On Feedback Strategies in Control Problems and Differential Games

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Contents

Introduction

1 General Introduction

This thesis consists of two parts. In the first part we study the existence and uniqueness of Nash equilibrium solutions for a class of infinite horizon, non-cooperative differential games. The second part is concerned with the construction of nearly-optimal patchy feedbacks, for problems of optimal control.

To the former is devoted Chapter 1. Postponing to the following Section 2 a complete introduction to this topic, we briefly summarize here the results proven in [10, 27]. We study a non-cooperative differential game for two players, with infinite horizon and exponentially discounted payoffs. The existing literature on the subject has been mainly concerned either with zero-sum games, or with a special class of non-zero sum games having linear dynamics and quadratic cost functionals. In the first case, optimal feedback strategies can be found in terms of the viscosity solution of the corresponding scalar Hamilton-Jacobi equation. In the second (linear-quadratic) case, the corresponding system of Hamilton-Jacobi equations for the value functions reduces to a finite-dimensional system of Riccati equations. Our present goal is to push the analysis a few steps beyond these two basic cases, and study a more general nonlinear system of H-J equations, describing non-cooperative Nash equilibrium. The main results refer to a non-cooperative differential game for two players in one space dimension. Roughly speaking, the following holds:

- If the cost functions is a smooth and small perturbation of a linear cost, as shown in [10], then the game admits a Nash equilibrium solution in feedback form. Moreover, if the sum of the derivatives of the two cost functions is bounded away from zero, these feedback strategies are unique.

- On the other hand, if players' costs are not smooth, then various instabilities can arise. This means that some games can have infinitely many Nash equilibria, while some slightly different games can have no equilibrium at all, as described in [27].

These negative results, in the non-smooth case, indicate that the problem of finding solutions within the framework of Nash equilibria is not well posed, in general. Similar results were found in [13], for finite horizon problems. They provide at least a glimpse of the great complexity of this problem for a general non-cooperative game with nonlinear costs.

Chapter 2 is devoted to the construction of nearly optimal patchy feedbacks. The main results presented here first appeared in [11]. We consider a nonlinear control system of the form

$$\dot{x} = f(x, u)\,,$$

and a cost functional

$$J = \int_0^\tau L(x(t), u(t))\, dt + \psi(x(\tau))\,, \tag{1.1}$$

which we wish to minimize among all controls $t \mapsto u(t)$ taking values in some given set $\mathbf{U} \subset \mathbb{R}^m$. In an ideal situation, one can find a terminal set S and a continuous feedback control $x \mapsto U(x)$ defined for $x \notin S$, such that every trajectory of the corresponding O.D.E.

$$\dot{x} = f(x, U(x)) \tag{1.2}$$

is optimal for the cost criterion (1.1). It is well known, however, that in general no continuous optimal feedback exists. On the other hand, according to the results in [4], it is possible to construct a piecewise constant feedback which is nearly optimal. As shown in [1], controls in the class of "patchy feedback" enjoy various useful properties. In particular, even if the right hand side of (1.2) is discontinuous, the Cauchy problem has at least one forward solution and at most one backward solution, in Carathéodory sense. Up to now, patchy feedbacks have been constructed in two different ways. Either, as in [1], by patching together families of open-loop controls. Or else, as in [4], starting with a suitable regularization of the value function, which was assumed known a-priori. The main purpose of the present analysis, taken from [11], is to construct patchy feedback *from scratch*, i.e. without a priori knowledge of the value function. Our basic procedure constructs at the same time a piecewise constant nearly optimal feedback, together with a piecewise smooth approximation to the value function. The basic step is repeated inductively on higher and higher level sets, eventually covering the entire domain.

2 Infinite Horizon Noncooperative Differential Games

Problems of optimal control, or zero-sum differential games, have been the topic of an extensive literature. In both cases, an effective tool for the analysis of optimal solutions is provided by the *value function*, which satisfies a scalar Hamilton-Jacobi equation. Typically, this first order P.D.E. is highly non-linear and solutions may not be smooth. However, thanks to a very effective comparison principle, the existence and stability of solutions can be achieved in great generality by the theory of viscosity solutions, see [6] and references therein.

In comparison, much less is known about non-cooperative differential games. In a Nash equilibrium solution, the value functions for the various players now satisfy not a scalar but a system of Hamilton-Jacobi equations [19]. For this type of nonlinear systems, no general theorems on the existence or uniqueness of solutions are yet known. A major portion of the literature is concerned with games having linear dynamics and quadratic costs, see [18] for a comprehensive treatment and a complete bibliography. In this case, solutions are sought among quadratic functions. This approach effectively reduces the P.D.E. problem to a quadratic system of O.D.E's. Well known results on Riccati equations can then be applied. However, it does not provide any insight on the stability (or instability) of the solutions w.r.t. small non-linear perturbations.

On the other hand, in the present more general case one has no established techniques to rely on. We recall here some of the few known results.

In [12], a class of non-cooperative games with general terminal payoff was studied, in one space dimension. Relying on recent advances in the theory of hyperbolic systems of conservation laws (see [8]), some results on the existence and stability of Nash equilibrium solutions could be obtained. On the other hand, for games in several space dimensions and also in various one-dimensional cases, the analysis in [13] shows that the corresponding H-J system is not hyperbolic, hence ill posed.

In the present work we begin exploring a class of non-cooperative differential games in infinite time horizon, with exponentially discounted costs. Namely, we consider a game with dynamics

$$\dot{x} = \sum_{i=1}^{m} \alpha_i \,, \qquad x(0) = y \,, \tag{2.1}$$

where each player acts on his control α_i to minimize an exponentially discounted cost of the form

$$J_i(\alpha_i) \doteq \int_0^\infty e^{-t}\, \psi_i\big(x(t),\, \alpha_i(t)\big)\, dt \,. \tag{2.2}$$

In one space dimension, the corresponding value functions satisfy a time-independent system of implicit O.D.E's. Global solutions are sought within a class of absolutely continuous functions, imposing certain growth conditions as $|x| \to \infty$, and suitable admissibility conditions at points where the gradient u_x has a jump.

The dynamics of our system is very elementary, and the cost functions we consider are small perturbations of linear ones. However, already in this simple setting we find cases where the problem has unique solution, and cases where infinitely many solutions exist.

This richness of different situations reflects in some sense the results found in [15]. Indeed, the exact same dynamics was studied, in the finite horizon case, with only exit costs. Main differences between [15] and [10, 12, 13] lay in the concept of solution. The authors of [15] look for discontinuous feedback controls that not only leads to Nash equilibria, but also satisfies a sort of programming principle. This resulted in (uncountable) infinitely many solutions, at price of stronger assumptions on the final costs.

All these difficulties provide a glimpse of the extreme complexity of the problem, for general non-cooperative N-player games with non-linear cost functions.

A first attempt to study this problem in the infinite horizon setting, for two players, was made in [10]. The same simple game was considered, and it was proved that, depending on the monotonicity of the cost functions, very different situations could arise. Indeed, the HJ system in this case takes the following form

$$\begin{cases} u_1(x) = h_1(x) - u_1' u_2' - (u_1')^2/2 \,, \\[2ex] u_2(x) = h_2(x) - u_1' u_2' - (u_2')^2/2 \,. \end{cases} \tag{2.3}$$

But with a system of this form, we can end up with too many solutions. We find not only value functions u that leads to Nash equilibria in feedback form, but also solutions that does not represent equilibria of the game. It is then necessary to introduce a suitable concept of admissibility. In particular we say that a solution u is *admissible*, if u is a Carathéodory solution of (2.3), which grows at most linearly as $|x| \to \infty$ and satisfies suitable jump conditions in points where its derivatives are discontinuous. For such a kind of solutions, a *verification theorem* was proved: given an admissible solution u and denoted by u_i' the components of its derivatives, then $\alpha_i = -u_i'$ provide a Nash equilibrium solution in feedback form. In [10], it turned out that existence and uniqueness of admissible solution for (2.3) heavily depend on the choice of the costs.

First, suppose that both the cost functionals are increasing (resp. decreasing). This means that both players would like to steer the game in the same direction, namely the direction along which their costs decreases. In this case an admissible solution always exists, and it is also unique, provided a *small oscillations* assumption is satisfied. This existence result was in some sense expected, since this case corresponds, in the finite horizon setting, to the hyperbolic one studied in [12].

Suppose now that the cost functionals have opposite monotonicity. This means that the players have conflicting interests, since they would like the game to go in different directions. In this case it is known, see [13], that the finite horizon problem is in general ill-posed. On the same line, for our game, it is enough to consider two linear functionals with opposite slopes (say $k, -k$, for any real number $k \neq 0$) to find infinitely many admissible solutions, and hence infinitely many Nash equilibria in feedback form. Nevertheless, quite surprisingly, it's still possible to recover existence and uniqueness of admissible solutions to (2.3) in the case of costs that are small perturbation of linear ones, but with slopes that are not exactly opposite.

While the cost functionals considered in [10] were a small perturbation of affine costs, in [27] we studied a wider class of cost functions. Motivated by the theory of hyperbolic systems [8], we considered piecewise linear cost functionals, whose derivative has jumps. This setting is a natural first step towards the analysis of existence and uniqueness of Nash equilibrium solutions for non-linear costs.

Again, as in [10], we reached different results depending on the signs chosen for h_i'. Indeed, as it will be showed in the second part of Chapter 1, if we are in the cooperative situation for all x, we can still recover a unique admissible solution for (2.3). On the other hand, any change in the behavior of the costs will translate in some sort of instability of the game, leading either to infinitely many admissible solutions, or to one unique admissible solution, or even to no admissible solution at all, only depending on the particular choices of the slopes h_i'.

In conclusion, this great variety of arising situations seems to suggest that the presented approach is not the most suitable one to deal with the intrinsic issues of the problem. In particular, we can provide examples of very simple differential games where no Carathéodory solution with sublinear growth at infinity exists. Recalling that, in the case of smooth costs (see [10]), this class of solutions was exactly the right one to find Nash equilibria in feedback form, our study strongly suggest that a different approach is needed: either to look for Pareto optima, as in [13], or to introduce some other relaxed concept of equilibrium.

The outline of Chapter 1 is the following. In Section 1, we introduce the class of differential games we deal with. In Section 2, the concept of admissible solution is presented and it is proved that to any admissible solution there corresponds a Nash equilibrium strategy in feedback form. In Section 3 we specialize our study to 2-players games in order to address, in Sections 4–5, the question of existence and uniqueness of admissible solutions in the case of smooth cost functionals. In Section 6, we approach the case of non-smooth costs. Namely, we consider piecewise linear costs, with a finite number of jumps in their derivatives. Finally, Sections 7–8–9 deal with existence and uniqueness of admissible solutions in these cases of non-smooth costs.

3 Nearly optimal patchy controls in feedback form

Consider an optimization problem for a nonlinear control system of the form

$$\begin{aligned} \dot{x} &= f(x,u) \\ x(0) &= y \end{aligned} \qquad u(t) \in \mathbf{U}\,, \tag{3.1}$$

where $x \in \mathbb{R}^n$ describes the state of the system, the upper dot denotes a derivative w.r.t. time, while $\mathbf{U} \subset \mathbb{R}^m$ is the set of admissible control values.

A classical problem is the existence of *optimal* feedback controls, i.e. if it is possible to construct a feedback control $u = U(x)$ such that all the trajectories of

$$\dot{x} = f(x, U(x))\,, \tag{3.2}$$

are optimal for the problem

$$\min \left\{ \psi\big(x(T)\big) + \int_0^T L\big(x(t),\, u(t)\big)\, dt \right\}\,. \tag{3.3}$$

We assume that $f \colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz continuous and satisfies

$$\big|f(x,u)\big| \leq C\big(1 + |x|\big), \qquad \forall\, u \in U\,, \tag{3.4}$$

that both the terminal cost $\psi : \mathbb{R}^n \mapsto \mathbb{R}$ and the running cost $L : \mathbb{R}^n \times \mathbf{U} \mapsto \mathbb{R}$ are continuous and strictly positive, say

$$\psi(x) \geq c_0 > 0\,, \qquad L(x,u) \geq c_0 > 0 \qquad \forall\; x \in \mathbb{R}^n, \quad u \in \mathbf{U}\,. \tag{3.5}$$

Moreover,

$$\lim_{|x| \to \infty} \psi(x) = \infty\,. \tag{3.6}$$

A strategy to attack this optimal control problem (see [24, 32]), is to investigate an optimal "synthesis", which is just a collection of optimal trajectories not necessarily arising from a feedback control. The existence and the structure of an optimal synthesis has been the subject of a large body of literature on nonlinear control. At present, a complete description is known for time optimal planar systems of the form

$$\dot{x} = f(x) + g(x)\,u \qquad u \in [-1,1]\,, \qquad x \in \mathbb{R}^2\,,$$

see [7] and the references therein. For more general classes of optimal control problems, or in higher space dimensions, the construction of an optimal synthesis faces severe difficulties.

On one hand, the optimal synthesis can have an extremely complicated structure, and only few regularity results are presently known (see [21]). Already for systems in two space dimensions, an accurate description of all generic singularities of a time optimal synthesis involves the classification of eighteen topological equivalence classes of singular points [24, 25]. In higher dimensions, an even larger number of different singularities arises, and the optimal synthesis can exhibit pathological behavior such as the famous "Fuller phenomenon" (see [22, 33]), where every optimal control has an infinite number of switchings.

On the other hand, even in cases where a regular synthesis exists, the performance achieved by the optimal synthesis may not be robust. In other words, small perturbations can greatly affect the behavior of the synthesis (e.g. see Example 5.3 in [26]).

Because of the difficulties faced in the construction of an optimal syntheses, it seems natural to slightly relax our requirements, and look for *nearly-optimal feedback controls* instead. Within this wider class, one can hope to find feedback laws with a simpler structure and better robustness properties than a regular synthesis.

But we immediately face a new theoretical obstacle. Indeed, it is well known that, in general, it is not possible to construct a nearly-optimal feedback law $u = U(x)$ among a class of continuous control. As shown in Example 1.1 in [23] or Example 2 in [9], restricting the study to continuous feedbacks leads to topological obstructions completely similar to the ones found in controllability problems [31, 30, 14].

Hence, we necessarily have to deal with discontinuous feedbacks $U(x)$ and corresponding discontinuous right hand side in (3.2). Therefore, it becomes essential to provide suitable definitions of "generalized solutions" for discontinuous O.D.E.s. Notice that the concepts of Filippov and Krasovskii generalized solutions [5], frequently encountered in the literature, are not suitable in this setting: indeed, these generalized solutions form a closed and connected set; thus the same obstructions to the existence of a continuous feedback are recovered.

Two approaches were followed in the literature to overcome this issue.

1. One can allow any arbitrary feedback control $u = U(x)$. In this case, we cannot be sure to still have Carathéodory solutions and hence a new concept of generalized solutions must be introduced for O.D.E.s with an arbitrary measurable right hand side.

2. Otherwise one can restrict the problem to a particular class of discontinuous feedbacks so that Carathéodory forward solutions always exist. Then we have to prove that a nearly-optimal feedback control exists within this class.

The first approach lead, in particular, to the concept of "sample-and-hold" solutions and Euler solutions (limits of sample-and-hold solutions), which were successfully implemented both within the context of stabilization problems [16, 28, 29] and of nearly-optimal feedbacks [17, 20, 23]. A drawback of this approach is that, as illustrated by Example 5.3 and Example 5.4 in [26], *arbitrary* discontinuous feedback can generate too many trajectories, some of which fail to be optimal. In fact, Example 5.3 in [26] shows that the set of Carathéodory solutions of the optimal closed-loop equation (3.2) contains, in addition to all optimal trajectories, some other arcs that are not optimal. Moreover, Example 5.4 in [26] exhibits an optimal control problem in which the optimal trajectories are Euler solutions, but the closed-loop equation (3.2) has many other Euler solutions which are not optimal.

The second approach was followed in [1, 2, 3, 4], through the introduction of *patchy feedbacks*. These controls, that are piecewise constant in the state space $\mathbb{R}^n$, were first introduced in [1] in order to study asymptotic stabilization problems. It turned out that the corresponding Cauchy problem always has at least one forward and at most one backward Carathéodory solutions.

Moreover, these solutions enjoy important robustness properties [2, 3], which are particularly relevant in many practical situations. Indeed, one of the main reasons for using a state feedback is precisely the fact that open loop controls are usually very sensitive to disturbances. Namely, it was proven in [2] that a patchy feedback is “fully robust” with respect to perturbation of the external dynamics, and to measurement errors having sufficiently small total variation, so to avoid the chattering behavior that may arise at discontinuity points.

Finally, it was proven in [4] that time nearly-optimal patchy feedbacks exist. Indeed, if we set $T(y)$ the minimum time needed to steer the system from the state $y \in \mathbb{R}^n$ to the origin, then every initial state y can be steered inside an ε-neighborhood of the origin within time $T(y) + \varepsilon$ using a patchy feedback, for any fixed $\varepsilon > 0$.

In all previous works, patchy feedbacks were constructed either by patching together piecewise constant open-loop controls as in [1], or, as in [4], relying on the a-priori knowledge of the value function V. We recall that this is defined as

$$V(y) \doteq \inf_{u(\cdot)} \left\{ \psi\big(x(T)\big) + \int_0^T L\big(x(t),\, u(t)\big)\, dt \right\} ,$$

where the minimization is taken over all $T \geq 0$ and all control functions $u : [0, T] \mapsto \mathbf{U}$ such that the trajectory of (3.1) satisfies the terminal constraint $|x(T)| < \varepsilon$.

Aim of [11] and of the results in Chapter 2 is to develop an algorithm that produces a nearly-optimal patchy feedback “starting from scratch”, i.e. without any a-priori information about the optimal trajectories. Both the patchy feedback and an approximate value function will be constructed simultaneously, working iteratively on higher and higher level sets.

This provides a very useful construction, that may be used in all those applications in which it is particularly difficult to derive explicitly the value function of the optimal control problem.

The outline of Chapter 2 will be the following. In Section 1 we recall the optimality result from [4] and state our main result on the construction of a general nearly optimal patchy feedback. Section 2 will be focused on the introduction of the basic concepts of patchy vector fields and patchy feedback controls, and on a brief overview of the known robustness results from [1, 2, 3]. Finally, in Section 3 we sketch the proof of the main result, which is still a work in progress and it will soon appear in [11].

Chapter 1

Infinite Horizon Noncooperative Differential Games

1 Basic definitions

Consider an m-persons non-cooperative differential game, with dynamics

$$\dot{x} = \sum_{i=1}^{m} f_i(x, \alpha_i), \qquad \alpha_i(t) \in A_i\,, \qquad x \in \mathbb{R}^n. \tag{1.1}$$

Here $t \mapsto \alpha_i(t)$ is the control chosen by the i-th player, within a set of admissible control values $A_i \subseteq \mathbb{R}^k$. We will study the **discounted, infinite horizon problem**, where the game takes place on an infinite interval of time $[0, \infty[$, and each player has only a running cost, discounted exponentially in time. More precisely, for a given initial data

$$x(0) = y \in \mathbb{R}^n\,, \tag{1.2}$$

the goal of the i-th player is to minimize the functional

$$J_i(\alpha_i) \doteq \int_0^\infty e^{-t}\, \psi_i\big(x(t),\, \alpha_i(t)\big)\, dt\,, \tag{1.3}$$

where $t \mapsto x(t)$ is the trajectory of (1.1). By definition, an m-tuple of feedback strategies $\alpha_i = \alpha_i^*(x)$, $i = 1, \ldots, m$, represents a *Nash non-cooperative equilibrium solution* for the differential game (1.1)-(1.2) if the following holds. For every $i \in \{1, \ldots, m\}$, the feedback control $\alpha_i = \alpha_i^*(x)$ provides a solution to the optimal control problem for the i-th player,

$$\min_{\alpha_i(\cdot)} \; J_i(\alpha_i)\,, \tag{1.4}$$

where the dynamics of the system is

$$\dot{x} = f_i(x, \alpha_i) + \sum_{j \neq i} f_j(x, \alpha_j^*(x)), \qquad \alpha_i(t) \in A_i\,. \tag{1.5}$$

More precisely, we require that, for every initial data $y \in \mathbb{R}$, the Cauchy problem

$$\dot{x} = \sum_{j=1}^{m} f_j\big(x, \alpha_j^*(x)\big)\,, \qquad x(0) = y\,, \tag{1.6}$$

should have at least one Carathéodory solution $t \mapsto x(t)$, defined for all $t \in [0, \infty[$. Moreover, for every such solution and each $i = 1, \ldots, m$, the cost to the i-th player should provide the minimum for the optimal control problem (1.4)–(1.5). We recall that a Carathéodory solution is an absolutely continuous function $t \mapsto x(t)$ which satisfies the differential equation in (1.6) at almost every $t > 0$.

Nash equilibrium solutions in feedback form can be obtained by studying a related system of P.D.E's. Assume that a value function $u(y) = (u_1, \ldots, u_n)(y)$ exists, so that $u_i(y)$ represents the cost for the i-th player when the initial state of the system is $x(0) = y$ and the strategies $\alpha_1^*, \ldots, \alpha_m^*$ are implemented. By the theory of optimal control, see for example [6], on regions where u is smooth, each component u_i should provide a solution to the corresponding scalar Hamilton-Jacobi-Bellman equation. The vector function u thus satisfies the stationary system of equations

$$u_i(x) = H_i(x,\, \nabla u_1, \ldots, \nabla u_m), \tag{1.7}$$

where the Hamiltonian functions H_i are defined as follows. For each $p \in \mathbb{R}^n$, assume that there exists an optimal control value $\alpha_j^*(x, p)$ such that

$$p \cdot f_j\big(x,\, \alpha_j^*(x,p)\big) + \psi_j\big(x,\, \alpha_j^*(x,p)\big) = \min_{a \in A_j} \big\{p \cdot f_j(x,a) + \psi_j(x,a)\big\}. \tag{1.8}$$

Then

$$H_i(x,\, p_1, \ldots, p_m) \doteq p_i \cdot \sum_{j=1}^{m} f_j\big(x,\, \alpha_j^*(x,p_j)\big) + \psi_i\big(x,\, \alpha_i^*(x,p_i)\big). \tag{1.9}$$

A rich literature is currently available on optimal control problems and on viscosity solutions to the corresponding scalar H-J equations. However, little is yet known about non-cooperative differential games, apart from the linear-quadratic case. Here, we begin a study of this class of differential games, with two players in one space dimension. Our main interest is in the existence, uniqueness and stability of Nash equilibrium solutions in feedback form.

When x is a scalar variable, (1.7) reduces to a system of implicit O.D.E's:

$$u_i = H_i(x, u_1', \ldots, u_m'). \tag{1.10}$$

In general, this system will have infinitely many solutions defined on the whole $\mathbb{R}$. To single out a (hopefully unique) admissible solution, corresponding to a Nash equilibrium for the differential game, additional requirements must be imposed. These are of two types:

(i) Asymptotic growth conditions as $|x| \to \infty$.

(ii) Jump conditions, at points where the derivative u' is discontinuous.

To fix the ideas, consider a game with the simple dynamics

$$\dot{x}(t) = \alpha_1(t) + \cdots + \alpha_m(t), \tag{1.11}$$

and with cost functionals of the form

$$J_i(\alpha) \doteq \int_0^\infty e^{-t}\left[h_i\big(x(t)\big) + k_i\big(x(t)\big)\,\frac{\alpha_i^2(t)}{2}\right] dt. \tag{1.12}$$

We shall assume that the functions h_i, k_i are smooth and satisfy

$$\big|h_i'(x)\big| \leq C\,, \qquad\qquad \frac{1}{C} \leq k_i(x) \leq C\,, \tag{1.13}$$

for some constant $C > 0$. Notice that in this case (1.8) yields $\alpha_i^* = -p_i/k_i$, hence (1.10) becomes

$$u_i = \left(\frac{u_i'}{2k_i(x)} - \sum_{j=1}^m \frac{u_j'}{k_j(x)}\right) u_i' + h_i(x)\,. \tag{1.14}$$

For a globally defined solution to the system of H-J equations (1.14), a natural set of admissibility conditions is formulated below.

Definition 1.1 *A function $u : \mathbb{R} \mapsto \mathbb{R}^m$ is called an admissible solution to the implicit system of O.D.E's (1.14) if the following holds.*
(A1) u is absolutely continuous. Its derivative u' satisfies the equations (1.14) at a.e. point $x \in \mathbb{R}$.
(A2) u has sublinear growth at infinity. Namely, there exists a constant C such that, for all $x \in \mathbb{R}$,

$$\big|u(x)\big| \leq C\,\big(1 + |x|\big)\,. \tag{1.15}$$

(A3) At every point $y \in \mathbb{R}$, the derivative u' admits right and left limits $u'(y+)$, $u'(y-)$. At points where u' is discontinuous, these limits satisfy at least one of the conditions

$$\sum_{i=1}^m \frac{u_i'(y+)}{k_i(y)} \leq 0 \qquad \textit{or} \qquad \sum_{i=1}^m \frac{u_i'(y-)}{k_i(y)} \geq 0\,. \tag{1.16}$$

Because of the assumption (1.13), the cost functions h_i are globally Lipschitz continuous. It is thus natural to require that the value functions u_i be absolutely continuous, with sub-linear growth as $x \to \pm\infty$. The motivation for the assumption (A3) is quite simple. Recalling that the feedback controls are $\alpha_i^* = -u_i'/k_i$, the condition (1.16) provides the existence of a local solution to the Cauchy problem

$$\dot{x} = -\sum_{i=1}^m \frac{u_i'(x)}{k_i(x)}\,, \qquad\qquad x(0) = y \tag{1.17}$$

forward in time. In the opposite case, solutions of the O.D.E. would approach y from both sides, and be trapped. As will be proved in the next section, the assumptions (A1)–(A3) together yield the existence of a global solution to (1.17), for every initial data $y \in \mathbb{R}$. Therefore, the functions $u_1, \ldots, u_m$ are indeed the costs for the various players, if the feedback strategies $\alpha_1^*, \ldots, \alpha_m^*$, are implemented.

Next, call $p_i^\pm \doteq u_i'(y\pm)$. By the equations (1.14) and the continuity of the functions u_i, h_i, k_i, one obtains the identities

$$\frac{(p_i^+)^2}{2k_i(y)} + \sum_{j\neq i} \frac{p_i^+ p_j^+}{k_j(y)} \;=\; \frac{(p_i^-)^2}{2k_i(y)} + \sum_{j\neq i} \frac{p_i^- p_j^-}{k_j(y)} \qquad\qquad i = 1, \ldots, m\,. \tag{1.18}$$

These are certainly satisfied when

$$u_i'(y+) + u_i'(y-) = 0 \qquad i = 1, \ldots, m\,. \tag{1.19}$$

In the case $m = 2$, it is easy to check that (1.19) yields the only non-trivial solution to the jump conditions (1.18). In this case, the assumptions (A3) yield

$$\frac{u_1'(y+)}{k_1(y)} + \frac{u_2'(y+)}{k_2(y)} \leq 0\,, \qquad u_i'(y-) = -u_i'(y+) \qquad (i = 1, 2)\,. \tag{1.20}$$

By (A1), the derivatives $p_i = u_i'$ are defined at a. e. point $x \in \mathbb{R}$. The optimal feedback controls $\alpha_i^* = -p_i/k_i$ are thus defined almost everywhere. We can use the further assumption (A3) and extend these functions to the whole real line by taking limits from the right:

$$\alpha^*(x) \doteq -\frac{u_i'(x+)}{k_i(x)}\,. \tag{1.21}$$

In this way, all feedback control functions will be right-continuous.

2 Solutions of the differential game

In this Section, we prove a *verification theorem*, i.e. we show that admissible solutions to the H-J equations yield a solution to the differential game. Moreover, we give a couple of examples showing the relevance of the assumptions (A2) and (A3).

Theorem 1.1 *Consider the differential game (1.11)–(1.12), with the assumptions (1.13). Let $u : \mathbb{R} \mapsto \mathbb{R}^m$ be an admissible solution to the systems of H-J equations (1.14), so that the conditions (A1)–(A3) hold. Then the controls (1.21) provide a Nash equilibrium solution in feedback form.*

Proof. The theorem will be proved in several steps.
1. First of all, setting

$$g(x) \;\doteq\; \sum_i \alpha_i^*(x) \;=\; -\sum_i \frac{u_i'(x)}{k_i(x)}\,, \tag{2.1}$$

we need to prove that the Cauchy problem

$$\dot{x}(t) = g\big(x(t)\big)\,, \qquad x(0) = y\,, \tag{2.2}$$

has a globally defined solution, for every initial data $y \in \mathbb{R}$. This is not entirely obvious, because the function g may be discontinuous. We start by proving the local existence of solutions.

CASE 1: $g(y) = 0$. In this trivial case $x(t) \equiv y$ is the required solution.

CASE 2: $g(y) > 0$. By right continuity, we then have $g(x) > 0$ for $x \in [y,\ y+\delta]$, for some $\delta > 0$. This implies the existence of a (unique) strictly increasing solution $x : [0, \varepsilon] \mapsto \mathbb{R}$, for some $\varepsilon > 0$.

CASE 3: $g(y) < 0$. By the admissibility conditions (1.16), this implies that g is strictly negative in a left neighborhood of y. Therefore the Cauchy problem (2.2) admits a (unique) strictly decreasing solution $x : [0, \varepsilon] \mapsto \mathbb{R}$, for some $\varepsilon > 0$.

2. Next, we prove that the local solution can be extended to all positive times. For this purpose, we need to rule out the possibility that $|x(t)| \to \infty$ in finite time. We first observe that each trajectory is monotone, i.e., either non-increasing, or non-decreasing, for $t \in [0,\infty[$. To fix the ideas, let $t \mapsto x(t)$ be strictly increasing, with $x(t) \to \infty$ as $t \to T-$. A contradiction is now obtained as follows. For each $\tau > 0$, using (1.14) we compute

$$\begin{aligned}
\sum_i u_i\big(x(\tau)\big) - \sum_i u_i\big(x(0)\big) &= \int_0^\tau \left\{ \frac{d}{dt} \sum_i u_i\big(x(t)\big) \right\} dt \\
&= \int_0^\tau - \left\{ \sum_i u_i'\big(x(t)\big) \cdot \sum_j \frac{u_j'\big(x(t)\big)}{k_j\big(x(t)\big)} \right\} dt \\
&= \int_0^\tau \sum_i \left\{ u_i\big(x(t)\big) - \frac{\big|u_i'(x(t))\big|^2}{2k_i(x(t))} - h_i\big(x(t)\big) \right\} dt \qquad (2.3)
\end{aligned}$$

By assumptions, the functions u_i and h_i have sub-linear growth. Moreover, each k_i is uniformly positive and bounded above. Using the elementary inequality

$$\big|x(\tau) - x(0)\big| \le \int_0^\tau 1 \cdot \big|\dot x(t)\big|\, dt \le \left(\int_0^\tau 1\, dt \right)^{1/2} \cdot \left(\int_0^\tau \big|\dot x(t)\big|^2 dt \right)^{1/2},$$

from (2.3) we thus obtain

$$\begin{aligned}
\frac{\big|x(\tau) - x(0)\big|^2}{\tau} &\le \int_0^\tau \big|\dot x(t)\big|^2 dt \ \le 4C \int_0^\tau \sum_i \frac{\big|u_i'(x(t))\big|^2}{2k_i(x(t))}\, dt \\
&\le 4C \left\{ \left| \sum_i u_i\big(x(\tau)\big) - \sum_i u_i\big(x(0)\big) \right| + \int_0^\tau \sum_i \big|u_i(x(t))\big|\, dt \right. \\
&\qquad \left. + \int_0^\tau \sum_i \big|h_i(x(t))\big|\, dt \right\} \\
&\le C_0\,(1+\tau) \Big\{ 2 + \big|x(\tau)\big| + \big|x(0)\big| \Big\},
\end{aligned}$$

for some constant C_0. Therefore, either $\big|x(\tau)\big| \le 2 + 3\big|x(0)\big|$, or else

$$\big|x(\tau)\big| \le \big|x(0)\big| + 2\tau \cdot C_0\,(1+\tau)\,. \qquad (2.4)$$

In any case, blow-up cannot occur at any finite time T.

3. To complete the proof, for each fixed $i \in \{1,\ldots,m\}$, we have to show that the feedback α_i^* in (1.21) provides solution to the optimal control problem for the i-th player:

$$\min_{\alpha_i(\cdot)} \int_0^\infty e^{-t} \left[h_i\big(x(t)\big) + k_i\big(x(t)\big)\, \frac{\alpha_i^2(t)}{2} \right] dt\,, \qquad (2.5)$$

where the system has dynamics

$$\dot x = \alpha_i + \sum_{j \ne i} \alpha_j^*(x)\,. \qquad (2.6)$$

Given an initial state $x(0) = y$, by the assumptions on u it follows that the feedback strategy $\alpha_i = \alpha_i^*(x)$ achieves a total cost given by $u_i(y)$. Now consider any absolutely continuous trajectory $t \mapsto x(t)$, with $x(0) = y$. Of course, this corresponds to the control

$$\alpha_i(t) \doteq \dot{x}(t) - \sum_{j\neq i} \alpha_j^*(x) \tag{2.7}$$

implemented by the i-th player. We claim that the corresponding cost satisfies

$$\int_0^\infty e^{-t}\left[h_i\big(x(t)\big) + \frac{k_i}{2}\Big(\dot{x}(t) - \sum_{j\neq i}\alpha_j^*\big(x(t)\big)\Big)^2\right] dt \ \geq\ u_i(y)\,. \tag{2.8}$$

To prove (2.8), we first observe that (2.4) implies

$$\lim_{t\to\infty} e^{-t}u_i\big(x(t)\big) = 0 \qquad\qquad i = 1,\ldots,n\,.$$

Hence

$$u_i(y) = u_i\big(x(0)\big) = -\int_0^\infty \frac{d}{dt}\Big[e^{-t}u_i\big(x(t)\big)\Big]\,dt\,.$$

The inequality (2.8) can now be established by checking that

$$e^{-t}\left[h_i\big(x(t)\big) + \frac{k_i}{2}\Big(\dot{x}(t) - \sum_{j\neq i}\alpha_j^*\big(x(t)\big)\Big)^2\right] \geq e^{-t}u_i\big(x(t)\big) - e^{-t}u_i'\big(x(t)\big)\cdot\dot{x}(t)\,. \tag{2.9}$$

Equivalently, letting α_i be as in (2.7),

$$u_i \leq \left(\alpha_i - \sum_{j\neq i}\frac{u_j'}{k_j}\right)u_i' + \frac{k_i}{2}\alpha_i^2 + h_i\,.$$

This is clearly true because, by (1.8),

$$u_i(x) = \min_a\left\{\frac{k_i}{2}a^2 + au_i' - \sum_{j\neq i}\frac{u_j'u_i'}{k_j} + h_i(x)\right\}.$$

□

We now give two examples showing that, if the growth assumptions (1.15) or if the jump conditions (1.16) are not satisfied, then the feedbacks (1.21) may not provide a Nash equilibrium solution. This situation is already well known in the context of control problems.

Example 1.1 Consider the game for two players, with dynamics

$$\dot{x} = \alpha_1 + \alpha_2\,, \tag{2.10}$$

and cost functionals

$$J_i = \int_0^\infty e^{-t} \cdot \frac{\alpha_i^2(t)}{2}\, dt\,.$$

In this case, if $u_i' = p_i$, the optimal control for the i-th player is

$$\alpha_i^*(p_i) \;=\; \arg\,\min_\omega \left\{ p_i\,\omega + \frac{\omega^2}{2}\right\} \;=\; -\,p_i\,.$$

The system of H-J takes the simple form

$$\begin{cases} u_1 = -\left(\frac{u_1'}{2} + u_2'\right) u_1'\,, \\ u_2 = -\left(u_1' + \frac{u_2'}{2}\right) u_2'\,. \end{cases} \tag{2.11}$$

The obvious admissible solution is $u_1 \equiv u_2 \equiv 0$, corresponding to identically zero controls, and zero cost. We now observe that the functions

$$u_1(x) = \begin{cases} 0 & \text{if} \quad |x| \geq 1\,, \\ -\frac{1}{2}\big(1-|x|\big)^2 & \text{if} \quad |x| < 1\,, \end{cases} \qquad\qquad u_2(x) = 0\,,$$

provide a solution to (2.11), which is not admissible because the conditions (1.16) fail at $x = 0$.

Next, the functions

$$u_1(x) = -\frac{1}{2}x^2, \qquad\qquad u_2(x) = 0\,,$$

provide yet another solution, which does not satisfy the growth conditions (1.15).

In the above two cases, the corresponding feedbacks $\alpha_i^*(x) = -u_i'(x)$ do not yield a solution to the differential game.

3 Two-Players Games

We consider here a game for two players, with dynamics

$$\dot{x} = \alpha_1 + \alpha_2\,, \qquad\qquad x(0) = y\,, \tag{3.1}$$

and cost functionals of the form

$$J_i(\alpha_i) \doteq \int_0^\infty e^{-t}\left[h_i\big(x(t)\big) + \frac{\alpha_i^2(t)}{2}\right] dt\,. \tag{3.2}$$

Notice that, for any positive constants k_1, k_2, λ, the more general case

$$J_i(\alpha_i) \doteq \int_0^\infty e^{-\lambda t}\left[\tilde{h}_i\big(x(t)\big) + \frac{\alpha_i^2(t)}{2k_i}\right] dt$$

can be reduced to (3.2) by a linear change of variables.

The system of H-J equations for the value functions now takes the form

$$\begin{cases} u_1(x) = h_1(x) - u_1'u_2' - (u_1')^2/2\,, \\ u_2(x) = h_2(x) - u_1'u_2' - (u_2')^2/2\,, \end{cases} \tag{3.3}$$

and the optimal feedback controls on the whole $\mathbb{R}$ are given by

$$\alpha_i^*(x) = -u_i'(x)\,. \tag{3.4}$$

Differentiating (3.3) and setting $p_i = u_i'$ one obtains the system

$$\begin{cases} h_1' - p_1 = (p_1 + p_2)p_1' + p_1 p_2'\,, \\ h_2' - p_2 = p_2 p_1' + (p_1 + p_2)p_2'\,. \end{cases} \tag{3.5}$$

Set

$$\Lambda(p) \doteq \begin{cases} p_1 + p_2 & p_1 \\ p_2 & p_1 + p_2\,, \end{cases} \qquad\qquad \Delta(p) \doteq \det \Lambda(p)\,.$$

Notice that

$$\frac{1}{2}\,(p_1^2 + p_2^2) \le \Delta(p) \le 2(p_1^2 + p_2^2)\,. \tag{3.6}$$

In particular, $\Delta(p) > 0$ for all $p = (p_1, p_2) \neq (0, 0)$. From (3.5), we can then deduce

$$\begin{cases} p_1' = \Delta(p)^{-1}\big[-p_1^2 + (h_1' - h_2')p_1 + h_1' p_2\big]\,, \\ p_2' = \Delta(p)^{-1}\big[-p_2^2 + (h_2' - h_1')p_2 + h_2' p_1\big]\,. \end{cases} \tag{3.7}$$

We can now simplify the equation by a suitable rescaling of the space variable. Define a new variable s such that $ds/dx = \Delta(p)^{-1}$. Using s as a new independent variable we write $p_i = p_i(s)$ and $h_i = h_i(x(s))$ and study the equivalent system

$$\begin{cases} \frac{d}{ds}p_1 = (h_1' - h_2')p_1 + h_1' p_2 - p_1^2\,, \\ \frac{d}{ds}p_2 = (h_2' - h_1')p_2 + h_2' p_1 - p_2^2\,. \end{cases} \tag{3.8}$$

We underline that it is possible to choose the rescaling in order to map 0 to 0. This choice will be assumed in the following, so that $s(0) = 0$.
In this case the condition (1.16), coming from (A3) is equivalent to those in (1.19). Therefore here jumps for piecewise smooth solutions are only allowed from any point (p_1^-, p_2^-) with

$$p_1^- + p_2^- \ge 0 \tag{3.9}$$

to the symmetric point

$$(p_1^+, p_2^+) = (-p_1^-,\ -p_2^-)\,. \tag{3.10}$$

4 Smooth costs: cooperative situations

Theorem 1.2 *Let the cost functions h_1, h_2 be smooth, and assume that their derivatives satisfy*

$$\frac{1}{C} \le h_i'(x) \le C \tag{4.1}$$

for some constant $C > 1$ and all $x \in \mathbb{R}$. Then the system (3.3) has an admissible solution. The corresponding functions α_i^* in (3.4) provide a Nash equilibrium solution to the non-cooperative game.

Here we talk of cooperative situation because (4.1) implies that both costs have the same monotone behavior. Hence it's convenient for both players to push the game in the same direction, and in this sense they cooperate. Of course a similar theorem can be proved if there holds

$$-\,C \le h_i'(x) \le -\,\frac{1}{C} \qquad \forall x \in \mathbb{R}\,.$$

Similarly in Section 5, where h_1' and h_2' have opposite signs, we'll talk of competitive situation and of players which have conflicting interests, in the sense that it is convenient for them to push the state of the system in opposite directions.

Proof. Write the O.D.E. (3.7) in the more compact form

$$\frac{dp}{dx} = f(p)\,. \tag{4.2}$$

To show the existence of at least one admissible solution of (3.3), for every $\nu \ge 1$ let $p^{(\nu)} : [-\nu, \infty[\to \mathbb{R}^2$ be the solution of the Cauchy problem

$$\frac{dp^{(\nu)}}{dx} = f\big(p^{(\nu)}\big)\,, \qquad\qquad p^{(\nu)}(-\nu) = (1,1). \tag{4.3}$$

It is easy to check that the polygon

$$\Gamma \doteq \Big\{(p_1,p_2)\,; \quad p_1,p_2 \in [0,2C]\,, \quad p_1+p_2 \ge 1/2C\Big\}$$

is positively invariant for the flow of (3.7). Hence $p^{(\nu)}(x) \in \Gamma$ for all $\nu \ge 1$ and $x \ge -\nu$. We can extend each function $p^{(\nu)}$ to the whole real line by setting

$$p^{(\nu)}(x) = (1,1) \qquad\qquad \text{for} \quad x < -\nu\,.$$

By uniform boundedness and equicontinuity, the sequence $p^{(\nu)}$ admits a subsequence converging to a uniformly continuous function $p : \mathbb{R} \mapsto \Gamma$. Clearly this limit function provides a continuous, globally bounded solution of (3.7). We then define the controls $\alpha_i^*(x) \doteq -p_i(x)$ and the cost functions

$$u_i(y) \doteq \int_0^\infty e^{-t}\Big[h_i\big(x(t,y)\big) + \frac{1}{2}\big(\alpha_i^*\big(x(t,y)\big)\big)^2\Big]\,dt\,, \tag{4.4}$$

where $t \mapsto x(t,y)$ denotes the solution to the Cauchy problem

$$\dot{x} = \alpha_1^*(x) + \alpha_2^*(x)\,, \qquad\qquad x(0) = y\,. \tag{4.5}$$

This function provides a globally Lipschitz, smooth solution of the system (3.3). □

In the case where the oscillation of the derivatives h_i' is sufficiently small, we can also prove the uniqueness of the Nash feedback solution.

Theorem 1.3 *Let the cost functions be smooth, with derivatives satisfying (4.1), for some constant C. Assume that the oscillation of their derivatives satisfies*

$$\sup_{x,y\in\mathbb{R}} \big|h_i'(x) - h_i'(y)\big| \le \delta \qquad i = 1,2 \tag{4.6}$$

for some $\delta > 0$ sufficiently small (depending only on C). Then the admissible solution of the system (3.3) is unique.

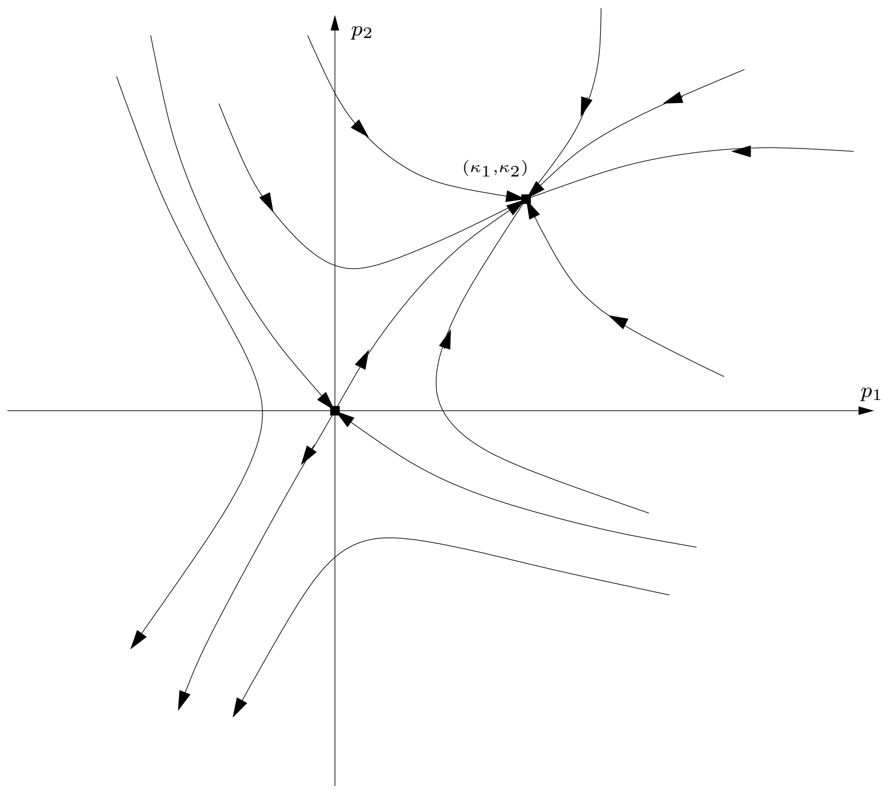

Figure 1

Before giving details of the proof, we sketch the main ideas. In the case of linear cost functions, where $h_i(x) = \kappa_i\, x$, $h_i' \equiv \kappa_i$, the phase portrait of the planar O.D.E. (3.8) is depicted in Figure 1. We observe that

- Unbounded trajectories of (3.8), with $\big|p(s)\big| \to \infty$ as $s \to \bar{s}$, correspond to solutions $p = p(x)$ of (3.7) with $\big|p(x)\big| \to \infty$, $\big|p'(x)\big| \to \infty$ as $x \to \pm\infty$. Indeed, because of the rescaling and of (3.6), as the parameter s approaches a finite limit $\bar{s}$, we have $|x| \to \infty$. This yields a solution $u(x) = \int_*^x p(x)\, dx$ which does not satisfy the growth restrictions (1.15).

- The heteroclinic orbit, joining the origin with the point (κ_1, κ_2), corresponds to a trajectory of (3.7) defined on a half line, say $[\bar{x}, \infty[$. To prolong this solution for $x < \bar{x}$ one needs a trajectory of (3.8) which approaches the origin as $s \to \infty$. But the two available solutions are both unbounded, hence not acceptable.

- Finally, one must examine solutions whose gradient has one or more jumps. Recalling that (A3) imposes a selection among admissible jumps, here we have to consider all the discontinuities from a point $P = (p_1, p_2)$ with $p_1 + p_2 \geq 0$ to its symmetric point $-P = (-p_1,\, -p_2)$. However, a direct inspection shows that, even allowing these jumps, one still cannot construct any new globally bounded trajectory.

In the end, in the linear case, one finds that the only admissible solution is $(p_1, p_2) \equiv (\kappa_1, \kappa_2)$. A perturbative argument shows that this conclusion remains valid if a small $\mathcal{C}^1$ perturbation is added to the cost functions.

Proof. *First Step.* We begin with the case $h_i'(x) \equiv \kappa_i$ and assume, without any loss of generality, that $\kappa_1 \leq \kappa_2$.

Let $\tilde{p}$ be a smooth solution of (3.8), as shown in Figure 1.
We observe that the following facts hold (see Figure 2):

1. Both sets $A = \{(p_1, p_2) \neq (0,0)\ :\ p_1 \leq 0\,,\ p_2 \leq 0\}$ and $\{(p_1, p_2) \neq (0,0)\ :\ p_1 \geq 0\,,\ p_2 \geq 0\}$ are positively invariant for the flow of (3.8) and both $B = \{(p_1, p_2)\ :\ p_1 > 0\,,\ p_2 < 0\}$ and $C = \{(p_1, p_2)\ :\ p_1 < 0\,,\ p_2 > 0\}$ are negatively invariant.

2. If $\tilde{p}(s_o) \in A = \{(p_1, p_2) \neq (0,0)\ :\ p_1 \leq 0\,,\ p_2 \leq 0\}$ for some s_o, then $|\tilde{p}| \to +\infty$ as s increases. Indeed, since

$$\frac{d}{ds}(\tilde{p}_1 + \tilde{p}_2) = -\tilde{p}_1^2 - \tilde{p}_2^2 + \kappa_1 \tilde{p}_1 + \kappa_2 \tilde{p}_2 \leq -\frac{1}{2}(\tilde{p}_1 + \tilde{p}_2)^2 < 0\,,$$

we can assume there exist $\bar{s} \geq s_o$ and $\varepsilon > 0$ such that $\tilde{p}_1(\bar{s}) + \tilde{p}_2(\bar{s}) < -\varepsilon$. Moreover, the following holds for any $\sigma > \bar{s}$:

$$\frac{d}{ds}(\tilde{p}_1+\tilde{p}_2)(\sigma) \leq -\frac{1}{2}(\tilde{p}_1(\sigma)+\tilde{p}_2(\sigma))^2 < -\frac{1}{2}(\tilde{p}_1(\bar{s})+\tilde{p}_2(\bar{s}))(\tilde{p}_1(\sigma)+\tilde{p}_2(\sigma)) < \frac{\varepsilon}{2}(\tilde{p}_1(\sigma)+\tilde{p}_2(\sigma))\,.$$

Hence, an integration yields $(\tilde{p}_1 + \tilde{p}_2)(s) \leq -\eta e^{\frac{\varepsilon}{2}s}$ for $s > \bar{s}$ (and $\eta > 0$). This means $(\tilde{p}_1 + \tilde{p}_2) \to -\infty$ as s increases, eventually reaching $+\infty$.

3. If $\tilde{p}(s_o) \in B = \{(p_1, p_2)\ :\ p_1 > 0\,,\ p_2 < 0\}$ for some s_o, then $|\tilde{p}| \to +\infty$ as s decreases. Indeed, let $\varepsilon > 0$ such that $\tilde{p}_1(s_o) > \varepsilon$. Since

$$\frac{d}{ds}\tilde{p}_1 = -\tilde{p}_1^2 + (\kappa_1 - \kappa_2)\tilde{p}_1 + \kappa_1 \tilde{p}_2 \leq -(\tilde{p}_1 + \kappa_2 - \kappa_1)\tilde{p}_1 < 0\,,$$

it is sufficient to observe that, for $\sigma < s_o$,

$$\frac{d}{ds}\tilde{p}_1(\sigma) < -(\tilde{p}_1(s_o) + \kappa_2 - \kappa_1)\tilde{p}_1(\sigma) \leq -(\varepsilon + \kappa_2 - \kappa_1)\tilde{p}_1(\sigma)\,.$$

Hence, an integration yields $\tilde{p}_1(s) \geq \eta e^{-(\varepsilon+\kappa_2-\kappa_1)s}$ for $s < s_o$ (and $\eta > 0$) and $\tilde{p}_1$ is bounded below by a term tending to $+\infty$ as $s \to -\infty$.

4. If $\tilde{p}(s_o) \in C_1 = \{(p_1, p_2)\ :\ p_1 < 0\,,\ p_2 > \kappa_2 - \kappa_1\}$ for some s_o, then $|\tilde{p}| \to +\infty$ as s decreases. Here the argument is exactly the same as in the previous case with $\tilde{p}_2$ in place of $\tilde{p}_1$.

5. If $\tilde{p}(s_o) \in C_2 = \{(p_1, p_2)\ :\ p_1 < 0\,,\ 0 < p_2 \leq \kappa_2 - \kappa_1\}$ then there exists $\bar{s} < s_o$ such that $\tilde{p}(\bar{s})$ is in C_1 as in case **4.** above. Indeed there could be only two situations.
If $-\tilde{p}_1(s_o)^2 + (\kappa_1 - \kappa_2)\tilde{p}_1(s_o) + \kappa_1 \tilde{p}_2(s_o) \geq 0$, then, by negative invariance, $\tilde{p}$ could only have reached this region from C_1, hence there exists $\bar{s} \leq s_o$ such that $\tilde{p}(\bar{s})$ is as in case **4** above. Otherwise, using again negative invariance and the fact that there are no equilibria in C_2, either there exists $\bar{s} \leq s_o$ such that $\tilde{p}(\bar{s})$ is in case **4** above, or there exists $s_1 < s_o$ such that $-\tilde{p}_1(s_1)^2 + (\kappa_1 - \kappa_2)\tilde{p}_1(s_1) + \kappa_1 \tilde{p}_2(s_1) \geq 0$ and then, by the previous case, the existence of such a $\bar{s} < s_1 < s_o$ follows.

6. If $\tilde{p}(s_o) \in D = \{(p_1, p_2)\ :\ p_1 \geq 0\,,\ p_2 \geq 0\,,\ p_1 + p_2 \geq 2\kappa_2\}$ for some s_o, then $|\tilde{p}| \to +\infty$ as s decreases. Indeed, since

$$\frac{d}{ds}(\tilde{p}_1 + \tilde{p}_2) = -\tilde{p}_1^2 - \tilde{p}_2^2 + \kappa_1 \tilde{p}_1 + \kappa_2 \tilde{p}_2 \leq -\frac{1}{2}(\tilde{p}_1 + \tilde{p}_2 - 2\kappa_2)(\tilde{p}_1 + \tilde{p}_2) \leq 0$$

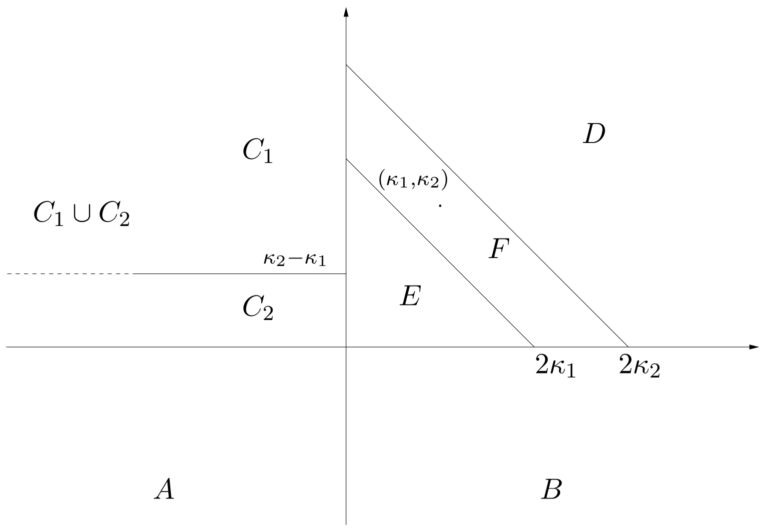

Figure 2

(and the inequality is actually strict when $p_1 + p_2 = 2\kappa_2$), we can assume that there exist $\bar{s} \le s_o$ and $\varepsilon > 0$ such that $\tilde{p}_1(\bar{s}) + \tilde{p}_2(\bar{s}) > 2\kappa_2 + \varepsilon$. Moreover, the following holds for any $\sigma < \bar{s}$:

$$\frac{d}{ds}(\tilde{p}_1 + \tilde{p}_2)(\sigma) < -\frac{1}{2}(\tilde{p}_1(\bar{s}) + \tilde{p}_2(\bar{s}) - 2\kappa_2)(\tilde{p}_1(\sigma) + \tilde{p}_2(\sigma)) < -\frac{\varepsilon}{2}(\tilde{p}_1(\sigma) + \tilde{p}_2(\sigma))\,.$$

Hence by integrating we find $(\tilde{p}_1 + \tilde{p}_2)(s) \ge \eta e^{-\frac{\varepsilon}{2}s}$ for $s < \bar{s}$ and $\eta > 0$. Therefore $(\tilde{p}_1 + \tilde{p}_2) \to +\infty$ as s decreases, eventually reaching $-\infty$.

7. If $\tilde{p}(s_o) \in E = \{(p_1, p_2) \ne (0,0) \,:\, p_1 \ge 0\,,\ p_2 \ge 0\,,\ p_1 + p_2 \le 2\kappa_1\}$ for some s_o, then from

$$\frac{d}{ds}(\tilde{p}_1 + \tilde{p}_2) = -\tilde{p}_1^2 - \tilde{p}_2^2 + \kappa_1\tilde{p}_1 + \kappa_2\tilde{p}_2 \ge -\frac{1}{2}(\tilde{p}_1 + \tilde{p}_2 - 2\kappa_1)(\tilde{p}_1 + \tilde{p}_2) \ge 0\,,$$

it follows, as above, that either $\tilde{p} \to 0$ for $s \to -\infty$ or there exists $\bar{s} \le s_o$ such that $\tilde{p}(\bar{s})$ satisfies one of the previous cases **3-4-5**. While the latter case has already been treated, we will deal with the former one in **10** .

8. If $\tilde{p}(s_o) \in F = \{(p_1, p_2) \,:\, p_1 \ge 0\,,\ p_2 \ge 0\,,\ 2\kappa_1 < p_1 + p_2 < 2\kappa_2\}$ for some s_o and $p \ne \tilde{p}$, then there exists a small circle V (say with radius smaller than $|\tilde{p}(s_o) - p(s_o)|$) around the stable focus $p \equiv (\kappa_1, \kappa_2)$ such that $\tilde{p} \notin V$ for $s < s_o$. But then, looking at the signs of the derivatives of $\tilde{p}_i$, as s decreases our solution $\tilde{p}$ must go away from the whole region F and there exists $\bar{s} < s_o$ such that $\tilde{p}(\bar{s})$ is in one of the previous cases.

9. We can now provide more accurate estimates on blow-up. Indeed, by previous analysis, blow-up of $|\tilde{p}|$ can occur either because $\tilde{p}_i \to -\infty$ as s increases or $\tilde{p}_i \to +\infty$ as s decreases, for some index $i \in \{1, 2\}$. But these results has been obtained by estimating $\tilde{p}_i$ with suitable exponential functions, hence it is still not clear whether there exists a finite s_o such that the blow-up occurs when $s \to s_o$ or $|\tilde{p}| \to \infty$ as $|s| \to \infty$. To fix

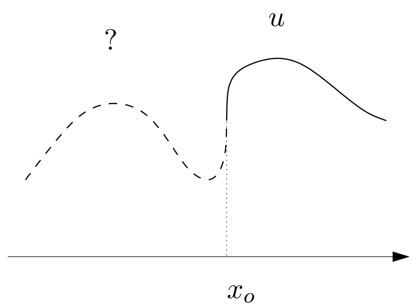

Figure 3a

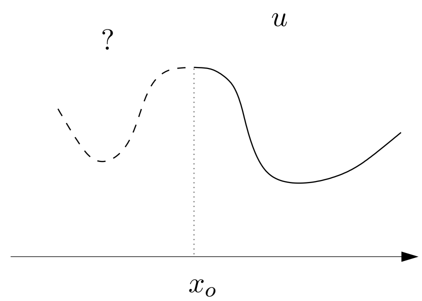

Figure 3b

the ideas, assume it holds $|\tilde{p}_1| \to \infty$, being $|\tilde{p}_2| \to \infty$ entirely similar. For s sufficiently large, we have

$$-\frac{\tilde{p}_1^2}{2} + (\kappa_1 - \kappa_2)\tilde{p}_1 + \kappa_1\tilde{p}_2 < 0\,.$$

Hence, integrating the inequality $\frac{d}{ds}\tilde{p}_1 < -\frac{\tilde{p}_1^2}{2}$, one can conclude that $|\tilde{p}| \to \infty$ as $s \to s_o$, for some finite s_o. In particular, for this $s_o \in \mathbb{R}$ (and $\eta > 0$), $\tilde{p}$ satisfies $|\tilde{p}(s)| \geq \frac{\eta}{|s-s_o|}$.

In terms of the original variable x, one may guess that the corresponding function $\tilde{p} = \tilde{p}(x)$ could be as in Figure 3a and that u may be continued beyond the point where $\tilde{p}$ blows-up (say $x_o = x(s_o)$). But this is not the case since such a trajectory yields a solution defined on the whole real line. Indeed by (3.6)

$$\left|\frac{dx}{ds}\right| = \Delta\big(\tilde{p}(s)\big) \geq \frac{c_o}{(s_o^{-} s)^2}\,. \tag{4.7}$$

for some $c_o > 0$, and therefore either $x(s) \to +\infty$ as $s \to s_o^-$ or $x(s) \to -\infty$ as $s \to s_o^+$. Therefore, the solution $\tilde{u}(x)$, corresponding to $\tilde{p}(x)$, violates the growth assumptions (1.15) and it is not admissible.

10. We remark that in case **7**, the solution $\tilde{p}$ can tend to 0 as $s \to -\infty$. But then for some $c_o > 0$

$$|\tilde{p}| \leq \tilde{p}_1 + \tilde{p}_2 \leq e^{c_o s}\,.$$

Recalling (3.6) we obtain, in terms of the variable x,

$$\left|\frac{dx}{ds}\right| = \Delta\big(\tilde{p}(s)\big) = \mathcal{O}(1)\cdot e^{2c_o s}\,, \tag{4.8}$$

$$\lim_{s\to-\infty} x(s) = x_o < \infty\,,$$

for some $x_o \in \mathbb{R}$. Therefore, to the entire trajectory $s \mapsto \tilde{p}(s)$, there corresponds only a portion of the trajectory $x \mapsto \tilde{p}(x)$, namely for $x > x_o$.

To prolong the solution $\tilde{u}$ for $x < x_o$, we need to construct another trajectory $s \mapsto p(s)$ such that $\lim_{s\to+\infty} p(s) = 0$. But this trajectory, by previous analysis (see **3.-4.-5.**), will be unbounded for negative s, hence the corresponding $\tilde{u}(x)$, will not be admissible.

11. Next, we consider the case where $\tilde{p}(s)$ is a discontinuous solution with admissible jumps. In this case, first of all we can say that $\tilde{p}$ has no more than 2 jumps. Indeed, the set $\Xi_1 = \{(p_1, p_2) : p_2 < 0, \ p_1 + p_2 \leq 0\}$ is positively invariant and $\Xi_2 = \{(p_1, p_2) : p_2 < 0, \ p_1 + p_2 > 0\}$ is negatively invariant. Hence, if a jump occurs at s_o, either $\tilde{p}(s_o^+) \in \Xi_1$ or $\tilde{p}(s_o^-) \in \Xi_2$. In the former case $\tilde{p}(s)$ has no jumps for $s > s_o$; in the latter case $\tilde{p}$ has no jumps for $s < s_o$. This means that there could be at most two jumps when there exist $s_1 < s_2 \leq s_3$ such that

- a first jump occurs at s_1 and $\tilde{p}(s_1-) \in \Xi_2$,
- $\tilde{p}$ crosses the line $p_1 + p_2 = 0$ at s_2,
- a last jump occurs at s_3 and $\tilde{p}(s_3+) \in \Xi_1$.

In any case, the corresponding solution $\tilde{u}$ does not satisfy (1.15) and is not admissible. Indeed, we can have only three situations for a $\tilde{p}$ with an admissible jump at s_o:

(a) if $\tilde{p}(s_o^-) \in \Xi_2$, then $|\tilde{p}| \to \infty$ as s decreases;

(b) if $\tilde{p}(s_o^+) \in \Xi_1$ and $\tilde{p}_1(s_o^+) > 0$, then either $\tilde{p}(s)$ is continuous for $s < s_o$ (and therefore $|\tilde{p}| \to \infty$ as s decreases) or $\tilde{p}$ has another jump at $\bar{s}$ such that $\tilde{p}(\bar{s}-) \in \Xi_2$ (and therefore again $|\tilde{p}| \to \infty$ as s decreases);

(c) if $\tilde{p}(s_o^+) \in \Xi_1$ and $\tilde{p}_1(s_o^+) \leq 0$, then $|\tilde{p}| \to \infty$ as s increases.

Second Step. We now extend the proof, in the presence of a sufficiently small perturbation. By (4.6), there exist constants $\kappa_1, \kappa_2 > 0$ such that

$$\left|h_1'(x) - \kappa_1\right| \leq \delta\,, \qquad \left|h_2'(x) - \kappa_2\right| \leq \delta \qquad \text{for all } x \in \mathbb{R}\,. \tag{4.9}$$

Let $u(\cdot)$ be the solution constructed in Theorem 2, and let $\tilde{u}$ be any other smooth solution of (3.8). Call $p = u'$, $\tilde{p} = \tilde{u}'$ the corresponding gradients, rescaled as before, and let V be a small open bounded set containing the whole image of p and the point (κ_1, κ_2). Of course it is not restrictive to consider V as circular, say with radius $\rho > 0$.

Now we split the proof in three cases.

CASE 1: $\tilde{p}(s) \in V$ for every s. In this case we look at the difference $w(s) = \tilde{p}(s) - p(s)$. We can write a linear evolution equation for w:

$$\frac{dw}{ds} = A(s)\, w(s)\,, \tag{4.10}$$

where the matrix A is the "average" matrix

$$A(s) = \int_0^1 Df(\theta p(s) + (1-\theta)\tilde{p}(s))d\theta\,, \tag{4.11}$$

and f is the vector field at (4.2).

Since $p, \tilde{p} \in V$, the matrix $A(s)$ is very close to the Jacobian matrix $Df(\kappa_1, \kappa_2)$. Therefore

$$\frac{d}{ds}\left|w(s)\right| \leq -K\left|w(s)\right|\,, \tag{4.12}$$

for some constant $K > 0$. Indeed $Df(\kappa_1, \kappa_2)$ is negative definite and, provided δ (and then ρ) is small enough, $A(s)$ is negative definite too. Hence

$$2\big|w(s)\big|\frac{d}{ds}\big|w(s)\big| = \frac{d}{ds}\big|w(s)\big|^2 = 2\frac{d}{ds}w(s)\cdot w(s) = 2A(s)w(s)\cdot w(s) \le -2K\big|w(s)\big|^2 .$$

Now integrating (4.12), we have for $s < 0$,

$$2\rho \ge \big|w(s)\big| \ge e^{-Ks}\big|w(0)\big|$$

and, letting $s \to -\infty$, find

$$\big|w(0)\big| \le \lim_{s\to-\infty}\big|w(s)\big|e^{Ks} \le \lim_{s\to-\infty} 2\rho e^{Ks} = 0\,. \tag{4.13}$$

This implies $p(0) = \tilde{p}(0)$, hence $p = \tilde{p}$ by the uniqueness of the Cauchy problem.
CASE 2: $\tilde{p}(s_o) \notin V$ for some s_o and, in particular, $\tilde{p}(s_o)$ in a small neighborhood W of the origin. Consider the linearized system near $(0,0)$

$$\begin{pmatrix} p_1' \\ p_2' \end{pmatrix} = H\cdot\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \qquad H = \begin{pmatrix} h_1' - h_2' & h_1' \\ h_2' & h_2' - h_1' \end{pmatrix},$$

and notice that the origin is a saddle point for this system. Indeed H has eigenvalues λ_1, λ_2 such that

$$0 < \sqrt{\frac{3}{4C^2}} \le |\lambda_i| = \sqrt{(h_1')^2 + (h_2')^2 - h_1'h_2'} \le \sqrt{2C^2 - \frac{1}{C^2}}\,, \tag{4.14}$$

where C is the constant in (4.1). Moreover its eigenvectors v_1, v_2 form angles α_1, α_2 with the positive direction of the p_1-axis such that

$$0 < \frac{1}{C}\left(\sqrt{d^2 + \frac{1}{C^2}} - d\right) \le |\tan\alpha_i| = \left|\frac{\lambda_i + (h_2' - h_1')}{h_1'}\right| \le C\left(\sqrt{d^2 + C^2} + d\right), \tag{4.15}$$

where $d = (C - \frac{1}{C}) > 0$ and C is again from (4.1).
Hence, exactly as one can do with saddle points in the autonomous case, we can prove that there exist four sectors S_i, $i = 1, \ldots, 4$ (see Figure 4), where the following facts hold:
(a) If $\tilde{p}(s_o)$ is in S_1 or S_3, then $|\tilde{p}(s)|$ grows for $s < s_o$ and the solution moves away from W;
(b) Both boundaries of S_2 and S_4 allow orbits to only exit from those sectors for $s < s_o$;
(c) If $\tilde{p}(s_o) \notin S_i$ for all $i = 1, \ldots, 4$, then for $s < s_o$ the angle between the vector $(\tilde{p}_1, \tilde{p}_2)$ and the p_1-axis is strictly monotone, forcing the solution either to reach S_1 or S_3, or to move away from W;
(d) Finally, if $\tilde{p}(s_o)$ is in S_2 or S_4, then for $s < s_o$ the solution can tend to the origin. But, since

$$\frac{d}{ds}(\tilde{p}_1 + \tilde{p}_2) = -\tilde{p}_1^2 - \tilde{p}_2^2 + h_1'\tilde{p}_1 + h_2'\tilde{p}_2 \ge -\frac{1}{2}(\tilde{p}_1 + \tilde{p}_2 - 2C)(\tilde{p}_1 + \tilde{p}_2) > 0\,,$$

as in the constant case, one obtains an estimate of exponential type of the decay of $|\tilde{p}|$.

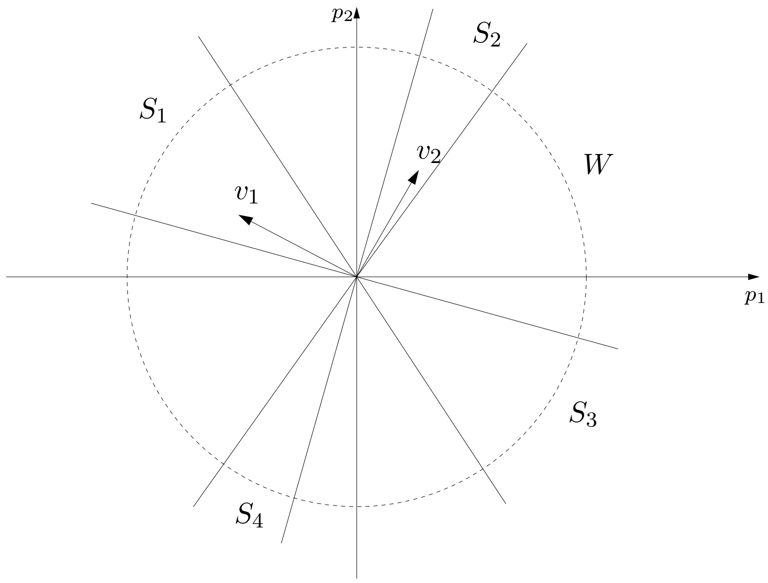

Figure 4

CASE 3: $\tilde{p}(s_o) \notin V$ and $\tilde{p}(s_o)$ not in a neighborhood of the origin. In this case, combining (4.9) and the continuous dependence of solutions with the estimates of the constant case (indeed, using (4.1), they remain true), we can prove that $|\tilde{p}| \to \infty$ for finite s and that the rate of blow-up of $|\tilde{p}|$ can be estimated in the same way we did in the case of $h_i' \equiv \kappa_i$.
In any case either $\tilde{u} \equiv u$ or, in the original coordinates x, $\tilde{u}$ fails to satisfy (1.15).

It remains to prove what happens if $\tilde{u}$ is an admissible solution with discontinuous (rescaled) gradient $\tilde{p}(s)$. Assume $\tilde{p}$ has an admissible jump at s_o. Using (4.9) it holds, for $\tilde{p}_1 > 0$,

$$\frac{d}{ds}(\tilde{p}_1 + \tilde{p}_2)\bigg|_{\tilde{p}_1+\tilde{p}_2=0} = -2\tilde{p}_1^2 + (h_1' - h_2')\tilde{p}_1 < -2\tilde{p}_1^2 + (\kappa_1 - \kappa_2 + 2\delta)\tilde{p}_1$$

and hence, provided δ small enough, the region Ξ_1 (resp. Ξ_2) defined in the *First Step* is positively (resp. negatively) invariant also in this setting. Then conclusions made in the constant case still hold and $\tilde{u}$ corresponding to $\tilde{p}$ is not admissible, since it violates (1.15).
□

5 Smooth costs: players with conflicting interests

We consider here a game for two players, with dynamics (3.1) and cost functionals as in (3.2). Contrary to the previous section, we now assume that the player have conflicting interest. Namely, their running costs h_i satisfy

$$h_1'(x) \le 0 \le h_2'(x). \tag{5.1}$$

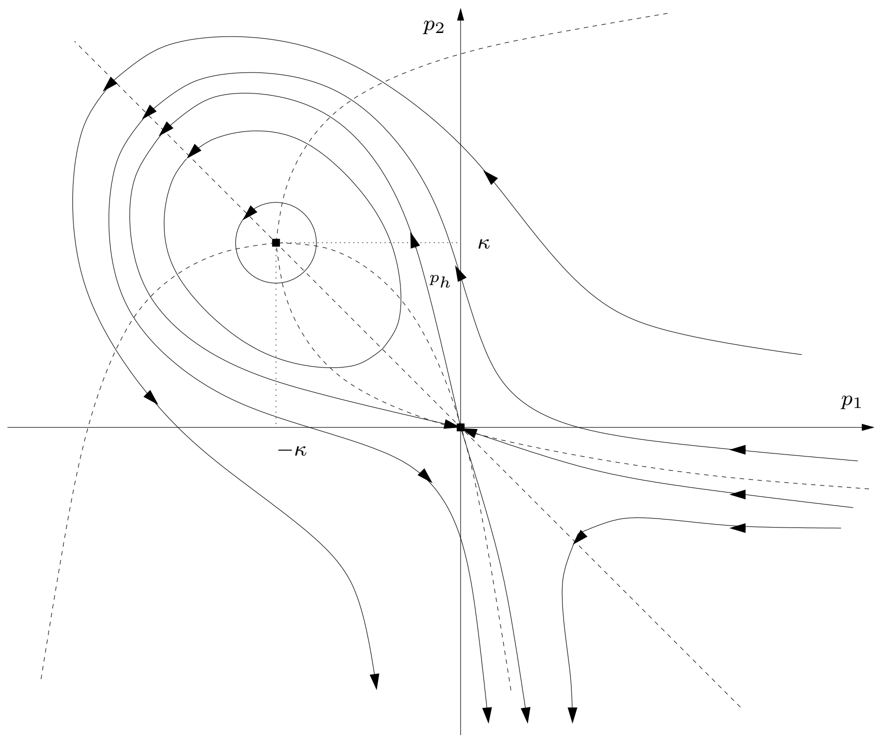

Figure 5

We begin with an example showing that in this case the H-J system can have infinitely many admissible solutions. Each of these determines a different Nash equilibrium solution to the differential game.

Example 1.2 Consider the game (3.1)–(3.2), with

$$h_1(x) = -\kappa x\,, \qquad\qquad h_2(x) = \kappa x\,, \tag{5.2}$$

for some constant $\kappa > 0$ (see Figure 5).

In this special case, the equations (3.8) reduce to

$$\begin{cases} p_1' = -2\kappa p_1 - \kappa p_2 - p_1^2\,, \\ p_2' = \quad \kappa p_1 + 2\kappa p_2 - p_2^2\,. \end{cases} \tag{5.3}$$

The point $\overline{P} \doteq (-\kappa, \kappa)$ is stationary for the flow of (5.3). Setting $q_1 \doteq p_1 + \kappa$, $q_2 \doteq p_2 - \kappa$, the local behavior of the system near $\overline{P}$ is described by

$$\begin{cases} q_1' = -\kappa q_2 - q_1^2\,, \\ q_2' = \quad \kappa q_1 - q_2^2\,. \end{cases} \tag{5.4}$$

Notice that

$$\frac{dp_2}{dp_1} = \frac{dq_2}{dq_1} = 0 \qquad \text{if} \qquad q_1 = \frac{q_2^2}{\kappa}\,,$$

$$\frac{dp_1}{dp_2} = \frac{dq_1}{dq_2} = 0 \qquad \text{if} \qquad q_2 = -\frac{q_1^2}{\kappa}\,,$$

$$\frac{dp_1}{dp_2} = \frac{dq_1}{dq_2} = 1 \qquad \text{if} \qquad p_1 = -p_2 \,.$$

By symmetry across the line $p_1 + p_2 = 0$, any trajectory passing through a point $P_\alpha \doteq (-\alpha, \alpha)$ with $0 < \alpha < \kappa$ is a closed orbit. We thus have infinitely many solutions of the H-J equations (5.3), having bounded, periodic gradients. Therefore, all of these solutions are globally Lipschitz continuous and satisfy the growth condition (1.15). Notice that the homoclinic orbit $p_h(\cdot)$ starting and ending at the origin also yields a periodic solution to the original equation (3.7). Indeed, to a solution $p = p(s)$ of (5.3) with

$$\lim_{s \to -\infty} p(s) = \lim_{s \to +\infty} p(s) = 0 \,,$$

through the reparametrization $x = x(s)$ there corresponds a solution $p = p(x)$ defined on some bounded interval $]\ell_o, \ell_1[$. This yields a periodic solution $p = p(x)$ with period $\ell = \ell_1 - \ell_o$.

The main result of this section is concerned with the existence and uniqueness of admissible solutions.

Theorem 1.4 *Let any two constants κ_1, κ_2 be given, with*

$$\kappa_1 < 0 < \kappa_2 \,, \qquad\qquad \kappa_1 + \kappa_2 \neq 0 \,. \tag{5.5}$$

Then there exists $\delta > 0$ such that the following holds. If h_1, h_2 are smooth functions whose derivatives satisfy

$$\big|h_1'(x) - \kappa_1\big| \leq \delta \,, \qquad\qquad \big|h_2'(x) - \kappa_2\big| \leq \delta \,, \tag{5.6}$$

for all $x \in \mathbb{R}$, then the system of H-J equations (3.3) has a unique admissible solution.

The structure of the proof will be the following. We first prove the result in the case of linear costs: the phase portrait in this case is depicted in Figure 6. We observe that:

- As in Theorem 1.3, unbounded trajectories of (3.8), with $\big|p(s)\big| \to \infty$ as $s \to \bar{s}$, correspond to solutions $p = p(x)$ of (3.7) with $\big|p(x)\big| \to \infty$, $\big|p'(x)\big| \to \infty$ as $x \to \pm\infty$. This yields a solution $u(x) = \int_*^x p(x)\,dx$ which does not satisfy the growth restrictions (1.15).

- The heteroclinic orbit, joining the origin with the point (κ_1, κ_2), corresponds to a trajectory of (3.7) defined on a half line, say $[\bar{x}, \infty[$. As in Theorem 1.3, there exists no admissible solution to prolong this one for $x < \bar{x}$.

- Finally, even considering solutions whose gradient has one or more jumps, it is impossible to find more admissible solutions.

Therefore, in the linear case, one finds that the only admissible solution is $(p_1, p_2) \equiv (\kappa_1, \kappa_2)$. As in Theorem 1.3, a perturbative argument allows to conclude the same holds if a small $\mathcal{C}^1$ perturbation is added to the cost functions.

Proof. We will first consider the linear case, where $h_i' \equiv \kappa_i$ is constant. Then we recover the more general case by a perturbation argument.

Existence. Assume that $h_i(x) = \kappa_i\, x$ with $\kappa_1 + \kappa_2 > 0$, which is not restrictive. The existence of an admissible solution for (3.8) is trivial, since we have the constant solution $p \equiv (\kappa_1, \kappa_2)$, which corresponds to

$$\big(u_1(x),\, u_2(x)\big) = \Big(\kappa_1 x + \kappa_1\kappa_2 + \frac{\kappa_1^2}{2}\,,\ \kappa_2 x + \kappa_1\kappa_2 + \frac{\kappa_2^2}{2}\Big)\,. \tag{5.7}$$

Consider now the case of h_1', h_2' small perturbations of the constants κ_1, κ_2. Notice that, in the previous case, every ball $B(\kappa, R)$ around $\kappa = (\kappa_1, \kappa_2)$ with radius $R < \frac{\sqrt{2}}{2}(\kappa_1 + \kappa_2)$ was positively invariant for the flow of (3.8).
Indeed, setting $q_i = p_i - \kappa_i$, the system becomes

$$\begin{cases} q_1' = -(\kappa_1 + \kappa_2)q_1 + \kappa_1 q_2 - q_1^2\,, \\ q_2' = \kappa_2 q_1 - (\kappa_1 + \kappa_2)q_2 - q_2^2\,. \end{cases} \tag{5.8}$$

and it holds

$$\frac{d}{ds}\frac{|q|^2}{2} = -q_1^3 - q_2^3 - (\kappa_1 + \kappa_2)(q_1^2 + q_2^2 - q_1 q_2) = -(q_1^2 + q_2^2 - q_1 q_2)(\kappa_1 + \kappa_2 + q_1 + q_2)\,.$$

Now, since $|q| \le R < \frac{\sqrt{2}}{2}(\kappa_1 + \kappa_2)$ ensures $\kappa_1 + \kappa_2 + q_1 + q_2 > 0$, one can conclude that

$$\frac{d}{ds}\frac{|q|^2}{2} = -(q_1^2 + q_2^2 - q_1 q_2)(\kappa_1 + \kappa_2 + q_1 + q_2) \le -\frac{|q|^2}{2}(\kappa_1 + \kappa_2 + q_1 + q_2) < 0\,, \tag{5.9}$$

and this prove the positively invariance of such a ball B.

Then, provided δ is small enough, we can choose one of these balls as a neighborhood U of (κ_1, κ_2) positively invariant also for the perturbed system (i.e. $h_i' \not\equiv \kappa_i$). Once we found such a compact, positively invariant set $\overline{U}$, we can repeat the existence proof of Theorem 2:

a. Consider $p^{(\nu)}\colon\ [-\nu, \infty[\to \mathbb{R}^2$ solution of the Cauchy problem with initial datum $p^{(\nu)}(-\nu) = (\kappa_1, \kappa_2)$;

b. By positive invariance, $p^{(\nu)}(x) \in U$ for $x > -\nu$. We then extend the function $p^{(\nu)}$ to the whole real line by setting $p^{(\nu)}(x) \equiv (\kappa_1, \kappa_2)$ for $x < -\nu$;

c. By uniform boundedness and equicontinuity, the sequence $p^{(\nu)}$ admits a subsequence converging to a uniformly continuous function $p : \mathbb{R} \mapsto U$. Clearly, this limit function $p(\cdot)$ provides a global, bounded solution to the system (3.8). In turn, this yields an admissible solution $u(\cdot)$ to (3.7).

Uniqueness. *First Step.* Let $h_i' \equiv \kappa_i$ and $\kappa_1 + \kappa_2 > 0$. In order to prove that the previously found solution is the only one that satisfies (A1)-(A3), we assume that $\tilde{u}$ is another solution of the system (3.3), whose gradient will be denoted by $\tilde{p}$. Figure 6 depicts possible trajectories $s \mapsto \tilde{p}(s)$ of the planar system (3.8). We remark that:

1. The regions $\{(p_1, p_2) \neq (0,0)\ :\ p_1 \ge 0\,,\ p_2 \le 0\,,\ p_1 + p_2 \le 0\}$ and $\{(p_1, p_2) \neq (0,0)\ :\ p_1 \ge 0\,,\ p_2 \le 0\,,\ p_1 + p_2 \ge 0\}$ are positively and negatively invariant for the flow of (3.8), respectively.

2. If $\tilde{p}(s_o) \in \{(p_1, p_2) \neq (0,0)\ :\ p_1 \ge 0\,,\ p_2 \le 0\,,\ p_1 + p_2 \le 0\}$ for some s_o, then $|\tilde{p}| \to +\infty$ as s increases. Indeed, since

$$\frac{d}{ds}(\tilde{p}_1 + \tilde{p}_2) = -\tilde{p}_1^2 - \tilde{p}_2^2 + \kappa_1 \tilde{p}_1 + \kappa_2 \tilde{p}_2 < -(\tilde{p}_1 + \tilde{p}_2)^2 < 0\,,$$

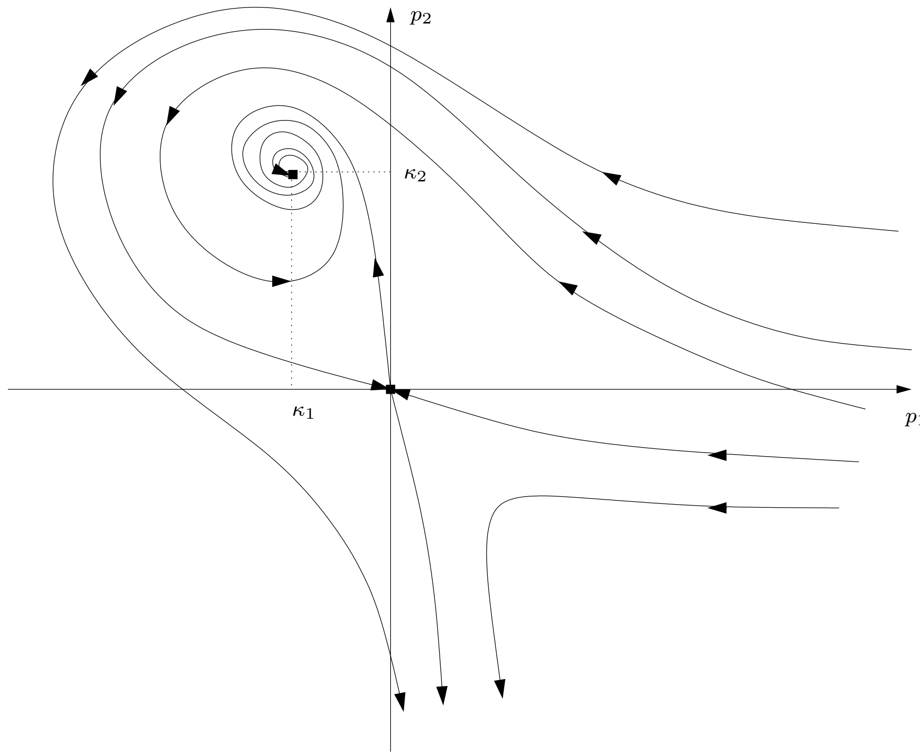

Figure 6

we can assume there exist $\bar{s} \geq s_o$ and $\varepsilon > 0$ such that $\tilde{p}_1(\bar{s}) + \tilde{p}_2(\bar{s}) < -\varepsilon$. Moreover, for any $\sigma > \bar{s}$ we have

$$\frac{d}{ds}(\tilde{p}_1+\tilde{p}_2)(\sigma) \leq -(\tilde{p}_1(\sigma)+\tilde{p}_2(\sigma))^2 < -(\tilde{p}_1(\bar{s})+\tilde{p}_2(\bar{s}))(\tilde{p}_1(\sigma)+\tilde{p}_2(\sigma)) < \varepsilon(\tilde{p}_1(\sigma)+\tilde{p}_2(\sigma)) .$$

After an integration, we find $(\tilde{p}_1 + \tilde{p}_2)(s) \leq -\eta e^{\varepsilon s}$ for $s > \bar{s}$ (and $\eta > 0$) and hence $(\tilde{p}_1 + \tilde{p}_2) \to -\infty$ as s increases, eventually reaching $+\infty$.

3. If $\tilde{p}(s_o) \in \{(p_1, p_2) \neq (0,0) \ : \ p_1 \geq 0\,,\ p_2 \leq 0\,,\ p_1 + p_2 \geq 0\}$ for some s_o, then $|\tilde{p}| \to +\infty$ as s decreases. Indeed, reasoning as above, we can assume there exist $\bar{s} \leq s_o$ and $\varepsilon > 0$ such that $\tilde{p}_1(\bar{s}) + \tilde{p}_2(\bar{s}) > \varepsilon$ and the following holds for any $\sigma < \bar{s}$:

$$\frac{d}{ds}(\tilde{p}_1+\tilde{p}_2)(\sigma) \leq -(\tilde{p}_1(\sigma)+\tilde{p}_2(\sigma))^2 < -(\tilde{p}_1(\bar{s})+\tilde{p}_2(\bar{s}))(\tilde{p}_1(\sigma)+\tilde{p}_2(\sigma)) < -\varepsilon(\tilde{p}_1(\sigma)+\tilde{p}_2(\sigma)) .$$

This implies $(\tilde{p}_1 + \tilde{p}_2)(s) \geq \eta e^{-\varepsilon s}$ for $s < \bar{s}$ (and $\eta > 0$), hence $(\tilde{p}_1 + \tilde{p}_2) \to +\infty$ as s decreases, eventually reaching $-\infty$.

4. If $\tilde{p}(s_o) \in \{(p_1, p_2) \ : \ p_1 > 0\,,\ p_2 > 0\}$ for some s_o, then $|\tilde{p}| \to +\infty$ as s decreases. Indeed, let $\varepsilon > 0$ such that $\tilde{p}_1(s_o) > \varepsilon$. Since

$$\frac{d}{ds}\tilde{p}_1 = -\tilde{p}_1^2 + (\kappa_1 - \kappa_2)\tilde{p}_1 + \kappa_1\tilde{p}_2 < -\tilde{p}_1^2 < 0\,,$$

it is sufficient to observe that for $\sigma < s_o$

$$\frac{d}{ds}\tilde{p}_1(\sigma) \leq -\tilde{p}_1(s_o)\tilde{p}_1(\sigma) < -\varepsilon\tilde{p}_1(\sigma)\,.$$

Hence, integrating, $\tilde{p}_1(s) \geq \eta e^{-\varepsilon s}$ for $s < s_o$ (and $\eta > 0$) and either $\tilde{p}_1 \to +\infty$ as s decreases, or there exists $\bar{s} < s_o$ such that $\tilde{p}$ is in the previous case.

5. If $\tilde{p}(s_o) \in \{(p_1, p_2) \ : \ p_1 < 0\,,\ p_2 < 0\}$ for some s_o, then $|\tilde{p}| \to +\infty$ as s increases. Here we can repeat the argument of **4.** with $\tilde{p}_2$ in place of $\tilde{p}_1$.

6. Let $\tilde{p}(s_o) \in \{(p_1, p_2) \neq (0,0) \ : \ p_1 \leq 0\,,\ p_2 \geq 0\}$ for some s_o and set $\hat{p}$ as the unique solution in this region that tends to the origin as $s \to +\infty$. Notice that, as s decreases, either $\hat{p}(s)$ crosses the p_2-axis or $\hat{p}_2 \to +\infty$. Then:

- if $\tilde{p} = \hat{p}$, then as stated above either there exists $\bar{s} < s_o$ such that $\tilde{p}(\bar{s})$ is in the case **4**, or $\tilde{p}_2 \to \infty$ as $s \to -\infty$. In both cases $|\tilde{p}| \to \infty$ as $s \to -\infty$.
- if $\tilde{p}(s_o)$ belongs to the region between $\hat{p}$ and the p_2-axis, then there could be only three possibilities: either $\tilde{p}$ is the unique solution that tends to the origin as $s \to -\infty$, or $\tilde{p}_2 \to \infty$ as s decreases, but $\tilde{p}$ does not cross the p_2-axis (and, of course, this can only happen if $\hat{p}$ does not cross it too), or there exists $\bar{s} < s_o$ such that $\tilde{p}(\bar{s})$ is in the case **4.** above. In the former case we will estimate the decay of $|\tilde{p}|$ in **8**; in the latter ones $|\tilde{p}| \to \infty$ as s decreases.
- if $\tilde{p}(s_o)$ doesn't belong to the region between $\hat{p}$ and the p_2-axis, then either $\tilde{p}_2 \to \infty$ as s decreases, or there exists $\bar{s} < s_o$ such that $\tilde{p}(\bar{s})$ is in case **5** above (and this is possible only if also $\hat{p}(s)$ crosses the p_2-axis). In both situations, again, $|\tilde{p}| \to \infty$ as s decreases.

7. We can now provide more accurate estimates on blow-up. Indeed, by previous analysis, blow-up of $|\tilde{p}|$ can occur either because $\tilde{p}_i \to -\infty$ as s increases or $\tilde{p}_i \to +\infty$ as s decreases, for some index $i \in \{1,2\}$. But these results has been obtained by estimating $\tilde{p}_i$ with suitable exponential functions, hence it is still not clear whether there exists a finite s_o such that the blow-up occurs when $s \to s_o$ or $|\tilde{p}| \to \infty$ as $|s| \to \infty$. However, applying the same ideas used in Theorem 1.3, we can prove that there exists $s_o \in \mathbb{R}$ (and $\eta > 0$) such that $|\tilde{p}(s)| \geq \frac{\eta}{|s-s_o|}$. In terms of the original variable x, such a trajectory yields a solution defined on the whole real line, because by (3.6)

$$\left|\frac{dx}{ds}\right| = \Delta\big(\tilde{p}(s)\big) \geq \frac{c_o}{(s_o^- s)^2}\,. \tag{5.10}$$

for some $c_o > 0$ and therefore either $x(s) \to +\infty$ as $s \to s_o^-$ or $x(s) \to -\infty$ as $s \to s_o^+$. In conclusion, the solution $\tilde{u}(x)$ which corresponds to $\tilde{p}(x)$ violates the growth condition (1.15), and hence it is not admissible.

8. Notice that only in case **6**-(ii), where $\tilde{p}$ is the unique solution that tends to 0 as $s \to -\infty$, we have a solution that could remain bounded in the whole $\mathbb{R}$. But in this case, we shall have as $s \to -\infty$

$$|\tilde{p}|(s) \leq (\tilde{p}_2 - \tilde{p}_1)(s) \leq \gamma e^{c_o s}\,, \tag{5.11}$$

for some $\gamma, c_o > 0$. Indeed studying the linearized system near the origin we see that $\tilde{p}$ tends to $(0,0)$ along the direction $(1, \frac{\kappa_2-\kappa_1+\sqrt{\kappa_1^2+\kappa_2^2-\kappa_1\kappa_2}}{\kappa_1})$. Then there exists $\bar{s}$ such that for $s < \bar{s}$ the following holds:

$$\tilde{p}_2(s) > (1 + \frac{\sqrt{2}}{2})\frac{\kappa_2 - \kappa_1}{\kappa_1}\,\tilde{p}_1(s) = \beta\,\frac{\kappa_2 - \kappa_1 + \sqrt{\kappa_1^2 + \kappa_2^2 - \kappa_1\kappa_2}}{\kappa_1}\,\tilde{p}_1(s)\,, \tag{5.12}$$

where

$$\beta = \frac{(1 + \frac{\sqrt{2}}{2})(\kappa_2 - \kappa_1)}{\kappa_2 - \kappa_1 + \sqrt{\kappa_1^2 + \kappa_2^2 - \kappa_1\kappa_2}}\,, \qquad \beta \in (0,1)\,. \tag{5.13}$$

Notice that, setting

$$\alpha = \frac{\kappa_1 - 2\kappa_2 - (\kappa_2 - 2\kappa_1)\,\beta\,\frac{\kappa_2-\kappa_1+\sqrt{\kappa_1^2+\kappa_2^2-\kappa_1\kappa_2}}{\kappa_1}}{1 - \beta\,\frac{\kappa_2-\kappa_1+\sqrt{\kappa_1^2+\kappa_2^2-\kappa_1\kappa_2}}{\kappa_1}} =$$
$$= \frac{\kappa_1 - 2\kappa_2 - (\kappa_2 - 2\kappa_1)(1 + \frac{\sqrt{2}}{2})\frac{\kappa_2-\kappa_1}{\kappa_1}}{1 - (1 + \frac{\sqrt{2}}{2})\frac{\kappa_2-\kappa_1}{\kappa_1}} > 0\,,$$

we obtain exactly

$$\beta\,\frac{\kappa_2 - \kappa_1 + \sqrt{\kappa_1^2 + \kappa_2^2 - \kappa_1\kappa_2}}{\kappa_1} = \frac{\kappa_1 - 2\kappa_2 - \alpha}{\kappa_2 - 2\kappa_1 - \alpha}\,. \tag{5.14}$$

Hence, for $s < \bar{s}$,

$$\tilde{p}_2(s) > \frac{\kappa_1 - 2\kappa_2 - \alpha}{\kappa_2 - 2\kappa_1 - \alpha}\,\tilde{p}_1(s)\,, \tag{5.15}$$

i.e. $(\kappa_2 - 2\kappa_1)\tilde{p}_2 - (\kappa_1 - 2\kappa_2)\tilde{p}_1 > \alpha(\tilde{p}_2 - \tilde{p}_1)$. Recalling that $|\tilde{p}| \to 0$ as $s \to -\infty$, which implies the existence of $c_o > 0$ and $\hat{s}$ such that $\alpha - \tilde{p}_1(s) - \tilde{p}_2(s) > c_o$ for any $s < \hat{s}$, we find

$$\frac{d}{ds}(\tilde{p}_2 - \tilde{p}_1) = \tilde{p}_1^2 - \tilde{p}_2^2 + (\kappa_2 - 2\kappa_1)\tilde{p}_2 - (\kappa_1 - 2\kappa_2)\tilde{p}_1 > (\alpha - \tilde{p}_1 - \tilde{p}_2)(\tilde{p}_2 - \tilde{p}_1) > c_o(\tilde{p}_2 - \tilde{p}_1)\,,$$

for s small enough (namely $s < \min\{\bar{s}, \hat{s}\}$). Integrating we find $(\tilde{p}_2 - \tilde{p}_1)(s) \le \gamma e^{c_o s}$ ($\gamma > 0$) and hence (5.11) is proved. Next, recalling (3.6), in terms of the variable x we obtain

$$\left|\frac{dx}{ds}\right| = \Delta\big(\tilde{p}(s)\big) = \mathcal{O}(1) \cdot e^{2c_o s}\,, \tag{5.16}$$

$$\lim_{s \to -\infty} x(s) = x_o < \infty\,,$$

for some $x_o \in \mathbb{R}$. Therefore, to the entire trajectory $s \mapsto \tilde{p}(s)$, there corresponds only a portion of the trajectory $x \mapsto \tilde{p}(x)$, namely the values for $x > x_o$. To extend this trajectory also on the half line $]-\infty, x_o]$, we need to construct another trajectory $s \mapsto p(s)$ with $\lim_{s\to+\infty} p(s) = 0$. But any such trajectory, by previous analysis, will yield a solution $\tilde{u}(x)$, which violates the sublinear growth condition (1.15) as $x \to -\infty$ and is not admissible.

Second Step. Next, we prove uniqueness of the admissible solution the case where h_i' is not constant. Let $u(\cdot)$ be the solution constructed before, with $p = u'$ remaining in a small disc V, centered at (κ_1, κ_2) with radius $\rho > 0$, positively invariant for the flow of (3.8). Moreover, let $\tilde{u}$ be any other smooth solution of (3.3). We split the proof in three cases.

CASE 1: $\tilde{p}(s) \in V$ for every s. In this case, as in Theorem 3, we look at the difference $w(s) = \tilde{p}(s) - p(s)$ and at the linear evolution equation for w:

$$\frac{dw}{ds} = A(s)\, w(s)\,, \tag{5.17}$$

where A is the averaged matrix

$$A(s) = \int_0^1 Df(\theta p(s) + (1-\theta)\tilde{p}(s))d\theta\,, \tag{5.18}$$

and f is the vector field describing our system, as in (4.2). Since $p, \tilde{p} \in V$, the matrix $A(s)$ is very close to the Jacobian matrix $Df(\kappa_1, \kappa_2)$, therefore

$$\frac{d}{ds}\big|w(s)\big| \leq -K\,\big|w(s)\big|\,, \tag{5.19}$$

for some constant $K > 0$. Indeed,

$$Df(\kappa_1, \kappa_2)x \cdot x < -\frac{\kappa_1 + \kappa_2}{2}|x|^2. \tag{5.20}$$

Provided that $\delta, \rho > 0$ are small enough, there will exist $K > 0$ such that $A(s)x \cdot x < -K|x|^2$. But then, exactly as in Theorem 3, (5.19) implies $p(0) = \tilde{p}(0)$ and hence $p = \tilde{p}$ by the uniqueness of the Cauchy problem.

CASE 2: $\tilde{p}(s_o) \notin V$ for some s_o and, in particular, $\tilde{p}(s_o)$ in a small neighborhood W of the origin. Consider the linearized system

$$\begin{pmatrix} p_1' \\ p_2' \end{pmatrix} = H \cdot \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}\,, \qquad\qquad H = \begin{pmatrix} h_1' - h_2' & h_1' \\ h_2' & h_2' - h_1' \end{pmatrix}\,,$$

and notice that the origin is again a saddle point for this system. Indeed H has eigenvalues λ_1, λ_2 such that, recalling (5.6) and provided $\delta < \frac{1}{2}\min\{-\kappa_1, \kappa_2\}$,

$$0 < \frac{\sqrt{2}}{2}(\kappa_2 - \kappa_1 - 2\delta) \leq |\lambda_i| = \sqrt{(h_1')^2 + (h_2')^2 - h_1'h_2'} \leq \kappa_2 - \kappa_1 + 2\delta\,. \tag{5.21}$$

Moreover its eigenvectors v_1, v_2 form angles α_1, α_2 with the positive direction of the p_1-axis such that

$$0 \geq \tan\alpha_1 = \frac{\lambda_1 + (h_2' - h_1')}{h_1'} > \tan\alpha_2 = \frac{\lambda_2 + (h_2' - h_1')}{h_1'}\,.$$

More precisely, for δ small enough, we have

$$0 \geq \tan\alpha_1 > (1 - \frac{\sqrt{2}}{2})\,\frac{\kappa_2 - \kappa_1 + 2\delta}{\kappa_1 + \delta} > (1 + \frac{\sqrt{2}}{2})\,\frac{\kappa_2 - \kappa_1 - 2\delta}{\kappa_1 - \delta} > \tan\alpha_2\,. \tag{5.22}$$

Hence, as in the previous proof of Theorem 3, we show the existence of four sectors S_i, $i = 1, \ldots, 4$ (see Figure 7), where the following holds:

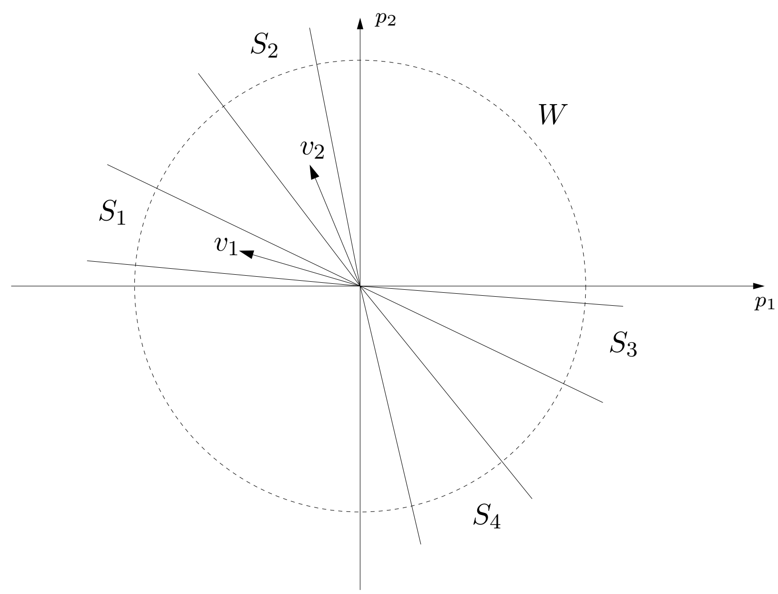

Figure 7

(a) If $\tilde{p}(s_o)$ is in S_1 or S_3, then $|\tilde{p}(s)|$ grows for $s < s_o$ and the solution moves away from W;
(b) Both boundaries of S_2 and S_4 allow orbits to only exit from those sectors for $s < s_o$;
(c) If $\tilde{p}(s_o) \notin S_i$ for all $i = 1, \ldots, 4$, then for $s < s_o$ the angle between the vector $(\tilde{p}_1, \tilde{p}_2)$ and the p_1-axis is strictly monotone, forcing the solution either to reach S_1 or S_3, or to move away from W;
(d) Finally, if $\tilde{p}(s_o)$ is in S_2 or S_4, then for $s < s_o$ the solution can tend to the origin. But

$$\begin{aligned}\frac{d}{ds}(\tilde{p}_2 - \tilde{p}_1) &= \tilde{p}_1^2 - \tilde{p}_2^2 + (h_2' - 2h_1')\tilde{p}_2 - (h_1' - 2h_2')\tilde{p}_1 > \\ &> \tilde{p}_1^2 - \tilde{p}_2^2 + (\kappa_2 - 2\kappa_1 - 3\delta)\tilde{p}_2 - (\kappa_1 - 2\kappa_2 - 3\delta)\tilde{p}_1 \,,\end{aligned}$$

and, provided δ is small enough, we can use (5.22) to find $\alpha > 0$ such that

$$\frac{d}{ds}(\tilde{p}_2-\tilde{p}_1) > \tilde{p}_1^2-\tilde{p}_2^2+(\kappa_2-2\kappa_1-3\delta)\tilde{p}_2-(\kappa_1-2\kappa_2-3\delta)\tilde{p}_1 > (\alpha-\tilde{p}_1-\tilde{p}_2)(\tilde{p}_2-\tilde{p}_1)\,. \quad (5.23)$$

Hence an estimate of exponential type of the decay of $|\tilde{p}|$ follows as in (5.11).
CASE 3: $\tilde{p}(s_o) \notin V$ and $\tilde{p}(s_o)$ not in a neighborhood of the origin. In this case, combining (5.6) and the continuous dependence of solutions with the estimates of the constant case (indeed they are true also in this more general setting), we can prove that $|\tilde{p}| \to \infty$ for finite s and that the rate of blow-up of $|\tilde{p}|$ can be estimated in the same way we did in the *First Step*.
In any case either $\tilde{u} \equiv u$ or, in the original coordinates x, $\tilde{u}$ fails to satisfy (1.15).

Finally, we rule out the possibility that the gradient $\tilde{p} = \tilde{u}'$ has jumps. Looking at the phase portrait in Figure 6, we see that after one or at most two admissible jumps, the

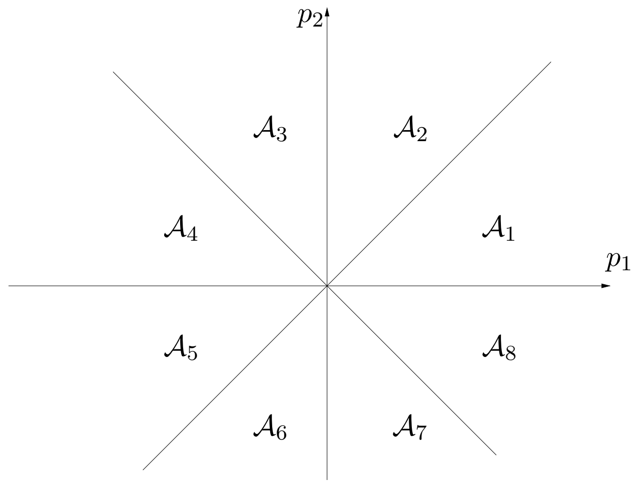

Figure 8

values of $\tilde{p}$ must fall within the positively invariant region $\{(p_1, p_2) \,:\, p_2 < 0\,,\ p_1 + p_2 < 0\}$. It follows that $\tilde{p}$ cannot have any more jumps, and the estimates in **2**, **5** and **7** (together with their analogs in the non-constant case) imply that $\tilde{u}(x)$ violates (1.15) as $x \to +\infty$. Therefore, $\tilde{u}$ is not an admissible solution. $\square$

6 Towards non-smooth costs

In the remaining part of this Chapter, we want to look for admissible solutions when smoothness of functions h_i is relaxed. Namely we consider functions h_i that are piecewise linear, with a finite number of discontinuity in their derivatives. In other words we require that there exists a finite subdivision

$$x_o = -\infty < x_1 < \ldots < x_N < x_{N+1} = +\infty$$

of $[-\infty, +\infty]$ and two $(N+1)$-tuple of constants $(\kappa_i^1, \ldots, \kappa_i^{N+1})$, $i = 1, 2$, such that

$$h_i'(x) = \kappa_i^j \quad \text{if } x \in \ ]x_j, x_{j+1}[\qquad\qquad i = 1, 2, \qquad j = 0, \ldots, N\,. \tag{6.1}$$

Could be of use to remark that this assumption on h_i' means that the system (3.8) follows different dynamics in each interval $\mathcal{I}_j \doteq\ ]x_j, x_{j+1}[$: indeed, in each $\mathcal{I}_j$, (3.8) will have an equilibrium in $(0, 0)$ and a second one in the point $K^j = (\kappa_1^j, \kappa_2^j)$.

We also introduce the following notation (see Figure 8)

$$\mathcal{A}_i = \left\{ \rho(\cos\theta, \sin\theta) \in \mathbb{R}^2 \,\middle|\, \rho > 0,\ \theta \in \ \left](i-1)\frac{\pi}{4}, i\frac{\pi}{4}\right[\ \right\}, \tag{6.2}$$

to label regions in $\mathbb{R}^2$, where we put our non-zero equilibria $K^j = (\kappa_1^j, \kappa_2^j)$.

Finally, we state a few easy properties we will need in the following. They provide expressions for both eigenvalues and eigenvectors of the system obtained linearizing (3.8) around the origin. These expressions were already found in [10], and they follow from simple linear algebra.

Proposition 1.1 *The linearized system near* $(0,0)$, *corresponding to (3.8), has the following form*

$$\begin{pmatrix} p_1' \\ p_2' \end{pmatrix} = H \cdot \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \qquad H = \begin{pmatrix} \kappa_1 - \kappa_2 & \kappa_1 \\ \kappa_2 & \kappa_2 - \kappa_1 \end{pmatrix}. \tag{6.3}$$

Moreover the eigenvalues of the matrix H *are*

$$\lambda_- = -\sqrt{(\kappa_1)^2 + (\kappa_2)^2 - \kappa_1\kappa_2}\,, \qquad \lambda_+ = \sqrt{(\kappa_1)^2 + (\kappa_2)^2 - \kappa_1\kappa_2}\,, \tag{6.4}$$

with corresponding eigenvectors

$$\begin{aligned} v_- &= \left(1,\ \frac{\kappa_2 - \kappa_1 - \sqrt{(\kappa_1)^2 + (\kappa_2)^2 - \kappa_1\kappa_2}}{\kappa_1}\right), \\ v_+ &= \left(1,\ \frac{\kappa_2 - \kappa_1 + \sqrt{(\kappa_1)^2 + (\kappa_2)^2 - \kappa_1\kappa_2}}{\kappa_1}\right). \end{aligned} \tag{6.5}$$

One can immediately see that the eigenvectors in (6.5) depend actually by the ratio between κ_2 and κ_1 only. Moreover it turns out that this kind of dependence is indeed monotone increasing, as proved in the following Proposition.

Proposition 1.2 *Set* $\alpha = \frac{\kappa_2}{\kappa_1}$. *Then the directions corresponding to the eigenvectors* v_- *and* v_+ *are given (respectively) by the maps*

$$G_-(\alpha)\colon \begin{cases} \,]0,\infty[& \to \ ]-2,-\frac{1}{2}[\\ \alpha & \mapsto \ \alpha - 1 - \sqrt{\alpha^2 - \alpha + 1} \end{cases}$$

$$g_-(\alpha)\colon \begin{cases} \,]-\infty,0[& \to \ ]-\infty,-2[\\ \alpha & \mapsto \ \alpha - 1 - \sqrt{\alpha^2 - \alpha + 1} \end{cases}$$

$$G_+(\alpha)\colon \begin{cases} \,]0,\infty[& \to \ ]0,\infty[\\ \alpha & \mapsto \ \alpha - 1 + \sqrt{\alpha^2 - \alpha + 1} \end{cases}$$

$$g_+(\alpha)\colon \begin{cases} \,]-\infty,0[& \to \ ]-\frac{1}{2},0[\\ \alpha & \mapsto \ \alpha - 1 + \sqrt{\alpha^2 - \alpha + 1} \end{cases}$$

depending on the sign of α *(and hence of* $\kappa_1 \cdot \kappa_2$*). These maps satisfy*

$$\begin{aligned} &\tfrac{d}{d\alpha} G_- > 0\,, && \tfrac{d}{d\alpha} G_+ > 0\,, \\ &\tfrac{d}{d\alpha} g_- > 0\,, && \tfrac{d}{d\alpha} g_+ > 0\,. \end{aligned} \tag{6.6}$$

Proof. The properties follow from

$$G'_-(\alpha) = g'_-(\alpha) = 1 - \frac{2\alpha - 1}{2\sqrt{\alpha^2 - \alpha + 1}} = \frac{\sqrt{(2\alpha-1)^2+3} - (2\alpha - 1)}{2\sqrt{\alpha^2 - \alpha + 1}} > 0\,,$$

$$G'_+(\alpha) = g'_+(\alpha) = 1 + \frac{2\alpha - 1}{2\sqrt{\alpha^2 - \alpha + 1}} = \frac{\sqrt{(2\alpha-1)^2+3} + (2\alpha - 1)}{2\sqrt{\alpha^2 - \alpha + 1}} > 0\,,$$

and from

$$\lim_{\alpha\to 0^+} G_-(\alpha) = \lim_{\alpha\to 0^-} g_-(\alpha) = -1 - 1 = -2\,,$$

$$\lim_{\alpha\to 0^+} G_+(\alpha) = \lim_{\alpha\to 0^-} g_+(\alpha) = -1 + 1 = 0\,,$$

$$\lim_{\alpha\to +\infty} G_-(\alpha) = \lim_{\alpha\to +\infty} -\frac{\alpha}{\alpha - 1 + \sqrt{(\alpha-1)^2 + \alpha}} = -\frac{1}{2}\,,$$

$$\lim_{\alpha\to -\infty} g_+(\alpha) = \lim_{\alpha\to -\infty} -\frac{\alpha}{\alpha - 1 - \sqrt{(\alpha-1)^2 + \alpha}} = -\frac{1}{2}\,,$$

$$\lim_{\alpha\to +\infty} G_+(\alpha) = +\infty\,,$$

$$\lim_{\alpha\to -\infty} g_-(\alpha) = -\infty\,.$$

□

Next Proposition collects properties of the rescaling $s = s(x)$, introduced in Section 3, which were proved while proving Theorem 1.3 and Theorem 1.4.

Proposition 1.3 *In the rescaled variable* $s = s(x)$*, such that* $ds/dx = \Delta(p)^{-1}$*, the following holds.*
(i) *Every unbounded trajectory* $p(s)$ *of (3.8) actually blows up at finite* s_o*, and it corresponds to an unbounded trajectory* $p(x)$ *that tends to* ∞ *as* $|x| \to \infty$*. Moreover, since*

$$\left|\frac{dx}{ds}\right| = \Delta\big(p(s)\big) \geq \frac{c_o}{(s_o - s)^2}\,,$$

it follows that $u(x)$ *increases more than linearly as* $x \to \infty$*. Therefore,* u *is not admissible.*
(ii) *Trajectories of (3.8) that tend to the origin, i.e. to the point where our change of variables is singular, satisfy*

$$\left|\frac{dx}{ds}\right| = \Delta\big(p(s)\big) = \mathcal{O}(1)\cdot e^{-2c_o|s|}\,.$$

In the original variable x*, to the whole trajectory* $s \mapsto p(s)$ *there corresponds only a portion of trajectory* $x \mapsto p(x)$*, say either for* $x \in\ ]x_o, \infty[$ *or* $x \in\ ]-\infty, x_o[$*. Another trajectory* $s \mapsto \hat{p}(s)$ *has to be constructed to extend the solution to all* $x \in \mathbb{R}$*.*

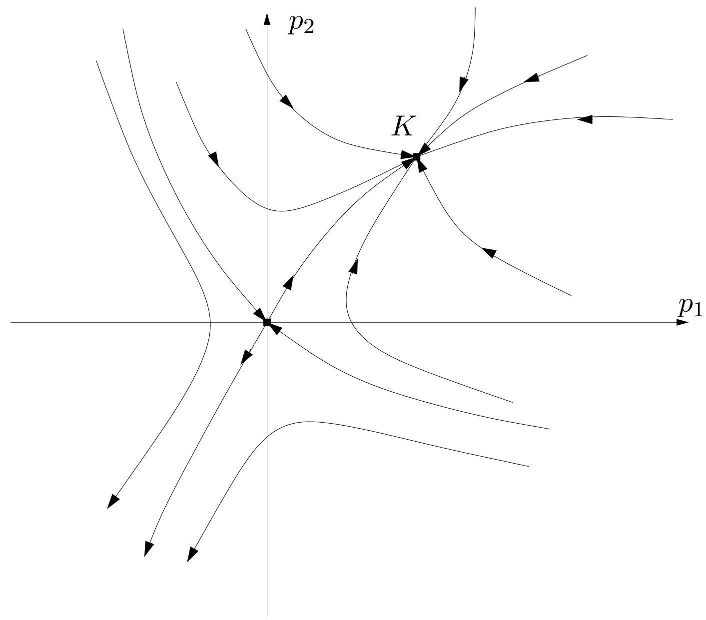

Figure 9

7 Non-smooth costs: cooperative situations

We start considering all $K^j = (\kappa_1^j, \kappa_2^j)$ in $\mathcal{A}_1 \cup \mathcal{A}_2$. Notice that a similar analysis, with straightforward adaptations, can be done if the K^j are in $\mathcal{A}_5 \cup \mathcal{A}_6$. This choice implies that our system follows the dynamics depicted in Figure 9.

Theorem 1.5 *Let the cost functions h_1, h_2 be as in (6.1), and assume that the constants (κ_1^j, κ_2^j) are all chosen in $\mathcal{A}_1 \cup \mathcal{A}_2$. Then the system (3.3) has a unique admissible solution and the corresponding functions $\alpha_i^* = -u_i'$ provide a Nash equilibrium solution to the non-cooperative game (3.1).*

Proof. **Existence.** The existence of an admissible solution is very easy to prove. Indeed, it is enough to glue together pieces of admissible solutions in each interval $\mathcal{I}_j$. We proceed as follows:

- in $\mathcal{I}_o$, we set $p^o \equiv K^o = (\kappa_1^o, \kappa_2^o)$;
- for $j \geq 1$, in $\mathcal{I}_j$ we set p^j the unique solution of the Cauchy problem for (3.8) with initial datum $p(s(x_j)) = p^{j-1}(s(x_j))$. Since the set

$$\Gamma^j = \left\{ (p_1, p_2) \,\middle|\, p_1, p_2 \in [0, 2C_1], p_1 + p_2 \geq \frac{C_2}{2} \right\},$$

  where

$$C_1 = \max\{\kappa_1^j, \kappa_2^j, \frac{p_1^{j-1}(s(x_j))}{2}, \frac{p_2^{j-1}(s(x_j))}{2}\},$$
$$C_2 = \min\{\kappa_1^j, \kappa_2^j, p_1^{j-1}(s(x_j)) + p_2^{j-1}(s(x_j))\},$$

  is positively invariant for (3.8), each p^j will exists up to $s(x^{j+1})$ without reaching $(0,0)$ and remaining bounded;

Then, it is well defined the continuous function $\bar{p}$ given by $\bar{p}(x) = p^j(x)$ whenever $x \in \mathcal{I}_j$. Its admissibility is an immediate consequence of its continuity and the admissibility of each p^j.

Uniqueness. To prove that the solution built above is the unique admissible solution to (3.8), we start proving uniqueness on $\mathcal{I}_o$.

We know from [10] that, for s negative small enough (eventually for $s \to -\infty$), the only solutions that remain bounded are the equilibrium K^o itself and the unstable orbits exiting from the origin. Therefore, these are the unique possible choices, in order to retain admissibility. If we choose an unstable orbit in place of K^o, in the original variable x it would correspond to a solution defined only for $x > x_o$ (for a suitable x_o). To define the solution also for $x < x_o$, we should need a solution to

$$\begin{cases} p_1' = (\kappa_1^o - \kappa_2^o)p_1 + \kappa_1^o p_2 - p_1^2\,, \\[2ex] p_2' = (\kappa_2^o - \kappa_1^o)p_2 + \kappa_2^o p_1 - p_2^2\,, \end{cases}$$

that tends to the origin as $s \to +\infty$ and remains bounded for all negative s. But we know from [10] that no solution with both these properties exists. Hence the uniqueness of the solution follows on $\mathcal{I}_o$.

For $s > s(x_1)$, the smoothness of the right hand side of (3.8) in each interval $\mathcal{I}_j$ ensures that $\bar{p}$ is the unique continuous solution.

It remains to prove that there exists no solution with admissible jumps in $s > s(x_1)$. But this property follows from (3.9)–(3.10) and from the positive invariance of the sets

$$\Gamma^+ \doteq \left\{(p_1, p_2) \in \mathbb{R}^2 \middle| p_1 \geq 0,\ p_2 \geq 0\right\}\,,$$
$$\Gamma^- \doteq \left\{(p_1, p_2) \in \mathbb{R}^2 \middle| p_1 \leq 0,\ p_2 \leq 0\right\}\,.$$

Indeed, for $s > s(x_1)$ a solution can have only jumps from Γ^+ to Γ^-. Hence, recalling [10], after a first jump the solution would be forced to remain in Γ^- and to tend towards ∞. In the x variable, this would translate into a solution $u(x)$ that grows more than linearly as $|x| \to \infty$, and this would contradict admissibility. □

In light of Theorem 1.5, on the same line of [10], it is natural to ask whether the result still hold for perturbations of (6.1) or it fails. Actually, we can prove the following Theorem.

Theorem 1.6 *Let the cost functions h_1', h_2' in (3.2) be smooth in each $\mathcal{I}_n$, and assume that:*

(1) their derivatives satisfy

$$\frac{1}{C} \leq h_i'(x) \leq C$$

for some constant $C > 1$ and all $x \in \mathbb{R}$;

(2) on $\mathcal{I}_o$, the following additional assumption is satisfied

$$\sup_{\xi,\eta \in \mathcal{I}_o} \left|h_i'(\xi) - h_i'(\eta)\right| \leq \delta \qquad i = 1, 2. \tag{7.1}$$

for some $\delta > 0$ sufficiently small (depending only on C).

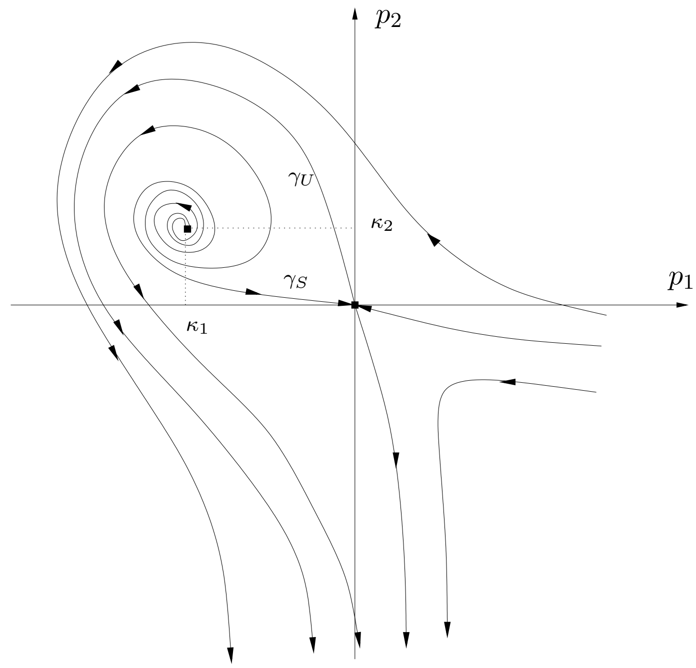

Figure 10

Then the system (3.3) has a unique admissible solution.

Proof. We can proceed as in Theorem 1.5, using Theorem 1.3 to deal with the perturbations. Indeed, for $s < s(x_1)$ Theorem 1.3 implies that there exists a unique admissible solution, say p^o. Hence, an admissible solution on the whole real line can be built as in the previous case: for $x \in]x_j, x_{j+1}[$, $j \geq 1$, we define $p(x) = p^j(x)$ where p^j is the unique solution to (3.3) with initial datum $p(s(x_j)) = p^{j-1}(s(x_j))$. Exactly as in Theorem 1.5, this function is well defined and is a continuous admissible solution to (3.8). Since the sets Γ^+ and Γ^- are still positively invariant, also uniqueness can be proved by means of the same arguments used in Theorem 1.5. □

Remark 1.1 We underline that the presence of the small oscillations assumption (7.1) is uniquely motivated by the use of Theorem 1.3, which requires (7.1) to provide a unique admissible solution for $s < s(x_o)$.

8 Non-smooth costs: players with conflicting interests

In this section we assume that the two players have conflicting interests, i.e. their costs satisfy $h_1'(x) \cdot h_2'(x) < -C < 0$ for all $x \in \mathbb{R}$. For particular choices of smooth costs, this situation can produce infinitely many Nash equilibria to the game (see Example 1.2). Nevertheless Theorem 1.4 shows that, for costs which are not exactly opposite and under suitable assumptions of small oscillations, it is possible to recover existence and uniqueness of Nash equilibria. This is not the case for costs as in (6.1).

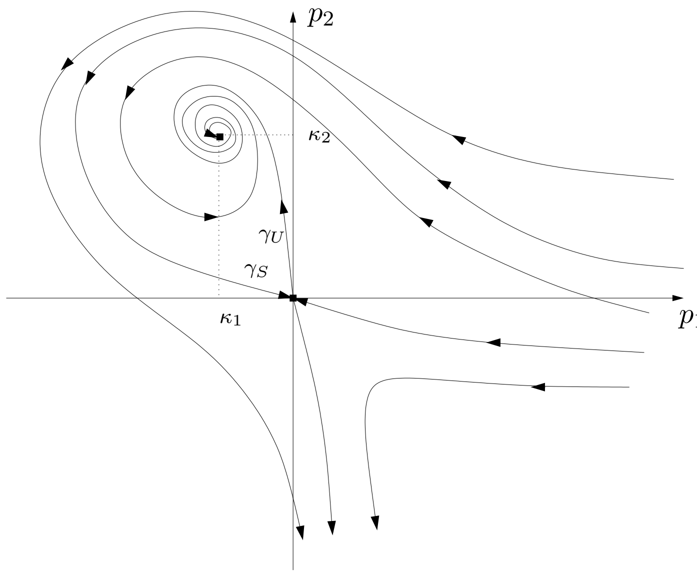

Figure 11

8.1 Infinitely many Nash equilibria

Let us consider $j = 1$ in (6.1), i.e. let us consider cost functionals that have a single jump in their derivatives. In particular, assume this jump is located at $x = s(x) = 0$. Moreover, let us choose the constants $K^j = (\kappa_1^j, \kappa_2^j)$, $j = 0, 1$, so that $K^o \in \mathcal{A}_4$ and $K^1 \in \mathcal{A}_3$.

Under these assumptions, the dynamics followed by the system are depicted in Figure 10 (for $x < 0$) and Figure 11 (for $x > 0$). We now prove that we could find infinitely many solutions to our problem. Indeed, consider an initial datum $p^{\text{in}} = (p_1^{\text{in}}, p_2^{\text{in}})$ such that $p_1^{\text{in}} + p_2^{\text{in}} = 0$ and $p_1^{\text{in}} < 0 < p_2^{\text{in}}$. Recalling Proposition 1.2 and setting $\alpha^o = \frac{\kappa_2^o}{\kappa_1^o}, \alpha^1 = \frac{\kappa_2^1}{\kappa_1^1}$, we have

$$g_-(\alpha^1) < -2 < -1 = \frac{p_2^{\text{in}}}{p_1^{\text{in}}} < -\frac{1}{2} < g_+(\alpha^o)\,,$$

i.e. p^{in} belongs to the region between the stable orbit for the negative system (say γ_S^-) and the unstable one for the positive system (say γ_U^+), provided it's been chosen sufficiently near the origin. Therefore to any choice of p^{in} there corresponds an admissible solution tending respectively to either K^1 or K^o as $s \to \pm\infty$.

Moreover, if the unstable orbit for the dynamics in Figure 10 (say γ_U^-) intersects the stable one for the dynamics in Figure 11 (say γ_S^+), we can obtain an additional solution considering as initial datum that point of intersection. Indeed the function given by the juxtaposition of γ_U^- and γ_S^+ corresponds, in the original variable x, to a solution defined on a bounded interval $[x_-, x_+]$, with $x_- < 0 < x_+$ by the choice of the rescaling. This solution can then be extended to an admissible trajectory defined on the whole real line by using γ_S^- for $x < x_-$ and γ_U^+ for $x > x_+$.

Remark 1.2 The same construction can be applied when $K^o \in \mathcal{A}_8$ and $K^1 \in \mathcal{A}_7$.

8.2 No admissible solutions

Now we want to show, by means of a second example, how a simple change between the positive and negative behaviors of the costs, can lead to completely different result. Namely, we consider costs with a single jump in their derivatives, located in $x = s(x) = 0$, and $K^o \in \mathcal{A}_3$, $K^1 \in \mathcal{A}_4$. This choice produce a game with no admissible solutions to (3.8).

We proceed by contradiction. Assume that an admissible solution $\tilde{p} = (\tilde{p}_1, \tilde{p}_2)$ exists, for a Cauchy problem with initial datum $\tilde{p}(0) = p^{\text{in}}$. Then, recalling the results in [10], we have that

$$\lim_{s \to +\infty} |\tilde{p}(s)| < +\infty$$

actually implies

$$\lim_{s \to +\infty} |\tilde{p}(s)| = 0\,,$$

and hence $\tilde{p}$ is one of the stable orbits of the positive system. Now we underline that this means $p^{\text{in}} \notin \gamma_U^+$. Then, we can repeat the proof of Theorem 1.4, given in [10], and find

$$\lim_{s \to s_o+} |\tilde{p}(s)| = +\infty\,,$$

for a suitable $s_o < 0$, eventually $s_o = -\infty$. Therefore the solution cannot be admissible, and we have a contradiction.

Notice that the previous calculations hold even if the unstable orbit for the dynamics in Figure 10 (say γ_U^-) intersects the stable one for the dynamics in Figure 11 (say γ_S^+). This means there is no solution as the one built in the previous case, using more trajectories in the s variable: this is obviously due to the fact that we cannot find solutions bounded at $+\infty$ (resp. $-\infty$) to extend a possible $\tilde{p}$ when $x > x_+$ (resp. $x < x_-$).

Remark 1.3 The same result can be obtained when $K^o \in \mathcal{A}_7$ and $K^1 \in \mathcal{A}_8$.

Remark 1.4 Actually, one can still construct particular cases so that there exist admissible solutions. Fixed K^o, K^1 as above, assume that the trajectories γ_U^- and γ_S^+ intersect in a point. Moreover, set x_- and x_+ the values introduced in the previous example, $\ell = |x_+ - x_-|$ and $\mathcal{J}_n =]x_- + n\ell, x_+ + n\ell[$, $n \in \mathbb{Z}$. We can define piecewise linear costs on the whole $\mathbb{R}$ by repeating on each $\mathcal{J}_n$ the same 2-value piecewise linear cost. In other words, $\forall n \in \mathbb{Z}$ set

$$h_i'(x)\Big|_{\mathcal{J}_n} = \begin{cases} \kappa_i^o & \text{if } x \in \ ]x_- + n\ell, n\ell[\\ \kappa_i^1 & \text{if } x \in \ ]n\ell, x_+ + n\ell[\end{cases} \qquad i = 1, 2, \tag{8.1}$$

Then, we find a solution by simply gluing together periodically γ_U^- and γ_S^+. This solution is admissible, being bounded in the p_1, p_2 plane.

Anyway no general results as Theorem 1.4 is possible.

9 Non-smooth costs: a mixed case

In this section we end our presentation of ill-posed problems, with a last example presenting costs that can switch from a situation with conflicting interests into a cooperative

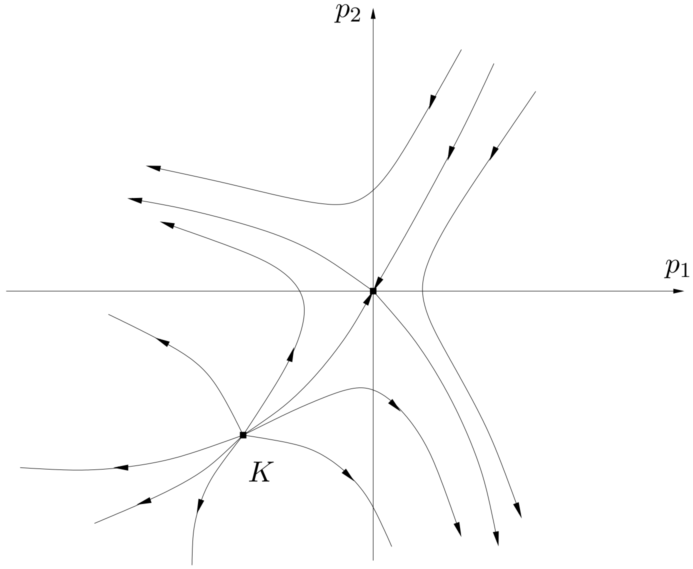

Figure 12

one. More precisely, we consider costs with a single jump in their derivative, located again in $x = s(x) = 0$, and $K^o \in \mathcal{A}_5 \cup \mathcal{A}_6$, $K^1 \in \mathcal{A}_1 \cup \mathcal{A}_2$. Moreover, let us assume

$$\alpha^1 = \frac{\kappa_2^1}{\kappa_1^1} \neq \frac{\kappa_2^o}{\kappa_1^o} = \alpha^o \,. \tag{9.1}$$

With these assumptions, the system follows the dynamics depicted in Figure 12 (resp. Figure 9) for $x < 0$ (resp. $x > 0$) and K^o, K^1 are not on the same line through the origin.

Again, we observe the existence of infinitely many Nash equilibria. Assume it holds $\alpha^o < \alpha^1$ in (9.1) (the opposite inequality leading to a similar analysis). Then, we can consider the non-empty region

$$\Omega = \left\{ (p_1, p_2) \in \mathbb{R}^2 \;\middle|\; p_1 < 0 < p_2,\ G_-(\alpha^o) < \frac{p_2}{p_1} < G_-(\alpha^1) \right\} .$$

This region is, at least near the origin, say in a neighborhood $\mathcal{O}$, exactly the region between the stable orbit for the positive system and the unstable one for the negative system. Taking as initial datum any point $p^{\rm in}$ both in Ω and in $\mathcal{O}$, we can construct an admissible solution in the following way. We take for $s < 0$ the unique solution to the negative system, passing through $p^{\rm in}$ at $s = 0$ and tending to K^o as $s \to -\infty$. In an analogous way, we take for $s > 0$ the unique solution to the positive system, passing through $p^{\rm in}$ at $s = 0$ and tending to K^1 as $s \to +\infty$. Every such a solution, being continuous and bounded in s, corresponds to an admissible solution $u(x)$.

Chapter 2

Nearly Optimal Patchy Feedback Controls

1 Statement of the main result

We consider a general optimization problem

$$\min \left\{ \psi\big(x(T)\big) + \int_0^T L\big(x(t),\, u(t)\big)\, dt \right\} , \tag{1.1}$$

for a nonlinear control system of the form

$$\dot{x} = f(x,u) \qquad\qquad u(t) \in \mathbf{U}\,. \tag{1.2}$$

Here $x \in \mathbb{R}^n$ describes the state of the system, the upper dot denotes a derivative w.r.t. time, and $\mathbf{U} \subset \mathbb{R}^m$ is the set of admissible control values.

In the literature, several results are available, which provide the existence of an optimal control $t \mapsto u^{opt}(t)$ in open-loop form, for each initial condition

$$x(0) = y\,, \tag{1.3}$$

On the other hand, the existence and regularity of an optimal control in feedback form is a far more difficult issue. In an ideal situation, one would like to construct a (sufficiently regular) feedback $u = U(x)$ such that all trajectories of the corresponding O.D.E.

$$\dot{x} = f\big(x, U(x)\big) \tag{1.4}$$

are optimal w.r.t. the cost criterion (1.1). Only few general results are yet known in this direction.

A possible strategy (see [24, 32]) is to investigate an optimal "synthesis", which is a collection of optimal trajectories not necessarily arising from a feedback control. The existence and the structure of an optimal synthesis has been the subject of a large body of literature on nonlinear control. At present, a complete description is known for time optimal planar systems of the form

$$\dot{x} = f(x) + g(x)\,u \qquad\qquad u \in [-1,1]\,, \qquad x \in \mathbb{R}^2\,,$$

see [7] and the references therein. For more general classes of optimal control problems, or in higher space dimensions, the construction of an optimal synthesis faces severe difficulties: the optimal synthesis can have an extremely complicated structure and even in cases where a regular synthesis exists, the performance achieved by the optimal synthesis may not be robust. About the former, already for systems in two space dimensions, an accurate description of all generic singularities of a time optimal synthesis involves the classification of eighteen topological equivalence classes of singular points [24, 25]. In higher dimensions, an even larger number of different singularities arises, and the optimal synthesis can exhibit pathological behavior such as the famous "Fuller phenomenon" (see [22, 33]), where every optimal control has an infinite number of switchings. About the latter, small perturbations can greatly affect the behavior of the synthesis (e.g. see Example 5.3 in [26]).

An alternative strategy, pursued in [17, 20, 23, 4], is to construct sub-optimal feedbacks, trading off the full optimality in favor of a simpler structure of the control and the robustness of the resulting system.

Anyway, this approach has its own difficulties. First of all, it is known that no continuous optimal feedback can be constructed [23, 9]. Therefore, one has to deal with a discontinuous right hand side in (1.4) and to introduce a suitable concept of solution for this kind of ODEs.

In literature, two different attacks to this theoretical obstacle can be found: either to introduce a new concept of generalized solution without any restriction on the choice of $U(x)$, or to allow only sufficiently tame discontinuities in $U(x)$, so that one can still have Carathódory trajectories. The first one was followed in the [16, 17, 20, 23, 28], where "sample-and-hold" solutions and Euler solutions are used to solve both controllability and sub-optimality problems. The second approach was studied in [1, 2, 3, 4], choosing *patchy feedbacks* as the class of allowed discontinuous controls. A *patchy control* in feedback form has a particularly simple structure, since it is a function $u = U(x)$ that is piecewise constant on the state space $\mathbb{R}^n$. One can prove that, for patchy feedbacks, forward Carathéodory solutions always exists and that this controls are robust (see [1, 2, 3]). Moreover, the sub-optimal control problem can be solved using these patchy feedbacks (see [4]).

Indeed, let $T(y) \leq \infty$ be the minimum time needed to steer the system (1.2) from the state y to the origin. Then the analysis in [4] shows that, for every $\varepsilon > 0$, there exists a patchy feedback which steers any initial state $y \in \mathbb{R}^n$ to some point inside the ball $B(0, \varepsilon)$ of radius ε around the origin, within time $(1 + \varepsilon)\, T(y)$.

In all cited works, patchy feedbacks were constructed either by patching together piecewise constant open-loop controls as in [1, 2, 3], or, as in [4], relying on the a-priori knowledge of the value function V. We recall that this is defined as

$$V(y) \doteq \inf_{u(\cdot)} \left\{ \psi\big(x(T)\big) + \int_0^T L\big(x(t),\, u(t)\big)\, dt \right\}, \tag{1.5}$$

where the minimization is taken over all $T \geq 0$ and all control functions $u : [0, T] \mapsto \mathbf{U}$ such that the trajectory of (1.2) reaches a sufficiently small neighborhood of the origin at time T.

Aim of the present Chapter, based on [11], is to develop an algorithm that produces a nearly-optimal patchy feedback "starting from scratch", i.e. without any a-priori information about the optimal trajectories. Both the patchy feedback and an approximate value function will here be constructed simultaneously, working iteratively on higher and higher level sets. For convenience, we list here the basic assumptions used throughout the Chapter.

(H) The set of admissible control values $U \subset \mathbb{R}^m$ is a compact, the function $f : \mathbb{R}^n \times U \mapsto \mathbb{R}^n$ is Lipschitz continuous and satisfies the sublinear growth condition

$$\big|f(x,u)\big| \leq C\big(1+|x|\big) \qquad\qquad \forall\, u \in U\,. \tag{1.6}$$

Both the terminal cost $\psi : \mathbb{R}^n \mapsto \mathbb{R}$ and the running cost $L : \mathbb{R}^n \times \mathbf{U} \mapsto \mathbb{R}$ are continuous and strictly positive, say

$$\psi(x) \geq c_0 > 0\,, \qquad\qquad L(x,u) \geq c_0 > 0 \qquad\qquad \forall\; x \in \mathbb{R}^n, \quad u \in \mathbf{U}\,. \tag{1.7}$$

Moreover,

$$\lim_{|x|\to\infty} \psi(x) = \infty\,. \tag{1.8}$$

Our main result can be stated as follows.

Theorem 2.1 *Let the system (1.2) satisfy the assumptions* **(H)***, and let $\varepsilon > 0$ be given. Then there exist a closed terminal set $S \subseteq \mathbb{R}^n$, a continuous function $W : \mathbb{R}^n \to \mathbb{R}_+$ and a patchy feedback $u = U(x)$ defined on the complement $\mathbb{R}^n \setminus S$ such that the following holds.*

(i) For every $y \in \mathbb{R}^n$, every Carathéodory solution of

$$\dot{x} = f\big(x,\, U(x)\big)\,, \qquad\qquad x(0) = y \tag{1.9}$$

eventually reaches the set S.

(ii) Calling $\tau = \tau(y)$ the first time where $x(\tau) \in S$, we have

$$\psi\big(x(\tau)\big) + \int_0^\tau L\big(x(t),\, U(x(t))\big)\, dt \;\leq (1+\varepsilon)\, W(y)\,. \tag{1.10}$$

(iii) Outside S, one has

$$-1-\varepsilon \leq \min_{\omega\in\mathbf{U}} \nabla W(x)\cdot f(x,\omega) \leq \nabla W(x)\cdot f(x,U(x)) \leq -1+\varepsilon\,. \tag{1.11}$$

2 Basic definitions

In this Section we recall basic definitions and properties connected to patchy controls, as introduced in [1, 2, 3, 4].

Definition 2.1 *A couple $\big(\Omega,\, g\big)$ is said to be a* patch *if $\Omega \subset \mathbb{R}^n$ is an open domain with smooth boundary $\partial\Omega$, and g is a smooth vector field defined on a neighborhood of the closure $\overline{\Omega}$ of Ω, which points strictly inward at each boundary point $x \in \partial\Omega$.*

Setting $\mathbf{n}(x)$ the outer normal at the boundary point x, we thus require

$$\big\langle g(x),\; \mathbf{n}(x)\big\rangle < 0 \qquad \forall t \quad x \in \partial\Omega. \tag{2.1}$$

Definition 2.2 *We say that* $g : \Omega \mapsto \mathbb{R}^n$ *is a* patchy vector field *on the open domain* Ω *if there exists a family of patches* $\{(\Omega_\alpha,\ g_\alpha);\quad \alpha \in \mathcal{A}\}$ *such that*
- $\mathcal{A}$ *is a totally ordered set of indices,*
- *the open sets* Ω_α *form a locally finite covering of* Ω*,*
- *the vector field* g *can be written in the form*

$$g(x) = g_\alpha(x) \qquad \textit{if} \qquad x \in \Omega_\alpha \setminus \bigcup_{\beta > \alpha} \Omega_\beta. \tag{2.2}$$

We shall occasionally adopt the longer notation $\big(\Omega,\ g,\ (\Omega_\alpha,\ g_\alpha)_{\alpha\in\mathcal{A}}\big)$ *to indicate a patchy vector field, when we want to specify both the domain and the single patches.*

Setting

$$\alpha^*(x) \doteq \max\big\{\alpha \in \mathcal{A}\ ;\quad x \in \Omega_\alpha\big\}, \tag{2.3}$$

we can rewrite (2.2) in the following form

$$g(x) = g_{\alpha^*(x)}(x) \qquad \forall t \quad x \in \Omega. \tag{2.4}$$

Remark 2.1 Notice that the patches $(\Omega_\alpha,\ g_\alpha)$ are not uniquely determined by a patchy vector field $(\Omega,\ g)$. Indeed, whenever $\alpha < \beta$, by (2.2) the values of g_α on the set $\Omega_\alpha \cap \Omega_\beta$ are irrelevant. Therefore, whenever we have open sets Ω_α which form a locally finite covering of Ω and vector fields g_α so that, for each $\alpha \in \mathcal{A}$, (2.1) is satisfied at every point $x \in \partial\Omega_\alpha \setminus \bigcup_{\beta>\alpha} \Omega_\beta$, then the vector field g defined using (2.2) is again a patchy vector field.
To see this, it suffices to construct new vector fields $\tilde{g}_\alpha$ (still defined on a neighborhood of $\overline{\Omega}_\alpha$ as g_α) which satisfy the inward pointing property (2.1) at every point $x \in \partial\Omega_\alpha$ and such that $\tilde{g}_\alpha = g_\alpha$ on $\Omega_\alpha \setminus \bigcup_{\beta>\alpha} \Omega_\beta$ (cfr. Remark 2.1 in [1]). In fact, with the same arguments one deduces that, to guarantee that a vector field g defined on an open domain Ω according with (2.2) be a patchy vector field, it is sufficient to require that each vector field g_α satisfy (2.1) at every point $x \in \partial\Omega_\alpha \setminus \big(\big(\bigcup_{\beta>\alpha} \Omega_\beta\big) \cup \partial\Omega\big)$.

If g is a patchy vector field, the differential equation

$$\dot{x} = g(x) \tag{2.5}$$

has several useful properties. In particular, in [1] it was proved that the set of Carathéodory solutions of (2.5) is closed (in the topology of uniform convergence) but possibly not connected. This allows to circumvent the topological obstructions present in [23, 9]. Moreover, given an initial condition

$$x(t_0) = x_0, \tag{2.6}$$

the Cauchy problem (2.5)–(2.6) has at least one forward solution, and at most one backward solution, in the Carathéodory sense. For every Carathéodory solution $x = x(t)$ of (2.5), the map $t \mapsto \alpha^*(x(t))$ is left continuous and non-decreasing.

More precisely the following theorem can be proved

Theorem 2.2 *Let* $\big(\Omega,\ g,\ (\Omega_\alpha,\, g_\alpha)_{\alpha\in\mathcal{A}}\big)$ *be a patchy vector field on an open domain* Ω. *Then the following holds.*

(i) If $t \mapsto x(t)$ *is a Carathéodory solution of (2.5) on an open interval* J, *then* $t \mapsto \dot x(t)$ *is piecewise smooth and has a finite set of jumps on any compact subinterval* $J' \subset J$. *The function* $t \mapsto \alpha^*(x(t))$ *defined by (2.3) is piecewise constant, left continuous and nondecreasing. Moreover, there holds*

$$\dot x(-t) = g(x(t)) \qquad \text{for all } t \in J\,.$$

(ii) For each $\bar x \in \Omega$, *the Cauchy problem for (2.5) with initial condition* $x(0) = \bar x$ *has at least one local forward Carathéodory solution and at most one local backward Carathéodory solution.*

(iii) The set of Carathéodory solutions of (2.5) is closed.

Definition 2.3 *Let* $\big(\Omega,\ g,\ (\Omega_\alpha,\, g_\alpha)_{\alpha\in\mathcal{A}}\big)$ *be a patchy vector field. Assume that there exist control values* $v_\alpha \in \mathbf{U}$ *such that, for each* $\alpha \in \mathcal{A}$, *there holds*

$$g_\alpha(x) = f(x,\, v_\alpha) \qquad\qquad \forall\ x \in D_\alpha \doteq \Omega_\alpha \setminus \bigcup_{\beta>\alpha} \Omega_\beta. \tag{2.7}$$

Then, the piecewise constant map

$$u = U(x) \doteq v_\alpha \qquad \text{if} \qquad x \in D_\alpha \tag{2.8}$$

is called a patchy feedback *control for the system* $\dot x = f(x,u)$ *on* Ω, *and referred to as*

$$\big(\Omega,\ U,\ (\Omega_\alpha,\, v_\alpha)_{\alpha\in\mathcal{A}}\big)$$

.

Remark 2.2 By Definitions 2.1 and 2.2, the vector field

$$g(x) = f\big(x,\, U(x)\big)$$

defined in connection with a given patchy feedback $\big(\Omega,\ U,\ (\Omega_\alpha,\, v_\alpha)_{\alpha\in\mathcal{A}}\big)$ is precisely the patchy vector field $\big(\Omega,\ g,\ (\Omega_\alpha,\, g_\alpha)_{\alpha\in\mathcal{A}}\big)$ associated with a family of fields $\big\{g_\alpha : \alpha \in \mathcal{A}\big\}$ satisfying (2.1). Notice that, recalling the notation (2.3), for all $x \in \Omega$ we have

$$U(x) = v_{\alpha^*(x)}\,. \tag{2.9}$$

As observed in Remark 2.1, the values of the vector fields $f(x,\, v_\alpha)$ on the set $\Omega_\alpha \cap \Omega_\beta$ are irrelevant whenever $\alpha < \beta$, and it is not necessary that $f(x,\, v_\alpha)$ satisfy the inward-pointing condition (2.1) at the points of $\partial\Omega_\alpha \cap \big(\bigcup_{\beta>\alpha} \Omega_\beta\big)$. Moreover, all the properties of a patchy feedback continue to hold even in the case where we assume that the inward-pointing condition (2.1) fails to be satisfied at the points of $(\partial\Omega_\alpha \cap \Sigma) \setminus \bigcup_{\beta>\alpha} \Omega_\beta$, for some region Σ of the boundary $\partial\Omega$. Clearly, in this case every Carathéodory trajectory of the patchy vector field g can eventually reach the boundary $\partial\Omega$ only crossing points of Σ.

The following results hold for patchy vector fields and patchy controls.

Theorem 2.3 *If the system $\dot{x} = f(x,u)$ is asymptotically controllable, then it admits an asymptotically stabilizing patchy feedback.*

Theorem 2.4 *The patchy feedback control found in Theorem 2.3, is robust with respect to both external disturbances and measurement errors. Namely, if g is a patchy vector field on an open domain Ω and w is a left continuous function with bounded total variation, then for any closed $A \subset \Omega$, any compact $K \subset A$ and any T, ε there exists $\delta = \delta(A, K, T, \varepsilon) > 0$ such that the following holds. If $\xi\colon [0,T] \to A$ is a solution of the perturbed system*

$$\dot{\xi} = g(\xi) + \dot{w}\,, \tag{2.10}$$

with $\xi(0) \in K$ and $Tot.Var.\{w\} < \delta$, then there exists a solution $x\colon [0,T] \to \Omega$ of $\dot{x} = g(x)$ with

$$||x - y||_{L^\infty([0,T])} < \varepsilon\,. \tag{2.11}$$

Remark 2.3 Consider a control system where we allow both external disturbance e_2 and measurement errors e_1, i.e. a system of the form

$$\dot{x} = g(x + e_1(t)) + e_2(t)$$

where g is a suitable bounded vector field, in this case a *patchy* vector field. Then, the map $y(t) = x(t) + e_1(t)$ satisfies

$$\dot{y} = g(y) + e_2(t) + \dot{e}_1(t) = g(y) + \dot{w}\,,$$

setting

$$w(t) = e_1(t) + \int_{t_o}^{t} e_2(s)ds\,.$$

Therefore, any perturbed system can be reduced to an impulsive system as (2.10).

On the other side, the hypothesis of small $Tot.Var.\{w\}$ is necessary to avoid chattering behaviors.

Remark 2.4 In some situations it is useful to adopt a more general definition of patchy vector field than the one formulated in Definition 2.2. Indeed, one can consider patches $(\Omega_\alpha,\ g_\alpha)$ where the domain Ω_α has a piecewise smooth boundary (see [3]). In this case, the inward-pointing condition (2.1) can be expressed requiring that

$$g(x) \in \overset{\circ}{T}_\Omega(x) \tag{2.12}$$

where $\overset{\circ}{T}_\Omega(x)$ denotes the interior of the tangent cone to Ω at the point x, defined by

$$T_\Omega(x) \doteq \left\{ v \in \mathbb{R}^n \ :\ \liminf_{t \downarrow 0} \frac{d\big(x + tv,\ \Omega\big)}{t} = 0 \right\}. \tag{2.13}$$

Clearly, at any regular point $x \in \partial\Omega$, the interior of the tangent cone $T_\Omega(x)$ is precisely the set of all vectors $v \in \mathbb{R}^n$ that satisfy $\big\langle v,\ \mathbf{n}(x)\big\rangle < 0$ and hence (2.12) coincides with the inward-pointing condition (2.1). One can easily see that all the results concerning patchy vector fields established in [1, 2] remain true within this more general formulation. On the other hand this generalization allows to better estimate the rate of convergence in (2.11). Indeed, for patches with smooth boundary, one would expect nothing more than an estimate like

$$||x - y||_{L^\infty([0,T])} < \mathcal{O}(1)\, Tot.Var.\{w\}^{1/\rho}\,,$$

for a suitable $\rho > 1$ (see Example 1.4 in [3]). However, slightly generalizing the definition of patch in the way above, it is possible to recover patchy vector fields defined on *polyhedral* coverings. In this new setting, one can prove a linear estimate. Namely, it holds

$$||x - y||_{L^\infty([0,T])} < \mathcal{O}(1)\, Tot.Var.\{w\}\,.$$

3 Proof of the main result

We are now ready to prove the result stated in Section 1, about the construction of a nearly optimal patchy feedback control without any use of the value function. Actually, some part of the construction is still a work in progress, so we sketch here only the main passages of the proof. For missing details (especially Steps **8.** and **15.**), we refer to [11].

1. Various reductions can be performed. By approximating the cost functions ψ, L, it is not restrictive to assume that ψ is piecewise quadratic, while $L \in \mathcal{C}^\infty$. Next, replacing $f(x,u)$ by

$$g(x,u) \doteq L^{-1}(x,u)\, f(x,u)\,, \tag{3.1}$$

the problem takes the form

$$\min_{\tau,\, u(\cdot)} \left\{\tau + \psi(x(\tau))\right\}, \tag{3.2}$$

with dynamics

$$\dot{x} = g(x,u)\,, \qquad\qquad x(0) = y\,. \tag{3.3}$$

In the following, we thus assume without loss of generality that the running cost is simply $L(x,u) \equiv 1$.

2. For the problem (3.1)–(3.3) with smooth coefficients, our eventual goal is to construct a closed **terminal set** $S \subseteq \mathbb{R}^n$ such that

$$\tau(y) = \min\left\{t \geq 0\,,\quad x(t) \in S\right\}. \tag{3.4}$$

In other words, as soon as the system enters S, the evolution stops.

Moreover, we construct a continuous, piecewise smooth, Lyapunov function $W : \mathbb{R}^n \mapsto R_+$ and a patchy feedback $u = U(x)$ on the complement $\mathbb{R}^n \setminus S$, with the following properties.

$$\min_{\omega\in\mathbf{U}} \nabla W(x)\cdot f(x,\omega) \geq -(1+\varepsilon)\,. \tag{3.5}$$

Moreover, if x lies in the interior of a region where U is constant, then

$$\nabla W(x)\cdot f\big(x,\, U(x)\big) \leq -1+\varepsilon\,. \tag{3.6}$$

3. We now sketch the inductive construction procedure, on progressively higher level sets of the approximate value function. Let

$$\min_x \psi(x) \doteq m_0 > 0\,.$$

Take a set S_0 with piecewise smooth boundary such that

$$S_0 = \{x\,;\quad \psi(x) \leq \lambda_0\}\,,$$

with

$$\Big(1+\frac{\varepsilon}{3}\Big)m_0 \leq \lambda_0 \leq \Big(1+\frac{2\varepsilon}{3}\Big)m_0\,.$$

Since ψ is piecewise quadratic, the boundary ∂S_0 consists of finitely many portions of spheres, say $\Gamma_1^0, \ldots, \Gamma_{N_0}^0$. We assume in the following that S_0 is connected, being the general case a simple adaptation of this one (see [11]).

We begin by defining the approximate value function

$$W(x) = \psi(x) \qquad\qquad x \in S_0\,.$$

Starting from ∂S_0 we solve the equations for characteristics corresponding to an Hamiltonian of the form

$$H^\eta(x,p) = \max_{y\in F(x)} \big\{ -p\cdot y - 1\big\}\,,$$

where

$$F(x) \doteq \big\{f(x,u)\,;\quad u \in \mathbf{U}\big\}$$

is the set of admissible velocities. Unfortunately, the multifunction $F(x)$ could depend in a non-smooth way from x, therefore we need to change slightly the admissible velocities. This will be achieved in two steps.

4. First, we replace the velocity sets

$$F(x) \doteq \big\{f(x,u)\,;\quad u \in \mathbf{U}\big\}$$

with slightly larger, uniformly convex sets. For this purpose, we call $B(z,r)$ the closed ball centered at z with radius r, and write $B(A,r)$ for the closed neighborhood of radius r around the set $A \subset \mathbb{R}^n$. For a suitably small constant $\eta > 0$, we define

$$F^\eta(x) \doteq \bigcap_{B(z,\eta^{-1})\supseteq B(F(x),\eta)} B(z,\eta^{-1})\,.$$

In other words, $F^\eta(x)$ is the intersection of all closed balls of radius η^{-1} which contain the η-neighborhood $B\big(F(x),\eta\big)$.

Claim 1. For any η sufficiently small, we claim that the above definition of the sets F^η yields

$$F(x) \subseteq F^\eta(x)\,, \tag{3.7}$$

$$d_H\big(F(x)\,,\ F^\eta(x)\big) \leq C\,\eta\,, \tag{3.8}$$

$$d_H\big(F^\eta(x)\,,\ F^\eta(x')\big) \leq 2L\,|x-x'|\,, \tag{3.9}$$

where the constant C depends only on an upper bound for the diameters of the sets $B\big(F(x),\eta\big)$, and L is a Lipschitz constant for the original multifunction F. Moreover, if p_1, p_2 are any unit vectors and

$$y_1 = \arg\max_{y\in F^\eta(x)} p_1\cdot y\,, \qquad\qquad y_2 = \arg\max_{y\in F^\eta(x)} p_2\cdot y\,,$$

we claim that

$$|y_1 - y_2| \le \frac{|p_1 - p_2|}{\eta}\,. \tag{3.10}$$

Notice that (3.10) is obvious if F^η is exactly a ball of radius η^{-1}.

5. *Proof of Claim 1.* We give here a proof of the above claims. Of course, (3.7) is obvious.

a. Recall that, setting

$$\psi^A(p) \doteq \sup_{y\in A} \big\langle y,p\big\rangle\,,$$

the Hausdorff distance between two convex sets A, B can be expressed by

$$d_H(A,B) = \max_{|p|=1} \big|\psi^A(p) - \psi^B(p)\big|\,. \tag{3.11}$$

Since

$$\psi^{B(F(x),\eta)}(p) = \psi^{F(x)}(p) + \eta\,,$$

it is clear that the Lipschitz constant of the map $x \mapsto \hat F(x) \doteq B\big(F(x),\eta\big)$ is the same as for the original multifunction F.

To establish (3.8), fix any unit vector $p\in\mathbb{R}^n$ and choose a point $y\in F(x)$ such that $\psi^{F(x)}(p) = \big\langle y,\, p\big\rangle$. Then $\hat F(x) = B\big(F(x),\eta\big)$ is contained in the half-ball centered at $y+\eta p$ with radius

$$R \ge 2\eta + \mathrm{diam}\, F(x) \ge \mathrm{diam}\, \hat F(x).$$

By Pythagoras' theorem, the ball with radius η^{-1} centered at the point $y + \big(\eta - \sqrt{\eta^{-2} - R^2}\big)p$ contains $\hat F(x)$. Therefore, for all $\eta > 0$ sufficiently small,

$$\begin{aligned} \psi^{F^\eta(x)}(p) &\le \big\langle y + (\eta - \sqrt{\eta^{-2} - R^2})p + \eta^{-1}p, p\big\rangle = \psi^{F(x)}(p) + \eta + \eta^{-1} - \sqrt{\eta^{-2} - R^2} \\ &\le \psi^{F(x)}(p) + \eta + \frac{\eta R^2}{2\sqrt{1 - R^2\eta^2}} \le \psi^{F(x)}(p) + C\eta\,. \end{aligned} \tag{3.12}$$

Since the unit vector $p\in\mathbb{R}^n$ is arbitrary, from (3.12) it follows (3.8).

b. To prove (3.9), fix again a unit vector p. Take a ball $B(z,\eta^{-1}) \supset \hat F(x)$ such that

$$\psi^{F^\eta(x)}(p) = \psi^{B(z,\eta^{-1})}(p) = \big\langle z,p\big\rangle + \eta^{-1}\,.$$

For $\eta^{-1} >> \mathrm{diam}\,\hat F(x)$, we have

$$\hat F(x) \subset B(z,\eta^{-1}) \cap \big\{y\,;\;\; \big\langle y - z, p\big\rangle \ge \eta^{-1}/2\big\}\,.$$

Therefore, for $r > 0$ small

$$B\big(\hat{F}(x)\,;\ r\big) \subset B(z + 2rp\,;\ \eta^{-1})\,.$$

For x' close to x, if

$$d_H\big(F(x'),\, F(x)\big) = d_H\big(\hat{F}(x'),\, \hat{F}(x)\big) \le L\,|x' - x|\,,$$

using the above estimate with $r = L\,|x' - x|$ we obtain

$$\begin{aligned}\psi^{F^\eta(x')}(p) &\le \psi^{B(\hat{F}(x),\, L|x'-x|)}(p) \le \psi^{B(z+2L|x'-x|p,\,\eta^{-1})}(p)\\ &\le \big\langle z,\, p\big\rangle + 2L|x' - x| + \eta^{-1} = \psi^{F^\eta(x)}(p) + 2L|x' - x|\,. \end{aligned}\tag{3.13}$$

Since p is arbitrary, from (3.13) it follows

$$F^\eta(x') \subseteq B\big(F^\eta(x)\,,\ 2L|x' - x|\big)\,.$$

Reversing the roles of x, x' we obtain (3.9).

c. Finally, we prove (3.10). Consider any unit vector p_1 and let $y_1 = \mathrm{argmax}_{y \in F^\eta(x)}\, p_1 \cdot y$. We claim that

$$F^\eta(x) \subseteq B_1 \doteq B(y_1 - \eta^{-1}p_1, \eta^{-1}) \tag{3.14}$$

Assuming for the time being that (3.14) holds, we easily reach the desired conclusion. Indeed, consider a second unit vector p_2 and let $y_2 = \mathrm{argmax}_{y \in F^\eta(x)}\, p_2 \cdot y$. In particular, this implies

$$y_2 \in \Big\{y \in B_1\,;\ \ \big\langle y, p_2\big\rangle \ge \big\langle y_1\,,\ p_2\big\rangle\Big\} \subset B\big(y_1\,,\ \eta^{-1}|p_2 - p_1|\big)\,,$$

proving (3.10).

It now remains to establish the inclusion (3.14). We argue by contradiction, assuming there exists a point $\xi \in F^\eta(x) \setminus B_1$. By assumption, every ball of radius η^{-1} containing $\hat{F}(x)$ also contains the two points y_1 and ξ. In particular, it must also contain the arc of circumference γ with radius η^{-1}, with endpoints y_1 and ξ, in a two-dimensional plane Π_1 parallel to p_1. But then γ is contained in F^η. Therefore,

$$\max_{y \in F^\eta(x)} \big\langle y, p_1\big\rangle \ge \max_{y \in \gamma} \big\langle y, p_1\big\rangle > \big\langle y_1, p_1\big\rangle\,, \tag{3.15}$$

obtaining a contradiction. The following computations better explain (3.15). Let π_1 be the hyperplane orthogonal to p_1 through y_1 and define

$$\begin{aligned}\pi_1^- &\doteq \{\zeta \in \mathbb{R}^n \mid p_1 \cdot \zeta < p_1 \cdot y_1\}\,,\\ \pi_1^+ &\doteq \{\zeta \in \mathbb{R}^n \mid p_1 \cdot \zeta > p_1 \cdot y_1\}\,.\end{aligned}$$

Then $F^\eta(x) \subset (\pi_1^- \cup \pi_1)$ and B_1 is the sphere tangent to π_1 through y_1. Hence, any sphere B' with radius η^{-1}, passing through both y_1 and ξ, intersects π_1 in more than a single point. Now if we prove that

$$\Xi \doteq \left(\bigcap_{\substack{\{y_1,\xi\} \subset B'\\ B' = B(z,\eta^{-1})}} B'\right) \cap \pi_1^+ \neq \emptyset\,, \tag{3.16}$$

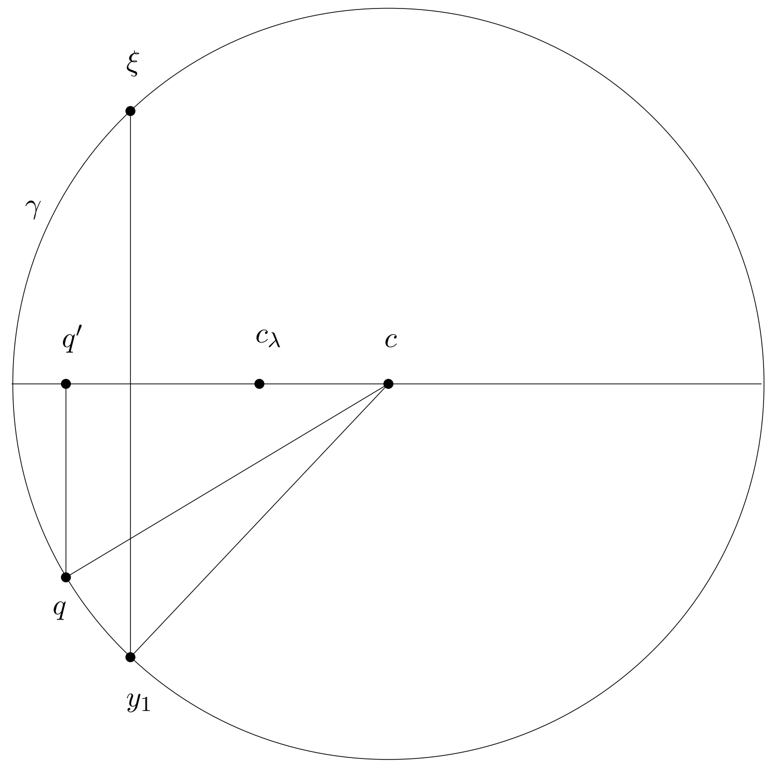

Figure 13

then we would have a contradiction. Indeed, for any ζ in that intersection, we would recover the same inequalities of (3.15) on Ξ.

In order to prove (3.16), we restrict our study to a suitable two-dimensional plane. We fix a center c of a sphere of radius η^{-1}, passing through y_1 and ξ. Clearly c belongs to the hyperplane

$$\{\zeta \in \mathbb{R}^n \mid |x - y_1| = |x - \xi|\}$$

and the three points y_1, ξ and c determine a plane, which we will indicate with Π. Notice that the circles given by each sphere with radius η^{-1} through y_1 and ξ intersected by Π, have centers $c_\lambda = \frac{y_1+\xi}{2} + \lambda(c - \frac{y_1+\xi}{2})$ and radii $R_\lambda = \sqrt{(\frac{1}{\eta})^2 + (|\lambda| - 1)|c - \frac{y_1+\xi}{2}|^2}$ for $\lambda \in [-1, 1]$. Our goal is to prove that

$$\left(\bigcap_{\lambda \in [-1,1]} (B(c_\lambda, R_\lambda) \cap \Pi) \right) \cap \pi_1^+ = \Xi \cap \Pi \neq \emptyset \,. \tag{3.17}$$

At this point, we just need to show that any $q \in \gamma$ belongs to $\bigcap_{\lambda\in[-1,1]}(B(c_\lambda, R_\lambda) \cap \Pi)$, where γ is the shorter arc of $\partial B(c, R_1) \cap \Pi$, joining y_1 and ξ (see Figure 13). Indeed, near y_1, these points q belongs also to π_1^+ (due to all B' not being tangent to π_1) and hence to $\Xi \cap \Pi$, which results non-empty.
For such a point q and for any $\lambda \in [-1, 1]$, setting q' the projection of q on the line on Π joining c and c_λ (we still refer to Figure 13 for a picture of the situation), it holds

$$
\begin{aligned}
|q-c_\lambda|^2 &= |q-q'|^2 + |q'-c_\lambda|^2 = \\
&= |q-c|^2 - (|q'-c_\lambda| + |c_\lambda - c|)^2 + |q'-c_\lambda|^2 = \\
&= \eta^{-2} - 2|q'-c_\lambda| \cdot |c_\lambda - c| - |c_\lambda - c|^2 \,,
\end{aligned}
$$

$$
\begin{aligned}
R_\lambda^2 &= \eta^{-2} - |c - \frac{y_1+\xi}{2}|^2 + |c_\lambda - \frac{y_1+\xi}{2}|^2 = \\
&= \eta^{-2} - (|c-c_\lambda| + |c_\lambda - \frac{y_1+\xi}{2}|)^2 + |c_\lambda - \frac{y_1+\xi}{2}|^2 = \\
&= \eta^{-2} - 2|c-c_\lambda| \cdot |c_\lambda - \frac{y_1+\xi}{2}| - |c-c_\lambda|^2 \,,
\end{aligned}
$$

and finally

$$
\begin{aligned}
|q-c_\lambda|^2 &= \eta^{-2} - 2(|q' - \frac{y_1+\xi}{2}| + |\frac{y_1+\xi}{2} - c_\lambda|) \cdot |c_\lambda - c| - |c_\lambda - c|^2 = \\
&= R_\lambda - 2|q' - \frac{y_1+\xi}{2}| \cdot |c_\lambda - c| < R_\lambda \,.
\end{aligned}
$$

Therefore we have (3.16) and the required contradiction.

6. At this point, we have obtained regularity w.r.t. p, and we have sets of admissible velocity that satisfy both the inner and the outer sphere condition. It is still not clear the regularity w.r.t. x. Hence, for any fixed x, set

$$\phi_x(z) \doteq d(z, F^\eta(x)) - d(z, \mathbb{R}^n \setminus F^\eta(x))$$

i.e. the signed distance from the boundary of $F^\eta(x)$.

The idea is to consider a regularizing functions ρ_h whose support is contained in $B(0, 1/h)$, $h > \eta^{-1}$, and the set

$$F_h^\eta(x) \doteq \{ y \; ; \; \rho_h * \phi_x(y) \le 0 \} \,, \tag{3.18}$$

in place of $F^\eta(x)$. In this way, we will have H^η smooth in both its arguments.

Claim 2. Replacing F^η with F_h^η, the following properties still hold. $F_h^\eta(x)$ is strictly convex,

$$F(x) \subseteq F_h^\eta(x) \,, \tag{3.19}$$

$$d_H\big(F(x) \,,\; F_h^\eta(x)\big) \le (C+1)\,\eta \,, \tag{3.20}$$

$$d_H\big(F_h^\eta(x) \,,\; F_h^\eta(x')\big) \le 2L\,|x - x'| \,, \tag{3.21}$$

where the constants C, L are as in (3.8)–(3.9).
Moreover, for any p_1, p_2 unit vectors and

$$y_1 = \arg \max_{y \in F_h^\eta(x)} \; p_1 \cdot y \,, \qquad\qquad y_2 = \arg \max_{y \in F_h^\eta(x)} \; p_2 \cdot y \,,$$

we still have that

$$|y_1 - y_2| \le \frac{|p_1 - p_2|}{\eta} \,. \tag{3.22}$$

7. *Proof of Claim 2.* Strict convexity of $F_h^\eta(x)$ follows from strict convexity of ϕ_x.

To prove (3.19), notice that if $\xi \in F(x)$, then $B(\xi, 1/h) \subset \hat{F}(x) \subset F^\eta$ by choice of h. Hence, $\rho_h * \phi_x(\xi) \le 0$ and $\xi \in F_h^\eta(x)$.

Now, for any small $\alpha > 0$, consider $\xi \in \mathbb{R}^n \setminus B(F^\eta(x), \alpha + 1/h)$. Then, $B\big(\xi, \frac{\alpha}{2} + \frac{1}{h}\big) \cap F^\eta(x) = \emptyset$ and $\rho_h * \phi_x(\xi) > \alpha/2 > 0$. Therefore, $F_h^\eta(x) \subset B(F^\eta(x), 1/h)$ and, for any unit vector p,

$$\psi^{F_h^\eta(x)}(p) \le \psi^{F^\eta(x)}(p) + 1/h\,.$$

On the other hand, $G_h(x) = \{y \in \mathbb{R}^n \ ; \ \phi_x(y) \le -1/h\}$ is contained in $F_h^\eta(x)$. Therefore, for any unit vector p, from $\psi^{G_h(x)}(p) = \psi^{F^\eta(x)}(p) + 1/h$, it follows

$$\psi^{F_h^\eta(x)}(p) \ge \psi^{F^\eta(x)}(p) - 1/h\,.$$

These two inequalities, since p is arbitrary, imply $d_H(F_h^\eta(x), F^\eta(x)) \le 1/h < \eta$. Hence,

$$d_H(F_h^\eta(x), F(x)) \le \ (C+1)\eta\,,$$

where C is the constant in (3.8), and (3.20) is proved.

To prove (3.21), recalling $1/h < \eta$, one can use $B(F(x), 1/h) \subset \hat{F}(x)$ and repeat the proof of (3.9) with $F(x)$ in place of $\hat{F}(x)$. Namely, using a ball of radius η^{-1} containing $F(x)$ and passing through y_h such that $\psi^{F_h^\eta(x)}(p) = p \cdot y_h$, one can follow the procedure in Claim 2 to conclude.

Similarly, one can prove (3.22) repeating the step used for (3.10), with simple adaptations.

8. Having replaced the sets $F(x)$ with the larger, uniformly convex, sets $F_h^\eta(x)$, the Hamiltonian function

$$H^\eta(x,p) = \max_{y \in F_h^\eta(x)} \big\{ -p \cdot y - 1 \big\} \tag{3.23}$$

is smooth w.r.t. both its arguments. We now take a look at the equations for the characteristics.

$$\dot{x} \ = \ \frac{\partial H^\eta}{\partial p} \ = \ \arg\max_{y \in F_h^\eta(x)} \ p \cdot y\,, \tag{3.24}$$

$$\dot{p} \ = \ -\frac{\partial H^\eta}{\partial x}\,. \tag{3.25}$$

Notice that the right hand side of (3.24) is Lipschitz continuous because of (3.22). Moreover, the right hand side of (3.25) is homogeneous of degree one w.r.t. p. We thus have an estimate

$$\big|p(t)\big| \le \big|p(t_0)\big| \cdot e^{\kappa|t-t_0|} \tag{3.26}$$

uniformly valid on compact sets, with κ independent of η.

For each boundary point $\bar{x} \in \partial S_0$, the initial conditions for (3.24)–(3.25) are $x(0) = \bar{x}$, while $p(0)$ is a suitable vector perpendicular to ∂S_0 at the point $\bar{x}$. More precisely, $p(0) = \alpha\, \nabla\psi(\bar{x})$, where the constant $\alpha \in [0,1]$ is such that

$$\max_{y \in F_h^\eta(\bar{x})} \big\{ -p(0) \cdot y - 1 \big\} = 0\,. \tag{3.27}$$

Notice that, if no $\alpha \in [0,1]$ exists such that (3.17) holds, this means that

$$\max_{y \in F_h^\eta(\bar{x})} \Big\{ -\nabla\psi(\bar{x}) \cdot y - 1 \Big\} < 0\,. \tag{3.28}$$

In this case, in a whole neighborhood of $\bar{x}$ it is convenient to stop immediately, and pay the cost ψ, rather than try to move to a point with lower cost. Indeed, in a suitably small neighborhood N of $\bar{x}$, we would still have the strict inequality

$$\max_{y \in F_h^\eta(\xi)} \Big\{ -\nabla\psi(\xi) \cdot y - 1 \Big\} < 0\,, \qquad \xi \in N\,.$$

and for $t < 0$ small enough (so that $x(t)$ remains in N) the cost to reach S_0, equal to $\lambda_0 - t$, would satisfy

$$\begin{aligned} \lambda_0 - t - \psi(x(t)) &= -t - \big(\psi(x(t)) - \psi(\bar{x})\big) = -t - \int_0^t \nabla\psi(x(s)) \cdot \dot{x}(s) ds \\ &\geq -\int_t^0 \max_{y \in F_h^\eta(x(s))} \{-1 - \nabla\psi(x(s)) \cdot y\}\, ds > 0\,. \end{aligned}$$

In other words moving from points of N would cost more than remain there. Characteristics starting at such points need not be constructed.

By choosing a sufficiently short time interval $[0,\delta]$, any two trajectories of (3.24)–(3.25) originating from point on the same sphere Γ_j^0 will not intersect. Notice that δ depend on η.

These trajectories, for $t \in [0,\delta]$, form lens-shaped domains around each $\Gamma_i^0 \subset S_0$ in the sense of [4]. Setting Γ_i^δ these domains, we define

$$K \doteq \left(\cup_i \Gamma_i^\delta\right) \setminus \mathring{S}_0\,. \tag{3.29}$$

We underline that K is compact, since the union of Γ_i^δ is contained in a sufficiently large sphere. Namely, for any $t \in [0,\delta]$, we have

$$|x(t) - x(0)| \leq C\left(\delta + \int_0^t |x(s)| ds\right)\,,$$

and, therefore, by Gronwall's Lemma

$$|x(t) - x(0)| \leq C\delta(1 + |x(0)|)\, e^{C\delta}\,.$$

We can conclude that, if $S_0 \subset B(0,R)$, then any $x(t)$ will remain in $B(0, R + C\delta(1 + R)e^{C\delta})$.

By solving (3.24)–(3.25), we obtain a function V^η corresponding to

$$\min_{\tau, x(\cdot)} \Big\{\tau + \psi\big(x(\tau)\big)\Big\}$$

$$\dot{x} \in F_h^\eta(x)\,.$$

Here τ is the first time t where $x(t) \in S_0$.

We can recover this value function V^η as the minimum among N_0 scalar functions. More precisely, calling $t \mapsto x(t,\bar{x})$ the trajectory of (3.24)–(3.25) starting at $\bar{x}$, for a given point z there will be at most N_0 initial points $\bar{x}_j \in \Gamma_j^0$ and times t_j such that $z = x(t_j, \bar{x}_j)$. We then set

$$V^\eta(z) = \min\left\{\lambda_0 + t_1, \ldots, \quad \lambda_0 + t_{N_0}\right\}, \tag{3.30}$$

where λ_0 is the constant value of ψ on the boundary of S_0, and

$$\tilde{V}^\eta(z) = \min\left\{\psi(z)\,, \; V^\eta(z)\right\}. \tag{3.31}$$

Outside the terminal set S_0 the value function V^η satisfies

$$H^\eta(x, V^\eta) = 0 \qquad\qquad x \notin S_0\,. \tag{3.32}$$

Notice that, by (3.26) and since along trajectories we have

$$\nabla V^\eta = p\,, \tag{3.33}$$

the size of the gradient $|\nabla V^\eta|$ is bounded, uniformly w.r.t. η. This, in particular, implies that V^η is differentiable almost everywhere. The same holds for $\widetilde{V}^\eta$.

Moreover, we assume V^η semiconcave on the same set where the uniform bound on its gradient holds. Hence, there exists $\lambda > 0$ such that for any y where V^η is differentiable

$$V^\eta(x) \le V^\eta(y) + \nabla V^\eta(y) \cdot (x - y) + \lambda |x - y|^2\,. \tag{3.34}$$

9. As in [4], we now approximate the value function V^η with a piecewise quadratic function $\widetilde{W}$ such that

$$-1 - \frac{\varepsilon}{2} \le \min_{y \in F_h^\eta(x)} \nabla\widetilde{W} \cdot y \le -1 + \frac{\varepsilon}{2} \tag{3.35}$$

Namely, we set, for any $y \in K$ in which V^η is differentiable,

$$V_y^\eta(x) = V^\eta(y) + \nabla V^\eta(y) \cdot (x - y) + (1 + \lambda)|x - y|^2\,. \tag{3.36}$$

where $\lambda > 0$ comes from the semiconcavity of V^η. Therefore we have

$$V^\eta(x) + |x - y|^2 \le V_y^\eta(x)\,. \tag{3.37}$$

Since $\min\limits_{\zeta \in F_h^\eta(y)} \{\nabla V^\eta(y) \cdot \zeta\} = -1$, for every fixed $\varepsilon_1 > 0$, there exists $u_{\varepsilon_1} = u(\varepsilon_1, y) \in \mathbf{U}$ such that $f(y, u_{\varepsilon_1}) \in F_h^\eta(y)$ and

$$\nabla V^\eta(y) \cdot f(y, u_{\varepsilon_1}) < -1 + \varepsilon_1\,. \tag{3.38}$$

Now we claim that there exists $\rho > 0$ sufficiently small, so that for any $z \in B(y, \rho)$ and $\omega \in \mathbf{U}$

$$\nabla V_y^\eta(z) \cdot f(z, u_{\varepsilon_1}) < -1 + 2\varepsilon_1\,, \tag{3.39}$$

$$\nabla V_y^\eta(z) \cdot f(z, \omega) \ge -1 - 2\varepsilon_1\,. \tag{3.40}$$

This follows from

$$\begin{aligned}\nabla V^\eta_y(z)\cdot f(z,u_{\varepsilon_1}) &= \nabla V^\eta(y)\cdot f(y,u_{\varepsilon_1}) + \nabla V^\eta(y)\cdot[f(z,u_{\varepsilon_1})-f(y,u_{\varepsilon_1})]\\ &\qquad + 2(1+\lambda)(z-y)\cdot f(z,u_{\varepsilon_1})\\ &< -1+\varepsilon_1+||V^\eta||Lip(f)|z-y|\\ &\qquad + 2(1+\lambda)\left\{\sup_K f\cdot|z-y| + Lip(f)|z-y|^2\right\},\end{aligned}$$

and

$$\begin{aligned}\nabla V^\eta_y(z)\cdot f(z,\omega) &= \nabla V^\eta(y)\cdot f(y,\omega) + \nabla V^\eta(y)\cdot[f(z,\omega)-f(y,\omega)]\\ &\qquad + 2(1+\lambda)(z-y)\cdot f(z,\omega)\\ &\geq -1-||V^\eta||Lip(f)|z-y|\\ &\qquad - 2(1+\lambda)\left\{\sup_K f\cdot|z-y| + Lip(f)|z-y|^2\right\}.\end{aligned}$$

We underline that ρ does not depend on y and that we can also assume $\rho \leq ||\nabla V^\eta||_\infty$. Moreover, the first inequality will be crucial in the introduction of the patchy control, since it will give the inward-pointing condition.

Using notations from **4.-5.** and choosing $\varepsilon_1 < \varepsilon/4$, we can rewrite the previous inequalities as follow. There exists ρ such that for any $y \in K$ in which V^η is differentiable, and any $z \in B(y,\rho)$, setting $z_{\varepsilon_1} \doteq f(z,u_{\varepsilon_1})$, one has

$$-1-\frac{\varepsilon}{2} \leq \min_{\zeta\in F^\eta_h(z)} \nabla V^\eta_y(z)\cdot\zeta \leq \nabla V^\eta_y(z)\cdot z_{\varepsilon_1} \leq -1+\frac{\varepsilon}{2}\,. \tag{3.41}$$

10. Now we consider the compact set K given in **8.**. We recall that we defined it as $\bigcup_i \Gamma^\delta_i \setminus \mathring{S}_0$, where Γ^δ_i were the lens-shaped domain around S_0. Let $y_1,\ldots,y_{N_1}$ be points of K such that V^η is differentiable in y_i and the balls $B(y_i,\rho')$ covers completely K. Here, we choose the radius ρ' sufficiently small so that for any $x \in B(y_i,\rho')$ it holds

$$V^\eta(x) \leq V^\eta_{y_i}(x) \leq V^\eta(x)+\rho^2\,, \tag{3.42}$$

where ρ is the value found in **9**. In particular $\rho' < \rho$. Then, we set $V_i \doteq V^\eta_{y_i}$, $u_i = u(\varepsilon_1, y_i)$ and finally

$$\widetilde{W}(x) = \min_i V_i(x)\,, \quad \forall x\in K\,. \tag{3.43}$$

With this definition, (3.5)–(3.6) easily hold, since each V_i verifies them. Moreover, the same bounds given above for $||\nabla V^\eta||$ can be achieved also for $||\nabla\widetilde{W}||$.
Indeed, for any $x \in K$ in which $\widetilde{W}$ is differentiable, if $x \in K\setminus B(y_i,\rho)$, then there exists j such that $x \in B(y_j,\rho')\setminus B(y_i,\rho)$ and hence, recalling (3.42),

$$\widetilde{W}(x) \leq V_j(x) \leq V^\eta(x)+\rho^2 < V_i(x)\,. \tag{3.44}$$

This implies that $\widetilde{W}(x) = V_i(x)$ can hold only if $x \in B(y_i, \rho)$. Therefore

$$\begin{aligned} |\nabla\widetilde{W}(x)| = |\nabla V_i(x)| &= |\nabla V^\eta(y_i)| + 2(1+\lambda)|x - y_i| \\ &\leq ||\nabla V^\eta||_\infty + 2(1+\lambda)\rho \leq (3+2\lambda)\cdot ||\nabla V^\eta||_\infty \,. \end{aligned}$$

Assuming $|\nabla\widetilde{W}|C\eta \leq \varepsilon/2$, from (3.8) and (3.35) we deduce

$$-1-\varepsilon \leq \min_{y\in F(x)} \nabla\widetilde{W}\cdot y \leq -1+\varepsilon \,. \tag{3.45}$$

For the suitably small $\delta > 0$ found in , we set

$$\lambda_1 = \lambda_0 + \delta \,, \tag{3.46}$$

and we define

$$W(x) \doteq \min\{\psi(x)\,,\, \widetilde{W}(x)\} \qquad \text{for all } x \in K \cup \{y\ ;\ \psi(y) \leq \lambda_1\} \,.$$

We now use this approximate value function W to construct a patchy feedback on a sub-level set

$$S_1 \doteq \big\{x \in \mathbb{R}^n\,; \quad W(x) \leq \lambda_1\big\}$$

so that (3.5)–(3.6) hold.

Notice that $S_1 \setminus \overset{\circ}{S_0}$ is contained in $K \cup \{\psi \leq \lambda_1\}$, which represent the sub-level for $\min\{V^\eta, \psi\}$, corresponding to λ_1. More precisely, $S_1 \setminus \overset{\circ}{S_0}$ is contained in the union of finitely many spheres $\Gamma^1_0, \Gamma^1_1, \ldots, \Gamma^1_{N_1}$, where Γ^1_0 is the sub-level of ψ and each Γ^1_i is the sub-level of V_i.

Observe that, since all functions V_i, $1 \leq i \leq N_1$, have the same coefficient of the quadratic term, it follows that, for each couple of indices $k \neq i$, the set

$$\pi_{k,i} \doteq \pi_{i,k} \doteq \big\{x \in \mathbb{R}^n\,; \quad V_k(x) = V_i(x)\big\} \tag{3.47}$$

is an hyperplane, and the difference of the gradients $\nabla V_i(x) - \nabla V_k(x)$ is a constant vector on $\pi_{k,i}$. Then, letting $\mathbf{n}_{k,i}$ denote the unit normal to $\pi_{k,i}$, pointing towards the half space

$$\pi^+_{k,i} \doteq \pi^-_{i,k} \doteq \big\{x \in \mathbb{R}^n\,; \quad V_k(x) > V_i(x)\big\} \,, \tag{3.48}$$

one has

$$\nabla V_i(x) - \nabla V_k(x) = -c\,\mathbf{n}_{k,i} \qquad \forall\, x \in \pi_{k,i} \,, \tag{3.49}$$

for some constant $c = c_{k,i} \geq 0$. Denote as $\pi^-_{k,i}$ the other half space determined by $\pi_{k,i}$, i.e. set

$$\pi^-_{k,i} \doteq \pi^+_{i,k} \doteq \big\{x \in \mathbb{R}^n\,; \quad V_k(x) < V_i(x)\big\} \,. \tag{3.50}$$

Now we introduce some more notation, related to the level set of V_i and W

Set

$$\mathcal{P}_0 \doteq \big\{x \in \mathbb{R}^n\,; \quad \psi(x) = W(x)\big\} \,, \qquad \mathcal{P}_i \doteq \big\{x \in \mathbb{R}^n\,; \quad V_i(x) = W(x)\big\} \,, \tag{3.51}$$

$$\mathcal{I} \doteq \Big\{i \in \{1,\ldots,N_1\}\ ;\ \Big(\mathcal{P}_i \setminus \bigcup_{j\neq i} \mathcal{P}_j\Big) \cap S_1 \neq \emptyset\,,\Big\} \,, \tag{3.52}$$

$$\Sigma_i \doteq \big\{x \in \mathbb{R}^n\,;\ \ V_i(x) = W(x) \le \lambda_0 + \delta\big\} = \mathcal{P}_i \cap \Gamma_i^1 \qquad \forall\, i \in \mathcal{I} \cup \{0\}\,. \tag{3.53}$$

Notice that in $\mathcal{P}_0$ the evolution stops. Moreover, this definition of $\mathcal{I}$ implies that, in the following, we ignore any index $\bar{k}$ such that the sub-level of the corresponding $V_{\bar{k}}$ is completely contained in the sub-levels of other V_i. With these notations, and recalling the definition of $\pi_{i,k}$ above,

$$\begin{array}{ll} \pi_{k,i} \cap \partial\Gamma_k^1 = \pi_{k,i} \cap \Gamma_i^1 = \Gamma_k^1 \cap \Gamma_i^1\,, & \\ \Sigma_k \subset \pi_{k,i} \cup \pi_{k,i}^-\,, \qquad\qquad \Sigma_i \subset \pi_{k,i} \cup \pi_{k,i}^+ & \end{array} \qquad \forall\; k, i \in \mathcal{I}\,. \tag{3.54}$$

Notice, also, that

$$\bigcup_i \Sigma_i = S_1 = \bigcup_i \Gamma_i^1\,.$$

On each Σ_i we have a constant control u_i given in (3.38) such that

$$\nabla W(z) \cdot f(z, u_i) < -1 + \varepsilon$$

holds for any $z \in \overline{\Sigma}_i$. Indeed $\Sigma_i \subset \mathcal{P}_i \subset B(y_i, \rho)$.

11. So far we have defined a constant control on S_1. Nevertheless, we cannot simply set $U(x) \doteq u_i$ for $x \in \Sigma_i \setminus \big(\bigcup_{j>i} \Sigma_j\big)$. Indeed, every trajectory $x(t)$ of $\dot{x} = f(x, u_i)$, passing only through points of Σ_i, satisfies $\forall\; t > s$

$$\begin{aligned} W\big(x(t)\big) = V_i\big(x(t)\big) \\ &= V_i\big(x(s)\big) + \int_s^t \big\langle \nabla V_i(x(\sigma)),\ f(x(\sigma), u_i)\big\rangle\; d\sigma \\ &\le V_i\big(x(s)\big) + (-1+\varepsilon)\cdot(t-s) \\ &= W\big(x(s)\big) + (-1+\varepsilon)\cdot(t-s) \end{aligned} \tag{3.55}$$

However, there may well be points $x(t) \in \Gamma_i^1$ where $V_i(x(t)) > W(x(t))$. Near these points there is no guarantee that (3.55), and therefore sub-optimality, should hold. To address this difficulty, we will consider the set of all indices $i \ne k$ such that $V_i(\overline{x}) < V_k(\overline{x})$ for some $\overline{x} \in \Gamma_k^1$, and such that

$$\min_{x \in \overline{\Gamma_k^1}} \big\langle \nabla V_k(x) - \nabla V_i(x),\ f(x, u_k)\big\rangle < 0\,. \tag{3.56}$$

domain

$$\Gamma_k^1 \cap \big\{x \in \mathbb{R}^n\,;\ \ V_k(x) < V_i(x)\big\}\,.$$

More rigorously, setting

$$\mathcal{J}_k \doteq \Big\{i \in \{1, \dots, N\} \setminus \{k\}\,;\ \Sigma_i \cap \Gamma_k^1 \ne \emptyset\,,\ \ \min_{x \in \overline{\Gamma_k^1}} \big\langle \nabla V_k(x) - \nabla V_i(x),\ f(x, u_k)\big\rangle < 0\Big\}\,, \tag{3.57}$$

we consider the domains

$$\widetilde{\Gamma}_k^1 \doteq \begin{cases} \Gamma_k^1 \cap \bigcap\limits_{i \in \mathcal{J}_k} \pi_{k,i}^- & \text{if} \qquad \mathcal{J}_k \ne \emptyset\,, \\ \Gamma_k^1 & \text{otherwise.} \end{cases} \tag{3.58}$$

Then we can prove the following.

Claim 3. For any $k \in \mathcal{I}$, the domains $\widetilde{\Gamma}^1_k$, defined in (3.58), enjoy the inward-pointing condition. Namely, the vector field $f(x, u_k)$ points strictly inward at every point of the upper boundary

$$\partial^- \widetilde{\Gamma}^1_k \doteq \partial \widetilde{\Gamma}^1_k \setminus \big(S_0 \cup \mathcal{P}_0\big) . \tag{3.59}$$

Moreover, for any $y \in \overline{\widetilde{\Gamma}^1_k} \setminus \big(S_0 \cup \mathcal{P}_0\big)$, there exists a time $\mathcal{T}_k(y) > 0$ so that, letting $x(\cdot)$ be the solution of $\dot{x} = f(x, u_k)$ through y, one has

$$x\big(\mathcal{T}_k(y);\, y, u_k\big) \in S_0 \cup \mathcal{P}_0 \,, \tag{3.60}$$

$$x(t;\, y, u_k) \in \widetilde{\Gamma}^1_k \qquad \forall\, t \in \,]0,\, \mathcal{T}_k(y)] \,, \tag{3.61}$$

and there holds

$$W\big(x(t)\big) \leq W\big(x(s)\big) + (-1+\varepsilon)\cdot(t-s) \qquad \forall\; 0 \leq s < t \leq \mathcal{T}_k(y) \,, \tag{3.62}$$

where ε is the constant satisfying (3.45).

12. *Proof of Claim 3.* We know that, for every $k \in \mathcal{I}$, the vector field $f(\cdot, u_k)$ is inward-pointing on the region $\partial^- \widetilde{\Gamma}^1_k \cap \big(\partial \Gamma^1_k \setminus \big(S_0 \cup \mathcal{P}_0\big)\big)$. On the other hand, recalling (3.49), the inequality (3.56) guarantees that $f(\cdot, u_k)$ enjoys the inward-pointing condition also at the boundary points $x \in \partial^- \widetilde{\Gamma}^1_k \cap \overset{\circ}{\Gamma}{}^1_k \cap \pi_{k,i}$, $i \in \mathcal{J}_k$. Then, observing that

$$\partial^- \widetilde{\Gamma}^1_k \setminus \partial^- \Gamma^1_k = \partial^- \widetilde{\Gamma}^1_k \cap \overset{\circ}{\Gamma}{}^1_k \cap \bigcup_{i \in \mathcal{J}_k} \pi_{k,i} \,,$$

by continuity it follows that $f(x, u_k) \in \overset{\circ}{T}_{\widetilde{\Gamma}^1_k}(x)$ at every point $x \in \partial^- \widetilde{\Gamma}^1_k$. Here, $\overset{\circ}{T}_{\widetilde{\Gamma}^1_k}$ denotes the interior of the tangent cone to $\widetilde{\Gamma}^1_k$, defined in (2.13). This completes the proof on $\partial^- \widetilde{\Gamma}^1_k$.

Moreover, for any x in $\widetilde{\Gamma}^1_k$,

$$\big\langle \mathbf{n}_i, f(x,u)\big\rangle \leq \frac{\big\langle \nabla V_i(x), f(x,u)\big\rangle}{|\nabla V_i(x)|} \leq \frac{-1+\varepsilon}{||\nabla V_i||} < -\beta < 0 \,, \tag{3.63}$$

for a suitable $\beta > 0$.

Hence, we have

$$|f(x,u)| \geq |\big\langle \mathbf{n}_i, f(x,u_k)\big\rangle| \geq \beta > 0 \,. \tag{3.64}$$

Therefore, any trajectory starting from a point of $\overline{\widetilde{\Gamma}^1_k}$, cannot remain in $\widetilde{\Gamma}^1_k$. At the same time, due to the inward-pointing condition, it can escape only through $\partial\big(S_0 \cup \mathcal{P}_0\big)$.

Setting $\mathcal{T}_k(x)$ the minimum time to reach $S_0 \cup \mathcal{P}_0$, we have (3.60)–(3.61). Finally we prove (3.62). For any fixed $0 \leq s < t \leq \mathcal{T}_k(x)$ there exists $i(s)$ such that $W(x(s)) = V_{i(s)}(x(s))$ and it holds

$$\begin{aligned} W(x(t)) \leq V_{i(s)}(x(t)) &= V_{i(s)}(x(s)) + \int_s^t \big\langle \nabla V_{i(s)}(x(\sigma)), f\big(x(\sigma), u_k\big)\big\rangle d\sigma \\ &< W(x(s)) + (-1+\varepsilon)\cdot(t-s) \,. \end{aligned}$$

13. Relying on the properties in Claim 3, we shall construct now a patchy feedback on the open region

$$\Omega \doteq \overset{\circ}{S}_1 \setminus \big(S_0 \cup \mathcal{P}_0\big)\,.$$

To this end we first need to slightly enlarge some of the domains defined in (3.58). Namely, for every $k \in \mathcal{I}$, consider the set

$$\widehat{\mathcal{J}}_k \doteq \big\{ i \in \mathcal{J}_k \cap \mathcal{I}\,;\ i > k\,,\quad k \in \mathcal{J}_i \big\}\,, \tag{3.65}$$

fix some positive constant $\sigma \ll \varepsilon$, denote by $\pi^{\sigma}_{k,i}$ the hyperplane parallel to $\pi_{k,i}$ that lies in the half space $\pi^+_{k,i} = \{x \in \mathbb{R}^n\,;\ \ V_k(x) > V_i(x)\}$ at a distance σ from $\pi_{k,i}$, and call $\pi^{\sigma,-}_{k,i}$ the half space determined by $\pi^{\sigma}_{k,i}$ that contains $\pi_{k,i}$. Then, set

$$\widehat{\Gamma}^1_k \doteq \begin{cases} \Gamma^1_k \cap \displaystyle\bigcap_{i \in \mathcal{J}_k \setminus \widehat{\mathcal{J}}_k} \pi^-_{k,i} \cap \bigcap_{i \in \widehat{\mathcal{J}}_k} \big(\pi^{\sigma,-}_{k,i} \cap \Gamma_i\big) & \text{if} \qquad \mathcal{J}_k \neq \widehat{\mathcal{J}}_k,\ \ \widehat{\mathcal{J}}_k \neq \emptyset\,, \\ \Gamma^1_k \cap \displaystyle\bigcap_{i \in \widehat{\mathcal{J}}_k} \big(\pi^{\sigma,-}_{k,i} \cap \Gamma_i\big) & \text{if} \qquad \mathcal{J}_k = \widehat{\mathcal{J}}_k \neq \emptyset\,, \\ \widetilde{\Gamma}^1_k & \text{if} \qquad \widehat{\mathcal{J}}_k = \emptyset\,, \end{cases} \tag{3.66}$$

$$\Omega_k \doteq \widehat{\Gamma}^1_k \setminus \big(S_0 \cup \mathcal{P}_0\big)\,, \tag{3.67}$$

and observe that, by definitions (3.58), (3.59), (3.65), (3.66), (3.67), one has

$$\partial\,\widehat{\Gamma}^1_k \setminus \bigcup_{\substack{h\in\mathcal{I}\\ h>k}} \widehat{\Gamma}^1_h \subset \partial\,\widetilde{\Gamma}^1_k\,,$$

$$\partial\,\Omega_k \setminus \Big(S_0 \cup \mathcal{P}_0 \cup \bigcup_{\substack{h\in\mathcal{I}\\ h>k}} \Omega_h\Big) \subset \partial^-\,\widetilde{\Gamma}^1_k\,.$$

Thus, by choice of σ and Claim 3, it follows that the vector field $f(x, u_k)$ still satisfies the inward-pointing condition at every point $x \in \partial\,\Omega_k \setminus \Big(S_0 \cup \mathcal{P}_0 \cup \bigcup_{\substack{h\in\mathcal{I}\\ h>k}} \Omega_h\Big)$. Then, we can finally setting on S_1,

$$g(x) \doteq f(x, u_k) \qquad \text{if} \qquad x \in \Delta_k \doteq \Omega_k \setminus \bigcup_{\substack{h\in\mathcal{I}\\ h>k}} \Omega_h\,, \tag{3.68}$$

and considering the map $U\colon \Omega \to \mathbf{U}$ defined by

$$U(x) \doteq u_k \qquad \text{if} \qquad x \in \Delta_k\,, \tag{3.69}$$

we deduce that the triple $\big(\Omega,\, g,\, (\Omega_k,\, g_k)_{k\in\mathcal{I}}\big)$ is a patchy vector field on Ω associated to the patchy feedback $\big(\Omega,\, U,\, (\Omega_k,\, u_k)_{k\in\mathcal{I}}\big)$.

Notice that, by definitions (3.58), (3.59), (3.65), (3.66), (3.67), (3.68), one has

$$\Delta_k \subset \overline{\widetilde{\Gamma}^1_k} \setminus \big(S_0 \cup \mathcal{P}_0\big) \qquad \forall\ k \in \mathcal{I}\,,$$

and hence we may apply Claim 3 to a trajectory of g passing through the domain Δ_k.

Claim 4. The patchy vector field g on the domain Ω, defined in (3.68) enjoys the following property. For any $y \in \Omega$, and for every Carathéodory trajectory $\gamma_y(\cdot)$ of

$$\dot{x} = g(x) \tag{3.70}$$

starting at y, there exists a time $\mathcal{T}(y, \gamma_y) > 0$ so that one has

$$\gamma_y\big(\mathcal{T}(y, \gamma_y)\big) \in S_0 \cup \mathcal{P}_0\,, \tag{3.71}$$

and there holds

$$W\big(\gamma_y(t)\big) \leq W(y) + (-1+\varepsilon)\cdot t \qquad \forall\; 0 \leq t \leq \mathcal{T}(y, \gamma_y)\,. \tag{3.72}$$

14. *Proof of Claim 4.* We proceed as it was done to prove Claim 3 of Theorem 1 in [4]. Given $y \in \Omega$, let γ_y be a trajectory of (3.70) starting at y, and set

$$t_{\max}(\gamma_y) \doteq \sup\big\{t > 0\;;\quad \gamma_y \;\text{ is defined on }\; [0,t]\big\}\,. \tag{3.73}$$

By the properties of the patchy vector fields and Claim 3 above, one can recursively construct two increasing sequences of times $0 = t_0 < t_1 < \ldots < t_{\overline{\nu}} = t_{\max}$, and of indices $i_1 < i_2 < \ldots < i_{\overline{\nu}} \in \mathcal{I}$ with the following properties:

a. γ_y is a solution of $\dot{x} = g_{i_\nu}(x)$ taking values in Δ_{i_ν} for all $t \in]t_{\nu-1},\, t_\nu]$, $1 \leq \nu \leq \overline{\nu}$;

b. $\gamma_y(t_\nu) \in \partial\,\Omega_{i_{\nu+1}}$ for all $1 \leq \nu < \overline{\nu}\,,$ and $\gamma_y(t_{\overline{\nu}}) \in S_0 \cup \mathcal{P}_0\,;$

c. $t_\nu - t_{\nu-1} < \mathcal{T}_{i_\nu}\big(\gamma_y(t_{\nu-1})\big)$ for all $1 \leq \nu < \overline{\nu}\,,$ and $t_{\overline{\nu}} - t_{\overline{\nu}-1} \leq \mathcal{T}_{i_{\overline{\nu}}}\big(\gamma_y(t_{\overline{\nu}-1})\big)\,.$

Hence (3.71) is proved. Next, applying repeatedly the estimate (3.62) of Claim 3, and recalling that $\gamma_y(0) = y$, we derive

$$\begin{aligned} W\big(\gamma_y(t)\big) &\leq W\big(\gamma_y(t_\nu)\big) + (-1+\varepsilon)\cdot\big(t - t_\nu\big) \\ &\leq W(y) + (-1+\varepsilon)\cdot t \qquad \forall\; t \in]t_{\nu-1}, t_\nu]\,,\quad 0 < \nu \leq \widehat{\nu}\,, \end{aligned}$$

which yields (3.72).

15. The inductive step can now be repeated. Assume that a patchy feedback and a piecewise quadratic value function W has been constructed on a set S_k, with

$$W(x) = \lambda_k = \lambda_0 + k\delta \qquad \forall\;\; x \in \partial S_k\,. \tag{3.74}$$

Since the boundary ∂S_k is the union of finitely many spheres, say $\Gamma_1^k, \ldots, \Gamma_{N_k}^k$, we can repeat the earlier construction and extend the patchy feedback and the approximate value function to a larger set S_{k+1}, etc... □

Remark. The upper bounds on the gradients $|\nabla V^\eta|$, $|\nabla W|$ of the value functions do not depend on η. On the other hand, the step δ in (3.46), (3.74) must be chosen suitably small, depending on η.

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