathcal{M}_{g,k}$ (for $k = 1, \dots, n$).

Notation 4.43. Let Y be a k-base twisted sector of rational type (see the last definition), we will also denote Y by $Y_{(\alpha(1),\dots,\alpha(n))}$ when we want to keep track of the marked points. Once g' and N are fixed, if we want to keep track of the dependence of Y on the discrete data a_i (cfr. Remark 4.24 for the equivalence of these data with the d_i), we write ${}_{(a_1,\dots,a_d)}Y_{(\alpha(1),\dots,\alpha(n))}$.

4.2.3 A description for the Inertia Stack of $\overline{\mathcal{M}}_{g,n}$

We want to say something general about the task of determining the Inertia Stack of stable curves.

Let $X \to \mathcal{M}_{g,n}$ be a twisted sector. We have given a modular interpretation of it in 4.39 and 4.41 as $\mathcal{M}'_N(g', d_1, \dots, d_{N-1}, \alpha)$ for some discrete invariants. We see how this stack can naturally be compactified by means of admissible covers. Admissible covers of genus 0 curves were first constructed by Harris and Mumford in [HMu82], this construction was then generalized in [ACV03, Definition 4.1.1] as a special case of compactification of the space of stable maps to a stack (the compactification of the space of maps to a stack by means of twisted stable maps was constructed by Abramovich and Vistoli in [AV02]).

Definition 4.44. ([ACV03, Definition 4.1.1]) Let $(C', x_1, \dots, x_n)$ be a family of nodal curves. An *admissible cover of degree d* is a finite morphism $p : C \to C'$, satisfying the conditions that:

1. C is a family of nodal curves,
2. every node of C maps to a node of C',
3. the map p is generically étale of degree d,
4. for every node q of D there is an integer $e(q)$, such that étale locally in a neighbourhood of q, the map $p : C \to C' \to S$ is induced by the map of rings:

$$A \to A[x, y]/(xy - a^e) \to A[\xi, \eta]/(\xi\eta - a)$$

   sending $a \to a$, $x \to \xi^e$, $y \to \eta^e$,

5. for every marked point x_i there is an integer $e(x_i)$, such that étale locally the map $p : C \to C' \to S$ is induced by:

$$A \to A[x] \to a[\xi]$$

   sending $a \to a$ and $x \to \xi^e$.

Definition 4.45. ([ACV03, Definition 4.3.1]) Let G be a finite group. An admissible cover $p : C \to C'$ is an *admissible G-cover* such that:

1. the restriction of p to the general locus of C is a G-principal bundle,
2. for each geometric nodal point $q \in C$, the action of G on q is balanced (2.53).

Admissible $G-$covers form a category fibered in groupoids where arrows are fibered diagrams. We call this fibered category $\mathrm{Adm}_{g,n}(G)$.

Theorem 4.46. *([ACV03, Theorem 4.3.2], [AV02, Theorem 1.4.1]) The fibered category* $\mathrm{Adm}_{g,n}(G)$ *is a proper Deligne–Mumford stack, with projective coarse moduli space.*

We can now define the compactification of $\mathcal{M}_N(g', d_1, \ldots, d_{N-1}, \alpha)$. We take Definition 4.35 and extend it, allowing the base curves to be nodal, and taking admissible covers:

Definition 4.47. Let $\mathcal{M}'_N(g', d_1, \ldots, d_{N-1}, \alpha)$ be a twisted sector of the Inertia Stack of $[\mathcal{M}_{g,n}/S]$ where S is a product of disjoint permutations of S_n (see 4.35, 4.39). Let $G(N, d_1, \ldots, d_{N-1}, \alpha)$ be as in 4.37. We define $\overline{\mathcal{M}}_N(g', d_1, \ldots, d_{N-1}, \alpha)$ as the closure of $\mathcal{M}'_N(g', d_1, \ldots, d_{N-1}, \alpha)$ inside the quotient stack: $\left[\mathrm{Adm}_{g,n}/G(N, d_1, \ldots, d_{N-1}, \alpha)\right]$.

Proposition 4.48. *With the notation introduced in 4.35, let* $\mathcal{M}'_N(g', d_1, \ldots, d_{N-1}, \alpha)$ *be a twisted sector of* $[\mathcal{M}_{g,n}/S]$. *Then* $\overline{\mathcal{M}}_N(g', d_1, \ldots, d_{N-1}, \alpha)$ *is a twisted sector of* $[\overline{\mathcal{M}}_{g,n}/S]$.

Proof. We have just to prove that the stack $\overline{\mathcal{M}}_N(g', d_1, \ldots, d_{N-1}, \alpha)$ is connected. This follows easily from the fact that it contains an open dense substack that is connected (Theorem 4.33). □

Definition 4.49. Let X be a twisted sector of the Inertia Stack of $\overline{\mathcal{M}}_{g,n}$. We define ∂X as the fiber product illustrated below:

$$\begin{array}{ccc} \partial X & \longrightarrow & X \\ \downarrow & & \downarrow \\ \partial\overline{\mathcal{M}}_{g,n} & \longrightarrow & \overline{\mathcal{M}}_{g,n} \end{array}$$

We say that the twisted sector *lies in the boundary* if ∂X equals X or, equivalently, if the image of X under the canonical projection of the Inertia Stack lies in the boundary of the moduli space.

So far, we have described all the twisted sectors not lying in the boundary. The inclusion:

$$i : \mathcal{M}_{g,n} \to \overline{\mathcal{M}}_{g,n}$$

induces a pull-back i^* on the Inertia Stacks (see Definition 4.14), and $i^*(I(\overline{\mathcal{M}}_{g,n})) = I(\mathcal{M}_{g,n})$. Nevertheless, there are many sectors in the Inertia Stack of the compactification, whose pull-back via i is empty. All these sectors lie in the boundary according to the last definition.

Proposition 4.50. *Let* X *be a twisted sector of* $I(\partial\overline{\mathcal{M}}_{g,n})$. *This twisted sector lies in the boundary (cfr Definition 4.49) if and only if the general point of* X*, which corresponds to a couple* (C, β) *for* C *a stable curve and* β *an automorphism of it, is not smoothable (cfr. Definition 2.53).*

Theorem 4.51. *The automorphisms of a stable curve (of compact type) that come from an automorphism of the graph are all smoothable (according to Definition 2.53).*

Proof. The action of G simply permutes some of the nodes and some of the irreducible components of the curve. The induced action of G on the vector space $H^0(\mathcal{E}xt^1(\Omega_C, \mathcal{O}_C))$ has at least an invariant subspace of dimension one. The sheaf whose global sections we are computing is supported on the nodes (cfr. Section 1.3 and [ACG2]). The subspace of constant sections is invariant under the action of G. □

We give here a combinatorial recipe for the construction of the twisted sectors of the Inertia Stack of moduli of genus g, $n-$pointed, stable curves.

Definition 4.52. If $G \in \mathcal{G}_{g,n}$ (Definition 2.33), let $T(G)$ be a choice of a subset of $\mathrm{Aut}(G)$ that contains one element for each conjugacy class of $\mathrm{Aut}(G)$. We define

$$\mathcal{A}_{g,n} := \{(G, \beta) \mid G \in \mathcal{G}_{g,n},\ \beta \in T(\mathrm{Aut}(G))\}$$

The choice of one element for each conjugacy class in the subsequent construction is motivated by Example 4.3.

Construction 4.53. (Combinatorial description of the twisted sectors of $\overline{\mathcal{M}}_{g,n}$) Let g and n be fixed natural numbers such that $g \geq 0$ and $n \geq 1$ if $g = 1$. If $(G, \beta) \in \mathcal{A}_{g,n}$ (see 4.52), let $B := \langle \beta \rangle$ be the subgroup of $\mathrm{Aut}(G)$ generated by β, and s be the smallest integer such that $\beta' := \beta^s$ fixes all the vertices of G. The group B acts in particular on the set V of vertices, and we can choose $v_1, \ldots, v_k$ as a set of representatives for each orbit in V of the action of B (so that $V = Bv_1 \sqcup \ldots \sqcup Bv_k$). For each vertex v_i, let g_i be its geometric genus and n_i be its valence. Notice that B' acts on the set of (half)-edges H as a product of cyclic disjoint subgroups of the permutation group on the half-edges: S_H. Thus we choose, for each vertex v_i, a sector X_i (possibly untwisted) of the Inertia Stack of:

$$I\left(\left[\mathcal{M}_{g_i,n_i}/B'\right]\right)$$

whose elements were described in Proposition 4.40.

We define

$$\Sigma(\beta) := \{\sigma \in \mathrm{Aut}(G) \mid \sigma \circ \beta = \beta \circ \sigma\}$$

The elements of $\Sigma(\beta)$ induce permutations on the set of representatives $v_1, \ldots, v_k$.

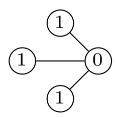
Definition 4.54. Let $(X_1, \ldots, X_k)$ and $(X'_1, \ldots, X'_k)$ be constructed as above. We say that they are equivalent if there exists a permutation $\sigma \in \Sigma(\beta)$ such that $X_i = X'_{\sigma(i)}$.

We let $\overline{X_i}$ be the compactification with admissible covers of X_i (Definition 4.47).

Definition 4.55. If G and β are fixed, and for all i we have fixed X_i, a connected component of $I\left(\left[\mathcal{M}_{g_i,n_i}/B'\right]\right)$ (as explained in the above construction), then an object of $(X_1, \ldots, X_k)$ is a couple (C, α). The curve C is a stable curve whose dual graph is G, and α is an automorphism of C, which we explain below. Every irreducible component corresponding to vertices Bv_i corresponds to an object $(x_i, \alpha_i) \in \overline{X_i}$. These components are glued and given marked points as prescribed by the graph G (see Construction 2.34). The automorphism α is the automorphism of C such that the action of α on the set of components corresponds to the action of β on the set of vertices, and such that α^s acts on the curve x_i as the automorphism α_i. All the spaces $(X_1, \ldots, X_k)$ thus described will bel called *the twisted sectors of* $\overline{\mathcal{M}}_G$. Two equivalent lists $(X_1, \ldots, X_k)$ and $(X'_1, \ldots, X'_k)$ (see Definition 4.54) are associated with the same couple (C, α).

Remark 4.56. The definition above exhibits $(X_1, \ldots, X_k)$ as a connected, closed substack of $I(\partial(\overline{\mathcal{M}}_{g,n}))$. Let B' be the normal subgroup of B generated by β', and $A := B/B'$. We observe that $(X_1, \ldots, X_k)$ is an A-gerbe on the stack $\prod_{i=1}^{k} \overline{X_i}$.

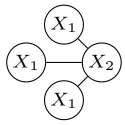
Example 4.57. Let G be the stable graph of genus 3 without marked points:



and consider β the cyclic automorphism of the graph that exchanges the genus 1 components. Then β is of order 3. The quotient set of vertices is:

$$\{v_1, v_2\}$$

where v_1 is a vertex of genus 1 with valence 1, and v_2 is a vertex of genus 0 with valence 3. So let X_1 be a connected component of $I(\mathcal{M}_{1,1})$ and X_2 be a connected component of $I([\mathcal{M}_{0,3}/(123)])$. In the rest of the thesis, we shall refer to this twisted sector by giving the stable graph, inserting in each vertex the twisted sector chosen, and specifying the automorphism of the graph. So in the present case, the twisted sector we are considering is:



with the cyclic automorphism that permutes the three components X_1.

Construction 4.58. (A set of indices for the twisted sectors of $\overline{\mathcal{M}}_{g,n}$) If $(G, \beta) \in \mathcal{A}_{g,n}$, not all the twisted sectors of $\overline{\mathcal{M}}_G$ are twisted sectors of $\overline{\mathcal{M}}_{g,n}$. We say that the locus $(X_1, \ldots, X_k)$ corresponding to (G, β) is smoothable iff the general curve of this locus is smoothable (cfr. Definition 2.53). A combinatorial way to construct the twisted sectors of $I(\overline{\mathcal{M}}_{g,n})$ that lie in the boundary (4.49) is:

- Consider all the elements $(G, \beta) \in \mathcal{A}_{g,n}$;
- If V is the set of vertices of G, consider a subset $\{v_1, \ldots, v_k\}$ of V made up of one representative in V for each orbit of $B = \langle \beta \rangle$. Call g_i the genus of v_i and n_i its valence. Let B' be the subgroup of B that stabilizes the vertices. Then for all i, B' acts naturally on $[n_i]$ as a product of cyclic disjoint permutations $S_i \subset S_{n_i}$;
- Consider the set T'' of all the possible assignments:

  $\xi : \{v_1, \ldots, v_k\} \to \{$Set of all possible generalized admissible data 4.34 with parameters g_i, n_i, $S_i\}$
- Consider the subset T' of T'', comprising of assignments ξ whose corresponding curves have the general element which is not smoothable, preserving the automorphism group (see construction above and 2.53);
- Consider the quotient set of T' by the equivalence relation induced by $\Sigma(\beta)$ (explained above), and call this T.

Then the set of all the triples (G, β, T), where $(G, \beta) \in \mathcal{A}_{g,n}$ and T is the set constructed in 4.58, is a set of indices (i.e. it is in natural bijection) for the twisted sectors of $\overline{\mathcal{M}}_{g,n}$.

Remark 4.59. In the notation introduced in the above construction, if there is an i such that the twisted sector X_i has, among the discrete data that identify it (4.43) a map α_i whose image contains 0 then the whole sector $(X_1, \ldots, X_k)$ constructed is smoothable (see Definition 2.53, Proposition 4.51).

The next Proposition is now an easy corollary of the constructions we have introduced in this Section:

Proposition 4.60. *The twisted sectors of $I(\overline{\mathcal{M}}_{g,n})$ are all constructed following the recipe given in the above constructions 4.53 and 4.58.*

4.3 The Inertia Stack and the forgetful map

We gave a description in Theorem 4.60 for all the twisted sectors of $\overline{\mathcal{M}}_{g,n}$. We now want to assume the following point of view. The following diagram is not $2-$cartesian:

$$\begin{tikzcd} I(\overline{\mathcal{M}}_{g,n+1}) \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{g,n+1} \arrow[d] \\ I(\overline{\mathcal{M}}_{g,n}) \arrow[r] & \overline{\mathcal{M}}_{g,n} \end{tikzcd}$$

since the forgetful morphism does not give an isomorphism on the stabilizers of points (see Proposition 4.15). In this section, we want to study the relation between the Inertia Stack of $\overline{\mathcal{M}}_{g,n+1}$ and that of $\overline{\mathcal{M}}_{g,n}$. If $C, x_1, \ldots, x_n$ is a stable curve, and β is an automorphism of it, then there are three different ways of adding a marked point to the curve in such a way that β still is an automorphism of it. The simplest option we have is: we choose to mark a point that is stabilized by β, and that is not one among the former x_i's. A second option is, if C admits at least one node and this node is stabilized by β, we can blow it up and put a marked point on the rational component (in fact, a rational bridge) thus obtained. On such a curve there is an unique automorphism induced by β in the obvious way. A third way is to choose one among the x_i's, to forget that marking, to blow that point up, and to put two marked points on the resulting rational component: x_i and the new one x_{n+1}. We refer to these procedures by saying that we add the marked point to a vertex, on a node, and on a former marked point, respectively. In this section, we see that performing these three operations in this order produces all the twisted sectors (after possibly reordering the marked points).

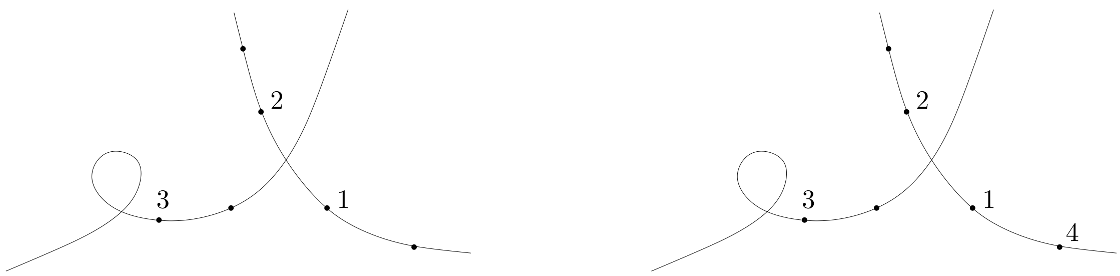


Figure 4.1: Adding a marked point to a vertex

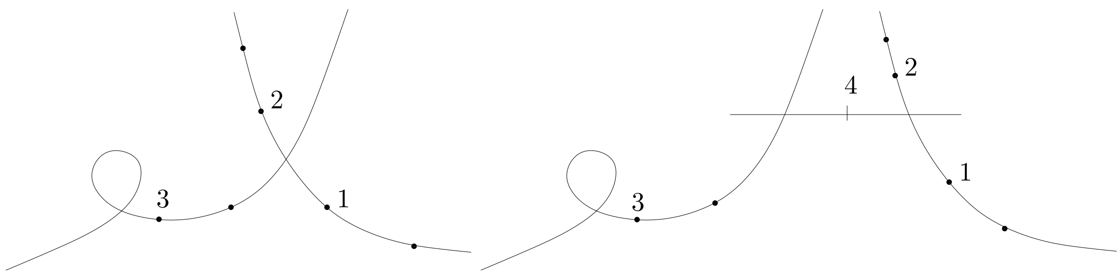


Figure 4.2: Adding a marked point on a node

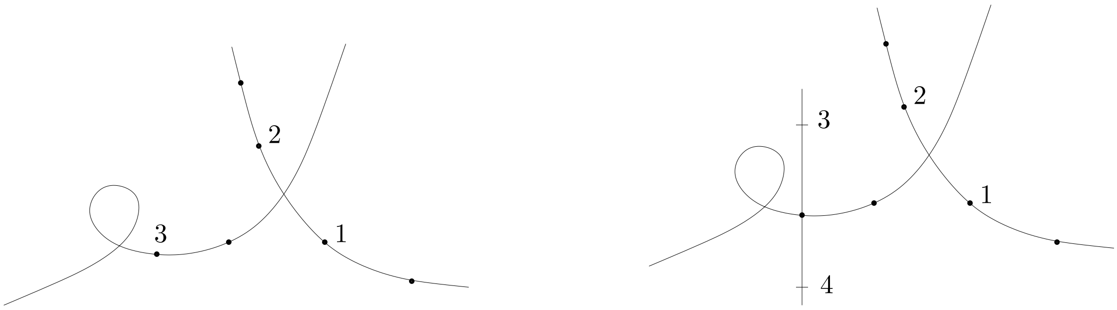


Figure 4.3: Adding a marked point on a former marked point

In the pictures, we have fixed a nodal curve with three marked points (C, x_1, x_2, x_3) and an automorphism α of it. The bullet points are the points of the curve that are fixed by α.

We now study the last two operations, starting from the third one, because it is the easiest to describe.

Theorem 4.61. *The following diagram of categories fibered in groupoids is* 2*-cartesian:*

$$\begin{tikzcd} I(\overline{\mathcal{M}}_{g,n}) \arrow[r, two heads] \arrow[d] & \coprod_{k=1}^{n} I(\overline{\mathcal{M}}^R_{g,k}) \arrow[r, hook] \arrow[d] & \coprod_{k=1}^{n} I(\overline{\mathcal{M}}_{g,k}) \arrow[d] \\ \overline{\mathcal{M}}_{g,n} \arrow[r, two heads, "\pi^{rt}"] & \coprod_{k=1}^{n} \overline{\mathcal{M}}^R_{g,k} \arrow[r, hook, "i"] & \coprod_{k=1}^{n} \overline{\mathcal{M}}_{g,k} \end{tikzcd}$$

Proof. Since the forgetful map induces an isomorphism on the stabilizers of all the objects, this is a simple application of Proposition 4.15. ☐

Remark 4.62. We observe that the map π^{rt} restricts to a map (which by abuse of notation we call with the same name):

$$\pi^{rt} : \mathcal{M}^{rt}_{g,n} \to \coprod_{k=1}^{n} \mathcal{M}_{g,k}$$

whenever $g, n \geq 1$. Since here we have only rational tails, the map could be simply defined as the one that forgets all the marked points that lie on all the rational components, contracts the rational components and puts a marking in the locus where the rational component was contracted. The marked point it puts in that point is x_i, where i is the minimum among the j such that x_j was a marked point in that rational component.

Now, as a corollary of Theorem 4.61, we obtain:

Corollary 4.63. *The following square of categories fibered in groupoids is* 2*-cartesian too:*

$$\begin{tikzcd} I(\mathcal{M}^{rt}_{g,n}) \arrow[r] \arrow[d] & \coprod_{k=1}^{n} I(\mathcal{M}_{g,k}) \arrow[d] \\ \mathcal{M}^{rt}_{g,n} \arrow[r, "\pi^{rt}"] & \coprod_{k=1}^{n} \mathcal{M}_{g,k} \end{tikzcd}$$

Corollary 4.64. *The twisted sectors of $\mathcal{M}^{rt}_{g,n}$ are all products of base twisted sectors of rational type and moduli of stable genus* 0 *pointed curves via the following isomorphism:*

$$\coprod_{k=1}^{n} \coprod_{\{I_1,\dots,I_k\}|\ [n]=\sqcup I_j} I(\mathcal{M}_{g,k}) \times \overline{\mathcal{M}}_{0,I_1+1} \times \ldots \times \overline{\mathcal{M}}_{0,I_k+1} \to I(\mathcal{M}^{rt}_{g,n})$$

where all the subsets in the partition of $[n]$ *are non empty.*

Proof. By Theorem 4.61, the bottom square of the following diagram is 2−cartesian, while the big square is 2−cartesian on the nose (the map pr_1 denotes the first projection):

$$\begin{tikzcd}
\coprod\limits_{k=1}^{n} \coprod\limits_{\{I_1,\dots,I_k\}|\ [n]=\sqcup I_j} I(\mathcal{M}_{g,k}) \times \overline{\mathcal{M}}_{0,I_1+1} \times \cdots \times \overline{\mathcal{M}}_{0,I_k+1} \arrow[r] \arrow[d, dashed, "\psi"] \arrow[dd, bend right, "pr_1"'] & \coprod\limits_{k=1}^{n} \coprod\limits_{\{I_1,\dots,I_k\}|\ [n]=\sqcup I_j} \mathcal{M}_{g,k} \times \overline{\mathcal{M}}_{0,I_1+1} \times \cdots \times \overline{\mathcal{M}}_{0,I_k+1} \arrow[d, "\phi^{RT}"] \arrow[dd, bend left, "pr_1"] \\
I(\mathcal{M}^{rt}_{g,n}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{M}^{rt}_{g,n} \arrow[d] \\
\coprod\limits_{k=1}^{n} I(\mathcal{M}_{g,k}) \arrow[r] & \coprod\limits_{k=1}^{n} \mathcal{M}_{g,k}
\end{tikzcd}$$

The map ϕ^{RT} was defined in Proposition 2.41. By the definition of the Inertia Stack, it is then easy to check that ψ is an equivalence of categories, and hence an isomorphism of stacks. $\square$

We now want to count the number of the twisted sectors of $\mathcal{M}^{rt}_{g,n}$, namely to count the elements of the set of all such twisted sectors.

Notation 4.65. Note that if Y is a twisted sector of $\mathcal{M}_g$ (see Notation 4.43):

$$\left(\coprod_{\text{all the assignments } \alpha} Y_{\alpha(1),\dots,\alpha(k)} \right) \subset I(\mathcal{M}_{g,k})$$

and $\{I_1,\dots,I_k\}$ is a partition of $[n]$, we can define:

$$\left(\sqcup_\alpha Y_{\alpha(1),\dots,\alpha(k)}\right)^{\{I_1,\dots,I_k\}} := \left(\sqcup_\alpha Y_{\alpha(1),\dots,\alpha(k)}\right) \times \overline{\mathcal{M}}_{0,I_1+1} \times \ldots \times \overline{\mathcal{M}}_{0,I_k+1}$$

In this way, the fiber under ψ of $\sqcup_\alpha Y_{\alpha(1),\dots,\alpha(k)}$ is $\left(\sqcup_\alpha Y_{\alpha(1),\dots,\alpha(k)}\right)^{\{I_1,\dots,I_k\}}$.

If Y is a twisted sector of $\mathcal{M}_g$, $I_1,\dots,I_k$ is a partition of $[n]$, then the set of twisted sectors that under ψ lie over $Y_{\alpha(1),\dots,\alpha(k)}$ is in bijection with the set of all the possible assignments $\lambda : \{1,\dots,k\} \to \mathbb{Z}/\mathbb{Z}^*_N$, such that $|\lambda^{-1}(j)| \leq |\alpha^{-1}(j)|$ for all j.

Let Y be a $k-$base twisted sector (Definition 4.42) identified by the discrete data: $(g', N, d_1, \dots, d_{N-1}, \alpha)$. We define the parameters $b_i := |\alpha^{-1}(i)|$ for all $i \in \mathbb{Z}/\mathbb{Z}^*_N$. Let $I_1,\dots,I_k$ be a partition of $[n]$. For all $l \in \{1,\dots,k-1\}$, we can form the following sets:

$$K_1 := \left\{ I_1, \dots, I_{b_1} \right\}$$

$$K_l := \left\{ I_{1+\sum_{i<l} b_i}, \dots, I_{\sum_{i\leq l} b_i} \right\}$$

$$K_{k-1} := \left\{ I_{1+\sum_{i<k-2} b_i}, \dots, I_{\sum b_i} \right\}$$

Assuming that $I_i \in K_{\lambda(i)}$ for all i's.

Definition 4.66. With the notation introduced above, we define $Y^{\{K_1,\ldots,K_{k-1}\}}_{(\alpha(1),\ldots,\alpha(k))}$, where K_j is the set whose elements are some of the subsets in the partition of $[n]$, as explained above. It is the twisted sector just constructed and described above. It is isomorphic to the product:

$$Y_{(\alpha(1),\ldots,\alpha(k))} \times \overline{\mathcal{M}}_{0,I_1+1} \times \ldots \overline{\mathcal{M}}_{0,I_k+1}$$

In this case, we say that $Y^{\{K_1,\ldots K_{n-1}\}}_{(\alpha(1),\ldots,\alpha(k))}$ *has* $Y_{(\alpha(1),\ldots,\alpha(k))}$ *as an associated base twisted sector.*

We now fix a base twisted sector Y (cf. Definition 4.42).

Proposition 4.67. *The number of twisted sectors of* $\mathcal{M}^{rt}_{g,n}$ *whose associated* $k-$*base twisted sector (cfr. Notation 4.66) is* $Y_{(\alpha(1),\ldots,\alpha(k))}$ *is:*

$$\prod_{j=1}^{N-1} \left(|\alpha^{-1}(j)|!\right) \left(\sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (k-i)^n\right)$$

where the discrete parameters (N, α) *are (part of) the discrete parameters that identify the base twisted sector* Y*, as in Notation 4.43 .*

Proof. This is now simply an exercise in combinatorics. □

We now study the behaviour of the twisted sectors under the map that forgets rational bridges.

Remark 4.68. As we have already explained in Definition 2.37 and in Definition 2.39, the following diagram is not 2-cartesian:

$$\begin{tikzcd}
I(\overline{\mathcal{M}}^{R}_{g,n}) \arrow[r, two heads] \arrow[d] & \coprod I(\overline{\mathcal{M}}^{RB}_{g,k}) \arrow[d] \\
\overline{\mathcal{M}}^{R}_{g,n} \arrow[r, two heads, "\pi^{rt}"] & \coprod \overline{\mathcal{M}}^{RB}_{g,k}
\end{tikzcd}$$

Another bad behaviour of adding rational bridges with respect to automorphisms, is that it may happen that the blow up of a smoothable node can be non smoothable and vice versa (cf. Definition 2.53).

Let G be a stable graph with m edges without rational bridges. Let $J_1, \ldots, J_m$ be a partition of $[n]$. We choose T to be a set in bijection with the set of twisted sectors of $I(\mathcal{M}^{RB}_G)$ (smoothable 2.53 or not) that induce an automorphism on the graph G which is also an automorphism of $G(J_1, \ldots, J_m)$. Namely we choose only the twisted sectors whose corresponding automorphism β of the graph is in the image of the canonical inclusion $\mathrm{Aut}(G)(J_1, \ldots, J_m) \subset \mathrm{Aut}(G)$ (in other words β fixes all the edges e_i such that the corresponding J_i is not empty).

Lemma 4.69. *The Inertia Stack* $I(\overline{\mathcal{M}}_G(J_1, \ldots, J_m))$ *(see Definition 2.39, Proposition 2.43) is isomorphic to:*

$$\coprod_{t \in T} X_t \times \left(\coprod_{i \in D(\beta(t))} \overline{\mathcal{M}}_{0,J_i+2} \right) \left(\coprod_{i \in U(\beta(t))} I(\overline{\mathcal{M}}_{0,J_i+2}/S_2) \right)$$

where $i \in D(\beta(t))$ *iff the edge* $e(i)$ *is stabilized as a directed edge by* β*, while* $i \in U(\beta(t))$ *iff the edge* $e(i)$ *is stabilized by* β *as an undirected edge (cfr. Definition 2.38).*

Remark 4.70. Special cases of this are when $\mathrm{Aut}(G) = \mathrm{Aut}(G)(J_1, \ldots, J_m)$, and when $\mathrm{Aut}(G)$ is the trivial group; in the latter case in addition D coincides with the set of all edges while U is the empty set.

We can easily compute the twisted sectors of the Inertia Stack of $\left[\overline{\mathcal{M}}_{0,n+2}/S_2\right]$.

Lemma 4.71. *The twisted sectors of the Inertia Stack of* $\left[\overline{\mathcal{M}}_{0,n+2}/S_2\right]$ *are isomorphic to:*

$$B\mu_2 \times \coprod_{L_1 \sqcup L_2 = [n]} \overline{\mathcal{M}}_{0,L_1+1} \times \overline{\mathcal{M}}_{0,L_2+1}$$

Proof. On $\mathbb{P}^1$ with three markings in 0, 1, ∞ there are two points that are stabilized under the map that exchanges 0 and ∞:

$$z \to \frac{1}{z}$$

that are: 1 and -1. Therefore, the twisted sectors of the thesis are all the possible ways of distributing all the marked points in two sets and "clutch" them on the first and the second stabilized point. $\square$

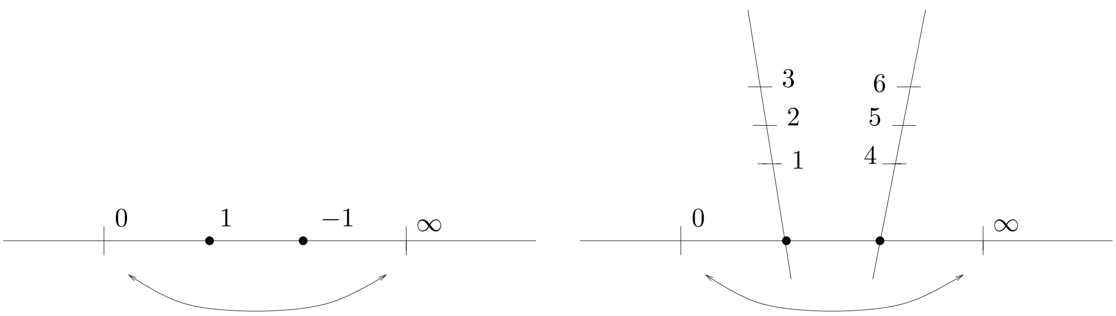


Figure 4.4: Adding marked points on a rational curve, preserving the automorphism that exchanges 0 and ∞.

We now show how we can construct inductively the twisted sectors of $\overline{\mathcal{M}}_{g,n}$ once the twisted sectors are known without marked points. We need to construct all the twisted sectors of $\overline{\mathcal{M}}^{RB}_{g,n}$, assuming knowledge of the twisted sectors of $\overline{\mathcal{M}}_g$. We perform the operation that we called at the beginning of this section "adding points on vertices".

Construction 4.72. Let X be a twisted sector of $\overline{\mathcal{M}}_G$, where G is a stable graph without marked points (see Definition 4.55). It is given by a couple $(G, \beta) \in \mathcal{A}_{g,n}$ (4.52), and an assignment for each vertex of admissible data $(g', N, d_1, \ldots, d_{N-1}, \alpha)$. If $B := \langle\beta\rangle$, this assignment is constant on the orbits of B when this group acts on the set of vertices. A bunch of marked points labelled by I may be added to a vertex v when v is stabilized by the action of B. The admissible data $(g', N, d_1, \ldots, d_{N-1}, \alpha)$, are then modified by extending α (a function on the set $H(v)$ of half-edges on v, according to Definition 2.33) to $\tilde{\alpha}$, where:

$$\tilde{\alpha} : H(v) \sqcup I \to \{0, \ldots, N-1\}, \ \tilde{\alpha}_{|H(v)|} = \alpha,$$

and $\tilde{\alpha}_{|I}$ has image contained in $\mathbb{Z}/N\mathbb{Z}^*$.

Proposition 4.73. *The twisted sectors of $\overline{\mathcal{M}}_{g,n}^{RB}$ are all obtained starting from stable graphs without marked points, and adding marked points on the twisted sectors where possible as explained in Construction 4.72.*

Theorem 4.74. *(Construction theorem for twisted sectors of $\overline{\mathcal{M}}_{g,n}$ once the twisted sectors of $\overline{\mathcal{M}}_g$ are given)*

1. *Take all the possible X twisted sectors of $\overline{\mathcal{M}}_G$ for all the G stable genus g graphs without marked points (see Definition 4.55).*

2. *Add marked points to the vertices, and construct all the twisted sectors X' of $\overline{\mathcal{M}}_{g,k}^{RB}$ for all $k \leq n$ as described in Proposition 4.73 and Construction 4.72.*

3. *Construct all the possible twisted sectors of $\overline{\mathcal{M}}_{g,k}^{R}$ out of all the possible twisted sectors of $\overline{\mathcal{M}}_{g,k}^{RB}$. For these, partition the subsets B of $[n] \setminus [k]$ in $J_1 \sqcup \ldots \sqcup J_m$ and construct all the twisted sectors of $\overline{\mathcal{M}}_G(J_1, \ldots, J_m)$ as in Lemma 4.69.*

4. *Take away from the list just constructed all the smoothable ones.*

5. *Lastly, add all the rational tails, constructing the twisted sectors of Definition 4.66.*

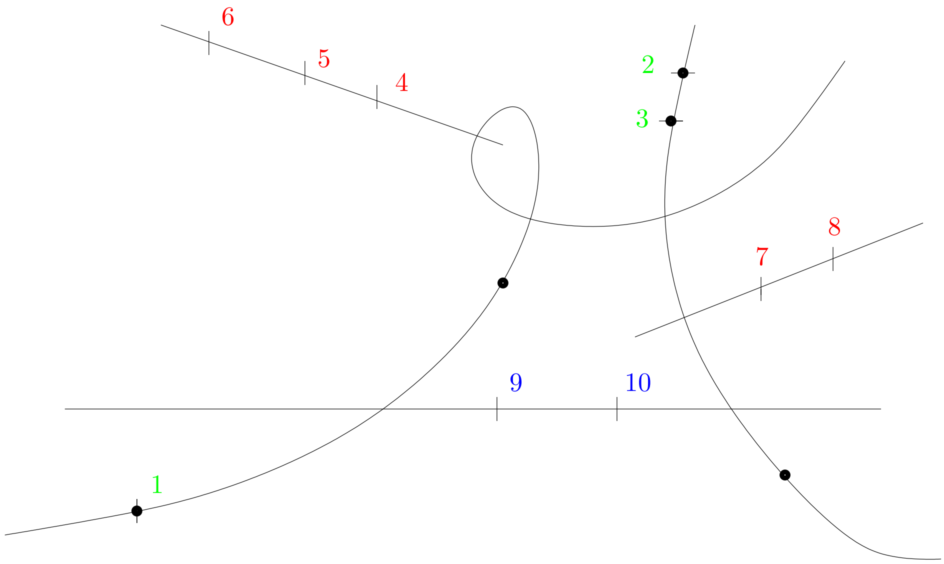


Figure 4.5: A curve C with an automorphism α where we added some marked points, on vertices, on nodes and on former marked points. The bullets are the points fixed by α.

Chapter 5

The Inertia Stack in Low Genus

In this chapter, we study explicitly the Inertia Stack for $\mathcal{M}_{1,n}$, $\mathcal{M}_{2,n}$, $\mathcal{M}_3$, $\overline{\mathcal{M}}_{1,n}$, $\overline{\mathcal{M}}_{2,n}$, and sketch some results on $\overline{\mathcal{M}}_3$. The construction of the Inertia Stacks uses the results of the previous chapter. In the case at hand, the combinatorics become easier than in the general picture described in the previous chapter. So we introduce a simpler notation. The results in genus 1 were studied in [P08], while the results in genus 2 without marked points were already studied in [S04]. Our notation agrees with these papers in the overlapping cases.

5.1 Genus 1 case

5.1.1 Moduli of smooth and rational tail genus 1 curves

This topic was studied in [P08, Theorem 3.17]. We give a naive geometrical description of the twisted sectors of $\mathcal{M}_{1,n}$ following [P08].
First of all, recall that every curve of the form:

$$\tilde{C} = \{[x : y : z]| \quad zy^2 = x^3 + az^2x + bz^3,\ \Delta := 4a^3 + 27b^2 \neq 0\} \subset \mathbb{P}^2$$

is a smooth genus 1 curve. If, instead,

$$\tilde{C} = \{[x : y : z]| \quad zy^2 = x^3 + az^2x + bz^3,\ \Delta := 4a^3 + 27b^2 = 0\} \subset \mathbb{P}^2$$

then $\tilde{C}$ is a nodal curve of arithmetic genus 1, geometric genus 0 and one node. We can describe all genus 1 curves with a marked point in this way.

Theorem 5.1. *(Weierstrass representation) Let (C, P) be a nodal elliptic curve. Then there exist $(a, b) \in \mathbb{C}^2$ such that (C, P) is isomorphic to $(C', [0 : 1 : 0])$, where*

$$C' := \{[x : y : z]| \quad zy^2 = x^3 + az^2x + bz^3\} \subset \mathbb{P}^2 \tag{5.2}$$

If β is an isomorphism of (C, P) with (D, Q) then up to the above isomorphism, β is of the form:

$$a \to \lambda^4 a$$
$$b \to \lambda^6 b$$
$$x \to \lambda^2 x$$
$$y \to \lambda^3 y$$

We call the marked point $[0:1:0]$ point at infinity of the elliptic curve. A simple analysis of the automorphisms tells us that all curves in $\overline{\mathcal{M}}_{1,1}$ are stabilized by the action of a group isomorphic to μ_2. There are two elements of $\overline{\mathcal{M}}_{1,1}$ that are stabilized by the action of a group respectively isomorphic to μ_4 and μ_6, we call them respectively C_4 and C_6. These are classes of curves whose Weierstrass representation can be chosen respectively as:

$$C_4 := \{[x:y:z] | \quad y^2z = x^3 + xz^2\} \subset \mathbb{P}^2$$

and

$$C_6 := \{[x:y:z] | \quad y^2z = x^3 + z^3\} \subset \mathbb{P}^2$$

We observe a simple corollary of this result:

Corollary 5.3. *For every elliptic curve (C, P), the automorphism group $\mathrm{Aut}(C, P)$ is canonically isomorphic to μ_2, μ_4 or μ_6.*

Proof. The action of $\mathrm{Aut}(C, P)$ is effective on $T_P^*(C)$, the cotangent space of C at P, which is canonically isomorphic to $\mathbb{C}$. This gives the canonical isomorphism. □

The twisted sectors in case $n = 1$ are well known as a direct consequence of the Weierstrass Theorem 5.1:

Proposition 5.4. *With the notation introduced in Notation 4.13, the decomposition of the Inertia Stack of $\mathcal{M}_{1,1}$ in twisted sectors is:*

$$I(\mathcal{M}_{1,1}) = (\mathcal{M}_{1,1}, 1) \coprod (\mathcal{M}_{1,1}, -1) \coprod (C_4, i/-i) \coprod (C_6, \epsilon/\epsilon^2/\epsilon^4/\epsilon^5)$$

Studying the fixed points of the action on the curves C_4 and C_6 of the two groups μ_4 and μ_6, by Theorem 5.1 we can obtain the following simple result:

Corollary 5.5. *Here we describe the points fixed by the action of $(i, -i)$ on C_4 and of $(\epsilon, \epsilon^2, \epsilon^4, \epsilon^5)$ on C_6*

- *i and $-i$ act on C_4 with $(0, 0)$ as the only fixed point different from infinity.*
- *ϵ, ϵ^5 act on C_6 with no fixed points different from infinity.*
- *ϵ^2, ϵ^4 act on C_6 with two fixed points: $(0, 1)$ and $(0, -1)$. The automorphisms i and $-i$ exchange the two C_6 curves with the two different possible marked points.*

Definition 5.6. We call C_4' the point in $\mathcal{M}_{1,2}$ stabilized by i or $-i$, C_6' the point in $\mathcal{M}_{1,2}$ stabilized by ϵ^2 or ϵ^4, and finally C_6'' the point in $\mathcal{M}_{1,3}$ stabilized by ϵ^2 or ϵ^4.

Thanks to Corollary 2.22, the automorphism groups of the fiber of the map

$$\pi_1 : \mathcal{M}_{1,n} \to \mathcal{M}_{1,1}$$

can only be subgroups of μ_2, μ_4 and μ_6.

We can now describe the twisted sectors. We compute the twisted sectors of $\mathcal{M}_{1,n}$ and $\overline{\mathcal{M}}_{1,n}$ thanks to the Weierstrass embedding theorem (recall that for $n > 4$ $\mathcal{M}_{1,n}$ is a scheme). In this lemma we compute the twisted sectors whose projection via π is an isolated point. We stick to the notation introduced in Definition 5.6.

Lemma 5.7. *Let $n > 1$. Here we describe all twisted sectors in $\mathcal{M}_{1,n}$ with automorphisms $\epsilon/\epsilon^2/\epsilon^4/\epsilon^5, i/-i$.*

- *there are no twisted sectors in $\mathcal{M}_{1,n}$ labelled by ϵ/ϵ^5,*
- *the Inertia Stack $I(\mathcal{M}_{1,2})$ contains the following twisted sectors:*

$$(C_4', i/-i)\coprod(C_6', \epsilon^2/\epsilon^4)$$

- *the Inertia Stack $I(\mathcal{M}_{1,3})$ contains the following twisted sectors:*

$$(C_6'', \epsilon^2/\epsilon^4)$$

- *$I(\mathcal{M}_{1,n})$ has no twisted sectors with automorphisms different from -1 if $n > 3$.*

Proof. This is a consequence of Corollary 5.5. $\square$

Remark 5.8. The following stack isomorphisms hold:

$$C_4' \cong B\mu_4$$

$$C_6' \cong C_6'' \cong B\mu_3$$

Now we describe the closed substacks of $\mathcal{M}_{1,n}$, which are twisted sectors with the automorphism $-1 \in \mu_2$.

Definition 5.9. Let $0 \leq i \leq 3$. A_i is the closed substack of $\mathcal{M}_{1,i}$ whose objects $A_i(S)$ are i-marked smooth genus 1 curves over S such that the sections are stabilized by the elliptic involution.

Remark 5.10. The stack A_1 is the whole $\mathcal{M}_{1,1}$. The forgetful map $\pi_{n+1} : \mathcal{M}_{1,n+1} \to \mathcal{M}_{1,n}$ restricts to $\pi_{n+1}|_{A_n} : A_n \to A_{n-1}$.

Lemma 5.11. *Each A_i is a global quotient by the action of $\mathbb{C}^*$ on an open subscheme of $\mathbb{A}^2_0$. The forgetful morphisms restricted to the spaces A_i are étale finite maps of the degree given in the following pictures:*

$$\begin{tikzcd}
 & & & & & A_2 \arrow[dllll, "\pi_{\{1,2\}}", "2:1"'] \arrow[d, "\pi_{\{1,3\}}"', "2:1"] \arrow[drrrr, "\pi_{\{2,3\}}", "2:1"'] & & & & & \\
 & A_1 \arrow[dl, "\pi_1"', "3:1"] \arrow[dr, "\pi_2", "3:1"'] & & & & A_1 \arrow[dl, "\pi_1"', "3:1"] \arrow[dr, "\pi_3", "3:1"'] & & & & A_1 \arrow[dl, "\pi_2"', "3:1"] \arrow[dr, "\pi_3", "3:1"'] & \\
A_0 & & A_0 & & A_0 & & A_0 & & A_0 & & A_0
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & A_3 \arrow[dlll, "\pi_1", "1:1"'] \arrow[dl, "\pi_2", "1:1"'] \arrow[dr, "\pi_3"', "1:1"] \arrow[drrr, "\pi_4"', "1:1"] & & & \\
A_2 & & A_2 & & A_2 & & A_2
\end{tikzcd}$$

Proof. The A_i's are smooth closed algebraic substacks of $\mathcal{M}_{1,i+1}$ since they are images of twisted sectors under the canonical map from the Inertia Stack to the original stack. The following isomorphism holds:

$$A_1 = [B_1/\mathbb{C}^*]$$

where $B_1 = \{(a,b)| \quad 4a^3 + 27b^2 \neq 0\}$ and $\mathbb{C}^*$ acts with weights 4 and 6 respectively. Moreover the following isomorphism holds:

$$A_2 = [B_2/\mathbb{C}^*]$$

where $B_2 = \{(x_1, a)| \quad b = -x_1^3 - ax_1, 4a^3 + 27b^2 \neq 0\}$ and $\mathbb{C}^*$ acts with weights 4 and 2 respectively. The pull-back $\tilde{\pi_1}$ of the forgetful map $A_2 \to A_1$ on the charts B_2 and B_1:

$$\begin{tikzcd} B_2 \arrow[r] \arrow[d] & A_2 \arrow[d, "\pi_1"] \\ B_1 \arrow[r] & A_1 \end{tikzcd}$$

is given by:

$$x_1 \to -x_1^3 - ax_1 \quad a \to a$$

This is a finite étale morphism of degree 3, therefore the forgetful map restricted to the stack A_2 is finite étale over A_1 as well. Finally the following isomorphism holds:

$$A_3 = [B_3/\mathbb{C}^*]$$

where $B_3 = \{(x_1, x_2)| \quad a = -(x_1^2 + x_2^2 + x_1x_2), b = x_1^2x_2 + x_2^2x_1, 4a^3 + 27b^2 \neq 0\}$ and $\mathbb{C}^*$ acts with weights 2 and 2 respectively. The pull-back of the forgetful map $A_3 \to A_2$ on the charts B_3 and B_2 is given by:

$$x_1 \to x_1 \quad x_2 \to -(x_1^2 + x^2 + x_1x_2)$$

This is a finite étale morphism of degree 2, therefore the forgetful map restricted to the stack A_3 is finite étale over A_2 as well. $\square$

Remark 5.12. The forgetful morphisms from A_4 to A_3 is an étale bijective morphism of smooth stacks, therefore it is an isomorphism. If we consider the morphisms induced over the coarse moduli spaces of the A_i's, they are ramified, as explained in the following pictures. In Figure 5.1 we present a picture of the ramification profile of A_2 over A_1. In Figure 5.2, we present a picture of the ramification profile of

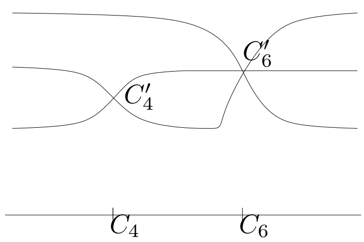


Figure 5.1: The upper curve is A_2, the forgetful morphism is represented as the projection onto the line down, A_1.

A_3 over A_2.

Note that the coarse space of A_4 is known in the literature as the full level-2 structure [Sc77].

Remark 5.13. The stack A_2 has generic stabilizer μ_2. It has a point with stabilizer μ_4: the unique one in the fiber of C_4. This is a point with stabilizer μ_4. A_3 has stabilizer μ_2 in all points.

We conclude this subsection with the final description of all twisted sectors of the Inertia Stack of $\mathcal{M}_{1,n}$.

Theorem 5.14. *We recollect all twisted sectors of $\mathcal{M}_{1,n}$ in the following table. Different rows correspond to different automorphisms, while the i-th column corresponds to the twisted sectors inside $\mathcal{M}_{1,i}$. Remember that $\mathcal{M}_{1,n}$ is a smooth scheme for $n > 4$.*

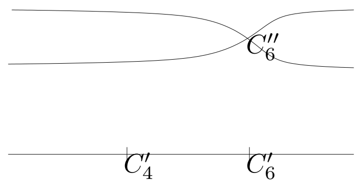


Figure 5.2: The upper curve is A_3, the forgetful morphism is represented as the projection onto the line down, A_2.

	1	2	3	4
-1	A_1	A_2	A_3	A_4
ϵ^2/ϵ^4	C_6	C_6'	C_6''	$\emptyset$
$i/-i$	C_4	C_4'	$\emptyset$	$\emptyset$
ϵ, ϵ^5	C_6	$\emptyset$	$\emptyset$	$\emptyset$

Proof. This is a consequence of Lemma 5.7 and of Definition 5.9. Earlier in this section we have given a geometric description of the twisted sectors that are not zero dimensional, namely the A_i's. □

Here we want to give a description of $I(\mathcal{M}_{1,n})$ according to the framework of [F09]. In the following table we give all the solutions of the equations 4.21 for $g = 1$ and $n = 1, \dots, 4$, and we recollect the results given above and in [P08] in the framework of [F09]. For $n \geq 5$, $\mathcal{M}_{1,n}$ is a scheme. Therefore its Inertia Stack coincides with the untwisted sector.

We start by presenting the twisted sectors of $\mathcal{M}_{1,1}$ in a table:

g'	N	$(d_1, \dots d_{N-1})$	$(\alpha(1))$	Description	Name in [P08]
0	2	(4)	1	μ_2-gerbe on $[\mathcal{M}_{0,4}/S_3]$	$(A_1, -1)$
0	3	$(3, 0)$	1	μ_3-gerbe on $[\mathcal{M}_{0,3}/S_2]$	(C_6, ϵ^4)
0	3	$(0, 3)$	2	μ_3-gerbe on $[\mathcal{M}_{0,3}/S_2]$	(C_6, ϵ^2)
0	4	$(2, 1, 0)$	1	μ_4-gerbe on $\mathcal{M}_{0,3}$	$(C_4, -i)$
0	4	$(0, 1, 2)$	3	μ_4-gerbe on $\mathcal{M}_{0,3}$	(C_4, i)
0	6	$(1, 1, 1, 0, 0)$	1	μ_6-gerbe on $\mathcal{M}_{0,3}$	(C_6, ϵ^5)
0	6	$(0, 0, 1, 1, 1)$	5	μ_6-gerbe on $\mathcal{M}_{0,3}$	(C_6, ϵ)

Here we describe the twisted sectors of $\mathcal{M}_{1,2}$:

g'	N	$(d_1, \dots d_{N-1})$	$(\alpha(1), \alpha(2))$	Description	Name in [P08]
0	2	(4)	$(1, 1)$	μ_2-gerbe on $[\mathcal{M}_{0,4}/S_2]$	$(A_2, -1)$
0	3	$(3, 0)$	$(1, 1)$	μ_3-gerbe on $\mathcal{M}_{0,3}$	(C_6', ϵ^4)
0	3	$(0, 3)$	$(2, 2)$	μ_3-gerbe on $\mathcal{M}_{0,3}$	(C_6', ϵ^2)
0	4	$(2, 1, 0)$	$(1, 1)$	μ_4-gerbe on $\mathcal{M}_{0,3}$	$(C_4', -i)$
0	4	$(0, 1, 2)$	$(3, 3)$	μ_4-gerbe on $\mathcal{M}_{0,3}$	(C_4', i)

And the twisted sectors of $\mathcal{M}_{1,3}$:

g'	N	$(d_1, \dots d_{N-1})$	$(\alpha(1), \alpha(2), \alpha(3))$	Description	Name in [P08]
0	2	(4)	$(1, 1, 1)$	μ_2-gerbe on $[\mathcal{M}_{0,4}/S_2]$	$(A_3, -1)$
0	3	$(3, 0)$	$(1, 1, 1)$	μ_3-gerbe on $\mathcal{M}_{0,3}$	(C_6'', ϵ^4)
0	3	$(0, 3)$	$(2, 2, 2)$	μ_3-gerbe on $\mathcal{M}_{0,3}$	(C_6'', ϵ^2)

And finally, the twisted sectors of $\mathcal{M}_{1,4}$:

g'	N	$(d_1,\dots d_{N-1})$	$(\alpha(1),\alpha(2),\alpha(3),\alpha(4))$	Description	Name in [P08]
0	2	(4)	$(1,1,1,1)$	μ_2-gerbe on $\mathcal{M}_{0,4}$	$(A_4,-1)$

The paper [P08] does not deal explicitly with the Inertia Stack of $\mathcal{M}^{rt}_{1,n} = \mathcal{M}^{ct}_{1,n}$. The main reason is that, as will become clear with Proposition 5.19, there is no difference in the combinatorics of the twisted sectors of $\mathcal{M}^{rt}_{1,n}$ and $\overline{\mathcal{M}}_{1,n}$.

We use the result of Corollary 4.64, together with the notation of Definition 4.66 to represent the twisted sectors.

Remark 5.15. In the genus 1 case a simplification occurs. Indeed, as one can see by observing the tables after Theorem 5.14, in genus 1 it always happens that:

$$\forall i,j \ \alpha(i)=\alpha(j)$$

Therefore, following the notation introduced in Definition 4.66, there is only one set of family of sets K_1 in this case. Hence we can forget about the set K_1 and introduce the simpler notation for all the twisted sectors:

$$Y^{\{I_1\dots I_k\}} := Y \times \overline{\mathcal{M}}_{0,I_1+1} \times \dots \times \overline{\mathcal{M}}_{0,I_k+1}$$

where Y is a base twisted sector of $\mathcal{M}_{g,n}$. Observe also that in this case k can be at most 4, because of Theorem 5.14.

Notation 5.16. Let $\sigma \in S_k$. Then $\overline{Z}^{(I_1,\dots,I_k)} = Z^{(I_{\sigma(1)},\dots,\sigma(k))}$. The twisted sector is identified up to isomorphism by Z and the partition $\{I_1,\dots I_k\}$ where the ordering of the $I_i's$ does not matter. From now on we will simply denote this twisted sector in $\overline{\mathcal{M}}_{1,n}$ as $\overline{Z}^{\{I_1,\dots,I_k\}}$: the elements of the set of parameters for the twisted sectors whose base space is Z is given by the set of the k partitions of $[n]$. To simplify notation, we will usually write $\overline{Z}^{I_1,\dots,I_k}$ to mean $\overline{Z}^{\{I_1,\dots,I_k\}}$. If $I_1=\{1\},\dots,I_k=\{k\}$, then $\overline{Z}$ is isomorphic to $\overline{Z}^{\{I_1,\dots,I_k\}}$ for every Z twisted sector (see Definition 2.35).

At this point it is convenient to use the names introduced in Theorem 5.14 for the base twisted sectors (they are clearer than the ones defined in Notation 4.43). We are now ready to state the following result, which is a consequence of Corollary 4.63 and Corollary 4.64:

Theorem 5.17. *The decomposition of $I(\mathcal{M}^{rt}_{1,n})$ in twisted sectors is:*

$$(\mathcal{M}^{rt}_{1,n}) \coprod \left(A_1^{[n]},-1\right) \coprod_{\{I_1,I_2\},\ I_1\sqcup I_2=[n]} \left({A_2}^{I_1,I_2},-1\right) \coprod_{\{I_1,I_2,I_3\},\ I_1\sqcup I_2\sqcup I_3=[n]} \left({A_3}^{I_1,I_2,I_3},-1\right)$$

$$\coprod_{\{I_1,I_2,I_3,I_4\},\ I_1\sqcup I_2\sqcup I_3\sqcup I_4=[n]} \left({A_4}^{I_1,I_2,I_3,I_4},-1\right)$$

$$\coprod (C_4^{[n]}, i/-i) \coprod_{\{I_1,I_2\},\ I_1\sqcup I_2=[n]} \left(C_4^{I_1,I_2}, i/-i\right)$$

$$\coprod_{\{I_1,I_2\},\ I_1\sqcup I_2=[n]} \left(C_6^{I_1,I_2}, \epsilon^2/\epsilon^4\right) \coprod_{\{I_1,I_2,I_3\},\ I_1\sqcup I_2\sqcup I_3=[n]} \left(C_6^{I_1,I_2,I_3}, \epsilon^2/\epsilon^4\right) \coprod (C_6^{[n]}, \epsilon/\epsilon^2/\epsilon^4/\epsilon^5)$$

where all the sets I_i are non empty.

5.1.2 Moduli of stable genus 1 curves

In this section, we study the Inertia Stack of $I(\overline{\mathcal{M}}_{1,n})$. There is an enormous simplification appearing in genus 1 that we shall see immediately:

Proposition 5.18. *The map $I(X) \to X$, (see Definition 4.1) is a closed embedding when restricted to each connected component when $X = \mathcal{M}_{1,n},\ \mathcal{M}_{1,n}^{rt},\ \overline{\mathcal{M}}_{1,n}$*

Proof. This follows from Proposition 4.4 and the fact that all the objects of these stacks have abelian stabilizers. □

So the compactification of thw twisted sectors may be described in a much simpler way than Definition 4.47.

Corollary 5.19. *The twisted sectors of $\overline{\mathcal{M}}_{1,n}$ are the compactifications of the twisted sectors of $\mathcal{M}_{1,n}^{rt}$ inside $\overline{\mathcal{M}}_{1,n}$.*

Proof. The claim is meaningful thanks to the above proposition. In genus 1, the curves not of rational type are curves C contained in the divisor D_{irr}, whose general element is a n-marked curve of geometric genus 0 with one node. Smooth genus 0 curves are rigid, therefore automorphisms of C are in a bijection with automorphisms of its dual graph. These automorphisms are all smoothable (4.51). □

Therefore, all that we need to deduce a formula for the twisted sectors of $\overline{\mathcal{M}}_{1,n}$ from the ones of $\mathcal{M}_{1,n}^{rt}$ is to compactify the base twisted sectors. The base twisted sectors of dimension 0 need no compactification. We have only to compactify the ones of dimension 1, namely the A_i's.

Definition 5.20. We define $\overline{A_i}$ as the closure of the respective spaces A_i, in $\overline{\mathcal{M}}_{1,i}$.

Remark 5.21. The stack $\overline{A_1}$ is the whole $\overline{\mathcal{M}}_{1,1}$.

We now study the geometry of the compactifications just introduced, as well as the forgetful maps between them.

Lemma 5.22. *In the following cartesian diagram of stacks:*

$$\begin{tikzcd}
A_3 \arrow[r, hook] \arrow[d, "2:1"'] & \overline{A_3} \cong \mathbb{P}(2,2) \arrow[d, "\pi_{\{1,2\}}"] \\
A_2 \arrow[r, hook] \arrow[d, "3:1"'] & \overline{A_2} \cong \mathbb{P}(2,4) \arrow[d, "\pi_1"] \\
\mathcal{M}_{1,1} \arrow[r, hook] & \overline{\mathcal{M}}_{1,1} \cong \mathbb{P}(4,6)
\end{tikzcd}$$

the finite étale morphisms on the left extend to ramified finite morphisms on the compactifications.

Proof. We can find charts for the spaces as we did in the smooth case. We use a similar notation as in Lemma 5.11. A smooth chart for $\overline{A_1}$=$\overline{\mathcal{M}}_{1,1}$ is $\mathbb{A}^2_0$, where we use coordinates (a, b). A chart for $\overline{A_2}$ is:

$$\overline{B}_2 := \{(a, x_1) | \quad (a, x_1) \neq (0, 0)\} = \mathbb{A}^2_0$$

The action of $\mathbb{C}^*$ has weights 4 and 2 once again. Subsequently this, the proof of the isomorphism $\overline{A_2}$≅ $\mathbb{P}(2, 4)$ is identical to the proof of the isomorphism of $\overline{\mathcal{M}}_{1,1} \cong \mathbb{P}(4, 6)$. The pull-back of the forgetful morphism to the charts sends $(a, x_1) \to (a, -x_1^3 - ax_1)$. It is ramified on the locus $\{4a^3 + 27b^2 = 0\}$. These (a, b)'s are the points corresponding to the nodal curve. The ramification profile on the fiber is $(2, 1)$.

Note that the fiber over the nodal curve in $\overline{\mathcal{M}}_{1,1}$ is made of the two points:

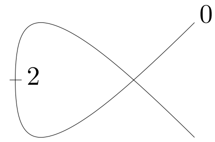


Figure 5.3: Nodal curve with smooth 2-torsion marked point.

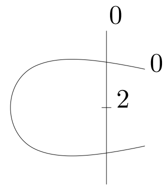


Figure 5.4: Nodal curve with 2-torsion marked point "on the node"

The forgetful morphism restricted to A_2 is ramified on the second point, and unramified on the first.

The vertical rational component in the picture has four special points. If we fix coordinates on the rational component, i.e. the two nodes have coordinates 0 and ∞, and the point marked with 2 has coordinate 1, then the third marked point must have coordinate equal to -1. Indeed, this point is obtained as the limit of marked curves that are stabilized by the elliptic involution. In the stable limit, the elliptic involution exchanges the two nodes, and it must fix the remaining two points.
The atlas is therefore again:

$$\overline{B}_3 := \{(x_1, x_2) | \quad (x_1, x_2) \neq (0,0)\} = \mathbb{A}^2_0$$

The action has weights 2 and 2 respectively.
The pull-backs of the forgetful morphisms restricted to $\overline{A_4}$ to the charts are given by the equations:

$$(x_1, x_2) \to (-x_1^2 - x_2^2 - x_1x_2, x_1)$$

$$(x_1, x_2) \to (-x_1^2 - x_2^2 - x_1x_2, x_1^2x_2 + x_1x_2^2)$$

The first morphism is ramified onto the points of the atlas $\overline{B}_2$ corresponding to the curve in $\overline{\mathcal{M}}_{1,2}$:

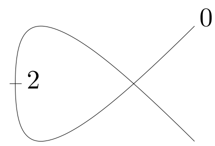


Figure 5.5: Nodal curve with smooth 2-torsion marked point

with ramification 2. The fiber is given in Figure 5.6.

□

Corollary 5.23. *The decomposition of $I(\overline{\mathcal{M}}_{1,n})$ in twisted sectors is the same as in Theorem 5.17, after substituting each A_i with its compactification $\overline{A_i}$.*

Proof. The stacks A_i are the only twisted sectors of positive dimension. The result than follows from Corollary 5.19 □

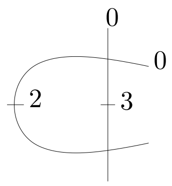


Figure 5.6: Nodal curve with smooth 2-torsion marked point and a marked point "on the node"

5.1.3 Genus 1 quotients

This section is needed to complete the program explained in Proposition 4.60. The construction for the twisted sectors of $\overline{\mathcal{M}}_{g,n}$ given there uses the result of Proposition 4.40. In this section we produce the list of all the twisted sectors of $\left[\mathcal{M}_{1,n}/S\right]$ (cfr. 4.35 and 4.60), which may appear in non smoothable (2.53), twisted sectors obtained following 4.60. These twisted sectors are the relevant ones for the subsequent chapters. If $S = \sigma_1 \ldots \sigma_k$ is a subgroup of S_n generated by products of cyclic disjoint permutations σ_i, the value of $\alpha(i)$ corresponds to the cyclic permutation σ_i (see Section 3.2.2).

The relevant twisted sectors of $\left[\mathcal{M}_{1,2}/(12)\right]$:

g'	N	$(d_1, \ldots d_{N-1})$	$(\alpha(1))$	Description	Name proposed
0	4	$(1,1,2)$	(2)	μ_4-gerbe on $\mathcal{M}_{0,3}$	$(\tilde{C}_4, -i)$
0	4	$(2,3,3)$	(2)	μ_4-gerbe on $\mathcal{M}_{0,3}$	$(\tilde{C}_4, i)$
0	6	$(1,2,3)$	(3)	μ_6-gerbe on $\mathcal{M}_{0,3}$	$(\tilde{C}_6, \epsilon^5)$
0	6	$(3,4,5)$	(3)	μ_6-gerbe on $\mathcal{M}_{0,3}$	$(\tilde{C}_6, \epsilon)$

Similarly, for the twisted sectors of $\left[\mathcal{M}_{1,3}/(12)\right]$:

g'	N	$(d_1, \ldots d_{N-1})$	$(\alpha(1))$	Description	Name proposed
0	4	$(1,1,2)$	$(2,1)$	μ_4-gerbe on $\mathcal{M}_{0,3}$	$(\tilde{C}'_4, -i)$
0	4	$(2,3,3)$	$(2,3)$	μ_4-gerbe on $\mathcal{M}_{0,3}$	$(\tilde{C}'_4, i)$
0	6	$(1,2,3)$	$(3,1)$	μ_6-gerbe on $\mathcal{M}_{0,3}$	$(\tilde{C}'_6, \epsilon^5)$
0	6	$(3,4,5)$	$(3,5)$	μ_6-gerbe on $\mathcal{M}_{0,3}$	$(\tilde{C}'_6, \epsilon)$

For $\left[\mathcal{M}_{1,4}/(12)\right]$:

g'	N	$(d_1, \ldots d_{N-1})$	$(\alpha(1), \alpha(2), \alpha(3))$	Description	Name proposed
0	4	$(1,1,2)$	$(2,1,1)$	μ_4-gerbe on $\mathcal{M}_{0,3}$	$(\tilde{C}''_4, -i)$
0	4	$(2,3,3)$	$(2,3,3)$	μ_4-gerbe on $\mathcal{M}_{0,3}$	$(\tilde{C}''_4, i)$

For $\left[\mathcal{M}_{1,3}/(123)\right]$:

g'	N	$(d_1, \ldots d_{N-1})$	$(\alpha(1))$	Description	Name proposed
0	6	$(1,2,3)$	(2)	μ_6-gerbe on $\mathcal{M}_{0,3}$	$(\tilde{C}'''_6, \epsilon^5)$
0	6	$(3,4,5)$	(4)	μ_6-gerbe on $\mathcal{M}_{0,3}$	$(\tilde{C}'''_6, \epsilon)$

For $[\mathcal{M}_{1,4}/(123)]$:

g'	N	$(d_1,\dots d_{N-1})$	$(\alpha(1))$	Description	Name proposed
0	6	$(1,2,3)$	$(2,1)$	μ_6-gerbe on $\mathcal{M}_{0,3}$	$(\tilde{C_6}^{(4)},\epsilon^5)$
0	6	$(3,4,5)$	$(4,5)$	μ_6-gerbe on $\mathcal{M}_{0,3}$	$(\tilde{C_6}^{(4)},\epsilon)$

For $[\mathcal{M}_{1,5}/(123),(45)]$:

g'	N	$(d_1,\dots d_{N-1})$	$(\alpha(1))$	Description	Name proposed
0	6	$(1,2,3)$	$(2,3)$	μ_6-gerbe on $\mathcal{M}_{0,3}$	$(\tilde{C_6}^{(5)},\epsilon^5)$
0	6	$(3,4,5)$	$(4,3)$	μ_6-gerbe on $\mathcal{M}_{0,3}$	$(\tilde{C_6}^{(5)},\epsilon)$

And finally for $[\mathcal{M}_{1,6}/(123),(45)]$:

g'	N	$(d_1,\dots d_{N-1})$	$(\alpha(1))$	Description	Name proposed
0	6	$(1,2,3)$	$(2,3,1)$	μ_6-gerbe on $\mathcal{M}_{0,3}$	$(\tilde{C_6}^{(6)},\epsilon^5)$
0	6	$(3,4,5)$	$(4,3,1)$	μ_6-gerbe on $\mathcal{M}_{0,3}$	$(\tilde{C_6}^{(6)},\epsilon)$

5.2 Genus 2 case

The problem of determining the Chen–Ruan cohomology (or the stringy Chow ring) of $\mathcal{M}_2$ and $\overline{\mathcal{M}}_2$ was studied by J. Spencer in his PhD thesis [S04]. We will follow his approach for smooth curves, simply correcting some minor mistakes and comparing our notation with his. Notice that in his thesis [S04], he deals with the case of cohomologies with integral coefficients for the case of $\mathcal{M}_2$. We will not discuss such a case, sticking instead to the simpler situation of rational coefficients. In this section we address and solve completely the problem of determining the list of all the twisted sectors of rational type (cfr 4.42). The problem of determining explicitly the number of twisted sectors for $\overline{\mathcal{M}}_{2,n}$ is then solved using 4.64, 4.66, 4.67.

5.2.1 Moduli of smooth and rational tail genus 2 curves

The relevant cases to be studied are $\mathcal{M}_2$, $\mathcal{M}_{2,1}$, $\mathcal{M}_{2,2}$, $\mathcal{M}_{2,3}$, $\mathcal{M}_{2,4}$, $\mathcal{M}_{2,5}$ and $\mathcal{M}_{2,6}$, since a smooth genus 2 curve admits at most 6 points fixed by an automorphism.

In the case with zero marked points, one can obtain the same results that we present, by studying the loci fixed by automorphisms in the hyperelliptic representation. This, which is completely analogous to what is carried out in [P08] for genus 1, was the approach followed by [S04].

Another way to study the Inertia Stack without marked points is using the description of $\mathcal{M}_2$ as a μ_2 gerbe on $[\mathcal{M}_{0,6}/S_6]$. This is a special case of the general problem of studying the Chen–Ruan cohomology of an abelian gerbe over an abelian DM stack X, assuming knowledge of the Chen–Ruan cohomology of the stack X. Now we follow the description of [F09], comparing the twisted sectors with the ones found by Spencer in [S04], and giving new names (in his spirit) to the ones missing in his description.

We solved the Equations 4.21 with $g = 2$ and $g' = 0, 1$ taking advantage of the C++ program [MP1]. Here we present the table with the twisted sectors of $\mathcal{M}_2, \mathcal{M}_{2,1}, \mathcal{M}_{2,2}, \mathcal{M}_{2,3}, \mathcal{M}_{2,4}$:

g'	N	$(d_1,\dots d_{N-1})$	Description	Name in [S04]
0	2	(6)	μ_2-gerbe on $[\mathcal{M}_{0,6}/S_6]$	(τ)
0	3	$(2,2)$	μ_3-gerbe on $[\mathcal{M}_{0,4}/(S_2\times S_2)]$	(III)
0	4	$(1,2,1)$	μ_4-gerbe on $[\mathcal{M}_{0,4}/S_2]$	(IV)
0	5	$(2,0,1,0)$	μ_5-gerbe on $[\mathcal{M}_{0,3}/S_2]$	$(X.4)$
0	5	$(0,1,0,2)$	μ_5-gerbe on $[\mathcal{M}_{0,3}/S_2]$	$(X.6)$
0	5	$(1,2,0,0)$	μ_5-gerbe on $[\mathcal{M}_{0,3}/S_2]$	$(X.2)$
0	5	$(0,0,2,1)$	μ_5-gerbe on $[\mathcal{M}_{0,3}/S_2]$	$(X.8)$
0	6	$(2,0,0,1,0)$	μ_6-gerbe on $[\mathcal{M}_{0,3}/S_2]$	$(V.1)$
0	6	$(0,1,0,0,2)$	μ_6-gerbe on $[\mathcal{M}_{0,3}/S_2]$	$(V.2)$
0	6	$(0,1,2,1,0)$	μ_6-gerbe on $[\mathcal{M}_{0,4}/S_2]$	(VI)
0	8	$(1,0,1,1,0,0,0)$	μ_8-gerbe on $\mathcal{M}_{0,3}$	$(VIII.1)$
0	8	$(0,0,0,1,1,0,1)$	μ_8-gerbe on $\mathcal{M}_{0,3}$	$(VIII.2)$
0	10	$(0,1,1,0,1,0,0,0,0)$	μ_{10}-gerbe on $\mathcal{M}_{0,3}$	$(X.7)$
0	10	$(0,0,0,0,1,0,1,1,0)$	μ_{10}-gerbe on $\mathcal{M}_{0,3}$	$(X.3)$
0	10	$(1,0,0,1,1,0,0,0,0)$	μ_{10}-gerbe on $\mathcal{M}_{0,3}$	$(X.1)$
0	10	$(0,0,0,0,1,1,0,0,1)$	μ_{10}-gerbe on $\mathcal{M}_{0,3}$	$(X.9)$
1	2	(2)	μ_2-gerbe on a $4:1$ covering on $[\mathcal{M}_{1,2}/S_2]$	(II)

g'	N	$(d_1,\dots d_{N-1})$	$(\alpha(1))$	Description	Name proposed
0	2	(6)	(1)	μ_2-gerbe on $[\mathcal{M}_{0,6}/S_5]$	τ_1
0	3	$(2,2)$	(1)	μ_3-gerbe on $[\mathcal{M}_{0,4}/(S_2]$	III_1
0	3	$(2,2)$	(2)	μ_3-gerbe on $[\mathcal{M}_{0,4}/(S_2]$	III_2
0	4	$(1,2,1)$	(1)	μ_4-gerbe on $[\mathcal{M}_{0,4}/S_2]$	IV_1
0	4	$(1,2,1)$	(3)	μ_4-gerbe on $[\mathcal{M}_{0,4}/S_2]$	IV_3
0	5	$(2,0,1,0)$	(1)	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.4_1$
0	5	$(2,0,1,0)$	(3)	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.4_3$
0	5	$(0,1,0,2)$	(4)	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.6_4$
0	5	$(0,1,0,2)$	(2)	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.6_2$
0	5	$(1,2,0,0)$	(1)	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.2_1$
0	5	$(1,2,0,0)$	(2)	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.2_2$
0	5	$(0,0,2,1)$	(4)	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.8_4$
0	5	$(0,0,2,1)$	(3)	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.8_3$
0	6	$(2,0,0,1,0)$	(1)	μ_6-gerbe on $\mathcal{M}_{0,3}$	$V.1_1$
0	6	$(0,1,0,0,2)$	(5)	μ_6-gerbe on $\mathcal{M}_{0,3}$	$V.2_5$
0	8	$(1,0,1,1,0,0,0)$	(1)	μ_8-gerbe on $\mathcal{M}_{0,3}$	$VIII_1$
0	8	$(1,0,1,1,0,0,0)$	(3)	μ_8-gerbe on $\mathcal{M}_{0,3}$	$VIII_3$
0	8	$(0,0,0,1,1,0,1)$	(7)	μ_8-gerbe on $\mathcal{M}_{0,3}$	$VIII_7$
0	8	$(0,0,0,1,1,0,1)$	(5)	μ_8-gerbe on $\mathcal{M}_{0,3}$	$VIII_5$
0	10	$(0,1,1,0,1,0,0,0,0)$	(3)	μ_{10}-gerbe on $\mathcal{M}_{0,3}$	$X.7$
0	10	$(0,0,0,0,1,0,1,1,0)$	(7)	μ_{10}-gerbe on $\mathcal{M}_{0,3}$	$X.3$
0	10	$(1,0,0,1,1,0,0,0,0)$	(1)	μ_{10}-gerbe on $\mathcal{M}_{0,3}$	$X.1$
0	10	$(0,0,0,0,1,1,0,0,1)$	(5)	μ_{10}-gerbe on $\mathcal{M}_{0,3}$	$X.9$
1	2	(2)	(1)	μ_2-gerbe on $4:1$ covering on $\mathcal{M}_{1,2}$	II_1

g'	N	$(d_1,\dots d_{N-1})$	$(\alpha(1),\alpha(2))$	Description	Name proposed
0	2	(6)	$(1,1)$	μ_2-gerbe on $[\mathcal{M}_{0,6}/S_4]$	τ_{11}
0	3	$(2,2)$	$(1,1)$	μ_3-gerbe on $[\mathcal{M}_{0,4}/(S_2]$	III_{11}
0	3	$(2,2)$	$(2,2)$	μ_3-gerbe on $[\mathcal{M}_{0,4}/(S_2]$	III_{22}
0	3	$(2,2)$	$(1,2)$	μ_3-gerbe on $\mathcal{M}_{0,4}$	III_{12}
0	3	$(2,2)$	$(2,1)$	μ_3-gerbe on $\mathcal{M}_{0,4}$	III_{21}
0	4	$(1,2,1)$	$(1,3)$	μ_4-gerbe on $[\mathcal{M}_{0,4}/S_2]$	IV_{13}
0	4	$(1,2,1)$	$(3,1)$	μ_4-gerbe on $[\mathcal{M}_{0,4}/S_2]$	IV_{31}
0	5	$(2,0,1,0)$	$(1,3)$	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.4_{13}$
0	5	$(2,0,1,0)$	$(3,1)$	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.4_{31}$
0	5	$(0,1,0,2)$	$(4,2)$	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.6_{42}$
0	5	$(0,1,0,2)$	$(2,4)$	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.6_{24}$
0	5	$(1,2,0,0)$	$(1,2)$	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.2_{12}$
0	5	$(1,2,0,0)$	$(2,1)$	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.2_{21}$
0	5	$(0,0,2,1)$	$(4,3)$	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.8_{43}$
0	5	$(0,0,2,1)$	$(3,4)$	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.8_{34}$
0	5	$(2,0,1,0)$	$(1,1)$	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.4_{11}$
0	5	$(0,1,0,2)$	$(4,4)$	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.6_{44}$
0	5	$(1,2,0,0)$	$(2,2)$	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.2_{22}$
0	5	$(0,0,2,1)$	$(3,3)$	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.8_{33}$
0	6	$(2,0,0,1,0)$	$(1,1)$	μ_6-gerbe on $\mathcal{M}_{0,3}$	$V.1_{11}$
0	6	$(0,1,0,0,2)$	$(5,5)$	μ_6-gerbe on $\mathcal{M}_{0,3}$	$V.2_{55}$
0	8	$(1,0,1,1,0,0,0)$	$(1,3)$	μ_8-gerbe on $\mathcal{M}_{0,3}$	$VIII_{13}$
0	8	$(1,0,1,1,0,0,0)$	$(3,1)$	μ_8-gerbe on $\mathcal{M}_{0,3}$	$VIII_{31}$
0	8	$(0,0,0,1,1,0,1)$	$(7,5)$	μ_8-gerbe on $\mathcal{M}_{0,3}$	$VIII_{75}$
0	8	$(0,0,0,1,1,0,1)$	$(5,7)$	μ_8-gerbe on $\mathcal{M}_{0,3}$	$VIII_{57}$
1	2	(2)	$(1,1)$	μ_2-gerbe on $4:1$ covering on $\mathcal{M}_{1,2}$	II_{11}

g'	N	$(d_1, \ldots d_{N-1})$	$(\alpha(1), \alpha 2, \alpha(3))$	Description	Name proposed
0	2	(6)	$(1,1,1)$	μ_2-gerbe on $[\mathcal{M}_{0,6}/S_3]$	τ_{111}
0	3	$(2,2)$	$(1,1,2)$	μ_3-gerbe on $\mathcal{M}_{0,4}$	III_{112}
0	3	$(2,2)$	$(2,2,1)$	μ_3-gerbe on $\mathcal{M}_{0,4}$	III_{221}
0	3	$(2,2)$	$(1,2,2)$	μ_3-gerbe on $\mathcal{M}_{0,4}$	III_{122}
0	3	$(2,2)$	$(2,1,1)$	μ_3-gerbe on $\mathcal{M}_{0,4}$	III_{211}
0	3	$(2,2)$	$(1,2,1)$	μ_3-gerbe on $\mathcal{M}_{0,4}$	III_{121}
0	3	$(2,2)$	$(2,1,2)$	μ_3-gerbe on $\mathcal{M}_{0,4}$	III_{212}
0	4	$(1,2,1)$	$(1,3)$	μ_4-gerbe on $[\mathcal{M}_{0,4}/S_2]$	IV_{13}
0	4	$(1,2,1)$	$(3,1)$	μ_4-gerbe on $[\mathcal{M}_{0,4}/S_2]$	IV_{31}
0	5	$(2,0,1,0)$	$(1,3,1)$	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.4_{131}$
0	5	$(2,0,1,0)$	$(3,1,1)$	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.4_{311}$
0	5	$(0,1,0,2)$	$(4,2,4)$	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.6_{424}$
0	5	$(0,1,0,2)$	$(2,4,4)$	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.6_{244}$
0	5	$(1,2,0,0)$	$(1,2,2)$	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.2_{122}$
0	5	$(1,2,0,0)$	$(2,1,2)$	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.2_{212}$
0	5	$(0,0,2,1)$	$(4,3,3)$	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.8_{433}$
0	5	$(0,0,2,1)$	$(3,4,3)$	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.8_{343}$
0	5	$(2,0,1,0)$	$(1,1,3)$	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.4_{113}$
0	5	$(0,1,0,2)$	$(4,4,2)$	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.6_{442}$
0	5	$(1,2,0,0)$	$(2,2,1)$	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.2_{221}$
0	5	$(0,0,2,1)$	$(3,3,4)$	μ_5-gerbe on $\mathcal{M}_{0,3}$	$X.8_{334}$

g'	N	$(d_1, \ldots d_{N-1})$	$(\alpha(1), \alpha 2, \alpha(3), \alpha(4))$	Description	Name proposed
0	2	(6)	$(1,1,1,1)$	μ_2-gerbe on $[\mathcal{M}_{0,6}/S_2]$	τ_{1111}
0	3	$(2,2)$	$(1,1,2,2)$	μ_3-gerbe on $\mathcal{M}_{0,4}$	III_{1122}
0	3	$(2,2)$	$(2,2,1,1)$	μ_3-gerbe on $\mathcal{M}_{0,4}$	III_{2211}
0	3	$(2,2)$	$(1,2,2,1)$	μ_3-gerbe on $\mathcal{M}_{0,4}$	III_{1221}
0	3	$(2,2)$	$(2,1,1,2)$	μ_3-gerbe on $\mathcal{M}_{0,4}$	III_{2112}
0	3	$(2,2)$	$(1,2,1,2)$	μ_3-gerbe on $\mathcal{M}_{0,4}$	III_{1212}
0	3	$(2,2)$	$(2,1,2,1)$	μ_3-gerbe on $\mathcal{M}_{0,4}$	III_{2121}

The cases of $\mathcal{M}_{2,5}$ and $\mathcal{M}_{2,6}$ are easier to describe, since there is only one twisted sector which we call τ_{11111} and τ_{111111} respectively. The first one is made of a $2:1$ covering of genus 0 curves with 6 branch points, and in this case five among the six ramification points are marked. In contrast, in τ_{111111} all the six ramification points are marked.

As for the genus 1 case, the machinery of Corollary 4.64 and the notation introduced in Definition 4.66, allow us to give an explicit description of the twisted sectors of $\mathcal{M}_{2,n}^{rt}$.

5.2.2 Moduli of stable genus 2 curves of compact type

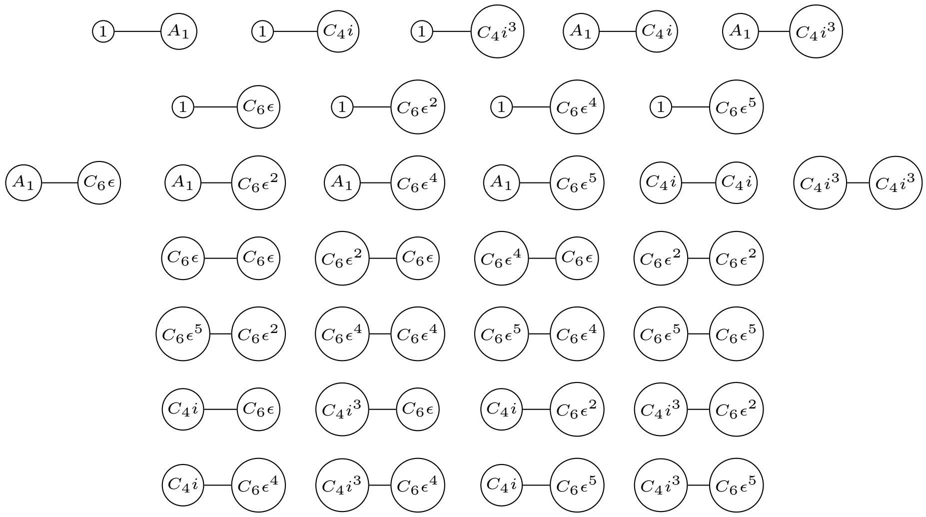
The curves of compact type but not rational tail of genus 2, without marked points, are those represented by the dual graph (Definition 2.33, Construction 2.34):

$$\textcircled{1} \text{——} \textcircled{1} \tag{5.24}$$

We can give here an example on how to use the construction of the twisted sectors described in 4.60. Note that since there is only one node, any automorphism of a compact type curve of genus 2 must fix it. Hence the smoothability condition (2.53, 4.58) reduces to the condition that the node be balanced (Example 2.54).

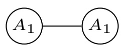
We call ρ the non trivial automorphism of the graph: it is an element of order two exchanging the two vertices of the graph.

Construction 5.25. (The twisted sectors of $\mathcal{M}_2^{ct}$ lying in the boundary) Our Construction 4.60 leads to the following twisted sectors associated with the graph 5.24, with the identity automorphism:

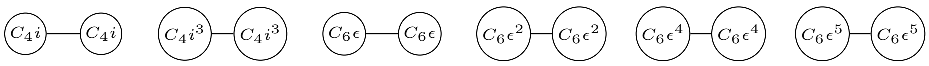


We have eliminated from the list the smoothable sectors, namely those such that the action of the automorphism is balanced on the node (Example 2.54).

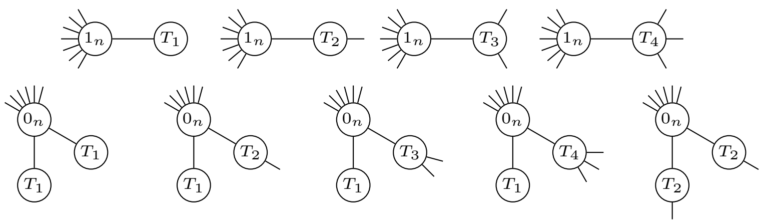
Associated with the same graph but with the automorphism element of it being ρ there is a 1 dimensional twisted sector (now the orbit of the group generated by ρ on the set of vertices is only one):

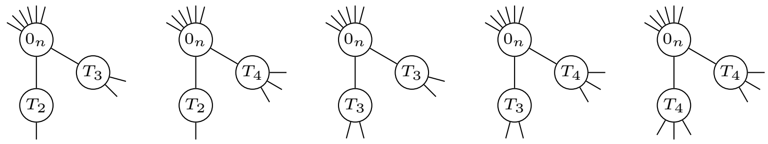


where the two elliptic curves are isomorphic. There are then several zero dimensional sectors:



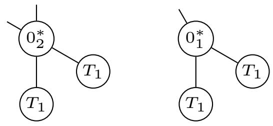
Construction 5.26. (Twisted sector in genus 2 with marked points of compact type) In the following, we list all the twisted sectors of the Inertia Stack of $\overline{\mathcal{M}}_{2,n}^R$ whose graph corresponds to 5.24. First we fix the automorphism of the graph as the identity (here T_i is to be chosen among all the compactified base twisted sectors of $\mathcal{M}_{1,i}$):





(Note that all these sectors are not smoothable (2.53, 4.58) if the number of marked points n on the genus 0 component is greater than 0. When $n = 0$ the genus 0 component contracts, and some of the elements in the list might be smoothable.)

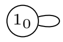
Then the list of the sectors corresponding to the graph of compact type 5.24 but with the non identical automorphism ρ is:



Here again T_1 varies among all the compactified twisted sectors of $\mathcal{M}_{1,1}$. The vertices 0_1^* and 0_2^* correspond to the twisted sectors of $[\overline{\mathcal{M}}_{0,3}/S_2]$ and $[\overline{\mathcal{M}}_{0,4}/S_2]$ (see Lemma 4.71).

5.2.3 Moduli of stable genus 2 curves

Without marked points, the stable genus 2 curves that are not of compact type lie inside the divisor whose dual graph corresponds to:

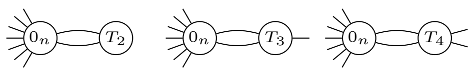


Construction 5.27. We list now the twisted sectors of $\overline{\mathcal{M}}_2$, which are not of compact type.

The ones associated with the identity as automorphism of the graph are:

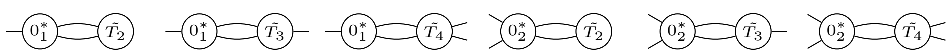
Those associated with the automorphism that exchanges the two sides of the edge are:

Construction 5.28. We here construct all the twisted sectors of $\overline{\mathcal{M}}_{2,n}^R$, corresponding to the graph 5.2.3. First fixing the automorphism of the graph to be the identity, we have:



where T_i is a compactified base twisted sector of $\mathcal{M}_{1,i}$ (see 4.42 and 4.47).

Then, chosing the automorphism that switches the two edges, we find:



where T_2' belongs to the set of twisted sectors of $\left[\mathcal{M}_{1,2}/S_2\right]$. The set of such (non smoothable) twisted sectors is:

$$\{(\tilde{C_4}, i), (\tilde{C_4}, -i), (\tilde{C_6}, \epsilon), (\tilde{C_6}, \epsilon^5)\}$$

while

$$T_3' \text{ is an element in the set } \{(\tilde{C_4'}, i), (\tilde{C_4'}, -i), (\tilde{C_6'}, \epsilon), (\tilde{C_6'}, \epsilon^5)\}$$

and

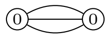
$$T_4' \text{ is an element in the set } \{(\tilde{C_4''}, i), (\tilde{C_4''}, -i)\}$$

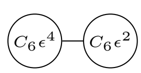
(see the section on genus 1 quotients). Once again the vertices 0_1^* and 0_2^* correspond to the twisted sectors of $\left[\overline{\mathcal{M}}_{0,3}/S_2\right]$ and $\left[\overline{\mathcal{M}}_{0,4}/S_2\right]$ (see Lemma 4.71).

As a corollary of the above construction, we know all the base twisted sectors of rational type (cfr Definition 4.42) of the Inertia Stack for $\overline{\mathcal{M}}_{2,n}$:

Now we want to give an example of how to compactify the base twisted sectors of rational type. For simplicity, we describe only the case without marked points.

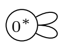
Example 5.29. If we indicate by $\overline{III}$ the compactification of the twisted sector III, then one has:

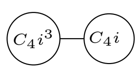
$\overline{III} = III \sqcup$ 

$\sqcup$ 

where in the first graph the automorphism is the automorphism of order 3 coming from the graph that permutes cyclically the three edges.

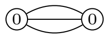
Example 5.30. We now indicate by $\overline{IV}$ the compactification of the twisted sector IV. Then

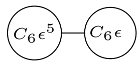
$\overline{IV} = IV \sqcup$ 

$\sqcup$ 

where in the first graph (which represents a locus of dimension 1) we choose a single point. This point is such that if three among the four special points on the normalization are chosen to be 0, 1 and ∞ on $\mathbb{P}^1$, then the last point is forced to be -1. In such a way, the automorphism of the normalization $z \to \frac{1}{z}$ descends to an automorphism of the curve. Then the automorphism of the curve that compactifies the twisted sector IV is represented on the normalization as $z \to \frac{z-1}{z+1}$. In the picture of the graph, it corresponds to exchanging the two loops composed with exchanging the two sides of one loop.

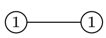
Example 5.31. Let $\overline{VI}$ be the compactification of VI. Then:

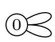
$\overline{VI} = VI \sqcup$ 

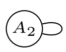
$\sqcup$ 

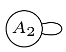
where the first graph has the automorphism of order six that is the composition of the automorphism of order two that switches the vertices, and the automorphism of order three that permutes cyclically the edges.

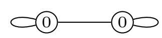
Example 5.32. We study the compactification of the only sector of $I(\mathcal{M}_2)$ of dimension 2, the one that we called II. We call $\overline{II}$ its compactification. Then:

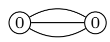
$\overline{II} = II \sqcup$ 

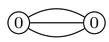
$\sqcup$ 

$\sqcup$ 

$\sqcup$ 

$\sqcup$ 

$\sqcup$ 

$\sqcup$ 

Where the automorphisms of the first four graphs are (in order): the one that exchanges the two vertices, the one that exchanges the two loops, the elliptic involution, the one that switches the two sides of the edge. The fifth graph comes with the two automorphisms that specialize the ones of the second graph, and the sixth graph has the automorphism that exchanges two of the three edges. The automorphism of the last graph exchanges the vertices and two of the three edges.

5.3 Genus 3 case

5.3.1 Moduli of smooth genus 3 curves

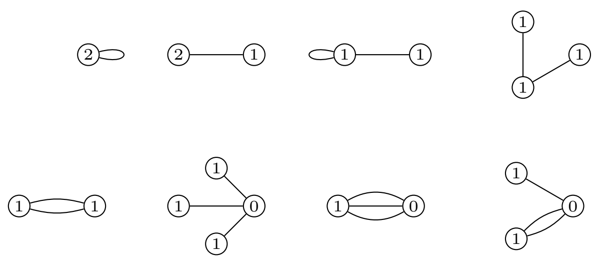
Again, we solved the Equations 4.21 with $g = 3$ and $g' = 0, 1, 2$ taking advantage of the C++ program [MP1].

Proposition 5.33. *We list the twisted sectors of $\mathcal{M}_3$, found using the program [MP1]. We use the notation of 4.22:*

1. *coverings of genus* 2 *curves:* $\mathcal{M}_2(2;0)$
2. *coverings of genus* 1 *curves:* $\mathcal{M}_2(1;4)$, $\mathcal{M}_3(1;1,1)$, $\mathcal{M}_4(1;0,2,0)$.
3. *coverings of genus* 0 *curves:* $\mathcal{M}_2(0;8)$, $\mathcal{M}_3(0;4,1)$, $\mathcal{M}_3(0;1,4)$, $\mathcal{M}_4(0;4,0,0)$, $\mathcal{M}_4(0;2,3,0)$, $\mathcal{M}_4(0;2,0,2)$, $\mathcal{M}_4(0;0,3,2)$, $\mathcal{M}_4(0;0,0,4)$, $\mathcal{M}_6(0;1,0,1,2,0)$, $\mathcal{M}_6(0;0,2,1,0,1)$, $\mathcal{M}_6(0;1,0,2,0,1)$, $\mathcal{M}_7(0;0,2,1,0,0,0)$, $\mathcal{M}_7(0;1,0,2,0,0,0)$, $\mathcal{M}_7(0;1,1,0,1,0,0)$, $\mathcal{M}_7(0;2,0,0,0,1,0)$, $\mathcal{M}_7(0;0,0,0,1,2,0)$, $\mathcal{M}_7(0;0,0,0,2,0,1)$, $\mathcal{M}_7(0;0,0,1,0,1,1)$, $\mathcal{M}_7(0;0,1,0,0,0,2)$, $\mathcal{M}_8(0;0,1,2,0,0,0,0)$, $\mathcal{M}_8(0;1,1,0,0,1,0,0)$, $\mathcal{M}_8(0;2,0,0,0,0,1,0)$, $\mathcal{M}_8(0;0,0,0,0,2,1,0)$, $\mathcal{M}_8(0;0,0,1,0,0,1,1)$, $\mathcal{M}_8(0;0,1,0,0,0,0,2)$, $\mathcal{M}_9(0;0,1,1,1,0,0,0,0)$, $\mathcal{M}_9(0;1,0,1,0,1,0,0,0)$, $\mathcal{M}_9(0;1,1,0,0,0,1,0,0)$, $\mathcal{M}_9(0;0,0,0,0,1,1,1,0)$, $\mathcal{M}_9(0;0,0,0,1,0,1,0,1)$, $\mathcal{M}_9(0;0,0,1,0,0,0,1,1)$, $\mathcal{M}_{12}(0;00111000000)$, $\mathcal{M}_{12}(0;10001100000)$, $\mathcal{M}_{12}(0;10100001000)$, $\mathcal{M}_{12}(0;00000011100)$, $\mathcal{M}_{12}(0;00000110001)$, $\mathcal{M}_{12}(0;00010000101)$, $\mathcal{M}_{14}(0;0011001000000)$, $\mathcal{M}_{14}(0;0100101000000)$, $\mathcal{M}_{14}(0;1000011000000)$, $\mathcal{M}_{14}(0;0000001001100)$, $\mathcal{M}_{14}(0;0000001010010)$, $\mathcal{M}_{14}(0;0000001100001)$.

5.3.2 Moduli of stable genus 3 curves without marked points

There are 32 stable graphs of genus 3 without marked points (including the trivial graph). Many of them, with our construction 4.60, give rise only to smoothable twisted sectors. The graphs that give rise to twisted sectors in $I(\partial\overline{\mathcal{M}}_3)$ are:



5.4 Higher genus

The program written with Stefano Maggiolo, [MP1], computes the solutions to the Equations 4.21. In particular, we compute the solutions and we exclude those that do not contain any base twisted sector of rational type; cf. Definition 4.42 (when $n = 0$, according to Proposition 4.33). The program uses Wiman's bound 2.28 to terminate. This improves a program written by Cornalba in [Co87] (where he finds algorithmically the solutions to the same equations only for N a prime number). Then we compute, for fixed g and n, the number of twisted sectors for $\mathcal{M}_{g,n}^{rt}$ associated to all the base twisted sectors of rational type, using Proposition 4.67. The program works in reasonable time for $g < 60$, and $n < 100$.

In another program [MP2] Stefano Maggiolo and I compute all the dual graphs of genus g and n marked points. This program, used with genus g and 0 marked points, together with Theorem 4.60

allows us in principle to find all the twisted sectors of $\overline{\mathcal{M}}_{g,n}$. Unfortunately, this program is much slower and works effectively only when $g \leq 7$.

We have chosen not to present all the data here because they take much space. Moreover we will not deal any further with the higher genus cases, since a stratification of $\mathcal{M}_g$ by automorphism group is available only for genera 2 and 3.

Chapter 6

The Cohomology of the Inertia Stacks

In this chapter, we define the Chen-Ruan cohomology group, simply as the ordinary cohomology of the Inertia Stack. We compute the dimension of these vector spaces for $\overline{\mathcal{M}}_{g,n}$. We will see that the computation of the cohomology of the twisted sectors constructed as admissible coverings of genus 0 curves, simply amounts to finding the invariant part of the cohomology $H^*(\overline{\mathcal{M}}_{0,n'})$ under the action of certain subgroups of $S_{n'}$. Those twisted sectors happen to be the large majority among all the twisted sectors of $\overline{\mathcal{M}}_{g,n}$. When the twisted sectors are constructed as coverings of higher genus curves, we have no general recipe. Nevertheless, we manage to solve the problem in some cases when the genus of the covered curves is 1. The cases of $\overline{\mathcal{M}}_{g,n}$ for $g \leq 2$ are covered, while we cannot complete the genus 3 case.

Let X be a Deligne–Mumford stack.

Definition 6.1. We define $H^*_{CR}(X, \mathbb{Q})$ and $A^*_{st}(X, \mathbb{Q})$ as rational vector spaces, respectively as $H^*(I(X), \mathbb{Q})$ and $A^*_{\mathbb{Q}}(I(X))$. They will be said to be the *Chen–Ruan cohomology group* ([CR04, Definition 3.2.3]) and the *Stringy Chow group* ([AGV02, Par. 6]).

We will deal mainly with cohomology. The theory for the two objects that we want to study develops in parallel. We will emphasize whenever the results that we give on the cohomology, are different from the ones that concern the Chow ring.

Since one connected component of $I(X)$, the untwisted sector, is isomorphic to the original stack X, the ordinary cohomology of X is a direct summand of the Chen–Ruan cohomology group.

6.1 General results

As we have seen in Corollary 4.64, the Inertia Stack of $\mathcal{M}_{g,n}$ and of $\mathcal{M}^{rt}_{g,n}$ is made up of (products of) base twisted sector of rational type (cfr. Definition 4.42) and moduli of stable genus zero pointed curves.

We can say something about the cohomology of the twisted sectors via the description given in [F09] (cfr. Remark 4.26).

Proposition 6.2. *Let $[X/G]$ be a Deligne–Mumford stack. Then the cohomology of the constant sheaf $H^*([X/G], \mathbb{Q})$ is isomorphic to $H^*(X, \mathbb{Q})^G$: the $G-$invariant part of the cohomology.*

Proof. The coarse moduli space has the same cohomology by Proposition 1.10. For the coarse moduli space this formula holds true. ☐

Amongst the spaces whose cohomology groups we will be calculating, most are spaces $\overline{\mathcal{M}}_N(g', d_1, \ldots, d_{N-1}, \alpha)$ defined in Definition 4.47) having g' equal to 0. For those, a result of Bayer and Cadman holds (see Remark 4.26 and Remark 4.24 for a conversion of the Bayer–Cadman notation into the one we are using):

Theorem 6.3. *([BC07, p.2]) The space $\overline{\mathcal{M}}_N(0, d_1, \ldots, d_{N-1}, \alpha)$ is a μ_N-gerbe over the quotient stack $[X/G(d_1, \ldots, d_{N-1}, \alpha)]$, where X is the stack constructed starting from $\overline{\mathcal{M}}_{0,\sum d_i}$ by successively applying the root construction (see [BC07, Section 2]).*

We now meet a stack $X_{D,r}$, called *root of line bundle with a section*, where X is a scheme, D is an effective Cartier divisor, and r is a natural number. This was introduced firstly in [Ca07] and [AGV06]. The only thing we shall need in this context of this construction is the following result:

Proposition 6.4. *([Ca07, Corollary 2.3.7]) Let X be a scheme. If $X_{D,r}$ is obtained from X applying the root construction, the canonical map $X_{D,r} \to X$ exhibits X as the coarse moduli space of $X_{D,r}$.*

Corollary 6.5. *The stack $\overline{\mathcal{M}}_N(0, d_1, \ldots, d_{N-1}, \alpha)$ has the same cohomology groups with rational coefficients of $\left[\overline{\mathcal{M}}_{0,\sum d_i}/G(d_1, \ldots, d_{N-1}, \alpha)\right]$.*

Proof. This follows thanks to Proposition 1.10, Theorem 6.3 and the fact that a stack and a gerbe over it share the same coarse moduli spaces. □

Let λ be a partition of n. Let S_λ be the irreducible representation associated with λ. Let $n = a+b$ and $S_a \times S_b$ be a subgroup of S_n. The restricted representation $S_{\lambda|S_a \times S_b}$ splits into irreducible representations of S_a and S_b, counted with multiplicities:

$$S_{\lambda|S_a \times S_b} = \sum_{\mu,\nu} c^{\lambda}_{\mu\nu}\ S_\mu \otimes S_\nu$$

where μ and ν are partitions respectively of a and b
The Littlewood–Richardson rule ([Sa00, Theorem 4.9.4]) gives a combinatorial way to compute the coefficients $c^{\lambda}_{\mu\nu}$.

Corollary 6.6. *Let V be an irreducible representation of S_n. Let Let $a_1 + \ldots + a_k = n$. The $S_{a_1} \times \ldots \times S_{a_k}$-invariant part of V is the direct summand of the restricted representation:*

$$S[a_1] \otimes \ldots \otimes S[a_k]$$

where $S[a_i]$ denotes the trivial representation on S_{a_i}.

The Littlewood–Richardson rule therefore allows one to compute the dimension of invariant part of a representation of S_n under a subgroup of kind $S_{a_1} \times \ldots \times S_{a_k}$.

From our description of the twisted sectors of $\mathcal{M}_{g,n}$, $\mathcal{M}^{rt}_{g,n}$ (cfr. Corollary 4.64), $\mathcal{M}^{ct}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ (cfr. Corollary 4.66), the Chen–Ruan cohomology of those spaces is known once one knows:

1. the cohomology of the base twisted sectors;
2. the combinatorics: namely how many twisted sectors are associated to a base twisted sector.

Remark 6.7. From the previous considerations, and due to the fact that the equivariant Poincaré polynomials of $\mathcal{M}_{0,n}$ and $\overline{\mathcal{M}}_{0,n}$ are known (Theorems 3.18 3.19), we can deduce that the rational cohomologies of the base twisted sectors of rational type (cfr. Definition 4.42) with associated $g' = 0$ (cfr. Remark 4.43) are known.

In genus 2 without marked points, there is only one base twisted sector of rational type whose associated discrete data g' is not 0, as we saw in Section 5.2.1. In genus 3 there are four such twisted sectors: three having discrete datum $g' = 1$ and one having $g' = 2$. We tabulate the number of base twisted sectors of rational type with discrete data $g' = 0$ and $g' > 0$ to show their prevalence among the set of all the base twisted sectors (4.42) of moduli of curves (these results were obtained using the program twistedrational.cpp [MP1]).

genus	$g'=0$	$g'>0$
2	16	1
3	43	4
4	65	7
5	64	12
6	193	10
7	163	33
8	207	18
9	372	43
10	485	52

6.2 The genus 1 case

We recall the result obtained in the previous section (see also [P08]).

Theorem 6.8. *(Theorem 5.17, Theorem 5.23) Each twisted sector of $\overline{\mathcal{M}}_{1,n}$ is isomorphic to a product:*

$$A \times \overline{\mathcal{M}}_{0,n_1} \times \overline{\mathcal{M}}_{0,n_2} \times \overline{\mathcal{M}}_{0,n_3} \times \overline{\mathcal{M}}_{0,n_4}$$

where $n_1, \ldots, n_4 \geq 3$ are integers and A is in the set:

$$\{B\mu_3, B\mu_4, B\mu_6, \mathbb{P}(4,6), \mathbb{P}(2,4), \mathbb{P}(2,2)\}$$

Thanks to this observation, we do not need to use the results of the previous chapter, and we can directly compute the cohomology of the twisted sectors.

We now address the problem of giving a compact formula for the dimension of the ordinary cohomology of the twisted sectors. From now on we call $TS(n)$ the stack of twisted sectors of $I(\overline{\mathcal{M}}_{1,n})$. Namely we can decompose the Inertia Stack in the obvious way:

$$I(\overline{\mathcal{M}}_{1,n}) = (\overline{\mathcal{M}}_{1,n}, 1) \sqcup TS(n) \tag{6.9}$$

Notice that this dimension equals the Euler characteristic, since all the Betti numbers are even. We collect together the cohomology of the twisted sectors whose base twisted sectors belong to the same space $\mathcal{M}_{1,m}$ (since their dimensions have a similar shape). In this way, a compact formula for the dimension of the cohomology of the twisted sectors can be written as:

$$\begin{aligned}\dim(H^*(TS(n))) = e(TS(n)) = (e(\overline{A_1}) + 2e(C_4) + 4e(C_6))\ e(\overline{\mathcal{M}}_{0,n+1}) + \\ + \frac{1}{2}(e(\overline{A_2}) + 2e(C_4') + 2e(C_6')) \sum \binom{n}{i,j} e(\overline{\mathcal{M}}_{0,i+1} \times \overline{\mathcal{M}}_{0,j+1}) + \\ + \frac{1}{6}(e(\overline{A_3}) + 2e(C_6'')) \sum \binom{n}{i,j,k} e(\overline{\mathcal{M}}_{0,i+1} \times \overline{\mathcal{M}}_{0,j+1} \times \overline{\mathcal{M}}_{0,k+1}) + \\ + \frac{1}{24} e(\overline{A_4}) \sum \binom{n}{i,j,k,l} e(\overline{\mathcal{M}}_{0,i+1} \times \overline{\mathcal{M}}_{0,j+1} \times \overline{\mathcal{M}}_{0,k+1} \times \overline{\mathcal{M}}_{0,l+1})\end{aligned} \tag{6.10}$$

where we sum over $1 \leq i,j,k,l \leq n$ whose sum is n. After computing the Euler characteristics of the base spaces, and after calling

$$h(n) := \dim H^*(\overline{\mathcal{M}}_{0,n+1}) = \sum_k a^k(n)$$

(the latter notation is the one of [Ke92, p. 550] shifted by 1), Formula 6.10 and decomposition 6.9, reduce to:

$$\dim(H^*_{CR}(\overline{\mathcal{M}}_{1,n})) = \dim(H^*(\overline{\mathcal{M}}_{1,n})) + \dim(H^*(TS(n))) = \quad \dim(H^*(\overline{\mathcal{M}}_{1,n})) + \tag{6.11}$$

$$+8h(n)+3\sum\binom{n}{i,j}h(i)h(j)+\frac{2}{3}\sum\binom{n}{i,j,k}h(i)h(j)h(k)+\frac{1}{12}\sum\binom{n}{i,j,k,l}h(i)h(j)h(k)h(l)$$

Here again we sum over $1 \leq i,j,k,l \leq n$ such that their sum is n . Note that this formula is true for all n. The formula gives the following numbers for low n:

	1	2	3	4	5
$\dim(H^*(TS(n))) =$	8	14	38	148	762

Next, we introduce the generating polynomials:

$$P_0(s) := \sum_{n=0}^{\infty} \frac{Q_0(n)}{n!} s^n \tag{6.12}$$

$$P_1(s) := \sum_{n=0}^{\infty} \frac{Q_1(n)}{n!} s^n \tag{6.13}$$

$$P_1^{CR}(s) := \sum_{n=0}^{\infty} \frac{Q_1^{CR}(n)}{n!} s^n \tag{6.14}$$

where:

$$Q_0(n) := \dim H^*(\overline{\mathcal{M}}_{0,n+1}) = h(n)$$
$$Q_1(n) := \dim H^*(\overline{\mathcal{M}}_{1,n})$$
$$Q_1^{CR}(n) := \dim H^*_{CR}(\overline{\mathcal{M}}_{1,n})$$

with the convention that when the right hand side is not defined, the left hand side is equal to 1.

Formula 6.11 can now be written compactly.

Theorem 6.15. *The following equality between power series relates the dimensions of the cohomology group of $\overline{\mathcal{M}}_{0,n}$ and $\overline{\mathcal{M}}_{1,n}$ with the dimension of the Chen–Ruan cohomology group of $\overline{\mathcal{M}}_{1,n}$.*

$$P_1^{CR}(s) = P_1(s) + 8P_0(s) + 3P_0(s)^2 + \frac{2}{3}P_0(s)^3 + \frac{1}{12}P_0(s)^4 \tag{6.16}$$

In complete analogy, one can compose a similar formula for the case of rational tail. Remember that in the rational tail case there is odd cohomology in the cohomologies of the base twisted sectors of rational type (namely in A_2, $A_3 = A_4$).

Theorem 6.17. *The following equality between power series relates the dimensions of the cohomology group of $\overline{\mathcal{M}}_{0,n}$ and $\mathcal{M}_{1,n}^{rt}$ with the dimension of the Chen–Ruan cohomology group of $\mathcal{M}_{1,n}^{rt}$.*

$$P_{1rt}^{CR}(s) = P_{1rt}(s) + 8P_0(s) + 3P_0(s)^2 + \frac{5}{6}P_0(s)^3 + \frac{1}{8}P_0(s)^4 \tag{6.18}$$

6.3 The genus 2 case

In this section we study the dimension of the Chen–Ruan cohomology groups $H^*_{CR}(\overline{\mathcal{M}}_{2,n})$. We compute the cohomology of all the compactified base twisted sectors (Definition 4.42, Definition 4.47). This is enough in principle, up to the combinatorics of the graphs, to find all the dimensions. We have not been able to find a compact way to write down the generating series for the Poincaré polynomials in genus 2. Instead, we explicitly write down the results only for $n = 0, 1$

The cohomologies of the zero dimensional base twisted sectors is computed on the nose. The one dimensional base twisted sectors when compactified have coarse moduli spaces all isomorphic to $\mathbb{P}^1$.

We then study the non trivial cases. The cohomologies of the spaces constructed as admissible coverings of genus 0 curves are computed using Remark 6.7:

Proposition 6.19. *The ordinary Poincaré polynomials of $\tau, \tau_1, \tau_{11}, \tau_{111}, \tau_{1111}, \tau_{11111}, \tau_{111111}$ are given in the sequel:*

1. *The Poincaré polynomial is $P(\tau) = t^3 + 2t^2 + 2t + 1$.*
2. *The Poincaré polynomial is $P(\tau_1) = t^3 + 3t^2 + 3t + 1$.*
3. *The Poincaré polynomial is $P(\tau_{11}) = t^3 + 5t^2 + 5t + 1$.*
4. *The Poincaré polynomial is $P(\tau_{111}) = t^3 + 8t^2 + 8t + 1$.*
5. *The Poincaré polynomial is $P(\tau_{1111}) = t^3 + 14t^2 + 14t + 1$.*
6. *The Poincaré polynomial is $P(\tau_{11111}) = t^3 + 16t^2 + 16t + 1$.*
7. *The Poincaré polynomial is $P(\tau_{111111}) = t^3 + 16t^2 + 16t + 1$.*

Proof. To obtain all these we apply the discussion at the beginning of the chapter; in particular Corollary 6.7 and Theorem 3.19 with 6 marked points. □

To complete the computation of the dimension of the Chen–Ruan cohomology of $\overline{\mathcal{M}}_{2,n}$, one needs to determine the ordinary cohomology of the few compactified base twisted sectors (4.42, 4.47) which appear as coverings of genus 1 curves, *i.e.* the stacks that we called II, II_1 and $II_{1,1}$ in Section 5.2.1.

Lemma 6.20. *The twisted sector II has the same coarse moduli space as the quotient stack $[\mathcal{M}_{0,5}/S_3]$.*

Proof. We show a morphism from II to $[\mathcal{M}_{0,5}/S_3]$, that induces a bijection on objects. Let C be a genus 2 curve with an automorphism ϕ of order 2 that is not the hyperelliptic involution. We call $E := C/\langle\phi\rangle$, it is an elliptic curve and the projection is ramified in two points. Let $\pi_C : C \to \mathbb{P}^1$ be the $2:1$ covering that induces the hyperelliptic structure on C. According to [Sc90, Lemma 1.1], there exists exactly one elliptic structure $\pi_E : E \to \mathbb{P}^1$, such that the following diagram commmutes:

$$\begin{tikzcd} C \arrow[r] \arrow[d, "\pi_C"] & E \arrow[d, "\pi_E"] \\ \mathbb{P}^1 \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

If we call $0, 1, \infty, \lambda$ the branching point of E on $\mathbb{P}^1$, and p_1, p_2 the branching point of the projection $C \to E$, then it is easy to see that one between $\pi_E(p_1)$ and $\pi_E(p_2)$ must coincide with one among $0, 1, \infty, \lambda$. Without loss of generality, we assume $\lambda = \pi_E(p_1)$. Therefore, we obtain the data of 5 points on $\mathbb{P}^1$: $0, 1, \infty, \lambda, q$. The map just defined from II to $\mathcal{M}_{0,5}$ induces a map on $[\mathcal{M}_{0,5}/S_3]$ by composition with the quotient map. If we assume that S_3 acts on the first three points, the inverse morphism from $[\mathcal{M}_{0,5}/S_3]$ to II is obtained as follows. Vice versa, from a genus 0 curve with five marked points such that the first three are undistinguished, one can construct first a $2:1$ covering γ branched in the last two points and then a genus 2 curve as a $2:1$ covering whose branching points are the fibers of the three undistinguished marked points under γ. The genus 2 curve thus constructed admits a $2:1$ map on the elliptic curve that can be constructed as a $2:1$ covering of the genus 0 curve we started with (ramified in the first four points). □

We observe that the map constructed in the last lemma exhibits II as a $\mu_2 \times \mu_2-$gerbe over its rigidification, $[\mathcal{M}_{0,5}/S_3]$.

Lemma 6.21. *The twisted sectors II_1 and II_{11} are isomorphic. They have the same coarse moduli space of $\mathcal{M}_{1,2} \setminus A_2$ (see Definition 5.9 for the definition of A_2).*

Proof. The stack II_1 shares the same coarse space with its rigidification introduced in Remark 4.27, whose objects are:

$$\{C, p_1, p_2, L\}$$

where p_i are closed points of C and L is a line bundle such that $2L$ is linearly equivalent to $p_1 + p_2$. Since C is genus 1, $L = \mathcal{O}(q)$. Our argument follows [Be98, Lemma 2.1.3]. If $f : \mathcal{C}_{1,2} \to \mathcal{M}_{1,2}$ denotes the universal curve, the stack II_1 is isomorphic to the closed locus in $\mathcal{C}_{1,2}$ given by the equation:

$$2q = p_1 + p_2$$

To the point (C, p_1, p_2, q) in $\mathcal{C}_{1,2}$ we associate the point $(C, p_1, q) \in \mathcal{M}_{1,2}$. The inverse map sends $(C, p_1, q) \in \mathcal{M}_{1,2}$ to $(C, p_1, 2q - p_1, q)$. This correspondence is bijective provided that $2q - p_1 \neq p_1$. □

Proposition 6.22. *The Poincaré polynomials of $\overline{II}$, $\overline{II}_1$, $\overline{II}_{11}$ are $t^2 + 3t + 1$.*

Proof. The Euler characteristic in $e(MHS)$ of $\left[\mathcal{M}_{0,5}/S_3\right]$ is $L^2 - L + 1$. By the additivity of the Euler characteristic, since we have seen that the coarse space of $\overline{II} \backslash II$ is made up of four irreducible components isomorphic to $\mathbb{P}^1$, we obtain the result for $\overline{II}$:

$$L^2 + 3L + 1 = (L^2 - L + 1) + (4L - 3) + 3$$

The Euler characteristic in $e(MHS)$ of $\mathcal{M}_{1,2}$ is L^2. Using the additivity of $e(MHS)$ we can compute the Euler characteristic $e(II_1)$ and then $e(II_{11})$.

The conclusion then follows from the fact that, since $\overline{II}$, $\overline{II}_1$ and $\overline{II}_{11}$ are proper Deligne–Mumford smooth stacks, the knowledge of the Euler characteristic in MHS determines the Poincaré polynomial. □

It is now possible to compute the dimension of the cohomologies of $\overline{\mathcal{M}}_{2,n}$. We give only the first numbers:

Corollary 6.23. *The dimension of $H^*_{CR}(\overline{\mathcal{M}}_2)$ is 97. The dimension of $H^*_{CR}(\overline{\mathcal{M}}_{2,1})$ is 163.*

6.4 The genus 3 case

We give the results on the cohomologies of the positive dimensionals base twisted sectors that are constructed by compactifying coverings of genus 0 curves in 4.22, 4.47). We use the techniques developed in the beginning of the Chapter, (see Remark 6.7).

Lemma 6.24.
1. *The Poincaré polynomial of $\mathcal{M}_2(0; 8)$, $t^5 + 3t^4 + 6t^3 + 6t^2 + 3t + 1$,*
2. *The Poincaré polynomial of $\mathcal{M}_3(0; 4, 1)$ and $\mathcal{M}_3(0; 1, 4)$ is $t^2 + 5t + 1$,*
3. *The Poincaré polynomial of $\mathcal{M}_4(0; 4, 0, 0)$ and $\mathcal{M}_4(0; 0, 0, 4)$ is $t + 1$,*
4. *The Poincaré polynomial of $\mathcal{M}_4(0; 2, 3, 0)$ and $\mathcal{M}_4(0; 0, 3, 2)$ is $t^2 + 2t + 1$,*
5. *The Poincaré polynomial of $\mathcal{M}_4(0; 2, 0, 2)$, is $t + 1$.*

For the base twisted sectors that we have described as coverings of curves of genus greater than 1, the two Poincaré polynomials of:

$$\overline{\mathcal{M}}_2(1; 4) \quad \overline{\mathcal{M}}_3(1; 1, 1)$$

can be computed using the same techniques used in Lemma 6.20, 6.21 and Proposition 6.22. However, it seems to us that the computations of the cohomologies of:

$$\overline{\mathcal{M}}_4(1; 0, 2, 0) \quad \overline{\mathcal{M}}_2(2; 0)$$

two stacks of dimensions 2 and 3 respectively, that appear as twisted sectors $I(\overline{\mathcal{M}}_3)$, would require further techniques. This last two cohomologies would be required to complete the computation of the Chen–Ruan cohomology groups of $\overline{\mathcal{M}}_3$.

6.5 Euler characteristics

In this section, we prove that our results are consistent with the existing results in the literature concerning the Euler Characteristics. We write in detail this consistency check in genus 1. We have also performed the check for the cases of $\mathcal{M}_{2,n}$, $\mathcal{M}_3$, $\mathcal{M}_{3,1}$ and $\overline{\mathcal{M}}_{2,0}$ and $\overline{\mathcal{M}}_{2,1}$.

We call e the Euler characteristic and χ the orbifold Euler characteristic, also called virtual Euler characteristic. The first one is simply the topological Euler characteristic of the coarse moduli space, while the second is the Euler characteristic of the stack. In order to compute the latter for our base twisted sectors, we recall that:

1. if X is a scheme and G is a finite group acting on X, then $\chi([X/G])$ equals $\frac{e(X)}{|G|}$,
2. if X is a Deligne–Mumford stack and $X = A \sqcup B$ where A is a closed substack, then $\chi(X)$ equals $\chi(A) + \chi(B)$.

Example 6.25. The orbifold Euler characteristic of the base twisted sector C_4 is $\frac{1}{4}$, as C_4 is the quotient stack of a point under the group μ_4.

Let us then introduce Behrend's formula ([B04], p. 21. Warning: in his notation χ and e are exchanged):

$$e(X) = \chi(I(X))$$

In the cases we are studying it becomes:

$$e(\mathcal{M}_{1,n}) = \chi(I(\mathcal{M}_{1,n})) \tag{6.26}$$

and

$$e(\overline{\mathcal{M}}_{1,n}) = \chi(I(\overline{\mathcal{M}}_{1,n})) \tag{6.27}$$

We first recall the Bini-Gaiffi-Polito ([BGP01] pag. 15) formula for $e(\mathcal{M}_{1,n})$:

1. $e(\mathcal{M}_{1,1}) = 1$
2. $e(\mathcal{M}_{1,2}) = 1$
3. $e(\mathcal{M}_{1,3}) = 0$
4. $e(\mathcal{M}_{1,4}) = 0$
5. $e(\mathcal{M}_{1,n}) = (-1)^n \frac{(n-1)!}{12}$ for $n \geq 5$.

When $n \geq 5$ this result coincides with the Harer-Zagier formula for the orbifold Euler characteristic since the coarse moduli scheme represents the moduli problem (there are no automorphisms). Indeed, the Harer-Zagier formula ([HZ86]), when the genus is one, reduces for all n to:

$$\chi(\mathcal{M}_{1,n}) = (-1)^n \frac{(n-1)!}{12}$$

As for the cases $n \leq 4$, then Equation 6.26 presents us with a check on our identification of the twisted sectors:

1. $1 = 2\chi(\mathcal{M}_{1,1}) + 2\chi(C_4) + 4\chi(C_6) = -\frac{1}{6} + \frac{1}{2} + \frac{4}{6}$;
2. $1 = \chi(\mathcal{M}_{1,2}) + \chi(A_2) + 2\chi(C_4') + 2\chi(C_6') = \frac{1}{12} + (-\frac{1}{2} + \frac{1}{4}) + \frac{1}{2} + \frac{2}{3}$;
3. $0 = \chi(\mathcal{M}_{1,3}) + \chi(A_3) + 2\chi(C_6'') = -\frac{1}{6} - \frac{1}{2} + \frac{2}{3}$;
4. $0 = \chi(\mathcal{M}_{1,4}) + \chi(A_4) = \frac{1}{2} - \frac{1}{2}$;

For the compact case, we no longer have the Harer-Zagier formula for the orbifold Euler characteristic. However, for low genus and marked points, we can compute the latter using [BH06, p.4, Formula 11]:

1. $\chi(\overline{\mathcal{M}}_{1,1}) = \frac{5}{12}$
2. $\chi(\overline{\mathcal{M}}_{1,2}) = \frac{1}{2}$
3. $\chi(\overline{\mathcal{M}}_{1,3}) = \frac{5}{24}$

4. $\chi(\overline{\mathcal{M}}_{1,4}) = \frac{35}{6}$

Meanwhile, the Euler characteristic was computed by Getzler in [Ge98, p.10, Appendix] up to 15 marked points:

1. $e(\overline{\mathcal{M}}_{1,1}) = 2$
2. $e(\overline{\mathcal{M}}_{1,2}) = 4$
3. $e(\overline{\mathcal{M}}_{1,3}) = 12$
4. $e(\overline{\mathcal{M}}_{1,4}) = 49$

We also use the facts that:

1. $\chi(\overline{\mathcal{M}}_{0,n})$ is known by Keel [Ke92]. If $n = 5$ it is 7.
2. $\chi(\mathbb{P}(4,6)) = \frac{5}{12}$
3. $\chi(\mathbb{P}(4,2)) = \frac{3}{4}$
4. $\chi(\mathbb{P}(2,2)) = 1$

Finally we can check that Equation 6.27 in our cases becomes:

1. $2 = \chi(\overline{\mathcal{M}}_{1,1}) + \chi(\overline{A}_1) + 2\chi(C_4) + 4\chi(C_6)$
2. $4 = \chi(\overline{\mathcal{M}}_{1,2}) + \chi(\overline{\mathcal{M}}_{1,1}) + \chi(\overline{A}_2) + 2\chi(C_4) + 2\chi(C_4') + 4\chi(C_6) + 2\chi(C_6')$
3. $12 = \chi(\overline{\mathcal{M}}_{1,3})+\chi(\overline{\mathcal{M}}_{1,1})\chi(\overline{\mathcal{M}}_{0,4})+3\chi(\overline{A}_2)+\chi(\overline{A}_3)+2\chi(\overline{\mathcal{M}}_{0,4})\chi(C_4)+2*3\chi(C_4')+4\chi(\overline{\mathcal{M}}_{0,4})\chi(C_6)+ 2*3\chi(C_6') + 2\chi(C_6'')$
4. $49 = \chi(\overline{\mathcal{M}}_{1,4}) + 43 + \frac{1}{6}$.

We now discuss the general (stable) case of $\overline{\mathcal{M}}_{1,n}$ for $n \geq 5$. The generating series:

$$\sum_{n=1}^{\infty} e(\overline{\mathcal{M}}_{1,n}) \ \frac{s^n}{n!} = P_1(s) \tag{6.28}$$

is described in [Ge98, Theorem 4.1]. The generating series :

$$F_1(s) := \sum_{n=1}^{\infty} \chi(\overline{\mathcal{M}}_{1,n}) \ \frac{s^n}{n!} \tag{6.29}$$

is described in [BH06, Theorem 3.2]. The description of the twisted sectors given in this thesis, makes it possible to compute the orbifold Euler characteristic of $I(\overline{\mathcal{M}}_{1,n})$. This is (see Formula 6.10):

$$\begin{aligned}
\chi(TS(n)) = (\chi(\overline{A_1}) + 2\chi(C_4) + 4\chi(C_6)) \ \chi(\overline{\mathcal{M}}_{0,n+1})+ \\
+\frac{1}{2}(\chi(\overline{A_2}) + 2\chi(C_4') + 2\chi(C_6')) \sum \binom{n}{i,j} \chi(\overline{\mathcal{M}}_{0,i+1} \times \overline{\mathcal{M}}_{0,j+1})+ \\
+\frac{1}{6}(\chi(\overline{A_3}) + 2\chi(C_6'')) \sum \binom{n}{i,j,k} \chi(\overline{\mathcal{M}}_{0,i+1} \times \overline{\mathcal{M}}_{0,j+1} \times \overline{\mathcal{M}}_{0,k+1})+ \\
+\frac{1}{24}\chi(\overline{A_4}) \sum \binom{n}{i,j,k,l} \chi(\overline{\mathcal{M}}_{0,i+1} \times \overline{\mathcal{M}}_{0,j+1} \times \overline{\mathcal{M}}_{0,k+1} \times \overline{\mathcal{M}}_{0,l+1})
\end{aligned} \tag{6.30}$$

where we sum over all $1 \leq i,j,k,l \leq n$ whose sum is n. After computing the Euler characteristics of the base twisted sectors (4.42), and after calling

$$h(n) := \dim H^*(\overline{\mathcal{M}}_{0,n+1}) = \sum_k a^k(n)$$

(the latter notation is the one of [Ke92, p. 550] shifted by 1), Formula 6.30 reduces to:

$$\chi(TS(n)) = \frac{19}{12}h(n) + \frac{23}{24}\sum \binom{n}{i,j} h(i)h(j) + \tag{6.31}$$
$$\frac{5}{18}\sum \binom{n}{i,j,k} h(i)h(j)h(k) + \frac{1}{24}\sum \binom{n}{i,j,k,l} h(i)h(j)h(k)h(l)$$

Here we always have $i+j+k+l=n$ and $1 \leq i,j,k,l \leq n$.

Now, Behrend's formula tells us that:

$$\sum_{n=1}^{\infty} e(\overline{\mathcal{M}}_{1,n})\ \frac{s^n}{n!} = \sum_{n=1}^{\infty} \chi(\overline{\mathcal{M}}_{1,n})\ \frac{s^n}{n!} + \sum_{n=1}^{\infty} \chi(TS(n))\ \frac{s^n}{n!}$$

So, writing 6.31 in the compact language of generating series introduced in 6.12, and putting this together with 6.28 and 6.29, the above equality becomes simply the equality of power series:

$$P_1 = F_1 + \frac{19}{12}P_0 + \frac{23}{24}P_0^2 + \frac{5}{18}P_0^3 + \frac{1}{24}P_0^4 \tag{6.32}$$

This equality can easily be checked with a computer program (we have checked it in low degrees with Maple).

Remark 6.33. We have checked the equality: 6.26 for all the stacks $\mathcal{M}_{2,n}$ and for $\mathcal{M}_{3,0}$ and $\mathcal{M}_{3,1}$. The equality 6.27 has been checked for $\overline{\mathcal{M}}_2$ and $\overline{\mathcal{M}}_{2,1}$. For the ordinary Euler characteristics of $\mathcal{M}_{2,n}$ and $\mathcal{M}_{3,n}$ we follow [BGP01]. For the orbifold Euler characteristics of $\overline{\mathcal{M}}_{2,n}$ we use [Ar05]. The ordinary Euler characteristics of $\overline{\mathcal{M}}_{2,n}$ with a small number of marked points is computed for instance in [Get98].

Chapter 7

The Chen-Ruan Cup Product for Moduli of Curves

In this chapter we define the ring structure on the Chen–Ruan cohomology groups. These rings will be Poincaré duality rings by shifting the ordinary grading on the cohomology of the twisted sectors of the Inertia Stack by a suitable rational number (one for each twisted sector). This number is called *degree shifting number*, or *fermionic shift*, or *age*. After recalling the definition of the age, we recall the definition of the second Inertia Stack, and then the definition of the excess intersection bundle. This information will be needed in the definition of the Chen-Ruan cup product. In the chapter on age, we have included some general results that compute this degree shifting number for the twisted sectors of all $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$. In the last sections of the chapter, we explicitly compute everything in the genus 1 compact case, and the age for the twisted sectors of $I(\mathcal{M}_{2,n})$. We conclude with the Chen-Ruan Poincaré polynomial of $\overline{\mathcal{M}}_2$.

7.1 Definitions

7.1.1 The age grading

We define the degree shifting number for the twisted sectors of the Inertia Stack of a smooth stack X. In the following $R\mu_r$ is the representation ring of μ_r, and ζ_r is a choice of a generator for the group of the μ_r-roots of 1. We work over the complex numbers, so there is a canonical isomorphism $\mu_r \to \mu_r^\vee$, and by abuse of notation, we can also consider ζ_r a generator for the group $\mu_r^\vee$.

Definition 7.1. Let $\rho : \mu_r \to \mathbb{C}^*$ be a group homomorphism. It is determined by an integer $0 \leq k \leq r-1$ as $\rho(\zeta_r) = \zeta_r^k$. We define a function *age*:

$$\mathrm{age}(\rho) = k/r$$

This function extends to a unique group homomorphism:

$$age : R\mu_r \to \mathbb{Q}$$

We now define the age of a twisted sector Y.

Definition 7.2. Let Y be a twisted sector and p a point of Y. It induces a morphism $p \to \overline{I}(X)$, that is, according to Definition 4.5, a representable morphism $f : B\mu_r \to X$. Then the pull–back via f of the tangent sheaf, $f^*(T_X)$, is a representation of μ_r (see 1.9). We define:

$$a(Y) := \mathrm{age}(f^*(T_X))$$

Proposition 7.3. *([CR04]) Let $X_{(g)}$, $X_{(g^{-1})}$ be two connected components of the Inertia Stack of an algebraic stack X, which are exchanged by the involution of the Inertia Stack. Then if $c = codim(X_{(g)}, X)$, the following holds:*

$$a(X_{(g)}) + a(X_{(g^{-1})}) = c$$

Remark 7.4. If $i : Y \to X$ is the restriction of the map $I(X) \to X$ to the twisted sector Y, and $x \in Y$ is a point, then the following splitting holds:

$$T_x X = T_x Y \bigoplus N_Y X|_x$$

If G is the stabilizer group of x, then $T_x X$ is a representation of G that splits as a sum of two representations: $T_x Y$ and $N_Y X|_x$. The first of these representations is trivial by definition of the twisted sector. Therefore what is needed in order to compute the age of a twisted sector are all the characters of the representation of $G \cong \mu_N$ on $N_Y X$.

We can then define the orbifold, or Chen–Ruan, degree.

Definition 7.5. We define the $d-th$ degree Chen–Ruan cohomology group as follows:

$$H^d_{CR}(X, \mathbb{Q}) := \bigoplus_i H^{d-2a(X_i, g_i)}(X_i, \mathbb{Q})$$

where the sum is over all twisted sectors.

To compute the age of a twisted sector of $\overline{\mathcal{M}}_{g,n}$, we follow the steps (we are building the twisted sectors following Proposition 4.60 without marked points, and then following Proposition 4.74):

1. The age of $\mathcal{M}_{g,n}$ was computed in [F09]. We review those results.
2. We study the age of the twisted sectors of $\overline{\mathcal{M}}_g$.
3. We study the relation between the age of $Y_{\alpha(1),\dots,\alpha(k)}$ of $\mathcal{M}_{g,k}$ and that of a corresponding twisted sector obtained by attaching rational tails (cfr 4.63),
4. We study the age of the twisted sectors of $\overline{\mathcal{M}}_{g,n}$.

Note that the age of a twisted sector X and of its compactified $\overline{X}$ (see 4.47) coincide.

We start by describing the results concerning age that are contained in [F09]. The results in genus 1 were also obtained in [P08]. In fact, to compute the age of the base twisted sectors of rational type we distinguish between the case of genus 1 and the cases of genus greater than 1, the difference being that the genus 1 curves without marked points are not stable.

Proposition 7.6. *([F09, Corollary 5.3]) Let $g > 1$ and let Y be the twisted sector of $\mathcal{M}_g$ corresponding to discrete data $(g', N, d_1, \dots, d_{N-1})$ (Definition 4.42). Then its age is equal to:*

$$a(Y) = \frac{3g'(N-1)(N-2)}{2N} + \frac{3(1-N)}{2N} + \frac{1}{N}\sum_{i=1}^{N-1} d_i \sum_{k=1}^{N-1} k \left(\left\{ \frac{ki}{N} \right\} + \sigma(k,i) \right) \tag{7.7}$$

where $\sigma(k,i) = 1$ iff $i \in S_{\chi_k}$ and 0 otherwise. The set S_{χ_k} is posed by definition:

$$S_{\chi_k} = \left\{ i \in \{1, \dots, N-1\} | \ \frac{ki}{n} + \frac{1}{m_k} \notin \mathbb{Z} \right\}$$

and $m_k = \mathrm{ord}_N(k)$.

Remark 7.8. In paper [F09], the author proves that this formula can be rearranged as:

$$a(Y) = \frac{(3g-3)(N-1)}{2} + \sum_{i=1}^{N-1} d_i A_i = \frac{1}{N}\sum_{k=1}^{N-1} k \left(3g - 3 + \sum_{i \in S_{\chi_k}} d_i + \deg(L_k) \right) \tag{7.9}$$

where
$$A_i := \sum_{k=1}^{N-1} k \ \left(\left\{\frac{ki}{N}\right\} + \sigma(k,i)\right)$$
and L_k are the building data of the covering ([Pa91]) whose degree is computed as:
$$\deg(L_k) = \sum_{i=1}^{N-1} \left\{\frac{ki}{N}\right\} \ d_i$$

Corollary 7.10. *Let $g > 1$ and let Y be a twisted sector of $\mathcal{M}_g$. If $Y_{\alpha(1),\dots,\alpha(n)}$ is a twisted sector of $\mathcal{M}_{g,n}$, obtained by adding marked points to Y (cfr. 4.43), then the following relation holds between the ages of the two sectors:*
$$a(Y_{\alpha(1),\dots,\alpha(n)}) = a(Y) + \frac{1}{N}\sum_{l=1}^{n} \ \alpha(l) \tag{7.11}$$

Proof. The proof is by induction on n. We show the first step since the others are analogous. Let $\pi : C_{g,1} \to \mathcal{M}_g$ be the universal curve. Let $C \in \mathcal{M}_g$ be a curve with an automorphism ϕ of order N. Suppose that p is a point in C stabilized by ϕ (and hence it is a total ramification point with respect to the quotient map $C \to C/\langle\phi\rangle$). Then $(C, p) \in C_{g,1}$ and the tangent space splits $\langle\phi\rangle$-equivariantly:
$$T_{(C,p)} = \pi^* T_C \mathcal{M}_g \bigoplus T_p C$$
The age of the first representation was computed in 7.7. The representation of the cyclic group μ_N on T_pC is the multiplication $e^{\frac{2\pi i \alpha(l)}{N}}$ (see 4.35, 4.24, 4.43), so the term we are adding is $\frac{\alpha(l)}{N}$. □

Proposition 7.12. *([P08, Proposition 5.5, Lemma 5.6]) The age for the connected components of the twisted sectors of $\mathcal{M}_{1,n}$ are:*

- *If $n = 1$:*
  1. $a(C_4, i) = 1 - a(C_4, -i) = \frac{1}{2}$;
  2. $a(C_6, \epsilon) = 1 - a(C_6, \epsilon^5) = \frac{2}{3} = a(C_6, \epsilon^4) = 1 - a(C_6, \epsilon^2)$;
  3. $a(A_1, -1) = 0$.
- *If $n = 2$:*
  1. $a(C_4', i) = \frac{5}{4} = 2 - a(C_4', -i)$;
  2. $a(C_6', \epsilon^2) = 1 = 2 - a(C_6', \epsilon^4)$;
  3. $a(A_2, -1) = \frac{1}{2}$.
- *If $n = 3$:*
  1. $a(C_6'', \epsilon^2) = \frac{5}{3} = 3 - a(C_6'', \epsilon^4)$;
  2. $a(A_3, -1) = 1$.
- *if $n = 4$, $a(A_4, -1) = \frac{3}{2}$.*

Proof. The proof is the same as in 7.10, the only difference being that we cannot start from $\mathcal{M}_1$ but from $\mathcal{M}_{1,1}$. The age of the twisted sectors in the case with one marked point is easily computed using the description of $\mathcal{M}_{1,1}$ as a global quotient stack. □

Note that such results agree with the results of Corollary 7.10. Namely, even though there are no twisted sectors corresponding to $I(\mathcal{M}_1)$, one can write down Formula 7.11 and obtain correct results, possibly having $a(Y)$ negative for some Y. We have seen in 4.60 how to construct all the base twisted sectors out of $\overline{\mathcal{M}}_{g,n}$ knowing the generalized base twisted sectors of rational type with $g' < g$.

Lemma 7.13. *([Mu83]) Let* $X = \overline{\mathcal{M}}_{g_1,n_1+1}$, $Y = \overline{\mathcal{M}}_{g_2,n_2+1}$, *and* $Z = \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$. *Let* $j : X \times Y \to Z$ *be the gluing morphism defined in Definition 2.32. Then* $N_{j_2(X\times Y)}Z \cong p_1^*(\mathbb{L}_{n_1+1}^\vee) \otimes_{\mathbb{C}} p_2^*(\mathbb{L}_{n_2+1}^\vee)$.

We want to compute the age of a general twisted sector of $\overline{\mathcal{M}}_g$. So we use the description of the twisted sectors given in 4.60. Let $(G, \beta) \in \mathcal{A}_{g,n}$ (4.52) and $v_1, \ldots, v_k$ be a set of representatives of vertices of G, one for each orbit under the action of $\langle\beta\rangle$. We call λ_i the number of elements in the orbit of v_i under the action of $\langle\beta\rangle$. Let then $(X_1, \ldots, X_k)$ be the twisted sector constructed in 4.60. Since $v_1, \ldots, v_k$ are one for each orbit under the action of G, the assignment $v_i \to X_i$ uniquely determines an assignment of a base twisted sector X_i to each vertex v of the graph G. So every vertex v is assigned a set of admissible data that we call $(g'_v, N_v, \ldots, \alpha_v)$. Every X_i is a twisted sector of the Inertia Stack of some $\left[\mathcal{M}_{g_i,n_i}/S\right]$ for S a given subgroup of S_{n_i}, which is a product of cyclic disjoint permutations. If e is an edge of G, we write $e = h + h'$ to mean that the edge e is made of the two half-edges h and h'.

Proposition 7.14. *With the notation introduced above, the age of the twisted sector* $(X_1, \ldots, X_k)$ *is:*

$$a(X_1, \ldots, X_k) = \sum_{i=1}^{k} a(X'_i) + \sum_{i=1}^{k} \lambda_i \left(\frac{3g_i - 3 + n_i - 1}{2} \right) + \sum_{e=h+h'} \left\{ \frac{\alpha_v(h)}{N_v} + \frac{\alpha_{v'}(h')}{N_{v'}} \right\} \tag{7.15}$$

Proof. (sketch) We use the sequence 2.48. If C is a general element of the twisted sector we are considering, the tangent space in C to $\overline{\mathcal{M}}_{g,n}$ splits equivariantly with respect to the action of the automorphism in the H^1 and H^0 terms. The sheaf $\mathcal{E}xt^1$ of 2.49 is supported on the nodes of C and the action of the automorphism on the nodes can be computed using 7.13 in complete analogy with the computation of Corollary 7.10. This corresponds to the third term in the summation.

The first two terms come from the computation of the age of the action of the automorphism on the term $H^1(C, \mathcal{H}om(\omega(\sum x_i), \mathcal{O}_C))$. Suppose for simplicity that $k = 1$, so that the set of orbits $\{v_1, \ldots, v_k\}$ is reduced to one element v. Let c be $3g_v - 3 + n_v$ and λ be the number of irreducible components that are permuted by the automorphism β of the graph. The group acts by permuting the components and acting on one of them as studied in 7.7. The linearized action on H^1 becomes the product of the two matrices of dimensions $\lambda * c \times \lambda * c$:

$$\begin{pmatrix} 0_c & \mathbb{I}_c & 0_c & \ldots & 0_c \\ 0_c & 0_c & \mathbb{I}_c & \ldots & 0_c \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \mathbb{I}_c & 0_c & 0_c & 0_c & 0_c \end{pmatrix} \quad \begin{pmatrix} \mathbb{I}_c & 0_c & \ldots & 0_c & 0_c \\ 0_c & \mathbb{I}_c & \ldots & 0_c & 0_c \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0_c & 0_c & \ldots & \mathbb{I}_c & 0_c \\ 0_c & 0_c & \ldots & 0_c & A \end{pmatrix} \tag{7.16}$$

where A is the linearized action on one component Y (whose age is given in 7.7). The age of the first matrix is then computed as:

$$\lambda \frac{c-1}{2}$$

So for each vertex v_i, we have a term $a(X_i) + \lambda_i \left(\frac{3g_i - 3 + n_i - 1}{2}\right)$, thus proving the assertion. □

Now we compute the age of the twisted sector constructed from base twisted sectors, adding rational tails (4.61)

Proposition 7.17. *Let* $Y_{(\alpha(1),\ldots,\alpha(k))}$ *be a* $k-$*base twisted sector of rational type (Definition 4.42). Let* $I_1 \sqcup \ldots \sqcup I_k$ *be a partition of* $[n]$ *with all the* I_i*'s non empty. Let* $K_1, \ldots, K_l, \ldots, K_{k-1}$ *be sets as in Notation 4.65. The twisted sector* $Y^{K_1,\ldots,K_{k-1}}_{(\alpha(1),\ldots,\alpha(k))}$ *has age equal to:*

$$a(Y) + \frac{1}{N} \sum_{l \in \mu_N^*} l \left(b_l - \sum_{I_i \in K_l} \delta(I_i) \right)$$

where the parameters b_l *are defined in 4.65.*

Proof. (sketch) These twisted sectors are obtained by simply gluing base twisted sectors with rational marked curves. The age of these new sectors is then the old one, plus the age of the representation of the automorphism on $H^0(C, \mathcal{E}xt^1(\omega(\sum x_i), \mathcal{O}_C))$ of 2.49. This action is computed thanks to Lemma 7.13, as we have seen at the beginning of the proof of Proposition 7.14. □

Lemma 7.18. *Let $n \geq 2$. The age of the twisted sectors of $\left[\overline{\mathcal{M}}_{0,n+2}/S_2\right]$ is 1 if they are of codimension 2, and $\frac{1}{2}$ if they are of codimension 1.*

Proof. This follows directly from Proposition 7.3. □

Let (C, α) be a general element of a twisted sector Y of $\overline{\mathcal{M}}_g$, and suppose that C has a node p fixed by α. We can perform the operation that we called "adding a rational bridge" in Section 4.3 by blowing up the node p and adding marked points on the resulting rational component. We have seen that this operation produces one new twisted sector, if the node is stabilized without switching the two components (stabilized as directed graph 2.38), or several twisted sectors, if the node is stabilized by switching the two components (4.69). We call the twisted sector obtained by adding a rational bridgeo in this way Y'. Following the same argument sketched in Proposition 7.14, and using Lemma 7.18 we can prove:

Lemma 7.19. *Let Y' and Y be two twisted sectors as above, then if the node p is stabilized as a directed graph (2.38), the age of Y' is obtained by replacing the term $\left\{\frac{\alpha_v(h)}{N_v} + \frac{\alpha_{v'}(h')}{N_{v'}}\right\}$ in Formula 7.14, which corresponds to the node $e = h + h'$, with $\frac{\alpha_v(h)}{N_v} + \frac{\alpha_{v'}(h')}{N_{v'}}$. If the node is stabilized as an undirected graph (2.38), then the same term must be substituted by $\frac{\alpha_v(h)}{N_v} + \frac{\alpha_{v'}(h')}{N_{v'}} + 1$*

7.1.2 The second Inertia Stack

The definition of the Chen–Ruan product involves the second Inertia Stack.

Definition 7.20. Let X be an algebraic stack. The *second Inertia Stack* $I_2(X)$ is defined as:

$$I_2(X) = I(X) \times_X I(X)$$

Remark 7.21. An object in $I_2(X)$ is a triplet (x, g, h) where x is an object of X and $g, h \in \mathrm{Aut}(x)$. It can equivalently be given as $(x, g, h, (gh)^{-1})$.

Remark 7.22. $I_2(X)$ comes with three natural morphisms to $I(X)$: p_1 and p_2, the two projections of the fiber product, and p_3 which acts on points sending (x, g, h) to (x, gh).

$$\begin{tikzcd}
 & (x,g) \\
(x,g,h) \arrow[ur, bend left, "p_1"] \arrow[r, "p_2"] \arrow[dr, bend right, "p_3"] & (x,h) \\
 & (x,gh)
\end{tikzcd}$$

This gives the following diagram, where $(Y, g, h, (gh)^{-1})$ is a double twisted sector and (X_1, g),

(X_2, h), $(X_3, (gh))$ are twisted sectors:

$$\begin{tikzcd} & (X_1, g) \\ (Y, g, h) \arrow[ur, bend left, "p_1"] \arrow[r, "p_2"] \arrow[dr, bend right, "p_3"'] & (X_2, h) \\ & (X_3, gh) \end{tikzcd} \tag{7.23}$$

Remark 7.24. If X is a Deligne–Mumford stack, the space $\mathcal{K}_{g,n}(X, \beta)$ (see for instance [AGV06, 4.3]) of twisted stable maps is introduced. Evaluation maps e_i are then defined:

$$e_i : \mathcal{K}_{g,n}(X, \beta) \to \bar{I}(X)$$

As first observed in [AGV02, 4.5], in general there are no liftings of such maps to the Inertia Stack $I(X)$. However, in the special case that we are dealing with, $g = 0$, $n = 3$, $\beta = 0$, it is easy to prove that $\mathcal{K}_{0,3}(X, 0) \cong I_2(X)$, and there does exist lifted evaluation maps (see [AGV02, 6.2.1]):

$$\tilde{e}_i : \mathcal{K}_{g,n}(X, \beta) \to I(X)$$

Let now i be the involution in the Inertia Stack, defined in 4.9. One can check that $p_1 = e_1$, $p_2 = e_2$ and $p_3 = i \circ e_3$.

7.1.3 The excess intersection bundle

We review the definition of the excess intersection bundle over $I_2(X)$, for X an algebraic smooth stack. Let $(Y, g, h, (gh)^{-1})$ be a twisted sector of $I_2(X)$. Let H be the group generated by g and h.

Construction 7.25. Let $\gamma_0, \gamma_1, \gamma_\infty$ be three small loops around $0, 1, \infty \subset \mathbb{P}^1$. Any map $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \to H$ corresponds to an $H-$principal bundle on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Let $\pi^0 : C^0 \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ be the $H-$principal bundle given by the map $\gamma_0 \to g, \gamma_1 \to h, \gamma_\infty \to (gh)^{-1}$. It can be uniquely extended to a ramified $H-$Galois covering $C \to \mathbb{P}^1$ (see [FG03, Appendix]), where C is a smooth compact curve. Note that H acts on $H^1(C, \mathcal{O}_C)$.

Let $f : Y \to X$ be the restriction of the canonical map $I_2(X) \to X$ to the twisted sector Y.

Definition 7.26. [CR04] With the same notation as in the previous paragraph, the *excess intersection bundle* over Y is defined as:

$$E_Y = \left(H^1(C, \mathcal{O}_C) \otimes_{\mathbb{C}} f^*(T_X)\right)^H$$

i.e. the H-invariant subbundle of the expression between parenthesis.

Remark 7.27. Since $H^1(C, \mathcal{O}_C)^H = 0$, it is the same to consider in the previous definition:

$$\left(H^1(C, \mathcal{O}_C) \otimes N_Y X\right)^H$$

where N_Y is the coker of $T_Y \to f^*(T_X)$ (Y is smooth and $f|Y$ is an étale covering of a closed substack of X).

We can now review the definition of the Chen–Ruan product.

Definition 7.28. Let $\alpha \in H^*_{CR}(X)$, $\beta \in H^*_{CR}(X)$. We define:

$$\alpha *_{CR} \beta = p_{3*}\left(p_1^*(\alpha) \cup p_2^*(\beta) \cup c_{top}(E)\right)$$

We sometimes use the notation $\alpha * \beta$ instead of $\alpha *_{CR} \beta$.

Theorem 7.29. *([CR04]) With the age grading defined in the previous section, $(H^*_{CR}(X,\mathbb{Q}), *_{CR})$ is a graded $(H^*(X,\mathbb{Q}),\cup)$-algebra.*

Theorem 7.29 allows us to compute the rank of the excess intersection bundle in terms of the already computed age grading. If $(Y,(g,h,(gh)^{-1}))$ is a sector of the second Inertia Stack, the rank of the excess intersection bundle is (here we stick to the notation introduced in Remark 7.22):

$$\operatorname{rk}(E_{(Y,g,h)}) = a(X_1,g) + a(X_2,h) + a(X_3,(gh)^{-1}) - \operatorname{codim}_Y X \tag{7.30}$$

where the codimension is taken in X.

Corollary 7.31. *The excess intersection bundle over double twisted sectors when either g,h, or $(gh)^{-1}$ is the identity, is the zero bundle.*

One other useful formula that follows from Proposition 7.3 relates the rank of the excess bundle over a double twisted sector and the rank of the excess bundle over the double twisted sector obtained inverting the automorphisms that label the sector:

$$\operatorname{rk}(E_{(Y,g^{-1},h^{-1})}) = \operatorname{codim}(X_1) + \operatorname{codim}(X_2) + \operatorname{codim}(X_3) - 2\operatorname{codim}(Y) - \operatorname{rk}(E_{(Y,g,h)}) \tag{7.32}$$

Let $(Y,(g,h,(gh)^{-1}))$ be a double twisted sector in $I_2(\overline{\mathcal{M}}_{1,n})$, and let H be the group generated by $(g,h,(gh)^{-1})$. We want to study $N_Y X$ and $H^1(C,\mathcal{O}_C)$ as representations of H. So we want an explicit decomposition pointwise (which extends locally on open neighbouroods):

$$N_Y X(x) = \bigoplus_{\chi\in H^*} N_\chi(x)$$

and a decomposition of the vector space:

$$H^1(C,\mathcal{O}_C) = \bigoplus_{\chi\in H^*} H_\chi$$

Note that in the first decomposition $\operatorname{rk}(N_\chi)$ does not depend on the point x. Moreover, by construction the trivial character does not appear in either of the two direct sums.

7.2 The genus 1 case

We now describe the age in the compact case. We use the convention that $\delta(I) = \delta_{1,|I|}$, the Kronecker delta.

Lemma 7.33. *The following table gives the age of all twisted sectors of the Inertia Stack of $\overline{\mathcal{M}}_{1,n}$:*

Comp	Aut	Codimension	Age
$\overline{A_1}^{[n]}$	-1	1	$\frac{1}{2}$
$\overline{A_2}^{I_1,I_2}$	-1	$3-\delta(I_1)-\delta(I_2)$	$\frac{1}{2}(3-\delta(I_1)-\delta(I_2))$
$\overline{A_3}^{I_1,I_2,I_3}$	-1	$5-\delta(I_1)-\delta(I_2)-\delta(I_3)$	$\frac{1}{2}(5-\delta(I_1)-\delta(I_2)-\delta(I_3))$
$\overline{A_4}^{I_1,I_2,I_3,I_4}$	-1	$7-\delta(I_1)-\delta(I_2)-\delta(I_3)-\delta(I_4)$	$\frac{1}{2}(7-\delta(I_1)-\delta(I_2)-\delta(I_3)-\delta(I_4))$
$C_4^{[n]}$	i	2	$\frac{5}{4}$
$C_4^{[n]}$	$-i$	2	$\frac{3}{4}$
$C_4^{I_1,I_2}$	i	$4-\delta(I_1)-\delta(I_2)$	$\frac{11}{4}-\frac{3}{4}(\delta(I_1)+\delta(I_2))$
$C_4^{I_1,I_2}$	$-i$	$4-\delta(I_1)-\delta(I_2)$	$\frac{5}{4}-\frac{1}{4}(\delta(I_1)+\delta(I_2))$
$C_6^{I_1,I_2}$	ϵ^2	$4-\delta(I_1)-\delta(I_2)$	$\frac{7}{3}-\frac{2}{3}(\delta(I_1)+\delta(I_2))$
$C_6^{I_1,I_2}$	ϵ^4	$4-\delta(I_1)-\delta(I_2)$	$\frac{5}{3}-\frac{1}{3}(\delta(I_1)+\delta(I_2))$
$C_6^{I_1,I_2,I_3}$	ϵ^2	$6-\delta(I_1)-\delta(I_2)-\delta(I_3)$	$\frac{11}{3}-\frac{2}{3}(\delta(I_1)+\delta(I_2)+\delta(I_3))$
$C_6^{I_1,I_2,I_3}$	ϵ^4	$6-\delta(I_1)-\delta(I_2)-\delta(I_3)$	$\frac{7}{3}-\frac{1}{3}(\delta(I_1)+\delta(I_2)+\delta(I_3))$
$C_6^{[n]}$	ϵ	2	$\frac{3}{2}$
$C_6^{[n]}$	ϵ^2	2	1
$C_6^{[n]}$	ϵ^4	2	1
$C_6^{[n]}$	ϵ^5	2	$\frac{1}{2}$

Proof. This is a consequence of Proposition 7.12 and Proposition 7.17, after using the simplified notation of 5.15. □

If Y is a twisted sector that is obtained by attaching rational tails to a zero dimensional base twisted sector, in $N_Y X$ we discovered some subbundles that are trivial subeigenbundles with respect to the action of G. Namely, if we call Z the base twisted sector of Y and Z is an element of $\mathcal{M}_{1,k}$, then there is a subeigenbundle of $N_Y X$ that is the pull-back of $N_Z\ \mathcal{M}_{1,k}$. The list of all the couples (N, ψ), where N is a trivial subbundle of $N_Y X$ and ψ is the character that identifies the action of the group G on N will be very useful in the sequel. Writing N_χ for χ a character, we are assuming that N_χ exists as a subeigenbundle of N of complex dimension 1, and that it carries the character χ.

Remark 7.34. Let Y be a twisted sector whose base twisted sector (according to Definition 4.42) Z is a gerbe over a point in $\mathcal{M}_{1,k}$. We will use in the following that some subeigenbundles of $N_Y\ \mathcal{M}_{1,k}$ are trivial. We list them here, when we write $N_{\zeta_N^a}$ we mean the subeigenbundle corresponding to the irreducible representation ζ_N^a of μ_N:

- $G = \mu_3$,
  1. $Y = C_6^{[n]}$, N_{ζ_3};
  2. $Y = C_6^{I_1,I_2}$, $N_{\zeta_3} \bigoplus N_{\zeta_3^2}$;
  3. $Y = C_6^{I_1,I_2,I_3}$, $N_{\zeta_3} \bigoplus N_{\zeta_3^2}^{\oplus 2}$;
- $G = \mu_4$,
  1. $Y = C_4^{[n]}$, $N_{\zeta_4^2}$;
  2. $Y = C_4^{I_1,I_2}$, $N_{\zeta_4^2} \bigoplus N_{\zeta_4^3}$;
- $G = \mu_6$, $Y = C_6^{[n]}$, $N_{\zeta_6^4} \bigoplus N_{\zeta_6^5}$;

We can then give a formula for the vector spaces of $m-th$ Chen–Ruan cohomology (with t possibly rational).

Notation 7.35. We will now compute the dimension of $H^{2(m+\alpha)}(TS(n))$. It is obviously a function of t and n. The parameter α, which takes into account fractional contributions to the grading coming from the (fractional part of) the age, is indicated on the left of the table. All the sums over indices i, j, k, l have the convention that the sum of the indices of the coefficients a, b and c equal n, considering the 1's as well. Moreover, here the indices i,j,k,l always satisfy $2 \leq i \leq j \leq k \leq l$. The cases when some of these indices are equal to 1 are considered separately, since their contribution to the dimension can belong to a different grading of the Chen–Ruan cohomology. The sum over the indices p,q,r,s instead ranges over all the possible natural values of those indices (including zero). Once again, as in [Ke92, p.550], shifted by 1:

$$a^m(n) := H^{2m}(\overline{\mathcal{M}}_{0,n+1})$$

The table holds true for all m's when $n \geq 5$ (the stable range). It stays true when $n \geq 2$ for $m \geq 2$. To obtain correct formulae for $\overline{\mathcal{M}}_{1,2}$ one has to add 1 to the dimensions of $H^{2*\frac{1}{2}}$, $H^{2*\frac{3}{4}}$, H^{2*1}, H^{2*1} and $H^{2*\frac{5}{4}}$. For $\overline{\mathcal{M}}_{1,3}$ one has to add, in addition to the previous cases, 1 to the dimensions of H^{2*1}, $H^{2*\frac{4}{3}}$, $H^{2*\frac{5}{3}}$, $H^{2*\frac{3}{2}}$. For $\overline{\mathcal{M}}_{1,4}$, one adds in addition 1 to the dimension of $H^{2*\frac{3}{2}}$.

We agree with the reader that these formulae are nasty:

0	$2n\left(a^{m-1}(n-1)+a^{m-2}(n-1)\right)+a^{m-1}(n)+a^{m-2}(n)+\frac{n(n-1)(n-2)}{6}\left(a^{m-2}(n-3)+a^{m-3}(n-3)\right)$ $2\sum\binom{n}{1,i,j}\ b_{1ij}\left(\sum_{p+q=m-3}a^p(i)a^q(j)+\sum_{p+q=m-2}a^p(i)a^q(j)+2\right)+$ $\sum\binom{n}{1,i,j,k}\ c_{1ijk}\left(\sum_{p+q+r=m-3}a^p(i)a^q(j)a^r(k)+\sum_{p+q+r=m-4}a^p(i)a^q(j)a^r(k)\right)$
$\frac{1}{4}$	$a^{m-1}(n)+\sum\binom{n}{i,j}\ a_{ij}\sum_{p+q=m-1}a^p(i)a^q(j)$
$\frac{1}{3}$	$na^{m-1}(n-1)+\frac{n(n-1)}{2}a^{m-2}(n-2)+$ $\left(\sum\binom{n}{i,j}\ a_{ij}\sum_{p+q=m-2}a^p(i)a^q(j)\right)+\left(\sum\binom{n}{i,j,k}\ b_{ijk}\sum_{p+q+r=m-2}a^p(i)a^q(j)a^r(k)\right)$
$\frac{1}{2}$	$2\left(a^m(n)+a^{m-1}(n)\right)+\frac{n(n-1)}{2}\left(a^{m-1}(n-2)+a^{m-2}(n-2)\right)$ $\sum\binom{n}{i,j}\ a_{ij}\left(\sum_{p+q=m-1}a^p(i)a^q(j)\sum_{p+q=m-2}a^p(i)a^q(j)\right)$ $\sum\binom{n}{i,j,k}\ b_{ijk}\left(\sum_{p+q+r=m-2}a^p(i)a^q(j)a^r(k)+\sum_{p+q+r=m-3}a^p(i)a^q(j)a^r(k)\right)$ $\sum\binom{n}{i,j,k,l}\ c_{ijkl}\left(\sum_{p+q+r+s=m-3}a^p(i)a^q(j)a^r(k)a^s(l)+\sum_{p+q+r=m-4}a^p(i)a^q(j)a^r(k)a^s(l)\right)+$ $\left(\sum\binom{n}{1,1,i,j}\ c_{11ij}\ \sum_{p+q=m-2}a^p(i)a^q(j)\right)$
$\frac{2}{3}$	$na^{m-1}(n-1)+\frac{n(n-1)}{2}a^{m-1}(n-2)+$ $\left(\sum\binom{n}{i,j}\ a_{ij}\sum_{p+q=m-1}a^p(i)a^q(j)\right)+\left(\sum\binom{n}{i,j,k}\ b_{ijk}\sum_{p+q+r=m-3}a^p(i)a^q(j)a^r(k)\right)$
$\frac{3}{4}$	$a^m(n)+\sum\binom{n}{i,j}\ a_{ij}\sum_{p+q=m-2}a^p(i)a^q(j)$

We can make a power series out of this table. We define:

$$P_0(s,t) := \sum_{n=0}^{\infty}\frac{Q_0(n,m)}{n!}s^n t^m \tag{7.36}$$

$$P_1(s,t) := \sum_{n=0}^{\infty}\frac{Q_1(n,m)}{n!}s^n t^m \tag{7.37}$$

$$P_{1,\alpha}^{CR}(s,t) := \sum_{n=0}^{\infty}\frac{Q_{1,\alpha}^{CR}(n,m)}{n!}s^n \tag{7.38}$$

where:

$$Q_0(n,m) := \dim H^{2m}(\overline{\mathcal{M}}_{0,n+1}) = a^m(n)$$
$$Q_1(n,m) := \dim H^{2m}(\overline{\mathcal{M}}_{1,n})$$
$$Q^{CR}_{1,\alpha}(n) := \dim H^{2m+\alpha}_{CR}(\overline{\mathcal{M}}_{1,n})$$

The relevant values of α are $0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$.

We recall that the power series of 7.36 are described in [Ge94, Theorem 5.9], [Ge98, Theorem 2.6]. There, the author describes the cohomology of the moduli of genus 0 and genus 1, n-pointed stable curves as a representation of S_n. The same could be done for the Chen-Ruan cohomology, though we choose not to explicitly write the representations of S_n in our power series, for the sake of simplicity.

Theorem 7.39. *The following equality between power series relates the dimension of the m-th Chen–Ruan cohomology group of $\overline{\mathcal{M}}_{1,n}$ with the dimensions of the m-th cohomology group of $\overline{\mathcal{M}}_{0,n}$ and $\overline{\mathcal{M}}_{1,n}$.*

$$\begin{aligned}
P^{CR}_{1,0}(s,t) &= P_1 + (t+t^2)P_0 + 3s(t^2+t^3)P_0^2 + \tfrac{s}{24}(t^3+t^4)P_0^3 + 2s(t+t^2)\tfrac{\partial}{\partial s}(sP_0) + \tfrac{s^3}{6}(t^2+t^3)\tfrac{\partial^3}{\partial s^3}(s^3P_0)\\
P^{CR}_{1,\frac{1}{4}}(s,t) &= tP_0 + \tfrac{t^2}{2}P_0^2\\
P^{CR}_{1,\frac{1}{3}}(s,t) &= \tfrac{t^2}{2}P_0^2 + \tfrac{t^2}{6}P_0^3 + ts\tfrac{\partial}{\partial s}(sP_0) + \tfrac{s^2t^2}{2}\tfrac{\partial^2}{\partial s^2}(s^2P_0)\\
P^{CR}_{1,\frac{1}{2}}(s,t) &= 2(1+t)P_0 + \tfrac{t+t^2}{2}P_0^2 + \tfrac{t^2+t^3}{6}P_0^3 + \tfrac{t^3+t^4}{24}P_0^4 + \tfrac{s^2}{2}(t+t^2)\tfrac{\partial^2}{\partial s^2}(s^2P_0)\\
P^{CR}_{1,\frac{2}{3}}(s,t) &= \tfrac{t}{2}P_0^2 + \tfrac{t^3}{6}P_0^3 + st\tfrac{\partial}{\partial s}(sP_0) + \tfrac{s^2}{2}t\tfrac{\partial^2}{\partial s^2}(s^2P_0)\\
P^{CR}_{1,\frac{3}{4}}(s,t) &= P_0 + \tfrac{t}{2}P_0^2
\end{aligned}$$

The second Inertia Stack of moduli of stable genus 1 curves

We here study the second Inertia Stack of $\overline{\mathcal{M}}_{1,n}$. We will see that this case is especially easy.

Remark 7.40. We label each sector of $I_2(X)$ via the triplet $(g,h,(gh)^{-1})$. There are two automorphism groups acting on $I_2(X)$: an involution sending a sector labelled with $(g,h,(gh)^{-1})$ into $(g^{-1},h^{-1},(gh))$, and S_3 which permutes the three automorphisms. Up to permutations and involution, the following are all possible labels of the sectors in $I_2(X)$:

- $(1,1,1)$, generated group μ_1;
- $(1,-1,-1)$, generated group μ_2;
- $(\epsilon^2,\epsilon^2,\epsilon^2)$, generated group μ_3;
- $(1,\epsilon^2,\epsilon^4)$, generated group μ_3;
- $(1,i,-i)$ generated group μ_4;
- $(i,i,-1)$, generated group μ_4;
- $(\epsilon,\epsilon,\epsilon^4)$, generated group μ_6;
- $(\epsilon,\epsilon^2,-1)$, generated group μ_6;
- $(1,\epsilon,\epsilon^5)$, generated group μ_6.

We now describe the sectors of the double Inertia Stack. We do so up to the automorphisms described in the previous remark, and up to the permutations of the marked points.

Proposition 7.41. *Up to permutation of the automorphisms, and up to involution, the following are the twisted sectors of $I_2(\overline{\mathcal{M}}_{1,n})$:*

$$\left(\overline{A_1}^{[n]},(1,-1,-1)\right),\quad \left(\overline{A_2}^{I_1,I_2},(1,-1,-1)\right),\quad \left(\overline{A_3}^{I_1,I_2,I_3},(1,-1,-1)\right),\quad \left(\overline{A_4}^{I_1,I_2,I_3,I_4},(1,-1,-1)\right)$$

$$\left(C_6^{I_1,I_2},(1,\epsilon^2,\epsilon^4)/(\epsilon^2,\epsilon^2,\epsilon^2)\right),\quad \left(C_6^{I_1,I_2,I_3},(1,\epsilon^2,\epsilon^4)/(\epsilon^2,\epsilon^2,\epsilon^2)\right),$$
$$\left(C_4^{[n]},(1,i,-i)/(i,i,-1)\right),\quad \left(C_4^{I_1,I_2},(1,i,-i)/(i,i,-1)\right)$$
$$\left(C_6^{[n]},(1,\epsilon,\epsilon^5)/(\epsilon,\epsilon,\epsilon^4)/(\epsilon,\epsilon^2,-1)/(\epsilon^2,\epsilon^2,\epsilon^2)\right)$$

Proof. This follows from Theorem 5.23, once one observes that no point in $\overline{\mathcal{M}}_{1,n}$ is stable under the action of both ϵ and i. □

From this, a very easy consideration follows:

Corollary 7.42. *Let $(Z,g,h,(gh)^{-1})$ be a double twisted sector of $\overline{\mathcal{M}}_{1,n}$. Then either (Z,g) or (Z,h) or $(Z,(gh)^{-1})$ is a twisted sector of the Inertia Stack of $\overline{\mathcal{M}}_{1,n}$.*

The excess intersection bundle in genus 1

Thanks to Corollary 7.31, the double twisted sectors whose excess intersection bundles have non zero rank are labelled by:

$$(\epsilon^2,\epsilon^2,\epsilon^2),(i,i,-1),(\epsilon,\epsilon,\epsilon^4),(\epsilon,\epsilon^2,-1) \tag{7.43}$$

up to permutation and involution. The top Chern classes of the excess intersection bundles for $\mathcal{M}_{1,n}$ are always 0 or 1, since the coarse moduli spaces of the double twisted sectors labelled by these automorphisms are points.

The rank of the excess intersection bundles for the twisted sectors labelled by 7.43 can be given thanks to formulae 7.30 and 7.32:

Proposition 7.44. *In the following table we present the ranks for the excess intersection bundles over all double twisted sectors:*

(g,h)	Double twisted sector	$\mathrm{rk}(E)$	(g^{-1},h^{-1})	$\mathrm{rk}(E)$
(ϵ^2,ϵ^2)	$C_6^{[n]}$	1	(ϵ^4,ϵ^4)	1
(ϵ^2,ϵ^2)	$C_6^{I_1,I_2}$	3	(ϵ^4,ϵ^4)	1
(ϵ^2,ϵ^2)	$C_6^{I_1,I_2,I_3}$	5	(ϵ^4,ϵ^4)	1
(i,i)	$C_4^{[n]}$	1	$(-i,-i)$	0
(i,i)	$C_4^{I_1,I_2}$	3	$(-i,-i)$	0
(ϵ,ϵ)	$C_6^{[n]}$	2	(ϵ^5,ϵ^5)	0
(ϵ,ϵ^2)	$C_6^{[n]}$	1	(ϵ^4,ϵ^4)	0

Remark 7.45. Corollary 7.31, together with the proposition above, tells us that a lot of top Chern classes of excess intersection bundles are 1, namely all top Chern classes of excess intersection bundles whose rank is 0.

Now we want to compute explicitly the excess intersection bundles and their top Chern classes for $\overline{\mathcal{M}}_{1,n}$. The following two propositions give the results we need in this direction.

Firstly, we give the decomposition of $H^1(C,\mathcal{O}_C)$ as a representation of H in the cases corresponding to non zero ranks in Proposition 7.44.

Proposition 7.46. *Here are the non zero dimensions $h_\chi = \dim(H_\chi)$ in the cases contemplated in Proposition 7.44, when the rank of the excess intersection bundle is not zero:*

- $H=\mu_3$*:* $(\epsilon^2,\epsilon^2,\epsilon^2)$*,* $p_g(C)=1$*,* $h_{\zeta_3}=1$*;*
- $H=\mu_3$*:* $(\epsilon^4,\epsilon^4,\epsilon^4)$*,* $p_g(C)=1$*,* $h_{\zeta_3^2}=1$*;*
- $H=\mu_4$*:* $(i,i,-1)$*,* $p_g(C)=1$*,* $h_{\zeta_4}=1$*;*

- $H = \mu_6$: $(\epsilon, \epsilon, \epsilon^4)$, $p_g(C) = 2$, $h_{\zeta_6} = h_{\zeta_6^2} = 1$;
- $H = \mu_6$: $(\epsilon, \epsilon^2, -1)$, $p_g(C) = 1$, $h_{\zeta_6} = 1$.

Proof. This is a direct computation which uses the tools developed in [Pa91]. The full computation is available in one example in [P08, Proposition 7.11]. $\square$

Secondly we give the description of $N_Y\overline{\mathcal{M}}_{1,n}$, where $Y \in I_2(\overline{\mathcal{M}}_{1,n})$ as a representation of H, in the relevant cases:

Proposition 7.47. *Here are the non zero dimensions $n_\chi = \mathrm{rk}(N_\chi)$, where $N = N_Y\overline{\mathcal{M}}_{1,n}$ in the cases listed in Proposition 7.44 where the rank of the excess intersection bundle is non zero:*

- $H = \mu_3$, $(\epsilon^2, \epsilon^2, \epsilon^2)$, $(\epsilon^4, \epsilon^4, \epsilon^4)$;
  1. $Y = C_6^{[n]}$, $n_{\zeta_3} = 1$, $n_{\zeta_3^2} = 1$,
  2. $Y = C_6^{I_1, I_2}$, $n_{\zeta_3} = 1$, $n_{\zeta_3^2} = 3 - \delta(I_1) - \delta(I_2)$,
  3. $Y = C_6^{I_1, I_2, I_3}$, $n_{\zeta_3} = 1$, $n_{\zeta_3^2} = 5 - \delta(I_1) - \delta(I_2) - \delta(I_3)$,
- $H = \mu_4$, $(i, i, -1)$;
  1. $Y = C_4^{[n]}$, $n_{\zeta_4^2} = 1$, $n_{\zeta_4^3} = 1$,
  2. $Y = C_4^{I_1, I_2}$, $n_{\zeta_4^2} = 1$, $n_{\zeta_4^3} = 3 - \delta(I_1) - \delta(I_2)$,
- $H = \mu_6$, $(\epsilon, \epsilon, \epsilon^4)$; $Y = C_6^{[n]}$, $n_{\zeta_6^4} = 1$, $n_{\zeta_6^5} = 1$,
- $H = \mu_6$, $(\epsilon, \epsilon^2, -1)$; $Y = C_6^{[n]}$, $n_{\zeta_6^4} = 1$, $n_{\zeta_6^5} = 1$.

Proof. The computation of the characters has already been carried out in Lemma 7.33, when we calculated the age. $\square$

With all this, and thanks to Remark 7.34, we can compute the excess intersection bundles and their respective top Chern classes. In the following Corollary, we call respectively pr_1 and pr_2 the projections onto the first and second factor of a product.

Corollary 7.48. *The following are all double twisted sectors whose top Chern classes of the excess intersection bundles are* 0*:*

- *All double twisted sectors with automorphisms* $(\epsilon^4, \epsilon^4, \epsilon^4)$;
- *All double twisted sectors with automorphisms* $(\epsilon, \epsilon, \epsilon^4)$;
- *All double twisted sectors* $C_4^{I_1, I_2}$ *with automorphisms* $(i, i, -1)$;
- *All double twisted sectors* $C_6^{I_1, I_2}$ *and* $C_6^{I_1, I_2, I_3}$ *with both the automorphisms* $(\epsilon^2, \epsilon^2, \epsilon^2)$ *and* $(\epsilon^4, \epsilon^4, \epsilon^4)$.

Proof. We can use now the study of Remark 7.34, where we listed some trivial subbundles of the normal bundles to double twisted sectors. In the two Propositions 7.46 and 7.47 we studied the H-invariant part of $H^1(C, \mathcal{O}_C) \otimes N_Y X$ (notation as in Definition 7.26), and we pointed out that these subbundles are subeigenbundles with respect to the action of H. This implies that all excess intersection bundles over the double twisted sectors listed in the statement of this corollary have a trivial subbundle, and from this the triviality of the top Chern class follows. $\square$

In the following diagram and in the following lemma, we identify the isomorphic spaces in order to simplify the notation for the projection maps.

$$\begin{tikzcd}
 & & B\mu_a \\
C_a^{[n]} \arrow[r] \arrow[urr, bend left, "pr_1"] \arrow[drr, bend right, "pr_2"'] & B\mu_a \times \overline{\mathcal{M}}_{0,n\sqcup\bullet} \arrow[ur, "pr_1"] \arrow[dr, "pr_2"'] & \\
 & & \overline{\mathcal{M}}_{0,n\sqcup\bullet}
\end{tikzcd}$$

where a can be 4 or 6. Remember that a bundle over a trivial gerbe is given as in Remark 1.9.

Corollary 7.49. *The only excess intersection bundles over double twisted sectors of $\overline{\mathcal{M}}_{1,n}$ that are non zero and that do not have any trivial subbundle are:*

- *over $(C_6^{[n]}, (\epsilon^2, \epsilon^2, \epsilon^2))$ the bundle is the pair $(\mathbb{L}_\bullet^\vee, \zeta_3^2)$;*
- *over $(C_4^{[n]}, (i, i, -1))$ the bundle is the pair $(\mathbb{L}_\bullet^\vee, \zeta_4^2)$;*
- *over $(C_6^{[n]}, (\epsilon, \epsilon^2, -1))$ the bundle is the pair $(\mathbb{L}_\bullet^\vee, \zeta_6^4)$.*

In the following corollary pr_2 is the projection onto the second factor of the product.

Corollary 7.50. *The only top Chern classes of the excess intersection bundles over double twisted sectors of $\overline{\mathcal{M}}_{1,n}$ that are not* 0 *nor* 1 *are:*

- *$(C_6^{[n]}, (\epsilon^2, \epsilon^2, \epsilon^2)) \cong B_{\mu_6} \times \overline{\mathcal{M}}_{0,n\sqcup\bullet}$, where the top Chern class of the excess intersection bundle is $-pr_2^*(\psi_\bullet)$;*
- *$(C_4^{[n]}, (i, i, -1)) \cong B_{\mu_4} \times \overline{\mathcal{M}}_{0,n\sqcup\bullet}$, where the top Chern class of the excess intersection bundle is $-pr_2^*(\psi_\bullet)$;*
- *$(C_6^{[n]}, (\epsilon, \epsilon^2, -1)) \cong B_{\mu_6} \times \overline{\mathcal{M}}_{0,n\sqcup\bullet}$, where the top Chern class of the excess intersection bundle is $-pr_2^*(\psi_\bullet)$.*

Note that when $n = 2$ the top Chern classes in the corollary above are 0 since the sectors involved are all points.

To conclude, we summarize the result we have obtained in this section:

Theorem 7.51. *All top Chern classes of excess intersection bundles over all double twisted sectors are explicitly given. They can be:*

1. *either* 1*, for all sectors listed in Proposition 7.41 (including inverses and permutations) such that one of the three automorphisms of the labelling is* 1*, and for all sectors in Remark 7.45;*
2. *either* 0*, for all sectors listed in Corollary 7.48;*
3. *or a pullback of a ψ class over a component $\overline{\mathcal{M}}_{0,n}$, for the sectors listed in Corollary 7.50.*

7.3 The genus 2 case

Following Proposition 7.7 and Corollary 7.10, it is easy to compute the degree shifting number for all the twisted sectors of $\mathcal{M}_{2,n}$.

$\mathcal{M}_{2,0}$	$\mathcal{M}_{2,1}$	$\mathcal{M}_{2,2}$	$\mathcal{M}_{2,3}$	$\mathcal{M}_{2,4}$
$a(\tau)=0$	$a(\tau_1)=\frac{1}{2}$	$a(\tau_{11})=1$	$a(\tau_{111})=\frac{3}{2}$	$a(\tau_{1111})=2$
$a(III)=1$	$a(III_1)=\frac{4}{3}$	$a(III_{11})=\frac{5}{3}$	$a(III_{112})=\frac{7}{3}$	$a(III_{1122})=3$
		$a(III_{12})=2$	$a(III_{121})=\frac{7}{3}$	$a(III_{1212})=3$
	$a(III_2)=\frac{5}{3}$	$a(III_{21})=2$	$a(III_{211})=\frac{7}{3}$	$a(III_{2112})=3$
		$a(III_{22})=\frac{7}{3}$	$a(III_{221})=\frac{8}{3}$	$a(III_{2211})=3$
$a(IV)=1$	$a(IV_1)=\frac{5}{4}$	$a(IV_{13})=2$		
	$a(IV_3)=\frac{7}{4}$	$a(IV_{31})=2$		
$a(X.4)=\frac{6}{5}$	$a(X.4_1)=\frac{7}{5}$	$a(X.4_{11})=\frac{8}{5}$	$a(X.4_{113})=\frac{11}{5}$	
		$a(X.4_{13})=2$	$a(X.4_{131})=\frac{11}{5}$	
	$a(X.4_3)=\frac{9}{5}$	$a(X.4_{31})=2$	$a(X.4_{311})=\frac{11}{5}$	
$a(X.6)=\frac{9}{5}$	$a(X.6_2)=\frac{11}{5}$	$a(X.6_{24})=3$	$a(X.6_{244})=\frac{19}{5}$	
	$a(X.6_4)=\frac{13}{5}$	$a(X.6_{42})=3$	$a(X.6_{424})=\frac{19}{5}$	
		$a(X.6_{44})=\frac{17}{5}$	$a(X.6_{442})=\frac{19}{5}$	
$a(X.2)=\frac{8}{5}$	$a(X.2_1)=\frac{9}{5}$	$a(X.2_{12})=\frac{11}{5}$	$a(X.2_{122})=\frac{13}{5}$	
	$a(X.2_2)=2$	$a(X.2_{21})=\frac{11}{5}$	$a(X.2_{212})=\frac{13}{5}$	
		$a(X.2_{22})=\frac{12}{5}$	$a(X.2_{221})=\frac{13}{5}$	
$a(X.8)=\frac{7}{5}$	$a(X.8_3)=2$	$a(X.8_{33})=\frac{13}{5}$	$a(X.8_{334})=\frac{17}{5}$	
		$a(X.8_{34})=\frac{14}{5}$	$a(X.8_{343})=\frac{17}{5}$	
	$a(X.8_4)=\frac{11}{5}$	$a(X.8_{43})=\frac{14}{5}$	$a(X.8_{433})=\frac{17}{5}$	
$a(V.1)=\frac{3}{2}$	$a(V.1_1)=\frac{5}{3}$	$a(V.1_{11})=\frac{11}{6}$		
$a(V.2)=\frac{3}{2}$	$a(V.2_5)=\frac{7}{3}$	$a(V.2_{55})=\frac{19}{6}$		
$a(VI)=1$				
$a(VIII.1)=\frac{3}{2}$	$a(VIII.1_1)=\frac{13}{8}$	$a(VIII.1_{13})=2$		
	$a(VIII.1_3)=\frac{15}{8}$	$a(VIII.1_{31})=2$		
$a(VIII.2)=\frac{3}{2}$	$a(VIII.2_5)=\frac{17}{8}$	$a(VIII.2_{57})=3$		
	$a(VIII.2_7)=\frac{19}{8}$	$a(VIII.2_{75})=3$		
$a(X.7)=\frac{8}{5}$	$a(X.7_2)=2$	$a(X.7_{23})=\frac{13}{5}$		
	$a(X.7_3)=\frac{11}{5}$	$a(X.7_{32})=\frac{13}{5}$		
$a(X.3)=\frac{7}{5}$	$a(X.3_7)=\frac{14}{5}$	$a(X.7_{78})=\frac{22}{5}$		
	$a(X.3_8)=3$	$a(X.3_{87})=\frac{22}{5}$		
$a(X.1)=\frac{9}{5}$	$a(X.1_1)=2$	$a(X.1_{14})=\frac{14}{5}$		
	$a(X.1_4)=\frac{13}{5}$	$a(X.1_{41})=\frac{14}{5}$		
$a(X.9)=\frac{6}{5}$	$a(X.9_6)=\frac{12}{5}$	$a(X.9_{69})=\frac{21}{5}$		
	$a(X.9_9)=3$	$a(X.9_{96})=\frac{21}{5}$		
$a(II)=\frac{1}{2}$	$a(II_1)=1$	$a(II_{11})=\frac{3}{2}$		

The two missing twisted sectors are τ_{11111}, whose age is $\frac{5}{2}$, and τ_{111111}, whose age is 3.

It is now a straightforward application of Propositions 7.14, 7.17, 7.19 to compute the degree shifting numbers for all the remaining twisted sectors of $\overline{\mathcal{M}}_{2,n}$. We will not write down all these results.

From all this, one can explicitly determine the Chen–Ruan Poincaré polynomials of $\overline{\mathcal{M}}_{2,n}$ defined as:

$$P^{CR}_{2,n}(t) := \sum_m \dim H^{2m}_{CR}(\overline{\mathcal{M}}_{2,n})\ t^m$$

We write the results for $n = 0$.

Theorem 7.52. *The Chen–Ruan Poincaré polynomial of $\overline{\mathcal{M}}_2$ is:*

$$P_2^{CR}(t) = 2+4t^{\frac{1}{2}}+2t^{\frac{3}{4}}+16t^1+2t^{\frac{6}{5}}+8t^{\frac{5}{4}}+2t^{\frac{4}{3}}+2t^{\frac{7}{5}}+21t^{\frac{3}{2}}+2t^{\frac{8}{5}}+2t^{\frac{5}{3}}+8t^{\frac{7}{4}}+2t^{\frac{9}{5}}+16t^2+2t^{\frac{9}{4}}+4t^{\frac{5}{2}}+2t^3$$

Chapter 8

The Chen-Ruan Cohomology as an Algebra on the Cohomology

In this chapter, we focus on the multiplicative structure of the Chen-Ruan cohomology ring of moduli spaces of curves. From the additive generators of the Chen-Ruan cohomology groups, we want to select a (smaller) set of multiplicative generators, and then to study their relations. Our point of view is to study the ring $H^*_{CR}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ as an algebra over the ordinary cohomology ring $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$. We found the multiplicative generators and all the relations among these generators. In this chapter and in the following one, we have restricted our attention to the special cases of genus 1 (with marked points). The results we mention have appeared in [P08]. In this special case, the map $I(X) \to X$ restricted to a twisted sector is a closed embedding (5.18).

In order to compute the Chen–Ruan product, one has to compute pull–backs from the twisted sectors to the double twisted sectors and push–forwards from the double twisted sectors to the twisted sectors. Thanks to Corollary 7.31, it is necessary and sufficient to compute the push–forward and the pull–back between twisted sectors of the Inertia Stack.

In this section we fix n and call $X := \overline{\mathcal{M}}_{1,n}$. Let (Y, g) be a twisted sector of X, and $f : Y \to X$ be the closed embedding of the twisted sector.

Lemma 8.1. *The cycle map:*

$$A^*(Y, \mathbb{Q}) \to H^{2*}(Y, \mathbb{Q})$$

is a graded ring isomorphism. Moreover the Chow ring of all twisted sectors is generated by divisors.

Proof. All factors of all twisted sectors have Chow ring isomorphic to the even cohomology. The cohomology ring of $\overline{\mathcal{M}}_{0,n}$ is generated by divisors due to the work of Keel [Ke92]. The spaces $\overline{A_1}$, $\overline{A_2}$, $\overline{A_3}$, $\overline{A_4}$ all have coarse moduli space isomorphic to $\mathbb{P}^1$, hence their cohomology is generated by divisors. □

We can now state and prove the result announced in the introduction. For some results needed in the proof we refer to the following two subsections on pull–back and push–forward.

Theorem 8.2. *The Chen–Ruan cohomology ring of $\overline{\mathcal{M}}_{1,n}$ is generated as an $H^*(\overline{\mathcal{M}}_{1,n}, \mathbb{Q})$-algebra by the fundamental classes of the twisted sectors with explicit relations.*

Proof. Since f^* is surjective, as we will prove in Corollary 8.18, using the projection formula, one only has to compute the Chen–Ruan products of the fundamental classes of the twisted sectors. Here is how to do so.

Let (X_1, g), (X_2, h) and (X_3, gh) be three twisted sectors of $\overline{\mathcal{M}}_{1,n}$, f_i are the closed embeddings of X_i in X. Let $\alpha_1 \in H^*(X_1, \mathbb{Q}), \alpha_2 \in H^*(X_2, \mathbb{Q})$. We want to compute $\alpha_1 *_{CR} \alpha_2$. We call $\tilde{\alpha_i}$ the two liftings of α_i to $H^*(X, \mathbb{Q})$ using the surjectivity of f_i. Let $p_3 : X_1 \times_X X_2 \to X_3$ be the third projection of the double twisted sector as in Formula 7.23. Let E be the excess intersection bundle on $X_1 \times_X X_2$, and $\gamma = p_{3*}(c_{top}(E))$. Then we have:

$$\alpha *_{CR} \beta = (\tilde{\alpha_1} * \tilde{\alpha_2}) * 1_{X_1} * 1_{X_2} = f_3^*(\tilde{\alpha_1} \cup \tilde{\alpha_2}) \cup \gamma \tag{8.3}$$

Thanks to this, the computation of $H^*_{CR}(\overline{\mathcal{M}}_{1,n}, \mathbb{Q})$ is determined by:

- the cup product in $H^*(\overline{\mathcal{M}}_{1,n}, \mathbb{Q})$;
- the map f_3^*, which is studied in the following subsection on pull–backs;
- the computation of the γ's, which involves push–forward computations.

The pull–back morphism is determined in Proposition 8.17 for all cohomology classes (using also the result of Theorem 8.12). As follows from Theorem 7.51, all top Chern classes of excess intersection bundles are pull–backs via $f_3 \circ p_3$ of ψ classes over boundary strata classes on $\overline{\mathcal{M}}_{1,n}$ (usually this object is referred to as a decorated boundary strata class).

In 8.25 we fix a candidate, for every couple X_1, X_2 of twisted sectors, of a cohomology class β such that $p_{3*}(c_{top}(E)) = f_3^*(\beta)$. Finally, we obtain the formula for the Chen–Ruan product:

$$\alpha_1 *_{CR} \alpha_2 = f_3^*(\tilde{\alpha_1} \cup \tilde{\alpha_2} \cup \beta)$$

This formula gives us knowledge of the Chen–Ruan product, once the cup product on the usual cohomology of $\overline{\mathcal{M}}_{1,n}$ is known.

□

This theorem reduces the computation of the Chen–Ruan product to the Chen–Ruan product of the fundamental classes of twisted sectors. The Chen–Ruan product of these fundamental classes is explicitly computed in Section 8.4.

8.1 The twisted sectors as linear combinations of products of divisors

In this section, we want to express the classes $[Y]$ for all Y a twisted sector of $\overline{\mathcal{M}}_{1,n}$, as linear combinations of elements in $R^*(\overline{\mathcal{M}}_{1,n})$. In fact it is possible to express them as linear combinations of products of divisor classes in $\overline{\mathcal{M}}_{1,n}$. This is due to the fact that there are base twisted sectors (4.42) in genus 1 only up to 4 marked points, and:

Theorem 8.4. *[Be98] The Chow ring of $\overline{\mathcal{M}}_{1,n}$ is generated by the divisor classes when $n \leq 5$.*

Notation 8.5. If Y is a base twisted sector, we can write $[Y] \in A^*(\overline{\mathcal{M}}_{1,n}) = R^*(\overline{\mathcal{M}}_{1,n}) = H^{ev}(\overline{\mathcal{M}}_{1,n}, \mathbb{Q})$ (since $n \leq 5$). If $i : Y \to \overline{\mathcal{M}}_{1,n}$ is the restriction of the map from the Inertia Stack, $[Y]$ is nothing but the push–forward via i of the fundamental class of the twisted sector Y. Moreover, due to Proposition 5.18 again, if Y is a twisted sector we can identify it with its image $i(Y)$ and write $Y \subset \overline{\mathcal{M}}_{1,n}$. The notation for the divisors is explained in 3.3

We stress that our result is manifestly symmetric under the action of S_n. We use the notation for the divisors introduced in Notation 3.3.

Theorem 8.6. *Let Y be the compactification of a base twisted sector of $\mathcal{M}_{1,n}$ (Definition 4.42, Definition 4.47, Theorem 5.23). We express it as a linear combination of products of divisor classes.*

- *Base space classes coming from $\overline{\mathcal{M}}_{1,1}$:*
  1. *$[\overline{A_1}] = 1$, the fundamental class of $\overline{\mathcal{M}}_{1,1}$;*
  2. *$[C_4] = \frac{1}{2} D_{irr}$*
  3. *$[C_6] = \frac{1}{3} D_{irr}$.*
- *Base space classes coming from $\overline{\mathcal{M}}_{1,2}$:*

1. $[\overline{A_2}] = \frac{1}{4}D_{irr} + 3D_{\{1,2\}}$;
2. $[C_4'] = \frac{1}{2}D_{irr}D_{\{1,2\}}$;
3. $[C_6'] = \frac{2}{3}D_{irr}D_{\{1,2\}}$.

- *Base space classes coming from $\overline{\mathcal{M}}_{1,3}$:*
    1. $[\overline{A_3}] = \frac{1}{4}D_{irr}(\sum_{\{i,j\}\subset\{1,2,3\}} D_{\{i,j\}}) + \frac{1}{4}D_{irr}D_{\{1,2,3\}} + 2D_{\{1,2,3\}}(\sum_{\{i,j\}\subset\{1,2,3\}} D_{\{i,j\}})$;
    2. $[C_6''] = \frac{2}{9}D_{irr}D_{\{1,2,3\}}(\sum_{\{i,j\}\subset\{1,2,3\}} D_{\{i,j\}})$.
- *Base space classes coming from $\overline{\mathcal{M}}_{1,4}$:*
    1. $[\overline{A_4}] = 2D_{\{1,2,3,4\}}(\sum_{\{i,j,k,l\}\subset\{1,2,3,4\}} D_{\{i,j\}}D_{\{k,l\}})+$
    $+\frac{1}{12}\sum_{\{i,j,k\}\subset\{1,2,3,4\}} D_{\{i,j,k\}}(\sum_{\{l,m\}\subset\{i,j,k\}} D_{\{i,j\}})+\frac{1}{12}D_{irr}D_{\{1,2,3,4\}}(\sum_{\{i,j\}\subset\{1,2,3,4\}} D_{\{i,j\}})$.

Proof. For the classes of the points the result is trivial. There is a little care involved in writing $[C_6'']$ as a linear combination that is S_3-invariant. We show how to obtain the result for the classes of the spaces $\overline{A_i}$.

We refer to [Be98] for all the bases of the Chow groups of $\overline{\mathcal{M}}_{1,n}$ that we use in the following. We have modified the bases that Belorousski finds in such a way that the sets of the elements of the bases are closed under the action of S_n.

First of all, a basis of $A^1(\overline{\mathcal{M}}_{1,2})$ is given by D_{irr} and $D_{\{1,2\}}$. Therefore:

$$[\overline{A_2}] = aD_{irr} + bD_{\{1,2\}} \tag{8.7}$$

taking the push–forward via $\pi_1 : \overline{\mathcal{M}}_{1,2} \to \overline{\mathcal{M}}_{1,1}$, and using that the forgetful morphism restricted to $\overline{A_2}$ is $3:1$ (5.22), gives that $b = 3$. Now taking on both sides of 8.7 the intersection product with $D_{\{1,2\}}$, and using that $D_{\{1,2\}}D_{\{1,2\}} = \frac{1}{24}$, we obtain $a = \frac{1}{4}$.

A basis of $A^2(\overline{\mathcal{M}}_{1,3})$ is given by:

$$D_{irr}D_{\{1,2\}},\ D_{irr}D_{\{1,3\}},\ D_{irr}D_{\{2,3\}},\ D_{irr}D_{\{1,2,3\}},\ D_{\{1,2,3\}}(\sum_{\{i,j\}\subset\{1,2,3\}} D_{\{i,j\}})$$

therefore $[\overline{A_3}]$ can be uniquely written as:

$$[\overline{A_3}] = aD_{irr}D_{\{1,2\}} + bD_{irr}D_{\{1,3\}} + c\ D_{irr}D_{\{2,3\}} + dD_{irr}D_{\{1,2,3\}} + eD_{\{1,2,3\}}(\sum_{\{i,j\}\subset\{1,2,3\}} D_{\{i,j\}}) \tag{8.8}$$

Taking the push-forwards via $\pi_{\{1,2\}}, \pi_{\{1,3\}}, \pi_{\{2,3\}}$, and using that these forgetful morphisms restricted to $\overline{A_3}$ are $2:1$ (Lemma 5.22), gives:

$$a + b = a + c = b + c = \frac{1}{2}, \quad e = 2$$

Now to determine d, intersect both sides of 8.8 with $D_{\{1,2\}}$ to find $d = c$.

To conclude, we have to work out the case of $\overline{A_4}$. A basis for $A^3(\overline{\mathcal{M}}_{1,4})$ is given by:

$$D_{\{1,2,3,4\}}\left(\sum_{\{i,j,k,l\}=\{1,2,3,4\}} D_{\{i,j\}}D_{\{k,l\}}\right) \quad D_{irr}D_{\{i,j,k\}}\left(\sum_{\{l,m\}\subset\{i,j,k\}} D_{\{l,m\}}\right) \ \{i,j,k\}\subset[4]$$

$$D_{irr}D_{\{1,2,3,4\}}D_{\{i,j,k\}}\ \{i,j,k\}\subset[4] \quad D_{irr}D_{\{1,2,3,4\}}\left(\sum_{\{i,j\}\subset\{1,2,3,4\}} D_{\{i,j\}}\right)$$

$$D_{irr}D_{\{1,2,3,4\}}D_{\{1,2\}} \quad D_{irr}D_{\{1,2,3,4\}}D_{\{1,3\}}$$

This set of classes is not closed under the action of S_4. Since the last two coordinates of $[\overline{A_4}]$ with respect to this basis will turn out to be zero, our result will again be symmetric under S_4. We can write it in an unique way:

$$[\overline{A_4}] = \sum_{i=1}^{12} b_i v_i$$

where the v_i's are the vectors of the basis just introduced, taken in the order of the previous list. Observe that $v_2, \ldots v_5$ do not have a precise position in the list, and nor do $v_6, \ldots, v_{10}$. The fact that the construction of $\overline{A}_4$ is S_4-equivariant means that this is not important, because for any possible choice of an ordering of these v_i's it turns out that:

$$b_2 = b_3 = b_4 = b_5 \quad b_6 = b_7 = b_8 = b_9 = b_{10}$$

Using the same trick as before, taking the four push-forwards via $\pi_{\{1,2,3\}}, \pi_{\{1,2,4\}}, \pi_{\{1,3,4\}}, \pi_{\{2,3,4\}}$, plus the fact that the forgetful morphism is an isomorphism when restricted to $\overline{A}_4$ (5.22), one can determine $b_1, b_2, b_3, b_4, b_5, b_{11}, b_{12}$. Moreover in this way one finds relations like:

$$b_i + 3b_{10} = \frac{1}{4} \quad 6 \leq i \leq 9$$

To finish the computation, it is enough to intersect everything with $D_{\{1,2,3,4\}}$: this gives $b_{10} = \frac{1}{12}$ and therefore concludes the proof of the last equality of the statement. □

Corollary 8.9. *Let Y be a twisted sector of $\overline{\mathcal{M}}_{1,n}$. Then $[Y]$ is in the subalgebra generated by the divisors of $\overline{\mathcal{M}}_{1,n}$.*

Proof. We proved in Theorem 5.23 that every twisted sector is $j_{g,k*}p^*([Z])$, where Z is a base twisted sector in $\overline{\mathcal{M}}_{1,k}$ (whose class in the Chow ring was studied in Theorem 8.6) and the maps fit the diagram:

$$\begin{tikzcd}
\overline{\mathcal{M}}_{1,k} \times \overline{\mathcal{M}}_{0,I_1+1} \times \ldots \times \overline{\mathcal{M}}_{0,I_k+1} \arrow[r, "j_k"] \arrow[d, "pr_1"] & \overline{\mathcal{M}}_{1,n} \\
\overline{\mathcal{M}}_{1,k} &
\end{tikzcd}$$

where $j_{1,k}$ is the gluing map defined in 2.35, and pr_1 is the projection onto the first factor. From this one can compute explicitly the twisted sectors as linear combinations of products of divisors. □

8.2 Pull–backs

Let now $Y = Z^{I_1,I_2,I_3,I_4}$ be a twisted sector of $\overline{\mathcal{M}}_{1,n}$. Suppose that $Z \subset \overline{\mathcal{M}}_{1,k}$. Here we identify the twisted sector Z^{I_1,I_2,I_3,I_4} with the product of $\overline{Z} \times \overline{\mathcal{M}}_{1,I_1+1} \times \ldots \times \overline{\mathcal{M}}_{1,I_k+1}$. From now on in this section, the projections onto the $k+1$ factors are called $pr_1, \ldots pr_{k+1}$.

We want to study the pull–back morphism:

$$f^* : H^*(\overline{\mathcal{M}}_{1,n}, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$$

The main results of this section, are:

1. that the pull–back morphism f^* is determined by its restriction to the subalgebra of the cohomology generated by the divisors (Theorem 8.12);
2. the explicit description of the pull–back of the divisor classes (Theorem 8.17);

Getzler's claims imply the following:

Theorem 8.10. *If we write the cohomology ring $H^*(\overline{\mathcal{M}}_{1,n}) = H^{even}(\overline{\mathcal{M}}_{1,n}) \bigoplus H^{odd}(\overline{\mathcal{M}}_{1,n})$, then:*

$$R^*(\overline{\mathcal{M}}_{1,n}) \cong RH^*(\overline{\mathcal{M}}_{1,n}) = H^{even}(\overline{\mathcal{M}}_{1,n})$$

Remark 8.11. Here, the difference between the algebraic setting (Chow ring) and the topological one (cohomology ring) here is mainly that in the second instance we have a canonical candidate for a splitting of the ring into two parts: a tautological one and a "purely non tautological" one. Namely, the decomposition is simply the decomposition into even and odd parts. This is the reason why we choose to describe everything in terms of cohomology instead of using the Chow ring.

It is clear that the pull–back morphism f^* is zero when restricted to the odd cohomology classes. Hence to study f^*, it is enough to study its restriction to the Tautological Ring.

Now we state the following theorem:

Theorem 8.12. *The pull–back morphism f^* is determined by its restriction on the subalgebra of $H^*(\overline{\mathcal{M}}_{1,n})$ generated by the divisors.*

We want a description of the cycles that are not in the subalgebra of the Tautological Ring which is generated by divisor classes. Let B be the class of the point of $\overline{\mathcal{M}}_{1,2}$ given by the following picture. Note that B is a boundary strata class:

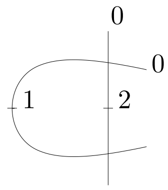


Figure 8.1: Banana cycle in $\overline{\mathcal{M}}_{1,2}$

Definition 8.13. Let $I_1 \sqcup I_2$ be a partition of $[n]$. We define the *banana cycle* B^{I_1,I_2}, following [Be98], by:

$$j_{1,2*}(\pi^*(B))$$

where the maps fit in the following diagram:

$$\begin{tikzcd}
\overline{\mathcal{M}}_{1,2} \times \overline{\mathcal{M}}_{0,I_1+1} \times \overline{\mathcal{M}}_{0,i_1+1} \arrow[r, "j_{1,2}"] \arrow[d, "\pi"] & \overline{\mathcal{M}}_{1,n} \\
\overline{\mathcal{M}}_{1,2} &
\end{tikzcd}$$

and $j_{1,2}$ is the map defined in 2.35.

Lemma 8.14. *The boundary strata classes that are not expressible as product of divisor classes are closed substacks of the banana cycles.*

Proof. See [Be98] for a proof of this. ☐

Lemma 8.15. *The square of D_{irr} (Notation 3.3) is zero in the Tautological Ring of $\overline{\mathcal{M}}_{1,n}$, for every n.*

Proof. (of Theorem 8.12) Thanks to Lemma 8.14, all that we have to prove is that the product of a banana cycle $[B^{I_1,I_2}]$ and $[Y]$ vanishes. We use Corollary 8.9. Therefore we have Y expressed as linear combination of product of divisor classes in $\overline{\mathcal{M}}_{1,n}$. The product of B^{I_1,I_2} with all the summands that contain a factor D_{irr} is zero, thanks to Lemma 8.15 (because B^{I_1,I_2} is a substack of the D_{irr} in $\overline{\mathcal{M}}_{1,n}$). All the summands whose generic point contains a smooth genus 1 component are easily checked to have product zero with the banana cycle, because the set theoretic intersection of the substacks of $\overline{\mathcal{M}}_{1,n}$ that they describe is empty.

☐

Thanks to this result, it is enough to compute the pull–back morphism for the divisor classes. The notation for the divisors in $\overline{\mathcal{M}}_{1,n}$ is explained in Notation 3.3.

Proposition 8.16. *The pull–back $f^*(D_{irr})$ is zero when the base space is C_a, for $a = 4$ or $a = 6$.*

1. *It is $\frac{1}{2}[pt] \times \overline{\mathcal{M}}_{0,n+1}$, when the space is $\overline{A_1}^{[n]}$;*
2. *It is $\frac{3}{2}[pt] \times \overline{\mathcal{M}}_{0,I_1+1} \times \overline{\mathcal{M}}_{0,I_2+1}$, when the space is $\overline{A_2}^{I_1,I_2}$;*
3. *It is $3[pt] \times \overline{\mathcal{M}}_{0,I_1+1} \times \overline{\mathcal{M}}_{0,I_2+1} \times \overline{\mathcal{M}}_{0,I_3+1}$, when the space is $\overline{A_3}^{I_1,I_2,I_3}$;*
4. *It is $3[pt] \times \overline{\mathcal{M}}_{0,I_1+1} \times \overline{\mathcal{M}}_{0,I_2+1} \times \overline{\mathcal{M}}_{0,I_3+1} \times \overline{\mathcal{M}}_{0,I_4+1}$, when the space is $\overline{A_4}^{I_1,I_2,I_3,I_4}$;*

Proof. Here the intersection of the loci inside $\overline{\mathcal{M}}_{1,n}$ is transversal, and therefore the intersection is the set theoretic intersection. Another way to compute this, is by using Theorem 8.6. □

Let $M \subset [n]$ with $|M| \geq 2$. We describe the pull–back $f^*([D_M])$.

Proposition 8.17. *The pullback f^* is zero whenever M is not contained in any of the I_i's. If it is contained in (wlog) I_1, then there are two cases. If M is a proper subset of I, then*

$$f^*([D_M]) \cong [Z \times \overline{\mathcal{M}}_{0,M+1} \times \overline{\mathcal{M}}_{0,I_1 \setminus M+1} \times \overline{\mathcal{M}}_{0,I_2+1} \times \overline{\mathcal{M}}_{0,I_3+1} \times \overline{\mathcal{M}}_{0,I_4+1}]$$

Otherwise, if $I = M$, then:

$$f^*([D_M]) = pr_2^*(-\psi_{I+1})$$

Proof. One simply observes that when M is strictly contained in I_1, then the intersection is proper. When $M = I_1$ there is an excess intersection whose result is obtained thanks to Lemma 7.13. Another way to see the same result is by using Theorem 8.6 □

Now a corollary of our description of the pull–back morphism, gives us a very important theoretical result:

Corollary 8.18. *The morphisms $f^* : R^*(\overline{\mathcal{M}}_{1,n}, \mathbb{Q}) \to A^*(Y, \mathbb{Q})$ are surjective. The same holds for the induced map in cohomology.*

Proof. Thanks to Lemma 8.1, it is sufficient to prove that the morphism $f^* : R^1(\overline{\mathcal{M}}_{1,n}, \mathbb{Q}) \to A^1(Y, \mathbb{Q})$ is surjective. The Kunneth formula allows us to reduce the problem to proving that one can obtain all divisors of each single factor of each twisted sector by pull–back from $R^1(\overline{\mathcal{M}}_{1,n})$. The set of the divisor classes described in Proposition 8.17 surjects onto $A^1(Y, \mathbb{Q})$.

□

Corollary 8.19. *If Y is a twisted sector, then the cohomology $H^*(Y, \mathbb{Q})$ is an $H^*(\overline{\mathcal{M}}_{1,n}, \mathbb{Q})$-module generated by the fundamental class $[Y]$. Indeed $H^*(Y, \mathbb{Q})$ is cyclic also as an $R^*(\overline{\mathcal{M}}_{1,n})$-module.*

Remark 8.20. This is no longer true for the twisted sectors in higher genera. Already in genus 2, the second Betti number $h^2(\overline{II})$ is equal to 3, and since the $h^2(\overline{\mathcal{M}}_2) = 2$, it is no longer possible for the map f^* to be surjective.

8.3 Push–forwards

We now begin to study the push–forward morphism. Let:

$$g : Z \to Y \quad f : Y \to X$$

be respectively the inclusion of a double twisted sector in a twisted sector and of a twisted sector inside $X = \overline{\mathcal{M}}_{1,n}$. We now study the induced push–forward morphism. We start with the case $Y = \overline{\mathcal{M}}_{1,n}$ itself.

Lemma 8.21. *The push–forward morphism $f_* : A^*(Z, \mathbb{Q}) \to A^*(\overline{\mathcal{M}}_{1,n}, \mathbb{Q})$ has image in the Tautological Ring. The same holds for the induced map in cohomology.*

Proof. Thanks to Corollary 8.19, one has only to prove that the fundamental classes of the twisted sectors belong to the Tautological Ring.

The fact that the base space classes $[Z]$ belong to the Tautological Ring was proved in 8.6. Now to prove that all the classes of the twisted sectors $[Z^{I_1,\dots,I_k}]$ belong to the Tautological Ring, one observes the following:

- Let $j_{1,k}$ be the morphism defined in 2.35. If $j_{1,k*}$ is the morphism induced in cohomology, then $j_{1,k*}(\overline{\mathcal{M}}_{1,k}\times\overline{\mathcal{M}}_{0,I_1+1}\times\ldots\times\overline{\mathcal{M}}_{0,I_k+1})$ is in the Tautological Ring because the latter is closed under push–forwards of gluing maps.
- Suppose without loss of generality that $1\in I_1,\ldots k\in I_k$. Let $\pi_{\{1,\ldots,k\}}:\overline{\mathcal{M}}_{1,n}\to\overline{\mathcal{M}}_{1,k}$ be the forgetful map. Then if $[Z]$ is the class of a base twisted sector (4.42) of $\overline{\mathcal{M}}_{1,k}$, then $\pi^*_{\{1,\ldots,k\}}([Z])$ is in the Tautological Ring, since the latter is closed under pull–back via the all the forgetful maps (Remark 3.5).
- The Tautological Ring is by definition closed under the cup product, and $Z^{I_1,\dots,I_k}$ is exactly the product of the two classes constructed in the two previous points.

□

The two lemmas 8.18 and 8.21 give meaning to the definition:

Definition 8.22. We define the *orbifold Tautological Ring* of $\overline{\mathcal{M}}_{1,n}$ as:

- $R^*_{CR}(\overline{\mathcal{M}}_{1,n}):=R^*(\overline{\mathcal{M}}_{1,n})\oplus\bigoplus H^*(X_i,\mathbb{Q})$ as $\mathbb{Q}$-vector space, where X_i are all twisted sectors;
- the graduation is inherited by $H^*_{CR}(\overline{\mathcal{M}}_{1,n},\mathbb{Q})$;
- the product is the product $*_{CR}$ restricted to the previously defined rationally graded $\mathbb{Q}$-algebra.

We want to show how $g_*([Z])$ can be obtained as a pull–back of a class in X in a canonical way.

Definition 8.23. If $a=4$ or $a=6$, we define C^*_a via the following pull–back diagram:

$$\begin{tikzcd} C^*_a \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal{M}}_{1,n} \arrow[d, "\pi_1"] \\ C_a \arrow[r, hook] & \overline{\mathcal{M}}_{1,1} \end{tikzcd}$$

Note that the equality:

$$[C^*_a]=\frac{2}{a}D_{irr}$$

holds.

Proposition 8.24. *With the notation introduced in this section, for Z a double twisted sector and Y a twisted sector, there is a canonical choice of W closed substack of $\overline{\mathcal{M}}_{1,n}$, such that $g_*([Z])=f^*([W])$.*

Proof. The only cases are, thanks to Proposition 7.41:

1. it happens that $Z=Y$ or $Y=X$. In all these cases we choose $[W]:=[X]$;
2. either $Z=C_a^{[n]}$ for $a=4,6$ and $Y=\overline{A_1}^{[n]}$. In these cases we choose $[W]:=[C^*_a]$;
3. or $Z=C_4^{I_1,I_2}$ and $Y=\overline{A_2}^{I_1,I_2}$. In these cases we again choose $[W]:=[C^*_4]$.

One can easily check that these are all the cases that occur, and that all the intersections are transversal.

□

We have just fixed the cohomology classes that represent via pull–back the push–forward of all the fundamental classes. This choice determines the top Chern class of the excess intersection bundles via projection formula.

Corollary 8.25. *Let now E be the excess intersection bundle over the double twisted sector Z. Once the choice of 8.24 is fixed, a cohomology class β on $\overline{\mathcal{M}}_{1,n}$ is determined such that:*

$$g_*(c_{top}(E)) = f^*(\beta)$$

Proof. If the top Chern class of E is zero, we choose $\beta := 0$. When the top Chern class is 1, the choice of Proposition 8.24 determines the class β of this corollary too. The list of non trivial top Chern classes of excess intersection bundles (non zero and non 1), is given in 7.50. So, if the top Chern class is a ψ class, there are two distinct cases: either $[Z] = [Y]$, or $[Z] = C_a^{[n]}$ and $Y = \overline{A_1}^{[n]}$. In the first case, we choose $\beta := D_{[n]}$, and in the second case we choose

$$\beta := D_{[n]} \cup [C_a^*]$$

where $[C_a^*]$ was defined above. □

8.4 Products of fundamental classes of twisted sectors

If X_i, X_j are twisted sectors, we have understood in Theorem 8.2 that, in order to determine the product of the orbifold Tautological Ring, we need to compute all products $1_{X_i} * 1_{X_j}$.

Remark 8.26. An explicit computation of all intersections of twisted sectors, shows that besides the orbifold intersections of the kinds $(1_{X_i}, \alpha) * (1_{X_i}, \beta)$, and besides the trivial products $1_{\overline{\mathcal{M}}_{1,n}} * 1_{X_i}$, the only pairs of twisted sectors whose fundamental classes give rise to non zero Chen–Ruan products are in the following list:

1. $(\overline{A_1}^{[n]}, (C_4^{[n]}, i/-i))$;
2. $(\overline{A_1}^{[n]}, (C_6^{[n]}, \epsilon/\epsilon^2/\epsilon^4/\epsilon^5))$;
3. $(\overline{A_2}^{I_1,I_2}, (C_4^{I_1,I_2}, i/-i))$.

We now compute the products of the pairs just described. Here if (X, α) is a twisted sector, we write $H^*((X,\alpha),\mathbb{Q})$, which is a direct summand of $H^*_{CR}(\overline{\mathcal{M}}_{1,n},\mathbb{Q})$ with its own graduation. In other words, we assume implicitly the inclusion

$$i : H^*((X,\alpha),\mathbb{Q}) \subset H^*_{CR}(\overline{\mathcal{M}}_{1,n},\mathbb{Q})$$

shifts the degree by twice the age of (X, α). As usual, $-pr_2^*(\psi_\bullet)$ is the Chern class of Corollary 7.50 and Theorem 7.51.

Corollary 8.27. *With our usual notation for the twisted sectors, and with the notation introduced above, here is the explicit result of all Chen–Ruan products described in Remark 8.26:*

1. $[(C_4^{[n]}, i)] *_{CR} [\overline{A_1}^{[n]}] = pr_2^*(-\psi_\bullet) \ \in H^2((C_4^{[n]}, -i),\mathbb{Q})$;
2. $[(C_4^{[n]}, -i)] *_{CR} [\overline{A_1}^{[n]}] = [C_4^{[n]}] \ \in H^0((C_4^{[n]}, i),\mathbb{Q})$;
3. $[(C_6^{[n]}, \epsilon)] *_{CR} [\overline{A_1}^{[n]}] = pr_2^*(-\psi_\bullet) \ \in H^2((C_6^{[n]}, \epsilon^4),\mathbb{Q})$;
4. $[(C_6^{[n]}, \epsilon^2)] *_{CR} [\overline{A_1}^{[n]}] = pr_2^*(-\psi_\bullet) \ \in H^2((C_6^{[n]}, \epsilon^5),\mathbb{Q})$;
5. $[(C_6^{[n]}, \epsilon^4)] *_{CR} [\overline{A_1}^{[n]}] = [C_6^{[n]}] \ \in H^0((C_6^{[n]}, \epsilon),\mathbb{Q})$;
6. $[(C_6^{[n]}, \epsilon^5)] *_{CR} [\overline{A_1}^{[n]}] = [C_6^{[n]}] \ \in H^0((C_6^{[n]}, \epsilon^2),\mathbb{Q})$;
7. $[\overline{A_2}^{I_1,I_2}] *_{CR} [(C_4^{I_1,I_2}, i)] = 0 \in H^2((C_4^{I_1,I_2}, -i),\mathbb{Q})$;
8. $[\overline{A_2}^{I_1,I_2}] *_{CR} [(C_4^{I_1,I_2}, -i)] = [C_4^{I_1,I_2}] \in H^2((C_4^{I_1,I_2}, i),\mathbb{Q})$;

Corollary 8.28. *With our usual notation for the twisted sectors, and with the notation introduced above, here we have all products of the kind $(1_{X_i}, \alpha) * (1_{X_i}, \beta)$:*

1. $[(X,\alpha)] *_{CR} [(X,\alpha^{-1})] = [X] \ \in H^*(\overline{\mathcal{M}}_{1,n},\mathbb{Q})$;
2. $[(C_4^{[n]},i)] *_{CR} [(C_4^{[n]},i)] = pr_2^*(-\psi_\bullet) \ \in H^4(\overline{A_1}^{[n]},\mathbb{Q})$;
3. $[(C_4^{[n]},-i)] *_{CR} [(C_4^{[n]},-i)] = [C_4^{[n]}] \ \in H^2(\overline{A_1}^{[n]},\mathbb{Q})$;
4. $[(C_4^{I_1,I_2},i)] *_{CR} [(C_4^{I_1,I_2},i)] = 0 \ \in H^4(\overline{A_2}^{I_1,I_2},\mathbb{Q})$;
5. $[(C_4^{I_1,I_2},-i)] *_{CR} [(C_4^{I_1,I_2},-i)] = [C_4^{I_1,I_2}] \ \in H^2(\overline{A_2}^{I_1,I_2},\mathbb{Q})$;
6. $[(C_6^{[n]},\epsilon)] *_{CR} [(C_6^{[n]},\epsilon)] = 0 \in H^4((C_6^{[n]},\epsilon^2),\mathbb{Q})$;
7. $[(C_6^{[n]},\epsilon)] *_{CR} [(C_6^{[n]},\epsilon^2)] = pr_2^*(-\psi_\bullet) \ \in H^4(\overline{A_1}^{[n]},\mathbb{Q})$;
8. $[(C_6^{[n]},\epsilon)] *_{CR} [(C_6^{[n]},\epsilon^4)] = 0 \in H^4((C_6^{[n]},\epsilon^5),\mathbb{Q})$;
9. $[(C_6^{[n]},\epsilon^2)] *_{CR} [(C_6^{[n]},\epsilon^2)] = pr_2^*(-\psi_\bullet) \in H^2((C_6^{[n]},\epsilon^4),\mathbb{Q})$;
10. $[(C_6^{[n]},\epsilon^2)] *_{CR} [(C_6^{[n]},\epsilon^5)] = [C_6^{[n]}] \in H^0((C_6^{[n]},\epsilon),\mathbb{Q})$;
11. $[(C_6^{[n]},\epsilon^4)] *_{CR} [(C_6^{[n]},\epsilon^4)] = pr_2^*(-\psi_\bullet) \in H^2((C_6^{[n]},\epsilon^2),\mathbb{Q})$;
12. $[(C_6^{[n]},\epsilon^4)] *_{CR} [(C_6^{[n]},\epsilon^5)] = [C_6^{[n]}] \ \in H^2(\overline{A_1}^{[n]},\mathbb{Q})$;
13. $[(C_6^{[n]},\epsilon^5)] *_{CR} [(C_6^{[n]},\epsilon^5)] = [C_6^{[n]}] \ \in H^0((C_6^{[n]},\epsilon^4),\mathbb{Q})$;
14. $[(C_6^{I_1,I_2,I_3},\epsilon^2)] *_{CR} [(C_6^{I_1,I_2,I_3},\epsilon^2)] = 0 \ \in H^4((C_6^{I_1,I_2,I_3},\epsilon^4),\mathbb{Q})$ *and the result still holds when* $K=\emptyset$;
15. $[(C_6^{I_1,I_2,I_3},\epsilon^4)] *_{CR} [(C_6^{I_1,I_2,I_3},\epsilon^4)] = 0 \ \in H^2((C_6^{I_1,I_2,I_3},\epsilon^2),\mathbb{Q})$ *and the result still holds when* $K=\emptyset$;

8.5 The orbifold Tautological Ring

The aim of this section is to speculate a little on the orbifold Tautological Ring. Of course, it is clear that the results that we obtained in genus 1 are somehow too trivial to give any possibility to speculating about a system of orbifold Tautological Rings for all g and n's. One conjecture that one would like to be able express is:

Conjecture 8.29. *(orbifold Faber Conjectures) (cfr. [Fa08])*

1. *The Ring* $R^*_{CR}(\mathcal{M}^{rt}_{g,n})$ *is Gorenstein with socle in degree* $g-2+n-\delta_{0g}$.
2. *The Ring* $R^*_{CR}(\mathcal{M}^{ct}_{g,n})$ *is Gorenstein with socle in degree* $2g-3+n$.
3. *The Ring* $R^*_{CR}(\overline{\mathcal{M}}_{g,n})$ *is Gorenstein with socle in degree* $3g-3+n$.

It is trivial to observe that, in genus 1, Faber conjectures 3.9 are equivalent to orbifold Faber conjectures 8.29, if the definition of the orbifold Tautological Ring is taken as in 8.22.

Definition 8.30. We define the *orbifold Tautological Ring* of $\mathcal{M}^{rt}_{1,n}$ as:

- $R^*_{CR}(\overline{\mathcal{M}}_{1,n}) := R^*(\overline{\mathcal{M}}_{1,n}) \oplus \bigoplus H^*(X_i,\mathbb{Q})$ as $\mathbb{Q}$-vector space, where X_i are all twisted sectors *of positive dimension*;
- the graduation is inherited by $H^*_{CR}(\overline{\mathcal{M}}_{1,n},\mathbb{Q})$;
- the product is the product $*_{CR}$ restricted to the previously defined rationally graded $\mathbb{Q}$-algebra.

It is crucial to exclude the zero dimensional twisted sectors in the definition of this last ring. In fact, if α is a class of a zero dimensional sector, the map:

$$\alpha *_{CR} : R^*_{CR}(\mathcal{M}^{rt}_{1,n}) \to R^*_{CR}(\mathcal{M}^{rt}_{1,n})$$

is the zero map. It is possible to modify accordingly Definition 8.22. Observe that when $n \geq 7$ all the twisted sectors of $\overline{\mathcal{M}}_{1,n}$ or $\mathcal{M}^{rt}_{1,n}$ are positive dimensional, so the two notions are different only when

$n \leq 6$. We also want to underline that, whatever definition should be taken of system R^*_{CR}, reasons of symmetry induce one to think that it must be closed via pullback and pushforward via the involution map i^*, i_* (Definition 4.9).

A final consideration: one other requirement that one could try to insist on for a "nice" subring of the Chen–Ruan cohomology of moduli of marked curves, is functoriality of the product under the natural maps 2.32. We explain this point more. We have seen in Definition 4.14, that if $f : X \to Y$ is a morphism of stacks, there are induced maps:

$$I(X) \to f^*(I(Y)) \to I(Y) \tag{8.31}$$

so if $\alpha \in H^*_{CR}(Y)$ it is possible to define a naive pullback $f^*_{CR}(\alpha)$, by taking pullback via the two maps in 8.31. This pullback does not induce an homomorphism of rings, as can be easily checked in trivial examples. One could hope the orbifold Tautological Ring, in the spirit of the Definition of [FP05], is such that *at least the natural map* induce a ring homomorphism, when restricted to this subring. Unfortunately, it is possible to see that:

Theorem 8.32. *Let $R'_n \subset H^*_{CR}(\overline{\mathcal{M}}_{1,n})$ be a system of subrings, such that if π is the forgetful map, the image of the Chen–Ruan pullback via π is contained in R':*

$$\pi^*_{CR} : R'_n \to R'_{n+1}$$

and moreover π^*_{CR} is a ring homomorphism and it is closed under i^* where i is the canonical involution on the Inertia Stack (4.9). Then R'_n is contained in ordinary cohomology $H^*(\overline{\mathcal{M}}_{1,n})$ for all n.

Proof. Let $\alpha \in H^*_{CR}(\overline{\mathcal{M}}_{1,n})$, then:

$$\gamma := \alpha *_{CR} i^*(\alpha) \in H^*(\overline{\mathcal{M}}_{1,n})$$

So one can compare $\pi^*(\gamma)$ and $\pi^*_{CR}(\alpha) *_{CR} \pi^*_{CR}(i^*(\alpha))$, to discover that they are always different if $\alpha \notin H^*(\overline{\mathcal{M}}_{1,n})$. We study an example to fix the notation. Let α be the fundamental class of $(C_4^{I,J}, i)$. Then γ is the class of $C_4^{I,J}$ inside the ordinary cohomology of $\overline{\mathcal{M}}_{1,n}$. The pullback:

$$\pi^*(\alpha) = [C_4^{I+1,J}] + [C_4^{I,J+1}] + [D^{I,J}]$$

where D is the locus in $\overline{\mathcal{M}}_{1,3}$ defined by the fiber product:

$$\begin{tikzcd} D \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{1,3} \arrow[d, "\pi"] \\ C'_4 \arrow[r] & \overline{\mathcal{M}}_{1,2} \end{tikzcd}$$

and $D^{I,J}$ is the locus in $\overline{\mathcal{M}}_{1,n+1}$ obtained by gluing $\overline{\mathcal{M}}_{0,I+1}$ on the first marked point of D and $\overline{\mathcal{M}}_{0,J+1}$ on the second marked point of D.

On the contrary, $\pi^*_{CR}(\alpha) *_{CR} \pi^*_{CR}(i^*(\alpha))$ is:

$$([C_4^{I+1,J}, i] + [C_4^{I,J+1}, i]) *_{CR} ([C_4^{I+1,J}, -i] + [C_4^{I,J+1}, -i]) = [C_4^{I+1,J}] + [C_4^{I,J+1}]$$

The two computations differ exactly by the class $[D^{I,J}]$, which is not zero. This concludes the proof. $\square$

Corollary 8.33. *Let R'_n be a system of $\mathbb{Q}$-subalgebras of $H^*_{CR}(\overline{\mathcal{M}}_{1,n})$ that is closed under i^* (i being the involution 4.9) and under the pullback via the forgetful map π^*_{CR}, and such that π^*_{CR} induces a ring homomorphism:*

$$R'_n \to R'_{n+1}$$

If R'_n contains the ordinary Tautological Ring R^*, then it coincides with R^*.

Chapter 9

Examples

9.1 The Chen–Ruan cohomology ring of $\overline{\mathcal{M}}_{1,1}$

With our notation, the product in $\overline{\mathcal{M}}_{1,1}$ becomes:

$*_{CR}$	$(\overline{A_1},-1)$	(C_4,i)	$(C_4,-i)$	(C_6,ϵ)	(C_6,ϵ^2)	(C_6,ϵ^4)	(C_6,ϵ^5)
$(\overline{A_1},-1)$	$\overline{\mathcal{M}}_{1,1}$	$(C_4,-i)$	(C_4,i)	(C_6,ϵ^4)	(C_6,ϵ^5)	(C_6,ϵ)	(C_6,ϵ^2)
(C_4,i)		$C_4<\overline{A_1}$	$C_4<\overline{\mathcal{M}}_{1,1}$	0	0	0	0
$(C_4,-i)$			$C_4<\overline{A_1}$	0	0	0	0
(C_6,ϵ)				0	$C_6<\overline{A_1}$	0	$C_6<\overline{\mathcal{M}}_{1,1}$
(C_6,ϵ^2)					(C_6,ϵ^4)	$C_6<\overline{\mathcal{M}}_{1,1}$	(C_6,ϵ)
(C_6,ϵ^4)						0	$C_6<\overline{A_1}$
(C_6,ϵ^5)							(C_6,ϵ^4)

Here $A<B$ means the fundamental class $[A]$ of A inside the cohomology of the space B.

We can explicitly write the inclusion:

$$H^*(\overline{\mathcal{M}}_{1,1},\mathbb{Q}) \to H^*_{CR}(\overline{\mathcal{M}}_{1,1},\mathbb{Q})$$

as:

$$\frac{\mathbb{Q}[t]}{(t^2)} \to \frac{\mathbb{Q}[x_0,y_0,z_0]}{(x_0^2-1,2y_0^2-3z_0^3,y_0z_0)}$$

$$t \to 2x_0y_0^2$$

Where $x_0=[\overline{A_1}^{[n]}]$, $y_0=[C_4,-i]$, and $z_0=[C_6,\epsilon^5]$.

9.2 The Chen–Ruan cohomology ring of $\overline{\mathcal{M}}_{1,2}$

We first review the result for the age of the sectors in the Inertia Stack:

Sector	Age	Sector	Age
$(\overline{A_1}^{[n]},-1)$	$\frac{1}{2}$	$(C_6^{[2]},\epsilon)$	$\frac{3}{2}$
$(\overline{A_2},-1)$	$\frac{1}{2}$	$(C_6^{[2]},\epsilon^2)$	1
$(C_4^{[2]},i)$	$\frac{5}{4}$	$(C_6^{[2]},\epsilon^4)$	1
$(C_4^{[2]},-i)$	$\frac{3}{4}$	$(C_6^{[2]},\epsilon^5)$	$\frac{1}{2}$
(C_4',i)	$\frac{5}{4}$	(C_6',ϵ^2)	$\frac{4}{3}$
$(C_4',-i)$	$\frac{3}{4}$	(C_6',ϵ^4)	$\frac{2}{3}$

Note that in $\overline{\mathcal{M}}_{1,2}$ the double twisted sectors either involve the identical automorphism, or are of dimension 0. As a consequence, the top Chern class of the excess intersection bundle can be either 1 or 0, respectively when the dimension of the bundle is 0 or greater than 0.

For typographical reasons, in the following table we denote $X := \overline{\mathcal{M}}_{1,2}$.

	-1	-1	i	$-i$	i	$-i$	ϵ	ϵ^2	ϵ^4	ϵ^5	ϵ^2	ϵ^4
$\overline{\mathcal{M}}_{1,2}$	$\overline{A_1}^{[n]}$	$\overline{A_2}$	$C_4^{[2]}$	$C_4^{[2]}$	C_4'	C_4'	$C_6^{[2]}$	$C_6^{[2]}$	$C_6^{[2]}$	$C_6^{[2]}$	C_6'	C_6'
$(\overline{A_1}^{[n]}, -1)$	$\overline{A_1}^{[n]} < X$	$\emptyset$	0	$(C_4^{[2]}, i)$	$\emptyset$	$\emptyset$	0	0	$(C_6^{[2]}, \epsilon)$	$(C_6^{[2]}, \epsilon^2)$	$\emptyset$	$\emptyset$
$(\overline{A_2}, -1)$		$\overline{A_2} < X$	$\emptyset$	$\emptyset$	0	(C_4', i)	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(C_4^{[2]}, i)$			0	$C_4^{[2]} < X$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(C_4^{[2]}, -i)$				$C_4^{[2]} < \overline{A_1}^{[n]}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
(C_4', i)					0	$C_4' < X$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(C_4', -i)$						$C_4' < \overline{A_2}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(C_6^{[2]}, \epsilon)$							0	0	0	$C_6^{[2]} < X$	$\emptyset$	$\emptyset$
$(C_6^{[2]}, \epsilon^2)$								0	$C_6^{[2]} < X$	$(C_6^{[2]}, \epsilon)$	$\emptyset$	$\emptyset$
$(C_6^{[2]}, \epsilon^4)$									0	$C_6^{[2]} < \overline{A_1}^{[n]}$	$\emptyset$	$\emptyset$
$(C_6^{[2]}, \epsilon^5)$										$(C_6^{[2]}, \epsilon^4)$	$\emptyset$	$\emptyset$
(C_6', ϵ^2)											0	$C_6' < X$
(C_6', ϵ^4)												(C_6', ϵ^2)

We can explicitly write the inclusion:

$$H^*(\overline{\mathcal{M}}_{1,2}, \mathbb{Q}) \to H^*_{CR}(\overline{\mathcal{M}}_{1,2}, \mathbb{Q})$$

as:

$$\frac{\mathbb{Q}[t_0, t_1]}{(t_0^2, t_0 t_1 + 12 t_1^2)} \to \frac{\mathbb{Q}[x_0, y_0, z_0, x_1, y_1, w]}{I}$$
$$t_0 \to -12x_0^2 + 4x_1^2$$
$$t_1 \to x_0^2$$

where I is the ideal defined as:

$$I := (x_0^2 y_0, 6x_0^3 + y_0^2, x_0^2 z_0, 3z_0^3 - 2y_0^2, y_0 z_0$$
$$x_1^2 y_1, 2x_1^3 - 3y_1^2, 9x_0^4 + x_1^4,$$
$$x_0 x_1, x_0 y_1, x_1 y_0, y_0 y_1, z_0 x_1, z_0 y_1$$
$$w x_0, w y_0, w z_0, w x_1, w y_1, w^3 + 4x_0^4)$$

The generators for the ordinary cohomology are taken to be $t_0 := \delta_{irr}$ and $t_1 := \delta_{1,1,2}$.

The generators for the Chen–Ruan cohomology are taken to be $x_0 = [\overline{A_1}^{[n]}]$, $y_0 = [C_4^{[2]}, -i]$, $z_0 = [C_6^{[2]}, \epsilon^5]$, $y_0 = [\overline{A_2}]$, $y_1 = [C_4', -i]$ and $w = [C_6', \epsilon^4]$.

9.3 The Chen–Ruan cohomology ring of $\overline{\mathcal{M}}_{1,3}$

From now on we use a uniform notation for the spaces Z^{I_1, I_2, I_3, I_4} (i.e. all base twisted sectors of Theorem 5.14 and Definition 4.42 will be referred to using the notation of Definition 4.42, see Remark 5.16). To simplify the notation, we write (for example) $C_6^{|I_1|,|I_2|,|I_3|}$ for $C_6^{I_1, I_2, I_3}$. We make an analogue convention for all twisted sectors, since all the ones collected under the same name act in a similar fashion (for instance, they have same age). There is no ambiguity nor loss of information in the tables, since two sectors with different superscripts have empty intersection (example: $C_4^{\{1,2,3\},\{4\}} \cap C_4^{\{1,3,4\},\{2\}} = \emptyset$). Moreover, the same general rule holds for the intersection of two different kinds of sectors. We present an example. With the contracted notation, the expression:

$$\overline{A_2}^{1,2} \cap C_4^{1,2} = C_4^{1,2}$$

means:

- $\overline{A_2}^{\{i\},\{j,k\}} \cap C_4^{\{i\},\{j,k\}} = C_4^{\{i\},\{j,k\}}$, where $\{i,j,k\} = \{1,2,3\}$;
- $\overline{A_2}^{\{i\},\{j,k\}} \cap C_4^{\{i'\},\{j',k'\}} = \emptyset$ whenever $\{i,j\} \neq \{i',j'\}$.

Sector	Age	Sector	Age
$(\overline{A_1}^{[n]}, -1)$	$\frac{1}{2}$	$(C_6^{[3]}, \epsilon^2)$	1
$(\overline{A_2}^{1,2}, -1)$	1	$(C_6^{[3]}, \epsilon^4)$	1
$(\overline{A_3}^{1,1,1})$	1	$(C_6^{[3]}, \epsilon^5)$	$\frac{1}{2}$
$(C_4^{[3]}, i)$	$\frac{5}{4}$	$(C_6^{1,2}, \epsilon^2)$	$\frac{5}{3}$
$(C_4^{[3]}, -i)$	$\frac{3}{4}$	$(C_6^{1,2}, \epsilon^4)$	$\frac{4}{3}$
$(C_4^{1,2}, i)$	$\frac{5}{4}$	$(C_6^{1,1,1}, \epsilon^2)$	$\frac{5}{3}$
$(C_4^{1,2}, -i)$	$\frac{3}{4}$	$(C_6^{1,1,1}, \epsilon^4)$	$\frac{4}{3}$
$(C_6^{[3]}, \epsilon)$	$\frac{3}{2}$		

We adopt the same notation as in the case of $\overline{\mathcal{M}}_{1,1}$ for the $<$ and we define $X = \overline{\mathcal{M}}_{1,3}$. We need the following divisor class several times: $\psi_4 \cap [\overline{\mathcal{M}}_{0,4}]$. If

$$i : X_i \to \overline{\mathcal{M}}_{0,4} \times A$$

is an isomorphism (X_i is a twisted sector), then we call θ the class $i^*(-\psi_4 \cap [\overline{\mathcal{M}}_{0,4}] \times [A])$. Note that $\theta \in A^1(X_i)$.

	-1	-1	i	$-i$	i	$-i$	ϵ	ϵ^2	ϵ^4	ϵ^5
$\overline{\mathcal{M}}_{1,3}$	$\overline{A_1}^{[n]}$	$\overline{A_2}^{1,2}$	$C_4^{[3]}$	$C_4^{[3]}$	$C_4^{1,2}$	$C_4^{1,2}$	$C_6^{[3]}$	$C_6^{[3]}$	$C_6^{[3]}$	$C_6^{[3]}$
$(\overline{A_1}^{[n]}, -1)$	$\overline{A_1}^{[n]} < X$	$\emptyset$	$\theta < \overline{A_1}^{[n]}$	$(C_4^{[3]}, i)$	$\emptyset$	$\emptyset$	θ	θ	$(C_6^{[3]}, \epsilon)$	$(C_6^{[3]}, \epsilon^2)$
$(\overline{A_2}^{1,2}, -1)$		$\overline{A_2}^{1,2} < X$	$\emptyset$	$\emptyset$	0	$(C_4^{1,2}, i)$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(C_4^{[3]}, i)$			θ	$C_4^{[3]} < X$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(C_4^{[3]}, -i)$				$C_4^{[3]} < \overline{A_1}^{[n]}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(C_4^{1,2}, i)$					0	$C_4^{1,2} < X$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(C_4^{1,2}, -i)$						$C_4^{1,2} < \overline{A_2}^{1,2}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(C_6^{[3]}, \epsilon)$							0	$\theta < \overline{A_1}^{[n]}$	0	$C_6^{[3]} < X$
$(C_6^{[3]}, \epsilon^2)$								θ	$C_6^{[3]} < X$	$(C_6^{[3]}, \epsilon)$
$(C_6^{[3]}, \epsilon^4)$									0	$C_6^{[3]} < \overline{A_1}^{[n]}$
$(C_6^{[3]}, \epsilon^5)$										$(C_6^{[3]}, \epsilon^4)$

$\overline{\mathcal{M}}_{1,3}$	$(\overline{A_3}^{1,1,1}, -1)$	$(C_6^{1,2}, \epsilon^2)$	$(C_6^{1,2}, \epsilon^4)$	$(C_6^{1,1,1}, \epsilon^2)$	$(C_6^{1,1,1}, \epsilon^4)$
$(\overline{A_3}^{1,1,1}, -1)$	$\overline{A_3}^{1,1,1} < X$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(C_6^{1,2}, \epsilon^2)$		0	$C_6^{1,2} < X$	$\emptyset$	$\emptyset$
$(C_6^{1,2}, \epsilon^4)$			$(C_6^{1,2}, \epsilon^2)$	$\emptyset$	$\emptyset$
$(C_6^{1,1,1}, \epsilon^2)$				0	$C_6^{1,1,1} < X$
$(C_6^{1,1,1}, \epsilon^4)$					$(C_6^{1,1,1}, \epsilon^2)$

9.4 The Chen–Ruan cohomology ring of $\overline{\mathcal{M}}_{1,4}$

We use all conventions adopted in the previous section for $\overline{\mathcal{M}}_{1,3}$. We firstly review the result for the age of the sectors in the Inertia Stack:

Sector	Age	Sector	Age
$(\overline{A_1}^{[n]}, -1)$	$\frac{1}{2}$	$(C_6^{[4]}, \epsilon)$	$\frac{3}{2}$
$(\overline{A_2}^{1,3}, -1)$	1	$(C_6^{[4]}, \epsilon^2)$	1
$(\overline{A_2}^{2,2}, -1)$	$\frac{3}{2}$	$(C_6^{[4]}, \epsilon^4)$	1
$(\overline{A_3}^{1,1,2}, -1)$	$\frac{3}{2}$	$(C_6^{[4]}, \epsilon^5)$	$\frac{1}{2}$
$(\overline{A_4}^{1,1,1,1}, -1)$	$\frac{3}{2}$	$(C_6^{1,3}, \epsilon^2)$	$\frac{5}{3}$
$(C_4^{[4]}, i)$	$\frac{5}{4}$	$(C_6^{1,3}, \epsilon^4)$	$\frac{4}{3}$
$(C_4^{[4]}, -i)$	$\frac{3}{4}$	$(C_6^{2,2}, \epsilon^2)$	$\frac{7}{3}$
$(C_4^{1,3}, i)$	2	$(C_6^{2,2}, \epsilon^4)$	$\frac{5}{3}$
$(C_4^{1,3}, -i)$	1	$(C_6^{1,1,2}, \epsilon^2)$	$\frac{7}{3}$
$(C_4^{2,2}, i)$	$\frac{11}{4}$	$(C_6^{1,1,2}, \epsilon^4)$	$\frac{5}{3}$
$(C_4^{2,2}, -i)$	$\frac{5}{4}$		

To write down the table of the product, we split it into four parts, all other products being zero due to empty intersection of the twisted sectors:

1. products of $\overline{A_i}^{I_1,I_2}$ and $C_4^{I_1,I_2}$ by themselves where $i = 1, 2$;
2. products of $\overline{A_1}^{[n]}$ by all $C_6^{[4]}$;
3. products of $\overline{A_i}^{I_1,I_2}$ by themselves, where $i = 3, 4$;
4. products of $C_6^{I_1,I_2,I_3}$ by themselves.

We write $X := \overline{\mathcal{M}}_{1,4}$. We need the following divisor class several times: $\psi_5 \cap [\overline{\mathcal{M}}_{0,5}]$. If

$$i : X_i \to \overline{\mathcal{M}}_{0,5} \times A$$

is an isomorphism (X_i is a twisted sector), then we call θ the class $i^*(-\psi_5 \cap [\overline{\mathcal{M}}_{0,5}] \times [A])$. Note that $\theta \in A^1(X_i)$.

1	$(\overline{A_1}^{[n]}, -1)$	$(\overline{A_2}^{1,3}, -1)$	$(\overline{A_2}^{2,2}, -1)$	$(C_4^{[4]}, i)$	$(C_4^{[4]}, -i)$	$(C_4^{1,3}, i)$	$(C_4^{1,3}, -i)$	$(C_4^{2,2}, i)$	$(C_4^{2,2}, -i)$
$(\overline{A_1}^{[n]}, -1)$	$\overline{A_1}^{[n]} < X$	$\emptyset$	$\emptyset$	$\theta < (C_4^{[4]}, -i)$	$(C_4^{[4]}, i)$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(\overline{A_2}^{1,3}, -1)$		$\overline{A_2}^{1,3} < X$	$\emptyset$	$\emptyset$	$\emptyset$	0	$(C_4^{1,3}, i)$	$\emptyset$	$\emptyset$
$(\overline{A_2}^{2,2}, -1)$			$\overline{A_2}^{2,2} < X$	$\emptyset$	$\emptyset$	$\emptyset$	0	$\emptyset$	$(C_4^{2,2}, i)$
$(C_4^{[4]}, i)$				$\theta < \overline{A_1}^{[n]}$	$C_4^{[4]} < X$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(C_4^{[4]}, -i)$					$C_4^{[4]} < \overline{A_1}^{[n]}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(C_4^{1,3}, i)$						0	$C_4^{1,3} < X$	$\emptyset$	$\emptyset$
$(C_4^{1,3}, -i)$							$C_4^{1,3} < \overline{A_2}^{1,3}$	$\emptyset$	$\emptyset$
$(C_4^{2,2}, i)$								0	$C_4^{2,2} < X$
$(C_4^{2,2}, -i)$									$C_4^{2,2} < \overline{A_2}^{2,2}$

2	$(C_6^{[4]}, \epsilon)$	$(C_6^{[4]}, \epsilon^2)$	$(C_6^{[4]}, \epsilon^4)$	$(C_6^{[4]}, \epsilon^5)$
$(\overline{A_1}^{[n]}, -1)$	$\theta < (C_6^{[4]}, \epsilon^4)$	$\theta < (C_6^{[4]}, \epsilon^5)$	$(C_6^{[4]}, \epsilon)$	$(C_6^{[4]}, \epsilon^2)$

3	$(\overline{A_3}^{1,1,2}, -1)$	$(\overline{A_4}^{1,1,1,1}, -1)$
$(\overline{A_3}^{1,1,2}, -1)$	$\overline{A_3}^{1,1,2} < X$	$\emptyset$
$(\overline{A_4}^{1,1,1,1}, -1)$		$\overline{A_4}^{1,1,1,1} < X$

4	$(C_6^{[4]}, \epsilon)$	$(C_6^{[4]}, \epsilon^2)$	$(C_6^{[4]}, \epsilon^4)$	$(C_6^{[4]}, \epsilon^5)$	$(C_6^{1,3}, \epsilon^2)$	$(C_6^{1,3}, \epsilon^4)$	$(C_6^{2,2}, \epsilon^2)$	$(C_6^{2,2}, \epsilon^4)$	$(C_6^{1,1,2}, \epsilon^2)$	$(C_6^{1,1,2}, \epsilon^4)$
$(C_6^{[4]}, \epsilon)$	0	$\theta < \overline{A_1}^{[n]}$	0	$C_6^{[4]} < X$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(C_6^{[4]}, \epsilon^2)$		θ	$C_6^{[4]} < X$	$(C_6^{[4]}, \epsilon)$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(C_6^{[4]}, \epsilon^4)$			0	$C_6^{[4]} < \overline{A_1}^{[n]}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(C_6^{[4]}, \epsilon^5)$				$(C_6^{[4]}, \epsilon^4)$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(C_6^{1,3}, \epsilon^2)$					0	$C_6^{1,3} < X$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(C_6^{1,3}, \epsilon^4)$						0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(C_6^{2,2}, \epsilon^2)$							0	$C_6^{2,2} < X$	$\emptyset$	$\emptyset$
$(C_6^{2,2}, \epsilon^4)$								0	$\emptyset$	$\emptyset$
$(C_6^{1,1,2}, \epsilon^2)$									0	$C_6^{1,1,2} < X$
$(C_6^{1,1,2}, \epsilon^4)$										0

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