Extended Toda hierarchy and its Hamiltonian structure

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Chapter 1

Introduction

The Toda lattice equation is a nonlinear evolution equation describing an infinite system of points on a line that interact through an exponential force; it was introduced by Toda [46] in the search for a simple explanation of the Fermi-Pasta-Ulam phenomenon, one of the most important events leading to the development of soliton theory.

The Hamiltonian is given by

$$H = \sum_n \frac{1}{2} P_n^2 + e^{Q_{n-1}-Q_n} \tag{1.1}$$

where the canonical coordinates are Q_n, the displacement of the n-th mass from equilibrium, and P_n, its momentum. The corresponding equation, in the coordinates $v_n = Q_{n-1} - Q_n$ and $u_n = -P_n$, can be written as a system

$$\frac{\partial}{\partial t} u_n = e^{v_{n+1}} - e^{v_n} \tag{1.2a}$$

$$\frac{\partial}{\partial t} v_n = u_n - u_{n-1}, \tag{1.2b}$$

with $n \in \mathbb{Z}$.

It was soon realized that this equation is an example of completely integrable system: it admits an infinite number of conserved quantities [30, 21], can be solved for rapidly decreasing boundary conditions through the method of inverse scattering [22] and admits explicit quasi-periodic solutions by algebro-geometric methods [16, 5, 34].

The Toda lattice equation can be seen as the first element of a hierarchy of commuting flows, the Toda lattice hierarchy[1]. These commuting flows can be defined by the Lax pair formalism

$$\epsilon \frac{d}{dt_q} L = [A_q, L], \tag{1.3}$$

hence they are isospectral deformations of the Lax operator

$$L = \Lambda + u(x) + e^{v(x)} \Lambda^{-1} \tag{1.4}$$

[1] We will also call it Toda chain hierarchy, to distinguish it from the bigraded and the two-dimensional Toda hierarchies described in the following.

where we have used the continuous notation $u(n\epsilon) = u_n$ and Λ is the shift operator, $\Lambda f(x) = f(x+\epsilon)$. The difference operators A_q are obtained by taking the positive part of powers of L, $A_q = \frac{1}{(q+1)!}(L^{q+1})_+$.

It is well known that this hierarchy admits a bihamiltonian structure [35], i.e. a pair of compatible Poisson brackets $\{,\}_i$, $i = 1, 2$ and a set of Hamiltonians h_q that give the flows defined above. The Hamiltonians can be expressed in terms of the Lax operator by

$$h_q = \frac{1}{(q+2)!} \int dx \ \operatorname{Res} L^{q+2} \tag{1.5}$$

where the residue of a difference operator $A = \sum_k a_k \Lambda^k$ is given by $\operatorname{Res} A = a_0$ and the flows are obtained through the first bracket by $\frac{d}{dt_q} \cdot = \{\cdot, h_q\}_1$. The pencil of Poisson brackets actually defines the whole set of flows since it gives a recursion relation

$$\{\cdot, h_{q-1}\}_2 = (q+1)\{\cdot, h_q\}_1 \tag{1.6}$$

from which all the Hamiltonians can be implicitly obtained beginning from a Casimir of the first bracket.

A new set of non-local flows was recently introduced independently by Zhang [48] and Getzler [27] by providing an ansatz for the first nontrivial Hamiltonian and then using the Lenard-Magri recursion relation. One actually expects that such flows exist, since the first Poisson bracket has two Casimirs from which the recursion relation could start; however the second Casimir cannot be used as a starting point since it is also a Casimir for the second bracket. We call the set of the usual Toda flows and of the new non-local flows *Extended Toda hierarchy*.

Let us look for a Lax pair formulation of the non-local flows. The form of the Lax pairs is suggested by an extension of the dispersionless Toda hierarchy that was obtained in [20] in relation with the genus zero approximation of the topological $C\mathrm{P}^1$ model.

Our first result is a construction of the logarithm of the difference operator (1.4); we define

$$\log L = -\frac{\epsilon}{2}(P_x P^{-1} - Q_x Q^{-1}) \tag{1.7}$$

where $P = 1 + p_1(x)\Lambda^{-1} + \ldots$ and $Q = q_0(x) + q_1(x)\Lambda + \ldots$ are the dressing operators defined by

$$L = P\Lambda P^{-1} = Q\Lambda^{-1}Q^{-1}. \tag{1.8}$$

We show that this logarithm is a difference operator of the form

$$\log L = \sum_{k\in\mathbb{Z}} w_k \Lambda^k \tag{1.9}$$

where w_k are power series in ϵ with coefficients given by differential polynomials in u, v, e^v and e^{-v} (i.e. polynomials in these symbols and their derivatives).

The new isospectral deformations of L are written in the Lax form by $\epsilon \frac{d}{d\tilde{t}_q} L = [\tilde{A}_q, L]$, where

$$\tilde{A}_q = \frac{2}{q!}(L^q(\log L - c_q))_+ \tag{1.10}$$

and $c_q = \sum_{k=1}^{q} \frac{1}{k}$, $c_0 = 0$.

We prove that these flows are Hamiltonian with respect to both Poisson structures of the Toda chain and the Hamiltonians can be expressed in terms of L and $\log L$ by

$$\tilde{h}_q = \frac{2}{(q+1)!} \int dx \ Res(L^{q+1}(\log L - c_{q+1})). \tag{1.11}$$

Moreover, they satisfy the recursion relation

$$\{\cdot, \tilde{h}_{q-1}\}_2 = q\{\cdot, \tilde{h}_q\}_1 + 2\{\cdot, h_{q-1}\}_1. \tag{1.12}$$

From this formula and the recursion relation (1.6) it follows that all the Hamiltonians h_q and $\tilde{h}_q$ are in involution among themselves.

An important object connected with the integrable hierarchies is the tau function, from which all the relevant quantities like the Hamiltonians and the dependent variables can be obtained by derivation. We show that for any solution of the extended Toda hierarchy there exists a tau function τ; then we obtain, for example, that the dependent variables can be expressed in terms of the tau function by

$$v = \Lambda^{-1}(\Lambda - 1)^2 \log \tau \qquad u = \epsilon(\Lambda - 1)\frac{\partial \log \tau}{\partial t_0}. \tag{1.13}$$

The wave operators are differential-difference operators $\hat{P}$, $\hat{Q}$ defined in analogy with the wave matrices in the context of two-dimensional Toda [47]. We show that all the equations of the extended Toda hierarchy can be encapsulated in a single bilinear relation

$$\hat{P}(\mathbf{t}) \cdot \hat{P}^{-1}(\mathbf{t}') = \hat{Q}(\mathbf{t}) \cdot \hat{Q}^{-1}(\mathbf{t}') \tag{1.14}$$

where $\mathbf{t} = (t_0, t_1, \dots; \tilde{t}_0, \dots)$, plus a constraint

$$\hat{P}\Lambda\hat{P}^{-1} = \hat{Q}\Lambda^{-1}\hat{Q}^{-1}. \tag{1.15}$$

Moreover, we find that similar bilinear relations hold for the wave functions $\hat{\psi}$, $\hat{\psi}^*$, $\hat{\phi}$, $\hat{\phi}^*$.

Finally, we consider the Darboux transformations of the Lax operator L and we obtain explicit soliton solutions for the extended Toda hierarchy.

A natural generalization of the Toda chain hierarchy is given by the *bigraded Toda hierarchy.* The Lax operator in this case is

$$L = \Lambda^N + u_{N-1}\Lambda^{N-1} + \cdots + u_{-M}\Lambda^{-M} \tag{1.16}$$

for two positive integers N, M. We define two fractional powers of L by

$$(L^{\frac{1}{N}})^N = L \qquad \text{and} \qquad (L^{\frac{1}{M}})^M = L \tag{1.17}$$

of the form $L^{\frac{1}{N}} = \Lambda + a_0 + a_1\Lambda^{-1} + \dots$ and $L^{\frac{1}{M}} = b_{-1}\Lambda^{-1} + b_0 + \dots$. Then we define the logarithm of L by the same formula used in the Toda chain case (1.7), where now the dressing operators are defined by

$$L = P\Lambda^N P^{-1} = Q\Lambda^{-M}Q^{-1}. \tag{1.18}$$

We prove that the coefficients w_k in (1.9) and the coefficients a_k, b_k are ϵ-power series of differential polynomials in $u_N, \dots, u_{-M+1}$, $(u_{-M})^{\frac{1}{M}}$, $(u_{-M})^{-\frac{1}{M}}$ and $\log u_{-M}$.

We then define the flows of the extended bigraded hierarchy through the Lax pair formalism

$$\epsilon \frac{\partial L}{\partial t^{\alpha,q}} = [A_{\alpha,q}, L] \tag{1.19}$$

with

$$A_{\alpha,q} = \frac{\Gamma(2-\frac{\alpha}{N})}{\Gamma(q+2-\frac{\alpha}{N})}(L^{q+1-\frac{\alpha}{N}})_+ \quad \text{for} \quad \alpha = N-1, \dots, 0 \tag{1.20a}$$

$$A_{\alpha,q} = \frac{-\Gamma(2+\frac{\alpha}{M})}{\Gamma(q+2+\frac{\alpha}{M})}(L^{q+1+\frac{\alpha}{M}})_- \quad \text{for} \quad \alpha = 0, \dots, -M+1 \tag{1.20b}$$

$$A_{-M,q} = \frac{1}{q!}[L^q(\log L - \frac{1}{2}(\frac{1}{M}+\frac{1}{N})c_q)]_+ \tag{1.20c}$$

and $q \geqslant 0$.

To obtain the associated bihamiltonian structure we make use of the R-matrix techniques developed first by Semenov-Tian-Shansky [42] and then generalized to the non-unitary case in [40, 36]. Given an associative algebra $\mathcal{G}$ with an invariant non-degenerate inner product we say that the linear endomorphism $R \in \operatorname{End} \mathcal{G}$ satisfies the *modified Yang-Baxter equation* if

$$[R(X), R(Y)] - R([X,Y]_R) = -[X,Y] \tag{1.21}$$

for every $X, Y \in \mathcal{G}$, where $[X,Y]_R := [R(X), Y] + [X, R(Y)]$. In particular if R satisfies the modified Yang-Baxter equation then $[,]_R$ gives a Lie algebra structure on $\mathcal{G}$ (different from the natural one given by the commutator) and hence defines on $\mathcal{G}^* = \mathcal{G}$ the usual Lie-Poisson (Kirillov-Konstant) linear bracket

$$\{f,g\}_1(L) = (L, [df, dg]_R). \tag{1.22}$$

The theorems proved in [40] state that if moreover the skew-symmetric part $S = \frac{1}{2}(R - R^*)$ of R satisfies the modified Yang-Baxter equation then we can define on $\mathcal{G}$ a second Poisson bracket $\{,\}_2$ compatible with the first one and quadratic in L.

We apply this construction to the algebra A^+ of formal difference operators of the form

$$\sum_{k<+\infty} u_k \Lambda^k; \tag{1.23}$$

the splitting $A^+ = (A^+)_+ \oplus (A^+)_-$ of the algebra A^+ gives an $R \in \operatorname{End} \mathcal{G}$ defined by $R(X) = X_+ - X_-$ such that both R and its skew-symmetric part S satisfy the modified Yang-Baxter equation. Thus we obtain a pencil of Poisson brackets on A^+.

We then perform a Dirac reduction of these Poisson brackets to the affine subspace of A^+ given by operators of the form (1.16) and obtain the following pair of compatible Poisson brackets in the variables $u_{N-1}, \dots, u_{-M}$

$$\{u_n(x), u_m(y)\}_1 = C_{n,m}[u_{n+m}(\Lambda^n \delta(x-y)) - (\Lambda^{-m} u_{n+m}\delta(x-y))], \tag{1.24}$$

$$
\begin{aligned}
\{u_n(x), u_m(y)\}_2 &= 2u_n((\Lambda^n + 1)(\Lambda^{-m} - 1)u_m\delta(x-y)) \\
&+ 4\sum_{l<m}\big(u_{n+m-l}(\Lambda^{n-l}u_l\delta(x-y)) - u_l(\Lambda^{l-m}u_{n+m-l}\delta(x-y))\big) \\
&- 2\big(u_n(1+\Lambda^{-N})(1+\Lambda^N)(\Lambda^{-N}-\Lambda^N)^{-1}(\Lambda^n - 1)(1-\Lambda^{-m})u_m\delta(x-y)\big)
\end{aligned} \tag{1.25}
$$

where $C_{n,m} = \begin{cases} -1 & n \leqslant 0 \\ 1 & n > 0 \end{cases} + \begin{cases} -1 & m \leqslant 0 \\ 1 & m > 0. \end{cases}$

We relate the Lax and the Hamiltonian representations of the flows using the first Poisson bracket, i.e. we obtain that

$$
\frac{d}{dt^{\alpha,q}} u_n = \{u_n, \bar{h}_{\alpha,q}\}_1 \tag{1.26}
$$

where $\bar{h}_{\alpha,q} = \int dx\ h_{\alpha,q}$ and the Hamiltonian densities are given by

$$
h_{\alpha,q} = \frac{1}{2}\frac{\Gamma(2-\frac{\alpha}{N})}{\Gamma(q+3-\frac{\alpha}{N})} Res(L^{q+2-\frac{\alpha}{N}}) \qquad \text{for} \qquad \alpha = N-1, \cdots, 0 \tag{1.27a}
$$

$$
h_{\alpha,q} = \frac{1}{2}\frac{\Gamma(2+\frac{\alpha}{M})}{\Gamma(q+3+\frac{\alpha}{M})} Res(L^{q+2+\frac{\alpha}{M}}) \qquad \text{for} \qquad \alpha = 0, \cdots, -M+1 \tag{1.27b}
$$

$$
h_{-M,q} = \frac{1}{2}\frac{1}{(q+1)!} Res\left(L^{q+1}(\log L - \frac{1}{2}(\frac{1}{M} + \frac{1}{N})c_{q+1})\right). \tag{1.27c}
$$

for $c_q = \sum_{j=1}^{q} \frac{1}{j}$ and $c_0 = 0$. For $N = M = 1$ we prove that this bihamiltonian structure coincides with that of the standard Toda chain.

Finally we show that these Hamiltonian densities satisfy the tau symmetry

$$
\{h_{\alpha,p-1}, \bar{h}_{\beta,q}\}_1 = \{h_{\beta,q-1}, \bar{h}_{\alpha,p}\}_1. \tag{1.28}
$$

This gives a possibility to define the tau function for an arbitrary solution to the hierarchy.

A further important step for the description of the bihamiltonian structure of this hierarchy should be the determination of the recursion relations, i.e. the generalization of formulas (1.6) and (1.12). The reduction of the second Poisson bracket to an affine subspace needs a Dirac correction term, that is expected to produce some non-standard recursion relation like (1.12). However the explicit approach used to derive the recursion relations in the Toda chain case could not be generalized to the general bigraded hierarchy.

In the case $M = 0$, Poisson structures related to those defined here for the bigraded Toda were considered by Frenkel and Reshetikhin [24, 23], in the context of deformations of W-algebras (see Remark 47) . The R-matrix approach has been applied to the theory of the Hamiltonian structures associated with difference operators also in [37, 1]. However the structure of the corresponding integrable hierarchy was not considered.

We then consider the *two-dimensional Toda hierarchy* and its generalizations. In its simplest form this hierarchy has been first studied by Ueno and Takasaki in [47]. In this case one has two Lax operators

$$
L = \Lambda + u_0 + u_{-1}\Lambda^{-1} + \dots \qquad \bar{L} = \bar{u}_{-1}\Lambda^{-1} + \bar{u}_0 + \bar{u}_1\Lambda + \dots \tag{1.29}
$$

and two sets of times t_q, $\bar{t}_q$.

Two sets of flows, denoted by the times t_q and $\bar{t}_q$ with $q > 0$, are usually defined by the following Lax equations

$$L_{t_q} = [(L^q)_+, L] \qquad \bar{L}_{t_q} = [(L^q)_+, \bar{L}] \tag{1.30}$$

and

$$L_{\bar{t}_q} = [-(\bar{L}^q)_-, L] \qquad \bar{L}_{\bar{t}_q} = [-(\bar{L}^q)_-, \bar{L}]. \tag{1.31}$$

To obtain a bihamiltonian structure in this case we introduce the algebra $A^+ \oplus A^-$ of pairs of difference operators of the form

$$(\sum_{k<+\infty} u_k \Lambda^k, \sum_{k>-\infty} \bar{u}_k \Lambda^k) \tag{1.32}$$

and then define $R \in \mathrm{End}(A^+ \oplus A^-)$ by

$$R(X, \bar{X}) = (X_+ - X_- + 2\bar{X}_-, \bar{X}_- - \bar{X}_+ + 2X_+) \tag{1.33}$$

where $(X, \bar{X}) \in A^+ \oplus A^-$.

This R-matrix comes from the following non-trivial splitting of the algebra

$$A^+ \oplus A^- = \Big(\mathrm{diag}(A^0 \oplus A^0)\Big) \oplus \Big((A^+)_- \oplus (A^-)_+\Big) \tag{1.34}$$

i.e., it is given by $R = P - \tilde{P}$, where P and $\tilde{P}$ are the projections operators given by

$$P(X, \bar{X}) = (X_+ + \bar{X}_-, X_+ + \bar{X}_-) \qquad \tilde{P}(X, \bar{X}) = (X_- - \bar{X}_-, \bar{X}_+ - X_+); \tag{1.35}$$

hence R automatically satisfies the modified Yang-Baxter equation. We show that also its skew-symmetric part satisfies the same equation, hence we obtain, by the general theorems mentioned above, two compatible Poisson structures on the algebra $A^+ \oplus A^-$.

Finally, by Dirac reduction to the affine subspace of couples of operators $(L, \bar{L})$ of the form

$$L = \Lambda^N + u_{N-1}\Lambda^{N-1} + \dots \qquad \bar{L} = \bar{u}_{-M}\Lambda^{-M} + \bar{u}_{-M+1}\Lambda^{-M+1} + \dots \tag{1.36}$$

for two positive integers N, M, we obtain a Poisson pencil on the variables u_n for $n < N$ and $\bar{u}_m$ for $m \geqslant -M$.

The (bigraded) Toda hierarchy fits in the general framework of classification of bihamiltonian integrable systems starting from their dispersionless limit developed by Dubrovin and Zhang in [19]. When the dispersionless limit of the bihamiltonian structure is given by a pencil of Poisson brackets of hydrodynamic type (as in the Toda case), then it is in one to one correspondence with a Frobenius manifold [13].

It was show in [18] that a Frobenius manifold structure can be constructed on the orbit space of the extended affine Weyl groups associated to a root system (which in turn is labelled by a Dynkin diagram) with a fixed root. In particular the extended (N, M)-bigraded Toda hierachy turns out to be the dispersive hierarchy corresponding to the Dynkin diagram A_{N+M-1} with the N-th vertex fixed.

An interesting problem is the determination of the full dispersive hierarchies corresponding to all the Dynkin diagrams (A_l, B_l, C_l, ...) considered in [18].

In the last Chapter we study the dispersionless limit of the hierarchies introduced before and briefly consider the structure of the related Frobenius manifolds.

We first consider the dispersionless bigraded Toda hierarchy. After writing down the explicit form of the dispersionless brackets and of the associated metrics, we find their generating functions. For example, the generating functions of the first and second (contravariant) metrics can be written as

$$(d\lambda(p), d\lambda(q))_1 = 2\frac{\lambda'(q) - \lambda'(p)}{p^{-1} - q^{-1}} \tag{1.37}$$

and

$$(d\lambda(p), d\lambda(q))_2 = \frac{4}{N} pq\lambda'(p)\lambda'(q) + \frac{4}{p^{-1} - q^{-1}} \left(\lambda(p)\lambda'(q) - \lambda(q)\lambda'(p)\right) \tag{1.38}$$

where

$$\lambda(p) = p^N + \cdots + u_{-M} p^{-M} \tag{1.39}$$

and

$$d\lambda(p) = du_{N-1} p^{N-1} + \cdots + du_{-M} p^{-M}. \tag{1.40}$$

To provide a concrete realization of the Frobenius manifold associated to this pencil of flat contravariant metrics we consider a particular case of Hurwitz space. A general structure of Frobenius manifold was defined on such spaces in [10]. We prove that the pencil of metrics obtained from the bihamiltonian structure of the bigraded Toda hierarchy is equal to the pencil naturally defined on the Hurwitz space $\widetilde{M}_{0;N-1,M-1}$; hence the Frobenius manifold associated to the dispersionless limit of the bigraded Toda hierarchy coincides with the one defined on this Hurwitz space.

It was shown in [18] that this Frobenius structure is moreover isomorphic with the one that has been defined on the orbit space of the extended affine Weyl group $\widetilde{W}^{(N)}(A_{N+M-1})$ associated to the irreducible reduced root system A_{N+M-1} with the N-th root fixed.

We then consider the dispersionless limit of the two-dimensional Toda hierarchy. As in the previous case we derive the generating functions for the Poisson brackets and the associated metrics.

We prove that the first metric is non-degenerate and find a new set of coordinates w_0, w_{-1} and v_k, $k \in \mathbb{Z}$ in which it has the form

$$\sum_{n,m\in\mathbb{Z}} (n+m) v_{n+m} \frac{\partial}{\partial v_n}\frac{\partial}{\partial v_m} + w_{-1}\left(\frac{\partial}{\partial w_0}\frac{\partial}{\partial w_{-1}} + \frac{\partial}{\partial w_{-1}}\frac{\partial}{\partial w_0}\right) \tag{1.41}$$

i.e. it splits in the orthogonal sum of two blocks. The first dispersionless Poisson bracket also splits in two independent parts: for the coordinates w_k it coincides with the first Poisson structure of the Toda chain hierarchy while the part corresponding to the coordinates v_k has the form

$$\{v_n(x), v_m(y)\}_1^{disp} = -2[nv_{n+m}(x) + mv_{n+m}(y)]\delta'(x-y) \quad n, m \in \mathbb{Z}. \tag{1.42}$$

If we pass to the coordinates given by the Fourier series with coefficients $v_n(x)$

$$v(x,y) = \sum_{n\in\mathbb{Z}} v_n(x)e^{iny} \tag{1.43}$$

we obtain the following expression for the infinite dimensional part of the first dispersionless Poisson bracket

$$\begin{aligned}\{v(x_1,y_1), v(x_2,y_2)\}_1^{disp} = -4\pi i\big[&\partial_{x_1}v(x_1,y_1)\cdot\delta(x_1-x_2)\delta'(y_1-y_2) \\ &-\partial_{y_1}v(x_1,y_1)\cdot\delta'(x_1-x_2)\delta(y_1-y_2)\big].\end{aligned} \tag{1.44}$$

We finally show that this bracket is the one naturally associated with the algebra of divergence-free vector fields on the cylinder, i.e. with the dynamics of a two dimensional incompressible fluid.

Layout of the thesis The layout of the thesis is the following.

In Chapter 2 we introduce the Toda chain hierarchy and its extended version. We motivate this extension by examining the dispersionless limit and the structure of the Casimirs of the hierarchy. We introduce a formal logarithm of the difference Lax operator L and define the new flows through the Lax pair formalism. In Theorem 7 we show that the coefficients of the logarithm are uniquely determined power series in ϵ of differential polynomials. Then we prove that these flows have a bihamiltonian formulation (Theorem 12), we show the tau-symmetry (Theorem 17) of the Hamiltonian densities and hence derive the existence of a tau-function for the hierarchy, expressed in terms of the Lax operator. We then derive the bilinear relations for the wave operators and the wave functions and finally we obtain the soliton solutions by the method of Darboux transformations of the Lax operator L.

In Chapter 3 we define the bigraded Toda hierarchy. First we generalize the definition of the logarithm $\log L$ and introduce two fractional powers $L^{\frac{1}{N}}$ and $L^{\frac{1}{M}}$. In Theorems 27 and 28 we show that these operators have uniquely defined coefficients that are power series in ϵ of differential polynomials. Then we introduce the flows by the Lax pair formalism. After a brief summary of some results of the R-matrix theory, we define an R-matrix on the algebra A^+ of difference operators and we obtain three compatible Poisson brackets on A^+. Then we perform a Dirac reduction to an affine subspace of A^+, getting a pencil of Poisson brackets for the bigraded Toda (Theorem 45). We finally obtain in Theorem 52 the Hamiltonian representation of the Lax flows through the first Poisson structure, and in Theorem 57 we prove the tau-symmetry of the Hamiltonian densities.

In Chapter 4 we obtain the bihamiltonian structure for the two-dimensional Toda hierarchy by applying again the R-matrix construction. We first define the relevant algebra $A^+ \oplus A^-$ and the define an R-matrix on it, showing that it comes from a non trivial splitting. We obtain explicit expressions for the pencil of Poisson brackets on $A^+ \oplus A^-$ and, by Dirac reduction to an affine subspace, we derive a pencil of Poisson brackets for the (M, N)-bigraded two-dimensional Toda hierarchy (Theorem 63).

In Chapter 5 we consider the dispersionless limit of the bigraded and of the two-dimensional Toda hierarchies. We write explicitly the Poisson brackets and the associated metrics and we obtain their generating functions. We then briefly recall the definition of the Hurwitz spaces and of the natural Frobenius manifold structure on them and show that it coincides with the Frobenius manifold structure associated with the Poisson pencil of the bigraded Toda hierarchy. In the case of the two-dimensional Toda hierarchy, after having obtained the generating functions for the Poisson brackets and for the metrics, we prove that the first metric is non-degenerate. We then define a new set of coordinates in which the first Poisson bracket has a simple structure and we show that it is actually given by the direct sum of the first bracket of the bigraded hierarchy and of the Poisson bracket associated with the dynamics of an incompressible fluid.

In the final Chapter we review open problems and perspectives.

Chapter 2

Extended Toda hierarchy

In this chapter we introduce the Toda (chain) hierarchy and we extend it by adding an infinite number of non-local commuting flows.

The Toda chain hierarchy is associated to the discrete Lax operator

$$L_{nm} = \delta_{n+1,m} + u_n \delta_{n,m} + e^{v_n} \delta_{n-1,m} \tag{2.1}$$

where the dynamical variables are given by u_n and v_n for $n \in \mathbb{Z}$ or $n \in \mathbb{Z}_k$ in the case of periodic boundary conditions.

The flows of the hierarchy are given by the isospectral deformations of such operator, i.e. by the Lax pair formalism $L_t = [A, L]$; this equation gives exactly the Toda equation (1.2) if we take $A_{nm} = \delta_{n+1,m} + u_n \delta_{n,m}$.

Instead of using discrete variables u_n and v_n we will adopt a continuous notation $v(x)$, $u(x)$ and we will denote by ϵ the lattice spacing; of course the two notations will be related by $u(n\epsilon) = u_n$. Since the usual equations of the Toda hierarchy are difference evolutionary equations, we will write them using the shift operator $\Lambda^k f(x) = f(x+\epsilon k)$.

These results are obtained in collaboration with B. Dubrovin and Y. Zhang [3] (see also [2]).

2.1 The Toda chain hierarchy

In this section we define the usual Toda chain hierarchy in the Lax pair formulation and consider its bihamiltonian structure. Since the Hamiltonians can be constructed recursively using the Lenard-Magri recursion procedure starting from a Casimir of the first Poisson bracket, one expects that another infinite set of Hamiltonians could be constructed starting from the second Casimir of the first bracket. We comment on the fact that this is impossible due to the "resonance" of the Poisson pencil.

We introduce first the definition of the Toda chain hierarchy flows using the Lax pair formalism.

Let's denote with $u(x)$ and $v(x)$ the two dependent variables of the hierarchy. Recall that the original dependent variables of the Toda hierarchy are u_n and v_n, where n identifies the n-th site of a lattice with spacing ϵ; in the continuous limit described above $u_n = u(\epsilon n)$ and $v_n = v(\epsilon n)$. The functions $u(x)$ and $v(x)$ can be taken in the space of periodic functions ($x \in S^1$) or on the real line ($x \in \mathbb{R}$) but for the moment we disregard the boundary conditions.

The Lax operator, that acts on the space of functions $f(x)$ of one variable x, is

$$L = \Lambda + u(x) + e^{v(x)}\Lambda^{-1}, \tag{2.2}$$

where $\Lambda f(x) = f(x+\epsilon)$ is the shift operator.

Given any difference operator $A = \sum_{k\in\mathbb{Z}} a_k \Lambda^k$ we denote by A_+ and A_- the positive and negative parts respectively, i.e. $A_+ = \sum_{k\geqslant 0} a_k\Lambda^k$, $A_+ + A_- = A$; we indicate the commutator of difference operators with $[A,B] = AB - BA$. Moreover we define the residue of A by $\operatorname{Res} A = a_0$.

The Toda chain hierarchy is given by the system of flows

$$\epsilon\frac{\partial L}{\partial t^{2,q}} = [A_{2,q}, L] \qquad q \geqslant 0 \tag{2.3}$$

where the operators $A_{2,q}$ are defined by

$$A_{2,q} = \frac{1}{(q+1)!}[L^{q+1}]_+. \tag{2.4}$$

Example 1 *Let's give some explicit examples of Lax operators and associated flows. The first example is simply*

$$A_{2,0} = \Lambda + u \tag{2.5}$$

that gives, by (2.3), the $t^{2,0}$-equations

$$\begin{aligned} \epsilon u_{t^{2,0}} &= ((\Lambda-1)e^v) &(2.6)\\ \epsilon v_{t^{2,0}} &= ((1-\Lambda^{-1})u); &(2.7)\end{aligned}$$

these are just the usual Toda chain equations, written in the continuos formalism.

The following example is given by

$$A_{2,1} = \frac{1}{2}(\Lambda^2 + ((\Lambda u) + u)\Lambda + u^2 + (\Lambda e^v) + e^v), \tag{2.8}$$

and the $t^{2,1}$-equations are

$$\begin{aligned} \epsilon u_{t^{2,1}} &= \frac{1}{2}(((\Lambda u) + u)(\Lambda e^v) - ((\Lambda^{-1}u) + u)e^v) &(2.9)\\ \epsilon v_{t^{2,1}} &= \frac{1}{2}((\Lambda e^v) - (\Lambda^{-1}e^v) + u^2 - (\Lambda^{-1}u^2)). &(2.10)\end{aligned}$$

Remark 2 *We will use the following notation: whenever the shift operator Λ appears inside a parenthesis, like in (Λf), it is supposed to act only on the function f inside the parenthesis; in the other cases it must be considered as an operator that acts on everything on the right. More explicitly (Λf) represents the function $f(x+\epsilon)$, while Λf represents the operator $(\Lambda f)\Lambda$.*

It is well-known that the hierarchy under consideration admits a bihamiltonian formulation. This essentially means that the Lax flows defined above can be written as Hamilton equations with respect to two compatible Poisson structures. We summarize these facts in the following

Theorem 3 *The flows $t^{2,q}$ defined above can be expressed in bihamiltonian form*

$$v_{t^{2,q}} = \{v, \bar{h}_{2,q}\}_1 = \frac{1}{q+1}\{v, \bar{h}_{2,q-1}\}_2, \tag{2.11}$$

$$u_{t^{2,q}} = \{u, \bar{h}_{2,q}\}_1 = \frac{1}{q+1}\{u, \bar{h}_{2,q-1}\}_2. \tag{2.12}$$

The first Poisson brackets are given by

$$\{u(x), u(y)\}_1 = \{v(x), v(y)\}_1 = 0 \tag{2.13a}$$

$$\{u(x), v(y)\}_1 = \frac{1}{\epsilon}(\delta(x-y+\epsilon) - \delta(x-y)) \tag{2.13b}$$

$$\{v(x), u(y)\}_1 = \frac{1}{\epsilon}(\delta(x-y) - \delta(x-y-\epsilon)) \tag{2.13c}$$

and the second Poisson brackets by

$$\{u(x), u(y)\}_2 = \frac{1}{\epsilon}(e^{v(x+\epsilon)}\delta(x-y+\epsilon) - e^{v(x)}\delta(x-y-\epsilon)) \tag{2.14a}$$

$$\{v(x), v(y)\}_2 = \frac{1}{\epsilon}(\delta(x-y+\epsilon) - \delta(x-y-\epsilon)) \tag{2.14b}$$

$$\{u(x), v(y)\}_2 = \frac{1}{\epsilon}u(x)(\delta(x-y+\epsilon) - \delta(x-y)) \tag{2.14c}$$

$$\{v(x), u(y)\}_2 = \frac{1}{\epsilon}(u(x)\delta(x-y) - u(x-\epsilon)\delta(x-y-\epsilon)). \tag{2.14d}$$

The Hamiltonian densities are given by

$$h_{2,q} = \frac{1}{(q+2)!} Res L^{q+2} \qquad q \geqslant -1, \tag{2.15}$$

from which the Hamiltonians $\bar{h}_{2,q}$ are obtained by integration: $\bar{h}_{2,q} = \int h_{2,q} dx$. All these Hamiltonians are in involution with respect to both Poisson brackets, i.e.

$$\{\bar{h}_{2,q}, \bar{h}_{2,p}\}_i = 0 \tag{2.16}$$

for $p, q \geqslant 0$ and $i = 1, 2$.

The proof will be given in the following sections together with the proof of the Hamiltonian theorem for the extended flows; however for the usual Toda flows it can be found e.g. in [35].

Example 4 *Some examples of Hamiltonian densities are*

$$h_{2,-1} = u \tag{2.17}$$
$$h_{2,0} = \frac{1}{2}((\Lambda e^v) + e^v + u^2). \tag{2.18}$$

Consider now the Hamiltonian $\bar{h}_{2,-1} = \int u(x)dx$. One can easily check that, through the first Poisson bracket, it gives a trivial flow. Indeed it is a Casimir of the first bracket, i.e. it commutes with any other functional of the variables $u(x)$ and $v(x)$.

It is a well-known fact that, in the presence of a bihamiltonian structure, one can construct recursively a sequence of Hamiltonians using the Lenard-Magri recursion relations given by (2.11) i.e.

$$(q+1)\{\cdot, \bar{h}_{2,q}\}_1 = \{\cdot, \bar{h}_{2,q-1}\}_2. \tag{2.19}$$

In this case one starts from the Casimir $\bar{h}_{2,-1}$ and builds all the Hamiltonians $\bar{h}_{2,q}$. In particular one might expect to perform the same procedure starting from every Casimir of the first bracket. The first Poisson bracket (2.13) actually admits a second Casimir

$$\int v(x)dx; \tag{2.20}$$

starting from this Hamiltonian one would like to obtain a second set of Hamiltonians in involution with $\bar{h}_{2,q}$ and between themselves. However we cannot start from the Casimir (2.20), since it is a Casimir also for the *second* bracket (2.14). This phenomenon is called "resonance" of the bihamiltonian structure (2.13), (2.14).

2.2 Dispersionless limit and extended hierarchy

Here we recall the Lax formulation of the dispersive limit of the Toda chain hierarchy. We show that in this limit new flows can be defined, using the logarithm of the Lax function. This is a first hint that analogous flows should exist in the dispersive hierarchy.

The dispersionless limit is obtained by putting $\epsilon \to 0$. It can be easily shown [45] that the Lax representation is simply obtained by substituing Λ with p and the commutator of operators with the canonical Poisson bracket between functions of the variables x, p. More precisely the dispersionless flows corresponding to (2.3) are

$$\frac{\partial \mathcal{L}}{\partial t^{2,q}} = \{\mathcal{A}_{2,q}, \mathcal{L}\}, \qquad \mathcal{A}_{2,q} = \frac{1}{(q+1)!}(\mathcal{L}^{q+1})_+, \tag{2.21}$$

where the Lax operator L is replaced by a Lax function

$$\mathcal{L}(x,p) = p + u(x) + e^{v(x)}p^{-1}, \tag{2.22}$$

and the bracket is

$$\{\mathcal{B},\mathcal{C}\} = p\frac{\partial \mathcal{B}}{\partial p}\frac{\partial \mathcal{C}}{\partial x} - p\frac{\partial \mathcal{C}}{\partial p}\frac{\partial \mathcal{B}}{\partial x} \tag{2.23}$$

for any two functions $\mathcal{B}$ and $\mathcal{C}$ of x and p. $(\mathcal{B})_+$ means that only non-negative powers of p, in the power series expansion of $\mathcal{B}$, are considered.

The dispersionless Hamiltonians $\bar{h}^{disp}_{2,q}$ and Poisson brackets $\{,\}^{disp}_i$ are obtained from their dispersive counterparts (2.15), (2.13)-(2.14) as the leading term in ϵ in the $\epsilon \to 0$ limit, i.e. for the Poisson brackets

$$\{f,g\}_i = \epsilon\{f,g\}^{disp}_i + O(\epsilon^2). \tag{2.24}$$

One finds that the only non-zero terms of the dispersionless Poisson brackets (we drop the superscript) are

$$\{u(x),v(y)\}_1 = \delta'(x-y) \tag{2.25a}$$

$$\{u(x),u(y)\}_2 = 2e^{v(x)}\delta'(x-y) + e^{v(x)}v_x(x)\delta(x-y) \tag{2.25b}$$

$$\{v(x),v(y)\}_2 = 2\delta'(x-y) \tag{2.25c}$$

$$\{u(x),v(y)\}_2 = u(x)\delta'(x-y). \tag{2.25d}$$

In particular the same recursion relation as above (2.19) holds in the dispersionless case.

Considering the genus zero approximation of the topological $C\mathrm{P}^1$ model, in [20] it was noted that new flows, that we denote with times $t^{1,q}$, can be added to the usual dispersionless flows given above; their Lax representation is

$$\frac{\partial \mathcal{L}}{\partial t^{1,q}} = \{\mathcal{A}_{1,q},\mathcal{L}\}, \qquad \mathcal{A}_{1,q} = \frac{2}{q!}(\mathcal{L}^q(\log\mathcal{L} - c_q))_+ \tag{2.26}$$

where $c_q = \sum_{k=1}^{q}\frac{1}{k}$, $c_0 = 0$. The logarithm of $\mathcal{L}$ must be understood in the following way

$$\log\mathcal{L} = \frac{1}{2}v + \frac{1}{2}\log(1+up^{-1}+e^vp^{-2}) + \frac{1}{2}\log(1+ue^{-v}p+e^{-v}p^2) \tag{2.27}$$

where the first logarithm on the RHS is seen as an expansion in negative powers of p while the second one in positive powers of p.

These flows can be expressed in Hamiltonian form by

$$\frac{\partial}{\partial t^{1,q}}\cdot = \{\cdot,\bar{h}^{disp}_{1,q}\}^{disp}_1 \tag{2.28}$$

where the dispersionless Hamiltonians are given by

$$h^{disp}_{1,q} = \frac{2}{(q+1)!}Res_{p=0}[\mathcal{L}^{q+1}(\log\mathcal{L} - c_{q+1})\frac{dp}{p}]. \tag{2.29}$$

These Hamiltonians however satisfy a recursion relation that is different from the previous one (2.19)

$$\{\cdot,\bar{h}_{1,q-1}\}_2 = q\{\cdot,\bar{h}_{1,q}\}_1 + 2\{\cdot,\bar{h}_{2,q-1}\}_1. \tag{2.30}$$

One can use the standard argument for a Lenard-Magri chain to show that all the Hamiltonians $\bar{h}_{\alpha,q}$ are in involution with respect to both brackets. Essentially, applying twice the recursion relation (2.19) one obtains that

$$\{\bar{h}_{2,p}, \bar{h}_{2,q}\}_1 = \frac{p+3}{q+1}\{\bar{h}_{2,p+1}, \bar{h}_{2,q-1}\}_1; \tag{2.31}$$

then it is clear that applying this formula multiple times one reaches the Casimir $\bar{h}_{2,-1}$, hence the Hamiltonians $\bar{h}_{2,q}$ are in involution with respect to $\{,\}_1$. Using this fact one obtains an analogous relation

$$\{\bar{h}_{2,p}, \bar{h}_{1,q}\}_1 = \frac{q+1}{p+1}\{\bar{h}_{2,p-1}, \bar{h}_{1,q+1}\}_1 \tag{2.32}$$

from which the involutivity of $\bar{h}_{1,q}$ and $\bar{h}_{2,p}$ is proved. Using again this fact and the recursion relation (2.30) one gets

$$\{\bar{h}_{1,p}, \bar{h}_{1,q}\}_1 = \frac{p+1}{q}\{\bar{h}_{1,p+1}, \bar{h}_{1,q-1}\}_1 \tag{2.33}$$

and by repeated application of this formula one obtains that $\{\bar{h}_{1,p}, \bar{h}_{1,q}\}_1$ is proportional to

$$\{\bar{h}_{1,0}, \bar{h}_{1,q+p}\}_1. \tag{2.34}$$

Here we don't reach the Casimir $\bar{h}_{1,-1}$; anyway the Hamiltonian $\bar{h}_{1,0} = \int u(x)v(x)dx$ is the generator of x-translations and hence acts trivially on any integrated quantity. We conclude that all the dispersionless Hamiltonians $\bar{h}_{\alpha,p}$ are in involution. An analogous procedure is used to show involution with respect to the second Poisson brackets.

Thus in the dispersionless case the Toda hierarchy has two perfectly well defined sequences of flows all commuting between themselves, denoted by the times $t^{\alpha,q}$ for $q \geqslant 0$, $\alpha = 1, 2$. We call this the *Extended dispersionless Toda chain hierarchy*. The classical dispersive flows corresponding to the times $t^{2,q}$ defined above reduce, for $\epsilon \to 0$, to the corresponding flows in the dispersionless hierarchy; on the other hand in the classical dispersive formulation there is apparently no flow reducing for $\epsilon \to 0$ to the dispersionless flows corresponding to the times $t^{1,q}$. However the Lenard-Magri recursion relation (2.30) for the second set of dispersionless flows starts not from the Casimir $\bar{h}_{1,-1}$ (resonance problem) but from the Hamiltonian $\bar{h}_{1,0}$.

This suggests to define a dispersive Hamiltonian $\bar{h}_{1,0}$ generating the x-translations under the full dispersive Poisson brackets and then to define the dispersive counterparts of the Hamiltonians $\bar{h}_{1,q}$ using the recursion relation (2.30). This was actually done by Y. Zhang in [48] by providing the ansatz

$$\bar{h}_{1,0} = \int u(x)(1 - \Lambda^{-1})^{-1}\epsilon v_x(x). \tag{2.35}$$

This Hamiltonian, containing the inverse of the discrete derivative $1 - \Lambda^{-1}$, is non-local so we expect that in general all the Hamiltonians $\bar{h}_{1,q}$ will be non-local, too. These Hamiltonians will be in involution between themselves and with all the usual Toda

Hamiltonians (2.15) with respect to both brackets (2.13) and (2.14). This can be seen by exactly the same proof just given above for the dispersionless hamiltonians. We call this system of commuting flows the *Extended Toda chain hierarchy*.

While in principle we can construct all the hierarchy of Hamiltonians $\bar{h}_{1,q}$ by recursion, in practice it is impossible to write down the Hamiltonians beyond the first few ones. A much better computational tool is given by the Lax representation. We will show in the following, by defining a logarithm of L, how to obtain the Lax representation for these flows and an explicit form of the non-local Hamiltonians.

2.3 Logarithm of L

In this section we define the logarithm of the operator L through the use of the dressing operators. Then we show that if we extend the space of functions to be the space of power series $\sum_{k\geqslant 0} f_k(x)\epsilon^k$ then $\log L$ is a well-defined infinite difference operator having for coefficients series in powers of ϵ of differential polynomials in the variables $u(x)$, $v(x)$ and $e^{v(x)}$.

It is well known [47] that one can write the Lax operator (2.2) as the dressing of the shift operators Λ and Λ^{-1}

$$L = P\Lambda P^{-1} = Q\Lambda^{-1}Q^{-1} \tag{2.36}$$

where the dressing operators P, Q have the form

$$P = \sum_{k\geqslant 0} p_k(x)\Lambda^{-k} \qquad p_0 = 1, \tag{2.37a}$$

$$Q = \sum_{k\geqslant 0} q_k(x)\Lambda^{k}. \tag{2.37b}$$

By substituting in the definition (2.36), the functions p_k, q_k can be found in terms of u, v. These dressing operators P and Q are defined up to the multiplication from the right by operators of the form $1 + \sum_{k\geqslant 1} c_k\Lambda^{-k}$ and $\sum_{k\geqslant 0} \hat{c}_k\Lambda^{k}$ respectively, with constant coefficients.

Since the shift operator can be written as $\Lambda = e^{\epsilon\partial_x}$ one is led to define two different logarithms in the following way

$$\log_+ L := P\epsilon\partial P^{-1} = \epsilon\partial + P\epsilon P_x^{-1} \tag{2.38a}$$

$$\log_- L := -Q\epsilon\partial Q^{-1} = -\epsilon\partial - Q\epsilon Q_x^{-1}. \tag{2.38b}$$

Notice that the ambiguity in the definition of the dressing operators is cancelled in the definition of these logarithms. They are differential-difference operators of the form

$$\log_+ L = \epsilon\partial + 2\sum_{k>0} w_{-k}(x)\Lambda^{-k} \tag{2.39a}$$

$$\log_- L = -\epsilon\partial + 2\sum_{k\geqslant 0} w_k(x)\Lambda^{k}. \tag{2.39b}$$

Since we want to write an expression like (2.29) and we need to make sense of the $(\cdot)_+$ part, we would like to have a purely difference operator for the logarithm, that we define by

$$\log L = \frac{1}{2}\log_+ L + \frac{1}{2}\log_- L = -\frac{\epsilon}{2}(P_x P^{-1} - Q_x Q^{-1}); \tag{2.40}$$

in this definition the derivative drops out and we get a difference operator of the form

$$\log L = \sum_{k\in\mathbb{Z}} w_k(x)\Lambda^k. \tag{2.41}$$

Now we would like to find explicit expressions of the coefficients w_k in terms of the basic variables $u(x)$, $v(x)$.

We will need in particular to invert the discrete derivative operator $\Lambda - 1$ that appears in the recursive definition of the coefficients w_k; however there is no way to find explicit formulas for such inversion if we work on the space of functions $f(x)$. We can consider instead functions that are power series in ϵ, i.e. of the form

$$f(x,\epsilon) = \sum_{k\geqslant 0} f_k(x)\epsilon^k, \tag{2.42}$$

and impose that the shift operator acts on these functions as the exponential of the x-derivative

$$\Lambda f = e^{\epsilon\frac{d}{dx}} f = \sum_{k\geqslant 0} \frac{\epsilon^k}{k!}(\frac{d}{dx})^k f; \tag{2.43}$$

then we have an explicit inversion formula in terms of the Bernoulli numbers B_k

$$(\Lambda^m - 1)^{-1}\epsilon\frac{d}{dx} f = \frac{1}{m}\sum_{k\geqslant 0}\frac{B_k}{k!}(m\epsilon\frac{d}{dx})^k f; \tag{2.44}$$

the Bernoulli numbers are defined by

$$\frac{x}{e^x - 1} = \sum_{k\geqslant 0}\frac{B_k}{k!}x^k. \tag{2.45}$$

In (2.44) the operator $\Lambda^m - 1$ acts on the derivative $\epsilon\frac{d}{dx}f$ since $\mathrm{Im}(\Lambda^m - 1) = \mathrm{Im}\epsilon\frac{d}{dx}$. Indeed it is easy to check that $\mathrm{Ker}(\Lambda - 1) = \mathbb{C}[[\epsilon]]$ and $\mathrm{Im}(\Lambda - 1) = \mathrm{Im}\epsilon\frac{d}{dx}$; these give also the kernel and the image of $\Lambda^m - 1$, since $\Lambda^m - 1 = (\Lambda - 1)(\Lambda^{m-1} + \cdots + 1)$ and $\Lambda^{m-1} + \cdots + 1$ is an automorphism of the space of functions of the form (2.42).

Now consider the case where the coefficients f_k are differential polynomials

Definition 5 *We denote by $\mathcal{A}$ the algebra of differential polynomials in u, v, e^v and e^{-v} with differential $\frac{d}{dx}$; $\hat{\mathcal{A}} := \mathcal{A}[[\epsilon]]$ is the differential algebra of formal power series in ϵ with coefficients in $\mathcal{A}$.*

Remark 6 *In the approach of some authors (e.g. [7], [35] and [28]), these objects are given a purely algebric definition, in the attempt to give a rigorous treatment without* *specifying any boundary condition for the space of functions. E.g. the algebra $\mathcal{A}$ can be seen as the quotient of the polynomial algebra in the symbols $u^{(m)}$, $v^{(m)}$, e^v and e^{-v} for $m \geqslant 0$, plus the constants, by the ideal generated by $e^v e^{-v} - 1$; it becomes a differential algebra if we define the action of the differential $\frac{d}{dx}$ on the generators by $\frac{d}{dx}u^{(m)} = u^{(m+1)}$, $\frac{d}{dx}v^{(m)} = v^{(m+1)}$ and $\frac{d}{dx}e^v = e^v \frac{d}{dx}v$.*

Now we can state the important

Theorem 7 *The coefficients w_k that appear in the definition of the operators* $\log_+ L$, $\log_- L$ *and* $\log L$ *are uniquely determined elements of $\hat{\mathcal{A}}$.*

Proof By dressing with the operator P the relation $[\epsilon\partial, \Lambda^m] = 0$ we clearly have $[\log_+ L, L^m] = 0$; spelling out this relation and taking the residue we obtain

$$((\Lambda^m - 1)w_{-m}) = \frac{1}{2}\epsilon\frac{d}{dx}p_0(m) + \sum_{l=1}^{m-1}[(1-\Lambda^l)(w_{-l}\Lambda^{-l}p_l(m))] \tag{2.46}$$

with $L^m = \sum_{k=-\infty}^{m} p_k(m)\Lambda^k$ and $p_m(m) = 1$. Since the RHS is in the image of $\Lambda^m - 1$, it's clear that we can invert such operator and obtain w_m for $m < 0$ in terms of w_k, $m < k < 0$; hence $w_m \in \hat{\mathcal{A}}$. A priori each w_m is determined up to an element of $\mathrm{Ker}(\Lambda^m - 1) = \mathbb{C}[[\epsilon]]$ i.e. of the form $\sum_{k\geqslant 0} c_k\epsilon^k$. On the other hand the definition in terms of the dressing operator shows that these constants are zero, hence w_m for $m \leqslant -1$ are uniquely determined; indeed from (2.38a) and (2.39a) we get

$$w_{-n} = -\frac{1}{2}\sum_{k=1}^{n}\epsilon(\frac{d}{dx}p_k)(\Lambda^{-k}p^*_{n-k}) \tag{2.47}$$

where $P^{-1} = \sum_{k\geqslant 0} p^*_k\Lambda^{-k}$. If we put $u = 0 = e^v$ in (2.36) we obtain $(\Lambda - 1)p_k = 0$ hence $p_k \in \mathbb{C}[[\epsilon]]$; this implies that $\epsilon\frac{d}{dx}p_k = 0$ hence, by the previous formula, the constant in w_{-n} is zero.

Now we have to repeat the same arguments for the coefficients w_m for $m \geqslant 0$; we define

$$\tilde{Q} := q_0^{-1}Q = 1 + \frac{q_1}{q_0}\Lambda + \ldots \tag{2.48}$$

and

$$\tilde{L} := q_0^{-1}Lq_0. \tag{2.49}$$

Using the fact that, from (2.36), we have

$$q_0^{-1}(\Lambda^{-1}q_0) = e^{-v} \tag{2.50}$$

then, it follows

$$\tilde{L} = \Lambda^{-1} + u + (\Lambda e^v)\Lambda. \tag{2.51}$$

Moreover we have that

$$\tilde{L} = \tilde{Q}\Lambda^{-1}\tilde{Q}^{-1} \tag{2.52}$$

and

$$\log_- L = -\epsilon\partial + \epsilon(\frac{d}{dx}q_0)q_0^{-1} + \epsilon q_0(\frac{d}{dx}\tilde{Q})\tilde{Q}^{-1}q_0^{-1}. \tag{2.53}$$

Hence $w_0 = \frac{\epsilon}{2}(\frac{d}{dx}q_0)q_0^{-1} = \frac{\epsilon}{2}(1-\Lambda^{-1})^{-1}(e^v)_x$ from (2.50) and, defining

$$\epsilon(\frac{d}{dx}\tilde{Q})\tilde{Q}^{-1} = 2\sum_{k>0}\tilde{w}_k\Lambda^k, \tag{2.54}$$

we can find w_k in terms of $\tilde{w}_k$

$$w_k = \tilde{w}_k q_0(\Lambda^k q_0^{-1}). \tag{2.55}$$

Since it easily follows from (2.50) that

$$q_0\Lambda^k q_0^{-1} = \prod_{j=1}^{k}\Lambda^j e^{-v} \in \hat{\mathcal{A}} \tag{2.56}$$

then we just need to show that $\tilde{w}_k \in \hat{\mathcal{A}}$. This is easily done as before by dressing with $\tilde{Q}$ the relation $[-\epsilon\partial, \Lambda^{-m}] = 0$. □

Example 8 *The first few examples of coefficients of* $\log L$ *are*

$$w_{-1} = \frac{1}{2}((\Lambda-1)^{-1}\epsilon u_x) \tag{2.57}$$

$$w_0 = \frac{1}{2}(\Lambda(\Lambda-1)^{-1}\epsilon v_x) \tag{2.58}$$

$$w_1 = \frac{1}{2}(\Lambda e^{-v}(\Lambda-1)^{-1}\epsilon u_x). \tag{2.59}$$

Observe that also the coefficients $p_k(x)$, $q_k(x)$ of the dressing operators are in general non-local functionals of the variables $u(x)$, $v(x)$; however a result similar to Theorem 7 does not hold, i.e. it is not possible to express them as power series in ϵ of differential polynomials in $\mathcal{A}$.

2.4 Lax representation for the non-local flows

Using the logarithm of L defined in the previous section we now define the additional flows by giving their Lax representation. We essentially follow the Lax form of the dispersionless flows $t^{1,q}$.

Definition 9 *The flows defined by the following Lax pair formalism*

$$\epsilon\frac{\partial L}{\partial t^{1,q}} = [A_{1,q}, L] \qquad q \geqslant 0 \tag{2.60}$$

with

$$A_{1,q} = \frac{2}{q!}[L^q(\log L - c_q)]_+ \tag{2.61}$$

where $c_q = \sum_{k=1}^{q}\frac{1}{k}$, $c_0 = 0$ *define, together with the usual flows (2.3), the* Extended Toda chain hierarchy*.*

Remark 10 *Notice that the difference Lax operators $A_{1,q}$ have in general an infinite number of terms; alternatively we can use the operator*

$$\tilde{A}_{1,q} = \frac{2}{q!}[L^q(\log L - c_q)]_+ - \frac{1}{q!}[L^q(\log_- L - c_q)] \tag{2.62}$$

which gives the same flows, since it differs by a part that commutes with L, but contains only a finite number of terms. However it is a differential–difference operator as one can see easily by writing it in the form

$$\tilde{A}_{1,q} = \frac{1}{q!}L^q\epsilon\partial + \frac{1}{q!}[(L^q(2\sum_{k<0} w_k\Lambda^k - c_q))_+ - (L^q(2\sum_{k\geqslant 0} w_k\Lambda^k - c_q))_-]. \tag{2.63}$$

Example 11 *The first example of Lax operator is*

$$\tilde{A}_{1,0} = \epsilon\partial \tag{2.64}$$

and the associated equations of motion are simply given by x-translation, i.e. $L_{t^{1,0}} = L_x$.

The following example is

$$\tilde{A}_{1,1} = \Lambda(\epsilon\partial - 1) + \Lambda(\Lambda-1)^{-1}\epsilon u^1_x + u^1(\epsilon\partial - 1) + e^{u^2}(\epsilon\partial + 1 - (\Lambda-1)^{-1}\epsilon u^2_x)\Lambda^{-1} \tag{2.65}$$

that through (2.60) gives the following non-local equations

$$\epsilon u_{t^{1,1}} = (\Lambda-1)(-e^v(\Lambda^{-1}-1)^{-1}\epsilon v_x) - 2(\Lambda-1)e^v \tag{2.66a}$$
$$+ \frac{\epsilon}{2}(u)^2_x + \epsilon(e^v)_x \tag{2.66b}$$
$$\epsilon v_{t^{1,1}} = ((\Lambda^{-1}-1)^{-1}\epsilon v_x)(\Lambda^{-1}-1)u + \epsilon v_x(\Lambda^{-1}u) \tag{2.66c}$$
$$+ \Lambda^{-1}\epsilon u_x + \epsilon u_x + 2(\Lambda^{-1}-1)u. \tag{2.66d}$$

2.5 Hamiltonian formulation

Here we prove the bihamiltonian theorem for all the flows of the Extended Toda chain hierarchy. This in particular shows that they coincide with the flows introduced by Zhang in [48] using bihamiltonian recursion relation starting from (2.35).

Theorem 12 *The flows defined above can be expressed in Hamiltonian form with respect to the first Poisson bracket*

$$u_{t^{\alpha,q}} = \{u, \bar{h}_{\alpha,q}\}_1, \qquad v_{t^{\alpha,q}} = \{v, \bar{h}_{\alpha,q}\}_1 \tag{2.67}$$

with $\alpha = 1, 2$ and $q \geqslant 0$. The recursion relation is given by

$$\{\cdot, \bar{h}_{\alpha,q-1}\}_2 = (q + \mu_\alpha + \frac{1}{2})\{\cdot, \bar{h}_{\alpha,q}\}_1 + R^\gamma_\alpha\{\cdot, \bar{h}_{\gamma,q-1}\}_1 \tag{2.68}$$

where $\mu_1 = -\mu_2 = -\frac{1}{2}$, $R^{\gamma}_{\beta} = 2\delta^{\gamma}_{2}\delta_{\beta,1}$. *The Poisson brackets were define previously by (2.13) and (2.14). The Hamiltonian densities are in involution with respect to both brackets and are given by*

$$h_{2,q} = \frac{1}{q+2!} Res L^{q+2}, \qquad h_{1,q} = \frac{2}{q+1!} Res[L^{q+1}(\log L - c_{q+1})] \tag{2.69}$$

and $\bar{h}_{\alpha,q} = \int h_{\alpha,q} dx$.

Example 13 *Some examples of Hamiltonian densities are*

$$h_{1,-1} = \Lambda(\Lambda-1)^{-1}\epsilon v_x \tag{2.70a}$$
$$h_{1,0} = (\Lambda+1)(\Lambda-1)^{-1}\epsilon u_x - 2u + u\Lambda(\Lambda-1)^{-1}\epsilon v_x; \tag{2.70b}$$

notice that, up to total derivatives, $h_{1,-1} \sim v$ *(the second Casimir for the first Poisson bracket), while*

$$h_{1,0} \sim u\Lambda(\Lambda-1)^{-1}\epsilon v_x \tag{2.71}$$

is the Hamiltonian (2.35) corresponding to x*-translations. Hence the times defined here in the Lax formalism correspond to those defined by Zhang in [48] by the recursion relation (2.68) starting from (2.71).*

Remark 14 *In the following chapter we will see that these Poisson brackets have an algebraic origin: they are respectively the Poisson-Lie (or Kirillov-Konstant) brackets on the dual of a Lie algebra and the Sklyanin brackets (naturally defined on a Lie group). Here however we prefer to introduce them by explicit formulas.*

Proof The fact that the Poisson brackets (2.13),(2.14) form a compatible pair i.e. that they give a bihamiltonian structure is well-known and can be found e.g. in [35].

To prove (2.67) we expand the Lax operators $A_{\beta,q}$ defined in (2.4) and (2.61) in the form

$$A_{\beta,q} = \sum_{k\geqslant 0} \Lambda^k a_{\beta,q,k} \qquad \beta = 1,2 \quad q \geqslant 0. \tag{2.72}$$

Since $A_{\beta,q} = (B_{\beta,q})_+$ for operators $B_{\beta,q}$ that commute with L, it follows that, since $[(B_{\beta,q})_+, L] = -[(B_{\beta,q})_-, L]$, the only nonzero terms in $[A_{\beta,q}, L]$ are the those with Λ^0 and Λ^{-1}; hence the Lax equations are well-defined.

Expliciting the Lax equations (2.3) and (2.60) we obtain

$$\epsilon u_{t^{\beta,q}} = (\Lambda e^v a_{\beta,q,1}) - e^v a_{\beta,q,1} \tag{2.73a}$$
$$\epsilon v_{t^{\beta,q}} = a_{\beta,q,0} - (\Lambda^{-1} a_{\beta,q,0}). \tag{2.73b}$$

Hence formulas (2.67) are proved if we show that

$$\frac{\delta \bar{h}_{\beta,q}}{\delta u} = a_{\beta,q,0} \qquad \frac{\delta \bar{h}_{\beta,q}}{\delta v} = a_{\beta,q,1} e^v. \tag{2.74}$$

Consider the space of Lax operators of the form (2.2); it is parametrized by two functions $u(x)$ and $v(x)$. The differential of L is

$$dL = du(x) + dv(x)e^v\Lambda^{-1} \tag{2.75}$$

and $du(x)$, $dv(x)$ are the basic differentials of the coordinates on such space. For a functional

$$\bar{h}_{\beta,q} = \int h_{\beta,q}dx \tag{2.76}$$

the differential will be obtained by functional derivation

$$d\bar{h}_{\beta,q} = \int dx \left[\frac{\delta\bar{h}_{\beta,q}}{\delta u(x)}du(x) + \frac{\delta\bar{h}_{\beta,q}}{\delta v(x)}dv(x)\right]. \tag{2.77}$$

Let's consider the case $\beta = 2$. Using the fact that for any difference operator $A = \sum_k A_k\Lambda^k$ it is easy to check that

$$\int Res[A, dL]dx = 0, \tag{2.78}$$

we have that

$$d\bar{h}_{2,q} = \int dx \ \frac{1}{(q+1)!}\text{Res}(L^{q+1}dL) \tag{2.79a}$$

$$= \int dx \ \Big(a_{2,q,0}(x)du(x) + a_{2,q,1}(x)e^{v(x)}dv(x)\Big) \tag{2.79b}$$

and since $du(x)$ and $dv(x)$ are independent, comparing with (2.77) we obtain the equations (2.74) in the $\beta = 2$ case.

For $\beta = 1$ we start by showing that

$$\int dx \ \text{Res}(L^p d\log L) = \int dx \ \text{Res}(L^{p-1}dL); \tag{2.80}$$

it is sufficient to show this for $\log_\pm L$. Using the formula

$$e^{p\log_\pm L} = L^p \tag{2.81}$$

we have

$$pRes(L^{p-1}dL) \sim Res \ dL^p \tag{2.82a}$$

$$= Res \ de^{p\log_\pm L} \tag{2.82b}$$

$$\sim Res(\sum_{n=1}^{\infty}\frac{1}{(n-1)!}(p\log_\pm L)^{n-1}p \ d\log_\pm L) \tag{2.82c}$$

$$= pRes(e^{p\log_\pm L}d\log_\pm L) \tag{2.82d}$$

$$= pRes(L^p d\log_\pm L) \tag{2.82e}$$

where by $\sim$ we mean that we have equality up to total derivatives that vanish under integration in x.

Now

$$d\bar{h}_{1,q} = \int dx \; \frac{2}{(q+1)!} \operatorname{Res}(dL^{q+1}(\log L - c_{q+1})) \tag{2.83a}$$

$$+ \int dx \; \frac{2}{(q+1)!} \operatorname{Res}(L^{q+1} d \log L) dx \tag{2.83b}$$

$$= \int dx \; \frac{2}{q!} \operatorname{Res}(L^d(\log L - c_q) dL) \tag{2.83c}$$

$$= \int dx \; \left(a_{2,q,0}(x) du(x) + a_{2,q,1}(x) e^{v(x)} dv(x) \right) \tag{2.83d}$$

hence we obtain equations (2.74) in the $\beta = 1$ case.

We have thus completed the proof for the Hamiltonian representation (2.67) using the first bracket.

The recursion relations (2.68) follow from the identities

$$(q+1)\frac{1}{(q+1)!}L^{q+1} = L\frac{1}{q!}L^q = \frac{1}{q!}L^q L \tag{2.84a}$$

$$q\frac{2}{q!}L^q(\log L - c_q) = L\frac{2}{(q-1)!}L^{q-1}(\log L - c_{q-1}) - 2\frac{1}{q!}L^q \tag{2.84b}$$

$$= \frac{2}{(q-1)!}L^{q-1}(\log L - c_{q-1})L - 2\frac{1}{q!}L^q; \tag{2.84c}$$

in particular one expands both sides using (2.72) and expresses $a_{\alpha,q,k}$ in terms of $a_{\alpha,q-1,k}$. Substituting such expressions in (2.73) one finds the recursion relation (2.68). The involutiveness of the Hamiltonians has been already shown, after (2.35), using the standard recursion relation argument. □

2.6 The tau structure

In this section we show that the Hamiltonian densities defined above are normalized in such a way that the so-called tau symmetry holds; from this property we derive the existence of a tau function for the Extended Toda hierarchy. Moreover we express the Hamiltonian densities in terms of derivatives of the tau function.

First we introduce a gradation on the algebra $\hat{\mathcal{A}}$ and we show that Hamiltonians and the vector fields of the extended Toda hierarchy are homogeneous elements. Let's define

$$\deg \partial^m u = 1 - m \qquad \deg \partial^m v = -m \qquad \deg e^v = 2 \qquad \deg \epsilon = 1 \tag{2.85}$$

where $\partial = \frac{d}{dx}$. Then from the definition of the extended Toda hierarchy and of the densities of the Hamiltonians it is easy to check that

$$\deg \frac{\partial u^\alpha}{\partial t^{\beta,q}} = q + \mu_\beta - \mu_\alpha \qquad \deg h_{\beta,q} = q + \frac{3}{2} + \mu_\beta \tag{2.86}$$

where for simplicity we have introduced the notation $u^1 := u$ and $u^2 := v$ and the constant μ_α is defined by $\mu_2 = -\mu_1 = \frac{1}{2}$. We denote by $\tilde{\mathcal{A}}$ the subring of $\hat{\mathcal{A}}$ that consists of homogeneous elements of the form

$$f = \sum_{k\geqslant 0} f_k \epsilon^k \tag{2.87}$$

where f_k are homogeneous polynomials of u, $e^{\pm v}$, $\partial^m u$, $\partial^m v$ for $m \geqslant 1$ and $\deg f_k = \deg f - k$. Then we have

$$\frac{\partial u^\alpha}{\partial t^{\beta,q}} \in \tilde{\mathcal{A}} \qquad h_{\beta,q} \in \tilde{\mathcal{A}}. \tag{2.88}$$

Now we prove the following simple

Lemma 15 *The following formula holds*

$$\frac{\partial h_{\alpha,p-1}}{\partial t^{\beta,q}} = \begin{cases} \frac{2}{p!}\operatorname{Res}[A_{\beta,q}, L^p(\log L - c_p)], & \alpha = 1; \\ \frac{1}{(p+1)!}\operatorname{Res}[A_{\beta,q}, L^{p+1}], & \alpha = 2. \end{cases} \tag{2.89}$$

Proof The only non trivial thing is to show that

$$\frac{\partial \log L}{\partial t^{\beta,q}} = [A_{\beta,q}, \log L]. \tag{2.90}$$

It is sufficient to prove this formula for $\log_\pm L$; from $[\log_+ L, L^m] = 0$ we have

$$[\frac{\partial \log_+ L}{\partial t^{\beta,q}} - [A_{\beta,q}, \log_+ L], L^m] = 0 \tag{2.91}$$

where we have used the Jacobi identity and (2.3)-(2.60). From the fact that $\frac{\partial \log_+ L}{\partial t^{\beta,q}} - [A_{\beta,q}, \log_+ L]$ is homogeneous and has the form $\sum_{k\leqslant -1} g_k \Lambda^k$ we conclude from the last equality that it is equal to zero. Similarly the same procedure holds for $\log_- L$, hence we conclude. □

Now we can define the coefficients $\Omega_{\alpha,p;\beta,q}$

Definition 16 *The functions $\Omega_{\alpha,p;\beta,q}$ are defined by*

$$\frac{1}{\epsilon}(\Lambda - 1)\Omega_{\alpha,p;\beta,q} = \frac{\partial h_{\alpha,p-1}}{\partial t^{\beta,q}} \tag{2.92}$$

and by the homogeneity condition

$$\Omega_{\alpha,p;\beta,q} \in \tilde{\mathcal{A}} \qquad \deg \Omega_{\alpha,p;\beta,q} = p + q + \mu_\alpha + \mu_\beta + 1. \tag{2.93}$$

The following Theorem shows that the coefficients $\Omega_{\alpha,p;\beta,q}$ are symmetric with respect to the pairs of indices (α, p) and (β, q)

Theorem 17 *The Hamiltonians densities $h_{\alpha,q}$ give a tau–structure compatible with spatial translations for the Poisson pencil (2.13)-(2.14), i.e. the following identities hold*

$$\{h_{\alpha,p-1},\bar{h}_{\beta,q}\}_1=\{h_{\beta,q-1},\bar{h}_{\alpha,p}\}_1 \qquad \alpha,\beta=1,2 \quad p,q\geqslant 0 \tag{2.94}$$

$$\{\cdot,\bar{h}_{1,0}\}_1=\frac{\partial}{\partial x} \tag{2.95}$$

Proof Begin from the case $\alpha=\beta=2$; by Lemma 15

$$\frac{\partial h_{2,p-1}}{\partial t^{2,q}}=Res[\frac{1}{(q+1)!}(L^{q+1})_+,\frac{1}{(p+1)!}L^{p+1}]; \tag{2.96}$$

then simply use the fact that given two commuting difference operators A, B it follows that

$$Res[A_+,B]=Res[B_+,A]. \tag{2.97}$$

The other cases follow in a similar fashion. □

This Theorem and definition (2.92) imply that $\frac{\partial\Omega_{\alpha,p;\beta,q}}{\partial t^{\sigma,k}}$ is symmetric with respect to the three pairs of indices (α,p), (β,q) and (σ,k). This property justifies the following definition of the tau function for the extended Toda hierarchy

Definition 18 *For any solution of the extended Toda hierarchy there exists a function τ such that*

$$\Omega_{\alpha,p;\beta,q}=\epsilon^2\frac{\partial^2\log\tau}{\partial t^{\alpha,p}\partial t^{\beta,q}} \tag{2.98}$$

holds true for any $\alpha,\beta=1,2$ and $p,q\geqslant 0$.

Since the first flow $\frac{\partial}{\partial t^{1,0}}$ of the extended Toda hierarchy coincides with the translation in x, i.e. $\frac{\partial\cdot}{\partial t^{1,0}}=\frac{\partial\cdot}{\partial x}$, we can moreover require that the tau function satisfies

$$\frac{\partial\log\tau}{\partial t^{1,0}}=\frac{\partial\log\tau}{\partial x}. \tag{2.99}$$

Corollary 19 *The densities of the Hamiltonians of the extended Toda hierarchy are expressed in terms of the tau function in the following form*

$$h_{\alpha,p}=\epsilon(\Lambda-1)\frac{\partial\log\tau}{\partial t^{\alpha,p+1}} \tag{2.100}$$

for $\alpha,\beta=1,2$ and $p,q\geqslant -1$.

Proof From the definition of $\Omega_{\alpha,p;1,0}$ we get

$$h_{\alpha,p-1}=\sum_{k\geqslant 1}\frac{\epsilon^{k-1}}{k!}\partial^{k-1}\Omega_{\alpha,p;1,0}=\sum_{k\geqslant 1}\frac{\epsilon^{k+1}}{k!}\partial^{k-1}\frac{\partial^2\log\tau}{\partial t^{\alpha,p}\partial t^{1,0}}=\epsilon(\Lambda-1)\frac{\partial\log\tau}{\partial t^{\alpha,p}}, \tag{2.101}$$

where we have used (2.99). □

Remark 20 *From the Corollary it follows that the variables u, v can be expressed in terms of the tau function by*

$$v = \Lambda^{-1}(\Lambda - 1)^2 \log \tau \tag{2.102a}$$

$$u = \epsilon(\Lambda - 1)\frac{\partial \log \tau}{\partial t^{2,0}}; \tag{2.102b}$$

these were obtained by putting $p = -1$ in (2.100) and $\alpha = 1, 2$ respectively.

2.7 Bilinear relations for the wave operators

All the equations of the extended Toda hierarchy can be encoded in a single bilinear expression for the wave operators (plus a constraint), in analogy with the analogous formula for the two-dimensional Toda hierarchy obtained in [47].

The equations of the extended Toda hierarchy can be obtained as a compatibility condition of the linear system

$$L\hat{P} = \hat{P}\Lambda \qquad L\hat{Q} = \hat{Q}\Lambda^{-1} \tag{2.103}$$

$$\epsilon\frac{\partial}{\partial t^{1,q}}\hat{P} = \tilde{A}_{1,q}\hat{P} \qquad \epsilon\frac{\partial}{\partial t^{1,q}}\hat{Q} = \tilde{A}_{1,q}\hat{Q} \tag{2.104}$$

$$\epsilon\frac{\partial}{\partial t^{2,q}}\hat{P} = \frac{1}{q+1!}(L^{q+1})_+\hat{P} \qquad \epsilon\frac{\partial}{\partial t^{2,q}}\hat{Q} = \frac{1}{q+1!}(L^{q+1})_+\hat{Q} \tag{2.105}$$

for $q = 0, 1, 2, \dots$ and [1]

$$\tilde{A}_{1,q} = [\frac{2}{q!}(L^q(\log L - c_q))_+ - \frac{1}{q!}L^q(\log_- L - c_q)] \tag{2.106}$$

$$= [-\frac{2}{q!}(L^q(\log L - c_q))_- + \frac{1}{q!}L^q(\log_+ L - c_q)]; \tag{2.107}$$

here $\hat{P}$, $\hat{Q}$ are differential–difference operators i.e. of the form

$$\sum_{k\in\mathbb{Z}}\sum_{l\geqslant 0} a_{k,l}\Lambda^k(\epsilon\partial)^l. \tag{2.108}$$

The solutions have the explicit expression given by the

Theorem 21 *If L is a solution of the extended Toda hierarchy then exist differential–difference operators $\hat{P}$, $\hat{Q}$, solutions of the previous equations, of the following form*

$$\hat{P} = P \exp \sum_{q=0}^{\infty} \frac{1}{\epsilon}[\frac{1}{q+1!}t^{2,q}\Lambda^{q+1} + \frac{1}{q!}t^{1,q}\Lambda^q(\epsilon\partial - c_q)] \tag{2.109}$$

[1] $A_{1,q}$, $\hat{A}_{1,q}$ and $\tilde{A}_{1,q}$ are all valid Lax operators for the same flow; $\tilde{A}_{1,q}$ anyway has a finite number of terms.

$$P = \sum_{k=0}^{\infty} p_k \Lambda^{-k} \qquad p_0 = 1 \tag{2.110}$$

$$\hat{Q} = Q \exp \sum_{q=0}^{\infty} \frac{1}{\epsilon} \frac{1}{q!} t^{1,q} \Lambda^{-q} (\epsilon \partial + c_q) \tag{2.111}$$

$$Q = \sum_{k=0}^{\infty} q_k \Lambda^k \tag{2.112}$$

These solutions are called *wave operators*. They are determined up to

$$\hat{P} \mapsto \hat{P} \sum_{k,l \geqslant 0} c_{k,l} \Lambda^{-k} (\epsilon \partial)^l \tag{2.113}$$

$$\hat{Q} \mapsto \hat{Q} \sum_{k,l \geqslant 0} \tilde{c}_{k,l} \Lambda^{k} (\epsilon \partial)^l \tag{2.114}$$

for constants coefficients c_k, $\tilde{c}_k$.

Proof The proof is based on the following Lemmas

Lemma 22 (ZS representation) *The extended Toda hierarchy is equivalent to*

$$[\partial_{t^{\alpha,q}} - A_{\alpha,q}, \partial_{t^{\beta,p}} - A_{\beta,p}] = 0 \tag{2.115}$$

for $\alpha, \beta = 1, 2$ *and* $p, q \geqslant 0$. *It is also equivalent to*

$$[\partial_{t^{\alpha,q}} - \hat{A}_{\alpha,q}, \partial_{t^{\beta,p}} - \hat{A}_{\beta,p}] = 0 \tag{2.116}$$

where

$$\hat{A}_{2,q} = -\frac{1}{q+1!}(L^{q+1})_- \tag{2.117}$$

$$\hat{A}_{1,q} = -\frac{2}{q!}(L^q(\log L - c_q))_- \tag{2.118}$$

Proof The proof is essentially the same as in [47]; one must be careful with commutators involving more than one $\log L$. □

Lemma 23 *If* L *is a solution of the extended Toda hierarchy then exist* P, Q *of the form (2.110),(2.112) satisfying*

$$L = P \Lambda P^{-1} = Q \Lambda^{-1} Q^{-1} \tag{2.119}$$

$$\epsilon \frac{\partial}{\partial t^{1,q}} P = -\frac{2}{q!}(L^q(\log L - c_q))_- P \qquad \epsilon \frac{\partial}{\partial t^{1,q}} Q = \frac{2}{q!}(L^q(\log L - c_q))_+ Q \tag{2.120}$$

$$\epsilon \frac{\partial}{\partial t^{2,q}} P = -\frac{1}{q+1!}(L^{q+1})_- P \qquad \epsilon \frac{\partial}{\partial t^{2,q}} Q = \frac{1}{q+1!}(L^{q+1})_+ Q \tag{2.121}$$

Proof This proof is essentially based on the fact that the compatibility conditions for these equations are given by the ZS equations (2.115)-(2.116). A solution of (2.119) can be easily shown to exist for $\mathbf{t} = 0$, and it can be extended for all times $\mathbf{t}$ by solving the Cauchy problem. The fact that (2.119) continues to hold for all $\mathbf{t}$ simply follows as in [47]. □

One can now take P, Q as in the previous lemma and show, by direct substitution, that $\hat{P}$,$\hat{Q}$ satisfy equations (2.104)-(2.105); the theorem 21 is proved. □

The wave operators of the extended Toda hierarchy can be characterized by a single bilinear relation plus a constraint. If $\hat{P}$, $\hat{Q}$ are wave operators (i.e. they solve the linear equations (2.103)-(2.105)) then it follows that

$$\epsilon\partial_{t^{\alpha,q}}\hat{P}\cdot\hat{P}^{-1} = \epsilon\partial_{t^{\alpha,q}}\hat{Q}\cdot\hat{Q}^{-1} \tag{2.122}$$

for $\alpha = 1,2,\ q \geqslant 0$; then by induction one can show that this holds for a generic multiindex $\underline{\alpha} = ((\alpha_1, q_1), (\alpha_2, q_2), \dots)$

$$\partial_t^{\underline{\alpha}}\hat{P}\cdot\hat{P}^{-1} = \partial_t^{\underline{\alpha}}\hat{Q}\cdot\hat{Q}^{-1} \tag{2.123}$$

where

$$\partial_t^{\underline{\alpha}} = \frac{\partial}{\partial t^{\alpha_1,q_1}}\frac{\partial}{\partial t^{\alpha_2,q_2}}\cdots \tag{2.124}$$

This infinite number of equations can be encapsulated in the single expression

$$\hat{P}(\mathbf{t})\cdot\hat{P}^{-1}(\mathbf{t}') = \hat{Q}(\mathbf{t})\cdot\hat{Q}^{-1}(\mathbf{t}'); \tag{2.125}$$

by a Taylor expansion one obtains the expressions above. Notice also that, from (2.103), we have

$$\hat{P}\Lambda\hat{P}^{-1} = \hat{Q}\Lambda^{-1}\hat{Q}^{-1}. \tag{2.126}$$

Viceversa the following theorem holds

Theorem 24 *Suppose $\hat{P}$, $\hat{Q}$ are operators of the form (2.109)-(2.112) and suppose they satisfy the bilinear relation (2.125) and the constraint (2.126); then they are wave operators. More explicitly, by defining $L = \hat{P}\Lambda\hat{P}^{-1} = \hat{Q}\Lambda^{-1}\hat{Q}^{-1}$ it follows that L has the form (2.2), and defining then $\log L$, $\log_+ L$, $\log_- L$ as usual, we have that $\hat{P}$ and $\hat{Q}$ satisfy the equations (2.104)-(2.105).*

Proof From (2.126) it follows $P\Lambda P^{-1} = Q\Lambda^{-1}Q^{-1}$, hence L has the tridiagonal form (2.2). Consider first the $t^{1,q}$ flow, the $t^{2,q}$ case is done similarly. From the bilinear relation, by Taylor expansion, we have

$$\epsilon\partial_{t^{1,q}}\hat{P}\cdot\hat{P}^{-1} = \epsilon\partial_{t^{1,q}}\hat{Q}\cdot\hat{Q}^{-1} \tag{2.127}$$

that explicitly gives

$$\epsilon\partial_{t^{1,q}}P\cdot P^{-1} + P\frac{1}{q!}\Lambda^q(\epsilon\partial - c_q)P^{-1} = \epsilon\partial_{t^{1,q}}Q\cdot Q^{-1} + Q\frac{1}{q!}\Lambda^q(\epsilon\partial - c_q)Q^{-1}, \tag{2.128}$$

hence

$$\epsilon\partial_{t^{1,q}}P\cdot P^{-1}-\epsilon\partial_{t^{1,q}}Q\cdot Q^{-1}=-\frac{2}{q!}L^q(\log L-c_q). \tag{2.129}$$

Notice that in the last expression there are only difference operators, hence it makes sense to take the $(\cdot)_+$ and $(\cdot)_-$ parts. Since $\epsilon\partial_{t^{1,q}}P\cdot P^{-1}$ and $\epsilon\partial_{t^{1,q}}Q\cdot Q^{-1}$ contain respectively only negative and positive (or zero) powers of Λ, by taking the $(\cdot)_+$ and $(\cdot)_-$ part of the last expression we obtain the equations (2.120) for P,Q and finally the correct equations for $\hat{P}$, $\hat{Q}$. □

2.8 Bilinear relations for the wave functions

In this section we show that we can encode all the equations of the extended Toda hierarchy in a single equation for the wave functions that is essentially a rewriting of the previous bilinear formula for the wave operators.

Define the coefficients p_k^*, q_k^* by

$$P^{-1}(x)=\sum_{k=0}^{\infty}\Lambda^{-k}p_k^*(x+1)\qquad Q^{-1}(x)=\sum_{k=0}^{\infty}\Lambda^{k}q_k^*(x+1) \tag{2.130}$$

and the wavefunctions $\hat{\psi}$, $\hat{\phi}$, $\hat{\psi}^*$, $\hat{\phi}^*$ in the following way

$$\hat{\psi}=\psi\lambda^{\frac{x}{\epsilon}}\exp\sum_{q=0}^{\infty}\frac{1}{\epsilon}[\frac{1}{q+1!}t^{2,q}\lambda^{q+1}+\frac{1}{q!}t^{1,q}\lambda^q(\log\lambda-c_q)] \tag{2.131}$$

$$\hat{\psi}^*=\psi^*\lambda^{-\frac{x}{\epsilon}}\exp-\sum_{q=0}^{\infty}\frac{1}{\epsilon}[\frac{1}{q+1!}t^{2,q}\lambda^{q+1}+\frac{1}{q!}t^{1,q}\lambda^q(\log\lambda-c_q)] \tag{2.132}$$

$$\psi=\sum_{k=0}^{\infty}p_k\lambda^{-k}\qquad \psi^*=\sum_{k=0}^{\infty}p_k^*\lambda^{-k} \tag{2.133}$$

$$\hat{\phi}=\phi\lambda^{\frac{x}{\epsilon}}\exp(\sum_{q=0}^{\infty}\frac{1}{\epsilon}[\frac{1}{q!}t^{1,q}\lambda^{-q}(\log\lambda+c_q)]) \tag{2.134}$$

$$\hat{\phi}^*=\phi^*\lambda^{-\frac{x}{\epsilon}}\exp(-\sum_{q=0}^{\infty}\frac{1}{\epsilon}[\frac{1}{q!}t^{1,q}\lambda^{-q}(\log\lambda+c_q)]) \tag{2.135}$$

$$\phi=\sum_{k=0}^{\infty}q_k\lambda^{k}\qquad \phi^*=\sum_{k=0}^{\infty}q_k^*\lambda^{k} \tag{2.136}$$

then the equations of the extended Toda hierarchy are given by

$$\oint[\epsilon\psi_{t^{2,q}}(x)\psi^*(x')\lambda^{x-x'}+\frac{1}{q+1!}\psi(x)\psi^*(x')\lambda^{q+1}\lambda^{x-x'}]d\lambda=\oint\phi_{t^{2,q}}(x)\phi^*(x')\lambda^{x-x'}d\lambda \tag{2.137}$$

$$\oint[\epsilon\psi_{t^{1,q}}(x)\psi^*(x')\lambda^{x-x'}+\frac{1}{q!}\psi(x)\epsilon\psi^*_{x'}(x')\lambda^{q+x-x'}-\frac{1}{q!}c_q\psi(x)\psi^*(x')\lambda^{q+x-x'}]d\lambda=$$
$$=\oint[\epsilon\phi_{t^{1,q}}(x)\phi^*(x')\lambda^{x-x'}+\frac{1}{q!}\phi(x)\epsilon\phi^*_{x'}(x')\lambda^{-q+x-x'}+\frac{1}{q!}\phi(x)\phi(x')\lambda^{-q+x-x'}c_q]d\lambda \tag{2.138}$$

and

$$\oint[\psi(x)\psi^*(x')\lambda^{q+x-x'}]d\lambda=\oint[\phi(x)\phi^*(x')\lambda^{-q+x-x'}]d\lambda. \tag{2.139}$$

The proof is obtained simply by expanding this formulas in the different cases and comparing with the bilinear relation for the wave operators considered in the previous section.

2.9 Darboux transformations and soliton solutions

Here we introduce the Darboux transformation for Lax operator (2.2) and we obtain that the usual solitonic solutions are stable under the non-local Toda flows.

The Darboux transformation for the equation $L\psi=\lambda\psi$, i.e.

$$\Lambda\psi+u\psi+e^v\Lambda^{-1}\psi=\lambda\psi \tag{2.140}$$

is given by

$$\psi[1]=\psi-\frac{\psi_1}{\Lambda^{-1}\psi_1}\Lambda^{-1}\psi \tag{2.141}$$

$$v[1]=\Lambda^{-1}v+(1-\Lambda^{-1})^2\log\psi_1 \tag{2.142}$$

$$u[1]=u+(\Lambda-1)\frac{\psi_1}{\Lambda^{-1}\psi_1} \tag{2.143}$$

where ψ_1 is a solution of (2.140) for $\lambda=\lambda_1$; this means that $\psi[1]$ will satisfy

$$\Lambda\psi[1]+u[1]\psi[1]+e^{v[1]}\Lambda^{-1}\psi[1]=\lambda\psi[1]. \tag{2.144}$$

Moreover if ψ and ψ_1 satisfy

$$\epsilon\psi_{t^{\alpha,q}}=A_{\alpha,q}\psi \tag{2.145}$$

we assume that the Darboux transformed $\psi[1]$ satisfies

$$\epsilon\psi[1]_{t^{\alpha,q}}=A_{\alpha,q}[1]\psi[1] \tag{2.146}$$

where $A_{\alpha,q}[1]$ is obtained from $A_{\alpha,q}$ by substituting u with $u[1]$ and v with $v[1]$. This implies that the $u[1]$, $v[1]$ are a new solution of the compatibility condition of the linear equations (2.144)-(2.146), i.e. a new solution of the extended Toda hierarchy.

One–soliton solutions are obtained by starting from a constant potential, e.g. $u = 0 = v$; the first equations for ψ in this case reduce to

$$\Lambda\psi + \Lambda^{-1}\psi = \lambda\psi \tag{2.147}$$

$$\epsilon\psi_{t^{1,1}} = (\Lambda + \Lambda^{-1})\epsilon\partial - (\Lambda - \Lambda^{-1}) \tag{2.148}$$

while $t^{1,0}$ is as usual identified with the x-translation. The generic solution of (2.147)-(2.148) is a linear combination of

$$\psi_{\pm} = \exp(\frac{1}{\epsilon}(x + \lambda t^{1,1}) \log z_{\pm} + \frac{1}{\epsilon} t^{1,1}(-z_{\pm} + 1/z_{\pm})) \tag{2.149}$$

with $z_{\pm}$ roots of $z + z^{-1} = \lambda$. We can choose, for example

$$\psi_1 = 2\cosh(\frac{1}{\epsilon}(x + \lambda_1 t^{1,1}) \log z_1 + \frac{1}{\epsilon} t^{1,1}(-z_1 + 1/z_1)) \tag{2.150}$$

with $z_1 + z_1^{-1} = \lambda_1$. The Darboux transformed potentials $u[1]$, $v[1]$ are

$$u[1] = (\Lambda - 1)\frac{\cosh(\frac{1}{\epsilon}((x + \lambda_1 t^{1,1}) \log z_1 + t^{1,1}(-z_1 + 1/z_1)))}{\cosh(\frac{1}{\epsilon}((x + \lambda_1 t^{1,1}) \log z_1 + t^{1,1}(-z_1 + 1/z_1)) - \log z_1)} \tag{2.151}$$

$$v[1] = (1 - \Lambda^{-1})^2 \log[2\cosh(\frac{1}{\epsilon}(x + \lambda_1 t^{1,1}) \log z_1 + \frac{1}{\epsilon} t^{1,1}(-z_1 + 1/z_1))] \tag{2.152}$$

These are solutions of the flow $\frac{\partial}{\partial t^{1,1}}$, that in explicit form is

$$\begin{aligned} \epsilon u_{t^{1,1}} &= (\Lambda - 1)(-e^v(\Lambda^{-1} - 1)^{-1}\epsilon v_x) - 2(\Lambda - 1)e^v && (2.153)\\ &\quad + \frac{\epsilon}{2}(u)^2_x + \epsilon(e^v)_x && (2.154)\\ \epsilon v_{t^{1,1}} &= ((\Lambda^{-1} - 1)^{-1}\epsilon v_x)(\Lambda^{-1} - 1)u + \epsilon v_x(\Lambda^{-1}u) && (2.155)\\ &\quad + \Lambda^{-1}\epsilon u_x + \epsilon u_x + 2(\Lambda^{-1} - 1)u. && (2.156) \end{aligned}$$

As a further example we can consider the same solution with explicit $t^{2,0}$ dependence

$$u[1] = (\Lambda - 1)\frac{\psi_1}{\Lambda^{-1}\psi_1} \tag{2.157}$$

$$v[1] = (1 - \Lambda^{-1})^2 \log \psi_1 \tag{2.158}$$

with

$$\psi_1 = \psi(z) + \psi(z^{-1}) \tag{2.159}$$

$$\psi(z) = \exp(\frac{x}{\epsilon} \log z + \frac{1}{\epsilon} t^{1,1}(-z + z^{-1} + \lambda \log z) + \frac{1}{\epsilon} t^{2,0} z) \tag{2.160}$$

i.e. these are also solutions of

$$\epsilon u_{t^{2,0}} = (\Lambda - 1)e^v \qquad \epsilon v_{t^{2,0}} = (1 - \Lambda^{-1})u. \tag{2.161}$$

The multisoliton solution with explicit dependence on all the times $t^{\alpha,q}$ is given by iterating the Darboux transformation; choose N solutions $\psi_1 \dots \psi_N$ of the Lax equations (2.140)-(2.145) with $u = v = 0$:

$$\psi_k = a_k \psi(z_k) + b_k \psi(z_k^{-1}) \tag{2.162}$$

with $z_k + z_k^{-1} = \lambda_k$ and

$$\psi(z) = z^{\frac{x}{\epsilon}} \exp\{\frac{1}{\epsilon}\sum_{q\geqslant 0} t^{1,q}[-\frac{1}{q!}c_q((\lambda^q)_+ - (\lambda^q)_-) + \frac{1}{q!}\lambda^q \log z] + \frac{1}{\epsilon}\sum_{q\geqslant 0} t^{2,q}\frac{1}{q+1!}(\lambda^{q+1})_+\} \tag{2.163}$$

where $(\cdot)_+$ and $(\cdot)_-$ denote the part of a polynomial in z, z^{-1} where z^k compares with $k \geqslant 0$ and $k < 0$ respectively. The multisoliton solution is then given by the iterated Darboux transform which can be expressed as

$$v[N] = \Lambda^{-N} v + (1-\Lambda^{-1})^2 \log W(\psi_1 \dots \psi_N) \tag{2.164}$$

$$u[N] = \Lambda^{-N} u + (1-\Lambda^{-1})\epsilon\partial_{t^{2,0}} \log W(\psi_1 \dots \psi_N) \tag{2.165}$$

(in this case $u = 0 = v$) in terms of the discrete Wronskian

$$W(\psi_1 \dots \psi_N) = \det(\Lambda^{-j+1}\psi_{N+1-i})_{1\leqslant i,j\leqslant N}. \tag{2.166}$$

Remark 25 *There are other classes of solutions of the extended hierarchy that deserve further investigation. First of all consider the similarity solutions. The extended Toda hierarchy admits the so called Galilean symmetry, i.e. the vector field*

$$\frac{\partial v}{\partial s} = 1 + \sum_{p=1}^{\infty} t^{\alpha,p}\frac{\partial v}{\partial t^{\alpha,p-1}} \qquad \frac{\partial u}{\partial s} = \sum_{p=1}^{\infty} t^{\alpha,p}\frac{\partial u}{\partial t^{\alpha,p-1}} \tag{2.167}$$

commutes with all the flows of the hierarchy, $\frac{\partial}{\partial s}\frac{\partial v}{\partial t^{\alpha,p}} = \frac{\partial}{\partial t^{\alpha,p}}\frac{\partial v}{\partial s}$. This is a general feature of the bihamiltonian integrable hierarchies considered by Dubrovin and Zhang [19]. The solutions of the hierarchy that are invariant under this symmetry satisfy the so-called string equation

$$\sum_{p=0}^{\infty} t^{\alpha,p}\frac{\delta \bar{h}_{\alpha,p-1}}{\delta u^\gamma} = 0 \tag{2.168}$$

and are called similarity solutions. We consider an example putting $t^{\alpha,p} = 0$ for $p > 1$, so we obtain the system

$$t^{2,0} + t^{2,1}u + t^{1,1}(1-\Lambda^{-1})^{-1}\epsilon v_x = 0 \tag{2.169}$$

$$t^{2,1}e^v + t^{1,0} + t^{1,1}(1-\Lambda)^{-1}(-\epsilon)u_x = 0, \tag{2.170}$$

that can be rewritten as a single equation for the variable v

$$t^{2,1}e^v + t^{1,0} + \frac{(t^{1,1})^2}{t^{2,1}}(1-\Lambda)^{-1}(1-\Lambda^{-1})^{-1}\epsilon^2 v_{xx} = 0 \tag{2.171}$$

or equivalently as a differential–difference equation

$$(t^{2,1})^2(1-\Lambda^{-1})(1-\Lambda)e^v + (t^{1,1})^2\epsilon^2 v_{xx} = 0. \tag{2.172}$$

The nonlinear equations obtained from similarity reductions of integrable hierarchies usually have nice analytic properties since they can be also seen as isomonodromic deformation equations [31]. In the present case we obtain differential–difference Painlevé type equations.

Another important class of solutions is given by the algebro-geometric quasi periodic solutions. They correspond to stationary reductions of the hierarchy, that in the extended case are given again by ordinary differential-difference equations. It should be possible to obtain such solutions through the method of the Baker-Akhiezer function on a Riemann surface as was already done for the standard Toda chain. The work in this direction is in progress.

Chapter 3

Extended bigraded Toda hierarchy

In this chapter we introduce the bigraded Toda hierarchy, a generalization of the Toda chain hierarchy where the Lax operator is a difference operator of the form

$$L = \Lambda^N + u_{N-1}\Lambda^{N-1} + \cdots + u_{-M}\Lambda^{-M} \tag{3.1}$$

for two integers $N, M > 0$; thus the Toda chain corresponds to $N = M = 1$.

First we define two fractional powers and the logarithm of the operator L and use them to define the flows of the hierarchy in the Lax formulation.

Then we derive the bihamiltonian structure for this hierarchy, using the R-matrix approach. We review the main theorems from the literature and we apply the general construction on suitable algebras of difference operators, thus obtaining two Poisson structures on these algebras; finally we obtain the Poisson brackets on operators of the form (3.1) by a Dirac reduction on an affine subspace.

By direct calculation we derive the connection between the Hamiltonian and the Lax formulations (through the first Poisson bracket).

3.1 Extended bigraded Toda hierarchy in the Lax formulation

Here we define the flows of the bigraded Toda hierarchy in the Lax formulation. To this purpose we define the fractional powers and the logarithm of L and we show that they are uniquely expressed in terms of powers series in ϵ of differential polynomials in the coefficients of L.

Let's first define the fractional powers $L^{\frac{1}{N}}$ and $L^{\frac{1}{M}}$ of the Lax operator (3.1); these are two operators of the form

$$L^{\frac{1}{N}} = \Lambda + \sum_{k\leqslant 0} a_k \Lambda^k \tag{3.2a}$$

$$L^{\frac{1}{M}} = \sum_{k\geqslant -1} b_k \Lambda^k \tag{3.2b}$$

defined by the relations

$$(L^{\frac{1}{N}})^N = L \qquad (L^{\frac{1}{M}})^M = L. \tag{3.3}$$

We stress that we consider $L^{\frac{1}{N}}$ and $L^{\frac{1}{M}}$ as two different operators, even if $N = M$.

As in the Toda chain case the operator (3.1) can be written as the dressed shift operators

$$L = P\Lambda^N P^{-1} = Q\Lambda^{-M}Q^{-1} \tag{3.4}$$

where the dressing operators P, Q have the form

$$P = \sum_{k\geqslant 0} p_k(x)\Lambda^{-k} \qquad p_0 = 1, \tag{3.5a}$$

$$Q = \sum_{k\geqslant 0} q_k(x)\Lambda^{k}. \tag{3.5b}$$

The logarithms of the operator L are then defined by

$$\log_+ L = PN\epsilon\partial P^{-1} = N\epsilon\partial + N\epsilon P P_x^{-1} \tag{3.6a}$$

$$\log_- L = -QM\epsilon\partial Q^{-1} = -M\epsilon\partial - M\epsilon Q Q_x^{-1}. \tag{3.6b}$$

These are differential-difference operators of the form

$$\log_+ L = N\epsilon\partial + 2N\sum_{k>0} w_{-k}(x)\Lambda^{-k} \tag{3.7a}$$

$$\log_- L = -M\epsilon\partial + 2M\sum_{k\geqslant 0} w_{k}(x)\Lambda^{k}. \tag{3.7b}$$

As before we define

$$\log L = \frac{1}{2N}\log_+ L + \frac{1}{2M}\log_- L = \sum_{k\in\mathbb{Z}} w_k\Lambda^k \tag{3.8}$$

that is a purely difference operator since the derivatives cancel.

Of course an equivalent definition of the operators $L^{\frac{1}{N}}$ and $L^{\frac{1}{M}}$ can be given in terms of the dressing operators

$$L^{\frac{1}{N}} = P\Lambda P^{-1} \qquad L^{\frac{1}{M}} = Q\Lambda^{-1}Q^{-1}. \tag{3.9}$$

As in the Toda chain case we would like to find explicit expressions for $\log L$, $L^{\frac{1}{N}}$ and $L^{\frac{1}{M}}$ in terms of the coefficients of L. Let's give the definition of the proper algebra of differential polynomials in this case

Definition 26 *We denote by $\mathcal{A}$ the algebra of differential polynomials in $u_{N-1},\dots,u_{-M+1}$, $(u_{-M})^{\frac{1}{M}}$, $(u_{-M})^{-\frac{1}{M}}$ and $\log u_{-M}$; $\hat{\mathcal{A}} := \mathcal{A}[[\epsilon]]$ is the differential algebra of formal power series in ϵ with coefficients in $\mathcal{A}$.*

First we prove that

Theorem 27 *The coefficients of the operators $L^{\frac{1}{N}}$ and $L^{\frac{1}{M}}$ are uniquely determined elements of $\hat{\mathcal{A}}$.*

Proof Spelling out the coefficient of Λ^{N-m-1} of the relation $(L^{\frac{1}{N}})^N = L$ that defines $L^{\frac{1}{N}}$ we have for $m \geqslant 0$

$$(\Lambda^{N-1} + \dots + 1)a_{-m} = f_m(a_0, \dots, a_{-m+1}) + u_{N-m-1} \tag{3.10}$$

where f_m is a difference polynomial in the variables $a_0, \dots, a_{-m+1}$. Since $\Lambda^{N-1}+\dots+1$ is invertible on $\hat{\mathcal{A}}$ we find that $a_m \in \hat{\mathcal{A}}$ for $m \leqslant 0$.

Now consider the operator $L^{\frac{1}{M}}$. Define the operator $L = q_0^{-1} L q_0$ where q_0 is the leading term in the expansion (3.5b) of the dressing operator Q. The coefficients $\tilde{u}_k$ of $\tilde{L}$ are clearly elements of $\hat{\mathcal{A}}$ since they are expressed as

$$\tilde{u}_k = u_k (\Lambda^k q_0) q_0^{-1} \tag{3.11}$$

and

$$\frac{\Lambda^k q_0}{q_0} = e^{(1-\Lambda^{-M})^{-1}(\Lambda^k - 1)\log u_{-M}}. \tag{3.12}$$

Indeed from the definition (3.4) of Q we have

$$u_{-M} = q_0 (\Lambda^{-M} q_0^{-1}) \tag{3.13}$$

from which (3.12) follows. Moreover, since $(1-\Lambda^{-M})^{-1}(\Lambda^k - 1)\log u_{-M} = g_0 + g_1\epsilon + \dots$ where g_k are differential polynomials and in particular $g_0 = \frac{k}{M}\log u_{-M}$, we have that (3.12) equals

$$(u_{-M})^{\frac{k}{M}} (\sum_{l \geqslant 0} \frac{1}{l!}(g_1\epsilon + g_2\epsilon^2 + \dots)^l \tag{3.14}$$

hence $\frac{\Lambda^k q_0}{q_0}$ is in $\hat{\mathcal{A}}$.

For the same reason the coefficients b_k of $L^{\frac{1}{M}}$ are elements of $\hat{\mathcal{A}}$ if we can show that the coefficients of $\tilde{L}^{\frac{1}{M}}$ can be expressed as ϵ-series of differential polynomials in $\tilde{u}_k$. Since $(\tilde{L}^{\frac{1}{M}})^M = \tilde{L}$, this is shown in the same way as we did before for the a_k coefficients. The theorem is proved. □

Finally we prove the generalization of the Theorem 7

Theorem 28 *The coefficients of $\log L$ are uniquely determined elements of $\hat{\mathcal{A}}$.*

Proof For the coefficients w_k with $k \leqslant -1$ we start by dressing with P the relation $[\epsilon\partial, \Lambda^m] = 0$. The proof simply follows the steps of that of Theorem 7 but in this case L^m is substituted with $L^{\frac{m}{N}}$; however, since we know from the previous theorem that the coefficients of $L^{\frac{1}{M}}$ are in $\hat{\mathcal{A}}$, the proof remains unchanged.

For the coefficients w_k with $k \geqslant 0$ we introduce the operators $\tilde{Q}$ and $\tilde{L}$ as in (2.48) and (2.49); then the coefficients $\tilde{u}_k$ are related to u_k by (3.11). One then follows the same steps as in Theorem 7. □

Example 29 *Some examples of coefficients w_k are*

$$w_{-1} = \frac{\epsilon}{2}(\Lambda^N - 1)^{-1}(u_{N-1})_x \tag{3.15a}$$

$$w_0 = \frac{\epsilon}{2}(1 - \Lambda^{-M})^{-1}\frac{(u_{-M})_x}{u_{-M}} \tag{3.15b}$$

$$w_1 = \frac{\epsilon}{2}\left(e^{(1-\Lambda^{-M})^{-1}(1-\Lambda)\log u_{-M}}\right)(1 - \Lambda^{-M})^{-1}(u_{-M+1})_x. \tag{3.15c}$$

Finally we can define the flows of the extended bigraded Toda hierarchy

Definition 30 *The* Extended bigraded Toda hierarchy *consists of the system of flows given by the following Lax pair formalism*

$$\epsilon\frac{\partial L}{\partial t^{\alpha,q}} = [A_{\alpha,q}, L] \tag{3.16}$$

for $\alpha = N-1, \ldots, -M$ and $q \geqslant 0$. The operators $A_{\alpha,q}$ are defined by

$$A_{\alpha,q} = \frac{\Gamma(2 - \frac{\alpha}{N})}{\Gamma(q + 2 - \frac{\alpha}{N})}(L^{q+1-\frac{\alpha}{N}})_+ \quad \textit{for} \quad \alpha = N-1, \ldots, 0 \tag{3.17a}$$

$$A_{\alpha,q} = \frac{-\Gamma(2 + \frac{\alpha}{M})}{\Gamma(q + 2 + \frac{\alpha}{M})}(L^{q+1+\frac{\alpha}{M}})_- \quad \textit{for} \quad \alpha = 0, \ldots, -M+1 \tag{3.17b}$$

$$A_{-M,q} = \frac{1}{q!}[L^q(\log L - \frac{1}{2}(\frac{1}{M} + \frac{1}{N})c_q)]_+ \tag{3.17c}$$

The choice of the normalization of the coefficients of $A_{\alpha,q}$ comes from the requirement of the tau symmetry for the associated Hamiltonians. We are however free to multiplicate the Gamma functions $\Gamma(x)$ in the denominators by a function f such that $f(x+1) = f(x)$ without losing the tau symmetry.

Remark 31 *Most of the definitions above continue to hold for the case $M = 0$ i.e. for a Lax operator of the form*

$$L = \Lambda^N + u_{N-1}\Lambda^{N-1} + \cdots + u_0. \tag{3.18}$$

However notice that in this case we will have only one dressing operator P

$$L = P\Lambda^N P^{-1} \tag{3.19}$$

and correspondingly only the fractional power $L^{\frac{1}{N}}$ *defined by (3.3) and only the logarithm* $\log_+ L := PN\epsilon\partial P^{-1}$. *The main difference however is that we cannot define a logarithm given by a difference operator like (3.8) since we cannot eliminate the derivative; hence we cannot define with such logarithm new flows by their Lax representations.*

We can however define the other flows by

$$\epsilon \frac{\partial L}{\partial t^{\alpha,q}} = [(L^{q+1-\frac{\alpha}{N}})_+, L]. \tag{3.20}$$

3.2 Background on R-matrix theory

In this section we introduce the basic facts about R-matrix theory that will be used to obtain compatible Poisson structures on certain Lie algebras of difference operators.

The classical R-matrix method was introduced by Sklyanin [44] as a by-product of the quantum inverse-scattering method.

In general a (unitary) R-matrix on a Lie algebra produces a "compatible" Lie algebra structure on the dual of the Lie algebra; this "Lie bialgebra" is the natural object that describes the infinitesimal structure of a Poisson-Lie group (a Lie group with a Poisson structure such that the multiplication is a Poisson map, see [8]).

The connection with the theory of integrable systems is mainly due to Semenov-Tian-Shansky that in [42] introduced the modified Yang–Baxter equation, defined linear and quadratic Poisson brackets on generic associative algebras with an R-matrix and related these Poisson structures with the Lax formalism.

After reviewing some facts about the unitary case considered by Semenov-Tian-Shansky we will recall the basic facts of the generalization to the non-unitary case developed in [40, 36].

3.2.1 R matrix and the modified Yang-Baxter equation

Let $\mathcal{G}$ be a Lie algebra. A linear mapping $R : \mathcal{G} \to \mathcal{G}$ is called a (classical) R-matrix if the bracket

$$[X, Y]_R := [R(X), Y] + [X, R(Y)] \quad \text{for} \quad X, Y \in \mathcal{G} \tag{3.21}$$

is a Lie bracket, i.e. if it satisfies the Jacobi identity.

A sufficient condition for $R \in \mathrm{End}(\mathcal{G})$ to be an R-matrix is that

$$[R(X), R(Y)] - R([X, Y]_R) = -[X, Y]; \tag{3.22}$$

equation (3.22) is called *modified Yang–Baxter equation.*[1]

[1] More generally the equation (3.22) can be substituted with

$$[R(X), R(Y)] - R([X, Y]_R) = -\alpha[X, Y] \tag{3.23}$$

with $\alpha \in \mathbb{R}$, that for $\alpha = 0$ corresponds to the original Yang-Baxter equation; by rescaling R one can always reduce it to one of the only two relevant cases $\alpha = 0$ and $\alpha = 1$.

The typical example of R-matrix comes from the splitting of the algebra $\mathcal{G}$; indeed if we can split $\mathcal{G}$ into the direct sum of two subalgebras: $\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_-$, then the difference of the projection operators onto these subalgebras

$$R = P_+ - P_- \tag{3.24}$$

satisfies the modified Yang-Baxter equation (3.22) and hence gives an R-matrix on $\mathcal{G}$.[2] The Lie bracket in this case is

$$[X, Y]_R = 2[P_+(X), P_+(Y)] - 2[P_-(X), P_-(Y)]. \tag{3.25}$$

The importance of the R-matrix in the theory of integrable systems lies in the fact that it can be used to define non-trivial Poisson structures. Let's recall the origin of the linear and of the quadratic brackets.

3.2.2 Linear brackets

It is well known that we can always define a Poisson bracket on the dual of a Lie algebra; such construction is due to Lie. Let $\mathcal{G}$ be a Lie algebra and f, g two functions on the dual $\mathcal{G}^*$; then the Lie-Poisson bracket $\{\ ,\ \} : C^\infty(\mathcal{G}^*) \times C^\infty(\mathcal{G}^*) \to C^\infty(\mathcal{G}^*)$ is given by

$$\{f, g\}(\xi) =< \xi, [df, dg] > \tag{3.28}$$

for any $\xi \in \mathcal{G}^*$; here the differentials df, dg are identified with elements of the algebra $\mathcal{G}$ in the obvious way and $<,>$ denotes the pairing of $\mathcal{G}$ with its dual.

Hence, given an R-matrix on $\mathcal{G}$, we can define a linear Poisson bracket on $\mathcal{G}^*$ associated with $[,]_R$

$$\{f, g\}_1(\xi) =< \xi, [df, dg]_R > . \tag{3.29}$$

If we define the Poisson tensor $P_1(\xi) : \mathcal{G} \to \mathcal{G}^*$ by

$$\{f, g\}_1(\xi) =< P_1(\xi)dg, df > \tag{3.30}$$

then the Hamiltonian vector field on $\mathcal{G}^*$ associated to some Hamiltonian H is given by

$$\frac{d\xi}{dt} = P_1(\xi)dH. \tag{3.31}$$

Among all the possible Hamiltonian functions on $\mathcal{G}^*$ a particular role is played by the functions invariant under the coadjoint action of $\mathcal{G}$ on $\mathcal{G}^*$.

---

[2]The splitting of an associative algebra $\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_-$ gives a class of solutions of the so called Poincaré-Bertrand formula

$$R(X)R(Y) = R(R(X)Y + XR(Y)) - XY \tag{3.26}$$

which is the analogous of the modified Yang-Baxter for associative algebras. It gives a sufficient condition for the product

$$X \times_R Y = R(X)Y + YR(X) \tag{3.27}$$

to be associative.

Recall that the adjoint and coadjoint representations of $\mathcal{G}$

$$ad : \mathcal{G} \to \mathrm{End}\, \mathcal{G} \qquad ad^* : \mathcal{G} \to \mathrm{End}\, \mathcal{G}^* \tag{3.32}$$
$$X \mapsto ad_X \qquad X \mapsto ad^*_X \tag{3.33}$$

are defined respectively by

$$ad_X L = [X, L] \tag{3.34}$$

and

$$< ad^*_X \xi, L >= - < \xi, ad_X L >=< \xi, [L, X] > \tag{3.35}$$

for $X, L \in \mathcal{G}$ and $\xi \in \mathcal{G}^*$.

A function $f \in C^\infty(\mathcal{G}^*)$ is ad^*-invariant if

$$ad^*_{df(\xi)} \xi = 0 \qquad \forall \xi \in \mathcal{G}^* \tag{3.36}$$

where $df(\xi) \in T^*_\xi \mathcal{G}^* \simeq \mathcal{G}$.

The ad^*-invariant functions on $\mathcal{G}^*$ coincide (by definition) with the Casimirs of the bracket $\{,\}$.

The main relation with the Lax formalism is given by the following theorem by Semenov-Tian-Shansky

Theorem 32 ([42]) *Let $\mathcal{G}$ be a Lie algebra with an R-matrix $R : \mathcal{G} \to \mathcal{G}$; then*
(i) The ad^*-invariant functions on $\mathcal{G}$ are in involution with respect to $\{\ ,\ \}_1$.
(ii) The vector field $\frac{d\xi}{dt} = P_1(\xi)dH$ on $\mathcal{G}^*$ associated to an invariant Hamiltonian H can be written

$$\frac{d\xi}{dt} = ad^*_{R(dH(\xi))} \xi = \widetilde{ad}^*_{dH(\xi)} \xi. \tag{3.37}$$

where $\widetilde{ad}^*$ is the coadjoint representation of $\mathcal{G}$ with the bracket $[,]_R$.

If we identify $\mathcal{G}$ and $\mathcal{G}^*$ through a non degenerate invariant inner product $(,)$ on $\mathcal{G}$ then formula (3.29) defines a Poisson bracket on the $\mathcal{G}$

$$\{f, g\}_1(L) = (L, [df, dg]_R) \tag{3.38}$$

and the Theorem simply says that the Casimirs of the bracket $\{f, g\}(L) = (L, [df, dg])$ are in involution with respect to $\{,\}_1$ and that the Hamiltonian vector field associated to one of such Casimirs can be written as

$$\frac{dL}{dt} = [RdH, L] \tag{3.39}$$

i.e. in the Lax pair formalism.

3.2.3 Quadratic brackets

The quadratic Poisson brackets are naturally defined on a Lie group (the so-called Sklyanin brackets) rather than on a Lie algebra. However their definition makes sense also on a generic associative algebra, as observed in [42].

If $\mathcal{G}$ is an associative algebra with a symmetric non-degenerate invariant inner product $(,)$ (by which we can identify $\mathcal{G}$ with its dual), and $R : \mathcal{G} \to \mathcal{G}$ is skew-symmetric and satisfies the modified Yang-Baxter equation (3.22), then the bracket

$$\{f, g\}_2^{Skl}(L) = (L, [df, R(Ldg)] - [dg, R(dfL)]) \tag{3.40}$$

is a Poisson bracket on $\mathcal{G}$; moreover it is compatible with the linear bracket (3.29), seen as a bracket on $\mathcal{G}$, i.e. any linear combination of $\{,\}_1$ and $\{,\}_2$ is still a Poisson bracket.

As before the Casimirs of $\{,\}$ are in involution with $\{,\}_2$ and for a Casimir H the Hamiltonian flow $\frac{dL}{dt} = P_2 dH$ can be written in Lax form

$$\frac{dL}{dt} = [L, R(dHL)]. \tag{3.41}$$

3.2.4 Non-unitary case

In some cases, for example in discrete systems like Toda, the R-matrix turns out to be not skew-symmetric. However, even in this "non-unitary" case, it is still possibile to define Poisson brackets on a Lie algebra; these results are due to [40] and [36].

Let $\mathcal{G}$ be an associative algebra, with the natural Lie bracket given by the commutator. Assume on $\mathcal{G}$ the existence of a symmetric non-degenerate trace-form $Tr : \mathcal{G} \to \mathbb{C}$ with an associated invariant inner product

$$(L_1, L_2) := \mathrm{Tr}(L_1 L_2); \tag{3.42}$$

using this inner product we identify $\mathcal{G}$ and $\mathcal{G}^*$. Let $R : \mathcal{G} \to \mathcal{G}$ be a linear map and define the following three brackets on $C^\infty(\mathcal{G})$

$$\{f_1, f_2\}_1(L) := (L, [df_1, df_2]_R) = ([L, df_1], R(df_2)) - ([L, df_2], R(df_1)) \tag{3.43a}$$
$$\{f_1, f_2\}_2(L) := ([L, df_1], R(Ldf_2 + df_2 L)) - ([L, df_2], R(Ldf_1 + df_1 L)) \tag{3.43b}$$
$$\{f_1, f_2\}_3(L) := ([L, df_1], R(Ldf_2 L)) - ([L, df_2], R(Ldf_1 L)). \tag{3.43c}$$

The first bracket is simply the linear bracket that we have considered above, hence we already know that it is a Poisson bracket on the algebra $\mathcal{G}$ when

The previous results are generalized for the non-unitary case by the following

Proposition 33 ([40, 36]) *(1) For any R-matrix R, $\{,\}_1$ is a Poisson bracket. (2) If both R and its skew-symmetric part $A = \frac{1}{2}(R - R^*)$ satisfy the modified Yang-Baxter equation (3.22) then $\{,\}_2$ is a Poisson bracket.*
(3) If R solves the modified Yang-Baxter equation (3.22) then $\{,\}_3$ is a Poisson bracket. Moreover the three brackets are compatible.

The Poisson tensors corresponding to the brackets (3.43) are defined by

$$\{f_1, f_2\}_i(L) =: (df_1, P_i(L)df_2) \quad i = 1, 2, 3 \tag{3.44}$$

and are explicitly given by

$$\begin{aligned}
P_1(L)df &= -[L, R(df)] - R^*([L, df]) && (3.45)\\
P_2(L)df &= -[L, R(Ldf + dfL)] - LR^*([L, df]) - R^*([L, df])L && (3.46)\\
P_3(L)df &= -[L, R(LdfL)] - LR^*([L, df])L. && (3.47)
\end{aligned}$$

As before the Casimir functions of $\{,\}$ are in involution with respect to these Poisson brackets and the associated flows admit a simple Lax representation

Proposition 34 *The Casimir functions of $\{,\}$ are in involution with respect to the three Poisson brackets (3.43). If H is a Casimir the associated Hamilton equations have the following Lax form*

$$\begin{aligned}
P_1 dH &= -[L, R(dH)] && (3.48)\\
P_2 dH &= -[L, R(LdH + dHL)] && (3.49)\\
P_3 dH &= -[L, R(LdHL)] && (3.50)
\end{aligned}$$

3.2.5 A lemma on Dirac reduction of Poisson brackets

We will need to reduce the Poisson brackets defined on the whole algebra on a certain submanifold, following the general Dirac prescription. Since we will always deal with reductions to affine subspaces all we need is summarized in the following

Lemma 35 ([40]) *Given two linear spaces U, V with coordinates u, v, let*

$$P(u, v) = \begin{pmatrix} P_{uu} & P_{uv} \\ P_{vu} & P_{vv} \end{pmatrix} : U^* \oplus V^* \to U \oplus V \tag{3.51}$$

be a Poisson tensor on $U \oplus V$. If the component $P_{vv} : V^* \to V$ is invertible then, for an arbitrary $v \in V$, the map $P^{rid}(u; v) : U^* \to U$ given by

$$P^{rid}(u; v) = P_{uu}(u, v) - P_{uv}(u, v)(P_{vv}(u, v))^{-1} P_{vu}(u, v) \tag{3.52}$$

is a Poisson tensor on the affine space $v + U \in U \oplus V$.

The condition of invertibility of P_{vv} can be relaxed by asking that P_{vv} be invertible on the image of $P_{vu} : U^* \to V$. Moreover it is easy to check that if two Poisson tensors are compatible then their reductions to an affine subspace will remain compatible.

3.3 Bigraded Toda bihamiltonian structure

In this section we introduce the simplest algebras of difference operators and using the R-matrix constructions of the previous section we define Poisson structures on such algebras. Moreover by reduction to a suitable affine subspace we obtain the bihamiltonian structure for the bigraded Toda hierarchy.

We will compute the reductions also for the case of L given by a semi-infinite operator

$$L = \Lambda^N + u_{N-1}\Lambda^{N-1} + \dots \tag{3.53}$$

thus obtaining its bihamiltonian structure; this can however be easily obtained from the two-dimensional Toda case considered in the next Chapter, by putting $\bar{u}_k = 0$.

3.3.1 Algebras of difference operators

We will consider the following linear spaces of (formal) difference operators with coefficients in $C^\infty(\mathbb{R})$ or $C^\infty(S^1)$

$$A^+ = \left\{ \sum_{k<+\infty} a_k(x)\Lambda^k \right\} \tag{3.54}$$

$$A^- = \left\{ \sum_{k>-\infty} a_k(x)\Lambda^k \right\}. \tag{3.55}$$

It is convenient to introduce also the spaces

$$A^\infty = \left\{ \sum_{k\in\mathbb{Z}} a_k(x)\Lambda^k \right\} \quad \text{and} \quad A^0 = A^+ \cap A^-. \tag{3.56}$$

The spaces A^+, A^- and A^0 are associative algebras with the usual multiplication defined by $\Lambda f(x) = f(x+\epsilon)\Lambda$; hence they are Lie algebras with Lie bracket given by the commutator. On the other hand the product is not well defined in A^∞ due to the presence of infinite sums; however it is possible to multiply elements of A^∞ by elements of A^0.

On these spaces we have the natural projections on the positive and negative parts, defined by

$$\left(\sum_{k\in\mathbb{Z}} a_k(x)\Lambda^k \right)_+ = \sum_{k\geqslant 0} a_k(x)\Lambda^k \quad \text{and} \quad X_+ + X_- = X \tag{3.57}$$

for X any difference operator; we will also use the notations $X_{>0}$, $X_{\leqslant 0}$ with the obvious meaning.

The residue is defined by

$$\operatorname{Res} X := X_0 \quad \text{where} \quad X = \sum_{k\in\mathbb{Z}} X_k\Lambda^k; \tag{3.58}$$

from translation invariance of the integral $\int f dx = \int (\Lambda f) dx$ it follows that $\mathrm{Tr}[X,Y]=0$ where the trace-form of a difference operator is defined by

$$\mathrm{Tr}\, X := \int \mathrm{Res}\, X dx. \tag{3.59}$$

The bilinear pairing

$$(X,Y) := \mathrm{Tr}\, XY \tag{3.60}$$

gives a non-degenerate symmetric inner product on A^+, A^- and A^0; moreover it gives a pairing between A^0 and A^∞.

3.3.2 Poisson tensors

The Poisson structures of the Toda lattice are related to the splitting of a difference operator in its positive and negative parts; moreover, since we need to use the R-matrix theorems, we need an associative algebra with a non-degenerate inner product. Hence we may consider the algebra A^+ with the bilinear pairing (3.60) and the splitting

$$A^+ = (A^+)_+ \oplus (A^+)_-. \tag{3.61}$$

The naturally associated linear endomorphism $R : A^+ \to A^+$

$$R(X) = X_+ - X_- \tag{3.62}$$

automatically satisfies the modified Yang-Baxter equation (3.22).

The splitting (3.61) is not isotropic with respect to the natural inner product on A^+; equivalently R is not skew-symmetric i.e. $R \neq -R^*$ where

$$R^*(X) = X_{\leqslant 0} - X_{>0}. \tag{3.63}$$

However we have that

Lemma 36 *The skew-symmetric part $A = \frac{1}{2}(R - R^*)$ of R satisfies the modified Yang-Baxter equation (3.22).*

Proof The skew-symmetric part is given by $A(X) = X_{>0} - X_{<0}$: then check (3.22) by direct substitution. □

Hence by Proposition 33 we have

Proposition 37 *There are three compatible Poisson structures on A^+ given by*

$$P_1(L)df = -2[L, df_+] + 2[L, df]_{>0} = 2[L, df_-] - 2[L, df]_{\leqslant 0} \tag{3.64a}$$

$$P_2(L)df = -2[L, (Ldf + dfL)_+] + 2L[L, df]_{>0} + 2[L, df]_{>0}L \tag{3.64b}$$

$$= 2[L, (Ldf + dfL)_-] - 2L[L, df]_{\leqslant 0} - 2[L, df]_{\leqslant 0}L \tag{3.64c}$$

$$P_3(L)df = 2[L, (LdfL)_-] - 2L[L, df]_{\leqslant 0}L \tag{3.64d}$$

$$= -2[L, (LdfL)_+] + 2L[L, df]_{>0}L. \tag{3.64e}$$

Notice that the same considerations can be done for the algebra A^- in a completely analogous way. It will be sufficient for us to consider the algebra A^+, since we are mainly interested in the resulting Poisson structures for the bigraded Toda hierarchy.

3.3.3 Explicit form of Poisson brackets

Here we express the Poisson brackets in the more usual form $\{u_n(x), u_m(y)\}$, using the following

Lemma 38 *The Poisson bracket (3.44) defined by a Poisson tensor $P(L)$ can be written as*

$$\{u_n(x), u_m(y)\}(L) = P_{nm}[\delta(x-y)] \tag{3.65}$$

where P_{nm} is given by

$$P_{nm}[\delta(x-y)] = (P(L)\Lambda^{-m}\delta(x-y))_n \quad \text{and} \quad P(L)df = (P(L)df)_l\Lambda^l. \tag{3.66}$$

Proof This easily follow from the definitions, if we start with the functional on A^+ given by

$$f(L) = u_l(y); \tag{3.67}$$

it follows

$$\frac{\delta f}{\delta u_s(x)} = \delta_{sl}\delta(x-y) \tag{3.68}$$

and

$$df(L) = \Lambda^{-l}\delta(x-y). \tag{3.69}$$

Substituting in (3.44) we conclude. □

We emphasize that there is no reduction involved here, we are simply calculating the explicit form of the brackets in some point L of the algebra A^+.

First bracket

The explicit form of the first Poisson bracket calculated in $L = \sum_{k\leqslant N} u_k\Lambda^k$ is

$$\{u_n(x), u_m(y)\}_1(L) = C_{n,m}\big[u_{n+m}(x)\delta(x-y+n\epsilon) - u_{n+m}(x-m\epsilon)\delta(x-y-m\epsilon)\big] \tag{3.70}$$

where $n, m \in \mathbb{Z}$ and the constant $C_{n,m}$ is given by

$$C_{n,m} = \begin{cases} -1 & n \leqslant 0 \\ 1 & n > 0 \end{cases} + \begin{cases} -1 & m \leqslant 0 \\ 1 & m > 0; \end{cases} \tag{3.71}$$

in formula (3.70) u_k is assumed to be 0 for $k > N$.

The same formula clearly holds true also if we calculate the bracket in $L = u_N\Lambda^N + \cdots + u_{-M}\Lambda^{-M}$; in that case $u_k = 0$ even if $k < -M$.

In the following we will use the abbreviated notation $(\Lambda^k f) = f(x+\epsilon x)$ where Λ acts on all the functions on its right, inside the parenthesis, and the variables u_k are functions of x; in this notation formula (3.70) becomes

$$\{u_n(x), u_m(y)\}_1(L) = C_{n,m}[u_{n+m}(\Lambda^n\delta(x-y)) - (\Lambda^{-m}u_{n+m}\delta(x-y))]. \tag{3.72}$$

Second bracket

The explicit form of the second Poisson bracket calculated in $L = \sum_{k\leqslant N} u_k\Lambda^k$ is

$$\begin{aligned}\{u_n(x), u_m(y)\}_2(L) &= -2u_n(\Lambda^n u_m\delta(x-y)) + 2u_n(\Lambda^{-m}u_m\delta(x-y))\\ &- \sum_{n+m-N\leqslant l\leqslant N} c_{l,m}\left[u_{n+m-l}(\Lambda^{n-l}u_l\delta(x-y)) - u_l(\Lambda^{l-m}u_{n+m-l}\delta(x-y))\right]\end{aligned} \quad (3.73)$$

where $n, m \in \mathbb{Z}$ and $u_k = 0$ for $k > N$ and the constant is given by

$$c_{l,m} = \begin{cases} 2 & l > m, \\ 0 & l = m, \\ -2 & l < m. \end{cases} \quad (3.74)$$

The same formula holds in the case $L = u_N\Lambda^N + \cdots + u_{-M}\Lambda^{-M}$ but the sum must be taken with the limits $n+m-N \leqslant l \leqslant N$ and $-M \leqslant l \leqslant n+m+M$ at the same time; this is equivalent to say that u_k is nonzero only for $-M \leqslant k \leqslant N$.

Remark 39 *One can easily show that the following is equivalent to (3.73)*

$$\begin{aligned}\{u_n(x), u_m(y)\}_2(L) &= -2u_n(\Lambda^n u_m\delta(x-y)) + 2u_n(\Lambda^{-m}u_m\delta(x-y))\\ &+ 2u_n(\Lambda^{n-m}u_m\delta(x-y)) - 2u_m u_n\delta(x-y)\\ &+ \sum_{l<m}[4u_{n+m-l}(\Lambda^{n-l}u_l\delta(x-y)) - 4u_l(\Lambda^{l-m}u_{n+m-l}\delta(x-y))].\end{aligned} \quad (3.75)$$

This observation is based on the identity

$$\sum_{l=m}^{n}[u_{n+m-l}\Lambda^{n-l}u_l - u_l\Lambda^{l-m}u_{n+m-l}] = 0, \quad (3.76)$$

hence the sum in (3.75) can be replaced with $\sum_{l\leqslant n}$. An alternative way of writing the sum in (3.75) is

$$\sum_{l<m} \to \sum_{l=\max(-M,n+m-N)}^{\min(N,m-1)} \quad (3.77)$$

or

$$\sum_{l\leqslant n} \to \sum_{l=\max(-M,n+m-N)}^{\min(N,n)}. \quad (3.78)$$

Third bracket

The explicit form of the third Poisson bracket calculated in $L = \sum_{k\leqslant N} u_k\Lambda^k$ is

$$\begin{aligned}\{u_n(x), u_m(y)\}_3(L) = \sum_{\substack{k,l\leqslant N\\ k+l\geqslant n+m-N}} &\Big[c_1 u_k(\Lambda^k u_l)(\Lambda^{k+l-m}u_{n+m-k-l}\delta(x-y))\\ &- c_2 u_l(\Lambda^{n-k}u_k)(\Lambda^{l-m}u_{n+m-k-l}\delta(x-y))\\ &+ c_3 u_k(\Lambda^{k+l-m}u_{n+m-k-l})(\Lambda^{k-m}u_l\delta(x-y))\Big]\end{aligned} \quad (3.79)$$

where the constants are given by

$$c_1 = c_2 - c_3, \quad c_2 = \begin{cases} 2 & n < k, \\ 0 & n \geqslant k, \end{cases} \qquad c_3 = \begin{cases} 0 & m < l, \\ 2 & m \geqslant l. \end{cases} \tag{3.80}$$

The same formula is true, considering u_k nonzero only for $-M \leqslant k \leqslant N$, in the case $L = u_N \Lambda^N + \cdots + u_{-M}\Lambda^{-M}$.

3.3.4 Reductions

To obtain the Poisson brackets for the bigraded Toda hierarchy we need to reduce the brackets, found so far on the algebra A^+, to the affine subspace of that algebra given by operators of the form

$$L = \Lambda^N + u_{N-1}\Lambda^{N-1} + \cdots + u_{-M}\Lambda^{-M}. \tag{3.81}$$

We will perform such reduction in two steps, first to a linear subspace of A^+, then to the affine subspace given by fixing $u_N = 1$.

Linear subspaces

Here we consider reductions of the Poisson brackets to the linear subspace of A^+ with elements of the form

$$L = u_N\Lambda^N + u_{N-1}\Lambda^{N-1} + \ldots \qquad \text{with } N > 0 \text{ fixed}, \tag{3.82}$$

and to the linear subspace with elements (bigraded case)

$$L = u_N\Lambda^N + \cdots + u_{-M}\Lambda^{-M} \qquad \text{with } N > 0,\ M > 0 \text{ fixed}. \tag{3.83}$$

In the case of the first and second Poisson brackets this reduction is very simple due to the follow observation

Lemma 40 *For both the linear ($i = 1$) and the quadratic brackets ($i = 2$) defined above, we have*

$$\{u_n(x), u_m(y)\}_i = 0 \qquad \text{for } n \leqslant N \text{ and } m > N \tag{3.84}$$

if $L = u_N\Lambda^N + \ldots$ and

$$\{u_n(x), u_m(y)\}_i = 0 \qquad \text{for } -M \leqslant n \leqslant N \text{ and } m \notin [-M, N] \tag{3.85}$$

for $L = u_N\Lambda^N + \cdots + u_{-M}\Lambda^{-M}$.

This means that the off-diagonal part P_{uv} (in the notations of Lemma 35) of both Poisson tensors P_1 and P_2 is zero, hence the correction term in (3.52) vanishes and the reduced brackets have simply the form (3.70) and (3.73).

Remark 41 *The third Poisson bracket does not admit this simple kind of reduction; indeed the analogous of Lemma 40 doesn't hold: for example $\{u_N(x), u_{N+1}\}_3$ is not zero since it always contains the non-zero term $-2u_N(\Lambda^N u_N)(\Lambda^{N-1}u_1\delta(x-y))$ (when $k = N$, $l = N$). One should hence apply a Dirac reduction, that however we won't develop here, since it is enough for us to consider the bihamiltonian structure.*

Affine subspaces

Here we want to reduce the Poisson brackets to the affine subspace obtained from the previous reduction by setting $u_N = 1$ i.e. either to the case

$$L = \Lambda^N + u_{N-1}\Lambda^{N-1} + \dots \tag{3.86}$$

or to the bigraded case

$$L = \Lambda^N + u_{N-1}\Lambda^{N-1} + \dots + u_{-M}\Lambda^{-M}. \tag{3.87}$$

In the case of the first Poisson bracket it is easy to check that

$$\{u_N, u_m\} = 0 \tag{3.88}$$

for each m, i.e. u_N is a Casimir, hence we can simply restrict the bracket without any correction, just by imposing $u_N = 1$.

The second Poisson bracket needs however a correction term

Proposition 42 *The explicit form of the reduced second Poisson bracket is*

$$\begin{aligned} \{u_n(x), u_m(y)\}_2(L) &= -2u_n(\Lambda^n u_m\delta(x-y)) + 2u_n(\Lambda^{-m}u_m\delta(x-y)) \\ &\quad - \sum_l c_{l,m}\big(u_{n+m-l}(\Lambda^{n-l}u_l\delta(x-y)) - u_l(\Lambda^{l-m}u_{n+m-l}\delta(x-y))\big) \\ &\quad - 2\big(u_n(1+\Lambda^{-N})(1+\Lambda^N)(\Lambda^{-N}-\Lambda^N)^{-1}(\Lambda^n-1)(1-\Lambda^{-m})u_m\delta(x-y)\big), \end{aligned} \tag{3.89}$$

where $u_N = 1$; moreover u_k is assumed to be zero for $k > N$ in the case of the reduction (3.86) and for $k \notin [-M, N]$ for the reduction (3.87). The constant $c_{l,m}$ si given by (3.74).

Proof We essentially need to apply Lemma 35. Consider the case $L = u_N\Lambda^N + \dots$, the other case being completely analogous. In the notations of Lemma 35, $P_2 : U^* \oplus V^* \to U \oplus V$ where

$$V = \{u_N\Lambda^N\} \qquad U = \{u_{N-1}\Lambda^{N-1} + \dots\} \tag{3.90}$$

$$V^* = \{\Lambda^{-N}X_{-N}\} \qquad V = \left\{\Lambda^k X_k + \dots + \Lambda^{-N+1}X_{-n+1}, k \text{ arbitrary}\right\}. \tag{3.91}$$

Then using the definition of P_2 one finds that the correction term in (3.52) is given by

$$-(P_{uv}\circ P_{vv}^{-1}\circ P_{vu})(\tilde X) = -2[\tilde L, ((1+\Lambda^{-N})(\Lambda^{-N}-\Lambda^N)^{-1}(\Lambda^N+1)[L,\tilde X]_0)] \tag{3.92}$$

where $\tilde X \in U^*$, $\tilde L = u_{N-1}\Lambda^{N-1} + \dots \in U$. One then concludes by substituting $\tilde X = \Lambda^{-m}\delta(x-y)$. □

Remark 43 *We can isolate in the sum in (3.89) the terms that contain u_N: these give*

$$4(\Lambda^{N-m}u_{n+m-N}\delta(x-y)) - 4u_{n+m-N}(\Lambda^{n-N}\delta(x-y)) \tag{3.93}$$

and the sum must be taken on all the other terms.

Remark 44 *The correction term in general is non-local due to the presence of* $(\Lambda^{-N}-\Lambda^{N})^{-1}$*. As in the previous chapter we can set* $\Lambda = e^{\epsilon\partial}$ *and ask whether* $(\Lambda^{-N}-\Lambda^{N})^{-1}$ *makes sense as a formal power series in* ϵ*. This amounts to check if it acts on something that is in the image of* $\Lambda^{-N}-\Lambda^{N}$*; this image coincides with the image of* $\Lambda^{-1}-1$ *since*

$$(\Lambda^{-N}-\Lambda^{N}) = (\Lambda^{N}+1)(\Lambda^{-N+1}+\cdots+1)(\Lambda^{-1}-1). \tag{3.94}$$

This is clearly always true hence the bracket is always well-defined as a formal power series in ϵ*.*

3.3.5 Bihamiltonian structure for (M,N)-difference operators

For clarity we summarize here the form of the Poisson pencil on the space of operators of the form (3.81)

Theorem 45 *The following brackets give two compatible Poisson structures in the variables* $u_{N-1}(x),\ldots,u_{-M}(x)$ *for* $N, M > 0$*:*

$$\{u_n(x), u_m(y)\}_1 = C_{n,m}[u_{n+m}(\Lambda^n\delta(x-y)) - (\Lambda^{-m}u_{n+m}\delta(x-y))], \tag{3.95}$$

$$\begin{aligned}\{u_n(x), u_m(y)\}_2 = {} & 2u_n((\Lambda^n+1)(\Lambda^{-m}-1)u_m\delta(x-y)) \\ & + 4\sum_{l<m}\big(u_{n+m-l}(\Lambda^{n-l}u_l\delta(x-y)) - u_l(\Lambda^{l-m}u_{n+m-l}\delta(x-y))\big) \\ & - 2\big(u_n(1+\Lambda^{-N})(1+\Lambda^{N})(\Lambda^{-N}-\Lambda^{N})^{-1}(\Lambda^n-1)(1-\Lambda^{-m})u_m\delta(x-y)\big)\end{aligned} \tag{3.96}$$

where

$$C_{n,m} = \begin{cases}-1 & n\leqslant 0\\ 1 & n>0\end{cases} + \begin{cases}-1 & m\leqslant 0\\ 1 & m>0.\end{cases} \tag{3.97}$$

Remark 46 *A further version of the formula (3.96) is obtained by semplification of the first and third terms on the RHS*

$$\begin{aligned}\{u_n(x), u_m(y)\}_2 = {} & 4\sum_{l<m}\big(u_{n+m-l}(\Lambda^{n-l}u_l\delta(x-y)) - u_l(\Lambda^{l-m}u_{n+m-l}\delta(x-y))\big) \\ & + 4\big(u_n(1-\Lambda^{-N})^{-1}(\Lambda^{n-N}-1)(1-\Lambda^{-m})u_m\delta(x-y)\big).\end{aligned} \tag{3.98}$$

Remark 47 *The reduction procedure works also for* $M = 0$ *(see Remark 31 for the definition of the corresponding hierarchy), hence these are compatible Poisson brackets in this case, too. Frenkel and Reshetikhin [24] have considered related Poisson structures in the context of deformations of the classical* W*-algebras.*

The second Poisson bracket of Frenkel and Reshetikhin (see [23])

$$\{t_i(z), t_j(w)\}_2 = \sum_{m\in\mathbb{Z}} \left(\frac{w}{z}\right)^m \frac{(1-q^{im})(1-q^{m(N-j)})}{1-q^{mN}} t_i(z)t_j(w) + \sum_{r=1}^{\min(i,N-j)} \delta\left(\frac{wq^r}{z}\right) t_{i-r}(w)t_{j+r}(z) - \sum_{r=1}^{\min(i,N-j)} \delta\left(\frac{w}{zq^{j-i+r}}\right) t_{i-r}(z)t_{j+r}(w). \tag{3.99}$$

reduces to (3.96) (multiplied by a factor $-\frac{1}{4}$*) in the case* $M = 0$ *if we identify* $w = q^y$*,* $z = q^x$*,* $t_k = (-1)^k u_{N-k}$ *and substitute* $\delta(q^x) \to \delta(x)$*,* $i \to n$*,* $j \to m$*. The infinite sum in the first term gives the Dirac correction; to evaluate it one uses the definition* $\delta(q^x) = \sum_{m\in\mathbb{Z}} q^{mx}$*. For example it is easy to show (multiplying on the left by* $1 - \Lambda^N$ *that*

$$(1-\Lambda^N)^{-1}\delta(q^x) = \sum_m \frac{q^{xm}}{1-q^{mN}}. \tag{3.100}$$

In the case of the first bracket, however, there is a discrepancy since the bracket proposed in [24] is quadratic in the fields, while our bracket (3.95) is linear.

3.3.6 Examples

Example 48 (N=1) *Let's start from the simplest case* $N = 1$*; restricting the Poisson brackets to the linear subspace of elements of the form*

$$L = u_1\Lambda + u_0 + \dots \tag{3.101}$$

we don't have to add any corrective term, as explained above. In this case we have an infinite number of fields u_k *hence there's an infinite number of non-zero brackets between them; spelling out the first entries of formulas (3.70) and (3.89) we obtain*

First bracket

$$\{u_1(x), u_k(y)\}_1 = 0 \quad k \leqslant 1 \tag{3.102a}$$
$$\{u_0(x), u_0(y)\}_1 = 0 \tag{3.102b}$$
$$\{u_0(x), u_{-1}(y)\}_1 = -2[u_{-1}\delta(x-y) - (\Lambda u_{-1}\delta(x-y))] \tag{3.102c}$$
$$\{u_0(x), u_{-2}(y)\}_1 = -2[u_{-2}\delta(x-y) - (\Lambda^2 u_{-2}\delta(x-y))] \tag{3.102d}$$
$$\{u_{-1}(x), u_{-1}(y)\}_1 = -2[u_{-2}(\Lambda^{-1}\delta(x-y)) - (\Lambda u_{-2}\delta(x-y))] \tag{3.102e}$$
$$\{u_{-1}(x), u_{-2}(y)\}_1 = -2[u_{-3}(\Lambda^{-1}\delta(x-y)) - (\Lambda^2 u_{-3}\delta(x-y))] \tag{3.102f}$$
$$\{u_{-2}(x), u_{-2}(y)\}_1 = -2[u_{-4}(\Lambda^{-2}\delta(x-y)) - (\Lambda^2 u_{-4}\delta(x-y))] \tag{3.102g}$$

Second bracket

$$\{u_1(x), u_1(y)\}_2 = -2u_1(\Lambda u_1\delta(x-y)) + 2u_1(\Lambda^{-1}u_1\delta(x-y)) \tag{3.103a}$$

$$\{u_1(x), u_0(y)\}_2 = 0 \tag{3.103b}$$

$$\{u_1(x), u_{-1}(y)\}_2 = -2u_{-1}u_1\delta(x-y) + 2u_1(\Lambda^2 u_{-1}\delta(x-y)) \tag{3.103c}$$

$$\{u_0(x), u_0(y)\}_2 = 4u_1(\Lambda u_{-1}\delta(x-y)) - 4u_{-1}(\Lambda^{-1}u_1\delta(x-y)) \tag{3.103d}$$

$$\begin{aligned}\{u_0(x), u_{-1}(y)\}_2 &= -4u_0u_{-1}\delta(x-y) + 4u_0(\Lambda u_{-1}\delta(x-y)) \\ &\quad + 4u_1(\Lambda^2 u_{-2}\delta(x-y)) - 4u_{-2}(\Lambda^{-1}u_1\delta(x-y))\end{aligned} \tag{3.103e}$$

$$\begin{aligned}\{u_{-1}(x), u_{-1}(y)\}_2 &= -2u_{-1}(\Lambda^{-1}u_{-1}\delta(x-y)) + 2u_{-1}(\Lambda u_{-1}\delta(x-y)) \\ &\quad + 4u_1(\Lambda^2 u_{-3}\delta(x-y)) - 4u_{-3}(\Lambda^{-2}u_1\delta(x-y)) \\ &\quad + 4u_0(\Lambda u_{-2}\delta(x-y)) - 4u_{-2}(\Lambda^{-1}u_0\delta(x-y))\end{aligned} \tag{3.103f}$$

Recall that the third bracket doesn't have a simple reduction to the $N=1$ subspace.

Let's consider now the reduction to the affine subspace $u_1 = 1$. The first bracket doesn't need any correction term, as we have seen before, hence the reduced bracket is obtained just by setting $u_1 = 1$. For the second bracket, we have to add to $\{u_n(x), u_m(y)\}_2$ a correction term that is given by

$$-2u_n(1+\Lambda^{-1})(\Lambda^n - 1)\Lambda(1 + \Lambda + \cdots + \Lambda^{-m-1})u_m\delta(x-y) \tag{3.104}$$

for $m \leqslant 0$ and is 0 if $m=0$; hence in the $N=1$ case the nonlocal terms $(\Lambda^{-1}-\Lambda)^{-1}$ are not present.

Example 49 (N=1, M=1 and the Toda chain) *The reduced brackets in the case*

$$L = u_1\Lambda + u_0 + u_{-1}\Lambda^{-1} \tag{3.105}$$

are simply obtained by putting $u_k = 0$ for $k < -1$ in equations (3.102) and (3.103); the only non-zero elements are

First bracket

$$\{u_0(x), u_{-1}(y)\}_1 = -2[u_{-1}\delta(x-y) - (\Lambda u_{-1}\delta(x-y))] \tag{3.106}$$

Second bracket

$$\{u_1(x), u_1(y)\}_2 = -2u_1(\Lambda u_1\delta(x-y)) + 2u_1(\Lambda^{-1}u_1\delta(x-y)) \tag{3.107a}$$

$$\{u_1(x), u_{-1}(y)\}_2 = -2u_{-1}u_1\delta(x-y) + 2u_1(\Lambda^2 u_{-1}\delta(x-y)) \tag{3.107b}$$

$$\{u_0(x), u_0(y)\}_2 = 4u_1(\Lambda u_{-1}\delta(x-y)) - 4u_{-1}(\Lambda^{-1}u_1\delta(x-y)) \tag{3.107c}$$

$$\{u_0(x), u_{-1}(y)\}_2 = -4u_0u_{-1}\delta(x-y) + 4u_0(\Lambda u_{-1}\delta(x-y)) \tag{3.107d}$$

$$\{u_{-1}(x), u_{-1}(y)\}_2 = -2u_{-1}(\Lambda^{-1}u_{-1}\delta(x-y)) + 2u_{-1}(\Lambda u_{-1}\delta(x-y)) \tag{3.107e}$$

If we now perform the reduction $u_1 = 1$ the only term that gets a correction is $\{u_{-1}(x), u_{-1}(y)\}_2$ to which we have to add

$$2u_{-1}(\Lambda - \Lambda^{-1})u_{-1}\delta(x-y), \tag{3.108}$$

finally obtaining the Toda chain Poisson brackets whose non-zero elements are

First bracket

$$\{u_0(x), u_{-1}(y)\}_1 = -2[u_{-1}\delta(x-y) - (\Lambda u_{-1}\delta(x-y))] \tag{3.109}$$

Second bracket

$$\{u_0(x), u_0(y)\}_2 = 4(\Lambda u_{-1}\delta(x-y)) - 4u_{-1}(\Lambda^{-1}\delta(x-y)) \tag{3.110a}$$
$$\{u_0(x), u_{-1}(y)\}_2 = -4u_0u_{-1}\delta(x-y) + 4u_0(\Lambda u_{-1}\delta(x-y)) \tag{3.110b}$$
$$\{u_{-1}(x), u_{-1}(y)\}_2 = -4u_{-1}(\Lambda^{-1}u_{-1}\delta(x-y)) + 4u_{-1}(\Lambda u_{-1}\delta(x-y)). \tag{3.110c}$$

Remark 50 *These Poisson brackets are respectively equal to (2.13) multiplied by 2ϵ and (2.14) multiplied by 4ϵ.*

Example 51 (N=2, M=2) *Now consider the bigraded case with*

$$L = u_2\Lambda^2 + u_1\Lambda + u_0 + u_{-1}\Lambda^{-1} + u_{-2}\Lambda^{-2}; \tag{3.111}$$

the only non-zero entries of the first and second brackets are

First bracket

$$\{u_1(x), u_1(y)\}_1 = 2u_2(\Lambda\delta(x-y)) - 2(\Lambda^{-1}u_2\delta(x-y)) \tag{3.112a}$$
$$\{u_0(x), u_{-1}(y)\}_1 = -2[u_{-1}\delta(x-y) - (\Lambda u_{-1}\delta(x-y))] \tag{3.112b}$$
$$\{u_0(x), u_{-2}(y)\}_1 = -2[u_{-2}\delta(x-y) - (\Lambda^2 u_{-2}\delta(x-y))] \tag{3.112c}$$
$$\{u_{-1}(x), u_{-1}(y)\}_1 = -2[u_{-2}(\Lambda^{-1}\delta(x-y)) - (\Lambda u_{-2}\delta(x-y))] \tag{3.112d}$$

Second bracket

$$\{u_2(x),u_2(y)\}_2 = 2u_2(\Lambda^{-2}u_2\delta(x-y)) - 2u_2(\Lambda^2 u_2\delta(x-y)) \tag{3.113a}$$

$$\begin{aligned}\{u_2(x),u_1(y)\}_2 &= 2u_2(\Lambda^{-1}u_1\delta(x-y)) - 2u_2u_1\delta(x-y)\\ &+ 2u_2(\Lambda u_1\delta(x-y)) - 2u_2(\Lambda^2 u_1\delta(x-y))\end{aligned} \tag{3.113b}$$

$$\begin{aligned}\{u_2(x),u_{-1}(y)\}_2 &= -2u_2u_{-1}\delta(x-y) + 2u_2(\Lambda u_{-1}\delta(x-y))\\ &- 2u_2(\Lambda^2 u_{-1}\delta(x-y)) + 2u_2(\Lambda^3 u_{-1}\delta(x-y))\end{aligned} \tag{3.113c}$$

$$\{u_2(x),u_{-2}(y)\}_2 = -2u_{-2}u_2\delta(x-y) + 2u_2(\Lambda^4 u_{-2}\delta(x-y)) \tag{3.113d}$$

$$\begin{aligned}\{u_1(x),u_1(y)\}_2 &= -2u_1(\Lambda u_1\delta(x-y)) + 2u_1(\Lambda^{-1}u_1\delta(x-y))\\ &- 4u_0(\Lambda^{-1}u_2\delta(x-y)) + 4u_2(\Lambda u_0\delta(x-y))\end{aligned} \tag{3.113e}$$

$$\{u_1(x),u_0(y)\}_2 = -4u_{-1}(\Lambda^{-1}u_2\delta(x-y)) + 4u_2(\Lambda^2 u_{-1}\delta(x-y)) \tag{3.113f}$$

$$\begin{aligned}\{u_1(x),u_{-1}(y)\}_2 &= -2u_{-1}u_1\delta(x-y) + 2u_1(\Lambda^2 u_{-1}\delta(x-y))\\ &- 4u_{-2}(\Lambda^{-1}u_2\delta(x-y)) + 4u_2(\Lambda^3 u_{-2}\delta(x-y))\end{aligned} \tag{3.113g}$$

$$\begin{aligned}\{u_1(x),u_{-2}(y)\}_2 &= -2u_1u_{-2}\delta(x-y) - 2u_1(\Lambda u_{-2}\delta(x-y))\\ &+ 2u_1(\Lambda^2 u_{-2}\delta(x-y)) + 2u_1(\Lambda^3 u_{-2}\delta(x-y))\end{aligned} \tag{3.113h}$$

$$\begin{aligned}\{u_0(x),u_0(y)\}_2 &= 4u_1(\Lambda u_{-1}\delta(x-y)) - 4u_{-1}(\Lambda^{-1}u_1\delta(x-y))\\ &- 4u_{-2}(\Lambda^{-2}u_2\delta(x-y)) + 4u_2(\Lambda^2 u_{-2}\delta(x-y))\end{aligned} \tag{3.113i}$$

$$\begin{aligned}\{u_0(x),u_{-1}(y)\}_2 &= -4u_0u_{-1}\delta(x-y) + 4u_0(\Lambda u_{-1}\delta(x-y))\\ &+ 4u_1(\Lambda^2 u_{-2}\delta(x-y)) - 4u_{-2}(\Lambda^{-1}u_1\delta(x-y))\end{aligned} \tag{3.113j}$$

$$\{u_0(x),u_{-2}(y)\}_2 = -4u_0u_{-2}\delta(x-y) + 4u_0(\Lambda^2 u_{-2}\delta(x-y)) \tag{3.113k}$$

$$\begin{aligned}\{u_{-1}(x),u_{-1}(y)\}_2 &= -2u_{-1}(\Lambda^{-1}u_{-1}\delta(x-y)) + 2u_{-1}(\Lambda u_{-1}\delta(x-y))\\ &+ 4u_0(\Lambda u_{-2}\delta(x-y)) - 4u_{-2}(\Lambda^{-1}u_0\delta(x-y))\end{aligned} \tag{3.113l}$$

$$\begin{aligned}\{u_{-1}(x),u_{-2}(y)\}_2 &= -2u_{-1}(\Lambda^{-1}u_{-2}\delta(x-y)) - 2u_{-1}u_{-2}\delta(x-y)\\ &+ 2u_{-1}(\Lambda u_{-2}\delta(x-y)) + 2u_{-1}(\Lambda^2 u_{-2}\delta(x-y))\end{aligned} \tag{3.113m}$$

$$\{u_{-2}(x),u_{-2}(y)\}_2 = -2u_{-2}(\Lambda^{-2}u_{-2}\delta(x-y)) + 2u_{-2}(\Lambda^2 u_{-2}\delta(x-y) \tag{3.113n}$$

The first bracket reduced to the affine subspace $u_2 = 1$, i.e. with

$$L = \Lambda^2 + u_1\Lambda + u_0 + u_{-1}\Lambda^{-1} + u_{-2}\Lambda^{-2} \tag{3.114}$$

is simply

First bracket

$$\{u_1(x),u_1(y)\}_1 = 2(\Lambda\delta(x-y)) - 2(\Lambda^{-1}\delta(x-y)) \tag{3.115a}$$

$$\{u_0(x),u_{-1}(y)\}_1 = -2[u_{-1}\delta(x-y) - (\Lambda u_{-1}\delta(x-y))] \tag{3.115b}$$

$$\{u_0(x),u_{-2}(y)\}_1 = -2[u_{-2}\delta(x-y) - (\Lambda^2 u_{-2}\delta(x-y))] \tag{3.115c}$$

$$\{u_{-1}(x),u_{-1}(y)\}_1 = -2[u_{-2}(\Lambda^{-1}\delta(x-y)) - (\Lambda u_{-2}\delta(x-y))]; \tag{3.115d}$$

For the second bracket the non-local correction term

$$-2u_n(1+\Lambda^{-2})(1+\Lambda^2)(\Lambda^{-2}-\Lambda^2)^{-1}(\Lambda^n-1)(1-\Lambda^{-m})u_m\delta(x-y) \tag{3.116}$$

must be added to $\{u_n(x), u_m(y)\}_2$*; the non-zero elements thus obtained are*

Second bracket

$$
\begin{aligned}
\{u_1(x), u_1(y)\}_2 &= -2u_1(\Lambda u_1\delta(x-y)) + 2u_1(\Lambda^{-1}u_1\delta(x-y)) \\
&\quad - 4u_0(\Lambda^{-1}\delta(x-y)) + 4(\Lambda u_0\delta(x-y)) \\
&\quad + 2u_1(1+\Lambda^{-2})(1+\Lambda^2)(\Lambda^{-1}+\Lambda)^{-1}(\Lambda+1)^{-1}(\Lambda-1)u_1\delta(x-y) && (3.117a) \\
\{u_1(x), u_0(y)\}_2 &= -4u_{-1}(\Lambda^{-1}\delta(x-y)) + 4(\Lambda^2 u_{-1}\delta(x-y)) && (3.117b) \\
\{u_1(x), u_{-1}(y)\}_2 &= -2u_{-1}u_1\delta(x-y) + 2u_1(\Lambda^2 u_{-1}\delta(x-y)) \\
&\quad - 4u_{-2}(\Lambda^{-1}\delta(x-y)) + 4(\Lambda^3 u_{-2}\delta(x-y)) \\
&\quad - 2u_1(1+\Lambda^{-2})(1+\Lambda^2)(\Lambda^{-1}+\Lambda)^{-1}(\Lambda+1)^{-1}(\Lambda-1)\Lambda u_{-1}\delta(x-y) && (3.117c) \\
\{u_1(x), u_{-2}(y)\}_2 &= -2u_1u_{-2}\delta(x-y) - 2u_1(\Lambda u_{-2}\delta(x-y)) \\
&\quad + 2u_1(\Lambda^2 u_{-2}\delta(x-y)) + 2u_1(\Lambda^3 u_{-2}\delta(x-y)) \\
&\quad + 2u_1(1+\Lambda^{-2})(1+\Lambda^2)(\Lambda^{-1}+\Lambda)^{-1}\Lambda(1-\Lambda)u_{-2}\delta(x-y) && (3.117d) \\
\{u_0(x), u_0(y)\}_2 &= 4u_1(\Lambda u_{-1}\delta(x-y)) - 4u_{-1}(\Lambda^{-1}u_1\delta(x-y)) \\
&\quad - 4u_{-2}(\Lambda^{-2}\delta(x-y)) + 4(\Lambda^2 u_{-2}\delta(x-y)) && (3.117e) \\
\{u_0(x), u_{-1}(y)\}_2 &= -4u_0u_{-1}\delta(x-y) + 4u_0(\Lambda u_{-1}\delta(x-y)) \\
&\quad + 4u_1(\Lambda^2 u_{-2}\delta(x-y)) - 4u_{-2}(\Lambda^{-1}u_1\delta(x-y)) && (3.117f) \\
\{u_0(x), u_{-2}(y)\}_2 &= -4u_0u_{-2}\delta(x-y) + 4u_0(\Lambda^2 u_{-2}\delta(x-y)) && (3.117g) \\
\{u_{-1}(x), u_{-1}(y)\}_2 &= -2u_{-1}(\Lambda^{-1}u_{-1}\delta(x-y)) + 2u_{-1}(\Lambda u_{-1}\delta(x-y)) \\
&\quad + 4u_0(\Lambda u_{-2}\delta(x-y)) - 4u_{-2}(\Lambda^{-1}u_0\delta(x-y)) \\
&\quad - 2u_{-1}(1+\Lambda^{-2})(1+\Lambda^2)(\Lambda^{-1}+\Lambda)^{-1}(\Lambda+1)^{-1}(1-\Lambda)u_{-1}\delta(x-y) && (3.117h) \\
\{u_{-1}(x), u_{-2}(y)\}_2 &= -2u_{-1}(\Lambda^{-1}u_{-2}\delta(x-y)) - 2u_{-1}u_{-2}\delta(x-y) \\
&\quad + 2u_{-1}(\Lambda u_{-2}\delta(x-y)) + 2u_{-1}(\Lambda^2 u_{-2}\delta(x-y)) \\
&\quad - 2u_{-1}(1+\Lambda^{-2})(1+\Lambda^2)(\Lambda^{-1}+\Lambda)^{-1}(1-\Lambda)u_{-2}\delta(x-y) && (3.117i) \\
\{u_{-2}(x), u_{-2}(y)\}_2 &= -2u_{-2}(\Lambda^{-2}u_{-2}\delta(x-y)) + 2u_{-2}(\Lambda^2 u_{-2}\delta(x-y)) \\
&\quad - 2u_{-2}(1+\Lambda^{-2})(1+\Lambda^2)(\Lambda^{-1}+\Lambda)^{-1}(\Lambda^{-1}+1)(1-\Lambda)u_{-2}\delta(x-y). && (3.117j)
\end{aligned}
$$

3.4 Relation between Lax and Hamiltonian formulation of the Extended bigraded Toda hierarchy

In this section we provide the explicit form of the Hamiltonian theorem, i.e. we show that the flows previously defined in the Lax pair formalism can be expressed in terms of suitable Hamiltonians through the first Poisson brackets defined above.

The main result is the following

Theorem 52 *The Lax equations (3.16) can be written in the following Hamiltonian* *form*

$$\frac{d}{dt^{\alpha,q}}u_n = \{u_n, \bar{h}_{\alpha,q}\}_1 \tag{3.118}$$

where the Poisson bracket has the form (3.95) and the Hamiltonians are given by

$$h_{\alpha,q} = \frac{1}{2}\frac{\Gamma(2-\frac{\alpha}{N})}{\Gamma(q+3-\frac{\alpha}{N})}Res(L^{q+2-\frac{\alpha}{N}}) \qquad \alpha = N-1,\cdots,0 \tag{3.119a}$$

$$h_{\alpha,q} = \frac{1}{2}\frac{\Gamma(2+\frac{\alpha}{M})}{\Gamma(q+3+\frac{\alpha}{M})}Res(L^{q+2+\frac{\alpha}{M}}) \qquad \alpha = 0,\cdots,-M+1 \tag{3.119b}$$

$$h_{-M,q} = \frac{1}{2}\frac{1}{(q+1)!}Res\left(L^{q+1}(\log L - \frac{1}{2}(\frac{1}{M}+\frac{1}{N})c_{q+1})\right). \tag{3.119c}$$

where $c_q = \sum_{j=1}^{q}\frac{1}{j}$ *and* $c_0 = 0$.

Proof (of Theorem 52) The proof is based on the following Lemmas. Lemma 53 essentially gives the form of the first Hamiltonian operator that coincides with that found before. Then one has to express p_m in terms of the functional derivatives of the Hamiltonians which is done in the following three lemmas.

Lemma 53 *Let* $A = \sum_{m\in\mathbb{Z}}\Lambda^m p_m$ *such that* $[A,L]=0$; *then* $L_t = [A_+, L]$ *is equivalent to*

$$(u_n)_t = \sum_{m=0}^{n+M}(\Lambda^m u_{-m+n} - u_{-m+n}\Lambda^n)p_m \tag{3.120a}$$

$$(u_n)_t = -\sum_{m=n-N}^{-1}(\Lambda^m u_{-m+n} - u_{-m+n}\Lambda^n)p_m \tag{3.120b}$$

for $-M \leqslant n \leqslant N-1$ *and* $u_N = 1$.

Proof From $[A_+,L] = -[A_-,L]$ it follows that the only nonzero terms in $[A_+,L]$ are the coefficients of Λ^k with $-M \leqslant k < N$; hence the Lax equations $L_t = [A_+,L]$ are well defined.

We have

$$[A_+,L] = [A_+,\Lambda^N] + [A_+,\sum_{k=-M}^{N-1}u_k\Lambda^k] \tag{3.121}$$

the first term on the RHS gives only contributions of order Λ^k with $k \geqslant N$, while the second term gives

$$\sum_{n\geqslant -M}\sum_{\substack{m\geqslant 0\\ -M\leqslant k\leqslant N-1\\ k+m=n}}[\Lambda^m(p_m u_k) - u_k(\Lambda^n p_m)]\Lambda^n; \tag{3.122}$$

collecting powers of Λ we conclude.

The second formula is obtained exactly in the same way, starting from $L_t = -[A_-, L]$. □

If we could express p_m, defined by $A = \sum_m \Lambda^m p_m$, in terms of the functional derivative of some functional H

$$p_{-m} = \frac{\delta}{\delta u_m} H \tag{3.123}$$

we would obtain by the previous Lemma that the Lax equation $L_t = [A_+, L]$ would become

$$(u_n)_t = \sum_{m=-M}^{N-1} B^{(1)}_{nm} \frac{\delta}{\delta u_m} \frac{1}{2} H \qquad n = N-1, \ldots, -M \tag{3.124}$$

where

$$B^{(1)}_{nm} = 2 \cdot \begin{cases} \tilde{\Lambda}^{-m} u_{n+m} - u_{n+m} \tilde{\Lambda}^n & \text{if } n \leqslant 0 \text{ and } m \leqslant 0, \\ -(\tilde{\Lambda}^{-m} u_{n+m} - u_{n+m} \tilde{\Lambda}^n) & \text{if } n > 0 \text{ and } m > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.125}$$

If we define

$$\{\bar{f}, \bar{g}\}_1 := \sum_{n,m} \frac{\delta \bar{f}}{\delta u_n} B^{(1)}_{nm} \frac{\delta \bar{g}}{\delta u_m} \tag{3.126}$$

this is easily seen to be precisely the first Poisson bracket defined before.

In the following three Lemmas we show how to relate p_m with the functional derivative of some Hamiltonian. We consider first the usual case of integer powers of L, that give the standard Toda flows.

Lemma 54 *If $A = L^n = \sum_{m\in\mathbb{Z}} \Lambda^m p_m$ then*

$$p_{-m} = \frac{\delta}{\delta u_m} \left(\frac{1}{n+1} \int dx \; ResL^{n+1} \right) \qquad m = N-1, \cdots, -M. \tag{3.127}$$

Proof The proof of this and of the subsequent Lemmas essentially follows the proof of (2.74) for the Toda chain hierarchy. As in that case we observe that the differential of L is

$$dL = \sum_{k=-M}^{N-1} du_k \Lambda^k \tag{3.128}$$

where du_k for $k = N-1, \ldots, -M$ are the basic differentials of the "coordinates" $u_k(x)$ on the space of operators L of the form (3.1). The differential of a functional H on such space is obtained by functional derivation

$$dH = \int dx \; \sum_k \frac{\delta H}{\delta u_k(x)} du_k(x). \tag{3.129}$$

For a functional H

$$H = \int dx \; \frac{1}{n+1} \operatorname{Res} L^{n+1} \tag{3.130}$$

we have

$$dH = \int dx \ \mathrm{Res}(L^n dL) \tag{3.131}$$

since for any difference operator $B = \sum_k B_k \Lambda^k$ it is easy to check that

$$\int dx \ \mathrm{Res}[B, dL]dx = 0. \tag{3.132}$$

On the other hand

$$\begin{aligned} Res(L^n dL) &= Res(\sum_m \Lambda^m p_m \sum_{k=-M}^{N-1} du_k \Lambda^k) && (3.133)\\ &= \sum_{k=-M}^{N-1} \Lambda^{-k}(p_{-k} du_k) && (3.134)\\ &\sim \sum_{k=-M}^{N-1} p_{-k} du_k && (3.135)\end{aligned}$$

up to total derivatives, hence

$$dH = \int dx \sum_{k=-M}^{N-1} p_{-k} du_k. \tag{3.136}$$

Finally comparing with (3.129) we obtain

$$\int dx \sum_{k=-M}^{N-1} p_{-k} du_k = \int dy \ \frac{\delta}{\delta u_k(y)} \left(\frac{1}{n+1} \int dx \ Res L^{n+1} \right) du_k(y) \tag{3.137}$$

and, by the independence of the differentials du_k, we conclude. □

Lemma 55 *If* $A = L^n \log L = \sum_{m \in \mathbb{Z}} \Lambda^m p_m$ *then*

$$p_{-k} = \frac{1}{n+1} \frac{\delta}{\delta u_k} \int dx \ Res(L^{n+1}(\log L - \frac{1}{2} \frac{1}{n+1}(\frac{1}{M} + \frac{1}{N}))) \qquad k = N-1, \ldots, -M. \tag{3.138}$$

Proof First we show the following formula

$$\int dx \ Res(L^p d \log_{\pm} L) = \int dx \ Res(L^{p-1} dL); \tag{3.139}$$

this implies

$$\int dx \ Res(L^p d \log L) = \int dx \ \frac{1}{2}(\frac{1}{M} + \frac{1}{N}) Res(L^{p-1} dL). \tag{3.140}$$

Using the formula

$$e^{p\log_\pm L} = L^p \tag{3.141}$$

we have

$$\begin{aligned} pRes(L^{p-1}dL) &\sim Res\ dL^p & (3.142)\\ &= Res\ de^{p\log_\pm L} & (3.143)\\ &\sim Res(\sum_{n=1}^\infty \frac{1}{(n-1)!}(p\log_\pm L)^{n-1}p\ d\log_\pm L) & (3.144)\\ &= pRes(e^{p\log_\pm L}d\log_\pm L) & (3.145)\\ &= pRes(L^p d\log_\pm L) & (3.146)\end{aligned}$$

up to total derivatives.

Using (3.140) one shows

$$d\int dx\ Res(L^{n+1}(\log L - \frac{1}{2}(\frac{1}{M}+\frac{1}{N})\frac{1}{n+1})) = \tag{3.147}$$

$$= \int dx\ [Res(L^{n+1}d\log L) + (n+1)Res(L^n dL(\log L - \frac{1}{2}\frac{1}{n+1}(\frac{1}{M}+\frac{1}{N})))] \tag{3.148}$$

$$= \int dx\ (n+1)Res(L^n \log L\ dL). \tag{3.149}$$

Essentially as in the previous lemma, one obtains that

$$\int dx\ (n+1)\sum_{k=-M}^{N-1} p_{-k}du_k =$$

$$= \int dx \sum_{k=-M}^{N-1}\frac{\delta}{\delta u_k(x)}\left(\int dy\ Res[L^{n+1}(\log L - \frac{1}{2}\frac{1}{n+1}(\frac{1}{M}+\frac{1}{N}))]\right)du_k(x) \tag{3.150}$$

and then concludes. □

Lemma 56 *If* $A = L^n L^{\frac{q}{N}} = \sum_{m\in\mathbb{Z}}\Lambda^m p_m$ *then*

$$p_{-m} = \frac{1}{n+1+\frac{q}{N}}\frac{\delta}{\delta u_m}\int dx\ Res(L^{n+1}L^{\frac{q}{N}}) \qquad m = N-1,\dots,-M. \tag{3.151}$$

The same formula holds also in the case $N \to M$*.*

Proof We have

$$d\int dx\ Res(L^{n+1+\frac{q}{N}}) = d\int dx\ Res(L^{\frac{nN+N+q}{N}}) \tag{3.152a}$$

$$= (nN+N+q)\int dx\ \mathrm{Res}(L^{\frac{nN+N+q-1}{N}}dL^{\frac{1}{N}}) \tag{3.152b}$$

$$= \int dx\ \frac{nN+N+q}{N}\mathrm{Res}(L^{\frac{nN+q}{N}}dL) \tag{3.152c}$$

where we have used the definition $(L^{\frac{1}{N}})^N = L$; as in previous lemmas

$$d\int dx\ Res(L^{n+1}L^{\frac{q}{N}}) = \int dx\ (n+1+\frac{q}{N})\sum_{k=-M}^{N-1} p_k du_k \tag{3.153}$$

and the Lemma is proved. □

Combining these three Lemmas with the observations made above one finally proves the Theorem. □

3.5 Tau symmetry

In this section we prove that the Hamiltonian densities defined above satisfy the tau symmetry.

Theorem 57 *The Hamiltonian densities $h_{\alpha,q}$ satisfy the tau symmetry, i.e. the following identities hold*

$$\{h_{\alpha,p-1}, \bar{h}_{\beta,q}\}_1 = \{h_{\beta,q-1}, \bar{h}_{\alpha,p}\}_1 \tag{3.154}$$

for $\alpha, \beta = N-1, \dots, -M$ and $p, q \geqslant 0$.

Proof The normalization of the Hamiltonians has been chosen such that

$$h_{\alpha,p} = \mathrm{Res}\, B_{\alpha,p+1} \qquad A_{\alpha,p} = (B_{\alpha,p})_+ \tag{3.155}$$

for suitable difference operators $B_{\alpha,p}$ that commute among themselves: $[B_{\alpha,p}, B_{\beta,q}] = 0$. It follows that

$$\frac{\partial h_{\alpha,p-1}}{\partial t^{\beta,q}} = \mathrm{Res}[(B_{\beta,q})_+, B_{\alpha,p}] \tag{3.156a}$$

$$= \mathrm{Res}[(B_{\alpha,p})_+, B_{\beta,q}] \tag{3.156b}$$

$$= \frac{\partial h_{\beta,q-1}}{\partial t^{\alpha,p}} \tag{3.156c}$$

since $\mathrm{Res}[(\cdot)_+, (\cdot)_+] = 0$ and $\mathrm{Res}[(\cdot)_-, (\cdot)_-] = 0$; the Theorem is proved. □

Moreover, using the terminology of [19], we have that this tau structure is compatible with spatial translations (up to an irrelevant factor $\frac{1}{2}$), i.e. the Hamiltonian $\bar{h}_{-M,0}$ corresponds to the x-translations

$$\frac{\partial}{\partial t^{-M,0}} \cdots = \{\cdot, \bar{h}_{-M,0}\}_1; \tag{3.157}$$

this follows from the fact that $A_{-M,0} = \frac{\epsilon}{2}\partial_x$.

In complete analogy with the case of the Toda chain hierarchy, one can define the functions $\Omega_{\alpha,p;\beta,q}$ and then use the previous Theorem to show the existence of the tau function for this hierarchy.

Chapter 4

Two-dimensional Toda hierarchy

In this chapter we provide an R-matrix formulation for the two-dimensional Toda hierarchy. We obtain a bihamiltonian structure on the algebra $A^+ \oplus A^-$ by providing an R-matrix associated to a non-trivial splitting. After suitable reductions to affine subspaces we obtain the Poisson brackets for the two-dimensional Toda hierarchy of Ueno-Takasaki [47] and for a generalized bigraded two-dimensional Toda hierarchy.

4.1 Ueno-Takasaki formulation

The two-dimensional Toda lattice hierarchy has been formulated in [47] by Ueno and Takasaki. We will recall their definition, using the language of difference operators.

Consider two (formal) difference operators of the form

$$L = \Lambda + u_0 + u_{-1}\Lambda^{-1} + \dots \tag{4.1}$$

$$\bar{L} = \bar{u}_{-1}\Lambda^{-1} + \bar{u}_0 + \bar{u}_1\Lambda + \dots \tag{4.2}$$

where the coefficients, following our usual notations, are functions of the continuous variable x. We define two sets of flows, denoted by the times t_q and $\bar{t}_q$ with $q > 0$, by the following Lax equations

$$L_{t_q} = [(L^q)_+, L] \qquad \bar{L}_{t_q} = [(L^q)_+, \bar{L}] \tag{4.3}$$

and

$$L_{\bar{t}_q} = [-(\bar{L}^q)_-, L] \qquad \bar{L}_{\bar{t}_q} = [-(\bar{L}^q)_-, \bar{L}]. \tag{4.4}$$

All the commutators are well-defined since the operators $(L^p)_+$ and $(\bar{L}^q)_-$ are of bounded order; the following obvious observation is useful when trying to find the R-matrix formulation of the hierarchy.

Remark 58 *In the equation (4.3a) (an analogous observation holds for the equation (4.4b)) we can equivalently use $-(L^q)_-$ instead of $(L^q)_+$ since L^q commutes with L;*

however in the equation (4.3b) (and in (4.4a) analogously) we must use $(L^q)_+$, otherwise the commutator is not well-defined.

These equations are compatible: this follows from the zero-curvature or Zakharov-Shabat representation given by the following

Proposition 59 ([47]) *The Toda lattice hierarchy (4.3), (4.4) is equivalent to the system of equations*

$$\partial_{t_p}(L^q)_+ - \partial_{t_q}(L^p)_+ + [(L^q)_+, (L^p)_+] = 0 \tag{4.5}$$

$$\partial_{\bar{t}_p}(\bar{L}^q)_- - \partial_{\bar{t}_q}(\bar{L}^p)_- - [(\bar{L}^q)_-, (\bar{L}^p)_-] = 0 \tag{4.6}$$

$$\partial_{\bar{t}_p}(L^q)_+ + \partial_{t_q}(\bar{L}^p)_- - [(L^q)_+, (\bar{L}^p)_-] = 0. \tag{4.7}$$

4.2 R-matrix formulation

In this section we first define an algebra of difference operators which is naturally associated with the two-dimensional Toda hierarchy. Then we introduce an R-matrix and we show that it comes from a non trivial splitting of the algebra. We check that the skew-symmetric part of the R-matrix satisfies the modified Yang-Baxter equation and thus defines two compatible Poisson structures.

Since the two-dimensional Toda hierarchy is characterized by two Lax operators L and $\bar{L}$ that are respectively elements of A^+ and A^-, it is natural to consider $A^+ \oplus A^-$ as the correct algebra in this case.

The natural inner product on $A^+ \oplus A^-$ is defined in the obvious way from the trace form

$$\operatorname{Tr} X \oplus \bar{X} = \operatorname{Tr} X + \operatorname{Tr} \bar{X} \tag{4.8}$$

for $X \oplus \bar{X} \in A^+ \oplus A^-$.

We can guess the form of the R-matrix (a linear operator on $A^+ \oplus A^-$) by comparing equations (4.3)-(4.4) with (3.39); we obtain

$$R(X, \bar{X}) = (X_+ - X_- + 2\bar{X}_-, \bar{X}_- - \bar{X}_+ + 2X_+) \tag{4.9}$$

where $(X, \bar{X}) \in A^+ \oplus A^-$.

We emphasize that this R-matrix is not simply given by a direct sum of the previously considered R-matrices on A^+ and A^-, since equations (4.3b) and (4.4a) give a coupling between L and $\bar{L}$, that must be taken into account.

The operator R satisfies the modified Yang-Baxter equation (3.22) since it is given by a splitting of the Lie algebra $A^+ \oplus A^-$; indeed we can write

$$R = P - \tilde{P} \tag{4.10}$$

where P and $\tilde{P}$, defined by

$$P(X, \bar{X}) = (X_+ + \bar{X}_-, X_+ + \bar{X}_-) \qquad \tilde{P}(X, \bar{X}) = (X_- - \bar{X}_-, \bar{X}_+ - X_+), \tag{4.11}$$

are projections operators, i.e. $P^2 = P$, $\tilde{P}^2 = \tilde{P}$, $\tilde{P}P = 0 = P\tilde{P}$ and $P + \tilde{P} = Id$. Thus the splitting associated to the R-matrix above is

$$A^+ \oplus A^- = \Big(\mathrm{diag}(A^0 \oplus A^0)\Big) \oplus \Big((A^+)_- \oplus (A^-)_+\Big). \tag{4.12}$$

As in the "one-dimensional" case (given by the algebra A^+ with the splitting $(A^+)_+ \oplus (A^*)_-$) the splitting is not isotropic with respect to the natural inner product; indeed the R matrix is not skew-symmetric since the adjoint R^* is given by

$$R^*(X, \bar{X}) = (X_{\leqslant 0} - X_{>0} + 2\bar{X}_{\leqslant 0}, \bar{X}_{>0} - \bar{X}_{\leqslant 0} + 2X_{>0}) = P^* - \tilde{P}^* \tag{4.13}$$

where the dual projections are

$$P^*(X, \bar{X}) = (X_{\leqslant 0} + \bar{X}_{\leqslant 0}, X_{>0} + \bar{X}_{>0}) \qquad \tilde{P}^*(X, \bar{X}) = (X_{>0} - \bar{X}_{\leqslant 0}, \bar{X}_{\leqslant 0} - X_{>0}). \tag{4.14}$$

Moreover one can verify[1] that

Proposition 60 *The skew-symmetric part A of the R-matrix (4.9)*

$$A(X, \bar{X}) = (X_{>0} - X_{<0} - \bar{X}_0, \bar{X}_{<0} - \bar{X}_{>0} + X_0) \tag{4.15}$$

satisfies the modified Yang-Baxter equation (3.22).

Proof It is checked by substitution in equations (3.21) and (3.22). □

Thus, by the previous general theorems, there are on $A^+ \oplus A^-$ three compatible Hamiltonian structures (3.43). We will consider only the first two structures since the third one doesn't behave well under restriction to a subspace. We summarize this result and the explicit form of the Poisson tensors in the following

Proposition 61 *On the Lie algebra $A^+ \oplus A^-$ there are two compatible Poisson structures given by*

$$\begin{aligned} P_1(L, \bar{L})X \oplus \bar{X} = \Big(&2[L, X_- - \bar{X}_-] - 2([L, X] + [\bar{L}, \bar{X}])_{\leqslant 0}, \\ &2[\bar{L}, \bar{X}_+ - X_+] - 2([L, X] + [\bar{L}, \bar{X}])_{>0}\Big) \end{aligned} \tag{4.18a}$$

$$\begin{aligned} P_2(L, \bar{L})X \oplus \bar{X} = \Big(&2[L, (LX + XL)_- - (\bar{L}\bar{X} + \bar{X}\bar{L})_-] \\ &- 2L([L, X]_{\leqslant 0} + [\bar{L}, \bar{X}]_{\leqslant 0}) - 2([L, X]_{\leqslant 0} + [\bar{L}, \bar{X}]_{\leqslant 0})L, \\ &2[\bar{L}, (\bar{L}\bar{X} + \bar{X}\bar{L})_+ - (LX + XL)_+] \\ &- 2\bar{L}([L, X]_{>0} + [\bar{L}, \bar{X}]_{>0}) - 2([L, X]_{>0} + [\bar{L}, \bar{X}]_{>0})\bar{L}\Big). \end{aligned} \tag{4.18b}$$

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[1] More generally one can prove that

$$R(X \oplus \bar{X}) = (X_+ - X_- - 2a\bar{X}_-, -a(\bar{X}_- - \bar{X}_+) + 2X_+) \tag{4.16}$$

satisfies the modified Yang-Baxter equation for $a = \pm 1$; however, given the adjoint

$$R^*(X \oplus \bar{X}) = (X_{\leqslant 0} - X_{>0} + 2\bar{X}_{\leqslant 0}, -2aX_{>0} + a\bar{X}_{\leqslant 0} - a\bar{X}_{>0}) \tag{4.17}$$

the skew-symmetric part A satisfies the modified Yang-Baxter equation only for $a = -1$.

4.3 Explicit form of Poisson brackets

Here we calculate the explicit form of the Poisson brackets above in the usual notation. We have not reduced the brackets yet.

Let N, M be two positive integers. The explicit form of the first Poisson structure for two-dimensional Toda calculated in $(L,\bar{L})$ for $L=\sum_{k\leqslant N}u_k\Lambda^k$ and $\bar{L}=\sum_{l\geqslant -M}\bar{u}_l\Lambda^l$ is

First bracket

$$\{u_n(x),u_m(y)\}_1(L,\bar{L})=2(c(m)+c(n)-1)[u_{n+m}(\Lambda^n\delta(x-y))-(\Lambda^{-m}u_{n+m}\delta(x-y))] \tag{4.19a}$$

$$\begin{aligned}\{u_n(x),\bar{u}_m(y)\}_1(L,\bar{L})&=2c(m)[(\Lambda^{-m}u_{m+n}\delta(x-y))-u_{m+n}(\Lambda^n\delta(x-y))]\\&+2(1-c(n))[(\Lambda^{-m}\bar{u}_{m+n}\delta(x-y))-\bar{u}_{m+n}(\Lambda^n\delta(x-y))]\end{aligned} \tag{4.19b}$$

$$\{\bar{u}_n(x),\bar{u}_m(y)\}_1(L,\bar{L})=2(1-c(n)-c(m))[\bar{u}_{m+n}(\Lambda^n\delta(x-y))-(\Lambda^{-m}\bar{u}_{m+n}\delta(x-y))] \tag{4.19c}$$

where the constant $c(n)$ is defined by

$$c(n)=\begin{cases}1 & n>0\\ 0 & n\leqslant 0.\end{cases} \tag{4.20}$$

The second Poisson structure calculated in $(L,\bar{L})$ for $L=\sum_{k\leqslant N}u_k\Lambda^k$ and $\bar{L}=\sum_{l\geqslant -M}\bar{u}_l\Lambda^l$ is given by

Second bracket

$$\begin{aligned}\{u_n(x),u_m(y)\}_2(L,\bar{L})&=-2u_n(\Lambda^n u_m\delta(x-y))+2u_n(\Lambda^{-m}u_m\delta(x-y))\\&+2u_n(\Lambda^{n-m}u_m\delta(x-y))-2u_mu_n\delta(x-y)\\&+\sum_{l<m}[4u_{n+m-l}(\Lambda^{n-l}u_l\delta(x-y))-4u_l(\Lambda^{l-m}u_{n+m-l}\delta(x-y))]\end{aligned} \tag{4.21a}$$

$$\begin{aligned}\{u_n(x),\bar{u}_m(y)\}_2(L,\bar{L})&=2u_n(\Lambda^{n-m}\bar{u}_m\delta(x-y))-2\bar{u}_mu_n\delta(x-y)\\&-2u_n(\Lambda^n\bar{u}_m\delta(x-y))+2u_n(\Lambda^{-m}\bar{u}_m\delta(x-y))\\&+\sum_{l<m}[4(\Lambda^{l-m}u_{n+m-l})(\Lambda^{-m}\bar{u}_l\delta(x-y))-4u_{n+m-l}(\Lambda^{n+m-l}\bar{u}_l)(\Lambda^n\delta(x-y))]\end{aligned} \tag{4.21b}$$

$$\begin{aligned}\{\bar{u}_n(x),\bar{u}_m(y)\}_2(L,\bar{L})&=2\bar{u}_n(\Lambda^n\bar{u}_m\delta(x-y))-2\bar{u}_n(\Lambda^{-m}\bar{u}_m\delta(x-y))\\&+2\bar{u}_n(\Lambda^{n-m}\bar{u}_m\delta(x-y))-2\bar{u}_m\bar{u}_n\delta(x-y)\\&+\sum_{k>m}[4\bar{u}_{n+m-k}(\Lambda^{n-k}\bar{u}_k\delta(x-y))-4\bar{u}_k(\Lambda^{k-m}\bar{u}_{n+m-k}\delta(x-y))].\end{aligned} \tag{4.21c}$$

4.4 Reductions

In this section we calculate the reduction of the Poisson brackets to affine subspaces of operators of the form

$$L = \Lambda^N + u_{N-1}\Lambda^{N-1} + \dots \qquad \bar{L} = \bar{u}_{-M}\Lambda^{-M} + \bar{u}_{-M+1}\Lambda^{-M+1} + \dots \tag{4.22}$$

hence obtaining a pencil of Poisson brackets for each of these bigraded two-dimensional Toda hierarchies. In particular for $N = M = 1$ we obtain the bihamiltonian structure for the usual two-dimensional Toda hierarchy.

As in the "one dimensional" case the reduction to the linear subspaces given by operators of the form

$$L = \sum_{k\leqslant N} u_k\Lambda^k \qquad \text{and} \qquad \bar{L} = \sum_{l\geqslant -M} \bar{u}_l\Lambda^l \tag{4.23}$$

is trivial.

Let's perform the reduction to the affine subspace given by $u_N = 1$; using Lemma 35 we obtain that the reduced second Poisson tensor in this case is given by

$$\begin{aligned} P^{rid}(\tilde{X}\oplus\bar{X}) = \Big(&2[L,(L\tilde{X}+\tilde{X}L)_- - (\bar{L}\bar{X}+\bar{X}\bar{L})_-] \\ &- 2L[L,\tilde{X}]_{\leqslant 0} - 2L[\bar{L},\bar{X}]_{\leqslant 0} - 2[L,\tilde{X}]_{\leqslant 0}L - 2[\bar{L},\bar{X}]_{\leqslant 0}L \\ &- 2[L,(\Lambda^{-N}+1)(\Lambda^N+1)(\Lambda^{-N}-\Lambda^N)^{-1}([L,\tilde{X}]_0+[\bar{L},\bar{X}]_0)], \\ &2[\bar{L},(\bar{L}\bar{X}+\bar{X}\bar{L})_+ - (L\tilde{X}+\tilde{X}L)_+] \\ &- 2\bar{L}([L,\tilde{X}]_{>0}+[\bar{L},\bar{X}]_{>0}) - 2([L,\tilde{X}]_{>0}+[\bar{L},\bar{X}]_{>0})\bar{L} \\ &- 2[\bar{L},((\Lambda^{-N}+1)(\Lambda^N+1)(\Lambda^{-N}-\Lambda^N)^{-1}([L,\tilde{X}]_0+[\bar{L},\bar{X}]_0))];\Big) \end{aligned} \tag{4.24}$$

the first Poisson structure does not need any correction term and its explicit form is simply obtained by putting $u_N = 1$ in (4.19).

The explicit form of the second reduced Poisson bracket is

Second bracket

$$\{u_n(x), u_m(y)\}_2(L,\bar{L}) = -2u_n(\Lambda^n u_m \delta(x-y)) + 2u_n(\Lambda^{-m} u_m \delta(x-y)) + 2u_n(\Lambda^{n-m} u_m \delta(x-y)) - 2u_m u_n \delta(x-y) + \sum_{l<m} [4u_{n+m-l}(\Lambda^{n-l} u_l \delta(x-y)) - 4u_l(\Lambda^{l-m} u_{n+m-l} \delta(x-y))] + 2u_n(\Lambda^N+1)(\Lambda^{-N}+1)(\Lambda^{-N}-\Lambda^N)^{-1}(\Lambda^n-1)(\Lambda^{-m}-1)u_m \delta(x-y) \tag{4.25a}$$

$$\{u_n(x), \bar{u}_m(y)\}_2(L,\bar{L}) = 2u_n(\Lambda^{n-m} \bar{u}_m \delta(x-y)) - 2u_n \bar{u}_m \delta(x-y) - 2u_n(\Lambda^n \bar{u}_m \delta(x-y)) + 2u_n(\Lambda^{-m} \bar{u}_m \delta(x-y)) + \sum_{l<m} [4(\Lambda^{l-m} u_{n+m-l})(\Lambda^{-m} \bar{u}_l \delta(x-y)) - 4u_{n+m-l}(\Lambda^{n+m-l} \bar{u}_l)(\Lambda^n \delta(x-y))] + 2u_n(\Lambda^N+1)(\Lambda^{-N}+1)(\Lambda^{-N}-\Lambda^N)^{-1}(\Lambda^n-1)(\Lambda^{-m}-1)\bar{u}_m \delta(x-y) \tag{4.25b}$$

$$\{\bar{u}_n(x), \bar{u}_m(y)\}_2(L,\bar{L}) = 2\bar{u}_n(\Lambda^n \bar{u}_m \delta(x-y)) - 2\bar{u}_n(\Lambda^{-m} \bar{u}_m \delta(x-y)) + 2\bar{u}_n(\Lambda^{n-m} \bar{u}_m \delta(x-y)) - 2\bar{u}_m \bar{u}_n \delta(x-y) + \sum_{k>m} [4\bar{u}_{n+m-k}(\Lambda^{n-k} \bar{u}_k \delta(x-y)) - 4\bar{u}_k(\Lambda^{k-m} \bar{u}_{n+m-k} \delta(x-y))] + 2\bar{u}_n(\Lambda^N+1)(\Lambda^{-N}+1)(\Lambda^{-N}-\Lambda^N)^{-1}(\Lambda^n-1)(\Lambda^{-m}-1)\bar{u}_m \delta(x-y) \tag{4.25c}$$

Remark 62 *The second Poisson brackets can be also rewritten in the following form*

$$\{u_n(x), u_m(y)\}_2 = 4\sum_{l=1}^{N-n} [u_{n+l}(\Lambda^{n-m+l} u_{m-l} \delta(x-y)) - u_{m-l}(\Lambda^{-l} u_{n+l} \delta(x-y))] + 4u_n(\Lambda^{-m}-1)(1-\Lambda^{n-N})(1-\Lambda^{-N})^{-1} u_m \delta(x-y) \tag{4.26a}$$

$$\{u_n(x), \bar{u}_m(y)\}_2 = +4u_n(\Lambda^{-m}-1)(1-\Lambda^{n-N})(1-\Lambda^{-N})^{-1} \bar{u}_m \delta(x-y) \tag{4.26b}$$

$$+4\sum_{k=1}^{\min(M+m,N-n)} [(\Lambda^{-k} u_{n+k})(\Lambda^{-m} \bar{u}_{m-k} \delta(x-y)) - u_{n+k}(\Lambda^{n+k} \bar{u}_{m-k})\Lambda^n \delta(x-y)]$$

$$\{\bar{u}_n(x), \bar{u}_m(y)\}_2 = 4\sum_{l=1}^{n+M} [\bar{u}_{n-l}\Lambda^{n-m-l} \bar{u}_{l+m} \delta(x-y) - \bar{u}_{l+m}\Lambda^l \bar{u}_{n-l} \delta(x-y)] + 4\bar{u}_n(\Lambda^n-1)(1-\Lambda^{-m-N})(1-\Lambda^{-N})^{-1} \bar{u}_m \delta(x-y) \tag{4.26c}$$

We summarize the results in the following

Theorem 63 *The brackets (4.19) and (4.25) give two compatible Poisson structures in the variables u_n for $n < N$ and $\bar{u}_m$ for $m \geqslant -M$.*

We will call these *Poisson brackets for the (M,N)-bigraded two-dimensional Toda hierarchy*. In particular for $N = M = 1$ we should obtain the bihamiltonian structure for the usual two-dimensional Toda. Notice however that to have a complete description of the bihamiltonian structure of the hierarchy we should relate the Hamiltonian flows with the Lax pair definition given above and in particular we should obtain the recursion relation for the Hamiltonians.

Remark 64 *The definition of logarithm makes sense even in the two-dimensional Toda case; the dressing operators are defined by*

$$L = P\Lambda^N P^{-1} \qquad \bar{L} = Q\Lambda^{-M} Q^{-1} \tag{4.27}$$

and the two logarithms by

$$\log L = N\epsilon P\partial P^{-1} \qquad \log \bar{L} = -M\epsilon Q\partial Q^{-1}. \tag{4.28}$$

In this case however we cannot use the same trick as before to obtain a logarithm that is a difference operator like in (3.8). So we cannot define additional logarithmic flows as before.

4.5 Hamiltonian representation

We obtain now the Hamiltonian representation of the flows defined in (4.3) and (4.4). On the algebra $A^+ \oplus A^-$ we can define the functions

$$h_p = \frac{1}{2}\frac{1}{p+1}\int dx \ \operatorname{Res} L^{p+1} \qquad \tilde{h}_p = \frac{1}{2}\frac{1}{p+1}\int dx \ \operatorname{Res} \ \bar{L}^{p+1} \tag{4.29}$$

where $(L, \bar{L})$ is a point in $A^+ \oplus A^-$. We clearly have

$$dh_p = (\frac{1}{2}L^p, 0) \qquad d\tilde{h}_p = (0, \frac{1}{2}\bar{L}^p). \tag{4.30}$$

From the R-matrix construction, since h_p and $\bar{h}_p$ are invariant functions on the algebra, we have that (3.48) gives

$$\frac{\partial \hat{L}}{\partial t_p} = [Rdh_p, \hat{L}] = P_1 dh_p \tag{4.31a}$$

$$\frac{\partial \hat{L}}{\partial \bar{t}_p} = [Rd\tilde{h}_p, \hat{L}] = P_1 d\tilde{h}_p \tag{4.31b}$$

where $\hat{L} = (L, \bar{L})$. Spelling out this relations, using the R-matrix (4.9), one obtains the Hamiltonian formulation of the flows (4.3) and (4.4). These formulas continue to hold when we restrict to an affine subspace. Hence we obtain

Theorem 65 *The flows (4.3) and (4.4) admit the Hamiltonian formulation*

$$\frac{\partial}{\partial t_p}\cdot = \{\cdot, h_p\}_1 \qquad \frac{\partial}{\partial \bar{t}_p}\cdot = \{\cdot, \tilde{h}_p\}_1 \tag{4.32}$$

where the Hamiltonians are defined by (4.29).

Chapter 5

Dispersionless limit

In this chapter we consider the dispersionless limit of the bigraded Toda and of the two-dimensional Toda hierarchies.

The process of taking the dispersionless limit $\epsilon \to 0$ of a bihamiltonian system is the simplest instance of a general procedure of averaging that associates to the dispersive brackets and Hamiltonians their averaged counterparts. In general these averaged systems are expected to be in correspondence with Frobenius manifolds; moreover in the simplest case of the $\epsilon \to 0$ limit the reconstruction of the whole dispersive hierarchy from the associated Frobenius manifold has been developed in [19].

After having obtained the form of the dispersionless brackets and of the associated metrics we will derive their generating functions. In the case of the bigraded Toda we will show that the associated Frobenius manifold is given by a Hurwitz space of meromorphic functions with two poles. In the case of the two-dimensional Toda hierarchy we will show that the first metric is non-degenerate and that the first bracket is given by the direct sum of the first bracket for the bigraded Toda plus a bracket associated to the algebra of divergence-free vector fields on the cylinder.

5.1 Dispersionless bigraded Toda hierarchy

In this section we consider the dispersionless limit of the first and second Poisson brackets of the bigraded Toda hierarchy. Since these are Poisson brackets of hydrodynamic type, they are naturally associated to a flat pencil of contravariant metrics. We write down the explicit form of the metric and the associated Christoffel symbols. Moreover we obtain the generating functions associated to these quantities. Finally we show that this hierarchy is associated to the Frobenius manifold given by the Hurwitz space of meromorphic functions on the Riemann sphere with two poles.

5.1.1 Poisson brackets

The dispersionless brackets $\{,\}_i^{disp}$ are obtained as the leading term of the dispersive brackets (equations (3.95) and (3.96)) in the $\epsilon \to 0$ limit, i.e.

$$\{u_n(x), u_m(y)\}_i = \epsilon\{u_n(x), u_m(y)\}_i^{disp} + O(\epsilon^2). \tag{5.1}$$

In the case of the bigraded Toda hierarchy the variables are $u_{N-1}, \dots, u_{-M}$ with $u_N = 1$ and $N, M > 0$. The explicit form of the brackets is

First bracket

$$\{u_n(x), u_m(y)\}_1^{disp} = 2(c(n)+c(m)-1)\left[(n+m)u_{n+m}\delta'(x-y)+mu'_{n+m}\delta(x-y)\right] \tag{5.2}$$

Second bracket

$$\begin{aligned}\{u_n(x), u_m(y)\}_2^{disp} &= -4mu_nu_m\delta'(x-y) - 4mu_nu'_m\delta(x-y)\\ &+ 4\sum_{l<m}\big[(n+m-2l)u_{n+m-l}u_l\delta'(x-y)\\ &\qquad + (n-l)u_{n+m-l}u'_l\delta(x-y) + (m-l)u'_{n+m-l}u_l\delta(x-y)\big]\\ &+ \frac{4}{N}nmu_nu_m\delta'(x-y) + \frac{4}{N}nmu_nu'_m\delta(x-y).\end{aligned} \tag{5.3}$$

The last line in (5.3) is given by the correction term obtained by Dirac reduction. In these equations, on the LHS the indices span the range $-M \leqslant n, m \leqslant N-1$, while on the RHS it is understood that $u_k = 0$ for $k > N$ or $k < -M$, and $u_N = 1$. We recall that here and in the following the costant $c(n)$ is defined to be 1 for $n > 0$ and 0 otherwise.

5.1.2 Associated metrics and Christoffel symbols

The dispersionless Poisson brackets (5.2) and (5.3) are of hydrodynamic type, i.e. they are of the form

$$\{u_n(x), u_m(y)\}_1^{disp} = g^{nm}\delta'(x-y) + \Gamma_k^{nm}u'_k\delta(x-y), \tag{5.4a}$$

$$\{u_n(x), u_m(y)\}_2^{disp} = \tilde{g}^{nm}\delta'(x-y) + \tilde{\Gamma}_k^{nm}u'_k\delta(x-y). \tag{5.4b}$$

It is a well-known result of Dubrovin and Novikov (see [15]) that such brackets satisfy the Jacobi identity if and only if the coefficients g^{nm} define a flat contravariant metric and the coefficients $\Gamma_{ij}^k := -g_{in}\Gamma_j^{nk}$ are the Christoffel symbols of the Levi-Civita connection associated with g_{ij}

$$\Gamma_{ij}^k = \frac{1}{2}g^{kl}(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij}) \tag{5.5}$$

where $\partial_i := \frac{\partial}{\partial u_i}$.

For reference we write the explicit forms of the metric and the Christoffel symbols for the first bracket

$$g^{nm} = 2(c(n) + c(m) - 1)(n + m)u_{n+m} \tag{5.6}$$
$$\Gamma^{nm}_{k} = 2(c(n) + c(m) - 1)m\delta_{k,n+m} \tag{5.7}$$

and for the second bracket

$$\tilde{g}^{nm} = 4(\frac{n}{N} - 1)mu_n u_m + 4\sum_{l<m}(n + m - 2l)u_l u_{n+m-l} \tag{5.8}$$
$$\tilde{\Gamma}^{nm}_{k} = 4(\frac{n}{N} - 1)mu_n\delta_{k,m} + 4\sum_{l<m}\Big[(n - l)u_{n+m-l}\delta_{k,l} + (m - l)u_l\delta_{k,n+m-l}\Big]. \tag{5.9}$$

In these equations hold the same considerations on the ranges of the indices that were given after (5.3).

5.1.3 Generating functions for the Poisson brackets

Here we want to obtain the generating functions for the dispersionless brackets (5.2) and (5.3).

Let's define the function

$$\lambda(p, x) = p^N + u_{N-1}p^{N-1} + \cdots + u_{-M}p^{-M}. \tag{5.10}$$

The generating function for the bracket $\{,\}^{disp}_i$ is an expression $\{\lambda(p, x), \lambda(q, y)\}^{disp}_i$ such that

$$\{\lambda(p, x), \lambda(q, y)\}^{disp}_i = \sum_{n,m}\{u_n(x), u_m(y)\}^{disp}_i p^n q^m, \tag{5.11}$$

at least for the powers of p and q for which the RHS is defined.

Let's start from the first bracket. Using the relation $f(y)\delta'(x-y) = f(x)\delta'(x-y) + f'(x)\delta(x-y)$ we can rewrite (5.2) in the form

$$\{u_n(x), u_m(y)\}^{disp}_1 = 2(c(n)+c(m)-1)\big[nu_{n+m}(x)\delta'(x-y)+mu_{n+m}(y)\delta'(x-y)\big]; \tag{5.12}$$

substituting in (5.11) one finds

$$\{\lambda(p, x), \lambda(q, y)\}^{disp}_1 = 2p\frac{\partial}{\partial p}\left(\sum_{n,m}(c(n) + c(m) - 1)u_{n+m}(x)p^n q^m\right)\delta'(x - y)+$$
$$+ 2q\frac{\partial}{\partial q}\left(\sum_{n,m}(c(n) + c(m) - 1)u_{n+m}(y)p^n q^m\right)\delta'(x - y) \tag{5.13}$$

and, using the identity

$$\sum_{n,m}(c(n) + c(m) - 1)u_{n+m}(x)p^n q^m = \frac{\lambda(q, x)q^{-1} - \lambda(p, x)p^{-1}}{p^{-1} - q^{-1}}, \tag{5.14}$$

finally obtains the desired generating function

Generating function for the first bracket

$$\{\lambda(p,x),\lambda(q,y)\}_1^{disp} = 2p\frac{\partial}{\partial p}\left(\frac{\lambda(q,x)q^{-1}-\lambda(p,x)p^{-1}}{p^{-1}-q^{-1}}\right)\delta'(x-y)+ \\ +2q\frac{\partial}{\partial q}\left(\frac{\lambda(q,y)q^{-1}-\lambda(p,y)p^{-1}}{p^{-1}-q^{-1}}\right)\delta'(x-y). \quad (5.15)$$

For the second dispersionless Poisson bracket we first consider the preliminary case where u_N is not yet fixed to 1. The bracket can be rewritten in the simpler form

$$\{u_n(x),u_m(y)\}_2^{disp} = -4mu_n(x)u_m(y)\delta'(x-y)+ \\ +4\sum_{l<m}\left[(n-l)u_{n+m-l}(x)u_l(y)+(m-l)u_l(x)u_{n+m-l}(y)\right]\delta'(x-y). \quad (5.16)$$

Then one has to substitute this in (5.11) and multiply both sides by $(p^{-1}-q^{-1})^2$; one then expresses the RHS in terms of products of λ and derivatives; this calculation is quite involved and we don't report it here. Eventually we have to add the term due to Dirac reduction to the affine subspace with $u_N=1$ that is given by

$$\frac{4}{N}pq\frac{\partial}{\partial p}\lambda(p,x)\frac{\partial}{\partial q}\lambda(q,y); \quad (5.17)$$

finally we obtain the

Generating function for the second bracket

$$\{\lambda(p,x),\lambda(q,y)\}_2^{disp} = \frac{4}{N}pq\frac{\partial}{\partial p}\lambda(p,x)\frac{\partial}{\partial q}\lambda(q,y)\delta'(x-y)+ \\ +\frac{4}{p^{-1}-q^{-1}}\left(\lambda(p,x)\frac{\partial}{\partial q}\lambda(q,y)-\lambda(q,y)\frac{\partial}{\partial p}\lambda(p,x)\right)\delta'(x-y)+ \\ +\frac{4p^{-1}q^{-1}}{(p^{-1}-q^{-1})^2}\left(\lambda(p,y)\lambda(q,x)-\lambda(p,x)\lambda(q,y)\right)\delta'(x-y). \quad (5.18)$$

Remark 66 *As we already observed, the first bracket can be obtained as the linear part of the second one. In particular the following relation holds between the generating functions*

$$\frac{1}{2}\{\lambda(p,x),\lambda(q,y)\}_2^{disp}\Big|_{\lambda\to\lambda+\varepsilon} = \frac{1}{2}\{\lambda(p,x),\lambda(q,y)\}_2^{disp}+\varepsilon\{\lambda(p,x),\lambda(q,y)\}_1^{disp}. \quad (5.19)$$

This suggests to write the generating function for the linear bracket as

$$\{\lambda(p,x),\lambda(q,y)\}_1^{disp} = \frac{2}{p^{-1}-q^{-1}}\left(\frac{\partial}{\partial q}\lambda(q,y)-\frac{\partial}{\partial p}\lambda(p,x)\right)\delta'(x-y)+ \\ +\frac{2p^{-1}q^{-1}}{(p^{-1}-q^{-1})^2}\left(\lambda(q,x)+\lambda(p,y)-\lambda(p,x)-\lambda(q,y)\right)\delta'(x-y). \quad (5.20)$$

5.1.4 Generating functions for the metrics

Using the generating functions for the Poisson brackets here we derive the generating functions for the associated contravariant metrics.

The bilinear forms $(,)_i$ associated to the metrics g^{nm} and $\tilde{g}^{nm}$ are extended on differentials $d\lambda(p)$ of the form

$$d\lambda(p) = du_{N-1}p^{N-1} + \cdots + du_{-M}p^{-M} \tag{5.21}$$

simply by

$$(d\lambda(p), d\lambda(q))_1 = \sum_{nm} g^{nm} p^n q^m, \qquad (d\lambda(p), d\lambda(q))_2 = \sum_{nm} \tilde{g}^{nm} p^n q^m. \tag{5.22}$$

Then, from (5.18) and (5.20), we easily get the expressions

$$(d\lambda(p), d\lambda(q))_1 = 2\frac{\lambda'(q) - \lambda'(p)}{p^{-1} - q^{-1}} \tag{5.23}$$

and

$$(d\lambda(p), d\lambda(q))_2 = \frac{4}{N} pq\lambda'(p)\lambda'(q) + \frac{4}{p^{-1} - q^{-1}} \left(\lambda(p)\lambda'(q) - \lambda(q)\lambda'(p)\right). \tag{5.24}$$

Remark 67 *The formula (5.23) is an analogue of the formula of Saito, Yano and Sekiguchi [41] that provides an invariant quadratic form on a space of polynomials associated to finite Coxeter groups.*

5.1.5 Hurwitz spaces and Frobenius manifold associated to the dispersionless bigraded Toda hierarchy

After briefly recalling the definition of the Hurwitz spaces and of their coverings we show that the pencil of metrics defining the Frobenius structure of such spaces coincides with the one obtained from the dispersionless limit of the Poisson pencil of the bigraded Toda hierarchy.

The Hurwitz spaces are moduli spaces of Riemann surfaces C of genus g with an $n+1$ branched covering λ of $\mathbb{C}\mathrm{P}^1$ with fixed ramification type over $\infty \in \mathbb{C}\mathrm{P}^1$. More precisely a point in the Hurwitz space $M_{g;n_0,\ldots,n_m}$ is given by an equivalence class of pairs (C, λ) where C is a compact Riemann surface of genus g and $\lambda : C \to \mathbb{C}\mathrm{P}^1$ a meromorphic function of degree $n+1$ such that the degrees of the ramification at the points $\infty_0, \ldots, \infty_m \in C$ over the point at infinity $\infty \in \mathbb{C}\mathrm{P}^1$ are respectively $n_0+1, \ldots, n_m+1$. Two pairs (C, λ) and $(\tilde{C}, \tilde{\lambda})$ are identified if there exists an analytic isomorphism $\theta : C \to \tilde{C}$ such that $\lambda \circ \theta = \tilde{\lambda}$.

In [10] it is shown how to construct a Frobenius manifold structure on a covering of $M_{g;n_0,\ldots,n_m}$ corresponding to a fixation of a symplectic basis of cycles in the first homology group of C and to a choice of primary differential dp. Factorization by the group of changes of the basis gives a twisted Frobenius manifold structure on the Hurwitz space.

For the general construction of the Frobenius manifold structure on the Hurwitz space see Lecture 5 in [10]. Here we will simply consider the Hurwitz space corresponding to the case of the bigraded Toda hierarchy.

Consider the case with $g = 0$, $m = 1$, $n_0 = M - 1$ and $n_1 = N - 1$ for two positive integers N, M. The (covering of the) corresponding Hurwitz space is given by the space of functions

$$\lambda(z) = z^N + u_{N-1}z^{N-1} + \cdots + u_{-M}z^{-M} \tag{5.25}$$

with $u_{-M} \neq 0$ and $z \in \mathbb{C}$ and the primary differential is defined by $dp = \frac{dz}{z}$.

As is well known from [13], the Frobenius structure on a manifold is uniquely specified by a flat pencil of metrics. From the general construction of the Frobenius manifold on a Hurwitz space we have that the corrisponding flat pencil of metrics is given by

$$(\partial', \partial'')_1 = \frac{1}{2} \sum_{|\lambda|<\infty} \mathrm{Res}_{d\lambda=0} \frac{\partial'(\lambda dz)\partial''(\lambda dz)}{z^2 d\lambda} \tag{5.26}$$

and by

$$(\partial', \partial'')_2 = \frac{1}{4} \sum_{|\lambda|<\infty} \mathrm{Res}_{d\lambda=0} \frac{\partial'(\log\lambda\ dz)\partial''(\log\lambda\ dz)}{z^2 d\log\lambda}. \tag{5.27}$$

Another pencil of flat metrics has been defined on a space with coordinates $u_{N-1}, \ldots, u_{-M}$ by the dispersionless limit of the bihamiltonian structure of the bigraded Toda hierarchy considered previously.

We now show that these two pencils coincide.

Proposition 68 *The (covariant) metrics (5.26) and (5.27) are the inverse of the (contravariant) metrics (5.6) and (5.8).*

Proof Let's start from the first metric. We want to show that

$$\sum_{m=-M}^{N-1} g^{nm} g_{mk} = \delta_{nk} \tag{5.28}$$

where g^{nm} is given by (5.6) and, from (5.26)

$$g_{mk} = \frac{1}{2} \sum_{|\lambda|<\infty} \mathrm{Res}_{\lambda_z=0} \frac{z^{m+k-1}}{z\lambda_z} dz. \tag{5.29}$$

Now multiply (5.28) by w^n, sum on $-M \leqslant n \leqslant N-1$ and then the generating function (5.23); then we have to show that

$$\sum_{|\lambda|<\infty} \mathrm{Res}_{\lambda_z=0} \frac{(\lambda_w(w) - \lambda_z(z))z^{k-1}}{(z^{-1} - w^{-1})z\lambda_z} dz = w^k. \tag{5.30}$$

In the LHS the term $\lambda_z(z)$ in the numerator doesn't contribute, since it cancels with the denominator and gives a function without poles in $\lambda_z = 0$. Hence the LHS is given by the following sum of three residues

$$(\mathrm{Res}_{z=0} + \mathrm{Res}_{z=\infty} + \mathrm{Res}_{z=w}) \frac{w\lambda_w(w) z^k}{(z-w)z\lambda_z} dz. \tag{5.31}$$

The residue in $z = w$ gives exactly the desired result w^k, while it is easy to show, using the fact that $-M \leqslant k \leqslant N-1$, that the other two residues are 0.

The analogous result for the second metric is obtained in the same way: multiplying the product of the two metrics as before by w^n and summing on n, one needs to show, after substitution of the generating function (5.24), that

$$\sum_{|\lambda|<\infty} \mathrm{Res}_{\lambda_z=0} \frac{w\lambda_w(w) z^{k-1}}{N\lambda} dz - \sum_{|\lambda|<\infty} \mathrm{Res}_{\lambda_z=0} \frac{\lambda(w) z^{k-1}}{(z^{-1}-w^{-1})z\lambda} dz + \\ + \sum_{|\lambda|<\infty} \mathrm{Res}_{\lambda_z=0} \frac{\lambda_w(w) z^{k-1}}{(z^{-1}-w^{-1})z\lambda_z} dz = w^k. \tag{5.32}$$

The first two terms on the LHS vanish since they don't have poles in $\lambda_z = 0$. The third term gives exactly the same sum of residues (5.31) as before. □

From this result it actually follows that the Frobenius manifold associated to the dispersionless limit of the bigraded Toda hierarchy is given by the Hurwitz space $M_{0;M-1,N-1}$.

An observation based on the proof of the first part of the previous Proposition is the following

Proposition 69 *The first metric (5.26) has the form*

$$(\partial', \partial'')_1 = -\mathrm{Res}_{z=\infty} \frac{\partial'\lambda_{>0}\partial''\lambda_{>0}}{z^2(\lambda_{>0})_z} dz - \mathrm{Res}_{z=0} \frac{\partial'\lambda_{\leqslant 0}\partial''\lambda_{\leqslant 0}}{z^2(\lambda_{\leqslant 0})_z} dz. \tag{5.33}$$

Proof Consider, in the coordinates u_i, the term

$$\mathrm{Res}_{z=0} \frac{\frac{\partial}{\partial u_i}\lambda \frac{\partial}{\partial u_j}\lambda}{z^2(\lambda)_z} dz; \tag{5.34}$$

expanding close to $z = 0$ it gives

$$\sim \mathrm{Res}_{z=0} \frac{dz}{z}\ z^{i+j}\ z^M(1 + O(z)) \tag{5.35}$$

hence it is non zero only for $i + j \leqslant -M$, i.e. it is necessary that both i, j are $\leqslant 0$. Then it is clear that only the $\lambda_{\leqslant 0}$ part is relevant. An analogous proof holds for the second term in (5.33). □

Remark 70 *Finally we recall, from [18], the prescription for the flat coordinates of the first metric. These are given by*

$$t^\alpha = \frac{1}{2}\operatorname{Res}_{z=\infty} \lambda^{\frac{N-\alpha}{N}} \frac{dz}{z} \qquad 1 \leqslant \alpha \leqslant N-1 \tag{5.36a}$$

$$t^\alpha = \frac{1}{2}\operatorname{Res}_{z=0} \lambda^{\frac{M+\alpha}{M}} \frac{dz}{z} \qquad -M+1 \leqslant \alpha \leqslant 0 \tag{5.36b}$$

$$t^{-M} = \frac{1}{4M} \log u_{-M}. \tag{5.36c}$$

Notice that these are the Casimirs of the first dispersionless Poisson bracket.

5.2 Dispersionless two-dimensional Toda hierarchy

In this section we consider the dispersionless limit of the first and second Poisson brackets of the two-dimensional Toda hierarchy. In this case the hierarchy has an infinite number of independent variables; thus to these hydrodynamic brackets we can associate a pencil of infinite dimensional contravariant metrics. We write down the explicit forms of the metrics and the Christoffel symbols. We obtain generating functions for the brackets and the metrics.

Then we prove that the first metric is non-degenerate; this gives a first hint that an infinite dimensional Frobenius manifold should be associated to this pencil of metrics.

We finally make a change of variables and show that the first Poisson bracket splits in two parts: a finite dimensional part that corresponds exactly to the first bracket of the Toda chain and an infinite dimensional part that, in Fourier coordinates, is the Poisson-Lie bracket on the dual of the algebra of potentials associated to divergence-free vector fields.

We have considered here only the case $N = M = 1$, however the results are easily modified to hold in the general N, M case.

5.2.1 Poisson brackets

As in the bigraded case the dispersionless brackets are obtained in the limit $\epsilon \to 0$ (see (5.1)) of the brackets (4.19) and (4.25). In this case the variables are u_k with $k < 1$ and $\bar{u}_l$ with $l \geqslant -1$. The explicit form of the brackets is

First bracket

$$\{u_n(x), u_m(y)\}_1^{disp} = -2\big[(n+m)u_{n+m}\delta'(x-y) + mu'_{n+m}\delta(x-y)\big] \tag{5.37a}$$

$$\begin{aligned}\{u_n(x), \bar{u}_m(y)\}_1^{disp} &= -2c(m)\big[(n+m)u_{n+m}\delta'(x-y) + mu'_{n+m}\delta(x-y)\big] \\ &\quad - 2\big[(n+m)\bar{u}_{n+m}\delta'(x-y) + m\bar{u}'_{n+m}\delta(x-y)\big]\end{aligned} \tag{5.37b}$$

$$\begin{aligned}\{\bar{u}_n(x), u_m(y)\}_1^{disp} &= -2c(n)\big[(n+m)u_{n+m}\delta'(x-y) + mu'_{n+m}\delta(x-y)\big] \\ &\quad - 2\big[(n+m)\bar{u}_{n+m}\delta'(x-y) + m\bar{u}'_{n+m}\delta(x-y)\big]\end{aligned} \tag{5.37c}$$

$$\{\bar{u}_n(x), \bar{u}_m(y)\}_1^{disp} = 2(1-c(m)-c(n))\big[(n+m)\bar{u}_{n+m}\delta'(x-y) + m\bar{u}'_{n+m}\delta(x-y)\big], \tag{5.37d}$$

Second bracket

$$\begin{aligned}\{u_n(x), u_m(y)\}_2^{disp} &= 4m(n-1)u_n u_m\delta'(x-y) + 4m(n-1)u_n u'_m\delta(x-y) \\ &\quad + 4\sum_{l<m}\big[(n+m-2l)u_{n+m-l}u_l\delta'(x-y) \\ &\quad + (n-l)u_{n+m-l}u'_l\delta(x-y) + (m-l)u'_{n+m-l}u_l\delta(x-y)\big]\end{aligned} \tag{5.38a}$$

$$\begin{aligned}\{u_n(x), \bar{u}_m(y)\}_2^{disp} &= 4m(n-1)u_n\bar{u}_m\delta'(x-y) + 4m(n-1)u_n\bar{u}'_m\delta(x-y) \\ &\quad + 4\sum_{l<m}[-(n+m)u_{n+m-l}\bar{u}_l\delta'(x-y) \\ &\quad + (l-m)u'_{n+m-l}\bar{u}_l\delta(x-y) - (n+2m-l)u_{n+m-l}\bar{u}'_l\delta(x-y)]\end{aligned} \tag{5.38b}$$

$$\begin{aligned}\{\bar{u}_n(x), u_m(y)\}_2^{disp} &= 4n(m-1)u_m\bar{u}_n\delta'(x-y) + 4n(m-1)u'_m\bar{u}_n\delta(x-y) \\ &\quad - 4\sum_{l<n}\big[(n+m)u_{n+m-l}\bar{u}_l\delta'(x-y) \\ &\quad + (l+m)u'_{n+m-l}\bar{u}_l\delta(x-y) - (n-l)u_{n+m-l}\bar{u}'_l\delta(x-y)\big]\end{aligned} \tag{5.38c}$$

$$\begin{aligned}\{\bar{u}_n(x), \bar{u}_m(y)\}_2^{disp} &= 4n(1+m)\bar{u}_n\bar{u}_m\delta'(x-y) + 4n(1+m)\bar{u}_n\bar{u}'_m\delta(x-y) \\ &\quad + 4\sum_{k>m}\big[(n+m-2k)\bar{u}_{n+m-k}\bar{u}_k\delta'(x-y) \\ &\quad + (n-k)\bar{u}_{n+m-k}\bar{u}'_k\delta(x-y) + (m-k)\bar{u}'_{n+m-k}\bar{u}_k\delta(x-y)\big].\end{aligned} \tag{5.38d}$$

It is understood that on the RHS of these formulas $\bar{u}_k = 0$ for $k < -1$ and $u_k = 0$ for $k > 0$ and u_1 is set to 1. In each of the quadratic brackets one can easily recognize the term due to Dirac reduction to the affine subspace $u_1 = 1$.

5.2.2 Associated metrics and Christoffel symbols

We define the metric $g^{\hat{n}\hat{m}}$ and the Christoffel symbols $\Gamma_{\hat{k}}^{\hat{n}\hat{m}}$ associated to the first metric by

$$\{u_{\hat{n}}(x), u_{\hat{m}}(y)\}_1^{disp} = g^{\hat{n}\hat{m}}\delta'(x-y) + \Gamma_{\hat{k}}^{\hat{n}\hat{m}}u'_{\hat{k}}\delta(x-y). \tag{5.39}$$

As in the previous formula, an index with hat, like $\hat{m}$, will be sometimes used to indicate that it spans both the values of m and $\bar{m}$. In (5.39) it is understood that $u_{\bar{n}} = \bar{u}_n$.

We obtain that the metric is

$$g^{nm} = -2(n+m)u_{n+m} \tag{5.40a}$$
$$g^{n\bar{m}} = -2c(m)(n+m)u_{n+m} - 2(n+m)\bar{u}_{n+m} \tag{5.40b}$$
$$g^{\bar{n}m} = -2c(n)(n+m)u_{n+m} - 2(n+m)\bar{u}_{n+m} \tag{5.40c}$$
$$g^{\bar{n}\bar{m}} = 2(1-c(n)-c(m))(n+m)\bar{u}_{n+m} \tag{5.40d}$$

and the Christoffel symbols

$$\Gamma_k^{nm} = -2m\delta_{k,n+m} \qquad \Gamma_{\bar{k}}^{nm} = 0 \tag{5.41a}$$
$$\Gamma_k^{n\bar{m}} = -2c(m)m\delta_{k,n+m} \qquad \Gamma_{\bar{k}}^{n\bar{m}} = -2m\delta_{k,n+m} \tag{5.41b}$$
$$\Gamma_k^{\bar{n}m} = -2c(n)m\delta_{k,n+m} \qquad \Gamma_{\bar{k}}^{\bar{n}m} = -2m\delta_{k,n+m} \tag{5.41c}$$
$$\Gamma_k^{\bar{n}\bar{m}} = 0 \qquad \Gamma_{\bar{k}}^{\bar{n}\bar{m}} = 2(1-c(n)-c(m))m\delta_{k,n+m}. \tag{5.41d}$$

For the second bracket we have

$$\{u_{\hat{n}}(x), u_{\hat{m}}(y)\}_2^{disp} = \tilde{g}^{\hat{n}\hat{m}}\delta'(x-y) + \tilde{\Gamma}_{\hat{k}}^{\hat{n}\hat{m}} u'_{\hat{k}}\delta(x-y) \tag{5.42}$$

and we find that the metric is

$$\tilde{g}^{nm} = 4m(n-1)u_n u_m + 4\sum_{l<m}(n+m-2l)u_{n+m-l}u_l \tag{5.43a}$$
$$\tilde{g}^{n\bar{m}} = 4m(n-1)u_n\bar{u}_m - 4\sum_{l<m}(n+m)u_{n+m-l}\bar{u}_l \tag{5.43b}$$
$$\tilde{g}^{\bar{n}\bar{m}} = 4n(m+1)\bar{u}_n\bar{u}_m + 4\sum_{k>m}(n+m-2k)\bar{u}_{n+m-k}\bar{u}_k \tag{5.43c}$$

and the Christoffel symbols

$$\tilde{\Gamma}_k^{nm} = 4m(n-1)u_n\delta_{k,m} + 4\sum_{l<m}[(n-l)u_{n+m-l}\delta_{k,l} + (m-l)u_l\delta_{k,n+m-l}] \tag{5.44a}$$
$$\tilde{\Gamma}_{\bar{k}}^{nm} = 0 \tag{5.44b}$$
$$\tilde{\Gamma}_k^{n\bar{m}} = 4\sum_{l<m}(l-m)\bar{u}_l\delta_{k,n+m-l} \tag{5.44c}$$
$$\tilde{\Gamma}_{\bar{k}}^{n\bar{m}} = 4m(n-1)u_n\delta_{k,m} - 4\sum_{l<m}(n+2m-l)u_{n+m-l}\delta_{k,l} \tag{5.44d}$$
$$\tilde{\Gamma}_k^{\bar{n}m} = 4n(m-1)\bar{u}_n\delta_{k,m} - 4\sum_{l<n}(l+m)\bar{u}_l\delta_{k,n+m-l} \tag{5.44e}$$
$$\tilde{\Gamma}_{\bar{k}}^{\bar{n}m} = 4\sum_{l<n}(n-l)u_{n+m-l}\delta_{k,l} \tag{5.44f}$$
$$\tilde{\Gamma}_k^{\bar{n}\bar{m}} = 0 \tag{5.44g}$$
$$\tilde{\Gamma}_{\bar{k}}^{\bar{n}\bar{m}} = 4n(1+m)\bar{u}_n\delta_{k,m} + 4\sum_{l>m}[(n-l)\bar{u}_{n+m-l}\delta_{k,l} + (m-l)\bar{u}_l\delta_{k,n+m-l}]. \tag{5.44h}$$

In all these formulas the indices on the LHS span the ranges $n, m \leqslant 0$ and $\bar{n}, \bar{m} \geqslant -1$; moreover the same considerations that were given after (5.38) apply.

5.2.3 Generating functions for the Poisson brackets

Here we write the generating functions for the dispersionless Poisson brackets (5.37) and (5.38).

Let's define the functions λ and $\bar\lambda$

$$\lambda(p,x)=\sum_{k\leqslant 1}u_k(x)p^k,\qquad \bar\lambda(p,x)=\sum_{k\geqslant -1}\bar u_k(x)p^k. \tag{5.45}$$

As in the bigraded Toda case we have to rewrite the first Poisson brackets in a form similar to (5.12); then, essentially applying identities like (5.14), one obtains

Generating functions for the first bracket

$$\begin{aligned}
\{\lambda(p,x),\lambda(q,y)\}_1 &= 2p\frac{\partial}{\partial p}\left[\frac{\lambda(q,x)q^{-1}-\lambda(p,x)p^{-1}}{p^{-1}-q^{-1}}\right]\delta'(x-y)\\
&+2q\frac{\partial}{\partial q}\left[\frac{\lambda(q,y)q^{-1}-\lambda(p,y)p^{-1}}{p^{-1}-q^{-1}}\right]\delta'(x-y), \qquad (5.46a)\\
\{\lambda(p,x),\bar\lambda(q,y)\}_1 &= 2p\frac{\partial}{\partial p}\left[\frac{\bar\lambda(q,x)q^{-1}+\lambda(p,x)p^{-1}}{p^{-1}-q^{-1}}\right]\delta'(x-y)\\
&+2q\frac{\partial}{\partial q}\left[\frac{\bar\lambda(q,y)q^{-1}-\lambda(p,y)p^{-1}}{p^{-1}-q^{-1}}\right]\delta'(x-y), \qquad (5.46b)\\
\{\bar\lambda(p,x),\bar\lambda(q,y)\}_1 &= 2p\frac{\partial}{\partial p}\left[\frac{\bar\lambda(p,x)p^{-1}-\bar\lambda(q,x)q^{-1}}{p^{-1}-q^{-1}}\right]\delta'(x-y)\\
&+2q\frac{\partial}{\partial q}\left[\frac{\bar\lambda(p,y)p^{-1}-\bar\lambda(q,y)q^{-1}}{p^{-1}-q^{-1}}\right]\delta'(x-y). \qquad (5.46c)
\end{aligned}$$

For the second dispersionless Poisson brackets one has essentially to follow the same steps as in the bigraded Toda case: first write the brackets in a form similar to (5.16) and then substitute in (5.11) to obtain the generating functions.

The main difference is in the equation

$$\begin{aligned}
\{u_n(x),\bar u_m(y)\}_2^{disp} &= 4m(n-1)u_n(x)\bar u_m(y)\delta'(x-y)+\\
&+4\sum_{l<m}\big[(l-m)\bar u_l(x)u_{n+m-l}(y)+(l-n-2m)u_{n+m-l}(x)\bar u_l(y)+\\
&+2(m-l)u_{n+m-l}(x)\bar u_l(x)\big]\delta'(x-y) \qquad (5.47)
\end{aligned}$$

since the last term has both factors evaluated in x.

Thus we obtain

Generating functions for the second bracket

$$\{\lambda(p,x),\lambda(q,y)\}_2^{disp} = \frac{4}{p^{-1}-q^{-1}}\left(\lambda(p,x)\frac{\partial}{\partial q}\lambda(q,y)-\lambda(q,y)\frac{\partial}{\partial p}\lambda(p,x)\right)\delta'(x-y)+ \\ +\frac{4p^{-1}q^{-1}}{(p^{-1}-q^{-1})^2}\left(\lambda(q,x)\lambda(p,y)-\lambda(p,x)\lambda(q,y)\right)\delta'(x-y)+ \\ +4pq\frac{\partial}{\partial p}\lambda(p,x)\frac{\partial}{\partial q}\lambda(q,y)\delta'(x-y) \tag{5.48a}$$

$$\{\lambda(p,x),\bar\lambda(q,y)\}_2^{disp} = \frac{4}{p^{-1}-q^{-1}}\left(\lambda(p,x)\frac{\partial}{\partial q}\bar\lambda(q,y)+\bar\lambda(q,y)\frac{\partial}{\partial p}\lambda(p,x)\right)\delta'(x-y)+ \\ +\frac{4p^{-1}q^{-1}}{(p^{-1}-q^{-1})^2}\left(2\lambda(p,x)\bar\lambda(q,x)-\bar\lambda(q,x)\lambda(p,y)-\lambda(p,x)\bar\lambda(q,y)\right)\delta'(x-y)+ \\ +4pq\frac{\partial}{\partial p}\lambda(p,x)\frac{\partial}{\partial q}\bar\lambda(q,y)\delta'(x-y) \tag{5.48b}$$

$$\{\bar\lambda(p,x),\bar\lambda(q,y)\}_2^{disp} = \frac{4}{p^{-1}-q^{-1}}\left(-\bar\lambda(p,x)\frac{\partial}{\partial q}\bar\lambda(q,y)+\bar\lambda(q,y)\frac{\partial}{\partial p}\bar\lambda(p,x)\right)\delta'(x-y)+ \\ +\frac{4p^{-1}q^{-1}}{(p^{-1}-q^{-1})^2}\left(-\bar\lambda(q,x)\bar\lambda(p,y)+\bar\lambda(p,x)\bar\lambda(q,y)\right)\delta'(x-y)+ \\ +4pq\frac{\partial}{\partial p}\bar\lambda(p,x)\frac{\partial}{\partial q}\bar\lambda(q,y)\delta'(x-y). \tag{5.48c}$$

Remark 71 *As in the bigraded case the first brackets (5.46) can be obtained as linear part of the second bracket after the shift $\lambda \to \lambda+\varepsilon$ and $\bar\lambda \to \bar\lambda+\varepsilon$.*

Remark 72 *The same generating functions (5.46) for the first brackets hold in the general N, M case. For the generating functions (5.48) of the second bracket one needs to divide by N the Dirac correction term (the last line in each of the equations (5.48)).*

5.2.4 Generating functions for the metrics

As was done for the bigraded Toda, we can extend the bilinear forms $(\,,)_i$ associated to the contravariant metrics $g^{\hat n\hat m}$ and $\tilde g^{\hat n\hat m}$ on the differentials

$$d\lambda(p) = \sum_{k<1} du_k p^k, \qquad d\bar\lambda(p) = \sum_{k\geqslant -1} d\bar u_k p^k. \tag{5.49}$$

Then from (5.46) and (5.48) we get

Generating functions for the first metric

$$(d\lambda(p), d\lambda(q))_1 = 2\frac{\lambda'(q)-\lambda'(p)}{p^{-1}-q^{-1}} \tag{5.50a}$$

$$(d\lambda(p), d\bar\lambda(q))_1 = 2\frac{\lambda'(p)+\bar\lambda'(q)}{p^{-1}-q^{-1}} \tag{5.50b}$$

$$(d\bar\lambda(p), d\bar\lambda(q))_1 = 2\frac{\bar\lambda'(p)-\bar\lambda'(q)}{p^{-1}-q^{-1}} \tag{5.50c}$$

and

Generating functions for the second metric

$$(d\lambda(p), d\lambda(q))_2 = \frac{4}{p^{-1} - q^{-1}} \big(\lambda(p)\lambda'(q) - \lambda(q)\lambda'(p)\big) + 4pq\lambda'(p)\lambda'(q) \tag{5.51a}$$

$$(d\lambda(p), d\bar{\lambda}(q))_2 = \frac{4}{p^{-1} - q^{-1}} \big(\lambda(p)\bar{\lambda}'(q) + \bar{\lambda}(q)\lambda'(p)\big) + 4pq\lambda'(p)\bar{\lambda}'(q) \tag{5.51b}$$

$$(d\bar{\lambda}(p), d\bar{\lambda}(q))_2 = \frac{4}{p^{-1} - q^{-1}} \big(-\bar{\lambda}(p)\bar{\lambda}'(q) + \bar{\lambda}(q)\bar{\lambda}'(p)\big) + 4pq\bar{\lambda}'(p)\bar{\lambda}'(q). \tag{5.51c}$$

5.2.5 The first metric is non-degenerate

Here we show that the metric associated to the first Poisson bracket of the dispersionless 2-dimensional Toda hierarchy is non-degenerate; this gives a first hint of the existence of an associated infinite dimensional Frobenius manifold.

Proposition 73 *The metric $g^{\hat{n}\hat{m}}$ is non-degenerate.*

Proof We essentially want to show that

$$\sum_{\hat{m}} g^{\hat{n}\hat{m}} v_{\hat{m}} = 0 \implies v_{\hat{m}} = 0 \tag{5.52}$$

for generic values of the entries of the metric. Here $v_{\hat{m}}$ is a vector with components v_m for $m < 1$ and $\bar{v}_{\bar{m}}$ with $\bar{m} \geqslant -1$. If we explicit the above equation for $\bar{n} = -1$ and $\bar{n} = 0$ we obtain

$$v_0 = \bar{v}_0 \quad \text{and} \quad v_{-1} = \bar{v}_{-1} \tag{5.53}$$

respectively.

If we introduce the variable w_m such that

$$w_m = \begin{cases} v_m & m \leqslant 0 \\ \bar{v}_m & m > 0 \end{cases} \tag{5.54}$$

then the equation (5.52) becomes

$$\sum_{n\in\mathbb{Z}} (n+m)(u_{n+m} + \bar{u}_{n+m}) w_m = 0. \tag{5.55}$$

Using the Fourier series

$$u(x) = \sum_{n\in\mathbb{Z}} e^{inx} n u_n, \quad w(x) = \sum_{n\in\mathbb{Z}} e^{-inx} w_n \tag{5.56}$$

one finds that (5.55) is equivalent to

$$u(x)w(x) = 0 \tag{5.57}$$

thus implying, for generic $u(x)$ and together with (5.53), that all the components $v_{\hat{m}}$ are zero. □

5.2.6 A new set of coordinates

We introduce a new set of coordinates obtained from the splitting that gives the R-matrix for the two-dimensional Toda. The first dispersionless Poisson bracket splits into two independent parts and the associated metric splits in diagonal blocks.

Let's consider the coordinates w_0, w_{-1} and v_k, $k \in \mathbb{Z}$, defined by

$$\begin{cases} w_0 = \bar{u}_0 \\ w_{-1} = \bar{u}_{-1} \end{cases} \qquad v_k = \begin{cases} u_k & k < -1 \\ u_{-1} + \bar{u}_{-1} & k = -1 \\ u_0 + \bar{u}_0 & k = 0 \\ 1 + \bar{u}_1 & k = 1 \\ \bar{u}_k & k > 1. \end{cases} \tag{5.58}$$

The first Poisson bracket (5.37) in these coordinated becomes

$$\{w_0(x), w_0(y)\}_1^{disp} = \{w_{-1}(x), w_{-1}(y)\}_1^{disp} = 0 \tag{5.59a}$$

$$\{w_0(x), w_{-1}(y)\}_1^{disp} = -2w_{-1}(y)\delta'(x-y) \tag{5.59b}$$

$$\{w_{-1}(x), w_0(y)\}_1^{disp} = -2w_{-1}(x)\delta'(x-y) \tag{5.59c}$$

$$\{w_n(x), v_m(y)\}_1^{disp} = 0 \qquad n = 0, 1 \quad m \in \mathbb{Z} \tag{5.60}$$

$$\{v_n(x), v_m(y)\}_1^{disp} = -2[nv_{n+m}(x) + mv_{n+m}(y)]\delta'(x-y) \quad n, m \in \mathbb{Z} \tag{5.61}$$

The coordinates w_k give exactly the first bracket for the first bracket of the Toda chain (i.e. bigraded Toda with $N = M = 1$).

The metric is given by

$$\sum_{n,m\in\mathbb{Z}} (n+m)v_{n+m}\frac{\partial}{\partial v_n}\frac{\partial}{\partial v_m} + w_{-1}\left(\frac{\partial}{\partial w_0}\frac{\partial}{\partial w_{-1}} + \frac{\partial}{\partial w_{-1}}\frac{\partial}{\partial w_0}\right). \tag{5.62}$$

Since the w-block of the metric exactly coincides with the metric of the Toda chain we can readily write down the first two flat coordinates: w_0 and $\log w_{-1}$.

5.2.7 Generating function in Fourier coordinates

First let's write down a generating function for the v-block of the metric. Defining

$$v(x) = \sum_n v_n e^{inx} \tag{5.63}$$

it easily follows

$$\begin{aligned}\sum_{nm} v_{n+m} e^{inx+imy} &= \sum_{nm} \frac{1}{2\pi} \int e^{-i(n+m)z} v(z) dz e^{inx+imy} \\ &= \frac{1}{2\pi} \int v(z) \sum_n e^{in(x-z)} \sum_m e^{im(y-z)} dz \\ &= 2\pi \int v(z) \delta(x-z) \delta(y-z) dz \\ &= 2\pi \int v(z) \delta(y-z) dz \; \delta(x-y) \\ &= 2\pi v(x) \delta(x-y). \end{aligned} \tag{5.64}$$

Now we have

$$\sum_{nm} (n+m) v_{n+m} e^{inx} e^{imy} = -i \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \sum_{nm} v_{n+m} e^{inx+imy} \tag{5.65}$$

$$= -2\pi i \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) v(x) \delta(x-y) \tag{5.66}$$

$$= -2\pi i v'(x) \delta(x-y). \tag{5.67}$$

Hence the generating function for the metric $(dv_n, dv_m) = g^{nm}$ is given by

$$(dv(x), dv(y)) = -2\pi i v'(x) \delta(x-y) \tag{5.68}$$

with $dv(x) = \sum_n dv_n e^{inx}$.

We can also obtain the generating function for the infinite dimensional part of the first Poisson bracket; defining

$$v(x,y) = \sum_n v_n(x) e^{iny} \tag{5.69}$$

and following essentially the same steps above we obtain

$$\begin{aligned}\{v(x_1,y_1), v(x_2,y_2)\}_1^{disp} = -4\pi i \big[&\partial_{x_1} v(x_1,y_1) \cdot \delta(x_1-x_2) \delta'(y_1-y_2) \\ &- \partial_{y_1} v(x_1,y_1) \cdot \delta'(x_1-x_2) \delta(y_1-y_2) \big]. \end{aligned} \tag{5.70}$$

5.2.8 Poisson brackets of divergence-free vector fields

In this section we want to show that the generating function for the first bracket in the coordinates v is the natural Poisson-Lie bracket associated to the potentials of divergence-free vector fields on a cylinder.

Let's consider, first of all, the Lie algebra $\mathcal{V}$ of vector fields $v = (v_1(x), \ldots, v_N(x))$ on $\mathbb{R}^N$ with coordinates $x_1, \ldots, x_N$. The commutator of two vector fields is simply

$$[v,w]_j(x) = v_i \frac{\partial}{\partial x_i} w_j - w_i \frac{\partial}{\partial x_i} v_j. \tag{5.71}$$

We denote an element of the dual space $\mathcal{V}^*$ by $p = (p_1(x), \ldots, p_n(x))$; the pairing between elements of the algebra and the dual is given by

$$< p, v >= \int d^N x (p_1(x) v_1(x) + \cdots + p_N(x) v_N(x)). \tag{5.72}$$

The commutator induces on $\mathcal{V}^*$ the Poisson-Lie bracket by the usual formula

$$\{f, g\}(p) =< p, [df, dg] >, \tag{5.73}$$

where f and g are functionals on $\mathcal{V}^*$. If we choose functionals of the form $f[p] = p_j(y)$ we obtain

$$\{p_i(x), p_j(y)\} = p_j(x) \frac{\partial}{\partial x_i} \delta(x - y) + p_i(y) \frac{\partial}{\partial x_j} \delta(x - y) \tag{5.74}$$

where, of course, $\delta(x - y) = \delta(x_1 - y_1) \cdots \delta(x_N - y_N)$.

Now consider the divergence-free vector fields on the plane ($N = 2$) with coordinates x, y. For the vector field $v = (v_x, v_y)$ the condition $0 = \mathrm{div} v = \frac{\partial}{\partial x} v_x + \frac{\partial}{\partial y} v_y$ implies the existence of a potential $f(x, y)$ such that

$$v_x = -\frac{\partial f}{\partial y} \qquad v_y = \frac{\partial f}{\partial x}. \tag{5.75}$$

In this case the commutator of the vector fields induces a Lie algebra structure on the space $\mathcal{P}$ of potentials

$$[f, g] = f_x g_y - f_y g_x \tag{5.76}$$

such that

$$[V_f, V_g] = V_{[f,g]} \tag{5.77}$$

where by V_f we indicate the vector field associated to the potential f by (5.75).

The dual space $\mathcal{P}^*$ will be given by functions ω with a pairing with $\mathcal{P}$ given by

$$< \omega, f >= \int dx\, dy\, \omega f \tag{5.78}$$

for $\omega \in \mathcal{P}^*$ and $f \in \mathcal{P}$.

The transpose of the map $f \mapsto V_f$ from $\mathcal{P}$ to $\mathcal{V}$ is given by the map that associates to $p = (p_x, p_y) \in \mathcal{V}^*$ the vorticity $\omega_p \in \mathcal{P}^*$

$$\omega_p := \frac{\partial p_x}{\partial y} - \frac{\partial p_y}{\partial x} \tag{5.79}$$

since we have

$$< \omega_p, f >=< p, V_f > . \tag{5.80}$$

The Lie algebra structure (5.76) on $\mathcal{P}$ defines a Poisson-Lie bracket on the $\mathcal{P}^*$; a straightforward calculation shows that this bracket is given by

$$\{v(x_1, y_1), v(x_2, y_2)\} = v_{x_1}(x_1, y_1) \delta(x_1 - x_2) \delta'(y_1 - y_2) - v_{y_1}(x_1, y_1) \delta'(x_1 - x_2) \delta(y_1 - y_2) \tag{5.81}$$

for $v(x,y) \in \mathcal{P}^*$, i.e. exactly the generating function obtained in the previous section (up to a factor).

Of course one can obtain the same bracket on $\mathcal{P}^*$ simply from (5.74) substituting the formula (5.79).

Notice that in the case of an incompressible fluid the space $\mathcal{V}$ and the dual $\mathcal{V}^*$ are identified through

$$p_x = \rho v_x \tag{5.82a}$$

$$p_y = \rho v_y \tag{5.82b}$$

where ρ is the constant density, thus obtaining the construction of [39].

We can now state the following theorem on the structure of the first dispersionless Poisson bracket

Theorem 74 *The dispersionless limit of the first Poisson bracket of the two-dimensional Toda is isomorphic to the direct sum of the first Poisson bracket of the dispersionless Toda chain with the brackets (5.81) associated to an incompressible fluid on the cylinder.*

Chapter 6

Conclusions

In this thesis we have studied three related integrable hierarchies of Toda type. Let's summarize our results in each case.

We have considered first the **extended Toda chain hierarchy**. We have defined the logarithm of the difference operator L and used it to obtain a Lax representation for additional non-local flows. Then we have introduced the bihamiltonian formalism, expressing the non-local Hamiltonians in terms of traces of L and $\log L$. We have shown the existence of a tau function and we have obtained the bilinear relations for the wave operators and the wave functions. Finally we have obtained the soliton solutions.

We have then generalized some of these results to the case of the **bigraded Toda hierarchy**. We have first defined the logarithm and two fractional powers of L and introduced the flows of the hierarchy through their Lax representation. We have then obtained a pair of Poisson brackets on the space of bigraded difference operators and expressed the flows of the hierarchy in Hamiltonian form using the first bracket. Finally we have shown that also in this case a tau function can be defined.

On the algebra of pairs of difference operators $A^+ \oplus A^-$, we have introduced an R-matrix associated to a non-trivial splitting and we have used it to obtain a bihamiltonian structure for the **two-dimensional Toda hierarchy**. Then we have expressed the well-known Lax flows of this hierarchy in Hamiltonian form through the first Poisson bracket.

Finally we have considered the **dispersionless limit** of the Poisson pencils associated to the bigraded and the two-dimensional Toda hierarchies. In both cases we have obtained the generating functions for such pencils of Poisson brackets and also for the associated pencils of metrics. We have then related the dispersionless bigraded Toda hierarchy to the Frobenius manifold structure on the Hurwitz space $M_{0;M-1,N-1}$. In the case of the two-dimensional hierarchy we have shown that the first Poisson bracket splits in the direct sum of the first bracket of the Toda chain with the Poisson bracket associated to an incompressible fluid on the cylinder.

We recall the main open points and lines of future research.

- Starting from the simpler case of the extended Toda chain, one should consider the behaviour of interesting classes of solutions under the non-local flows. In particular the **similarity solutions** satisfy ordinary differential-difference equations that should be analogues of the Painlevé equations. The **algebro-geometric quasiperiodic solutions** should be tractable through some modification of the usual method of the Baker-Akhiezer function on a Riemann surface.

- Still in the case of the extended Toda chain, one needs to obtain the **Hirota bilinear relation** for the tau function. To this purpose one should use the bilinear relations for the wave operators and the wave functions that were obtained in Chapter 2. This problem is also connected with the covariance under the Darboux transformations that we have assumed in the derivation of the soliton solutions.

- In the extended bigraded Toda hierarchy case, the main missing point is the determination of the **recursion relation** for the Hamiltonians. The explicit method used in the Toda chain case fails to work for the general bigraded hierarchy.

- The construction of a Poisson-Lie group of pseudo-differential operators done in [32, 33] should be possibly extended to the case of difference operators. One of course doesn't expect to have a Poisson-Lie group in this case, but a twisted Poisson structure [36] on a Lie group of difference operators with complex leading exponent. This framework should provide a natural characterization of the logarithm as the inverse of the exponential map connecting the Lie algebra and the group.

- It is important, in particular, to complete the study of the **Frobenius manifold** associated to the bigraded two-dimensional Toda hierarchy. This turns out to be an infinite dimensional manifold that naturally splits in a part corresponding to the bigraded Toda hierarchy plus a infinite dimensional part. One expects that an explicit realization of such manifold, in analogy with the realization as an Hurwitz space of the Frobenius manifold associated to the bigraded Toda hierarchy, could be linked to a space of conformal maps. Work in this direction is in progress.

- A general line of research, that we have already stressed in the introduction, is to construct the full dispersive hierarchies associated to the Dynkin diagrams B_l, C_l, ... In this thesis we have shown that the bigraded Toda hierarchy is associated with the A_l root system with a fixed root. A possible hint on the integrable systems associated to the other root systems is given by the construction of Frenkel-Reshetikhin of deformations of W-algebras associated to simple Lie algebras [26].

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