The Infinite Dimensional Frobenius Manifold of $2D$ Toda Hierarchy

Luca Philippe Mertens

Mathematical Physics Sector
SISSA - International School for Advanced Studies

Supervisor: Prof. Boris Dubrovin

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Introduction

Frobenius manifolds are a geometric framework to describe conformal solutions of WDVV equations [20]. They were introduced by B. Dubrovin [23] in order to study the relationship between 2D topological field theories and integrable systems. The theory was inspired by the discovery of E. Witten and M. Kontsevich that the logarithm of the τ function of the KdV hierarchy coincides with the generating function of the intersection numbers of the so called tautological classes in $\mathbf{H}^*(\bar{M}_{g,n})$, the ϵ expansion corresponding to the genus expansion of the generating function [61]. Frobenius manifolds turned out to be a key tool for the study of sophisticated topological objects. For example, they appear as natural structures over quantum cohomologies of smooth projective varieties, encoding the structure of genus zero Gromov-Witten invariants.

A n-dimensional Frobenius manifold is a manifold M of dimension n equipped with a flat metric η and an associative, commutative, algebra structure on the tangent space such that, at every point $p \in M$, the metric is *invariant* with respect to the product, i.e.

$$\eta(x, y \cdot z) = \eta(x \cdot y, z) \qquad \forall x, y, z \in \mathrm{T}_p M$$

A finite dimensional unital associative algebra together with a non degenerate invariant bilinear form is called a Frobenius algebra.

Let c be the symmetric 3 tensor defined by $c(x, y, z) := \eta(x, y \cdot z)$. In the physical jargon, this tensor is called the *3-point correlator function.* If we require the covariant derivative $(\nabla_w c)(x, y, z)$ to be a 4 symmetric tensor, then in a system of local flat coordinates (v^α) the components of c are given by:

$$c_{\alpha,\beta,\gamma} = \partial_\alpha \partial_\beta \partial_\gamma F \qquad \alpha, \beta, \gamma = 1, \ldots, n.$$

The function F is called the *potential* of the Frobenius manifold. A Frobenius manifold is completely determined by its potential and by two vector fields: the identity vector field e and the Euler vector field E. The former is the identity of the algebra structure on the tangent space, while the latter describes certain conformal properties of the Frobenius structure.

Writing the *associativity equations* for the algebra in terms of the potential one results in a system of non linear equations for $F(v)$ which turn out to be the celebrated WDVV equations:

$$\frac{\partial^3 F}{\partial v^\alpha \partial v^\beta \partial v^\lambda} \eta^{\lambda,\mu} \frac{\partial^3 F}{\partial v^\mu \partial v^\gamma \partial v^\delta} = \frac{\partial^3 F}{\partial v^\gamma \partial v^\beta \partial v^\lambda} \eta^{\lambda,\mu} \frac{\partial^3 F}{\partial v^\mu \partial v^\alpha \partial v^\delta} \qquad \alpha, \beta, \gamma = 1, \ldots, n. \quad (1)$$

Following the axiomatic formulation given by M.F. Atiyah, it is possible to establish an equivalence of categories between 2D topological field theories and commutative Frobenius algebras [5]. Given a topological conformal field theory, one can construct the canonical moduli space of deformations parametrized by the *coupling constants* $\mathbf{v} = (v^1, v^2, \dots v^n)$ of the theory. From a physical viewpoint, a Frobenius manifold is the moduli space of these deformations. The flat coordinates are the coupling constants, and for a given point $v = (v^\alpha) \in M$, the Frobenius algebra structure at the tangent space $\mathrm{T}_v M$ is the algebraic formulation of the corresponding physical theory.

Given a n-dimensional Frobenius manifold M, one can construct a (dispersionless) bihamiltonian integrable system on the (formal) loop space[1] $\mathcal{L}M$, known as the *principal hierarchy*. The bihamiltonian structure is determined by a pencil of metrics naturally defined on the manifold [17], while a basis of the first integrals of the hierarchy can be efficiently computed in terms of the flat sections of the canonical *deformed flat connection*

$$
\begin{aligned}
\left(\tilde{\nabla}_\kappa\right)_u v &= \nabla_u v + \kappa u \cdot v \\
\left(\tilde{\nabla}_\kappa\right)_{\frac{d}{d\kappa}} v &= \partial_\kappa v + U.v - \frac{1}{\kappa}\mathcal{V} v
\end{aligned}
$$

A flat section is a function $f(v,\kappa)$ such that $\tilde{\nabla} df(v,\kappa) = 0$. The basis of first integrals $\{\theta_{\alpha,k}\}$ is constructed by taking the power series expansion in the deformation parameter κ of the analytic part $\theta_\alpha(\kappa) = \sum_{k\geqslant 0} \theta_{\alpha,k}\kappa^k$ of a basis of flat sections $v_\alpha(\kappa)$, for $\alpha = 1, \dots, n$. In a system of flat coordinates v^α the resulting system of differential equations for the first integrals $\{\theta_{\alpha,k}\}$ is:

$$
\begin{aligned}
\partial_\lambda, \partial_\mu \theta_{\alpha,p} &= \kappa c^\nu_{\lambda,\mu} \partial_\nu \theta_{\alpha,(p-1)} \qquad p > 0 &\quad (2)\\
\theta_{\alpha,0} &= v_\alpha = \eta_{\alpha,\beta} v^\beta &\quad (3)
\end{aligned}
$$

with the additional constraint for the matrix $\Theta(\kappa) := \eta^{\alpha,\nu}\partial_\nu \theta_\beta(\kappa)$ given by:

$$
\kappa \partial_\kappa \Theta(\kappa) + [\Theta(\kappa), \mathcal{V}] = \kappa \mathcal{U}\Theta(\kappa) - \Theta(\kappa) R(\kappa) \quad (4)
$$

here $c^\nu_{\lambda,\mu}$ are the structure constants of the algebra structure, $\mathcal{U}$ is the multiplication by E operator, while $R(\kappa) = \sum_{k>0} R_k \kappa^k$ is a polynomial in κ with coefficients in the constant matrices ring.

The Hamiltonians

$$
H_{\alpha,p} = \int_{S^1} \theta_{\alpha,p+1}(v(x))\, dx
$$

commute pairwise with respect to both the Poisson structures associated with the flat pencil of metrics on M. These Hamiltonians satisfy certain recursion relations with respect to the bihamiltonian structure. The Hamiltonians $H_{\alpha,-1}$ are Casimirs of the first Poisson structure while the Hamiltonians $H_{\alpha,0}$ generate the *primary flows*. A

[1] The loop space of M is the space of maps from $\mathbb{S}^1$ to M. The formal loop space is a scheme-theoretic definition of this space, where one does not define the space itself, but the algebra of functionals over the space.

simple computation shows that these flows can be written in terms of the structure constants of the Frobenius algebra:

$$\frac{\partial v^\gamma}{\partial t^{\alpha,0}} = \sum_{\beta=1}^{n} c^{\gamma}_{\alpha\beta}(v)\, \frac{\partial v^\beta}{\partial x}, \quad \gamma = 1, \dots, n \tag{5}$$

Among the τ functions of the principal hierarchy, the one selected by the so called *topological solution* is of particular interest. One can prove that $\log\tau$ restricted to the primary times $t^{\alpha,0}$, once the primary times are identified with the corresponding flat coordinates, coincides with the potential of the Frobenius manifold under consideration (see for instance [24, 28]).

Frobenius manifolds are an efficient tool to study integrable systems of PDEs with one spatial dimension. For an integrable system of n evolutionary PDEs

$$\partial_t u_i = K_i(u; u_x, u_{xx}, \dots; \epsilon), \quad i = 1, \dots, n \tag{6}$$

depending on a small parameter ϵ the structure of the small dispersion limit $\epsilon \to 0$ (if the limit exists), under very general assumptions of existence of a bihamiltonian structure and a tau-function, is completely described by a suitable n-dimensional Frobenius manifold (see details in [28]).

This thesis is aimed at extending the theory of Frobenius manifold to integrable evolutionary PDEs in two spatial dimensions, the so-called 2+1 dimensional systems. In particular, $2+1$ dimensional systems, as integrable hierarchies, have an infinite number of dependent variables u_i (see for instance the Lax symbols of KP or 2D Toda hierarchies). This implies a generalization of the theory to infinite dimensional manifolds. Several reasons motivate the interest in this construction. From the viewpoint of integrable systems, the Frobenius manifold is a tool to describe a complete set of first integrals of an integrable dispersionless system also for the 2+1 dimensional case. This result can be used to study properties of general solutions to these systems under very mild analytic assumptions.

Our main result is the construction of the first example of infinite dimensional Frobenius manifold. As the principal hierarchy associated to this Frobenius manifold is an extension of the $2D$ Toda hierarchy, we will call it the $2D$ Toda Frobenius manifold.

Definition 0.1 *Let $\mathcal{H}(D_0)$ be the set of functions on the closed unitary disk*[2] *that are holomorphic on the punctured disc and have a simple pole at 0. Let $\mathcal{H}(D_\infty)$ be the set of functions on the complementary disk (i.e. $|z| \geqslant 1$) that are holomorphic and with a simple pole at ∞.*

$$M_{2DT} := \left\{(\lambda, \bar\lambda) \in \mathcal{H}(D_\infty) \times \mathcal{H}(D_0) \;\; | \; \lambda + \bar\lambda \textit{ is injective on } S^1,\ \mathrm{wind}_0(\lambda + \bar\lambda)(z) = 1\right\}$$

[2] Holomorphic in this case means holomorphic in some open set containing the closed disk

Theorem 0.2 *M_{2DT} has a Frobenius Manifold structure with potential F given by:*

$$
\begin{aligned}
F(\mathbf{t},u,v) =& \frac{1}{2}\left(\frac{1}{2\pi i}\right)^2 \oint_\Gamma \oint_\Gamma \mathrm{Li}_3\left(\frac{w_1 e^{t(w_1)}}{w_2 e^{t(w_2)}}\right) dw_1\, dw_2 + \\
& + \frac{1}{2\pi i}\oint_\Gamma \left(\frac{e^u}{w e^{t(w)}} - w e^{t(w)}\right) dw + \frac{(2v+t_{-1})}{4}\frac{1}{2\pi i}\oint_\Gamma (t(w))^2\, dw + \\
& + \frac{1}{2}v^2 u - e^u
\end{aligned}
\tag{7}
$$

where the Euler and identity vector fields are given by

$$
E = -\sum_{k\in\mathbb{Z}} k t^k \partial_k + v\partial_v + 2\partial_u \qquad e = \partial_v \tag{8}
$$

Here for a given point $(\bar{\lambda},\lambda) \in M_{2DT}$ we have that $t(w) := \log\left(\frac{z(w)}{w}\right)$, where $z(w)$ is the inverse function of the map $w(z) = \bar{\lambda}(z) + \lambda(z)$. Γ is the image of the unit circle under the map $w(z)$. The coefficients of the function $\mathbf{t}(w) = \sum_{\alpha\in\mathbb{Z}} t^\alpha w^\alpha$ are, together with u and v, a system of flat coordinates for M_{2DT}.

In particular, the 3-point correlator function is given by the following "Landau – Ginsburg type" formula:

$$
\begin{aligned}
&< \partial_1 \cdot \partial_2, \partial_3 >= \\
&= \frac{1}{4\pi i}\oint_{|z|=1} \frac{\partial_1 w\, \partial_2 w\, \partial_3 s + \partial_1 w\, \partial_2 s\, \partial_3 w + \partial_1 s\, \partial_2 w\, \partial_3 w - s'\, \partial_1 w\, \partial_2 w\, \partial_3 w}{z^2 w'}\, dz \\
&-\mathrm{Res}_{z=0} \frac{\partial_1(\bar{\lambda}-l)\, \partial_2 l\, \partial_3 l + \partial_1 l\, \partial_2(\bar{\lambda}-l)\, \partial_3 l + \partial_1 l\, \partial_2 l\, \partial_3(\bar{\lambda}-l) + \partial_1 l\, \partial_2 l\, \partial_3 l}{z^2 \bar{\lambda}'}\, dz
\end{aligned}
\tag{9}
$$

where all differentiations of the functions $w(z) := (\lambda + \bar{\lambda})(z)$, $s(z) := \bar{\lambda}(z) - \lambda(z)$, $l(z) := -z + v + \frac{e^u}{z}$ have to be done keeping $z = \text{const}$.

Let us recall that the 2D Toda equation is the differential difference equation

$$
\partial_t^2 u_n - \partial_y^2 u_n = e^{u_{n+1}} - 2e^{u_n} + e^{u_{n-1}}. \tag{10}
$$

In this case we have two spatial variables: a continuous variable y and a discrete one $n \in \mathbb{Z}$. The 1+1 reduction $\partial_y u_n = 0$ of (2.20) gives the classical Toda lattice

$$
\ddot{q}_n = e^{q_{n+1}-q_n} - e^{q_n - q_{n-1}}, \quad u_n = q_{n+1} - q_n
$$

i.e., an infinite system of points on the line with exponential interaction of the nearest neighbors. The dispersionless limit is the PDE

$$
u_{tt} - u_{yy} = (e^u)_{xx} \tag{11}
$$

for the function $u = u(x, y, t)$ obtained by interpolating

$$u_n(y, t) = u(\epsilon\, n, y, t)$$

rescaling

$$y \mapsto \epsilon\, y, \quad t \mapsto \epsilon t$$

and then setting $\epsilon \to 0$.

Following K.Ueno and K.Takasaki [58], the 2D Toda equation can be embedded into the 2D Toda hierarchy. The dispersionless limit of this hierarchy was introduced by K. Takasaki ant T. Takebe [55]. Let $\lambda(p), \bar{\lambda}(p)$ be two formal power series in p^{-1}, p defined by

$$\begin{aligned}\lambda(p) &= \sum_{k\leqslant 1} u_k p^k \\ \bar{\lambda}(p) &= \sum_{k\geqslant -1} \bar{u}_k p^k\end{aligned}$$

where $u_1 = 1, \bar{u}_{-1} \neq 0$. The dispersionless 2D Toda hierarchy is described by the following set of Lax equations:

$$\begin{array}{ccc}\partial_{t_n}\lambda = \{\mathcal{A}_n, \lambda\} & , & \partial_{\bar{t}_n}\lambda = \{\mathcal{B}_n, \lambda\} \\ \partial_{t_n}\bar{\lambda} = \left\{\mathcal{A}_n, \bar{\lambda}\right\} & , & \partial_{\bar{t}_n}\bar{\lambda} = \{\mathcal{B}_n, \bar{\lambda}\}\end{array}$$

Where $\mathcal{A}_n(p) := (\lambda^n(p))_{<0}$ and $\mathcal{B}_n(p) := (\bar{\lambda}^n(p))_{\geqslant 0}$, while $\{f, g\} := p\frac{df}{dp}\frac{dg}{dx} - p\frac{dg}{dp}\frac{df}{dx}$ is the canonical Poisson bracket.

In particular the coefficient

$$u(x) = \log \bar{u}_{-1}(x)$$

viewed as a function of $t = s_1 - \bar{s}_1$, $y = s_1 + \bar{s}_1$, satisfies the dispersionless Toda equation.

The 2D Toda hierarchy owns all usual properties of 1+1 systems: the flows commute pairwise, and admit a bihamiltonian description provided by G. Carlet [11]. These equations have been extensively studied [39, 41, 50, 56, 62] after the discovery, due to M.Mineev-Weinstein, P.B.Wiegmann and A.Zabrodin [51, 60] of a remarkable connection between the dispersionless 2D Toda hierarchy and the theory of conformal maps.

In [8, 42] it was shown that the logarithm of the tau-function $\tau(\mathbf{t}, \bar{\mathbf{t}})$ of any solution to the dispersionless 2D Toda hierarchy satisfies WDVV equation. This gives solutions to WDVV depending on an infinite number of variables, however, so far no one proved that there is a Frobenius manifold structure related to these solutions. In particular, from a Frobenius manifold viewpoint, the higher times $t_n, \bar{t}_n$ for $n > 0$ are not primary flows, i.e. flows which correspond to variables on the Frobenius manifold. These flows appear in the principal hierarchy, but the solution we seek must be defined on a set of *new* flows, which are the ones we present in Theorem 0.3. Let us remark that a particular tau-function admits an elegant realization on the space of simply connected plane domains bounded by simple analytic contours assuming the possibility to locally parametrize the domains by their exterior harmonic moments (see also [56]). Such an assumption has been rigorously justified in [35] for the class of polynomial boundary curves in which case all harmonic moments but a finite number are equal to zero. It is clear that the two WDVV solutions - the one proposed by Krichever and the one we propose - are defined on different spaces. A connection between our solution of WDVV given in Theorem 0.2 and the one of [8, 42] has to be clarified yet. In particular, extending our solution to all the hiegher times of the principal hierarchy and then restricting it to the classical 2D Toda times, we should get, according to Krichever, another solution of WDVV equations. We remark that, to our best knowledge, this new solution of WDVV is not the potential of a Frobenius manifold. Nevertheless, it would be interesting to understand if also other restrictions give solutions of WDVV equations, and if these solutions are potentials of Frobenius manifolds.

One novelty in our construction of the infinite dimensional Frobenius manifold associated with the 2D Toda hierarchy is that the symbols λ, $\bar\lambda$ are no more formal power series, but a pair of analytic functions which are both defined in a neighborhood of $|z| = 1$. Let us point that this assumption is fundamental in our construction, as it becomes evident when we give the Lax formulation of the primary flows associated to M_{2DT}:

Theorem 0.3 *The primary flows of the principal hierarchy associated to M_{2DT} have the following Lax form*

$$\begin{aligned}
&\frac{\partial\lambda(z)}{\partial t^{\alpha,0}} = \frac{1}{\alpha+1}\left\{w^{\alpha+1}(z)_{<0}, \lambda(z)\right\}, \quad \frac{\partial\bar\lambda(z)}{\partial t^{\alpha,0}} = -\frac{1}{\alpha+1}\left\{w^{\alpha+1}(z)_{\geqslant 0}, \bar\lambda(z)\right\}\\
&\alpha\in\mathbb{Z}, \quad \alpha\neq -1,\\
&\frac{\partial\lambda(z)}{\partial t^{-1,0}} = \left\{\left(\log\frac{w(z)}{z}\right)_{<0} + \log z, \lambda(z)\right\}, \quad \frac{\partial\bar\lambda(z)}{\partial t^{-1,0}} = -\left\{\left(\log\frac{w(z)}{z}\right)_{\geqslant 0}, \bar\lambda(z)\right\}\\
&\frac{\partial}{\partial t^{v,0}} = \frac{\partial}{\partial x}\\
&\frac{\partial}{\partial t^{u,0}} = -\frac{\partial}{\partial \bar s_1}
\end{aligned} \tag{12}$$

The function $w^{\alpha+1}(z) = (\lambda+\bar\lambda)^{\alpha+1}(z)$ has no meaning at the formal level, but it is well defined as product of functions holomorphic in a neighborhood of $\mathbb{S}^1$.

Another difference is with respect to the widely accepted scheme of [15], which suggests to use the symbol of the Lax operator as the "Landau – Ginsburg superpotential" in order to construct the Frobenius manifold (also known as the small phase space of the two-dimensional topological field theory). For the 2D Toda case one has to deal with a pair of "Landau – Ginsburg superpotentials" treating them on equal footing.

The second main result is that, starting from our infinite dimensional Frobenius manifold, we were able to construct a new integrable hierarchy which extends 2D Toda.

Theorem 0.4 *The following Hamiltonian densities:*

$$\theta_{j,p} \quad := \quad -\oint_\Gamma \frac{(\lambda+\bar\lambda)(z)^{j+1}}{j+1}\frac{(\bar\lambda-\lambda)^p}{2^p p!}\frac{dz}{z} \qquad j \neq -1, p \geqslant 0 \tag{13}$$

$$\theta_{u,p} \quad := \quad \oint_\Gamma \frac{\bar\lambda(z)^{(p+1)}}{(p+1)!}\frac{dz}{z} \qquad p \geqslant 0 \tag{14}$$

$$\theta_{\lambda,p} \quad := \quad \oint_\Gamma \frac{\lambda(z)^{(p+1)}}{(p+1)!}\frac{dz}{z} \qquad p \geqslant 0 \tag{15}$$

define a set of mutually commuting Hamiltonians with respect to the bihamiltonain structure induced by M_{2DT}. The resulting integrable system is an extension of the $2D$-Toda hierarchy.

In [1, 2] M.Adler and P. van Moerbeke proposed an extension of the 2D Toda hierarchy by adding the flows with the Lax representation of the form

$$\frac{\partial L}{\partial s_{ij}} = [P_{\geqslant 0}, L], \quad \frac{\partial \bar L}{\partial s_{ij}} = -\left[P_{<0}, \bar L\right] \tag{16}$$

where L, $\bar L$ are the Lax operators of the $2D$ Toda hierarchy, P is an operator defined by

$$P = L^i \bar L^j, \quad i,\, j \geqslant 0, \tag{17}$$

while $P_{\geqslant 0}$ and $P_{<0}$ are the positive and negative part of the operator (see Definitions 2.4). They argued that these flows, if well-defined, should commute pairwise. Note that the dispersionless limits of these flows make sense on our infinite dimensional Frobenius manifold since the products $\lambda^i(z)\bar\lambda^j(z)$ are well defined for all nonnegative integers i, j. One can check that all these dispersionless flows are linear combinations of the flows of the principal hierarchy associated with the Frobenius manifold M_{2DT}.

Outline of the thesis. The first chapter is a brief review on Frobenius manifolds and integrable systems. In the second chapter we recall the construction of the bihamiltonian structure for the 2D Toda hierarchy using R-matrix theory. Although the procedure is the same as in [11], a new R-matrix is proposed to provide a new bihamiltonian structure in the dispersionless limit. In the third chapter the Frobenius manifold M_{2DT} is defined. We provide explicit formulae for the 3-point correlator function and the intersection form. Moreover, we prove that M_{2DT} is semi simple by defining the canonical coordinates. The last chapter is devoted to the principal hierarchy.

Chapter 1

Frobenius manifolds and integrable systems

1.1 What is a Frobenius manifold?

Definition 1.1 *A* Frobenius structure $(M, \eta, \cdot, e, E)$ *of charge* d *on a manifold* M *is given by:*

- *A non degenerate symmetric* $(0,2)$ *tensor*[1] η*. We require* η *to be flat in the usual metric sense (vanishing of the curvature tensor).*
- *An associative, commutative algebra structure with unity* $(\mathrm{T}M, \cdot)$ *over the tangent space. Let* c *be the 3-tensor defined by* $c(x,y,z) := \eta(x, y\cdot z)$ *for* x,y,z *in* $\mathrm{T}M$*. We require* c *to be symmetric in* x,y,z*, and* $\nabla_w c(x,y,z)$ *to be symmetric in* x,y,z,w*.*
- *A covariantly constant vector field* e*, which is the unity of the algebra structure.*
- *An affine vector field* E*, i.e.* $\nabla\nabla E = 0$*, called the* Euler vector field*. The Euler vector field acts conformally on the structure coefficients of the product and the metric:*

$$\begin{aligned} [E, x\cdot y] - [E,x]\cdot y - x\cdot [E,y] &= x\cdot y \\ E < x,y > - < [E,x]\,, y > - < x, [E,y] > &= (2-d) < x,y > \end{aligned}$$

The symmetry of c is equivalent to the requirement that η is *invariant* with respect to the product: $\eta(a\cdot b, c) = \eta(a, b\cdot c)$ for every $a,b,c \in \mathrm{T}M$. For every $\mathbf{t} \in M$ we have that $(\mathrm{T}_{\mathbf{t}}M, \cdot, \eta)$ is a *Frobenius algebra*: a finite dimensional associative unital algebra equipped with a symmetric non degenerate invariant bilinear form.

Let $\{t^\alpha\}_{\alpha\in I}$ be a set of local flat coordinates. The requirement that ∇c is a symmetric tensor implies that locally $c_{\alpha,\beta,\gamma} = \nabla_\alpha\nabla_\beta\nabla_\gamma F$. The function F is called the

[1] We will refer to it as a metric, although we don't require η to be positive definite.

potential of the Frobenius manifold. In the flat coordinates t^α, the *associativity equations* for the structure coefficients of the algebra give the following set of non linear PDEs for F:

$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\lambda}\eta^{\lambda,\mu}\frac{\partial^3 F}{\partial t^\mu \partial t^\gamma \partial t^\delta} = \frac{\partial^3 F}{\partial t^\gamma \partial t^\beta \partial t^\lambda}\eta^{\lambda,\mu}\frac{\partial^3 F}{\partial t^\mu \partial t^\alpha \partial t^\delta} \qquad \alpha,\beta,\gamma,\delta \in I \tag{1.1}$$

These are the celebrated WDVV equations [25].

Theorem 1.2 *A Frobenius manifold structure is locally equivalent to the existence of a potential F satisfying the WDVV equations and the two properties:*

Normalization: $F_{1,\lambda,\mu} = \eta_{\lambda,\mu}$ *where* $\frac{\partial}{\partial v^1} = e$ *is a marked coordinate*

Conformal structure: $\mathcal{L}_E F = (3-d)F +$ *(quadratic terms)*

WDVV equations firstly appeared in the physics literature as a set of equations to describe the partition function $F(\mathbf{t})$ of a 2D TCFT (topological conformal field theory) together with its deformations preserving topological invariance [61]. Following the axiomatic formulation of a TFT (topological field theory) given by M.F. Atiyah [5], one can establish an equivalence of categories between 2D TFT and commutative Frobenius algebras [44]. For any TCFT one can construct a canonical moduli space of deformations parametrized by the *coupling constants* $\mathbf{t} = (t^\alpha, t^\beta, t^\gamma, \ldots)$ of the theory. The function $F(\mathbf{t})$ is the partition function of the theory. In this framework, one can think of a Frobenius manifold as a moduli space of 2D topological field theories. M is the space of parameters of the model, and for every fixed $\mathbf{t} \in M$ the Frobenius algebra $(\mathrm{T}_{\mathbf{t}}M, \cdot, \eta)$ describes the corresponding perturbation of the theory.

1.2 From Frobenius Manifolds to Integrable Hierarchies

In this section we show how to associate an integrable system to a given Frobenius manifold. Loosely speaking, an integrable system is an infinite set of compatible PDEs of evolutionary type. We will construct this set on the formal loop space of the Frobenius manifold. The standard reference here is the paper of B. Dubrovin and Y. Zhang - Normal forms of integrable PDEs, Frobenius manifolds and Gromov – Witten invariants [28], but we also invite the reader interested in the formal loop space definition, or more generally in the algebraic approach to integrable systems to refer to the excellent book of L.A. Dickey - Soliton Equations and Hamiltonian Systems - [14].

1.2.1 Formal loop spaces

Definition 1.3 *Let $n \in \mathbb{N}$. The ring of* differential polynomials *in the variables u^i for $i = 1, \ldots, n$ is the ring of polynomials $\mathcal{A}$ in the variables u^i_m, where $i = 1, \ldots, n$ and*

$m \in \mathbb{N}$, $m \geqslant 1$ with coefficients in the algebra of smooth functions in the variables u^i for $i = 1, \ldots, n$. This ring has a natural differential ∂_x which is defined on the generators as

$$\begin{array}{rccc} \partial_x : & \mathcal{A} & \longrightarrow & \mathcal{A} \\ & u^i_m & \longmapsto & u^i_{(m+1)} \end{array} \tag{1.2}$$

and extends to the whole ring $\mathcal{A}$ via Leibniz rule.

The variable u^i_m is the formal m-th derivative of u^i. For this reason, for small values of m the variables u^i_m are also denoted with the alternative notation $u^i := u^i_0$, $u^i_x := u^i_1$, $u^i_{xx} := u^i_2$ and so on.

Definition 1.4 *A* vector field *∂_t is a derivation of $\mathcal{A}$ which commutes with ∂_x.*

We will refer to a vector field also as a PDE of evolutionary type. A vector field is determined by a set of n differential polynomials $P^i \in \mathcal{A}$, where $i = 1, \ldots, n$. These polynomials describe the evolution with respect to a formal time variable t of the variables u^i:

$$\partial_t u^i := P^i(u^j, u^j_x, u^j_{xx}, \ldots)$$

Since the derivation in t commutes with the derivation in x we have that:

$$\partial_t u^i_m = \partial_t \partial_x^m u^i = \partial_x^m \partial_t u^i = \partial_x^m P^i$$

hence given an element $Q \in \mathcal{A}$ we have that:

$$\partial_t Q := \sum_{i=1}^{n} \sum_{m \geqslant 0} (\partial_x^m P^i) \frac{\partial Q}{\partial u^i_m} \qquad \forall Q \in \mathcal{A}$$

Two vector fields ∂_s and ∂_t *commute* when $\partial_s \partial_t u^i = \partial_t \partial_s u^i$ for $i = 1, \ldots, n$.

We define the formal integration as the cokernel of the map ∂_x, hence

$$\int h(u^i, u^i_x, u^i_{xx}, \ldots) dx$$

is just the equivalence class of elements in $\mathcal{A}$ which differ by a total derivative.

Definition 1.5 *Let $\Lambda_0 := \mathcal{A} / \partial_x \mathcal{A}$. We define the* formal loop space $\mathcal{L}(M)$ *of the manifold $M = \{u^i\}$ to be the ring:*

$$\mathcal{F} := \Lambda_0 \oplus \Lambda_0^{2\otimes} \oplus \Lambda_0^{3\otimes} \oplus \ldots \tag{1.3}$$

An element in $\mathcal{F}$ is also called a *local functional.* Let us explain briefly how to interpret $\mathcal{F}$ as a ring of functionals of a loop space. A loop space of a manifold M is a space of maps from the unit circle S^1 to M. Usually the maps are required to have a certain regularity, like being continuous, smooth, etc. A point of the loop space will be represented by a set of functions $u_i(x)$, where x is the independent variable on the unit circle. Given a differential polynomial $P \in \mathcal{A}$ we can evaluate it in $u^i(x)$ and then integrate the resulting function to get a scalar. In this way, every differential polynomial defines a functional on the loop space, but clearly polynomials differing by a total derivative give the same functional, hence we have to look at equivalence classes. The tensor powers $\Lambda_0^{n\otimes}$ are needed just to complete Λ_0 to a ring structure (i.e. to define multiplication of functionals).

Suppose we have a system of PDEs of evolutionary type, i.e. of vector fields. Each vector field describes the evolution with respect to a formal "time variable" t. By definition, a vector field ∂_t commutes with ∂_x, hence ∂_t induces a PDE of evolutionary type on Λ_0 by:

$$\partial_t \int h(u^i, u^i_x, u^i_{xx}, \ldots)dx := \int \partial_t h(u^i, u^i_x, u^i_{xx}, \ldots)dx$$

A *Hamiltonian structure* for a system of PDEs is a Poisson bracket $\{\ ,\ \}$ defined on the space $\mathcal{F}$ such that each flow can be written in the form $\partial_{t_n}\bar{u}^i = \left\{\bar{H}_n, \bar{u}^i\right\}$, where $\bar{u}^i := \int u^i dx$. The local functional $\bar{H}_n$ are called the *Hamiltonians* of the flows, while the differential polynomials H_n inducing them are the *Hamiltonian densities*. A *bihamiltonian structure* is the data of two Poisson structures which are compatible, i.e. $\{\ ,\ \}_\lambda := \{\ ,\ \}_2 - \lambda\{\ ,\ \}_1$ is a Poisson bracket for every $\lambda \in \mathbb{C}$.

Given a system of PDEs of evolutionary type, one can compute its *dispersionless limit* to obtain an auxiliary system of quasilinear PDEs. This system is usually easier to study, and gives information on the behavior of the solutions of the original *dispersive* system. The intuitive idea is to replace $t \mapsto \epsilon t, x \mapsto \epsilon x$ and then take the limit $\epsilon \to 0$. Note that one can perform the limit procedure to all the machinery introduced so far (local functionals, Hamiltonian structures,...). The algebraic way to do this is to extend the above theory to the ring of formal power series $\mathcal{A}[[\epsilon]]$ (see [28] for details).

1.2.2 The Principal hierarchy of a Frobenius manifold

Given a n-dimensional Frobenius Manifold M, the so called *principal hierarchy* is a dispersionless bihamiltonian integrable sytem on the formal loop space[2] $\mathcal{L}(M)$, i.e. an infinite system of compatible PDEs of evolutionary type possesing a bihamiltonian structure.

We first give the bihamiltonian structure. Given a flat contravariant metric $g^{i,j}$ over M with Christoffel symbols $\Gamma^{i,j}_k$ one can define a Poisson bracket of Dubrovin-Novikov type over $\mathcal{L}(M)$:

$$\left\{\bar{P}, \bar{Q}\right\} := \int \frac{\delta P}{\delta u^i(x)} \left\{u^i(x), u^j(y)\right\} \frac{\delta Q}{\delta u^j(y)} dy \tag{1.4}$$

where: $\left\{u^i(x), u^j(y)\right\} = g^{i,j}\delta'(x-y) + \Gamma^{i,j}_k u^k \delta(x-y)$, while $\frac{\delta P}{\delta u^i(x)}$ is the (formal) variational derivative. The bihamiltonian structure of the principal hierarchy is given

[2] We suppose here to have a chart of global local coordinates u^i, which is actually the case for many classes of Frobenius manifolds. Clearly the construction can be generalized to the general case

by two contravariant flat metrics. The first one is just the dual of η on the cotangent space. The second, called the *intersection form*, is defined by $g^{i,j} = E^k c_k^{i,j}$. One can prove that the two induced Poisson structures are compatible (details can be found in [17]).

To construct the Hamiltonian densities we have to introduce the *deformed flat connection*. This is a contravariant connection defined on $\pi^*\mathrm{T}M$, where $\pi : M \times \mathbb{CP}^1 \longrightarrow M$, depending on a complex parameter κ. Let $\mathcal{V}$ be the operator defined by

$$\mathcal{V} := \frac{2-d}{2} - \nabla E.$$

This is an antisymmetric operator on $\mathrm{T}M$ with respect to η. Let here $\mathcal{U}$ be the multiplication by E operator. The deformed flat connection is defined by

$$\begin{aligned}\left(\tilde{\nabla}_\kappa\right)_u v &= \nabla_u v + \kappa u \cdot v \\ \left(\tilde{\nabla}_\kappa\right)_{\frac{d}{d\kappa}} v &= \partial_\kappa v + \mathcal{U}.v - \frac{1}{\kappa}\mathcal{V}v\end{aligned}$$

A flat section is a function $f(v,\kappa) \in C^\infty(M \times \mathbb{CP}^1)$ such that $\tilde{\nabla} df(v,\kappa) = 0$. The basis of first integrals $\{\theta_{\alpha,k}\}$ is constructed by taking the power series expansion in the deformation parameter κ of the analytic part $\theta_\alpha(\kappa) = \sum_{k\geqslant 0} \theta_{\alpha,k}\kappa^k$ of a basis of flat sections $v_\alpha(\kappa)$, for $\alpha = 1,\ldots,n$. In a system of flat coordinates v^α the resulting system of differential equations for the first integrals $\{\theta_{\alpha,k}\}$ is:

$$\partial_\lambda, \partial_\mu \theta_{\alpha,p} = \kappa c^\nu_{\lambda,\mu} \partial_\nu \theta_{\alpha,(p-1)} \qquad p > 0 \tag{1.5}$$

$$\theta_{\alpha,0} = v_\alpha = \eta_{\alpha,\beta} v^\beta \tag{1.6}$$

with the additional constraint for the matrix $\Theta(\kappa) := \eta^{\alpha,\nu}\partial_\nu\theta_\beta(\kappa)$ given by:

$$\kappa\partial_\kappa\Theta(\kappa) + [\Theta(\kappa), \mathcal{V}] = \kappa\mathcal{U}\Theta(\kappa) - \Theta(\kappa)R(\kappa) \tag{1.7}$$

here $R(\kappa) = \sum_{k>0} R_k\kappa^k$ is a polynomial in κ with coefficients in the constant matrices ring.

The Hamiltonians

$$H_{\alpha,p} = \int_{S^1} \theta_{\alpha,p+1}(v(x))\, dx$$

commute pairwise with respect to both the Poisson structures associated with the flat pencil of metrics on M. These Hamiltonians satisfy the following recursion relations with respect to the bihamiltonian structure:

$$\{\cdot\ , H_{\alpha,p-1}\}_2 = \sum_{0\leqslant q\leqslant p} R^{\beta,q}_{\alpha,p}\{\cdot\ , H_{\beta,q}\}_1 \qquad p \geqslant 0 \tag{1.8}$$

where $R^p_p := p + \mathcal{V} + \frac{1}{2}$, $R^q_p := R_{p-q}$.

The Hamiltonians $H_{\alpha,-1}$ are Casimirs of the first Poisson structure while the Hamiltonians $H_{\alpha,0}$ generate the *primary flows*. A simple computation shows that these flows can be written in terms of the structure constants of the Frobenius algebra:

$$\frac{\partial v^\gamma}{\partial t^{\alpha,0}} = \sum_{\beta=1}^n c^\gamma_{\alpha\beta}(v)\,\frac{\partial v^\beta}{\partial x}, \quad \gamma = 1,\ldots,n \tag{1.9}$$

Many integrable systems admit a reformulation in terms of a unique equation, called the Hirota equation [13]. The solutions of this auxiliary differential equation are called τ *functions*, and they allow to give a solution of the whole integrable hierarchy in terms of a single function. More precisely one can define the functions $u^i(x, \mathbf{t})$ in terms of logarithmic derivatives of the τ function, and these functions u^i will satisfy all the equations of the hierarchy. Among the τ functions of the principal hierarchy, the one selected by the so called *topological solution* is of particular interest. One can prove that $\log \tau$ restricted to the primary times $t^{\alpha,0}$, once the primary times are identified with the corresponding flat coordinates, coincides with the potential of the Frobenius manifold under consideration (see for instance [24, 28]).

Chapter 2

2D Toda Hierarchy

The two dimensional Toda hierarchy is an integrable system of differential-difference equations, i.e. a system of equations for a set of variables $u_k, \bar{u}_l$ depending on a discrete parameter n and a set of continuous variables $s^i, \bar{s}^i$. The standard way to describe the flows of the hierarchy is with the Lax formulation. In order to do this one has to introduce an appropriate algebra of difference operators $\mathcal{D}$. We briefly recall its definition, essentially following that of U. Kimio and K. Takasaki, who introduced it in [58].

One can define a bihamiltonian structure for the $2D$ Toda hierarchy using the method of R-matrix theory. The R matrix theory techniques for integrable systems were mainly developed by Semenov-Tian-Shansky [53], and applied to the 2D Toda case by G. Carlet [11]. Basically, given an endomorphism $R : \mathcal{D} \longrightarrow \mathcal{D}$ with certain properties we can construct two compatible Poisson brackets and relate them to the Lax formalism.

The construction we expose here is very similar to the one done by G. Carlet: we first introduce an algebra splitting, then the associated R- matrix, and finally we compute the two Poisson brackets. What changes in our construction is the algebra structure of the space of difference operators. We define a new product which differs from the usual one used in the litterature (like for example in [1], [2]) for a sign in the second component (see Definition 2.1). This sign change modifies the final formula of the second Poisson bracket that one gets in [11]. Remarkably, the hint that it was possible to define this new $R - matrix$ came from the Frobenius manifold M_{2DT}. The advantage in constructing this R matrix (when actually we already had the Poisson structures!) is twofold. We don't need anymore to prove compatibility, since this is guaranteed from the general theory, and moreover we can write the bihamiltonian structure at the dispersive level.

2.1 Dispersive and dispersionless $2D$ Toda Hierarchy

Let $F := \mathbb{C}^{\mathbb{Z}}$ be the space of maps defined on the integer lattice with values in $\mathbb{C}$. We define the *shift operator* $\Lambda : F \longrightarrow F$:

$$(\Lambda f)(n) := f(n+1) \qquad \forall f \in F$$

Definition 2.1 *We introduce the following associative unital algebras of formal power series in Λ^{-1} and Λ with coefficients in F:*

$$\mathcal{A} \;:=\; \left\{ \sum_{k=-\infty}^{n} a_k \Lambda^k \mid a_k \in F, \quad n \in \mathbb{Z} \right\} \tag{2.1}$$

$$\mathcal{B} \;:=\; \left\{ \sum_{k=n}^{+\infty} a_k \Lambda^k \mid a_k \in F, \quad n \in \mathbb{Z} \right\} \tag{2.2}$$

We endow $\mathcal{A}$ with the standard algebra structure, i.e.

$$\sum_{k=-\infty}^{m} a_k \Lambda^k + \sum_{k=-\infty}^{n} b_k \Lambda^k := \sum_{k=-\infty}^{\sup(m,n)} (a_k + b_k) \Lambda^k$$

$$\sum_{k=-\infty}^{m} a_k \Lambda^k \cdot \sum_{k=-\infty}^{n} b_k \Lambda^k := \sum_{k=-\infty}^{m+n} \Big(\sum_{i+j=k} a_i b_j \Big) \Lambda^k$$

while in $\mathcal{B}$ we keep the same addition rule, and change the product:

$$\sum_{k=m}^{+\infty} a_k \Lambda^k \cdot \sum_{k=n}^{+\infty} b_k \Lambda^k := - \sum_{k=m+n}^{+\infty} \Big(\sum_{i+j=k} a_i b_j \Big) \Lambda^k$$

Clearly the identity of $\mathcal{A}$ is 1, while the identity of $\mathcal{B}$ is -1. Elements in $\mathcal{A}$ and $\mathcal{B}$ are usually called *difference operators*, note however that the action of a difference operator $A \in \mathcal{A}$ on a function $f \in F$ is not well defined in general.

One can easily check that these algebras are not commutative, hence we can give them a non trivial Lie algebra structure using the bracket:

$$[A, B] := A \cdot B - B \cdot A \tag{2.3}$$

We introduce the subalgebras:

$$\mathcal{A}_{\geqslant 0} \;:=\; \left\{ \sum_{k=0}^{n} a_k \Lambda^k \mid a_k \in F, \quad n \in \mathbb{Z} \right\} \tag{2.4}$$

$$\mathcal{A}_{<0} \;:=\; \left\{ \sum_{k=-\infty}^{-1} a_k \Lambda^k \mid a_k \in F, \quad n \in \mathbb{Z} \right\} \tag{2.5}$$

$$\mathcal{B}_{\geqslant 0} \;:=\; \left\{ \sum_{k=0}^{+\infty} a_k \Lambda^k \mid a_k \in F, \quad n \in \mathbb{Z} \right\} \tag{2.6}$$

$$\mathcal{B}_{<0} \;:=\; \left\{ \sum_{k=n}^{-1} a_k \Lambda^k \mid a_k \in F, \quad n \in \mathbb{Z} \right\} \tag{2.7}$$

Given an element $L \in \mathcal{A}$, we will denote respectively with $L_{\geqslant 0}$ and $L_{<0}$ the projection of L on $\mathcal{A}_{\geqslant 0}$ and $\mathcal{A}_{<0}$. We will use the same notation for elements in $\mathcal{B}$.

Definition 2.2

$$\mathcal{D} := \mathcal{A} \oplus \mathcal{B} \tag{2.8}$$

We are ready to recall the Lax formulation of the 2D Toda hierarchy:

Definition 2.3 *Let $(L, \bar{L}) \in \mathcal{D}$ such that:*

$$L = \Lambda + u_0 + u_{-1}\Lambda^{-1} + u_{-2}\Lambda^{-2} + \ldots \tag{2.9}$$

$$\tag{2.10}$$

$$\bar{L} = \bar{u}_{-1}\Lambda^{-1} + \bar{u}_0 + \bar{u}_1\Lambda + \bar{u}_2\Lambda^2 + \ldots . \tag{2.11}$$

where we require that $\bar{u}_{-1} \neq 0$. The 2D hierarchy is the set of evolutionary equations in the times $\{s_k, \bar{s}_k\}_{k\geqslant 1}$ for the functions $u_k, \bar{u}_l$

$$\begin{gathered} \frac{\partial L}{\partial s_k} = [A_k, L]\,, \quad \frac{\partial \bar{L}}{\partial s_k} = \left[A_k, \bar{L}\right] \\ \frac{\partial L}{\partial \bar{s}_k} = [B_k, L]\,, \quad \frac{\partial \bar{L}}{\partial \bar{s}_k} = \left[B_k, \bar{L}\right] \end{gathered} \tag{2.12}$$

where:

$$A_k := (L^k)_{\geqslant 0} \qquad B_k := (\bar{L}^k)_{<0} \qquad \forall\ k \geqslant 1$$

Theorem 2.4 *The 2D Toda Hierarchy is equivalent to the system of equations of the Zakharov–Shabat type:*

$$\partial_{s_k} A_l - \partial_{s_l} A_k + [A_l, A_k] = 0 \tag{2.13}$$

$$\partial_{\bar{s}_k} A_l - \partial_{s_l} B_k + [A_l, B_k] = 0 \tag{2.14}$$

$$\partial_{\bar{s}_k} B_l - \partial_{\bar{s}_l} B_k + [B_l, B_k] = 0 \tag{2.15}$$

for $k, l \geqslant 0$.

Proof See reference [58] $\square$

By taking the equation 2.14 with $k = l = 1$ we get the equations:

$$\partial_x \bar{u}_{-1}(n) = \bar{u}_{-1}(n)(u_0(n) - u_0(n-1)) \tag{2.16}$$

$$\partial_y u_0(n) = \bar{u}_{-1}(n) - u_{-1}(n+1) \tag{2.17}$$

where we have relabeled the time variables $x \equiv s_1$, $y \equiv \bar{s}_1$. Since $\bar{u}_{-1} \neq 0$ we can rewrite the system using the variable $u(n) := \log \bar{u}_{-1}$:

$$\partial_x u(n) = u_0(n) - u_0(n-1) \tag{2.18}$$

$$\partial_y u_0(n) = e^{u(n)} - e^{u(n+1)} \tag{2.19}$$

Finally, by writing the compatibility condition for ∂_x, ∂_y we get the 2D Toda equation:

$$\partial_x \partial_y u(n) = e^{u(n)} - e^{u(n+1)} - e^{u(n-1)} + e^{u(n)} = -\Delta e^{u(n)} \tag{2.20}$$

where Δ is the discrete Laplace operator [1] on the lattice $\mathbb{Z}$.

The dispersionless limit of the 2D Toda hierarchy is an integrable system of quasi-linear PDEs of evolutionary type which can be obtained from the 2D Toda hierachy performing a certain limit procedure. This system was introduced by K. Takasaki and T. Takebe [55], and then extensively studied after the discovery, due to M.Mineev-Weinstein, P.B.Wiegmann and A.Zabrodin [51, 60] of a remarkable connection between the dispersionless 2D Toda hierarchy and the theory of conformal maps.

To perform the dispersionless limit we have to replace difference operators with their symbols. Given a difference operator $A := \sum a_k(n)\Lambda^k$ its symbol is the formal function in the z variable $\mathcal{A} := \sum a_k(x)z^k$. The coefficients $a_k(x)$ are functions in the continuous variable x obtained by interpolating $a_k(n)$: first we define $a_k(\epsilon; n)$ to be a smooth function such that $a_k(\epsilon; \epsilon n) := a_k(n)$, and then we take the limit $\epsilon \to 0$. Also the shift operator has to be replaced with the ϵ-shift operator $e^{\epsilon\partial_x}$. One can easily prove then that the limit of the Lie bracket gives:

$$\frac{1}{\epsilon}[A, B] \longrightarrow \{\mathcal{A}, \mathcal{B}\} \tag{2.21}$$

where $\mathcal{A}, \mathcal{B}$ are the symbols of A and B, while the curly brackets are the canonical Poisson brackets in the variables $\log(z)$ and x

$$\{\mathcal{A}, \mathcal{B}\} := z\partial_z\mathcal{A}\partial_x\mathcal{B} - z\partial_z\mathcal{B}\partial_x\mathcal{A} \tag{2.22}$$

Definition 2.5 *Let $\lambda(z)$ and $\bar{\lambda}(z)$ be two formal power series in the variables z^{-1} and z of type:*

$$\begin{aligned} \lambda &= z + u_0 + u_{-1}z^{-1} + u_{-2}z^{-2} + \dots && (2.23) \\ & && (2.24) \\ \bar{\lambda} &= \bar{u}_{-1}z^{-1} + \bar{u}_0 + \bar{u}_1 z + \bar{u}_2 z^2 + \dots . && (2.25) \end{aligned}$$

The dispersionless 2D Toda hierarchy is the set of evolutionary equations in the times $\{s_k, \bar{s}_k\}_{k\geqslant 1}$ for the functions $u_k(x), \bar{u}_l(x)$

$$\begin{aligned} \frac{\partial \lambda}{\partial s_k} &= \{\mathcal{A}_k, \lambda\}, \quad \frac{\partial \bar{\lambda}}{\partial s_k} = \{\mathcal{A}_k, \bar{\lambda}\} \\ \frac{\partial \lambda}{\partial \bar{s}_k} &= \{\mathcal{B}_k, \lambda\}, \quad \frac{\partial \bar{\lambda}}{\partial \bar{s}_k} = \{\mathcal{B}_k, \bar{\lambda}\} \end{aligned} \tag{2.26}$$

where:

$$\mathcal{A}_k := (\lambda^k)_{\geqslant 0} \qquad \mathcal{B}_k := (\bar{\lambda}^k)_{<0} \qquad \forall\ k \geqslant 1$$

[1] Given a function $\varphi : \mathbb{Z} \longrightarrow \mathbb{C}$ the discrete Laplace operator of φ is defined as $(\Delta\varphi)(n) := \varphi(n+1) - \varphi(n) + \varphi(n-1) - \varphi(n)$ for $n \in \mathbb{Z}$.

2.2 R-matrix theory and bihamiltonian structure

Let us introduce the R-matrix theory that we will use to define a new bihamiltonian structure for the dispersionless $2D$ Toda hierarchy. A detailed exposition of the theory we recall here can be found in G. Carlet's Ph.D. thesis [9]. We should stress that, although our construction is very similar to the one presented by G. Carlet, the bihamiltonian structure we get differs from the one given in [9] (some signs are changed in the formulae for second Poisson bracket). We were able to obtain this new bihamiltonian structure by changing the algebra structure on the space of Lax operators. The reason why we needed to do this modification is that this new bihamiltonian structure is naturally selected by the Frobenius manifold we will define in the next chapter. In particular, the dispersionless limit of the second Poisson structure is determined by the potential and the Euler vector field of the Frobenius manifold.

Let $\mathcal{G}$ be a Lie algebra. An endomorphism $R : \mathcal{G} \longrightarrow \mathcal{G}$ is called an R-matrix if the bracket

$$[X, Y]_R := [R(X), Y] + [X, R(Y)] \tag{2.27}$$

is a Lie bracket. A sufficient condition for R to be an R-matrix is that R satisfies the *modified Yang-Baxter* equation:

$$[R(X), R(Y)] - R([X, Y]_R) = -[X, Y] \tag{2.28}$$

Let $\mathcal{G}_1, \mathcal{G}_2$ be two subalgebras of $\mathcal{G}$ such that $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2$, then the difference $R := \pi_1 - \pi_2$ of the projectors on the subalgebras is an R-matrix for $\mathcal{G}$.

Let $\mathcal{A}$ be an associative algebra with a symmetric non degenerate trace-form $\mathrm{Tr} : \mathcal{A} \longrightarrow \mathbb{C}$. We can identify $\mathcal{A}$ with $\mathcal{A}^*$ using the invariant inner product:

$$(X, Y) := \mathrm{Tr}(XY) \qquad \forall X, Y \in \mathcal{A} \tag{2.29}$$

Let R be an endomorphism of $\mathcal{A}$. We can define on $C^\infty(A)$ the brackets:

$$\begin{aligned} \{f_1, f_2\}_1 &:= ([L, df_1], R(df_2)) - ([L, df_2], R(df_1)) && (2.30) \\ \{f_1, f_2\}_2 &:= ([L, df_1], R(Ldf_2 + df_2L)) - ([L, df_2], R(Ldf_1 + df_1L)) && (2.31) \end{aligned}$$

Theorem 2.6 *Let $R : \mathcal{A} \longrightarrow \mathcal{A}$ and $A := \frac{1}{2}(R - R^*)$ (the skew symmetric part of R) satisfy the modified Yang–Baxter equation, then the two brackets $\{\ ,\ \}_1$, $\{\ ,\ \}_2$ defined in 2.30 are two compatible Poisson brackets.*

The Poisson tensors corresponding to the brackets 2.30 are defined by:

$$\{f_1, f_2\}_i (X) =: (df_1, P_i(X) df_2) \tag{2.32}$$

and are explicitly given by:

$$\begin{aligned} P_1(X)df &= -[L, R(df)] - R^*([L, df]) && (2.33) \\ P_2(X)df &= -[L, R(Ldf + dfL)] - LR^*([L, df]) - R^*([L, df])L && (2.34) \end{aligned}$$

Going back to our specific case, we introduce the following subalgebras of $\mathcal{D}$:

$$\mathcal{D}_1 \quad := \quad \{(X, -X) \in \mathcal{D} \mid X \in \mathcal{A} \cap \mathcal{B}\} \tag{2.35}$$

$$\mathcal{D}_2 \quad := \quad \{(X, Y) \in \mathcal{D} \mid X \in \mathcal{A}_{<0} \quad Y \in \mathcal{B}_{\geqslant 0}\} \tag{2.36}$$

and the projectors:

$$\begin{array}{rccc} \pi_1 : & \mathcal{D} & \longrightarrow & \mathcal{D}_1 \\ & (X, \bar{X}) & \longmapsto & (X_{\geqslant 0} - \bar{X}_{<0}, -X_{\geqslant 0} + \bar{X}_{<0}) \end{array} \tag{2.37}$$

$$\begin{array}{rccc} \pi_2 : & \mathcal{D} & \longrightarrow & \mathcal{D}_2 \\ & (X, \bar{X}) & \longmapsto & (X_{<0} + \bar{X}_{<0}, X_{\geqslant 0} + \bar{X}_{\geqslant 0}) \end{array} \tag{2.38}$$

Clearly $\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2$, hence the difference of the projectors 2.37 gives the following R-matrix:

$$R(X, \bar{X}) := (X_{\geqslant 0} - \bar{X}_{<0} - 2\bar{X}_{<0}, \bar{X}_{<0} - \bar{X}_{\geqslant 0} - 2X_{\geqslant 0}) \tag{2.39}$$

By construction, R satisfies the modified Yang–Baxter equation. One can prove that also the skew symmetric part of R satisfies the Yang–Baxter equation, so we have two Poisson brackets. At this point, one has to do a Dirac reduction of the brackets on the space of Lax operators (i.e. on the affine subspace of elements $(L, \bar{L})$ described in 2.9) and then compute the dispersionless limit of this two brackets. What we get are two compatible Poisson brackets of Dubrovin-Novikov type with Poisson tensors given by:

$$\begin{aligned} P_1(\hat{\omega}) = \quad & \big(\{\lambda, (z\omega - z\bar{\omega})_-\} - (\{\lambda, z\omega\} + \{\bar{\lambda}, z\bar{\omega}\})_{\leqslant 0}, \\ & -\{\bar{\lambda}, (z\omega - z\bar{\omega})_+\} - (\{\lambda, z\omega\} + \{\bar{\lambda}, z\bar{\omega}\})_{>0}\big) \end{aligned} \tag{2.40}$$

$$\begin{aligned} P_2(\hat{\omega}) = \quad & \big(\{\lambda, (z\lambda\omega + z\bar{\lambda}\bar{\omega})_-\} - \lambda(\{\lambda, z\omega\} + \{\bar{\lambda}, z\bar{\omega}\})_{\leqslant 0} + z\lambda_z\varphi_x, \\ & -\{\bar{\lambda}, (z\bar{\lambda}\bar{\omega} + z\lambda\omega)_+\} + \bar{\lambda}(\{\lambda, z\omega\} + \{\bar{\lambda}, z\bar{\omega}\})_{>0} + z\bar{\lambda}_z\varphi_x\big). \end{aligned} \tag{2.41}$$

The function φ is given by

$$\varphi = \frac{1}{2\pi i} \oint_{|z|=1} \big(z\lambda_z\omega + z\bar{\lambda}_z\bar{\omega}\big) dz, \tag{2.42}$$

but only its x-derivative actually enters the formulas above

$$\varphi_x = \frac{1}{2\pi i} \oint_{|z|=1} \big(\{\lambda, z\omega\} + \{\bar{\lambda}, z\bar{\omega}\}\big) \frac{dz}{z}. \tag{2.43}$$

In the next chapter, we will introduce a Frobenius manifold structure on the space of Lax symbols of $2D$ Toda (see Theorem 3.90). In the previous chapter, we have recalled how, given a Frobenius manifold, one can construct a (dispersionless) bihamiltonian integrable hierarchy on its loop space. In the last chapter of this thesis we will prove that the bihamiltonian strucutre given in 2.40, 2.41 coincides with the one induced by our Frobenius manifold structure (see Proposition 4.1).

Chapter 3

2D Toda Frobenius manifold

In this chapter we state and prove the main result of this thesis: the construction of M_{2DT}, the first example of infinite dimensional Frobenius manifold. We begin by defining an appropriate functional space M_0, this space could be interpreted as the space of holomorphic Lax symbols of the $2D$ Toda hierarchy. Basically, the idea is to substitute the formal power series $\lambda, \bar{\lambda}$ introduced in [55] with a pair of holomorphic functions with simple poles at ∞ and 0 defined in a neighborhood of $\mathbb{S}^1$. Dealing with functions rather than formal power series allow us to combine together these two objects to define new functions (like for example $(\lambda+\bar{\lambda})^n$) which are needed in the definition of the Frobenius manifold. Once a suitable space is defined, we give the Frobenius algebra structure on the tangent space, and finally, we prove that the 3-point correlator function components can be integrated to a potential. We conclude the chapter by proving semi semplicity of the Frobenius manifold and discussing some interesting reductions.

3.1 The manifolds M and M_0

Let S^1 be the unit circle $|z| = 1$ on the complex z-plane. Denote:

$$D_0 := \left\{z \in \mathbb{CP}^1 \mid |z| \leqslant 1\right\} \tag{3.1}$$

$$D_\infty := \left\{z \in \mathbb{CP}^1 \mid |z| \geqslant 1\right\} \tag{3.2}$$

the inner and outer parts of S^1 on the Riemann sphere.

We recall that a function on the punctured disk $D_0 \setminus \{0\}$ is holomorphic if it is the restriction of a holomorphic function defined on $0 < |z| < 1+\rho$ for some positive ρ; and similarly a function is holomorphic on $D_\infty \setminus \{\infty\}$ if it is the restriction of a holomorphic function on $|z| > 1-\rho$, for some positive ρ.

Definition 3.1 *We introduce the following spaces of meromorphic functions:*

$$\begin{aligned}
\mathcal{H}(D_0) &:= \{\textit{holomorphic functions on } D_0 \setminus \{0\} \textit{ with a simple pole} \\
&\qquad \textit{at } 0, \textit{ and non vanishing residue}\} &(3.3)\\
\mathcal{H}(D_\infty) &:= \{\textit{holomorphic functions on } D_\infty \setminus \{\infty\} \textit{ with a simple pole} \\
&\qquad \textit{at } \infty, \textit{ and residue equal to } 1\} &(3.4)\\
\dot{\mathcal{H}}(D_0) &:= \{\textit{holomorphic functions on } D_0 \setminus \{0\} \textit{ with a simple pole} \\
&\qquad \textit{at } 0, \textit{ or regular at } 0\} &(3.5)\\
\dot{\mathcal{H}}(D_\infty) &:= \{\textit{holomorphic functions on } D_\infty\} &(3.6)
\end{aligned}$$

Let $f(z)$ be a holomorphic function defined in a neighborhood of $|z| = 1$. Clearly we can compute the Fourier expansion:

$$f = f(z) = \sum_{n\in\mathbb{Z}} f_n z^n$$

We use the notations $(f)_{\geqslant k}$ and $(f)_{\leqslant k}$ to denote the projections:

$$(f)_{\geqslant k} = \sum_{n\geqslant k} f_n z^n = \frac{z^k}{2\pi i}\oint_{|z|<|\zeta|} \frac{\zeta^{-k} f(\zeta)}{\zeta - z} d\zeta \tag{3.7}$$

$$(f)_{\leqslant k} = \sum_{n\leqslant k} f_n z^n = -\frac{z^{k+1}}{2\pi i}\oint_{|z|>|\zeta|} \frac{\zeta^{-k-1} f(\zeta)}{\zeta - z} d\zeta \tag{3.8}$$

for a given integer k. It is worth to observe the following identity that will be used in subsequent calculations:

Lemma 3.2 *Given two holomorphic functions $f(z)$, $g(z)$ defined on a neighborhood of $|z| = 1$, and an integer $k \in \mathbb{Z}$, we have that:*

$$\oint f(z)\ (g(z))_{\geqslant k}\ dz = \oint (f(z))_{\leqslant -(k+1)}\, g(z)\ dz \tag{3.9}$$

Proof The identity follows from the fact that:

$$\frac{1}{2\pi i}\oint f(z)g(z)\ dz = \sum_{i,j\in\mathbb{Z}\ i+j=-1} f_i\ g_j$$

where $f(z) = \sum_{i\in\mathbb{Z}} f_i z^i$, $g(z) = \sum_{j\in\mathbb{Z}} g_j z^j$. The proof of 3.9 is now straightforward:

$$\begin{aligned}
&\frac{1}{2\pi i}\oint f(z)(g(z))_{\geqslant k}\ dz = \sum_{i+j=-1,\ j\geqslant k} f_i\ g_j = \\
&= \sum_{i+j=-1,\ i\leqslant -(k+1)} f_i\ g_j = \frac{1}{2\pi i}\oint (f(z))_{\leqslant -(k+1)} g(z)\ dz
\end{aligned} \tag{3.10}$$

□

Note that a function $\lambda \in \mathcal{H}(D_\infty)$ is determined by its power series expansion at ∞:

$$\lambda(z) = z + u_0 + \frac{u_{-1}}{z} + \dots \quad \text{at } \infty; \tag{3.11}$$

in an analogous way, a function $\bar{\lambda} \in \mathcal{H}(D_0)$ is determined by its power series expansion at 0:

$$\bar{\lambda}(z) = \frac{\bar{u}_{-1}}{z} + \bar{u}_0 + \bar{u}_1 z + \dots \quad \text{at } 0. \tag{3.12}$$

Definition 3.3 **($u, \bar{u}$-coordinates)** *Let u_k, for $k \leqslant 0$, be the linear functional:*

$$\begin{array}{rccc} u_k: & \mathcal{H}(D_\infty) & \longrightarrow & \mathbb{C} \\ & \lambda & \longmapsto & \frac{1}{2\pi i}\oint \lambda(z) z^{-(k+1)}\, dz \end{array} \tag{3.13}$$

and $\bar{u}_l$, for $l \geqslant -1$, be the linear functional:

$$\begin{array}{rccc} \bar{u}_l: & \mathcal{H}(D_0) & \longrightarrow & \mathbb{C} \\ & \bar{\lambda} & \longmapsto & \frac{1}{2\pi i}\oint \bar{\lambda}(z) z^{-(l+1)}\, dz \end{array} \tag{3.14}$$

Definition 3.4 *We define the infinite dimensional manifold M to be the space:*

$$M := \mathcal{H}(D_\infty) \oplus \mathcal{H}(D_0). \tag{3.15}$$

endowed with the set of (global) coordinates $\mathcal{U} := \left\{u^k, \bar{u}^l\right\}$ given in Definition 3.3.

Remark 3.5 *In this thesis bar never stands for complex conjugation unless the opposite is explicitly stated, hence for example $\bar{u}_0$ is not the conjugate of u_0. The choice of this notation is in agreement with the literature on the 2D Toda Hierarchy, where $\lambda(z)$ and $\bar{\lambda}(z)$ are usually used to denote the two Lax symbols of the hierarchy.*

Let $\hat{\lambda} = (\lambda, \bar{\lambda}) \in M$. The tangent space $\mathrm{T}_{\hat{\lambda}}M$ is isomorphic to the direct sum:

$$\mathrm{T}_{\hat{\lambda}}M = \dot{\mathcal{H}}(D_\infty) \oplus \dot{\mathcal{H}}(D_0). \tag{3.16}$$

identifying first order linear differential operators with the derivatives of the functions $\lambda(z)$, $\bar{\lambda}(z)$:

$$\partial \mapsto (\partial\lambda(z), \partial\bar{\lambda}(z)) \in \dot{\mathcal{H}}(D_\infty) \oplus \dot{\mathcal{H}}(D_0) \tag{3.17}$$

(the differentiation with $z = \text{const}$). Similarly, the cotangent space is identified with

$$\mathrm{T}^*_{\hat{\lambda}}M = \dot{\mathcal{H}}(D_0) \oplus \dot{\mathcal{H}}(D_\infty). \tag{3.18}$$

The duality between the tangent and cotangent spaces is established by the residue pairing:

$$\langle \hat{\omega}, \hat{\alpha} \rangle = \frac{1}{2\pi i}\oint_{|z|=1} \left[\alpha(z)\omega(z) + \bar{\alpha}(z)\bar{\omega}(z)\right] dz, \tag{3.19}$$

where $\hat{\alpha} = (\alpha, \bar{\alpha}) \in \mathrm{T}_{\hat{\lambda}}M$, $\hat{\omega} = (\omega, \bar{\omega}) \in \mathrm{T}^*_{\hat{\lambda}}M$.

Definition 3.6 *There is a natural basis on* $\mathrm{T}_{\hat{\lambda}}M$ *which is given by*

$$\frac{\partial}{\partial u^k} = (z^k, 0) \qquad \frac{\partial}{\partial \bar{u}^l} = (0, z^l), \tag{3.20}$$

where $k \leqslant 0$, $l \geqslant -1$. *The pairing 3.19 induces a dual basis on* $\mathrm{T}^*_{\hat{\lambda}}M$

$$du^k = (z^{-(k+1)}, 0) \qquad d\bar{u}^l = (0, z^{-(l+1)}). \tag{3.21}$$

We introduce another set of 1-forms. This is not a basis, but a set of generators for the cotangent space that will be useful in computations.

Definition 3.7 *Let* $\mathcal{L} := \left\{d\lambda(p), d\bar{\lambda}(p)\right\}_{p\in\mathbb{S}^1}$, *where:*

$$\langle d\lambda(p), \hat{\alpha}\rangle = \alpha(p), \quad \langle d\bar{\lambda}(p), \hat{\alpha}\rangle = \bar{\alpha}(p), \quad \hat{\alpha} = (\alpha, \bar{\alpha}) \in \mathrm{T}_{\hat{\lambda}}M. \tag{3.22}$$

Lemma 3.8 *Any covector* $\hat{\omega} = (\omega(z), \bar{\omega}(z))$ *can be represented as a linear combination of elements in* $\mathcal{L}$*:*

$$\hat{\omega} = \frac{1}{2\pi i}\oint_{|p|=1} \left(\omega(p)d\lambda(p) + \bar{\omega}(p)d\bar{\lambda}(p)\right),$$

Proof This is a direct application of Cauchy integral formula, and of the realization of the covectors in $\mathcal{L}$ as elements of the space $\dot{\mathcal{H}}(D_0) \oplus \dot{\mathcal{H}}(D_\infty)$

$$d\lambda(p) = \left(\frac{p}{z}\frac{1}{p-z}, 0\right), \quad d\bar{\lambda}(p) = \left(0, \frac{z}{p}\frac{1}{z-p}\right). \tag{3.23}$$

When looking at these formulae, the reader should keep in mind that $\frac{p}{z}\frac{1}{p-z}$, as function in the z variable, is defined on D_0, while $\frac{z}{p}\frac{1}{z-p}$ is a function over D_∞. In the first case, the power series expansion in z is determined by the inequivalence $|p| > |z|$, i.e.:

$$\frac{p}{z}\frac{1}{p-z} = \sum_{n\leqslant 0} p^n z^{-(n+1)} \tag{3.24}$$

while in the second one we have $|z| > |p|$, hence:

$$\frac{z}{p}\frac{1}{z-p} = \sum_{n\geqslant -1} p^n z^{-(n+1)} \tag{3.25}$$

With these power series expansion formulae 3.23 can be easily proved, for instance:

$$\begin{aligned} \langle d\lambda(p), \hat{\alpha}\rangle &= \frac{1}{2\pi i}\oint_{|z|=1} \alpha(z)\frac{p}{z}\frac{1}{p-z}\, dz = \\ &= \sum_{n\leqslant 0} p^n \frac{1}{2\pi i}\oint_{|z|=1} \alpha(z) z^{-(n+1)}\, dz = \alpha(p) \end{aligned} \tag{3.26}$$

□

We end this section by introducing the manifold $M_0 \subseteq M$. In the next session we will define an algebra structure over T^*M and a contravariant metric η which is invariant with respect to the product. In order to have a Frobenius algebra structure we need η to be non degenerate, and we will prove that this happens when we restrict to M_0.

Definition 3.9 *Given a point $(\lambda, \bar{\lambda}) \in M$, let $w(z) : \mathbb{S}^1 \longrightarrow \mathbb{C}$ be defined by $w(z) := \lambda(z) + \bar{\lambda}(z)$. We set M_0 to be:*

$$M_0 := \left\{ (\lambda, \bar{\lambda}) \in M \ \mid w(z) \text{ is injective, } \mathrm{wind}_0(w(z)) = 1 \right\} \tag{3.27}$$

where $\mathrm{wind}_0(w(z))$ is the winding number of w(z) around 0.

Note that, according to Definition 3.3, both λ and $\bar{\lambda}$ are defined on an annular neighborhood of $1 - \rho < |z| < 1 + \rho$ of the unit circle S^1, hence $w(z)$ is a well defined holomorphic map. This map describes a non-self intersecting positively oriented closed curve encircling the origin $w = 0$ that we denote $\Gamma := \mathrm{Im}(w)$. We recall that a holomorphic map f defined on a domain D is said to be *univalent* if it is injective over D. One can prove that the derivative of a univalent map is never 0, the map is always invertible, and the inverse map is also holomorphic [29].

3.2 Frobenius algebra structure on $\mathrm{T}M_0$

We begin by introducing a Frobenius algebra[1] structure on the cotangent space T^*M_0. Our ultimate goal is to dualize this construction to the tangent space to define the Frobenius algebra structure of the Frobenius manifold M_{2DT}. The reader might wonder why we define this structure starting from cotangent space. The reason is that the product admits a very simple formulation on the cotangent space in terms of the generators $d\lambda(p)$, $d\bar{\lambda}(p)$. We will use this formulation to prove associativity and invariance of the metric η. In order to do certain computations, it's worth to introduce another representation of these geometric structures in terms of generating functions. For instance, this formulation is useful to compute the 3-point correlator function. Finally, we will prove flatness of the metric by giving a set of local flat coordinates. These coordinates will be the one we will use in the next session to describe the full Frobenius Manifold structure.

Suppose we have defined a (1,2)-tensor on M. The components will be functions on M labeled by three indices which can take value in two different *infinite* sets:

$$\begin{array}{cccc} c_k^{i,j}, & c_k^{\bar{i},j}, & c_k^{i,\bar{j}}, & c_k^{\bar{i},\bar{j}}, \\ c_{\bar{k}}^{i,j}, & c_{\bar{k}}^{\bar{i},j}, & c_{\bar{k}}^{i,\bar{j}}, & c_{\bar{k}}^{\bar{i},\bar{j}}. \end{array} \tag{3.28}$$

[1]Note that here we are relaxing the standard definition of Frobenius algebra, since obviously can't require anymore the algebra to be finite dimensional.

The corresponding generating functions will be a *finite* set of functions in three variables, one for each index, obtained by the following formulae:

$$c_z^{x,y} := \sum_{i,j,k\leqslant 0} c_k^{i,j} x^i y^j z^{-(k+1)}, \qquad c_z^{\bar{x},y} := \sum_{j,k\leqslant 0, i\geqslant -1} c_k^{i,j} x^i y^j z^{-(k+1)}, \tag{3.29}$$

and so on. The summation of the index is done over the appropriate set, while we use the convention to multiply the $i-th$ component to x^i for higher indices, and to $x^{-(i+1)}$ for lower indices. In this way an infinite summation over an index can be replaced by a simple contour integral:

$$\sum_{j,k\leqslant 0} \left(\sum c_k^{i,j} d_i + \sum c_k^{\bar{i},j} d_{\bar{i}}\right) y^j z^k = \frac{1}{2\pi i}\oint c_z^{x,y} d_x dx + \frac{1}{2\pi i}\oint c_z^{\bar{x},y} d_{\bar{x}} d\bar{x} \tag{3.30}$$

Definition 3.10 *Let $H(j) : \mathbb{Z} \longrightarrow \{0,1\}$ be the discrete Heaviside function, i.e. $H(j) = 1$ for $j \geqslant 0$ and $H(j) = 0$ for $j < 0$. We define a symmetric bilinear form η and a multiplication on the cotangent space $\mathrm{T}^*_{\hat{\lambda}} M$ by:*

$$\begin{aligned}
\eta^{i,j} &:= -(i+j)u_{i+j} && (3.31)\\
\eta^{i,\bar{j}} &:= (H(-j)-1)(i+j)u_{i+j} - (i+j)\bar{u}_{i+j} && (3.32)\\
\eta^{\bar{i},j} &:= (H(-i)-1)(i+j)u_{i+j} - (i+j)\bar{u}_{i+j} && (3.33)\\
\eta^{\bar{i},\bar{j}} &:= (H(-i)+H(-j)-1)(i+j)\bar{u}_{i+j} && (3.34)
\end{aligned}$$

$$\begin{aligned}
c_k^{i,j} &:= (1-H(k-i)-H(k-j))\,H(1+k-i-j)(i+j-k)u_{i+j-k} && (3.35)\\
c_k^{i,\bar{j}} &:= -H(k-i)H(i+j-k+1)(i+j-k)\bar{u}_{i+j-k} && (3.36)\\
c_{\bar{k}}^{\bar{i},j} &:= H(k+1-i-j)H(j-k-1)(i+j-k)u_{i+j-k} && (3.37)\\
c_{\bar{k}}^{\bar{i},\bar{j}} &:= (1-H(k-i)-H(k-j))\,H(i+j-k+1)(i+j-k)\bar{u}_{i+j-k} && (3.38)\\
c_k^{\bar{i},\bar{j}} &:= c_{\bar{k}}^{i,j} = 0 && (3.39)
\end{aligned}$$

One can easily compute the corresponding generating functions:

$$\begin{aligned}
\eta^{p,q} &= \frac{pq}{p-q}(\lambda'(p)-\lambda'(q)) && (3.40)\\
\eta^{p,\bar{q}} &= \frac{pq}{p-q}(\lambda'(p)+\bar{\lambda}'(q)) && (3.41)\\
\eta^{\bar{p},q} &= -\frac{pq}{p-q}(\bar{\lambda}'(p)+\lambda'(q)) && (3.42)\\
\eta^{\bar{p},\bar{q}} &= -\frac{pq}{p-q}(\bar{\lambda}'(q)-\bar{\lambda}'(p)) && (3.43)
\end{aligned}$$

$$c_z^{p,q} = \frac{pq}{p-q}\frac{1}{z}\left(\frac{\lambda'(p)q}{q-z} - \frac{\lambda'(q)p}{p-z}\right) \tag{3.44}$$

$$c_z^{p,\bar{q}} = \frac{pq}{p-q}\frac{1}{z}\left(-\frac{\lambda'(q)p}{p-z}\right) \tag{3.45}$$

$$c_{\bar{z}}^{\bar{p},q} = \frac{pq}{p-q}z\left(\frac{\lambda'(p)q}{q-z} - \frac{\lambda'(q)p}{p-z}\right) \tag{3.46}$$

$$c_{\bar{z}}^{\bar{p},\bar{q}} = \frac{pq}{p-q}z\left(\frac{\bar{\lambda}'(p)}{q(z-q)} - \frac{\bar{\lambda}'(q)}{p(z-p)}\right) \tag{3.47}$$

For instance, to compute $\eta^{p,q}$ we proceed as follows:

$$\begin{aligned}\eta^{p,q} &= -\sum_{i,j\leqslant 0}(i+j)u_{i+j}p^i q^j = -\sum_{k\leqslant 0} k u_k \sum_{i,j\leqslant 0; i+j=k} p^i q^j = \\ &= -\sum_{k\leqslant 0} k u_k \, pq\frac{q^{k-1}-p^{k-1}}{p-q} = \frac{pq}{p-q}(\lambda'(p)-\lambda'(q))\end{aligned} \tag{3.48}$$

all other formulae are obtained in a very similar way.

Finally, if we compute the bilinear form η and the multiplication for two generic covectors in $\mathcal{L}$ we get:

$$< d\alpha(p), d\beta(q) >_* = \frac{p\,q}{p-q}\left(\epsilon(\alpha)\,\beta'(q) - \epsilon(\beta)\,\alpha'(p)\right) \tag{3.49}$$

and

$$d\alpha(p)\cdot d\beta(q) = \frac{p\,q}{p-q}\left[\alpha'(p)\,d\beta(q) - \beta'(q)\,d\alpha(p)\right]. \tag{3.50}$$

Here $d\alpha(p)$, $d\beta(q)$ stand for one of the symbols $d\lambda(p)$ or $d\bar{\lambda}(p)$, the signs $\epsilon(\alpha)$, $\epsilon(\beta)$ are defined as follows

$$\epsilon(\alpha) = 1 \quad \text{if} \quad \alpha = \lambda \quad \text{and} \quad \epsilon(\alpha) = -1 \quad \text{if} \quad \alpha = \bar{\lambda}.$$

Introduce a linear map

$$\eta : T^*_{(\lambda,\bar{\lambda})}M \to T_{(\lambda,\bar{\lambda})}M$$

by the formula

$$\begin{aligned}&\eta(\hat{\omega}) = z^2\left((\lambda'\omega + \bar{\lambda}'\bar{\omega})_{\leqslant -2} - \lambda'(\omega-\bar{\omega})_{\leqslant -2}, (\lambda'\omega + \bar{\lambda}'\bar{\omega})_{\geqslant -1} + \bar{\lambda}'(\omega-\bar{\omega})_{\geqslant -1}\right) \\ &\hat{\omega} = (\omega,\bar{\omega}) \in \mathrm{T}^*_{\hat{\lambda}}M\end{aligned} \tag{3.51}$$

The associated bilinear form on $\mathrm{T}^*_{\hat{\lambda}}M$

$$< \hat{\omega}_1, \hat{\omega}_2 >_* := \langle(\omega_1,\bar{\omega}_1), \eta(\omega_2,\bar{\omega}_2)\rangle, \quad \hat{\omega}_1 = (\omega_1,\bar{\omega}_1), \quad \hat{\omega}_2 = (\omega_2,\bar{\omega}_2) \in T^*_{(\lambda,\bar{\lambda})}M \tag{3.52}$$

coincides with (3.49).

Let $\hat{\alpha}, \hat{\beta} \in \mathrm{T}^*_{\hat{\lambda}} M$, then $\hat{\gamma} := \hat{\alpha} \cdot \hat{\beta}$ is given by:

$$\gamma = z^2 \left(\alpha(\lambda' \beta + \bar{\lambda}' \bar{\beta})_{\geqslant -1} + \beta(\lambda' \alpha + \bar{\lambda}' \bar{\alpha})_{\geqslant -1} - [\lambda' \alpha \beta + \bar{\lambda}'(\alpha \bar{\beta} + \bar{\alpha} \beta)]_{\geqslant -3} \right)$$

$$\begin{aligned} \hat{\gamma} &= z^2 \left(-\bar{\alpha}(\lambda' \beta + \bar{\lambda}' \bar{\beta})_{\leqslant -2} - \bar{\beta}(\lambda' \alpha + \bar{\lambda}' \bar{\alpha})_{\leqslant -2} + \right. \\ &\quad \left. + [\bar{\lambda}' \bar{\alpha} \bar{\beta} + \lambda'(\omega_1 \bar{\beta} + \bar{\alpha} \beta)]_{\leqslant -2} \right) \end{aligned}$$

Lemma 3.11 *For any $\hat{\lambda} \in M$ the formula* (3.50) *defines on $\mathrm{T}^*_{\hat{\lambda}} M$ a structure of a commutative associative algebra with unity $e^* := (0, \frac{1}{\bar{u}_{-1}})$.*

Proof Commutativity of the product (3.50) is obvious. In order to prove associativity let us compute the product of three 1-forms of the form (3.22). An easy computation shows that this product can be written in the following manifestly symmetric way

$$\begin{aligned} &[d\alpha(p) \cdot d\beta(q)] \cdot d\gamma(r) = \\ &= \frac{p\, r}{p - r} \frac{p\, q}{p - q} \beta'(q) \gamma'(r)\, d\alpha(p) + \frac{q\, p}{q - p} \frac{q\, r}{q - r} \alpha'(p) \gamma'(r)\, d\beta(q) \\ &+ \frac{r\, p}{r - p} \frac{r\, q}{r - q} \alpha'(p) \beta'(q)\, d\gamma(r). \end{aligned} \tag{3.53}$$

□

Remark 3.12 *The following formula generalizing* (3.50) *can be easily derived by induction:*

$$d\alpha_1(p_1) \cdot \dots \cdot d\alpha_n(p_n) = \sum_{i=1}^{n} \frac{\alpha_1'(p_1)}{p_i^{-1} - p_1^{-1}} \frac{\alpha_2'(p_2)}{p_i^{-1} - p_2^{-1}} \dots d\alpha_i(p_i) \dots \frac{\alpha_n'(p_n)}{p_i^{-1} - p_n^{-1}}. \tag{3.54}$$

Lemma 3.13 *The bilinear form* (3.49) *is invariant with respect to the multiplication* (3.50)*:*

$$< \hat{\omega}_1 \cdot \hat{\omega}_2, \hat{\omega}_3 >_* = < \hat{\omega}_1, \hat{\omega}_2 \cdot \hat{\omega}_3 >_* . \tag{3.55}$$

for any $\hat{\omega}_1$, $\hat{\omega}_2$, $\hat{\omega}_3 \in \mathrm{T}^*_{\hat{\lambda}} M$.

Proof As in the proof of Lemma 3.11 let us compute $< \hat{\omega}_1 \cdot \hat{\omega}_2, \hat{\omega}_3 >_*$ choosing the three 1-forms $\hat{\omega}_1$, $\hat{\omega}_2$, $\hat{\omega}_3$ among $d\lambda(p)$ and $d\bar{\lambda}(p)$. Using the formula (3.49) one easily obtains the following symmetric expression

$$\begin{aligned} &< d\alpha(p) \cdot d\beta(q), d\gamma(r) >_* = \\ &= -p\, q\, r \left[\frac{\epsilon(\alpha) \beta'(q) \gamma'(r) p}{(p - q)(p - r)} + \frac{\epsilon(\beta) \alpha'(p) \gamma'(r) q}{(q - p)(q - r)} + \frac{\epsilon(\gamma) \alpha'(p) \beta'(q) r}{(r - p)(r - q)} \right]. \end{aligned}$$

□

As a by-product of this proof, we get the formula of the 3-point correlator function on the generators $d\lambda(p)$ and $d\bar{\lambda}(p)$. We recall that this is the symmetric (0,3) tensor defined by $c(\alpha, \beta, \gamma) =< \alpha, \beta \cdot \gamma >_*$.

Proposition 3.14 *The 3-point correlator function generating functions are:*

$$\begin{aligned}
c^{x,y,z} &= -\frac{x^2 y\lambda'(y) z\lambda'(z)}{(x-y)(x-z)} - \frac{y^2 x\lambda'(x) z\lambda'(z)}{(y-x)(y-z)} - \frac{z^2 x\lambda'(x) y\lambda'(y)}{(z-x)(z-y)} && (3.56)\\
c^{\bar{x},y,z} &= +\frac{x^2 y\lambda'(y) z\lambda'(z)}{(x-y)(x-z)} - \frac{y^2 x\bar{\lambda}'(x) z\lambda'(z)}{(y-x)(y-z)} - \frac{z^2 x\bar{\lambda}'(x) y\lambda'(y)}{(z-x)(z-y)} && (3.57)\\
c^{\bar{x},\bar{y},z} &= -\frac{x^2 y\bar{\lambda}'(y) z\lambda'(z)}{(x-y)(x-z)} + \frac{y^2 x\bar{\lambda}'(x) z\lambda'(z)}{(y-x)(y-z)} - \frac{z^2 x\bar{\lambda}'(x) y\bar{\lambda}'(y)}{(z-x)(z-y)} && (3.58)\\
c^{\bar{x},\bar{y},\bar{z}} &= +\frac{x^2 y\bar{\lambda}'(y) z\bar{\lambda}'(z)}{(x-y)(x-z)} + \frac{y^2 x\bar{\lambda}'(x) z\bar{\lambda}'(z)}{(y-x)(y-z)} + \frac{z^2 x\bar{\lambda}'(x) y\bar{\lambda}'(y)}{(z-x)(z-y)} && (3.59)
\end{aligned}$$

all other functions are obtained by variables permutation.

Proof We simply have to rise an index using the generating functions 3.40, 3.44, for instance:

$$c^{\bar{x},y,z} := \frac{1}{2\pi i}\oint \eta^{\bar{x},p} c_p^{y,z} dp + \frac{1}{2\pi i}\oint \eta^{\eta x,\bar{p}} c_{\bar{p}}^{y,z} d\bar{p} \qquad (3.60)$$

□

Remark 3.15 *By looking at the power expansion of the generating functions 3.56 we get the 3-point correlator function components. On the other hand, the direct computation of these components with formulae 3.31, 3.35 is rather tricky.*

Let us now prove non degeneracy of the symmetric bilinear form (3.52) on M_0.

Lemma 3.16 *For any* $(\lambda, \bar{\lambda}) \in M_0$ *the linear operator* (3.51) *is an isomorphism.*

Proof Resolving the equation

$$\eta(\hat{\omega}) = \hat{\alpha}, \quad \hat{\alpha} = (\alpha, \bar{\alpha}) \in \mathrm{T}_{(\lambda,\bar{\lambda})} M_0$$

one obtains

$$\begin{aligned}
\omega &= \frac{1}{z^2}\left(\frac{\alpha(z)+\bar{\alpha}(z)}{w'(z)}\right)_{\geqslant 1}\\
\bar{\omega} &= \frac{1}{z^2}\left(\frac{\alpha(z)+\bar{\alpha}(z)}{w'(z)}\right)_{\leqslant 2} + \frac{1}{\bar{u}_{-1}}\left(\frac{\bar{\alpha}_{-1}}{z} + \bar{\alpha}_0\right).
\end{aligned} \qquad (3.61)$$

which is a well posed map since $w'(z) \neq 0$ for $z \in S^1$. □

Remark 3.17 *The algebra structure 3.35 together with the metric 3.31 gives a Frobenius algebra on* T^*M_0*, but not on* T^*M *(were both structures are still defined), since* η *must be non degenerate.*

Corollary 3.18 *For $(\lambda, \bar{\lambda}) \in M_0$ the non-degenerate symmetric bilinear form* (3.52) *on* $\mathrm{T}^*_{\hat{\lambda}} M$ *induces a non-degenerate symmetric bilinear form on* $\mathrm{T}_{\hat{\lambda}} M$*. The latter can be written in the following form*

$$< \partial_1, \partial_2 >= \frac{1}{2\pi i} \oint_{|z|=1} \frac{\partial_1 w(z)\, \partial_2 w(z)}{z^2 w'(z)} dz - \mathrm{Res}_{z=0} \frac{\partial_1 l(z)\, \partial_2 l(z)}{z^2 l'(z)} dz \tag{3.62}$$

for any two tangent vectors $\partial_1,\, \partial_2 \in \mathrm{T}_{\hat{\lambda}} M$ *where*

$$l(z) = -z + v + \frac{e^u}{z}. \tag{3.63}$$

We will now prove flatness of η by constructing a system of local flat coordinates.

Definition 3.19 *We introduce the following set of functionals on M_0:*

$$t^k := \frac{1}{2\pi i} \oint \frac{w(z)^{-k}}{k} \frac{dz}{z} \qquad \text{for } k \in \mathbb{Z} \setminus \{0\} \tag{3.64}$$

$$t^0 := \frac{1}{2\pi i} \oint \ln \frac{z}{w(z)} \frac{dz}{z} \tag{3.65}$$

$$v := \bar{u}_0 \tag{3.66}$$

$$u := \ln \bar{u}_{-1} \tag{3.67}$$

Note that the t^k functionals, for $k \neq 0$, could be extended to M, while the definition of t^0 is well posed only over M_0. In particular for a point $(\lambda, \bar{\lambda})$ in M_0 we have that $w(z) := \lambda(z) + \bar{\lambda}(z)$ has winding number 1 (see definition 3.9), hence $\frac{z}{w(z)}$ has winding number 0 and we can find a branch of the logarithm such that $\ln \frac{z}{w(z)}$ is a single valued function.

Lemma 3.20 *The set $(\mathbf{t}, u, v)$ constitute a system of flat coordinates on M_0 for the metric η.*

Proof We first prove that these are indeed a set of local coordinates. Let $(\lambda, \bar{\lambda}) \in M_0$. Consider the inverse function of the map $w(z) := \lambda(z) + \bar{\lambda}(z)$:

$$z = z(w) : \Gamma \to S^1.$$

It is holomorphic on some neighborhood of the curve Γ and satisfies

$$|z(w)|_{w \in \Gamma} = 1.$$

Introduce the Riemann–Hilbert factorization of this function

$$z(w) = f_0^{-1}(w) f_\infty(w) \quad \text{for} \quad w \in \Gamma \tag{3.68}$$

where the functions $f_0(w)$ and $f_\infty(w)/w$ are holomorphic and non-vanishing inside/outside the curve Γ (in both cases holomorphicity can be assumed in a bigger domain containing the curve itself). The factorization will be uniquely defined by normalizing

$$f_\infty(w) = w + O(1), \quad |w| \to \infty.$$

Denote t^n the coefficients of Taylor expansions of the logarithms of these functions

$$\log f_0(w) = -t^0 - t^1 w - t^2 w^2 - \ldots, \quad |w| \to 0$$

$$\log \frac{f_\infty(w)}{w} = \frac{t^{-1}}{w} + \frac{t^{-2}}{w^2} + \ldots, \quad |w| \to \infty. \tag{3.69}$$

Let $t(w) := \sum_{k\in\mathbb{Z}} t^k w^k$. This series converge in a neighborhood of Γ, and $z(w) = w \exp t(w)$. Let $k \neq 0$, by integrating by parts we get:

$$\begin{aligned} t^k &= \tfrac{1}{2\pi i} \oint \ln\left(\tfrac{z(w)}{w}\right) w^{-(k+1)} dw &&= \\ &= -\tfrac{1}{2\pi i} \oint \left(\tfrac{z'(w)}{z(w)} - \tfrac{1}{w}\right) \tfrac{w^{-k}}{k} dw &&= -\frac{1}{2\pi i} \oint \frac{w(z)^{-k}}{k} \frac{dz}{z} \end{aligned}$$

while for $k = 0$:

$$\begin{aligned} t^0 &= \tfrac{1}{2\pi i} \oint \ln\left(\tfrac{z(w)}{w}\right) \tfrac{dw}{w} &&= \\ &= \tfrac{1}{2\pi i} \oint \ln\left(\tfrac{z(w)}{w}\right) \tfrac{z'(w)}{z(w)} dw &&= \frac{1}{2\pi i} \oint \ln\left(\frac{z}{w(z)}\right) \frac{dz}{z} \end{aligned}$$

So, starting from the $\mathbf{t}$ coordinates, we can recover $w(z)$. To get $\lambda(z)$ and $\bar{\lambda}(z)$ we must solve the additive Riemann–Hilbert problem: $w(z) = \lambda(z) + \bar{\lambda}(z)$:

$$\lambda(z) := \frac{1}{2\pi i} \oint \frac{w(x)}{(z-x)_{|z|>|x|}} \frac{z}{x} dx - l(z) \tag{3.70}$$

$$\bar{\lambda}(z) := \frac{1}{2\pi i} \oint \frac{w(x)}{(x-z)_{|x|>|z|}} \frac{z}{x} dx + l(z) \tag{3.71}$$

$$\tag{3.72}$$

where we recall that $l := -z + v + \frac{e^u}{z}$. Hence we have a bijective map $(\lambda, \bar{\lambda}) \leftrightarrow (\mathbf{t}, \mathbf{u}, \mathbf{v})$. It remains to prove that these are indeed flat coordinates. We first observe that:

$$\frac{\partial z(w)}{\partial t_n} = w^n z(w) \tag{3.73}$$

if we derive with respect to $\frac{\partial}{\partial t_n}$ the identity: $w(z(w)) = w$ we get:

$$\frac{\partial w}{\partial t_n}(z(w)) + w'(z(w)) \frac{\partial z}{\partial t_n}(w) = 0 \tag{3.74}$$

hence performing the coordinate change $w \mapsto w(z)$ we get:

$$\frac{\partial w(z)}{\partial t_n} = -z\, w^n(z) w'(z) \tag{3.75}$$

(note that in the formula (3.73) we differentiate keeping $w = \text{const}$ while in (3.75) $z = \text{const}$). By plugging this into the formula (3.62) we obtain:

$$\left\langle \frac{\partial}{\partial t^k}, \frac{\partial}{\partial t^l} \right\rangle = \frac{1}{2\pi i} \oint w^{k+l} w' dz = \frac{1}{2\pi i} \oint w^{k+l} dw = \delta_{k+l,-1}.$$

The remaining part of the Gram matrix is computed in a similar way, and it's a constant matrix. $\square$

Corollary 3.21 *The Gram matrix of the metric* (3.62) *becomes constant in the coordinates* $(\mathbf{t}, u, v)$*, namely*

$$\left\langle \frac{\partial}{\partial t^k}, \frac{\partial}{\partial t^l} \right\rangle = \delta_{k+l,-1}, \quad \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\rangle = 1, \tag{3.76}$$

all other inner products vanish.

We can easily compute the derivatives of λ, $\bar\lambda$ with respect to the flat coordinates:

$$\begin{aligned} \frac{\partial \lambda(z)}{\partial t_n} &= -z \left[w^n(z) w'(z)\right]_{\leqslant -1} \\ \frac{\partial \bar\lambda(z)}{\partial t_n} &= -z \left[w^n(z) w'(z)\right]_{\geqslant 0} . \end{aligned} \tag{3.77}$$

$$\begin{aligned} &\frac{\partial \lambda(z)}{\partial v} = -1, \quad \frac{\partial \bar\lambda(z)}{\partial v} = 1 \\ &\frac{\partial \lambda(z)}{\partial u} = -\frac{e^u}{z}, \quad \frac{\partial \bar\lambda(z)}{\partial u} = \frac{e^u}{z}. \end{aligned} \tag{3.78}$$

Remark 3.22 *The flat coordinates suggest to look at* M_0 *as a fibered space over* M_{red} *of parametrized simple analytic curves:*

$$M_0 \ni (\lambda(z), \bar\lambda(z)) \mapsto \{z \to w(z) \,|\, |z| = 1\} \in M_{\text{red}} \tag{3.79}$$

with a two dimensional fiber with coordinates u, v.

We have introduced a Frobenius algebra structure on the cotangent space T^*M_0, and since the metric η is non degenerate, we now have automatically a dual one on the tangent space $\mathrm{T}M_0$. In the next session we will need to write down the components of this tensor in the flat coordinates, hence we conclude this section by giving a formula for the 3-point correlator function on the tangent space $\mathrm{T}M_0$ for a triple of general vectors $\partial_1 \cdot \partial_2$ and ∂_3.

Proposition 3.23 *The 3-point correlator function on the tangent space* $\mathrm{T}_{\hat\lambda} M_0$ *is given by the formula*

$$\begin{aligned} &< \partial_1 \cdot \partial_2, \partial_3 >= \\ &= \frac{1}{4\pi i} \oint_{|z|=1} \frac{\partial_1 w\, \partial_2 w\, \partial_3 s + \partial_1 w\, \partial_2 s\, \partial_3 w + \partial_1 s\, \partial_2 w\, \partial_3 w - s'\, \partial_1 w\, \partial_2 w\, \partial_3 w}{z^2 w'} dz \\ &-\mathrm{Res}_{z=0} \frac{\partial_1(\bar\lambda - l)\, \partial_2 l\, \partial_3 l + \partial_1 l\, \partial_2(\bar\lambda - l)\, \partial_3 l + \partial_1 l\, \partial_2 l\, \partial_3(\bar\lambda - l) + \partial_1 l\, \partial_2 l\, \partial_3 l}{z^2 \bar\lambda'} dz \end{aligned} \tag{3.80}$$

where all differentiations of the functions $w = w(z)$, $s = s(z) := \bar\lambda(z) - \lambda(z)$, $l = l(z)$, $\bar\lambda = \bar\lambda(z)$ *have to be done keeping* $z = const$.

Proof Raising the indices i, j, k in the formula

$$\left\langle \frac{\partial}{\partial t_i} \cdot \frac{\partial}{\partial t_j}, \frac{\partial}{\partial t_k} \right\rangle =$$

$$= -\frac{1}{4\pi i} \oint z\, w' \left[w^{i+j} \Pi(w^k w') + w^{j+k} \Pi(w^i w') + w^{k+i} \Pi(w^j w') - w^{i+j+k}\, \Pi w' \right] dz$$

$$-\frac{1}{2\pi i} \oint \left(z + \frac{e^u}{z} \right) w^{i+j+k} w'\, dz$$

we obtain

$$< dt_i \cdot dt_j, dt_k >_* =$$

$$= -\frac{1}{4\pi i} \oint z\, w' \left[w^{-i-j-2} \Pi(w^{-k-1} w') + w^{-j-k-2} \Pi(w^{-i-1} w') + w^{-k-i-2} \Pi(w^{-j-1} w') \right.$$

$$\left. -w^{-i-j-k-3}\, \Pi w' \right] dz - \frac{1}{2\pi i} \oint \left(z + \frac{e^u}{z} \right) w^{-i-j-k-3} w'\, dz.$$

where the operator Π is the difference of two projectors:

$$\Pi\,(f(z)) = (f)_{\geqslant 0} - (f)_{\leqslant -1}. \tag{3.81}$$

We will now derive the same formula by using the 3-point correlator function on the cotangent bundle 3.56.

We will need the formula for the Jacobi matrix of the coordinate transformation $(u_k, \bar{u}_k) \mapsto (\mathbf{t}, u, v)$. Recall that $w(z) = \lambda(z) + \bar{\lambda}(z)$, hence $\frac{\partial w(z)}{\partial u_k} = z^k$, $\frac{\partial w(z)}{\partial \bar{u}_l} = z^l$ for $k \leqslant 0$, $l \geqslant -1$. We now plug this into formulae 3.64. For $n \neq 0$ we immediately get:

$$\frac{\partial t^n}{\partial u_m} = \frac{\partial t^n}{\partial \bar{u}_m} = -\frac{1}{2\pi i} \oint w^{-n-1}(z) z^{m-1} dz \tag{3.82}$$

while for $n = 0$:

$$\frac{\partial t^0}{\partial u_m} = \frac{\partial t^0}{\partial \bar{u}_m} = \frac{1}{2\pi i} \oint \left(\frac{1}{z} \frac{\partial z}{\partial \bar{u}_m} - \frac{1}{w(z)} \frac{\partial w(z)}{\partial \bar{u}_m} \right) \frac{dz}{z} \tag{3.83}$$

Clearly $\frac{\partial z}{\partial u_m} = \frac{\partial z}{\partial \bar{u}_m} = 0$ (since z here is an independent variable), hence formula 3.82 is valid also for $n = 0$. Next we compute the generating functions:

$$\frac{\partial t_n}{\partial u_x} = \sum_{m \leqslant 0} \frac{\partial t_n}{\partial u_m} x^{-(m+1)} = -\frac{1}{2\pi i} \oint w(z)^{-(n+1)} \frac{z}{x} \frac{1}{z-x} \frac{dz}{z} \tag{3.84}$$

$$\frac{\partial t_n}{\partial \bar{u}_x} = \sum_{m \geqslant -1} \frac{\partial t_n}{\partial \bar{u}_m} x^{-(m+1)} = -\frac{1}{2\pi i} \oint w(z)^{-(n+1)} \frac{z}{x} \frac{1}{z-x} \frac{dz}{z} \tag{3.85}$$

at this point, with a lengthy but straight forward computation we can check that:

$$\begin{aligned} < dt_i \cdot dt_j, dt_k >_* = (\frac{1}{2\pi i})^3 \oint \oint \oint \frac{\partial t_n}{\partial u_x} \frac{\partial t_n}{\partial u_x} \frac{\partial t_n}{\partial u_z} c^{x,y,x} dx\ dy\ dz + \\ + (\frac{1}{2\pi i})^3 \oint \oint \oint \frac{\partial t_n}{\partial u_{\bar{x}}} \frac{\partial t_n}{\partial u_x} \frac{\partial t_n}{\partial u_z} c^{\bar{x},y,x} d\bar{x}\ dy\ dz + \dots \\ \dots + (\frac{1}{2\pi i})^3 \oint \oint \oint \frac{\partial t_n}{\partial \bar{u}_{\bar{x}}} \frac{\partial t_n}{\partial \bar{u}_{\bar{y}}} \frac{\partial t_n}{\partial \bar{u}_{\bar{z}}} c^{\bar{x},\bar{y},\bar{z}} d\bar{x}\ d\bar{y}\ d\bar{z} \end{aligned} \tag{3.86}$$

To complete the proof, one should check the identity also with the 1-forms du, dv. Here the Jacobi matrix is given by

$$\frac{\partial v}{\partial u_k} = \frac{\partial u}{\partial u_k} = 0 \quad \frac{\partial v}{\partial \bar{u}_k} = \delta_{k,0} \quad \frac{\partial u}{\partial \bar{u}_k} = \delta_{-1,k} \frac{1}{\bar{u}_{-1}} \tag{3.87}$$

□

Note that using formulae 3.84 we can derive the following representation for the pair of functions $\hat{dt_n} \in \dot{\mathcal{H}}(D_0) \oplus \hat{\mathcal{H}}(D_\infty) = T^*_{\hat{\lambda}} M$ representing the 1-form dt_n

$$\hat{dt_n} = -\frac{1}{z} \left(w^{-n-1}(z)_{\geqslant 0}, w^{-n-1}(z)_{\leqslant 1} \right). \tag{3.88}$$

3.3 Frobenius manifold structure on M_0

3.3.1 M_{2DT} Frobenius manifold

We are now ready to define the Frobenius manifold M_{2DT}. Recall that a Frobenius manifold must be equipped with a Frobenius algebra structure on the tangent bundle such that the associated non degenerate symmetric invariant bilinear form $< \ , \ >$ is a metric of vanishing curvature and the 3-point correlator function satisfies:

$$< \partial_1 \cdot \partial_2, \partial_3 >= \partial_1 \partial_2 \partial_3 F. \tag{3.89}$$

Here ∂_1, ∂_2, ∂_3 are three arbitrary flat vector fields, the function F is the *potential* of the Frobenius manifold. Besides the above conditions there must be a flat unit field and an Euler vector field involved in the quasi homogeneity condition.

Theorem 3.24 *M_0 has a Frobenius manifold structure of central charge $d = 1$ with potential F given by:*

$$\begin{aligned} F(\mathbf{t}, u, v) =& \frac{1}{2} \left(\frac{1}{2\pi i} \right)^2 \oint_\Gamma \oint_\Gamma \mathrm{Li}_3 \left(\frac{w_1 e^{t(w_1)}}{w_2 e^{t(w_2)}} \right) dw_1 \, dw_2 + \\ &+ \frac{1}{2\pi i} \oint_\Gamma \left(\frac{e^u}{w e^{t(w)}} - w e^{t(w)} \right) dw + \frac{(2v + t_{-1})}{4} \frac{1}{2\pi i} \oint_\Gamma (t(w))^2 \, dw \\ &+ \frac{1}{2} v^2 u - e^u \end{aligned} \tag{3.90}$$

where the Euler and identity vector fields are given by

$$E = -\sum_{k \in \mathbb{Z}} k t^k \partial_k + v \partial_v + 2\partial_u \qquad e = \partial_v \tag{3.91}$$

Note that the function Li_3 is the tri-logarithm, defined by its Taylor expansion

$$\mathrm{Li}_3(x) = \sum_{k \geqslant 1} \frac{x^k}{k^3}, \quad |x| < 1.$$

hence the double integral must be regularized in such a way that $|z(w_1)| < |z(w_2)|$. We will denote by M_{2DT} the manifold M_0 endowed with this Frobenius structure.

Proof Let ∂_1, ∂_2, ∂_3 be among the flat vector fields $\partial/\partial t_i$, $\partial/\partial u$ or $\partial/\partial v$. We must show that the 3-point correlator function (3.80) coincides with the triple derivatives of the potential (3.90):

$$< \partial_1 \cdot \partial_2, \partial_3 >= \partial_1 \partial_2 \partial_3 F. \tag{3.92}$$

and that $\mathcal{L}_E F = (3-d)F$ plus quadratic terms.

Computation of triple derivatives of the potential by applying (3.73) is straightforward. To keep the notation more compact, we use $z(w) = w \exp(t(w))$, remembering that z(w) now is a function depending on t:

$$\frac{\partial^3 F}{\partial t_i \partial t_j \partial t_k} = \frac{1}{2} \frac{1}{(2\pi i)^2} \oint_\Gamma \oint_\Gamma \frac{z(w_1)}{z(w_2) - z(w_1)} (w_1^i - w_2^i)(w_1^j - w_2^j)(w_1^k - w_2^k)\, dw_1\, dw_2 \tag{3.93}$$

$$-\frac{1}{2\pi i} \oint_\Gamma \left(z(w) + \frac{e^u}{z(w)} \right) w^{i+j+k} dw + \frac{1}{2} \left[\delta_{i,-1}\delta_{j+k,-1} + \delta_{j,-1}\delta_{k+i,-1} + \delta_{k,-1}\delta_{i+j,-1} \right]$$

$$\frac{\partial^3 F}{\partial t_i \partial t_j \partial v} = \delta_{i+j,-1}$$

$$\frac{\partial^3 F}{\partial v^2 \partial u} = 1$$

$$\frac{\partial^3 F}{\partial t_i \partial t_j \partial u} = \frac{e^u}{2\pi i} \oint_\Gamma \frac{w^{i+j}}{z(w)} dw$$

$$\frac{\partial^3 F}{\partial t_i \partial u^2} = -\frac{e^u}{2\pi i} \oint_\Gamma \frac{w^i}{z(w)} dw$$

$$\frac{\partial^3 F}{\partial u^3} = \frac{e^u}{2\pi i} \oint_\Gamma \frac{dw}{z(w)} - e^u = \bar{u}_1\, e^u$$

all other triple derivatives vanish.

Let us start with the first integral. We open the brackets and return to the integration in $z_1 = z(w_1)$, $z_2 = z(w_2)$ in order to obtain the representation

$$\frac{1}{2} \frac{1}{(2\pi i)^2} \oint_\Gamma \oint_\Gamma \frac{z(w_1)}{z(w_2) - z(w_1)} (w_1^i - w_2^i)(w_1^j - w_2^j)(w_1^k - w_2^k)\, dw_1\, dw_2$$

$$= I_1(i,j,k) + I_2(i,j,k) + I_3(i,j,k) + I_4(i,j,k)$$

where

$$I_1(i,j,k) = \frac{1}{2} \frac{1}{(2\pi i)^2} \oint \oint_{|z_1|<|z_2|} \frac{z_1}{z_2 - z_1} w_1^{i+j+k} w_1' w_2' dz_1\, dz_2$$

$$I_2(i,j,k) = -\frac{1}{2} \frac{1}{(2\pi i)^2} \oint \oint_{|z_1|<|z_2|} \frac{z_1}{z_2 - z_1} \left(w_1^{i+j} w_2^k + w_1^{j+k} w_2^i + w_1^{i+k} w_2^j \right) w_1' w_2' dz_1\, dz_2$$

$$I_3(i,j,k) = \frac{1}{2} \frac{1}{(2\pi i)^2} \oint \oint_{|z_1|<|z_2|} \frac{z_1}{z_2 - z_1} \left(w_1^i w_2^{j+k} + w_1^j w_2^{k+i} + w_1^k w_2^{i+j} \right) w_1' w_2' dz_1\, dz_2$$

$$I_4(i,j,k) = -\frac{1}{2} \frac{1}{(2\pi i)^2} \oint \oint_{|z_1|<|z_2|} \frac{z_1}{z_2 - z_1} w_2^{i+j+k} w_1' w_2' dz_1\, dz_2.$$

Here we denote

$$w_1 = w(z_1), \quad w_2 = w(z_2).$$

Integrating in z_2 we represent the first integral in the form

$$I_1(i,j,k) = \frac{1}{4\pi i} \oint_{|z|=1} z\, w^{i+j+k} w'(w')_{\geqslant 0} dz.$$

Similarly,

$$I_2(i,j,k) = -\frac{1}{4\pi i} \oint_{|z|=1} z\, w' \left[w^{i+j}(w^k w')_{\geqslant 0} + w^{j+k}(w^i w')_{\geqslant 0} + w^{k+i}(w^j w')_{\geqslant 0} \right] dz$$

etc. Using the identity (3.9) we rewrite

$$I_1(i,j,k) + I_4(i,j,k) = \frac{1}{4\pi i} \oint z\, w' w^{i+j+k} (w')_{\geqslant 0}\, dz - \frac{1}{4\pi i} \oint z\, w' w^{i+j+k}\, (w')_{\leqslant -2}\, dz$$

$$= \frac{1}{4\pi i} \oint z\, w' w^{i+j+k}\, \Pi w'\, dz$$

In a similar way we find that

$$I_2(i,j,k) + I_3(i,j,k) =$$
$$-\frac{1}{4\pi i} \oint z\, w' \left[w^{i+j} \Pi(w^k w') + w^{j+k} \Pi(w^i w') + w^{k+i} \Pi(w^j w') \right] dz$$
$$-\frac{1}{2} \left[\delta_{i+j,-1}\delta_{k,-1} + \delta_{j+k,-1}\delta_{i,-1} + \delta_{k+i,-1}\delta_{j,-1} \right].$$

Thus

$$\frac{\partial^3 F}{\partial t_i \partial t_j \partial t_k} =$$
$$= -\frac{1}{4\pi i} \oint z\, w' \left[w^{i+j} \Pi(w^k w') + w^{j+k} \Pi(w^i w') + w^{k+i} \Pi(w^j w') - w^{i+j+k}\, \Pi w' \right] dz$$
$$-\frac{1}{2\pi i} \oint \left(z + \frac{e^u}{z} \right) w^{i+j+k} w'\, dz.$$

On the other side, evaluation of the expression (3.80) with

$$\partial_1 = \frac{\partial}{\partial t_i}, \quad \partial_2 = \frac{\partial}{\partial t_j}, \quad \partial_3 = \frac{\partial}{\partial t_k}$$

using

$$\frac{\partial s(z)}{\partial t_n} = -z\, \Pi(w^n w')$$

(see (3.77)) yields

$$< \partial_1 \cdot \partial_2, \partial_3 >= -\frac{1}{4\pi i} \oint z\, w' \left[w^{i+j} \Pi(w^k w') + w^{j+k} \Pi(w^i w') + w^{k+i} \Pi(w^j w') \right] dz$$
$$+ \frac{1}{4\pi i} \oint z\, s'(z)\, w^{i+j+k} w'\, dz.$$

Since

$$s'(z) = \Pi w' - 2\left(1 + \frac{e^u}{z^2}\right)$$

one finally obtains

$$< \partial_1 \cdot \partial_2, \partial_3 >= \frac{\partial^3 F}{\partial t_i \partial t_j \partial t_k}.$$

Next, taking

$$\partial_1 = \frac{\partial}{\partial t_i}, \quad \partial_2 = \frac{\partial}{\partial t_j}, \quad \partial_3 = \frac{\partial}{\partial v}$$

gives

$$< \partial_1 \cdot \partial_2, \partial_3 >= \frac{1}{4\pi i} \oint \frac{\partial_1 w\, \partial_2 w\, \partial_3 s}{z^2 w'}\, dz = \frac{1}{2\pi i} \oint \frac{\partial_1 w\, \partial_2 w}{z^2 w'}\, dz =< \partial_1, \partial_2 >= \frac{\partial^3 F}{\partial t_i \partial t_j \partial v}$$

(we use that $\partial_v s(z) = 2$). A similar computation works for $\partial_1 = \partial_2 = \partial/\partial u$, $\partial_3 = \partial/\partial v$. For the choice

$$\partial_1 = \frac{\partial}{\partial t_i}, \quad \partial_2 = \frac{\partial}{\partial t_j}, \quad \partial_3 = \frac{\partial}{\partial u}$$

using

$$\frac{\partial s(z)}{\partial u} = 2\,\frac{e^u}{z}$$

we obtain

$$< \partial_1 \cdot \partial_2, \partial_3 >= \frac{e^u}{2\pi i} \oint \frac{w^{i+j} w'}{z}\, dz = \frac{\partial^3 F}{\partial t_i \partial t_j \partial u}.$$

In order to perform a similar computation for

$$\partial_1 = \partial_2 = \frac{\partial}{\partial u}, \quad \partial_3 = \frac{\partial}{\partial t_i}$$

one has to use the second line in the formula (3.80). In this case

$$< \partial_1 \cdot \partial_2, \partial_3 >= \mathrm{Res}_{z=0} \frac{e^{2u} (w^i w')_{\geqslant 0}}{z^3 \bar{\lambda}'} dz = e^{2u}\, \mathrm{Res}\; w^i w' \left(\frac{1}{z^3 \bar{\lambda}'}\right)_{\leqslant -1} dz$$
$$= -e^u\; \mathrm{Res}\; \frac{w^i w'}{z} dz = \frac{\partial^3 F}{\partial u^2 \partial t_i}.$$

In the remaining cases the computation is even simpler. The last step of the proof is in verifying the quasi homogeneity identity

$$E\,F = 2F + v^2 \tag{3.94}$$

To do this, we first show that:

$$E.z(w) = z(w) - w z'(w) \tag{3.95}$$

since:

$$E.z(w) = \sum_{k \in \mathbb{Z}} -k t^k \frac{\partial z(w)}{\partial t^k} = \sum_{k \in \mathbb{Z}} -k t^k z(w) w^k = -z(w) w t'(w) \tag{3.96}$$

but $z(w) = we^t(w)$, hence $-wt'(w) = 1 - w\frac{z'(w)}{z(w)}$. By plugging this identity into the previous formula we get formula 3.95. In a similar way, one can prove that:

$$E.\frac{1}{z(w)} = -\frac{1}{z(w)} + \frac{wz'(w)}{z(w)^2} \tag{3.97}$$

We can now proceed to prove formula 3.94. Hereafter we rewrite, for the sake of the reader, the formula 3.90 of the potential:

$$\begin{aligned} F(\mathbf{t}, u, v) =& \frac{1}{2}\left(\frac{1}{2\pi i}\right)^2 \oint_\Gamma \oint_\Gamma \mathrm{Li}_3\left(\frac{z(a)}{z(b)}\right) \, da\, db + \\ &+ \frac{1}{2\pi i}\oint_\Gamma \left(\frac{e^u}{z(w)} - z(w)\right) \, dw + \frac{(2v + t_{-1})}{4}\frac{1}{2\pi i}\oint_\Gamma (t(w))^2 \, dw \\ &+ \frac{1}{2}v^2 u - e^u \end{aligned} \tag{3.98}$$

We prove quasi homogeneity term by term:

$$\begin{aligned} &E.\oint_\Gamma \oint_\Gamma \mathrm{Li}_3\left(\frac{z(a)}{z(b)}\right) \, da\, db = \oint_\Gamma \oint_\Gamma \mathrm{Li}_2\left(\frac{z(a)}{z(b)}\right)\left(\frac{E.z(a)}{z(a)} - \frac{E.z(b)}{z(b)}\right) \, da\, db = \\ &= \oint_\Gamma \oint_\Gamma \mathrm{Li}_2\left(\frac{z(a)}{z(b)}\right)\left(1 - \frac{az'(a)}{z(a)} - 1 + \frac{bz'(b)}{z(b)}\right) \, da\, db = \\ &= -\oint_\Gamma \oint_\Gamma a\ \mathrm{Li}_2\left(\frac{z(a)}{z(b)}\right)\frac{az'(a)}{z(a)} \, da\, db + \oint_\Gamma \oint_\Gamma b\ \mathrm{Li}_2\left(\frac{z(a)}{z(b)}\right)\frac{bz'(b)}{z(b)} \, da\, db \end{aligned} \tag{3.99}$$

Now observe that $\partial_a \mathrm{Li}_3\left(\frac{z(a)}{z(b)}\right) = \mathrm{Li}_2\left(\frac{z(a)}{z(b)}\right)\frac{z'(a)}{z(a)}$, while $\partial_b \mathrm{Li}_3\left(\frac{z(a)}{z(b)}\right) = -\mathrm{Li}_2\left(\frac{z(a)}{z(b)}\right)\frac{z'(b)}{z(b)}$. Integrating by parts the two terms in the last line of the previous computation we get:

$$E.\oint_\Gamma \oint_\Gamma \mathrm{Li}_3\left(\frac{z(a)}{z(b)}\right) \, da\, db = 2\left(\oint_\Gamma \oint_\Gamma \mathrm{Li}_3\left(\frac{z(a)}{z(b)}\right) \, da\, db\right) \tag{3.100}$$

We proceed with the second term:

$$\begin{aligned} &E.\oint_\Gamma \left(\frac{e^u}{z(w)} - z(w)\right) \, dw = \\ &2\oint_\Gamma \frac{e^u}{z(w)} \, dw - \oint_\Gamma \frac{e^u}{z(w)} \, dw + \oint_\Gamma e^u \frac{wz'(w)}{z(w)^2} \, dw - \oint z(w) \, dw + \oint wz'(w) dw \end{aligned} \tag{3.101}$$

where we used formulae 3.95 and 3.97. We integrate by parts the third and the fifth term:

$$\oint_\Gamma e^u \frac{wz'(w)}{z(w)^2} \, dw = \oint_\Gamma \frac{e^u}{z(w)^2} \, dw \tag{3.102}$$

$$\oint_\Gamma wz'(w) dw = -\oint_\Gamma z(w) dw \tag{3.103}$$

and simplifying the resulting formula we get:

$$E.\oint_\Gamma \left(\frac{e^u}{z(w)} - z(w)\right) \, dw = 2\left(\oint_\Gamma \left(\frac{e^u}{z(w)} - z(w)\right) \, dw\right) \tag{3.104}$$

For the remaining terms the proof is similar. We just observe that the cubic term $\frac{1}{2}v^2u$ of the potential is the one responsible for the appearance of a quadratic term in formula 3.94:

$$E.\frac{1}{2}v^2u = (v\partial_v + 2\partial_u)\frac{1}{2}v^2u = 2(\frac{1}{2}v^2u) + v^2 \tag{3.105}$$

This completes the proof of the Theorem. $\square$

Remark 3.25 *We may rewrite the potential in the $u_k, \bar{u}_k$ coordinates:*

$$F = \frac{1}{2}\frac{1}{(2\pi i)^2}\oint\oint_{|z_1|<|z_2|}\frac{w(z_1)}{z_1}\frac{w(z_2)}{z_2}\log\frac{z_2-z_1}{z_2}\,dz_1 + \tag{3.106}$$

$$+ \frac{1}{2}(\bar{u}_0 - u_0)\left[\frac{1}{2\pi i}\oint_{|z|=1}\frac{w(z)}{z}\log\frac{w(z)}{z}\,dz - u_0 - \bar{u}_0\right] + \tag{3.107}$$

$$+ \frac{1}{2}\bar{u}_0^2\log\bar{u}_{-1} + \bar{u}_{-1}(1+\bar{u}_1).$$

the identity is given by $e = (-1, 1)$, while the Euler vector field

$$E = \big(\lambda(z) - z\,\lambda'(z), \bar{\lambda}(z) - z\,\bar{\lambda}'(z)\big) \tag{3.108}$$

3.3.2 Intersection form

Recall [25] that on the cotangent bundle of an arbitrary Frobenius manifold there exists another important symmetric bilinear form with zero curvature defined by the formula

$$(\omega_1, \omega_2)_* = \mathrm{i}_E(\omega_1 \cdot \omega_2) \tag{3.109}$$

(the so-called *intersection form* of the Frobenius manifold). It does not degenerate outside a closed analytic subset. For the case under consideration the intersection form admits the following explicit expression:

Proposition 3.26 *The intersection form of the Frobenius manifold M_0 is given by the formula*

$$(d\alpha(p), d\beta(q))_* = \frac{p\,q}{p-q}\big[\alpha'(p)\beta(q) - \beta'(q)\alpha(p)\big] + p\,q\,\alpha'(p)\beta'(q). \tag{3.110}$$

It does not degenerate on the open subset of M_0 defined by the conditions

$$\lambda'(z) \neq 0, \quad \bar{\lambda}'(z) \neq 0, \quad \lambda(z)\bar{\lambda}'(z) - \bar{\lambda}(z)\lambda'(z) \neq 0 \quad \text{for} \quad |z| = 1. \tag{3.111}$$

On this subset it defines a metric of zero curvature given by the following inner product on the tangent space

$$(\partial_1, \partial_2) = \frac{1}{2\pi i}\oint_{|z|=1}\frac{\left(\frac{\partial_1\lambda}{\lambda'} - \frac{\partial_1\bar{\lambda}}{\bar{\lambda}'}\right)\left(\frac{\partial_2\lambda}{\lambda'} - \frac{\partial_2\bar{\lambda}}{\bar{\lambda}'}\right)}{\frac{\lambda}{\lambda'} - \frac{\bar{\lambda}}{\bar{\lambda}'}}\frac{dz}{z^2}. \tag{3.112}$$

The complement locus to the subset (3.111) is the discriminant of the infinite dimensional Frobenius manifold.

Proof The intersection form of the Frobenius manifold M_0 reads

$$(\hat{\omega}_1, \hat{\omega}_2)_* = \langle \hat{\omega}_1, \gamma(\hat{\omega}_2)\rangle \tag{3.113}$$

where the linear map $\gamma : \mathrm{T}^*_{\hat{\lambda}} M \to \mathrm{T}_{\hat{\lambda}} M_0$ is defined by

$$\begin{aligned} \gamma(\hat{\omega}) \quad &= z^2 \big(\lambda'(\varepsilon\omega + \bar{\varepsilon}\bar{\omega})_{\leqslant -2} - \varepsilon(\lambda'\omega + \bar{\lambda}'\bar{\omega})_{\leqslant -2}, && (3.114)\\ &\quad -\bar{\lambda}'(\varepsilon\omega + \bar{\varepsilon}\bar{\omega})_{\geqslant -1} + \bar{\varepsilon}(\lambda'\omega + \bar{\lambda}'\bar{\omega})_{\geqslant -1}\big) && (3.115)\end{aligned}$$

where $\varepsilon = \lambda(z) - z\lambda'(z)$ and $\bar{\varepsilon} = \bar{\lambda}(z) - z\bar{\lambda}'(z)$ are the components of the Euler vector field $E = (\varepsilon, \bar{\varepsilon})$.

The linear operator γ is invertible on the open subset of M defined by the conditions (3.111). The inverse operator $\hat{\omega} = (\omega, \bar{\omega}) = \gamma^{-1}\hat{\alpha}, \quad \hat{\alpha} = (\alpha, \bar{\alpha})$ reads

$$\omega = \frac{1}{z^2}\left(\frac{1}{\lambda'}\left[\frac{\bar{\lambda}'\alpha - \lambda'\bar{\alpha}}{\lambda\bar{\lambda}' - \bar{\lambda}\lambda'}\right]_{\geqslant 1}\right)_{\geqslant 1}, \quad \bar{\omega} = -\frac{1}{z^2}\left(\frac{1}{\bar{\lambda}'}\left[\frac{\bar{\lambda}'\alpha - \lambda'\bar{\alpha}}{\lambda\bar{\lambda}' - \bar{\lambda}\lambda'}\right]_{\leqslant 0}\right)_{\leqslant 0}. \tag{3.116}$$

Computing the pairing

$$\langle \hat{\alpha}, \gamma^{-1}(\hat{\beta})\rangle =: (\hat{\alpha}, \hat{\beta})$$

we obtain the expression (3.112) for the intersection form on the tangent bundle. This completes the proof of Proposition 3.26. □

3.3.3 Canonical coordinates

A point of a finite-dimensional Frobenius manifold is said to be *semi simple* when the Frobenius algebra defined on the tangent space admits a basis of idempotents [25]. Given a semi simple point, one can find a set of local *canonical* coordinates $u_1, \ldots, u_n$ which satisfy:

$$\begin{aligned} \frac{\partial}{\partial u_i} \cdot \frac{\partial}{\partial u_j} &= \delta_{ij}\frac{\partial}{\partial u_i}\\ \left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right\rangle &= \eta_{ii}(u)\,\delta_{ij}. \end{aligned}$$

These coordinates can be chosen in such a way that

$$\langle du_i, E\rangle = u_i, \quad i = 1, \ldots, n.$$

The multiplication and inner products of the differentials of the canonical coordinates satisfy

$$\begin{aligned} < du_i, du_j >_* &= \eta_{ii}^{-1}(u)\delta_{ij}\\ du_i \cdot du_j &= \eta_{ii}^{-1}(u)\delta_{ij} du_i. \end{aligned}$$

Using these finite-dimensional hints one arrives at the following construction of the canonical coordinates for M_{2DT}. Consider the curve Σ:

$$\Sigma := \{S^1 \ni p \mapsto \sigma(p) = \frac{\lambda'(p)}{\lambda'(p) + \bar{\lambda}'(p)}\}. \tag{3.117}$$

Denote $M_{s\,s} \subset M_0$ the subset consisting of pairs $(\lambda, \bar{\lambda})$ such that the curve Σ is smooth non-self intersecting, and $(\lambda, \bar{\lambda})$ satisfy the condition:

$$\lambda'(p)\bar{\lambda}''(p) - \bar{\lambda}'(p)\lambda''(p) \neq 0 \quad \text{for any} \quad p \in S^1. \tag{3.118}$$

For a given curve Σ introduce the following functional on $M_{s\,s}$ depending on the point of the curve

$$u_\sigma := \left[\sigma\, \bar{\lambda}(p) + (\sigma - 1)\, \lambda(p)\right]_{p=p(\sigma)}, \quad \sigma \in \Sigma \tag{3.119}$$

where $p = p(\sigma) \in S^1$ is determined from the equation

$$\left[\sigma\, \bar{\lambda}'(p) + (\sigma - 1)\, \lambda'(p)\right]_{p=p(\sigma)} = 0, \quad \sigma \in \Sigma. \tag{3.120}$$

Varying the curve Σ we obtain the variation of the point of the Frobenius manifold defined by the equation (3.120). Note that the analytic curve Σ is smooth, i.e. $\sigma'(p) \neq 0$ due to the assumptions (3.118).

Proposition 3.27 *The functionals u_σ are the canonical coordinates on $M_{s\,s}$, and the 1-forms $du(p)$ are idempotents of the Frobenius algebra on $T^*M_{s\,s}$*

Proof Taking the differential of (3.119) one obtains, due to the equation (3.120) the 1-form (3.122)

$$du_\sigma = du(p)_{p=p(\sigma)}.$$

It is convenient to change normalization of these 1-forms introducing, for every $p \in S^1$, the linear functionals $d\mu(p)$:

$$\langle d\mu(p), \hat{\alpha}\rangle = \frac{\alpha(p)}{\lambda'(p)} - \frac{\bar{\alpha}(p)}{\bar{\lambda}'(p)}. \tag{3.121}$$

A simple computation shows that:

$$du(p) = -\frac{\lambda'(p)\bar{\lambda}'(p)}{\lambda'(p) + \bar{\lambda}'(p)} d\mu(p) = \frac{\lambda'(p)\, d\bar{\lambda}(p) - \bar{\lambda}'(p)\, d\lambda(p)}{\lambda'(p) + \bar{\lambda}'(p)}. \tag{3.122}$$

The inner products and multiplication of the 1-forms $du(p)$ is given by the following expressions

$$< du(p), du(q) >_* = f(p)\delta(p - q) \tag{3.123}$$

$$du(p) \cdot du(q) = f(p)\delta(p - q)\, du(p) \tag{3.124}$$

$$f(p) = -p^2 \frac{\lambda'(p)\bar{\lambda}'(p)}{\lambda'(p) + \bar{\lambda}'(p)} \tag{3.125}$$

where the delta-function on the circle is defined by

$$\delta(p-q)=\sum_{k\in\mathbb{Z}}\frac{p^k}{q^{k+1}},\quad \frac{1}{2\pi i}\oint_{|q|=1} f(q)\delta(p-q)\,dq=f(p).$$

The functionals $d\mu(p)$ span the cotangent space to M_{ss}. Indeed, the vector $\hat{\alpha}=(\alpha(z),\bar{\alpha}(z))$ can be reconstructed from knowing all the values

$$a(p):=\langle d\mu(p),\hat{\alpha}\rangle \quad \text{for all} \quad p\in S^1$$

by the following procedure:

$$\begin{aligned}\alpha(z)&=\quad\lambda'(z)[a(z)]_{\leqslant 0}\\ \bar{\alpha}(z)&=-\bar{\lambda}'(z)[a(z)]_{\geqslant 1}.\end{aligned}\tag{3.126}$$

□

Remark 3.28 *For the dispersionless limit of the 1+1 Lax equations the well known prescription suggests to take the critical values of the symbol of the Lax operator in order to obtain the Riemann invariants (aka the canonical coordinates) of the dispersionless equations. This rule extends also to the 1+1 Whitham equations, where the Riemann invariants are given by the ramification points of the spectral curve [33]. Our procedure* (3.119), (3.120) *looks very similarly. The main difference is that now the Riemann invariants are labeled by a continuous parameter running through the curve* Σ.

3.3.4 Reductions

We conclude the chapter by exposing two interesting examples of reductions. The first case is given by the constraint $w(z)=z$, i.e. $t^k=0$ for every $k\in\mathbb{Z}$. The second one is the famous Toda reduction $\lambda(z)=\bar{\lambda}(z)$. It is interesting to observe that, among all the possible Toda bigraded reductions [12] of the 2D Toda hierarchy ($\lambda(z)^m=\bar{\lambda}(z)^n$), the Toda reduction is the only possible one at the Frobenius manifold level. The reason for this is that the other reductions take place on other functional spaces, where $\lambda,\bar{\lambda}$ are not single valued functions, but multivalued ones.

Example 3.29 *Let us compute the Frobenius algebra structure on the two-dimensional locus* $M_0^2\subset M_0$ *defined by*

$$\lambda=z-v-\frac{e^u}{z},\quad \bar{\lambda}=\frac{e^u}{z}+v.\tag{3.127}$$

The curve $\Gamma=w(S^1)$ *in this case is the unit circle with the standard parametrization. So the tangent space to* M_0 *at the points of* M_0^2 *coincides with the Cartesian product of the space of vector fields on the circle spanned by*

$$X_n=z^{n+1}\frac{\partial}{\partial z}=-\frac{\partial}{\partial t_n},\quad n\in\mathbb{Z}\tag{3.128}$$

and the two-dimensional space with the basis $e = \partial/\partial v$ and $\partial/\partial u$. The multiplication table of these vector fields reads

$$
\begin{aligned}
X_i \cdot X_j &= \frac{1}{2}\left[\theta(i) + \theta(j) + \theta(-i-j-2) + 1\right] X_{i+j+1} + \delta_{i+j,-1}\frac{\partial}{\partial u} \\
&\quad + e^u \left[X_{i+j-1} + \delta_{i+j,0}\frac{\partial}{\partial v}\right] \\
\frac{\partial}{\partial u} \cdot X_i &= e^u \left[X_{i-1} + \delta_{i,0}\frac{\partial}{\partial v}\right] \\
\frac{\partial}{\partial u} \cdot \frac{\partial}{\partial u} &= e^u X_{-1}
\end{aligned}
\tag{3.129}
$$

where θ is the step function,

$$
\theta(n) = \begin{cases} 1, & n \geqslant 0 \\ -1, & n < 0 \end{cases}
$$

Remark 3.30 *A combination of the limit*

$$
\operatorname{Re} u \to -\infty
$$

and the projector

$$
\mathrm{pr} : TM_0 \to TM_0, \quad \mathrm{pr}\left(\frac{\partial}{\partial u}\right) = \mathrm{pr}\left(\frac{\partial}{\partial v}\right) = 0, \quad \mathrm{pr}\left(\frac{\partial}{\partial t_i}\right) = \frac{\partial}{\partial t_i}
$$

provides the tangent planes to the space M_{red} of parametrized analytic curves $\{z \mapsto w(z),\ |z| = 1\} \in M_{\mathrm{red}}$ with a structure of Frobenius algebra. The non degenerate invariant inner product of tangent vectors is given by the first term in the formula (3.62)*, the trilinear symmetric form $< \partial_1 \cdot \partial_2, \partial_3 >_{M_{\mathrm{red}}}$ is given by the triple derivatives of the reduced potential*

$$
F_{\mathrm{red}}(\mathbf{t}) = \frac{1}{2}\frac{1}{(2\pi i)^2}\oint_\Gamma \oint_\Gamma \mathrm{Li}_3\left(\frac{z(w_1)}{z(w_2)}\right) dw_1\, dw_2 - \frac{1}{2\pi i}\oint_\Gamma z(w)\, dw \tag{3.130}
$$

For example, specializing the Frobenius algebra at the point $w(z) \equiv z$ one obtains the following graded Frobenius algebra with no unit

$$
\begin{aligned}
X_i \cdot X_j &= \frac{1}{2}\left[\theta(i) + \theta(j) + \theta(-i-j-2) + 1\right] X_{i+j+1} \\
< X_i, X_j > &= \delta_{i+j,-1}
\end{aligned}
\tag{3.131}
$$

(cf. (3.129) *above),* $\deg X_i = i + 1$.

Example 3.31 *Let us consider a two-dimensional locus $M^2 \subset M_0$ defined by the equation*

$$
\lambda(z) = \bar{\lambda}(z).
$$

From the explicit formulae (3.93) *one conclude that M^2 is a Frobenius submanifold isomorphic to the quantum cohomology of $\mathbf{P}^1$*

$$
\frac{\partial}{\partial u} \cdot \frac{\partial}{\partial u} = e^u \frac{\partial}{\partial v}, \quad \frac{\partial}{\partial u} \cdot \frac{\partial}{\partial v} = \frac{\partial}{\partial u}, \quad \frac{\partial}{\partial v} \cdot \frac{\partial}{\partial v} = \frac{\partial}{\partial v}, \quad \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\rangle = 1
$$

(also describing the dispersionless limit of the standard 1+1 Toda lattice).

Chapter 4

Extended 2D Toda Hierarchy

Recall that given a Frobenius manifold M, one can construct a bihamiltonian integrable system defined on the loop space $\mathcal{L}M$ of the Frobenius manifold which is called the *principal hierarchy*. The Hamiltonian densities of the principal hierarchy are given by the expansion of the deformed flat coordinates in the deformation parameter. In a nutshell, to every flat coordinate v^k of the manifold corresponds a series of flows $t^{k,n}$, where $n \geqslant 0$. All these flows mutually commute. The flows $t^{k,0}$ are called the *primary flows*. The peculiarity of the primary flows is that they are determined by the structure coefficients of the Frobenius algebra: $\frac{\partial v^k}{\partial t^{i,0}} = c^k_{i,j} v^j_x$.

In this last chapter we study the principal hierarchy associated to M_{2DT}. We prove that this hierarchy is an extension of $2D$ Toda hierarchy. The construction of this hierarchy was one of the main motivations to introduce the Frobenius manifold M_{2DT}. In the $2D$ Toda hierarchy we just have two series of flows: $\frac{\partial}{\partial s_n}$ and $\frac{\partial}{\partial \bar{s}_n}$. According to the theory of Frobenius manifolds we can define a series of flows for each flat coordinate, hence an infinite set. In the first section we prove that the bihamiltonian structure introduced in 2.40 coincides with the one determined by the flat pencil of metrics of M_{2DT}. We then give an explicit Lax formulation of the primary flows. The last section is dedicated to the principal hierarchy extension. This last result is incomplete, since two series of flows, namely those corresponding to the flat coordinates t_{-1} and t_v, are missing. We plan to give a full description of the principal hierarchy in the near future.

4.1 From M_{2DT} to 2D Toda Hierarchy

In this section we show how to construct the $2D$ Toda hierarchy on the loop space $\mathcal{L}M_{2DT}$ and prove that the bihamiltonian structure induced by the metrics 3.49, 3.110 coincide with the one given in (2.40)

Let $\mathcal{L}M_{2DT}$ be the loop space of maps from S^1 to the manifold M. A point in $\mathcal{L}M$ is given by a pair of maps $(\lambda(z,x), \bar{\lambda}(z,x))$, where $x \in S^1$. A tangent vector at a point $(\lambda, \bar{\lambda}) \in \mathcal{L}M$ is naturally identified with a map from S^1 to $\dot{\mathcal{H}}(D_\infty) \oplus \dot{\mathcal{H}}(D_0)$ and a covector with a map from S^1 to $\dot{\mathcal{H}}(D_0) \oplus \dot{\mathcal{H}}(D_\infty)$. The pairing between a vector

$\hat{\alpha} = (\alpha, \bar{\alpha})$ and a covector $\hat{\omega} = (\omega, \bar{\omega})$ is the natural extension of the pairing (3.19), i.e.

$$\langle \hat{\omega}, \hat{\alpha} \rangle = \frac{1}{2\pi i} \oint_{S^1} \oint_{|z|=1} [\alpha(z,x)\omega(z,x) + \bar{\alpha}(z,x)\bar{\omega}(z,x)] \, dz \, dx. \tag{4.1}$$

The dispersionless two-dimensional Toda hierarchy is composed of two sequences of commuting vector fields on $\mathcal{L}M$, denoted by times s_n and $\bar{s}_n$ for $n > 0$. They are defined by the Lax equations

$$\frac{\partial \lambda}{\partial s_n} = \{(\lambda^n)_+, \lambda\} \qquad \frac{\partial \bar{\lambda}}{\partial s_n} = \{(\lambda^n)_+, \bar{\lambda}\} \tag{4.2}$$

$$\frac{\partial \lambda}{\partial \bar{s}_n} = \{(\bar{\lambda}^n)_-, \lambda\} \qquad \frac{\partial \bar{\lambda}}{\partial \bar{s}_n} = \{(\bar{\lambda}^n)_-, \bar{\lambda}\}. \tag{4.3}$$

The curly bracket stands for the standard Poisson bracket on the cylinder $(z, x) \in S^1 \times \mathbb{R}$:

$$\{f(z,x), g(z,x)\} = z\frac{\partial f}{\partial z}\frac{\partial g}{\partial x} - z\frac{\partial g}{\partial z}\frac{\partial f}{\partial x}.$$

A Poisson structure on $\mathcal{L}M$ is defined by a map P_i from the cotangent to the tangent space of $\mathcal{L}M$ at each point of the loop space, such that the associated Poisson bracket between local functionals on $\mathcal{L}M$

$$\{F, G\}_i = \langle dF, P_i(dG) \rangle \tag{4.4}$$

is skew-symmetric and satisfies the Jacobi identity.

Proposition 4.1 *The Poisson brackets induced by the contravariant metric* (3.49) *and the intersection form* (3.110) *on* $\mathcal{L}M_{2DT}$ *coincide with the two Poisson brackets defined in* (2.40)*.*

Proof We recall the formulae (2.40) of the Poisson brackets induce by the R-matrix:

$$\begin{aligned} P_1(\hat{\omega}) = \quad & \big(\{\lambda, (z\omega - z\bar{\omega})_-\} - (\{\lambda, z\omega\} + \{\bar{\lambda}, z\bar{\omega}\})_{\leqslant 0}, \\ & -\{\bar{\lambda}, (z\omega - z\bar{\omega})_+\} - (\{\lambda, z\omega\} + \{\bar{\lambda}, z\bar{\omega}\})_{>0}\big) \end{aligned} \tag{4.5}$$

$$\begin{aligned} P_2(\hat{\omega}) = \quad & \big(\{\lambda, (z\lambda\omega + z\bar{\lambda}\bar{\omega})_-\} - \lambda(\{\lambda, z\omega\} + \{\bar{\lambda}, z\bar{\omega}\})_{\leqslant 0} + z\lambda_z\varphi_x, \\ & -\{\bar{\lambda}, (z\bar{\lambda}\bar{\omega} + z\lambda\omega)_+\} + \bar{\lambda}(\{\lambda, z\omega\} + \{\bar{\lambda}, z\bar{\omega}\})_{>0} + z\bar{\lambda}_z\varphi_x\big). \end{aligned} \tag{4.6}$$

Let us introduce 1-forms $d\lambda(p, y)$, $d\bar{\lambda}(p, y)$ at a point of $\mathcal{L}M$ such that

$$\langle d\lambda(p,y), \hat{\alpha} \rangle = \alpha(p,y), \qquad \langle d\bar{\lambda}(p,y), \hat{\alpha} \rangle = \hat{\alpha}(p,y) \tag{4.7}$$

for any element $\hat{\alpha} = (\alpha, \bar{\alpha})$ of the tangent at the same point. Clearly they are the differentials of the functionals on $\mathcal{L}M$ that evaluate λ ($\bar{\lambda}$ respectively) at a point (p, y) with $y \in S^1$ and $p \in D_\infty$ (D_0 respectively). As before they can be realized as

$$d\lambda(p,y) = (\frac{p}{z}\frac{1}{p-z}\delta(x-y), 0) \qquad d\bar{\lambda}(p,y) = (0, \frac{z}{p}\frac{1}{z-p}\delta(x-y)). \tag{4.8}$$

The following expressions are obtained by substitution of (4.8) in (4.4) and (2.40). The explicit form of the first Poisson bracket is

$$
\begin{aligned}
\{\alpha(p,x),\beta(q,y)\}_1 &= \tfrac{pq}{p-q}(\epsilon(\beta)\alpha_p(p,x)-\epsilon(\alpha)\beta_q(q,x))\delta'(x-y) && (4.9)\\
&\quad +pq\tfrac{\partial}{\partial q}\left(\tfrac{\epsilon(\beta)\alpha_x(p,x)-\epsilon(\alpha)\beta_x(q,x)}{p-q}\right)\delta(x-y) && (4.10)\\
& && (4.11)
\end{aligned}
$$

and that of the second Poisson bracket is

$$
\begin{aligned}
\{\alpha(p,x),\beta(q,y)\}_2 &= pq\left[\tfrac{\alpha_p\beta-\beta_q\alpha}{p-q}+\alpha_p\beta_q\right]\delta'(x-y) && (4.12)\\
&\quad +pq\left[\tfrac{\partial}{\partial q}\left(\tfrac{\alpha_x\beta-\beta_x\alpha}{p-q}\right)+\tfrac{\alpha_p\beta_x-\beta_q\alpha_x}{p-q}+\alpha_p\beta_{qx}\right]\delta(x-y). && (4.13)
\end{aligned}
$$

As before α, β can take the values λ, $\bar\lambda$ and by definition $\epsilon(\lambda)=1$ and $\epsilon(\bar\lambda)=-1$. In the right-hand side of the last formula we have assumed $\alpha=\alpha(p,x)$ and $\beta=\beta(q,x)$.

These are Poisson brackets of hydrodynamic type. The metrics can be read easily as coefficients of $\delta'(x-y)$. □

The flows (4.2) are Hamiltonian with respect to both Poisson structures (2.40)

$$
\begin{aligned}
\tfrac{\partial}{\partial s_n}\cdot &= \{\cdot,H_n\}_1=\{\cdot,H_{n-1}\}_2 && (4.14)\\
\tfrac{\partial}{\partial \bar s_n}\cdot &= \{\cdot,\bar H_n\}_1=-\{\cdot,\bar H_{n-1}\}_2 && (4.15)
\end{aligned}
$$

with Hamiltonians

$$
H_n=\frac{1}{2\pi i}\oint_{S^1}\oint_{|z|=1}\frac{\lambda^{n+1}}{n+1}\frac{dz}{z}\,dx \qquad \bar H_n=\frac{1}{2\pi i}\oint_{S^1}\oint_{|z|=1}\frac{\bar\lambda^{n+1}}{n+1}\frac{dz}{z}\,dx. \tag{4.16}
$$

Remark 4.2 *The map*

$$
M_0\to M_{\mathrm{red}},\quad (\lambda(z),\bar\lambda(z))\mapsto w(z)=\lambda(z)+\bar\lambda(z)
$$

induces a map of the loop spaces

$$
\mathcal{L}M_0\to\mathcal{L}M_{\mathrm{red}}.
$$

Let us equip the second loop space $\mathcal{L}M_{\mathrm{red}}$ with the Poisson structure of the two-dimensional incompressible fluid on the two-dimensional torus $\mathbb{T}=\{(x_1,x_2)\sim(x_1+2\pi m,x_2+2\pi n)\}$:

$$
\begin{aligned}
\{w(x),w(y)\} = {} & \partial_{x_1}w(x)\delta(x_1-x_2)\delta'(y_1-y_2)- && (4.17)\\
& -\partial_{x_2}w(x)\delta'(x_1-x_2)\delta(y_1-y_2) && (4.18)\\
x=(x_1,x_2), \quad & y=(y_1,y_2)\in\mathbb{T}.
\end{aligned}
$$

that is, with the Lie – Poisson bracket on the dual space to the Lie algebra $\mathcal{V}$ of divergence-free vector fields (see, e.g., [4]). The loop space $\mathcal{L}M_0$ will be considered as a Poisson manifold with respect to the first *Poisson bracket* (4.9)*. Then the map* (4.2) *is a morphism of Poisson manifolds.*

Indeed, the Poisson brackets (4.9) of the point-functionals $w(z,x) = \lambda(z,x)+\bar{\lambda}(z,x)$ after the substitution

$$p = e^{i\,x_1}, \quad x = x_2$$

reduce to (4.17). The functionals $w(z,x)$ commute with $u(x)$, $v(x)$ while the brackets of these two are familiar from the Hamiltonian description of the dispersionless limit $u_{tt} = (e^u)_{xx}$ of the standard 1+1 Toda lattice:

$$\{u(x), v(y)\} = \delta'(x-y), \quad \text{other brackets vanish.}$$

4.2 Primary flows

Theorem 4.3 *The primary flows of the Principal Hierarchy associated to M_{2DT} have the following Lax form*

$$\begin{aligned}
&\frac{\partial\lambda(z)}{\partial t^{\alpha,0}} = \frac{1}{\alpha+1}\left\{w^{\alpha+1}(z)_{<0}, \lambda(z)\right\}, \quad \frac{\partial\bar{\lambda}(z)}{\partial t^{\alpha,0}} = -\frac{1}{\alpha+1}\left\{w^{\alpha+1}(z)_{\geqslant 0}, \bar{\lambda}(z)\right\}\\
&\alpha\in\mathbb{Z}, \quad \alpha\neq -1,\\
&\frac{\partial\lambda(z)}{\partial t^{-1,0}} = \left\{\left(\log\frac{w(z)}{z}\right)_{<0} + \log z, \lambda(z)\right\}, \quad \frac{\partial\bar{\lambda}(z)}{\partial t^{-1,0}} = -\left\{\left(\log\frac{w(z)}{z}\right)_{\geqslant 0}, \bar{\lambda}(z)\right\}\\
&\frac{\partial}{\partial t^{v,0}} = \frac{\partial}{\partial x}\\
&\frac{\partial}{\partial t^{u,0}} = -\frac{\partial}{\partial \bar{s}_1}
\end{aligned} \tag{4.19}$$

Note that only two of these flows are covered by the dispersionless limit of the 2D Toda equations (2.12):

$$\frac{\partial}{\partial s_1} = -\frac{\partial}{\partial t^{0,0}} + \frac{\partial}{\partial t^{u,0}}, \quad \frac{\partial}{\partial \bar{s}_1} = -\frac{\partial}{\partial t^{u,0}}.$$

All these flows are symmetries of the dispersionless limit (4.2) of the 2D Toda hierarchy (See Corollary 4.13).

Proof We prove the theorem for the flows $\frac{\partial}{\partial t^{\alpha,0}}$, $\alpha\in\mathbb{Z}$. The remaining cases $\frac{\partial}{\partial t^{u,0}}$ and $\frac{\partial}{\partial t^{v,0}}$ can be easily proved in a similar way. We will first compute the derivatives $\frac{\partial}{\partial t^{\alpha,0}}$ of the variables $w(z), u, v$ and then show that they coincide with the Lax flows (4.19).

Step 1. We compute $\frac{\partial z(w)}{\partial t^{\alpha,0}}$.

$$\frac{\partial z(w)}{\partial t^{\alpha,0}} = \sum_i z(w)w^i\frac{\partial t_i}{\partial t^{\alpha,0}} = z(w)w^i\left[c_{-(i+1),\alpha,\beta}t_x^\beta + c_{-(i+1),\alpha,u}u_x + c_{-(i+1),\alpha,v}v_x\right] \tag{4.20}$$

Before plugging the triple derivatives $c_{\alpha,\beta,\gamma}$ of the potential (3.93) in (4.20), we observe that they can be rewritten as follows:

$$\begin{aligned} c_{\alpha,\beta,\gamma} &= \frac{1}{2\pi i}\oint \frac{z(a)a^\alpha - z(b)b^\alpha}{z(a)-z(b)} a^\gamma b^\beta da\ db + && (4.21)\\ &+\frac{1}{2\pi i}\oint \frac{z(a)}{z(a)-z(b)} a^{\beta+\gamma} b^\alpha da\ db - && (4.22)\\ &-\frac{1}{2\pi i}\oint \left(z(a) + \frac{e^u}{a} + \oint \frac{z(a)}{z(a)-z(r)} dr \right) a^{\alpha+\beta+\gamma} da && (4.23) \end{aligned}$$

Substituting this expression into (4.20) we get:

$$\begin{aligned} \frac{\partial z(w)}{\partial t^{\alpha,0}} &= z(w)w^\alpha \oint \frac{1}{z(w)-z(b)}\frac{z(w)}{z(b)} z_x(b) db - z(w)\oint \frac{b^\alpha}{z(w)-z(b)} z_x(b) db + \\ &+z(w)z_x(w)\oint \frac{b^\alpha}{z(w)-z(b)} db - \\ &-\left(z(w) + \frac{e^u}{z}(w) + \oint \frac{z(w)}{z(w)-z(r)} dr \right) z_x(w)w^\alpha + \\ &+e^u u_x w^\alpha + z(w)w^\alpha v_x \end{aligned} \tag{4.24}$$

Note that this computation holds for every $\alpha \in \mathbb{Z}$.

Using simple identities

$$\begin{aligned} \frac{\partial w(z)}{\partial t^{\alpha,0}} &= -w'(z)\frac{\partial z(w)}{\partial t^{\alpha,0}}(w(z)) && (4.25)\\ w_x(z) &= -w'(z)z_x(w(z)) && (4.26) \end{aligned}$$

along with (4.24) we get

$$\begin{aligned} \frac{\partial w(z)}{\partial t^{\alpha,0}} &= -w'(z)w^\alpha(z) z\oint \frac{1}{z-z(b)}\frac{z}{z(b)} z_x(b) db + zw'(z)\oint \frac{b^\alpha}{z-z(b)} z_x(b) db + \\ &+zw_x(z)\oint \frac{b^\alpha}{z-z(b)} db - w_x(z)w^\alpha(z)\left(z + \frac{e^u}{z} + \oint \frac{z}{z-z(r)} dr \right) - \\ &-e^u u_x w'(z) w^\alpha(z) - zw' z w^\alpha(z) v_x \end{aligned} \tag{4.27}$$

Perform the change of variables $a = z(b)$ in the integrals, using (4.26) for the first two terms:

$$\frac{\partial w(z)}{\partial t^{\alpha,0}} = w'(z)w^{\alpha}(z)z\oint\frac{1}{z-a}\frac{z}{a}w_x(a)da - zw'(z)\oint\frac{w^{\alpha}(a)}{z-a}w_x(a)db + $$
$$+zw_x(z)\oint\frac{w^{\alpha}(a)}{z-a}w'(a)da - w_x(z)w^{\alpha}(z)\oint\frac{z}{z-a}w'(a)da -$$
$$-w_x(z)w^{\alpha}(z)\left(z+\frac{e^u}{z}\right) -$$
$$-e^u u_x w'(z)w^{\alpha}(z) - zw'zw^{\alpha}(z)v_x \tag{4.28}$$

All this integrals are projections of the type (3.8). Computing them explicitly we finally get:

$$\frac{\partial w(z)}{\partial t^{\alpha,0}} = w^{\alpha}\left\{w(z), w(z)_{\leqslant 0}\right\} + w^{\alpha}\left\{w^{\alpha+1}(z), z - v - \tfrac{e^u}{z}\right\} + \tag{4.29}$$
$$+\left(zw'(z)w^{\alpha}(z)\right)_{<0} w_x(z) - zw'(z)\left(w^{\alpha}(z)w_x(z)\right)_{<0}$$

The primary time derivatives of the coordinates u, v are given directly by (3.93):

$$\frac{\partial v}{\partial t^{\alpha,0}} = e^u\partial_x\left(-\oint\frac{w^{\alpha}}{z(w)}dw\right) + e^u u_x\left(-\oint\frac{w^{\alpha}}{z(w)}dw\right) \tag{4.30}$$

$$\frac{\partial u}{\partial t^{\alpha,0}} = \partial_x\left(t^{-(\alpha+1)}\right) \tag{4.31}$$

Step 2. We write the Lax flows $\frac{\partial}{\partial t^{\alpha,0}}$ for $\alpha \neq -1$ in the $w(z), u, v,$ coordinates and show that they coincide with (4.29),(4.30) and (4.31). Plugging formulas (3.70) into (4.19) we get:

$$\frac{\partial w(z)}{\partial t^{\alpha,0}} = \frac{1}{\alpha+1}\left\{w^{\alpha+1}(z)_{<0}, w(z)_{\leqslant 0}\right\} - \frac{1}{\alpha+1}\left\{w^{\alpha+1}(z)_{\geqslant 0}, w_{>0}\right\} + \tag{4.32}$$
$$+\frac{1}{\alpha+1}\left\{w^{\alpha+1}(z), z - v - \frac{e^u}{z}\right\}$$

$$\frac{\partial v}{\partial t^{\alpha,0}} = \left\{\left(-\frac{w^{\alpha+1}(z)}{\alpha+1}\right)_1 z, \frac{e^u}{z}\right\} \tag{4.33}$$

$$\frac{e^u}{z}\frac{\partial u}{\partial t^{\alpha,0}} = \left\{\left(-\frac{w^{\alpha+1}(z)}{\alpha+1}\right)_0, \frac{e^u}{z}\right\} \tag{4.34}$$

Here as above $(f(z,x))_n$ is the n-th coefficient of the Laurent expansion of $f(z,x)$ in the z variable. Adding

$$\frac{1}{\alpha+1}\left\{w^{\alpha+1}(z)_{\geqslant 0}, w(z)_{\leqslant 0}\right\} - \frac{1}{\alpha+1}\left\{w^{\alpha+1}(z)_{\geqslant 0}, w(z)_{\leqslant 0}\right\} = 0$$

to (4.32) we get:

$$\frac{\partial w(z)}{\partial t^{\alpha,0}} = \frac{1}{\alpha+1}\left\{w^{\alpha+1}(z), w(z)_{\leqslant 0}\right\} + \frac{1}{\alpha+1}\left\{w^{\alpha+1}(z)_{<0}, w(z)\right\} + \tag{4.35}$$
$$+\frac{1}{\alpha+1}\left\{w^{\alpha+1}(z), z - v - \frac{e^u}{z}\right\}$$

Clearly $z\partial_z\left(f(z)\right)_{<0}=\left(z\partial_z f(z)\right)_{<0}$, $\partial_x\left(f(z)\right)_{<0}=\left(\partial_x f(z)\right)_{<0}$, hence we can rewrite the flow in the final form:

$$
\begin{aligned}
\frac{\partial w(z)}{\partial t^{\alpha,0}} &= w^\alpha\left\{w(z),w(z)_{\leqslant 0}\right\}+w^\alpha\left\{w^{\alpha+1}(z),z-v-\frac{e^u}{z}\right\}+ \\
&\quad+\left(zw'(z)w^\alpha(z)\right)_{<0}w_x(z)-zw'(z)\left(w^\alpha(z)w_x(z)\right)_{<0}
\end{aligned}
\tag{4.36}
$$

The formula (4.36) can be proven in a similar manner also for the exceptional Lax flow $\frac{\partial}{\partial t^{-1,0}}$. This formula coincides with primary flow evaluation over $w(z)$ (formula (4.29)).

To conclude the proof we rewrite (4.33) and (4.34) in the more convenient form:

$$
\frac{\partial v}{\partial t^{\alpha,0}} = \left(-\frac{w^{\alpha+1}(z)}{\alpha+1}\right)_1 e^u u_x+e^u\partial_x\left(-\frac{w^{\alpha+1}(z)}{\alpha+1}\right)_1 \tag{4.37}
$$

$$
\frac{\partial u}{\partial t^{\alpha,0}} = \partial_x\left(-\frac{w^{\alpha+1}(z)}{\alpha+1}\right)_0 \tag{4.38}
$$

Also this formulas have an analog in the $\alpha=-1$ case, one just has to replace the function $\left(-\frac{w^{\alpha+1}(z)}{\alpha+1}\right)$ with $\left(-\frac{\log w(z)}{z}\right)$.

One can easily see that $\left(-\oint\frac{w^\alpha}{z(w)}dw\right)=\left(-\frac{w^{\alpha+1}(z)}{\alpha+1}\right)_1$ for $\alpha\neq-1$, while $\left(-\oint\frac{w^{-1}}{z(w)}dw\right)=\left(-\frac{\log w(z)}{z}\right)_1$. Hence (4.37) coincides with the evaluation of the primary flow over v (formula (4.30)). Proving that (4.38) and (4.31) coincide only uses the definition of the flat coordinates t_i in terms of $w(z)$. □

Proposition 4.4 *The primary flows* (4.19) *in the canonical coordinates* (3.119) *take the following diagonal form*

$$
\begin{aligned}
&\frac{\partial u_\sigma}{\partial t^{i,0}}=A_i(\sigma)\,\frac{\partial u_\sigma}{\partial x},\quad i\in\mathbb{Z},\quad \sigma\in\Sigma\\
&A_i(\sigma)=-p(\sigma)\left[\sigma\left(w^i(p)w'(p)\right)_{\geqslant 0}+(\sigma-1)\left(w^i(p)w'(p)\right)_{\leqslant -1}\right]_{p=p(\sigma)}\\
&\frac{\partial u_\sigma}{\partial t^{u,0}}=A_u(\sigma)\,\frac{\partial u_\sigma}{\partial x},\quad A_u(\sigma)=\frac{e^u}{p(\sigma)}.
\end{aligned}
\tag{4.39}
$$

Proof We derive the diagonal form (4.39) of the primary flows using the canonical coordinates (3.119). By the general definition [25] the primary time derivative of an arbitrary function f on M_0 can be written in the form [25]

$$
\frac{\partial f}{\partial t^{i,0}}=\frac{\partial}{\partial t_i}\cdot\frac{\partial f}{\partial x}
$$

where the product of tangent vectors $\partial/\partial t_i$ and $\partial f/\partial x$ has to be computed in the right hand side. The operator of multiplication by $\partial/\partial t_i$ becomes diagonal in the canonical coordinates with the eigenvalues

$$
\langle du(p),\frac{\partial}{\partial t_i}\rangle.
$$

Computation of this pairing with the help of (3.77) gives the needed expression. In a similar way, using (3.78) we prove the second formula in (4.39). This completes the proof of Proposition 4.4. □

Remark 4.5 *As it follows from the theory of Frobenius manifolds, the primary Hamiltonians are given by:*

$$H_{\alpha,0} = \int_{S^1} \frac{\partial F}{\partial t_\alpha}\, dx, \quad \alpha \in \mathbb{Z}$$
$$H_{u,0} = \int_{S^1} \frac{\partial F}{\partial u}\, dx, \quad H_{v,0} = \int_{S^1} \frac{\partial F}{\partial v}\, dx$$

Remark 4.6 *One can also easily obtain the diagonal form of the dispersionless Toda equations:*

$$\begin{aligned}
&\frac{\partial u_\sigma}{\partial s_n} = C_n(\sigma)\frac{\partial u_\sigma}{\partial x}\\
&\frac{\partial u_\sigma}{\partial \bar{s}_n} = \bar{C}_n(\sigma)\frac{\partial u_\sigma}{\partial x}\\
&C_n(\sigma) = \Big[\big(p\,\lambda'(p)\big)_{\geqslant 0}\Big]_{p=p(\sigma)}, \quad \bar{C}_n(\sigma) = \Big[\big(p\,\bar{\lambda}'(p)\big)_{<0}\Big]_{p=p(\sigma)}\\
&n = 1, 2, \ldots, \quad \sigma \in \Sigma.
\end{aligned} \tag{4.40}$$

4.3 The Principal Hierarchy

Let $\mathcal{V}$ be the operator defined by

$$\mathcal{V} := \frac{2-d}{2} - \nabla E$$

Given a set of flat coordinates v^α, the deformed flat coordinates $\theta_\alpha(\kappa)$ are determined by the set of equations:

$$\partial_\lambda, \partial_\mu \theta_\alpha(\kappa) = \kappa c^\nu_{\lambda,\mu} \partial_\nu \theta_\alpha(\kappa) \tag{4.41}$$
$$\theta_\alpha(0) = v_\alpha = \eta_{\alpha,\beta} v^\beta \tag{4.42}$$

with the additional constraint for the matrix $\Theta(\kappa) := \eta^{\alpha,\nu}\partial_\nu\theta_\beta(\kappa)$ given by:

$$\kappa\partial_\kappa\Theta(\kappa) + [\Theta(\kappa), \mathcal{V}] = \kappa\mathcal{U}\Theta(\kappa) - \Theta(\kappa)R(\kappa) \tag{4.43}$$

here $\mathcal{U}$ is the multiplication by E operator, while $R(\kappa) = \sum_{k>0} R_k\kappa^k$ is a polynomial in κ with coefficients in the constant matrices ring.

We want to write the equations for the deformed flat coordinates $\theta_j(\kappa), \theta_u(\kappa)$ and $\theta_v(\kappa)$ corresponding to the deformation of t_j, u, v. We first need the following

Lemma 4.7 *For M_{2DT} the operator $\mathcal{V}$ in the flat coordinates $(\mathbf{t}, u, v)$ is described by a diagonal matrix with coefficients:*

$$\mathcal{V}_u^u = \frac{1}{2} \quad \mathcal{V}_v^v = -\frac{1}{2} \quad \mathcal{V}_k^k = \frac{1}{2} + k \tag{4.44}$$

while the operator $\mathcal{U}$ has components given by the following generating functions:

$$\begin{aligned}
\mathcal{U}_y^x &= -yt'(y)\phi(x,y) - xt'(x)\phi(y,x) + (\pi(x) - x\pi'(x))\delta(x-y) && (4.45)\\
\mathcal{U}_x^u &= \mathcal{U}_v^x = -xt'(x) && (4.46)\\
\mathcal{U}_u^x &= \mathcal{U}_x^v = -\frac{e^u}{z(x)}(xt'(x)+2) && (4.47)\\
\mathcal{U}_v^u &= 2 \quad \mathcal{U}_u^v = 2e^u(\frac{1}{2\pi i}\oint \frac{dw}{z(w)} - 1) && (4.48)\\
\mathcal{U}_u^u &= \mathcal{U}_v^v = v && (4.49)
\end{aligned}$$

where:

$$\phi(x,y) := \frac{1}{2}\left(\mathrm{Li}_0\left(\frac{z(z(x))}{z(z(y))}\right) - \mathrm{Li}_0\left(\frac{z(z(y))}{z(z(x))}\right) + 1\right) \tag{4.50}$$

$$\pi(w) := \frac{1}{2}\oint \left(\mathrm{Li}_1\left(\frac{z(w)}{z(\sigma)}\right) + \mathrm{Li}_1\left(\frac{z(\sigma)}{z(w)}\right)\right) d\sigma + \frac{1}{2}t^{-1} - z(w) + v + \frac{e^u}{z(w)} \tag{4.51}$$

Next we define the generating functions:

$$\begin{array}{ll}
\theta_j(\kappa)_x := \sum_{\lambda\in\mathbb{Z}} \frac{\partial\theta_j(\kappa)}{\partial t^\lambda} x^{-(\lambda+1)} & \theta_j(\kappa)_u := \frac{\partial\theta_j(\kappa)}{\partial u}\\
\theta_j(\kappa)_v := \frac{\partial\theta_j(\kappa)}{\partial v} & \theta_j(\kappa)_{x,y} := \sum_{\lambda,\mu\in\mathbb{Z}} \frac{\partial^2\theta_j(\kappa)}{\partial t^\lambda \partial t^\mu} x^{-(\lambda+1)} y^{-(\mu+1)}\\
\theta_j(\kappa)_{x,u} := \sum_{\lambda\in\mathbb{Z}} \frac{\partial^2\theta_j(\kappa)}{\partial t^\lambda \partial u} x^{-(\lambda+1)} & \theta_j(\kappa)_{x,v} := \sum_{\lambda\in\mathbb{Z}} \frac{\partial^2\theta_j(\kappa)}{\partial t^\lambda \partial v} x^{-(\lambda+1)}\\
\theta_j(\kappa)_{u,u} := \frac{\partial^2\theta_j(\kappa)}{\partial u \partial u} & \theta_j(\kappa)_{u,v} := \frac{\partial^2\theta_j(\kappa)}{\partial u \partial v}\\
\theta_j(\kappa)_{v,v} := \frac{\partial^2\theta_j(\kappa)}{\partial v \partial v} &
\end{array} \tag{4.52}$$

and analogous generating functions for $\theta_u(\kappa)$ and $\theta_v(\kappa)$.

Proposition 4.8 *The deformed flat coordinates $\theta_j(\kappa), \theta_u(\kappa), \theta_v(\kappa)$ are determined by the equations:*

$$\theta_j(\kappa)_{\lambda,\mu} = \kappa \oint c_{\lambda,\mu}^x \theta_j(\kappa)_x dx + \kappa c_{\lambda,\mu}^u \theta_j(\kappa)_u + \kappa c_{\lambda,\mu}^v \theta_j(\kappa)_v \quad \lambda,\mu \in \{x,u,v\} \tag{4.53}$$

$$\begin{aligned}
[j + \kappa\partial_\kappa - x\partial_x]\,\theta_j(\kappa)_x &= \kappa \oint \mathcal{U}_y^x \theta_j(\kappa)_y dy + \kappa \mathcal{U}_u^x \theta_j(\kappa)_v + \kappa \mathcal{U}_v^x \theta_j(\kappa)_u && (4.54)\\
[j + \kappa\partial_\kappa]\,\theta_j(\kappa)_v &= \kappa \oint \mathcal{U}_y^u \theta_j(\kappa)_y dy + \kappa \mathcal{U}_u^u \theta_j(\kappa)_v + \kappa \mathcal{U}_v^u \theta_j(\kappa)_u && (4.55)\\
[j + 1 + \kappa\partial_\kappa]\,\theta_j(\kappa)_u &= \kappa \oint \mathcal{U}_y^v \theta_j(\kappa)_y dy + \kappa \mathcal{U}_u^v \theta_j(\kappa)_v + \kappa \mathcal{U}_v^v \theta_j(\kappa)_u. && (4.56)
\end{aligned}$$

$$
\begin{aligned}
[\kappa\partial_\kappa - x\partial_x]\,\theta_u(\kappa)_x &= \kappa\oint \mathcal{U}^x_y\theta_u(\kappa)_y dy + \kappa\mathcal{U}^x_u\theta_u(\kappa)_v + \kappa\mathcal{U}^x_v\theta_u(\kappa)_u && (4.57)\\
[\kappa\partial_\kappa]\,\theta_u(\kappa)_v &= \kappa\oint \mathcal{U}^u_y\theta_u(\kappa)_y dy + \kappa\mathcal{U}^u_u\theta_u(\kappa)_v + \kappa\mathcal{U}^u_v\theta_u(\kappa)_u && (4.58)\\
[1+\kappa\partial_\kappa]\,\theta_u(\kappa)_u &= \kappa\oint \mathcal{U}^v_y\theta_u(\kappa)_y dy + \kappa\mathcal{U}^v_u\theta_u(\kappa)_v + \kappa\mathcal{U}^v_v\theta_u(\kappa)_u. && (4.59)
\end{aligned}
$$

$$
\begin{aligned}
[-1+\kappa\partial_\kappa - x\partial_x]\,\theta_v(\kappa)_x &= \kappa\oint \mathcal{U}^x_y\theta_v(\kappa)_y dy + \kappa\mathcal{U}^x_u\theta_v(\kappa)_v + \kappa\mathcal{U}^x_v\theta_v(\kappa)_u && (4.60)\\
[-1+\kappa\partial_\kappa]\,\theta_v(\kappa)_v &= \kappa\oint \mathcal{U}^u_y\theta_v(\kappa)_y dy + \kappa\mathcal{U}^u_u\theta_v(\kappa)_v + \kappa\mathcal{U}^u_v\theta_v(\kappa)_u && (4.61)\\
[\kappa\partial_\kappa]\,\theta_v(\kappa)_u &= \kappa\oint \mathcal{U}^v_y\theta_v(\kappa)_y dy + \kappa\mathcal{U}^v_u\theta_v(\kappa)_v + \kappa\mathcal{U}^v_v\theta_v(\kappa)_u. && (4.62)
\end{aligned}
$$

Proof Simply rewrite equations 4.41 using formulae 4.44.

To write these formulae we need the generating functions for the structure coefficients in the flat coordinates:

$$
\begin{aligned}
c^p_{x,y} &= \phi(y,x)\delta(x-p)+\phi(x,y)\delta(y-p)+\phi(p,x)\delta(x-y)+ && (4.63)\\
&\quad +\pi'(x)\frac{z(x)}{z'(x)}\delta(x-p)\delta(y-p) && (4.64)\\
c^v_{x,y} &= c^y_{x,u} = \frac{e^u}{z(x)}\delta(x-y) && (4.65)\\
c^u_{x,y} &= c^y_{v,x} = \delta(x-y) && (4.66)\\
c^x_{u,u} &= c^v_{x,u} = -\frac{e^u}{z(x)} && (4.67)\\
c^v_{u,u} &= e^u\frac{1}{2\pi i}\oint\frac{dp}{z(p)} - e^u && (4.68)\\
c^v_{v,v} &= c^u_{v,u} = 1 && (4.69)
\end{aligned}
$$

All other generating functions are equal to zero. These generating functions are computed by taking the triple derivatives of the potential in the flat coordinates 3.93, raising an index, and then applying the scheme described in 3.29. For example:

$$
\frac{\partial^3 F}{\partial t_i\partial t_j\partial u} = \frac{e^u}{2\pi i}\oint_\Gamma \frac{w^{i+j}}{z(w)}dw
$$

raising the first index we get:

$$
c^i_{j,u} = \frac{e^u}{2\pi i}\oint_\Gamma \frac{w^j w^{-(i+1)}}{z(w)}dw
$$

and finally:

$$c^y_{x,u} = \sum_{i,j\in\mathbb{Z}} c^i_{j,u} x^{-(j+1)} y^i = \tag{4.70}$$

$$= \sum_{i,j\in\mathbb{Z}} \frac{e^u}{2\pi i} \oint_\Gamma \frac{x^{-(j+1)} w^j y^i w^{-(i+1)}}{z(w)} dw = \tag{4.71}$$

$$= \frac{e^u}{2\pi i} \oint_\Gamma \frac{\delta(x-w)\delta(y-w)}{z(w)} dw = \tag{4.72}$$

$$= \frac{e^u}{z(x)} \delta(x-y) \tag{4.73}$$

We now look for a solution to these equations

Definition 4.9 *We define the functions $\theta_j(\kappa)$ for $j \neq -1$:*

$$\theta_j(\kappa) := -\oint \frac{w^{j+1}}{j+1} \exp\left[\kappa\pi(w)\right] \frac{z'(w)}{z(w)} dw \qquad j \neq -1 \tag{4.74}$$

The generating functions of the derivatives of $\theta_j(\kappa)$ are given by:

$$\theta_j(\kappa)_x = +x^j \exp\left[\kappa\pi(x)\right] - \kappa \oint \frac{w^{j+1}}{j+1} \exp\left[\kappa\pi(w)\right] \phi(x,w) \frac{z'(w)}{z(w)} dw \tag{4.75}$$

$$\theta_j(\kappa)_u = -\kappa e^u \oint \frac{w^{j+1}}{j+1} \exp\left[\kappa\pi(w)\right] \frac{z'(w)}{z(w)^2} dw \tag{4.76}$$

$$\theta_j(\kappa)_v = +\kappa\theta_j(\kappa) \tag{4.77}$$

$$\theta_j(\kappa)_{u,u} = -\kappa e^u \oint \frac{w^{j+1}}{j+1} \exp\left[\kappa\pi(w)\right] \frac{z'(w)}{z(w)^2} dw - \tag{4.78}$$

$$-\kappa^2 e^{2u} \oint \frac{w^{j+1}}{j+1} \exp\left[\kappa\pi(w)\right] \frac{z'(w)}{z(w)^3} dw \tag{4.79}$$

$$\theta_j(\kappa)_{x,u} = +\kappa \frac{e^u}{z(x)} x^j \exp\left[\kappa\pi(x)\right] - \tag{4.80}$$

$$-\kappa^2 e^u \oint \frac{w^{j+1}}{j+1} \exp\left[\kappa\pi(w)\right] \phi(x,w) \frac{z'(w)}{z(w)^2} dw \tag{4.81}$$

$$\theta_j(\kappa)_{x,y} = +\kappa x^j \exp\left[\kappa\pi(x)\right] \phi(y,x) + \kappa y^j \exp\left[\kappa\pi(y)\right] \phi(x,y) + \tag{4.82}$$

$$+\kappa x^j \exp\left[\kappa\pi(x)\right] \left(\oint \phi(x,\sigma) d\sigma + z(x) - \frac{e^u}{z(x)} \right) \delta(x-y) - \tag{4.83}$$

$$-\kappa \oint \frac{w^{j+1}}{j+1} \exp\left[\kappa\pi(w)\right] \mathrm{Li}_{-1}\left(\frac{z(x)}{z(w)}\right) \frac{z'(w)}{z(w)} dw \delta(x-y) - \tag{4.84}$$

$$-\kappa \oint \frac{w^{j+1}}{j+1} \exp\left[\kappa\pi(w)\right] \mathrm{Li}_{-1}\left(\frac{z(w)}{z(x)}\right) \frac{z'(w)}{z(w)} dw \delta(x-y) - \tag{4.85}$$

$$-\kappa^2 \oint \frac{w^{j+1}}{j+1} \exp\left[\kappa\pi(w)\right] \phi(x,w)\phi(y,w) \frac{z'(w)}{z(w)} dw \tag{4.86}$$

Theorem 4.10 *The functions $\theta_j(\kappa)$ defined in 4.9 satisfy the equations 4.41, i.e. they are the deformed flat coordinates over the coordinates t_j, for $j \neq -1$.*

Proof The only equation that requires some attention is 4.53. Writing down the equation, after a lenghtly calculation one ends up with the equation:

$$\begin{aligned}&\frac{1}{2\pi i}\oint\frac{w^{j+1}}{j+1}\exp\left[\kappa\pi(w)\right]\left(\phi(x,w)\phi(y,w)-\right.\\&\left.-\phi(y,x)\phi(x,w)-\phi(x,y)\phi(y,w)\right)\frac{z'(w)}{z(w)}dw=\\=&\frac{1}{2\pi i}\oint\frac{w^{j+1}}{j+1}\exp\left[\kappa\pi(w)\right]\left(-\phi(x,w)\phi(\tau,x)+\right.\\&\left.+\phi(\tau,w)\phi(\tau,x)-\phi(w,x)\phi(\tau,w)\right)\frac{z'(w)}{z(w)}dw\ d\tau\delta(x-y)\end{aligned}\tag{4.87}$$

Using the identity:

$$\phi(x,w)\phi(y,w)-\phi(y,x)\phi(x,w)-\phi(x,y)\phi(y,w)=-\frac{1}{4}\left(\frac{z(w)}{z'(w)}\right)^2\delta(x-w)\delta(y-w)\tag{4.88}$$

we prove that both terms of the equation coincide with:

$$-\frac{1}{4}\frac{x^{j+1}}{j+1}\exp\left[\kappa\pi(x)\right]\frac{z(x)}{z'(x)}\delta(x-y)\tag{4.89}$$

and this concludes the proof. □

Theorem 4.11 *The $\theta_{j,p}$ for $p\geqslant 0$, $j\neq 1$ satisfy the following bihamiltonian recursion relation:*

$$\left\{,\bar\theta_{j,p}\right\}_2=(p+j+1)\left\{,\bar\theta_{j,p+1}\right\}_1\qquad\theta_{j,0}=t_j=\eta_{j,\lambda}t^\lambda\tag{4.90}$$

Lemma 4.12 *The Hamiltonians of the principal hierarchy associated with the Frobenius manifold M_{2DT} satisfy the following recursion*

$$\begin{aligned}\{\ \cdot\ ,H_{\alpha,p-1}\}_2&=(p+\alpha+2)\ \{\ \cdot\ ,H_{\alpha,p}\}_1\\\{\ \cdot\ ,H_{u,p-1}\}_2&=(p+1)\ \{\ \cdot\ ,H_{u,p}\}_1\\\{\ \cdot\ ,H_{v,p-1}\}_2&=p\ \{\ \cdot\ ,H_{v,p}\}_1+2\ \{\ \cdot\ ,H_{u,p}\}_1\,.\end{aligned}\tag{4.91}$$

Proof follows from the standard formalism of the theory of Frobenius manifolds [28, 25] taking into account the quasi homogeneity degrees of the flat coordinates

$$\deg t_\alpha=-(\alpha+1),\quad \deg v=1,\quad \deg u=0,\ \ \deg e^u=2.$$

□

Corollary 4.13 *The Hamiltonians* (4.16) *of the dispersionless 2D Toda hierarchy commute, with respect to both the Poisson brackets with all Hamiltonians of the Principal Hierarchy.*

Proof Let us prove that $\{H_n, H_{\alpha,p}\}_1 = 0$. Using recursions (4.14) and (4.91) we obtain

$$\{H_n, H_{\alpha,p}\}_1 = \{H_{n-1}, H_{\alpha,p}\}_2 = (p+\alpha+3)\ \{H_n, H_{\alpha,p+1}\}_1 .$$

Iterating we arrive at the equation

$$\{H_n, H_{\alpha,p}\}_1 = \text{const}\,\{H_{-1}, H_{\alpha,q}\}_1$$

for some constant coefficient and some $q > p$. The Poisson bracket in the right hand side vanishes since

$$H_{-1} = -\int_{S^1} (t_{-1} + v)dx$$

is a Casimir of the first Poisson bracket. Similarly, using

$$\bar{H}_{-1} = \int_{S^1} v\, dx$$

we prove that

$$\{\bar{H}_n, H_{\alpha,p}\}_1 = 0.$$

Commutativity of the Hamiltonians H_n, $\bar{H}_n$ with other Hamiltonians of the principal hierarchy with respect to both Poisson brackets can be proved in a similar manner. This completes the proof of the Corollary. □

Corollary 4.14 *The following Hamiltonian densities:*

$$\theta_{j,p} \quad := \quad -\oint_\Gamma \frac{(\lambda+\bar{\lambda})(z)^{j+1}}{j+1} \frac{(\bar{\lambda}-\lambda)^p}{2^p p!} \frac{dz}{z} \qquad j \neq -1, p \geqslant 0 \tag{4.92}$$

$$\theta_{\bar{\lambda},p} \quad := \quad \oint_\Gamma \frac{\bar{\lambda}(z)^{(p+1)}}{(p+1)!} \frac{dz}{z} \qquad p \geqslant 0 \tag{4.93}$$

$$\theta_{\lambda,p} \quad := \quad \oint_\Gamma \frac{\lambda(z)^{(p+1)}}{(p+1)!} \frac{dz}{z} \qquad p \geqslant 0 \tag{4.94}$$

define a set of mutually commuting Hamiltonians with respect to the bihamiltonain structure induced by M_{2DT}. The resulting integrable system is an extension of the 2D-Toda hierarchy.

Proof One simply has to take the power series expansion in κ of the deformed flat coordinate given in Definition 4.9. □

Chapter 5

Conclusions and Outlook

In the present thesis we introduced a structure of infinite-dimensional semi simple Frobenius manifold on the space of pairs of symbols of Lax operators of the 2D Toda hierarchy provided validity of certain analyticity conditions for the symbols. We demonstrated that the rich geometry known from the finite-dimensional theory extends to the infinite-dimensional case. The analytic conditions for the symbols were crucial in order to establish the main properties of these geometrical structures. We also showed that, starting from our infinite dimensional Frobenius manifold, a new hierarchy can be defined which extends the classical 2D Toda hierarchy. We conclude by suggesting possible future research projects:

- Project 1: Computation of the full principal hierarchy. This will give the complete set of first integrals of 2D Toda. I would expect that this requires an extension of the algebra of Lax symbols to symbols where the $\log p$ appears (symbols of differential-difference operators, which already appeared in the Extended Toda hierarchy formulation [10]). Properties of solutions of the hierarchy and their tau-functions.

- Project 2: Study of the dispersive corrections to the hierarchy. In particular we plan to apply this tools to the Laplacian Growth Problem (interface dynamics of a domain filled with water in a plane filled with oil). In this setting, Wiegmann, Zabrodin and collaborators [60, 51] have proved that the idealized Laplacian Growth is described by a reduction of the dispersionless 2D Toda hierarchy. One issue is that in finite time for generic smooth initial data, the idealized dynamics develops singularities on the border. There is a growing attention around the possibility to add dispersive terms corrections to overcome this problem.

- Project 3: Reductions of the 2D Toda Frobenius Manifold structure to Toda bigraded submanifolds. By a reduction of type $\lambda^m = \bar{\lambda}^n$ for m, n natural numbers, the 2D Toda hierarchy reduces to the so called Toda bigraded hierarchies [12]. These hierarchies were related to a Frobenius manifold in [9]. Note that the Lax symbol of a bigraded Toda hierarchy is a multivalued function on the unit circle. This imply that if we want to include these reductions in our scheme we should first generalize the functional space M_{2DT} to a suitable space of multivalued functions. It would be interesting to see then if by reducing the Frobenius manifold structure of 2D toda we get the known Frobenius manifold structures for the bigraded Toda hierarchies.

- Project 4: Frobenius manifold theory for the Genus 0 universal Whitham hierarchy. It is well known that the 2D Toda hierarchy is a particular case of the genus universal 0 Whitham hierarchy [47]. I expect that our construction would be extended in order to define a Frobenius manifold associated to the Whitham hierarchy. The formulae for the metric and the product seem to have a straightforward generalization to an arbitrary number of Lax symbols.

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