Variational Techniques for Quasistatic Evolutionary Models

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Thesis submitted for the degree of
"Doctor Philosophiae"

Academic Year 2005/2006

Ringraziamenti

Desidero ringraziare il mio relatore Gianni Dal Maso poiché i suoi consigli, la sua disponibilità, il suo entusiasmo e la sua incredibilmente vasta e profonda conoscenza della matematica mi hanno guidata e sostenuta in questi anni. In particolare il suo intuito, la sua capacità di ascoltare, dar forma e far funzionare anche idee appena abbozzate, e la sua abilità a tradurre (quasi) istantaneamente le idee in parole, e poi le parole in precise formulazioni matematiche, sono stati per me un grande incentivo e insegnamento.

Ringrazio Rodica Toader con cui ho condiviso gioie e dolori di quest'ultimo anno di lavoro in SISSA, e Fabio Zanolin che ha trovato il tempo, l'energia e l'entusiasmo per continuare la nostra collaborazione dopo la tesi di laurea fino a produrre alcuni lavori nuovi sull'esistenza e molteplicità di soluzioni periodiche in un modello di fibra nervosa.

Un ringraziamento a tutti gli amici della SISSA che hanno condiviso con me questi indimenticabili anni. Un grazie particolare a Laura, Paolo e Fabio per la loro nutriente vicinanza.

Ringrazio i miei familiari per il loro insostituibile affetto, la fiducia e il continuo sostegno. Desidero ringraziare in modo speciale il mio papà Luigi: i suoi consigli mi hanno sempre pacificata consentendomi così di liberare nuove energie.

Un grazie dal profondo del cuore a mio marito Cristian per il suo amore, la sua presenza e disponibilità, la sua infinita pazienza e bontà, il suo modo unico e prezioso di andare incontro alla Vita.

Trieste, 8 Settembre 2006.

Ai miei nonni

Contents

Introduction

0.1 Origin and development of the problem and its mathematical formulation

The idea of consider the crack propagation as the result of the competition between the volume energy and the surface energy (needed to create new fracture) is the basis of the mathematical formulation for irreversible quasistatic evolutionary models that will be used in the present thesis.

The fracture process is assumed to be irreversible, so that the crack set can only increase in time, and quasistatic, i.e., at each time the configuration describing the body is in equilibrium.

The link between the crack propagation and surface and bulk energies of the body is due to Griffith [21]. In 1920, making some precise experiments on a metallic body subjected to alternating or repeated loads, he pointed out the necessity of modify the hypotheses of rupture commonly used for elastic solids, by which a fracture may be expected if either the maximum tensile stress or the maximum extension exceeds a certain critical value. Hence, he proposed a new criterion, the celebrated Griffith's criterion, linking the crack process with a decrease in the potential energy. More precisely, he added to the known statement that the potential energy of an elastic body in equilibrium, deformed by surface forces, is a minimum, the following one:

> *Rupture has occurred if the system can pass from the unbroken to the broken condition by a process involving a continuous decrease in potential energy.*

Moreover, under the assumption that the surfaces of the crack are traction-free, he proposed to express the increase of surface energy, due to the crack propagation, by the product of the increment of surface with the surface tension (toughness) of the material.

In 1998 Francfort and Marigo in [18] proposed a mathematical formulation inspired by Griffith's criterion, and then based on energy minimization, for the study of quasistatic crack growth in elastic bodies.

To be more specific, let $\Omega \subset \mathbb{R}^2$ represent the crack-free reference configuration of a linearly elastic, isotropic and homogeneous body, let $\partial_d\Omega$ be a part of its boundary and let $\psi\colon \partial_d\Omega \to \mathbb{R}^2$ be the displacement at the points of $\partial_d\Omega$. Then, for a given crack set $\Gamma \subset \overline{\Omega}$ of finite 1-dimensional Hausdorff measure, and for a boundary displacement ψ, the set of admissible displacements is given by

$$AD(\psi,\Gamma) := \{u \in H^1(\Omega\setminus\Gamma;\mathbb{R}^2) : u = \psi \ \text{ on } \ \partial_d\Omega\setminus\Gamma\}\,,$$

where the equality has to be considered in the sense of traces. According to [18], the total energy associated to a boundary displacement ψ and a crack Γ is given by

$$\mathscr{E}(\psi,\Gamma) = \min_v \left\{\mathcal{W}(Ev) + k\mathscr{H}^1(\Gamma) : v \in AD(\psi,\Gamma)\right\} \tag{0.1.1}$$

where Ev denotes the symmetrized gradient of v and the bulk energy is defined by

$$\mathcal{W}(Ev) := \frac{1}{2}\int_{\Omega\setminus\Gamma} A(x)Ev\cdot Ev\,dx$$

where $A(x)$ is the elasticity tensor. In the surface energy, $k\mathscr{H}^1(\Gamma)$, k denotes the toughness (surface energy density) of the material and $\mathscr{H}^1$ is the 1-dimensional Hausdorff measure.

The evolution is driven by time-dependent imposed boundary displacements $\psi(t)$ for $t \in [0,T]$, and in [18] is studied in the case of monotonically increasing loadings

$$\psi(t)(x) := \begin{cases} t\psi_0(x) & \text{for } t > 0 \\ 0 & \text{for } t \le 0\,, \end{cases}$$

with $\psi_0\colon \partial_d\Omega \to \mathbb{R}^n$ an H^1-function.

The mathematical formulation of Francfort and Marigo consists of finding a time-dependent map $t \mapsto \Gamma(t)$ describing the evolution of the crack during the loading process generated by the imposed boundary displacement. More in detail, the map $t \mapsto \Gamma(t)$ shall satisfy the following three conditions:

(a) *global stability:*

$$\mathscr{E}(\psi(t),\Gamma(t)) \le \mathscr{E}(\psi(t),K) \qquad \text{for every } K \supset \bigcup_{s<t}\Gamma(s);$$

(b) *irreversibility:* the map $t \mapsto \Gamma(t)$ is monotonically increasing;

(c) *energy inequality:*

$$\mathscr{E}(\psi(t),\Gamma(t)) \le \mathscr{E}(\psi(s),\Gamma(s)) \qquad \text{for every } 0 \le s < t \le T\,.$$

This formulation, as condition (a) shows, deals with *global* minimizers for the energy functional. In [18] the authors pointed out that a more realistic approach would be to investigate *local* minimizers, but because of the mathematical difficulties, this approach was not considered at that moment.

Condition (b) expresses the irreversibility of the crack process, that is, the fracture can only increase, while condition (c) is a constraint in order to bound the number of solutions.

The authors indicated also how to obtain the existence of this evolution, that is by a time-discretization process. Indeed, let us fix a sequence of subdivisions $(t_k^i)_{0\le i\le k}$ of the interval $[0,T]$, with

$$0 = t_k^0 < t_k^1 < \cdots < t_k^k = T\,,$$
$$\lim_{k\to+\infty} \max_{1\le i\le k} (t_k^i - t_k^{i-1}) = 0\,.$$

For $i = 1,\ldots,k$ we set $\psi_k^i = \psi(t_k^i)$. For every $k \in \mathbb{N}$, we may define the discretized evolution Γ_k^i by induction as follows. Let Γ_0 be given and set $\Gamma_k^0 := \Gamma_0$. Then Γ_k^i is defined through the two conditions written below:

$$\begin{aligned} &\Gamma_k^i \supset \Gamma_k^{i-1} \\ &\mathscr{E}(\psi_k^i, \Gamma_k^i) \le \mathscr{E}(\psi_k^i, \Gamma) \qquad \text{for every } \Gamma \supset \Gamma_k^{i-1}\,. \end{aligned}$$

As a next step, for every $t \in [0,T]$, we can define the piecewise constant functions

$$\Gamma_k(t) = \Gamma_k^i, \qquad \psi_k(t) = \psi_k^i\,,$$

where i is the greatest integer such that $t_k^i \le t$. Therefore, the real evolution $\Gamma(t)$ will be obtained as the limit as the time step goes to zero (i.e., $k \to +\infty$) of the discrete evolutions $\Gamma_k(t)$.

This is the "abstract" scheme of the proof. The "real" one needs a suitable mathematical environment in order to get the existence of the discretized evolution Γ_k^i, some apriori bounds on $\Gamma_k(t)$ in order to deal with a convergent subsequence, and the right notion of convergence, togheter eventually with some mathematical tool, which guarantees that the minimality properties at the discrete level of the sequence $\Gamma_k(t)$ are transferred to the limit function $\Gamma(t)$.

Now we shall see how Griffith's criterion, in the case when the crack path is prescribed, is written through this notion of evolution. Let us suppose that the crack path is a rectifiable curve Γ parametrized by its arc-length $\gamma(s)$. We define

$$\Gamma(\ell) := \Gamma_0 \cup \{\gamma(s) : 0 \le s \le \ell\}$$

and

$$F(\ell) := \min_v \left\{ \frac{1}{2} \int_{\Omega\setminus\Gamma(\ell)} A(x)Ev\cdot Ev\,dx : v \in AD(\psi_0, \Gamma(\ell)) \right\}.$$

The Griffith's criterion is given in terms of the trajectory of the crack along its path, i.e., the map $t \mapsto \ell(t)$, and states the following.

(1) $\dot\ell(t) \ge 0$;

(2) $-t^2 \frac{d}{d\ell} F(\ell(t)) \leq k$;

(3) $\dot{\ell}(t)\big(k + t^2 \frac{d}{d\ell} F(\ell(t))\big) = 0$;

where $\dot{\ell}(t)$ is the time derivative of the map $\ell(t)$. Indeed, if the energy release rate, $-t^2 \frac{d}{d\ell} F(\ell(t))$, equals the toughness k, then the crack propagation will take place, and it will not if on the contrary in condition (2) the strict inequality holds.

As noticed before, this formulation needs to be treated as a well-posed mathematical problem in order to be meaningful. In [18] the authors pointed out the correspondence of the minimization problem (in its discrete form) with the model proposed by Mumford and Shah [37] in the context of image segmentation, where the well-posed minimization problem is studied in the space of special functions of bounded variation, $SBV(\Omega)$, and they indicate this same space as the right one for their formulation.

However, when the crack path is prescribed, the mathematical problem is well-posed in a suitable subspace of Sobolev functions, which will be the actual setting of the present thesis.

Another assumption in order to deal with Sobolev spaces was taken into account by Dal Maso and Toader in [9]. In this paper they obtained a precise mathematical formulation of the problem and proved an existence result for an irreversible quasistatic evolution, in the antiplane case and for general imposed boundary displacements, in the context of linear elasticity. The admissible cracks are assumed to be connected or with a uniform bound on the number of connected components. With this restriction the cracks are assumed to be closed and then the deformation belongs to a suitable Sobolev space, simplifying the mathematical formulation of the problem. In this paper, the notion of evolution satisfies the analogous of conditions (a) and (b) while clarifies in some sense condition (c) of the definition of evolution law in [18], which is here expressed as

(c) *energy balance:* the function $t \mapsto \mathscr{E}(\psi(t), \Gamma(t))$ is absolutely continuous on $[0, T]$ and

$$\frac{d}{dt}\mathscr{E}(\psi(t), \Gamma(t)) = 2 \int_{\Omega \setminus \Gamma(t)} \nabla u(t) \cdot \nabla \dot{\psi}(t)\, dx \qquad \text{for a.e. } t \in [0, T],$$

where $u(t)$ is a solution of a minimum problem similar to (0.1.1) defining $\mathscr{E}(\psi(t), \Gamma(t))$, and $\dot{\psi}(t)$ is the time derivative of $\psi(t)$.

This condition states that the increment in stored energy plus the surface energy dissipated in the crack process is equal to the work done by the loadings on $\partial_d\Omega$ to produce the imposed displacement. Indeed, if $\partial_d\Omega$ is sufficiently regular, an integration by parts of the right-hand side of equality in (c) gives

$$\frac{d}{dt}\mathscr{E}(\psi(t), \Gamma(t)) = 2 \int_{\partial_d\Omega \setminus \Gamma(t)} \partial_\nu u(t) \dot{\psi}(t)\, d\mathscr{H}^1 \qquad \text{for a.e. } t \in [0, T],$$

where ν is the outer unit normal to the boundary of Ω. This last equality expresses the conservation of energy in this quasistatic model, since the right-hand side is the power of the force exerted on $\partial_d\Omega$ in order to obtain the prescribed displacement $\psi(t)$.

The authors also described and proved the validity of Griffith's criterion for their model. For simplicity let us consider here the case of connected cracks. Let then $t \mapsto \Gamma(t)$ be the map describing the evolution during the loading process. Assume that there exists a simple arc K contained in Ω and parametrized by the regular path $\gamma\colon [\sigma_0, \sigma_1] \to \Omega$ such that $\Gamma(t) = \Gamma(0) \cup K(\sigma(t))$, where $K(\sigma) = \{\gamma(\tau) : \sigma_0 \le \tau \le \sigma\}$, and $\sigma\colon [0,T] \to [\sigma_0, \sigma_1]$ is a nondecreasing function such that $\sigma(0) = \sigma_0$, and $\sigma_0 < \sigma(t) < \sigma_1$ for every $t \in (0,T)$. Assume also that $K \cap \Gamma(0) = \{\gamma(\sigma_0)\}$.

Let $u(t)$ be a solution to the minimum problem defining $\mathscr{E}(\psi(t), \Gamma(t))$ (quite similar to (0.1.1)). Finally, let $\kappa(u(t), \sigma(t))$ be the stress intensity factor of $u(t)$ at the tip $\gamma(\sigma(t))$ (for its definition see Proposition 3.2.2 in Chapter 3). Then Griffith's criterion can be written here by the following three conditions:

(1) $\dot\sigma(t) \ge 0$ for a.e. $t \in [0,T]$;

(2) $1 - \kappa(u(t), \sigma(t))^2 \ge 0$ for every $t \in [0,T]$;

(3) $(1 - \kappa(u(t), \sigma(t))^2)\dot\sigma(t) \ge 0$ for a.e. $t \in [0,T]$.

Indeed, the first condition expresses the irreversibility of the process, while the second condition, thanks to the link between the stress intensity factor and the energy release rate (see, e.g., [23]), gives an upper bound for the energy release rate. Finally, the third condition states that this upper bound is reached at almost every time in which the tip of the crack moves with a positive velocity.

Chambolle in [3] extended then the existence result of [9] to the case of planar linear elasticity, under the same restrictive assumptions on the number of the connected components but imposed to $\Gamma \cup \partial_N\Omega$, instead of the fracture set Γ alone, where $\partial_N\Omega$ is the part of the boundary that we have to add to the Dirichlet part of the boundary $\partial_D\Omega$ (where some displacement is prescribed) to obtain the whole boundary $\partial\Omega$ of the domain $\Omega \subset \mathbb{R}^2$. The author proposed another improvement to the model, that is to consider $\mathscr{H}^1(\Gamma \setminus \partial_N\Omega)$ (instead of simply $\mathscr{H}^1(\Gamma)$) as the energy that must be spent to open the crack Γ, following the idea that the possible parts of Γ touching $\partial_N\Omega$ cannot really be considered as "cracks".

Subsequently, the paper [17] of Francfort and Larsen removed the restriction on the connected components of the cracks and introduced, in an n-dimensional setting, a weak formulation of the problem on the set $SBV(\Omega)$ of special functions with bounded variation. The main mathematical tool developed by the authors in this paper is the so called jump transfer theorem, which enabled the authors to show how to obtain the global stability condition for the evolution from the equivalent condition of the approximating evolution.

In [8], Dal Maso, Francfort and Toader provided the mathematical tools necessary to solve the problem in the most general situation, without any simplifying hypothesis. For this reason, we want to describe more in detail their formulation.

The deformation u is this time vector-valued, and maps a subset Ω of $\mathbb{R}^n$ into $\mathbb{R}^m$ (with $m \geq 1$). Due to this assumption, the mathematical formulation shall be given on a subset of $GSBV(\Omega;\mathbb{R}^m)$, the space of generalized special functions of bounded variation.

The authors studied the case of nonlinear elasticity, considering an arbitrary bulk energy $\mathcal{W}(\xi)$, with energy density $W(x,\xi)$ quasiconvex with respect to ξ and satisfying suitable polynomial growth and regularity conditions. They took into account a large class of time-dependent body and surface forces, whose work on the deformation u is here denoted by $\mathcal{L}(t)(u)$.

In their work, a crack is any rectifiable set $\Gamma \subset \overline{\Omega}$ with $\mathscr{H}^{n-1}(\Gamma) < +\infty$, where $\mathscr{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. The work done to produce the crack Γ is given by

$$\mathcal{K}(\Gamma) = \int_\Gamma \kappa(x, \nu_\Gamma(x))\, d\mathscr{H}^{n-1}(x),$$

where ν_Γ is a unit normal vector field. The function $\kappa(x,\nu)$ depends on the material and satisfies the standard hypotheses which guarantee the lower semicontinuity of $\mathcal{K}$.

An admissible configuration is a pair (u,Γ), where Γ is an admissible crack and u is an admissible deformation with jump set $S(u)$ contained in Γ. The total energy of the admissible configuration (u,Γ) at time t is therefore given by

$$\mathscr{E}(t)(u,\Gamma) := \mathcal{W}(\nabla u) + \mathcal{K}(\Gamma) - \mathcal{L}(t)(u)\,.$$

On a part of the boundary, $\partial_d\Omega$, they prescribed also a time-dependent deformation $\psi(t)$ so that the set $AD(\psi(t),\Gamma(t))$ of admissible deformations with crack $\Gamma(t)$ and boundary deformation $\psi(t)$ is defined now as the set of functions u in $GSBV(\Omega;\mathbb{R}^m)$ whose jump set is contained in $\Gamma(t)$ and such that $u = \psi(t)$ in the sense of traces on $\partial_d\Omega$. An irreversible quasistatic evolution of minimum energy configurations is then defined as a map $t \mapsto (u(t),\Gamma(t))$ satisfying the following three conditions:

(a) *global stability:* for every $t \in [0,T]$

$$\mathscr{E}(t)(u(t),\Gamma(t)) \leq \mathscr{E}(t)(u,\Gamma) \qquad \text{for every } \Gamma \supset \Gamma(t) \text{ and every } u \in AD(\psi(t),\Gamma);$$

(b) *irreversibility:* the map $t \mapsto \Gamma(t)$ is increasing;

(c) *energy balance:* the increment in stored energy plus the work done to produce the crack is equal to the work of the external loadings.

In the spirit of Griffith's theory, a minimum energy configuration at time t, is any pair $(u(t),\Gamma(t))$, with $u(t) \in AD(\psi(t),\Gamma(t))$ which satisfies the global stability condition at time t.

The authors proved the following existence result: if (u_0, Γ_0) is a minimum energy configuration at time $t = 0$, then there exists an irreversible quasistatic evolution $t \mapsto (u(t), \Gamma(t))$ starting from (u_0, Γ_0). The proof is obtained also here by time-discretization.

In the same years, Mielke together with his collaborators (see [34], [31], and [32]) developed a general scheme of continuous-time energetic formulation of rate-independent processes (see also the more recent study [36] and references therein for an excellent survey on the subject). Rate-independence means that if $t \mapsto y(t)$ represents an irreversible quasistatic evolution for the load $\mathcal{L}(t)$, then for each strictly monotone time-reparameterization $\tau(t)$, the function $y(\tau(t))$ represents the correspondent evolution for the load $\mathcal{L}(\tau(t))$.

Let us describe now this formulation: in an abstract space $\mathcal{Y}$, the unknown is a function $y\colon [0, +\infty] \to \mathcal{Y}$, whose evolution is governed by some energy storage potential $\mathcal{E}\colon [0, T] \times \mathcal{Y} \to \mathbb{R}$ and some dissipation potential $\Delta\colon \mathcal{Y} \to [0, +\infty)$. The continuous-time energetic formulation consists of finding a map $y\colon [0, T] \to \mathcal{Y}$ satisfying a suitable initial condition (as $y(0) = y_0$, with y_0 given), and the following two conditions:

(S) *Global Stability:*

$$\mathcal{E}(t, y(t)) \leq \mathcal{E}(t, \hat{y}) + \Delta(\hat{y} - y(t)) \qquad \text{for every } \hat{y} \in \mathcal{Y}\,;$$

(E) *Energy Equality:*

$$\mathcal{E}(t, y(t)) + \int_0^t \Delta(\dot{y}(s))\, ds = \mathcal{E}(0, y_0) + \int_0^t \partial_s \mathcal{E}(s, y(s))\, ds\,.$$

The previous definition of irreversible quasistatic evolution given in [8] fits actually this formulation. Indeed, taking $y(t) := (u(t), \Gamma(t))$ with $\Gamma(t)$ admissible crack and $u(t) \in AD(\psi(t), \Gamma(t))$ (this shall define the state space $\mathcal{Y}$), and $\mathcal{E}(t, u, \Gamma) := \mathcal{W}(\nabla u) - \mathcal{L}(t)(u)$, we deduce that the Global Stability condition (S) coincides with item (a) in the definition of irreversible quasistatic evolution and the Energy Equality condition (E) coincides with item (c), while (b) can be obtained by using a nonsymmetric dissipation distance.

At this point, the precise mathematical formulation with a good notion of irreversible quasistatic evolution and the way to obtain existence results seem to be clarified in the case of elasticity, even if the majority of the results remains inserted in the context of global minimizers. But further steps have to be done in the study evolutionary models, performing an analysis of the mathematical tools developed so far in order to generalize them, and taking into account some additional physical or mechanical features, in order to represent in a more realistic way the fracture process.

Also this part is very difficult, since to any mechanical property added to the model has to correspond the right mathematical tool in order to obtain the well-posedness of the problem and then some existence result. Anyway, in the past few years, several contributes appeared. We cite for example the paper [20] of Giacomini, where he studied the size effect for quasistatic crack growth in linearly isotropic elastic bodies under antiplanar shear. Instead of the Griffith's energy functional

$$\mathscr{E}(u,\Gamma) = \int_{\Omega\setminus\Gamma} |\nabla u|^2\,dx + \mathscr{H}^{n-1}(\Gamma) \tag{0.1.2}$$

the author considered a surface energy of the form

$$\int_\Gamma \varphi(|[u]|(x))d\mathscr{H}^{n-1}\,,$$

where $[u](x) := u^\oplus(x) - u^\ominus(x)$ is the difference of the traces of u on both sides of the crack Γ, and $\varphi\colon [0,+\infty[\, \to [0,+\infty[$ is increasing, concave and constant after a critical length, and is such that $\varphi(0) = 0$. Therefore in this model we point out the attempt to take into account the cohesive forces exerted between the lips of the crack through this energy term. Indeed, we associate the opening of the crack to a nonzero jump $[u]$ of the deformation and through such a function φ we may get the following physical interpretation: the interaction between the two lips of the crack Γ decreases as the opening increases, and disappears when the opening is larger than a critical length.

The author proved an asymptotic result, that is the following: as the size of the body increases, despite of the cohesive form of the surface energy, under suitable boundary displacements the fracture propagates following the Griffith's functional (0.1.2).

0.2 Contents

The purpose of the present thesis is to continue this investigation, in the case of elasticity, taking into account different aspects of the fracture process.

Accordingly, in Chapter 1 we study a variational model for which the new feature is that the fracture energy term depends on the opening of the crack. This term is similar to the one studied in [20], but here we do not perform an asymptotic analysis and prove on the contrary an existence result via an energy functional containing this surface term.

The formulation of the problem and the plan of the proof (via a time-discretization process) fit the scheme proposed in the previous section, including the condition on global minimizers, but the dependence on the opening of the crack prevents us from applying directly the tools developed so far in the applications to fracture mechanics of the theory of free discontinuity problems (see [18], [9], [10], [3], [17], [7], [8]).

The formulation of the problem studied in Chapter 2 is completely different and is the result of a preliminary study (performed in finite dimension) in order to deal with local minimizers and with a notion of approximable evolution which will be considered in Chapter 3.

To be more specific, let us recall that for a given Banach space X, and for an energy functional $E\colon [0,T]\times X \to \mathbb{R}$, a quasistatic evolution $t \mapsto u(t)$ may be thought as a solution of the system

$$\nabla_x E(t, u(t)) = 0\,, \tag{0.2.1}$$

so that $u(t)$ is a critical point for $E(t, \cdot)$, and, possibly, a local minimizer.

In Chapter 2 we consider the limit behavior, as $\varepsilon \to 0$, of the solutions u_ε of the ε-gradient flow

$$\varepsilon \dot{u}_\varepsilon(t) + \nabla_x E(t, u_\varepsilon(t)) = 0\,,$$

and we prove (in the case when $X = \mathbb{R}^n$) that, under very general assumptions on E, the limit function $u(t)$ does exist and solves (0.2.1). In addition, by uniqueness, on its continuity intervals $]t_{i-1}, t_i[$, the function $u(t)$ is actually defined via the Implicit Function Theorem. As the main new feature of this chapter, the connection between the limits $u(t_i^-)$ and $u(t_i^+)$ is analyzed.

Inspired by the previous two ideas, that are to define the evolution map as a local minimizer (as also suggested by Francfort and Marigo in their pioneering paper [18]) and to characterize the evolution on its continuity intervals via the Implicit Function Theorem, we study in Chapter 3 a new model of irreversible quasistatic evolution which satisfies a local stability criterion rather than the "usual" global one.

To simplify the mathematical difficulties, along the thesis we make the assumption that the crack path is prescribed, and this allows us to use Sobolev spaces in the formulation of the variational problem. This hypothesis appears to be natural in the study of the *past* evolution of a crack. It can also be used to predict the *future* evolution of a crack assuming that the body has some natural weaker parts. For example, as already pointed out by Mainik and Mielke in [32], we may assume that the two sides of the body are glued together along the prescribed crack path, and that the glue is softer than the material itself, so that, upon loading, the body can fracture only along the glue.

Let us describe now more in detail the content of this thesis.

In Chapter 1 we present a variational model for quasistatic crack growth in the presence of a cohesive force exerted between the lips of the crack.

The evolution of the crack for our model is governed by an energy which is, (like, e.g., in [8]), the sum of three terms: the bulk energy of the uncracked part, the energy dissipated in the fracture process, and the work of the external loads, but here, as already announced, the fracture energy depends on the opening of the crack.

The cracks are constrained to belong to a compact C^1-orientable $(n-1)$-dimensional manifold M contained in the reference configuration $\Omega \subset \mathbb{R}^n$, such that $\Omega \backslash M$ is connected. This restriction allows us to consider very general bulk and crack energies, which may include constraints on the crack opening, related to the infinitesimal noninterpenetration of matter.

To be more specific, we assume that the uncracked part of the body is hyperelastic and that its bulk energy relative to the deformation $u \in W^{1,p}(\Omega \setminus M; \mathbb{R}^m)$ is of the form

$$\int_{\Omega\setminus M} W(x, \nabla u)\, dx,$$

where $W\colon (\Omega \setminus M) \times \mathbb{M}^{m\times n} \to \mathbb{R}$ is a given Carathéodory function such that $W(x,\xi)$ is quasiconvex with respect to ξ and satisfies suitable p-polynomial growth and regularity conditions (see Section 1.1).

The work done by the external time-dependent loads $\mathscr{L}(t)$ on the deformation u is denoted by $\langle \mathscr{L}(t), u\rangle$, being $\langle\cdot,\cdot\rangle$ the duality pairing between $(W^{1,p}(\Omega \setminus M; \mathbb{R}^m))'$ and $W^{1,p}(\Omega \setminus M; \mathbb{R}^m)$. For the general form of the work done by the external loads see (1.1.5).

In order to obtain the work done to produce a crack for this model, we need some preliminary discussion. If we neglect for a first moment the problem of irreversibility, we may then assume that the work done to produce a crack can be written in the form

$$\int_M \varphi(x, [u])\, d\mathscr{H}^{n-1},$$

where $\varphi\colon M \times \mathbb{R}^m \to [0,+\infty]$ is a Borel function such that $\varphi(x,0) = 0$ and the map $y \mapsto \varphi(x,y)$ is lower semicontinuous on $\mathbb{R}^m$ for $\mathscr{H}^{n-1}$-a.e. $x \in M$.

Suppose now that the deformation u depends on time, i.e., we have a map $t \mapsto u(t)$ from $[0,T]$ into $W^{1,p}(\Omega \setminus M; \mathbb{R}^m)$. If no crack is present until time 0 and

$$\varphi(x, [u(s)](x)) \le \varphi(x, [u(t)](x)) \qquad \mathscr{H}^{n-1}\text{-a.e. on } M$$

for every $s \in [0,t]$, then the energy dissipated in the crack process in the time interval $[0,t]$ is given, in our model, by

$$\int_M \varphi(x, [u(t)](x))\, d\mathscr{H}^{n-1}\,.$$

However, in the general case, the irreversibility of the fracture process leads to introduce an auxiliary time-dependent function $t \mapsto \gamma(t)$ (see Section 1.1), defined on the prescribed crack path, which takes into account the local history of the crack up to time t. More in detail we assume that $t \mapsto \gamma(t)$ is the smallest increasing in time function such that $\gamma(t)(x) \ge \varphi(x, [u(t)](x))$ for $\mathscr{H}^{n-1}$-a.e. $x \in M$ and for every $t \in [0,T]$, and that the total energy spent in the process of the crack production in the time interval $[0,t]$ is given by

$$\int_M \gamma(t)(x)\, d\mathscr{H}^{n-1}\,.$$

The notion of evolution of the crack (see Definition 1.2.4) is given in the framework of Mielke's approach to a variational theory of rate-independent processes (see [35], [32]), and satisfies a global stability condition, an irreversibility condition and an energy balance condition which correspond to the analogous conditions for the evolutionary models presented in the previous section.

We prove an existence result (Theorem 1.2.10) for the quasistatic evolution following the scheme already introduced by Francfort and Marigo, i.e., by approximating the continuous-time problem by discrete-time problems, for which the evolution is defined by solving incremental minimum problems. The main mathematical difficulty in the proof is the compactness of the approximating functions $t \mapsto \gamma_k(t)$. This is solved by introducing a new notion of convergence of functions related to the problem, with good compactness and semicontinuity properties (see Section 1.3).

In Chapter 2 we use a different approach with the aim of investigating whether there is the possibility to characterize the jumps of an evolution defined using some criterion different from the global minimality. The analysis is carried on in a model case where the main feature is that the dimension of the Banach space is finite.

As already observed in the first part of this section, we recall that the study of quasistatic rate-independent evolutionary models may lead to consider gradient flow-type problems. Indeed, suppose that X is a given Banach space and $f\colon [0,T] \times X \to \mathbb{R}$ a time-dependent energy functional. Then one can regard a quasistatic evolution as a solution $u(t)$ to the problem

$$\nabla_x f(t, u(t)) = 0\,. \tag{0.2.2}$$

In order to obtain such an evolution, it seems natural to study the limit as $\varepsilon \to 0$ of the perturbed problem

$$\varepsilon \dot{u}_\varepsilon(t) + \nabla_x f(t, u_\varepsilon(t)) = 0\,, \tag{0.2.3}$$

which is actually a gradient flow problem. The intention is to prove that if the functional f satisfies suitable assumptions, then the solutions u_ε converge to a limit function u solving problem (0.2.2) and that this method selects in a sense the most interesting solutions u of (0.2.2) (see [15], [6]).

In Chapter 2 we study a model case with $X = \mathbb{R}^n$. We shall see that, under very general assumptions on f, the limit function $u(t)$ is a local minimum of $f(t, \cdot)$. Moreover, it may admit some discontinuity times, while the approximating solutions $u_\varepsilon(t)$ of the ε-gradient system (0.2.3) are always continuous.

The first work on similar subjects was written by Efendiev and Mielke [15], who add to the energy functional a dissipation term, which is crucial in the proof of the compactness of u_ε. In our work, we do not have dissipative terms, but the assumptions on f are stronger (for the precise assumptions see Section 2.1).

Let us describe now more in detail our model case. We consider a smooth energy function $f\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}$ satisfying a suitable coerciveness condition (see Assumption 1). We suppose also that $u_\varepsilon(0) \to u(0)$, $\nabla_x f(0, u(0)) = 0$, and that $\nabla_x^2 f(0, u(0))$ is positive definite. We will prove that u_ε converges, as ε goes to zero, to a piecewise regular function $u\colon [0,T] \to \mathbb{R}^n$, defined via the Implicit Function Theorem, such that $\nabla_x f(t, u(t)) = 0$ and $\nabla_x^2 f(t, u(t))$ is positive definite on each continuity interval $]t_{i-1}, t_i[$. Moreover the limits from the left and from the right, $u(t_i^-)$ and $u(t_{i-1}^+)$, exist. It turns out that at each discontinuity time t_i, the value of $u(t_i^-)$ is located at a degenerate critical point of $f(t_i, \cdot)$, i.e., at a point $x \in \mathbb{R}^n$ where the Hessian matrix $\nabla_x^2 f(t_i, x)$ possesses at least one zero eigenvalue.

To conclude this analysis we have to establish the connection between the limits $u(t_i^-)$ and $u(t_i^+)$. This will be done by passing to the fast dynamics, i.e., the dynamics governed by the rescaled system of differential equations

$$\dot{v}(s) = -\nabla_x f(t_i, v(s)) \,. \tag{0.2.4}$$

In a generic situation we may assume that $\nabla_x^2 f(t_i, u(t_i^-))$ has exactly one zero eigenvalue, while the other eigenvalues are positive. To discuss the behavior of (0.2.4), we are led to consider the autonomous system of differential equations $\dot{v}(s) = -\nabla_x f(t, v(s))$ where t is close to t_i and plays the role of a parameter. Under very general hypotheses, the following happens: before t_i the vector field $\nabla_x f(t, \cdot)$ has two zeroes, a saddle and a node, at $t = t_i$ there is only one zero (the node and the saddle coalesce), and for $t > t_i$ these zeroes of the vector field no longer exist. This corresponds to an abrupt change in the phase portrait as the parameter varies, and it is known in the literature as saddle-node bifurcation of codimension one (see [24], [42], [27]).

In Section 2.1 we list the technical assumptions which permit to obtain the main result of the chapter, Theorem 2.2.7. Without entering all technical details, the setting obtained from our assumptions is the following one. For every $t \in [0, T]$ there is a finite number of critical points $x \in \mathbb{R}^n$ of $f(t, \cdot)$ and among them at most one is degenerate. Moreover there exists only a finite number of pairs (t, ξ) such that ξ is a degenerate critical point of $f(t, \cdot)$. On the degenerate critical points with only nonnegative eigenvalues, the Hessian matrix $\nabla_x^2 f(t, \xi)$ has only one zero eigenvalue and satisfies two transversality conditions (see (b) and (c) in Assumption 3). Although we do not prove that Assumptions 1–4 of Section 2.1 are generic in any technical sense, they cover a wide class of interesting examples.

If ξ is a degenerate critical point of $f(t, \cdot)$ satisfying all conditions considered above, then we prove that there is a unique heteroclinic solution $v(s)$ of $\dot{v}(s) = -\nabla_x f(t, v(s))$ issuing from the degenerate critical point ξ, and we suppose that $v(s)$ tends, as $s \to +\infty$, to a nondegenerate critical point y of $f(t, \cdot)$, with $\nabla_x^2 f(t, y)$ positive definite. The existence of such heteroclinic solution is standard. Since we have not been able to find the proof of uniqueness in the literature, we give the complete proof in Lemma 2.1.5.

This analysis leads to a more precise construction of the function u mentioned above. Accordingly, the main result of this chapter, Theorem 2.2.7, states that if $u_\varepsilon(0) \to u(0)$, $\nabla_x f(0, u(0)) = 0$, and $\nabla_x^2 f(0, u(0))$ is positive definite, and Assumptions 1–4 of Section 2.1 are satisfied, then $u_\varepsilon(t)$ converges to $u(t)$ uniformly on compact sets of $[0, T] \setminus \{t_1, \ldots, t_{k-1}\}$, where t_i are the discontinuity times for u. Moreover in a small neighborhood of t_i, a rescaled version of $u_\varepsilon(t)$ converges to the heteroclinic solution $v(s)$, connecting $u(t_i^-)$ and $u(t_i^+)$. Finally, the graph of u_ε approaches the completion of the graph of u obtained by using the heteroclinic trajectories.

The previous chapter may suggest the following idea, that it may be possible to characterize the regular paths of an evolution $u(t)$ using the Implicit Function Theorem. This actually has become the key property for the irreversible quasistatic evolutionary model studied in Chapter 3 of the present thesis.

For this model, the prescribed crack path, Γ, is a regular arc with one endpoint on the boundary of the reference configuration Ω and the other inside Ω. We assume in addition that there exists an initial connected crack starting from the boundary point, and that the crack remains connected during the evolution. Hence, such a crack will be completely determined by its length σ. Here a configuration is a pair (u, σ) where u is a scalar function representing the displacement orthogonal to the plane of Ω, and σ represents the length of the crack.

The evolution is driven by time-dependent imposed boundary displacements $\psi(t)$ on a part $\partial_D\Omega$ of the boundary, and applied boundary forces $g(t)$ on the remaining part $\partial_N\Omega$. The total energy, $\mathscr{E}(t)(u, \sigma)$, of a configuration (u, σ) at time t, is the sum of the bulk energy and the surface energy minus the work of the applied forces $g(t)$.

For this model we are interested in obtaining a local stability criterion for the energy functional rather than a global one. Hence, we focus on this aspect and keep the rest of the model as simple as possible.

Therefore, the bulk part of the energy is given by the square of the L^2-norm of the gradient of u, while, according to Griffith's theory, we assume the surface energy to be proportional to the length σ of the crack, the constant of proportionality being given by the toughness of the material.

For a given crack length σ and for a boundary displacement $\psi(t)$, let $AD(\psi(t), \sigma)$ be the set of admissible displacements, i.e., displacements with finite bulk energy, compatible with $\psi(t)$ and σ.

Note that for this model, given t and σ, there exists a unique minimizer $u_{t,\sigma}$ of the energy $\mathscr{E}(t)(u, \sigma)$ in $AD(\psi(t), \sigma)$. Then let us consider the minimal energy $E(t, \sigma)$ corresponding to the boundary data $\psi(t)$ and to the crack length σ, i.e., $E(t, \sigma) := \mathscr{E}(t)(u_{t,\sigma}, \sigma)$. The derivative $\partial_\sigma E(t, \sigma)$ can be computed (see Proposition 3.2.4) and it is related to the stress intensity factor of the displacement $u_{t,\sigma}$ at the tip of the crack. It plays a crucial rôle in the Griffith's criterion for the propagation of cracks, as reminded in the previous section (see also, e.g., [9]).

Let us define now the notion of evolution we are interested in. The *irreversible quasistatic evolution problem* consists in finding a left-continuous function of time $t \mapsto (u(t), \sigma(t))$ such that the displacement $u(t)$ at time t belongs to the set $AD(\psi(t), \sigma(t))$, and the following three conditions are satisfied:

(a) *local stability:* at every time $t \geq 0$

$$\begin{aligned} &\mathscr{E}(t)(u(t), \sigma(t)) \leq \mathscr{E}(t)(v, \sigma(t)) \qquad \forall\, v \in AD(\psi(t), \sigma(t)) \\ &\partial_\sigma E(t, \sigma(t)) \geq 0\,, \end{aligned}$$

(b) *irreversibility:* the map $t \mapsto \sigma(t)$ is increasing;

(c) *energy inequality:* for every $0 \leq s < t$ we have

$$\mathscr{E}(t)(u(t), \sigma(t)) \leq \mathscr{E}(s)(u(s), \sigma(s)) + \mathrm{Work}(u; s, t)\,,$$

where $\mathrm{Work}(u; s, t)$ denotes the work of external forces.

A solution to this problem will be called an *irreversible quasistatic evolution.*

The main difference with respect to previous models stays in condition (a), where the pair $(u(t), \sigma(t))$ shall satisfy the first order necessary conditions for the global stability, but not the sufficient ones. In condition (c) three terms contribute to the work of the external forces: two of them are due to the surface forces generated by the imposed boundary displacement and the third one comes from the applied surface loads.

Despite of the new features, we will prove that conditions (a)-(c) are enough to ensure that at almost every time t a weak version of Griffith's criterion is satisfied (see Proposition 3.3.2). We observe also that the globally stable irreversible quasistatic evolution problem studied in [18], [9], [3], [17], [8] is actually a particular case of the previous one. Therefore, this model contains the previously studied globally stable evolutionary models and continue to ensure the validity of Griffith's criterion.

In order to obtain an existence result we adopt a selection criterion different from global stability and based on an approximation procedure with a regularizing effect. This is performed directly in the time-continuous formulation, in the sense that there is no need of a preliminary study of time-discrete problems.

We thus propose the notion of approximable irreversible quasistatic evolution (see Definition 3.3.6) defined as an irreversible quasistatic evolution which is the limit, along a suitable sequence, of some approximating more regular evolutions $(u_\varepsilon, \sigma_\varepsilon)$ obtained by solving suitable nonlinear PDEs (see (0.2.5)). We prove that if $(u(t), \sigma(t))$ is a limit of such an approximation, then the following property holds.

($\mathcal{P}$) if on a certain time interval $[t_0, t_1]$ there exists a regular function $\sigma^0(t)$ such that

$$\partial_\sigma E(t, \sigma^0(t)) = 0 \quad \text{and} \quad \partial^2_\sigma E(t, \sigma^0(t)) > 0 \qquad \forall\, t \in [t_0, t_1]\,,$$

and if $\dot\sigma_\varepsilon(t) > 0$ for every $t \in [t_0, t_1]$, then the equality $\sigma(t_0) = \sigma^0(t_0)$ implies that $\sigma(t) = \sigma^0(t)$ for every $t \in [t_0, t_1]$.

Since the regular function $t \mapsto \sigma^0(t)$ can be obtained, under suitable standard assumptions, by applying the Implicit Function Theorem to $\partial_\sigma E(t,\sigma) = 0$, we deduce that our evolution can be characterized on the convexity intervals of the energy functional through this regular function $\sigma^0(t)$, provided that some additional assumptions are satisfied. This feature makes the difference with the globally stable evolution, which is expected to move abruptly toward the absolute minimum of the energy, while our approximable evolution is expected to propagate continuously at least on every time interval where property $(\mathcal{P})$ holds.

Let us now describe more in detail the construction of the approximating evolutions. First of all, we fix an initial condition: assume that at time $t = 0$ the crack length is equal to $\sigma_0 > 0$ and the displacement is equal to u_0, in such a way that the initial configuration (u_0, σ_0) is in equilibrium. Then, for every σ between σ_0 and $\overline{\sigma}$, where $\overline{\sigma}$ is the length of Γ, we consider a diffeomorphism Φ_σ of Ω that transforms the crack of length σ into the one of length σ_0. Using Φ_σ, we change variables in the expression of the energy functional $\mathscr{E}$ and transform it into a functional $\mathscr{F}$ depending on the time t, the crack length σ, and the modified displacement v, which takes the form

$$\mathscr{F}(t, v, \sigma) = \int_{\Omega \setminus \Gamma(\sigma_0)} (A(\sigma, x) Dv(x) | Dv(x))\, dx + \dots\,,$$

where $\Gamma(\sigma_0)$ is the crack of length σ_0, $A(\sigma, x)$ is a $2{\times}2$ symmetric matrix of smooth coefficients coming from the change of variables, Dv is the distributional gradient of v with respect to the spatial variables x, and $(\cdot|\cdot)$ denotes the scalar product in $\mathbb{R}^2$. The advantage of this change of variables is that now the set of admissible functions v does not depend on t, nor on the crack length σ. Therefore our argument is developed in terms of the functional $\mathscr{F}$ (after having proved the equivalence of the critical points of $\mathscr{F}$ and of $\mathscr{E}$, see Proposition 3.2.1).

The same change of variables is considered, in a suitable small neighbourhood of the crack tip, in order to compute the derivative $\partial_\sigma E(t, \sigma)$ (see also [23], [2], [28]).

The approximating evolution $(v_\varepsilon, \sigma_\varepsilon)$ is obtained as the solution of a suitably modified ε-gradient flow for the functional $\mathscr{F}$ which starts from the initial data (u_0, σ_0):

$$\begin{cases} \varepsilon \dot{v}_\varepsilon = -\mathrm{grad}_v \mathscr{F}(t, v_\varepsilon, \sigma_\varepsilon) \\ \varepsilon \dot{\sigma}_\varepsilon = (-\partial_\sigma \mathscr{F}(t, v_\varepsilon, \sigma_\varepsilon))^+ \lambda(\sigma_\varepsilon) \\ v_\varepsilon(0) = u_0 \\ \sigma_\varepsilon(0) = \sigma_0\,. \end{cases} \tag{0.2.5}$$

Here $\mathrm{grad}_v \mathscr{F}(t, v, \sigma)$ denotes the gradient of the function $v \mapsto \mathscr{F}(t, v, \sigma)$ considered as a function defined on the Sobolev space $H^1(\Omega \setminus \Gamma(\sigma_0))$ with suitable boundary conditions. The positive part in the second equation guarantees the irreversibility of the evolution, while λ is a Lipschitz continuous positive cut-off function that becomes zero for $\sigma = \overline{\sigma}$, so that only increasing solutions with crack length less than $\overline{\sigma}$ are considered.

If we are interested in the evolution until a certain crack length σ_1, with $\sigma_0 < \sigma_1 < \overline{\sigma}$, is reached, then we choose $\lambda(\sigma) = 1$ for $\sigma_0 \leq \sigma \leq \sigma_1$. In this way for crack lengths less than σ_1, the regularized evolution law is proportional to the gradient flow for $\mathscr{F}$, with the constraint that the crack length is increasing, while it is distorted by λ for crack lengths between σ_1 and $\overline{\sigma}$. Therefore the evolution is considered meaningful only until the crack reaches the length σ_1.

Note that, using the form of the functional $\mathscr{F}$, the first equation in (0.2.5) can be written as

$$\varepsilon \Delta_x \dot{v}_\varepsilon(t, x) = -\mathrm{div}_x(A(\sigma_\varepsilon(t), x) D v_\varepsilon(t, x)) + \dots$$

with suitable boundary conditions. We preferred the evolution problem in H^1 to the usual parabolic one

$$\varepsilon \dot{v}_\varepsilon(t, x) = -\mathrm{div}_x(A(\sigma_\varepsilon(t), x) D v_\varepsilon(t, x)) + \dots$$

which corresponds to the gradient flow in L^2, because it helped us to prove property $(\mathcal{P})$, see Theorem 3.4.1. Note also that in this way the first equation in (0.2.5) becomes an ODE and thus the existence of the solution for this modified ε-gradient flow follows from classical existence and uniqueness results for ordinary differential equations in Banach spaces.

We prove in Theorem 3.3.7 the existence of an approximable irreversible quasistatic evolution, while in Theorem 3.4.1 we obtain property $(\mathcal{P})$ for our evolution.

Let us remark that this model is not suited for the study of the crack initiation problem. We also note that the approximating evolutions we consider have been chosen on the basis of their mathematical simplicity and do not seem to have any mechanical interpretation. Nevertheless, we think that the notion of approximable irreversible quasistatic evolution proposed here could be the starting point for the study of different approximations with a mechanical justification.

The results of Chapter 1 are obtained in collaboration with Gianni Dal Maso and will appear in [11], while the content of Chapter 2 corresponds to the paper [44]. The results of Chapter 3 are achieved in collaboration with Rodica Toader and correspond to paper [41].

Chapter 1

Quasistatic crack growth for a cohesive zone model

In this chapter we present a variational model for quasistatic crack growth in the presence of a cohesive force exerted between the lips of the crack.

We assume that the crack path is prescribed, and, more precisely, that it consists of a compact C^1-orientable $(n-1)$-dimensional manifold M contained in the reference configuration $\Omega \subset \mathbb{R}^n$, such that $\Omega \setminus M$ is connected. We want to study the time evolution of the crack in the framework of Mielke's approach to a variational theory of rate-independent processes (see [35], [32]).

The evolution of the crack is governed by an energy which is the sum of three terms: the bulk energy of the uncracked part, the energy dissipated in the fracture process, and the work of the external loads. The main mathematical difficulty is given by the fact that the fracture energy depends on the opening of the crack. For this reason we cannot apply directly the tools developed so far in the applications to fracture mechanics of the theory of free discontinuity problems (see [18], [9], [10], [3], [17], [7], [8]).

We prove an existence result for the quasistatic evolution (see Theorem 1.2.10), by approximating the continuous-time problem by discrete-time problems, for which the evolution is defined by solving incremental minimum problems. The irreversibility of the crack process leads to introduce an auxiliary time-dependent function $t \mapsto \gamma(t)$ (see Section 1.1 below), defined on the prescribed crack path, which takes into account the local history of the crack up to time t. The main mathematical difficulty in the proof is the compactness of the approximating functions $t \mapsto \gamma_k(t)$. This is solved by introducing a new notion of convergence of functions related to the problem, with good compactness and semicontinuity properties.

The chapter is organized as follows. In Section 1.1 we describe the setting of the problem introducing all the mathematical quantities we need to define, in Section 1.2, the total energy of an admissible configuration (u, γ), where $u \in W^{1,p}(\Omega \setminus M; \mathbb{R}^m)$ represents the deformation and γ belongs to a suitable subset of $L^1(M)$, named $L^1(M)^+$, and represents the internal variable due to the irreversibility of the process. Then we define the notion of irreversible quasistatic evolution (see Definition 1.2.4) we are interested in and prove some properties of it. In Section 1.3 we develop the mathematical tools in order to prove, in Section 1.4, the main result of this chapter, Theorem 1.2.10. In Section 1.5 we study the Euler conditions satisfied by globally stable pairs $(u,\gamma) \in W^{1,p}(\Omega \setminus M;\mathbb{R}^m) \times L^1(M)^+$. Finally, in Section 1.6, we prove that, with some modifications, the main theorem continue to hold also in the case where the uncracked part of the body in linearly elastic.

1.1 Setting

The reference configuration is a bounded open set Ω of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$, which can be written as the union of two disjoint Borel sets $\partial_0\Omega$ and $\partial_1\Omega$, with $\mathscr{H}^{n-1}(\partial_0\Omega) > 0$ and $\partial_1\Omega$ relatively open. Here and henceforth $\mathscr{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. On $\partial_0\Omega$, the Dirichlet part of the boundary, we will assign the boundary deformation, while on $\partial_1\Omega$, the Neumann part of the boundary, we will prescribe surface forces.

We assume that the cracks are contained in a compact C^1-orientable $(n-1)$-dimensional manifold $M \subset \Omega$ with boundary ∂M, such that $\Omega \setminus M$ is connected. Therefore it is reasonable to take the deformation u as a function in the space $W^{1,p}(\Omega \setminus M;\mathbb{R}^m)$, so that the essential discontinuity points of u are contained in M. Although the natural choice is $m = n$, there are no mathematical difficulties in considering an arbitrary $m \geq 1$. The case $m = 1$ is used in the study of antiplane shears. The number $p > 1$ depends on the bounds on the energy density considered below.

We take into account prescribed time-dependent boundary deformations $t \mapsto \psi(t)$, with $\psi(t) \in W^{1,p}(\Omega;\mathbb{R}^m)$, in the sense that for each time $t \in [0,T]$ we consider only deformations $u \in W^{1,p}(\Omega \setminus M;\mathbb{R}^m)$ such that

$$u = \psi(t) \quad \text{on } \partial_0\Omega,$$

where the previous equality has to be considered in the sense of traces. We assume also that, as a function of time, $t \mapsto \psi(t)$ is absolutely continuous from $[0,T]$ into $W^{1,p}(\Omega;\mathbb{R}^m)$.

Thus the time derivative $t \mapsto \dot{\psi}(t)$ belongs to the space $L^1([0,T];W^{1,p}(\Omega;\mathbb{R}^m))$ and its spatial gradient $t \mapsto \nabla\dot{\psi}(t)$ belongs to the space $L^1([0,T];L^p(\Omega;\mathbb{M}^{m\times n}))$.

We assume that the uncracked part of the body is hyperelastic and that its bulk energy relative to the deformation $u \in W^{1,p}(\Omega \setminus M;\mathbb{R}^m)$ is of the form

$$\int_{\Omega\setminus M} W(x,\nabla u)\,dx,$$

where $W(x,\xi)$ is a given Carathéodory function $W\colon (\Omega \setminus M) \times \mathbb{M}^{m\times n} \to \mathbb{R}$ such that

(W_1) $\xi \mapsto W(x,\xi)$ is quasiconvex and C^1 for every $x \in \Omega \setminus M$;

(W_2) there are two positive constants a_0, a_1 and two nonnegative functions $b_0, b_1 \in L^1(\Omega \setminus M)$ such that

$$a_0 \, |\xi|^p - b_0(x) \le W(x,\xi) \le a_1 \, |\xi|^p + b_1(x), \tag{1.1.1}$$

for every $(x,\xi) \in (\Omega \setminus M) \times \mathbb{M}^{m\times n}$.

Since $\xi \mapsto W(x,\xi)$ is rank-one convex on $\mathbb{M}^{m\times n}$ for every $x \in \Omega \setminus M$, we can deduce from (1.1.1) an estimate for the partial gradient of W with respect to ξ, $\partial_\xi W\colon (\Omega \setminus M) \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$. More precisely, there are a positive constant a_2 and a nonnegative function $b_2 \in L^1(\Omega \setminus M)$ such that

$$|\partial_\xi W(x,\xi)| \le a_2 \, |\xi|^{p-1} + b_2(x), \tag{1.1.2}$$

for every $(x,\xi) \in (\Omega \setminus M) \times \mathbb{M}^{m\times n}$.

To shorten the notation we introduce the function $\mathcal{W}\colon L^p(\Omega \setminus M; \mathbb{M}^{m\times n}) \to \mathbb{R}$ defined by

$$\mathcal{W}(\Psi) := \int_{\Omega\setminus M} W(x,\Psi)\, dx,$$

for every $\Psi \in L^p(\Omega \setminus M; \mathbb{M}^{m\times n})$. By (1.1.1) and (1.1.2) the functional $\mathcal{W}$ is of class C^1 on $L^p(\Omega \setminus M; \mathbb{M}^{m\times n})$ and its differential $\partial\mathcal{W}\colon L^p(\Omega \setminus M; \mathbb{M}^{m\times n}) \to L^q(\Omega \setminus M; \mathbb{M}^{m\times n})$, $p^{-1} + q^{-1} = 1$, is given by

$$\langle \partial\mathcal{W}(\Psi), \Phi\rangle = \int_{\Omega\setminus M} \partial_\xi W(x,\Psi) : \Phi \, dx,$$

for every Φ, $\Psi \in L^p(\Omega \setminus M; \mathbb{M}^{m\times n})$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between the spaces $L^q(\Omega \setminus M; \mathbb{M}^{m\times n})$ and $L^p(\Omega \setminus M; \mathbb{M}^{m\times n})$, and $\partial_\xi W(x,\Psi) : \Phi$ denotes the scalar product between the two matrices $\partial_\xi W(x,\Psi)$ and Φ.

By the assumptions on W, the functions $\mathcal{W}$ and $\partial\mathcal{W}$ satisfy the following properties: there are two positive constants α_0, α_1 and two nonnegative constants β_0, β_1 such that

$$\alpha_0 \, \|\Psi\|_p^p - \beta_0 \le \mathcal{W}(\Psi) \le \alpha_1 \, \|\Psi\|_p^p + \beta_1, \tag{1.1.3}$$

for every $\Psi \in L^p(\Omega \setminus M; \mathbb{M}^{m\times n})$, and there is a positive constant α_2 such that

$$\langle \partial\mathcal{W}(\Psi), \Phi\rangle \le \alpha_2 (1 + \|\Psi\|_p^{p-1}) \|\Phi\|_p, \tag{1.1.4}$$

for every Ψ, $\Phi \in L^p(\Omega \setminus M; \mathbb{M}^{m\times n})$.

For a fixed time $t \in [0,T]$, we assume that the external time-dependent loads $\mathscr{L}(t)$ belong to $(W^{1,p}(\Omega \setminus M; \mathbb{R}^m))'$, the dual space of $W^{1,p}(\Omega \setminus M; \mathbb{R}^m)$. The duality product $\langle \mathscr{L}(t), u\rangle$ is interpreted as the work done by the loads on the deformation u.

Let us fix an orientation of M and let $u^\oplus$ be the trace of u on the positive side of M, and $u^\ominus$ be the trace of u on the negative side of M. The most general form of the work done by the external loads is given by

$$\begin{aligned}\langle \mathscr{L}(t), u\rangle = &\int_{\Omega\setminus M} f(t)\, u\, dx + \int_{\Omega\setminus M} H(t) : \nabla u\, dx + \\ &+ \int_{\partial_1\Omega} g(t)\, u\, d\mathscr{H}^{n-1} + \int_M (g^{\oplus}(t)\, u^{\oplus} + g^{\ominus}(t)\, u^{\ominus})\, d\mathscr{H}^{n-1},\end{aligned} \tag{1.1.5}$$

where $f(t) \in L^q(\Omega \setminus M; \mathbb{R}^m)$, $H(t) \in L^q(\Omega \setminus M; \mathbb{M}^{m\times n})$, $g(t) \in L^q(\partial_1\Omega; \mathbb{R}^m)$, $g^{\oplus}(t)$ and $g^{\ominus}(t) \in L^q(M; \mathbb{R}^m)$, with $p^{-1} + q^{-1} = 1$. Actually the representation theorem for $(W^{1,p}(\Omega\setminus M; \mathbb{R}^m))'$ shows that it is enough to use just the terms of the first line of (1.1.5). The terms in the second line have been added in order to write in an explicit way the contribution of the surface forces acting on the Neumann part of the boundary and on one or both sides of M.

With these assumptions we do not exclude the possibility that $H(t)$ could be discontinuous on M. Moreover, observe that if $f(t), H(t), g(t), g^{\oplus}(t)$ and $g^{\ominus}(t)$ are sufficiently regular, then

$$f(t) - \operatorname{div} H(t)$$

plays the role of the volume forces on $\Omega \setminus M$,

$$g(t) + H(t)\nu$$

plays the role of the surface forces on $\partial_1\Omega$, and

$$g^{\oplus}(t) - H^{\oplus}(t)\nu \qquad \text{and} \qquad g^{\ominus}(t) + H^{\ominus}(t)\nu$$

play the role of the surface forces acting on the positive (respectively negative) side of M, where ν is the outer unit normal to $\partial(\Omega\setminus M)$. We observe that, by our positions, ν turns out to be the inner normal on the positive side of M; this is why in the last formula we take the minus sign in front of $H^{\oplus}(t)\nu$.

We assume that, as a function of time, $t \mapsto \mathscr{L}(t)$ is absolutely continuous from $[0, T]$ into $(W^{1,p}(\Omega \setminus M; \mathbb{R}^m))'$. Thus the time derivative $t \mapsto \dot{\mathscr{L}}(t)$ belongs to the space $L^1([0, T]; (W^{1,p}(\Omega \setminus M; \mathbb{R}^m))')$. If $\mathscr{L}(t)$ is represented by (1.1.5), then the absolute continuity of $t \mapsto \mathscr{L}(t)$ follows from the absolute continuity of the functions $t \mapsto f(t)$, $t \mapsto H(t)$, $t \mapsto g(t)$, $t \mapsto g^{\oplus}(t)$, and $t \mapsto g^{\ominus}(t)$.

If the deformation u has a nonzero jump $[u] = u^{\oplus} - u^{\ominus}$ on M, then the body has a crack on (part of) M. More precisely the crack is given by the set

$$\{x \in M : [u](x) \neq 0\}.$$

Let us consider now the work done to produce a crack. If we neglect for a moment the problem of irreversibility, we may assume that this work can be written in the form

$$\int_M \varphi(x, [u])\, d\mathscr{H}^{n-1},$$

where $\varphi\colon M \times \mathbb{R}^m \to [0, +\infty]$ satisfies the following properties

(φ_1) φ is a Borel function;

(φ_2) $\varphi(x,0)=0$ for $\mathscr{H}^{n-1}$-a.e. $x\in M$;

(φ_3) the function $y\mapsto\varphi(x,y)$ is lower semicontinuous on $\mathbb{R}^m$ for $\mathscr{H}^{n-1}$-a.e. $x\in M$.

A simple example is given by the function

$$\varphi(x,y) \ := \ \begin{cases} a+b|y| & \text{if } y\in\mathbb{R}^m\setminus\{0\},\\ 0 & \text{if } y=0,\end{cases} \tag{1.1.6}$$

where $a\geq 0$ and $b\geq 0$ are real constants. The constant a plays the role of an activation energy; if $b>0$, there is also an energy term proportional to the amplitude of the crack opening. The classical Griffith's model corresponds to the case $a>0$ and $b=0$.

Let $L^0(M)$ be the set of extended real valued measurable functions on M and let $L^0(M)^+$ be the set of functions $w\in L^0(M)$ such that $w\geq 0$ $\mathscr{H}^{n-1}$-a.e. on M.

We introduce the function $\phi\colon L^p(M;\mathbb{R}^m)\to L^0(M)^+$ defined by

$$\phi(w)(x):=\varphi(x,w(x)),$$

for every $w\in L^p(M;\mathbb{R}^m)$ and for $\mathscr{H}^{n-1}$-a.e. $x\in M$.

Given an arbitrary family $(w_i)_{i\in I}$ in $L^0(M)^+$ the essential supremum

$$w=\operatorname*{ess\,sup}_{i\in I}\, w_i$$

of the family is defined as the unique (up to $\mathscr{H}^{n-1}$-equivalence) function in $L^0(M)^+$ such that

- $w\geq w_i$ $\mathscr{H}^{n-1}$-a.e. on M for all $i\in I$;
- if $z\in L^0(M)^+$ and $z\geq w_i$ $\mathscr{H}^{n-1}$-a.e. on M, then $z\geq w$ $\mathscr{H}^{n-1}$-a.e. on M.

For the existence of such a function see, for instance, [39, Proposition VI-**1**-1].

Suppose now that the deformation u depends on time, i.e., we have a map $t\mapsto u(t)$ from $[0,T]$ into $W^{1,p}(\Omega\setminus M;\mathbb{R}^m)$. If no crack is present until time 0 and

$$\phi([u(s)])\leq\phi([u(t)]) \qquad \mathscr{H}^{n-1}\text{-a.e. on } M$$

for every $s\in[0,t]$, then the energy dissipated in the crack process in the time interval $[0,t]$ is given, in our model, by

$$\int_M \phi([u(t)])\,d\mathscr{H}^{n-1}.$$

This happens for instance when $s\mapsto\phi([u(s)])$ is monotonically increasing $\mathscr{H}^{n-1}$-a.e. on M.

In the general case, the irreversibility of the fracture process leads to introduce an auxiliary function $t \mapsto \beta(t)$ from $[0,T]$ to $L^1(M)$, which takes into account the history of the system up to time t. We assume that for every $0 \le t_1 \le t_2 \le T$ we have

$$\beta(t_2) = \beta(t_1) \vee \operatorname*{ess\,sup}_{t_1 \le s \le t_2} \phi([u(s)]) \quad \mathscr{H}^{n-1}\text{-a.e. on } M, \tag{1.1.7}$$

so that

$$\beta(t_2) - \beta(t_1) = \operatorname*{ess\,sup}_{t_1 \le s \le t_2} (\phi([u(s)]) - \beta(t_1))^+ \quad \mathscr{H}^{n-1}\text{-a.e. on } M,$$

where for every $a \in \mathbb{R}$, $a^+ := a \vee 0$ denotes the positive part of a.

In particular

- $t \mapsto \beta(t)$ is increasing, i.e., $\beta(t_1) \le \beta(t_2)$ $\mathscr{H}^{n-1}$-a.e. on M for $0 \le t_1 \le t_2 \le T$;
- $\beta(t) \ge \phi([u(t)])$ $\mathscr{H}^{n-1}$-a.e. on M for every $t \in [0,T]$.

In our model the energy dissipated in the time interval $[t_1, t_2]$ is given by

$$\|\beta(t_2) - \beta(t_1)\|_{1,M} := \int_M (\beta(t_2) - \beta(t_1))\, d\mathscr{H}^{n-1}.$$

According to this assumption there is no dissipation in the intervals $[t_1, t_2]$ where $\phi([u(s)]) \le \beta(t_1)$ $\mathscr{H}^{n-1}$-a.e. on M for every $s \in [t_1, t_2]$, while the dissipation is given by

$$\int_M \big(\phi([u(t_2)]) - \phi([u(t_1)])\big)\, d\mathscr{H}^{n-1}$$

whenever $\beta(t_1) \le \phi([u(s)]) \le \phi([u(t_2)])$ for every $s \in [t_1, t_2]$.

It follows from (1.1.7) that $\beta(t)$ is uniquely determined by $\beta(0)$ and by the history of the deformation $s \mapsto u(s)$ in the interval $[0,t]$. Since it is difficult to deal with (1.1.7) directly, we prefer to define the notion of quasistatic evolution by considering a more general internal variable $t \mapsto \gamma(t)$ which is assumed to satisfy the following weaker conditions:

- $t \mapsto \gamma(t)$ is increasing, i.e., $\gamma(t_1) \le \gamma(t_2)$ $\mathscr{H}^{n-1}$-a.e. on M for $0 \le t_1 \le t_2 \le T$;
- $\gamma(t) \ge \phi([u(t)])$ $\mathscr{H}^{n-1}$-a.e. on M for every $t \in [0,T]$.

We do not assume from the beginning that $t \mapsto \gamma(t)$ satisfies (1.1.7). This property will be a nontrivial consequence of the other conditions considered in the definition of quasistatic evolution (see Theorem 1.2.7).

Given functions $\psi \in W^{1,p}(\Omega; \mathbb{R}^m)$ and $\gamma \in L^0(M)^+$, it is convenient to introduce the set $AD(\psi, \gamma)$ of *admissible deformations* with boundary value ψ on $\partial_0\Omega$ and internal variable γ. It is defined by

$$AD(\psi, \gamma) := \{u \in W^{1,p}(\Omega \setminus M; \mathbb{R}^m) : \phi([u]) \le \gamma \text{ on } M, \text{ and } u = \psi \text{ on } \partial_0\Omega\},$$

where equalities and inequalities are considered $\mathscr{H}^{n-1}$-a.e., and the last equality refers to the traces of u and ψ on $\partial_0\Omega$.

An *admissible configuration* with boundary value ψ on $\partial_0\Omega$ is a pair (u, γ), with $\gamma \in L^1(M)^+ := L^1(M) \cap L^0(M)^+$ and $u \in AD(\psi, \gamma)$.

1.2 Definition and properties of quasistatic evolutions

For every $t \in [0,T]$, the *total energy* of an admissible configuration (u,γ) at time t is defined as

$$\mathscr{E}(t)(u,\gamma) := \mathcal{W}(\nabla u) - \langle \mathscr{L}(t), u \rangle + \|\gamma\|_{1,M},$$

where $\|\cdot\|_{1,M}$ denotes the L^1-norm on M.

We now introduce the following definition in the spirit of Griffith's original theory on the crack propagation.

Definition 1.2.1. A pair $(u,\gamma) \in W^{1,p}(\Omega \smallsetminus M;\mathbb{R}^m) \times L^1(M)^+$ is *globally stable* at time $t \in [0,T]$ if $u \in AD(\psi(t),\gamma)$ and

$$\mathscr{E}(t)(u,\gamma) \le \mathscr{E}(t)(v,\delta) \tag{1.2.1}$$

for every $\delta \ge \gamma$ and for every $v \in AD(\psi(t),\delta)$.

In other words, the total energy of (u,γ) at time t cannot be reduced by increasing the internal variable γ or by choosing a new admissible deformation with the same boundary condition.

Remark 1.2.2. For every $t \in [0,T]$ let $(u(t),\gamma(t)) \in W^{1,p}(\Omega \smallsetminus M;\mathbb{R}^m) \times L^1(M)^+$ be globally stable at time t. By Definition 1.2.1 we can deduce an a priori estimate on $u(t)$. Indeed, by comparing $\mathscr{E}(t)(u(t),\gamma(t))$ with $\mathscr{E}(t)(\psi(t),\gamma(t))$, which is bounded uniformly with respect to t, we get that $\mathcal{W}(\nabla u(t)) - \langle \mathscr{L}(t), u(t)\rangle$ is bounded uniformly in time. Next, by the assumption (1.1.3) on $\mathcal{W}$ and the boundedness of $\mathscr{L}(t)$ in $(W^{1,p}(\Omega \smallsetminus M;\mathbb{R}^m))'$, we obtain that the $W^{1,p}$-norm of $u(t)$, $\|u(t)\|_{1,p}$, is bounded uniformly with respect to t. Furthermore from this fact and by Definition 1.2.1 we get that the crack term $\|\gamma(t)\|_{1,M}$ is bounded uniformly in time, too.

Remark 1.2.3. Condition (1.2.1) is equivalent to

$$\mathscr{E}(t)(u,\gamma) \le \mathscr{E}(t)(v, \gamma \vee \phi([v])),$$

for every $v \in W^{1,p}(\Omega \smallsetminus M;\mathbb{R}^m)$ such that $v = \psi(t)$ $\mathscr{H}^{n-1}$-a.e. on $\partial_0\Omega$. This is equivalent to

$$\mathcal{W}(\nabla u) - \langle \mathscr{L}(t), u\rangle \le \mathcal{W}(\nabla v) - \langle \mathscr{L}(t), v\rangle + \|(\phi([v]) - \gamma)^+\|_{1,M} \tag{1.2.2}$$

for every $v \in W^{1,p}(\Omega \smallsetminus M;\mathbb{R}^m)$ such that $v = \psi(t)$ $\mathscr{H}^{n-1}$-a.e. on $\partial_0\Omega$. This implies that if (u,γ) is globally stable at time t and $\tilde\gamma \in L^1(M)^+$ satisfies $\phi([u]) \le \tilde\gamma \le \gamma$ $\mathscr{H}^{n-1}$-a.e. on M, then $(u,\tilde\gamma)$ is globally stable at time t.

Definition 1.2.4. An *irreversible quasistatic evolution* of minimum energy configurations is a function $t \mapsto (u(t),\gamma(t))$ from $[0,T]$ into $W^{1,p}(\Omega \smallsetminus M;\mathbb{R}^m) \times L^1(M)^+$ which satisfies the following conditions:

(a) *global stability*: for every $t \in [0, T]$ the pair $(u(t), \gamma(t))$ is globally stable at time t;

(b) *irreversibility*: $\gamma(s) \le \gamma(t)$ $\mathscr{H}^{n-1}$-a.e. on M for every $0 \le s \le t \le T$;

(c) *energy balance*: the function $t \mapsto \mathscr{E}(t)(u(t), \gamma(t))$ is absolutely continuous on $[0, T]$ and

$$\frac{d}{dt}(\mathscr{E}(t)(u(t), \gamma(t))) = \langle \partial \mathcal{W}(\nabla u(t)), \nabla \dot{\psi}(t) \rangle - \langle \mathscr{L}(t), \dot{\psi}(t) \rangle - \langle \dot{\mathscr{L}}(t), u(t) \rangle \,,$$

for a.e. $t \in [0, T]$.

Remark 1.2.5. Condition (c) is equivalent to the following one:

(c') *energy balance in integral form*: the function $t \mapsto \langle \partial \mathcal{W}(\nabla u(t)), \nabla \dot{\psi}(t) \rangle - \langle \dot{\mathscr{L}}(t), u(t) \rangle$ belongs to $L^1([0, T])$ and

$$\begin{aligned} \mathscr{E}(t)(u(t), \gamma(t)) - \mathscr{E}(0)(u(0), \gamma(0)) = \\ = \int_0^t \Big(\langle \partial \mathcal{W}(\nabla u(s)), \nabla \dot{\psi}(s) \rangle - \langle \mathscr{L}(s), \dot{\psi}(s) \rangle - \langle \dot{\mathscr{L}}(s), u(s) \rangle \Big) \, ds \end{aligned}$$

for every $t \in [0, T]$.

This can be written in the form

$$\begin{aligned} \mathcal{W}(\nabla u(t)) - \mathcal{W}(\nabla u(0)) + \|\gamma(t) - \gamma(0)\|_{1,M} = \\ = \int_0^t \Big(\langle \partial \mathcal{W}(\nabla u(s)), \nabla \dot{\psi}(s) \rangle - \langle \mathscr{L}(s), \dot{\psi}(s) \rangle \Big) \, ds + \\ + \langle \mathscr{L}(t), u(t) \rangle - \langle \mathscr{L}(0), u(0) \rangle - \int_0^t \langle \dot{\mathscr{L}}(s), u(s) \rangle \, ds, \end{aligned} \tag{1.2.3}$$

for every $t \in [0, T]$. The first line is the increment in stored energy plus a term which will be interpreted as the energy dissipated by the crack process in the time interval $[0, t]$, as we shall see in Remark 1.2.8. Using the divergence theorem we can show that the second line represents the work done in the same time interval by the forces which act on $\partial_0 \Omega$ to produce the imposed deformation. The third line represents the work done by the imposed forces in the interval $[0, t]$; this follows from an integration by parts when $t \mapsto u(t)$ is regular enough, and can be obtained by approximation in the other cases.

If $t \mapsto (u(t), \gamma(t))$ satisfies condition (a), then $(u(t), \gamma(t))$ is bounded in $W^{1,p}(\Omega \setminus M; \mathbb{R}^m) \times L^1(M)^+$ by Remark 1.2.2. Therefore in condition (c') it is enough to assume that $t \mapsto \langle \partial \mathcal{W}(\nabla u(t)), \nabla \dot{\psi}(t) \rangle - \langle \dot{\mathscr{L}}(t), u(t) \rangle$ is measurable.

In the following theorem we prove one inequality of the energy balance.

Theorem 1.2.6. *Let $t \mapsto (u(t), \gamma(t))$ be a function from $[0,T]$ into $W^{1,p}(\Omega \setminus M; \mathbb{R}^m) \times L^1(M)^+$ which satisfies the global stability condition (a) and the irreversibility condition (b) of Definition 1.2.4. Assume that $t \mapsto \langle \partial\mathcal{W}(\nabla u(t)), \nabla\dot{\psi}(t)\rangle - \langle \dot{\mathscr{L}}(t), u(t)\rangle$ is measurable. Then*

$$\mathscr{E}(t)(u(t), \gamma(t)) - \mathscr{E}(0)(u(0), \gamma(0)) \geq$$
$$\geq \int_0^t \Big(\langle \partial\mathcal{W}(\nabla u(s)), \nabla\dot{\psi}(s)\rangle - \langle \mathscr{L}(s), \dot{\psi}(s)\rangle - \langle \dot{\mathscr{L}}(s), u(s)\rangle \Big)\, ds$$

for every $t \in [0,T]$.

Proof. We note that $t \mapsto \langle \partial\mathcal{W}(\nabla u(t)), \nabla\dot{\psi}(t)\rangle - \langle \dot{\mathscr{L}}(t), u(t)\rangle$ belongs to $L^1([0,T])$ by the arguments of Remark 1.2.2. The result can now be obtained arguing as in [8] (see the proof of Lemma 7.1 and the final part of the proof of Theorem 3.15). □

Now we prove that for a quasistatic evolution $t \mapsto (u(t), \gamma(t))$, the internal variable $t \mapsto \gamma(t)$ satisfies a condition analogous to (1.1.7).

Theorem 1.2.7. *Let $t \mapsto (u(t), \gamma(t))$ be a quasistatic evolution. Then*

$$\gamma(t_2) = \gamma(t_1) \vee \operatorname*{ess\,sup}_{t_1 \leq s \leq t_2} \phi([u(s)]) \quad \mathscr{H}^{n-1}\text{-a.e. on } M, \tag{1.2.4}$$

for every $0 \leq t_1 \leq t_2 \leq T$.

Proof. It is enough to prove that

$$\gamma(t) = \gamma(0) \vee \operatorname*{ess\,sup}_{0 \leq s \leq t} \phi([u(s)]) \quad \mathscr{H}^{n-1}\text{-a.e. on } M, \tag{1.2.5}$$

for every $t \in [0,T]$. Let $\tilde{\gamma}(t)$ be the right-hand side of (1.2.5). Since $t \mapsto \gamma(t)$ is increasing and $\phi([u(t)]) \leq \gamma(t)$ $\mathscr{H}^{n-1}$-a.e. on M for every $t \in [0,T]$, it follows that $\tilde{\gamma}(t) \leq \gamma(t)$ $\mathscr{H}^{n-1}$-a.e. on M for every $t \in [0,T]$. As $\phi([u(t)]) \leq \tilde{\gamma}(t)$ $\mathscr{H}^{n-1}$-a.e. on M, by Remark 1.2.3 the pair $(u(t), \tilde{\gamma}(t))$ is globally stable at time t for every $t \in [0,T]$. Since $t \mapsto \tilde{\gamma}(t)$ is increasing, we can apply Theorem 1.2.6 and we obtain

$$\mathscr{E}(t)(u(t), \tilde{\gamma}(t)) - \mathscr{E}(0)(u(0), \gamma(0)) \geq$$
$$\geq \int_0^t \Big(\langle \partial\mathcal{W}(\nabla u(s)), \nabla\dot{\psi}(s)\rangle - \langle \mathscr{L}(s), \dot{\psi}(s)\rangle - \langle \dot{\mathscr{L}}(s), u(s)\rangle \Big)\, ds$$

for every $t \in [0,T]$. By the energy balance, item (c), it follows that the inequality $\mathscr{E}(t)(u(t), \tilde{\gamma}(t)) \geq \mathscr{E}(t)(u(t), \gamma(t))$ holds, i.e.,

$$\mathcal{W}(\nabla u(t)) - \langle \mathscr{L}(t), u(t)\rangle + \|\tilde{\gamma}(t)\|_{1,M} \geq \mathcal{W}(\nabla u(t)) - \langle \mathscr{L}(t), u(t)\rangle + \|\gamma(t)\|_{1,M},$$

which implies $\|\tilde{\gamma}(t)\|_{1,M} \geq \|\gamma(t)\|_{1,M}$. As $\tilde{\gamma}(t) \leq \gamma(t)$ $\mathscr{H}^{n-1}$-a.e. on M, we deduce that $\tilde{\gamma}(t) = \gamma(t)$ $\mathscr{H}^{n-1}$-a.e. on M for every $t \in [0,T]$, which concludes the proof. □

Theorem 1.2.7 can be used to explain the mechanical meaning of the internal variable γ in the model case $\varphi(x, y) := |y|$. Indeed, if $t \mapsto (u(t), \gamma(t))$ is a quasistatic evolution with $\gamma(0) = 0$ and $\varphi(x, y) := |y|$, then (1.2.4) shows that $\gamma(t)(x)$ coincides with the maximum modulus of the amplitude of the opening reached by the crack at x up to time t.

Remark 1.2.8. As $t \mapsto \gamma(t)$ satisfies (1.1.7) by Theorem 1.2.7, the mechanical interpretation given in Section 1.1 shows that the term $\|\gamma(t) - \gamma(0)\|_{1,M}$ in (1.2.3) represents the energy dissipated in the crack process in the time interval $[0, t]$.

Remark 1.2.9. In our model, the dissipation term in the energy functional comes from the expression $\|\gamma \vee \phi([v]) - \gamma\|_{1,M}$ and is nonlinear in γ. This turns out to be the main mathematical difference between our model and the model considered by Mielke and Mainik and Mielke in [35, Section 4.2] and [32, Section 6.2], where the dissipation term is linear.

We are now in a position to state our main result.

Theorem 1.2.10. *Let $(u_0, \gamma_0) \in W^{1,p}(\Omega \setminus M; \mathbb{R}^m) \times L^1(M)^+$ be globally stable at time $t = 0$. Then there exists an irreversible quasistatic evolution $t \mapsto (u(t), \gamma(t))$ such that $(u(0), \gamma(0)) = (u_0, \gamma_0)$.*

1.3 Some tools

We introduce a notion of convergence for the functions γ, which is the counterpart of the notion of convergence of sets introduced in [8]. The main property of this convergence is that, if u_k converges weakly in $W^{1,p}(\Omega \setminus M; \mathbb{R}^m)$ to some function u and $\phi([u_k]) \leq \gamma_k$ $\mathscr{H}^{n-1}$-a.e. on M, then $\phi([u]) \leq \gamma$ $\mathscr{H}^{n-1}$-a.e. on M.

Definition 1.3.1. Let $\gamma_k, \gamma \in L^0(M)^+$. We say that γ_k σ^p_φ-converges to γ if the following two conditions are satisfied:

(a) if $u_j \rightharpoonup u$ weakly in $W^{1,p}(\Omega \setminus M; \mathbb{R}^m)$ and $\phi([u_j]) \leq \gamma_{k_j}$ $\mathscr{H}^{n-1}$-a.e. on M for some sequence $k_j \to \infty$, then $\phi([u]) \leq \gamma$ $\mathscr{H}^{n-1}$-a.e. on M;

(b) there exist a sequence $u^i \in W^{1,p}(\Omega \setminus M; \mathbb{R}^m)$, with $\sup_i \phi([u^i]) = \gamma$ $\mathscr{H}^{n-1}$-a.e. on M, and, for every i, a sequence $u^i_k \in W^{1,p}(\Omega \setminus M; \mathbb{R}^m)$, converging to u^i weakly in $W^{1,p}(\Omega \setminus M; \mathbb{R}^m)$ as $k \to \infty$, such that $\phi([u^i_k]) \leq \gamma_k$ $\mathscr{H}^{n-1}$-a.e. on M for every i and k.

Notice that we do not require any upper bound in $L^1(M)^+$ for the functions γ_k.

Remark 1.3.2. If γ_k σ^p_φ-converges to γ, then in particular there are functions u^i_k and u^i in $W^{1,p}(\Omega \setminus M;\mathbb{R}^m)$ such that condition (b) in Definition 1.3.1 holds. We define for every k the following two quantities

$$\gamma^i := \sup_{j=1,\dots,i} \phi([u^j]) \quad \text{and} \quad \gamma^i_k := \sup_{j=1,\dots,i} \phi([u^j_k]).$$

With this notation it turns out that

$$\gamma = \lim_{i\to\infty} \gamma^i \qquad \text{and} \qquad \gamma_k \geq \sup_{i\in\mathbb{N}} \gamma^i_k,$$

for every k.

Remark 1.3.3. If γ_k σ^p_φ-converges to γ, then

$$\gamma \leq \limsup_{k\to\infty} \gamma_k, \quad \mathscr{H}^{n-1}\text{-a.e. on } M,$$

as we can see by modifing the proof of Lemma 1.3.4 below. Notice that the inequality can be strict, even when γ_k converges pointwise to a function $\tilde{\gamma}$. As an example, consider $n = 2$, $m = 1$, $p = 2$, $\Omega = \left]-2, 2\right[^2$ and $M = [0,1] \times \{0\}$. Let $\gamma_k \in L^0(M)^+$ be defined as follows:

$$\gamma_k(x) := \begin{cases} 1 & \text{for } x \in \left[\frac{i}{k}, \frac{i+1}{k} - \frac{1}{k^2}\right[; \\ \\ 0 & \text{for } x \in \left[\frac{i+1}{k} - \frac{1}{k^2}, \frac{i+1}{k}\right[; \end{cases} \qquad \text{for } i = 0,\dots,k-1.$$

It follows from homogenization theory (see [12], [38], [40]) that condition (a) in Definition 1.3.1 is satisfied with $\gamma = 0$, hence γ_k σ^2_φ-converges to 0. Furthermore γ_k converge in measure to 1, so up to a subsequence we have pointwise convergence to $1 =: \tilde{\gamma} > \gamma$.

We prove in the following lemma that the L^1-norm is lower semicontinuous with respect to σ^p_φ-convergence.

Lemma 1.3.4. *Let γ_k, $\gamma \in L^0(M)^+$. If γ_k σ^p_φ-converges to γ then*

$$\|\gamma\|_{1,M} \leq \liminf_{k\to\infty} \|\gamma_k\|_{1,M}. \tag{1.3.1}$$

Proof. From the hypothesis it follows in particular that there are functions u^i_k and u^i in $W^{1,p}(\Omega \setminus M;\mathbb{R}^m)$ which satisfy condition (b) in Definition 1.3.1. With notation from Remark 1.3.2, let us prove that for every i

$$\|\gamma^i\|_{1,M} \leq \liminf_{k\to\infty} \|\gamma^i_k\|_{1,M}. \tag{1.3.2}$$

Extracting a subsequence we may assume that $\liminf_k \|\gamma^i_k\|_{1,M}$ is a limit. As $[u^j_k] \to [u^j]$ strongly in $L^p(M;\mathbb{R}^m)$ for $j = 1,\dots,i$, we can extract a further subsequence such that

$[u_k^j] \to [u^j]$ pointwise $\mathscr{H}^{n-1}$-a.e. on M for $j = 1, \ldots, i$. By the lower semicontinuity assumption (φ_3) this implies

$$\gamma^i \leq \liminf_{k\to\infty} \gamma_k^i \quad \mathscr{H}^{n-1}\text{-a.e. on } M.$$

By the Fatou lemma we obtain (1.3.2), which yields

$$\|\gamma^i\|_{1,M} \leq \liminf_{k\to\infty} \|\gamma_k\|_{1,M}.$$

We then pass to the limit as i tends to infinity and obtain (1.3.1). □

We now prove a compactness result for the notion of σ_φ^p-convergence.

Lemma 1.3.5. *Every sequence in $L^0(M)^+$ has a σ_φ^p-convergent subsequence.*

Proof. Let us denote the L^p-norm by $\|\cdot\|_p$. Let $\gamma_k \in L^0(M)^+$, let $w_h \in L^\infty(\Omega \setminus M; \mathbb{R}^m)$ be dense in $L^p(\Omega \setminus M; \mathbb{R}^m)$, and, for every positive integers l, h, and k, let us consider the problem

$$\min \left\{ \|\nabla u\|_p^p + \ell \, \|u - w_h\|_p^p \right\}, \tag{1.3.3}$$

where the minimum is taken over all functions $u \in W^{1,p}(\Omega \setminus M; \mathbb{R}^m)$ such that $\phi([u]) \leq \gamma_k$ $\mathscr{H}^{n-1}$-a.e. on M.

To prove that the minimum is achieved, we take a minimizing sequence and we easily obtain that it is bounded in $W^{1,p}(\Omega \setminus M; \mathbb{R}^m)$. Then, up to a subsequence, we can pass to the limit and by using our lower semicontinuity assumption (φ_3) we can prove that the limit function is actually a solution to the minimum problem (1.3.3). This solution, which is unique by strict convexity, will be denoted by $u_k^{\ell,h}$. Notice that this function is bounded in $W^{1,p}(\Omega \setminus M; \mathbb{R}^m)$ uniformly with respect to k, thus, up to a subsequence, we can pass to the limit in k and get that there is a function $u^{\ell,h}$ such that $u_k^{\ell,h} \rightharpoonup u^{\ell,h}$ weakly in $W^{1,p}(\Omega \setminus M; \mathbb{R}^m)$. Further we define

$$\gamma := \sup_{\ell,h\in\mathbb{N}} \phi([u^{\ell,h}]) \quad \mathscr{H}^{n-1}\text{-a.e. on } M. \tag{1.3.4}$$

In this way point (b) of Definition 1.3.1 is automatically satisfied.

We need to prove point (a). To this aim, let $v_j \rightharpoonup v$ weakly in $W^{1,p}(\Omega \setminus M; \mathbb{R}^m)$ be such that $\phi([v_j]) \leq \gamma_{k_j}$ $\mathscr{H}^{n-1}$-a.e. on M for some sequence $k_j \to \infty$. We want to prove that $\phi([v]) \leq \gamma$ $\mathscr{H}^{n-1}$-a.e. on M. By density there is a subsequence of w_h, say w_{h_i}, which converges strongly to v in $L^p(\Omega \setminus M; \mathbb{R}^m)$. Let $\ell_i \to +\infty$ be such that $\ell_i \|v - w_{h_i}\|_p^p \to 0$ as i tends to infinity. By the minimality of $u_{k_j}^{\ell_i,h_i}$, we have

$$\|\nabla u_{k_j}^{\ell_i,h_i}\|_p^p + \ell_i \|u_{k_j}^{\ell_i,h_i} - w_{h_i}\|_p^p \leq \|\nabla v_j\|_p^p + \ell_i \|v_j - w_{h_i}\|_p^p.$$

Then $u_{k_j}^{\ell_i,h_i}$ is bounded in $W^{1,p}(\Omega \setminus M; \mathbb{R}^m)$ uniformly with respect to j, and passing to the limit as j tends to infinity we get

$$\|\nabla u^{\ell_i,h_i}\|_p^p + \ell_i \|u^{\ell_i,h_i} - w_{h_i}\|_p^p \leq \sup_{j\in\mathbb{N}} \|\nabla v_j\|_p^p + \ell_i \|v - w_{h_i}\|_p^p.$$

Since $\ell_i\|v - w_{h_i}\|_p^p \to 0$ as i tends to infinity, this inequality ensures that $\nabla u^{\ell_i,h_i}$ is bounded in $L^p(\Omega \setminus M;\mathbb{R}^m)$ uniformly with respect to i, and $u_{k_j}^{\ell_i,h_i} - w_{h_i} \to 0$ strongly in $L^p(\Omega \setminus M;\mathbb{R}^m)$. As $w_{h_i} \to v$ strongly in $L^p(\Omega \setminus M;\mathbb{R}^m)$, we deduce that u^{ℓ_i,h_i} converges weakly to v in $W^{1,p}(\Omega \setminus M;\mathbb{R}^m)$. Then $[u^{\ell_i,h_i}]$ converges strongly to $[v]$ in $L^p(M;\mathbb{R}^m)$. Passing to a subsequence, we may also obtain pointwise convergence $\mathscr{H}^{n-1}$-a.e. on M. By (1.3.4) we have $\phi([u^{\ell_i,h_i}]) \leq \gamma$ $\mathscr{H}^{n-1}$-a.e. on M, so that the lower semicontinuity assumption (φ_3) yields $\phi([v]) \leq \gamma$ $\mathscr{H}^{n-1}$-a.e. on M, which is precisely the conclusion to point (a) in the definition of σ^p_φ-convergence. ☐

We shall use the following Helly-type compactness result. We recall that a function $t \mapsto \gamma(t)$ from $[0,T]$ into $L^0(M)^+$ is said to be increasing if $\gamma(s) \leq \gamma(t)$ $\mathscr{H}^{n-1}$-a.e. on M, whenever $0 \leq s \leq t \leq T$.

Lemma 1.3.6. *Let $t \mapsto \gamma_k(t)$ be a sequence of increasing functions from $[0,T]$ into $L^0(M)^+$. Then there exist a subsequence γ_{k_j}, independent of t, and an increasing function $t \mapsto \gamma(t)$ from $[0,T]$ into $L^0(M)^+$, such that $\gamma_{k_j}(t)$ σ^p_φ-converges to $\gamma(t)$ for every $t \in [0,T]$.*

Proof. Let D be a countable dense subset of $[0,T]$ containing 0 and T. By Lemma 1.3.5, using a diagonal argument, we can extract a subsequence, still named $\gamma_k(t)$, and an increasing function $t \mapsto \gamma(t)$ from D into $L^0(M)^+$, such that $\gamma_k(t)$ σ^p_φ-converges to $\gamma(t)$ for every $t \in D$.

Let us define

$$\gamma(t+) := \inf_{s \geq t,\, s \in D} \gamma(s) \qquad \text{and} \qquad \gamma(t-) := \sup_{s \leq t,\, s \in D} \gamma(s),$$

for every $t \in [0,T]$. It is easy to prove that:

(1) $\gamma(t-) = \gamma(t) = \gamma(t+)$ for every $t \in D$;

(2) $\gamma(t-) \leq \gamma(t+)$ for every $t \in [0,T]$;

(3) if $s < t$, then $\gamma(s+) \leq \gamma(t-)$.

Define $E := \{t \in [0,T] : \gamma(t+) = \gamma(t-)\ \mathscr{H}^{n-1}\text{-a.e. in } M\}$ and $\gamma(t) := \gamma(t-) = \gamma(t+)$ for every $t \in E$. Note that by (1) D is contained in E and the definition of $\gamma(t)$ agrees with the original one on D. Then the definition of σ^p_φ-convergence and the monotonicity condition imply that $\gamma_k(t)$ σ^p_φ-converges to $\gamma(t)$ for every $t \in E$.

Let us show now that the set $E^c := [0,T] \setminus E$ is at most countable. For every pair of positive integers i,k we set $A_{i,k} := \{t \in [0,T] : \|(\gamma(t+) \wedge k) - (\gamma(t-) \wedge k)\|_{1,M} > 1/i\}$, so that E^c is the union of the sets $A_{i,k}$. Therefore it is enough to show that each set

$A_{i,k}$ is finite. Let $t_1 < \cdots < t_r \in A_{i,k}$. Since, by (3), $(\gamma(t_{j-1}+) \wedge k) \leq (\gamma(t_j-) \wedge k)$ for $j = 2, \dots, r$, we get

$$\frac{r}{i} \leq \sum_{j=1}^{r} \|(\gamma(t_j+) \wedge k) - (\gamma(t_j-) \wedge k)\|_{1,M} \leq \|\gamma(t_r+) \wedge k\|_{1,M} \leq k\mathscr{H}^{n-1}(M),$$

so that $r \leq ik\mathscr{H}^{n-1}(M)$, which implies that $A_{i,k}$ is finite. It follows that E^c is at most countable, thus we can conclude the proof of the lemma by applying again the compactness Lemma 1.3.5 for every $t \in E^c$, together with a diagonal argument. $\square$

The following result plays a crucial role in the proof of point (a) in the Definition 1.2.4 of quasistatic evolution.

Lemma 1.3.7. *Let γ_k, $\gamma \in L^0(M)^+$. Assume that γ_k σ^p_φ-converges to γ. Then for any $v \in W^{1,p}(\Omega \setminus M; \mathbb{R}^m)$ with $\phi([v]) \in L^1(M)^+$ the following inequality holds true:*

$$\limsup_{k\to\infty} \|(\phi([v]) - \gamma_k)^+\|_{1,M} \leq \|(\phi([v]) - \gamma)^+\|_{1,M}. \tag{1.3.5}$$

Proof. It is not restrictive to assume that the lim sup is a limit. Let u^i and u^i_k be the functions considered in point (b) of Definition 1.3.1. During the proof we shall use the notation introduced in Remark 1.3.2. As $\gamma^i_k \leq \gamma_k$ $\mathscr{H}^{n-1}$-a.e. on M, we have

$$(\phi([v]) - \gamma_k)^+ \leq (\phi([v]) - \gamma^i_k)^+,$$

hence

$$\lim_{k\to\infty} \|(\phi([v]) - \gamma_k)^+\|_{1,M} \leq \liminf_{k\to\infty} \|(\phi([v]) - \gamma^i_k)^+\|_{1,M}\,. \tag{1.3.6}$$

Passing to a subsequence, we may assume that $[u^i_k]$ converges to $[u^i]$ $\mathscr{H}^{n-1}$-a.e. on M. By the lower semicontinuity assumption (φ_3) we obtain

$$\gamma^i \leq \liminf_{k\to\infty} \gamma^i_k \quad \mathscr{H}^{n-1}\text{-a.e. on } M,$$

so that Fatou Lemma gives

$$\limsup_{k\to\infty} \|(\phi([v]) - \gamma^i_k)^+\|_{1,M} \leq \|(\phi([v]) - \gamma^i)^+\|_{1,M},$$

which, together with (1.3.6), yields

$$\lim_{k\to\infty} \|(\phi([v]) - \gamma_k)^+\|_{1,M} \leq \|(\phi([v]) - \gamma^i)^+\|_{1,M}.$$

As $\gamma^i \to \gamma$ $\mathscr{H}^{n-1}$-a.e. on M, inequality (1.3.5) can be obtained by passing to the limit as $i \to \infty$. $\square$

Remark 1.3.8. The conclusion of Lemma 1.3.7 does not hold, in general, when $\gamma_k, \gamma \in L^\infty(M)^+$ and $\gamma_k \rightharpoonup \gamma$ weakly* in $L^\infty(M)$. Consider, for instance, the case $n = 2$, $m = 1$, $\Omega =]-4, 4[^2$, $M = [-\pi, \pi] \times \{0\}$, and define $\gamma_k(x) := 1 + \sin(kx_1)$, where x_1 denotes the first coordinate of x. Then, γ_k converges to $\gamma(x) := 1$ weakly* in $L^\infty(M)$, but (1.3.5) is not satisfied for $\phi([v]) = 1$, since in this case $\|(\phi([v]) - \gamma_k)^+\|_{1,M} = 2$ for every k, while $\|(\phi([v]) - \gamma)^+\|_{1,M} = 0$.

1.4 The discrete-time problems and proof of the main result

In this section we prove Theorem 1.2.10 by a discrete-time approximation. We fix a sequence of subdivisions $(t_k^i)_{0\le i\le k}$ of the interval $[0,T]$, with

$$0 = t_k^0 < t_k^1 < \cdots < t_k^{k-1} < t_k^k = T, \tag{1.4.1}$$

$$\lim_{k\to\infty} \max_{1\le i\le k} (t_k^i - t_k^{i-1}) = 0. \tag{1.4.2}$$

For $i = 1, \dots, k$ we set $\mathscr{L}_k^i = \mathscr{L}(t_k^i)$, $\psi_k^i = \psi(t_k^i)$, $\mathscr{E}_k^i = \mathscr{E}(t_k^i)$.

For every $k \in \mathbb{N}$ we define u_k^i and γ_k^i by induction as follows. Let (u_0, γ_0) be a minimum energy configuration at time $t = 0$. We set $(u_k^0, \gamma_k^0) := (u_0, \gamma_0)$ and define (u_k^i, γ_k^i) as a solution of the minimum problem

$$\min \left\{ \mathscr{E}_k^i(u, \gamma) : \gamma \in L^1(M)^+,\ \gamma \ge \gamma_k^{i-1},\ u \in AD(\psi_k^i, \gamma) \right\}, \tag{1.4.3}$$

where the inequality means that $\gamma \ge \gamma_k^{i-1}$ $\mathscr{H}^{n-1}$-a.e. on M.

Remark 1.4.1. Consider the minimum problem

$$\min \Big\{ \mathcal{W}(\nabla u) - \langle \mathscr{L}_k^i, u\rangle + \|\phi([u]) \vee \gamma_k^{i-1}\|_{1,M} : u = \psi_k^i \text{ on } \partial_0\Omega \Big\}, \tag{1.4.4}$$

where u is assumed to belong to $W^{1,p}(\Omega \setminus M; \mathbb{R}^m)$. Then the following two conditions are equivalent:

(a) the pair (u_k^i, γ_k^i) is a solution to (1.4.3);

(b) u_k^i is a solution to (1.4.4) and $\gamma_k^i := \gamma_k^{i-1} \vee \phi([u_k^i])$ $\mathscr{H}^{n-1}$-a.e. on M.

The existence of a solution of (1.4.3) (or equivalently (1.4.4)) can be easily obtained by using the direct methods of the calculus of variations. The compactness of a minimizing sequence follows from (1.1.3) and positiveness of φ. The lower semicontinuity follows from (W_1), (W_2), (φ_3), and from the compactness of the trace operator.

For every $t \in [0,T]$ we define

$$\begin{gathered} \tau_k(t) = t_k^i,\ u_k(t) = u_k^i,\ \gamma_k(t) = \gamma_k^i,\ \psi_k(t) = \psi(t_k^i), \\ \mathscr{L}_k(t) = \mathscr{L}(t_k^i),\ \mathscr{E}_k(t) = \mathscr{E}(t_k^i), \end{gathered} \tag{1.4.5}$$

where i is the greatest integer such that $t_k^i \le t$. Note that $u_k(t) = u_k(\tau_k(t))$, $\gamma_k(t) = \gamma_k(\tau_k(t))$, $\psi_k(t) = \psi(\tau_k(t))$, $\mathscr{L}_k(t) = \mathscr{L}(\tau_k(t))$ and $\mathscr{E}_k(t) = \mathscr{E}(\tau_k(t))$.

Remark 1.4.2. Since $\psi_k^i \in AD(\psi_k^i, \gamma_k^{i-1})$, then by Remark 1.2.2 we deduce that the L^p-norms $\|\nabla u_k^i\|_p$ and $\|u_k^i\|_p$ are bounded uniformly with respect to i and k. Passing to the piecewise constant functions $t \mapsto \nabla u_k(t)$ and $t \mapsto u_k(t)$, we have that there exists a positive constant C such that

$$\|\nabla u_k(t)\|_p \leq C \quad \text{and} \quad \|u_k(t)\|_p \leq C \tag{1.4.6}$$

for every k and for every $t \in [0, T]$. Since $\mathscr{E}_k(t)(u_k(t), \gamma_k(t))$ is bounded uniformly with respect to k, we get also that

$$\|\gamma_k(t)\|_{1,M} \leq C, \tag{1.4.7}$$

for every k and for every $t \in [0, T]$.

We introduce now a sequence of functions which play an important role in our estimates. For a.e. $t \in [0, T]$ we set

$$\theta_k(t) := \langle \partial \mathcal{W}(\nabla u_k(t)), \nabla \dot{\psi}(t) \rangle - \langle \mathscr{L}_k(t), \dot{\psi}(t) \rangle - \langle \dot{\mathscr{L}}(t), u_k(t) \rangle. \tag{1.4.8}$$

In the following lemma we present the main energy estimate for the discrete process.

Lemma 1.4.3. *There exists a sequence $R_k \to 0$ such that*

$$\mathscr{E}(\tau_k(t))(u_k(t), \gamma_k(t)) \leq \mathscr{E}(0)(u_0, \gamma_0) + \int_0^{\tau_k(t)} \theta_k(s)\, ds + R_k,$$

for every k and for every $t \in [0, T]$.

Proof. We need to prove that there exists a sequence $R_k \to 0$ such that

$$\mathscr{E}_k^i(u_k^i, \gamma_k^i) \leq \mathscr{E}(0)(u_0, \gamma_0) + \int_0^{t_k^i} \theta_k(s)\, ds + R_k,$$

for any k and for any $i = 1, \ldots, k$.

Let us fix j and k with $1 \leq j \leq k$. Since $u_k^{j-1} = \psi_k^{j-1}$ on $\partial_0 \Omega$, and $[u_k^{j-1} + \psi_k^j - \psi_k^{j-1}] = [u_k^{j-1}]$ $\mathscr{H}^{n-1}$-a.e. on M, the function $u_k^{j-1} + \psi_k^j - \psi_k^{j-1}$ belongs to $AD(\psi_k^j, \gamma_k^{j-1})$, hence $\mathscr{E}_k^j(u_k^j, \gamma_k^j) \leq \mathscr{E}_k^j(u_k^{j-1} + \psi_k^j - \psi_k^{j-1}, \gamma_k^{j-1})$. The proof now can be concluded arguing as in the proof of [8, Lemma 6.1]. $\square$

We are now in a position to prove our main result.

Proof of Theorem 1.2.10. Let (t_k^i), $0 \leq i \leq k$, be a sequence of subdivisions of the interval $[0, T]$ satisfying (1.4.1) and (1.4.2). For any k consider the pairs (u_k^i, γ_k^i) inductively defined as solutions of the discrete problems (1.4.3) for $i = 1, \ldots, k$ with the initial condition $(u_k^0, \gamma_k^0) = (u_0, \gamma_0)$. Let $\tau_k(t)$, $u_k(t)$, $\gamma_k(t)$, and $\psi_k(t)$ be defined by (1.4.5) for any $t \in [0, T]$. By Lemma 1.3.6 there exists a subsequence of $\gamma_k(t)$, independent of t, which σ^p_φ-converges to $\gamma_\infty(t) \in L^0(M)^+$, for every $t \in [0,T]$. By (1.4.7) and Lemma 1.3.4 we have $\gamma_\infty(t) \in L^1(M)^+$.

Let $\theta_k(t)$ be defined by (1.4.8) for a.e. t and let

$$\theta_\infty(t) := \limsup_{k\to\infty} \theta_k(t).$$

By (1.1.4) and (1.4.6) we deduce that

$$|\theta_k(t)| \le \alpha_2(C^{p-1}+1)\|\nabla\dot\psi(t)\|_p + \|\mathscr{L}_k(t)\|_*\|\dot\psi(t)\|_{1,p} + C\|\dot{\mathscr{L}}(t)\|_*,$$

where $\|\cdot\|_*$ is the norm in the dual space of $W^{1,p}(\Omega\setminus M;\mathbb{R}^m)$. Since the right-hand side of previous formula belongs to $L^1([0,T])$, we deduce that θ_∞ belongs to $L^1([0,T])$, too, and using the Fatou lemma we get

$$\limsup_{k\to\infty}\int_0^{\tau_k(t)}\theta_k(s)\,ds \le \int_0^t \theta_\infty(s)\,ds. \tag{1.4.9}$$

For a.e. $t \in [0,T]$ we can extract a subsequence θ_{k_j} of θ_k, depending on t, such that

$$\theta_\infty(t) = \lim_{j\to\infty}\theta_{k_j}(t).$$

By (1.4.6) the sequence $u_{k_j}(t)$ is bounded in $W^{1,p}(\Omega\setminus M;\mathbb{R}^m)$, therefore we can extract a further subsequence, still denoted by $u_{k_j}(t)$, which converges weakly in $W^{1,p}(\Omega\setminus M;\mathbb{R}^m)$ to a function $u_\infty(t)$.

Since $\phi([u_{k_j}(t)]) \le \gamma_{k_j}(t)$ $\mathscr{H}^{n-1}$-a.e. on M, by point (a) in Definition 1.3.1 we have $\phi([u_\infty(t)]) \le \gamma_\infty(t)$ $\mathscr{H}^{n-1}$-a.e. on M. On the other hand, as $u_{k_j}(t) = \psi_{k_j}(t)$ $\mathscr{H}^{n-1}$-a.e. on $\partial_0\Omega$, we have also $u_\infty(t) = \psi(t)$ $\mathscr{H}^{n-1}$-a.e. on $\partial_0\Omega$, so that $u_\infty(t) \in AD(\psi(t),\gamma_\infty(t))$ for every $t \in [0,T]$.

The next step is to prove that the pair $(u_\infty(t),\gamma_\infty(t))$ satisfies property (a) of Definition 1.2.4. To this aim, let $\gamma \in L^1(M)^+$, $\gamma \ge \gamma_\infty(t)$ and $v \in AD(\psi(t),\gamma)$. By the minimality of the incremental solutions $(u_k(t),\gamma_k(t))$, we have that $\mathscr{E}_k(t)\,(u_k(t),\gamma_k(t)) \le \mathscr{E}_k(t)(v_k,\gamma_k(t)\vee\phi([v]))$, where $v_k := v+\psi_k(t)-\psi(t)$. Since the functional $u \mapsto \mathcal{W}(\nabla u)$ is weakly lower semicontinuous and strongly continuous, and the function $t \mapsto \mathscr{L}(t)$ is continuous, it follows immediately that

$$\mathcal{W}(\nabla u_\infty(t)) \le \liminf_{k\to\infty}\mathcal{W}(\nabla u_k(t)), \qquad \mathcal{W}(\nabla v) = \lim_{k\to\infty}\mathcal{W}(\nabla v_k), \tag{1.4.10}$$

$$\langle\mathscr{L}(t),u_\infty(t)\rangle = \lim_{k\to\infty}\langle\mathscr{L}_k(t),u_k(t)\rangle, \qquad \langle\mathscr{L}(t),v\rangle = \lim_{k\to\infty}\langle\mathscr{L}_k(t),v_k\rangle. \tag{1.4.11}$$

So far we have easily obtained that

$$\begin{aligned}\mathcal{W}(\nabla u_\infty(t)) - \langle\mathscr{L}(t),u_\infty(t)\rangle \le\\ \le \mathcal{W}(\nabla v) - \langle\mathscr{L}(t),v\rangle + \limsup_{k\to\infty}\|(\phi([v])-\gamma_k(t))^+\|_{1,M}\,,\end{aligned} \tag{1.4.12}$$

where the last term in right-hand side comes from the equality

$$(\gamma \vee \phi([v])) - \gamma = (\phi([v]) - \gamma)^+, \tag{1.4.13}$$

which holds for every $\gamma \in L^0(M)^+$. In order to obtain that the pair $(u_\infty(t), \gamma_\infty(t))$ satisfies point (a) in Definition 1.2.4 of quasistatic evolution we want to apply Lemma 1.3.7. To this aim we need to know that $\phi([u_\infty(t)]) \in L^1(M)^+$. By (1.4.7) in Remark 1.4.2 we have that $\|\gamma_k(t)\|_{1,M}$ is bounded uniformly with respect to k. As $u_k(t)$ belongs to $AD(\psi_k(t), \gamma_k(t))$, the sequence $\phi([u_k(t)])$ is bounded in $L^1(M)^+$, and by the lower semicontinuity assumption (φ_3) we obtain that $\phi([u_\infty(t)]) \in L^1(M)^+$ thanks to the Fatou lemma. Then we can apply Lemma 1.3.7 and we get

$$\begin{aligned} \mathcal{W}(\nabla u_\infty(t)) - \langle \mathscr{L}(t), u_\infty(t)\rangle \leq \\ \leq \mathcal{W}(\nabla v) - \langle \mathscr{L}(t), v\rangle + \|(\phi([v]) - \gamma_\infty(t))^+\|_{1,M}\,. \end{aligned} \tag{1.4.14}$$

Applying (1.4.13) to the last term in the right-hand side of (1.4.14) we conclude that $\mathscr{E}(t)(u_\infty(t), \gamma_\infty(t)) \leq \mathscr{E}(t)(v, \gamma_\infty(t) \vee \phi([v])) \leq \mathscr{E}(t)(v, \gamma)$ for every $t \in [0,T]$ and point (a) of Definition 1.2.4 is satisfied.

By the definition of the discrete problems, for every k the function $t \mapsto \gamma_k(t)$ is increasing. Passing to the σ^p_φ-limit, the same property holds for $t \mapsto \gamma_\infty(t)$, so that point (b) of Definition 1.2.4 is satisfied.

It remains to prove point (c). For a.e. t define

$$\theta(t) := \langle \partial\mathcal{W}(\nabla u_\infty(t)), \nabla\dot\psi(t)\rangle - \langle \mathscr{L}(t), \dot\psi(t)\rangle - \langle \dot{\mathscr{L}}(t), u_\infty(t)\rangle.$$

Arguing as in the proof of [8, Theorem 3.15] we get

$$\theta_\infty(t) = \theta(t), \tag{1.4.15}$$

for a.e. $t \in [0,T]$. This in particular means that the map $t \mapsto \theta(t)$ is measurable. Since we have proved that for every $t \in [0,T]$ the pair $(u_\infty(t), \gamma_\infty(t))$ satisfies points (a) and (b) of Definition 1.2.4, we are in a position to apply Theorem 1.2.6 and get

$$\mathscr{E}(t)(u_\infty(t), \gamma_\infty(t)) - \mathscr{E}(0)(u_0, \gamma_0) \geq \int_0^t \theta(s)\, ds.$$

By (1.3.1), (1.4.10), and (1.4.11) we get

$$\mathscr{E}(t)(u_\infty(t), \gamma_\infty(t)) \leq \liminf_{j\to\infty} \mathscr{E}_{k_j}(t)(u_{k_j}(t), \gamma_{k_j}(t)) \leq \limsup_{k\to\infty} \mathscr{E}_k(t)(u_k(t), \gamma_k(t)). \tag{1.4.16}$$

Using Lemma 1.4.3 and taking (1.4.9) and (1.4.15) into account, we obtain

$$\limsup_{k\to\infty} \mathscr{E}_k(t)(u_k(t), \gamma_k(t)) \leq \mathscr{E}(0)(u_0, \gamma_0) + \int_0^t \theta(s)\, ds. \tag{1.4.17}$$

By (1.4.16) and (1.4.17) we get that

$$\mathscr{E}(t)(u_\infty(t),\gamma_\infty(t)) \le \mathscr{E}(0)(u_0,\gamma_0) + \int_0^t \theta(s)\,ds$$

holds true for any $t \in [0,T]$, and this concludes the proof. $\square$

In the following theorem we prove that for every $t \in [0,T]$ the energy for the discrete-time problems converges to the energy for the continuous-time problem. We emphasize that the theorem is true for *any* irreversible quasistatic evolution $t \mapsto (u(t),\gamma(t))$ corresponding to a given $t \mapsto \gamma(t)$, not only for the one obtained as limit of the solutions of the discrete-time problems.

Theorem 1.4.4. *For every $t \in [0,T]$ let $u_k(t)$ and $\gamma_k(t)$ be defined as in the beginning of the proof of Theorem 1.2.10. Assume that $\gamma_k(t)$ σ^p_φ-converges to $\gamma(t) \in L^1(M)^+$ for any $t \in [0,T]$. Let $t \mapsto (u(t),\gamma(t))$ be an irreversible quasistatic evolution. For a.e. $t \in [0,T]$ let $\theta_k(t)$ be defined as in (1.4.8), and set*

$$\theta(t) := \langle \partial\mathcal{W}(\nabla u(t)), \nabla\dot\psi(t)\rangle - \langle \mathscr{L}(t), \dot\psi(t)\rangle - \langle \dot{\mathscr{L}}(t), u(t)\rangle.$$

Then

$$\begin{gathered}\mathcal{W}(\nabla u(t)) - \langle \mathscr{L}(t), u(t)\rangle = \lim_{k\to\infty}(\mathcal{W}(\nabla u_k(t)) - \langle \mathscr{L}_k(t), u_k(t)\rangle), \\ \|\gamma(t)\|_{1,M} = \lim_{k\to\infty}\|\gamma_k(t)\|_{1,M},\end{gathered} \tag{1.4.18}$$

for every $t \in [0,T]$. Furthermore

$$\theta_k \to \theta \quad \text{in } L^1([0,T]),$$

so that there exists a subsequence of θ_k which converges to θ a.e. in $[0,T]$.

Proof. For the proof we need to show that

$$\lim_{j\to\infty}\mathcal{W}(\nabla u_{k_j}(t)) = \mathcal{W}(\nabla u_\infty(t)), \tag{1.4.19}$$

for every $t \in [0,T]$, where $u_{k_j}(t)$ is the subsequence constructed in the proof of Theorem 1.2.10, and $u_\infty(t)$ is its limit. To this aim, let $v_j := u_\infty(t) + \psi_{k_j}(t) - \psi(t)$. By the minimality of the pair $(u_{k_j}(t),\gamma_{k_j}(t))$ we obtain that $\mathscr{E}_{k_j}(t)(u_{k_j}(t),\gamma_{k_j}(t)) \le \mathscr{E}_{k_j}(t)(v_j, \gamma_{k_j}(t) \vee \phi([u_\infty(t)]))$, and passing to the limit as j goes to infinity, we get by (1.2.2), (1.4.10), and (1.4.11)

$$\begin{aligned}&\limsup_{j\to\infty}\left[\mathcal{W}(\nabla u_{k_j}(t)) - \langle \mathscr{L}_{k_j}(t), u_{k_j}(t)\rangle\right] \le \\ &\le \limsup_{j\to\infty}\left[\mathcal{W}(\nabla v_j) - \langle \mathscr{L}_{k_j}(t), v_j\rangle + \|(\phi([u_\infty(t)]) - \gamma_{k_j}(t))^+\|_{1,M}\right] = \\ &= \mathcal{W}(\nabla u_\infty(t)) - \langle \mathscr{L}(t), u_\infty(t)\rangle + \limsup_{j\to\infty}\|(\phi([u_\infty(t)]) - \gamma_{k_j}(t))^+\|_{1,M}.\end{aligned} \tag{1.4.20}$$

Since $\gamma_{k_j}(t)$ σ^p_φ-converges to $\gamma_\infty(t)$, by Lemma 1.3.7 we have

$$\limsup_{j\to\infty} \|(\phi([u_\infty(t)]) - \gamma_{k_j}(t))^+\|_{1,M} \leq 0. \tag{1.4.21}$$

Taking into account (1.4.20) and (1.4.21) we get in particular that

$$\limsup_{j\to\infty} \mathcal{W}(\nabla u_{k_j}(t)) \leq \mathcal{W}(\nabla u_\infty(t)). \tag{1.4.22}$$

This, together with (1.4.10), gives (1.4.19).

To conclude the proof it is sufficient to follow the arguments of the proof of [8, Theorem 8.1]. □

The result can be improved under strict convexity assumption.

Theorem 1.4.5. *In addition to the hypotheses of Theorem 1.4.4, assume that* $\xi \mapsto W(x,\xi)$ *is strictly convex for a.e.* $x \in \Omega \smallsetminus M$ *and that* $y \mapsto \varphi(x,y)$ *is convex for* $\mathscr{H}^{n-1}$*-a.e.* $x \in M$*. Then* $u_k(t) \to u(t)$ *strongly in* $W^{1,p}(\Omega \smallsetminus M; \mathbb{R}^m)$*, for every* $t \in [0,T]$*.*

Proof. We observe that for every $t \in [0,T]$ and $\gamma \in L^1(M)^+$ the functional $v \mapsto \mathscr{E}(t)(v,\gamma)$ is strictly convex on the set of functions $v \in W^{1,p}(\Omega \smallsetminus M; \mathbb{R}^m)$ with $v = \psi(t)$ $\mathscr{H}^{n-1}$-a.e. on $\partial_0\Omega$. Therefore for every t there exists a unique function $u \in AD(\psi(t), \gamma(t))$ such that the pair (u,γ) is globally stable at time t. It follows that $u(t)$ coincides with the function $u_\infty(t)$ constructed in the proof of Theorem 1.2.10 and that the whole sequence $u_k(t)$ converge to $u(t)$ weakly in $W^{1,p}(\Omega \smallsetminus M; \mathbb{R}^m)$. Therefore (1.4.18) implies that $\mathcal{W}(\nabla u_k(t)) \to \mathcal{W}(\nabla u(t))$. Using [43, Theorem 3] we deduce that $\nabla u_k(t) \to \nabla u(t)$ in measure. As

$$|\nabla u_k(t) - \nabla u(t)|^p \leq 2^{p-1} a_0^{-1} [W(\nabla u_k(t)) + W(\nabla u(t))] + 2^{p-1} a_0^{-1} b_0,$$

the conclusion follows from the generalized dominated convergence theorem. □

1.5 Euler conditions

In this section we study the Euler conditions satisfied by globally stable pairs $(u,\gamma) \in W^{1,p}(\Omega \smallsetminus M; \mathbb{R}^m) \times L^1(M)^+$. Let us fix $t \in [0,T]$ and let $(u,\gamma) \in W^{1,p}(\Omega \smallsetminus M; \mathbb{R}^m) \times L^1(M)^+$ be globally stable at time t, and let $v \in W^{1,p}(\Omega \smallsetminus M; \mathbb{R}^m)$ be such that $v = 0$ $\mathscr{H}^{n-1}$-a.e. on $\partial_0\Omega$. Hence for every $\varepsilon > 0$ the function $u + \varepsilon v$ belongs to $AD(\psi(t), \gamma \vee \phi([u] + \varepsilon[v]))$, and by the global stability of the pair (u,γ) at time t, we have that $\mathscr{E}(t)(u,\gamma) \leq \mathscr{E}(t)(u + \varepsilon v, \gamma \vee \phi([u] + \varepsilon[v]))$, therefore

$$\liminf_{\varepsilon\to 0^+} \frac{\mathscr{E}(t)(u + \varepsilon v, \gamma \vee \phi([u] + \varepsilon[v])) - \mathscr{E}(t)(u,\gamma)}{\varepsilon} \geq 0. \tag{1.5.1}$$

The weak formulation of the Euler conditions will be obtained from this inequality. Without loss of generality, we assume that $\mathscr{L}(t)$ is given by (1.1.5), and we omit the dependence on time. After some standard calculation, one can express (1.5.1) in the following form

$$\begin{aligned}\int_{\Omega\setminus M}\big(\partial_\xi W(x,\nabla u)-H\big):\nabla v\,dx-\int_{\Omega\setminus M} fv\,dx-\int_{\partial_1\Omega} gv\,d\mathscr{H}^{n-1}+\\ -\int_M\big(g^{\oplus}v^{\oplus}+g^{\ominus}v^{\ominus}\big)\,d\mathscr{H}^{n-1}+\liminf_{\varepsilon\to0^+}\frac{\|(\phi([u]+\varepsilon[v])-\gamma)^+\|_{1,M}}{\varepsilon}\geq 0,\end{aligned}\tag{1.5.2}$$

for any $v\in W^{1,p}(\Omega\setminus M;\mathbb{R}^m)$ such that $v=0$ $\mathscr{H}^{n-1}$-a.e. on $\partial_0\Omega$.

To continue our analysis we need now to specify the form of the function φ. More precisely, we consider $\varphi\colon M\times\mathbb{R}^m\to[0,+\infty]$ defined by

$$\varphi(x,y):=\varphi_0(x)+\tilde{\varphi}(x,y)\quad\text{for } y\neq0\quad\text{and}\quad\varphi(x,0):=0\quad\text{for all } x\in M,\tag{1.5.3}$$

where $\varphi_0\in L^1(M)^+$ and $\tilde{\varphi}\colon M\times\mathbb{R}^m\to[0,+\infty]$ is a Borel function. We assume that for every $x\in M$ the following properties hold:

(1) $\varphi(x,y)=0$ if and only if $y=0$;

(2) the function $\tilde{\varphi}(x,\cdot)$ belongs to the space $C^0(\mathbb{R}^m)\cap C^1(\mathbb{R}^m\setminus\{0\})$;

(3) $\tilde{\varphi}(x,0)=0$;

(4) there exists an L^∞-function $\bar{\varphi}$ such that $|\partial_y\tilde{\varphi}(x,y)|\leq\bar{\varphi}(x)$ for any $y\neq0$, where $\partial_y\tilde{\varphi}(x,y)$ denotes the vector of the partial derivatives of $\tilde{\varphi}$ with respect to y;

(5) the limit
$$\tilde{\psi}(x,y):=\lim_{\varepsilon\to0^+}\partial_y\tilde{\varphi}(x,\varepsilon y)y\tag{1.5.4}$$
exists and is finite for any $y\neq0$.

Remark 1.5.1. By using de l'Hôpital Theorem, one obtain immediately that

$$\tilde{\psi}(x,y)=\lim_{\varepsilon\to0^+}\frac{\tilde{\varphi}(x,\varepsilon y)}{\varepsilon},$$

for any $x\in M$, $y\neq0$. It follows from the positiveness of $\tilde{\varphi}$ that $\tilde{\psi}\geq0$. Moreover, we get easily that $\tilde{\psi}$ is positively 1-homogeneous with respect to y, i.e., $\tilde{\psi}(x,\lambda y)=\lambda\tilde{\psi}(x,y)$, for every $\lambda>0$. Furthermore, by (1.5.4) and (4), we get also

$$|\tilde{\psi}(x,y)|\leq\bar{\varphi}(x)|y|\quad\text{for every } x\in M\text{ and } y\neq0.\tag{1.5.5}$$

The main result of this section is a theorem which makes explicit the Euler conditions obtained from (1.5.2) in the case of the function φ specified above. Before stating the theorem, we establish a general result concerning closed linear subspaces of $L^1_\mu(\Omega)$, for an arbitrary Radon measure μ on Ω. We will apply this result to the measure $\mu=\mathscr{H}^{n-1}\llcorner M$.

The characteristic function of any set E is denoted by 1_E, i.e., $1_E(x)=1$ if $x\in E$, $1_E(x)=0$ otherwise.

Lemma 1.5.2. *Let μ be a Radon measure in Ω and let Y be a closed linear subspace of $L^1_\mu(\Omega)$ with the following properties:*

(a) if $u, v \in Y$, then $u \vee v \in Y$;

(b) if $u \in Y$ and $\omega \in C^\infty_c(\Omega)$, then $\omega u \in Y$.

Then there exists a Borel set $E \subset \Omega$ such that $Y = \{u \in L^1_\mu(\Omega) : u = 0\ \mu\text{-a.e. on } E\}$.

Proof. We begin by proving that

$$\text{if } u \in L^1_\mu(\Omega) \text{ and } |u| \le |v| \text{ for some } v \in Y, \text{ then } u \in Y. \tag{1.5.6}$$

Indeed in this case there exists $\omega \in L^\infty_\mu(\Omega)$ such that $u = \omega v$ and there is a sequence $\omega_k \in C^\infty_c(\Omega)$ such that ω_k is bounded in $L^\infty_\mu(\Omega)$ and $\omega_k \to \omega$ μ-a.e. on Ω. By (b) we have $\omega_k v \in Y$, and by the Lebesgue dominated convergence theorem $\omega_k v \to \omega v = u$ in $L^1_\mu(\Omega)$. Since Y is closed, we conclude that $u \in Y$.

Now we prove that

$$\text{if } u \in Y \text{ and } t > 0, \text{ then } u \wedge t \in Y \text{ and } (u-t)^+ \in Y. \tag{1.5.7}$$

As $|u \wedge t| \le |u|$, we have $u \wedge t \in Y$ by (1.5.6). Since $(u-t)^+ = u - u \wedge t$, we obtain that $(u-t)^+ \in Y$.

Next we prove that

$$\text{if } u \in Y \text{ and } t > 0, \text{ then } 1_{\{u>t\}} \in Y, \tag{1.5.8}$$

where $\{u > t\} := \{x \in \Omega : u(x) > t\}$. By (1.5.7) we deduce that for every $k > 0$ we have $k(u-t)^+ \wedge 1 \in Y$. As $[k(u-t)^+] \wedge 1 \to 1_{\{u>t\}}$ pointwise and $[k(u-t)^+] \wedge 1 \le |u|/t$, the convergence takes place in $L^1_\mu(\Omega)$ and we conclude that $1_{\{u>t\}} \in Y$.

Let (u_k) be a sequence dense in Y and let E be the intersection of the sets $\{u_k = 0\}$. It is easy to prove by approximation that $u = 0$ μ-a.e. on E for every $u \in Y$. Conversely, let $u \in L^1_\mu(\Omega)$ with $u = 0$ μ-a.e. on E. For every k let

$$A_k := \{u_1 \vee u_2 \vee \cdots \vee u_k > 1/k\}.$$

By (a) and (1.5.8) we have $1_{A_k} \in Y$, so that $(k1_{A_k}) \wedge u^+$ and $(k1_{A_k}) \wedge u^-$ belong to Y, by (1.5.6). As $(k1_{A_k}) \wedge u^+ \to u^+$ and $(k1_{A_k}) \wedge u^- \to u^-$ in $L^1_\mu(\Omega)$ we conclude that $u \in Y$. ☐

Lemma 1.5.3. *Let $D \subset M$, let Y^m_D be the set of all functions of the form $[v]$, with $v \in W^{1,p}(\Omega \setminus M; \mathbb{R}^m)$ and $[v] = 0$ $\mathscr{H}^{n-1}$-a.e. on D, and let $\overline{Y}^m_D$ be the closure of Y^m_D in $L^1(M \setminus \partial M; \mathbb{R}^m)$. Then there exists a Borel set $\tilde{D}$ (unique up to $\mathscr{H}^{n-1}$-equivalence), containing D, such that $\overline{Y}^m_D = \{w \in L^1(M \setminus \partial M; \mathbb{R}^m) : w = 0\ \mathscr{H}^{n-1}\text{-a.e. on } \tilde{D}\}$.*

Proof. Let Y_D be the set corresponding to the case $m = 1$. It is easy to see that $Y_D^m = (Y_D)^m$. Therefore it suffices to prove the lemma in the case $m = 1$.

The conclusion follows from Lemma 1.5.2 applied to $\overline{Y}_D$. It is enough to verify that conditions (a) and (b) are satisfied by Y_D. Condition (b) is trivial. To prove (a) we consider an open set $U \subset \Omega \setminus M$, with C^1 boundary and $M \subset \partial U$, such that U lies on the negative side of M. Given two functions u and $v \in W^{1,p}(\Omega \setminus M)$ it is easy to check that $[u] \vee [v] = [u \vee (v - \tilde{v} + \tilde{u})]$, where $\tilde{u}$ and $\tilde{v} \in W^{1,p}(\Omega)$ coincide with u and v on U, respectively. □

In the following theorem we will consider a function $u \in W^{1,p}(\Omega \setminus M; \mathbb{R}^m)$ such that the divergence of the matrix field $\partial_\xi W(x, \nabla u) - H$ belongs to $L^q(\Omega \setminus M; \mathbb{R}^m)$. It turns out that its normal trace $(\partial_\xi W(x, \nabla u) - H)\nu$ is defined as an element of $(W^{1-\frac{1}{p},p}(\partial_1\Omega; \mathbb{R}^m))'$. Moreover, we have that the normal traces $(\partial_\xi W(x, \nabla u) - H)^{\oplus}\nu$ and $(\partial_\xi W(x, \nabla u) - H)^{\ominus}\nu$ (defined on the positive and negative side of M) are both elements of the space $(W^{1-\frac{1}{p},p}(M \setminus \partial M; \mathbb{R}^m))'$. The duality pairing between $(W^{1-\frac{1}{p},p}(M \setminus \partial M; \mathbb{R}^m))'$ and $W^{1-\frac{1}{p},p}(M \setminus \partial M; \mathbb{R}^m)$ will be denoted by $\langle \cdot, \cdot \rangle$.

Theorem 1.5.4. *Let $t \in [0, T]$ and $(u, \gamma) \in W^{1,p}(\Omega \setminus M; \mathbb{R}^m) \times L^1(M)^+$ be globally stable at time t. Assume that $\varphi\colon M \times \mathbb{R}^m \to [0, +\infty]$ is defined as above in (1.5.3) and it satisfies (1)–(5). Then*

$$-\mathrm{div}\big(\partial_\xi W(x, \nabla u) - H\big) = f \qquad \text{on } \Omega \setminus M, \tag{1.5.9}$$

$$(\partial_\xi W(x, \nabla u) - H)\nu = g \qquad \text{on } \partial_1\Omega, \tag{1.5.10}$$

$$(\partial_\xi W(x, \nabla u) - H)^{\oplus}\nu + g^{\oplus} = (\partial_\xi W(x, \nabla u) - H)^{\ominus}\nu - g^{\ominus} \qquad \text{on } M \setminus \partial M. \tag{1.5.11}$$

Let us define

$$\begin{aligned} A &:= \{x \in M : 0 < \phi([u])(x) = \gamma(x)\}, \\ B &:= \{x \in M : 0 = \phi([u])(x) \text{ and } \gamma(x) = \varphi_0(x)\}, \\ D &:= \{x \in M : \gamma(x) < \varphi_0(x)\}, \end{aligned}$$

and let $\tilde{D}$ be the set associated with D by Lemma 1.5.3.
Then there exists $h \in L^\infty(M \setminus \tilde{D}; \mathbb{R}^m)$ such that

$$\langle (\partial_\xi W(x, \nabla u) - H)^{\oplus}\nu + g^{\oplus}, [v] \rangle = \int_{M \setminus \tilde{D}} h[v]\, d\mathscr{H}^{n-1}, \tag{1.5.12}$$

for every $v \in W^{1,p}(\Omega \setminus M; \mathbb{R}^m)$ such that $[v] = 0$ $\mathscr{H}^{n-1}$-a.e. on D. Moreover

(a) for $\mathscr{H}^{n-1}$-a.e. $x \in A \setminus \tilde{D}$ the vector $h(x)$ belongs to the segment joining 0 and $\partial_y \tilde{\varphi}(x, [u](x))$;

(b) for $\mathscr{H}^{n-1}$-a.e. $x \in B \setminus \tilde{D}$ the vector $h(x)$ belongs to the bounded convex set $K(x) := \{a \in \mathbb{R}^m : ay \le \tilde{\psi}(x, y),\ \forall y \in \mathbb{R}^m\}$;

(c) for $\mathscr{H}^{n-1}$-a.e. $x \in M \setminus (A \cup B \cup \tilde{D})$ we have $h(x) = 0$.

Remark 1.5.5. It is easy to see that, if D is ($\mathscr{H}^{n-1}$-equivalent to) a closed set, then $\tilde{D} = D$ (up to $\mathscr{H}^{n-1}$-equivalence). A more difficult proof shows that the same result is true if D is ($\mathscr{H}^{n-1}$-equivalent to) a quasi closed set with respect to $(1,p)$-capacity.

It is clear that, if $\varphi_0 = 0$, then $\tilde{D} = D = \emptyset$.

Remark 1.5.6. For $\mathscr{H}^{n-1}$-a.e. $x \in M$ the vector $h(x)$, obtained in Theorem 1.5.4, represents the cohesive force exerted from the positive lip of the crack on the negative lip. The theorem shows the conditions satisfied by the cohesive force on the different regions of M determined by the respective relations between $\phi([u])$, γ and φ_0.

Proof of Theorem 1.5.4. Since $\phi([u]) \leq \gamma$ $\mathscr{H}^{n-1}$-a.e. on M, we have $(\phi([u]) - \gamma)^+ = 0$ $\mathscr{H}^{n-1}$-a.e. on M. If $[v] = 0$ $\mathscr{H}^{n-1}$-a.e. on M, then the lim inf in (1.5.2) is actually a limit and it is zero. Therefore (1.5.9), (1.5.10), and (1.5.11) can be obtained from (1.5.2) by standard argument involving integration by parts and a suitable choice of the test function $v \in W^{1,p}(\Omega; \mathbb{R}^m)$.

To shorten the notation, we set $\tilde{h} := (\partial_\xi W(x, \nabla u) - H)^{\oplus}\nu + g^{\oplus}$ on $M \setminus \partial M$. As explained before the statement of the theorem, we have $\tilde{h} \in (W^{1-\frac{1}{p},p}(M \setminus \partial M); \mathbb{R}^m))'$. So far, we may rewrite (1.5.2) as

$$\langle -\tilde{h}, [v] \rangle + \liminf_{\varepsilon \to 0^+} \frac{\|(\phi([u] + \varepsilon[v]) - \gamma)^+\|_{1,M}}{\varepsilon} \geq 0, \tag{1.5.13}$$

for any $v \in W^{1,p}(\Omega \setminus M; \mathbb{R}^m)$ such that $v = 0$ $\mathscr{H}^{n-1}$-a.e. on $\partial_0\Omega$.

Let us extend the definition of $\tilde{\psi}$ by setting $\tilde{\psi}(x, 0) = 0$ for every $x \in M$. Now we prove that

$$\begin{aligned} &\lim_{\varepsilon \to 0^+} \frac{\|(\phi([u] + \varepsilon w) - \gamma)^+\|_{1,M}}{\varepsilon} = \\ &= \int_M \Big((\partial_y \tilde{\varphi}(x, [u]) w)^+ \, 1_A + \tilde{\psi}(x, w) \, 1_B \Big) \, d\mathscr{H}^{n-1}, \end{aligned} \tag{1.5.14}$$

for every $w \in L^1(M \setminus \partial M; \mathbb{R}^m)$ with $w = 0$ $\mathscr{H}^{n-1}$-a.e. on D. To this aim, it is convenient to split the set M into the union of the following two disjoint subsets $A' := \{x \in M : [u](x) \neq 0\}$ and $B' := \{x \in M : [u](x) = 0\}$.

On A', as $\phi([u]) \leq \gamma$ $\mathscr{H}^{n-1}$-a.e. on M, we have that

$$\begin{aligned} \frac{(\phi([u] + \varepsilon w) - \gamma)^+}{\varepsilon} &\leq \frac{(\phi([u] + \varepsilon w) - \phi([u]))^+}{\varepsilon} = \frac{(\tilde{\varphi}(x, [u] + \varepsilon w) - \tilde{\varphi}(x, [u]))^+}{\varepsilon} \leq \\ &\leq (\bar{\varphi}(x) w)^+, \end{aligned}$$

$\mathscr{H}^{n-1}$-a.e. on M, where we used (1.5.3), and assumptions (3) and (4). Moreover, we have that

$$\frac{(\phi([u] + \varepsilon w) - \gamma)^+}{\varepsilon} \to (\partial_y \tilde{\varphi}(x, [u]) w)^+ \, 1_A \quad \mathscr{H}^{n-1}\text{-a.e. on } A',$$

because $A = \{0 < \phi([u]) = \gamma\}$. By the Lebesgue dominated convergence theorem we get

$$\int_{A'} \frac{(\phi([u] + \varepsilon w) - \gamma)^+}{\varepsilon} \, d\mathscr{H}^{n-1} \to \int_M (\partial_y \tilde{\varphi}(x, [u])w)^+ \, 1_A \, d\mathscr{H}^{n-1}, \tag{1.5.15}$$

as $\varepsilon \to 0^+$, for every $w \in L^1(M \smallsetminus \partial M; \mathbb{R}^m)$.

Let us consider now the integral over B'. If $w \in L^1(M \smallsetminus \partial M; \mathbb{R}^m)$ and $w = 0$ $\mathscr{H}^{n-1}$-a.e. on D, we have

$$\frac{(\phi(\varepsilon w) - \gamma)^+}{\varepsilon} = 0 \quad \mathscr{H}^{n-1}\text{-a.e. on } D,$$

thus we can focus on the set $B' \smallsetminus D$. As $\gamma \geq \varphi_0$ $\mathscr{H}^{n-1}$-a.e. on $M \smallsetminus D$, for every $w \in L^1(M \smallsetminus \partial M; \mathbb{R}^m)$ with $w = 0$ $\mathscr{H}^{n-1}$-a.e. on D, we obtain

$$\frac{(\phi(\varepsilon w) - \gamma)^+}{\varepsilon} \leq \frac{(\phi(\varepsilon w) - \varphi_0)^+}{\varepsilon} = \frac{\tilde{\varphi}(x, \varepsilon w)}{\varepsilon} \leq \bar{\varphi}(x)|w|$$

$\mathscr{H}^{n-1}$-a.e. on $M \smallsetminus D$, where we used (1.5.3), and assumptions (3) and (4). Moreover, by Remark 1.5.1 we get that

$$\frac{(\phi(\varepsilon w) - \gamma)^+}{\varepsilon} \to (\tilde{\psi}(x, w))^+ \, 1_B = \tilde{\psi}(x, w) 1_B \quad \mathscr{H}^{n-1}\text{-a.e. on } B',$$

as $\varepsilon \to 0^+$, for every $w \in L^1(M \smallsetminus \partial M; \mathbb{R}^m)$ with $w = 0$ $\mathscr{H}^{n-1}$-a.e. on D. We can apply again the Lebesgue dominated convergence theorem and obtain

$$\int_{B'} \frac{(\phi([u] + \varepsilon w) - \gamma)^+}{\varepsilon} \, d\mathscr{H}^{n-1} \to \int_M \tilde{\psi}(x, w) \, 1_B \, d\mathscr{H}^{n-1},$$

as $\varepsilon \to 0^+$, for every $w \in L^1(M \smallsetminus \partial M; \mathbb{R}^m)$ with $w = 0$ $\mathscr{H}^{n-1}$-a.e. on D. This concludes the proof of (1.5.14). We note that this equality cannot be true if the condition $w = 0$ $\mathscr{H}^{n-1}$-a.e. on D is violated, because in this case

$$\lim_{\varepsilon \to 0^+} \frac{\|(\phi(\varepsilon w) - \gamma)^+\|_{1,M}}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{\|(\varphi_0 + \tilde{\varphi}(\varepsilon w) - \gamma)^+\|_{1,M}}{\varepsilon} = +\infty.$$

Let Y_D^m be the space defined in Lemma 1.5.3. Notice that $Y_D^m \subset W^{1-\frac{1}{p},p}(M \smallsetminus \partial M); \mathbb{R}^m)$. By (1.5.13) and (1.5.14) we have

$$\langle -\tilde{h}, w \rangle + \int_M \left[(\partial_y \tilde{\varphi}(x, [u])w)^+ \, 1_A + \tilde{\psi}(x, w) \, 1_B \right] d\mathscr{H}^{n-1} \geq 0, \tag{1.5.16}$$

for any $w \in Y_D^m$. In order to localize this inequality, we prove first (1.5.12). Due to our assumption (4) and to (1.5.5), if we apply (1.5.16) to w and $-w$ we deduce that

$$|\langle \tilde{h}, w \rangle| \leq \|\bar{\varphi}\|_\infty \|w\|_{1,M \smallsetminus D}\,, \tag{1.5.17}$$

for every $w \in Y_D^m$. It follows that there exists a function $h \in L^\infty(M \setminus D; \mathbb{R}^m)$ such that

$$\langle \tilde{h}, w \rangle = \int_{M \setminus D} hw \, d\mathscr{H}^{n-1},$$

for every $w \in Y_D^m$. This implies that (1.5.12) is satisfied. By density from (1.5.16) we obtain

$$\int_{M \setminus D} \Big[-hw + (\partial_y \tilde{\varphi}(x, [u])w)^+ \, 1_A + \tilde{\psi}(x, w) \, 1_B \Big] \, d\mathscr{H}^{n-1} \geq 0, \tag{1.5.18}$$

for every $w \in \overline{Y}_D^m$. Since by Lemma 1.5.3 we have $\overline{Y}_D^m = \{w \in L^1(M \setminus \partial M; \mathbb{R}^m) : w = 0 \ \mathscr{H}^{n-1}\text{-a.e. on } \tilde{D}\}$, we conclude that

$$-h(x)y + (\partial_y \tilde{\varphi}(x, [u](x))y)^+ \, 1_A(x) + \tilde{\psi}(x, y) \, 1_B(x) \geq 0, \tag{1.5.19}$$

for every $y \in \mathbb{R}^m$ and for $\mathscr{H}^{n-1}$-a.e. $x \in M \setminus \tilde{D}$.

In particular, for $\mathscr{H}^{n-1}$-a.e. $x \in A \setminus \tilde{D}$ the equality $\partial_y \tilde{\varphi}(x, [u](x))y = 0$ implies that $h(x)y = 0$ (it is enough to use (1.5.19) with y and $-y$), so that for a given $x \in A \setminus \tilde{D}$ the two vectors $\partial_y \tilde{\varphi}(x, [u](x))$ and $h(x)$ are parallel, hence there exists $\lambda(x)$ such that

$$h(x) = \lambda(x) \, \partial_y \tilde{\varphi}(x, [u](x)) \quad \text{for } \mathscr{H}^{n-1}\text{-a.e. } x \in A \setminus \tilde{D}, \tag{1.5.20}$$

and it is easy to verify that $0 \leq \lambda(x) \leq 1$, by using again (1.5.19). In this way we get condition (a).

On $B \setminus \tilde{D}$, from (1.5.19) we obtain

$$-h(x)y + \tilde{\psi}(x, y) \geq 0 \quad \text{for } \mathscr{H}^{n-1}\text{-a.e. } x \in B \setminus \tilde{D}, \tag{1.5.21}$$

for every $y \in \mathbb{R}^m$, which is precisely condition (b), by the definition of K. On the remaining part of $M \setminus \tilde{D}$, from (1.5.19) we get condition (c). This concludes the proof. ☐

Remark 1.5.7. If $\varphi_0(x) > 0$ for $\mathscr{H}^{n-1}$-a.e. $x \in M$, and $(u, \gamma) = (u(t), \gamma(t))$ for an irreversible quasistatic evolution, then (1.2.4) implies that the set $B \setminus \tilde{D}$ is nonempty only if there exists $y \in \mathbb{R}^m \setminus \{0\}$ such that $\tilde{\varphi}(x, y) = 0$, for some $x \in M$. This happens, for instance, in the Griffith model, where φ is given by (1.1.6) with $a > 0$ and $b = 0$. In this special case, condition (b) becomes $h(x) = 0$ $\mathscr{H}^{n-1}$-a.e. on $B \setminus \tilde{D}$, because $K(x) = \{0\}$.

Remark 1.5.8. If for every x the functions $\xi \mapsto W(x, \xi)$ and $y \mapsto \varphi(x, y)$ are convex, then for any $t \in [0, T]$ and $\gamma \in L^1(M)^+$, the functional $u \mapsto \mathscr{E}(t)(u, \gamma \vee \varphi([u]))$ is convex. Therefore, it is possible to prove by standard arguments that conditions (a), (b), and (c) of Theorem 1.5.4 are equivalent to the inequality

$$-\int_M hw \, d\mathscr{H}^{n-1} + \lim_{\varepsilon \to 0^+} \frac{\|(\phi([u] + \varepsilon w) - \gamma)^+\|_{1,M}}{\varepsilon} \geq 0,$$

for every $w \in \overline{Y}_D^m$. Thus, Euler conditions (1.5.9), (1.5.10), (1.5.11), (a), (b), (c) are not only necessary, but also sufficient to global stability.

We show now an example of a scalar problem, where the Euler conditions of Theorem 1.5.4 lead to a simplified set of boundary conditions.

Example 1.5.9. Let $m = 1$, $p = 2$, $W(x, \xi) := \frac{1}{2}|\xi|^2$, $H(t) := 0$, $g^{\oplus}(t) = g^{\ominus}(t) := 0$, $\phi(y) := |y|$, which correspond to the energy functional:

$$\mathscr{E}(t)(u, \gamma) := \frac{1}{2}\int_{\Omega \smallsetminus M} |\nabla u|^2 \, dx + \int_M \gamma \, d\mathscr{H}^{n-1} - \int_{\Omega \smallsetminus M} f(t) u \, dx - \int_{\partial_1 \Omega} g(t) u \, d\mathscr{H}^{n-1}.$$

Let $t \in [0, T]$ and $(u, \gamma) \in W^{1,2}(\Omega \smallsetminus M) \times L^1(M)^+$ be globally stable at time t. Then we are in a position to apply Theorem 1.5.4 and the final part of Remark 1.5.5, obtaining

$$\begin{cases} -\Delta u = f(t) & \text{on } \Omega \smallsetminus M, \\ u = \psi(t) & \text{on } \partial_0 \Omega, \\ \dfrac{\partial u}{\partial \nu} = g(t) & \text{on } \partial_1 \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } M \cap \{0 \le |[u]| < \gamma\}, \\ \left|\dfrac{\partial u}{\partial \nu}\right| \le 1 \text{ and } \dfrac{\partial u}{\partial \nu}[u] \ge 0 & \text{on } M \cap \{|[u]| = \gamma\}. \end{cases}$$

By Remark 1.5.8 we have also that if u solves the previous boundary value problem for a given γ, then the pair (u, γ) is globally stable at time t.

1.6 The case of linear elasticity

In this section we show that, with some modifications, it is possible to consider also the case where the uncracked part of the body is linearly elastic, which is excluded by the first inequality in (1.1.1).

Let $p = 2$ and $m = n \ge 1$. We assume now that the bulk energy relative to the displacement $u \in W^{1,2}(\Omega \smallsetminus M; \mathbb{R}^n)$ has the form of linear elasticity

$$\int_{\Omega \smallsetminus M} A(x) Eu : Eu \, dx,$$

where $Eu := \frac{1}{2}(\nabla u + (\nabla u)^T)$ is the symmetric part of the gradient of u, and A satisfies the following properties:

(E_1) for every $x \in \Omega$, $A(x)$ is a linear symmetric operator from the space $\mathbb{M}^{n \times n}_{sym}$ of symmetric $n \times n$ matrices into itself, and the map $x \mapsto A(x)$ is measurable;

(E_2) there are two positive constants c_0 and c_1 such that

$$c_0 \, |\xi|^2 \le A(x)\xi : \xi \le c_1 \, |\xi|^2 \tag{1.6.1}$$

for every $x \in \Omega \smallsetminus M$ and $\xi \in \mathbb{M}^{n \times n}_{sym}$.

For the sake of simplicity in the notation we introduce the C^1 map $\mathcal{Q}\colon L^2(\Omega\setminus M;\mathbb{M}^{n\times n}_{sym})\to\mathbb{R}$ defined by

$$\mathcal{Q}(\Psi) := \int_{\Omega\setminus M} A(x)\Psi : \Psi\, dx$$

for every $\Psi\in L^2(\Omega\setminus M;\mathbb{M}^{n\times n}_{sym})$, whose differential $\partial\mathcal{Q}\colon L^2(\Omega\setminus M;\mathbb{M}^{n\times n}_{sym})\to L^2(\Omega\setminus M;\mathbb{M}^{n\times n}_{sym})$ is given by

$$\langle\partial\mathcal{Q}(\Psi),\Phi\rangle = 2\int_{\Omega\setminus M} A(x)\Psi : \Phi\, dx,$$

for every Φ, $\Psi\in L^2(\Omega\setminus M;\mathbb{M}^{n\times n}_{sym})$, where $\langle\cdot,\cdot\rangle$ denotes now the scalar product in the space $L^2(\Omega\setminus M;\mathbb{M}^{n\times n}_{sym})$.

For every $t\in[0,T]$ the total energy of an admissible configuration $(u,\gamma)\in W^{1,2}(\Omega\setminus M,\mathbb{R}^n)\times L^1(M)^+$ at time t is now defined as

$$\mathscr{E}(t)(u,\gamma) := \mathcal{Q}(Eu) - \langle\mathscr{L}(t),u\rangle + \|\gamma\|_{1,M}.$$

Once we have the energy functional, we introduce the notion of global stability as in Definition 1.2.1.

Since the $(n-1)$-dimension of $\partial_0\Omega$ is positive, Korn inequality holds (see, e.g., [5], [14]): there exists a constant $C=C(\Omega,\partial_0\Omega)$ such that

$$\|\nabla u\|_2 \le C\|Eu\|_2 \quad \text{for all } u\in W^{1,2}(\Omega;\mathbb{R}^n) \text{ such that } u=0 \text{ on } \partial_0\Omega.$$

As an immediate consequence, we get the following Korn-type inequality:

$$\|\nabla u\|_2 \le C\|Eu\|_2 + (C+1)\|\nabla\psi\|_2 \tag{1.6.2}$$

for every $u\in W^{1,2}(\Omega\setminus M;\mathbb{R}^n)$, and $\psi\in W^{1,2}(\Omega;\mathbb{R}^n)$ such that $u=\psi$ on $\partial_0\Omega$.

Thanks to (1.6.2), we still have an a priori bound for the displacement u as in Remark 1.2.2.

The definition of irreversible quasistatic evolution of minimum energy configurations is now given replacing $\langle\partial\mathcal{W}(\nabla u(t)),\nabla\dot\psi(t)\rangle$ by $\langle\partial\mathcal{Q}(Eu(t)),E\dot\psi(t)\rangle$ in Definition 1.2.4.

Thanks to the Korn-type inequality (1.6.2), Theorems 1.2.7, 1.2.10, 1.4.4, and 1.4.5 (and Remark 1.2.5) continue to hold, with essentially the same proofs, if we replace $\mathcal{W}(\nabla u(t))$ and $\langle\partial\mathcal{W}(\nabla u(t)),\nabla\dot\psi(t)\rangle$ by $\mathcal{Q}(Eu(t))$ and $\langle\partial\mathcal{Q}(Eu(t)),E\dot\psi(t)\rangle$, respectively, and a similar substitution is done for $u_k(t)$.

Chapter 2

Singular perturbations of finite dimensional gradient flows

In this chapter we give a description of the asymptotic behavior, as $\varepsilon \to 0$, of the ε-gradient flow in the finite dimensional case.

Under very general assumptions we prove that it converges to an evolution obtained by connecting some smooth branches of solutions to the equilibrium equation (slow dynamics) through some heteroclinic solutions of the gradient flow (fast dynamics).

The chapter is organized as follows. In Section 2.1 we fix the mathematical assumptions, while in Section 2.2 we prove some preliminaries and define the evolution we are interested in (see Definition 2.2.4). Section 2.3 consists in the proof of the main result of the chapter, Theorem 2.2.7.

2.1 Setting of the problem

Throughout the chapter, for fixed $T > 0$, we make the following assumption:

Assumption 1. $f\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}$ *is a* C^3*-function satisfying the property*

$$\nabla_x f(t,x) \cdot x \geq c_0 \, |x|^2 - a_0 \, ,$$

for some $a_0 \geq 0$ *and* $c_0 > 0$*,*

where $\nabla_x f = (f_{x_1}, \dots, f_{x_n})$ denotes the gradient of f with respect to its spatial variable $x \in \mathbb{R}^n$.

We may deduce from this assumption that there exist two positive constants M and $\tilde{c}$ (depending on a_0 and c_0), and a constant $\tilde{a}$ (depending also on f and T) such that

$$f(t,x) \geq \tilde{c}|x|^2 - \tilde{a} \quad \text{for every } |x| \geq M \text{ and every } t \in [0,T] \, . \tag{2.1.1}$$

For given $t \in [0,T]$, we say that a point $x \in \mathbb{R}^n$ is a critical point for $f(t,\cdot)$ if $\nabla_x f(t,x) = 0$.

Remark 2.1.1. Note that by Assumption 1 all critical points for the function $f(t,\cdot)$ belong to the compact $\overline{B}$, where $\overline{B} := \overline{B}(0, \sqrt{a_0 c_0^{-1}})$ is the closed ball in $\mathbb{R}^n$ centered at 0 and with radius $\sqrt{a_0 c_0^{-1}}$. Moreover, taking the minimum of $f(t,\cdot)$ in $\overline{B}$ it is immediate to get a critical point. Hence, for every $t \in [0,T]$, critical points for $f(t,\cdot)$ exist and belong to $\overline{B}$.

We denote the set of zeroes to the gradient of f by Γ_f, namely,

$$\Gamma_f := \{(t,x) \in [0,T] \times \mathbb{R}^n : \nabla_x f(t,x) = 0\}, \tag{2.1.2}$$

and observe that $\Gamma_f \subset [0,T] \times \overline{B}$, by Remark 2.1.1.

We recall that a critical point ξ for $f(t,\cdot)$ is said to be degenerate if the kernel of the Hessian matrix $\nabla_x^2 f(t,\xi) := (f_{x_i x_j}(t,\xi))_{ij}$ is nontrivial, i.e., $\det \nabla_x^2 f(t,\xi) = 0$.

In this chapter, a particular interest will be given to the set Z_f of all pairs (t,ξ) such that ξ is a degenerate critical point for $f(t,\cdot)$, i.e.,

$$Z_f := \{(t,\xi) \in \Gamma_f : \det \nabla_x^2 f(t,\xi) = 0\}. \tag{2.1.3}$$

We make the following assumption.

Assumption 2. *The number of all pairs (t,ξ), such that ξ is a degenerate critical point for $f(t,\cdot)$, is finite, i.e.,*

$$card(Z_f) = m < +\infty.$$

Moreover, let $\Pi\colon Z_f \to [0,T]$ denote the projection of Z_f on the time-segment $[0,T]$, then we assume that Π is injective and that $0, T \notin \Pi(Z_f)$.

Throughout this chapter we will focus on a particular class of degenerate critical points. More in detail, we make the following assumption.

Assumption 3. *For every $\tau \in [0,T]$ and for every degenerate critical point $\xi \in \mathbb{R}^n$ for $f(\tau,\cdot)$, such that $\nabla_x^2 f(\tau,\xi)$ is positive semidefinite, there exists $\ell \in \mathbb{R}^n \setminus \{0\}$ such that the following conditions are satisfied:*

(a) $\ker \nabla_x^2 f(\tau,\xi) = \operatorname{span}(\ell)$*;*

(b) $\nabla_x f_t(\tau,\xi) \cdot \ell \neq 0$*, where $f_t(\tau,\xi)$ denotes the partial derivative of f with respect to the time variable t;*

(c) $\sum_{i,j,k} f_{x_i x_j x_k}(\tau,\xi)\ell_i \ell_j \ell_k \neq 0$*.*

Notice that condition (a) means that 0 is a simple eigenvalue of $\nabla_x^2 f(\tau,\xi)$ with eigenvector ℓ, while the remaining $n-1$ eigenvalues are positive. Conditions (b) and (c) are known in the literature as transversality conditions (see, e.g., [24]).

Remark 2.1.2. Let $(\tau, \xi) \in Z_f$, with $\nabla_x^2 f(\tau, \xi)$ positive semidefinite. An argument based on the Implicit Function Theorem (see, e.g., [42]), implies that if (τ, ξ) satisfies Assumption 3, then there exists a smooth curve of solutions of $\nabla_x f(t(\lambda), x(\lambda)) = 0$, for λ in a neighborhood of zero, with $(t(0), x(0)) = (\tau, \xi)$.

More precisely, if conditions (b) and (c) have the same sign, then for every $t < \tau$ and near τ there are two solutions for the problem $\nabla_x f(t, x) = 0$, while for t near τ but $t > \tau$ there are no solutions. If conditions (b) and (c) have opposite sign, then the reverse is true.

Moreover, the curve of zeroes passing through (τ, ξ) possesses a vertical tangent at (τ, ξ).

Remark 2.1.3. From our assumptions it turns out that Γ_f is the union of a finite number of C^2-curves with end-points contained in $(\{0\} \times \mathbb{R}^n) \cup (\{T\} \times \mathbb{R}^n)$, see Figure 2.1 below for an example of the set Γ_f.

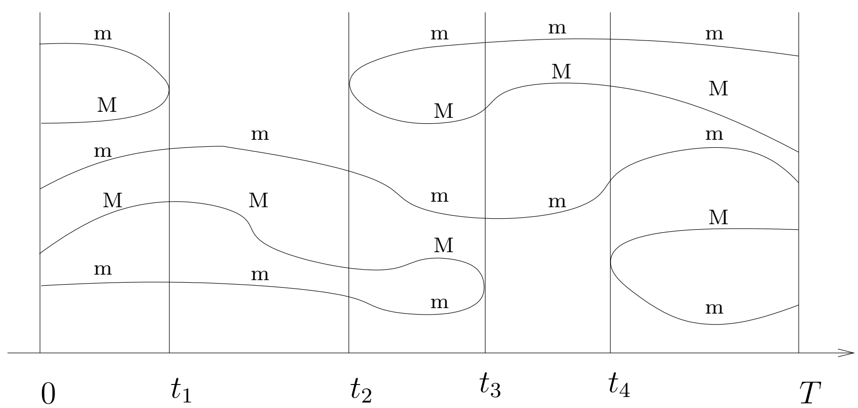

Figure 2.1: *An example for the set Γ_f: m and M stand for local minimum and maximum, respectively.*

Remark 2.1.4. The assumptions we made imply that for every $t \in [0, T]$ there exists a finite number of critical points for $f(t, \cdot)$. Indeed, if $t \notin \Pi(Z_f)$, then Assumption 2 ensures that there are only nondegenerate critical points for $f(t, \cdot)$. Assumption 1 implies that all critical points belong to the compact set $\overline{B}$, while by Assumption 3 it follows that they are isolated. On the other hand, by Assumption 2, at $t = \tau \in \Pi(Z_f)$ there is only one degenerate critical point ξ for $f(\tau, \cdot)$.

Let us freeze now a point $\tau \in \Pi(Z_f)$ and consider the autonomous system of differential equations in $\mathbb{R}^n$ (depending on the single parameter τ)

$$\dot{w}(s) = -\nabla_x f(\tau, w(s)). \tag{2.1.4}$$

This is obviously a gradient system and, thanks to Assumption 1, (since positive semiorbits are bounded) we may apply the well known result that the ω-limit set is contained into the set of equilibria of equation (2.1.4) (see, e.g., [26, Theorem 14.17]). Moreover, since the equilibrium points are isolated (see Remark 2.1.4), such an ω-limit set is a single equilibrium point.

The following lemma ensures the existence of a unique heteroclinic solution w issuing from $(\tau,\xi)\in Z_f$, while previous argument guarantees that w has limit as $s\to+\infty$, and this limit is a (nondegenerate) critical point for $f(\tau,\cdot)$.

Lemma 2.1.5. *Suppose that Assumption 1 and conditions (a) and (c) of Assumption 3 are satisfied. Let (τ,ξ) be a point of Z_f such that $\nabla_x^2 f(\tau,\xi)$ is positive semidefinite. Then there exists a unique (up to time-translations) solution of the problem*

$$\begin{cases}\dot w(s) = -\nabla_x f(\tau, w(s))\\ \lim\limits_{s\to-\infty} w(s)=\xi\,.\end{cases}\tag{2.1.5}$$

Proof. The proof is obtained by adapting a proof of the existence of the global center manifold, based on the Contraction Mapping Principle (see, e.g., [42]). The main difficulty is that usually, when the linearized part of a system of ordinary differential equations has some zero eigenvalue, there is, in general, existence of a heteroclinic solution, but not uniqueness (this is related to non-uniqueness of the local center manifold, see, e.g., [24], [42, §1.4]). Here the uniqueness is obtained thanks to the particular conditions (a) and (c) of Assumption 3.

During the proof, we will use the following notation: $g(x):=f(\tau,x)$, for every $x\in\mathbb{R}^n$. To simplify further the formulation we make a number of preliminary transformations: a translation to take ξ to the origin, and a linear transformation to bring $\nabla^2 g(0)$ in a diagonal form where the first eigenvalue is zero with eigenvector $e_1=(1,0,\dots,0)$. Therefore we are reduced to the following hypotheses:

$$\nabla g(0)=0,\qquad \nabla^2 g(0)=\begin{pmatrix}0&0\\0&A\end{pmatrix},\qquad\text{and}\qquad g_{x_1x_1x_1}(0)\neq 0\,,\tag{2.1.6}$$

where A is an $(n-1)\times(n-1)$ diagonal and invertible matrix. Moreover, by our assumption, the diagonal entries of A are all positive real numbers. In order to simplify the notation, we also suppose that $\frac{1}{2}g_{x_1x_1x_1}(0)=1$.

The existence and the uniqueness for the problem

$$\begin{cases}\dot w(t) = -\nabla g(w(t)),\\ \lim\limits_{t\to-\infty} w(t)=0\,,\end{cases}\tag{2.1.7}$$

will be obtained applying the Contraction Mapping Theorem. As a first step, for every $x\in\mathbb{R}^n$ we take the following decomposition $x=(x_1,\bar x)$, with $\bar x:=(x_2,\dots,x_n)\in\mathbb{R}^{n-1}$

and consider the space Y of all functions $y\colon (-\infty,0]\to\mathbb{R}^n$, $y(t)=(y_1(t),\bar y(t))$, such that

$$\|y_1\|_{Y_1} := \sup_{t\le 0} |(t-1)y_1(t)| < \infty\,, \quad \text{and} \quad \|\bar y\|_{\overline{Y}} := \sup_{t\le 0} |(t-1)^2\bar y(t)| < \infty\,,$$

endowed with the norm

$$\|y\|_Y := \|y_1\|_{Y_1} + \|\bar y\|_{\overline{Y}}\,.$$

For every $x\in\mathbb{R}^n$ let $\nabla g(x) = (D_1 g(x), \bar D g(x))$. Using now the Taylor expansion for x in a neighborhood of $0\in\mathbb{R}^n$, we get

$$\begin{aligned} D_1 g(x) &= x_1^2 + x_1 b\cdot\bar x + \varphi(\bar x,\bar x) + o(|x|^2) \\ \bar D g(x) &= A\bar x + x_1^2 b + x_1 B\bar x + \Phi(\bar x,\bar x) + o(|x|^2)\,, \end{aligned} \tag{2.1.8}$$

where b is a suitable vector in $\mathbb{R}^{n-1}$, $\varphi\colon \mathbb{R}^{n-1}\times\mathbb{R}^{n-1}\to\mathbb{R}$ and $\Phi\colon \mathbb{R}^{n-1}\times\mathbb{R}^{n-1}\to\mathbb{R}^{n-1}$ are bilinear symmetric forms (whose coefficients depend on the third derivative of g at the origin), A is the matrix which appears in (2.1.6), and B is a $(n-1)\times(n-1)$ matrix whose entries depend on the third derivative of g at 0.

More in detail, let $a_{ijk} := g_{x_ix_jx_k}(0)$, $i,j,k=1,\dots,n$. Then $b\in\mathbb{R}^{n-1}$ is defined by $b_i := 2a_{11(i+1)}$, $i=1,\dots,n-1$, $\varphi\colon \mathbb{R}^{n-1}\times\mathbb{R}^{n-1}\to\mathbb{R}$ is given by $\varphi(\bar x,\bar x) := \sum_{i,j=2}^n a_{ij1}x_ix_j$, the $(n-1)\times(n-1)$ matrix B is given by $B_{ij} := 2a_{1(i+1)(j+1)}$, $i,j=1,\dots,n-1$, and $\Phi\colon \mathbb{R}^{n-1}\times\mathbb{R}^{n-1}\to\mathbb{R}^{n-1}$ is the bilinear symmetric form $\Phi_k(\bar x,\bar x) := \sum_{i,j=2}^n a_{ij(k+1)}x_ix_j$, for $k=1,\dots,n-1$.

Moreover, for every $y\in Y$ we define

$$\begin{aligned} h_1(t) &:= -D_1 g(y(t)) + y_1(t)^2 \\ \bar h(t) &:= -\bar D g(y(t)) + A\bar y(t)\,, \end{aligned} \tag{2.1.9}$$

and observe that due to (2.1.8) the asymptotic behavior at $-\infty$ is $h_1(t)\sim(t-1)^{-2}$, and $\bar h(t)\sim(t-1)^{-2}$, respectively. For every $h=(h_1,\bar h)$ satisfying these estimates, let us consider the function $x=(x_1,\bar x)$ obtained by solving the following two problems depending on a parameter $\varepsilon>0$, which will be fixed later.

$$\begin{cases} \dot x_1 + \varepsilon x_1^2 = \varepsilon h_1(t) & \text{on } (-\infty,0], \\ x_1(0) = -\dfrac{1}{\varepsilon}\,, \end{cases} \tag{2.1.10}$$

and

$$\begin{cases} \dot{\bar x} + A\bar x = \varepsilon\bar h(t) & \text{on } (-\infty,0], \\ \lim\limits_{t\to-\infty}\bar x(t) = 0\,. \end{cases} \tag{2.1.11}$$

We shall prove that problem (2.1.11) has a unique solution with $\|\bar x\|_{\overline{Y}}$ finite, and that for ε sufficiently small the solution of problem (2.1.10) does exist and satisfies

$$\lim_{t\to-\infty} x_1(t) = 0\,. \tag{2.1.12}$$

Note that, if $h = (h_1, \bar{h})$ is defined by (2.1.9), and if $x = y$, then $w := \varepsilon x$ solves problem (2.1.7).

From the variation of constant formula it follows that the unique solution of problem (2.1.11) is

$$\bar{x}(t) = \varepsilon \int_{-\infty}^{t} e^{-A(t-s)} \bar{h}(s)\, ds\,. \tag{2.1.13}$$

Moreover $\|\bar{x}\|_{\overline{Y}} < \infty$.

Let us discuss now the existence of a solution of problem (2.1.10). For sake of intuition, let $s = -t$ and define $v(s) := x_1(t)$. Hence, problem (2.1.10) becomes, in terms of the function v,

$$\begin{cases} \dot{v} - \varepsilon v^2 = \tilde{h}(s) & \text{on } [0, +\infty) \\ v(0) = -\dfrac{1}{\varepsilon}\,, \end{cases}$$

where $\tilde{h}(s) := -h_1(-s)$ is such that

$$|\tilde{h}(s)| \leq \frac{M}{(1+s)^2}\,.$$

Thanks to this bound, the existence of the solution will be obtained using differential inequalities. Accordingly, let us begin with the auxiliary problem related to an upper solution:

$$\begin{cases} \dot{v} - \varepsilon v^2 = \varepsilon \dfrac{M}{(1+s)^2} & \text{on } [0, +\infty) \\ v(0) = -\dfrac{1}{\varepsilon}\,. \end{cases} \tag{2.1.14}$$

We observe that this is a particular case of the Riccati equation (see, e.g., [33]), and can be solved as follows. Putting $v(s) = \frac{u(s)}{\varepsilon(1+s)}$ we get

$$\frac{\dot{u}}{\varepsilon(1+s)} - \frac{u}{\varepsilon(1+s)^2} - \varepsilon \frac{u^2}{\varepsilon^2(1+s)^2} = \varepsilon \frac{M}{(1+s)^2}$$
$$\Leftrightarrow (1+s)\dot{u} = \varepsilon^2 M + u + u^2\,.$$

By separation of variables, since $u(0) = -1$, we obtain

$$\int_{-1}^{u(s)} \frac{du}{u^2 + u + \varepsilon^2 M} = \int_0^s \frac{dt}{1+t}\,.$$

Now, the second order equation $u^2 + u + \varepsilon^2 M = 0$ has the following two solutions

$$a := \frac{-1 - \sqrt{1 - 4\varepsilon^2 M}}{2} \qquad\qquad b := \frac{-1 + \sqrt{1 - 4\varepsilon^2 M}}{2}$$

provided that $\varepsilon < (2\sqrt{M})^{-1}$. Notice that $a < b < 0$. Hence,

$$\int_{-1}^{u(s)} \frac{du}{u^2+u+\varepsilon^2 M} = -\frac{1}{b-a}\int_{-1}^{u(s)} \frac{du}{u-a} + \frac{1}{b-a}\int_{-1}^{u(s)} \frac{du}{u-b} =$$
$$= \frac{1}{b-a}\log\Big(\frac{|u(s)-b|}{|u(s)-a|}\frac{|-1-a|}{|-1-b|}\Big).$$

Thus, being $-1 < a < b$, we have obtained that

$$\frac{|u(s)-b|}{|u(s)-a|}\frac{1+a}{1+b} = (1+s)^{b-a} \Leftrightarrow |u(s)-b| = C|u(s)-a|(1+s)^{b-a},$$

where $C := \frac{1+b}{1+a}$. Since $-1 < a$, we have $u(s) < a$ for every $s \geq 0$. Then we get

$$u(s) = \frac{b - aC(1+s)^{b-a}}{1 - C(1+s)^{b-a}},$$

and $u(s) \to a$ as $s \to +\infty$.

Therefore, we have obtained that for $\varepsilon < 1/(2\sqrt{M})$ the function $v(s) = u(s)(\varepsilon(1+s))^{-1}$ solves the auxiliary problem (2.1.14) and tends to zero as $s \to +\infty$.

Let us pass now to consider the auxiliary problem related to the lower solution:

$$\begin{cases} \dot v - \varepsilon v^2 = -\varepsilon\,\dfrac{M}{(1+s)^2} & \text{on } [0,+\infty) \\ v(0) = -\dfrac{1}{\varepsilon}. \end{cases} \tag{2.1.15}$$

Putting again $v(s) = \frac{u(s)}{\varepsilon(1+s)}$ and arguing as before, we are reduced to consider the following problem

$$\begin{cases} (1+s)\dot u = -\varepsilon^2 M + u + u^2 & \text{on } [0,+\infty) \\ u(0) = -1. \end{cases}$$

By separation of variables we obtain

$$\int_{-1}^{u(s)} \frac{du}{u^2+u-\varepsilon^2 M} = \int_0^s \frac{dt}{1+t}.$$

Now, the second order equation $u^2 + u - \varepsilon^2 M = 0$ has the following two solutions

$$a := \frac{-1-\sqrt{1+4\varepsilon^2 M}}{2} \qquad\qquad b := \frac{-1+\sqrt{1+4\varepsilon^2 M}}{2},$$

with $a < 0 < b$, so that in particular $u(s) < b$ for every $s \geq 0$, being $a < -1 < b$. Arguing as before, we deduce that the function $v(s) = u(s)(\varepsilon(1+s))^{-1}$ solves the auxiliary problem (2.1.15) and converges to zero as $s \to +\infty$, for every $\varepsilon > 0$.

Using now differential inequalities (see, e.g., [25, Theorem 6.1]), and then passing from $s \in [0,+\infty)$ to $t \in (-\infty, 0]$, we may conclude that there exists $\varepsilon_0 = \varepsilon_0(M)$ such that problem (2.1.10) admits a unique solution satisfying also the limit condition (2.1.12), for every $\varepsilon < \varepsilon_0$. Moreover, it is immediate to prove that the asymptotic behavior of $x_1(t)$ at $-\infty$ is like $(t-1)^{-1}$.

As a next step, for $\varepsilon < \varepsilon_0$ we define the map $\Gamma\colon Y \to Y$ by setting

$$\Gamma(y)(t) := (x_1(t), \bar{x}(t)) \,, \tag{2.1.16}$$

where $x_1(t)$ is the solution of (2.1.10)–(2.1.12), and $\bar{x}(t)$ is given by (2.1.13), with h_1 and $\bar{h}$ defined by (2.1.9). Obviously, $\Gamma(y)$ belongs to Y, while it remains to prove that the map $y \mapsto \Gamma(y)$ is a strict contraction, for ε sufficiently small.

Let us begin with the first component of $\Gamma(y)$. For every $y \in Y$, let $h(t) = (h_1(t), \bar{h}(t))$ be defined as in (2.1.9), and let us pass from t to $-t$, as before. Next, we notice that there exists $H_1 \in L^\infty(0,+\infty)$ such that $-h_1(-t) = \frac{H_1(t)}{(1+t)^2}$ for every $t \geq 0$. Let $v(t) := x_1(-t)$ be the solution to the following problem

$$\begin{cases} \dot{v} - \varepsilon v = \varepsilon \dfrac{H_1(t)}{(1+t)^2} & \text{on } [0,+\infty) \\ v(0) = -\dfrac{1}{\varepsilon} \,. \end{cases}$$

In the same way, starting from $y^* \in Y$, we define $h^* = (h_1^*, \bar{h}^*)$, $H_1^* \in L^\infty(0,+\infty)$, and $v^*(t)$ as the solution of an analogous problem having H_1^* in the right-hand side, instead of H_1. Put as before $v(t) = \frac{u(t)}{\varepsilon(1+t)}$, so that $u(t)$ solves the problem

$$\begin{cases} (1+t)\dot{u} = \varepsilon^2 H_1(t) + u^2 + u & \text{on } [0,+\infty) \\ u(0) = -1 \,. \end{cases}$$

By this choice of the initial datum, we deduce that

$$|u(t) + 1| \leq \varepsilon^2 M \,, \tag{2.1.17}$$

for every $t \geq 0$, being M an upper bound for the L^∞-norm of H_1. Arguing in the same manner for v^*, we define u^*. We want to prove now that

$$|u(t) - u^*(t)| \leq \varepsilon^2 C \|H_1 - H_1^*\|_\infty \,, \tag{2.1.18}$$

for every $t \geq 0$, so that, passing from t to $-t$ and setting $x_1^*(-t) := v^*(t)$, we will get

$$|x_1(t) - x_1^*(t)| \leq \frac{\varepsilon}{|t-1|} C \|y - y^*\|_Y \qquad \text{for every } t \leq 0 \,,$$

where we used the inequality $\|H_1 - H_1^*\|_\infty \leq C\|y - y^*\|_Y$, which follows from (2.1.9). We will obtain that

$$\|x_1 - x_1^*\|_{Y_1} \leq \frac{1}{2} \|y - y^*\|_Y \,, \tag{2.1.19}$$

having supposed that $\varepsilon C < \frac{1}{2}$.

Therefore, we are reduced to prove (2.1.18). Let $z(t) := u(t) - u^*(t)$, and $\alpha(t) := -u(t) - u^*(t) > 0$. Then $z(t)$ solves the problem

$$\begin{cases} (1+t)\dot{z} = \varepsilon^2(H_1(t) - H_1^*(t)) - \alpha(t)z + z & \text{on } [0, +\infty) \\ z(0) = 0\,. \end{cases}$$

By the variation of constant method, we deduce that the solution $z(t)$ can be represented by the following formula:

$$z(t) = \varepsilon^2 \int_0^t \frac{H_1(s) - H_1^*(s)}{1+s}\, e^{-\int_s^t \frac{\alpha(\sigma)-1}{1+\sigma}\,d\sigma}\, ds\,.$$

Due to (2.1.17), we obtain that $u(t) < -\frac{3}{4}$ for ε sufficiently small, and the same is true for $u^*(t)$. Hence, $\alpha(t) > \frac{3}{2}$, for ε sufficiently small. Then

$$\begin{aligned} |z(t)| &\le \varepsilon^2 \|H_1 - H_1^*\|_\infty \Big| \int_0^t \frac{1}{1+s}\, e^{-\frac{1}{2}\int_s^t \frac{d\sigma}{1+\sigma}}\, ds \Big| = \\ &= \varepsilon^2 \|H_1 - H_1^*\|_\infty \frac{1}{(1+t)^{\frac{1}{2}}} \Big| \int_0^t (1+s)^{-\frac{1}{2}}\, ds \Big| \le 2\varepsilon^2 \|H_1 - H_1^*\|_\infty\,. \end{aligned}$$

This last estimate gives (2.1.18), and therefore (2.1.19) is proved.

Let us consider now the second component of $\Gamma(y)$. Let $\Gamma(y)(t) = (x_1(t), \bar{x}(t))$ and $\Gamma(y^*)(t) = (x_1^*(t), \bar{x}^*(t))$. Therefore,

$$\bar{x}(t) - \bar{x}^*(t) = \varepsilon \int_{-\infty}^t (\bar{h}(s) - \bar{h}^*(s)) e^{-A(t-s)}\, ds\,.$$

Hence, using the fact that, by (2.1.9), $|\bar{h}(s) - \bar{h}^*(s)| \le C\|y - y^*\|_Y (s-1)^{-2}$, we get

$$(t-1)^2 |\bar{x}(t) - \bar{x}^*(t)| \le C\varepsilon (t-1)^2 \int_{-\infty}^t \frac{e^{-A(t-s)}}{(s-1)^2}\, ds \|y - y^*\|_Y\,.$$

Since

$$\sup_{t\le 0} \Big((t-1)^2 \int_{-\infty}^t \frac{e^{-A(t-s)}}{(s-1)^2}\, ds \Big) < +\infty\,,$$

we deduce that there exists a positive constant C^* such that $(t-1)^2|\bar{x}(t) - \bar{x}^*(t)| \le \varepsilon C^* \|y - y^*\|_Y$, i.e.,

$$\|\bar{x} - \bar{x}^*\|_{\bar{Y}} \le \frac{1}{2}\|y - y^*\|_Y\,, \tag{2.1.20}$$

having supposed that $\varepsilon C^* < \frac{1}{2}$.

This estimate, together with (2.1.19), guarantees that the inequality

$$\|\Gamma(y) - \Gamma(y^*)\|_Y \le \frac{1}{2}\|y - y^*\|_Y$$

holds true, and this concludes the proof. □

Throughout the chapter, we make the following assumption.

Assumption 4. *For every $(\tau, \xi) \in [0,T] \times \mathbb{R}^n$ such that ξ is a degenerate critical point for $f(\tau, \cdot)$, satisfying the assumptions of Lemma 2.1.5, let w be the unique solution of (2.1.5) corresponding to τ and ξ. Let $w_\infty := \lim_{s \to +\infty} w(s)$, then we assume that*

$$\nabla_x^2 f(\tau, w_\infty) \quad \textit{is positive definite}. \tag{2.1.21}$$

2.2 Preliminary results

Starting from a suitable point $(\bar{t}, \bar{x}) \in [0, T[\times \mathbb{R}^n$ we prove in the next lemma the existence of a maximal interval $[\bar{t}, \hat{t}[$, and of a regular function u, defined on $[\bar{t}, \hat{t}[$, such that $u(t)$ is a critical point for $f(t, \cdot)$, for every $t \in [\bar{t}, \hat{t}[$.

Lemma 2.2.1. *Let $0 \le \bar{t} < T$, and let $\bar{x} \in \mathbb{R}^n$ be such that $\nabla_x f(\bar{t}, \bar{x}) = 0$ and $\nabla_x^2 f(\bar{t}, \bar{x})$ is positive definite. Suppose that Assumptions 1 and 2 are satisfied. Then there exist a maximal interval of existence $[\bar{t}, \hat{t}[$, and a function $u \colon [\bar{t}, \hat{t}[\to \mathbb{R}^n$ of class C^2, such that $u(\bar{t}) = \bar{x}$ and $\nabla_x f(t, u(t)) = 0$ for every $t \in [\bar{t}, \hat{t}[$. Moreover, either $\hat{t} = T$ or $\hat{t}$ belongs to $\Pi(Z_f)$ (defined in Assumption 2).*

Proof. The Implicit Function Theorem ensures that there are a maximal interval of existence $[\bar{t}, \hat{t}[$ and a function $u \colon [\bar{t}, \hat{t}[\to \mathbb{R}^n$ of class C^2 such that

$$\nabla_x f(t, u(t)) = 0 \qquad \text{and} \qquad \nabla_x^2 f(t, u(t)) > 0 is positive definite on [\bar{t}, \hat{t}[.$$

The next step is to prove that $u(t)$ has limit as t approaches $\hat{t}$ to the left. This is trivial if $\hat{t} = T$. For the case $\hat{t} < T$, we introduce the following auxiliary result that will be proved later.

Lemma 2.2.2. *Under the same assumptions of Lemma 2.2.1, let us define the following set*

$$K := \{x \in \mathbb{R}^n | \exists s_k \nearrow \hat{t} : u(s_k) \to x\}. \tag{2.2.1}$$

Then K is a compact and connected set, composed only of critical points of $f(\hat{t}, \cdot)$. Moreover, if $\hat{t} < T$ then $\det \nabla_x^2 f(\hat{t}, x) = 0$ *for any $x \in K$.*

Proof of Lemma 2.2.1 (continued). Let us suppose that Lemma 2.2.2 is true, and let us prove that

$$\lim_{t \to \hat{t}^-} u(t) \tag{2.2.2}$$

does exist. Indeed, let K be the nonempty set defined by (2.2.1). We need to show that K reduces to just one point. Assume by contradiction that the limit (2.2.2) does not exist. Then there are at least two sequences $s_k^i \nearrow \hat{t}$, $i = 1, 2$ and two distinct points $w_1, w_2 \in \mathbb{R}^n$ such that $u(s_k^i) \to w_i$, $i = 1, 2$. But K is a connected set, thus there exists a continuous path $\gamma\colon [0,1] \to \mathbb{R}^n$, connecting w_1 to w_2, such that $\gamma([0,1]) \subset K$. The contradiction comes from the fact that by Lemma 2.2.2 $x \in K$ implies $(\hat{t}, x) \in Z_f$, which is finite by Assumption 2.

Finally, if $\hat{t} < T$, then by Lemma 2.2.2 $\det \nabla_x^2 f(\hat{t}, x) = 0$ for any $x \in K$, while by continuity $\nabla_x f(\hat{t}, x) = 0$, i.e., every $x \in K$ is a degenerate critical point for $f(\hat{t}, \cdot)$. Hence by Assumption 2, $\hat{t} \in \Pi(Z_f)$, and this concludes the proof of Lemma 2.2.1. $\square$

Proof of Lemma 2.2.2. We begin with compactness. By definition the set K is closed, while Assumption 1 guarantees that it is bounded (see Remark 2.1.1).

We continue by proving that K is connected. This can be done in two steps. The first one consists into prove that for any neighborhood U of the set K there exists $k > 0$ such that $u(s) \in U$ for any $s \in V_k := [\hat{t} - \frac{1}{k}, \hat{t}[$, that is, in other words, $u(s)$ converges to K whenever $s \to \hat{t}$. This can be done arguing by contradiction and using again Assumption 1. The second step consists in taking two closed and disjoint sets A and B and assuming by contradiction $B \cap K = K \setminus A$, and that distance$(A \cap K, B \cap K)$ is positive. Then the first step gives the contradiction. These two arguments are standard and we omit the details of them.

Last, let $x \in K$ and assume $\hat{t} < T$. Then by definition there exists $s_k \nearrow \hat{t}$ such that $u(s_k)$ converges to x as $k \to \infty$. By continuity, $\det \nabla_x^2 f(s_k, u(s_k))$ tends to $\det \nabla_x^2 f(\hat{t}, x)$, and, moreover, $\nabla_x f(\hat{t}, x) = 0$. If $\det \nabla_x^2 f(\hat{t}, x) \neq 0$, then the Implicit Function Theorem could be applied, a contradiction with the definition of $\hat{t}$. This concludes the proof. $\square$

Starting from $\bar{t} = 0$ and from a suitable point $y_0 \in \mathbb{R}^n$, we may repeatedly apply Lemma 2.2.1 and Lemma 2.1.5 obtaining the result stated in the following proposition.

Proposition 2.2.3. *Suppose that Assumptions 1–4 are satisfied. Let y_0 be such that $\nabla_x f(0, y_0) = 0$ and $\nabla_x^2 f(0, y_0)$ is positive definite. Then there exist a unique (and finite) family of times $0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = T$ and a unique family of functions $u_i\colon [t_{i-1}, t_i[\to \mathbb{R}^n$ of class C^2, for $i = 1, \ldots, k$, and a unique (up to time-translations) family of functions $v_i\colon \mathbb{R} \to \mathbb{R}^n$ of class C^2, $i = 1, \ldots, k-1$, such that*

(1) $u_1(0) = y_0$,

(2) for every $t \in [t_{i-1}, t_i[$, $\nabla_x f(t, u_i(t)) = 0$ and $\nabla_x^2 f(t, u_i(t))$ is positive definite,

(3) for every $i = 1, \ldots, k$, there exists $x_i := \lim_{s \to t_i^-} u_i(t)$, while for every $i = 1, \ldots, k-1$, $(t_i, x_i) \in Z_f$, $\nabla_x^2 f(t_i, x_i)$ is positive semidefinite and conditions (b) and (c) of Assumption 3 have the same sign,

(4) for every $i = 1, \ldots, k-1$, function $v_i(s)$ solves

$$\dot{v}_i(s) = -\nabla_x f(t_i, v_i(s)) \tag{2.2.3}$$

and satisfies

$$\lim_{s\to-\infty} v_i(s) = \lim_{t\to t_i^-} u_i(t) \qquad\qquad \lim_{s\to+\infty} v_i(s) = u_{i+1}(t_i). \tag{2.2.4}$$

Proof. We apply Lemma 2.2.1 with $(\bar t, \bar x) = (0, y_0)$ obtaining the existence of $\hat t =: t_1$, and of a function $u_1 \colon [0, t_1[\to \mathbb{R}^n$ of class C^2 such that $u_1(0) = y_0$, $\nabla_x f(t, u_1(t)) = 0$ on $[0, t_1[$, and $\nabla_x^2 f(t, u_1(t))$ is positive definite on $[0, t_1[$. This proves conditions (1) and (2) restricted to $[0, t_1[$.

Arguing as in the proof of Lemma 2.2.1 (using Lemma 2.2.2), we deduce that there exists $x_1 := \lim_{t\to t_1^-} u_1(t)$ and all eigenvalues of $\nabla_x^2 f(t_1, x_1)$ are nonnegative. Moreover, since for every $t < t_1$ the function $u_1(t)$ solves the problem $\nabla_x f(t, x) = 0$, then it follows from Remark 2.1.2 that the transversality conditions (b) and (c) of Assumption 3 have the same sign. Indeed, if on the contrary they had the opposite sign, then there should be no solutions for the problem $\nabla_x f(t, x) = 0$ for t belonging to a left neighborhood of t_1. Thus condition (3) (restricted to $[0, t_1[$) is satisfied.

By Lemma 2.1.5 there exists a unique (up to time-translations) heteroclinic solution v_1 issuing from x_1. In addition, as $s \to +\infty$, $v_1(s)$ tends to a critical point y_1 for $f(t_1, \cdot)$, and by Assumption 4, $\nabla_x^2 f(t_1, y_1)$ is positive definite, so that condition (4) for $i = 1$ is satisfied.

Finally, if $t_1 < T$, we apply Lemma 2.2.1 with $(\bar t, \bar x) = (t_1, y_1)$ and repeat the previous arguments. □

Definition 2.2.4. Suppose that Assumptions 1–4 are satisfied. For fixed $y_0 \in \mathbb{R}^n$ such that $\nabla_x f(0, y_0) = 0$ and $\nabla_x^2 f(0, y_0)$ is positive definite, let u_i, $i = 1, \ldots, k$ be the functions obtained in Proposition 2.2.3. We thus define the C^2-piecewise function $u \colon [0, T] \to \mathbb{R}^n$, such that $u(0) = y_0$, by

$$u\big|_{[t_{i-1}, t_i)} := u_i, \qquad \text{for every } i = 1, \ldots, k\,.$$

Hence, u is discontinuous at $t_1 < \cdots < t_{k-1}$ and satisfies

$$\nabla_x f(t, u(t)) = 0 \quad \text{for every } t \in [t_{i-1}, t_i), \quad u(t_{i-1}) = y_{i-1} \quad \text{and} \quad \lim_{t\to t_i^-} u(t) = x_i\,, \tag{2.2.5}$$

for every $i = 1, \ldots, k$ (cfr. Figure 2.2).

At the points (t_i, x_i), by Assumption 3, the Hessian matrix $\nabla_x^2 f(t_i, x_i)$ has one zero eigenvalue while, by construction, the remaining $n-1$ eigenvalues are positive, for $i = 1, \ldots, k-1$. By Remark 2.1.2 there exist $r_i > 0$ and $R_i > 0$ such that the following conditions hold true (see also Figure 2.3).

(C_l) There are two regular branches of solutions of $\nabla_x f(t, x) = 0$ for $t \in [t_i - r_i, t_i]$, $i = 1, \ldots, k-1$. Moreover, if $\ker \nabla_x f(t_i, x_i) = \mathrm{span}(\ell_i)$, then the two branches have common (vertical) tangent $(0, \ell_i)$ at (t_i, x_i), $i = 1, \ldots, k-1$;

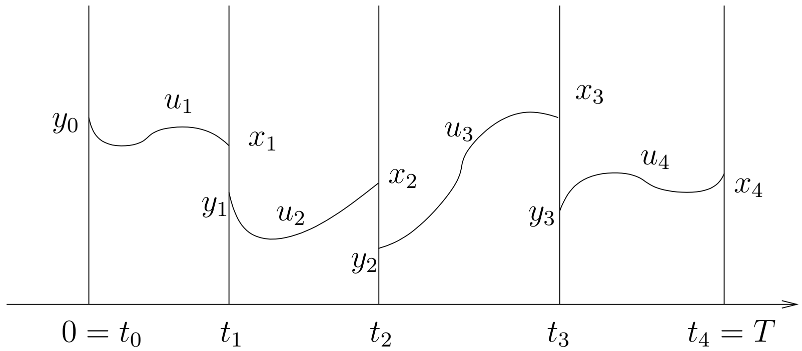

Figure 2.2: *The C^2-piecewise function u, expressed in terms of the functions u_i defined in Proposition 2.2.3, when $k = 4$.*

(C_r) We have

$$|\nabla_x f(t,x)| > 0, \qquad \text{on }]t_i, t_i + r_i] \times \overline{B}(x_i, R_i) \tag{2.2.6}$$

for every $i = 1, \ldots, k-1$.

By Definition 2.2.4 and condition (C_l) one of these two regular branches of solutions has graph contained in $\{(t, u(t)) | t \in [0,T] \setminus \{t_1, \ldots, t_k\}\}$. Throughout the chapter, the other one will be called $\bar{u}(t)$. We notice that u and $\bar{u}$ have the same limit as $t \to t_i^-$.

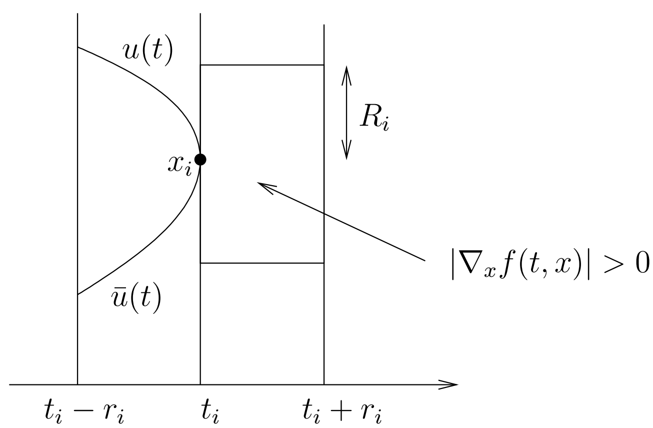

Figure 2.3: *The local structure of Γ_f near t_i.*

In the second part of this section we study some properties of the following ε-gradient system

$$\varepsilon \dot{u}_\varepsilon(t) = -\nabla_x f(t, u_\varepsilon(t)). \tag{2.2.7}$$

We start by proving the existence of global solutions to Cauchy problems associated to (2.2.7). By *global* we mean here a solution defined on the whole interval $[0,T]$.

Lemma 2.2.5. *Under Assumption 1, for any $x \in \mathbb{R}^n$ there exists a unique solution $t \mapsto u_\varepsilon(t)$ to equation (2.2.7), defined on the whole interval $[0,T]$, with the initial condition $u_\varepsilon(0) = x$. Moreover, $u_\varepsilon(t)$ is bounded uniformly with respect to t and ε.*

Proof. Since, by assumption, the function f is regular, it follows from standard arguments on ordinary differential equations that for every ε, the Cauchy problem associated to (2.2.7) has locally a unique solution $t \mapsto u_\varepsilon(t)$. Moreover, multiplying (2.2.7) by u_ε and using Assumption 1 we get

$$\frac{d}{dt}|u_\varepsilon(t)|^2 \leq 2\frac{a_0}{\varepsilon} - 2\frac{c_0}{\varepsilon}|u_\varepsilon(t)|^2\,,$$

which in particular implies that for every ε the solution u_ε is defined on $[0,T]$.

By a standard comparison argument it follows that

$$|u_\varepsilon(t)|^2 \leq \frac{a_0}{c_0} + e^{-2\frac{c_0}{\varepsilon}t}\big(|x|^2 - \frac{a_0}{c_0}\big) \leq \max\left\{\frac{a_0}{c_0}, |x|^2\right\},$$

which gives the uniform boundedness of u_ε with respect to t and ε. This concludes the proof. ☐

In the next proposition we deduce another important fact for the sequence $(u_\varepsilon)_\varepsilon$.

Proposition 2.2.6. *Suppose that Assumption 1 is satisfied. For every ε, let u_ε be the solution to a Cauchy problem associated to (2.2.7). Then*

$$\varepsilon \dot{u}_\varepsilon \to 0 \quad \textit{strongly in } L^2([0,T]),$$

as ε goes to zero.

Proof. Let us first notice that

$$-\nabla_x f(t,u_\varepsilon)\dot{u}_\varepsilon = -\frac{d}{dt}\,f(t,u_\varepsilon) + f_t(t,u_\varepsilon). \tag{2.2.8}$$

Multiplying equation (2.2.7) by $\dot{u}_\varepsilon$, integrating between 0 and T, and taking into account (2.2.8), we get

$$\varepsilon\int_0^T |\dot{u}_\varepsilon(t)|^2\,dt = f(0,u_\varepsilon(0)) - f(T,u_\varepsilon(T)) + \int_0^T f_t(t,u_\varepsilon(t))\,dt.$$

The conclusion follows now from the fact that the right-hand side is bounded uniformly with respect to ε. ☐

Now we are in a position to state the main result of this chapter.

Theorem 2.2.7. *Under Assumptions 1–4, let* $y_0 \in \mathbb{R}^n$ *be such that* $\nabla_x f(0, y_0) = 0$ *and* $\nabla_x^2 f(0, y_0)$ *is positive definite. Let* $u\colon [0,T] \to \mathbb{R}^n$ *be the* C^2*-piecewise function given by Definition 2.2.4 with* $u(0) = y_0$*, and let* $u_\varepsilon\colon [0,T] \to \mathbb{R}^n$ *be the solution of (2.2.7) starting from* $u_\varepsilon(0) =: y_\varepsilon \in \mathbb{R}^n$*. If* $y_\varepsilon \to y_0$*, then*

$$u_\varepsilon \to u \quad \text{uniformly on compact sets of } [0,T] \setminus \{t_1, \ldots, t_k\}. \tag{2.2.9}$$

Moreover, for every $i = 1, \ldots, k-1$ *let* v_i *be the heteroclinic solution of (2.2.3)–(2.2.4). Then there exists* t_ε^i *such that* $t_\varepsilon^i \to t_i$ *as* $\varepsilon \to 0$ *and*

$$v_\varepsilon^i(s) := u_\varepsilon(t_\varepsilon^i + \varepsilon s) \to v_i(s) \quad \text{uniformly on compact sets of } \mathbb{R}. \tag{2.2.10}$$

Finally, if $\gamma_i := v_i(\mathbb{R}) \cup \{x_i, y_i\}$ *represents the trajectory of* v_i *and*

$$G := \operatorname{graph}(u) \cup \bigcup_{i=1}^{k} (\{t_i\} \times \gamma_i), \tag{2.2.11}$$

then

$$\operatorname{dist}((t, u_\varepsilon(t)), G) \to 0 \quad \text{as } \varepsilon \to 0\,, \tag{2.2.12}$$

uniformly for $t \in [0,T]$*.*

Remark 2.2.8. In previous theorem the following three facts are established. First, that out of some small neighborhoods of the critical times t_i, the distance between the "perturbed" solution $u_\varepsilon(t)$ and the limit function $u(t)$ is small uniformly with respect to t. Next, that in a small neighborhood of t_i, the solution u_ε belongs to a tubular neighborhood of the trajectory of the heteroclinic solution v_i.

Notice that these two facts together imply that the graph of u_ε approaches the completion of the graph of u obtained by using the heteroclinic trajectories, defined in (2.2.11).

The third fact is that near the critical times t_i, a suitable rescaled version of u_ε converges to the heteroclinic solution v_i.

2.3 Proof of the main result

The proof of Theorem 2.2.7 follows from some intermediate lemmas which we are going to prove. For simplicity, we focus on the first subinterval $[0, t_1]$ and we start by showing in the next lemma that (2.2.9) holds true.

Lemma 2.3.1. *Under the assumptions of Theorem 2.2.7, if* $u_\varepsilon(0) \to u(0)$*, then* $u_\varepsilon \to u$ *uniformly on compact subsets of* $[0, t_1)$*.*

Proof. For every $0 \le \tau < t_1$, by construction of the function u, there exists $\alpha = \alpha(\tau)$ such that

$$\nabla_x^2 f(t, u(t)) y \cdot y \ge 2\alpha |y|^2$$

for every $y \in \mathbb{R}^n$ and every $0 \le t \le \tau$. (Indeed it is sufficient to take $\alpha(\tau)$ be equal to one half of the smallest (positive) eigenvalue of $\nabla_x^2 f(t, u(t))$ for $t \in [0, \tau]$).

By uniform continuity, there exists $\delta_0 > 0$ such that

$$\nabla_x^2 f(t,x) y \cdot y \ge \alpha |y|^2 \tag{2.3.1}$$

for every $y \in \mathbb{R}^n$ and every $t \in [0, \tau]$, provided that $|u(t) - x| < \delta_0$.

Since $u_\varepsilon(0)$ converges to $u(0)$ as $\varepsilon \to 0$, then there exists $\varepsilon_0 > 0$ such that $|u_\varepsilon(0) - u(0)| < \delta_0$, for every $\varepsilon < \varepsilon_0$. Let t_1^* be the largest time such that $|u_\varepsilon(t) - u(t)| < \delta_0$ for every $t \in [0, t_1^*)$, i.e.,

$$t_1^* := \sup\{t \in [0, \tau) : |u_\varepsilon(t) - u(t)| < \delta_0\}\,.$$

For every $t \in [0, t_1^*)$ and every $\varepsilon < \varepsilon_0$, subtracting $\varepsilon \dot u(t)$ to the ε-gradient system (2.2.7), we deduce that

$$\varepsilon(\dot u_\varepsilon(t) - \dot u(t)) = -\nabla_x f(t, u_\varepsilon(t)) + \nabla_x f(t, u(t)) - \varepsilon \dot u(t).$$

Let us multiply previous equation by $w_\varepsilon(t) := u_\varepsilon(t) - u(t)$. By the Mean Value Theorem, and using (2.3.1) and the Cauchy inequality, we obtain

$$\frac{\varepsilon}{2}\frac{d}{dt}|w_\varepsilon(t)|^2 \le -\alpha |w_\varepsilon(t)|^2 + \frac{\varepsilon}{2}\beta + \frac{\varepsilon}{2}|w_\varepsilon(t)|^2\,,$$

where β is an upper bound for $|\dot u(t)|^2$. Using differential inequalities, we deduce that

$$|w_\varepsilon(t)|^2 \le \Big(|w_\varepsilon(0)|^2 - \varepsilon\frac{\beta}{2\alpha - \varepsilon}\Big)e^{-(2\frac{\alpha}{\varepsilon}-1)t} + \varepsilon\frac{\beta}{2\alpha - \varepsilon}\,. \tag{2.3.2}$$

It follows from (2.3.2) that for ε small enough $|u_\varepsilon(t_1^*) - u(t_1^*)| < \delta_0$, which, by the definition of t_1^*, implies that $t_1^* = \tau$. Moreover, since by assumption $w_\varepsilon(0) \to 0$, we deduce from (2.3.2) that $w_\varepsilon(t) \to 0$ uniformly on $[0, \tau]$ as $\varepsilon \to 0$, and this concludes the proof. $\square$

In order to prove condition (2.2.10) in Theorem 2.2.7, we zoom in on a neighborhood of t_1 and discuss what happens. Let x_1 be defined by condition (3) in Proposition 2.2.3, and let $\Lambda := \min\{|x_1 - y| : \nabla_x f(t_1, y) = 0\,, y \ne x_1\}$ be the minimal distance between x_1 and the other critical points of $f(t_1, \cdot)$. Let $0 < \delta_1 < \min\{\Lambda, R_1\}$, where R_1 is the constant such that inequality (2.2.6) is satisfied for every $t \in\,]t_1, t_1 + r_1]$, and $|x - x_1| < R_1$.

By continuity, since $u(t)$ tends to x_1 as $t \to t_1^-$, there exists $\bar t < t_1$ such that

$$|u(t) - x_1| < \frac{\delta_1}{4} \quad \forall\, t \in (\bar t, t_1)\,. \tag{2.3.3}$$

Consider now an increasing sequence (τ_h) approaching t_1 to the left, with $\tau_1 > \bar t$. Since, for every h, Lemma 2.3.1 implies that $|u_\varepsilon(t) - u(t)| \to 0$ uniformly on $[0, \tau_h]$, we deduce that there exists $\varepsilon_h > 0$ such that

$$|u_\varepsilon(t) - u(t)| < \frac{\delta_1}{4} \quad \forall\, t \in [0, \tau_h],\ \forall\, 0 < \varepsilon < \varepsilon_h. \tag{2.3.4}$$

Let us define

$$t_\varepsilon^{\delta_1} := \inf\{t \geq \tau_1 : |u_\varepsilon(t) - x_1| \geq \delta_1\}, \tag{2.3.5}$$

i.e., $t_\varepsilon^{\delta_1}$ is the first time larger than τ_1 such that $|u_\varepsilon(t) - x_1| = \delta_1$.

Lemma 2.3.2. *Let $t_\varepsilon^{\delta_1}$ be defined by (2.3.5). Then*

$$t_\varepsilon^{\delta_1} \to t_1 \qquad \text{as } \varepsilon \to 0. \tag{2.3.6}$$

Proof. We begin by proving that

$$\liminf_{\varepsilon\to 0} t_\varepsilon^{\delta_1} \geq t_1. \tag{2.3.7}$$

Let $\bar{t} < t_1$ be such that (2.3.3) is satisfied. Since by definition $\tau_1 > \bar{t}$, we have that τ_h belongs to $(\bar{t}, t_1)$ for every h. Then for fixed τ_h it follows from (2.3.3), (2.3.4), and triangular inequality, that

$$|u_\varepsilon(t) - x_1| < \frac{\delta_1}{2} \quad \forall\, t \in (\bar{t}, \tau_h), \quad \text{and every } \varepsilon < \varepsilon_h.$$

Hence, $t_\varepsilon^{\delta_1} > \tau_h$, for every $0 < \varepsilon < \varepsilon_h$. Thus, $\liminf_{\varepsilon\to 0} t_\varepsilon^{\delta_1} \geq \tau_h$ for every h, which implies (2.3.7).

On the other hand, by Proposition 2.2.6 and by (2.2.7), for a.e. $t^* \in [0, T]$ we have that $|\nabla_x f(t^*, u_\varepsilon(t^*))|$ tends to 0 as $\varepsilon \to 0$ along a suitable sequence. In particular, this is true for a.e. t^* in a right-neighborhood of t_1. Condition (2.2.6) implies now that $|u_\varepsilon(t^*) - x_1| > R_1$ for ε sufficiently small. Let us take $\eta > 0$ and choose $t^* \in \,]t_1, t_1 + \eta[$. Since $R_1 > \delta_1$, from the definition of $t_\varepsilon^{\delta_1}$ and the regularity of u_ε, we deduce immediately that $t_\varepsilon^{\delta_1} < t^*$ for ε sufficiently small. This concludes the proof, since the result does not depend on the subsequence of ε chosen. $\square$

Let us observe now that for $s \in [-t_\varepsilon^{\delta_1}/\varepsilon, (T - t_\varepsilon^{\delta_1})/\varepsilon]$, function $v_1^\varepsilon(s) := u_\varepsilon(t_\varepsilon^{\delta_1} + \varepsilon s)$ solves the following problem

$$\begin{cases} \dot{v}_1^\varepsilon(s) = -\nabla_x f(t_\varepsilon^{\delta_1} + \varepsilon s, v_1^\varepsilon(s)) \\ v_1^\varepsilon(0) = u_\varepsilon(t_\varepsilon^{\delta_1}). \end{cases} \tag{2.3.8}$$

Moreover, since $u_\varepsilon(t_\varepsilon^{\delta_1})$ belongs to the compact set $\partial B(x_1, \delta_1)$, there exists $\kappa_1 \in \partial B(x_1, \delta_1)$ such that, passing to a subsequence, $u_\varepsilon(t_\varepsilon^{\delta_1}) \to \kappa_1$ as $\varepsilon \to 0$. Therefore, Lemma 2.3.2 and the Continuous Dependence Theorem imply that v_1^ε converges uniformly on compact sets of $\mathbb{R}$ to the solution $w(s)$ of the following problem:

$$\begin{cases} \dot{w}(s) = -\nabla_x f(t_1, w(s)) \\ w(0) = \kappa_1. \end{cases} \tag{2.3.9}$$

The next step consists in proving that w is precisely (up to time-translations) the heteroclinic solution v_1, defined in Proposition 2.2.3. To this aim we introduce a sequence $\delta_k \searrow 0$, where δ_1 is the constant already introduced (after the proof of Lemma 2.3.1), and define, for $k > 1$,

$$t_\varepsilon^{\delta_k} := \sup\{t \le t_\varepsilon^{\delta_1} : |u_\varepsilon(t) - x_1| \le \delta_k\}, \tag{2.3.10}$$

i.e., $t_\varepsilon^{\delta_k}$ is the last time before $t_\varepsilon^{\delta_1}$ such that $|u_\varepsilon(t) - x_1| = \delta_k$.

Lemma 2.3.3. *For $s \in [-t_\varepsilon^{\delta_k}/\varepsilon, (T - t_\varepsilon^{\delta_k})/\varepsilon]$, let $v_k^\varepsilon(s) := u_\varepsilon(t_\varepsilon^{\delta_k} + \varepsilon s)$, and let $t_\varepsilon^{\delta_1} = t_\varepsilon^{\delta_k} + \varepsilon S_\varepsilon^{1,k}$, for some $S_\varepsilon^{1,k} > 0$. Then $S_\varepsilon^{1,k} \to s_k < +\infty$ as $\varepsilon \to 0$ along a suitable sequence, and, for every k,*

$$v_k^\varepsilon(s) \to w(s - s_k) \qquad \textit{uniformly on compact subsets of } \mathbb{R}\,. \tag{2.3.11}$$

Moreover $s_k \to +\infty$ as $k \to \infty$. Finally, $\lim_{s\to-\infty} w(s) = x_1$.

Proof. We begin by observing that $t_\varepsilon^{\delta_k}$ converges to t_1 as $\varepsilon \to 0$. Indeed, arguing in the same manner as for $t_\varepsilon^{\delta_1}$ in the proof of Lemma 2.3.2, we get that $\liminf_{\varepsilon\to 0} t_\varepsilon^{\delta_k} \ge t_1$, while, since $t_\varepsilon^{\delta_k} \le t_\varepsilon^{\delta_1}$, and $t_\varepsilon^{\delta_1} \to t_1$, we deduce that $\limsup_{\varepsilon\to 0} t_\varepsilon^{\delta_k} \le t_1$.

Near (t_1, x_1), using the local structure of the set Γ_f (defined in (2.1.2)), given by (C_l) and (C_r), we can prove that for every k there exists $\eta_k > 0$ such that

$$\Gamma_f \cap \big([t_1 - \eta_k, t_1 + \eta_k] \times \overline{B}(x_1, R_1)\big) \subset [t_1 - \eta_k, t_1] \times \overline{B}\big(x_1, \frac{\delta_k}{2}\big)\,, \tag{2.3.12}$$

where R_1 is the constant such that (2.2.6) is satisfied for every $t \in \,]t_1, t_1 + r_1]$ and $|x - x_1| < R_1$. Next, we notice that for fixed k, for ε sufficiently small and for $t \in [t_\varepsilon^{\delta_k}, t_\varepsilon^{\delta_1}]$, we have

$$(t, u_\varepsilon(t)) \in \mathcal{S}_k := [t_1 - \eta_k, t_1 + \eta_k] \times \{x \in \mathbb{R}^n : \delta_k \le |x - x_1| \le \delta_1\}\,.$$

Moreover, we observe that $\mathcal{S}_k$ is closed and since, by (2.3.12), $(t, x) \in \mathcal{S}_k$ is quite distant from both $(t, u(t)$ and $(t, \overline{u}(t))$ (the two regular branches of Γ_f near (t_1, x_1)), we have $\mathcal{S}_k \cap \Gamma_f = \emptyset$, so that there exists a positive constant c_k such that $|\nabla_x f(t, x)| \ge c_k > 0$ for every $(t, x) \in \mathcal{S}_k$. By the fact that $(t, u_\varepsilon(t)) \in \mathcal{S}_k$ for $t \in [t_\varepsilon^{\delta_k}, t_\varepsilon^{\delta_1}]$, it follows

$$|\nabla_x f(t, u_\varepsilon(t))| \ge c_k > 0 \qquad \text{for every } t_\varepsilon^{\delta_k} \le t \le t_\varepsilon^{\delta_1}\,, \tag{2.3.13}$$

and for ε small. (See also Figure 2.4)

After these preliminaries, we prove now that $S_\varepsilon^{1,k} \to s_k < +\infty$, as $\varepsilon \to 0$ along a suitable sequence. Indeed, for $s \in [0, 1]$ let us define $w_\varepsilon(s) := u_\varepsilon(s(t_\varepsilon^{\delta_1} - t_\varepsilon^{\delta_k}) + t_\varepsilon^{\delta_k})$. Hence, w_ε is the solution of the following problem:

$$\begin{cases} \dfrac{1}{S_\varepsilon^{1,k}}\, \dot{w}_\varepsilon(s) = -\nabla_x f(s(t_\varepsilon^{\delta_1} - t_\varepsilon^{\delta_k}) + t_\varepsilon^{\delta_k}, w_\varepsilon(s)) \\ w_\varepsilon(0) = u_\varepsilon(t_\varepsilon^{\delta_k}). \end{cases} \tag{2.3.14}$$

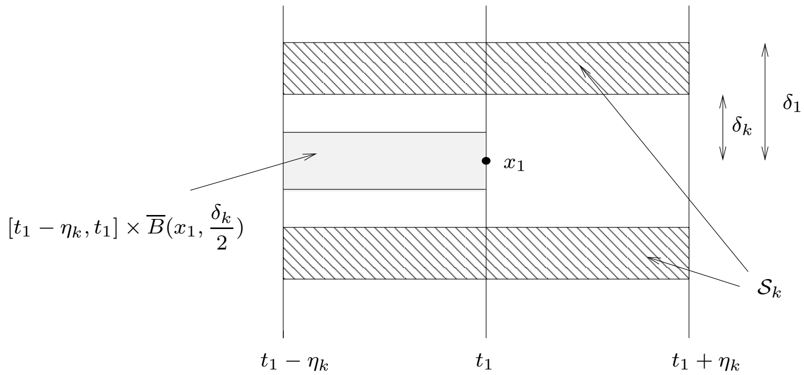

Figure 2.4: *The sets* $\mathcal{S}_k$ *and* $[t_1-\eta_k,t_1]\times\overline{B}(x_1,\frac{\delta_k}{2})$ *are disjoint.*

Multiplying by $\dot w_\varepsilon$ and integrating between 0 and 1 we get:

$$\frac{1}{S_\varepsilon^{1,k}}\int_0^1 |\dot w_\varepsilon(s)|^2\,ds = f(t_\varepsilon^{\delta_k},u_\varepsilon(t_\varepsilon^{\delta_k})) - f(t_\varepsilon^{\delta_1},u_\varepsilon(t_\varepsilon^{\delta_1}))+ \\ + (t_\varepsilon^{\delta_1}-t_\varepsilon^{\delta_k})\int_0^1 f_t(s(t_\varepsilon^{\delta_1}-t_\varepsilon^{\delta_k})+t_\varepsilon^{\delta_k},w_\varepsilon(s))\,ds,$$

where the right-hand side is bounded uniformly with respect to ε. If, by contradiction, $S_\varepsilon^{1,k}\to\infty$ as $\varepsilon\to 0$ along a suitable sequence, then $(S_\varepsilon^{1,k})^{-1}\dot w_\varepsilon\to 0$ strongly in $L^2(0,1)$, which in particular implies that

$$\frac{1}{S_\varepsilon^{1,k}}\dot w_\varepsilon(s)\to 0 \quad \text{ for a.e. } s\in[0,1]. \tag{2.3.15}$$

On the other hand, by (2.3.13) and the definition of w_ε, for ε sufficiently small, we obtain that

$$|\nabla_x f(s(t_\varepsilon^{\delta_1}-t_\varepsilon^{\delta_k})+t_\varepsilon^{\delta_k},w_\varepsilon(s))|\geq c_k>0\,,$$

for every $s\in[0,1]$, which contradicts (2.3.14) and (2.3.15).

Now we continue as for δ_1, and define $v_k^\varepsilon(s):=u_\varepsilon(t_\varepsilon^{\delta_k}+\varepsilon s)$, for $s\in[-t_\varepsilon^{\delta_k}/\varepsilon, T-t_\varepsilon^{\delta_k}/\varepsilon]$. It turns out that v_k^ε solves

$$\begin{cases}\dot v_k^\varepsilon(s)=-\nabla_x f(t_\varepsilon^{\delta_k}+\varepsilon s,v_k^\varepsilon(s)),\\ v_k^\varepsilon(0)=u_\varepsilon(t_\varepsilon^{\delta_k}).\end{cases}$$

Since $u_\varepsilon(t_\varepsilon^{\delta_k})$ belongs to the compact set $\partial B(x_1,\delta_k)$, we deduce that there exists $\kappa_k\in\partial B(x_1,\delta_k)$ such that $u_\varepsilon(t_\varepsilon^{\delta_k})\to\kappa_k$ as $\varepsilon\to 0$ along a suitable sequence. It follows that, if we define w_k as the solution of

$$\begin{cases}\dot w_k(s)=-\nabla_x f(t_1,w_k(s))\\ w_k(0)=\kappa_k,\end{cases}$$

then $v_k^\varepsilon(s) \to w_k(s)$ uniformly on compact subsets of $\mathbb{R}$, by the Continuous Dependence Theorem. Moreover, the equality $v_k^\varepsilon(S_\varepsilon^{1,k}) = u_\varepsilon(t_\varepsilon^{\delta_1})$ implies $w_k(s_k) = \kappa_1$. By the uniqueness of the solution of the Cauchy problem (2.3.9) we obtain $w_k(s) = w(s - s_k)$ and $w(-s_k) = \kappa_k$. It follows that $w(-s_k) \to x_1$ as $k \to +\infty$.

Let now s_∞ be such that $s_k \to s_\infty$, and assume by contradiction that $s_\infty < +\infty$. Then, by continuity, $w(s_\infty) = x_1$ and, since x_1 is an equilibrium point, from the uniqueness it should follow $w(s) \equiv x_1$, a contradiction.

It remains to prove that $\lim_{s\to-\infty} w(s) = x_1$. Indeed this follows from some standard facts on the α-limit set (see, e.g., [30], [1]).

More precisely, let $g(x) := f(t_1, x)$ for every $x \in \mathbb{R}^n$, and let E be the set of critical points of g, i.e., $E = \{x : \nabla g(x) = 0\}$. Let us denote the α-limit set of w by $\alpha(w)$. Then $x_1 \in \alpha(w)$, and we can prove that for every $y \in \alpha(w)$ we have $g(y) = g(x_1)$. Indeed, for $y \in \alpha(w)$ there exists a sequence $\hat{s}_k$ such that $w(-\hat{s}_k) \to y$ and $\hat{s}_k \to +\infty$. Moreover, the sequence $g(w(-\hat{s}_k))$ converges to $g(y)$ as $k \to \infty$. Since the map $s \mapsto g(w(s))$ is nonincreasing, there exists $a \in \mathbb{R} \cup \{+\infty\}$ such that $g(w(s)) \to a$, as $s \to -\infty$. But

$$\lim_{s\to-\infty} g(w(s)) = \lim_{k\to\infty} g(w(-s_k)) = \lim_{k\to\infty} g(\kappa_k) = g(x_1)\,,$$

so that $a = g(x_1) \in \mathbb{R}$ and $a = g(y)$, for every $y \in \alpha(w)$. It follows that $g(w(s)) \leq a$ for every $s \leq 0$. By the coerciveness of the function f (2.1.1), we deduce that the negative semiorbit is precompact. Then $\alpha(w)$ is connected, compact, and contained in E.

Since by assumption the points of E are isolated, we have $\alpha(w) = \{x_1\}$, therefore $\lim_{s\to-\infty} w(s) = x_1$. ☐

By Lemma 2.3.3 there exists a subsequence $\varepsilon_k \to 0$ such that

$$u_{\varepsilon_k}(t_{\varepsilon_k}^{\delta_1} + \varepsilon_k s) \to w(s)\,.$$

Let us choose now v_1 satisfying (2.2.3) and (2.2.4) of Proposition 2.2.3. Then there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\{s \in \mathbb{R} : v_1(s) \in \partial B(x_1, \delta_1)\} \subset [\alpha, \beta]\,. \tag{2.3.16}$$

From the fact that $w(0) = \kappa_1 \in \partial B(x_1, \delta_1)$ and by uniqueness, there exists unique $c \in \mathbb{R}$ (defined by $v_1(c) = w(0) = \kappa_1$) such that

$$w(s) = v_1(s + c)\,. \tag{2.3.17}$$

In order to prove that our main result does not depend on suitable subsequences of ε, we will use the following lemma (given in an abstract setting).

Lemma 2.3.4. *Let $\mathcal{C}$ be a nonempty set, let $(\mathcal{F}, d)$ be a metric space, and let us consider a function $G\colon]0, \varepsilon_0[\times \mathcal{C} \to \mathcal{F}$. Assume that there exists $g_0 \in \mathcal{F}$ such that*

$$\forall \varepsilon_k \to 0\ \exists \varepsilon_{k_j}, \exists c \in \mathcal{C} : G(\varepsilon_{k_j}, c) \to g_0\,. \tag{2.3.18}$$

Then for every $\varepsilon \in]0, \varepsilon_0[$ there exists $c_\varepsilon \in \mathcal{C}$ such that

$$G(\varepsilon, c_\varepsilon) \to g_0 \, .$$

Proof. It is sufficient to prove that

$$\inf_{c \in \mathcal{C}} d(G(\varepsilon, c), g_0) \to 0 \, . \tag{2.3.19}$$

Assume by contradiction that (2.3.19) is not true. Then, there exists $\eta > 0$ such that for every $k \in \mathbb{N}$ there exists $\varepsilon_k \in]0, \frac{1}{k}[$ with

$$\inf_{c \in \mathcal{C}} d(G(\varepsilon_k, c), g_0) > \eta \, . \tag{2.3.20}$$

But this contradicts the assumption (2.3.18), since for every subsequence ε_{k_j} of ε_k and for every $c \in \mathcal{C}$ we should get $d(G(\varepsilon_{k_j}, c), g_0) > \eta$. □

Now we are in a position to prove the main result of this chapter.

Proof of Theorem 2.2.7. Let us first concentrate on the time interval $[0, t_1]$.

Then Lemma 2.3.1 implies that condition (2.2.9) restricted to $[0, t_1]$ is satisfied.

Let us prove now condition (2.2.12). For fixed $\eta \in]0, \frac{t_1}{2}[$ the goal is to prove that there exists $\varepsilon_0 > 0$ such that

$$\operatorname{dist}((t, u_\varepsilon(t)), G) < \eta \qquad \text{for every } \varepsilon < \varepsilon_0 \, , \tag{2.3.21}$$

uniformly with respect to $t \in [0, t_1]$. Indeed, let $\tau := t_1 - \frac{\eta}{2}$. Let us take $\delta_1 < \eta$ and define $t_\varepsilon^{\delta_1}$ as in (2.3.5). We consider now ε belonging to a suitable sequence tending to zero such that κ_1 is defined (as the limit of $u_\varepsilon(t_\varepsilon^{\delta_1})$). Then by Lemma 2.3.1 there exists $\varepsilon_1 > 0$ such that (2.3.21) is satisfied for every $t \in [0, \tau]$ and every $\varepsilon < \varepsilon_1$. Moreover, from the definition of $t_\varepsilon^{\delta_1}$ we deduce that $|u_\varepsilon(t) - x_1| \leq \delta_1 < \eta$, for every $t \in]\tau, t_\varepsilon^{\delta_1}]$. Hence (2.3.21) is satisfied for every $t \in [0, t_\varepsilon^{\delta_1}]$ and every $\varepsilon < \varepsilon_1$.

On the other hand, by Lemma 2.3.3 and by (2.3.17), there exists $\varepsilon_2 > 0$ such that

$$|u_\varepsilon(t_\varepsilon^{\delta_1} + \varepsilon s) - v_1(s + c)| < \frac{\eta}{2} \qquad \text{for every } 0 \leq s \leq S_\eta \text{ and } \varepsilon < \varepsilon_2 \, ,$$

where $S_\eta \in \mathbb{R}$ is such that

$$|v_1(s) - y_1| < \frac{\eta}{2} \qquad \text{for every } s \geq S_\eta \, .$$

Let $\tau_\varepsilon^1 := t_\varepsilon^{\delta_1} + \varepsilon S_\eta - \varepsilon c$ and observe that it is not restrictive to assume that $\tau_\varepsilon^1 > t_\varepsilon^{\delta_1}$. Hence, for $s = S_\eta$ and ε small enough,

$$|u_\varepsilon(\tau_\varepsilon^1) - v_1(S_\eta)| < \frac{\eta}{2} \, .$$

We have thus obtained that

$$\operatorname{dist}((t,u_\varepsilon(t)),\{t_1\}\times\gamma_1)\le\eta \qquad \text{on } [t_\varepsilon^{\delta_1},\tau_\varepsilon^1]\,,$$

recalling that γ_1 is the trajectory of v_1. This, together with the fact that $\tau_\varepsilon^1\to t_1$, completes the proof of (2.3.21) and begins the proof of the uniform convergence in the interval $[t_1,t_2]$. After a finite number of steps we obtain (2.2.9) and (2.2.12).

Since (2.2.9) and (2.2.12) do not depend on the particular subsequence chosen, we deduce that the result holds true for the whole sequence ε.

More delicate to prove is condition (2.2.10), since the constant c introduced in (2.3.17) depends on the subsequence ε_k. But we recall that $u_{\varepsilon_k}(t_{\varepsilon_k}^{\delta_1}+\varepsilon_k s)\to w(s)=v_1(s+c)$, i.e.,

$$u_{\varepsilon_k}(t_{\varepsilon_k}^{\delta_1}-\varepsilon_k c+\varepsilon_k s)\to v_1(s)\,.$$

Therefore Lemma 2.3.4 applies with $\mathcal{C}:=[\alpha,\beta]$ (see (2.3.16)) and $\mathcal{F}$ be equal to the set of continuous functions endowed with the distance induced by the uniform convergence on compact sets, and we obtain (2.2.10). $\square$

Chapter 3

An artificial viscosity approach to quasistatic crack growth

In this chapter we study the crack growth in brittle materials when the prescribed crack path Γ is a regular arc with one endpoint on the boundary of the reference configuration Ω and the other inside Ω. We assume in addition that there exists an initial connected crack starting from the boundary point, and that the crack remains connected during the evolution. Hence, such a crack will be completely determined by its length σ.

For this model the evolution is driven by time-dependent imposed boundary displacements $\psi(t)$ on a part $\partial_D\Omega$ of the boundary, and applied boundary forces $g(t)$ on the remaining part $\partial_N\Omega$. The total energy, $\mathscr{E}(t)(u,\sigma)$, of a configuration (u,σ) (where u represents the displacement) at time t, is the sum of the bulk energy and the surface energy, minus the work of the applied forces.

Let us introduce now all the ingredients necessary to define the notion of irreversible quasistatic evolution we are interested in. The set of admissible displacements, i.e., displacements with finite bulk energy, compatible with the imposed boundary displacement $\psi(t)$ and with the crack length σ, will be denoted by $AD(\psi(t),\sigma)$. Since for this model, given t and σ, there exists a unique minimizer $u_{t,\sigma}$ of the energy $\mathscr{E}(t)(u,\sigma)$ in $AD(\psi(t),\sigma)$, we consider the minimal energy, $E(t,\sigma)$, corresponding to the boundary data $\psi(t)$ and to the crack length σ, i.e.,

$$E(t,\sigma) := \mathscr{E}(t)(u_{t,\sigma},\sigma)\,.$$

The derivative $\partial_\sigma E(t,\sigma)$ can be computed (see Proposition 3.2.4) and it is related to the stress intensity factor of the displacement $u_{t,\sigma}$ at the tip of the crack. It plays a crucial rôle in the Griffith's criterion for the propagation of cracks.

Now we are in a position to define the notion of evolution we are interested in. The *irreversible quasistatic evolution problem* consists in finding a left-continuous function of time $t \mapsto (u(t),\sigma(t))$ such that the displacement $u(t)$ at time t belongs to the set $AD(\psi(t),\sigma(t))$, and the following three conditions are satisfied:

(a) *local unilateral stability:* at every time $t \geq 0$

$$\begin{aligned} &\mathscr{E}(t)(u(t), \sigma(t)) \leq \mathscr{E}(t)(v, \sigma(t)) \qquad \forall\, v \in AD(\psi(t), \sigma(t)) \\ &\partial_\sigma E(t, \sigma(t)) \geq 0\,, \end{aligned}$$

(b) *irreversibility:* the map $t \mapsto \sigma(t)$ is increasing;

(c) *energy inequality:* for every $0 \leq s < t$ we have

$$\mathscr{E}(t)(u(t), \sigma(t)) \leq \mathscr{E}(s)(u(s), \sigma(s)) + \mathrm{Work}(u; s, t)\,,$$

where $\mathrm{Work}(u; s, t)$ denotes the work of external forces.

A solution to this problem will be called an *irreversible quasistatic evolution.*

We observe that condition (a) deals only with the first order necessary conditions for the minimality of the pair $(u(t), \sigma(t))$, neglecting completely the sufficient ones. Anyway the globally stable irreversible quasistatic evolutions considered in [18], [9], [3], [17], [8], fit the previous definition, and despite of the new feature, the three items ensure that a weak version of the Griffith's criterion is satisfied (see Proposition 3.3.2).

The aim of this chapter is to obtain an existence result for an evolution possibly different from the globally stable one. More precisely, the selection criterion we adopt is based on an approximation procedure with a regularizing effect. Accordingly, we give in Definition 3.3.6 the notion of approximable irreversible quasistatic evolution $(u(t), \sigma(t))$ defined as limit of solutions $(u_\varepsilon(t), \sigma_\varepsilon(t))$ of regular evolution problems (see (3.3.7) in Definition 3.3.3), after having performed a suitable change of variables in order to study the problem on a fixed Sobolev space (independent of σ), to simplify the mathematical difficulties.

Accordingly, the main result of this chapter, Theorem 3.3.7, states the existence of such an approximable evolution.

In addition, we prove that if $(u(t), \sigma(t))$ is an approximable irreversible quasistatic evolution, then the following property holds.

($\mathcal{P}$) if on a certain time interval $[t_0, t_1]$ there exists a regular function $\sigma^0(t)$ such that

$$\partial_\sigma E(t, \sigma^0(t)) = 0 \quad \text{and} \quad \partial^2_\sigma E(t, \sigma^0(t)) > 0 \qquad \forall\, t \in [t_0, t_1]\,,$$

and if $\dot\sigma_\varepsilon(t) > 0$ for every $t \in [t_0, t_1]$, then the equality $\sigma(t_0) = \sigma^0(t_0)$ implies that $\sigma(t) = \sigma^0(t)$ for every $t \in [t_0, t_1]$.

In this way, when some additional hypotheses are satisfied (like in ($\mathcal{P}$)) our evolution can be characterized on the continuity subintervals by the Implicit Function Theorem, which gives the regular evolution $t \mapsto \sigma^0(t)$ (see Theorem 3.4.1). On the other hand, we note in Remark 3.3.9 that the energy functional decreases when the evolution has a jump. This characterization makes the difference with the globally stable evolution that we expect to move directly to the absolute minimum for the energy functional, while our approximable evolution is expected to propagate continuously at least on every time interval where property $(\mathcal{P})$ holds.

Let us remark that this model is not suited for the study of the crack initiation problem. We also note that the approximating evolutions we consider have been chosen on the basis of their mathematical simplicity and do not seem to have any mechanical interpretation. Nevertheless, we think that the notion of approximable irreversible quasistatic evolution proposed here could be the starting point for the study of different approximations with a mechanical justification. For a different approach to the irreversible quasistatic crack growth see also [19].

The chapter is organized as follows. In Section 3.1 we fix our notations and make some preliminary calculations in order to write the problem on a fixed domain, independent of σ. Section 3.2 collects some (known) results on critical points of the energy, while in Section 3.3 we prove our main result, Theorem 3.3.7. In Section 3.4, property $(\mathcal{P})$ is obtained. In Section 3.5 we detail our results in the case of monotonically increasing in time imposed boundary displacements and compare this evolution with the one proposed by Francfort and Marigo in [18], while in Section 3.6 we provide an example where the energy, as function of the crack length, has at least a concavity interval.

3.1 Setting of the problem

3.1.1 The reference configuration and the crack.

Let Ω be a bounded connected open set of $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. The set $\overline{\Omega}$ represents the reference configuration of an isotropic, homogeneous elastic body. Let $\partial_D\Omega$ be a closed subset of $\partial\Omega$ with $\mathscr{H}^1(\partial_D\Omega) > 0$, where $\mathscr{H}^1$ denotes the one-dimensional Hausdorff measure, and let $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$. On the Dirichlet part of the boundary, $\partial_D\Omega$, we will impose the boundary displacements, while on the Neumann part of the boundary, $\partial_N\Omega$, we will prescribe the boundary forces.

Let Γ be a simple C^3-arc and let $\gamma\colon [0,\overline{\sigma}] \to \Gamma$ be its arc-length parametrization. We assume that $\gamma(0) \in \partial_N\Omega$ and $\gamma(\sigma) \in \Omega$ for $0 < \sigma \leq \overline{\sigma}$. For technical reasons it is convenient to extend Γ until it reaches another point in $\partial_N\Omega$, so that it cuts the reference configuration Ω into two subsets. The extension will still be called Γ, and its arc-length parametrization will now be $\gamma\colon [0,\sigma_{max}] \to \Gamma$. We assume that its intersection with the boundary $\partial\Omega$ is not tangential. Let ν be a unit normal vector field on Γ. Then we denote by Ω^+ the part of $\Omega \setminus \Gamma$ which is positively oriented with respect to ν, and by Ω^- the remaining part, so that $\Omega \setminus \Gamma = \Omega^+ \cup \Omega^-$. Both Ω^+ and Ω^- are bounded connected sets with Lipschitz boundary. We assume that $\mathscr{H}^1(\partial_D\Omega\cap\partial\Omega^+) > 0$ and $\mathscr{H}^1(\partial_D\Omega\cap\partial\Omega^-) > 0$.

We make the following simplifying assumption: all admissible cracks are of the form

$$\Gamma(\sigma) := \{\gamma(s) : 0 \leq s \leq \sigma\} \qquad \text{with } \sigma \leq \overline{\sigma}\,.$$

According to Griffith's theory we assume that the energy spent to produce the crack $\Gamma(\sigma)$ is proportional to the length of the crack, and, for simplicity, we take it to be equal to σ.

3.1.2 The bulk energy.

We consider here the case of antiplane shears. Given a crack $\Gamma(\sigma)$, an admissible displacement is any function $u \in H^1(\Omega\backslash\Gamma(\sigma))$, and the bulk energy associated to the displacement u is

$$\mathcal{W}(Du) := \int_{\Omega\backslash\Gamma(\sigma)} |Du(x)|^2 dx\,,$$

where Du is the distributional gradient of u and $|\cdot|$ denotes the norm in $\mathbb{R}^2$.

3.1.3 The boundary displacement.

In the sequel it will be convenient to work on a fixed time interval $[0,T]$ with $T > 0$. We impose a time-dependent Dirichlet boundary condition on $\partial_D\Omega$:

$$u = \psi(t) \quad \text{ on } \partial_D\Omega,$$

where equality on the boundary is considered in the sense of traces. We assume that $\psi(t)$ is the trace on $\partial_D\Omega$ of a bounded Sobolev function, still denoted by $t \mapsto \psi(t)$, with $\psi(t) \in H^1(\Omega) \cap L^\infty(\Omega)$.

We assume also that $\psi \in W^{1,\infty}(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega))$. Thus, the time derivative $t \mapsto \dot{\psi}(t)$ belongs to the space $L^\infty(0,T;H^1(\Omega))$ and its spatial gradient $t \mapsto D\dot{\psi}(t)$ belongs to the space $L^\infty(0,T;L^2(\Omega;\mathbb{R}^2))$.

3.1.4 The external loads.

We are interested in the case of time-dependent dead loads, in which the density, $g\colon [0,T]\times \partial_N\Omega \to \mathbb{R}$, of the applied surface force per unit area in the reference configuration does not depend on the displacement u. We assume that the function $t \mapsto g(t)(\cdot)$ belongs to $W^{1,\infty}(0,T;L^2(\partial_N\Omega,\mathscr{H}^1))$, with time derivative denoted by $t \mapsto \dot{g}(t)(\cdot)$. The associated potential, for a displacement u, is given by

$$\mathcal{G}(t)(u) := \int_{\partial_N\Omega} g(t)(x)u(x)d\mathscr{H}^1\,.$$

Moreover, we assume that for every $t \in [0,T]$ the support of $g(t)$ does not intersect the set Γ.

3.1.5 The admissible displacements and their total energy.

For every $t \in [0,T]$, the set $AD(\psi(t),\sigma)$ of admissible displacements in Ω with finite energy, corresponding to the crack $\Gamma(\sigma)$ and to the boundary data $\psi(t)$ is given by

$$AD(\psi(t),\sigma) := \{u \in H^1(\Omega \setminus \Gamma(\sigma)) : u = \psi(t) \text{ on } \partial_D\Omega\}\,,$$

where the last equality refers to the traces of u and $\psi(t)$ on $\partial_D\Omega$. The total energy of a configuration (u,σ) with $u \in AD(\psi(t),\sigma)$ is given by

$$\mathscr{E}(t)(u,\sigma) := \mathcal{W}(Du) + \sigma - \mathcal{G}(t)(u)\,.$$

Note that it does not depend on the particular extension $\psi(t)$ chosen, but only on its value on the Dirichlet part of the boundary.

3.1.6 Moving to a fixed domain.

Let $H^1_{\partial_D\Omega}(\Omega \setminus \Gamma(\sigma))$ denote the space of functions $u \in H^1(\Omega \setminus \Gamma(\sigma))$ whose trace on $\partial_D\Omega$ is zero. We may consider the energy as a functional defined on $H^1_{\partial_D\Omega}(\Omega \setminus \Gamma(\sigma))$ by simply writing $\tilde{u} = u + \psi(t)$ with $\tilde{u} \in AD(\psi(t),\sigma)$ and $u \in H^1_{\partial_D\Omega}(\Omega \setminus \Gamma(\sigma))$. Still the domain of the energy functional would depend on σ. To transform it into a functional defined on a fixed domain we consider the following change of variables.

Let $0 < \sigma_0 < \overline{\sigma}$. For $\sigma \in [\sigma_0,\overline{\sigma}]$, let $\Phi(\cdot,\sigma) = \Phi_\sigma(\cdot)\colon \Omega \to \Omega$ be a diffeomorphism which coincides with the identity near the boundary of Ω, leaves invariant both Ω^+ and Ω^- and transforms $\Gamma(\sigma)$ into $\Gamma(\sigma_0)$. Let $\Psi(\cdot,\sigma) = \Psi_\sigma(\cdot) := \Phi^{-1}(\cdot,\sigma)\colon \Omega \to \Omega$. Then letting $x = \Psi_\sigma(y)$ we get

$$\int_{\Omega\setminus\Gamma(\sigma)} |(Du + D\psi(t))(x)|^2 dx = \int_{\Omega\setminus\Gamma(\sigma_0)} |(Du + D\psi(t))(\Psi_\sigma(y))|^2 \det D\Psi_\sigma(y)dy\,.$$

For $u \in H^1_{\partial_D\Omega}(\Omega \setminus \Gamma(\sigma))$ define $v(y,\sigma) := u(\Psi_\sigma(y))$, so that v belongs to the fixed domain $H^1_{\partial_D\Omega}(\Omega \setminus \Gamma(\sigma_0))$, and let $\tilde{\psi}(t)(y,\sigma) := \psi(t)(\Psi_\sigma(y))$. With these notations

$$\int_{\Omega\setminus\Gamma(\sigma)} |(Du + D\psi(t))(x)|^2 dx = \int_{\Omega\setminus\Gamma(\sigma_0)} |((D\Psi_\sigma)^T)^{-1}(y)(Dv + D\tilde{\psi}(t))(y,\sigma)|^2 \det D\Psi_\sigma(y)dy,$$

and the last integral can be written also in the form

$$\int_{\Omega\setminus\Gamma(\sigma_0)} \sum_{i,j\in\{1,2\}} a_{ij}(\sigma)(y) D_j(v + \tilde{\psi}(t))(y,\sigma) D_i(v + \tilde{\psi}(t))(y,\sigma)dy\,,$$

with the coefficients a_{ij} given by the change of variables as follows. From

$$D\Psi_\sigma(x) := \begin{pmatrix} D_1\Psi_1(x) & D_1\Psi_2(x) \\ D_2\Psi_1(x) & D_2\Psi_2(x) \end{pmatrix}$$

and

$$D\Psi_\sigma^{-1}(x) = \frac{1}{\det D\Psi_\sigma(x)} \begin{pmatrix} D_2\Psi_2(x) & -D_1\Psi_2(x) \\ -D_2\Psi_1(x) & D_1\Psi_1(x) \end{pmatrix},$$

we obtain

$$\begin{aligned} |((D\Psi_\sigma)^T)^{-1} D(v+\tilde{\psi}(t))|^2 \det D\Psi_\sigma = \frac{1}{\det D\Psi_\sigma} \big[& (D_1(v+\tilde{\psi}(t)))^2((D_2\Psi_2)^2 + (D_1\Psi_2)^2) - \\ & - 2D_1(v+\tilde{\psi}(t))D_2(v+\tilde{\psi})(D_2\Psi_2 D_2\Psi_1 + D_1\Psi_2 D_1\Psi_1) + \\ & + (D_2(v+\tilde{\psi}(t)))^2((D_2\Psi_1)^2 + (D_1\Psi_1)^2)\big], \end{aligned}$$

which implies:

$$\begin{aligned} a_{11}(\sigma)(x) &:= \frac{1}{\det D\Psi_\sigma(x)} \big((D_2\Psi_2)^2 + (D_1\Psi_2)^2\big)(x), \\ a_{12}(\sigma)(x) = a_{21}(\sigma)(x) &:= -\frac{1}{\det D\Psi_\sigma(x)} \big(D_2\Psi_2 D_2\Psi_1 + D_1\Psi_2 D_1\Psi_1\big)(x), \\ a_{22}(\sigma)(x) &:= \frac{1}{\det D\Psi_\sigma(x)} \big((D_2\Psi_1)^2 + (D_1\Psi_1)^2\big)(x). \end{aligned}$$

Define $A(\sigma) := (a_{ij}(\sigma))_{ij}$ and note that $a_{ij}(\sigma) \in C(\overline{\Omega})$, for every $\sigma \in [\sigma_0, \overline{\sigma}]$, and every $i, j = 1, 2$.

We may assume that $0 < c < \|\det D\Phi_\sigma\|_\infty < C$ independently of $\sigma \in [\sigma_0, \overline{\sigma}]$, where $\|\cdot\|_\infty$ denotes the L^∞-norm on Ω. Since Γ is of class C^3, we may also choose $\Phi(\cdot, \sigma)$ (and hence $\Psi(\cdot, \sigma)$) to depend regularly on σ in such a way that, as functions of σ, the coefficients a_{ij} be of class C^2 on $[\sigma_0, \overline{\sigma}]$, uniformly in $\overline{\Omega}$. In particular, we shall use the fact that there exist five positive constants $\lambda, \Lambda, \Lambda', L, L' > 0$ independent of σ, such that

$$(A(\sigma)\xi|\xi) \geq \lambda|\xi|^2 \qquad \forall\, \xi \in \mathbb{R}^2, \quad \forall\, x \in \overline{\Omega}, \tag{3.1.1}$$

where $(\cdot|\cdot)$ denotes the scalar product in $\mathbb{R}^2$,

$$\|(A(\sigma)\xi|\eta)\|_\infty \leq \Lambda|\xi|\,|\eta| \qquad \forall\, \xi, \eta \in \mathbb{R}^2, \tag{3.1.2}$$

$$\|(\partial_\sigma A(\sigma)\xi|\eta)\|_\infty \leq \Lambda'|\xi|\,|\eta| \qquad \forall\, \xi, \eta \in \mathbb{R}^2, \tag{3.1.3}$$

$$\|a_{ij}(\sigma') - a_{ij}(\sigma'')\|_\infty \leq L|\sigma' - \sigma''| \qquad \text{and} \tag{3.1.4}$$

$$\|\partial_\sigma a_{ij}(\sigma') - \partial_\sigma a_{ij}(\sigma'')\|_\infty \leq L'|\sigma' - \sigma''| \tag{3.1.5}$$

for every $\sigma', \sigma'' \in [\sigma_0, \overline{\sigma}]$ and $i, j = 1, 2$.

Note that, since Ψ_σ coincides with the identity near the boundary of Ω, this change of variables does not have any effect on $\mathcal{G}$:

$$\mathcal{G}(t)(u + \psi(t)) = \mathcal{G}(t)(v + \tilde{\psi}(t)).$$

Moreover, we can neglect the dependence of $\tilde{\psi}$ on σ since, for every $\sigma \in [\sigma_0, \overline{\sigma}]$, Ψ_σ coincides with the identity near the boundary of Ω, and we may assume that the support of ψ is included in the set where, for every $\sigma \in [\sigma_0, \overline{\sigma}]$, Ψ_σ is the identity. Hence the change of variables influences only the bilinear term in v.

For brevity of notation, let

$$V := H^1_{\partial_D \Omega}(\Omega \setminus \Gamma(\sigma_0)) \,.$$

On V we consider the norm $\|\cdot\|_V$ defined by $\|v\|_V := \|Dv\|_2$, and the scalar product $(v, w)_V := (Dv, Dw)$, where $\|\cdot\|_2$ and $(\cdot,\cdot)$ denote the norm and, respectively, the scalar product in $L^2(\Omega)$ or $L^2(\Omega \setminus \Gamma(\sigma_0); \mathbb{R}^2)$, depending on the context. Let V' denote its dual space and let $\langle \cdot, \cdot \rangle$ denote the duality pairing between V' and V.

For every $t \in [0, T]$, $v \in V$, and $\sigma \in [\sigma_0, \overline{\sigma}]$ define

$$\begin{aligned}&\mathscr{F}(t, v, \sigma) := \\ &= \int_{\Omega \setminus \Gamma(\sigma_0)} \sum_{i,j \in \{1,2\}} a_{ij}(\sigma) D_j(v + \tilde{\psi}(t)) D_i(v + \tilde{\psi}(t)) dx + \sigma - \int_{\partial_N \Omega} g(t)(v + \tilde{\psi}(t)) d\mathscr{H}^1.\end{aligned}$$

Then the functional $\mathscr{F}$ can be also written as

$$\begin{aligned}\mathscr{F}(t, v, \sigma) := &\int_{\Omega \setminus \Gamma(\sigma_0)} (A(\sigma) Dv | Dv)\, dx + 2 \int_{\Omega \setminus \Gamma(\sigma_0)} (D\psi(t) | Dv)\, dx - \\ &- \int_{\partial_N \Omega} g(t) v \, d\mathscr{H}^1 + \sigma + \int_\Omega |D\psi(t)|^2 dx - \int_{\partial_N \Omega} g(t) \psi(t)\, d\mathscr{H}^1,\end{aligned}$$

or

$$\begin{aligned}&\mathscr{F}(t, v, \sigma) := \\ &= (A(\sigma) Dv, Dv) + 2(D\psi(t), Dv) - (g(t), v)_{\partial_N \Omega} + \sigma + \|D\psi(t)\|_2^2 - (g(t), \psi(t))_{\partial_N \Omega}\end{aligned}$$

where $(\cdot,\cdot)_{\partial_N \Omega}$ denotes the scalar product in $L^2(\partial_N \Omega, \mathscr{H}^1)$. Hence the elastic energy becomes $\mathscr{F}^{el}(t, v, \sigma) := \mathscr{F}(t, v, \sigma) - \sigma$, and there exist four positive constants $\lambda_{\mathscr{F}}$, $\Lambda_{\mathscr{F}}$, $\mu_{\mathscr{F}}$, and $M_{\mathscr{F}}$, independent of t and σ, such that for every $t \in [0, T]$ and every $\sigma \in [\sigma_0, \overline{\sigma}]$

$$\begin{aligned}\mathscr{F}^{el}(t, v, \sigma) &\geq \lambda_{\mathscr{F}} \|v\|_V^2 - \mu_{\mathscr{F}} \\ \mathscr{F}^{el}(t, v, \sigma) &\leq \Lambda_{\mathscr{F}} \|v\|_V^2 + M_{\mathscr{F}},\end{aligned}$$

for every $v \in V$. Indeed, this follows from the uniform ellipticity of the bilinear part and standard estimates (on Ω^+ and Ω^-).

3.2 Critical points of the energy

For every $t \in [0, T]$ the function $\mathscr{F}(t, \cdot, \cdot)\colon V \times [\sigma_0, \overline{\sigma}] \to \mathbb{R}$ is twice Fréchet partially differentiable with respect to (v, σ). In particular, the partial differential $\partial_v \mathscr{F}(t, v, \sigma)$

belongs to V', while the partial gradient $\mathrm{grad}_v \mathscr{F}(t,v,\sigma)$ is, by definition, the element of V given by

$$(\mathrm{grad}_v \mathscr{F}(t,v,\sigma), w)_V = 2(A(\sigma)Dv, Dw) + 2(\psi(t), w)_V - (g(t), w)_{\partial_N \Omega},$$

for every $w \in V$. The partial differential $\partial_\sigma \mathscr{F}(t,v,\sigma)$ is given by

$$\partial_\sigma \mathscr{F}(t,v,\sigma) = (\partial_\sigma A(\sigma)Dv, Dv) + 1\,.$$

For fixed $v \in V$ and $\sigma \in [\sigma_0, \overline{\sigma}]$, we have that $\mathscr{F}(\cdot, v, \sigma) \in W^{1,\infty}(0,T)$, with

$$\partial_t \mathscr{F}(t,v,\sigma) = 2(D\dot{\psi}(t), Dv + D\psi(t)) - (\dot{g}(t), v + \psi(t))_{\partial_N \Omega} - (g(t), \dot{\psi}(t))_{\partial_N \Omega}\,.$$

Note that by the regularity assumptions on ψ and g it follows also that the map

$$(t,v,\sigma) \mapsto (\mathrm{grad}_v \mathscr{F}(t,v,\sigma), \partial_\sigma \mathscr{F}(t,v,\sigma))$$

is continuous from $(0,T) \times V \times (\sigma_0, \overline{\sigma})$ into $V \times \mathbb{R}$.

The second order partial differentials with respect to (v,σ) are given by

$$\begin{aligned}\langle\langle \partial^2_{(v,\sigma)} \mathscr{F}(t,v,\sigma)(w_1,\tau_1),(w_2,\tau_2)\rangle\rangle &= 2(A(\sigma)Dw_1, Dw_2) + 2(\partial_\sigma A(\sigma)Dv, Dw_1)\tau_2 + \\ &\quad +2(\partial_\sigma A(\sigma)Dv, Dw_2)\tau_1 + (\partial^2_{\sigma\sigma} A(\sigma)Dv, Dv)\tau_1\tau_2\,,\end{aligned}$$

for every $(w_i, \tau_i) \in V \times \mathbb{R}$, $i = 1,2$, where $\langle\langle \cdot, \cdot \rangle\rangle$ denotes the duality product between $V' \times \mathbb{R}$ and $V \times \mathbb{R}$.

For fixed t and σ, the function $v \mapsto \mathscr{F}(t,v,\sigma)$, being strictly convex, has a unique critical point $v_{t,\sigma}$, and $v_{t,\sigma}$ is a minimum point. Also the function $u \mapsto \mathscr{E}(t)(u,\sigma)$ is strictly convex and its critical point is the unique minimum point $u_{t,\sigma} \in AD(\psi(t),\sigma)$ of $u \mapsto \mathscr{E}(t)(u,\sigma)$. The function $u_{t,\sigma}$ satisfies

$$2\int_{\Omega\setminus\Gamma(\sigma)} (Du_{t,\sigma}|Dw)dx = \int_{\partial_N\Omega} g(t,x)w\, d\mathscr{H}^1 \qquad \forall\, w \in H^1_{\partial_D\Omega}(\Omega \setminus \Gamma(\sigma))\,.$$

Proposition 3.2.1. *For fixed $t \in [0,T]$ critical points of $\mathscr{F}(t,\cdot,\cdot)$ correspond to critical points of $\mathscr{E}(t)$ in the following sense: minimum points $v_{t,\sigma} \in V$ of $v \mapsto \mathscr{F}(t,v,\sigma)$ correspond by the change of variables to minimum points $u_{t,\sigma} \in AD(\psi(t),\sigma)$ of $u \mapsto \mathscr{E}(t)(u,\sigma)$. Moreover, $\partial_\sigma \mathscr{F}(t, v_{t,\sigma}, \sigma) = \partial_\sigma E(t,\sigma)$, where $E(t,\sigma) := \mathscr{E}(t)(u_{t,\sigma},\sigma)$.*

Before giving the proof we discuss some properties of the minimizers $u_{t,\sigma}$. The following result provides a useful characterization of the "singular" part of the displacement $u_{t,\sigma}$ near the tip $\gamma(\sigma)$ of the crack. For the proof we refer to [22], [23].

Proposition 3.2.2. *Let $\sigma \in [\sigma_0, \overline{\sigma}]$ and $u \in H^1(\Omega \setminus \Gamma(\sigma))$ be such that*

$$\Delta u \in L^2(\Omega \setminus \Gamma(\sigma)) \quad \textit{and} \quad \partial_\nu u = 0 \ \ \textit{on}\ \Gamma(\sigma). \tag{3.2.1}$$

Then there exists $\kappa \in \mathbb{R}$ satisfying

$$u - \kappa\sqrt{\frac{2}{\pi}}\, r^{\frac{1}{2}} \sin\frac{\theta}{2} \in H^2(U \setminus \Gamma(\sigma)), \tag{3.2.2}$$

for every $U \subset\subset \Omega$ open. In (3.2.2), $r(x) := |x - \gamma(\sigma)|$ and $\theta(x)$ is the continuous function on $U \setminus \Gamma(\sigma)$ which coincides with the counterclockwise oriented angle between $\dot\gamma(\sigma)$ and $x - \gamma(\sigma)$, and vanishes on the points of the form $x = \gamma(\sigma) + h\dot\gamma(\sigma)$ for $h > 0$ sufficiently small.

The coefficient $\kappa\sqrt{2/\pi}$ represents the *stress intensity factor* associated to the displacement u at the tip $\gamma(\sigma)$. We shall use its following characterization.

Proposition 3.2.3. *Let $\sigma \in [\sigma_0, \overline{\sigma}]$, $u \in H^1(\Omega \setminus \Gamma(\sigma))$ satisfying (3.2.1), and let κ be defined by (3.2.2). Then for every $\phi = (\phi_1, \phi_2) \in C_c^\infty(\Omega; \mathbb{R}^2)$ we have*

$$\begin{aligned}\kappa^2 \phi(\gamma(\sigma))\dot\gamma(\sigma) = \int_\Omega &\Big[((D_1u)^2 - (D_2u)^2)(D_1\phi_1 - D_2\phi_2) + \\ &+ 2D_1u\, D_2u(D_1\phi_2 + D_2\phi_1)\Big]\, dx + 2\int_\Omega \Delta u(D_1u\phi_1 + D_2u\phi_2)\, dx.\end{aligned} \tag{3.2.3}$$

Proof. The proof follows the lines of [2, Proposition 2.2], but we present it for the sake of completeness. Let $\eta > 0$ be such that the closed ball centered at $\gamma(\sigma)$ with radius η, denoted by $\overline{B}(\gamma(\sigma), \eta)$, is contained in Ω and let us consider the following integration by parts:

$$\begin{aligned}&\int_{\Omega\setminus B(\gamma(\sigma),\eta)} \Big[((D_1u)^2 - (D_2u)^2)(D_1\phi_1 - D_2\phi_2) + 2D_1uD_2u(D_1\phi_2 + D_2\phi_1)\Big]\, dx = \\ &= \int_{\partial B(\gamma(\sigma),\eta)} \Big[((D_1u)^2 - (D_2u)^2)(\nu_1\phi_1 - \nu_2\phi_2) + 2D_1uD_2u(\nu_1\phi_2 + \nu_2\phi_1)\Big]\, d\mathscr{H}^1 + \\ &-2\int_{\Omega\setminus B(\gamma(\sigma),\eta)} \Delta u(D_1u\phi_1 + D_2u\phi_2)\, dx,\end{aligned} \tag{3.2.4}$$

where (ν_1, ν_2) is the inner normal to $\partial B(\gamma(\sigma), \eta)$, i.e., it is the outer normal to $\partial(\Omega \setminus B(\gamma(\sigma), \eta))$. Therefore, we obtain the following identity:

$$\begin{aligned}&\int_\Omega \Big[((D_1u)^2 - (D_2u)^2)(D_1\phi_1 - D_2\phi_2) + 2D_1uD_2u(D_1\phi_2 + D_2\phi_1)\Big]\, dx + \\ &+2\int_\Omega \Delta u(D_1u\phi_1 + D_2u\phi_2)\, dx = \\ &= \int_{\partial B(\gamma(\sigma),\eta)} \Big[((D_1u)^2 - (D_2u)^2)(\nu_1\phi_1 - \nu_2\phi_2) + 2D_1uD_2u(\nu_1\phi_2 + \nu_2\phi_1)\Big]\, d\mathscr{H}^1 + \\ &+ \int_{B(\gamma(\sigma),\eta)} \Big[((D_1u)^2 - (D_2u)^2)(D_1\phi_1 - D_2\phi_2) + 2D_1uD_2u(D_1\phi_2 + D_2\phi_1)\Big]\, dx + \\ &+2\int_{B(\gamma(\sigma),\eta)} \Delta u(D_1u\phi_1 + D_2u\phi_2)\, dx.\end{aligned} \tag{3.2.5}$$

Now, taking into account (3.2.1) and (3.2.2), we are going to prove that the limit in (3.2.5) when η tends to zero is the left-hand side in (3.2.3).

Thanks to (3.2.2) we may split u into the sum of two terms, the regular one and the singular one, as follows:

$$u = u_{\mathrm{reg}} + \kappa\sqrt{\frac{2}{\pi}}\, r^{\frac{1}{2}} \sin\frac{\theta}{2}, \tag{3.2.6}$$

where $u_{\mathrm{reg}} \in H^2(\Omega \setminus \Gamma(\sigma))$. Note that, taking into account (3.2.6),

$$\lim_{\eta\to 0} \int_{B(\gamma(\sigma),\eta)} \Delta u(D_1 u\phi_1 + D_2 u\phi_2)\, dx = 0\,. \tag{3.2.7}$$

Indeed,

$$\int_{B(\gamma(\sigma),\eta)} \Delta u(\partial_1 u\phi_1 + \partial_2 u\phi_2)\, dx \le \|\Delta u\|_2 \Big(\int_{B(\gamma(\sigma),\eta)} (\partial_1 u\phi_1 + \partial_2 u\phi_2)^2\, dx\Big)^{\frac{1}{2}}.$$

Here by (3.2.1) the first term in the right-hand side is bounded, and the second one is $O(\eta)$ (it is sufficient to pass to the polar coordinates and notice that $Du = O(r^{-\frac{1}{2}})$). Passing to the limit as η goes to zero, we get (3.2.7).

Using (3.2.6), we can prove also that

$$\lim_{\eta\to 0} \int_{B(\gamma(\sigma),\eta)} \Big[((D_1u)^2 - (D_2u)^2)(D_1\phi_1 - D_2\phi_2) + 2D_1uD_2u(D_1\phi_2 + D_2\phi_1)\Big]\, dx = 0. \tag{3.2.8}$$

Next, we prove that

$$\lim_{\eta\to 0} \int_{\partial B(\gamma(\sigma),\eta)} \Big[((D_1u)^2 - (D_2u)^2)(\nu_1\phi_1 - \nu_2\phi_2) + 2D_1uD_2u(\nu_1\phi_2 + \nu_2\phi_1)\Big]\, d\mathscr{H}^1 = \\ = \kappa^2\phi(\gamma(\sigma))\dot\gamma(\sigma)\,. \tag{3.2.9}$$

First of all, note that, by (3.2.6), we can split the left-hand side integrand in three parts. The first one contains the terms in u_{reg}, the second one contains mixed terms, and the third one is given only by the derivatives of $v := \kappa\sqrt{\frac{2}{\pi}} r^{\frac{1}{2}} \sin\frac{\theta}{2}$, the singular part of u. Since u_{reg} has bounded gradient, the first two integrands are bounded uniformly by $\mathrm{O}(\sqrt{r})$ and consequently their integrals tend to zero. Thus we have to study the third integral.

Let α be the counterclockwise oriented angle between the vector $\dot\gamma(\sigma)$ and the half line $\{(x_1, 0) : x_1 \ge 0\}$. Passing to the polar coordinates, after standard calculations we get

$$\int_{\partial B(\gamma(\sigma),\eta)} \Big[((D_1v)^2 - (D_2v)^2)(\nu_1\phi_1 - \nu_2\phi_2) + 2D_1vD_2v(\nu_1\phi_2 + \nu_2\phi_1)\Big]\, d\mathscr{H}^1 = \\ = \frac{1}{2\pi}\kappa^2 \int_0^{2\pi} \phi(\gamma(\sigma) + \eta(\cos(\theta-\alpha), \sin(\theta-\alpha)))\dot\gamma(\sigma)\, d\theta,$$

where we used the fact that $\dot\gamma(\sigma) = (\cos\alpha, -\sin\alpha)$, since $|\dot\gamma(\sigma)| = 1$. To be more precise we have $(x_1, x_2) = \gamma(\sigma) + r(\cos(\theta-\alpha), \sin(\theta-\alpha))$ and

$$(\partial_1 v, \partial_2 v) = \Big(\frac{1}{\sqrt{2\pi}}\kappa r^{-\frac12}\sin(\alpha-\frac{\theta}{2}), \frac{1}{\sqrt{2\pi}}\kappa r^{-\frac12}\cos(\alpha-\frac{\theta}{2})\Big). \tag{3.2.10}$$

Further, $(\nu_1, \nu_2) = (-\cos(\theta-\alpha), -\sin(\theta-\alpha))$ which, together with (3.2.10) gives

$$\begin{aligned}
&\int_{\partial B(\gamma(\sigma),\eta)} \Big[((\partial_1 v)^2 - (\partial_2 v)^2)(\nu_1\phi_1 - \nu_2\phi_2) + 2\partial_1 v\partial_2 v(\nu_1\phi_2 + \nu_2\phi_1)\Big]\, d\mathscr{H}^1 = \\
&= \frac{1}{2\pi}\kappa^2\eta \int_0^{2\pi} \frac{1}{\eta}(\sin^2(\alpha-\frac{\theta}{2}) - \cos^2(\alpha-\frac{\theta}{2}))(-\cos(\theta-\alpha)\phi_1 + \sin(\theta-\alpha)\phi_2) + \\
&\quad + 2\frac{1}{\eta}\sin(\alpha-\frac{\theta}{2})\cos(\alpha-\frac{\theta}{2})(-\cos(\theta-\alpha)\phi_2 - \sin(\theta-\alpha)\phi_1)\, d\theta = \\
&= \frac{1}{2\pi}\kappa^2 \int_0^{2\pi} -\cos(2\alpha-\theta)(-\cos(\theta-\alpha)\phi_1 + \sin(\theta-\alpha)\phi_2) + \\
&\quad - \sin(2\alpha-\theta)(\cos(\theta-\alpha)\phi_2 + \sin(\theta-\alpha)\phi_1)\, d\theta = \\
&= \frac{1}{2\pi}\kappa^2 \int_0^{2\pi} \phi_1(\cos(2\alpha-\theta)\cos(\theta-\alpha) - \sin(2\alpha-\theta)\sin(\theta-\alpha)) + \\
&\quad + \phi_2(-\cos(2\alpha-\theta)\sin(\theta-\alpha) - \sin(2\alpha-\theta)\cos(\theta-\alpha))\, d\theta = \\
&= \frac{1}{2\pi}\kappa^2 \int_0^{2\pi} \phi_1\cos\alpha - \phi_2\sin\alpha\, d\theta = \\
&= \frac{1}{2\pi}\kappa^2 \int_0^{2\pi} \phi(\gamma(\sigma) + \eta(\cos(\theta-\alpha), \sin(\theta-\alpha)))\dot\gamma(\sigma)\, d\theta\,.
\end{aligned}$$

Taking the limit as η tends to zero, we obtain (3.2.9).

By (3.2.5), (3.2.7), (3.2.8), and (3.2.9) we finally reach the conclusion (3.2.3).

□

Proposition 3.2.4. *The function $\sigma \mapsto E(t,\sigma)$ is differentiable on $[\sigma_0, \overline{\sigma}]$ and*

$$\partial_\sigma E(t,\sigma) = 1 - \kappa_{t,\sigma}^2\,, \tag{3.2.11}$$

where $\kappa_{t,\sigma}\sqrt{\frac{2}{\pi}}$ is the stress intensity factor associated to $u_{t,\sigma}$ at $\gamma(\sigma)$.

Proof. The proof follows the same arguments of the proof of [2, Theorem 3.3], but for the sake of completeness we make it.

In order to fix the notation, let us compute the derivative at $\sigma^* \in [\sigma_0, \overline{\sigma}]$. We consider $\phi \in C_c^2(\Omega; \mathbb{R}^2)$ such that $\phi(\gamma(\sigma)) = \dot\gamma(\sigma)$ for σ in a small neighborhood of σ^*. For $|\tilde\sigma|$ small enough, $\tilde\sigma > 0$ if $\sigma^* = \sigma_0$, and, conversely, $\tilde\sigma < 0$ if $\sigma^* = \overline{\sigma}$, we consider the application $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(y) := y + \tilde\sigma\phi(y)$. It is a C^2-diffeomorphism from Ω into Ω which maps Ω^+ into Ω^+, Ω^- into Ω^-, $\Gamma(\sigma^*)$ into $\Gamma(\sigma^* + \tilde\sigma)$, and does not vary $\partial\Omega$.

During the proof we use the following notation: for fixed σ, $z_\sigma(t)(x) := (u_{t,\sigma} + \psi(t))(x)$ and $\tilde z_\sigma(t)(x) := z_\sigma(t)(F(x))$. Hence, using the change of variable $y = F(x)$ and the approximations $DF(x)^{-1} = x - \tilde\sigma D\phi(x)$, $\det DF(x) = 1 + \tilde\sigma \operatorname{div}\phi$ we get

$$\int_{\Omega\setminus\Gamma(\sigma)} |Dz_\sigma(t)|^2\, dy = \int_{\Omega\setminus\Gamma(\sigma^*)} |(x - \tilde\sigma D\phi)D\tilde z_\sigma(t)|^2 (1 + \tilde\sigma \operatorname{div}\phi)\, dx\,.$$

Since $\tilde z_\sigma(t)(x) := z_\sigma(t)(F(x)) = z_{\sigma^*}(t)(x) + \tilde\sigma \dot U$, where $\dot U$ is the derivative of $\tilde z_\sigma$ at $\sigma = \sigma^*$, we obtain

$$\begin{aligned} &\frac{d}{d\sigma}\Big(\int_{\Omega\setminus\Gamma(\sigma)} |Dz_\sigma(t)|^2\, dy\Big)_{|\sigma=\sigma^*} = \\ &\int_{\Omega\setminus\Gamma(\sigma^*)} \big(-2(Dz_{\sigma^*}(t)|D\phi Dz_{\sigma^*}(t) + D\dot U) + |Dz_{\sigma^*}(t)|^2 \operatorname{div}\phi\big)\, dx\,. \end{aligned} \tag{3.2.12}$$

On the other hand,

$$\frac{d}{d\sigma}\Big(\int_{\partial_N\Omega} g(t) z_\sigma(t)\, d\mathscr{H}^1\Big)_{|\sigma=\sigma^*} = \int_{\partial_N\Omega} g(t)\dot U\, d\mathscr{H}^1\,. \tag{3.2.13}$$

In conclusion, from (3.2.12) and (3.2.13) it follows that

$$\begin{aligned} \frac{d}{d\sigma}E(t,\sigma)_{|\sigma=\sigma^*} = 1 &+ \int_{\Omega\setminus\Gamma(\sigma^*)} \big(-2(Dz_{\sigma^*}(t)|D\phi Dz_{\sigma^*}(t) + D\dot U) + |Dz_{\sigma^*}(t)|^2 \operatorname{div}\phi\big)\, dx - \\ &- \int_{\partial_N\Omega} g(t)\dot U\, d\mathscr{H}^1 = \\ = 1 + \int_{\Omega\setminus\Gamma(\sigma^*)} &\big(-2(Dz_{\sigma^*}(t)|D\phi Dz_{\sigma^*}(t)) + |Dz_{\sigma^*}(t)|^2 \operatorname{div}\phi\big)\, dx\,, \end{aligned}$$

because $\dot U$ is a good test function for the equation satisfied by $z_{\sigma^*}(t)$. This last equality concludes the proof, thanks to the characterization of κ_{t,σ^*} given in Proposition 3.2.3, since equality (3.2.3), expressed in vector notation, actually becomes:

$$\kappa^2_{t,\sigma^*}\phi(\gamma(\sigma^*))\dot\gamma(\sigma^*) = \int_\Omega (2(Du_{t,\sigma^*}|D\phi Du_{t,\sigma^*}) - |Du_{t,\sigma^*}|^2 \operatorname{div}\phi)\, dx + 2\int_\Omega \Delta u_{t,\sigma^*}(Du_{t,\sigma^*}|\phi)\, dx\,. \tag{3.2.14}$$

□

Similar computations have been recently done in [28] where the stored energy density W is a polyconvex function with $W(A) = \infty$ for every matrix A with $\det A \le 0$, and Γ is a segment.

Remark 3.2.5. Fix $t_0 \in\,]0, T[$. The map $\sigma \mapsto v_{t_0,\sigma}$ has the same regularity as $\sigma \mapsto A(\sigma)$, hence, under the regularity assumptions we made on $A(\sigma)$, it is of class $C^2([\sigma_0, \overline\sigma])$. Since in this case we are not interested in the dependence on t, let us simplify the notation and set $v_\sigma := v_{t_0,\sigma}$. Then standard arguments for elliptic PDE's allow us to obtain that for every $\sigma^* \in [\sigma_0, \overline\sigma]$ there exists $v'_{\sigma^*} \in V$ as strong limit in V of the difference quotient

$\frac{v_\sigma - v_{\sigma^*}}{\sigma-\sigma^*}$, and the map $\sigma \mapsto v'_\sigma$ is continuous in the strong topology of V. More precisely, from the equality

$$2(A(\sigma)Dv_\sigma, Dv_\sigma) + 2(D\psi(t), Dv_\sigma) - (g(t), v_\sigma)_{\partial_N\Omega} = 0\,,$$

it follows that there exists a positive constant C such that $\|Dv_\sigma\|_2 \le C$. Hence there exists $v_{\sigma^*} \in V$ with $v_\sigma \rightharpoonup v_{\sigma^*}$ as $\sigma \to \sigma^*$. Moreover, using $v_\sigma - v_{\sigma^*}$ as test function for the equation satisfied by v_σ and the equation satisfied by v_{σ^*}, respectively, we get also that $v_\sigma \to v_{\sigma^*}$ strongly in V. Indeed, the equality

$$\begin{aligned}(A(\sigma)Dv_\sigma - A(\sigma^*)Dv_{\sigma^*}, Dv_\sigma - Dv_{\sigma^*}) = {} & (A(\sigma)(Dv_\sigma - Dv_{\sigma^*}), Dv_\sigma - Dv_{\sigma^*}) + \\ & + ((A(\sigma) - A(\sigma^*))Dv_{\sigma^*}, Dv_\sigma - Dv_{\sigma^*})\end{aligned}$$

implies that $\|Dv_\sigma - Dv_{\sigma^*}\|_2^2 \to 0$ as $\sigma \to \sigma^*$, and therefore the map $\sigma \mapsto v_\sigma$ is continuous in the strong topology of V.

Now we take the difference of the equations satisfied by v_σ and v_{σ^*}, for a generic test function, obtaining

$$2(A(\sigma)Dv_\sigma - A(\sigma^*)Dv_{\sigma^*}, Dw) = 0 \quad \text{for every } w \in V\,.$$

Hence,

$$\Big(\frac{A(\sigma) - A(\sigma^*)}{\sigma - \sigma^*}\,Dv_\sigma + A(\sigma^*)\frac{Dv_\sigma - Dv_{\sigma^*}}{\sigma - \sigma^*}, Dw\Big) = 0 \quad \text{for every } w \in V\,. \tag{3.2.15}$$

Passing to the limit as $\sigma \to \sigma^*$ we obtain

$$\lim_{\sigma\to\sigma^*}\Big(\frac{Dv_\sigma - Dv_{\sigma^*}}{\sigma - \sigma^*}, A(\sigma^*)Dw\Big) = -(\partial_\sigma A(\sigma^*)Dv_{\sigma^*}, Dw) \quad \text{for every } w \in V\,.$$

Using $w = \frac{v_\sigma - v_{\sigma^*}}{\sigma - \sigma^*}$ in (3.2.15) we get

$$\Big(\frac{A(\sigma) - A(\sigma^*)}{\sigma - \sigma^*}\,Dv_\sigma + A(\sigma^*)\frac{Dv_\sigma - Dv_{\sigma^*}}{\sigma - \sigma^*}, \frac{Dv_\sigma - Dv_{\sigma^*}}{\sigma - \sigma^*}\Big) = 0$$

i.e., there exists a positive constant C such that

$$\|\frac{Dv_\sigma - Dv_{\sigma^*}}{\sigma - \sigma^*}\|_2 \le C\,.$$

Therefore, there exists $v'_{\sigma^*} \in V$ such that

$$\frac{Dv_\sigma - Dv_{\sigma^*}}{\sigma - \sigma^*} \rightharpoonup Dv'_{\sigma^*} \quad \text{as } \sigma \to \sigma^*\,,$$

and solves the following equation:

$$(A(\sigma^*)Dv'_{\sigma^*}, Dw) + (\partial_\sigma A(\sigma^*)Dv_{\sigma^*}, Dw) = 0 \quad \text{for every } w \in V\,.$$

In order to obtain the strong convergence of v'_σ to v'_{σ^*} in V as $\sigma \to \sigma^*$, we take the difference of the equations satisfied by v'_σ and v'_{σ^*} using $w := v'_\sigma - v'_{\sigma^*}$ as test function, obtaining

$$\begin{aligned}
&((A(\sigma) - A(\sigma^*)), Dv'_\sigma + A(\sigma^*)(Dv'_\sigma - Dv'_{\sigma^*}), Dv'_\sigma - Dv'_{\sigma^*})+\\
&+(\partial_\sigma A(\sigma)Dv_\sigma - \partial_\sigma A(\sigma^*)Dv_{\sigma^*}, Dv'_\sigma - Dv'_{\sigma^*}) = 0\,.
\end{aligned}$$

Now we pass to the limit as $\sigma \to \sigma^*$, which gives $\lim_{\sigma\to\sigma^*}(A(\sigma^*)(Dv'_\sigma - Dv'_{\sigma^*}), Dv'_\sigma - Dv'_{\sigma^*}) = 0$, hence the strong convergence of $v'_\sigma \to v'_{\sigma^*}$ in V is proved.

The same arguments can be repeated to obtain the existence of $v''_{\sigma^*} \in V$ as strong limit in V of the difference quotient $\frac{v'_\sigma - v'_{\sigma^*}}{\sigma-\sigma^*}$, and the continuity of the map $\sigma \mapsto v''_\sigma$ with respect to the strong topology in V. Note that v''_σ solves the following equation

$$(A(\sigma)Dv''_\sigma, Dw) + 2(\partial_\sigma A(\sigma)Dv'_\sigma, Dw) + (\partial^2_\sigma A(\sigma)Dv_\sigma, Dw) = 0 \qquad \forall\, w \in V\,.$$

Proof of Proposition 3.2.1. It follows from the change of variables, Proposition 3.2.3, and Proposition 3.2.4.

More precisely, taking the usual change of variables $y = \Psi_\sigma(x)$ and setting as before $v(y,\sigma) = u(\Psi_\sigma(y))$, we obtain

$$\begin{aligned}
&\langle \partial_u \mathscr{E}(t)(u,\sigma), w\rangle = 2\int_{\Omega\setminus\Gamma(\sigma)} (Du|Dw)\,dy - \int_{\partial_N\Omega} g(t,y)w\,d\mathscr{H}^1 =\\
&= \int_{\Omega\setminus\Gamma(\sigma_0)} \Big[2(A(\sigma)Dv|D\tilde{w}) + 2(D\psi(t))|D\tilde{w})\Big]\,dx - \int_{\partial_N\Omega} g(t,y)w\,d\mathscr{H}^1 =\\
&= (\mathrm{grad}_v\mathscr{F}(t,v,\sigma), \tilde{w})_V\,,
\end{aligned}$$

for every $w \in H^1_{\partial_D\Omega}(\Omega\setminus\Gamma(\sigma))$, where $\tilde{w}(x) = w(\Psi_\sigma(x))$.

On the other hand, let $u_{t,\sigma}$ be a minimum of $u \mapsto \mathscr{E}(t)(u,\sigma)$, $\phi \in C^1_c(\Omega;\mathbb{R}^2)$ be such that $\phi(\Psi_\sigma(x)) = \partial_\sigma\Psi_\sigma(x)$, and consider the usual change of variables $y = \Psi_\sigma(x)$. Proposition 3.2.3, Proposition 3.2.4 and the fact that $\phi(\gamma(\sigma)) = \dot\gamma(\sigma)$ imply

$$\begin{aligned}
&\partial_\sigma E(t,\sigma) = 1 - \kappa^2_{t,\sigma} =\\
&= 1 + \int_{\Omega\setminus\Gamma(\sigma)} \big(-2(Du_{t,\sigma}|D\phi Du_{t,\sigma}) + |Du_{t,\sigma}|^2\mathrm{div}\phi\big)\,dy =\\
&= 1 + \int_{\Omega\setminus\Gamma(\sigma_0)} (\partial_\sigma A(\sigma,x)Dv_{t,\sigma}|Dv_{t,\sigma})\,dx =\\
&= \partial_\sigma\mathscr{F}(t,v_{t,\sigma},\sigma)\,.
\end{aligned}$$

Indeed, from $Dv_{t,\sigma}(x) = (D\Psi_\sigma)^T Du_{t,\sigma}(\Psi_\sigma(x))$ we get

$$\begin{aligned}
&\int_{\Omega\setminus\Gamma(\sigma_0)} (\partial_\sigma A(\sigma,x)Dv_{t,\sigma}(x)|Dv_{t,\sigma}(x))\,dx =\\
&= \int_{\Omega\setminus\Gamma(\sigma_0)} (\partial_\sigma A(\sigma,x)(D\Psi_\sigma)^T Du_{t,\sigma}(\Psi_\sigma(x))|(D\Psi_\sigma)^T Du_{t,\sigma}(\Psi_\sigma(x)))\,dx\,.
\end{aligned}$$

To shorten the notation we denote by $Dz_{t,\sigma}(x)$ the term $Du_{t,\sigma}(\Psi_\sigma(x))$. Then:

$$(D\Psi_\sigma)^T Dz_{t,\sigma} = \begin{pmatrix} D_1\Psi_1 D_1 z_{t,\sigma} + D_2\Psi_1 D_2 z_{t,\sigma} \\ D_1\Psi_2 D_1 z_{t,\sigma} + D_2\Psi_2 D_2 z_{t,\sigma} \end{pmatrix},$$

and $\partial_\sigma A(D\Psi_\sigma)^T Dz_{t,\sigma}$ is equal to the vector

$$\begin{pmatrix} \partial_\sigma a_{11}(D_1\Psi_1 D_1 z_{t,\sigma} + D_2\Psi_1 D_2 z_{t,\sigma}) + \partial_\sigma a_{12}(D_1\Psi_2 D_1 z_{t,\sigma} + D_2\Psi_2 D_2 z_{t,\sigma}) \\ \partial_\sigma a_{12}(D_1\Psi_1 D_1 z_{t,\sigma} + D_2\Psi_1 D_2 z_{t,\sigma}) + \partial_\sigma a_{22}(D_1\Psi_2 D_1 z_{t,\sigma} + D_2\Psi_2 D_2 z_{t,\sigma}) \end{pmatrix}.$$

Hence

$$\begin{aligned}
&(\partial_\sigma A(D\Psi_\sigma)^T Dz_{t,\sigma} | (D\Psi_\sigma)^T Dz_{t,\sigma}) = \\
&= (D_1 z_{t,\sigma})^2 \big[\partial_\sigma a_{11}(D_1\Psi_1)^2 + 2\partial_\sigma a_{12} D_1\Psi_1 D_1\Psi_2 + \partial_\sigma a_{22}(D_1\Psi_2)^2\big] + \\
&+ 2D_1 z_{t,\sigma} D_2 z_{t,\sigma}\big[\partial_\sigma a_{11} D_1\Psi_1 D_2\Psi_1 + \partial_\sigma a_{12}(D_1\Psi_2 D_2\Psi_1 + D_2\Psi_2 D_1\Psi_1) + \\
&+ \partial_\sigma a_{22} D_1\Psi_2 D_2\Psi_2\big] + (D_2 z_{t,\sigma})^2\big[\partial_\sigma a_{11}(D_2\Psi_1)^2 + 2\partial_\sigma a_{12} D_2\Psi_2 D_2\Psi_1 + \partial_\sigma a_{22}(D_2\Psi_2)^2\big].
\end{aligned}$$

Developing the terms $\partial_\sigma a_{ij}$, $i, j = 1, 2$ we get

$$\begin{aligned}
&(\partial_\sigma A(D\Psi_\sigma)^T Dz_{t,\sigma} | (D\Psi_\sigma)^T Dz_{t,\sigma}) = (D_1 z_{t,\sigma})^2\big[D_1\Psi_1 D_2\tilde{\phi}_2 + D_2\Psi_1 D_1\tilde{\phi}_2 - D_1\Psi_2 D_2\tilde{\phi}_1 - \\
&- D_2\Psi_2 D_1\tilde{\phi}_1\big] + 2D_1 z_{t,\sigma} D_2 z_{t,\sigma}\big[D_2\Psi_1 D_2\tilde{\phi}_2 - D_1\Psi_1 D_1\tilde{\phi}_2 + D_1\Psi_2 D_1\tilde{\phi}_1 - D_2\Psi_2 D_2\tilde{\phi}_1\big] + \\
&+ (D_2 z_{t,\sigma})^2\big[D_2\Psi_2 D_1\tilde{\phi}_1 + D_1\Psi_2 D_2\tilde{\phi}_1 - D_2\Psi_1 D_1\tilde{\phi}_2 - D_1\Psi_1 D_2\tilde{\phi}_2\big],
\end{aligned}$$

where $\tilde{\phi}(x) := \partial_\sigma \Psi_\sigma(x) = \phi(\Psi_\sigma(x))$. In conclusion we have obtained the following equality:

$$\begin{aligned}
&\int_{\Omega\setminus\Gamma(\sigma_0)} (\partial_\sigma A(\sigma, x) Dv_{t,\sigma} | Dv_{t,\sigma})\, dx = \\
&= \int_{\Omega\setminus\Gamma(\sigma_0)} \Big\{\big[(D_2 z_{t,\sigma})^2 - (D_1 z_{t,\sigma})^2\big]\big[D_2\Psi_2 D_1\tilde{\phi}_1 + D_1\Psi_2 D_2\tilde{\phi}_1 - D_2\Psi_1 D_1\tilde{\phi}_2 - D_1\Psi_1 D_2\tilde{\phi}_2\big] - \\
&- 2D_1 z_{t,\sigma} D_2 z_{t,\sigma}(D_2\Psi_1 D_2\tilde{\phi}_2 - D_1\Psi_1 D_1\tilde{\phi}_2 - D_2\Psi_2 D_2\tilde{\phi}_1 + D_1\Psi_2 D_1\tilde{\phi}_1)\Big\}\, dx.
\end{aligned}$$

Now, for $y = \Psi_\sigma(x)$, $\tilde{\phi}(y) = \phi(\Psi_\sigma(x))$, we deduce $D\phi(\Psi_\sigma(x)) = D\tilde{\phi}(y)(D\Psi_\sigma)^{-1}$, where

$$D\tilde{\phi}(D\Psi_\sigma)^{-1} = \frac{1}{\det D\Psi_\sigma}\begin{pmatrix} D_2\Psi_2 D_1\tilde{\phi}_1 - D_2\Psi_1 D_1\tilde{\phi}_2 & D_1\Psi_1 D_1\tilde{\phi}_2 - D_1\Psi_2 D_1\tilde{\phi}_1 \\ D_2\Psi_2 D_2\tilde{\phi}_1 - D_2\Psi_1 D_2\tilde{\phi}_2 & D_1\Psi_1 D_2\tilde{\phi}_2 - D_1\Psi_2 D_2\tilde{\phi}_1 \end{pmatrix}.$$

Therefore,

$$\begin{aligned}
&\int_{\Omega\setminus\Gamma(\sigma_0)} (\partial_\sigma A(\sigma, x) Dv_{t,\sigma} | Dv_\sigma)\, dx = \\
&= \int_{\Omega\setminus\Gamma(\sigma_0)} \Big\{\big[(D_2 z_{t,\sigma})^2 - (D_1 z_{t,\sigma})^2\big](D_1\phi_1(\Psi_\sigma) - D_2\phi_2(\Psi_\sigma)) - \\
&\quad - 2D_1 z_{t,\sigma} D_2 z_{t,\sigma}(D_1\phi_2(\Psi_\sigma) + D_2\phi_1(\Psi_\sigma))\Big\} \det D\Psi_\sigma\, dx = \\
&= \int_{\Omega\setminus\Gamma(\sigma)} \Big\{[(D_2 u_{t,\sigma})^2 - (D_1 u_{t,\sigma})^2](D_1\phi_1 - D_2\phi_2) - \\
&\quad - 2D_1 u_{t,\sigma} D_2 u_{t,\sigma}(D_1\phi_2 + D_2\phi_1)\Big\}\, dy,
\end{aligned}$$

and this concludes the proof.

□

Remark 3.2.6. Fix $t_0 \in \,]0,T[$. With the same notation as in Remark 3.2.5 we set $v_\sigma := v_{t_0,\sigma}$, and note that the second order differential, $\partial^2_{(v,\sigma)}\mathscr{F}(t_0,v_\sigma,\sigma)$, of $\mathscr{F}$ with respect to (v,σ) is strictly positive definite if and only if the second order derivative of the function $\sigma \mapsto \mathscr{F}(t_0,v_\sigma,\sigma)$ is strictly positive, when both exist. Moreover, by Proposition 3.2.1, this is equivalent to the fact that the second order derivative of $\sigma \mapsto E(t_0,\sigma)$ is strictly positive.

Indeed, as $\partial_v\mathscr{F}(t_0,v_\sigma,\sigma)=0$, and $\sigma \mapsto v_\sigma$ is, by Remark 3.2.5, a C^2-function, we have

$$\langle \partial_\sigma\partial_v\mathscr{F}(t_0,v_\sigma,\sigma),w\rangle + \langle \partial^2_{vv}\mathscr{F}(t_0,v_\sigma,\sigma)v'_\sigma,w\rangle = 0 \quad \forall\, w\in V\,.$$

Assume that the second order derivative of the function $\sigma \mapsto \mathscr{F}(t_0,v_\sigma,\sigma)$ is strictly positive, i.e.,

$$\begin{aligned}0 < {} & \frac{d}{d\sigma}(\partial_\sigma\mathscr{F}(t_0,v_\sigma,\sigma)+\langle\partial_v\mathscr{F}(t_0,v_\sigma,\sigma),v'_\sigma\rangle) = \\ & = \partial^2_{\sigma\sigma}\mathscr{F}(t_0,v_\sigma,\sigma)+\langle\partial_\sigma\partial_v\mathscr{F}(t_0,v_\sigma,\sigma),v'_\sigma\rangle+\langle\partial_v\partial_\sigma\mathscr{F}(t_0,v_\sigma,\sigma),v'_\sigma\rangle + \\ & +\langle\partial^2_{vv}\mathscr{F}(t_0,v_\sigma,\sigma)v'_\sigma,v'_\sigma\rangle+\langle\partial_v\mathscr{F}(t_0,v_\sigma,\sigma),v''_\sigma\rangle\,.\end{aligned}$$

Hence

$$\partial^2_{\sigma\sigma}\mathscr{F}(t_0,v_\sigma,\sigma)+\langle\partial_v\partial_\sigma\mathscr{F}(t_0,v_\sigma,\sigma),v'_\sigma\rangle > 0\,,$$

which implies that

$$\partial^2_{\sigma\sigma}\mathscr{F}(t_0,v_\sigma,\sigma) > \langle\partial^2_{vv}\mathscr{F}(t_0,v_\sigma,\sigma)v'_\sigma,v'_\sigma\rangle$$

(recall that in our case $\langle\partial_\sigma\partial_v\mathscr{F},w\rangle = \langle\partial_v\partial_\sigma\mathscr{F},w\rangle$).

Therefore

$$\begin{aligned}&\langle\langle\partial^2_{(v,\sigma)}\mathscr{F}(t_0,v_\sigma,\sigma)(w,\tau),(w,\tau)\rangle\rangle = \\ &\quad = \partial^2_{\sigma\sigma}\mathscr{F}(t_0,v_\sigma,\sigma)\tau^2 + 2\langle\partial_\sigma\partial_v\mathscr{F}(t_0,v_\sigma,\sigma),w\rangle\tau + \langle\partial^2_{vv}\mathscr{F}(t_0,v_\sigma,\sigma)w,w\rangle > \\ &\quad > \langle\partial^2_{vv}\mathscr{F}(t_0,v_\sigma,\sigma)v'_\sigma,v'_\sigma\rangle\tau^2 - 2\langle\partial^2_{vv}\mathscr{F}(t_0,v_\sigma,\sigma)v'_\sigma,w\rangle\tau + \\ &\qquad +\langle\partial^2_{vv}\mathscr{F}(t_0,v_\sigma,\sigma)w,w\rangle \geq 0\,,\end{aligned}$$

which shows that $\partial^2_{(v,\sigma)}\mathscr{F}(t_0,v_\sigma,\sigma)$ is strictly positive definite.

It is also easy to see that if $\partial^2_{(v,\sigma)}\mathscr{F}(t_0,v_\sigma,\sigma)$ is strictly positive definite then the second order derivative of the function $\sigma \mapsto \mathscr{F}(t_0,v_\sigma,\sigma)$ is strictly positive.

3.3 Irreversible quasistatic evolution

Let σ_0 an initial crack length and let u_0 be an initial value of the displacement, such that the initial configuration (u_0,σ_0) is in unilateral equilibrium, i.e., unilateral with respect to the crack growth:

$$\begin{cases} \mathrm{grad}_v \mathscr{F}(0, u_0, \sigma_0) = 0\,, \\ \partial_\sigma \mathscr{F}(0, u_0, \sigma_0) \geq 0\,. \end{cases}$$

Our purpose is to study a quasistatic evolution of configurations (u, σ) which starts from (u_0, σ_0).

We are interested in the evolution until the crack length reaches the value σ_1. We cannot avoid the solution to have jumps (even at $t = 0$) to configurations with crack lengths larger than σ_1; if this is the case, then the boundary data are not compatible with a progressive crack growth on the interval $[\sigma_0, \sigma_1]$.

Definition 3.3.1. The *irreversible quasistatic evolution problem* consists in finding a left-continuous map $t \mapsto (u(t), \sigma(t))$, where $\sigma(t)$ represents the length of the crack up to time t, and the displacement $u(t)$ belongs to $AD(\psi(t), \sigma(t))$, which satisfies the following three conditions:

(a) *local unilateral stability:* for every t

$$\mathscr{E}(t)(u(t), \sigma(t)) \leq \mathscr{E}(t)(u, \sigma(t)) \qquad \forall u \in AD(\psi(t), \sigma(t)) \tag{3.3.1}$$

$$\partial_\sigma E(t, \sigma(t)) \geq 0, \tag{3.3.2}$$

where $E(t, \sigma)$ is defined in Proposition 3.2.1;

(b) *irreversibility:* the map $t \mapsto \sigma(t)$ is increasing;

(c) *energy inequality:* for every $0 \leq s < t$ we have

$$\mathscr{E}(t)(u(t), \sigma(t)) \leq \mathscr{E}(s)(u(s), \sigma(s)) + \\ + \int_s^t \Big(2 \int_{\Omega \setminus \Gamma(\sigma(\tau))} (Du(\tau) | D\dot{\psi}(\tau)) dx - \int_{\partial_N \Omega} g(\tau) \dot{\psi}(\tau) d\mathscr{H}^1 - \int_{\partial_N \Omega} \dot{g}(\tau) u(\tau) d\mathscr{H}^1 \Big) d\tau\,.$$

In terms of the functional $\mathscr{F}$, the irreversible quasistatic evolution problem consists in finding a left-continuous function $t \mapsto (v(t), \sigma(t))$ which satisfies the following three conditions:

($\mathrm{a}_{\mathscr{F}}$) local unilateral stability: for every t

$$\begin{cases} \mathrm{grad}_v \mathscr{F}(t, v(t), \sigma(t)) = 0, \\ \partial_\sigma \mathscr{F}(t, v(t), \sigma(t)) \geq 0; \end{cases}$$

($\mathrm{b}_{\mathscr{F}}$) irreversibility: the map $t \mapsto \sigma(t)$ is increasing;

($\mathrm{c}_{\mathscr{F}}$) energy inequality: for every $0 \leq s < t$ we have

$$\mathscr{F}(t, v(t), \sigma(t)) \leq \mathscr{F}(s, v(s), \sigma(s)) + \int_s^t \partial_t \mathscr{F}(\tau, v(\tau), \sigma(\tau)) d\tau\,.$$

A solution, $t \mapsto (v(t), \sigma(t))$, to this problem is called an *irreversible quasistatic evolution for* $\mathscr{F}$.

Let us remark that, by the very construction of the functional $\mathscr{F}$, an evolution for $\mathscr{F}$ is well-defined only for cracks whose length is less than or equal to $\overline{\sigma}$.

In terms of an irreversible quasistatic evolution $t \mapsto (v(t), \sigma(t))$ associated to the functional $\mathscr{F}$, the Griffith's criterion can be expressed as:

$$\begin{cases} \dot{\sigma}(t) \geq 0 \\ \partial_\sigma \mathscr{F}(t, v(t), \sigma(t)) \geq 0 \\ \partial_\sigma \mathscr{F}(t, v(t), \sigma(t)) \dot{\sigma}(t) = 0 \end{cases} \tag{3.3.3}$$

for a.e. t. Since the first two conditions are included in the definition of an irreversible quasistatic evolution, it remains to prove the last one.

Proposition 3.3.2. *Let $t \mapsto (v(t), \sigma(t))$ be an irreversible quasistatic evolution for $\mathscr{F}$. Then for a.e. t we have*

$$\partial_\sigma \mathscr{F}(t, v(t), \sigma(t)) \dot{\sigma}(t) = 0\,.$$

Proof. Since σ is increasing, $\dot{\sigma}$ exists at a.e. t. Fix t_0 such that $\dot{\sigma}(t_0)$ exists. As, given $\sigma(t)$, the function $v(t)$ is determined as the unique solution of $\mathrm{grad}_v \mathscr{F}(t, v, \sigma(t)) = 0$, the hypotheses we made on $A(\sigma)$ and on the data ψ and g imply that $\dot{v}(t_0)$ exists, as strong limit in V of the difference quotient $\frac{v(t)-v(t_0)}{t-t_0}$.

More in detail, from the continuity of the map $t \mapsto \mathrm{grad}_v \mathscr{F}(t, v(t), \sigma(t))$ and the weak convergence of $v(t)$ to $v(t_0)$ in V as $t \to t_0$ (which can be easily deduced from standard calculations), we obtain that

$$(A(\sigma(t))Dv(t), Dv(t)) \to (A(\sigma(t_0))Dv(t_0), Dv(t_0))\,.$$

Hence, using also the continuity of the map $t \mapsto A(\sigma(t))$ we get

$$\lim_{t \to t_0} (A(\sigma(t))(Dv(t) - Dv(t_0)), Dv(t) - Dv(t_0)) =$$
$$\lim_{t \to t_0} \big[(A(\sigma(t))Dv(t), Dv(t)) - 2(A(\sigma(t))Dv(t), Dv(t_0)) + (A(\sigma(t))Dv(t_0), Dv(t_0))\big] = 0\,,$$

so that $v(t) \to v(t_0)$ strongly in V, as $t \to t_0$. Thus $\dot{v}(t_0)$ belongs to V, as strong limit in V of the difference quotient $\frac{v(t)-v(t_0)}{t-t_0}$.

Next, let us observe that, as $\mathrm{grad}_v \mathscr{F}(t, v(t), \sigma(t)) = 0$, we have

$$\partial_t \mathscr{F}(t, v(t), \sigma(t)) = \frac{d}{dt} \mathscr{F}(t, v(t), \sigma(t)) - \partial_\sigma \mathscr{F}(t, v(t), \sigma(t)) \dot{\sigma}(t)\,,$$

and using this fact together with the energy estimate ($\mathrm{c}_{\mathscr{F}}$) we deduce that for a.e. t

$$\partial_\sigma \mathscr{F}(t, v(t), \sigma(t)) \dot{\sigma}(t) \leq 0\,. \tag{3.3.4}$$

Since $\dot{\sigma}(t) \geq 0$ and $\partial_\sigma \mathscr{F}(t, v(t), \sigma(t)) \geq 0$, (3.3.4) implies the equality to be proved.

□

Going back to the energy functional $\mathscr{E}$, the Griffith's criterion now reads

$$\begin{cases} \dot{\sigma}(t) \geq 0 \\ 1 - \kappa^2(t) \geq 0 \\ (1 - \kappa^2(t))\dot{\sigma}(t) = 0 \end{cases} \tag{3.3.5}$$

for a.e. t, where $\kappa(t)\sqrt{\frac{2}{\pi}}$ is the stress intensity factor associated to the displacement $u(t)$ at the tip $\sigma(t)$ (see Proposition 3.2.2). Since by the change of variables we made, $\partial_\sigma \mathscr{F}(t, v(t), \sigma(t)) = 1 - \kappa^2(t)$, the previous proposition shows that during an irreversible quasistatic evolution the Griffith's criterion is satisfied. Note that this can be proved directly for $\mathscr{E}$, following, for instance, the lines of [10, Theorem 6.1].

Let us return to our notion of irreversible quasistatic evolution. We remark that the globally stable evolutions of minimum energy configurations studied in [9], [3], [17], [8] satisfy the three axioms of our Definition 3.3.1. Moreover we recall that during a globally stable irreversible quasistatic evolution the total energy is an absolutely continuous function of time and the energy inequality (c) becomes an equality. However, this notion of evolution is not completely satisfactory since, in order to get the global stability, we have to compare, at each time, the energy of a configuration with the energy of all admissible configurations with larger crack lengths.

For this reason we adopt here a different selection criterion: among all irreversible quasistatic evolutions we choose the *approximable* ones, i.e., those that can be obtained as limits of solutions to a regularized evolution problem. In particular we consider here the approximation problem given by a modified ε-gradient flow for the functional $\mathscr{F}$.

Before giving the precise definition of the approximation problem for $\mathscr{F}$, we need some preliminary discussion. Since we are interested in an *irreversible* crack growth for σ varying in the interval $[\sigma_0, \sigma_1]$, we ask the function $\sigma(t)$ to be increasing. Hence, we are led to consider the positive part of the derivative of $\mathscr{F}$ with respect to σ.

In addition, we modify the evolution law for the crack length in such a way that it never reaches $\overline{\sigma}$. To this end we introduce a penalization factor $\lambda(\sigma)$ that can be any Lipschitz continuous function of σ which is equal to one for $\sigma \leq \sigma_1$, is strictly positive for $\sigma_1 < \sigma < \overline{\sigma}$, and is equal to zero for $\sigma = \overline{\sigma}$. For instance, let

$$\lambda(\sigma) := \frac{(\overline{\sigma} - (\sigma \vee \sigma_1))^+}{\overline{\sigma} - \sigma_1}\,. \tag{3.3.6}$$

In such a way the evolution is the one given by the ε-gradient flow, with the constraint that σ is increasing, on the interval $[\sigma_0, \sigma_1]$ that we are interested in, and it is modified by this artificial penalization term for $\sigma > \sigma_1$, so that we do not consider it meaningful for $\sigma > \sigma_1$.

Definition 3.3.3. A function $t \mapsto (v_\varepsilon(t), \sigma_\varepsilon(t))$ is called a *solution to the initial value problem for the modified ε-gradient flow* for the functional $\mathscr{F}$ on $[0,T]$

$$\begin{cases} \varepsilon \dot v_\varepsilon = -\mathrm{grad}_v \mathscr{F}(t, v_\varepsilon, \sigma_\varepsilon) \\ \varepsilon \dot\sigma_\varepsilon = (-\partial_\sigma \mathscr{F}(t, v_\varepsilon, \sigma_\varepsilon))^+ \lambda(\sigma_\varepsilon)\,, \\ v_\varepsilon(0) = u_0 \\ \sigma_\varepsilon(0) = \sigma_0\,, \end{cases} \tag{3.3.7}$$

where $\lambda(\sigma)$ is given by (3.3.6), if $v_\varepsilon \in C^1([0,T];V)$, σ_ε is a C^1-increasing function from $[0,T]$ into $[\sigma_0, \overline{\sigma}]$ and the first equation in (3.3.7) is satisfied in the following sense

$$(\varepsilon \dot v_\varepsilon, w)_V = -(\mathrm{grad}_v \mathscr{F}(t, v_\varepsilon, \sigma_\varepsilon), w)_V \qquad \forall\, w \in V \quad \forall\, t \in [0,T]\,.$$

Note that (3.3.7) is a Cauchy problem for an ordinary differential equation in $V \times \mathbb{R}$.

Theorem 3.3.4. *There exists a unique solution $(v_\varepsilon, \sigma_\varepsilon)$ to the initial value problem (3.3.7) with $\lambda(\sigma)$ given by (3.3.6), and the following energy estimate holds: for every s, $t \in [0,T]$ with $s < t$*

$$\begin{aligned} &\varepsilon \int_s^t \|\dot v_\varepsilon(\tau)\|_V^2 d\tau + \varepsilon \int_s^t \frac{|\dot\sigma_\varepsilon(\tau)|^2}{\lambda(\sigma_\varepsilon(\tau))}\, d\tau + \mathscr{F}(t, v_\varepsilon(t), \sigma_\varepsilon(t)) \le \\ &\le \mathscr{F}(s, v_\varepsilon(s), \sigma_\varepsilon(s)) + \int_s^t \partial_t \mathscr{F}(\tau, v_\varepsilon(\tau), \sigma_\varepsilon(\tau))\, d\tau\,. \end{aligned} \tag{3.3.8}$$

Proof. Taking into account the expressions of $\mathrm{grad}_v \mathscr{F}$ and $\partial_\sigma \mathscr{F}$ (see Section 3.2), the equations in (3.3.7) can be written as

$$\begin{cases} \varepsilon (\dot v_\varepsilon, w)_V = -2(A(\sigma_\varepsilon) D v_\varepsilon, Dw) - 2(\psi(t), w)_V + (g(t), w)_{\partial_N \Omega} \quad \forall\, w \in V \\ \varepsilon \dot\sigma_\varepsilon = (-(\partial_\sigma A(\sigma_\varepsilon) D v_\varepsilon, D v_\varepsilon) - 1)^+ \lambda(\sigma_\varepsilon)\,. \end{cases} \tag{3.3.9}$$

The vector field defining the equation (3.3.9) depends on t only through the boundary data ψ and g, therefore it is Lipschitz continuous in t. Moreover, for fixed t, standard estimates show that it is Lipschitz continuous and bounded on the bounded subsets of $V \times \mathbb{R}$. Hence classical results on ODE's (see, e.g. [13]) give the local existence and the uniqueness of the solution. Since there exist $\alpha \in C([0,T])$ and $\beta > 0$ such that

$$(-\mathrm{grad}_v \mathscr{F}(t, v, \sigma), v)_V + \sigma(-\partial_\sigma \mathscr{F}(t, v, \sigma))^+ \lambda(\sigma) \le \alpha(t)(\|v\|_V^2 + \sigma^2) + \beta$$

for every $(v, \sigma) \in V \times \mathbb{R}$, the solution is defined on the whole interval $[0,T]$.

The function $t \mapsto \mathscr{F}(t, v_\varepsilon(t), \sigma_\varepsilon(t))$ is then Lipschitz continuous on $[0,T]$ with derivative given for a.e. $t \in [0,T]$ by

$$\begin{aligned} \frac{d}{dt} \mathscr{F}(t, v_\varepsilon(t), \sigma_\varepsilon(t)) \quad = \quad & \partial_t \mathscr{F}(t, v_\varepsilon(t), \sigma_\varepsilon(t)) + (\mathrm{grad}_v \mathscr{F}(t, v_\varepsilon(t), \sigma_\varepsilon(t)), \dot v_\varepsilon(t))_V + \\ & + \partial_\sigma \mathscr{F}(t, v_\varepsilon(t), \sigma_\varepsilon(t)) \dot\sigma_\varepsilon(t)\,. \end{aligned}$$

Taking into account the equations satisfied by v_ε and σ_ε, for every s, $t \in [0,T]$ with $s < t$ we have

$$
\begin{aligned}
&\mathscr{F}(t, v_\varepsilon(t), \sigma_\varepsilon(t)) - \mathscr{F}(s, v_\varepsilon(s), \sigma_\varepsilon(s)) = \\
&\qquad = \int_s^t \left(\partial_t \mathscr{F}(\tau, v_\varepsilon(\tau), \sigma_\varepsilon(\tau)) - \varepsilon \|\dot v_\varepsilon(\tau)\|_V^2 - \varepsilon \frac{(\dot\sigma_\varepsilon(\tau))^2}{\lambda(\sigma_\varepsilon(\tau))} \right) d\tau \,,
\end{aligned}
$$

which implies (3.3.8) and the proof is complete. $\square$

Remark 3.3.5. Let $t \mapsto (v_\varepsilon(t), \sigma_\varepsilon(t))$ be a solution to problem (3.3.7). Assume $\|v_\varepsilon(t)\|_V \leq M$ for some positive constant M independent of t and ε. By (3.1.3) and the definition (3.3.6) of the map λ,

$$
\varepsilon \dot\sigma_\varepsilon(t) \leq (\Lambda' M^2 + 1)\lambda(\sigma_\varepsilon(t)) \leq C(\overline{\sigma} - \sigma_\varepsilon(t))^+ \,,
$$

for some constant $C > 0$. By classical results on differential inequalities (see, e.g. [25, Theorem I.6.1]) it follows that for every $t \in [0,T]$

$$
\sigma_\varepsilon(t) \leq \overline{\sigma} - e^{-C\frac{t}{\varepsilon}}(\overline{\sigma} - \sigma_0) \,,
$$

hence σ_ε never reaches $\overline{\sigma}$.

Note that, since the evolution is constrained to cracks with lengths less than or equal to $\overline{\sigma}$, Griffith's criterion is meaningful in this setting only until the length $\overline{\sigma}$ is reached. As the penalization factor $\lambda(\sigma)$ is strictly positive for $\sigma < \overline{\sigma}$, we may replace the expression (3.3.3) of Griffith's criterion by

$$
\begin{cases}
\dot\sigma(t) \geq 0 \\
\partial_\sigma \mathscr{F}(t, v(t), \sigma(t))\lambda(\sigma(t)) \geq 0 \\
\partial_\sigma \mathscr{F}(t, v(t), \sigma(t))\dot\sigma(t) = 0 \,.
\end{cases}
$$

for a.e. $t \in [0,T]$. Therefore, also the second line in the local stability condition ($\mathrm{a}_{\mathscr{F}}$) may be replaced by $\partial_\sigma \mathscr{F}(t, v(t), \sigma(t))\lambda(\sigma(t)) \geq 0$.

We introduce now the following notion of evolution.

Definition 3.3.6. The *approximable irreversible quasistatic evolution problem* on the interval $[0,T]$ with initial data (u_0, σ_0) consists in finding a left-continuous map $t \mapsto (v(t), \sigma(t))$ from $[0,T]$ into $V \times \mathbb{R}$ which satisfies the following conditions:

($\mathrm{a}'_{\mathscr{F}}$) for every $t \in [0,T]$

$$
\begin{aligned}
&\mathrm{grad}_v \mathscr{F}(t, v(t), \sigma(t)) = 0 \\
&\partial_\sigma \mathscr{F}(t, v(t), \sigma(t))\lambda(\sigma(t)) \geq 0 \,;
\end{aligned}
$$

($\mathrm{b}_{\mathscr{F}}$) the map $t \mapsto \sigma(t)$ is increasing;

($\mathrm{c}_{\mathscr{F}}$) for every $0 \leq s < t \leq T$

$$\mathscr{F}(t, v(t), \sigma(t)) \leq \mathscr{F}(s, v(s), \sigma(s)) + \int_s^t \partial_t \mathscr{F}(\tau, v(\tau), \sigma(\tau)) d\tau \,;$$

($\mathrm{d}_{\mathscr{F}}$) the pair $(v(t), \sigma(t))$ is the limit, along a suitable sequence, of solutions $(v_\varepsilon(t), \sigma_\varepsilon(t))$ of the *modified ε-gradient flow* for $\mathscr{F}$ with initial conditions $v_\varepsilon(0) = u_0$ and $\sigma_\varepsilon(0) = \sigma_0$, in the sense that for a.e. t

$$\begin{aligned} &\sigma_\varepsilon(t) \to \sigma(t), \\ &v_\varepsilon(t) \to v(t) \quad \text{strongly in } V. \end{aligned} \tag{3.3.10}$$

A solution $t \mapsto (v(t), \sigma(t))$ to this problem is called an *approximable quasistatic evolution* for $\mathscr{F}$.

We are now in a position to state the main result of this paper.

Theorem 3.3.7. *There exists a solution $t \mapsto (v(t), \sigma(t))$ to the approximable irreversible quasistatic evolution problem with initial condition (u_0, σ_0) on $[0, T]$.*

Remark 3.3.8. The fact that an approximable quasistatic evolution starts from (u_0, σ_0) means only that for every $\varepsilon > 0$, $v_\varepsilon(0) = u_0$ and $\sigma_\varepsilon(0) = \sigma_0$. We may always set $(v(0), \sigma(0)) := (u_0, \sigma_0)$, but in general v and σ are not continuous in $t = 0$. The only case in which (u_0, σ_0) is the initial value for the evolution in a "classical" sense, is when (u_0, σ_0) is the absolute minimum point of $\mathscr{F}(0, \cdot, \cdot)$. Indeed, in this case, by semicontinuity and by the energy inequality ($\mathrm{c}_{\mathscr{F}}$), it is easy to see that $t \mapsto \mathscr{F}(t, v(t), \sigma(t))$ is continuous in $t = 0$.

Proof of Theorem 3.3.7. For $\varepsilon > 0$ let $(v_\varepsilon, \sigma_\varepsilon)$ be the solution of the modified ε-gradient flow with initial data (u_0, σ_0). Let $t \in [0, T]$. From the ε-energy inequality (3.3.8) evaluated between $s = 0$ and $t \in (0, T]$, using the estimates we have on $\mathscr{F}$ (see Subsection 3.1.6) we get

$$\lambda_{\mathscr{F}} \|v_\varepsilon(t)\|_V^2 \leq \mu_{\mathscr{F}} + \mathscr{F}(0, u_0, \sigma_0) + \int_0^t (a(\tau)\|v_\varepsilon(\tau)\|_V^2 + b(\tau)) d\tau$$

for some functions $a, b \in L^\infty(0, T)$ which depend only on the data ψ and g. Then, by Gronwall's Lemma, there exists a positive constant $C > 0$ independent of t and ε such that

$$\|v_\varepsilon(t)\|_V \leq C \qquad \forall t \in [0, T] \,. \tag{3.3.11}$$

By the ε-energy estimate (3.3.8) we now get that there exists two positive constants C_1 and C_2 such that

$$\varepsilon \|\dot{v}_\varepsilon\|_{L^2(0,T;V)}^2 \leq C_1 \tag{3.3.12}$$

$$\varepsilon \|\dot{\sigma}_\varepsilon\|_{L^2(0,T)}^2 \leq C_2 \,. \tag{3.3.13}$$

Let $\varepsilon \to 0$. By Helly's Theorem, there exists a subsequence, still denoted by ε, and an increasing function $\sigma\colon [0,T] \to [\sigma_0, \overline{\sigma}]$ such that

$$\sigma_\varepsilon(t) \to \sigma(t) \qquad \text{for every } t \in [0,T]\,.$$

The estimate (3.3.11) implies that there exists a function $v \in L^2(0,T;V)$ such that

$$v_\varepsilon \rightharpoonup v \qquad \text{weakly in } L^2(0,T;V)\,,$$

while, by (3.3.12),

$$\varepsilon \dot{v}_\varepsilon \to 0 \qquad \text{strongly in } L^2(0,T;V)\,.$$

Hence

$$\varepsilon(\dot{v}_\varepsilon(t), w)_V = (-\mathrm{grad}_v \mathscr{F}(t, v_\varepsilon(t), \sigma_\varepsilon(t)), w)_V \to (-\mathrm{grad}_v \mathscr{F}(t, v(t), \sigma(t)), w)_V = 0\,,$$

for every $w \in V$ and for a.e. $t \in [0,T]$. It follows that

$$\int_0^T (A(\sigma_\varepsilon(t)) D v_\varepsilon(t), D v_\varepsilon(t)) dt \to \int_0^T (A(\sigma(t)) D v(t), D v(t)) dt\,,$$

which gives the strong convergence in V of $v_\varepsilon(t)$ to $v(t)$ for a.e. $t \in [0,T]$, by using the same argument proposed during the proof of Proposition 3.3.2.

By (3.3.13), $\varepsilon \dot{\sigma}_\varepsilon(t) \to 0$ for a.e. $t \in [0,T]$. Taking into account the equation satisfied by σ_ε, we obtain that $(-\partial_\sigma \mathscr{F}(t, v(t), \sigma(t)))^+ \lambda(\sigma(t)) = 0$ for a.e. $t \in [0,T]$, i.e., $\partial_\sigma \mathscr{F}(t, v(t), \sigma(t)) \lambda(\sigma(t)) \geq 0$.

When passing to the limit in the ε-energy estimate (3.3.8), we neglect the terms containing the norms of the time derivatives of v_ε and σ_ε, and thus get that for a.e. s, $t \in [0,T]$ with $s < t$

$$\mathscr{F}(t, v(t), \sigma(t)) \leq \mathscr{F}(s, v(s), \sigma(s)) + \int_s^t \partial_t \mathscr{F}(\tau, v(\tau), \sigma(\tau)) d\tau\,. \tag{3.3.14}$$

(By semicontinuity the estimate holds true for every $t \in [0,T]$.)

Since σ is increasing, for every $t \in [0,T]$ there exists the limit $\sigma^\ominus(t) := \lim_{s \to t^-} \sigma(s)$. Let $v^\ominus(t)$ be the unique solution to $\mathrm{grad}_v \mathscr{F}(t, v, \sigma^\ominus(t)) = 0$. Then $v(s) \to v^\ominus(t)$ strongly in V as $s \to t^-$, $\sigma(t) = \sigma^\ominus(t)$ and $v(t) = v^\ominus(t)$ for a.e. $t \in [0,T]$. By construction, the map $t \mapsto (v^\ominus(t), \sigma^\ominus(t))$ is left-continuous from $[0,T]$ into $V \times [\sigma_0, \overline{\sigma}]$. Moreover, $\partial_\sigma \mathscr{F}(t, v^\ominus(t), \sigma^\ominus(t)) \lambda(\sigma^\ominus(t)) \geq 0$ for every $t \in [0,T]$. Let s, $t \in [0,T]$ with $s < t$, and let $s_n \to s^-$, $t_n \to t^-$ be such that (3.3.14) holds for s_n and t_n. Passing to the limit as $n \to +\infty$ we obtain

$$\mathscr{F}(t, v^\ominus(t), \sigma^\ominus(t)) \leq \mathscr{F}(s, v^\ominus(s), \sigma^\ominus(s)) + \int_s^t \partial_t \mathscr{F}(\tau, v^\ominus(\tau), \sigma^\ominus(\tau)) d\tau\,,$$

so that we conclude that $(v^\ominus, \sigma^\ominus)$ is an approximable quasistatic evolution for $\mathscr{F}$ on $[0,T]$ which starts from (u_0, σ_0). $\square$

Remark 3.3.9. From (3.3.14) we deduce that if $\hat{t} \in [0,T]$ is a discontinuity point of $t \mapsto \mathscr{F}(t, v(t), \sigma(t))$ then

$$\lim_{t \to \hat{t}^+} \mathscr{F}(t, v(t), \sigma(t)) \leq \mathscr{F}(\hat{t}, v(\hat{t}\,), \sigma(\hat{t}\,))\,.$$

Indeed, note that at every time t the function $t \mapsto \sigma(t)$ has a right limit. Let $\sigma^{\oplus}(\hat{t}\,) := \lim_{t\to\hat{t}^+} \sigma(t)$, and let $v^{\oplus}(\hat{t}\,)$ be the solution to $\mathrm{grad}_v \mathscr{F}(\hat{t}, v, \sigma^{\oplus}(\hat{t}\,)) = 0$. By the regularity assumptions made on the data, we have that $v(t)$ converges to $v^{\oplus}(\hat{t}\,)$ strongly in V, and hence, using (3.3.14), we obtain

$$\lim_{t \to t^+} \mathscr{F}(t, v(t), \sigma(t)) = \mathscr{F}(\hat{t}, v^{\oplus}(\hat{t}\,), \sigma^{\oplus}(\hat{t}\,)) \leq \mathscr{F}(\hat{t}, v(\hat{t}\,), \sigma(\hat{t}\,))\,.$$

3.4 Quasistatic evolution and the Implicit Function Theorem

In this section we show that, under suitable regularity assumptions, the solution to the modified ε-gradient flow converges to the continuous solution for the quasistatic evolution problem given by the Implicit Function Theorem.

We recall that for fixed t and σ the map $u \mapsto \mathscr{E}(t)(u,\sigma)$ admits a unique minimum point $u_{t,\sigma}$ and we set $E(t,\sigma) := \mathscr{E}(t)(u_{t,\sigma}, \sigma)$ as in Proposition 3.2.1.

Theorem 3.4.1. *Assume that in $(t^0, \sigma^0) \in [0,T[\, \times [\sigma_0, \sigma_1[$ the following conditions are satisfied*

$$\begin{aligned} &\partial_\sigma E(t^0, \sigma^0) = 0 \\ &\partial_\sigma^2 E(t^0, \sigma^0) > 0\,. \end{aligned}$$

Then there exists a time interval $[t^0, t^1]$ and a unique Lipschitz continuous function $\sigma^0 : [t^0, t^1] \to [\sigma^0, \sigma_1]$ such that

$$\partial_\sigma E(t, \sigma^0(t)) = 0 \qquad \forall\, t \in [t^0, t^1]\,.$$

Moreover, if $(v_\varepsilon, \sigma_\varepsilon)$ is the solution to the modified ε-gradient flow and the following two conditions are satisfied:

$$\begin{aligned} &\dot{\sigma}_\varepsilon(t) > 0 \qquad \forall\, t \in [t^0, t^1] \\ &\sigma_\varepsilon(t^0) \to \sigma^0 \qquad \text{as } \varepsilon \to 0\,, \end{aligned}$$

then $\sigma_\varepsilon(t) \to \sigma^0(t)$ and $E(t, \sigma_\varepsilon(t)) \to E(t, \sigma^0(t))$ for every $t \in [t^0, t^1]$.

The first part of the theorem follows from the Implicit Function Theorem. As for the second part, let us remark that even if there are not at the moment general theorems guaranteeing the strict monotonicity of σ_ε during the approximation process, in many cases this will follow, for a suitable choice of the boundary data, from a symmetry argument.

We now prove the theorem in an equivalent form for the functional $\mathscr{F}$. Indeed, since $\partial_\sigma E(t,\sigma) = \partial_\sigma \mathscr{F}(t, v_{t,\sigma}, \sigma)$ (see Proposition 3.2.1), if the second order derivative $\partial^2_\sigma E(t^0, \sigma^0) > 0$, then also $\frac{d}{d\sigma}\partial_\sigma \mathscr{F}(t^0, v_{t^0,\sigma^0}, \sigma^0) > 0$, and this last condition is equivalent to the fact that the second order partial differential $\partial^2_{(v,\sigma)}\mathscr{F}(t^0, v_{t^0,\sigma^0}, \sigma^0)$ is strictly positive definite (see Remark 3.2.6).

Theorem 3.4.2. *Assume that in $(t^0, v^0, \sigma^0) \in [0,T[\times V \times [\sigma_0, \sigma_1[$ the following conditions are satisfied*

$$\begin{cases} \mathrm{grad}_v \mathscr{F}(t^0, v^0, \sigma^0) = 0\,, \\ \partial_\sigma \mathscr{F}(t^0, v^0, \sigma^0) = 0\,, \end{cases}$$

and the second order differential, $\partial^2_{(v,\sigma)}\mathscr{F}(t^0, v^0, \sigma^0)$, of $\mathscr{F}$ with respect to (v,σ) is strictly positive definite, i.e., there exists $\alpha > 0$ such that

$$\langle\langle \partial^2_{(v,\sigma)}\mathscr{F}(t^0, v^0, \sigma^0)(w,\tau),(w,\tau)\rangle\rangle \geq \alpha(\|w\|^2_V + |\tau|^2) \qquad \forall w \in V \quad \forall \tau \in \mathbb{R}\,. \tag{3.4.1}$$

Then there exist a time interval $[t^0, t^1]$ and a unique Lipschitz continuous function (v^0, σ^0) : $[t^0, t^1] \to V \times [\sigma^0, \sigma_1]$ such that

$$\begin{cases} \mathrm{grad}_v \mathscr{F}(t, v^0(t), \sigma^0(t)) = 0 \\ \partial_\sigma \mathscr{F}(t, v^0(t), \sigma^0(t)) = 0 \end{cases}$$

for every $t \in [t^0, t^1]$.

Moreover, let $(v_\varepsilon, \sigma_\varepsilon)$ be the solution of the modified ε-gradient flow for $\mathscr{F}$ given by Theorem 3.3.4 and assume that

$$\begin{aligned} &v_\varepsilon(t^0) \to v^0 \qquad \textit{strongly in } V \quad \textit{and} \\ &\sigma_\varepsilon(t^0) \to \sigma^0 \qquad \textit{as } \varepsilon \to 0\,. \end{aligned}$$

Assume in addition that $\dot\sigma_\varepsilon(t) > 0$ and $\sigma_\varepsilon(t) < \sigma_1$ for every $t \in [t^0, t^1]$. Then for every $t \in [t^0, t^1]$

$$\begin{aligned} &v_\varepsilon(t) \to v^0(t) \qquad \textit{strongly in } V \quad \textit{and} \\ &\sigma_\varepsilon(t) \to \sigma^0(t)\,, \end{aligned}$$

as ε tends to zero.

Proof. By our assumptions on the data, $\partial^2_{(v,\sigma)}\mathscr{F}(t,v,\sigma)$ (see Section 3.2) is continuous with respect to $(t,v,\sigma) \in [0,T] \times V \times [\sigma_0, \overline{\sigma}]$. Moreover, the function $t \mapsto \partial_t \mathrm{grad}_v \mathscr{F}(t,v,\sigma)$ belongs to $L^\infty(0,T;V)$, while $\partial_t\partial_\sigma \mathscr{F}(t,v,\sigma) = 0$. By the Implicit Function Theorem (see, e.g., [29]) applied in (t^0, v^0, σ^0) to

$$\begin{cases} \mathrm{grad}_v \mathscr{F}(t, v, \sigma) = 0 \\ \partial_\sigma \mathscr{F}(t, v, \sigma) = 0\,, \end{cases}$$

it follows that there exist a time interval $[t^0, t^1]$ and a unique Lipschitz continuous function $(v^0, \sigma^0)\colon [t^0, t^1] \to V \times [\sigma^0, \sigma_1)$ such that

$$\begin{cases} \mathrm{grad}_v \mathscr{F}(t, v^0(t), \sigma^0(t)) = 0 \\ \partial_\sigma \mathscr{F}(t, v^0(t), \sigma^0(t)) = 0 \end{cases} \tag{3.4.2}$$

for every $t \in [t^0, t^1]$. By a compactness argument, changing eventually the value of α, we may assume that there exist $\alpha > 0$ and $r > 0$ such that for every $t \in [t^0, t^1]$, for every $v \in B_r(v^0(t)) \subset V$, and for every $\sigma \in (\sigma^0(t) - r, \sigma^0(t) + r)$

$$\langle\langle \partial^2_{(v,\sigma)} \mathscr{F}(t, v, \sigma)(w, \tau), (w, \tau)\rangle\rangle \geq \alpha(\|w\|_V^2 + |\tau|^2) \qquad \forall w \in V \quad \forall \tau \in \mathbb{R}\,. \tag{3.4.3}$$

Restricting eventually the time interval, we have $\sigma^0(t) + r < \sigma_1$ for every $t \in [t^0, t^1]$.

Let $0 < r' < r$ be a number that we shall choose later. For every $\varepsilon > 0$ small enough we have $\|v_\varepsilon(t^0) - v^0\|_V < r'$ and $|\sigma_\varepsilon(t^0) - \sigma^0| < r'$. By continuity, there exists a time interval, depending on ε, on which these inequalities hold. Let τ_ε be the largest time such that for $t < \tau_\varepsilon$, $\|v_\varepsilon(t) - v^0(t)\|_V < r'$ and $|\sigma_\varepsilon(t) - \sigma^0(t)| < r'$. In addition, $\lambda(\sigma_\varepsilon(t)) = 1$ for $t < \tau_\varepsilon$.

We want to prove that $\tau_\varepsilon = t^1$. Assume by contradiction that $\tau_\varepsilon < t^1$. Taking $v_\varepsilon(t) - v^0(t)$ as test function in the equation satisfied by v_ε, multiplying by $\sigma_\varepsilon(t) - \sigma^0(t)$ the equation satisfied by σ_ε, and taking also into account system (3.4.2), we obtain

$$\begin{aligned}
&\frac{\varepsilon}{2}\frac{d}{dt}\|v_\varepsilon(t) - v^0(t)\|_V^2 + \frac{\varepsilon}{2}\frac{d}{dt}|\sigma_\varepsilon(t) - \sigma^0(t)|^2 = \\
&= -(\mathrm{grad}_v \mathscr{F}(t, v_\varepsilon(t), \sigma_\varepsilon(t)) - \mathrm{grad}_v \mathscr{F}(t, v^0(t), \sigma^0(t)), v_\varepsilon(t) - v^0(t))_V + \\
&\quad + \big(- \partial_\sigma \mathscr{F}(t, v_\varepsilon(t), \sigma_\varepsilon(t)) + \partial_\sigma \mathscr{F}(t, v^0(t), \sigma^0(t))\big)(\sigma_\varepsilon(t) - \sigma^0(t)) - \\
&\quad - \varepsilon(\dot v^0(t), v_\varepsilon(t) - v^0(t))_V - \varepsilon\dot\sigma^0(t)(\sigma_\varepsilon(t) - \sigma^0(t))\,.
\end{aligned}$$

Setting

$$\zeta_\varepsilon(t) := \|v_\varepsilon(t) - v^0(t)\|_V^2 + |\sigma_\varepsilon(t) - \sigma^0(t)|^2\,,$$

in the previous equality and using (3.4.3), it follows that

$$\begin{aligned}
\frac{\varepsilon}{2}\dot\zeta_\varepsilon(t) &\leq -\alpha\zeta_\varepsilon(t) - \varepsilon(\dot v^0(t), v_\varepsilon(t) - v^0(t))_V - \varepsilon\dot\sigma^0(t)(\sigma_\varepsilon(t) - \sigma^0(t)) \leq \\
&\leq -\alpha\zeta_\varepsilon(t) + \frac{\varepsilon}{2}\|\dot v^0(t)\|_V^2 + \frac{\varepsilon}{2}\|v_\varepsilon(t) - v^0(t)\|_V^2 + \frac{\varepsilon}{2}|\dot\sigma^0(t)|^2 + \frac{\varepsilon}{2}|\sigma_\varepsilon(t) - \sigma^0(t)|^2 \leq \\
&\leq (-\alpha + \frac{\varepsilon}{2})\zeta_\varepsilon(t) + \frac{\varepsilon}{2}\beta \qquad \forall t \in [t^0, \tau_\varepsilon)\,,
\end{aligned} \tag{3.4.4}$$

where β is an upper bound for $\|\dot v^0(t)\|_V^2 + |\dot\sigma^0(t)|^2$ on $[t^0, t^1]$.

Hence

$$\zeta_\varepsilon(t) \leq \Big(\zeta_\varepsilon(t^0) - \frac{\beta\varepsilon}{2\alpha - \varepsilon}\Big) e^{(-\frac{2\alpha}{\varepsilon}+1)(t-t^0)} + \frac{\beta\varepsilon}{2\alpha - \varepsilon} \qquad \forall\, t \in [t^0, \tau_\varepsilon)\,. \tag{3.4.5}$$

Therefore, choosing now r' small enough, from (3.4.5) we get that also $\|v_\varepsilon(\tau_\varepsilon)-v^0(\tau_\varepsilon)\|_V < r$ and $|\sigma_\varepsilon(\tau_\varepsilon)-\sigma^0(\tau_\varepsilon)| < r$. By continuity, these inequalities hold also for some $t > \tau_\varepsilon$, which contradicts the maximality of τ_ε, and so we deduce that $\tau_\varepsilon = t^1$. We observe that r' is independent of ε, since we can take r' and ε_0 such that

$$2(r')^2 + \frac{\beta\varepsilon_0}{2\alpha-\varepsilon_0} < r\,,$$

and observe that the previous inequality remains true for every $\varepsilon < \varepsilon_0$.

We have thus obtained that (3.4.5) holds for every $t \in [t^0, t^1]$. Passing now to the limit in (3.4.5) as $\varepsilon \to 0$, we reach the conclusion. □

By the change of variables that defines the functional $\mathscr{F}$, and by the uniqueness of the regular evolution given by the Implicit Function Theorem, it follows that the regular evolution in Theorem 3.4.2 corresponds to the one in Theorem 3.4.1.

Proof of Theorem 3.4.1 continued. Let $(v_\varepsilon, \sigma_\varepsilon)$ be the solution to the modified ε-gradient flow. By Theorem 3.4.2, $\sigma_\varepsilon(t) \to \sigma^0(t)$ and $v_\varepsilon(t) \to v^0(t)$ strongly in V for every $t \in [t^0, t^1]$. Since the function $v \mapsto \mathrm{grad}_v \mathscr{F}(t, v, \sigma)$ is continuous from V to V with respect to the strong topology, it follows that

$$\mathrm{grad}_v \mathscr{F}(t, v_\varepsilon(t), \sigma_\varepsilon(t)) \to \mathrm{grad}_v \mathscr{F}(t, v^0(t), \sigma^0(t)) = 0\,.$$

Let now $\overline{v}_\varepsilon(t)$ be the element of V associated to $u_{t,\sigma_\varepsilon(t)}$ by the change of variables,i.e., $\overline{v}_\varepsilon(t)(x) = u_{t,\sigma_\varepsilon(t)}(\Psi_{\sigma_\varepsilon(t)}(x))$. As $\mathrm{grad}_v \mathscr{F}(t, \overline{v}_\varepsilon(t), \sigma_\varepsilon(t)) = 0$ we deduce that $v_\varepsilon(t) - \overline{v}_\varepsilon(t) \to 0$ strongly in V. This implies that

$$\mathscr{F}(t, \overline{v}_\varepsilon(t), \sigma_\varepsilon(t)) - \mathscr{F}(t, v_\varepsilon(t), \sigma_\varepsilon(t)) \to 0\,,$$

On the other hand,

$$\begin{aligned} &\mathscr{F}(t, v_\varepsilon(t), \sigma_\varepsilon(t)) \to \mathscr{F}(t, v^0(t), \sigma^0(t)) = E(t, \sigma^0(t)) \\ &\mathscr{F}(t, \overline{v}_\varepsilon(t), \sigma_\varepsilon(t)) = E(t, \sigma_\varepsilon(t))\,, \end{aligned}$$

so that we conclude that $E(t, \sigma_\varepsilon(t)) \to E(t, \sigma^0(t))$ for every $t \in [t^0, t^1]$. □

3.5 Monotonically increasing in time boundary displacements

In this section we consider the setting proposed by Francfort and Marigo in [18] and compare the evolution defined therein with a solution to the irreversible quasistatic evolution problem.

Assume $\psi(t) := t\psi_0$, with $\psi_0 \in H^1(\Omega)$, and $g(t) = 0$, and define

$$E(\sigma) := \min\{\|Du\|_2^2 : u \in AD(\psi_0, \sigma)\}.$$

Since $H^1(\Omega \setminus \Gamma(\sigma')) \subset H^1(\Omega \setminus \Gamma(\sigma''))$ for $\sigma' < \sigma''$, we have that $E(\sigma') \geq E(\sigma'')$, so that the function $\sigma \mapsto E(\sigma)$ is decreasing.

As in Definition 4.13 of [18], we define a crack trajectory $t \mapsto \sigma_{FM}(t)$ by the following three properties:

(i) $t \mapsto \sigma_{FM}(t)$ is increasing;

(ii) $t^2 E(\sigma_{FM}(t)) + \sigma_{FM}(t) \leq t^2 E(\sigma) + \sigma$, for every $\sigma \geq \sigma_{FM}^-(t)$;

(iii) $t^2 E(\sigma_{FM}(t)) + \sigma_{FM}(t) \leq t^2 E(\sigma_{FM}(s)) + \sigma_{FM}(s)$, for every $s \leq t$.

The following result shows that if $\sigma \to E(\sigma)$ is concave in some subinterval of $(\sigma_0, \overline{\sigma})$ then $t \mapsto \sigma_{FM}(t)$ is discontinuous.

Proposition 3.5.1. *Let $t \mapsto \sigma_{FM}(t)$ be a crack trajectory which satisfies properties (i)–(iii) above. If there exists a subinterval $(a, b) \subset (\sigma_0, \overline{\sigma})$, with $a < b$, where $\sigma \mapsto E(\sigma)$ is concave, then $\sigma_{FM}(t)$ has some discontinuity points.*

Proof. Let $t_0 \geq 0$ be such that $\sigma_{FM}(t_0) < a$. We first prove that there exists $t > t_0$ such that $\sigma_{FM}(t) > a$. Indeed, assume by contradiction that $\sigma_{FM}(t) < a$ for every $t > t_0$. Then conditions (i), (ii) and the fact that $\sigma \mapsto E(\sigma)$ is decreasing imply the following inequalities:

$$t^2 E(b) + b \geq t^2 E(\sigma_{FM}(t)) + \sigma_{FM}(t) \geq t^2 E(a) + \sigma_{FM}(t) \geq t^2 E(a) + \sigma_{FM}(t_0).$$

In particular, we deduce that $t^2 \leq b(E(a) - E(b))^{-1}$, which, up to considering T large enough, represents a contradiction.

If $\sigma_{FM}(t) \neq a$ for every $t \in [0, T]$ then σ_{FM} is discontinuous and the proof is concluded. Otherwise, let $\bar{t}$ be the first time such that $\sigma_{FM}(\bar{t}) = a$. We claim that

$$\sigma_{FM}(t) = a \quad \text{for every } \bar{t} \leq t \leq t^*, \tag{3.5.1}$$

where

$$t^* := \sqrt{\frac{b - a}{E(a) - E(b)}}. \tag{3.5.2}$$

Indeed, fix $t \in (\bar{t}, t^*)$ and assume by contradiction that $\sigma_{FM}(t) \in]a, b]$. Then there exists $\alpha \in]0, 1]$ such that $\sigma_{FM}(t) = \alpha a + (1 - \alpha) b$. By condition (ii) and the concavity of $\sigma \mapsto E(\sigma)$ on (a, b) we have

$$t^2 E(b) + b \geq t^2 E(\sigma_{FM}(t)) + \sigma_{FM}(t) \geq t^2 \alpha E(a) + t^2(1 - \alpha) E(b) + \alpha a + (1 - \alpha) b,$$

that is

$$t^2E(a) + a \le t^2E(b) + b. \tag{3.5.3}$$

Therefore

$$t^2E(\sigma_{FM}(t)) + \sigma_{FM}(t) \ge \alpha(t^2E(a) + a) + (1-\alpha)(t^2E(b) + b) \ge t^2E(a) + a. \tag{3.5.4}$$

Since (3.5.4) is in contradiction with condition (iii), we deduce that $\sigma_{FM}(t) = a$.

Consider now the case $t = t^*$. Formula (3.5.3) becomes the identity

$$(t^*)^2E(a) + a = (t^*)^2E(b) + b.$$

Assume there exists $\alpha \in]0,1]$ such that $\sigma_{FM}(t^*) = \alpha a + (1-\alpha)b$; then, arguing as before, we obtain that

$$(t^*)^2E(\sigma_{FM}(t^*)) + \sigma_{FM}(t^*) \ge (t^*)^2E(a) + a,$$

which, by conditions (ii) and (iii), implies that $\sigma_{FM}(t^*) = a$.

To conclude, we prove that

$$\sigma_{FM}(t) \ge b \quad \text{for } t > t^*. \tag{3.5.5}$$

Indeed, let us fix $t > t^*$ and assume by contradiction that $\sigma_{FM}(t) < b$. Then there exists $\alpha \in]0,1]$ such that $\sigma_{FM}(t) = \alpha a + (1-\alpha)b$, and this fact together with condition (ii) implies that

$$t^2 \le \frac{b - \sigma_{FM}(t)}{E(\sigma_{FM}(t)) - E(b)} \le \frac{\alpha(b-a)}{\alpha E(a) + (1-\alpha)E(b) - E(b)} = (t^*)^2, \tag{3.5.6}$$

a contradiction. This fact concludes the proof, since we have shown that for $t \le t^*$ $\sigma_{FM}(t) = a$, while $\sigma_{FM}(t) \ge b$ for $t > t^*$. □

Let $(u(\cdot), \sigma(\cdot))$ be an irreversible quasistatic evolution. Recalling that $u(t)$ is the minimum point of $\|Du\|_2^2$ on $AD(t\psi_0, \sigma(t))$, we have that $\|Du(t)\|_2^2 = t^2E(\sigma(t))$. We may now express conditions (a), (b) and (c) of Definition 3.3.1 of an irreversible quasistatic evolution, in terms of $\sigma(t)$, and, in the case of this particular choice of the data, we obtain:

(a') $1 + t^2E'(\sigma(t)) \ge 0$ for every $t \ge 0$;

(b') the map $t \mapsto \sigma(t)$ is increasing;

(c') $t^2E(\sigma(t)) + \sigma(t) \le s^2E(\sigma(s)) + \sigma(s) + 2\int_s^t \tau E(\sigma(\tau))\,d\tau$, for every $0 \le s < t$,

where $E'(\sigma(t))$ denotes the derivative of E with respect to σ computed at $\sigma(t)$.

Since $E(\sigma(\tau)) \le E(\sigma(s))$ for $\tau \in [s,t]$, condition (c') implies condition (iii).

In terms of this evolution $t \mapsto \sigma(t)$, the Griffith's criterion can be expressed by conditions (a'), (b') and the following one: $(1 + t^2E'(\sigma(t)))\dot{\sigma}(t) = 0$ for a.e. $t \in [0,T]$.

Remark 3.5.2. Let $t \mapsto \sigma(t)$ be a left-continuous map on $[0, T]$ which satisfies condition (c') and define

$$\dot{\sigma}^{\ominus}(t) := \limsup_{s \to t^-} \frac{\sigma(t) - \sigma(s)}{t - s}\,.$$

Then

$$(1 + t^2 E'(\sigma(t)))\dot{\sigma}^{\ominus}(t) \leq 0 \tag{3.5.7}$$

for every $t \in [0, T]$. Indeed, let $t_k \nearrow t$ be such that

$$\lim_{k \to \infty} \frac{\sigma(t) - \sigma(t_k)}{t - t_k} = \dot{\sigma}^{\ominus}(t).$$

Then condition (c') between t_k and t can be written as

$$(t^2 - t_k^2)E(\sigma(t)) + t_k^2(E(\sigma(t)) - E(\sigma(t_k))) + \sigma(t) - \sigma(t_k) \leq 2\int_{t_k}^{t} \tau E(\sigma(\tau))\, d\tau,$$

and (3.5.7) follows dividing by $t - t_k$ and letting $k \to +\infty$.

In the last part of this section we analyze the behavior of the evolution $t \mapsto \sigma(t)$ distinguishing between the concavity and convexity intervals for the energy functional $E(\sigma)$.

Remark 3.5.3. Let $t \mapsto \sigma(t)$ be a left-continuous map on $[0, T]$ which satisfies conditions (a'), (b'), and (c'), and let $t \geq 0$ be such that $\dot{\sigma}^{\ominus}(t) > 0$. Then, by Remark 3.5.2 and conditions (a') and (b'), it follows that

$$E'(\sigma(t)) = \frac{d}{d\sigma}E(\sigma)|_{\sigma=\sigma(t)} = -\frac{1}{t^2}\,,$$

which implies that $\sigma(t)$ does not belong to the concavity intervals of $E(\sigma)$, since $t \mapsto \sigma(t)$ is increasing, and $t \mapsto E'(\sigma(t))$ would be decreasing, while the right-hand side is increasing. More precisely, let $(a, b) \subset [\sigma_0, \overline{\sigma}]$, with $a < b$, be such that $\sigma \mapsto E'(\sigma)$ is strictly decreasing on (a, b) and let $t_0 \geq 0$ such that $\sigma(t_0) \in (a, b)$ and $\dot{\sigma}(t_0) > 0$ (or $\dot{\sigma}^{\ominus}(t_0) > 0$). Let $t > t_0$ with $\sigma(t) \in (a, b)$. Then $\sigma(t) > \sigma(t_0)$, and condition (a') and our assumption on $E'(\sigma)$ imply

$$-\frac{1}{t^2} \leq E'(\sigma(t)) < E'(\sigma(t_0)) = -\frac{1}{t_0^2} < -\frac{1}{t^2},$$

a contradiction.

In order to specify better the monotonicity needed in the above remarks we introduce the following notion. We say that t_0 is a *local left-constancy point* for σ if there exists $\varepsilon > 0$ such that σ is constant on the interval $[t_0 - \varepsilon, t_0]$.

Proposition 3.5.4. *Let $\sigma\colon [0, T] \to [\sigma_0, \overline{\sigma}[$ be a left-continuous map which satisfies conditions (a'), (b'), and (c'), and let $t_0 \geq 0$. If*

(1) t_0 is not a local left-constancy point for σ and

(2) there exists $(a,b) \subset [\sigma_0, \overline{\sigma}[$ such that $E'(\sigma)$ is strictly decreasing on (a,b)

then $\sigma(t_0) \notin (a,b)$.

Proof. If t_0 is not a local left-constancy point for σ, then, given $\varepsilon > 0$, there are $t_\varepsilon^1, t_\varepsilon^2 \in [t_0 - \varepsilon, t_0]$ such that $\sigma(t_\varepsilon^1) \neq \sigma(t_\varepsilon^2)$. Therefore, there exists $t_\varepsilon \in [t_0 - \varepsilon, t_0[$ such that $\dot{\sigma}^{\ominus}(t_\varepsilon) > 0$. Then (3.5.7) together with (a') imply that $1 + t_\varepsilon^2 E'(\sigma(t_\varepsilon)) = 0$. By Remark 3.5.3, $\sigma(t_\varepsilon) \notin (a,b)$ and we conclude by passing to the limit as $\varepsilon \to 0$ (since σ is left-continuous). $\square$

Proposition 3.5.5. *Let $\sigma : [0,T] \to [\sigma_0, \overline{\sigma}[$ be a left-continuous map which satisfies conditions (a'), (b'), and (c'). Assume that $E(\sigma)$ is convex on $(a,b) \subset [\sigma_0, \overline{\sigma}]$. Then $\sigma(t)$ is continuous at every t with $\sigma(t) \in (a,b)$.*

Proof. Assume by contradiction that $\sigma(t) < \sigma(t^+)$. Then condition (c') and condition (a') imply

$$\frac{E(\sigma(t^+) - E(\sigma(t)))}{\sigma(t^+) - \sigma(t)} \leq -\frac{1}{t^2} \leq E'(\sigma(t)),$$

a contradiction. $\square$

3.6 Concavity and convexity intervals for the energy functional

In this section we consider the energy functional

$$\sigma \mapsto E(\sigma) := \min\{\|Du\|_2^2 : u \in AD(\psi, \sigma)\},$$

and construct an explicit example of Ω and ψ for which $E(\sigma)$ is concave on some subinterval. Let B_{-2} denote the ball of radius 1 centred in $(-2,0)$, let B_2 denote the ball of radius 1 centred in $(2,0)$, and let $\Gamma := [-3,3] \times \{0\}$.

For $\varepsilon > 0$ let

$$T_\varepsilon := \,]-2+\cos\varepsilon, 2-\cos\varepsilon[\, \times \,]-\sin\varepsilon, \sin\varepsilon[\,, \qquad \Omega_\varepsilon := B_{-2} \cup T_\varepsilon \cup B_2.$$

Further, for every $\sigma \in [-3,3]$ let

$$\Gamma(\sigma) := [-3, \sigma] \times \{0\}.$$

Let (ρ, θ) and $(\tilde{\rho}, \tilde{\theta})$ be polar coordinates around $(-2,0)$ and $(2,0)$, respectively, where the functions θ and $\tilde{\theta}$ are chosen, as in Proposition 3.2.2, such that $\theta(x_1,x_2) \to -\pi$ if $x_2 \to 0-$

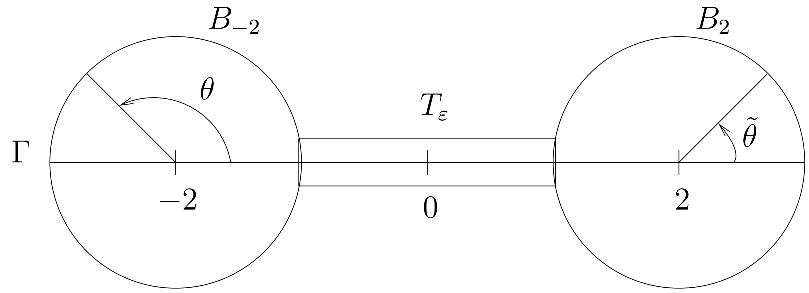

Figure 3.1: *The set* Ω_ε*.*

and $x_1 < -2$, $\theta(x_1, x_2) \to \pi$ if $x_2 \to 0+$ and $x_1 < -2$, and, analogously, $\tilde\theta(x_1, x_2) \to -\pi$ if $x_2 \to 0-$ and $x_1 < 2$, $\tilde\theta(x_1, x_2) \to \pi$ if $x_2 \to 0+$ and $x_1 < 2$.

On $\partial\Omega_\varepsilon$ we define the boundary data ψ_ε as follows:

$$\psi_\varepsilon(x) := \begin{cases} \sin\frac{\theta(x)}{2} & \text{on } \big(\partial B_{-2} \cap \partial\Omega_\varepsilon\big) \setminus \Gamma(\sigma), \\ \sin\frac{\tilde\theta(x)}{2} & \text{on } \big(\partial B_{2} \cap \partial\Omega_\varepsilon\big) \setminus \Gamma(\sigma), \\ \sin\frac{\varepsilon}{2} & \text{on }]-2+\cos\varepsilon, 0[\times \{\sin\varepsilon\}, \\ -\sin\frac{\varepsilon}{2} & \text{on }]-2+\cos\varepsilon, 0[\times \{-\sin\varepsilon\}, \\ \sin\frac{\varepsilon}{2} + \frac{x_1}{2-\cos\varepsilon}\big(\cos\frac{\varepsilon}{2} - \sin\frac{\varepsilon}{2}\big) & \text{on } [0, 2-\cos\varepsilon[\times \{\sin\varepsilon\}, \\ -\sin\frac{\varepsilon}{2} + \frac{x_1}{2-\cos\varepsilon}\big(\sin\frac{\varepsilon}{2} - \cos\frac{\varepsilon}{2}\big) & \text{on } [0, 2-\cos\varepsilon[\times \{-\sin\varepsilon\}. \end{cases} \tag{3.6.1}$$

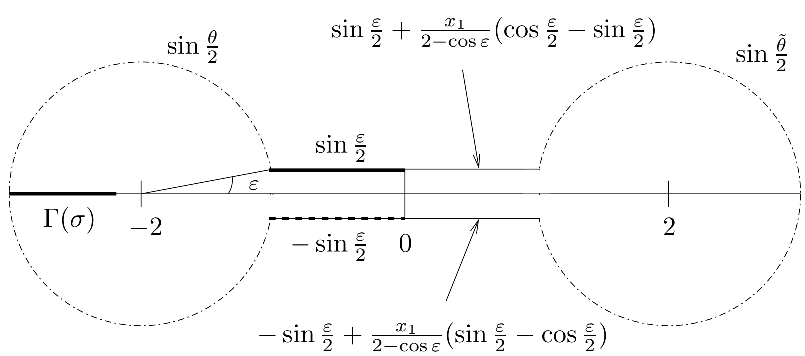

Figure 3.2: *The boundary datum* ψ_ε*.*

For every $\sigma \in]-3, 3[$, let $u^\varepsilon(\sigma) \in H^1(\Omega_\varepsilon \setminus \Gamma(\sigma))$ be the solution of the problem:

$$E_\varepsilon(\sigma) := \min\Big\{ \int_{\Omega_\varepsilon\setminus\Gamma(\sigma)} |Du|^2\,dx : u \in AD(\psi_\varepsilon, \sigma)\Big\}. \tag{3.6.2}$$

Our aim is to prove that for ε sufficiently small there exists a subinterval $[a, b]$ of $[-2, 2]$ such that $E_\varepsilon(\sigma)$ is concave on $[a, b]$.

As $\sigma \mapsto E_\varepsilon(\sigma)$ is a C^2-function, in order to prove that $E_\varepsilon(\sigma)$ cannot be convex on the whole interval $[-2,2]$, it is enough to show that the following three conditions are satisfied:

(a) $\limsup_{\varepsilon\to 0^+} E_\varepsilon(2)$ is finite;

(b) $\liminf_{\varepsilon\to 0^+} E_\varepsilon(-2) = \infty$;

(c) $\limsup_{\varepsilon\to 0^+} E'_\varepsilon(-2)$ is finite;

where we denote by $'$ the first derivative with respect to σ.

In order to prove condition (a) we construct an admissible function $\tilde{u}_\varepsilon$ for $E_\varepsilon(2)$ whose energy, $\|D\tilde{u}_\varepsilon\|_2^2$, is bounded uniformly with respect to ε. We define the open sets B_{-2}^+ and B_{-2}^- by

$$B_{-2}^+ = \{(x_1,x_2) \in B_{-2} : x_2 > 0\}$$
$$B_{-2}^- = \{(x_1,x_2) \in B_{-2} : x_2 < 0\}.$$

Let v^+ be the solution to the following problem:

$$\begin{cases} \Delta u = 0 & \text{on } B_{-2}^+; \\ u(x) = \sin\frac{\theta(x)}{2} & \text{on } \partial B_{-2}^+ \cap \partial B_{-2}; \\ \partial_\nu u = 0 & \text{on }]-3,-1[\times\{0\}. \end{cases}$$

Then the function $v^-(x_1,x_2) := -v^+(x_1,-x_2)$ solves the analogue problem on B_{-2}^-. Let $\tilde{u}_\varepsilon$ be the function which coincides with the harmonic functions that satisfy the boundary conditions on B_{-2}^+, on B_{-2}^-, and on B_2, respectively, that is, $\tilde{u}_\varepsilon := v^+$ on B_{-2}^+, $\tilde{u}_\varepsilon := v^-$ on B_{-2}^-, and $\tilde{u}_\varepsilon := \tilde{\rho}^{\frac{1}{2}} \sin\frac{\tilde{\theta}}{2}$ on B_2. On $T_\varepsilon \setminus (B_2 \cup B_{-2})$ we define $\tilde{u}_\varepsilon$ in the following way: on the horizontal line $x_2 = \sin\theta$, with $\theta \in [-\varepsilon,\varepsilon]$, we set $\tilde{u}_\varepsilon(x_1,x_2) := \sin\frac{\theta}{2}$ for $x_1 \in\,]-2+\cos\theta, 0]$ and then interpolate linearly with the boundary data on $\partial B_2 \cap T_\varepsilon$: $\tilde{u}_\varepsilon(x_1,x_2) := \sin\frac{\theta}{2} + \frac{x_1}{2-\cos\theta}(\cos\frac{\theta}{2} - \sin\frac{\theta}{2})$ for $x_1 \in [0, 2-\cos\theta[$, if $0 < \theta \le \varepsilon$, and $\tilde{u}_\varepsilon(x_1,x_2) := -\sin\frac{\theta}{2} + \frac{x_1}{2-\cos\theta}(-\sin\frac{\theta}{2} - \cos\frac{\theta}{2})$ for $x_1 \in [0,2-\cos\theta[$, if $-\varepsilon \le \theta < 0$. It is easy to check that $\tilde{u}_\varepsilon \in AD(\psi_\varepsilon, 2)$ and that $D\tilde{u}_\varepsilon$ is bounded in $L^2(\Omega_\varepsilon \setminus \Gamma; \mathbb{R}^2)$ uniformly with respect to ε. This implies that

$$\limsup_{\varepsilon\to 0^+} E_\varepsilon(2) \le \limsup_{\varepsilon\to 0^+} \int_{\Omega_\varepsilon\setminus\Gamma(2)} |D\tilde{u}_\varepsilon|^2\,dx < +\infty,$$

and condition (a) is satisfied.

We continue by proving condition (b), i.e., $E_\varepsilon(-2)$ tends to infinity as ε goes to zero. Let us first consider the model problem

$$\min\Big\{\int_{R_\varepsilon} |Du|^2\,dx : u \ge \frac{1}{2} \text{ on } \partial_1 R_\varepsilon, u \le -\frac{1}{2} \text{ on } \partial_2 R_\varepsilon\Big\} \tag{3.6.3}$$

where

$$R_\varepsilon :=]0,1[\times]-\varepsilon,\varepsilon[, \quad \partial_1 R_\varepsilon := [0,1]\times\{\varepsilon\}, \quad \partial_2 R_\varepsilon := [0,1]\times\{-\varepsilon\}. \tag{3.6.4}$$

It is easy to see that problem (3.6.3) admits a solution and that it is equivalent to

$$\min\Big\{\int_{R_\varepsilon} |Du|^2\,dx : u = \frac{1}{2} \text{ on } \partial_1 R_\varepsilon, u = -\frac{1}{2} \text{ on } \partial_2 R_\varepsilon\Big\}, \tag{3.6.5}$$

which admits the affine solution $u^a(x_1,x_2) := \frac{1}{2\varepsilon}\,x_2$ for every $x=(x_1,x_2)\in R_\varepsilon$.

Going back to the domain Ω_ε, let us consider the same problem with different constants: the rectangle R_ε is defined now by

$$R_\varepsilon :=]A_\varepsilon, 2-\cos\varepsilon[\times]-\sin\varepsilon,\sin\varepsilon[\subset T_\varepsilon\,,$$

where A_ε is a positive constant such that $\psi_\varepsilon(x)\geq\frac{1}{2}$ on $\partial_1 R_\varepsilon := [A_\varepsilon, 2-\cos\varepsilon]\times\{\sin\varepsilon\}$, (and $\psi_\varepsilon(x)\leq -\frac{1}{2}$ on $\partial_2 R_\varepsilon := [A_\varepsilon, 2-\cos\varepsilon]\times\{-\sin\varepsilon\}$), when ε is sufficiently small. Then

$$E_\varepsilon(-2) = \int_{\Omega_\varepsilon\setminus\Gamma(-2)} |Du^\varepsilon(-2)|^2 dx \geq \int_{R_\varepsilon} |Du^\varepsilon(-2)|^2 dx \geq \int_{R_\varepsilon} |Du^a|^2 dx\,.$$

Since $\int_{R_\varepsilon}|Du^a|^2dx\to+\infty$ as $\varepsilon\to 0$, condition (b) is proved.

It remains to show that condition (c) is satisfied, i.e., that the first derivative of $\sigma\mapsto E_\varepsilon(\sigma)$ at $\sigma=-2$ is bounded as ε goes to zero. Since

$$E'_\varepsilon(\sigma) = -\kappa^2_\varepsilon(\sigma)\,, \tag{3.6.6}$$

(for the proof see, e.g. [23, Theorem 6.4.1]), where $\kappa_\varepsilon(\sigma)\sqrt{\frac{2}{\pi}}$ is the stress intensity factor associated to $u^\varepsilon(\sigma)$ at the tip $(\sigma,0)$, see Proposition 3.2.2, it is enough to show that $\kappa_\varepsilon(\sigma)$ remains bounded when, for instance, $-\frac{5}{2}\leq\sigma\leq-\frac{3}{2}$.

For $\sigma\in[-5/2,-3/2]$, let $v(\sigma)$ be the solution of the following problem:

$$\min\Big\{\int_{B_{-2}\setminus\Gamma(\sigma)} |Du|^2\,dx : u\in H^1(B_{-2}\setminus\Gamma(\sigma)),\ u=\sin\frac{\theta}{2} \text{ on } \partial B_{-2}\setminus\Gamma(\sigma)\Big\}. \tag{3.6.7}$$

Let us extend $v(\sigma)$ to $\mathbb{R}\times[-1,1]$ constantly on the horizontal lines and denote now by $v(\sigma)$ this extension.

We claim that

$$u^\varepsilon(\sigma)\to v(\sigma) \quad \text{strongly in } H^1(B_{-2}\setminus\Gamma(\sigma)). \tag{3.6.8}$$

Assuming the claim true, we now use the following characterization of κ_ε (see Proposition 3.2.3):

$$\kappa^2_\varepsilon(\sigma) = \int_{B_{-2}\setminus\Gamma(\sigma)} \Big[((D_1u^\varepsilon)^2-(D_2u^\varepsilon)^2)D_1\varphi + 2D_1u^\varepsilon D_2u^\varepsilon D_2\varphi\Big]\,dx \tag{3.6.9}$$

with $\varphi \in C_c^1(B_{-2})$ such that $\varphi(\sigma,0)=1$. By (3.6.8) and the definition of $v(\sigma)$, we can pass to the limit in the right-hand side as $\varepsilon \to 0^+$ and define in such a way the quantity:

$$\kappa^2(\sigma) := \int_{B_{-2}\setminus\Gamma(\sigma)} \Big[((D_1 v(\sigma))^2 - (D_2 v(\sigma))^2) D_1\varphi + 2D_1 v(\sigma) D_2 v(\sigma) D_2\varphi\Big]\, dx. \qquad (3.6.10)$$

Therefore, by (3.6.6),

$$\limsup_{\varepsilon\to 0^+} E'_\varepsilon(\sigma) = -\kappa^2(\sigma) \quad \text{for every } -\frac{5}{2} \le \sigma \le -\frac{3}{2}\,. \qquad (3.6.11)$$

As, by (3.6.10), $\kappa(\sigma)$ is bounded, formula (3.6.11) concludes the proof of condition (c).
Proof of the claim. Let $\tilde\Omega_\varepsilon := T_\varepsilon \cup B_2$ and let w_ε be the solution of the following problem:

$$\min\Big\{ \int_{\tilde\Omega_\varepsilon} |Du|^2\, dx : u \in H^1(\tilde\Omega_\varepsilon),\ u = \psi_\varepsilon \ \text{ on } \partial\Omega_\varepsilon \cap \partial\tilde\Omega_\varepsilon \Big\}. \qquad (3.6.12)$$

We consider a cut-off function $\varphi \in C^\infty(\mathbb{R})$ such that $0 \le \varphi \le 1$, $\varphi(x_1) = 1$ for $x_1 \le -\frac{2}{3}$, and $\varphi(x_1) = 0$ for $x_1 \ge -\frac{1}{3}$. Then the function $\zeta := \varphi v(\sigma) + (1-\varphi) w_\varepsilon$ belongs to $AD(\psi_\varepsilon, \sigma)$ and

$$E_\varepsilon(\sigma) = \int_{\Omega_\varepsilon\setminus\Gamma(\sigma)} |Du^\varepsilon(\sigma)|^2\, dx \le \int_{\Omega_\varepsilon\setminus\Gamma(\sigma)} |D\zeta|^2\, dx. \qquad (3.6.13)$$

By convexity, we have

$$\begin{aligned}\int_{\Omega_\varepsilon\setminus\Gamma(\sigma)} |D\zeta|^2\, dx \le \int_{B_{-2}\setminus\Gamma(\sigma)} |Dv(\sigma)|^2\, dx + \int_{\tilde\Omega_\varepsilon} |Dw_\varepsilon|^2\, dx + \int_{T_\varepsilon} |Dv(\sigma)|^2\, dx + \\ + \int_{T_\varepsilon\cap(\mathrm{supp} D\varphi)} \big(2D\varphi(\varphi Dv(\sigma) + (1-\varphi)Dw_\varepsilon)(v(\sigma)-w_\varepsilon) + |D\varphi|^2 (v(\sigma)-w_\varepsilon)^2\big)\, dx.\end{aligned} \qquad (3.6.14)$$

Now

$$\begin{aligned}&\lim_{\varepsilon\to 0^+} \int_{T_\varepsilon} |Dv(\sigma)|^2\, dx = 0,\\ &\lim_{\varepsilon\to 0^+} \int_{T_\varepsilon\cap(\mathrm{supp} D\varphi)} |D\varphi|^2 (v(\sigma)-w_\varepsilon)^2\, dx = 0,\end{aligned} \qquad (3.6.15)$$

and, for any $\eta > 0$,

$$\begin{aligned}&\int_{T_\varepsilon\cap(\mathrm{supp} D\varphi)} 2D\varphi(\varphi Dv(\sigma) + (1-\varphi)Dw_\varepsilon)(v(\sigma)-w_\varepsilon)\, dx \le \\ &\qquad \le 2\int_{T_\varepsilon\cap(\mathrm{supp} D\varphi)} D\varphi\varphi Dv(\sigma)(v(\sigma)-w_\varepsilon)\, dx + \\ &+\frac{1}{\eta}\int_{T_\varepsilon\cap(\mathrm{supp} D\varphi)} |D\varphi|^2 |v(\sigma)-w_\varepsilon|^2\, dx + \eta \int_{T_\varepsilon\cap(\mathrm{supp} D\varphi)} |Dw_\varepsilon|^2 (1-\varphi)^2\, dx\,.\end{aligned} \qquad (3.6.16)$$

Since the first two terms in the right-hand side tend to zero, it remains to prove that

$$\lim_{\varepsilon\to 0^+} \int_{T_\varepsilon\cap(\mathrm{supp} D\varphi)} |Dw_\varepsilon|^2\, dx = 0\,. \qquad (3.6.17)$$

As in the proof of condition (b), we consider first a model problem. Similarly to (3.6.4), we now set

$$R_\varepsilon :=]-1,0[\times]-\varepsilon,\varepsilon[,\quad \partial_1 R_\varepsilon := [-1,0]\times\{\varepsilon\},\quad \partial_2 R_\varepsilon := [-1,0]\times\{-\varepsilon\},$$

and define h_ε as the solution to the following problem:

$$\begin{cases} \Delta h_\varepsilon = 0 & \text{on } R_\varepsilon, \\ h_\varepsilon = \frac{\varepsilon}{2} & \text{on } \partial_1 R_\varepsilon, \\ h_\varepsilon = -\frac{\varepsilon}{2} & \text{on } \partial_2 R_\varepsilon, \\ \|h_\varepsilon\|_\infty \le 1. \end{cases} \tag{3.6.18}$$

We claim that

$$\lim_{\varepsilon\to 0^+} \int_{\tilde{R}_\varepsilon} |Dh_\varepsilon|^2\, dx = 0, \tag{3.6.19}$$

where

$$\tilde{R}_\varepsilon := \left]-\frac{4}{5},-\frac{1}{5}\right[\times\left]-\varepsilon,\varepsilon\right[\subset R_\varepsilon.$$

Indeed, note that the function $z_\varepsilon(x_1,x_2) := \frac{1}{2}x_2$ solves (3.6.18) (for $\varepsilon \le 1$). By a Cacciopoli type estimate we obtain

$$\int_{\tilde{R}_\varepsilon} |D(h_\varepsilon - z_\varepsilon)|^2 dx \le C \int_{R_\varepsilon} |h_\varepsilon - z_\varepsilon|^2 dx \le C_1 |R_\varepsilon|\,,$$

for some positive constants C and C_1 which do not depend on ε, hence (3.6.19) holds.

Applying this argument with

$$R_\varepsilon = \left]-1,0\right[\times\left]-\sin\varepsilon,\sin\varepsilon\right[\quad \text{and} \quad \tilde{R}_\varepsilon = \left]-\frac{4}{5},-\frac{1}{5}\right[\times\left]-\sin\varepsilon,\sin\varepsilon\right[$$

it follows that (3.6.17) holds true.

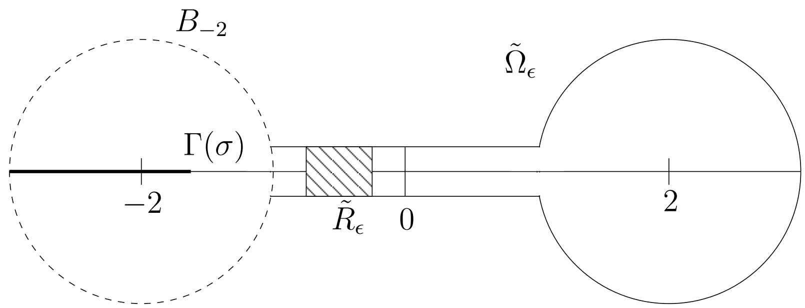

Figure 3.3: *The rectangle $\tilde{R}_\epsilon$ where we apply a Cacciopoli type estimate in order to obtain (3.6.17).*

From (3.6.13), (3.6.14), (3.6.15), (3.6.16), and (3.6.17) we deduce that

$$\int_{\Omega_\varepsilon\setminus\Gamma(\sigma)} |Du^\varepsilon(\sigma)|^2\, dx \le \int_{B_{-2}\setminus\Gamma(\sigma)} |Dv(\sigma)|^2\, dx + \int_{\tilde\Omega_\varepsilon} |Dw_\varepsilon|^2\, dx + o(1). \tag{3.6.20}$$

Since

$$\int_{\tilde\Omega_\varepsilon} |Du^\varepsilon(\sigma)|^2 dx \ge \int_{\tilde\Omega_\varepsilon} |Dw_\varepsilon|^2 dx$$

we obtain

$$\int_{B_{-2}\setminus\Gamma(\sigma)} |Du^\varepsilon(\sigma)|^2\, dx \le \int_{B_{-2}\setminus\Gamma(\sigma)} |Dv(\sigma)|^2\, dx + o(1) \le C \tag{3.6.21}$$

uniformly with respect to ε. Thus, there exists $u^*(\sigma) \in H^1(B_{-2} \setminus \Gamma(\sigma))$ such that

$$u^\varepsilon(\sigma) \rightharpoonup u^*(\sigma) \quad \text{weakly on } H^1(B_{-2} \setminus \Gamma(\sigma)), \tag{3.6.22}$$

and

$$u^*(\sigma) = \sin\frac{\theta}{2} \quad \text{on } \partial B_{-2} \setminus \Gamma(\sigma). \tag{3.6.23}$$

As $(Du^\varepsilon(\sigma), D\varphi) = 0$ for every $\varphi \in H^1(B_{-2}\setminus\Gamma(\sigma))$ with $\varphi = 0$ on $\partial B_{-2}\setminus\Gamma(\sigma)$, by (3.6.22) we obtain that $(Du^*(\sigma), D\varphi) = 0$. By (3.6.7), this fact, together with (3.6.23), implies that

$$u^*(\sigma) = v(\sigma)\,. \tag{3.6.24}$$

In addition, by the lower semicontinuity and by (3.6.21), we have

$$\int_{B_{-2}\setminus\Gamma(\sigma)} |Dv(\sigma)|^2\, dx \le \liminf_{\varepsilon\to 0^+} \int_{B_{-2}\setminus\Gamma(\sigma)} |Du^\varepsilon(\sigma)|^2\, dx \le \int_{B_{-2}\setminus\Gamma(\sigma)} |Dv(\sigma)|^2\, dx. \tag{3.6.25}$$

By (3.6.22), (3.6.24), and (3.6.25), we deduce that (3.6.8) holds.

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