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Quantum Principal Bundles
and
Instantons

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Introduction

Of particular interest for both mathematics and physics are non-abelian gauge theories introduced by C. N. Yang and R. Mills in 1954 [53]. They are generalization of Maxwell's equations of electromagnetism with the abelian group $U(1)$ of internal symmetries replaced by a non abelian one, the starting example being $SU(2)$.

Self-dual solutions of $SU(2)$-Yang-Mills equations, commonly called instantons, are connections with self-dual curvature on a smooth $SU(2)$-bundle over the four sphere S^4. These equations were previously studied by physicists in terms of minima of some Lagrangians; first solutions were introduced by A. Belavin, A. Polyakov, A. Schwartz and Y. Tyupkin [5] and G. 't Hooft [64]. Later, other solutions, called multinstantons, were provided [22], [23], [31], [38]. It is in the seventies that the relations between Yang-Mills theory and the mathematical theory of fibrations became clear. General solutions, not only for $SU(2)$, were constructed by M. Atiyah, V. G. Drinfeld, N. J. Hitchin and Yu I. Manin [2]. Important contributions were also given by S. K. Donaldson [27], G. Penrose [55], I. M. Singer [3], R. S. Ward [4].

The classification of such solutions on S^4 is equivalent to the classification of certain rank-2 holomorphic bundles on the complex projective space $\mathbb{P}^3(\mathbb{C})$ [4]. This was showed by using the Penrose twistor approach [55] to spacetime translating a problem on S^4 into a problem of complex variables. The identification $S^4 \simeq \mathbb{P}^1\mathbb{H}$ of the four sphere with the projective line over the quaternions was used.

In particular it was shown [3] that the parameter space of instantons on the $SU(2)$-bundle is a smooth manifold $M(k)$ of dimension $8k-3$. The integer k, called the instanton charge, is the second Chern class of the bundle. For $k = 1$, the moduli space $M(1)$ is isomorphic to an open ball in $\mathbb{R}^5$ [35]. In this construction, generic moduli are obtained from the so called basic instanton. The latter is though as a quaternion line bundle over $\mathbb{P}^1\mathbb{H}$ with connection induced from $\mathbb{H}^2$ by orthogonal projection. Others charge 1 instantons are then obtained from the basic one by the action of the conformal group $Sl(2, \mathbb{H})$, modulo the isometry group $Sp(2)$. The resulting homogeneous space is the space of quaternion norms in $\mathbb{H}^2$.

Recent years have seen a lot of activity in extending gauge theory (fibrations, connections, etc.) to noncommutative differential geometry. One of the motivation for developing such noncommutative geometry and gauge theories within comes from the desire to describe fundamental forces of nature; more precisely to unify commutative geometric Einstein's theory of gravity with noncommutative quantum-mechanical theories of nuclear interactions. Nevertheless, the beautiful mathematical techniques behind the study of these physical theories is itself source of great interest.

Noncommutative geometry is based on the algebraic reformulation of differential geometry which takes its origin from Gel'fand Naimark (commutative) theorem (1943) [30]. The basic idea is to use the correspondence between geometric spaces and commutative algebras, like it is done in algebraic geometry.

The attempt to generalize to the quantum case the ADHM construction of $SU(2)$ instantons together with their moduli space is the topic of this thesis. The content of the chapters which follow explains the research shared in these years with Prof. Landi, Prof. Reina and (in the last months) with Walter van Suijlekom and culminated in the two papers *"A Hopf bundle over a quantum four-sphere from the symplectic group"* (to appear in Commun. Math. Phys.) [46] and [47] in preparation.

In the noncommutative context vector bundles are replaced by projective modules of finite type. This reflects the classical correspondance between a vector bundle on a manifold and its module (over the algebra of continuous functions on the manifold itself) of sections (Serre Swan theorem [63]). Symmetries, which play a central role in gauge theories, at a deformed level, are implemented by means of quantum groups [51], [57], [67], and their coactions [15, 17].
Finally, good candidates for noncommutative principal fiber bundles are faithfully flat Hopf-Galois extensions, or more generally coalgebra-Galois extensions. Indeed in the commutative limit, Hopf-Galois extensions are objects dual to principal fibrations [61].

The general notion of quantum principal bundles was first introduced in [15] where the construction of the q-monopole on two dimensional quantum spheres was also presented. A step toward the construction of instantons and their principal bundles was taken only ten years later in [7], but the resulting bundle was only a coalgebra extension [8].

In our paper [46] we constructed a new quantum version of the Hopf bundle $S^7 \to S^4$ giving one of the first concrete examples of a Hopf-Galois extension with non abelian quantum structure group. The explanation of this construction is the subject of the first chapters of this thesis.

The fibration was obtained by deforming symplectic structures and groups entering into the classical fibration which provides one with the geometric interpretation of instantons:

$$S^7 \simeq Sp(2)/Sp(1) \longrightarrow S^4 \simeq Sp(2)/(Sp(1) \times Sp(1)) .$$

In analogy with the classical case [1], it is hence natural to start with the quantum version $A(Sp_q(n))$ of the (compact) symplectic groups (Ch. 1). These Hopf algebras are generated by the entries T_i^j of a $2n \times 2n$ matrix T; the commutation rules among these generators come from RTT equations, where the R matrix is the one of the C-series [57]. Furthermore, $A(Sp_q(n))$ is endowed with a antipode $S(T)$: $S(T)T = TS(T) = \mathbb{I}$. These quantum groups have comodule-subalgebras $A(S_q^{4n-1})$ yielding deformations of the algebras of polynomials over the spheres S^{4n-1}. These comodules are obtained by observing that the matrix elements of the first and the last columns of T generate a subalgebra of $A(Sp_q(n))$ and the condition $S(T)T = 1$ gives the sphere relation (Sect. 1.1.2).

The case in which we are more interested is for $n = 2$. The resulting symplectic quantum 7-sphere $A(S_q^7)$ turns out to be the quantum version of the homogeneous space $Sp(2)/Sp(1)$ (Sect. 2.2.1). Indeed we found a Hopf ideal I_q in $A(Sp_q(2))$ such that the corresponding quotient is isomorphic to $A(Sp_q(1)) \simeq A(SU_{q^2}(2))$. Then the restriction of the coproduct of $A(Sp_q(n))$ to this quotient yields a coaction of $A(Sp_q(1))$ with algebra of coinvariants given by the 7-sphere.
The injection $A(S_q^7) \hookrightarrow A(Sp_q(2))$ is a quantum principal bundle with "structure Hopf algebra" $A(Sp_q(1)) \simeq A(SU_{q^2}(2))$. Indeed we showed that the extension is Hopf Galois, giving another example of the general construction of principal bundles over quantum homogeneous spaces [15] (Sect. 3.3.1).

Most importantly, we showed that $A(S_q^7)$ is the total space of a quantum $SU_q(2)$ principal bundle over a quantum 4-sphere $A(S_q^4)$: $A(S_q^4) \hookrightarrow A(S_q^7)$ with a 'non abelian structure quantum group' given by the quantum group $SU_q(2)$ (Sect. 2.2). The algebra $A(S_q^4)$ is constructed as the subalgebra of $A(S_q^7)$ generated by the matrix elements of a self-adjoint projection p which generalizes the anti-instanton of charge -1. This projection is of the form vv^* with v a 4×2 matrix whose entries are made out of generators of $A(S_q^7)$. The naive generalization of the classical case produces a subalgebra with extra generators which vanish at $q = 1$. Luckily enough, there is just one alternative choice of v which gives the right number of generators of an algebra which deforms the algebra of polynomial functions of S^4. At $q = 1$ this gives a projection which is gauge equivalent to the standard one.

This good choice becomes even better because there is a natural coaction of $SU_q(2)$ on $A(S_q^7)$ with coinvariant algebra $A(S_q^4)$ and the injection $A(S_q^4) \hookrightarrow A(S_q^7)$ turns out to be a faithfully flat $A(SU_q(2))$-Hopf-Galois extension (Sect. 3.3). This is also shown by the explicit construction of a strong connection [32] (Sect. 3.3.2).

This quantum principal bundle dualizes the classical instanton of charge -1. Here the charge is computed through the pairing between K-homology and K-theory. Following a general strategy of noncommutative index theorem [18], we construct representations of the algebra $A(S_q^4)$ and the corresponding K-homology.
The analogue of the fundamental class of S^4 is given by a non trivial Fredholm module μ. The natural coupling between μ and the projection p is computed via the pairing of the corresponding Chern characters $\mathrm{ch}^*(\mu) \in HC^*[A(S_q^4)]$ and $\mathrm{ch}_*(p) \in HC_*[A(S_q^4)]$ in cyclic cohomology and homology respectively [18]. The result of this pairing is an integer by principle being the index of a Fredholm operator. The computation allows to us to conclude that the charge is -1 (Sect. 2.4). As a consequence of the non

vanishing of this pairing, we can also conclude that the bundle $A(S^4_q) \hookrightarrow A(S^7_q)$ is non trivial.

Another deformation of the instanton bundle was then provided in [48] with a similar procedure. A Hopf Galois extension $A(S^4_\theta) \hookrightarrow A(S^7_\theta)$ was constructed working over the Connes-Landi spheres $A(S^7_\theta)$ [21]. In both cases the construction leads to the (deformed) basic instanton. The construction is based on the requirement that the matrix v giving the projection is linear in the generators of the seven sphere and such that $v^*v = 1$. This last property is false even classically at generic moduli and charge greater than one. Hence a more elaborate strategy is needed to tackle the general case.

In paper [47] we are facing with this problem. The idea is to obtain generic charge 1 instantons by reproducing the action of the homogeneous space $SL(2,\mathbb{H})/Sp(2)$ which parametrizes norms in $\mathbb{H}^2$. Indeed, as said, classically instantons are constructed from the basic one by moving the norm on $\mathbb{H}^2$ [1]. This is possible for the θ-case, with the construction of θ-deformations $A(SL_\theta(2,\mathbb{H})$ and $A(Sp_\theta(2))$ of the corresponding classical groups (Sect. 4.2.2). This leads to a noncommutative 5-dimensional moduli space $\mathcal{M}_\theta$ and a 1 parameter family of 4-sphere of radius $\rho^2 = v^*v$. The 4-sphere $A(S^4_\theta)$ seats at the "boundary" of $\mathcal{M}_\theta$ like in the classical picture (Sect. 4.2.4). The corresponding projections are provided and the charge is explicitly computed (Sect. 4.2.5).

More difficult is the question of instantons of charge $k > 1$. Classically they are obtained by means of a map $v = Cx + Dy$ from $\mathbb{P}^1\mathbb{H}$ to the Stiefel variety $St(k, k+1)$. The matrices $C, D \in Mat((k+1) \times k), \mathbb{H})$ are suitable constant matrices satisfying some requirements. The $3k^2 + 13k$ parameters which enter in the construction are then reduced to $8k - 3$ by quotient by the action of $Sp(k+1)$ and $Gl(k,\mathbb{R})$. At a noncommutative level, the map v_θ has been constructed by means of suitable algebras C, D but the question of symmetries is still open (see Sect. 4.2.6).

The noncommutative symplectic case is more difficult from the point of view of the algebra structure of the algebras involved. Some steps have been made toward the construction of others charge 1 instantons but the problem is still open (Sect. 4.3).

The structure of the thesis

Chapter 1

The first chapter deals with some basic elements of the theory of noncommutative spheres. We describe in details symplectic quantum spheres $A(S^{2n-1}_q)$. These algebras are obtained as subalgebras of the compact real form $Sp_q(n)$ of the symplectic quantum groups and the sphere relation is obtained from the existence of the antipode. The algebra at $n = 2$ will be used in the construction of the quantum instanton bundle as the total space of the fibration. We then briefly recall some facts about so-called Connes-Landi spheres, S^N_θ [21].

We conclude with few facts about the construction of Fredholm modules for even quantum spheres following [36].

Chapter 2

This chapter contains the construction of the deformed version of the Hopf bundle

$$S^7 \simeq Sp(2)/Sp(1) \longrightarrow S^4 \simeq Sp(2)/(Sp(1) \times Sp(1)) \; ,$$

constructed from the quantum symplectic group. Here we limit ourself to describe the algebras involved in the construction while in Ch. 3 we will study the nature of the resulting bundle.
As said, following the common idea to replace spaces by algebras of functions, the basic ingredients for the formulation in noncommutative geometry of a theory of principal bundles will be two algebras corresponding to the (algebras of functions on the) total and base spaces and a Hopf algebra, or a quantum group, playing the role of the structure group. The chapter begins with a brief review of the classical (dual) picture. The algebra $A(S^7_q)$ introduced in the previous chapter becomes in Sect. 2.2 the total space of a quantum $SU_q(2)$-fibration in which the base space $A(S^4_q)$ is firstly given in terms of a projection and then described as the space of coinvariants of the $SU_q(2)$-coaction. (This fact is presented here with two proofs).
Furthermore we show that the algebra $A(S^7_q)$ can be made in a quantum homogeneous space. Indeed we show that it is the algebra of coinvairnats with respect to the coaction of $A(Sp_q(1))$. Here $A(Sp_q(1))$ is obtained as quotient of $A(Sp_q(2))$ by a Hopf ideal I_q.

Finally a Fredholm module is constructed over $A(S^4_q)$ in order to compute the Chern-Connes pairing between K-homolgy and K-theory giving the topological invariant of the bundle, the instanton charge.

Chapter 3

In the first section we discuss quantum bundles as introduced in [15]. In the second section we study principal bundles from the more algebraic point of view of Hopf-Galois extensions [61]. The overlap between these two construction is recovered when, in the first formulation, the algebras are endowed with the universal differential calculus. Finally, in Sect. 3.2.1 we recall the concept of connection on quantum principal bundles, concluding the review of the general theory.

Then we show that the two extensions $A(S^7_q) \subseteq A(Sp_q(2))$ and $A(S^4_q) \subseteq A(S^7_q)$ described in the previous chapter are examples of quantum principal bundles. For both of them we will define a strong connection, firstly on the generators of the structure groups and then on the whole algebras.

Chapter 4

We briefly review the construction of the deformation of the instanton bundle $S^7 \to S^4$ provided in [48] over the Connes-Landi θ sphere. In both the symplectic and the theta case (see Ch. 2 and Sect. 4.2.1) the construction was limited to the basic instanton of unit charge. The results obtained until now [47] in the attempt to generalise this picture to generic $SU(2)$-instantons is the topic of this chapter.

After a brief review of the classical situation [1] we try to reproduce it in the noncommutative case. We first deal with the instanton bundle $A(S^4_\theta) \hookrightarrow A(S^7_\theta)$ over the Connes-Landi sphere (Sect. 4.2). Then we address our attention to the more complicated case of the symplectic fibration $A(S^7_q) \hookrightarrow A(S^4_q)$ (Sect. 4.3).

Finally, **Appendix A** lists the complete commutation relations of the algebra $A(Sp_q(4,\mathbb{C}))$ computed from RTT equations.

Chapter 1

Noncommutative Spheres

This first chapter contains some basic elements of the theory of noncommutative spheres. Generalizing the classical correspondence between spaces and algebras of functions on them, spheres are described and studied in terms of deformations of the algebras of functions on the classical ones. These $*$-algebras are given in terms of generators x_i plus a sphere relation $\sum_{i=1}^{n} x_i^* x_i = 1$. The integer n becomes the dimension of the sphere and we refer to even/odd spheres accordingly to the parity of n. We recall here some general facts about such algebras we will need later on.

A first class of spheres was obtained considering suitable quotients of quantum groups. Into this class of examples we can cite (even and odd) quantum orthogonal spheres and (odd) quantum unitary spheres which are homogeneous spaces of R-matrix deformations of orthogonal and unitary group respectively, see resp. [57] and [65]. (Anyway in [36] it has been shown that odd orthogonal spheres and odd unitary spheres are the same.) Furthermore, two-dimensional spheres were obtained by Podleś [56] as homogeneous $SU_q(2)$-spaces. We refer to [24] for a reviewing list of quantum spheres.
In the following, our attention concentrates in (odd) symplectic quantum spheres that we discuss in details. The existence of these spheres was indicated in [57] and their structure described in [46]. These algebras are obtained as subalgebras of the compact real form of symplectic quantum groups and the sphere relation follows from the existence of the antipode.

The other class of noncommutative spheres, called the Connes-Landi spheres, S^N_θ, was introduced in [21]. These algebras are obtained by solving some equations in cyclic homology. They are part of a more wide construction of noncommutative manifolds from θ-deformations in which a central role is played by noncommutative tori. A noncommutative torus T^θ being the algebra generated by two unitaries, say U, V with the relation $UV = e^{2\pi i\theta} VU$ [58]. In Sect. 1.2 we will briefly discuss the general construction while we postpone to Ch. 4 the description of a concrete example consisting of the deformation $A(S^4_\theta)$ of the usual algebra of functions on the 4-sphere. Note that we refer to these θ-spheres as "manifolds" since they can be endowed with a noncommutative geometry structure consisting of a spectral triple, [21]. On the other hand, there is no a general way to construct a spectral triple on quantum spheres but Fredholm modules constitute a suitable substitute. We will recall few facts about the construction of Fredholm modules for (even) quantum spheres in Sect. 2.4 following [36].

1.1 Noncommutative spheres from R-matrix deformations: the case of symplectic spheres

Notations. In the following algebras are assumed over $\mathbb{C}$ and unadorned tensor product stands for $\otimes_{\mathbb{C}}$. Also, $\dot{\otimes}$ stands for tensor product of algebras and matrix multiplication. From here on, whenever no confusion arises, the sum over repeated indexes is understood.

The first examples of quantum groups, as R-matrix deformations of the algebras of polynomial functions of Lie groups, were provided by Drinfel'd [28] and Jimbo [43] in terms of quantum enveloping algebras. Important has also been the contribution of Manin [51]. In this brief introduction to the topic which follows we will refer to Reshetikhin, Takhtadzhyan and Faddeev's paper [57] in which the theory has been extended to every Lie group.

Let us first consider the free commutative algebra $\mathbb{C}[t_{ij}]$ of polynomials in the n^2 generators t_{ij}, that for convenience we arrange in a $N \times N$ matrix $T = (t_{ij})$. We will refer to T as the defining matrix. In addition to the algebra structure, $\mathbb{C}[t_{ij}]$ has a dual structure of bialgebra with coproduct and counit given respectively by

$$\begin{aligned} &\Delta : \mathbb{C}[t_{ij}] \longrightarrow \mathbb{C}[t_{ij}] \otimes \mathbb{C}[t_{ij}] \,, \qquad T \mapsto T \dot{\otimes} T \,; \\ &\varepsilon : \mathbb{C}[t_{ij}] \longrightarrow \mathbb{C} \,, \qquad T \mapsto I \end{aligned} \tag{1.1}$$

or, in components,

$$\Delta(t_{ij}) = \sum_k t_{ik} \otimes t_{kj} \qquad ; \qquad \varepsilon(t_{ij}) = \delta_{ij} \,.$$

This commutative algebra can be seen as a particular case of a more general classes of algebras. The so called R-matrix deformations. We assume that the elements t_{ij} satisfy RTT equations

$$RT_1T_2 = T_2T_1R \,, \tag{1.2}$$

where $T_1 = T \otimes 1,\ T_2 = 1 \otimes T$, or in components $(T \otimes 1)_{ij}{}^{kl} = T_i{}^k \delta_j{}^l$. R is a $N^2 \times N^2$ matrix whose entries depend on a parameter $q \in \mathbb{C}$-$\{0\}$.
The matrix R is also assumed to fulfil Yang-Baxter equations

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{1.3}$$

where $R_{12} = R \otimes R \otimes 1$ and analogous definitions for R_{13}, R_{23}.

Hence, let us consider the quotient A_R of the free algebra $\mathbb{C}[t_{ij}]$ under the two-sided ideal generated by RTT equations. This is a bialgebra with co-structures given like above in (1.1) and in addition it has a Hopf algebra structure given by introducing an antipode S, whose definition depends on the structure of R (which depends on the Lie group we are deforming). This leads to the notion of quantum groups. The commutative case is recovered at $q = 1$ when the matrix R becomes the identity matrix and therefore A_R reduces to the commutative algebra of coordinate functions on the matrix algebra $GL_n(\mathbb{C})$.

In the classical case different Lie groups can be obtained as subgroups of $M_n(\mathbb{C})$ by asking that some further structures are preserved. With the same procedure, allowing R to vary compatibly with the above requests and introducing further conditions, we recover noncommutative deformations of the Hopf algebras of smooth functions on the different Lie groups [57].

For our future needs, we discuss real forms of quantum groups. They are classified by $*$-anti involution. We remind that a $*$-structure, or $*$-conjugation, over a Hopf algebra H is an algebra anti-automorphism, $(\eta a\ b)^* = \overline{\eta} b^* a^*, \quad a, b \in H, \quad \eta \in \mathbb{C}$, a coalgebra automorphism, $\Delta * = (* \otimes *)\Delta$, and moreover $** = id$ and $(S*)^2 = id$. We say that two conjugation $*$ and $\star$ are equivalent if there exists an automorphism α of the algebra such that $\star = \alpha * \alpha^{-1}$.
Following [57], in general, there are at least two ways to define a conjugation on a quantum group of matrix type. Requiring the compatibility with RTT equations, in accordance with the value of q, we can define a conjugation by setting $T^* = T$ or using the antipode, as $T^* = US(T)^t U^{-1}$, where $U \in Mat_n(\mathbb{C})$ is a diagonal matrix depending on the quantum group.

Next we give the example of the symplectic case with the description of $A(Sp_q(4, \mathbb{C}))$ and its compact real form $Sp_q(2)$.

1.1.1 The symplectic quantum groups $Sp_q(N, \mathbb{C})$.

The algebra $A(Sp_q(N, \mathbb{C}))$, is the associative algebra generated over $\mathbb{C}$ by the entries $T_i{}^j, \ i, j = 1, \ldots, N$ of a matrix T. The integer N being even: $N = 2n$. The noncommutativity is introduced as said, by imposing that T satisfy RTT equations, where the $N^2 \times N^2$ matrix R is the one for the C_N series and has the form [57],

$$\begin{aligned} R &= q\sum_{i=1}^{N} e_i{}^i \otimes e_i{}^i + \sum_{\substack{i,j=1\\ i\neq j,j'}}^{N} e_i{}^i \otimes e_j{}^j + q^{-1}\sum_{i=1}^{N} e_{i'}{}^{i'} \otimes e_i{}^i \\ &\quad +(q-q^{-1})\sum_{\substack{i,j=1\\ i>j}}^{N} e_i{}^j \otimes e_j{}^i - (q-q^{-1})\sum_{\substack{i,j=1\\ i>j}}^{N} q^{\rho_i-\rho_j}\varepsilon_i\varepsilon_j\, e_i{}^j \otimes e_{i'}{}^{j'} \end{aligned} \tag{1.4}$$

Some explanation for notations is due:

- $i' = N+1-i$, for each $i = 1, \dots n$,
- the matrices $e_i^j \in M_n(\mathbb{C})$ have elements $(e_j{}^i)_l^k = \delta_{jl}\delta^{ik}$;
- $\varepsilon_i = 1$, for $i = 1, \dots, n$; $\varepsilon_i = -1$, for $i = n+1, \dots, N$
- finally, $(\rho_1, \dots, \rho_N) = (n, n-1, \dots, 1, -1, \dots, -n)$.

The elements of the matrix R will be denoted by $R_{ij,kl}$, with $i,j,k,l = 1, \dots N$ and $R_{AB} := R_{N(i-1)+j,N(k-1)+l} = R_{ij,kl}$.

As described in the general theory,

$$\Delta(T) = T \dot{\otimes} T\ , \quad \varepsilon(T) = I \tag{1.5}$$

endow $A(Sp_q(4,\mathbb{C}))$ with a bialgebra structure.
The symplectic group structure comes from the anti-diagonal matrix C,

$$C_i^j = q^{\rho_j}\varepsilon_i\delta_{ij'} \tag{1.6}$$

by imposing that C is preserved by T, i.e. requiring the additional relations

$$TCT^tC^{-1} = CT^tC^{-1}T = 1\ .$$

In accordance with the above equation, the bialgebra structure (1.5) can be completed to a Hopf algebra structure (Δ, ε, S) for $A(Sp_q(N,\mathbb{C}))$ by introducing the antipode

$$S(T) = CT^tC^{-1}\ ,$$

which in components explicitly reads

$$S(T)_i{}^j = -q^{\rho_{i'}+\rho_j}\varepsilon_i\varepsilon_{j'}T_{j'}{}^{i'}\ . \tag{1.7}$$

As said, the classical limit is recovered at $q = 1$ when the Hopf algebra $A(Sp_q(N,\mathbb{C}))$ reduces to the algebra of polynomial functions over the symplectic group $Sp(N,\mathbb{C})$.

We now discuss admissible real forms which depend on the range of the parameter q [57]. If $|q| < 1$, the $*$-structure is $T^* = T$, i.e. $t^*_{ij} = t_{ij}$, and the corresponding algebra is denoted by $A(Sp_q(n,\mathbb{R}))$. The compact real form $A(Sp_q(n))$ of the quantum group $A(Sp_q(N,\mathbb{C}))$ is given by taking $q \in \mathbb{R}$ and the anti-involution

$$T^* = S(T)^t = C^tT(C^{-1})^t\ . \tag{1.8}$$

On the compatibility between R and C

We notice that the matrices R and C are strictly related one to the other. In particular once one has fixed the symplectic structure, i.e. the matrix C, then R can not be generic.

Let us assume the invertibility of R, then RTT equations can be rewritten as

$$T_1T_2R^{-1} = R^{-1}T_2T_1 \tag{1.9}$$

Thus

$$RT_1T_2 = T_2T_1R \quad \Longrightarrow \quad \begin{array}{l} S(T_2)R\, T_1 = T_1R\, S(T_2) \\ S(T_1)S(T_2)\ R = R\ S(T_2)S(T_1) \end{array} \tag{1.10}$$

$$T_1T_2R^{-1} = R^{-1}T_2T_1 \quad \Longrightarrow \quad \begin{array}{l} S(T_1)R^{-1}T_2 = T_2R^{-1}S(T_1) \\ S(T_2)S(T_1)R^{-1} = R^{-1}S(T_1)S(T_2) \end{array} \tag{1.11}$$

where $S(T_1) = S(T) \otimes 1 = S(T)_1 = CTC_1^{-1} \otimes 1 = C_1T_1^tC_1^{-1}$.
Starting from eq. (1.11) $S(T_1)R^{-1}T_2 = T_2R^{-1}S(T_1)$ and substituting $S(T)$ as given in (1.24) we have

$$C_1T_1^{t_1}C_1^{-1}R^{-1}T_2 = T_2R^{-1}C_1T_1^{t_1}C_1^{-1} ,$$

where t_1 means transposition with respect to the first factor of the $\mathbb{C}^N \otimes \mathbb{C}^N$ tensor product and we used the fact that $T_1^t = T_1^{t_1}$. Then multipling on the left by C_1^{-1} and by C_1 on the right side and observing that T_2 and C_1 commute, we have

$$T_1^{t_1}C_1^{-1}R^{-1}C_1T_2 = T_2C_1^{-1}R^{-1}C_1T_1^{t_1} ,$$

that we can rewrite as

$$[(C_1^{-1}R^{-1}C_1)^{t_1}T_1]^{t_1}T_2 = T_2[T_1(C_1^{-1}R^{-1}C_1)^{t_1}]^{t_1} .$$

Finally we transpose t_1 and use the fact that $T_2^{t_1} = T_2$ concluding that

$$(C_1^{-1}R^{-1}C_1)^{t_1}T_1T_2 = T_2T_1(C_1^{-1}R^{-1}C_1)^{t_1} \tag{1.12}$$

Comparing this equation with RTTs we see that it is satisfied if

$$\nu R = (C_1^{-1}R^{-1}C_1)^{t_1}$$

that is

$$\nu RC_1R^{t_1} = C_1 , \tag{1.13}$$

with ν a constant parameter.

On the other hand, if we start by eq. (1.10) and we susbstitute the expression of $S(T)$, we have

$$C_2 T_2^{t_2} C_2^{-1} R\ T_1 = T_1 R\ C_2 T_2^{t_2} C_2^{-1}\ .$$

As done before, multipling on the left by C_2^{-1} and on the right by C_2 and using the commutativity between C_2 and T_1, we have

$$T_2^{t_2} C_2^{-1} R\ C_2 T_1 = T_1 C_2^{-1} R\ C_2 T_2^{t_2}\ ,$$

that we write in the form

$$[(C_2^{-1} R\ C_2)^{t_2} T_2]^{t_2} T_1 = T_1 [T_2 (C_2^{-1} R\ C_2)^{t_2}]^{t_2}\ .$$

Then we transpose t_2 :

$$(C_2^{-1} R C_2)^{t_2} T_2 T_1 = T_1 T_2 (C_2^{-1} R C_2)^{t_2}\ . \tag{1.14}$$

We may conclude that for

$$\mu^{-1} (R^{-1})^{t_2} C_2^{-1} R^{-1} = C_2^{-1}\ , \tag{1.15}$$

μ a constant paramter, the previous equation is satisfied. Putting eqs. (1.13) and (1.15) together we have

$$\begin{aligned} \nu R C_1 R^{t_1} = C_1 \quad &\Longrightarrow \quad R^{-1} = -C_1 R^{t_1} C_1 \nu \\ &\Longrightarrow \quad\ R^t = \mu \nu^{-1} C_1 C_2 R C_2 C_1 \end{aligned}$$

For $\mu = \nu = 1$ eq. (1.13), (1.15) reduce to eq. (1.10) of [57]. We write this condition in components:

$$(C_1 C_2 R C_2 C_1)_{ij,kl} = \sum_{a,b,c,d} C_{ia} C_{jb} R_{ab,cd} C_{dl} C_{ck}$$

If $C_{ij} \propto \delta ij'$ then

$$\mu \nu^{-1} R_{i'j',k'l'} \propto R_{kl,ij} \tag{1.16}$$

This force the matrix R to be of a particular form.

Yang-Baxter equations

The form of R as given in (1.4) automatically assures that Yang-Baxter equations are satisfied. Let R be of the form

$$R = \sum a_{ij}\, e_i{}^i \otimes e_j{}^j + \sum_{i>j} b_{ij}\, e_i{}^j \otimes e_j{}^i + \sum_{i>j} c_{ij}\, e_i{}^j \otimes e_{i'}{}^{j'}\ .$$

Then

$$R_{12} \;=\; \sum a_{ij}\, e_i{}^i \otimes e_j{}^j \otimes 1 + \sum_{i>j} b_{ij}\, e_i{}^j \otimes e_j{}^i \otimes 1 + \sum_{i>j} c_{ij}\, e_i{}^j \otimes e_{i'}{}^{j'} \otimes 1 \;;$$

and similarly

$$\begin{aligned} R_{13} &= \sum a_{ij}\, e_i{}^i \otimes 1 \otimes e_j{}^j + \sum_{i>j} b_{ij}\, e_i{}^j \otimes 1 \otimes e_j{}^i + \sum_{i>j} c_{ij}\, e_i{}^j \otimes 1 \otimes e_{i'}{}^{j'} \;; \\ R_{23} &= \sum a_{ij} \otimes e_i{}^i \otimes e_j{}^j + \sum_{i>j} b_{ij} \otimes e_i{}^j \otimes e_j{}^i + \sum_{i>j} c_{ij} \otimes e_i{}^j \otimes e_{i'}{}^{j'} \;. \end{aligned}$$

and hence it is straightforward to check that

$$R_{12}R_{13}R_{23} = \sum_{i,j,l} a_{ij}a_{il}a_{jl} e_i{}^i \otimes e_j{}^j \otimes e_l{}^l = R_{23}R_{13}R_{12}$$

The example of $A(Sp_q(2))$.

We discuss in details a concrete example setting $N = 4$. The corresponding algebra $A(Sp_q(2))$ will play a crucial role in the construction of the (deformed) instanton bundle in Ch. 2.

The algebra $A(Sp_q(4,\mathbb{C}))$ is generated by the elements t_{ij}, $i,j = 1,\dots 4$ subjected to RTT equations. Esplicitly the matrix R (1.4) is given by

$$R = \left(\begin{array}{cccc|cccc|cccc|cccc}
q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & \lambda & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{\lambda}{q} & 0 & 0 & \frac{1}{q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\lambda}{q^3} & 0 & 0 & \lambda+\frac{\lambda}{q^2} & 0 & 0 & \frac{1}{q} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \lambda+\frac{\lambda}{q^4} & 0 & 0 & \frac{\lambda}{q^3} & 0 & 0 & -\frac{\lambda}{q} & 0 & 0 & \frac{1}{q} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q
\end{array}\right) \qquad (1.17)$$

where $\lambda := q - q^{-1}$. It is an invertible matrix with inverse $R^{-1} = R(q \to q^{-1})$.

The symplectic structure is given by the C matrix

$$C = \begin{pmatrix} 0 & 0 & 0 & q^{-2} \\ 0 & 0 & q^{-1} & 0 \\ 0 & -q & 0 & 0 \\ -q^2 & 0 & 0 & 0 \end{pmatrix} \tag{1.18}$$

Remark 1.1. *In the identification between quaternions and reals, $\mathbb{H} \simeq \mathbb{R}^4$ we can associate three matrices to the quaternions units i, j, k. In the quantum case, the three matrices*

$$I_q = \begin{pmatrix} 0 & -q^{-a} & 0 & 0 \\ q^a & 0 & 0 & 0 \\ 0 & 0 & -q^{a-1} & 0 \\ 0 & 0 & 0 & q^{1-a} \end{pmatrix} \; ; \; J_q = \begin{pmatrix} 0 & 0 & q^{-1-a} & 0 \\ 0 & 0 & 0 & -q^{a-2} \\ -q^{a+1} & 0 & 0 & 0 \\ 0 & q^{2-a} & 0 & 0 \end{pmatrix} \; ; \; K_q = C \tag{1.19}$$

have the properties $I_q^2 = J_q^2 = K_q^2 = -\mathbb{I}$ and $I_q J_q = K_q$ and cyclic. Furthermore they reduce to the classical ones at $q = 1$.

The explicit commutation rules among the generators of this algebra obtained from RTT equations are listed in Appendix A.
The matrix $S(T) = CT^tC^{-1}$ as the following form

$$S(T) = \begin{pmatrix} t_{44} & q^{-1}t_{34} & -q^{-3}t_{24} & -q^{-4}t_{14} \\ qt_{43} & t_{33} & -q^{-2}t_{23} & -q^{-3}t_{13} \\ -q^3t_{42} & -q^2t_{32} & t_{22} & q^{-1}t_{12} \\ -q^4t_{41} & -q^3t_{31} & qt_{21} & t_{11} \end{pmatrix} \tag{1.20}$$

We restrict q to be real and we consider the comapct real form $A(Sp_q(2))$ of $A(Sp_q(4, \mathbb{C}))$ obtained by setting $T^* = S(T)^t$. In components

$$T^*_{ij} = S(T)_{ji} = -q^{\rho_{j'}+\rho_i}\varepsilon_j\varepsilon_{i'}T_{i'}{}^{j'} \ .$$

Hence T in $Sp_q(2)$ assumes the form

$$T = \begin{pmatrix} \bar{t}_{44} & q\bar{t}_{43} & t_{13} & t_{14} \\ q^{-1}\bar{t}_{34} & \bar{t}_{33} & t_{23} & t_{24} \\ -q^{-3}\bar{t}_{24} & -q^{-2}\bar{t}_{23} & t_{33} & t_{34} \\ -q^{-4}\bar{t}_{14} & -q^{-3}\bar{t}_{13} & t_{43} & t_{44} \end{pmatrix} \tag{1.21}$$

1.1.2 Odd spheres from quantum symplectic groups

Let us now finally come to the construction of spheres from symplectic quantum groups $A(Sp_q(n))$. We concentrate on the elements of the last column of the defining matrix T and those of the last row of $S(T)$:

$$x_i = T_i{}^N \ , \quad v^j = S(T)_N{}^j \ , \quad i, j = 1, \ldots, N \ .$$

Using RTT equations we can show that these generators give subalgebras of $A(Sp_q(N,\mathbb{C}))$ and furthermore that, with the natural involution (1.8), the algebra generated by the $\{x_i,\ v^j\}$ can be thought of as the algebra $A(S_q^{4n-1})$ of polynomial functions on a quantum sphere of 'dimension' $4n-1$.
In components the RTT equations are given by

$$R_{ij}{}^{kp}\ T_k{}^r\ T_p{}^s = T_j{}^p\ T_i{}^m\ R_{mp}{}^{rs}\ . \tag{1.22}$$

Hence

$$R_{ij}{}^{kl}\ T_k{}^r = T_j{}^p\ T_i{}^m\ R_{mp}{}^{rs} S(T)_s{}^l\ ,$$

and in turn

$$S(T)_p{}^j\ R_{ij}{}^{kl} = T_i{}^a\ R_{ap}{}^{rs}\ S(T)_s{}^l\ S(T)_r{}^k\ ,$$

so that

$$S(T)_a{}^i\ S(T)_p{}^j\ R_{ij}{}^{kl} = R_{ap}{}^{rs}\ S(T)_s{}^l\ S(T)_r{}^k\ . \tag{1.23}$$

Conversely, if we multiply $R_{ij}{}^{kp}\ T_k{}^r = T_j{}^l\ T_i{}^m\ R_{ml}{}^{rs}\ S(T)_s{}^p$ on the left by $S(T)$ we have

$$S(T)_l{}^j\ R_{ij}{}^{kp}\ T_k{}^r = T_i{}^m\ R_{ml}{}^{rs}\ S(T)_s{}^p\ . \tag{1.24}$$

We shall use equations (1.22), (1.23) and (1.24) to describe the algebra generated by the x_i's and by the v^i's.

Commutation rules for x_i

We start with eq.(1.22) $R_{ij,kp}\ T_{kr}\ T_{ps} = T_{jp}\ T_{im}\ R_{mp,rs}$ and show that the x_i generate an algebra. Let $r = s = N$:

$$R_{ij,kp}\ x_k x_p = T_{jp}\ T_{im}\ R_{mp,NN} \tag{1.25}$$

The only element $R_{mp,NN}\ \propto\ e_{mN}\otimes e_{pN},\ m,p\leq N$ different from zero is $R_{NN,NN} = q$:

$$R_{ij,kp}\ x_k x_p = q\ x_j x_i \tag{1.26}$$

We can write explicitly the commutation relations. We have

$$(e_{mn}\otimes e_{rs})_{ij,kp}\ x_k x_p = \delta_{mi}\delta_{rj}\ x_n x_s$$

and we consider each summand of R_q separately:

1.
$$q\sum_{\substack{a=1\\ a\neq a'}}^{N}(e_{aa}\otimes e_{aa})_{ij,kp}\ x_k x_p = q\ x_i x_i\ \delta_{ij}$$

2.
$$\sum_{\substack{a,b=1\\ a\neq b,b'}}^{N} (e_{aa}\otimes e_{bb})_{ij,kp}\ x_k x_p = x_i x_j, \quad i\neq j,j'$$

3.
$$q^{-1}\sum_{\substack{a=1\\ a\neq a'}}^{N} (e_{a'a'}\otimes e_{aa})_{ij,kp}\ x_k x_p = q^{-1} x_i x_j \delta_{ij'}$$

4.
$$(q-q^{-1})\sum_{\substack{a,b=1\\ a>b}}^{N} (e_{ab}\otimes e_{ba})_{ij,kp}\ x_k x_p = (q-q^{-1}) x_j x_i, \quad i>j$$

5.
$$-(q-q^{-1})\sum_{\substack{a,b=1\\ a>b}}^{N} q^{\rho_a-\rho_b}\varepsilon_a\varepsilon_b\ (e_{ab}\otimes e_{a'b'})_{ij,kp}\ x_k x_p =$$
$$-(q-q^{-1})\sum_{b=1}^{i-1} q^{\rho_i-\rho_b}\varepsilon_i\varepsilon_b\ \delta_{i'j}\ x_b x_{b'}$$

Now we can write down commutations relations between the x_i :

$$\begin{aligned} &x_i x_j = q x_j x_i\ , \quad i<j, \quad i\neq j'\ , \\ &x_{i'} x_i = q^{-2} x_i x_{i'} + (q^{-2}-1)\sum_{k=1}^{i-1} q^{\rho_i-\rho_k}\varepsilon_i\varepsilon_k\ x_k x_{k'}\ , \quad i<i'\ . \end{aligned} \tag{1.27}$$

Commutation rules for v^i

Take eq.(1.23): $S(T_{ai})\ S(T_{pj})\ R_{ij,kl} = R_{ap,rs}\ S(T_{sl})\ S(T_{rk})$ and let $a=p=N$:

$$v^i v^j R_{ij,kl} = R_{NN,rs} S(T_{sl}) S(T_{rk})$$

The sum on the right reduce to $R_{NN,NN}S(T_{Nl})S(T_{Nk})$ since
$R_{NN,rs}\sim e_{Nr}\otimes e_{Ns}$ and, by construction, the only term of this type in R is obtained for $r=s=N$.
The v^i's algebra is given by

$$v^l v^g R_{lg,ji} = q v^i v^j \tag{1.28}$$

As before we split R. Using

$$v^l v^g (e_{mn}\otimes e_{rs})_{lg,ji} = \delta_{si}\delta_{nj}\ v^m v^r$$

we have

1.
$$qv^l v^g \sum_{\substack{a=1 \\ a\neq a'}}^{N} (e_{aa} \otimes e_{aa})_{lg,ji} = qv^i v^i \delta_{ij}$$

2.
$$v^l v^g \sum_{\substack{a,b=1 \\ a\neq b,b'}}^{N} (e_{aa} \otimes e_{bb})_{lg,ji} = v^j v^i, \quad i \neq j, j'$$

3.
$$q^{-1} v^l v^g \sum_{\substack{a=1 \\ a\neq a'}}^{N} (e_{a'a'} \otimes e_{aa})_{lg,ji} = q^{-1} v^j v^i \delta_{ij'}$$

4.
$$(q-q^{-1}) v^l v^g \sum_{\substack{a,b=1 \\ a>b}}^{N} (e_{ab} \otimes e_{ba})_{lg,ji} = (q-q^{-1}) v^i v^j, \quad i > j$$

5.
$$-(q-q^{-1}) v^l v^g \sum_{\substack{a,b=1 \\ a>b}}^{N} q^{\rho_a - \rho_b} \ \varepsilon_a \varepsilon_b \ (e_{ab} \otimes e_{a'b'})_{lg,ji} =$$
$$-(q-q^{-1}) \sum_{b=j+1}^{N} q^{\rho_a - \rho_j} \ \varepsilon_a \varepsilon_j \delta_{i'j} \ v^a v^{a'}$$

The commutation relations between the v^i are the following: Explicitly

$$\begin{aligned} v^i v^j &= q^{-1} v^j v^i \ , \quad i < j, \quad i \neq j' \ , \\ v^{i'} v^i &= q^2 v^i v^{i'} + (q^2-1) \sum_{k=i'+1}^{N} q^{\rho_k - \rho_{i'}} \ \varepsilon_k \varepsilon_{i'} v^k v^{k'} \ , \quad i < i' \ . \end{aligned} \tag{1.29}$$

Commutation rules for x_i, v^j

Finally let $l = r = N$ in eq. (1.24):

$$v^j R_{ij,kp} \ x_k = T_{im} R_{mN,Ns} S(T_{sp})$$

Observing once more that the only term in the matrix R of the form $e_{mN} \otimes e_{Ns}$, $m \leq N$ is $e_{NN} \otimes e_{NN}$ we have

$$v^j R_{ij,kp} \ x_k = q \ x_i v^p \tag{1.30}$$

From

$$v^l (e_{mn} \otimes e_{rs})_{il,kj} \ x_k = v^r x_n \delta_{mi} \delta_{sj}$$

we have

1.
$$qv^l \sum_{\substack{a=1\\a\neq a'}}^{N} (e_{aa}\otimes e_{aa})_{il,kj}\ x_k = q\ v^i x_i\ \delta_{ij}$$

2.
$$v^l \sum_{\substack{a,b=1\\a\neq b,b'}}^{N} (e_{aa}\otimes e_{bb})_{il,kj}\ x_k = v^j x_i, \quad i\neq j,j'$$

3.
$$q^{-1}v^l \sum_{\substack{a=1\\a\neq a'}}^{N} (e_{a'a'}\otimes e_{aa})_{il,kj}\ x_k = q^{-1}v^j x_i\ \delta_{ij'}$$

4.
$$(q-q^{-1})v^l \sum_{\substack{a,b=1\\a>b}}^{N} (e_{ab}\otimes e_{ba})_{il,kj}\ x_k = (q-q^{-1})\sum_{b=1}^{i-1} v^b x_b\ \delta_{ij}$$

5.
$$-(q-q^{-1})v^l \sum_{\substack{a,b=1\\a>b}}^{N} q^{\rho_a-\rho_b}\varepsilon_a\varepsilon_b\ (e_{ab}\otimes e_{a'b'})_{il,kj} = \\ -(q-q^{-1})q^{\rho_i-\rho_{j'}}\varepsilon_i\varepsilon_{j'}\ v^{i'}x_{j'}, \qquad i>j'$$

Summarizing, explicitly the mixed commutation rules for the algebra $\mathbb{C}_q[x_i,v^j]$ read,

$$\begin{aligned}
&x_i v^i = v^i x_i + (1-q^{-2})\sum_{k=1}^{i-1} v^k x_k + \underbrace{(1-q^{-2})q^{\rho_i-\rho_{i'}}v^{i'}x_{i'}}_{if\quad i>i'}\ ,\\
&x_i v^{i'} = q^{-2}v^{i'}x_i\ ,\\
&x_i v^j = q^{-1}v^j x^i\ ,\quad i\neq j \quad \text{and} \quad i<j'\\
&x_i v^j = q^{-1}v^j x^i + (q^{-2}-1)q^{\rho_i-\rho_{j'}}\ \varepsilon_i\varepsilon_{j'}v^{i'}x_{j'}\ ,\quad i\neq j \quad \text{and} \quad i>j'\ .
\end{aligned} \tag{1.31}$$

The quantum spheres S_q^{4n-1}

Let us observe that with the anti-involution (1.8) $T^* = S(T)^t$ we have the identification

$$v^i = S(T)_N{}^i = \overline{x}^i\ ,$$

where $\overline{x}^i$ denotes the conjugate $(x_i)^*$ of x_i. The subalgebra $A(S_q^{4n-1})$ of $A(Sp_q(n))$ generated by $\{x_i, v^i = \overline{x}^i,\ i=1,\ldots,2n\}$ is the algebra of polynomial functions on a sphere. Indeed

$$S(T)T = I \ \Rightarrow\ \sum S(T)_N{}^i T_i{}^N = \delta_N^N = 1$$

i.e.

$$\sum_i \overline{x}^i x_i = 1 \ . \tag{1.32}$$

Note that the restriction of the comultiplication is a natural left coaction:

$$\Delta_L : A(S_q^{4n-1}) \longrightarrow A(Sp_q(n)) \otimes A(S_q^{4n-1}) \ .$$

The fact that Δ_L is an algebra map then implies that $A(S_q^{4n-1})$ is a comodule algebra over $A(Sp_q(n))$.
At $q = 1$ this algebra reduces to the algebra of polynomial functions over the sphere S^{4n-1} as homogeneous spaces of the symplectic group $Sp(n)$: $S^{4n-1} = Sp(n)/Sp(n-1)$.

In the first part of Ch. 2 we will elaborate more on the structure of these algebras at $n = 2$ with $A(S_q^7)$ entering into the construction of a quantum principal bundle.

1.2 Spheres from idempotents, θ deformations

We recall here some facts about the construction of noncommutative manifolds in terms of θ deformations as introduced in [21]. Here we are mainly interested in the case of spheres. We refer to [21], [45] for details. For selfconsistency we will briefly recall some basic notions of cyclic homology and cohomology in the appendix 1.A at the end of this Chapter.

Starting from general considerations of noncommutative differential topology, the algebra of functions A on such a noncommutative n-sphere is generated by the matrix components of a cycle of dimension n of the K-theory of A. This cycle being given by the Chern-Connes character of an idempotent e or a unitary $u \in Mat(A)$ in the even and odd cases respectively. We recall here only the even case, starting by the definition of the Chern character of an idempotent.

Definition 1.1. *Let A be an associative unital algebra over $\mathbb{C}$ and e an idempotent:*

$$e = (e_{ij}) \in Mat_r(A) \ , \qquad e^2 = e \ . \tag{1.33}$$

The component $ch_n(e)$ of the even (reduced) Chern-Connes character $ch_*(e) = \sum ch_n(e)$ (formal sum) of e is an element in $A \otimes \tilde{A}^{\otimes 2n}$ given by the formula

$$\begin{aligned} ch_n(e) &= \left\langle (e - \frac{1}{2} \cdot \mathbb{I}) \otimes \tilde{e} \otimes \ldots \otimes \tilde{e} \right\rangle \\ &= \lambda_n \sum (e_{i_0 i_1} - \frac{1}{2}\delta_{i_0 i_1}) \otimes \tilde{e}_{i_1 i_2} \otimes \tilde{e}_{i_2 i_3} \otimes \ldots \otimes \tilde{e}_{i_{2n} i_0} \ , \end{aligned} \tag{1.34}$$

where $\tilde{A}$ is the quotient of A by the scalar multiples of the unit and $\tilde{e}$ is the class of e in $\tilde{A}$. Here λ_n is a normalization constant.

The main point is that $ch_n(e)$ defines a cycle in the bicomplex of cyclic homology

$$Bch_n(e) = b\, ch_{n+1}(e)$$

and the map $e \mapsto ch_*(e)$ gives rise to a well defined map from the Grothendick group $K_0(A)$ of A to cyclic homology of A [18]. Spherical manifold are defined by imposing that only the top component of the Chern-Connes character of e does not vanish. More in details.

Let A_r be the algebra generated by the r^2 entries of an idempotent $e \in Mat_r(A)$ subjected to the condition $e^2 = e$. We introduce further relations in A_r by requiring that the components $ch_k(e)$ of the Chern-Connes character of e vanish if k is less than a certain integer m:

$$ch_k(e) = 0 \ , \quad \forall k < m \ . \tag{1.35}$$

Moreover, let us define an admissible morphism to be a map ρ from A_r to a generic algebra B such that

$$\rho(e)^2 = \rho(e) \ , \quad ch_k(\rho(e)) = 0 \ , \quad \forall k < m \ .$$

Then we denote by A_{mr} the quotient of A_r by the intersection of the kernels of the admissible morphisms.

It turns out that the elements of A_{mr} can be represented as polynomials P in the elements e_{ij}. This algebra can be endowed with a $*$-structure by requiring that e is self-adjoint:

$$e^* = e \ , \qquad e^*_{ij} = e_{ji} \ .$$

Furthermore A_{mr} can be endowed with a norm: if p is a polynomial $p = p(e_{ij})$ in A_{mr} we set

$$\|p\| := sup\|\pi(p)\|$$

where π ranges through all representations of A_{mr}. For each p this quantity is finite. We rename A_{mr} to be the completion with respect to the above norm.

Allowing to m and r to vary, one obtains different algebras. In particular for $m = 2$ and $r = 4$, equations (1.35) admits both a commutative and a noncommutative solution. The latter being a one-parameter family of noncommutative 4-spheres $A(S^4_\theta)$. The algebras $A(S^4_\theta)$ consitute deformations of the $*$-algebra of polynomial functions on the classical 4-sphere; the classical case is recovered at $\theta = 0$. Finally, we stress that these algebras can be endowed with a noncommutative structure given by a spectral triple, i.e. are noncommutative manifolds in the sense of [19], see [21].

We will remind the structure of the resulting algebra $A(S^4_\theta)$ in Sect. 4.2.1 and we refer to the original paper [21] for the detailed construction.

1.A Appendix: Some elements of cyclic homology and cohomology

We remind few elements of cyclic homology following [18], [49] to whom we refer for details.

Let A be an associative algebra over $\mathbb{C}$. For any integer n, let $C_n(A) := A^{\otimes n+1}$, i.e. the tensor product of A with itself $(n+1)$ times. Define the map

$$\begin{array}{rcl} b : C_n(A) & \longrightarrow & C_{n-1}(A) \\ a_0 \otimes \ldots \otimes a_n & \mapsto & \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1} \ . \end{array}$$

Since $b^2 = 0$, we have a chain complex $(C_*(A) := \oplus_n C_n(A), b)$. By definition, the *Hochschild homology* $HH_*(A)$ of A is the homology of this complex:

$$HH_n(A) := H_n(C_*(A), b) = Z_n/B_n$$

where Z_n, B_n are respectively the kernel and the image of the boundary map b:

$$Z_n = ker(b : C_n(A) \to C_{n-1}(A)) \quad , \quad B_n = im(b : C_{n+1}(A) \to C_n(A)) \ .$$

We can also introduce the map

$$B : C_n(A) \longrightarrow C_{n+1}(A) \ , \qquad B = B_0 \circ A$$

with

$$\begin{array}{rcl} B_0(a_0 \otimes a_1 \otimes \ldots \otimes a_n) & := & 1 \otimes a_0 \otimes a_1 \otimes \ldots \otimes a_n \\ A(a_0 \otimes a_1 \otimes \ldots \otimes a_n) & := & \frac{1}{n+1} \sum_{i=0}^{n} (-1)^{ni} a_i \otimes a_{i+1} \otimes \ldots \otimes a_{i-1} \ . \end{array}$$

We have $B^2 = 0$ and $b \circ B + B \circ b = 0$ and thus the bi-complex $(C_*(A), b, B)$. The *cyclic homology* $HC_*(A)$ of A is the homology of the total complex $(CC(A), b + B)$ in which the n-th term is given by $CC_n(A) := \oplus_{p+q=n} C_{p-q}(A)$ and

$$HC_n(A) := H_n(CC(A), b + B) = Z_n^\lambda / B_n^\lambda$$

with cyclic cycles and cocycles given respectively by

$$Z_n^\lambda = ker(b{+}B : CC_n(A) \to CC_{n-1}(A)) \quad , \quad B_n^\lambda = im(b{+}B : CC_{n+1}(A) \to CC_n(A)) \ .$$

Let us now dualize the above picture giving few notions of cyclic cohomology. Let $C^n(A) := Hom(A^{\otimes n+1}, \mathbb{C})$ the space of n-cochains, i.e. by definition a Hochschild cochain is a $(n+1)$-linear functional on A. We define the coboundary map

$$\begin{aligned} b : C^n(A) &\longrightarrow C^{n+1}(A) \\ f(a_0, \dots, a_{n+1}) &\mapsto \sum_{i=0}^{n} (-1)^i f(a_0, \dots, a_i a_{i+1}, \dots, a_{n+1}) + (-1)^{n+1} f(a_{n+1}a_0, a_1, \dots a_n) \ . \end{aligned}$$

Then $b^2 = 0$. The *Hochschild cohomology* $HH^*(A)$ of A is by definition the cohomology of the complex $(C^*(A) := \oplus_n C^n(A), b)$:

$$HH^n(A) := H^n(C^*(A), b) = Z^n / B^n$$

with cocycles $Z^n = ker(b : C^n(A) \to C^{n+1}(A))$ and coboundaries $B^n = im(b : C^{n-1}(A) \to C^n(A))$.

We now introduce the following definition [18]: a cochain $f \in C^n(A)$ is said to be *cyclic* if it satisfies

$$f(a_0, \dots, a_n) = (-1)^n f(a_n, a_0, \dots, a_{n-1}) \ .$$

A cyclic cocycle is a cyclic cochain f for which $bf = 0$.

These cyclic cochains form a sub-module of $C^n(A)$ that we denote by C^n_λ. Since the image under b of a cyclic cochain is still cyclic, then $(C^*_\lambda(A) := \oplus_n C^n_\lambda(A), b)$ is a well defined sub-complex of $(C^*(A) = \oplus_n C^n(A), b)$. We define the *cyclic cohomology* $HC^*(A)$ of the algebra A to be the cohomology of this sub-complex:

$$HC^n(A) := H^n(C^*_\lambda(A), b) = Z^n_\lambda / B^n_\lambda \ ,$$

with cyclic cocycles and coboundaries given respectively by

$$Z^n_\lambda = ker(b : C^n_\lambda(A) \to C^{n+1}_\lambda(A)) \quad , \quad B^n_\lambda = im(b : C^{n-1}_\lambda(A) \to C^n_\lambda(A)) \ .$$

We conclude by recalling the so called *periodicity operator* S that we will need later in the definition of the Chern character in K-homology. This is a map of degree 2 between cyclic cocycles defined by

$$\begin{aligned} S : Z^{n-1}_\lambda &\longrightarrow Z^{n+1}_\lambda \\ f(a_0, \dots a_{n+1}) &\longmapsto -\frac{1}{n(n+1)} \sum_{i=1}^{n} f(a_0, \dots a_{i-1} a_i a_{i+1}, \dots a_{n+1}) + \\ &\quad -\frac{1}{n(n+1)} \sum_{1 \le i < j \le n}^{n} (-1)^{i+j} f(a_0, \dots a_{i-1} a_i, \dots, a_j a_{j+1}, \dots a_{n+1}) \ . \end{aligned} \tag{1.36}$$

The induced map in cohomology will be denoted in the same way, $S : HC^n(A) \longrightarrow HC^{n+2}(A)$.

1.A.1 K-homology: Fredholm modules

In what follows we briefly recall few definitions and results concerning k-homology following once more [18], [49].

We remind that the K-homology of an involutive algebra A is given in terms of homotopy classes of Fredholm modules [18], [37]. We are interested here in *even* Fredholm modules:

Definition 1.2. *[18] An even Fredholm module $\mu := (\mathcal{H}, F, \gamma)$ over an involutive algebra A is given by*

1. *a representation Ψ of the algebra by means of bounded operators on a Hilbert space $\mathcal{H}$,*
2. *an operator F on $\mathcal{H}$ such that $F^2 = I$, $F^* = F$ and*
$$[F, \Psi(a)] \text{ is compact } \forall a \in A \ ,$$
3. *a $\mathbb{Z}_2$ grading γ of $\mathcal{H}$ such that $\gamma^* = \gamma$, $\gamma^2 = I$ and*
$$F\gamma + \gamma F = 0, \quad \Psi(a)\gamma - \gamma\Psi(a) = 0, \quad \forall a \in A \ .$$

We denote by $\mathcal{H}^{\pm}$, $\Psi^{\pm}$ the components of $\mathcal{H}$, Ψ with respect to the grading. The existence of such a grading γ distinguishes even Fredholm modules from odd ones.

We need also the following

Definition 1.3. *A Fredholm module is p-summable if for any element a of the algebra*

$$[F, \Psi(a)] \in \mathcal{L}^p(\mathcal{H}) \ ,$$

where $\mathcal{L}^p(\mathcal{H})$ is the Schatten ideal made of bounded operators T such that $\sum(\mu_n)^p < \infty$, μ_n eigenvalues of $|T|$.

Notice that given a Fredholm module over A, it is possible to construct a Fredholm module $(\mathcal{H}_r, F_r, \gamma_r)$ for the algebra $Mat_r(A) \simeq A \otimes Mat_r(\mathbb{C})$ (for generic $r \in \mathbb{N}$) by setting

$$\mathcal{H}_r = \mathcal{H} \otimes \mathbb{C}^r \ , \quad \Psi_r = \Psi \otimes id \ , \quad F_r = F \otimes \mathbb{I} \ , \quad \gamma_r = \gamma \otimes \mathbb{I} \ .$$

We come now to the definition of the Chern-Connes character of a Fredholm module, [18]. Given an operator T on $\mathcal{H}$ we define

$$Tr'(T) := \frac{1}{2} Tr(F[F, T])$$

Then let n be a generic integer. Assume that the even Fredholm module $(\mathcal{H}, F, \gamma)$ over A is (at least) $n+1$ summable. Then one defines the *character* of the Fredholm module $(\mathcal{H}, F, \gamma)$ to be the cyclic cocycle $\tau^n \in Z^n_\lambda(A)$

$$\tau^n(a_0, a_1, \ldots, a_n) := Tr'(\gamma a_0[F, a_1], \ldots [F, a_n]) . \tag{1.37}$$

It is defined in terms of *quantized differential forms*, i.e. operators of the form $\omega = a_0[F, a_1], \ldots [F, a_n]$, $a_i \in A$, see [18].

Notice that the integer n is not fixed but it is only required to fulfil $(n+1)$-summability $[F, \Psi(a)] \in \mathcal{L}(\mathcal{H})$, $\forall a \in A$. Hence, using the inclusion $\mathcal{L}^p \subseteq \mathcal{L}^{p'}$, $p \leq p'$, we get a sequence of cyclic cocycles τ^{n+2q}, $q \in \mathbb{N}$ (with the same parity). These cocylcles are related by means of the periodicity operator $S : HC^n(A) \longrightarrow HC^{n+2}(A)$:

$$\tau^{m+2} = -\frac{2}{m+2} S(\tau^m) , \quad \forall m = n + 2q , \; q \geq 0 .$$

This sequence $\{\tau^{n+2q}\}_{q\in\mathbb{N}}$ determines a class $[\tau^F]$ in the periodic cyclic cohomology of A defined as the inductive limit $H^*(A) :=\stackrel{lim}{\rightarrow} (HC^n(A), S)$, [18]. By definition, the *Connes-Chern character* $ch_*(\mathcal{H}, F, \gamma) \in H^*(A)$ is the periodic cyclic cohomology class $[\tau^F]$.

After these definitons, we finally recall the following important result [39] in the "even" case involving unitaries. Analogous result can be stated in the odd case in terms of unitaries.

Theorem 1.1. *Let $(\mathcal{H}, F, \gamma)$ be a Fredholm module over an algebra A, let $e \in Mat_r(A)$ be a self-adjoint idempotent. Then*

$$\Psi^-_r(e) F_r \Psi^+_r(e) : \Psi^+_r(e)\mathcal{H}_r \longrightarrow \Psi^-_r(e)\mathcal{H}_r$$

is a Fredholm operator whose index depends only on the class of e in K-theory, $[e] \in K_0(A)$. This leads to the additive map

$$\begin{array}{rcl} \varphi : K_0(A) & \longrightarrow & \mathbb{Z} \\ {[e]} & \mapsto & Index(\Psi^-_r(e) F_r \Psi^+_r(e)) . \end{array}$$

If A is a C^*-algebra, the map φ depends only on the homotopy class of the Fredholm module, [39].

The index pairing $\varphi([e])$ can be given using the Chern-Connes characters in cyclic cohomology and homology, respectively $ch^*(\mathcal{H}, F, \gamma) \in HC^*(A)$ and $ch_*([e]) \in HC_*(A)$, through [18]

$$\varphi([e]) = \langle ch^*(\mathcal{H}, F, \gamma), ch_*([e]) \rangle .$$

Chapter 2

The instanton bundle

The purpose of the research which moved us to the study of quantum symplectic spheres was the construction of a deformed version of the Hopf bundle

$$S^7 \simeq Sp(2)/Sp(1) \longrightarrow S^4 \simeq Sp(2)/(Sp(1) \times Sp(1)) \ .$$

The starting point for such a deformation is the rewritting of the classical theory from the algebraic point of view of associated vector bundles considered as finite projective modules. This dual picture illustrated in [44] is the topic of the first section of this Chapter.

The discussion then moves to noncommutative geometry. The algebra $A(S^7_q)$ introduced in the previous chapter becomes in Sect. 2.2 the total space of a quantum $SU_q(2)$-fibration in which the base space $A(S^4_q)$ is firstly given in terms of a projection and then described as the space of coinvariants of the $SU_q(2)$-coaction. (This fact is presented here with two proofs, one of those presented in App. 2.A.) Here we limit ourself to describe the algebras involved in the construction while we postpone to Ch. 3 the study of the nature of this bundle.

Finally a Fredholm module is constructed over $A(S^4_q)$ in order to compute the Chern-Connes pairing between K-homolgy and K-theory giving the "charge" of the bundle.

Note. The paper *"A Hopf bundle over a quantum four-sphere from the symplectic group"* by G. Landi, C. Pagani, C. Reina, [46] will be the common reference of this Chapter.

2.1 The (classical) Hopf fibration $S^7 \to S^4$

As said, in order to formulate a quantum version of the $SU(2)$-Hopf bundle $S^7 \to S^4$, the starting point is to dualise the classical picture. This consists in working with a finite projective module representing the module of sections of the vector bundle on which instantons live. Taking advatage of Serre-Swan theorem [63], this module is identified with the image of a suitable projection $p \in Mat_4(C^\infty(S^4, \mathbb{C}))$, i.e. a self adjoint idempotent matrix $p = p^2 = p^*$ whose entries are elements of the algebra $C^\infty(S^4, \mathbb{C})$ of smooth functions defined over the base space.

We review the classical construction of the basic anti-instanton bundle of charge -1 following [44]. We begin with some notations.

A quaternion $q \in \mathbb{H}$ is identified with $\mathbb{C}^2$ by

$$\begin{aligned} q = q_o + q_1 i + q_2 j + q_3 k &= (q_o + q_1 i) + (q_2 + q_3 i) j \\ &=: v_1 + v_2 j = v_1 + j\overline{v}_2 \simeq (v_1, v_2) \in \mathbb{C}^2 \ , \end{aligned}$$

with conjugate

$$\overline{q} = \overline{v}_1 - j\overline{v}_2 = \overline{v}_1 - v_2 j \simeq (\overline{v}_1, -v_2) \ .$$

The right multiplication of q by $q' = w_1 + w_2 j$ reads

$$(v_1, v_2) \longmapsto (v_1 w_1 - v_2 \overline{w}_2) + (v_1 w_2 + v_2 \overline{w}_1) j = (v_1, v_2) \begin{pmatrix} w_1 & w_2 \\ -\overline{w}_2 & \overline{w}_1 \end{pmatrix} \tag{2.1}$$

or equivalently, we can consider the left multiplication of q by q'

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \longmapsto \begin{pmatrix} w_1 & w_2 \\ -\overline{w}_2 & \overline{w}_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{2.2}$$

The equations (2.1), (2.2) give respectively the right and the left representation of $SU(2)$ on $\mathbb{C}^2$.

The generic element w of the group $SU(2)$, in accordance with the group isomorphism $SU(2) \simeq Sp(1)$ provided by (2.2), is written in the form

$$w = \begin{pmatrix} w_1 & w_2 \\ -\overline{w}_2 & \overline{w}_1 \end{pmatrix} \ ; \quad \overline{w} w = 1 \ . \tag{2.3}$$

The total space of the $SU(2)$ principal fibration over the sphere S^4 is

$$S^7 = \{ z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \ , \ \sum_{i=1}^{4} |z_i|^2 = 1 \} \ ,$$

with right block-diagonal action

$$S^7 \times SU(2) \ \rightarrow \ S^7 \ , \quad z \cdot w := (z_1, z_2, z_3, z_4) \begin{pmatrix} w_1 & w_2 & 0 & 0 \\ -\overline{w}_2 & \overline{w}_1 & 0 & 0 \\ 0 & 0 & w_1 & w_2 \\ 0 & 0 & -\overline{w}_2 & \overline{w}_1 \end{pmatrix} \ . \tag{2.4}$$

The bundle projection $\pi : S^7 \to S^4$ is just the Hopf projection and it can be explicitly given as $\pi(z_1, z_2, z_3, z_4) := (x, \alpha, \beta)$ with

$$\begin{aligned} &x = |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 = -1 + 2(|z_1|^2 + |z_2|^2) = 1 - 2(|z_3|^2 + |z_4|^2) \ , \\ &\alpha = 2(z_1\overline{z}_3 + z_2\overline{z}_4) \ , \qquad \beta = 2(-z_1 z_4 + z_2 z_3) \ . \end{aligned} \tag{2.5}$$

with

$$\begin{aligned} |\alpha|^2 + |\beta|^2 + x^2 &= 4(|z_1|^2|z_3|^2 + |z_2|^2|z_4|^2 + |z_1|^2|z_4|^2 + |z_2|^2|z_3|^2) + \sum(|z_i|^4) + \\ &\quad 2(|z_1|^2|z_2|^2 - |z_1|^2|z_3|^2 - |z_1|^2|z_4|^2 - |z_2|^2|z_3|^2 - |z_2|^2|z_4|^2 + |z_3|^2|z_4|^2) \\ &= (\sum_{i=1}^{4} |z_i|^2)^2 = 1 \ . \end{aligned}$$

We need the rank 2 complex vector bundle E associated with the defining left representation ρ of $SU(2)$ on $\mathbb{C}^2$. The quickest way to get this is to identify S^7 with the unit sphere in the 2-dimensional quaternionic (right) $\mathbb{H}$-module $\mathbb{H}^2$:

$$S^7 = \left\{(a, b) \in \mathbb{H}^2 / |a|^2 + |b|^2 = 1\right\}$$

and S^4 with the projective line $\mathbb{P}^1(\mathbb{H})$, i.e. the set of equivalence classes $(w_1, w_2)^t \simeq (w_1, w_2)^t\lambda$ with $(w_1, w_2) \in S^7$ and $\lambda \in Sp(1) \simeq SU(2)$. In other words, action (2.4) reads

$$S^7 \times Sp(1) \to S^7 \qquad (a, b)w = (aw, bw)$$

The corresponding equivariant functions $\varphi : S^7 \to \mathbb{H}$, $\varphi(p \cdot w) = w^{-1}\varphi(p)$ are of the form

$$\varphi(a, b) = \overline{a}f + \overline{b}g \ ,$$

with f and g $\mathbb{H}$-valued functions invariant under the above action: $f, g \in C^\infty(S^4, \mathbb{H})$.

As a general result, this right $C^\infty(S^4, \mathbb{H})$-module of equivariant functions is isomorphic to the right module of sections $\Gamma(S^4, E)$ of the associated bundle $E = S^7 \times_{Sp(1)} \mathbb{C}^2$. We are ready to introduce the projection p which give to $\Gamma(S^4, E)$ the projectivity property:

$$\Gamma(S^4, E) \simeq p(C^\infty(S^4, \mathbb{H}))^4 \tag{2.6}$$

Identifying $\mathbb{H} \simeq \mathbb{C}^2$, the vector $(w_1, w_2)^t \in S^7$ reads

$$v = \begin{pmatrix} z_1 & z_2 \\ -\overline{z}_2 & \overline{z}_1 \\ z_3 & z_4 \\ -\overline{z}_4 & \overline{z}_3 \end{pmatrix} . \tag{2.7}$$

This is actually a map from S^7 to the Stieffel variety of frames for E. In particular, notice that the two vectors $|\psi_1\rangle \, , |\psi_2\rangle$ given by the columns of v are orthonormal, indeed $v^*v = \mathbb{I}_2$. As a consequence, $p := vv^* = |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|$ is a self-adjoint idempotent (a projector), $p^2 = p$, $p^* = p$. Of course p is $SU(2)$ invariant and hence its entries are functions on S^4 rather than S^7. Indeed the $Sp(1)$ action is simply given on v by $v \mapsto v \cdot w$ and hence

$$p = vv^* \longmapsto v \underbrace{ww^*}_{1} v^*$$

is invariant.

An explicit computation yields

$$p = \frac{1}{2}\begin{pmatrix} 1+x & 0 & \alpha & \beta \\ 0 & 1+x & -\overline{\beta} & \overline{\alpha} \\ \overline{\alpha} & -\beta & 1-x & 0 \\ \overline{\beta} & \alpha & 0 & 1-x \end{pmatrix}, \tag{2.8}$$

where (x, α, β) are the coordinates (2.5) on S^4. Then $p \in \mathrm{Mat}_4(C^\infty(S^4, \mathbb{C}))$ is of rank 2 by construction.
Furthermore, the explicit isomorphism (2.6) is given by

$$\begin{array}{ccc} \Gamma(S^4, E) & \longleftrightarrow & p(C^\infty(S^4, \mathbb{H}))^4 \\ \sigma = p\begin{pmatrix} f \\ g \end{pmatrix} & \longleftrightarrow & \varphi_\sigma = (\overline{a}, \overline{b})\begin{pmatrix} f \\ g \end{pmatrix} = \overline{a}f + \overline{b}g \end{array}$$

with $f, g \in C^\infty(S^4, \mathbb{H})$.

Remark 2.1. *The matrix v in* (2.7) *is a particular example of the matrices $v = Cx + Dy$ given in [1], for $n = 1$, $k = 1$, $C_0 = 0$, $C_1 = 1$, $D_0 = 1$, $D_1 = 0$. This gives the (anti-) instanton of charge -1 centered at the origin and with unit scale. The only difference is that here we identify $\mathbb{C}^4$ with $\mathbb{H}^2$ as a right $\mathbb{H}$-module. This notwithstanding, the projections constructed in the two formalisms actually coincide.*

Proof. In Atiyah [1] the two column vectors which compose the matrix v, (which we denote in the following by v_A to distinguish it from (2.7)), are

$$|\varphi_1\rangle = (z_1, z_2, z_3, z_4)^t \quad , \quad |\varphi_2\rangle = (-\overline{z}_2, \overline{z}_1, -\overline{z}_4, \overline{z}_3)^t \; .$$

The second vector $(\sigma(x)^t$ in Atiyah's notations) is obtained after the identification $q = z_1 + z_2 j$ of quaternions with $\mathbb{C}^2$ and taking the transforamtion induced on $x = (z_1, z_2, z_3, z_4) \in \mathbb{CP}^3$ by the left multiplication of $(z_1 + z_2 i,\ z_3 + z_4 i) \in \mathbb{P}^1$ by the unit j. The corresponding projector $p_A = v_A v_A^*$ reads

$$p_A = \frac{1}{2}\begin{pmatrix} 1+y & 0 & a & b \\ 0 & 1+y & -\overline{b} & \overline{a} \\ \overline{a} & -b & 1-y & 0 \\ \overline{b} & a & 0 & 1-y \end{pmatrix}, \tag{2.9}$$

where (y, a, b) are given by

$$y = |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 = x \ , \quad a = 2(\overline{z}_1 z_3 + z_2 \overline{z}_4) \ , \quad b = 2(\overline{z}_1 z_4 - z_2 \overline{z}_3) \ .$$

Hence p and p_A are equivalent (in particular they coincide) and the algebras generated by their entries are isomorphic. $\square$

The canonical connection associated with the projector,

$$\nabla := p \circ d \ : \ \Gamma^\infty(S^4, E) \ \to \ \Gamma^\infty(S^4, E) \otimes_{C^\infty(S^4,\mathbb{C})} \Omega^1(S^4, \mathbb{C}), \tag{2.10}$$

corresponds to a Lie-algebra valued ($su(2)$) 1-form A on S^7 whose matrix components are given by

$$A_{ij} = \langle \psi_i | d\psi_j \rangle \ , \qquad i, j = 1, 2 \ . \tag{2.11}$$

This connection can be used to compute the Chern character of the bundle. Out of the curvature of the connection $\nabla^2 = p(dp)^2$ one has the Chern 2-form and 4-form given respectively by

$$\begin{aligned} C_1(p) &:= -\frac{1}{2\pi i} \ \mathrm{tr}(p(dp)^2) \ , \\ C_2(p) &:= -\frac{1}{8\pi^2} \ [\mathrm{tr}(p(dp)^4) - C_1(p) C_1(p)] \ , \end{aligned} \tag{2.12}$$

with the trace tr just an ordinary matrix trace. It turns out that the 2-form $p(dp)^2$ has vanishing trace so that $C_1(p) = 0$. As for the second Chern class, a straightforward calculation shows that,

$$\begin{aligned} C_2(p) &= -\frac{1}{32\pi^2}[(x_0 dx_4 - x_4 dx_0)(d\xi)^3 + 3 dx_0 dx_4 \ \xi \ (d\xi)^2] \\ &= -\frac{3}{8\pi^2}[x_0 dx_1 dx_2 dx_3 dx_4 + x_1 dx_2 dx_3 dx_4 dx_0 + \\ &\quad x_2 dx_3 dx_4 dx_0 dx_1 + x_3 dx_4 dx_0 dx_1 dx_2 + x_4 dx_0 dx_1 dx_2 dx_3] \\ &= -\frac{3}{8\pi^2} \ d(vol(S^4)) \ . \end{aligned} \tag{2.13}$$

The second Chern number is then given by

$$c_2(p) = \int_{S^4} C_2(p) = -\frac{3}{8\pi^2} \int_{S^4} d(vol(S^4)) = -\frac{3}{8\pi^2} \frac{8}{3} \pi^2 = -1 \ . \tag{2.14}$$

The connection A in (2.11) is (anti-)self-dual, i.e. its curvature $F_A := dA + A \wedge A$ satisfies (anti-)self-duality equations, $*_H F_A = -F_A$, with $*_H$ the Hodge map of the canonical (round) metric on the sphere S^4. It is indeed the basic Yang-Mills anti-instanton found in [5].

2.2 Quantum symplectic Hopf bundle

In this section we introduce the basic ingredients will enter into the quantization of the above picture. What we need in the first instance are quantum deformations of the algebras of functions on the seven and four-spheres. The latter being given by the $SU_q(2)$-equivariants entries of a projection.

2.2.1 The total space: the symplectic 7-sphere S^7_q

We take the total space $A(S^7_q)$ to be the algebra $A(S^{2n-1}_q)$ for $n = 2$ introduced in Sect. 1.1.2. We first recall the algebra structure of $A(S^7_q)$, then we describe it as the subalgebra of coinvariants of $A(Sp_q(2))$ dualising the classical picture $S^7 \simeq Sp(2)/Sp(1)$.

The algebra $A(S^7_q)$ is generated by the elements $x_i = T_i{}^4$ and $\overline{x}^i = S(T)_4{}^i = q^{2+\rho_i}\varepsilon_{i'}T_{i'}{}^1$, for $i = 1, \dots, 4$ of the defining 4×4 matrix T of the quantum group $Sp_q(2)$ explicitly described in pag. 13. As it happens for generic n, the equations $S(T)\ T = 1$ give the sphere relation

$$\sum_{i=1}^{4} \overline{x}^i x_i = 1 \ .$$

Since we will systematically use them in the following, we shall explicitly give the commutation relations among the generators.
From (1.27), the algebra of the x_i's is given by

$$\begin{aligned}
&x_1x_2 = qx_2x_1 \ , && x_1x_3 = qx_3x_1 \ , \\
&x_2x_4 = qx_4x_2 \ , && x_3x_4 = qx_4x_3 \ , \\
&x_4x_1 = q^{-2}x_1x_4 \ , && x_3x_2 = q^{-2}x_2x_3 + q^{-2}(q^{-1} - q)x_1x_4 \ ,
\end{aligned} \tag{2.15}$$

together with their conjugates (given for general n in (1.29)).
We have also the commutation relations between the x_i and the $\overline{x}^j$ deduced from (1.30):

$$\begin{aligned}
&x_1\overline{x}^1 = \overline{x}^1x_1 \ , \qquad x_1\overline{x}^2 = q^{-1}\overline{x}^2x_1 \ , \\
&x_1\overline{x}^3 = q^{-1}\overline{x}^3x_1 \ , \quad x_1\overline{x}^4 = q^{-2}\overline{x}^4x_1 \ , \\
&\\
&x_2\overline{x}^2 = \overline{x}^2x_2 + (1 - q^{-2})\overline{x}^1x_1 \ , \\
&x_2\overline{x}^3 = q^{-2}\overline{x}^3x_2 \ , \\
&x_2\overline{x}^4 = q^{-1}\overline{x}^4x_2 + q^{-1}(q^{-2} - 1)\overline{x}^3x_1 \ , \\
&\\
&x_3\overline{x}^3 = \overline{x}^3x_3 + (1 - q^{-2})[\overline{x}^1x_1 + (1 + q^{-2})\overline{x}^2x_2] \ , \\
&x_3\overline{x}^4 = q^{-1}\overline{x}^4x_3 + (1 - q^{-2})q^{-3}\overline{x}^2x_1 \ , \\
&\\
&x_4\overline{x}^4 = \overline{x}^4x_4 + (1 - q^{-2})[(1 + q^{-4})\overline{x}^1x_1 + \overline{x}^2x_2 + \overline{x}^3x_3] \ ,
\end{aligned} \tag{2.16}$$

again with their conjugates.

Now we come to show that the algebra $A(S_q^7)$ can be realized as the subalgebra of $A(Sp_q(2))$ generated by the coinvariants under the right-coaction of $A(Sp_q(1))$, in complete analogy with the classical homogeneous space $Sp(2)/Sp(1) \simeq S^7$.

Lemma 2.1. *The two-sided *-ideal in $A(Sp_q(2))$ generated as*

$$I_q = \{T_1{}^1 - 1, T_4{}^4 - 1, T_1{}^2, T_1{}^3, T_1{}^4, T_2{}^1, T_2{}^4, T_3{}^1, T_3{}^4, T_4{}^1, T_4{}^2, T_4{}^3\}$$

with the involution (1.8) *is a Hopf ideal.*

Proof. We remind that a subspace $I \subseteq H$ of a given coalgebra (H, Δ, ε) is a coideal if

$$\Delta I \subseteq I \otimes H + H \otimes I \ , \qquad \varepsilon(I) = 0 \ .$$

If in addition H is endowed with an antipode S, then the compatibility condition which gives to I the structure of a Hopf ideal is $S(I) \subseteq I$.
Firstly, since $S(T)_i{}^j \propto T_{j'}{}^{i'}$, then $S(I_q) \subseteq I_q$. This also proves that I_q is a $*$-ideal. Then, using $\varepsilon(T) = \mathbb{I}$, $\Delta(T) = T \dot{\otimes} T$ it is easy to show that $\varepsilon(I_q) = 0$ and $\Delta(I_q) \subseteq I_q \otimes A(Sp_q(2)) + A(Sp_q(2)) \otimes I_q$. $\square$

It is a well known result that the quotient of a Hopf algebra H by a Hopf ideal is still a Hopf algebra with co-structures induced from H. Then we have the following

Proposition 2.1. *The Hopf algebra $B_q := A(Sp_q(2))/I_q$ is isomorphic to the coordinate algebra $A(SU_{q^2}(2)) \cong A(Sp_q(1))$.*

Proof. Using $\overline{T} = S(T)^t$ and setting $T_2{}^2 = \alpha$, $T_3{}^2 = \gamma$, the algebra B_q can be described as the algebra generated by the entries of the matrix

$$T' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & -q^2\overline{\gamma} & 0 \\ 0 & \gamma & \overline{\alpha} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{2.17}$$

(Compare with the matrix in (1.21).)
The commutation relations deduced from RTT equations (1.22) read:

$$\begin{array}{lll} \alpha\overline{\gamma} = q^2\overline{\gamma}\alpha \quad , & \alpha\gamma = q^2\gamma\alpha \ , & \gamma\overline{\gamma} = \overline{\gamma}\gamma \ , \\ \overline{\alpha}\alpha + \overline{\gamma}\gamma = 1 \quad ; & \alpha\overline{\alpha} + q^4\gamma\overline{\gamma} = 1 \ . & \end{array} \tag{2.18}$$

Hence, as an algebra B_q is isomorphic to the algebra $A(SU_{q^2}(2))$. Furthermore, the restriction of the coproduct of $A(Sp_q(2))$ to B_q endows the latter with a coalgebra structure, $\Delta(T') = T' \dot{\otimes} T'$, which is the same as the one of $A(SU_{q^2}(2))$.

We can conclude that also as a Hopf algebra, B_q is isomorphic to the Hopf algebra $A(SU_{q^2}(2)) \cong A(Sp_q(1))$. $\square$

We come now to the following nice result stating the correspondence

$$S^7 \simeq Sp(2)/Sp(1) \qquad \rightsquigarrow \qquad A(S^7_q) = A(Sp_q(2))^{co(A(Sp_q(1))}$$

Proposition 2.2. *The algebra $A(S^7_q) \subset A(Sp_q(2))$ is the algebra of coinvariants with respect to the natural right coaction*

$$\Delta_R : A(Sp_q(2)) \to A(Sp_q(2)) \dot{\otimes} A(Sp_q(1)) \quad ; \quad \Delta_R(T) = T \dot{\otimes} T' \, . \tag{2.19}$$

Proof. It is straightforward to show that the generators of the algebra $A(S^7_q)$ are coinvariants:

$$\Delta_R(x_i) = \Delta_R(T^4_i) = x_i \otimes 1 \;\; ; \;\; \Delta_R(\overline{x}^i) = -q^{2+\rho_i}\varepsilon_i \, \Delta_R(T^1_i) = \overline{x}^i \otimes 1$$

thus the algebra $A(S^7_q)$ is made of coinvariants. There are no other coinvariants of degree one since each row of the submatrix of T made out of the two central columns is a fundamental comodule under the coaction of $A(SU_{q^2}(2))$. Other coinvariants arising at higher even degree are of the form $(T_{i2}T_{i3} - q^2 T_{i3}T_{i2})^n$; thanks to the commutation relations of $A(Sp_q(2))$, one checks these belong to $A(S^7_q)$ as well. It is an easy computation to check that similar expressions involving elements from different rows cannot be coinvariant. $\square$

Remark 2.2. *The previous construction gives one more example of the general construction [15] of a quantum principal bundle over a quantum homogeneous space, see Sect. 3.1.1. The latter is the datum of a Hopf quotient $\pi : A(G) \to A(K)$ with the right coaction of $A(K)$ on $A(G)$ given by the reduced coproduct $\Delta_R := (id \otimes \pi)\Delta$ where Δ is the coproduct of $A(G)$. The subalgebra $B \subset A(G)$ made of the coinvariants with respect to Δ_R is called a quantum homogeneous space. To prove that it is a quantum principal bundle one needs some more assumptions and we postpone this topic to Ch. 3. In our case $A(G) = A(Sp_q(2))$, $A(K) = A(Sp_q(1))$ with $\pi(T) = T'$. We will prove in Sec. 3.3 that the resulting inclusion $B = A(S^7_q) \hookrightarrow A(Sp_q(2))$ is indeed an Hopf Galois extension and hence a quantum principal bundle.*

2.2.2 The base space: the subalgebra $A(S^4_q)$

We now face the fundamental step which constists in making the sphere $A(S^7_q)$ itself into the total space of a quantum principal bundle over a deformed 4-sphere. Unlike what we saw in the previous section, this is not a quantum homogeneous space construction and it is not obvious that such a bundle exists at all. Nonetheless the notion of quantum bundle is more general and one only needs that the total space algebra is a comodule algebra over an Hopf algebra with additional suitable properties.

The first natural step would be to construct a map from S^7_q into a deformation of the Stieffel variety of unitary frames of 2-planes in $\mathbb{C}^4$ to parallel the classical construction as recalled in the Sec. 2.1. The naive choice we have is to take as generators the elements of two (conjugate) columns of the matrix T. We are actually forced to take the first and the last columns of the matrix T because the other choice (i.e. the second and the third columns) does not yield a subalgebra since commutation relations of their elements will involve elements from the other two columns (see App. A). If we set

$$v = \begin{pmatrix} \overline{x}^4 & x_1 \\ q^{-1}\overline{x}^3 & x_2 \\ -q^{-3}\overline{x}^2 & x_3 \\ -q^{-4}\overline{x}^1 & x_4 \end{pmatrix} , \tag{2.20}$$

we have $v^*v = \mathbb{I}_2$ and the matrix $p = v\, v^*$ is a self-adjoint idempotent, i.e. $p = p^* = p^2$. At $q = 1$ the entries of p are invariant for the natural action of $SU(2)$ on S^7 and generate the algebra of polynomials on S^4. This fails to be the case at generic q due to the occurrence of extra generators e.g.

$$p_{14} = (1 - q^{-2})x_1\overline{x}^4, \quad p_{23} = (1 - q^{-2})x_2\overline{x}^3, \tag{2.21}$$

which vanish only at $q = 1$.

These facts indicate that the naive quantum analogue of the quaternionic projective line as a homogeneous space of $Sp_q(2)$ has not the right number of generators. Rather surprisingly, we shall anyhow be able to select another subalgebra of $A(S^7_q)$ which is a deformation of the algebra of polynomials on S^4 having the same number of generators. These generators come from a better choice of a projection.

Obviously, in differential geometry a principle bundle is more than just a free and effective action of a Lie group. In our example, thanks to the fact that the "structure group" is $SU_q(2)$, from Th. I of [61] further nice properties can be established. We shall elaborate more on these points later on in Sect. 3.3 .

The quantum sphere S^4_q

Firstly we introduce some notations we will need later. On the free module $\mathcal{E} := \mathbb{C}^4 \otimes A(S^7_q)$ we consider the hermitean structure given by

$$h(|\xi_1\rangle\, , |\xi_2\rangle) = \sum_{j=1}^{4} \overline{\xi_1}^j \xi_2^j \, .$$

To every element $|\xi\rangle \in \mathcal{E}$ we associate an element $\langle\xi|$ in the dual module $\mathcal{E}^*$ by using the pairing

$$\langle\xi|\,(|\eta\rangle) := \langle\xi|\eta\rangle = h(|\xi\rangle\,,|\eta\rangle).$$

Following the classical (dual) construction illustrated in Sec. 2.1, we search two elements $|\phi_1\rangle\,,\ |\phi_2\rangle$ in $\mathcal{E}$ with the property that

$$\langle\phi_1|\phi_1\rangle = 1\ , \quad \langle\phi_2|\phi_2\rangle = 1\ , \quad \langle\phi_1|\phi_2\rangle = 0\ .$$

As a consequence, the matrix valued function defined by

$$p := |\phi_1\rangle\,\langle\phi_1| + |\phi_2\rangle\,\langle\phi_2|\ , \tag{2.22}$$

is a self-adjoint idempotent (a projection).
In principle, $p \in \mathrm{Mat}_4(A(S^7_q))$, but we can choose $|\phi_1\rangle\,,\ |\phi_2\rangle$ in such a way that the entries of p will generate a subalgebra $A(S^4_q)$ of $A(S^7_q)$ which is a deformation of the algebra of polynomial functions on the 4-sphere S^4. The two elements $|\phi_1\rangle\,,\ |\phi_2\rangle$ will be obtained in two steps as follows simply using the noncommutativity of the algebra.

Firstly we write the relation $1 = \sum \overline{x}^i x_i$ in terms of the quadratic elements $\overline{x}^1 x_1$, $x_2\overline{x}^2$, $\overline{x}^3 x_3$, $x_4\overline{x}^4$ by using the commutation relations (2.16). We have that

$$1 = \sum \overline{x}^i x_i = q^{-6}\overline{x}^1 x_1 + q^{-2} x_2 \overline{x}^2 + q^{-2}\overline{x}^3 x_3 + x_4\overline{x}^4\ .$$

Then we take,

$$|\phi_1\rangle = (q^{-3}x_1, -q^{-1}\overline{x}^2, q^{-1}x_3, -\overline{x}^4)^t\ , \tag{2.23}$$

(t denoting transposition) which by construction is such that $\langle\phi_1|\phi_1\rangle = 1$.

Next, we write $1 = \sum \overline{x}^i x_i = \sum \overline{x}^i x_i$ as a function of the quadratic elements $x_1\overline{x}^1, \overline{x}^2 x_2$, $x_3\overline{x}^3, \overline{x}^4 x_4$:

$$1 = q^{-2}x_1\overline{x}^1 + q^{-4}\overline{x}^2 x_2 + x_3\overline{x}^3 + \overline{x}^4 x_4\ .$$

By taking,

$$|\phi_2\rangle = (\pm q^{-2}x_2, \pm q^{-1}\overline{x}^1, \pm x_4, \pm\overline{x}^3)^t$$

we get $\langle\phi_2|\phi_2\rangle = 1$. The ambiguity in the choice of the signs allows to us to obtains also the orthogonality condition $\langle\phi_1|\phi_2\rangle = 0$: for

$$|\phi_2\rangle = (q^{-2}x_2, q^{-1}\overline{x}^1, -x_4, -\overline{x}^3)^t \tag{2.24}$$

this is satisfied.

The resulting matrix is

$$v = (|\phi_1\rangle\,,|\phi_2\rangle) = \begin{pmatrix} q^{-3}x_1 & q^{-2}x_2 \\ -q^{-1}\overline{x}^2 & q^{-1}\overline{x}^1 \\ q^{-1}x_3 & -x_4 \\ -\overline{x}^4 & -\overline{x}^3 \end{pmatrix}\ , \tag{2.25}$$

which by construction satisfies $v^* v = 1$. Hence $p = vv^*$ is a self-adjoint projection.

We can state (and prove) the following:

Proposition 2.3. *The entries of the projection $p = vv^*$, with v given in* (2.25)*, generate a subalgebra of $A(S^7_q)$ which is a deformation of the algebra of polynomial functions on the 4-sphere S^4.*

Proof. Let us compute explicitly the components of the projection p and their commutation relations.

1. The diagonal elements are given by

$$\begin{aligned} p_{11} &= q^{-6} x_1 \overline{x}^1 + q^{-4} x_2 \overline{x}^2 \ , & p_{22} &= q^{-2} \overline{x}^2 x_2 + q^{-2} \overline{x}^1 x_1 \ , \\ p_{33} &= q^{-2} x_3 \overline{x}^3 + x_4 \overline{x}^4 \ , & p_{44} &= \overline{x}^4 x_4 + \overline{x}^3 x_3 \ , \end{aligned}$$

and satisfy the relation

$$q^{-2} p_{11} + q^2 p_{22} + p_{33} + p_{44} = 2 \ . \tag{2.26}$$

Only one of the p_{ii}'s is independent; indeed by using the commutation relations and the equation $\sum \overline{x}^i x_i = 1$, we can rewrite the p_{ii}'s in terms of

$$t := p_{22} \ , \tag{2.27}$$

as

$$p_{11} = q^{-2} t \quad , \quad p_{22} = t \quad , \quad p_{33} = 1 - q^{-4} t \quad , \quad p_{44} = 1 - q^2 t \ .$$

Equation (2.26) is easily verified. Notice that t is self-adjoint: $\overline{t} = t$.

2. As in the classical case, the elements p_{12}, p_{34} (and their conjugates) vanish:

$$p_{12} = -q^{-4} x_1 x_2 + q^{-3} x_2 x_1 = 0 \ , \quad p_{34} = -q^{-1} x_3 x_4 + x_4 x_3 = 0 \ .$$

3. The remaining elements are given by

$$\begin{aligned} p_{13} &= q^{-4} x_1 \overline{x}^3 - q^{-2} x_2 \overline{x}^4 \ , & p_{14} &= -q^{-3} x_1 x_4 - q^{-2} x_2 x_3 \ , \\ p_{23} &= -q^{-2} \overline{x}^2 \overline{x}^3 - q^{-1} \overline{x}^1 \overline{x}^4 \ , & p_{24} &= q^{-1} \overline{x}^2 x_4 - q^{-1} \overline{x}^1 x_3 \ , \end{aligned}$$

with $p_{ji} = \overline{p}_{ij}$ when $j > i$.
By using the commutation relations of $A(S^7_q)$, one finds that only two of these are independent. We take them to be p_{13} and p_{14}; one finds $p_{23} = q^{-2} \overline{p}_{14}$ and $p_{24} = -q^2 \overline{p}_{13}$.

Finally, we also have the sphere relation,

$$(q^6 - q^8) p_{11}^2 + p_{22}^2 + p_{44}^2 + q^4 (p_{13} p_{31} + p_{14} p_{41}) + q^2 (p_{24} p_{42} + p_{23} p_{32}) = (\sum \overline{x}^i x_i)^2 = 1 \ . \tag{2.28}$$

Summing up, together with $t = p_{22}$, we set $a := p_{13}$ and $b := p_{14}$. Then the projection p takes the following form

$$p = \begin{pmatrix} q^{-2}t & 0 & a & b \\ 0 & t & q^{-2}\overline{b} & -q^{2}\overline{a} \\ \overline{a} & q^{-2}b & 1-q^{-4}t & 0 \\ \overline{b} & -q^{2}a & 0 & 1-q^{2}t \end{pmatrix} . \tag{2.29}$$

By construction $p^* = p$ and this means that $\overline{t} = t$, as observed, and that $\overline{a}, \overline{b}$ are conjugate to a, b respectively. Furthermore $p^2 = p$: this property gives the easiest way to compute the commutation relations between the generators. One finds,

$$\begin{array}{ll} ab = q^4 ba \; , & \overline{a}b = b\overline{a} \; , \\ ta = q^{-2} at \; , & tb = q^4 bt \; , \end{array} \tag{2.30}$$

together with their conjugates, and sphere relations

$$\begin{array}{ll} a\overline{a} + b\overline{b} = q^{-2}t(1-q^{-2}t) \; , & q^4\overline{a}a + q^{-4}\overline{b}b = t(1-t) \; , \\ b\overline{b} - q^{-4}\overline{b}b = (1-q^{-4})t^2 \; . & \end{array} \tag{2.31}$$

It is straightforward to check also the relation (2.28). □

We define the algebra $A(S^4_q)$ to be the algebra generated by the elements $a, \overline{a}, b, \overline{b}, t$ with the commutation relations (2.30) and (2.31). For $q = 1$ it reduces to the algebra of polynomial functions on the sphere S^4.

Before to address our attention to the research of a "quantum group structure" we make some observations.

Observation 2.1. *At $q = 1$, the projection p in (2.29) is conjugate to the classical one given in Sec. 2.1 by the matrix $diag[1, -1, 1, 1]$ (up to a renaming of the generators).*

Proof. We remind that two projectors p, $q \in Mat(A)$ are said to be equivalent (in the sense of Murray-von-Neumann) $p \sim q$ if there exists a matrix $u \in Mat(A)$ such that $p = u^*u$ and $q = uu^*$. Note that the condition to be equivalent in noncommutative geometry is weeker with respect to the classical one. If p, q are conjugate through a matrix B, $p = BqB^{-1}$ with $B^{-1} = B^*$, then $u = Bq$ is such that

$$p = BqB^{-1} = Bqq^*B^{-1} = uu^* \; , \qquad u^*u = q^*B^*Bq = q^*q = q \; ,$$

where we used $p = p^2 = p^*$, $q = q^2 = q^*$.

Now we prove that the projections in (2.29) and (2.8) are conjugate (and hence equivalent). At $q = 1$, the matrix v in (2.25) reads

$$v_{q=1} = \begin{pmatrix} x_1 & x_2 \\ -\overline{x}^2 & \overline{x}^1 \\ x_3 & -x_4 \\ -\overline{x}^4 & -\overline{x}^3 \end{pmatrix} . \tag{2.32}$$

Then the corresponding projector $p_{q=1} = v_{q=1} v^*_{q=1}$ is conjugated to p in (2.8) by means of $u := diag[1, -1, 1, 1]$: $u\, v_{q=1} v^*_{q=1} u^* = p$ if one sets $a = \alpha$, $b = \beta\, t = 1 + x$. □

Observation 2.2. *Let us consider the algebra $A(S^4_{q^{-1}})$ obtained from $A(S^4_q)$ by mapping $q \mapsto q^{-1}$. This algebra is isomorphic to the previous one. Indeed, we obtain exactly the same commutation rule as* (2.30), (2.31), *i.e. an algebra isomorphism, if we take as generators of $A(S^4_{q^{-1}})$ the elements*

$$t' = q^{-2} t \ , \qquad a' = q^2 \overline{a} \ , \qquad b' = q^{-2} \overline{b} \ .$$

Hence we can limit ourselves to $|q| < 1$.

Observation 2.3. *Our sphere S^4_q seems to be different from the one constructed in [7]. Two of our generators commute and most importantly, it does not come from a deformation of a subgroup (let alone coisotropic) of $Sp(2)$. However, at the continuous level these two quantum spheres are the same since the C^*-algebra completion of both polynomial algebras is the minimal unitization $\mathcal{K} \oplus \mathbb{C}\mathbb{I}$ of the compact operators on an infinite dimensional separable Hilbert space, a property shared with Podleś standard sphere as well [56]. This fact will be derived in Sect. 2.3 when we study the representations of the algebra $A(S^4_q)$. See also Remark 3.3*

2.2.3 The $SU_q(2)$-coaction and $A(S^4_q)$ as algebra of coinvariants

In this rewritting of the bundle $S^7 \to S^4$, the structure group is replaced by the algebra of functions on the deformed $SU(2)$. Indeed we now provide a coaction of the quantum group $SU_q(2)$ on the sphere $A(S^7_q)$. This coaction will be used later in Sect. 3.3 when analyzing the quantum principle bundle structure.

Let us observe that the two pairs of generators $(x_1, x_2), (x_3, x_4)$ both yield a quantum plane,

$$x_1 x_2 = q x_2 x_1 \ , \qquad \overline{x}^1 \overline{x}^2 = q^{-1} \overline{x}^2 \overline{x}^1 \ ,$$
$$x_3 x_4 = q x_4 x_3 \ , \qquad \overline{x}^3 \overline{x}^4 = q^{-1} \overline{x}^4 \overline{x}^3 \ .$$

Next, these couples of generators are exactly those which are present in the rows of the matrix v in (2.25) hence we look for a right-coaction of $SU_q(2)$ on $A(S^7_q)$ to be firstly defined on v itself. (Note that other pairs of generators yield quantum planes but the only choice which gives a projection with the right number of generators is the one given above. We postpone this discussion to pag. 41.)

We recall the structure of the quantum group $SU_q(2)$ also in order to fix the notations, see e.g. [67]. The defining matrix of the quantum group $SU_q(2)$ reads

$$\begin{pmatrix} \alpha & -q\overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix} \tag{2.33}$$

with commutation relations

$$\begin{array}{lll} \alpha\gamma = q\gamma\alpha \ , & \alpha\overline{\gamma} = q\overline{\gamma}\alpha \ , & \gamma\overline{\gamma} = \overline{\gamma}\gamma \ , \\ \alpha\overline{\alpha} + q^2\overline{\gamma}\gamma = 1 \ , & \overline{\alpha}\alpha + \overline{\gamma}\gamma = 1 \ . & \end{array} \tag{2.34}$$

We define a coaction of $SU_q(2)$ on the matrix (2.25) by,

$$\delta_R(v) := \begin{pmatrix} q^{-3}x_1 & q^{-2}x_2 \\ -q^{-1}\overline{x}^2 & q^{-1}\overline{x}^1 \\ q^{-1}x_3 & -x_4 \\ -\overline{x}^4 & -\overline{x}^3 \end{pmatrix} \dot{\otimes} \begin{pmatrix} \alpha & -q\overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix} \ . \tag{2.35}$$

We shall prove presently that this coaction comes from a coaction of $A(SU_q(2))$ on the sphere algebra $A(S^7_q)$. For the moment we remark that, by its form in (2.35) the entries of the projection $p = vv^*$ are automatically coinvariants, a fact that we shall also prove explicitly in the following.

On the generators, the coaction (2.35) is given explicitly by

$$\begin{array}{ll} \delta_R(x_1) = x_1 \otimes \alpha + q\ x_2 \otimes \gamma \ , & \delta_R(\overline{x}^1) = q\overline{x}^2 \otimes \overline{\gamma} + \overline{x}^1 \otimes \overline{\alpha} = \overline{\delta_R(x_1)} \ , \\ \delta_R(x_2) = -x_1 \otimes \overline{\gamma} + x_2 \otimes \overline{\alpha} \ , & \delta_R(\overline{x}^2) = \overline{x}^2 \otimes \alpha - \overline{x}^1 \otimes \gamma = \overline{\delta_R(x_2)} \ , \\ \delta_R(x_3) = x_3 \otimes \alpha - q\ x_4 \otimes \gamma \ , & \delta_R(\overline{x}^3) = -q\overline{x}^4 \otimes \overline{\gamma} + \overline{x}^3 \otimes \overline{\alpha} = \overline{\delta_R(x_3)} \ , \\ \delta_R(x_4) = x_3 \otimes \overline{\gamma} + x_4 \otimes \overline{\alpha} \ , & \delta_R(\overline{x}^4) = \overline{x}^4 \otimes \alpha + \overline{x}^3 \otimes \gamma = \overline{\delta_R(x_4)} \ , \end{array} \tag{2.36}$$

from which it is also clear its compatibility with the anti-involution, i.e. $\delta_R(\overline{x}^i) = \overline{\delta_R(x_i)}$. The map δ_R in (2.36) extends as an algebra homomorphism to the whole of $A(S^7_q)$. Then, as alluded to before, we have the following

Proposition 2.4. *The coaction* (2.36) *is a right coaction of the quantum group* $SU_q(2)$ *on the 7-sphere* S^7_q,

$$\delta_R : A(S^7_q) \longrightarrow A(S^7_q) \otimes A(SU_q(2)) \ . \tag{2.37}$$

Proof. By using the commutation relations of $A(SU_q(2))$ in (2.34), a lengthy but easy computation gives that the commutation relations of $A(S^7_q)$ are preserved. This fact also shows that extending δ_R as an algebra homomorphism yields a consistent coaction. □

This coaction allows to us to interpret the "base space" $A(S^4_q)$ as the (projective) algebra of equivariant functions.

Proposition 2.5. *The algebra $A(S^4_q)$ is the algebra of coinvariants under the coaction defined in* (2.36).

Proof. We have to show that $A(S^4_q) = \{f \in A(S^7_q) \mid \delta_R(f) = f \otimes 1\}$. By using the commutation relations of $A(S^7_q)$ and those of $A(SU_q(2))$, we first prove explicitly that the generators of $A(S^4_q)$ are coinvariants:

$$\begin{aligned}\delta_R(a) &= q^{-4}\delta_R(x_1)\delta_R(\overline{x}^3) - q^{-2}\delta_R(x_2)\delta_R(\overline{x}^4)\\ &= q^{-4}x_1\overline{x}^3 \otimes (\alpha\overline{\alpha} + q^2\overline{\gamma}\gamma) - q^{-2}x_2\overline{x}^4 \otimes (\gamma\overline{\gamma} + \overline{\alpha}\alpha)\\ &= (q^{-4}x_1\overline{x}^3 - q^{-2}x_2\overline{x}^4) \otimes 1 = a \otimes 1\end{aligned}$$

$$\begin{aligned}\delta_R(b) &= -q^{-3}\delta_R(x_1)\delta_R(x_4) - q^{-2}\delta_R(x_2)\delta_R(x_3)\\ &= -q^{-3}x_1x_4 \otimes (\alpha\overline{\alpha} + q^2\overline{\gamma}\gamma) - q^{-2}x_2x_3 \otimes (\gamma\overline{\gamma} + \overline{\alpha}\alpha)\\ &= -(q^{-3}x_1x_4 + q^{-2}x_2x_3) \otimes 1 = b \otimes 1\end{aligned}$$

$$\begin{aligned}\delta_R(t) &= q^{-2}\delta_R(\overline{x}^2)\delta_R(x_2) + q^{-2}\delta_R(\overline{x}^1)\delta_R(x_1)\\ &= q^{-2}\overline{x}^2x_2 \otimes (\alpha\overline{\alpha} + q^2\overline{\gamma}\gamma) + q^{-2}\overline{x}^1x_1 \otimes (\gamma\overline{\gamma} + \overline{\alpha}\alpha)\\ &= (q^{-2}\overline{x}^2x_2 + q^{-2}\overline{x}^1x_1) \otimes 1 = t \otimes 1\end{aligned}$$

By construction the coaction is compatible with the anti-involution so that

$$\delta_R(\overline{a}) = \overline{\delta_R(a)} = \overline{a} \otimes 1, \quad \delta_R(\overline{b}) = \overline{\delta_R(b)} = \overline{b} \otimes 1$$

In fact, this only shows that $A(S^4_q)$ is made of coinvariants but does not rule out the possibility of other coinvariants not in $A(S^4_q)$. However this does not happen for the following reason. From eq. (2.36) it is clear that $w_1 \in \{x_1, x_3, \overline{x}^2, \overline{x}^4\}$ (respectively $w_{-1} \in \{x_2, x_4, \overline{x}^1, \overline{x}^3\}$) are weight vectors of weight 1 (resp. -1) in the fundamental comodule of $SU_q(2)$. It follows that the only possible coinvariants are of the form $(w_1w_{-1} - qw_{-1}w_1)^n$. When $n = 1$ these are just the generators of $A(S^4_q)$. □

In the appendix 2.A we will provide another proof of this proposition by using the so called Diamond lemma [6].

Remark 2.3. *The last part of the proof above is also related to the quantum Plücker coordinates. For every* 2×2 *matrix of* (2.25)*, let us define the determinant by*

$$det\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} := a_{11}a_{22} - q\ a_{12}a_{21}\ . \tag{2.38}$$

(Note that a_{12}, a_{21} do not commute and so in the previous formula the ordering between them is fixed.) Let m_{ij} be the minors of (2.25) *obtained by considering the i, j rows. Then*

$$\begin{aligned} &m_{12} = q^2p_{11} = t\ , && m_{13} = p_{14} = b\ ,\\ &m_{14} = -q\ p_{13} = -q\ a\ , && m_{23} = p_{24} = -q^2\overline{a}\ ,\\ &m_{24} = -q\ p_{23} = -q^{-1}\overline{b}\ , && m_{34} = -q\ p_{33} = q^{-3}t - q\ .\end{aligned} \tag{2.39}$$

At $q = 1$, these give the classical Plücker coordinates [1].

The right coaction (2.35) of $SU_q(2)$ on the 7-sphere $A(S_q^7)$ can be written on the vector (x_1, x_2, x_3, x_4) made of generators as

$$\delta_R(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4) \dot{\otimes} \begin{pmatrix} \alpha & -\overline{\gamma} & 0 & 0 \\ q\gamma & \overline{\alpha} & 0 & 0 \\ 0 & 0 & \alpha & \overline{\gamma} \\ 0 & 0 & -q\gamma & \overline{\alpha} \end{pmatrix} , \qquad (2.40)$$

together with $\delta_R(\overline{x}_i) = \overline{\delta_R(x_i)}$. Once more, extended as an algebra map.
Notice that in the block-diagonal matrix which appears in (2.40) the second copy is given by $SU_q(2)$ while the first one is twisted as

$$\begin{pmatrix} \alpha & -\overline{\gamma} \\ q\gamma & \overline{\alpha} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha & \overline{\gamma} \\ -q\gamma & \overline{\alpha} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

A similar phenomenon occurs in [7].

Remark 2.4. *It is also interesting to observe that*

$$\delta_R(v^* v) = v^* v \otimes 1 = 1 \otimes 1 .$$

Indeed,

$$\begin{aligned} \delta_R(\langle \phi_1 | \phi_1 \rangle) &= \delta_R(q^{-6} \overline{x}^1 x_1 + q^{-2} x_2 \overline{x}^2 + q^{-2} \overline{x}^3 x_3 + x_4 \overline{x}^4) \\ &= (-q^{-5} \overline{x}^2 x_1 + q^{-2} x_1 \overline{x}^2 + q^{-1} \overline{x}^4 x_3 - x_3 \overline{x}^4) \otimes \overline{\gamma}\alpha \\ &\quad + (q^{-4} \overline{x}^2 x_2 + q^{-2} x_1 \overline{x}^1 + \overline{x}^4 x_4 + x_3 \overline{x}^3) \otimes \overline{\gamma}\gamma \\ &\quad + (q^{-6} \overline{x}^1 x_1 + q^{-2} x_2 \overline{x}^2 + q^{-2} \overline{x}^3 x_3 + x_4 \overline{x}^4) \otimes \overline{\alpha}\alpha \\ &\quad + (-q^{-5} \overline{x}^1 x_2 + q^{-2} x_2 \overline{x}^1 + q^{-1} \overline{x}^3 x_4 - x_4 \overline{x}^3) \otimes \overline{\alpha}\gamma \\ &= \langle \phi_2 | \phi_1 \rangle \otimes \overline{\gamma}\alpha + \langle \phi_2 | \phi_2 \rangle \otimes \overline{\gamma}\gamma + \langle \phi_1 | \phi_1 \rangle \otimes \overline{\alpha}\alpha + \langle \phi_1 | \phi_2 \rangle \otimes \overline{\alpha}\gamma \\ &= 1 \otimes (\overline{\gamma}\gamma + \overline{\alpha}\alpha) = 1 \otimes 1 , \end{aligned}$$

$$\begin{aligned} \delta_R(\langle \phi_2 | \phi_2 \rangle) &= \delta_R(q^{-2} x_1 \overline{x}^1 + q^{-4} \overline{x}^2 x_2 + x_3 \overline{x}^3 + \overline{x}^4 x_4) \\ &= (q^{-4} \overline{x}^2 x_1 - q^{-1} x_1 \overline{x}^2 - \overline{x}^4 x_3 + q x_3 \overline{x}^4) \otimes \alpha\overline{\gamma} \\ &\quad + (q^{-4} \overline{x}^2 x_2 + q^{-2} x_1 \overline{x}^1 + \overline{x}^4 x_4 + x_3 \overline{x}^3) \otimes \alpha\overline{\alpha} \\ &\quad + (q^{-4} \overline{x}^1 x_1 + x_2 \overline{x}^2 + \overline{x}^3 x_3 + q^2 x_4 \overline{x}^4) \otimes \gamma\overline{\gamma} \\ &\quad + (q^{-4} \overline{x}^1 x_2 - q^{-1} x_2 \overline{x}^1 - \overline{x}^3 x_4 + q x_4 \overline{x}^3) \otimes \gamma\overline{\alpha} \\ &= -q \langle \phi_2 | \phi_1 \rangle \otimes \alpha\overline{\gamma} + \langle \phi_2 | \phi_2 \rangle \otimes \alpha\overline{\alpha} + q^2 \langle \phi_1 | \phi_1 \rangle \otimes \gamma\overline{\gamma} - q \langle \phi_1 | \phi_2 \rangle \otimes \gamma\overline{\alpha} \\ &= 1 \otimes (\alpha\overline{\alpha} + q^2 \gamma\overline{\gamma}) = 1 \otimes 1 , \end{aligned}$$

$$\begin{aligned} \delta_R(\langle \phi_1 | \phi_2 \rangle) &= q^{-5} \delta_R(\overline{x}^1) \delta_R(x_2) - q^{-2} \delta_R(x_2) \delta_R(\overline{x}^1) \\ &\qquad - q^{-1} \delta_R(\overline{x}^3) \delta_R(x_4) + \delta_R(x_4) \delta_R(\overline{x}^3) = 0 \end{aligned}$$

since δ_R defines a coaction on S_q^7 and so preserves its commutation relations. $\square$

On the crucial choice of $|\phi_1\rangle, |\phi_2\rangle$.

As seen, the choice of $|\phi_1\rangle, |\phi_2\rangle$ has been particularly lucky allowing to use to construct a noncommutative principal bundle (see Sect. 3.3). Is it the only possible choice? The answer seems to be positive if one requires

- $|\phi_1\rangle, |\phi_2\rangle$ orthonormal: crucial to have p projection;
- quantum planes on the rows of v: necessary to define a right $SU_q(2)$-coaction directly on v (and hence, such that the entries of p are automatically coinvariant;)
- p with the "correct" number of generators.

Firstly, the choice similar to Atiyah's one would be with the elements x_i into a vector and their conjugates on the other one: we deduce these two orthonormal vectors in $A(S^7_q) \otimes \mathbb{C}^4$ from the sphere relation $\sum \overline{x}_i x_i = 1$ as before:

$$|\psi_1\rangle = (x_1, x_2, x_3, x_4) \qquad |\psi_2\rangle = (-q^{-3}\overline{x}^2, q^{-4}\overline{x}^1, -\overline{x}^4, q^{-1}\overline{x}^3) \ .$$

Unfortunately, this choice fails the conditions to have quantum planes on the rows of the matrix $(|\phi_1\rangle, |\phi_2\rangle)$.

The only possible ways to take couples of generators of $A(S^7_q)$ which generate quantum planes are, from page 30,

$$a) \left\{ \begin{array}{ll} (x_1\ , & x_2) \\ \\ (x_3\ , & x_4) \end{array} \right. , \qquad b) \left\{ \begin{array}{ll} (x_1\ , & x_3) \\ \\ (x_2\ , & x_4) \end{array} \right. , \qquad c) \left\{ \begin{array}{ll} (\overline{x}_4\ , & x_1) \\ \\ (\overline{x}_3\ , & x_2) \end{array} \right. ,$$

in the latter case with a q^2-plane.

Case $a)$ is exactly the one used in the construction of the instanton bundle. We know that it works.
In case $b)$ the first corresponding orthonormal vector would be

$$|\phi_1\rangle = (\pm q^{-3} x_1, \pm q^{-1}\overline{x}^3, \pm q^{-3} x_2, \pm \overline{x}^4)^t \ .$$

But then it is no longer possible to find out another orthonormal vector $|\phi_2\rangle$ such that $\langle\phi_1|\phi_2\rangle = 0$. This problem does not occur for the last choice $c)$, but here the corresponding projection generate an algebra with too many generators, not a 4-sphere.

Observation 2.4. *We remind that classically, under the identification $S^4 \simeq \mathbb{P}^1\mathbb{H}$, the group which acts on S^7 is $Sp(1)$. At a quantum level hence the natural choice would be the $c)$ (being $Sp_q(1) \simeq SU_{q^2}(2)$) but as said this choice fails to produce a 4-sphere .*

2.3 Representations of the algebra $A(S^4_q)$

We conclude this Chapter by computing the instanton charge. As recalled in App. 1.A, this is done by computing the pairing of the Chern-Connes characters $ch^*(\mu) \in HC^*[A(S^4_q)]$ and $ch_*(p) \in HC_*[A(S^4_q)]$ in cyclic homology and cohomology respectively. Here μ being a Fredholm module over $A(S^4_q)$ that we compute now.

What we need firstly are irreducible $*$-representations of $A(S^4_q)$ as bounded operators on a separable Hilbert space $\mathcal{H}$. For the moment, we denote in the same way the elements of the algebra and their images as operators in the given representation. As mentioned before in Obs. 2.2, since $q \mapsto q^{-1}$ gives an isomorphic algebra, we can restrict ourselves to $|q| < 1$. We will consider the representations which are t-finite [41], i.e. such that the eigenvectors of t span $\mathcal{H}$.

Since the self-adjoint operator t must be bounded due to the spherical relations, from the commutation relations $ta = q^{-2}at$, $t\bar{b} = q^{-4}\bar{b}t$, it follows that the spectrum should be of the form λq^{2k} and $a, \bar{b}$ (resp. $\bar{a}, b$) act as rising (resp. lowering) operators on the eigenvectors of t. Then boundedness implies the existence of an highest weight vector, i.e. there exists a vector $|0,0\rangle$ such that

$$t\,|0,0\rangle = t_{00}\,|0,0\rangle\,, \quad a\,|0,0\rangle = 0, \quad \bar{b}\,|0,0\rangle = 0\,. \tag{2.41}$$

By evaluating $q^4\bar{a}a + b\bar{b} = (1 - q^{-4}t)t$ on $|0,0\rangle$ we have

$$(1 - q^{-4}t_{00})t_{00} = 0$$

According to the values of the eigenvalue t_{00} we have two representations.

2.3.1 The representation β

The first representation, that we call β, is obtained for $t_{00} = 0$. Then, $t\,|0,0\rangle = 0$ implies $t = 0$. Moreover, using the commutation relations (2.30) and (2.31), it follows that this representation is the trivial one

$$t = 0, \quad a = 0, \quad b = 0\,, \tag{2.42}$$

the representation Hilbert space being just $\mathbb{C}$; of course, $\beta(1) = 1$.

2.3.2 The representation σ

The second representation, that we call σ, is obtained for $t_{00} = q^4$. This is infinite dimensional. We take the set $|m,n\rangle = N_{mn}\bar{a}^m b^n\,|0,0\rangle$ with $n, m \in \mathbb{N}$, to be an orthonormal basis of the representation Hilbert space $\mathcal{H}$, with $N_{00} = 1$ and $N_{mn} \in \mathbb{R}$

the normalizations, to be computed below.
Then

$$t\,|m,n\rangle = t_{mn}\,|m,n\rangle \ ,$$
$$\overline{a}\,|m,n\rangle = a_{mn}\,|m+1,n\rangle \ ,$$
$$b\,|m,n\rangle = b_{mn}\,|m,n+1\rangle \ .$$

By requiring that we have a $*$-representation we have also that

$$a\,|m,n\rangle = a_{m-1,n}\,|m-1,n\rangle \ , \quad \overline{b}\,|m,n\rangle = b_{m,n-1}\,|m,n-1\rangle \ ,$$

with the following recursion relations

$$a_{m,n\pm1} = q^{\pm2}a_{m,n} \ , \qquad b_{m\pm1,n} = q^{\pm2}b_{m,n} \ , \qquad b_{m,n} = q^2 a_{2n+1,m} \ .$$

By explicit computation, we find

$$\begin{cases} t_{m,n} = q^{2m+4n+4} \ , \\ a_{m,n} = N_{mn}N^{-1}_{m+1,n} = (1-q^{2m+2})^{\frac{1}{2}}q^{m+2n+1} \ , \\ b_{m,n} = N_{mn}N^{-1}_{m,n+1} = (1-q^{4n+4})^{\frac{1}{2}}q^{2(m+n+2)} \ . \end{cases} \tag{2.43}$$

Proof. of eqs. (2.43). Firstly

$$t\,|m,n\rangle = N_{mn}q^{2m+4n}\overline{a}^m b^n t\,|0,0\rangle = N_{mn}q^{2m+4n}f_{00}\overline{a}^m b^n\,|0,0\rangle = q^{2m+4(n+1)}\,|m,n\rangle \ .$$

Using $\overline{a}b = b\overline{a}$ and $|m+1,n\rangle = N_{m+1,n}\overline{a}^{m+1}b^n\,|0,0\rangle = N_{m+1,n}\overline{a}^m b^n\overline{a}\,|0,0\rangle$ we have

$$\overline{a}\,|m,n\rangle = N_{m,n}\overline{a}^{m+1}b^n\,|0,0\rangle = \frac{N_{m,n}}{N_{m+1,n}}\,|m+1,n\rangle$$

and similarly

$$b\,|m,n\rangle = \frac{N_{m,n}}{N_{m,n+1}}\,|m,n+1\rangle$$

so that in order to compute $a_{m,n}$, $b_{m,n}$ we only need to determine the quotients of the normalizing constants. We use the following

$$\begin{aligned} a^{m+1}\overline{a} &= a^m(a\overline{a}) = a^m(q^4\overline{a}a + (q^{-2}-1)t) = \\ &= (q^{-2}-1)a^m t + q^4 a^{m-1}(q^4\overline{a}a + (q^{-2}-1)t)a \\ &= (q^{-2}-1)(1+q^2)a^m t + q^8 a^{m-2}(q^4\overline{a}a + (q^{-2}-1)t)a^2 = \ldots \\ &= (q^{-2}-1)(1+q^2+q^4+\cdots+q^{2m})a^m t + (\cdots)\overline{a}a \end{aligned}$$

Then, using $\sum_{i=0}^m q^{2i} = \frac{1-q^{2m+2}}{1-q^2}$ and $a\,|0,0\rangle = 0$ we have

$$\begin{aligned} \langle m+1,n|m+1,n\rangle &= N^2_{m+1,n}\,\langle 0,0|\,\overline{b}^n a^{m+1}\overline{a}^{m+1}b^n\,|0,0\rangle \\ &= q^{-2}(1-q^{2m+2})N^2_{m+1,n}\,\langle 0,0|\,\overline{b}^n a^m t\overline{a}^m b^n\,|0,0\rangle \\ &= q^{-2}(1-q^{2m+2})q^{2m+4n+4}N^2_{m+1,n}\,\langle 0,0|\,\overline{b}^n a^m\overline{a}^m b^n\,|0,0\rangle \\ &= q^{-2}(1-q^{2m+2})q^{2m+4n+4}\frac{N^2_{m+1,n}}{N^2_{mn}}\,\langle m,n|m,n\rangle \ . \end{aligned}$$

Assuming the orthonormality of the bases vectors, we can conclude that

$$\frac{N^2_{m,n}}{N^2_{m+1,n}} = q^{2(m+2n+1)}(1-q^{2m+2})$$

and hence the a_{mn} have the expression given in (2.43).
For b_{mn} the computation is analogous: as a first step

$$\overline{b}^{n+1} b = (1-q^4)\sum_{i=0}^{n} q^{4i}\overline{b}^n t^2 + (\ )b\overline{b}$$

and then

$$\begin{aligned} 1 = \langle m, n+1 | m, n+1\rangle &= q^{-4n}(1-q^{4n+4})N^2_{m,n+1}\,\langle 0,0|\, a^m \overline{b}^n t^2 b^n \overline{a}^m\, |0,0\rangle \\ &= (1-q^{4n+4})q^{4(n+m+2)}\frac{N^2_{m,n+1}}{N^2_{m,n}} \end{aligned}$$

and so

$$b\,|m,n\rangle = \frac{N_{m,n}}{N_{m,n+1}}\,|m,n+1\rangle = (1-q^{4n+4})^{\frac{1}{2}}q^{2(m+n+2)}\,|m,n+1\rangle\ . \quad \square$$

Summarizing, we have the following action

$$\begin{aligned} t\,|m,n\rangle &= q^{2m+4n+4}\,|m,n\rangle\,, \\ \overline{a}\,|m,n\rangle &= (1-q^{2m+2})^{\frac{1}{2}}q^{m+2n+1}\,|m+1,n\rangle\,, \\ a\,|m,n\rangle &= (1-q^{2m})^{\frac{1}{2}}q^{m+2n}\,|m-1,n\rangle\,, \\ b\,|m,n\rangle &= (1-q^{4n+4})^{\frac{1}{2}}q^{2(m+n+2)}\,|m,n+1\rangle\,, \\ \overline{b}\,|m,n\rangle &= (1-q^{4n})^{\frac{1}{2}}q^{2(m+n+1)}\,|m,n-1\rangle\,. \end{aligned} \tag{2.44}$$

It is straightforward to check that all the defining relations (2.30) and (2.31) are satisfied.

In this representation the algebra generators are all trace class:

$$\begin{aligned} \mathrm{Tr}(t) &= q^4\sum_m q^{2m}\sum_n q^{4n} = \frac{q^4}{(1-q^2)(1-q^4)}\ , \\ \mathrm{Tr}(|a|) &= q\sum_{m,n}(1-q^{2m+2})^{\frac{1}{2}}q^{m+2n} = \frac{q}{1-q^2}\sum_m(1-q^{2m+2})^{\frac{1}{2}}q^m \\ &\leq \frac{q}{1-q^2}\sum_m q^m = \frac{q}{(1-q)(1-q^2)}\ , \\ \mathrm{Tr}(|b|) &= q^4\sum_{m,n}(1-q^{4n+4})^{\frac{1}{2}}q^{2(n+m)} = \frac{q^4}{1-q^2}\sum_n(1-q^{4n+4})^{\frac{1}{2}}q^{2n} \\ &\leq \frac{q^4}{1-q^2}\sum_n q^{2n} = \frac{q^4}{(1-q^2)^2}\ . \end{aligned} \tag{2.45}$$

From the sequence of Schatten ideals in the algebra of compact operators one knows [62] that the norm closure of trace class operators gives the ideal of compact operators $\mathcal{K}$. As a consequence, the closure of $A(S^4_q)$ is the C^*-algebra $\mathcal{C}(S^4_q) = \mathcal{K} \oplus \mathbb{C}\mathbb{I}$.

2.4 The index pairing: the charge

The 'defining' self-adjoint idempotent p in (2.29) determines a class in the K-theory of S^4_q, i.e. $[p] \in K_0[\mathcal{C}(S^4_q)]$. A way to prove its nontriviality is by pairing it with a nontrivial element in the dual K-homology, i.e. with (the class of) a nontrivial Fredholm module $[\mu] \in K^0[\mathcal{C}(S^4_q)]$. In fact, in order to compute the pairing of K-theory with K-homology, it is more convenient to first compute the corresponding Chern characters in the cyclic homology $\mathrm{ch}_*(p) \in HC_*[A(S^4_q)]$ and cyclic cohomology $\mathrm{ch}^*(\mu) \in HC^*[A(S^4_q)]$ respectively, and then use the pairing between cyclic homology and cohomology [18].

Like for the q-monopole [33], to compute the pairing and to prove the nontriviality of the bundle it is enough to consider $HC_0[A(S^4_q)]$ and dually to take a suitable trace of the projection.

The Chern character of the projection p in (2.29) has a component in degree zero $\mathrm{ch}_0(p) \in HC_0[A(S^4_q)]$ simply given by the matrix trace,

$$\mathrm{ch}_0(p) := \mathrm{tr}(p) = 2 - q^{-4}(1-q^2)(1-q^4)\, t \ \in A(S^4_q). \tag{2.46}$$

The higher degree parts of $\mathrm{ch}_*(p)$ are obtained via the periodicity operator S; not needing them here we shall not dwell more upon this point and refer to [18] for the relevant details.

As mentioned, the K-homology of an involutive algebra $\mathcal{A}$ is given in terms of homotopy classes of Fredholm modules. In the present situation we are dealing with a 1-summable Fredholm module $[\mu] \in K^0[\mathcal{C}(S^4_q)]$. This is in contrast to the fact that the analogous element of $K_0(S^4)$ for the undeformed sphere is given by a 4-summable Fredholm module, being the fundamental class of S^4.

The Fredholm module $\mu := (\mathcal{H}, \Psi, \gamma)$ is constructed as follows. The Hilbert space is $\mathcal{H} = \mathcal{H}_\sigma \oplus \mathcal{H}_\sigma$ and the representation is $\Psi = \sigma \oplus \beta$. Here σ is the representation of $A(S^4_q)$ introduced in (2.44) and β given in (2.42) is trivially extended to $\mathcal{H}_\sigma$. The grading operator is

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The corresponding Chern character $\mathrm{ch}^*(\mu)$ of the class of this Fredholm module has a component in degree 0, $\mathrm{ch}^0(\mu) \in HC^0[A(S^{2n}_q)]$. From the general construction [18], the element $\mathrm{ch}^0(\mu_{\mathrm{ev}})$ is the trace

$$\tau^1(x) := \mathrm{Tr}\,(\gamma\Psi(x)) = \mathrm{Tr}\,(\sigma(x) - \beta(x))\,. \tag{2.47}$$

The operator $\sigma(x) - \beta(x)$ is always trace class. Obviously $\tau^1(1) = 0$. The higher degree parts of $\mathrm{ch}^*(\mu_{\mathrm{ev}})$ can again be obtained via a periodicity operator.
A similar construction of the class $[\mu]$ and the corresponding Chern character were given in [52] for quantum two and three dimensional spheres.

We are ready to compute the pairing:

$$\begin{aligned}\langle[\mu],[p]\rangle &:= \left\langle \mathrm{ch}^0(\mu), \mathrm{ch}_0(p)\right\rangle = -q^{-4}(1-q^2)(1-q^4)\ \tau^1(t) \\ &= -q^{-4}(1-q^2)(1-q^4)\,\mathrm{Tr}(t) = -q^{-4}(1-q^2)(1-q^4)q^4(1-q^2)^{-1}(1-q^4)^{-1} \\ &= -1\ . \end{aligned} \tag{2.48}$$

This result shows also that the right $A(S^4_q)$-module $p[A(S^4_q)^4]$ is not free. Indeed, any free module is represented in $K_0[\mathcal{C}(S^4_q)]$ by the idempotent 1, and since $\langle[\mu],[1]\rangle = 0$, the evaluation of $[\mu]$ on any free module always gives zero.

We can extract the 'trivial' element in the K-homology $K^0[\mathcal{C}(S^4_q)]$ of the quantum sphere S^4_q and use it to measure the 'rank' of the idempotent p. This generator corresponds to the trivial generator of the K-homology $K_0(S^4)$ of the classical sphere S^4. The latter (classical) generator is the image of the generator of the K-homology of a point by the functorial map $K_*(\iota) : K_0(*) \to K_0(S^{N-1})$, where $\iota : * \hookrightarrow S^{N-1}$ is the inclusion of a point into the sphere. Now, the quantum sphere S^4_q has just one 'classical point', i.e. the 1-dimensional representation β constructed in Sect. 2.3.1. The corresponding 1-summable Fredholm module $[\varepsilon] \in K^0[\mathcal{C}(S^4_q)]$ is easily described: the Hilbert space is $\mathbb{C}$ with representation β; the grading operator is $\gamma = 1$. Then the degree 0 component $\mathrm{ch}^0(\varepsilon) \in HC^0[A(S^{2n}_q)]$ of the corresponding Chern character is the trace given by the representation itself (since it is a homomorphism to a commutative algebra),

$$\tau^0(x) = \beta(x)\ , \tag{2.49}$$

and vanishes on all the generators whereas $\tau^0(1) = 1$.
Not surprisingly, the pairing with the class of the idempotent p is,

$$\langle[\varepsilon],[p]\rangle := \tau^0(\mathrm{ch}_0(p)) = \beta(2) = 2\ . \tag{2.50}$$

The non triviality of this pairing will be used later in Sect. 3.3 to conclude that the bundle $A(S^4_q) \subset A(S^7_q)$ is non trivial.

2.A Appendix: An alternative proof of $A(S^4_q) = A(S^7_q)^{SU_q(2)}$

In this appendix we supply an alternative proof of the inclusion $A(S^4_q) \supseteq A(S^7_q)^{SU_q(2)}$ stated in Prop. 2.5 in Sect. 2.2.3:

Proposition 2.6. *The algebra $A(S_q^4)$ coincides with the subalgebra $A(S_q^7)^{co(SU_q(2))}$ of $A(S_q^7)$ made of the coinvariants with respect to the right coaction δ_R of the quantum group $A(SU_q(2))$:*

$$A(S_q^4) = A(S_q^7)^{co(SU_q(2))} := \{x \in A(S_q^7) \mid \delta_R(x) = x \otimes 1\}.$$

The proof uses the so called Diamond Lemma, Th. 1.2 [6]. Given an algebra defined in terms of generators and relations, the Diamond lemma provides a way to prove that a certain set of elements is a basis for that algebra. We briefly remind the points of the theorem that will be used. We adopt the notation of [6] and we refer to it for details.

Let k be a commutative associative ring with unity, X a set, $\langle X\rangle$ the free semigroup with unity on X and $k\langle X\rangle$ the free associative k-algebra generated by the elements of X. On $\langle X\rangle$ we introduce a partial order $\leq$ such that $A < B$ implies $CAD < CBD$ for all $A, B, C, D \in \langle X\rangle$.

A reduction system S is, by definition, a subset of $\langle X\rangle \times k\langle X\rangle$. Given any element $\sigma = (W_\sigma, f_\sigma) \in S$ and $A, B \in \langle X\rangle$, we define the reduction $r_{AW_\sigma B} : k\langle X\rangle \to k\langle X\rangle$ which maps $AW_\sigma B \mapsto Af_\sigma B$ and acts as the identity on the other elements. An element of $k\langle X\rangle$ is irreducible if every reduction acts trivially on it, i.e. leaves it unchanged. Observe that an element is unchanged under $r_{AW_\sigma B}$ if and only if the monomial $AW_\sigma B$ does not appear in it. Let $I_S =< W_\sigma - f_\sigma,\ \sigma \in S >$ be the two-sided ideal of $k\langle X\rangle$ associated to S. The order in $\langle X\rangle$ is called compatible with S if for each element σ of S, the monomials which constitute f_σ are $< W_\sigma$.

Let us introduce the notion of ambiguities. Two elements $\sigma, \tau \in S$ produce an overlap ambiguity if $W_\sigma = AB,\ W_\tau = BC$. The ambiguity can be solved if $r(f_\sigma C) = r'(Af_\tau)$ through compositions of reductions. We say that σ, τ produce an inclusion ambiguity if $W_\tau = BW_\sigma C$ and an analogous definition of resolution should be given.

After the assumption that the order is compatible with S and it has descending chain condition, then the Diamond lemma states that if (and only if) all ambiguities of S are resolvable, then the S-irreducible monomials of $\langle X\rangle$ are a (vector space) basis for $k\langle X\rangle/I_S$.

Proof. of Prop. 2.6. The idea is to show that there are as many coinvariants as classically and then to show that any coinvariant can be written in terms of elements of $A(S_q^4)$.

Let X be the set $\{x_i,\ \overline{x}_i,\ i = 1, \dots, 4\}$. Let $\mathbb{C}_{q,q^{-1}}\langle X\rangle$ be the corresponding free algebra and $\langle X\rangle$ the free semigroup with 1 on X, i.e. the set of monomials.

We introduce on $\langle X\rangle$ an ordering: monomials are ordered according to their lenght and for monomials of the same lenght we adopt the anti-lexicographic ordering induced by

$$\overline{x}_i < \overline{x}_j < x_i < x_j\ , \quad i > j$$

This ordering satisfies the requested descending chain condition.

We take the following reduction system given by elements of the form $\sigma = (W_\sigma, f_\sigma)$ with $W_\sigma \in \langle X\rangle$ and $f_\sigma \in \mathbb{C}_{q,q^{-1}}\langle X\rangle$:

$$S = \left\{ \begin{array}{lll}
(x_1x_2,\ qx_2x_1)\ , & (x_1x_3,\ qx_3x_1)\ , & (x_2x_4,\ qx_4x_2)\ , \\
(x_3x_4\ , qx_4x_3)\ , & (x_1x_4\ , q^2x_4x_1)\ , & (x_2x_3\ , q^2x_3x_2 + (q^3-q)x_4x_1)\ , \\
 & & \\
(\overline{x}^1\overline{x}^2,\ q^{-1}\overline{x}^2\overline{x}^1)\ , & (\overline{x}^1\overline{x}^3,\ q^{-1}\overline{x}^3\overline{x}^1)\ , & (\overline{x}^2\overline{x}^4,\ q^{-1}\overline{x}^4\overline{x}^2)\ , \\
(\overline{x}^3\overline{x}^4\ , q^{-1}\overline{x}^4\overline{x}^3)\ , & (\overline{x}^1\overline{x}^4\ , q^{-2}\overline{x}^4\overline{x}^1)\ , & (\overline{x}^2\overline{x}^3\ , q^{-2}\overline{x}^3\overline{x}^2 + (q^{-3}-q^{-1})\overline{x}^4\overline{x}^1)\ , \\
 & & \\
(x_1\overline{x}^1\ , \overline{x}^1x_1)\ , & (x_1\overline{x}^2\ , q^{-1}\overline{x}^2x_1)\ , & \\
(x_1\overline{x}^3\ , q^{-1}\overline{x}^3x_1)\ , & (x_1\overline{x}^4\ , q^{-2}\overline{x}^4x_1)\ , & \\
 & & \\
(x_2\overline{x}^1\ , q^{-1}\overline{x}^1x_2)\ , & (x_2\overline{x}^2\ , \overline{x}^2x_2 + (1-q^{-2})\overline{x}^1x_1)\ , & \\
(x_2\overline{x}^3\ , q^{-2}\overline{x}^3x_2)\ , & (x_2\overline{x}^4\ , q^{-1}\overline{x}^4x_2 + q^{-1}(q^{-2}-1)\overline{x}^3x_1)\ , & \\
 & & \\
(x_3\overline{x}^1\ , q^{-1}\overline{x}^1x_3)\ , & (x_3\overline{x}^3\ , \overline{x}^3x_3 + (1-q^{-2})[\overline{x}^1x_1 + (1+q^{-2})\overline{x}^2x_2])\ , & \\
(x_3\overline{x}^2\ , q^{-2}x_2\overline{x}^3)\ , & (x_3\overline{x}^4\ , q^{-1}\overline{x}^4x_3 + (1-q^{-2})q^{-3}\overline{x}^2x_1)\ , & \\
 & & \\
(x_4\overline{x}^1\ , q^{-2}\overline{x}^1x_4)\ , & (x_4\overline{x}^2\ , q^{-1}\overline{x}^2x_4 + q^{-1}(q^{-2}-1)\overline{x}^1x_3)\ , & \\
\multicolumn{3}{l}{(x_4\overline{x}^3\ , q^{-1}\overline{x}^3x_4 + (1-q^{-2})q^{-3}\overline{x}^1x_2)\ ,} \\
\multicolumn{3}{l}{(x_4\overline{x}^4\ , \overline{x}^4x_4 + (1-q^{-2})[(1+q^{-4})\overline{x}^1x_1 + \overline{x}^2x_2 + \overline{x}^3x_3])\ ,} \\
 & & \\
\multicolumn{3}{l}{(\overline{x}^1x_1\ , 1 - \overline{x}^4x_4 - \overline{x}^3x_3 - \overline{x}^2x_2)\ .}
\end{array} \right\}$$

This system is choosen in such a way that the corresponding two-sided ideal

$$I_S = \langle W_\sigma - f_\sigma\ ,\ \sigma \in S\rangle$$

coincides with the defining ideal for $A(S^7_q)$, i.e. $A(S^7_q) = \mathbb{C}_{q,q^{-1}}\langle X\rangle / I_S$.
The ordering introduced above is compatible with S, that is $\forall\sigma \in S$ f_σ is a linear combination of monomials $< W_\sigma$. There are only overlap ambiguities in S, which can be all resolved. For example, $(x_1x_2,\ qx_2x_1)$, $(x_2x_4,\ qx_4x_2)$ is an overlap ambiguity but there exist r, r' compositions of reductions such that $r(x_2x_1x_4) = r'(x_1x_4x_2)$. Indeed it is enough to take $r = r_{\sigma_2} \circ r_{\sigma_1}$ and $r = r_{\sigma_3} \circ r_{\sigma_1}$ where $\sigma_1 = (x_1x_4,\ q^2x_4x_1)$, $\sigma_2 = (x_2x_4,\ qx_4x_2)$ and $\sigma_3 = (x_1x_2,\ qx_2x_1)$.

Using the diamond lemma we can conclude that a vector space basis for $A(S^7_q)$ is given by the set of S-irreducible monomials of $\langle X\rangle$:

$$\overline{x}_4^{a_4}\ \overline{x}_3^{a_3}\ \overline{x}_2^{a_2}\ \overline{x}_1^{a_1}\ x_4^{b_4}\ x_3^{a_3}\ x_2^{b_2}\ x_1^{b_1}\ , \qquad a_i,\ b_i \in \mathbb{N}\ , \quad a_1b_1 = 0 \tag{2.51}$$

According to the degree $d = \sum(a_i + b_i)$ of monomials, we split $A(S^7_q)$ into subspaces $A_d(S^7_q)$ of homogeneous polynomials of degree d.

We observe that the subalgebra of coinvariants $A(S_q^7)^{co(SU_q(2))}$ coincides with the kernel of the map

$$\begin{array}{rcl} \tilde{\delta} : A(S_q^7) & \to & A(S_q^7) \otimes A(SU_q(2)) \\ x & \mapsto & \delta_R(x) - x \otimes 1 \end{array} \tag{2.52}$$

where δ_R is the coaction map (2.36) and 1 denotes the unit $\alpha\overline{\alpha} + q^2\overline{\gamma}\gamma$ of the algebra $A(SU_q(2))$.
By construction, the coaction map δ_R (and as a consequence the map $\tilde{\delta}$) preserves the degree of polynomials, that is $\delta_R : A_d(S_q^7) \to A_d(S_q^7) \otimes A(SU_q(2))$.
In particular we can observe that for d odd, the unit of $A(SU_q(2))$ cannot be in the image of δ_R and hencefore there cannot be coinvariants of odd degree.
For $d = 2m$, let $\tilde{\delta}_d$ be the restrictions of $\tilde{\delta}$ to $A_d(S_q^7)$. The dimension of the kernel of $\tilde{\delta}_d$,i.e. of the subspace of coinvariants does not depend on q and it is equal to the classical case $q = 1$. Indeed it is possible to rescale (by powers of q) the generators in the source and in the target of $\tilde{\delta}_d$ in such a way that the matrix representing $\tilde{\delta}_d$ has scalar entries which are independent of q.
In order to conclude our proof, we now exibit at each degree a basis of coinvariants showing that the elements which constitute this basis also generate the classical subspace of coinvariants.
For $d = 2$ this has already been checked. At higher degrees we use once more the diamond lemma. We choose the reduction system in such a way that the corresponding ideal of relations gives the commutation relations (2.30), (2.31) (and their conjugates) and it is the following one

$$S = \left\{ \begin{array}{lll} (ba, q^{-4}ab) \ , & (b\overline{a}, \overline{a}b) \ , & (\overline{b}b, q^4t(1-t) - q^8\overline{a}a) \\ (at, q^2ta) \ , & (\overline{a}t, q^{-2}t\overline{a}) \ , & (b\overline{b}, (1-q^{-4})t^2 + q^{-4}\overline{b}b) \\ (\overline{b}a, a\overline{b}) \ , & (\overline{b}\overline{a}, q^4\overline{a}\overline{b}) \ , & (b\overline{b}, q^{-2}t(1-q^{-2}t) - a\overline{a}) \\ (bt, q^{-4}tb) \ , & (\overline{b}t, q^4t\overline{b}) \ , & (a\overline{a}, (1-q^{-2})t - q^4\overline{a}a) \end{array} \right\}$$

We take the "lexicographic" ordering induced by the following ordering of the letters

$$t < \overline{a} < a < \overline{b} < b \ .$$

It is compatible with S and obviously satisfies the descending chain condition. As before, there are only overlap ambiguities in S and can all be resolved. We are in the hypothesis of the diamond lemma and we can conclude that a vector space basis for the space $A_d(S_q^4)$ is given by the S-irreducible elements

$$\{t^i\overline{a}^j a^{j'}\overline{b}^k b^{k'} \text{ with } kk' = 0; \ i + j + j' + k + k' = d\}.$$

These same elements give also a basis for the classical subspace of coinvariants and this conclude our proof. □

Chapter 3

Noncommutative principal bundles

In the previous Chapter we presented the two "bundles" $A(S^7_q) \subseteq A(Sp_q(2))$ and $A(S^4_q) \subseteq A(S^7_q)$ in terms of algebras, coactions and subalgebras of coinvariants. Here we make more clear and precise the concept of principal bundles in noncommutative geometry giving, in the first part of the chapter, a review of some elements of the general theory [15]. In the second one, we will elaborate on the two examples just mentioned. (In a certain sense, Sect. 3.1 conceptually precedes Ch. 2.)

As said, following the common idea to replace spaces by algebras of functions, the basic ingredients for the formulation in noncommutative geometry of a theory of principal bundles will be two algebras correspondent to the (algebras of functions on the) total and base spaces and a Hopf algebra, or a quantum group, playing the role of the structure group.

In the first section we discuss quantum bundles as introduced in [15] by Brzezinski and Majid in which the theory is developed with a particular attention to the differential calculus of which the algebras are endowed. In the second section we study principal bundles from the more algebric point of view of Hopf-Galois extensions [61]. The overlap between these two construction is recovered when, in the first formulation, the algebras are endowed with the universal differential calculus. (A quite different approach to differential geometry of quantum bundles was developed by Durdevich [29], in this case without refering to the theory of Galois extensions.) The concept of connection on quantum principal bundles was introduced in [15] and then developed in [32], [25]. This will be the topic of Sect. 3.2.1.

Finally we conclude by showing that the two extensions $A(S^7_q) \subseteq A(Sp_q(2))$ and $A(S^4_q) \subseteq A(S^7_q)$ are examples of quantum principal bundles. For both of them we will define a strong connection, firstly on the generators of the structure groups and then by extending these maps to the whole algebras.

3.1 Quantum principal bundles

In the approach [15] to quantum principal bundles, the theory is developed accordingly to the differential structure of the total space and the structure group. For self-consistency and in order to fix the notations, we begin by recalling some basic definitions of the theory of differential calculus on algebras.

Definition 3.1. *A first order differential calculus over an algebra A consists of a pair (Γ, d) such that Γ is a bimodule over A, $d : A \to \Gamma$ is a linear map which satisfies Leibnitz rule: $d(ab) = d(a)b + ad(b)$, $a, b \in A$ and every element of Γ is a finite sum $\sum_k a_k db_k$, $a_k, b_k \in A$.*

Given a first order differential calculus on A we can associate the external algebra $\Omega(A) = \oplus\, \Omega^n(A)$ constructed by defining

$$\Omega^0(A) = A \quad ; \qquad \Omega^n(A) \subset \Gamma \otimes_A \ldots \otimes_A \Gamma\,, \quad n > 0$$

as the span of elements

$$(a_0, a_1, \ldots a_n) := a_0 \otimes_A da_1 \otimes_A \ldots \otimes_A da_n \;\;, \quad \forall a_k \in A\,.$$

$\Omega(A)$ is endowed with a structure of a $\mathbb{Z}_2$-graded algebra by taking n mod 2 and setting

$$(a_0, \ldots a_n) \cdot (a_{n+1}, \ldots a_{n+m}) = \sum_{i=0}^{n} (a_0, \ldots a_{n-i-1}, a_{n-i}a_{n-i+1}, \ldots a_{n+m})$$

for each $(a_0, \ldots a_n) \in \Omega^n$, $(a_{n+1}, \ldots a_{n+m}) \in \Omega^{n+m-1}$. The map d is extended to the external algebra by setting

$$d(a_0, \ldots a_n) = (1, a_1, \ldots a_n) \qquad ; \qquad d(1, a_0, \ldots a_n) = 0\,.$$

Example 3.1. *Of particular interest is the so called universal differential calculus. In this case the module of one form is defined through the exact sequence*

$$0 \longrightarrow \Gamma \longrightarrow A \otimes A \xrightarrow{m} A \longrightarrow 0$$

that is Γ is the kernel of the multiplication map m of A. The module $\Gamma = ker(m)$ is usually denoted by A^2. The differential $d =: \boldsymbol{d}$ is defined by setting

$$\boldsymbol{d}(a) = 1 \otimes a - a \otimes 1 \in ker(m)$$

and endowing Γ with the A-bimodule structure

$$c(\sum a_k \otimes b_k) = \sum ca_k \otimes b_k \;; \quad (\sum a_k \otimes b_k)c = \sum a_k \otimes b_k c\,,$$

$\boldsymbol{d}$ satisfies the Leibnitz rule. Moreover, every element $a_k \otimes b_k \in A^2$ can be written in the form $a_k \otimes b_k = a_k \boldsymbol{d} b_k$.
The external algebra corresponding to the universal differential calculus is usually denoted by ΩA. Let us observe that for the universal differential calculus the space $\Omega^n A$ of n-forms coincides with

$$\Omega^n A = \{\rho \in A^{\otimes n+1} / \ \forall i = 1, \dots n \ , \ m_i(\rho) = 0\}$$

where $m_i : A^{\otimes n+1} \to A^{\otimes n}$ is the multiplication on the $i, i+1$ factors, $m_i = id \otimes \dots \otimes m \dots \otimes id$.

The importance of this differential calculus is the fact that every first order differential calculus can be obtained from this one. If N is a sub-bimodule of A^2, $\pi : A^2 \to \Gamma := A^2/N$, then $(\Gamma, \pi \circ \boldsymbol{d})$ is a first order differential calculus over A. Conversly, any first order differential calculus (Γ, d) over A can be obtained in this way. Define $\pi : a_k \otimes b_k \in A^2 \mapsto a_k db_k \in \Gamma$. Then Γ is isomorphic as bimodule to the quotient $A^2/ker\pi$.

Let us now introduce the building blocks for quantum principal bundles. The "total space" of the fibration will be an algebra P playing the role of the algebra of functions on the total space of an ordinary bundle. On this algebra we assume the existence of a right coaction of a "structure group" H that we assume to be a Hopf algebra, like the algebra of functions of a Lie group or a deformation consisting of a quantum group. We denote by $\Delta_R : P \to P \otimes H$ this coaction and we will use Sweedler-like notations $\Delta_R(p) = p_{(0)} \otimes p_{(1)}$. The base space B is constructed as the set of those functions which are coinvariant:

$$B = P^{coH} := \{a \in P / \Delta_R(a) = a \otimes 1\} \subseteq P \ .$$

This is a subalgebra of P: if $a, b \in B$, since the coaction map is an algebra homomorphism, we have

$$\Delta_R(ab) = \Delta_R(a)\Delta_R(b) = ab \otimes 1 \quad \Rightarrow ab \in B \ .$$

The inclusion $j : B \hookrightarrow P$ dualises the canonical projection π. It is also clear that P is both a left and a right B-module.

Remark 3.1. *We deal in the following with the case in which P is endowed with the universal differential calculus but we point out that the theory can be developed also in the case of a non universal differential calculus. In that case one should take care of the compatibility between the differential calculus of the total space and the coaction of the structure group and request some further covariant conditions no longer automatically granted. Since we don't need it here, we don't discuss this point and we refer to [15].*

We endowe P with the universal differential calculus, let us denote by Γ_P, Γ_B the bimodules of one forms on P and B respectively and by ΩP, ΩB the external algebras associated. The inclusion map extends to $j : \Omega B \hookrightarrow \Omega P$ since the definition of the spaces of n-forms depends only on the product on P. Furthermore, also the comultiplication map extends naturally to the space of one forms by

$$\Delta_R : \Gamma_P \to \Gamma_P \otimes H \ , \qquad \Delta_R(a_k db_k) := \Delta_R(a_k)(d \otimes id)\Delta_R(b_k)$$

and to the external algebra ΩP in analogous way.

We consider the space of *horizontal forms* Γ_{hor}, defined as the subspace

$$\Gamma_{hor} := Pj(\Gamma_B)P \subseteq \Gamma_P \ .$$

An element of Γ_{hor} is called an horizontal form. It is hence of the form $\sum_i p_i(db_i)p'_i$, $b_i \in B$, $p_i, p'_i \in P$. (Horizontal forms are defined in analogy with the classical case by pullback from the base, see [15] for a discussion.) The definiton is extended to higher orders through

$$\Omega^n P_{hor} = Pj(\Gamma_B)Pj(\Gamma_B)\ldots j(\Gamma_B)P \ ,$$

the space of horizontal n-forms, with $\Omega^0 P_{hor} := P$.

Definition 3.2. *[[15], Def 4.1] Let P, B, H defined as before, we say that P is a quantum principal bundle with universal differential calculus, structure group H and base B if the two following conditions are satisfied*

- *the map $(m \otimes id)(id \otimes \Delta_R) : P \otimes P \to P \otimes H$ is surjective;*
- $\Gamma_{hor} = ker\left((m \otimes id)(id \otimes \Delta_R)_{|_{P^2}}\right)$

We will refer to the previous first condition as the the *freeness condition* because it dualises the condition for the structure group to act freely. The second one, the *exactness condition*, is related to the fact that fibers are copies of the structure Lie group.

Note that the above points imply that the following sequence is exact

$$0 \longrightarrow \Gamma_{hor} \xrightarrow{j} \Gamma_P \longrightarrow P \otimes ker(\varepsilon) \longrightarrow 0 \ , \tag{3.1}$$

where ε is the counit of H and the penultimate row is the map $(m \otimes id)(id \otimes \Delta_R)_{|_{P^2}}$ whose image is in fact $P \otimes ker(\varepsilon)$ rather than the whole $P \otimes H$.

Definition 3.3. *Let P, B, H defined as before, we say that P is a quantum trivial principal bundle if there exists a convolution invertible map $\Psi : H \to P$ such that*

$$\Delta_R \circ \Psi = (\Psi \otimes id) \circ \Delta \ ; \quad \Psi(1_H) = 1_P \ .$$

The map Ψ is called a trivialization.

In this case, one can check explicitly that the above contruction is in the first instance a quantum principal bundle in the sense of Def. 3.2. Furthermore, the map

$$b \otimes h \mapsto j(b)\Psi(h)$$

realizes an isomorphism of linear spaces between $B \otimes H$ and P.

3.1.1 Quantum principal bundles over homogeneous spaces

A particular case of quantum principal bundle is when the total space is itself a quantum group and the structure group is a subgroup acting by restriction of the coproduct.

Definition 3.4. *Let P be a quantum group (endowed with a differential calculus, not necessarly the universal one), let $\pi : P \to H$ a Hopf-algebra map, that is H is obtained by P under quotient by a Hopf ideal. Consider the natural coaction of H on P given by the restriction of the coproduct Δ of P: $\Delta_R := (id \otimes \pi) \circ \Delta$. The algebra of coinvariants $B = \{b \in P / b_{(1)} \otimes \pi(b_{(2)}) = b \otimes 1\}$ is called a quantum homogeneous space.*

In [15] a sufficient condition under which P is a quantum principal bundle is discussed. In Sect. 3.3.1 we will describe the example given by the extension $A(S^7_q) \subseteq A(Sp_q(2))$ which perfectly fits in the above class of bundles over quantum homogeneous spaces.

3.2 Hopf-Galois extensions

If we consider principal bundles from the point of view of affine algebraic geometry rather than concentrate in the differential structure of the total space, the notion of quantum principle bundle translates in the one of Hopf-Galois extension [60]. The notion of Hopf-Galois extensions was introduced by Kreimer and Takeuchi [42].

We recall here some relevant definitions, see e.g.[54] and describe the overlap between Galois theory and quantum bundles as introduced in Def. 3.2. In section 3.3 we will prove that the extensions $A(S^7_q) \subseteq A(Sp_q(2))$ and $A(S^4_q) \subseteq A(S^7_q)$ are both Hopf Galois.

Definition 3.5. *Let H be a Hopf algebra and P a right H-comodule algebra with multiplication $m : P \otimes P \to P$ and coaction $\Delta_R : P \to P \otimes H$. We use Sweedler-like notation $\Delta_R p = p_{(0)} \otimes p_{(1)}$. Let $B \subseteq P$ be the subalgebra of coinvariants, i.e.*

$B = \{p \in P \mid \Delta_R(p) = p \otimes 1\}$. *The extension* $B \subseteq P$ *is called a right* H*-Hopf-Galois extension if the canonical map*

$$\begin{aligned} &\chi : P \otimes_B P \longrightarrow P \otimes H \ , \\ &\chi := (m \otimes id) \circ (id \otimes_B \Delta_R) \ , \quad p' \otimes_B p \mapsto \chi(p' \otimes_B p) = p' p_{(0)} \otimes p_{(1)} \end{aligned} \tag{3.2}$$

is bijective.

In the following we will skip writing "right".

Observation 3.1. *Note that if the antipode of* H *is bijective, one could equivalently consider the map* $\chi' = (m \otimes id) \circ (\Delta_R \otimes_B id)$*. Indeed in this case* χ *is injective or surjective if and only if* χ' *is respectively injective or surjective, [42].*

We notice the similarity between the definition of Hopf-Galois extension and the one of quantum principal bundles recalled in Def. 3.2. In particular, if χ is surjective then freenes and exactness conditions follow, [15]. The converse is also true, we have the following:

Proposition 3.1 ([32], Prop. 1.6)**.** *Let* P, B, H *as in Def. 3.5, then* P *is a* H*-Hopf Galois extension if and only if* $P(B, A)$ *is a quantum principal bundle with the universal differential calculus.*

By its definition, the canonical map is left P-linear and right H-colinear and is a morphism (an isomorphism for Hopf-Galois extensions) of left P-modules and right H-comodules. Moreover χ is determined by its values on the generators.
Note that, if χ is bijective, then also the inverse map χ^{-1} is P-linear and so it is completely determined once one knows its values on the elements $1 \otimes h$, being $h \in H$ a generator. Hence we introduce the restricted map

$$\tau : H \to P \otimes_B P \ , \quad \tau(h) := \chi^{-1}(1 \otimes h) \ . \tag{3.3}$$

The map τ is called the *translation map* . It was introduced in [9], where some properties and examples of τ were also discussed. See also [25]. Classically, if we suppose to have a group G which acts freely on a manifold, then a translation map is defined as the map which associates to any two point on a orbit, the element of G which relates these two points. In [9] the translation map is defined by dualisation.

Observation 3.2. *The injectivity of the canonical map dualizes the condition of a group action to be free. Let* G *acts on the right on a set,* $X \times G \to X$, $x \mapsto x \cdot g$, *let us denote by* α *the map* $\alpha : X \times G \to X \times_M X$, $(x, g) \mapsto (x, x \cdot g)$ *then* $\alpha^* = \chi$ *with* P, H *the algebras of functions on* X, G *respectively and* $M := X/G$ *being the space of orbits with projection map* $\pi : X \to M$, $\pi(x \cdot g) = \pi(x)$, *for all* $x \in X, g \in G$*. Then the action is free if and only if* α *is injective. Furthermore,* α *is surjective if and only if for all* $x \in X$, *the fibre* $\pi^{-1}(\pi(x))$ *of* $\pi(x)$ *is equal to the residue class* $x \cdot G$, *that is, if and only if* G *acts transitively on the fibres of* π*. [54].*

The notion of Hopf Galois extension is in some sense too general to be useful. In order to assure that some further good geometric properties at level of algebraic principal bundles are preserved, one has to require that the extension $B \subset P$, beside being Hopf-Galois, is also faithfully flat. We briefly recall this notion.

Denote by ${}_B\mathcal{M}$ (resp. $\mathcal{M}_B$) the category of left (resp. right) B-modules and by $\mathcal{M}_P^H$ (resp. ${}_P\mathcal{M}^H$) the category of (P, H)-Hopf modules, that is right P-modules and right (resp. left) H-comodule with P-linear comodule structure. Then

Definition 3.6. *A right module P over a ring R is faithfully flat if the functor $P \otimes_R \cdot$ is exact and faithful on the category ${}_R\mathcal{M}$ of left R-modules.*

Flatness means that the functor associates exact sequences of abelian groups to exact sequences of R-modules and the functor is faithful if it is injective on morphisms (and not necessarly on objects). Equivalently one could state that a right module P over a ring R is faithfully flat if a sequence $M' \to M \to M''$ in ${}_R\mathcal{M}$ is exact if and only if $P \otimes_R M' \to P \otimes_R M \to P \otimes_R M''$ is exact. See [66] for a discussion of faithfully flatness.

Among the contributions to Galois theory given by Schneider, there is a particular usefull Theorem which characterizes faithfully flat Hopf-Galois extensions. This result, reported below, assures that if the structure group has nice properties, then the surjectivity of the canonical map is enough to conclude that the extension is Hopf-Galois and faithfully flat:

Theorem 3.1. *[[61], Th. I]: Let H be a Hopf-algebra, let P be an algebra carrying a right H-comodule structure Δ_R with coinvariant algebra $B = \{p \in P \mid \Delta_R(p) = p \otimes 1\}$ and let*
$\chi : P \otimes_B P \to P \otimes H$ be the canonical map (see e.g. (3.2)). Assume that the antipode of H is bijective, then the following conditions are equivalent:

1. *P is injective as right H-comodule and χ is surjective;*
2. *P is faithfully flat as left B-module and χ is an isomorphism;*
3. *P is faithfully flat as right B-module and χ is an isomorphism.*
4. *the map $\mathcal{M}_B \to \mathcal{M}_P^H, \quad M \mapsto M \otimes_B P$ is an equivalence;*
5. *the map ${}_B\mathcal{M} \to_P \mathcal{M}^H, \quad M \mapsto P \otimes_B M$ is an equivalence;*

In particular, if the Hopf algebra H is also cosemisimple (as is the case for $A(SU_q(2))$ in the examples on Sect. 3.3), then any right H-comodule is injective [54] and the above theorem hence states that the surjectivity of the canonical map is enough to ensure that it is bijective (so that we have an H-Galois structure) and (left and right) faithfully flat. Usually to prove the injectivity of χ is much more complicated that proving the surjectivity and the above theorem is also particularly useful from the computational point of view.

Remark 3.2. *A generalization of Hopf-Galois extension is encoded in the notion of coalgebra Galois extensions obtained by giving up the condition for the coaction map to be an algebra homomorphism and relaxing the condition for the coalgebra H to be a Hopf algebra [17], [13]. Such a generalization was introduced in order to consider gauge theories on some important quantum homogeneous spaces, such as for example Podleś spheres. Anyway, as we will show in the next section, the two extensions constructed in Ch. 2 fit in the more restricted theory of Hopf-Galois extensions and therefore we are not interested here in deal with coalgebra extensions.*

We conclude by giving the following

Definition 3.7. *A H Hopf-Galois extension $B \subseteq P$ is called cleft if there exists a unital convolution invertible linear map $\Phi : H \to P$ which fulfils $\Delta_R \circ \Phi = (\Phi \otimes id) \circ \Delta$, where Δ, Δ_R are respectively the coproduct and the coaction on P. The map Φ is called a cleaving map.*

Note that by construction, a cleaving map is injective and moreover that the above properties are not enough to complitely characterise Φ. Moreover, notice that the condition of unitality is not necessary in the sense that if we have a map ϕ which satisfies all the above requests except that $\phi(1) = b$, then $\Phi := b^{-1}\phi$ is also a unital cleaving map. In other worlds, it is possible to normalise any such a map.

We have to stress that cleftness of an extension is not exactly the same of triviality of a bundle. A trivial bundle is cleft, but the converse in general is not true. This is why we call Φ a cleaving map rather than a trivialization. See [26] for a discussion and for equivalent notions of cleftness of extensions.

3.2.1 Strong connections

We deal now with the topic of connections on quantum principal bundles. As before, we treat here only the case of algebras endowed with the universal differential structure, that is of Hopf-Galois extensions. We follow [15], [32] and [25] and we also refer to them for connections on quantum principal bundles with general differential calculi.

Definition 3.8. *[15] Let $B \subseteq P$ a Hopf-Galois extension, with structure group H. A left P-module projection Π on Γ_P, $\Pi^2 = \Pi$, is called a connection if*

1. $ker \Pi = \Gamma_{hor} = P(\Gamma_B)P$,
2. $\Delta_R \circ \Pi = (\Pi \otimes id) \circ \Delta_R$ *(i.e. Π is right-invariant).*

We used the same notations of the previous section: Γ_P, Γ_B are respectively the spaces of one forms of P and B with respect to the universal differential calculus. Here Δ_R denotes the coaction on P and also its extension to the external algebra ΩP.

The space $Im\Pi$ is by definition the space of vertical forms, it is a left P-submodule $\Gamma_{vert} \subseteq \Gamma_P$ and a connection is an assignment of such a vertical space. Once one has introduced a connection, then the space Γ_P splits as $\Gamma_P = \Gamma_{vert} \oplus \Gamma_{hor}$ and every one-form can be written in a unique way as a sum of a horizontal form and a vertical one.

Given a connection, one can introduce the notion of connection form, [15]. For Hopf-Galois extensions there is a one to one correspondence between connection forms and connections.

Definition 3.9. *Given a connection* Π *as in Def. 3.8, the connection form of* Π *is the map* $\omega : H \longrightarrow \Gamma_P$ *defined by*

$$\omega(h) = \sigma(1 \otimes (h - \varepsilon(h))) ,$$

where ε *is the counit of* H *and* $\sigma : P \otimes ker(\varepsilon) \longrightarrow \Gamma_P$ *is the map such that*

$$\sigma \circ (m \otimes id)(id \otimes \Delta_R)|_{P^2} = id .$$

The existence of such a map σ is ensured by the fact that we are dealing with a Hopf-Galois extension. Indeed, as said, this implies that the sequence (3.1)

$$0 \longrightarrow \Gamma_{hor} \xrightarrow{j} \Gamma_P \longrightarrow P \otimes ker(\varepsilon) \longrightarrow 0 ,$$

is exact. The existence of Π is equivalent to the existence of a map $\sigma : P \otimes ker(\varepsilon) \longrightarrow \Gamma_P$ which splits the sequence. Moreover, by definition Π should be a right invariant left P-module map and hence σ has the same property and they are related by

$$\Pi = \sigma \circ (m \otimes id)(id \otimes \Delta_R)|_{P^2} .$$

We refer to [15] for details and relations with the classical picture.

In the space of connections, a particular class is the one of strong connections. The importance of such a kind of connections is related to the fact that for Hopf Galois extensions admitting a strong connection it is easier to study the projective structure of associated bundles [32]. Moreover, constructing a strong connection is an alternative way to prove that one has a Hopf Galois extension [25, 32].

Definition 3.10. *Let* Π *a connection on* $B \subseteq P$*, it is called strong if*

$$(id - \Pi)(dP) \subseteq (\Gamma_B)P .$$

The space $\Omega^1_{s-hor}P := (\Gamma_B)P$ is called the space of strongly horizontal one-forms. Note that $\Omega^1 B \otimes_B P = ker(m|_{B\otimes P}) = (\Gamma_B)P$, [16].

There are different equivalent characterisations of strong connections [32], [25]. In particular an equivalent description of a strong connection is the following

Definition 3.11. *A strong connection on $B \subseteq P$ is as a unital left B-linear right H-colinear (i.e. which preserves the H-comodule structure of P) splitting s of the multiplication map $B \otimes P \xrightarrow{m} P$. In this case we say that $B \subseteq P$ is equivariantly projective.*

In the following examples we will take advantage of the fact that a H-Hopf-Galois extension $B \subseteq P$ for which H is cosemisimple and has a bijective antipode is also equivariantly projective. More in general:

Theorem 3.2. *[60] Let H be a Hopf algebra and P a right H-comodule algebra with subalgebra of coinvariants $B = P^{coH}$. Assume that P is injective as a right H-comodule. Then*

1. *$B \subseteq P$ is equivariantly projective if and only if P is projective as a left B-module;*
2. *if the extension $B \subseteq P$ is Hopf-Galois and the antipode of H is bijective, then $B \subseteq P$ is equivariantly projective.*

The above statement follows by recalling that a Hopf algebra is cosemisimple if and only if any right H-comodule is injective.

Furthermore, if H has an invertible antipode S, an equivalent description of a strong connection s can be given in terms of a map $\ell : H \to P \otimes P$ satisfying the conditions we list below [50, 14] (see also [34, 12]). We denote by Δ the coproduct on H with Sweedler notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$, by $\Delta_R : P \to P \otimes H$ the right-comodule structure on P with notation $\Delta_R(p) = p_{(0)} \otimes p_{(1)}$, and $\Delta_l : P \to H \otimes P$ is the induced left H-comodule structure of P defined by

$$\Delta_l(p) = (S^{-1} \otimes id)(flip \circ \Delta_R(p)) = S^{-1}(p_{(1)}) \otimes p_{(0)} ,$$

where $flip : P \otimes H \to H \otimes P$ is given by $flip(p \otimes h) = h \otimes p$. Note that

$$\Delta_l(pq) = S^{-1}(p_{(1)}q_{(1)}) \otimes p_{(0)}q_{(0)} = S^{-1}(q_{(1)})S^{-1}(p_{(1)}) \otimes p_{(0)}q_{(0)} ,$$

so that $\Delta_l(pq) \neq \Delta_l(p)\Delta_l(q)$. Then, for the map ℓ one requires that $\ell(1) = 1 \otimes 1$ and that for all $h \in H$,

$$\begin{aligned} &\chi \circ \pi_B(\ell(h)) = 1 \otimes h , \\ &\ell(h_{(1)}) \otimes h_{(2)} = (id \otimes \Delta_R) \circ \ell(h) , \\ &h_{(1)} \otimes \ell(h_{(2)}) = (\Delta_l \otimes id) \circ \ell(h) , \end{aligned} \tag{3.4}$$

where π_B is the surjection $\pi_B : P \otimes P \to P \otimes_B P$. In the following we simply write $\chi(\ell(h))$.

The splitting s of the multiplication map is then given by [14]

$$s : P \to B \otimes P \,, \quad p \mapsto p_{(0)}\ell(p_{(1)}) \ ,$$

that is

$$s = (m \otimes id) \circ (id \otimes \ell) \circ \Delta_R \ .$$

Note that the first condition in (3.4) could also be written by means of the translation map (3.3) as $\pi_B \circ \ell = \tau$. Connections are liftings of the translation map. The translation map also allows to construct ℓ once one knows s: if $\tau(h) = h^{(1)} \otimes_B h^{(2)}$, then the map ℓ is given by $\ell(h) = h^{(1)}s(h^{(2)})$ [14].

If H has a PBW basis [40], one can try to extend the defintion of ℓ on the basis and then iteratively construct ℓ once one knows its value on the generators of H. In the next section we will make such a construction explicitly for the two examples given by the extensions illustrated in the previous chapter.

The importance of strong connection, as already pointed out, is also related to the fact that if a H-Hopf-Galois extension admits a strong connection, then the modules associated through a representation of H are projective and finitely generated (as vector bundles in Serre-Swan's spirit) [25].

3.3 Two examples.

After the brief introductory description of the general theory of Hopf-Galois extensions given above, we come now to illustrate two concrete examples which are provided by the algebras $A(S^7_q) \subseteq A(Sp_q(2))$ and $A(S^4_q) \subseteq A(S^7_q)$ constructed in the previous chapter. As mentioned before, for Hopf algebras which are cosemisimple and have bijective antipodes, as is the case for $SU_q(2)$, Th. 3.1 grants further nice properties. In particular the surjectivity of the canonical map implies bijectivity and faithfully flatness of the extension.

An additional useful result [59] is that the map χ is surjective whenever, for any generator h of H, the element $1 \otimes h$ is in its image. This follows from the properties of left P-linearity and right H-colinearity of the canonical map. Indeed, let $h,\ k$ be two elements of H and $\sum p'_i \otimes p_i,\ \sum q'_j \otimes q_j \in P \otimes P$ be such that $\chi(\sum p'_i \otimes_B p_i) = 1 \otimes h,\ \chi(\sum q'_j \otimes_B q_j) = 1 \otimes k$. Then $\chi(\sum p'_i q'_j \otimes_B q_j p_i) = 1 \otimes kh$, that is $1 \otimes kh$ is in the image of χ. But, since the map χ is left P-linear, this implies its surjectivity.

We need also the following

Definition 3.12. *Let P be a bimodule over the ring B. Given any two elements $|\xi_1\rangle$ and $|\xi_2\rangle$ in the free module $\mathcal{E} = \mathbb{C}^m \otimes P$, we shall define $\left\langle \xi_1 \dot{\otimes}_B \xi_2 \right\rangle \in P \otimes_B P$ by*

$$\left\langle \xi_1 \dot{\otimes}_B \xi_2 \right\rangle := \sum_{j=1}^{m} \overline{\xi}_1^j \otimes_B \xi_2^j \,. \tag{3.5}$$

Analogously, one can define quantities $\left\langle \xi_1 \dot{\otimes} \xi_2 \right\rangle \in P \otimes P$ with the same formula as above and tensor products taken over the ground field $\mathbb{C}$.

3.3.1 The Hopf-Galois extension $A(S_q^7) \subseteq A(Sp_q(2))$

Proposition 3.2. *The extension $A(S_q^7) \subset A(Sp_q(2))$ is a faithfully flat $A(Sp_q(1))$-Hopf-Galois extension.*

Proof. Now $P = A(Sp_q(2))$, $H = A(Sp_q(1))$ and $B = A(S_q^7)$ and the coaction Δ_R of H is given on the defining matrices $\Delta_R T = T \otimes T'$. Since $A(Sp_q(1)) \simeq A(SU_{q^2}(2))$ has a bijective antipode and is cosemisimple ([41], Chapter 11), from the general considerations given above in order to show the bijectivity of the canonical map

$$\chi : A(Sp_q(2)) \otimes_{A(S_q^7)} A(Sp_q(2)) \longrightarrow A(Sp_q(2)) \otimes A(Sp_q(1)) \,,$$

it is enough to show that all generators $\alpha, \gamma, \overline{\alpha}, \overline{\gamma}$ of $A(Sp_q(1))$ in (2.17) are in its image.
Let $|T^2\rangle, |T^3\rangle$ be the second and third columns of the defining matrix T of $Sp_q(2)$. We shall think of them as elements of the free module $\mathbb{C}^4 \otimes A(Sp_q(2))$. Obviously, $\langle T^i | T^j \rangle = \delta^{ij}$. Recalling that $A(Sp_q(2))$ is both a left and right $A(S_q^7)$-module and using Def. 3.12, we have that

$$\chi \begin{pmatrix} \left\langle T^2 \dot{\otimes}_{A(S_q^7)} T^2 \right\rangle & \left\langle T^2 \dot{\otimes}_{A(S_q^7)} T^3 \right\rangle \\ \left\langle T^3 \dot{\otimes}_{A(S_q^7)} T^2 \right\rangle & \left\langle T^3 \dot{\otimes}_{A(S_q^7)} T^3 \right\rangle \end{pmatrix} = 1 \otimes \begin{pmatrix} \alpha & -q^2\overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix} \,. \tag{3.6}$$

Indeed, using $\Delta_R(T_i^j) = T_i^k \otimes T'^j_k$, with $T = (T_i^j$ and $T' = T'^j_i$ the defining matrices of $A(Sp_q(2))$, $A(Sp_q(1)$ respectivley, we have

$$\Delta_R(T_i^2) = T_i^2 \otimes \alpha + T_i^3 \otimes \gamma \qquad \Delta_R(T_i^3) = -q^2 T_i^2 \otimes \overline{\gamma} + T_i^3 \otimes \overline{\alpha} \,.$$

Hence

$$\begin{aligned} \chi(\left\langle T^2 \dot{\otimes}_{A(S_q^7)} T^2 \right\rangle) &= \overline{T}_i^2 \Delta_R T_i^2 = \left\langle T^2 | T^2 \right\rangle \otimes \alpha + \left\langle T^2 | T^3 \right\rangle \otimes \gamma = 1 \otimes \alpha \,, \\ \chi(\left\langle T^3 \dot{\otimes}_{A(S_q^7)} T^2 \right\rangle) &= \overline{T}_i^3 \Delta_R T_i^2 = \left\langle T^3 | T^2 \right\rangle \otimes \alpha + \left\langle T^3 | T^3 \right\rangle \otimes \gamma = 1 \otimes \gamma; \end{aligned}$$

and similar computation giving the other two generators:

$$\begin{aligned}
\chi(\Big\langle T^2 \dot{\otimes}_{A(S_q^7)} T^3 \Big\rangle) &= \overline{T}_i^2 \Delta_R T_i^3 = -q^2 \left\langle T^2 | T^2 \right\rangle \otimes \overline{\gamma} + \left\langle T^3 | T^2 \right\rangle \otimes \overline{\alpha} = 1 \otimes (-q^2 \overline{\gamma}) \ , \\
\chi(\Big\langle T^3 \dot{\otimes}_{A(S_q^7)} T^3 \Big\rangle) &= \overline{T}_i^3 \Delta_R T_i^3 = -q^2 \left\langle T^3 | T^2 \right\rangle \otimes \overline{\gamma} + \left\langle T^3 | T^3 \right\rangle \otimes \overline{\alpha} = 1 \otimes \overline{\alpha}) \ .
\end{aligned}$$

□

As already pointed out, it is important to notice that this extension also provides an example of principal bundle on quantum homogeneous space as described in Sect. 3.1.1. We talk about *principal homogeneous Hopf-Galois extension* , i.e. a P/I Hopf-Galois extension given by a Hopf ideal I in a Hopf algebra P and coaction given by the restiction of the coproduct, as in (2.19), [26].

In this case we have another way to prove the bijectivity of the canonical map, that is by constructing directly its inverse. Following [9], let us consider the map

$$\tau : Sp_q(1) \to A(Sp_q(2)) \otimes_B A(Sp_q(2)) \ , \quad \tau([p]) = S(p_{(1)}) \otimes_B p_{(2)} \ , \tag{3.7}$$

where $[p]$ is the image of $p \in P$ under $\pi : P \to P/I$, that we remind was given by eq. (2.17)

$$\pi(T) = T' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & -q^2\overline{\gamma} & 0 \\ 0 & \gamma & \overline{\alpha} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{3.8}$$

Proposition 3.3. *The map τ in* (3.7) *is the translation map.*

Proof. We have to show that $\tilde{\chi} := (m \otimes id)(id \otimes \tau)$ is the inverse of the canonical map $\chi = (m \otimes id)(id \otimes \Delta_R)$. Using the fact that χ and $\tilde{\chi}$ are both left P-linear it is enough to show the following

$$\begin{aligned}
(\chi \circ \tilde{\chi})(1 \otimes [T_{ij}]) &= \chi(m \otimes id)(1 \otimes S(T_{il}) \otimes_B T_{lj}) = \\
& S(T_{il})\chi(1 \otimes_B T_{lj}) = S(T_{il})(m \otimes id)(1 \otimes_B \Delta_R(T_{lj})) = \\
& S(T_{il})(T_{lp} \otimes T'_{pj}) = \delta_{ip} \otimes T'_{pj} \ .
\end{aligned}$$

Thus $\chi \circ \tilde{\chi} = id_{P \otimes H}$ and similarly

$$\begin{aligned}
(\chi \circ \tilde{\chi})(1 \otimes_B T_{ij}) &= \tilde{\chi}(m \otimes id)(1 \otimes_B \Delta_R(T_{ij})) = \\
& \tilde{\chi}(T_{il} \otimes T'_{lj}) = T_{il}(m \otimes id)(id \otimes \tau)(1 \otimes T'_{lj}) = \\
& T_{il}(S(T_{lp}) \otimes_B T_{pj}) = \delta_{ip} \otimes_B T_{pj} \ .
\end{aligned}$$

which means $\tilde{\chi} \circ \chi = id_{P \otimes_B P}$. □

We can now construct a strong connection for $A(S^7_q) \subset A(Sp_q(2))$. As said, a H-Hopf-Galois extension for which the structure group is cosemisimple and has a bijective antipode always admits a left B-linear right H-colinear splitting s of the multiplication map. We can constuct s by means of the map ℓ (3.4). We do it by using the translation map τ. The map ℓ should satisfy $\pi_B \circ \ell = \tau$ where $\pi_B : P \otimes P \to P \otimes_B P$ is the surjection. We set $\ell(1) = 1 \otimes 1$ and on the generators

$$\begin{array}{ll} \ell(\alpha) = S(T_{2i}) \otimes T_{i2} & \ell(\gamma) = S(T_{3i}) \otimes T_{i2} \\ \ell(\overline{\alpha}) = S(T_{3i}) \otimes T_{i3} & \ell(\overline{\gamma}) = -q^{-2} S(T_{2i}) \otimes T_{i3}. \end{array} \tag{3.9}$$

Proposition 3.4. *The map $\ell : A(Sp_q(1)) \to A(Sp_q(2)) \otimes A(Sp_q(2))$ constructed in* (3.9) *defines a strong connection on the generators of $A(Sp_q(1))$.*

Proof. We have to show that all the properties (3.4) are satisfied. Firstly $\chi(\ell(h)) = 1 \otimes h$, for each generator h of $P = A(Sp_q(2))$:

$$\begin{aligned} \chi(\ell(\alpha)) &= S(T_{2i})(T_{i2} \otimes \alpha + T_{i3} \otimes \gamma) = 1 \otimes \alpha \ ; \\ \chi(\ell(\gamma)) &= S(T_{3i})(T_{i2} \otimes \alpha + T_{i3} \otimes \gamma) = 1 \otimes \gamma \ ; \\ \chi(\ell(\overline{\alpha})) &= S(T_{3i})(-q^{-2} T_{i2} \otimes \overline{\gamma} + T_{i3} \otimes \overline{\alpha}) = 1 \otimes \overline{\alpha} \ ; \\ \chi(\ell(\overline{\gamma})) &= -q^2 S(T_{2i})(-q^{-2} T_{i2} \otimes \overline{\gamma} + T_{i3} \otimes \overline{\alpha}) = 1 \otimes \overline{\gamma} \ . \end{aligned}$$

The second property $(id \otimes \Delta_R)\ell(h) = \ell(h_{(1)}) \otimes h_{(2)}$ holds:

$$\begin{aligned} (id \otimes \Delta_R)\ell(\alpha) &= S(T_{2i}) \otimes (T_{i2} \otimes \alpha + T_{i3} \otimes \gamma) = \ell(\alpha) \otimes \alpha - q^2 \ell(\overline{\gamma}) \otimes \gamma = \\ &\quad \ell(\alpha_{(1)}) \otimes \alpha_{(2)} \ ; \\ (id \otimes \Delta_R)\ell(\gamma) &= S(T_{3i}) \otimes (T_{i2} \otimes \alpha + T_{i3} \otimes \gamma) = \ell(\gamma) \otimes \alpha + \ell(\overline{\alpha}) \otimes \gamma = \\ &\quad \ell(\gamma_{(1)}) \otimes \gamma_{(2)} \ ; \\ (id \otimes \Delta_R)\ell(\overline{\alpha}) &= S(T_{3i}) \otimes (-q^{-2} T_{i2} \otimes \overline{\gamma} + T_{i3} \otimes \overline{\alpha}) = -q^2 \ell(\gamma) \otimes \overline{\gamma} + \ell(\overline{\alpha}) \otimes \overline{\alpha} = \\ &\quad \ell(\overline{\alpha}_{(1)}) \otimes \overline{\alpha}_{(2)} \ ; \\ (id \otimes \Delta_R)\ell(\overline{\gamma}) &= -q^{-2} S(T_{2i}) \otimes (-q^{-2} T_{i2} \otimes \overline{\gamma} + T_{i3} \otimes \overline{\alpha}) = \ell(\alpha) \otimes \overline{\gamma} + \ell(\overline{\gamma}) \otimes \overline{\alpha} = \\ &\quad \ell(\overline{\gamma}_{(1)}) \otimes \overline{\gamma}_{(2)} \ . \end{aligned}$$

Observe now that

$$\Delta_R(S(T_{ij})) = S(T_{kj}) \otimes S(T'_{ik})$$

Indeed, using (1.7),

$$\begin{aligned} S(T_{kj}) \otimes S(T'_{ik}) &= -q^{\rho_{k'}+\rho_j} \varepsilon_k \varepsilon_{j'} T_{j'k'} \otimes -q^{\rho_{i'}+\rho_k} \varepsilon_i \varepsilon_{k'} T'_{k'i'} = -q^{\rho_{i'}+\rho_j} \varepsilon_i \varepsilon_{j'} T_{j'k} \otimes T'_{ki'} = \\ &\quad -q^{\rho_{k'}+\rho_j} \varepsilon_k \varepsilon_{j'} \Delta_R(T_{j'i'}) = \Delta_R(S(T_{ij})) \ . \end{aligned}$$

Thus the induced left action Δ_l reads

$$\begin{aligned} \Delta_l(S(T_{2i})) &= T'_{2k} \otimes S(T_{ki})) = \alpha \otimes S(T_{2i}) - q^2 \overline{\gamma} \otimes S(T_{3i}) \ ; \\ \Delta_l(S(T_{3i})) &= T'_{3k} \otimes S(T_{ki})) = \gamma \otimes S(T_{2i}) + \overline{\alpha} \otimes S(T_{3i}) \ . \end{aligned}$$

We can now show that $h_{(1)} \otimes \ell(h_{(2)}) = (\Delta_l \otimes id)\ell(h)$:

$$
\begin{aligned}
(\Delta_l \otimes id)\ell(\alpha) &= \alpha \otimes S(T_{2i}) \otimes T_{i2} - q^2\overline{\gamma} \otimes S(T_{3i}) \otimes T_{i2} = \alpha \otimes \ell(\alpha) - q^2\overline{\gamma} \otimes \ell(\gamma) = \\
&\quad \alpha_{(1)} \otimes \ell(\alpha_{(2)}) \;; \\
(\Delta_l \otimes id)\ell(\gamma) &= \gamma \otimes S(T_{2i}) \otimes T_{i2} + \overline{\alpha} \otimes S(T_{3i}) \otimes T_{i2} = \gamma \otimes \ell(\alpha) + \overline{\alpha} \otimes \ell(\gamma) = \\
&\quad \gamma_{(1)} \otimes \ell(\gamma_{(2)}) \;; \\
(\Delta_l \otimes id)\ell(\overline{\alpha}) &= \gamma \otimes S(T_{2i}) \otimes T_{i3} + \overline{\alpha} \otimes S(T_{3i}) \otimes T_{i3} = -q^2\gamma \otimes \ell(\overline{\gamma}) + \overline{\alpha} \otimes \ell(\alpha) = \\
&\quad \overline{\alpha}_{(1)} \otimes \ell(\overline{\alpha}_{(2)}) \;; \\
(\Delta_l \otimes id)\ell(\overline{\gamma}) &= -q^{-2}\alpha \otimes S(T_{2i}) \otimes T_{i3} + \overline{\gamma} \otimes S(T_{3i}) \otimes T_{i3} = \alpha \otimes \ell(\overline{\gamma}) + \overline{\gamma} \otimes \ell(\alpha) = \\
&\quad \overline{\gamma}_{(1)} \otimes \ell(\overline{\gamma}_{(2)}) \;.
\end{aligned}
$$

□

We can now try to extend the definition of ℓ to the whole algebra, firstly defining ℓ on the product of two generators. Let $g, h \in \{\alpha, \gamma, \overline{\alpha}, \overline{\gamma}\}$, $\ell(g) = g^1 \otimes g^2$, $\ell(h) = h^1 \otimes h^2$. Observe that using the translation map as defined in (3.7), ℓ should satisfy

$$\pi_B \circ \ell(gh) = (gh)^1 \otimes_B (gh)^2 = \tau(gh) = S(h_{(1)})S(g_{(1)}) \otimes_B g_{(2)}h_{(2)} = h^1 g^1 \otimes_B g^2 h^2 \;.$$

This suggests to define

$$\ell(gh) := h^1 g^1 \otimes g^2 h^2 \tag{3.10}$$

on the product of any two generators. We show now that with this definition, properties (3.4) are satisfied:

Proof. Firstly observe that in general, if $g, h \in H$ satisfy $\chi(\ell(g)) = 1 \otimes g$, $\chi(\ell(h)) = 1 \otimes h$, so does $\ell(gh)$ defined by (3.10). Indeed, using the fact that χ restricted to $1 \otimes P$ is an homomorphism, we have

$$
\begin{aligned}
\chi(\ell(gh)) &= h^1 g^1 \chi(1 \otimes g^2 h^2) = h^1 g^1 \chi(1 \otimes g^2)\chi(1 \otimes h^2) = h^1(1 \otimes g)\chi(1 \otimes h^2) = \\
&\quad (1 \otimes g)h^1 \chi(1 \otimes h^2) = 1 \otimes gh \;.
\end{aligned} \tag{3.11}
$$

It is usefull to arrange the generators of H into the matrix T' so that the map ℓ given in Prop. 3.4 reads

$$\ell(T'_{ij}) = S(T_{im}) \otimes T_{mj}$$

and we now show that $(\ell \otimes id)\Delta = (id \otimes \Delta_R)\ell$ on the product $gh = T'_{ij}T'_{pq}$:

$$
\begin{aligned}
\ell(g_{(1)}h_{(1)}) \otimes g_{(2)}h_{(2)} &= \ell(T'_{ik}T'_{ph}) \otimes T'_{kj}T'_{hq} = S(T_{pm})S(T_{in}) \otimes T_{nk}T_{mh} \otimes T'_{kj}T'_{hq} \\
&= S(T_{pm})S(T_{in}) \otimes \Delta_R(T_{nj}T_{mq}) = (id \otimes \Delta_R)\ell(T'_{ij}T'_{pq}) \;.
\end{aligned}
$$

Moreover, using $\Delta_l(S(T_{ij})) = T'_{ik} \otimes S(T_{kj})$ we have

$$
\begin{aligned}
g_{(1)}h_{(1)} \otimes \ell(g_{(2)}h_{(2)}) &= T'_{il}T'_{pm} \otimes \ell(T'_{lj}T'_{mq}) = T'_{il}T'_{pm} \otimes S(T_{mr})S(T_{ls}) \otimes T_{sj}T_{rq} \\
&= (\Delta_l \otimes id)[S(T_{pr})S(T_{is}) \otimes T_{sj}T_{rq}] = (\Delta_l \otimes id)\ell(T'_{ij}T'_{pq}) \;.
\end{aligned}
$$

and this conclude our proof. □

We can now extend the definition of ℓ on the whole of $H = A(Sp_q(1))$ by using (3.10) on the PBW basis of H:

$$r^{k,l,m} := \alpha^k \gamma^l \overline{\gamma}^m \ , \ \text{for } k,l,m \geq 0 \quad ; \qquad s^{k,l,m} := \gamma^k \overline{\gamma}^l \overline{\alpha}^m \ , \ \text{for } k,l \geq 0 \, , m > 0 \ .$$

Let $g, h \in \{\alpha, \gamma, \overline{\alpha}, \overline{\gamma}\}$, $\ell(g) = g^1 \otimes g^2$, $\ell(h) = h^1 \otimes h^2$, then we set $\ell(gh) = h^1 g^1 \otimes g^2 h^2$, from which:

$$\begin{aligned}
\ell(r^{k+1,l,m}) &= q^{2(l+m)} \alpha^1 \ell(r^{k,l,m}) \alpha^2 = q^{2(l+m)} S(T_{2i}) \ell(r^{k,l,m}) T_{i2} \ , \\
\ell(r^{k,l+1,m}) &= \gamma^1 \ell(r^{k,l,m}) \gamma^2 = S(T_{3i}) \ell(r^{k,l,m}) T_{i2} \ , \\
\ell(r^{k,l,m+1}) &= \overline{\gamma}^1 \ell(r^{k,l,m}) \overline{\gamma}^2 = -q^{-2} S(T_{2i}) \ell(r^{k,l,m}) T_{i3}
\end{aligned}$$

and

$$\begin{aligned}
\ell(s^{k+1,l,m}) &= q^{2m} \gamma^1 \ell(s^{k,l,m}) \gamma^2 = q^{2m} S(T_{3i}) \ell(r^{k,l,m}) T_{i2} \ , \\
\ell(s^{k,l+1,m}) &= q^{2m} \overline{\gamma}^1 \ell(s^{k,l,m}) \overline{\gamma}^2 = -q^{2(m-1)} S(T_{2i}) \ell(r^{k,l,m}) T_{i3} \ , \\
\ell(s^{k,l,m+1}) &= \overline{\alpha}^1 \ell(s^{k,l,m}) \overline{\alpha}^2 = S(T_{3i}) \ell(r^{k,l,m}) T_{i3} \ ,
\end{aligned}$$

where the sum over i is suppressed and

$$S(T_{2i}) = q^{\rho_i - 1} \varepsilon_i T_{i'3} \quad ; \quad S(T_{3i}) = -q^{\rho_i + 1} \varepsilon_i T_{i'2} \quad .$$

This complete the definition of a strong connection for our extension $A(S^7_q) \subseteq A(Sp_q(2))$.

3.3.2 The Hopf-Galois extension $A(S^4_q) \subseteq A(S^7_q)$

In this section the notation can be misleading: here $\alpha, \gamma, \overline{\alpha}, \overline{\gamma}$ denote the generators of $A(SU_q(2))$ while in the previous section the same letters were used for the generators of $A(Sp_q(1)) \cong A(SU_{q^2}(2))$.

Proposition 3.5. *The extension $A(S^4_q) \subset A(S^7_q)$ is a faithfully flat $A(SU_q(2))$-Hopf-Galois extension.*

Proof. Now $P = A(S^7_q)$, $H = A(SU_q(2))$ and $B = A(S^4_q)$ and the coaction δ_R of H is given in Prop. 2.4. As already mentioned $A(SU_q(2))$ has a bijective antipode and is cosemisimple, then as before in order to show the bijectivity of the canonical map

$$\chi : A(S^7_q) \otimes_{A(S^4_q)} A(S^7_q) \longrightarrow A(S^7_q) \otimes A(SU_q(2)) \ ,$$

we have to show that all generators $\alpha, \gamma, \overline{\alpha}, \overline{\gamma}$ of $A(SU_q(2))$ in (2.33) are in its image. Recalling that $A(S_q^7)$ is both a left and right $A(S_q^4)$-module and using Def. 3.12, we have that

$$\chi \begin{pmatrix} \left\langle \phi_1 \dot{\otimes}_{A(S_q^4)} \phi_1 \right\rangle & \left\langle \phi_1 \dot{\otimes}_{A(S_q^4)} \phi_2 \right\rangle \\ \left\langle \phi_2 \dot{\otimes}_{A(S_q^4)} \phi_1 \right\rangle & \left\langle \phi_2 \dot{\otimes}_{A(S_q^4)} \phi_2 \right\rangle \end{pmatrix} = 1 \dot{\otimes} \begin{pmatrix} \alpha & -q\overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix} ,$$

where $|\phi_1\rangle, |\phi_2\rangle$ are the two vectors introduced in eqs. (2.23) and (2.24). Indeed

$$\begin{aligned} \chi(\left\langle \phi_1 \dot{\otimes}_{A(S_q^4)} \phi_1 \right\rangle) &= \chi \Big(q^{-6}\overline{x}^1 \otimes_{A(S_q^4)} x_1 + q^{-2} x_2 \otimes_{A(S_q^4)} \overline{x}^2 \\ &\qquad + q^{-2}\overline{x}^3 \otimes_{A(S_q^4)} x_3 + x_4 \otimes_{A(S_q^4)} \overline{x}^4 \Big) \\ &= q^{-6}\overline{x}^1 \delta_R(x_1) + q^{-2} x_2 \delta_R(\overline{x}^2) + q^{-2}\overline{x}^3 \delta_R(x_3) + x_4 \delta_R(\overline{x}^4) \\ &= q^{-6}\overline{x}^1 x_1 \otimes \alpha + q^{-5}\overline{x}^1 x_2 \otimes \gamma + q^{-2} x_2 \overline{x}^2 \otimes \alpha - q^{-2} x_2 \overline{x}^1 \otimes \gamma \\ &\quad + q^{-2}\overline{x}^3 x_3 \otimes \alpha - q^{-1}\overline{x}^3 x_4 \otimes \gamma + x_4 \overline{x}^4 \otimes \alpha + x_4 \overline{x}^3 \otimes \gamma \\ &= \langle \phi_1 | \phi_1 \rangle \otimes \alpha = 1 \otimes \alpha , \end{aligned}$$

$$\begin{aligned} \chi(\left\langle \phi_2 \dot{\otimes}_{A(S_q^4)} \phi_1 \right\rangle) &= q^{-5}\overline{x}^2 \delta_R(x_1) - q^{-2} x_1 \delta_R(\overline{x}^2) - q^{-1}\overline{x}^4 \delta_R(x_3) + x_3 \delta_R(\overline{x}^4) \\ &= q^{-5}\overline{x}^2 x_1 \otimes \alpha + q^{-4}\overline{x}^2 x_2 \gamma - q^{-2} x_1 \overline{x}^2 \otimes \alpha + q^{-2} x_1 \overline{x}^1 \otimes \gamma \\ &\quad - q^{-1}\overline{x}^4 x_3 \otimes \alpha + \overline{x}^4 x_4 \otimes \gamma + x_3 \overline{x}^4 \otimes \alpha + x_3 \overline{x}^3 \otimes \gamma \\ &= \langle \phi_2 | \phi_1 \rangle \otimes \alpha + \langle \phi_2 | \phi_1 \rangle \otimes \gamma = 1 \otimes \gamma , \end{aligned}$$

with similar computations for the conjugated generators:

$$\begin{aligned} \chi(\left\langle \phi_1 \dot{\otimes}_{A(S_q^4)} \phi_2 \right\rangle) &= q^{-5}\overline{x}^1 \delta_R(x_2) - q^{-2} x_2 \delta_R(\overline{x}^1) - q^{-1}\overline{x}^3 \delta_R(x_4) + x_4 \delta_R(\overline{x}^3) \\ &= -q^{-5}\overline{x}^1 x_1 \otimes \overline{\gamma} + q^{-5}\overline{x}^1 x_2 \otimes \overline{\alpha} - q^{-1} x_2 \overline{x}^2 \otimes \overline{\gamma} - q^{-2} x_2 \overline{x}^1 \otimes \overline{\alpha} \\ &\quad - q^{-1}\overline{x}^3 x_3 \otimes \overline{\gamma} - q^{-1}\overline{x}^3 x_4 \otimes \overline{\alpha} - q x_4 \overline{x}^4 \otimes \overline{\gamma} + x_4 \overline{x}^3 \otimes \overline{\alpha} \\ &= -q \langle \phi_1 | \phi_1 \rangle \otimes \overline{\gamma} + \langle \phi_1 | \phi_2 \rangle \otimes \overline{\alpha} = 1 \otimes (-q\overline{\gamma}) , \end{aligned}$$

$$\begin{aligned} \chi(\left\langle \phi_2 \dot{\otimes}_{A(S_q^4)} \phi_2 \right\rangle) &= q^{-2} x_1 \delta_R(\overline{x}^1) + q^{-4}\overline{x}^2 \delta_R(x_2) + x_3 \delta_R(\overline{x}^3) + \overline{x}^4 \delta_R(x_4) \\ &= q^{-1} x_1 \overline{x}^2 \otimes \overline{\gamma} + q^{-2} x_1 \overline{x}^1 \otimes \overline{\alpha} - q^{-4}\overline{x}^2 x_1 \otimes \overline{\gamma} + q^{-4}\overline{x}^2 x_2 \otimes \overline{\alpha} \\ &\quad - q x_3 \overline{x}^4 \otimes \overline{\gamma} + x_3 \overline{x}^3 \otimes \overline{\alpha} + \overline{x}^4 x_3 \otimes \overline{\gamma} + \overline{x}^4 x_4 \otimes \overline{\alpha} \\ &= \langle \phi_2 | \phi_2 \rangle \otimes \overline{\alpha} = 1 \otimes \overline{\alpha} . \end{aligned}$$

□

Remark 3.3. *It was proven in [8] that the bundle constructed in [7] is a coalgebra Galois extension and not a Hopf-Galois extension [17, 13]. The fact that our bundle* $A(S^4_q) \subset A(S^7_q)$ *is Hopf-Galois shows also that these two bundles cannot be the same. See also Rem. 2.3.*

As in the previous example, since the structure group $H = A(SU_q(2))$ is cosemisimple and has a bijective antipode, on our extension $A(S^4_q) \subset A(S^7_q)$ there is a strong connection. Also in this case, we provide a strong connection in terms of the map ℓ starting by giving a suitable definition on the generators of H. We set $\ell(1) = 1 \otimes 1$. Then, on the generators we set

$$\ell(\alpha) := \left\langle \phi_1 \dot{\otimes} \phi_1 \right\rangle , \qquad \ell(\overline{\alpha}) := \left\langle \phi_2 \dot{\otimes} \phi_2 \right\rangle ,$$
$$\ell(\gamma) := \left\langle \phi_2 \dot{\otimes} \phi_1 \right\rangle , \qquad \ell(\overline{\gamma}) := -q^{-1} \left\langle \phi_1 \dot{\otimes} \phi_2 \right\rangle .$$

The same expressions can be written in a more coincise way as

$$\ell(m_{ij}) = \left\langle \phi_i \dot{\otimes} \phi_j \right\rangle = (\overline{\phi}_i)_l \otimes (\phi_j)_l \ , \tag{3.12}$$

where $m = (m_{ij}) = \begin{pmatrix} \alpha & -q\overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix}$.

Proposition 3.6. *The above expressions for ℓ satisfy all the properties* (3.4)*.*

Proof. Firstly, $\chi(\ell(h)) = 1 \otimes h$, for h any generator of $A(SU_q(2))$, follows from the proof of Prop. 3.5. Then,

$$\begin{aligned}(id \otimes \delta) \circ \ \ell(\alpha) &= q^{-6}\overline{x}^1 \otimes \delta x_1 + q^{-2} x_2 \otimes \delta \overline{x}^2 + q^{-2}\overline{x}^3 \otimes \delta x_3 + x_4 \otimes \delta \overline{x}^4 \\ &= \left\langle \phi_1 \dot{\otimes} \phi_1 \right\rangle \otimes \alpha + \left\langle \phi_1 \dot{\otimes} \phi_2 \right\rangle \otimes \gamma \\ &= \ell(\alpha) \otimes \alpha - q\ell(\overline{\gamma}) \otimes \gamma = \ell(\alpha_{(1)}) \otimes \alpha_{(2)} \ .\end{aligned}$$

In the same way:

$$\begin{aligned}(id \otimes \delta) \circ \ \ell(\gamma) &= q^{-5}\overline{x}^2 \otimes \delta x_1 - q^{-2} x_1 \otimes \delta \overline{x}^2 - q^{-1}\overline{x}^4 \otimes \delta x_3 + x_3 \otimes \delta \overline{x}^4 \\ &= \left\langle \phi_2 \dot{\otimes} \phi_1 \right\rangle \otimes \alpha + \left\langle \phi_2 \dot{\otimes} \phi_2 \right\rangle \otimes \gamma \\ &= \ell(\gamma) \otimes \alpha + \ell(\overline{\alpha}) \otimes \gamma = \ell(\gamma_{(1)}) \otimes \gamma_{(2)} \ ,\end{aligned}$$

and so for $\overline{\alpha}, \overline{\gamma}$.

This property can be shown in a more coincise way by using (3.12). The right coaction (2.35) is

$$\delta((\phi_j)_i) = (\phi_l)_i \otimes m_{lj}$$

so that :

$$\begin{aligned}(id \otimes \delta)\ell(m_{ij}) &= (\overline{\phi}_i)_l \otimes \delta((\phi_j)_l) = (\overline{\phi}_i)_l \otimes (\phi_r)_l \otimes m_{rj} = \\ &\quad \ell(m_{ir}) \otimes m_{rj} = (\ell \otimes id)\Delta(m_{ij}) \ .\end{aligned}$$

In order to show the last property $(\delta_l \otimes id) \circ \ell(h) = h_{(1)} \otimes \ell(h_{(2)})$ we need the induced left coaction $\delta_l : A(S^7_q) \to A(SU_q(2)) \otimes A(S^7_q)$ defined by $\delta_l(p) = S^{-1}(p_{(1)}) \otimes p_{(0)}$: with the above notations is

$$\delta_l((\phi_j)_i) = S^{-1}(m_{lj}) \otimes (\phi_l)_i \ .$$

We can also write this map explicitly. We remind that the antipode for $A(SU_q(2))$ and its inverse are given respectively by

$$S \begin{pmatrix} \alpha & -q\overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix} = \begin{pmatrix} \overline{\alpha} & \overline{\gamma} \\ -q\gamma & \alpha \end{pmatrix} \quad ; \quad S^{-1} = *S* = \begin{pmatrix} \alpha & -q\overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix} = \begin{pmatrix} \overline{\alpha} & q^2\overline{\gamma} \\ -q^{-1}\gamma & \alpha \end{pmatrix}$$

hence the induced left coaction is given on the generators by [1]

$$\begin{array}{ll} \delta_l(x_1) = \overline{\alpha} \otimes x_1 - \ \gamma \otimes x_2 \ , & \delta_l(\overline{x}^1) = -q^2\overline{\gamma} \otimes \overline{x}^2 + \alpha \otimes \overline{x}^1 \ , \\ \delta_l(x_2) = q\overline{\gamma} \otimes x_1 + \alpha \otimes x_2 \ , & \delta_l(\overline{x}^2) = \overline{\alpha} \otimes \overline{x}^2 + q^{-1}\gamma \otimes \overline{x}^1 \ , \\ \delta_l(x_3) = \overline{\alpha} \otimes x_3 + \gamma \otimes x_4 \ , & \delta_l(\overline{x}^3) = q^2\overline{\gamma} \otimes \overline{x}^4 + \alpha \otimes \overline{x}^3 \ , \\ \delta_l(x_4) = -q\overline{\gamma} \otimes x_3 + \alpha \otimes x_4 \ , & \delta_l(\overline{x}^4) = \overline{\alpha} \otimes \overline{x}^4 - q^{-1}\gamma \otimes \overline{x}^3 \ . \end{array} \tag{3.13}$$

We can now compute the following

$$\begin{aligned} (\delta_l \otimes id) \circ \ \ell(m_{ij}) \ &= \ \delta_l((\overline{\phi}_i)_l) \otimes (\phi_j)_l = \overline{S(m_{ri})} \otimes (\overline{\phi}_r)_l \otimes (\phi_j)_l = \\ & \quad m_{ir} \otimes (\overline{\phi}_r)_l \otimes (\phi_j)_l = m_{ir} \otimes \ell(m_{rj}) \ , \end{aligned}$$

where we used the fact that $*S = S^{-1}*$ and $\overline{S(m_{ij})} = m_{ji}$, so that $\delta_l((\overline{\phi}_i)_l) = m_{ir} \otimes (\overline{\phi}_r)_l$. For example, explicitly,

$$\begin{aligned} (\delta_l \otimes id) \circ \ \ell(\alpha) \ &= \ q^{-6}(\alpha \otimes \overline{x}^1 - q^2\overline{\gamma} \otimes \overline{x}^2) \otimes x_1 + q^{-2}(q\overline{\gamma} \otimes x_1 + \alpha \otimes x_2) \otimes \overline{x}^2 \\ &+ \ q^{-2}(q^2\overline{\gamma} \otimes \overline{x}^4 + \alpha \otimes \overline{x}^3) \otimes x_3 + (-q\overline{\gamma} \otimes x_3 + \alpha \otimes x_4) \otimes \overline{x}^4 \\ &= \ \alpha \otimes \left\langle \phi_1 \dot{\otimes} \phi_1 \right\rangle - q\overline{\gamma} \otimes \left\langle \phi_2 \dot{\otimes} \phi_1 \right\rangle \\ &= \ \alpha \otimes \ell(\alpha) - q\overline{\gamma} \otimes \ell(\gamma) = \alpha_{(1)} \otimes \ell(\alpha_{(2)}) \ . \end{aligned}$$

□

Having defined ℓ on the generators, let us now extend it as done in (3.10):

$$\ell(gh) := h^1 g^1 \otimes g^2 h^2 \ .$$

On the product of two generators m_{ij}, m_{kh} this reads

$$\ell(m_{ij} m_{kh}) := (\overline{\phi}_k)_l (\overline{\phi}_i)_g \otimes (\phi_j)_g (\phi_h)_l \ . \tag{3.14}$$

We prove now that this map is well defined, that is, it satisfies properties (3.4):

---

[1] Notice that $\delta_l(\overline{x}_i) \neq \overline{\delta_l(x_i)}$.

Proof. The first property $\chi(\ell(m_{ij}m_{kh})) = 1 \otimes m_{ij}m_{kh}$ has showed in (3.11) holds. Then

$$\begin{aligned}(\ell \otimes id)\Delta(m_{ij}m_{kh}) &= \ell(m_{ir}m_{ks}) \otimes m_{rj}m_{sh} = (\overline{\phi}_k)_p(\overline{\phi}_i)_q \otimes (\phi_r)_q(\phi_s)_p \otimes m_{rj}m_{sh} = \\ &\quad (\overline{\phi}_k)_p(\overline{\phi}_i)_q \otimes \delta(\phi_j)_q(\phi_h)_p = (id \otimes \delta)\ell(m_{ij}m_{kh}) \ .\end{aligned}$$

Moreover

$$\begin{aligned}(id \otimes \ell)\Delta(m_{ij}m_{kh}) &= m_{ir}m_{ks} \otimes \ell(m_{rj}m_{sh}) = m_{ir}m_{ks} \otimes (\overline{\phi}_s)_l(\overline{\phi}_r)_p \otimes (\phi_j)_p(\phi_h)_l = \\ &\quad \delta_l((\overline{\phi}_k)_l(\overline{\phi}_i)_p) \otimes \delta(\phi_j)_p(\phi_h)_l = (\delta_l \otimes id)\ell(m_{ij}m_{kh}) \ .\end{aligned}$$

where we also used $\delta_l(pq) = S^{-1}(q_{(1)})S^{-1}(p_{(1)}) \otimes p_{(0)}q_{(0)}$. □

We can now use an iterative procedure constructed by using (3.14) on the PBW basis to construct a well defined ℓ on the whole of $H = A(SU_q(2))$. On the PBW basis

$$r^{k,l,m} := \alpha^k\gamma^l\overline{\gamma}^m \ , \text{ for } k,l,m \geq 0 \quad ; \qquad s^{k,l,m} := \gamma^k\overline{\gamma}^l\overline{\alpha}^m \ , \text{ for } k,l \geq 0 \ , m > 0$$

this reads

$$\begin{aligned}\ell(r^{k+1,l,m}) &= q^{l+m}\alpha^1\ell(r^{k,l,m})\alpha^2 = q^{l+m}\left[q^{-6}\overline{x}^1\ell(r^{k,l,m})x_1 + q^{-2}x_2\ell(r^{k,l,m})\overline{x}^2 + \right. \\ &\quad \left. q^{-2}\overline{x}^3\ell(r^{k,l,m})x_3 + x_4\ell(r^{k,l,m})\overline{x}^4 \ \right] \ , \\ \ell(r^{k,l+1,m}) &= \gamma^1\ell(r^{k,l,m})\gamma^2 = q^{-5}\overline{x}^2\ell(r^{k,l,m})x_1 - q^{-2}x_1\ell(r^{k,l,m})\overline{x}^2 + \\ &\quad -q^{-1}\overline{x}^4\ell(r^{k,l,m})x_3 + x_3\ell(r^{k,l,m})\overline{x}^4 \ , \\ \ell(r^{k,l,m+1}) &= \overline{\gamma}^1\ell(r^{k,l,m})\overline{\gamma}^2 = -q^{-6}\overline{x}^1\ell(r^{k,l,m})x_2 + q^{-3}x_2\ell(r^{k,l,m})\overline{x}^1 + \\ &\quad q^{-2}\overline{x}^3\ell(r^{k,l,m})x_4 - q^{-1}x_4\ell(r^{k,l,m})\overline{x}^3\end{aligned}$$

and

$$\begin{aligned}\ell(s^{k+1,l,m}) &= q^m\gamma^1\ell(s^{k,l,m})\gamma^2 = q^m\left[q^{-5}\overline{x}^2\ell(s^{k,l,m})x_1 - q^{-2}x_1\ell(s^{k,l,m})\overline{x}^2 + \right. \\ &\quad \left. -q^{-1}\overline{x}^4\ell(s^{k,l,m})x_3 + x_3\ell(s^{k,l,m})\overline{x}^4 \ \right] \ , \\ \ell(s^{k,l+1,m}) &= q^m\overline{\gamma}^1\ell(s^{k,l,m})\overline{\gamma}^2 = q^m\left[-q^{-6}\overline{x}^1\ell(s^{k,l,m})x_2 + q^{-3}x_2\ell(s^{k,l,m})\overline{x}^1 + \right. \\ &\quad \left. q^{-2}\overline{x}^3\ell(s^{k,l,m})x_4 - q^{-1}x_4\ell(s^{k,l,m})\overline{x}^3 \ \right] \ , \\ \ell(s^{k,l,m+1}) &= \overline{\alpha}^1\ell(s^{k,l,m})\overline{\alpha}^2 = q^{-4}\overline{x}^2\ell(s^{k,l,m})x_2 + q^{-2}x_1\ell(s^{k,l,m})\overline{x}^1 + \\ &\quad \overline{x}^4\ell(s^{k,l,m})x_4 + x_3\ell(s^{k,l,m})\overline{x}^3 \ .\end{aligned}$$

Observation 3.3. *Note that for both the examples, the definition of ℓ on the product through $\ell(gh) = \sum h^1g^1 \otimes g^2h^2$ give rise to a well defined map on the whole algebra. But this cannot be assumed as a general properties: if a map ℓ defined on the generators of an algebra satisfies* (3.4) *it is not possible in general to show that the same is true for its extension given above (in particular the property of right colinearity) [11].*

3.A Appendix: The associated bundle and the coequivariant functions

We now give some elements of the theory of associated quantum vector bundles [15] (see also [29]). Let $B \subset P$ be a H-Galois extension with Δ_R the coaction of H on P. Let $\rho : V \to H \otimes V$ be a corepresentation of H with V a finite dimensional vector space. A coequivariant map is an element φ in $P \otimes V$ with the property that

$$(\Delta_R \otimes id)\varphi = (id \otimes (S \otimes id)\rho)\varphi \, ; \tag{3.15}$$

where S is the antipode of H. The collection $\Gamma_\rho(P, V)$ of coequivariant maps is a right and left B-module.

The algebraic analogue of bundle nontriviality is translated in the fact that the Hopf-Galois extension $B \subset P$ is not cleft. On the other hand, it is known that for a cleft Hopf-Galois extension, the module of coequivariant maps $\Gamma_\rho(P, V)$ is isomorphic to the free module of coinvariant maps $\Gamma_0(P, V) = B \otimes V$ [15, 33].

For our $A(SU_q(2))$-Hopf-Galois extension $A(S^4_q) \subset A(S^7_q)$, let $\rho_1 : \mathbb{C}^2_q \to \mathbb{C}^2_q \otimes A(SU_q(2))$ be the fundamental corepresentation of $A(SU_q(2))$ with $\Gamma_1(A(S^7_q), \mathbb{C}^2_q)$ the right $A(S^4_q)$-module of corresponding coequivariant maps.

Now, the projection p in (2.29) determines a quantum vector bundle over S^4_q whose module of section is $p[A(S^4_q)^4]$, which is clearly a right $A(S^4_q)$-module. The following proposition in straightforward

Proposition 3.7. *The modules $\mathcal{E} := p[A(S^4_q)^4]$ and $\Gamma_1(A(S^7_q), \mathbb{C}^2_q)$ are isomorphic as right $A(S^4_q)$-modules.*

Proof. Remember that $p = vv^*$ with v in (2.25). The element $p(F) \in \mathcal{E}$, with $F = (f_1, f_2, f_3, f_4)^t$, corresponds to the equivariant map $v^*F \in \Gamma_1(A(S^7_q), \mathbb{C}^2_q)$. □

We expect that a similar construction extends to every irreducible corepresentation of $A(SU_q(2))$ by means of suitable projections giving the corresponding associated bundles.

Proposition 3.8. *The Hopf-Galois extension $A(S^4_q) \subset A(S^7_q)$ is not cleft.*

Proof. As mentioned, the cleftness of the extension does imply that all modules of coequivariant maps are free. On the other hand, the nontriviality of the pairing (2.48) between the defining projection p in (2.29) and the Fredholm module μ constructed in Sect. 2.4 also shows that the module $p[A(S^4_q)^4] \simeq \Gamma_\rho(A(S^7_q), \mathbb{C}^2)$ is not free. □

Chapter 4

Moving away from the basic instanton

In [46] and [48] two deformations of the instanton bundle $S^7 \to S^4$ over two different noncommutative 4-spheres, the symplectic and the θ one respectively, were constructed. In both cases (reviewed resp. in Ch. 2 and Sect. 4.2.1) the construction was limited to the basic instanton of charge -1. The attempt to generalise this picture to generic $SU(2)$-instantons is the topic of the paper in preparation [47]. Here we explain the results we have obtained until now.

The Chapter is organised as follows. The first section is a brief review of the classical situation [1]. We try then to reproduce it in the noncommutative case firstly for the instanton bundle $A(S^4_\theta) \hookrightarrow A(S^7_\theta)$ over the Connes-Landi sphere (Sect. 4.2) and then for the (more complicated) symplectic case $A(S^7_q) \hookrightarrow A(S^4_q)$ (Sect. 4.3).

4.1 The classical construction of instantons

We illustrate the procedure used by Atiyah et al. [1] [2] to construct all the solutions of the $SU(2)$-Yang-Mills theory on S^4. See in particular [1] Sect. II.3 for the geometric interpretation of what follows.

We use quaternionic notations

$$\mathbb{H} \ni x = x_1 + x_2 i + x_3 j + x_4 k \simeq (x_1 + x_2 i, x_3 + x_4 i) \in \mathbb{C}^2$$

with quaternion differential

$$dx = dx_1 + dx_2 i + dx_3 j + dx_4 k$$

and complex conjugate $d\overline{x} = d\overline{x}_1 - d\overline{x}_2 i - d\overline{x}_3 j - d\overline{x}_4 k$. In the identification of the algebra of quaternions as a subalgebra of complex 2×2 matrices, the group $SU(2)$ get identified with the group $Sp(1)$ of quaternions of unit norm, being $\|x\|^2 = x\overline{x} =$

$\overline{x}x = \sum x_i^2$.

Let (x, y) be a point of $S^4 \simeq \mathbb{P}^1\mathbb{H}$ with the identification $(x, y) \sim (xq, yq)$, $x, y, q \in \mathbb{H}$. The instanton connections on S^4 are induced from $\mathbb{H}^{1+k}$, by orthogonal projection, by suitable maps

$$v : \mathbb{P}^1\mathbb{H} \to St_{\mathbb{H}}(k, k+1) \ ,$$

with St being the Stiefel variety. Let

$$v(x, y) = C(x \cdot \mathbb{I}_k) + D(y \cdot \mathbb{I}_k) \ \in Mat((k+1) \times k, \mathbb{H}) \tag{4.1}$$

with C, D constants matrices of quaternions independent of x, y.[1](In the following we will write x instead of $x \cdot \mathbb{I}_k$).

We assume that this matrix v has maximal rank for all $(x, y) \neq (0, 0)$ in such a way that its columns span a subspace of $\mathbb{H}^{1+k}$ of dimension k. The projection onto this subspace is $Q = v\rho^{-2}v^*$, with $\rho^2 := v^*v$. The orthogonal complement $E_{(x,y)}$ has dimension 1 over $\mathbb{H}$ and $P = 1 - Q = uu^*$ is the complementary projection, with u the orthogonal matrix

$$u^*u = 1 \quad ; \quad u^*v = 0 \ .$$

The gauge potential corresponding to v is constructed by projection from $\mathbb{H}^{1+k}$ and it is explicitly expressed in terms of u as $A = u^*du$. One has also to assume that

$$\rho^2 = v^*v = x^*C^*Cx + y^*D^*Dy + (x^*C^*Dy + y^*D^*Cx)$$

is a real matrix for each $x, y \in \mathbb{H}$ (where real means that in the identification between quaternions and 2×2 complex matrices the entries v_{ij} are real two by two diagonal matrices). Under this assumption, the curvature $F = dA + A \wedge A$ involves only the selfdual expression $dx\, d\overline{x}$.

The topological invariant of the resulting bundle E is then proved to be k being $E^{\perp}$ a direct sum of k line-bundles of charge -1 [1].

In particular, it is important to recall that all $SU(2)$-instantons arise in this way [2]. The number of quaternionic parameters which occur in the matrices C, D in (4.1)

[1] In the general case of $Sp(n)$-instantons, the matrix v is in $Mat((k+n) \times k, \mathbb{H})$.

are $2 \cdot (k+1)k$. The condition ρ^2 real is equivalent to the following [2]

$$C^*C \text{ and } D^*D \text{ real} \quad \rightsquigarrow \quad 2 \cdot [\frac{1}{2}3k(k-1)] \text{ conditions;} \tag{4.2}$$

$$C^*D \text{ symmetric (as quaternion matrix)} \quad \rightsquigarrow \quad 4\,\frac{1}{2}k(k-1) \text{ conditions} \tag{4.3}$$

Hence we have $8k^2+8k$ real parameters with $5k(k-1)$ conditions. We obtain a gauge equivalence if we replace v (and u) by

$$v' = RvS \;,\; R \in Sp(k+1) \;,\; S \in Gl(k,\mathbb{R}) \;. \tag{4.4}$$

Indeed under this transformation the projections get conjugate by R. Hence, being $dim_{\mathbb{R}} Sp(k+1) = 2(k+1)^2 + (k+1)$ and $dim_{\mathbb{R}} Gl(k,\mathbb{R}) = k^2$ we can conclude that the number of effective real parameters is

$$8k^2 + 8k - [5k(k-1)] - [3k^2 + 5k + 3] = 8k - 3 \;.$$

When $k = 1$, the matrices C, D reduce to 2×1 matrices and eq. (4.1) can be rewritten as

$$v = A \begin{pmatrix} x \\ y \end{pmatrix} \quad , \quad A = (C,D) \in Mat(2 \times 2, \mathbb{H}) \;. \tag{4.5}$$

In this case

$$\rho^2 = v^*v = (x^*, y^*)A^*A \begin{pmatrix} x \\ y \end{pmatrix}$$

is a real 2×2 matrix by construction, i.e. conditions (4.2), (4.3) are automatically satisfied. Notice that the projection Q depends on C, D only through ρ^{-2}, therefore it is the datum of $(A^{-1})^*A^{-1}$. Then, if $B \in Mat(2 \times 2, \mathbb{H})$, $M := B^*B$ has the form

$$M = \begin{pmatrix} \mu & b \\ b^* & \nu \end{pmatrix} \tag{4.6}$$

with μ, $\nu \in \mathbb{R}$ and $b \in \mathbb{H}$. The gauge equivalence under left multiplication by $R \in Sp(2)$ is here evident: if $B \rightsquigarrow B' := RB$ then B^*B (and hence ρ^{-2}) is unchanged. Moreover the invariance under $Gl(1)$ allows to us to assume that $det(B) = 1$. Summarizing, the moduli space of equivalence classes of instantons of charge $|k| = 1$ on S^4 is the five-dimensional quotient manifold $\mathrm{SL}(2,\mathbb{H})/\mathrm{Sp}(2)$ parametrized by the 6 real parameters which enter in (4.6) subjected to the relation $\mu\nu - bb^* = 1$. [1].

[2] The condition v^*v real splits in three parts since x^*C^*Cx, y^*D^*Dy and $x^*C^*Dy + y^*D^*Cx$ contain different elements. Then, C^*C and D^*D has to be real. Let $C^*D = N$, then imposing that $x^*Ny + y^*N^*x$ is real means

$$(x^*Ny + y^*N^*x)_{ij,ab} = \begin{cases} 0 \text{ if } a \neq b \\ \\ (x^*Ny + y^*N^*x)_{ij,a'a'} \text{ if } a = b \end{cases}$$

The above two conditions imply the same result: using the commutativity between the complex entries of N and the one of x, y we have $N_{ij,aa} = N_{ji,aa}$ and $N_{ij,a'a} = N_{ji,a'a}$, i.e. N is symmetric.

Observation 4.1. *As illustrated above, charge 1 instantons are generated by the basic instanton, say w, by applying a matrix $A \in SL(2,\mathbb{H})/Sp(2)$: $v = Aw$, with $w^*w = 1$ and $v^*v = \rho^2$. The projection is the $*$selfadjoin idempotent $Q = v\rho^{-2}v^*$. One can also consider*

$$Q_M = vv^*M \ , \quad M = (A^{-1})^*A^{-1} \ .$$

This is an idempotent:

$$Q_M^2 = v\underbrace{v^*Mv}_{w^*w}v^*M = Q_M \ .$$

As matrices Q and Q_M coincide:

$$Q_M = Aww^*A^{-1} = Q = Aw\rho^{-2}w^*A^* \Longleftrightarrow \rho^2 = w^*A^*Aw \ ,$$

but they are $*$-self-adjoint in two different metrics. Indeed Q_M is self-adjoint in the new metric $\langle s,t\rangle := s^*Mt$ and not in $\langle s,t\rangle := s^*t$:

$$\langle s,\, Q_M t\rangle_M = \left\langle (Q_M)^{*M}s,\, t\right\rangle_M \quad \Rightarrow (Q_M)^{*M} = Q_M$$

while $Q_M^* \neq Q_M$ whenever $A^* \neq A^{-1}$.

4.2 Moduli space for the Connes-Landi sphere

We move to noncommutative geometry addressing our attention to the Hopf fibration $A(S^4_\theta) \hookrightarrow A(S^7_\theta)$ constructed in [48] over the Connes-Landi sphere $A(S^4_\theta)$. We begin by briefly reviewing the algebras involved and the construction of the basic instanton. We follow [48]. Then we will try to produce other instantons of charge $k \geq 1$.

4.2.1 The principal fibration $A(S^4_\theta) \hookrightarrow A(S^4_\theta)$

The algebra $A(S^7_\theta)$ is defined as the $*$-algebra generated by the elements z^μ $\overline{z}^\mu$, ($\mu = 1,\ldots 4$) with relations

$$z^\mu z^\nu = \lambda^{\mu\nu} z^\nu z^\mu \ ; \ \overline{z}^\mu z^\nu = \lambda^{\nu\mu} z^\nu \overline{z}^\mu \ ; \ \overline{z}^\mu \overline{z}^\nu = \lambda^{\mu\nu} \overline{z}^\nu \overline{z}^\mu \ ; \tag{4.7}$$

and spherical relation $\sum z^\mu \overline{z}^\mu = 1$. Here $\lambda^{\mu\nu}$ is taken to be of the following particular form

$$\lambda^{\mu\nu} = \begin{pmatrix} 1 & 1 & \mu & \mu \\ 1 & 1 & \mu & \mu \\ \overline{\mu} & \overline{\mu} & 1 & 1 \\ \overline{\mu} & \overline{\mu} & 1 & 1 \end{pmatrix} . \tag{4.8}$$

Indeed this choice allows the definition of a coaction of $SU(2)$ on $A(S^7_\theta)$ given on the generators of the algebra as [48]

$$\alpha_w : (z^1, z^2, z^3, z^4) \mapsto (z^1, z^2, z^3, z^4) \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix} \ , \quad w = \begin{pmatrix} w^1 & w^2 \\ -\overline{w}^2 & \overline{w}^1 \end{pmatrix} \in SU(2) \ .$$

The algebra of coinvariants under this coaction is identified with $A(S^4_\theta)$, the latter being generated as a $*$-algebra by a central element x and elements α, β satisfying

$$\alpha\beta = \mu^2\beta\alpha \ , \quad \alpha\beta^* = \overline{\mu}^2\beta^*\alpha \ , \quad \alpha\alpha^* = \alpha^*\alpha \ , \quad \beta\beta^* = \beta^*\beta \ , \tag{4.9}$$

and a spherical relation given by $\alpha\alpha^* + \beta\beta^* + x^2 = 1$. Explicitly, these elements can be expressed in terms of the generators of $A(S^7_\theta)$ as

$$\begin{aligned} &\alpha = 2(z^1\overline{z}^3 + z^2\overline{z}^4) \ , \qquad \beta = 2(-z^1z^4 + z^2z^3) \ , \\ &x = z^1\overline{z}^1 + z^2\overline{z}^2 - z^3\overline{z}^3 - z^4\overline{z}^4 \ . \end{aligned} \tag{4.10}$$

As it happens for the symplectic 4-sphere, the generators of $A(S^4_\theta)$ can be obtained as the entries of a projection p [21]. Let us consider the matrix

$$u = (|\psi_1\rangle \, , |\psi_2\rangle) = \begin{pmatrix} z^1 & z^2 \\ -\overline{z}^2 & \overline{z}^1 \\ z^3 & z^4 \\ -\overline{z}^4 & \overline{z}^3 \end{pmatrix} \tag{4.11}$$

where the vectors $|\psi_1\rangle \, , |\psi_2\rangle$ are elements in the right $A(S^7_\theta)$-module $\mathbb{C}^4 \otimes A(S^7_\theta)$. They are orthonormal with respect to the hermitean structure given by $\langle \xi, \eta \rangle = \sum \xi_j^* \eta_j$ so that $u^*u = \mathbb{I}_2$. The matrix

$$p = uu^* = |\psi_1\rangle \langle\psi_1| + |\psi_2\rangle \langle\psi_2|$$

is a projection with entries in $A(S^4_\theta)$:

$$p = \frac{1}{2} \begin{pmatrix} 1+x & 0 & \alpha & \beta \\ 0 & 1+x & -\mu\beta^* & \overline{\mu}\alpha^* \\ \alpha^* & -\overline{\mu}\beta & 1-x & 0 \\ \beta^* & \mu\alpha & 0 & 1-x \end{pmatrix} . \tag{4.12}$$

This projector describes the basic instanton on $A(S^4_\theta)$ in the sense that p defines a finitely generated projective module $\mathcal{E}$, and a connection on it, which has a anti-selfdual curvature in some proper sense. Moreover the construction of a noncommutative spin geometry on S^4_θ allows to compute the instanton charge as the index of a Dirac operator obtaining the value 1. We refer to [20] for the precise formulation.

Remark 4.1. *The inclusion $A(S^4_\theta) \to A(S^7_\theta)$ is another example of a principal Hopf-Galois extension, i.e. it is a principal bundle in noncommutative geometry in the sense of Ch. 3. We refer to [48] for more details.*

4.2.2 The quantum groups $\mathrm{SL}_\theta(2,\mathbb{H})$ and $\mathrm{Sp}_\theta(2)$.

In this section we focus on the question of how to generate other instantons of charge 1. Classically, as said before, (charge 1) instantons are generated from the basic instanton by the action of the conformal group $\mathrm{SL}(2,\mathbb{H})$ of S^4. The action of elements of the subgroup $Sp(2) \subset Sl(2,\mathbb{H})$ generates gauge equivalent instantons, hence in order to get new instantons we quotient $SL(2,\mathbb{H})$ by the isometry group $\mathrm{Sp}(2)$.

Here we will closely follow the above classical construction of instantons in our attempt to generate instantons on $A(S^4_\theta)$. In particular, we shall describe the quantum group $\mathrm{SL}_\theta(2,\mathbb{H})$ and its quantum subgroup $\mathrm{Sp}_\theta(2)$.

We look for the coaction of a Hopf algebra (to be determined) on $A(S^7_\theta)$ by considering a generic 4×4 matrix A_θ acting as

$$u \mapsto A_\theta \otimes u \ .$$

More explicitly, in components we have

$$\Delta_L : u_{ia} \mapsto A_{ij} \otimes u_{ja} =: u'_{ia} \tag{4.13}$$

with $i,j = 1,\dots,4$ and $a = 1,2$. We suppose that this transformation is a (not unit-preserving) $*$-algebra map from $A(S^7_\theta)$ to the tensor product $A(M_\theta(2,\mathbb{H}))\otimes A(S^7_\theta)$, where $\mathcal{A}_\theta := A(M_\theta(2,\mathbb{H}))$ is the algebra generated by the entries of A_θ.

From the condition that Δ_L respects the involution and the particular form of u, we obtain the following form of A_θ with "quaternion entries" (hence the notation $M_\theta(2,\mathbb{H})$):

$$A_\theta = \begin{pmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ -\overline{a}_2 & \overline{a}_1 & -\overline{b}_2 & \overline{b}_1 \\ c_1 & c_2 & d_1 & d_2 \\ -\overline{c}_2 & \overline{c}_1 & -\overline{d}_2 & \overline{d}_1 \end{pmatrix} . \tag{4.14}$$

The transformations induced on the generators of $A(S^7_\theta)$ reads

$$\begin{aligned} w^1 &:= \Delta_L(z^1) = a_1\otimes z^1 - a_2\otimes \overline{z}^2 + b_1\otimes z^3 - b_2\otimes \overline{z}^4 \\ w^2 &:= \Delta_L(z^2) = a_1\otimes z^2 + a_2\otimes \overline{z}^1 + b_1\otimes z^4 + b_2\otimes \overline{z}^3 \\ w^3 &:= \Delta_L(z^3) = c_1\otimes z^1 - c_2\otimes \overline{z}^2 + d_1\otimes z^3 - d_2\otimes \overline{z}^4 \\ w^4 &:= \Delta_L(z^4) = c_1\otimes z^2 + c_2\otimes \overline{z}^1 + d_1\otimes z^4 + d_2\otimes \overline{z}^3 \end{aligned} \tag{4.15}$$

Then we impose that Δ_L respects the algebra structure. It follows that the algebra generated by the a_{ij} is commutative, as well as the algebras generated by the b_{ij}, c_{ij}

and the d_{ij}. However, the whole algebra is not commutative:

$$
\begin{array}{lll}
a_1b_1 = \overline{\mu}\ b_1a_1 & a_2b_1 = \mu\ b_1a_2 & a_1c_1 = \mu\ c_1a_1 \\
a_1b_2 = \mu\ b_2a_1 & a_2b_2 = \overline{\mu}\ b_2a_2 & a_1c_2 = \mu\ c_2a_1 \\
a_1\overline{b}_1 = \mu\ \overline{b}_1a_1 & a_2\overline{b}_1 = \overline{\mu}\ \overline{b}_1a_2 & a_1\overline{c}_1 = \overline{\mu}\ \overline{c}_1a_1 \\
a_1\overline{b}_2 = \overline{\mu}\ \overline{b}_2a_1 & a_2\overline{b}_2 = \mu\ \overline{b}_2a_2 & a_1\overline{c}_2 = \overline{\mu}\ \overline{c}_2a_1 \\
\\
a_2c_1 = \mu\ c_1a_2 & a_1d_1 = d_1a_1 & a_2d_1 = \mu^2\ d_1a_2 \\
a_2c_2 = \mu\ c_2a_2 & a_1d_2 = \mu^2\ d_2a_1 & a_2d_2 = d_2a_2 \\
a_2\overline{c}_1 = \overline{\mu}\ \overline{c}_1a_2 & a_1\overline{d}_1 = \overline{d}_1a_1 & a_2\overline{d}_1 = \overline{\mu}^2\ \overline{d}_1a_2 \\
a_2\overline{c}_2 = \overline{\mu}\ \overline{c}_2a_2 & a_1\overline{d}_2 = \overline{\mu}^2\ \overline{d}_2a_1 & a_2\overline{d}_2 = \overline{d}_2a_2 \\
\\
b_1c_1 = \mu^2\ c_1b_1 & b_2c_1 = c_1b_2 & b_1d_1 = \mu\ d_1b_1 \\
b_1c_2 = c_2b_1 & b_2c_2 = \mu^2\ c_2b_2 & b_1d_2 = \mu\ d_2b_1 \\
b_1\overline{c}_1 = \overline{\mu}^2\ \overline{c}_1b_1 & b_2\overline{c}_1 = \overline{c}_1b_2 & b_1\overline{d}_1 = \overline{\mu}\ \overline{d}_1b_1 \\
b_1\overline{c}_2 = \overline{c}_2b_1 & b_2\overline{c}_2 = \overline{\mu}^2\ \overline{c}_2b_2 & b_1\overline{d}_2 = \overline{\mu}\ \overline{d}_2b_1 \\
\\
b_2d_1 = \mu\ d_1b_2 & c_1d_1 = \overline{\mu}\ d_1c_1 & c_2d_1 = \mu\ d_1c_2 \\
b_2d_2 = \mu\ d_2b_2 & c_1d_2 = \mu\ d_2c_1 & c_2d_2 = \overline{\mu}\ d_2c_2 \\
b_2\overline{d}_1 = \overline{\mu}\ \overline{d}_1b_2 & c_1\overline{d}_1 = \mu\ \overline{d}_1c_1 & c_2\overline{d}_1 = \overline{\mu}\ \overline{d}_1c_2 \\
b_2\overline{d}_2 = \overline{\mu}\ \overline{d}_2b_2 & c_1\overline{d}_2 = \overline{\mu}\ \overline{d}_2c_1 & c_2\overline{d}_2 = \mu\ \overline{d}_2c_2
\end{array}
\tag{4.16}
$$

together with their conjugates.

The above commutation rules can be rewritten in a condensed form as

$$A_{ij}A_{kl} = \eta^{ki}\eta^{jl}A_{kl}A_{ij} \tag{4.17}$$

with η the matrix:

$$(\eta^{ij}) = \begin{pmatrix} 1 & 1 & \overline{\mu} & \mu \\ 1 & 1 & \mu & \overline{\mu} \\ \mu & \overline{\mu} & 1 & 1 \\ \overline{\mu} & \mu & 1 & 1 \end{pmatrix} . \tag{4.18}$$

Observation 4.2. *In particular, the relations* (4.17) *could be computed by observing that the elements of the matrix u in* (4.11)*, due to eqs.* (4.7)*, satisfy*

$$u_{ia}u_{jb} = \eta_{ji}u_{jb}u_{ia}\ . \tag{4.19}$$

Then, if we impose that (4.13) *defines an algebra homomorphism, we have*

$$\sum (A_{ik}A_{jl} - \eta_{ji}\eta_{kl}A_{jl}A_{ik}) \otimes u_{ka}u_{lb} = 0$$

and equations (4.16) *follow by observing that for $a \leq b$ the elements $u_{ka}u_{lb}$ should be taken to be all independent. Then, $A_{ik}A_{jl} - \eta_{ji}\eta_{kl}A_{jl}A_{ik} = 0$ hold for each a, b, being independent of a, b.*

Observation 4.3. *The splitting homomorphism introduced in [20], allows us to reduce computations to the classical case. Let us briefly recall what this means in the case of the algebra $A(S^7_\theta)$. Firstly, we define an action σ (by automorphisms) of the torus $\mathbb{T}^2$ on the classical generators denoted $z^\mu_{(0)}$ by*

$$\sigma(s) : z^\mu_{(0)} \mapsto \begin{cases} e^{2\pi i s_1} z^\mu_{(0)}, & (\mu = 1, 2); \\ e^{2\pi i s_2} z^\mu_{(0)}, & (\mu = 3, 4), \end{cases} \tag{4.20}$$

with $s \in \mathbb{T}^2$. The noncommutative torus $A(\mathbb{T}^2_\theta)$ is the algebra generated by two unitaries U^1, U^2 satisfying

$$U^1 U^2 = \mu\; U^2 U^1. \tag{4.21}$$

This algebra carries a natural action τ of the torus $\mathbb{T}^2$, defined by

$$\tau(s) : U^\mu \mapsto e^{2\pi i s_\mu} U^\mu\ .$$

In [20], it was shown that the map

$$\mathrm{st} : A(S^7_\theta) \to A(S^7) \otimes A(\mathbb{T}^2_\theta) \tag{4.22}$$

$$z^\mu \mapsto \begin{cases} z^\mu_{(0)} \otimes U^1, & (\mu = 1, 2); \\ z^\mu_{(0)} \otimes U^2, & (\mu = 3, 4), \end{cases} \tag{4.23}$$

is an isomorphism onto the fixed points of the action $\sigma \otimes \tau^{-1}$ of $\mathbb{T}^2$ on $A(S^7) \otimes A(\mathbb{T}^2_\theta)$.

Also the matrix A can be expressed in terms of this splitting homomorphism st, very similar to the case of the matrix quantum groups introduced in [20].[3] We introduce two unitary elements U_1, U_2, satisfying

$$U_1 U_2 = \overline{\mu}\; U_2 U_1. \tag{4.24}$$

therefore generating the algebra $A(\mathbb{T}^2_{-\theta})$. We then define the splitting homomorphism as a map

$$\mathrm{st} : A(M_\theta(2, \mathbb{H})) \to A(M(2, \mathbb{H})) \otimes A(\mathbb{T}^2_\theta) \otimes A(\mathbb{T}^2_{-\theta}) \tag{4.25}$$

$$A_{ij} \mapsto A^{(0)}_{ij} \otimes U_{ij} \tag{4.26}$$

with $A^{(0)}_{ij}$ the classical coordinates of $M(2, \mathbb{H})$ and U_{ij} the following matrix

$$\big(U_{ij}\big) := \begin{pmatrix} U^1 \otimes U_1 & U^1 \otimes U_1^* & U^1 \otimes U_2 & U^1 \otimes U_2^* \\ U^{1*} \otimes U_1 & U^{1*} \otimes U_1^* & U^{1*} \otimes U_2 & U^{1*} \otimes U_2^* \\ U^2 \otimes U_1 & U^2 \otimes U_1^* & U^2 \otimes U_2 & U^2 \otimes U_2^* \\ U^{2*} \otimes U_1 & U^{2*} \otimes U_1^* & U^{2*} \otimes U_2 & U^{2*} \otimes U_2^* \end{pmatrix} \tag{4.27}$$

One can check that st *is indeed a homomorphism. Its image is again a fixed point subalgebra, under an action of $\mathbb{T}^2 \otimes \mathbb{T}^2$ defined as follows. We define the matrix*

[3]Indeed, we are considering a quotient of the quantum group $A(GL_\theta(8, \mathbb{R}))$ introduced therein.

$U^{(0)}_{ij}(s,t)$ as the classical analogue of the matrix U_{ij}, that is, with $U^{\mu}_{(0)}(s) := e^{2\pi i s_\mu}$ being the unitaries defining the torus. An action σ of $\mathbb{T}^2 \times \mathbb{T}^2$ on $A(M(2,\mathbb{H}))$ is then defined by

$$\sigma(s,t) : A^{(0)}_{ij} \mapsto U^{(0)}_{ij}(s,t) A^{(0)}_{ij} \tag{4.28}$$

whereas $A(\mathbb{T}^2_\theta) \otimes A(\mathbb{T}^2_{-\theta})$ carries a natural action $\tau \otimes \tau$ of $\mathbb{T}^2 \times \mathbb{T}^2$. It is not difficult to check that st *becomes an isomorphism onto the subalgebra of $A(M(2,\mathbb{H})) \otimes A(\mathbb{T}^2_\theta) \otimes A(\mathbb{T}^2_{-\theta})$, consisting of the fixed points with respect to the action $\sigma \otimes \tau^{-1} \otimes \tau^{-1}$.*

In terms of the differential calculus of [20], we can introduce an element called the determinant by setting

$$\Delta_L(\mathrm{d}z^1 \mathrm{d}\overline{z}^2 \mathrm{d}z^3 \mathrm{d}\overline{z}^4) = \det_\theta A \otimes \mathrm{d}z^1 \mathrm{d}\overline{z}^2 \mathrm{d}z^3 \mathrm{d}\overline{z}^4. \tag{4.29}$$

Explicitly, we find

$$\begin{aligned} \det_\theta A \;:=\;& a_1[\overline{a}_1(d_1\overline{d}_1 + d_2\overline{d}_2) + \overline{b}_2(\mu c_2\overline{d}_1 - d_2\overline{c}_1) - \overline{b}_1(\overline{\mu} c_2\overline{d}_2 + d_1\overline{c}_1)] \\ & -a_2[-\overline{a}_2(d_1\overline{d}_1 + d_2\overline{d}_2) + \overline{b}_2(\overline{\mu} c_1\overline{d}_1 + d_2\overline{c}_2) + \overline{b}_1(-\mu c_1\overline{d}_2 + d_1\overline{c}_2)] \\ & +b_1[-\overline{a}_2(c_2\overline{d}_1 - \overline{\mu} d_2\overline{c}_1) - \overline{a}_1(c_1\overline{d}_1 + \mu d_2\overline{c}_2) + \overline{b}_1(c_1\overline{c}_1 + c_2\overline{c}_2)] \\ & -b_2[\overline{a}_2(c_2\overline{d}_2 + \mu d_1\overline{c}_1) + \overline{a}_1(c_1\overline{d}_2 - \overline{\mu} d_1\overline{c}_2) - \overline{b}_2(c_1\overline{c}_1 + c_2\overline{c}_2)]. \end{aligned} \tag{4.30}$$

Observe that in the limit $\theta \to 0$ the element $\det_\theta A$ reduces to the determinant of the matrix.

The particular form of the matrix $\lambda^{\mu\nu}$ defining the relations in $A(S^7_\theta)$, implies that $\det_\theta A$ is a central element in the algebra generated by the entries of A. Hence we can take the quotient of this algebra by the two sided ideal generated by $\det_\theta A - 1$, which we will denote by $A(\mathrm{SL}_\theta(2,\mathbb{H}))$ and will be referred to as *the algebra of polynomials on the quantum group* $\mathrm{SL}_\theta(2,\mathbb{H})$. The image of A_{ij} in this quotient will again be denoted by A_{ij}. Note that in the limit $\theta \to 0$, we recover the algebra of polynomials on the Lie group $\mathrm{SL}(2,\mathbb{H})$.

Remark 4.2. *The determinant $det_\theta A$ can be expressed in a condensed form as*

$$det_\theta A = \sum_{\sigma \in S_4} (-1)^{|\sigma|} \varepsilon^{\sigma(1)\sigma(2)\sigma(3)\sigma(4)} A_{1,\sigma(1)} A_{2,\sigma(2)} A_{3,\sigma(3)} A_{4,\sigma(4)}$$

with

$$\varepsilon^{1324} = \varepsilon^{2413} = \varepsilon^{3241} = \varepsilon^{4132} = \mu \quad ; \quad \varepsilon^{1423} = \varepsilon^{cycl} = \overline{\mu}$$

and equal to 1 otherwise. (Note that $(\varepsilon^{ijkl})^2 = \eta^{ji}\eta^{lk}$, with η in (4.18).)

Let us now justify the name quantum group. Similar to [20], a coproduct is defined by $\Delta(A_{ij}) := \sum_k A_{ik} \otimes A_{kj}$, a counit by $\epsilon(A_{ij}) := \delta_{ij}$, whereas an antipode S can be constructed using the determinant $\det_\theta A$. However, we can also derive an explicit expression for the antipode, using the splitting homomorphism as in Obs. 4.3.[4]

Lemma 4.1. *The antipode in $A(\mathrm{SL}_\theta(2,\mathbb{H}))$ is given on $A_{ij} = A_{ij}^{(0)} \otimes U_{ij}$ by*

$$S(A_{ij}) := S^{(0)}(A_{ij}^{(0)}) \otimes U_{ji}^*, \tag{4.31}$$

where $S^{(0)}$ is the classical antipode of $A(\mathrm{SL}(2,\mathbb{H}))$.

Proof. Let us compute for example,

$$A_{ki}S(A_{ij}) = A_{ki}^{(0)} S^{(0)}(A_{ij}^{(0)}) \otimes U_{ki}U_{ji}^*. \tag{4.32}$$

For fixed k, j, the expression $U_{ki}U_{ji}^*$ is independent of the column index i, as can be easily seen from the form of the matrix U_{ij}. Therefore,

$$\sum_i A_{ki}S(A_{ij}) = \big(\sum_i A_{ki}^{(0)} S^{(0)}(A_{ij}^{(0)})\big) \otimes U_{k1}U_{j1}^* = \delta_{kj} \otimes U_{k1}U_{j1}^*, \tag{4.33}$$

following from the classical case. Since the matrix U_{ij} consists of unitaries, we conclude that indeed

$$\sum_i A_{ki}S(A_{ij}) = \delta_{kj} \tag{4.34}$$

□

Note that due to the particular form of the matrix A, we have the following form of S on conjugates:

$$S(\overline{A_{ij}}) = S^{(0)}(\overline{A_{ij}^{(0)}}) \otimes U_{ji}. \tag{4.35}$$

Let I denote the two-sided $*$-ideal in $A(\mathrm{SL}_\theta(2,\mathbb{H}))$ being generated by the elements $\sum_k \overline{A_{ki}}A_{kj} - \delta_{ij}$ for $i, j = 1, \dots, 4$. Recall that an ideal I in a Hopf algebra A is called a Hopf ideal if

$$\Delta(I) \subseteq I \otimes A + A \otimes I, \qquad \epsilon(I) = 0, \qquad S(I) \subseteq I. \tag{4.36}$$

Lemma 4.2. *The above ideal $I \subset A(\mathrm{SL}_\theta(2,\mathbb{H}))$ is a Hopf ideal.*

Proof. The first two properties follow easily from the definition of Δ and ϵ, whereas the third follows from Lemma 4.1 and (4.35):

$$S\big(\sum_i \overline{A_{ij}}A_{il} - \delta_{jl}\big) = S^{(0)}\big(\sum_i \overline{A_{ij}^{(0)}}A_{il}^{(0)} - \delta_{jl}\big) \otimes \overline{U}_{l1}U_{j1}, \tag{4.37}$$

using similar arguments as above. Hence, this property follows from the classical case. □

Let us summarize these results in the following proposition.

Proposition 4.1. *1. The collection $(A(\mathrm{SL}_\theta(2,\mathbb{H})), \Delta, \epsilon, S)$ is a Hopf algebra.*

2. The quotient $A(\mathrm{Sp}_\theta(2,\mathbb{H})) := A(\mathrm{SL}_\theta(2,\mathbb{H}))/I$ is a Hopf algebra with the induced Hopf algebra structure.

[4] At this point, we will not distinguish any more between an element in $A(\mathrm{SL}_\theta(2,\mathbb{H}))$ and its image under the splitting homomorphsim.

4.2.3 Coaction of $A(\mathrm{SL}_\theta(2,\mathbb{H}))$ on the Hopf fibration

The image of $A(S^7_\theta)$ under the coaction Δ_L in (4.15) of $A(\mathrm{SL}_\theta(2,\mathbb{H}))$ is a subalgebra of $A(\mathrm{SL}_\theta(2,\mathbb{H})) \otimes A(S^7_\theta)$, which we will denote by $A(\tilde{S_\theta}^7)$. We will now see that, in some sense, this is a θ-deformation of a family of inflated 7-dimensional sphere. First, consider the element $\sum_\mu z^\mu \overline{z}^\mu$, being the identity in $A(S^7_\theta)$. Since Δ_L is an algebra map from $A(S^7_\theta)$ to $A(\tilde{S_\theta}^7)$, we conclude that [5]

$$\rho^2 := \sum_\mu w^\mu \overline{w}^\mu = \Delta_L(\sum_\mu \overline{z}^\mu z^\mu) \tag{4.38}$$

is a central element in $A(\tilde{S_\theta}^7)$. In particular if $A \in A(Sp_\theta(2)$ then $\Delta_L(\sum_\mu \overline{z}^\mu z^\mu) = 1 \otimes \sum_\mu \overline{z}^\mu z^\mu$.
The element ρ^2 parametrizes a whole family of noncomutative 7-spheres $\tilde{S_\theta}^7$. Indeed, we can evaluate ρ^2 as any real number $r^2 \in \mathbb{R}$, to obtain an algebra $A(S^7_{\theta,r})$ which is a deformation of the the algebra of polynomials on a 7-sphere of radius r.

Also in this case, one can define a (right) coaction Δ_R of $A(SU(2))$ on $A(\tilde{S_\theta}^7)$ in such a way that the algebra of coinvariants forms an algebra of polynomials on some noncommutative 4-sphere. It is most natural to assume that Δ_R commutes with the abovely defined left coaction of $A(\mathrm{SL}_\theta(2,\mathbb{H}))$. As in [48], similarly, the algebra of coinvariants is generated by

$$\begin{aligned} &\tilde{\alpha} = 2(w^1\overline{w}^3 + w^2\overline{w}^4) \quad , \quad \tilde{\beta} = 2(-w^1 w^4 + w^2 w^3) \ , \\ &\tilde{x} = w^1\overline{w}^1 + w^2\overline{w}^2 - w^3\overline{w}^3 - w^4\overline{w}^4 \end{aligned} \tag{4.39}$$

together with ρ^2. The resulting algebra will be denoted by $A(\tilde{S_\theta}^4)$. It has the same commutation relations as $A(S^4_\theta)$ (cf. (4.9)), but a different spherical relation following from (4.38), which now reads:

$$\tilde{\alpha}\tilde{\alpha}^* + \tilde{\beta}\tilde{\beta}^* + \tilde{x}^2 = \rho^4. \tag{4.40}$$

We conclude that the coaction Δ_L of $A(\mathrm{SL}_\theta(2,\mathbb{H}))$ on the $SU(2)$ principal Hopf fibration $A(S^4_\theta) \to A(S^7_\theta)$, generates a family of $SU(2)$ principal Hopf fibrations $A(\tilde{S_\theta}^4) \to A(\tilde{S_\theta}^7)$. Indeed, by evaluating the central element ρ^2, we find a principal Hopf fibration $A(S^4_{\theta,r}) \to A(S^7_{\theta,r})$ for any $r \in \mathbb{R}$, consisting of spheres of radius r.

[5] The elements ρ^2 can also be expressed as

$$\rho^2 = \sum_{ijk} \overline{A}_{ij} A_{ik} \otimes \overline{u}_{j1} u_{k1} \ .$$

Since by construction $\tilde{\alpha}, \tilde{\beta}, \tilde{x}$ are the images of α, β, x, respectively, we can write explicitly:

$$\begin{aligned}\tilde{\alpha} &= 2(a_1\overline{c}_1 + a_2\overline{c}_2)\otimes x_{12} + 2(b_1\overline{d}_1 + b_2\overline{d}_2)\otimes x_{34} + (a_1\overline{d}_1 + \mu b_2\overline{c}_2)\otimes\alpha \\ &\quad +(a_1\overline{d}_2 - \overline{\mu} b_1\overline{c}_2)\otimes\beta + (b_1\overline{c}_1 + \overline{\mu} a_2\overline{d}_2)\otimes\alpha^* + (b_2\overline{c}_1 - \mu a_2\overline{d}_1)\otimes\beta^*\end{aligned} \tag{4.41}$$

$$\begin{aligned}\tilde{\beta} &= 2(a_2c_1 - a_1c_2)\otimes x_{12} + 2(b_2d_1 - b_1d_2)\otimes x_{34} + (-a_1d_2 + \mu b_2c_1)\otimes\alpha \\ &\quad +(a_1d_1 - \overline{\mu} b_1c_1)\otimes\beta + (-b_1c_2 + \overline{\mu} a_2d_1)\otimes\alpha^* + (-b_2c_2 + \mu a_2d_2)\otimes\beta^*\end{aligned} \tag{4.42}$$

$$\begin{aligned}\tilde{x} &= (a_1\overline{a}_1 + a_2\overline{a}_2)\otimes x_{12} + (b_1\overline{b}_1 + b_2\overline{b}_2)\otimes x_{34} + \frac{1}{2}(a_1\overline{b}_1 + \mu b_2\overline{a}_2)\otimes\alpha \\ &\quad +\frac{1}{2}(b_1\overline{a}_1 + \overline{\mu} a_2\overline{b}_2)\otimes\alpha^* + \frac{1}{2}(a_1\overline{b}_2 - \overline{\mu} b_1\overline{a}_2)\otimes\beta + \frac{1}{2}(b_2\overline{a}_1 - \mu a_2\overline{b}_1)\otimes\beta^* \\ &\quad -(c_1\overline{c}_1 + c_2\overline{c}_2)\otimes x_{12} - (d_1\overline{d}_1 + d_2\overline{d}_2)\otimes x_{34} - \frac{1}{2}(c_1\overline{d}_1 + \mu d_2\overline{c}_2)\otimes\alpha \\ &\quad -\frac{1}{2}(d_1\overline{c}_1 + \overline{\mu} c_2\overline{d}_2)\otimes\alpha^* - \frac{1}{2}(c_1\overline{d}_2 - \overline{\mu} d_1\overline{c}_2)\otimes\beta - \frac{1}{2}(d_2\overline{c}_1 - \mu c_2\overline{d}_1)\otimes\beta^*\end{aligned} \tag{4.43}$$

where $x_{12} = z^1\overline{z}^1 + z^2\overline{z}^2$ and $x_{34} = z^3\overline{z}^3 + z^4\overline{z}^4$.

We have thus established a coaction of $A(\mathrm{SL}_\theta(2,\mathbb{H}))$ on the noncommutative principal Hopf fibration $A(S^4_\theta) \to A(S^7_\theta)$. Note that the coaction of its quantum subgroup $A(\mathrm{Sp}_\theta(2))$ is such that it does not 'inflate the spheres', i.e. $\rho^2 = 1$.

4.2.4 The moduli space $\mathcal{M}_\theta$

Classically, the moduli space of instantons on S^4 is given as the quotient $\mathrm{SL}(2,\mathbb{H})/\mathrm{Sp}(2)$. In this section, we will construct an algebra that describes the moduli space M_θ of instantons on $A(S^4_\theta)$. The latter turns out to be a noncommutative space.

Motivated by the classical construction of instantons, let us consider the quantum quotient space as defined by:

$$A(\mathcal{M}_\theta) := \{a \in A(\mathrm{SL}_\theta(2,\mathbb{H})) | (\pi\otimes\mathrm{id})\circ\Delta a = 1\otimes a\} \tag{4.44}$$

where π denotes the quotient map from $A(\mathrm{SL}_\theta(2,\mathbb{H}))$ to $A(\mathrm{Sp}_\theta(2))$. Note that since $\mathrm{Sp}_\theta(2)$ is a quantum subgroup of $\mathrm{SL}_\theta(2,\mathbb{H})$ this quotient is well-defined.

Lemma 4.3. *The quantum quotient space $A(\mathcal{M}_\theta)$ is generated as an algebra by the elements $\sum_k \overline{A_{ki}}A_{kj}$.*

Proof. From the fact that the relations in the quotient $A(\mathrm{Sp}_\theta(2))$ are quadratic in a_{ij} and $\overline{a_{ij}}$, the generators of $A(\mathcal{M}_\theta)$ have to be at least quadratic in them. For the first leg of the tensor product $\Delta(a)$ to involve these relations, we need to have $a = \sum_i \overline{a_{ik}}a_{il}$, so that

$$(\pi\otimes\mathrm{id})\Delta(a) = \pi(\overline{a_{im}}a_{in})\otimes\overline{a_{mk}}a_{nl} = \delta_{mn}\otimes\overline{a_{mk}}a_{nl}, \tag{4.45}$$

giving the desired result. □

We will work with this set of generators $m_{ij} := \sum_k \overline{A_{ki}} A_{kj}$ and define the matrix $M := (m_{ij})$. The structure of the algebra $A(\mathcal{M}_\theta)$ can be deduced from the structure of $A(\mathrm{SL}_\theta(2,\mathbb{H}))$ as follows. First of all, the entries of M read:

$$M = \begin{pmatrix} m_{11} & 0 & m_{13} & m_{14} \\ 0 & m_{11} & -\overline{\mu}\,\overline{m}_{14} & \mu\,\overline{m}_{13} \\ \overline{m}_{13} & -\mu\, m_{14} & m_{33} & 0 \\ \overline{m}_{14} & \overline{\mu}\, m_{13} & 0 & m_{33} \end{pmatrix} =: \begin{pmatrix} m\mathbb{I}_2 & g \\ g^* & n\mathbb{I}_2 \end{pmatrix}, \tag{4.46}$$

with g of the following "θ-quaternionic" form:

$$g = \begin{pmatrix} g_1 & g_2 \\ -\overline{\mu}\overline{g}_2 & \mu\overline{g}_1, \end{pmatrix}.$$

We have the following relations between the entries:

$$\begin{aligned} m\ x = x\ m\ ; &\qquad n\ x = x\ n \quad \forall x \in \mathcal{M}_\theta \\ g_1\overline{g}_1 = \overline{g}_1 g_1\ ; &\qquad g_2\overline{g}_2 = \overline{g}_2 g_2\ ; \\ g_1 g_2 = \overline{\mu}^2 g_2 g_1\ ; &\qquad g_1\overline{g}_2 = \mu^2\overline{g}_2 g_1, \end{aligned} \tag{4.47}$$

together with

$$m\ n\mathbb{I}_2 - g\ g^* = \mathbb{I}_2, \tag{4.48}$$

coming from the condition $\det_\theta A = 1$ (as classically). Here $g\ g^* = (g_1\overline{g}_1 + g_2\overline{g}_2)\mathbb{I}_2$.

Proof of eq. (4.48): Firstly

$$\begin{aligned} m &= \overline{a}_1 a_1 + \overline{a}_2 a_2 + \overline{b}_1 b_1 + \overline{b}_2 b_2\ , \\ n &= \overline{c}_1 c_1 + \overline{c}_2 c_2 + \overline{d}_1 d_1 + \overline{d}_2 d_2\ , \\ g_1 &= \overline{a}_1 c_1 + \overline{a}_2 c_2 + \overline{b}_1 d_1 + \overline{b}_2 d_2\ , \\ g_2 &= -\overline{a}_1 \overline{c}_2 + \overline{a}_2 \overline{c}_1 - \overline{b}_1 \overline{d}_2 + \overline{b}_2 \overline{d}_1\ . \end{aligned}$$

Then, using the fact that the elements of the form $\overline{A_{ki}} A_{ki}$ are central in the algebra, we have that mn and $g_1\overline{g}_1 + g_2\overline{g}_2$ are both central elements in $A(\mathcal{M}_\theta)$ and

$$\begin{aligned} mn - gg^* = {} & \overline{a}_1 a_1 \overline{d}_1 d_1 + \overline{a}_1 a_1 \overline{d}_2 d_2 + \overline{a}_2 a_2 \overline{d}_1 d_1 + \overline{a}_2 a_2 \overline{d}_2 d_2 + \overline{b}_1 b_1 \overline{c}_1 c_1 + \overline{b}_1 b_1 \overline{c}_2 c_2 + \\ & \overline{b}_2 b_2 \overline{c}_1 c_1 + \overline{b}_2 b_2 \overline{c}_2 c_2 - \overline{a}_1 c_1 \overline{d}_1 b_1 - \overline{a}_1 c_1 \overline{d}_2 b_2 - \overline{a}_2 c_2 \overline{d}_1 b_1 - \overline{a}_2 c_2 \overline{d}_2 b_2 \\ & -\overline{b}_1 d_1 \overline{c}_1 a_1 - \overline{b}_1 d_1 \overline{c}_2 a_2 - \overline{b}_2 d_2 \overline{c}_1 a_1 - \overline{b}_2 d_2 \overline{c}_2 a_2 - \overline{a}_1 \overline{c}_2 d_2 b_1 + \overline{a}_1 \overline{c}_2 d_1 b_2 + \\ & \overline{a}_2 \overline{c}_1 d_2 b_1 - \overline{a}_2 \overline{c}_1 d_1 b_2 - \overline{b}_1 \overline{d}_2 c_2 a_1 + \overline{b}_1 \overline{d}_2 c_1 a_2 + \overline{b}_2 \overline{d}_1 c_2 a_1 - \overline{b}_2 \overline{d}_1 c_1 a_2 \end{aligned}$$

which can be shown to coincide with $det_\theta A$ given in (4.30) by using the commutation rules (4.16). $\square$

The geometric structure of the moduli space

The structure of the noncommutative space $\mathcal{M}_\theta$ becomes more clear if we introduce two central elements x and y, defined in terms of m, n by

$$x := m + n; \qquad y := m - n. \tag{4.49}$$

Relation (4.48) then reads

$$x^2\mathbb{I}_2 - y^2\mathbb{I}_2 - gg^* = \mathbb{I}_2 \tag{4.50}$$

so that $\mathcal{M}_\theta$ is a θ-deformation of a hyperboloid in 6 dimensions. Let us examine the structure at 'infinity'. We first adjoin the inverse of x to $A(\mathcal{M}_\theta)$ (hence removing the origin), and stereographically project onto the coordinates: $\tilde{y} := x^{-1}y$ and $\tilde{g} := x^{-1}g$. Then the above relation becomes:

$$\mathbb{I}_2 - \tilde{y}^2\mathbb{I}_2 - \tilde{g}\tilde{g}^* = \mathbb{I}_2 x^{-2}. \tag{4.51}$$

Evaluating x as a real number, and then taking the limit to infinity, we find the spherical relation:

$$\tilde{y}^2 + \tilde{g}_1\tilde{g}_1^* + \tilde{g}_2\tilde{g}_2^* = 1. \tag{4.52}$$

If we combine this with relations (4.47), we can conclude that at the 'boundary' of $\mathcal{M}_\theta$, we re-encounter the noncommutative 4-sphere $A(S^4_\theta)$. This fact exactly resembles the classical case, in which S^4 is found at the boundary of the moduli space.

4.2.5 The instanton projections

We will now construct the instanton projections in terms of the matrix $u' = A \otimes u$. For this, we first adjoin ρ^{-2} to $A(\tilde{S_\theta}^4) \subset A(\tilde{S_\theta}^7)$ as the inverse of ρ^2. Using the matrix u' we can construct a projection (as in Sect. 4.1) by

$$p' := u'\rho^{-2}(u')^* \tag{4.53}$$

or in components:

$$(p')_{ij} = u'_{ia}\rho^{-2}\overline{u'_{aj}} \tag{4.54}$$

The condition $(u')^*u' = \rho^2$ implies that p' is an idempotent. Moreover it is $*$-selfadjoint, and hence a projection.

By an argument similar to that of [48], one can prove that the entries of this matrix are coinvariants under the coaction of $A(SU(2))$, and hence elements in the algebra $A(\tilde{S_\theta}^4)$. As a check of this, one can compute the elements of p' explicitly. We have

$$p' = \frac{1}{2}\rho^{-2}\begin{pmatrix} \rho^2 + \tilde{x} & 0 & \tilde{\alpha} & \tilde{\beta} \\ 0 & \rho^2 + \tilde{x} & -\mu\tilde{\beta}^* & \overline{\mu}\tilde{\alpha}^* \\ \tilde{\alpha}^* & -\overline{\mu}\tilde{\beta} & \rho^2 - \tilde{x} & 0 \\ \tilde{\beta}^* & \mu\tilde{\alpha} & 0 & \rho^2 - \tilde{x} \end{pmatrix} \tag{4.55}$$

where the commutation rules (4.39), (4.40) of $A(\tilde{S_\theta}^4)$ can here be deduced from the condition $(p')^2 = p'$.

The topological charge

The relation of this projection with the basic instanton projection (4.12), can be seen by writing it in terms of elements in the tensor product $A(\mathrm{SL}_\theta(2,\mathbb{H})) \otimes A(S^4_\theta)$ using (4.41). However, expressing the projection p' in terms of $\tilde{\alpha}, \tilde{\beta}$ and $\tilde{x}$ allows to conclude that p' indeed describes another instanton on S^4_θ. Since $\tilde{\alpha}, \tilde{\beta}, \tilde{x}$ satisfy the same commutation relations as the α, β and x (differing only in the spherical relation), the splitting homomorphism only differs on the first leg, i.e. the classical part. For example, $\alpha = \alpha^{(0)} \otimes U^1 U^{2*}$, and $\tilde{\alpha} = \tilde{\alpha}^{(0)} \otimes U^1 U^{2*}$, where now $\tilde{\alpha}^{(0)}$ is an element in the tensor product $A(\mathrm{SL}(2,\mathbb{H})) \otimes A(S^4)$. The curvature $p'(\mathrm{d}p')^2$ of the connection $\nabla = p \circ \mathrm{d}$ then becomes an element in some fixed point subalgebra

$$\left(A(\mathrm{SL}(2,\mathbb{H})) \otimes \Omega^2(S^4) \otimes A(\mathbb{T}^2_\theta)\right)^{\text{f.p.}}. \tag{4.56}$$

in which the Hodge star operator can be defined using the splitting homomorphism. The projection p' turns out to describe a family of instantons on S^4_θ [47].

The rank of the instanton bundle is given by the zeroth Chern character, defined as the trace of the projector. In our case:

$$\mathrm{ch}_0(p') = \mathrm{tr}(p') = 2 \tag{4.57}$$

On the other hand, taking advantage of the possibility to endow θ-deformations of S^4 with a spin geometry, the charge of the instanton is given by:

$$\mathrm{tr}_\omega(\gamma_5 \pi'(p')[D, \pi'(p')]^4 |D|^{-4}) \tag{4.58}$$

where tr_ω denotes the Dixmier trace (cf. [18]), D the Dirac operator, γ_5 the grading of the Hilbert space, and π' a representation of the algebra $A(\tilde{S_\theta}^4)$ onto some Hilbert space. It is not difficult to see from the commutation relations and the spherical relation in $A(\tilde{S_\theta}^4)$ that this representation can be obtained from the representation of $A(S^4_\theta)$ on $L^2(S^4, S)$, with S the space of spinors (cf. [21]). Without changing the Hilbert space, it is simply given by a rescaling on the generators:

$$\pi'(\tilde{a}) := r\pi(a); \qquad \pi'(\rho^2) := r^2 \mathrm{id} \tag{4.59}$$

where $a = \alpha, \beta, x$ and $r \in \mathbb{R}$. Hence, from the definition of p' we see that $\pi'(p') = \pi(p)$. Of course, also the Dirac operator scales by a factor of r, but this is readily seen to cancel in the Dixmier trace. We conclude that the charge of the instanton defined by p' is equal to the charge of the basic instanton, which is one.

4.2.6 On the construction of instantons of generic charge

Let us now address our attention to instantons of generic charge k. As said before in Sect. 4.1, in the classical construction, [1], [2], instantons are constructed in terms of a $(k+1)\times k$ matrix of quaternions $v(x,y) = Cx + Dy$ with $(x,y)\in\mathbb{P}^1\mathbb{H}$ and $C,D\in Mat((k+1)\times k,\mathbb{H})$ constant matrices. The matrix v is assumed to be of maximal rank for each $(x,y)\neq(0,0)$ (hence its columns span a k-dimensional subspace of $\mathbb{H}^{k+1}$) and $\rho^2 := v^*v$ is assumed to be a real matrix. A gauge equivalent potential is obtained by considering the action on v of the groups $Sp(k+1)$ on the left and $GL(k,\mathbb{R})$ on the right. The moduli space of instantons reduces, as said, to a space of real dimension $8k-3$.

Let us try to reproduce this construction for $A(S^4_\theta)$.

Let us organise the elements of $A(S^7_\theta)$ into two "quaternions"

$$x = \begin{pmatrix} z^1 & z^2 \\ -\bar{z}^2 & \bar{z}^1 \end{pmatrix}, \quad y = \begin{pmatrix} z^3 & z^4 \\ -\bar{z}^4 & \bar{z}^3 \end{pmatrix}$$

with commutation relations (4.7) which read

$$x_{ab}y_{cd} = \varepsilon_{ca}y_{cd}x_{ab}, \qquad \varepsilon = \begin{pmatrix} \mu & \bar{\mu} \\ -\bar{\mu} & \mu \end{pmatrix}$$

and $x_{ab}x_{cd} = x_{cd}x_{ab}$, $y_{ab}y_{cd} = y_{cd}y_{ab}$, i.e. the elements inside a quaternion commute.

We consider

$$v(x,y) := C\,\dot{\otimes}\,x\cdot\mathbb{I}_k + D\,\dot{\otimes}\,y\cdot\mathbb{I}_k \tag{4.60}$$

a matrix whose quaternionic entries $v_{ij} = C_{ij}\otimes x + D_{ij}\otimes y$ are 2×2 matrix

$$v_{ij,ab} = C_{ij,al}\otimes x_{lb} + D_{ij,al}\otimes y_{lb}$$

where $i = 1,\dots,k+1$, $j = 1,\dots,k$, $a,b = 1,2$ and sum over $l = 1,2$ is understood. Let us also introduce the notation $a' = 3-a$ for $a = 1,2$.

Note that in this case the algebra generated by the entries of C and D and the one generated by the entries of v are both unkonwn. Anyway, following the classical case in which the columns of v are elements of $\mathbb{H}^{k+1}$, it is natural to assume that the $v^{ij,ab}$ satisfy commutation relations which generalise the ones of $\mathbb{H}^2$, i.e. the ones between the elments x,y of v at $k=1$. Firstly, the condition for v to have quaternionic entries means that $v^*_{ij,aa} = v_{ij,a'a'}$ and $v^*_{ij,aa'} = -v_{ij,a'a}$. The algebra generated by the entries of C and D, that we will denote with $\mathcal{C}$, is computed by assuming [6] that

$$v_{ij,ab}v_{kh,cd} = \eta_{ki,ca}\; v_{kh,cd}v_{ij,ab} \tag{4.61}$$

where the matrix η generalises the one given in (4.18):

$$\eta_{ii,ab} = 1\ ; \qquad \eta_{ij,aa} = \bar{\mu}\ \text{ if }\ i<j\ ; \qquad \eta_{ij,aa'} = \mu\ \text{ if }\ i<j$$

[6] Observe that in principle the commutation rules between the elements of v should also depend on the columns indices:

$$v_{ij,ab}v_{kh,cd} = \eta_{ki,ca}\eta_{jh,bd}\; v_{kh,cd}v_{ij,ab}\ ,$$

but this would cause immediately an absurd into the commutation rules of $\mathcal{C}$.

and $\eta_{ij,ab} = \overline{\eta}_{ji,ab}$. We have

$$\begin{aligned} C_{ij,al}C_{kh,cr} &= \eta_{ki,ca}C_{kh,cr}C_{ij,al} \quad ; \quad D_{ij,al}D_{kh,cr} = \eta_{ki,ca}D_{kh,cr}D_{ij,al} \\ C_{ij,al}D_{kh,cs} &= \eta_{ki,ca}\overline{\varepsilon}_{sl}D_{kh,cs}C_{ij,al} \quad ; \quad D_{kh,cs}C_{ij,al} = \eta_{ik,ca}\varepsilon_{sl}C_{ij,al}D_{kh,cs} \end{aligned} \tag{4.62}$$

Observation 4.4. *Note that when k reduces to 1, the above commutation relations are the one given in* (4.16) *with $C = (a_{ij}, c_{ij})^t$, $D = (b_{ij}, d_{ij})^t$ and $A = (C/D)$.*

The compatibility with the involution requires $C^*_{ij,aa} = C_{ij,a'a'}$, $C^*_{ij,aa'} = -C_{ij,a'a}$ and analogous for D, i.e. the matrices C and D have quaternionic entries and the above equations imply

$$\begin{aligned} C_{kl,cd}C^*_{ij,ab} &= \eta_{jk,b'c}C^*_{ij,ab}C_{kl,cd} \quad ; \quad D_{kl,cd}D^*_{ij,ab} = \eta_{jk,b'c}D^*_{ij,ab}D_{kl,cd} \; ; \\ C_{kl,cd}D^*_{ij,ab} &= \eta_{jk,b'c}\overline{\varepsilon}_{a'd}D^*_{ij,ab}C_{kl,cd} \; ; \quad ; \quad C^*_{ij,ab}D_{kl,cd} = \eta_{kj,cb'}\overline{\varepsilon}_{da'}D_{kl,cd}C^*_{ij,ab} \; . \end{aligned} \tag{4.63}$$

Dualising the classical assumption, we ask v^*v to be real:

$$(v^*v)_{ij,aa} = (v^*v)_{ij,a'a'} \quad ; (v^*v)_{ij,aa'} = 0 \; .$$

Using the fact that the elements in x^*x, y^*y and x^*y do not contain the same elements, also in this noncommutative case, the condition v^*v real splits in C^*C and D^*D real and C^*D symmetric. Imposing that the matrices C^*C, D^*D are real is not in contrast with the commutation relations among the elements of the algebra $\mathcal{C}$:

$$(C^*C)_{ij,aa} = (C^*C)_{ij,a'a'}, \qquad (C^*C)_{ij,aa'} = 0$$

are compatible with commutation relations (4.63) being $(C^*C)_{ij,aa}$ a central element in the algebra generated by the entries of C and D for each i, j, a. The same result holds for D^*D. The other condition C^*D symmetric (as a quaternion matrix) reads

$$(C^*D)_{ij,ab} = (C^*D)_{ji,ab}$$

and can be imposed without conflict with the algebra structure. Indeed the commutation relations between $(C^*D)_{ij,ab}$ and the generators of the algebra depends only on the indices $a, b = 1, 2$ into the block quaternion through $\overline{\varepsilon}$.

Remark 4.3. *We point out that the conditions imposed are motivated from the classical case but it is still to be understood if these requests are the correct ones also in the noncommutative case.*

Observation 4.5. *If $k = 1$, then the matrices C, D which constitute $A = (C, D)$ automatically satisfy these conditions, as it happens in the commutative case.*

4.2.7 Symmetries

Let us now try to reproduce the gauge invariance (4.4). Let us consider two matrices R, S (generating two algebras $\mathcal{R}$, $\mathcal{S}$ to be determined) such that the map

$$v \to R \dot{\otimes} v \dot{\otimes} S \tag{4.64}$$

defines a left-coaction of R and a right coaction of S and the resulting matrix is still of the form (4.60): $C' \dot{\otimes} x + D' \dot{\otimes} y$.

The two matrices R and S must have a "quaternionic structure" in order to preserve the "quaternionic form" of v. Let us analyse the two actions separately.

Let $\tilde{v} = v \dot{\otimes} S$. In order to have $\tilde{v} = C' \dot{\otimes} x + D' \dot{\otimes} y$ we need that S and x, y commute:

$$(x \dot{\otimes} S)_{ij,aa} = \tau(S \dot{\otimes} x)_{ij,aa} \qquad \Rightarrow \qquad S_{ij,aa'} = 0\ , \tag{4.65}$$

$$(x \dot{\otimes} S)_{ij,aa'} = \tau(S \dot{\otimes} x)_{ij,aa'} \qquad \Rightarrow \qquad S_{ij,aa} = S_{ij,a'a'}\ , \tag{4.66}$$

(τ being the flip) so that the matrix S is a real matrix, as in the classical case. Let us now impose that the commutation relations (4.61) are preserved. Here it is no longer necessary to split the quaternion entries of v. Let $\tilde{v} = (\tilde{v}_{ij})$, $i = 1, \ldots k+1$, $j = 1, \ldots k$, then $\forall i, j, k, l$

$$0 = \tilde{v}_{ij}\tilde{v}_{kl} - \eta_{ki}\tilde{v}_{kl}v_{ij} = \sum_{mn} v_{im}v_{kn} \otimes (S_{mj}S_{nl} - S_{nl}S_{mj}) \tag{4.67}$$

We introduce a basis (as a vector space) for the algebra generated by the entries of v. This is simply given by introducing a lexicographic order $v_{ik} < v_{kl}$ if $i < k$ or $i = k, j < l$ then extended to higher degrees. For degree two we may take $\{v_{ik}v_{kl}\ i < k\} \bigcup \{v_{ik}v_{il}\ j < l\}$ Hence if $i < k$ we may conclude that

$$S_{mj}S_{nl} = S_{nl}S_{mj}$$

and since it does not depend on i, k it holds also for generic i, k. Summarizing, the algebra $\mathcal{S}$ generated by the entries of S has to be commmutative.

On the other hand, if we assume a left coaction $R \dot{\otimes} v$, the algebra generated by the entries of R has to be noncommutative. Indeed, with a procedure similar to the above one, we can conclude that the entries of R have to satisfy

$$R_{il}R_{km} = \eta^{ki}\eta^{lm}R_{km}R_{il}\ .$$

4.3 On the moduli space for the symplectic principal fibration $A(S_q^4) \hookrightarrow A(S_q^7)$

Let us now consider the symplectic fibration $A(S_q^4) \hookrightarrow A(S_q^7)$ obtained in [46] from the quantum symplectic group $Sp_q(2)$ and described in Ch. 2. In this case the attempt to generalise the construction to generic instantons becomes immediatly difficult due to the complicated commutation rules among the elements of the algebras involved. We have made only few steps toward the construction of other instantons of charge 1 that we report below.

4.3.1 The algebra $\mathcal{A}_q$

We consider the transformation induced on the elements of $A(S^7_q)$ by appling a matrix transformation A on v:

$$\Delta_L : v_{ia} \mapsto A_{ij} \otimes v_{ja} =: v'_{ia} \tag{4.68}$$

with A a generic 4x4 matrix. The structure of the matrix A and relations between the entries of A can again be deduced by requiring that Δ_L be a (not unit preserving) $*$-algebra map. From the condition on Δ_L to be a $*$-map, we infer the following form of A:

$$A_q = \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ -q^3\overline{a}_2 & \overline{a}_1 & q^2\overline{b}_2 & -q\overline{b}_1 \\ c_1 & c_2 & d_1 & d_2 \\ q^2\overline{c}_2 & -q^{-1}\overline{c}_1 & -q\overline{d}_2 & \overline{d}_1 \end{pmatrix} \tag{4.69}$$

and hence the transformation reads on the generators of $A(S^7_q)$:

$$\begin{aligned} x'_1 &= a_1 \otimes x_1 - q^2 a_2 \otimes \overline{x}^2 + q^2 b_1 \otimes x_3 - q^3 b_2 \otimes \overline{x}^4 \\ x'_2 &= a_1 \otimes x_2 + q a_2 \otimes \overline{x}^1 - q^2 b_1 \otimes x_4 - q^2 b_2 \otimes \overline{x}^3 \\ x'_3 &= q^{-2} c_1 \otimes x_1 - c_2 \otimes \overline{x}^2 + d_1 \otimes x_3 - q d_2 \otimes \overline{x}^4 \\ x'_4 &= -q^{-2} c_1 \otimes x_2 - q^{-1} c_2 \otimes \overline{x}^1 + d_1 \otimes x_4 + d_2 \otimes \overline{x}^3 \end{aligned} \tag{4.70}$$

together with $(\overline{x}_i)' = (\overline{x'_i})$.

The algebra map condition entails the following complicated commutation relations:

$a_1 a_2 = q^{-1} a_2 a_1 + (q^2 - q^{-4}) b_2 b_1;$
$a_1 b_1 = q^{-3} b_1 a_1;$
$a_1 b_2 = b_2 a_1;$
$a_1 c_1 = q c_1 a_1;$
$a_1 c_2 = q^2 c_2 a_1 + (q - q^3) c_1 a_2 + (q^5 - q^3) d_2 b_1 + (q^4 - 1) d_1 b_2;$
$a_1 d_1 = d_1 a_1 + (q - q^3) c_1 b_1;$
$a_1 d_2 = q^3 d_2 a_1 + (q - q^3) c_1 b_2;$

$a_2 b_1 = b_1 a_2;$
$a_2 b_2 = q^3 b_2 a_2;$
$a_2 c_1 = q^2 c_1 a_2 + (1 - q^4) d_2 b_1 + (q^{-1} - q) d_1 b_2;$
$a_2 c_2 = q c_2 a_2;$
$a_2 d_1 = q d_1 a_2;$
$a_2 d_2 = q^4 d_2 a_2;$

$$
\begin{aligned}
b_1b_2 &= q^{-1}b_2b_1; & b_2c_1 &= qc_1b_2;\\
b_1c_1 &= q^4c_1b_1; & b_2c_2 &= c_2b_2 + (q-q^3)d_2a_2;\\
b_1c_2 &= q^3c_2b_1 + (q-q^3)d_1a_2; & b_2d_1 &= q^2d_1b_2;\\
b_1d_1 &= qd_1b_1; & b_2d_2 &= qd_2b_2;\\
b_1d_2 &= q^2d_2b_1 + (q-q^3)d_1b_2; & &
\end{aligned}
$$

$$
\begin{aligned}
c_1c_2 &= q^{-1}c_2c_1 + (q^2-q^{-4})d_2d_1; & c_2d_1 &= d_1c_2;\\
c_1d_1 &= q^{-3}d_1c_1; & c_2d_2 &= q^3d_2c_2;\\
c_1d_2 &= d_2c_1; & &
\end{aligned}
$$

$$
d_1d_2 = q^{-1}d_2d_1,
$$

together with their conjugates.
Moreover, we have

$$
\begin{aligned}
a_1\overline{a}_1 &= \overline{a}_1a_1 + (q^2-q^4)\overline{b}_1b_1 + (q^4-q^2)\overline{a}_2a_2 + (q^4-1)\overline{b}_2b_2;\\
a_1\overline{a}_2 &= q^{-1}\overline{a}_2a_1;\\
a_1\overline{b}_2 &= q^{-2}\overline{b}_2a_1 + (q^2-1)\overline{a}_2b_1;\\
a_1\overline{b}_1 &= q\overline{b}_1a_1 + (q-q^3)\overline{a}_2b_2;\\
a_1\overline{c}_2 &= q^{-2}\overline{c}_2a_1;\\
a_1\overline{c}_1 &= q^{-1}\overline{c}_1a_1 + (q-q^3)\overline{d}_1b_1 + (q^3-q)\overline{c}_2a_2 + (q^3-q^{-1})\overline{d}_2b_2;\\
a_1\overline{d}_2 &= q^{-3}\overline{d}_2a_1 + (q-q^{-1})\overline{c}_2b_1;\\
a_1\overline{d}_1 &= \overline{d}_1a_1 + (1-q^2)\overline{c}_2b_2;
\end{aligned}
$$

$$
\begin{aligned}
a_2\overline{a}_2 &= \overline{a}_2a_2 + (q^{-4}-1)\overline{b}_1b_1 + (q^{-2}-q^{-4})\overline{b}_2b_2;\\
a_2\overline{b}_2 &= q^{-1}\overline{b}_2a_2;\\
a_2\overline{b}_1 &= q^2\overline{b}_1a_2;\\
a_2\overline{c}_2 &= q^{-1}\overline{c}_2a_2 + (q^{-5}-q^{-1})\overline{d}_1b_1 + (q^{-3}-q^{-5})\overline{d}_2b_2;\\
a_2\overline{c}_1 &= q^{-2}\overline{c}_1a_2;\\
a_2\overline{d}_2 &= q^{-2}\overline{d}_2a_2;\\
a_2\overline{d}_1 &= q\overline{d}_1a_2;
\end{aligned}
$$

and

$$
\begin{aligned}
b_1\overline{b}_2 &= q^{-1}\overline{b}_2b_1; & &\\
b_1\overline{b}_1 &= \overline{b}_1b_1 + (q^2-1)\overline{b}_2b_2; & b_2\overline{b}_2 &= \overline{b}_2b_2;\\
b_1\overline{c}_2 &= q\overline{c}_2b_1; & b_2\overline{c}_2 &= q^{-2}\overline{c}_2b_2;\\
b_1\overline{c}_1 &= \overline{c}_1b_1 + (1-q^2)\overline{d}_2a_2; & b_2\overline{c}_1 &= q^{-3}\overline{c}_1b_2 + (q-q^{-1})\overline{d}_1a_2;\\
b_1\overline{d}_2 &= q^{-2}\overline{d}_2b_1; & b_2\overline{d}_2 &= q^{-1}\overline{d}_2b_2;\\
b_1\overline{d}_1 &= q^{-1}\overline{d}_1b_1 + (q-q^{-1})\overline{d}_2b_2; & b_2\overline{d}_1 &= q^{-2}\overline{d}_1b_2,
\end{aligned}
$$

and finally

$$
\begin{aligned}
c_1\overline{c}_2 &= q^{-1}\overline{c}_2c_1 + (q^{-1}-q)\overline{a}_2a_1;\\
c_1\overline{c}_1 &= \overline{c}_1c_1 + (q^2-q^4)\overline{d}_1d_1 + (q^4-q^2)\overline{c}_2c_2 + (q^4-1)\overline{d}_2d_2 + (q^4-q^2)\overline{a}_1a_1;\\
&\quad + (q^8+q^4-q^2-q^6)\overline{a}_2a_2 + (-q^4+q^2+q^8-q^6)\overline{b}_2b_2 + (-q^4-q^8+2q^6)\overline{b}_1b_1;\\
c_1\overline{d}_2 &= q^{-2}\overline{d}_2c_1 + (q^2-1)\overline{c}_2d_1 + (q^2-1)\overline{b}_2a_1 + (q^6-1-q^4+q^2)\overline{a}_2b_1;\\
c_1\overline{d}_1 &= q\overline{d}_1c_1 + (q-q^3)\overline{c}_2d_2 + (q^5-q^7+q-q^3)\overline{a}_2b_2 + (q^5-q^3)\overline{b}_1a_1;
\end{aligned}
$$

$$
\begin{aligned}
c_2\overline{c}_2 &= \overline{c}_2c_2 + (q^{-4}-1)\overline{d}_1d_1 + (q^{-2}-q^{-4})\overline{d}_2d_2 + (q^4-q^2)\overline{a}_2a_2;\\
&\quad + (1-q^{-4})\overline{a}_1a_1 + (q^{-2}+q^2-2)\overline{b}_2b_2 + (1-q^4-q^{-2}+q^2)\overline{b}_1b_1;\\
c_2\overline{d}_2 &= q^{-1}\overline{d}_2c_2 + (q^3-q)\overline{b}_2a_2 + (q^{-3}-q)\overline{a}_1b_1;\\
c_2\overline{d}_1 &= q^2\overline{d}_1c_2 + (q^6-q^4)\overline{b}_1a_2 + (q^2-q^{-2})\overline{a}_1b_2;
\end{aligned}
$$

$$
\begin{aligned}
d_1\overline{d}_2 &= q^{-1}\overline{d}_2d_1 + (q^{-1}-q)\overline{b}_2b_1;\\
d_1\overline{d}_1 &= \overline{d}_1d_1 + (q^2-1)\overline{d}_2d_2 + (q^4-q^2)\overline{b}_1b_1 + (q^6-q^4+q^2-1)\overline{b}_2b_2;
\end{aligned}
$$

$$
d_2\overline{d}_2 = \overline{d}_2d_2 + (q^4-q^2)\overline{b}_2b_2 + (q^2-q^{-2})\overline{b}_1b_1,
$$

again with their conjugates. We denote this algebra by $\mathcal{A}_q$.

Observation 4.6. *In an analogous way, we can look for a matrix B_q such that $v = B_qv'$. Of course the algebra given by the entries of B_q and the algebra $\mathcal{A}_q$ are isomorphic by construction. If we assume that the matrix A_q is invertible, then the matrix B_q would be the inverse, but this is still to be checked. In particular it remains to find (if there exists) a central element $det_q \in \mathcal{A}_q$ to be interpreted as a quantum determinant.*

4.3.2 The projection p_ρ

In spite of the fact that the commutation relations among the elements of the algebra $\mathcal{A}_q$ are much more complicated than the one of the algebra $\mathcal{A}_\theta$, the element $\rho^2 := \sum_i \overline{x'_i}x'_i$ is again a central element in the corresponding algebra, being the image of 1 under the algebra map Δ_L:

$$
\rho^2 := \Delta_L(1) = \langle \phi'_1|\phi'_1\rangle = \langle \phi'_2|\phi'_2\rangle = \sum_i \overline{x'_i}x'_i
$$

The observation of this property allows us to introduce the element ρ^{-2}, being defined by requiring that it is a central element in the algebra $\mathcal{A}_q$ such that $\rho^2\rho^{-2} = 1$.
The projection is constructed as in the classical (and θ-) case as $p_\rho = v'\rho^{-2}v'^*$ and it

is given by

$$p_\rho = \rho^{-2} \begin{pmatrix} q^{-2}\tilde{t} & 0 & \tilde{a} & \tilde{b} \\ 0 & \tilde{t} & q^{-2}\tilde{b}^* & -q^2\tilde{a}^* \\ \tilde{a}^* & q^{-2}\tilde{b} & \rho^2 - q^{-4}\tilde{t} & 0 \\ \tilde{b}^* & -q^2\tilde{a} & 0 & \rho^2 - q^2\tilde{t} \end{pmatrix} . \tag{4.71}$$

where

$$\begin{aligned} \tilde{t} &= q^{-2}\overline{x}^1{}' x_1' + q^{-2}\overline{x}^2{}' x_2' , \\ \tilde{a} &= q^{-4}x_1{}'\overline{x}^3{}' - q^{-2}x_2'\overline{x}^4{}' , \\ \tilde{b} &= -q^{-3}x_1'x_4' - q^{-2}x_2'x_3' \end{aligned}$$

are the elements obtained by applying the transformation A_q to the generators of $A(S^4_q)$. The commutation relations among these elements can be computed by using $p_\rho^2 = p_\rho$ and are given by

$$\begin{aligned} &\tilde{a}\tilde{b} = q^4\tilde{b}\tilde{a} , \quad \tilde{a}^*\tilde{b} = \tilde{b}\tilde{a}^* , \\ &\tilde{t}\tilde{a} = q^{-2}\tilde{a}\tilde{t} , \quad \tilde{t}\tilde{b} = q^4\tilde{b}\tilde{t} , \end{aligned} \tag{4.72}$$

together with their conjugates, and relations

$$\begin{aligned} &\tilde{a}\tilde{a}^* + \tilde{b}\tilde{b}^* = q^{-2}\tilde{t}(\rho^2 - q^{-2}\tilde{t}) , \qquad q^4\tilde{a}^*\tilde{a} + q^{-4}\tilde{b}^*\tilde{b} = \tilde{t}(\rho^2 - \tilde{t}) , \\ &\tilde{b}\tilde{b}^* - q^{-4}\tilde{b}^*\tilde{b} = (1 - q^{-4})\tilde{t}^2 . \end{aligned} \tag{4.73}$$

The algebra generated by the entries of p_ρ when evaluating ρ^2, i.e. in the quotient $\rho^2 = r^2 \in \mathbb{R}$, is a deformation of the algebra of polynomials on a 4-sphere of radius r. We denote this algebra with $A(S^4_q)_r$.

Representations of the algebra $A(S^4_q)_r$ and the charge

We compute now the instanton charge with a procedure analogous to the one used in Sect. 2.3. We start by associate a Fredholm module μ over $A(S^4_q)_r$ and we procede by computing the pairing of the Chern-Connes characters $ch^*(\mu) \in HC^*[A(S^4_q)_r]$ and $ch_*(p_\rho) \in HC_*[A(S^4_q)_r]$ in cyclic homology and cohomology respectively.

We consider irreducible $*$-representations of $A(S^4_q)_r$ as bounded operators on a separable Hilbert space $\mathcal{H}$. We denote in the same way the elements of the algebra and their images as operators in the given representation. We can restrict ourselves to $|q| < 1$ and we consider the representations which are t-finite [41], i.e. we assume that the eigenvectors of $\tilde{t}$ span $\mathcal{H}$.

With observations similar to the ones done in Sect. 2.3 we conclude that the spectrum should be of the form λq^{2k} and $\tilde{a}, \tilde{b}^*$ (resp. $\tilde{a}^*, \tilde{b}$) act as rising (resp. lowering) operators on the eigenvectors of $\tilde{t}$. The boundedness implies that there exists a highest weight vector $|0,0\rangle$ such that

$$\tilde{t}\,|0,0\rangle = \tilde{t}_{00}\,|0,0\rangle\,, \quad \tilde{a}\,|0,0\rangle = 0, \quad \tilde{b}^*\,|0,0\rangle = 0\ . \tag{4.74}$$

Using eqs. (4.73) we have $q^4\tilde{a}^*\tilde{a} + \tilde{b}\tilde{b}^* = (r^2 - q^{-4}\tilde{t})\tilde{t}$ that evalueted on $|0,0\rangle$ gives

$$(r^2 - q^{-4}\tilde{t}_{00})\tilde{t}_{00} = 0$$

According to the values of the eigenvalue $\tilde{t}_{00}$ we have two representations. The calculus which follow are analogous to the ones done at $\rho^2 = 1$, i.e. at $A_q = \mathbb{I}$.

The representation β

The first representation, say β, is obtained for $\tilde{t}_{00} = 0$. This representation is the trivial one

$$\tilde{t} = 0, \quad \tilde{a} = 0, \quad \tilde{b} = 0\,, \tag{4.75}$$

with representation Hilbert space being $\mathbb{C}$ and $\beta(1) = 1$.

The representation σ

The second representation, that we call σ, is obtained for $\tilde{t}_{00} = \rho^2 q^4$. This is infinite dimensional; we take the set $|m,n\rangle = N_{mn}\tilde{a}^{*m}\tilde{b}^n\,|0,0\rangle$ with $n,m \in \mathbb{N}$, to be an orthonormal basis of the representation Hilbert space $\mathcal{H}$, with $N_{00} = 1$ and $N_{mn} \in \mathbb{R}$ the normalizations, to be computed.

Then

$$\begin{aligned} \tilde{t}\,|m,n\rangle &= \tilde{t}_{mn}\,|m,n\rangle\ , \\ \tilde{a}^*\,|m,n\rangle &= \tilde{a}_{mn}\,|m+1,n\rangle\ , \\ \tilde{b}\,|m,n\rangle &= \tilde{b}_{mn}\,|m,n+1\rangle\ . \end{aligned}$$

By requiring that we have a $*$-representation we have also that

$$\tilde{a}\,|m,n\rangle = \tilde{a}_{m-1,n}\,|m-1,n\rangle\ , \quad \tilde{b}^*\,|m,n\rangle = \tilde{b}_{m,n-1}\,|m,n-1\rangle\ ,$$

with the following recursion relations

$$\tilde{a}_{m,n\pm 1} = q^{\pm 2}\tilde{a}_{m,n}\ , \qquad \tilde{b}_{m\pm 1,n} = q^{\pm 2}\tilde{b}_{m,n}\ , \qquad \tilde{b}_{m,n} = q^2\tilde{a}_{2n+1,m}\ .$$

We have

$$\begin{cases} \tilde{t}_{m,n} = \rho^2 q^{2m+4n+4}\ , \\ \\ \tilde{a}_{m,n} = N_{mn}N^{-1}_{m+1,n} = \rho^2(1-q^{2m+2})^{\frac{1}{2}}q^{m+2n+1}\ , \\ \\ \tilde{b}_{m,n} = N_{mn}N^{-1}_{m,n+1} = \rho^2(1-q^{4n+4})^{\frac{1}{2}}q^{2(m+n+2)}\ . \end{cases} \tag{4.76}$$

Proof. Firstly

$$\tilde{t}\,|m,n\rangle = N_{mn}q^{2m+4n}\tilde{a}^{*m}\tilde{b}^n\tilde{t}\,|0,0\rangle = N_{mn}q^{2m+4n}\tilde{f}_{00}\tilde{a}^{*m}\tilde{b}^n\,|0,0\rangle = r^2q^{2m+4(n+1)}\,|m,n\rangle\ .$$

In order to compute $\tilde{a}_{mn}$ and $\tilde{b}_{mn}$ we only need to compute the quotients of the normalizing constants being

$$\tilde{a}^* |m,n\rangle = N_{m,n}\tilde{a}^{*m+1}\tilde{b}^n |0,0\rangle = \frac{N_{m,n}}{N_{m+1,n}} |m+1,n\rangle$$

and

$$\tilde{b} |m,n\rangle = \frac{N_{m,n}}{N_{m,n+1}} |m,n+1\rangle \ .$$

We use $\tilde{a}\tilde{a}^* - q^4\tilde{a}^*\tilde{a} = \rho^2 t(q^{-2}-1)$, then

$$\begin{aligned}
\tilde{a}^{m+1}\tilde{a}^* &= \tilde{a}^m(\tilde{a}\tilde{a}^*) = \tilde{a}^m(q^4\tilde{a}^*\tilde{a} + \rho^2(q^{-2}-1)\tilde{t}) = \\
&= \rho^2(q^{-2}-1)\tilde{a}^m\tilde{t} + q^4\tilde{a}^{m-1}(q^4\tilde{a}^*\tilde{a} + (q^{-2}-1)\rho^2\tilde{t})\tilde{a} \\
&= \rho^2(q^{-2}-1)(1+q^2)\tilde{a}^m\tilde{t} + q^8\tilde{a}^{m-2}(q^4\tilde{a}^*\tilde{a} + \rho^2(q^{-2}-1)\tilde{t})\tilde{a}^2 = \ldots \\
&= \rho^2(q^{-2}-1)(1+q^2+q^4+\cdots+q^{2m})\tilde{a}^m\tilde{t} + (\cdots)\tilde{a} \\
&= \rho^2 q^{-2}(1-q^{2m+2})\tilde{a}^m\tilde{t} + (\cdots)\tilde{a}^*\tilde{a}
\end{aligned}$$

Then we have

$$\begin{aligned}
1 = \langle m+1,n|m+1,n\rangle &= N^2_{m+1,n}\langle 0,0|\,\tilde{b}^{*n}\tilde{a}^{m+1}\tilde{a}^{*m+1}\tilde{b}^n |0,0\rangle \\
&= \rho^2 q^{-2}(1-q^{2m+2})N^2_{m+1,n}\langle 0,0|\,\tilde{b}^{*n}\tilde{a}^m\tilde{t}\tilde{a}^{*m}\tilde{b}^n |0,0\rangle \\
&= \rho^4 q^{-2}(1-q^{2m+2})q^{2m+4n+4}N^2_{m+1,n}\langle 0,0|\,\tilde{b}^{*n}\tilde{a}^m\tilde{a}^{*m}\tilde{b}^n |0,0\rangle \\
&= \rho^4 q^{-2}(1-q^{2m+2})q^{2m+4n+4}\frac{N^2_{m+1,n}}{N^2_{mn}}\langle m,n|m,n\rangle \ .
\end{aligned}$$

Hence we can conclude that

$$\frac{N^2_{m,n}}{N^2_{m+1,n}} = \rho^4 q^{2(m+2n+1)}(1-q^{2m+2})$$

and hence the $\tilde{a}_{m,n}$ have the expression given in (4.76):

$$\tilde{a}_{m,n} = \rho^2(1-q^{2m+2})^{\frac{1}{2}}q^{m+2n+1} \ .$$

For $\tilde{b}_{mn}$ the computation is analogous: as a first step

$$\tilde{b}^{*n+1}\tilde{b} = (1-q^4)\sum_{i=0}^{n} q^{4i}\tilde{b}^{*n}\tilde{t}^2 + (\ )\tilde{b}\tilde{b}^*$$

and then

$$\begin{aligned}
1 = \langle m,n+1|m,n+1\rangle &= q^{-4n}(1-q^{4n+4})N^2_{m,n+1}\langle 0,0|\,\tilde{a}^m\tilde{b}^{*n}\tilde{t}^2\tilde{b}^n\tilde{a}^{*m}|0,0\rangle \\
&= \rho^4(1-q^{4n+4})q^{4(n+m+2)}\frac{N^2_{m,n+1}}{N^2_{m,n}}
\end{aligned}$$

and so

$$\tilde{b}\,|m,n\rangle = \frac{N_{m,n}}{N_{m,n+1}}\,|m+1,n\rangle = \rho^2(1-q^{4n+4})^{\frac{1}{2}}q^{2(m+n+2)}\,|m,n+1\rangle\ . \quad \square$$

Summarizing, the representation σ is given by

$$\begin{aligned}
\tilde{t}\,|m,n\rangle &= \rho^2 q^{2m+4n+4}\,|m,n\rangle\,, \\
\tilde{a}^*\,|m,n\rangle &= \rho^2(1-q^{2m+2})^{\frac{1}{2}}q^{m+2n+1}\,|m+1,n\rangle\,, \\
\tilde{a}\,|m,n\rangle &= \rho^2(1-q^{2m})^{\frac{1}{2}}q^{m+2n}\,|m-1,n\rangle\,, \\
\tilde{b}\,|m,n\rangle &= \rho^2(1-q^{4n+4})^{\frac{1}{2}}q^{2(m+n+2)}\,|m,n+1\rangle\,, \\
\tilde{b}^*\,|m,n\rangle &= \rho^2(1-q^{4n})^{\frac{1}{2}}q^{2(m+n+1)}\,|m,n-1\rangle\,.
\end{aligned} \tag{4.77}$$

We need also to observe that, once $\rho^2 = r^2$ is fixed, the algebra generators are all trace class operators:

$$\begin{aligned}
\mathrm{Tr}(\tilde{t}) &= r^2 q^4 \sum_m q^{2m} \sum_n q^{4n} = r^2 \frac{q^4}{(1-q^2)(1-q^4)} \quad , \\
\mathrm{Tr}(|\tilde{a}|) &= r^2 q \sum_{m,n} (1-q^{2m+2})^{\frac{1}{2}} q^{m+2n} = r^2 \frac{q}{1-q^2} \sum_m (1-q^{2m+2})^{\frac{1}{2}} q^m \\
&\leq \frac{r^2 q}{1-q^2} \sum_m q^m = \frac{r^2 q}{(1-q)(1-q^2)} \quad , \qquad\qquad (4.78) \\
\mathrm{Tr}(|\tilde{b}|) &= r^2 q^4 \sum_{m,n} (1-q^{4n+4})^{\frac{1}{2}} q^{2(n+m)} = r^2 \frac{q^4}{1-q^2} \sum_n (1-q^{4n+4})^{\frac{1}{2}} q^{2n} \\
&\leq r^2 \frac{q^4}{1-q^2} \sum_n q^{2n} = \frac{r^2 q^4}{(1-q^2)^2} \quad .
\end{aligned}$$

The closure of $A(S_q^4)_r$ is the C^*-algebra $\mathcal{C}(S_q^4)_r = \mathcal{K} \oplus \mathbb{C}\mathbb{I}$.

We can construct a nontrivial Fredholm module $[\mu_\rho] \in K^0[\mathcal{C}(S_q^4)_r]$ which we will need later on to compute the charge.

The Fredholm module $\mu_\rho := (\mathcal{H}, \Psi, \gamma)$ is constructed exactly as for $A(S_q^4)$. The Hilbert space is $\mathcal{H} = \mathcal{H}_\sigma \oplus \mathcal{H}_\sigma$ and the representation is $\Psi = \sigma \oplus \beta$. Here σ and β are the representations of $A(S_q^4)_r$ introduced in (4.77) and (4.75), respectively, with β trivially extended to $\mathcal{H}_\sigma$. The grading operator is

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The index pairing

As done in Sect. 2.4, in order to compute the pairing between K-theory and K-homology, we need to compute the Chern characters in the cyclic homology $\mathrm{ch}_*(p_\rho) \in HC_*[A(S^4_q)_r]$ and cyclic cohomology $\mathrm{ch}^*(\mu_\rho) \in HC^*[A(S^4_q)_r]$ respectively. Here p_ρ is the projection introduced in (4.71) and μ_ρ is the Fredholm module described above.

The idempotent p_ρ determines a class $[p_\rho] \in K_0[\mathcal{C}(S^4_q)_r]$ in K-theory. The component in degree zero $\mathrm{ch}_0(p_\rho) \in HC_0[A(S^4_q)_r]$ of the Chern character $\mathrm{ch}_*(p_\rho)$ is given by the matrix trace:

$$\mathrm{ch}_0(p_\rho) := \mathrm{tr}(p_\rho) = 2 - \rho^{-2}q^{-4}(1-q^2)(1-q^4)\ \tilde{t}\ \in A(S^4_q)_r. \tag{4.79}$$

The component in degree 0, $\mathrm{ch}^0(\mu_\rho) \in HC^0[A(S^{2n}_q)]$ of the Chern character $\mathrm{ch}^*(\mu_\rho)$ of the class of the Fredholm module μ_ρ is the trace

$$\tau^1(x) := \mathrm{Tr}\left(\gamma\Psi(x)\right) = \mathrm{Tr}\left(\sigma(x) - \beta(x)\right). \tag{4.80}$$

The operator $\sigma(x) - \beta(x)$ being always trace class and $\tau^1(1) = 0$.

Hence the pairing is given by

$$\begin{aligned}\langle [\mu_\rho], [p_\rho] \rangle &:= \left\langle \mathrm{ch}^0(\mu_\rho), \mathrm{ch}_0(p_\rho) \right\rangle = -\rho^{-2}q^{-4}(1-q^2)(1-q^4)\ \tau^1(\tilde{t}) \\ &= -\rho^{-2}q^{-4}(1-q^2)(1-q^4)\,\mathrm{Tr}(\tilde{t}) \\ &= -\rho^{-2}q^{-4}(1-q^2)(1-q^4)\rho^2 q^4(1-q^2)^{-1}(1-q^4)^{-1} = -1\ .\end{aligned} \tag{4.81}$$

4.3.3 The matrix $\mathbf{M}_q$

In the classical case we have seen that the 5-parameters family of instantons is described by the elments of $SL(2,\mathbb{H})/Sp(2)$, i.e. the space of quaternion norms on $\mathbb{H}^2$. Let us compute the elements of the matrix $M_q := B_q^* B_q = (\mathbf{h}_{ij})$, with B_q a matrix of the form (4.69) and generating an algebra isomorphic to $\mathcal{A}_q$. [7] We first observe that the transformation $v = B_q v'$ can be written also as $v^* = v'^* B_q^*$. This fact allows to deduce that

$$B_q^* = {}^t\overline{B}_q = \begin{pmatrix} \overline{\mathbf{a}}_1 & -q^3\mathbf{a}_2 & \overline{\mathbf{c}}_1 & q^2\mathbf{c}_2 \\ \overline{\mathbf{a}}_2 & \mathbf{a}_1 & \overline{\mathbf{c}}_2 & -q^{-1}\mathbf{c}_1 \\ \overline{\mathbf{b}}_1 & q^2\mathbf{b}_2 & \overline{\mathbf{d}}_1 & -q\mathbf{d}_2 \\ \overline{\mathbf{b}}_2 & -q\mathbf{b}_1 & \overline{\mathbf{d}}_2 & \mathbf{d}_1 \end{pmatrix} \tag{4.83}$$

[7] The two column vectors $|\phi'_1\rangle$, $|\phi'_2\rangle$ which constitute v' are orthonormal in the metric

$$\langle \xi | \eta \rangle_M := \xi^* \mathbf{M}_q \eta = \sum \xi_j^* \mathbf{h}_j{}^i \eta^i \tag{4.82}$$

We can compute explicitly the elements of the matrix $\mathbf{M}_q = B_q^* B_q$. They are given by

$$\begin{aligned}
\mathbf{h}_{11} &= \overline{\mathbf{a}}_1\mathbf{a}_1 + q^6\mathbf{a}_2\overline{\mathbf{a}}_2 + \overline{\mathbf{c}}_1\mathbf{c}_1 + q^4\mathbf{c}_2\overline{\mathbf{c}}_2\\
\mathbf{h}_{12} &= \overline{\mathbf{a}}_1\mathbf{a}_2 - q^3\mathbf{a}_2\overline{\mathbf{a}}_1 + \overline{\mathbf{c}}_1\mathbf{c}_2 - q\mathbf{c}_2\overline{\mathbf{c}}_1\\
\mathbf{h}_{21} &= \overline{\mathbf{a}}_2\mathbf{a}_1 - q^3\mathbf{a}_1\overline{\mathbf{a}}_2 + \overline{\mathbf{c}}_2\mathbf{c}_1 - q\mathbf{c}_1\overline{\mathbf{c}}_2\\
\mathbf{h}_{22} &= \overline{\mathbf{a}}_2\mathbf{a}_2 + \mathbf{a}_1\overline{\mathbf{a}}_1 + \overline{\mathbf{c}}_2\mathbf{c}_2 + q^{-2}\mathbf{c}_1\overline{\mathbf{c}}_1\\
\mathbf{h}_{13} &= \overline{\mathbf{a}}_1\mathbf{b}_1 - q^5\mathbf{a}_2\overline{\mathbf{b}}_2 + \overline{\mathbf{c}}_1\mathbf{d}_1 - q^3\mathbf{c}_2\overline{\mathbf{d}}_2\\
\mathbf{h}_{14} &= \overline{\mathbf{a}}_1\mathbf{b}_2 + q^4\mathbf{a}_2\overline{\mathbf{b}}_1 + \overline{\mathbf{c}}_1\mathbf{d}_2 + q^2\mathbf{c}_2\overline{\mathbf{d}}_1\\
\mathbf{h}_{23} &= \overline{\mathbf{a}}_2\mathbf{b}_1 + q^2\mathbf{a}_1\overline{\mathbf{b}}_2 + \overline{\mathbf{c}}_2\mathbf{d}_1 + \mathbf{c}_1\overline{\mathbf{d}}_2\\
\mathbf{h}_{24} &= \overline{\mathbf{a}}_2\mathbf{b}_2 - q\mathbf{a}_1\overline{\mathbf{b}}_1 + \overline{\mathbf{c}}_2\mathbf{d}_2 - q^{-1}\mathbf{c}_1\overline{\mathbf{d}}_1\\
\mathbf{h}_{33} &= \overline{\mathbf{b}}_1\mathbf{b}_1 + q^4\mathbf{b}_2\overline{\mathbf{b}}_2 + \overline{\mathbf{d}}_1\mathbf{d}_1 + q^2\mathbf{d}_2\overline{\mathbf{d}}_2\\
\mathbf{h}_{34} &= \overline{\mathbf{b}}_1\mathbf{b}_2 - q^3\mathbf{b}_2\overline{\mathbf{b}}_1 + \overline{\mathbf{d}}_1\mathbf{d}_2 - q\mathbf{d}_2\overline{\mathbf{d}}_1\\
\mathbf{h}_{44} &= \overline{\mathbf{b}}_2\mathbf{b}_2 + q^2\mathbf{b}_1\overline{\mathbf{b}}_1 + \overline{\mathbf{d}}_2\mathbf{d}_2 + \mathbf{d}_1\overline{\mathbf{d}}_1
\end{aligned}$$

with $\mathbf{h}_{ji} = \overline{\mathbf{h}}_{ij}$.
By using the commutation relations founded before, we have

$$\begin{aligned}
\mathbf{h}_{12} &= 0\,;\\
\mathbf{h}_{34} &= 0\,;\\
\overline{\mathbf{h}}_{23} &= q^{-2}\mathbf{h}_{14}\,;\\
\overline{\mathbf{h}}_{24} &= -\mathbf{h}_{13}\,;\\
\mathbf{h}_{44} &= \mathbf{h}_{33}\,;\\
\mathbf{h}_{11} &= q^2\mathbf{h}_{22} + (1-q^2)\mathbf{h}_{33}\,;\\
\mathbf{h}_{22} &= \mathbf{h}_{33} + q^2\overline{\mathbf{a}}_1\mathbf{a}_1 + q^6\overline{\mathbf{a}}_2\mathbf{a}_2 - q^6\overline{\mathbf{b}}_1\mathbf{b}_1 - q^2\overline{\mathbf{b}}_2\mathbf{b}_2 + q^{-2}\overline{\mathbf{c}}_1\mathbf{c}_1 +\\
&\quad q^2\overline{\mathbf{c}}_2\mathbf{c}_2 - q^2\overline{\mathbf{d}}_1\mathbf{d}_1 - q^{-2}\overline{\mathbf{d}}_2\mathbf{d}_2
\end{aligned}$$

Hence setting

$$\begin{aligned}
\mathbf{m} &:= \mathbf{h}_{22} - \mathbf{h}_{33} = q^2\overline{\mathbf{a}}_1\mathbf{a}_1 + q^6\overline{\mathbf{a}}_2\mathbf{a}_2 - q^6\overline{\mathbf{b}}_1\mathbf{b}_1 - q^2\overline{\mathbf{b}}_2\mathbf{b}_2 +\\
&\quad q^{-2}\overline{\mathbf{c}}_1\mathbf{c}_1 + q^2\overline{\mathbf{c}}_2\mathbf{c}_2 - q^2\overline{\mathbf{d}}_1\mathbf{d}_1 - q^{-2}\overline{\mathbf{d}}_2\mathbf{d}_2\\
\mathbf{n} &:= \mathbf{h}_{33} = q^4\overline{\mathbf{b}}_1\mathbf{b}_1 + q^6\overline{\mathbf{b}}_2\mathbf{b}_2 + \overline{\mathbf{d}}_1\mathbf{d}_1 + q^2\overline{\mathbf{d}}_2\mathbf{d}_2\\
\mathbf{g}_1 &:= \mathbf{h}_{13} = q^4\overline{\mathbf{a}}_1\mathbf{b}_1 - q^6\overline{\mathbf{b}}_2\mathbf{a}_2 + \overline{\mathbf{c}}_1\mathbf{d}_1 - q^2\overline{\mathbf{d}}_2\mathbf{c}_2\\
\mathbf{g}_2 &:= \mathbf{h}_{14} = q^4\overline{\mathbf{a}}_1\mathbf{b}_2 + q^8\overline{\mathbf{b}}_1\mathbf{a}_2 + \overline{\mathbf{c}}_1\mathbf{d}_2 + q^4\overline{\mathbf{d}}_1\mathbf{c}_2
\end{aligned}$$

we have

$$\mathbf{M}_q = \begin{pmatrix} q^2\mathbf{m}+\mathbf{n} & 0 & \mathbf{g}_1 & \mathbf{g}_2 \\ 0 & \mathbf{m}+\mathbf{n} & q^{-2}\overline{\mathbf{g}}_2 & -\overline{\mathbf{g}}_1 \\ \overline{\mathbf{g}}_1 & q^{-2}\mathbf{g}_2 & \mathbf{n} & 0 \\ \overline{\mathbf{g}}_2 & \mathbf{g}_1 & 0 & \mathbf{n} \end{pmatrix} \tag{4.84}$$

The surprising fact is that inspite of the fact that the algebra generated by the entries of B_q has commutations rules complicated, the resulting matrix M_q has a form which is similar to the classical ones (4.6) with two real parameters $\mathbf{m}$, $\mathbf{n}$ and two complex ones $\mathbf{g}_1$, $\mathbf{g}_2$ arranged in a quaternion. [8] This fact suggest that a description of the moduli space of charge 1 instantons as homogeneous space could be constructed also in this quantum case. Anyway it is important to notice that the entries of $\mathbf{M}$ fail to generate an algebra. The non-vanishing commutator between any two elements $\mathbf{m}$, $\mathbf{n}$, $\mathbf{g}_1$, $\mathbf{g}_2$, $\overline{\mathbf{g}}_1$, $\overline{\mathbf{g}}_2$ can not be expressed in terms of the product of other elements. This fact suggest that some further conditions should be imposed on the matrix A_q.

[8]Note that the central element ρ^2 can be written in terms of the elements of the algebra $A(S^4_q)$ and those of the matrix $M_q := A^*_q A_q$:

$$\rho^2 = mt + n + q(g_1\overline{a} + \overline{g}_1 a) + q^{-2}(\overline{g}_2 b + g_2\overline{b}) \ . \tag{4.85}$$

Here $m,\ n, ...$ denote the elements of the matrix M_q whose form is the same as that of $\mathbf{M}_q$ as well as the expression of its elements.
Also in this case, the element ρ^2 reduces to 1 when M_q becomes the identity matrix, i.e. $n = 1,\ m = g_1 = g_2 = 0$.

Appendix A

Commutation relations for the algebra $A(Sp_q(4,\mathbb{C}))$

In this appendix we list the commutation relations of the elements t_{ij} of the 4×4 defining matrix T for the algebra $A(Sp_q(4,\mathbb{C}))$. They are computed through RTT's equations (1.22) with the matrix R given in (1.17).

Furthermore at the end of this appendix, we write explicitly the sixteen equations $TS(T) = S(T)T = 1$.

Notations. The inscription $rtt_{i,j}$ beside each relation denotes that it has been obtained by setting $(RT_1T_2 - T_2T_1R)_{ij} = 0$, $\forall i, j = 1 \ldots 16$. Moreover $\lambda := q - q^{-1}$.

Commutation relations $t_{1,i}$ - $t_{1,j}$

$$
\begin{aligned}
rtt_{1,5} \qquad & t_{1,1}t_{1,2} = q\ t_{1,2}t_{1,1} \\
rtt_{1,9} \qquad & t_{1,1}t_{1,3} = q\ t_{1,3}t_{1,1} \\
rtt_{1,13} \qquad & t_{1,1}t_{1,4} = q^2\ t_{1,4}t_{1,1} \\
rtt_{1,10} \qquad & t_{1,2}t_{1,3} = q^2\ t_{1,3}t_{1,2} + \lambda t_{1,1}t_{1,4} \\
rtt_{1,14} \qquad & t_{1,2}t_{1,4} = q\ t_{1,4}t_{1,2} \\
rtt_{1,15} \qquad & t_{1,3}t_{1,4} = q\ t_{1,4}t_{1,3}
\end{aligned}
$$

Commutation relations $t_{1,i}$ - $t_{2,j}$

$$
\begin{aligned}
rtt_{2,1}\quad & t_{1,1}t_{2,1} = q\ t_{2,1}t_{1,1}\\
rtt_{2,2}\quad & t_{1,1}t_{2,2} = t_{2,2}t_{1,1} + (q-\frac{1}{q})t_{2,1}t_{1,2}\\
rtt_{2,3}\quad & t_{1,1}t_{2,3} = t_{2,3}t_{1,1} + (q-\frac{1}{q})t_{2,1}t_{1,3}\\
rtt_{2,4}\quad & t_{1,1}t_{2,4} = \frac{1}{q}t_{2,4}t_{1,1} - \frac{\lambda}{q}t_{2,3}t_{1,2} + \frac{\lambda}{q^3}t_{2,2}t_{1,3} + (\lambda+\frac{\lambda}{q^4})t_{2,1}t_{1,4}\\
rtt_{5,13}\quad & t_{1,1}t_{2,4} = (q-\frac{1}{q})t_{2,1}t_{1,4} + q\ t_{2,4}t_{1,1}\\
rtt_{2,5}\quad & t_{1,2}t_{2,1} = t_{2,1}t_{1,2}\\
rtt_{2,6}\quad & t_{1,2}t_{2,2} = q\ t_{2,2}t_{1,2}\\
rtt_{2,7}\quad & t_{1,2}t_{2,3} = t_{2,3}t_{1,2}\frac{1}{q} + t_{2,1}t_{1,4}(\frac{\lambda}{q^3}) + t_{2,2}t_{1,3}(\lambda+\frac{\lambda}{q^2})\\
rtt_{5,10}\quad & t_{1,3}t_{2,2}(q-\frac{1}{q}) + t_{2,3}t_{1,2} = t_{1,1}t_{2,4}(-\frac{\lambda}{q}) + t_{1,2}t_{2,3}\frac{1}{q}\\
rtt_{2,8}\quad & t_{1,2}t_{2,4} = t_{2,4}t_{1,2} + (q-\frac{1}{q})t_{2,2}t_{1,4}\\
rtt_{2,9}\quad & t_{1,3}t_{2,1} = t_{2,1}t_{1,3}\\
rtt_{2,10}\quad & q\ t_{1,3}t_{2,2} = -\lambda t_{2,1}t_{1,4} + t_{2,2}t_{1,3}\\
rtt_{5,7}\quad & t_{2,2}t_{1,3} = t_{1,3}t_{2,2}\frac{1}{q} + t_{1,1}t_{2,4}(\frac{\lambda}{q^3}) + t_{1,2}t_{2,3}(\frac{\lambda}{q^2})\\
rtt_{2,11}\quad & t_{1,3}t_{2,3} = q\ t_{2,3}t_{1,3}\\
rtt_{2,12}\quad & t_{1,3}t_{2,4} = t_{1,4}t_{2,3}(q-\frac{1}{q}) + t_{2,4}t_{1,3}\\
rtt_{2,13}\quad & q\ t_{1,4}t_{2,1} = t_{2,1}t_{1,4}\\
rtt_{5,4}\quad & q\ t_{2,1}t_{1,4} = -t_{1,3}t_{2,2} + (\frac{1}{q^2})t_{1,2}t_{2,3} + (\frac{1}{q^3})t_{1,1}t_{2,4}\\
rtt_{2,14}\quad & t_{1,4}t_{2,2} = t_{2,2}t_{1,4}\\
rtt_{2,15}\quad & t_{1,4}t_{2,3} = t_{2,3}t_{1,4}\\
rtt_{2,16}\quad & t_{1,4}t_{2,4} = q\ t_{2,4}t_{1,4}
\end{aligned}
$$

Commutation relations $t_{1,i}$ - $t_{3,j}$

$$
\begin{array}{rl}
rtt_{3,1} & t_{1,1}t_{3,1} = q\ t_{3,1}t_{1,1} \\
rtt_{3,2} & t_{1,1}t_{3,2} = t_{3,2}t_{1,1} + t_{3,1}t_{1,2}(q - \frac{1}{q}) \\
rtt_{3,3} & t_{1,1}t_{3,3} = t_{3,3}t_{1,1} + t_{3,1}t_{1,3}(q - \frac{1}{q}) \\
rtt_{3,4} & t_{1,1}t_{3,4} = t_{3,4}t_{1,1}\frac{1}{q} + t_{3,3}t_{1,2}(-\frac{\lambda}{q}) + t_{3,2}t_{1,3}(\frac{\lambda}{q^3}) + t_{3,1}t_{1,4}(\lambda + \frac{\lambda}{q^4}) \\
rtt_{9,13} & t_{3,4}t_{1,1} + t_{1,4}t_{3,1}(q - \frac{1}{q}) = t_{1,1}t_{3,4}\frac{1}{q} \\
rtt_{3,5} & t_{1,2}t_{3,1} = t_{3,1}t_{1,2} \\
rtt_{3,6} & t_{1,2}t_{3,2} = q\ t_{3,2}t_{1,2} \\
rtt_{3,7} & t_{1,2}t_{3,3} = t_{3,3}t_{1,2}\frac{1}{q} + t_{3,1}t_{1,4}(\frac{\lambda}{q^3}) + t_{3,2}t_{1,3}(\lambda + \frac{\lambda}{q^2}) \\
rtt_{9,10} & t_{3,3}t_{1,2} + t_{1,3}t_{3,2}(q - \frac{1}{q}) = t_{1,1}t_{3,4}(-\frac{\lambda}{q}) + t_{1,2}t_{3,3}\frac{1}{q} \\
rtt_{3,8} & t_{1,2}t_{3,4} = t_{3,4}t_{1,2} + t_{3,2}t_{1,4}(q - \frac{1}{q}) \\
rtt_{3,9} & t_{1,3}t_{3,1} = t_{3,1}t_{1,3} \\
rtt_{3,10} & q\ t_{1,3}t_{3,2} = -\lambda\ t_{3,1}t_{1,4} + t_{3,2}t_{1,3} \\
rtt_{9,7} & t_{3,2}t_{1,3} + t_{1,2}t_{3,3}(q - \frac{1}{q}) = t_{1,3}t_{3,2}\frac{1}{q} + t_{1,1}t_{3,4}(\frac{\lambda}{q^3}) + t_{1,2}t_{3,3}(\lambda + \frac{\lambda}{q^2}) \\
rtt_{3,11} & t_{1,3}t_{3,3} = q\ t_{3,3}t_{1,3} \\
rtt_{3,12} & t_{1,3}t_{3,4} = t_{3,3}t_{1,4}(q - \frac{1}{q}) + t_{3,4}t_{1,3} \\
rtt_{3,13} & q\ t_{1,4}t_{3,1} = t_{3,1}t_{1,4} \\
rtt_{9,4} & t_{3,1}t_{1,4} = -t_{1,3}t_{3,2} + t_{1,2}t_{3,3}(\frac{1}{q^2}) + t_{1,1}t_{3,4}(\frac{1}{q^3}) \\
rtt_{3,14} & t_{1,4}t_{3,2} = t_{3,2}t_{1,4} \\
rtt_{3,15} & t_{1,4}t_{3,3} = t_{3,3}t_{1,4} \\
rtt_{3,16} & t_{1,4}t_{3,4} = q\ t_{3,4}t_{1,4}
\end{array}
$$

Commutation relations $t_{1,i}$ - $t_{4,j}$ and $t_{2,i}$ - $t_{3,j}$

$$
\begin{aligned}
rtt_{4,1} \quad & t_{1,1}t_{4,1} = q^2 t_{4,1}t_{1,1} \\
rtt_{4,2} \quad & t_{1,1}t_{4,2} = q\ t_{4,2}t_{1,1} + (q - \frac{1}{q})t_{1,2}t_{4,1} \\
rtt_{13,1} \quad & t_{2,1}t_{3,1} = q^2 t_{3,1}t_{2,1} + \lambda t_{1,1}t_{4,1} \\
rtt_{4,6} \quad & t_{1,2}t_{4,2} = q^2 t_{4,2}t_{1,2} \\
rtt_{13,6} \quad & t_{2,2}t_{3,2} = q^2 t_{3,2}t_{2,2} + \lambda t_{1,2}t_{4,2} \\
rtt_{4,11} \quad & t_{1,3}t_{4,3} = q^2 t_{4,3}t_{1,3} \\
rtt_{13,11} \quad & t_{2,3}t_{3,3} = q^2 t_{3,3}t_{2,3} + \lambda t_{1,3}t_{4,3} \\
rtt_{4,16} \quad & t_{1,4}t_{4,4} = q^2 t_{4,4}t_{1,4} \\
rtt_{13,16} \quad & t_{2,4}t_{3,4} = q^2 t_{3,4}t_{2,4} + \lambda t_{1,4}t_{4,4} \\
rtt_{13,5} \quad & -t_{3,2}t_{2,1} + t_{2,2}t_{3,1}(\frac{1}{q^2}) + t_{1,2}t_{4,1}(\frac{1}{q^3}) = q t_{4,2}t_{1,1} \\
rtt_{4,5} \quad & t_{1,2}t_{4,1} = q\ t_{4,1}t_{1,2} \\
rtt_{13,2} \quad & -t_{4,1}t_{1,2} + t_{3,1}t_{2,2}(-\frac{1}{q}) + t_{2,1}t_{3,2}(\frac{1}{q^3}) + t_{1,1}t_{4,2}(\frac{1}{q^4}) = 0 \\
rtt_{7,2} \quad & t_{1,1}t_{4,2}(-\frac{\lambda}{q}) + t_{2,1}t_{3,2}\frac{1}{q} = t_{3,2}t_{2,1} + t_{3,1}t_{2,2}(q - \frac{1}{q}) \\
rtt_{10,5} \quad & t_{3,2}t_{2,1}\frac{1}{q} + t_{1,2}t_{4,1}(\frac{\lambda}{q^3}) + t_{2,2}t_{3,1}(\lambda + \frac{\lambda}{q^2}) = t_{2,1}t_{3,2} \\
rtt_{7,5} \quad & t_{2,2}t_{3,1} = q\ t_{3,1}t_{2,2} + \lambda\ t_{1,2}t_{4,1} \\
rtt_{10,2} \quad & -t_{3,1}t_{2,2} + t_{1,1}t_{4,2}(\frac{1}{q^3}) + t_{2,1}t_{3,2}(\frac{1}{q^2}) = t_{1,2}t_{4,1} \\
rtt_{4,3} \quad & t_{1,1}t_{4,3} = q\ t_{4,3}t_{1,1} + \lambda\ t_{1,3}t_{4,1} \\
rtt_{13,9} \quad & t_{3,3}t_{2,1}(-\frac{1}{q}) + t_{2,3}t_{3,1}(\frac{1}{q^3}) + t_{1,3}t_{4,1}(\frac{1}{q^4}) = t_{4,3}t_{1,1} \\
rtt_{4,9} \quad & t_{1,3}t_{4,1} = q\ t_{4,1}t_{1,3} \\
rtt_{13,3} \quad & q t_{4,1}t_{1,3} - t_{3,1}t_{2,3} = t_{2,1}t_{3,3}(\frac{1}{q^2}) + t_{1,1}t_{4,3}(\frac{1}{q^3}) \\
rtt_{7,3} \quad & t_{2,1}t_{3,3} = q\ t_{3,3}t_{2,1} + q\ \lambda\ t_{3,1}t_{2,3} + \lambda\ t_{1,1}t_{4,3} \\
rtt_{10,9} \quad & t_{3,3}t_{2,1}\frac{1}{q} + t_{1,3}t_{4,1}(\frac{\lambda}{q^3}) + t_{2,3}t_{3,1}(\lambda + \frac{\lambda}{q^2}) = t_{2,1}t_{3,3} \\
rtt_{7,9} \quad & t_{2,3}t_{3,1} = q\ t_{3,1}t_{2,3} + \lambda\ t_{1,3}t_{4,1} \\
rtt_{10,3} \quad & -t_{3,1}t_{2,3} + t_{1,1}t_{4,3}(\frac{1}{q^3}) + t_{2,1}t_{3,3}(\frac{1}{q^2}) = t_{1,3}t_{4,1}
\end{aligned}
$$

$$
\begin{aligned}
&rtt_{4,4} && t_{1,1}t_{4,4} = t_{4,4}t_{1,1} - \lambda t_{4,3}t_{1,2} + t_{4,2}t_{1,3}(\frac{\lambda}{q^2}) + t_{4,1}t_{1,4}q\ (\lambda + \frac{\lambda}{q^4})\\
&rtt_{13,13} && t_{1,1}t_{4,4} = t_{4,4}t_{1,1} - \lambda\ t_{3,4}t_{2,1} + t_{2,4}t_{3,1}(\frac{\lambda}{q^2}) + t_{1,4}t_{4,1}q\ (\lambda + \frac{\lambda}{q^4})\\
&rtt_{4,7} && t_{1,2}t_{4,3} = t_{4,3}t_{1,2} + t_{4,1}t_{1,4}(\frac{\lambda}{q^2}) + t_{4,2}t_{1,3}q\ (\lambda + \frac{\lambda}{q^2})\\
&rtt_{13,10} && t_{1,2}t_{4,3} = t_{4,3}t_{1,2} - \lambda\ t_{3,3}t_{2,2} + t_{2,3}t_{3,2}(\frac{\lambda}{q^2}) + t_{1,3}t_{4,2}q\ (\lambda + \frac{\lambda}{q^4}) + \lambda\ t_{1,1}t_{4,4}\\
&rtt_{4,10} && t_{1,3}t_{4,2} = t_{4,2}t_{1,3} - \lambda\ t_{4,1}t_{1,4}\\
&rtt_{13,7} && t_{4,2}t_{1,3} - \lambda\ t_{3,2}t_{2,3} + t_{2,2}t_{3,3}(\frac{\lambda}{q^2}) +\\
& && t_{1,2}t_{4,3}(\frac{\lambda}{q^3}) = t_{1,3}t_{4,2} + t_{1,1}t_{4,4}(\frac{\lambda}{q^2}) + t_{1,2}t_{4,3}(\frac{\lambda}{q})\\
&rtt_{4,13} && t_{1,4}t_{4,1} = t_{4,1}t_{1,4}\\
&rtt_{13,4} && -t_{3,1}t_{2,4} + t_{2,1}t_{3,4}(\frac{1}{q^2}) = -t_{1,3}t_{4,2} + t_{1,2}t_{4,3}(\frac{1}{q^2})\\
&rtt_{7,4} && -\lambda\ t_{1,1}t_{4,4} + t_{2,1}t_{3,4} = t_{3,4}t_{2,1} - \lambda\ t_{3,3}t_{2,2} + t_{3,2}t_{2,3}(\frac{\lambda}{q^2}) + t_{3,1}t_{2,4}q\ (\lambda + \frac{\lambda}{q^4})\\
&rtt_{10,13} && t_{2,1}t_{3,4} = t_{3,4}t_{2,1} + t_{1,4}t_{4,1}(\frac{\lambda}{q^2}) + t_{2,4}t_{3,1}q\ (\lambda + \frac{\lambda}{q^2})\\
&rtt_{7,7} && -\lambda\ t_{1,2}t_{4,3} + t_{2,2}t_{3,3} = t_{3,3}t_{2,2} + t_{3,1}t_{2,4}(\frac{\lambda}{q^2}) + t_{3,2}t_{2,3}q\ (\lambda + \frac{\lambda}{q^2})\\
&rtt_{10,10} && t_{2,2}t_{3,3} = t_{3,3}t_{2,2} + t_{1,3}t_{4,2}(\frac{\lambda}{q^2}) + t_{2,3}t_{3,2}q\ (\lambda + \frac{\lambda}{q^2}) + \lambda\ t_{2,1}t_{3,4}\\
&rtt_{7,10} && t_{2,3}t_{3,2} = t_{3,2}t_{2,3} + \lambda\ t_{3,1}t_{2,4} + \lambda\ t_{1,3}t_{4,2}\\
&rtt_{10,7} && t_{2,3}t_{3,2} = t_{3,2}t_{2,3} + t_{1,2}t_{4,3}(\frac{\lambda}{q^2}) - t_{2,1}t_{3,4}(\frac{\lambda}{q^2})\\
&rtt_{7,13} && t_{2,4}t_{3,1} = t_{3,1}t_{2,4} + \lambda t_{1,4}t_{4,1}\\
&rtt_{10,4} && t_{3,1}t_{2,4} + t_{1,1}t_{4,4}(\frac{\lambda}{q^2}) + t_{2,1}t_{3,4}(\frac{\lambda}{q}) = t_{2,4}t_{3,1} - \lambda\ t_{2,3}t_{3,2}\\
& && t_{2,2}t_{3,3}(\frac{\lambda}{q^2}) + t_{2,1}t_{3,4}(\frac{\lambda}{q^3})
\end{aligned}
$$

$$
\begin{aligned}
&rtt_{4,8} && t_{1,2}t_{4,4} = q\ t_{4,4}t_{1,2} + \lambda\ t_{1,4}t_{4,2}\\
&rtt_{13,14} && -t_{4,4}t_{1,2} + t_{3,4}t_{2,2}(-\frac{1}{q}) + t_{2,4}t_{3,2}(\frac{1}{q^3}) + t_{1,4}t_{4,2}(\frac{1}{q^4}) = 0\\
&rtt_{4,14} && t_{1,4}t_{4,2} = q\ t_{4,2}t_{1,4}\\
&rtt_{13,8} && -t_{4,2}t_{1,4} + t_{3,2}t_{2,4}(-\frac{1}{q}) + t_{2,2}t_{3,4}(\frac{1}{q^3}) + t_{1,2}t_{4,4}(\frac{1}{q^4}) = 0
\end{aligned}
$$

$$
\begin{aligned}
rtt_{7,8} \quad & t_{1,2}t_{4,4}(-\frac{\lambda}{q}) + t_{2,2}t_{3,4}\frac{1}{q} = t_{3,4}t_{2,2} + t_{3,2}t_{2,4}(q - \frac{1}{q}) \\
rtt_{10,14} \quad & t_{2,2}t_{3,4} = t_{3,4}t_{2,2}\frac{1}{q} + t_{1,4}t_{4,2}(\frac{\lambda}{q^3}) + t_{2,4}t_{3,2}(\lambda + \frac{\lambda}{q^2}) \\
rtt_{7,14} \quad & t_{2,4}t_{3,2} = q\ t_{3,2}t_{2,4} + \lambda\ t_{1,4}t_{4,2} \\
rtt_{10,8} \quad & -t_{3,2}t_{2,4} + t_{1,2}t_{4,4}(\frac{1}{q^3}) + t_{2,2}t_{3,4}(\frac{1}{q^2}) = t_{1,4}t_{4,2} \\
rtt_{4,12} \quad & t_{1,3}t_{4,4} = q\ t_{4,4}t_{1,3} + \lambda\ t_{1,4}t_{4,3} \\
rtt_{13,15} \quad & -t_{4,4}t_{1,3} + t_{3,4}t_{2,3}(-\frac{1}{q}) + t_{2,4}t_{3,3}(\frac{1}{q^3}) + t_{1,4}t_{4,3}(\frac{1}{q^4}) = 0 \\
rtt_{4,15} \quad & t_{1,4}t_{4,3} = q\ t_{4,3}t_{1,4} \\
rtt_{13,12} \quad & -t_{4,3}t_{1,4} + t_{3,3}t_{2,4}(-\frac{1}{q}) + t_{2,3}t_{3,4}(\frac{1}{q^3}) + t_{1,3}t_{4,4}(\frac{1}{q^4}) = 0 \\
rtt_{7,12} \quad & t_{1,3}t_{4,4}(-\frac{\lambda}{q}) + t_{2,3}t_{3,4}\frac{1}{q} = t_{3,3}t_{2,4}(q - \frac{1}{q}) + t_{3,4}t_{2,3} \\
rtt_{10,15} \quad & t_{3,4}t_{2,3}\frac{1}{q} + t_{1,4}t_{4,3}(\frac{\lambda}{q^3}) + t_{2,4}t_{3,3}(\lambda + \frac{\lambda}{q^2}) = t_{2,3}t_{3,4} \\
rtt_{7,15} \quad & t_{2,4}t_{3,3} = q\ t_{3,3}t_{2,4} + \lambda\ t_{1,4}t_{4,3} \\
rtt_{10,12} \quad & -t_{3,3}t_{2,4} + t_{1,3}t_{4,4}(\frac{1}{q^3}) + t_{2,3}t_{3,4}(\frac{1}{q^2}) = t_{1,4}t_{4,3}
\end{aligned}
$$

Commutation relations $t_{2,i}$ - $t_{2,j}$

$$
\begin{aligned}
rtt_{6,5} \quad & t_{2,1}t_{2,2} = q\ t_{2,2}t_{2,1} \\
rtt_{6,9} \quad & t_{2,1}t_{2,3} = q\ t_{2,3}t_{2,1} \\
rtt_{6,13} \quad & t_{2,1}t_{2,4} = q^2 t_{2,4}t_{2,1} \\
rtt_{6,10} \quad & t_{2,2}t_{2,3} = q^2 t_{2,3}t_{2,2} + \lambda\ t_{2,1}t_{2,4} \\
rtt_{6,14} \quad & t_{2,2}t_{2,4} = q\ t_{2,4}t_{2,2} \\
rtt_{6,15} \quad & t_{2,3}t_{2,4} = q\ t_{2,4}t_{2,3}
\end{aligned}
$$

Commutation relations $t_{2,i}$ - $t_{4,j}$

$$
\begin{array}{rl}
rtt_{8,1} & t_{2,1}t_{4,1} = qt_{4,1}t_{2,1} \\
rtt_{8,2} & t_{2,1}t_{4,2} = t_{4,2}t_{2,1} + t_{4,1}t_{2,2}(q - \frac{1}{q}) \\
rtt_{8,3} & t_{2,1}t_{4,3} = t_{4,3}t_{2,1} + t_{4,1}t_{2,3}(q - \frac{1}{q}) \\
rtt_{8,4} & t_{2,1}t_{4,4} = t_{4,4}t_{2,1}\frac{1}{q} + t_{4,3}t_{2,2}(-\frac{\lambda}{q}) + t_{4,2}t_{2,3}(\frac{\lambda}{q^3}) + t_{4,1}t_{2,4}(\lambda + \frac{\lambda}{q^4}) \\
rtt_{14,13} & t_{4,4}t_{2,1} + t_{2,4}t_{4,1}(q - \frac{1}{q}) = t_{2,1}t_{4,4}\frac{1}{q} \\
rtt_{8,5} & t_{2,2}t_{4,1} = t_{4,1}t_{2,2} \\
rtt_{8,6} & t_{2,2}t_{4,2} = t_{4,2}t_{2,2}q \\
rtt_{8,7} & t_{2,2}t_{4,3} = t_{4,3}t_{2,2}\frac{1}{q} + t_{4,1}t_{2,4}(\frac{\lambda}{q^3}) + t_{4,2}t_{2,3}(\lambda + \frac{\lambda}{q^2}) \\
rtt_{14,10} & t_{4,3}t_{2,2} + t_{2,3}t_{4,2}(q - \frac{1}{q}) = t_{2,1}t_{4,4}(-\frac{\lambda}{q}) + t_{2,2}t_{4,3}\frac{1}{q} \\
rtt_{8,8} & t_{2,2}t_{4,4} = t_{4,4}t_{2,2} + t_{4,2}t_{2,4}(q - \frac{1}{q}) \\
rtt_{8,9} & t_{2,3}t_{4,1} = t_{4,1}t_{2,3} \\
rtt_{8,10} & t_{2,3}t_{4,2} = t_{4,1}t_{2,4}(-\frac{\lambda}{q}) + t_{4,2}t_{2,3}\frac{1}{q} \\
rtt_{14,7} & t_{4,2}t_{2,3} = t_{2,3}t_{4,2}\frac{1}{q} + t_{2,1}t_{4,4}(\frac{\lambda}{q^3}) + t_{2,2}t_{4,3}(\frac{\lambda}{q^2}) \\
rtt_{8,11} & t_{2,3}t_{4,3} = t_{4,3}t_{2,3}q \\
rtt_{8,12} & t_{2,3}t_{4,4} = t_{4,3}t_{2,4}(q - \frac{1}{q}) + t_{4,4}t_{2,3} \\
rtt_{8,13} & t_{2,4}t_{4,1} = t_{4,1}t_{2,4}\frac{1}{q} \\
rtt_{14,4} & t_{4,1}t_{2,4} = t_{2,3}t_{4,2}(-\frac{1}{q}) + t_{2,2}t_{4,3}(\frac{1}{q^3}) + t_{2,1}t_{4,4}(\frac{1}{q^4}) \\
rtt_{8,14} & t_{2,4}t_{4,2} = t_{4,2}t_{2,4} \\
rtt_{8,15} & t_{2,4}t_{4,3} = t_{4,3}t_{2,4} \\
rtt_{8,16} & t_{2,4}t_{4,4} = qt_{4,4}t_{2,4}
\end{array}
$$

Commutation relations $t_{3,i}$ - $t_{3,j}$

$$
\begin{array}{rl}
rtt_{11,5} & t_{3,1}t_{3,2} = q\ t_{3,2}t_{3,1} \\
rtt_{11,9} & t_{3,1}t_{3,3} = q\ t_{3,3}t_{3,1} \\
rtt_{11,13} & t_{3,1}t_{3,4} = q^2 t_{3,4}t_{3,1} \\
rtt_{11,10} & t_{3,2}t_{3,3} = q^2 t_{3,3}t_{3,2} + \lambda t_{3,1}t_{3,4} \\
rtt_{11,14} & t_{3,2}t_{3,4} = q\ t_{3,4}t_{3,2} \\
rtt_{11,15} & t_{3,3}t_{3,4} = q\ t_{3,4}t_{3,3}
\end{array}
$$

Commutation relations $t_{3,i}$ - $t_{4,j}$

$$
\begin{array}{rl}
rtt_{12,1} & t_{3,1}t_{4,1} = qt_{4,1}t_{3,1} \\
rtt_{12,2} & t_{3,1}t_{4,2} = t_{4,2}t_{3,1} + t_{4,1}t_{3,2}(q - \frac{1}{q}) \\
rtt_{12,3} & t_{3,1}t_{4,3} = t_{4,3}t_{3,1} + t_{4,1}t_{3,3}(q - \frac{1}{q}) \\
rtt_{12,4} & t_{4,4}t_{3,1} = -\frac{1}{q}t_{4,3}t_{3,2} + \frac{1}{q^3}t_{4,2}t_{3,3} + \frac{1}{q^4}t_{4,1}t_{3,4} \\
rtt_{15,13} & t_{3,1}t_{4,4} = qt_{4,4}t_{3,1} + \lambda t_{4,1}t_{3,4} \\
rtt_{12,5} & t_{3,2}t_{4,1} = t_{4,1}t_{3,2} \\
rtt_{12,6} & t_{3,2}t_{4,2} = t_{4,2}t_{3,2}q \\
rtt_{15,10} & t_{3,2}t_{4,3} = q\ t_{4,3}t_{3,2} + q\ \lambda\ t_{3,3}t_{4,2} + \lambda t_{3,1}t_{4,4} \\
rtt_{12,8} & t_{3,2}t_{4,4} = t_{4,4}t_{3,2} + t_{4,2}t_{3,4}(q - \frac{1}{q}) \\
rtt_{12,9} & t_{3,3}t_{4,1} = t_{4,1}t_{3,3} \\
rtt_{12,10} & t_{4,2}t_{3,3} = qt_{3,3}t_{4,2} + \lambda q t_{3,4}t_{4,1} \\
rtt_{12,11} & t_{3,3}t_{4,3} = t_{4,3}t_{3,3}q \\
rtt_{12,12} & t_{3,3}t_{4,4} = t_{4,3}t_{3,4}(q - \frac{1}{q}) + t_{4,4}t_{3,3} \\
rtt_{12,13} & t_{4,1}t_{3,4} = qt_{3,4}t_{4,1} \\
rtt_{15,4} & t_{3,4}t_{4,1} = t_{3,3}t_{4,2}(-\frac{1}{q}) + t_{3,1}t_{4,4}(\frac{1}{q^4}) + t_{3,2}t_{4,3}\frac{1}{q^3} \\
rtt_{12,7} & t_{3,2}t_{4,3} = \frac{1}{q}t_{4,3}t_{3,2} + \frac{\lambda}{q^3}t_{4,1}t_{3,4} + (\lambda + \frac{\lambda}{q^2})t_{4,2}t_{3,3} \\
rtt_{12,14} & t_{3,4}t_{4,2} = t_{4,2}t_{3,4} \\
rtt_{12,15} & t_{3,4}t_{4,3} = t_{4,3}t_{3,4} \\
rtt_{12,16} & t_{3,4}t_{4,4} = qt_{4,4}t_{3,4}
\end{array}
$$

Commutation relations $t_{4,i}$ - $t_{4,j}$

$$\begin{aligned}
rtt_{16,5} \qquad & t_{4,1}t_{4,2} = q\ t_{4,2}t_{4,1} \\
rtt_{16,9} \qquad & t_{4,1}t_{4,3} = q\ t_{4,3}t_{4,1} \\
rtt_{16,13} \qquad & t_{4,1}t_{4,4} = q^2 t_{4,4}t_{4,1} \\
rtt_{16,10} \qquad & t_{4,2}t_{4,3} = q^2 t_{4,3}t_{4,2} + \lambda t_{4,1}t_{4,4} \\
rtt_{16,14} \qquad & t_{4,2}t_{4,4} = q\ t_{4,4}t_{4,2} \\
rtt_{16,15} \qquad & t_{4,3}t_{4,4} = q\ t_{4,4}t_{4,3}
\end{aligned}$$

We conclude this section by writing explicitly the conditions for $S(T)$ to be the antipode of $A(Sp_q(4,\mathbb{C}))$. However we can notice that these relations are already contained in (or can be deduced from) the above ones.

$$\begin{aligned}
& -q^4 t_{14}t_{41} - q^3 t_{13}t_{42} + q t_{12}t_{43} + t_{11}t_{44} = 1 \ ; \\
& -q^3 t_{14}t_{31} - q^2 t_{13}t_{32} + t_{12}t_{33} + q^{-1} t_{11}t_{34} = 0 \ ; \\
& q t_{14}t_{21} + t_{13}t_{22} - q^{-2} t_{12}t_{23} - q^{-3} t_{11}t_{24} = 0 \ ; \\
& t_{14}t_{11} + q^{-1} t_{13}t_{12} - q^{-3} t_{12}t_{13} - q^{-4} t_{11}t_{14} = 0 \ ; \\
& -q^4 t_{24}t_{41} - q^3 t_{23}t_{42} + q t_{22}t_{43} + t_{21}t_{44} = 0 \ ; \\
& q^3 t_{24}t_{31} - q^2 t_{23}t_{32} + t_{22}t_{33} + q^{-1} t_{21}t_{34} = 1 \ ; \\
& q t_{24}t_{21} + t_{23}t_{22} - q^{-2} t_{22}t_{23} - q^{-3} t_{21}t_{24} = 0 \ ; \\
& t_{24}t_{11} + q^{-1} t_{23}t_{12} - q^{-3} t_{22}t_{13} - q^{-4} t_{21}t_{14} = 0 \ ; \\
& -q^4 t_{34}t_{41} - q^3 t_{33}t_{42} + q t_{32}t_{43} + t_{31}t_{44} = 0 \ ; \\
& -q^3 t_{34}t_{31} - q^2 t_{33}t_{32} + t_{32}t_{33} + q^{-1} t_{31}t_{34} = 0 \ ; \\
& q t_{34}t_{21} + t_{33}t_{22} - q^{-2} t_{32}t_{23} - q^{-3} t_{31}t_{24} = 1 \ ; \\
& t_{34}t_{11} + q^{-1} t_{33}t_{12} - q^{-3} t_{32}t_{13} - q^{-4} t_{31}t_{14} = 0 \ ; \\
& -q^4 t_{44}t_{41} - q^3 t_{43}t_{42} + q t_{42}t_{43} + t_{41}t_{44} = 0 \ ; \\
& -q^3 t_{44}t_{31} - q^2 t_{43}t_{32} + t_{42}t_{33} + q^{-1} t_{41}t_{34} = 0 \ ; \\
& q t_{44}t_{21} + t_{43}t_{22} - q^{-2} t_{42}t_{23} - q^{-3} t_{41}t_{24} = 0 \ ; \\
& t_{44}t_{11} + q^{-1} t_{43}t_{12} - q^{-3} t_{42}t_{13} - q^{-4} t_{41}t_{14} = 1 \ .
\end{aligned}$$

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