**International School for Advanced Studies (ISAS)**

# Holographic Weyl Anomaly Matching & Black Holes in 3D Higher Spin Theories

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# Abstract

This thesis consists of two parts. The first part addresses the issue of conformal anomaly matching from the holographic perspective and is reported in chapter 2. The second part aims to the study of higher spin generalisations of black holes in 3D and is reported in chapters 3 and 4.

In chapter 2 we discuss various issues related to the understanding of the conformal anomaly matching in CFT from the dual holographic viewpoint. First, we act with a PBH diffeomorphism on a generic 5D RG flow geometry and show that the corresponding on-shell bulk action reproduces the Wess-Zumino term for the dilaton of broken conformal symmetry, with the expected coefficient $a_{UV} - a_{IR}$. Then, we consider a specific 3D example of RG flow whose UV asymptotics is normalizable and admits a 6D lifting. We promote a modulus $\rho$ appearing in the geometry to a function of boundary coordinates. In a 6D description $\rho$ is the scale of an $SU(2)$ instanton. We determine the smooth deformed background up to second order in the space-time derivatives of $\rho$ and find that the 3D on-shell action reproduces a boundary kinetic term for the massless field $\tau = \log \rho$ with the correct coefficient $\delta c = c_{UV} - c_{IR}$. We further analyze the linearized fluctuations around the deformed background geometry and compute the one-point functions $< T_{\mu\nu} >$ and show that they are reproduced by a Liouville-type action for the massless scalar $\tau$, with background charge due to the coupling to the 2D curvature $R^{(2)}$. The resulting central charge matches $\delta c$. We give an interpretation of this action in terms of the $(4, 0)$ SCFT of the D1-D5 system in type I theory.

The content of this chapter has been reported in arXiv:1307.3784v3 [hep-th] (JHEP 1311 (2013) 044).

In chapter 3 we address some issues of recent interest, related to the asymptotic symmetry algebra of higher spin black holes in $sl(3,\mathbb{R}) \times sl(3,\mathbb{R})$ Chern Simons (CS) formulation. In our analysis we resort to both, Regge-Teitelboim and Dirac bracket methods and when possible identify them. We compute explicitly the Dirac bracket algebra on the phase space, in both, diagonal and principal embeddings. The result for principal embedding is shown to be isomorphic to $W_3^{(2)} \times W_3^{(2)}$. Our revision complements the viewpoints of [1, 2]. The content of this chapter has been reported in arXiv:1407.8241 [hep-th].

In chapter 4 we present a class of 3D black holes based on flat connections which are polynomials in the BTZ $hs(\lambda) \times hs(\lambda)$-valued connection. We solve analytically the fluctuation equations of matter in their background and find the spectrum of their Quasi Normal Modes. We analyze the bulk to boundary two-point functions. We also relate our results and those arising in other backgrounds discussed recently in the literature on the subject. The content of this chapter has been reported in arXiv:1407.5203v2 [hep-th](submitted to JHEP).

# Contents

# 1

# Introduction

The proof of the a-theorem in D=4 CFT and the alternative proof of c-theorem in D=2 CFT [3], given in [4, 5], inspired by the anomaly matching argument of [6], has prompted several groups to address the issue of a description of the corresponding mechanism on the dual gravity side [7, 8]. While a sort of $a(c)$-"theorem" is known to hold for RG-flows in the context of gauged supergravity [9, 10], as a consequence of the positive energy condition, which guarantees the monotonic decrease of the $a(c)$ function from UV to IR [11][1], one of the aims of the renewed interest on the topic has been somewhat different: the field-theoretic anomaly matching argument implies the existence of an IR effective action for the conformal mode, which in the case of spontaneous breaking of conformal invariance is the physical dilaton, whereas for a RG flow due to relevant perturbations is a Weyl mode of the classical background metric ("spurion"). In any case, upon combined Weyl shifting of the conformal mode and the background metric, the effective action reproduces the conformal anomaly of amount $a_{UV} - a_{IR}$ ($c_{UV} - c_{IR}$), therefore matching the full conformal anomaly of the UV CFT. This effective action therefore is nothing but the Wess-Zumino local term corresponding to broken conformal invariance. So, one obvious question is how to obtain the correct Wess-Zumino term for the dilaton (or spurion) from the dual gravity side. One of the purposes of our study is to discuss this issue offering a different approach from those mentioned above. In known examples

[1] Different approaches have been discussed lately [12, 13, 14].

of 4D RG flows corresponding to spontaneous breaking of conformal invariance on the Coulomb branch of $N = 4$ Yang-Mills theory [15, 16, 17, 18], indeed the existence of a massless scalar identifiable with the CFT's dilaton (see also [8, 19]) has been shown. However, the background geometry is singular in the IR, so that one does not have a full control on the geometry all along the RG flow. It would be therefore desirable to have an explicit example which is completely smooth from UV to IR, and indeed we will discuss such an example in the $AdS_3/CFT_2$ context.

Before going to analyze in detail a specific example, we will generally ask what is the bulk mode representing the spurion field of the CFT. The spurion couples to field theory operators according to their scale dimension and transforms under conformal transformations by Weyl shifts. These properties point towards an identification of this mode with the PBH (Penrose-Brown-Henneaux)-diffeomorphism, which are bulk diffeomorphisms inducing Weyl transformations on the boundary metric, parametrized by the spurion field $\tau$. This identification has been first adopted in [20, 21] to study holographic conformal anomalies and also recently in [7, 8, 22] to address the anomaly matching issue from the gravity side.

As will be shown in section 2.1, for the case of a generic 4D RG flow, by looking at how PBH diffeomorphisms act on the background geometry at the required order in a derivative expansion of $\tau$, we will compute the regularized bulk action for the PBH-transformed geometry and show that it contains a finite contribution proportional to the Wess-Zumino term for $\tau$, with proportionality constant given by $a_{UV} - a_{IR}$.

In the case where conformal invariance is spontaneously broken, when $D > 2$, one expects to have a physical massless scalar on the boundary CFT, the dilaton, which is the Goldstone boson associated to the broken conformal invariance. As stressed in [8], one expects on general grounds that the dilaton should be associated to a normalizable bulk zero mode, and therefore cannot be identified with the PBH spurion, which is related to a non normalizable deformation of the background geometry.

In section 2.2 we will follow a different approach to the problem: starting from an explicit, smooth RG flow geometry in 3D gauged supergravity [23], we will promote some moduli appearing in the solution to space-time dependent fields. More specifically, we will identify a modulus which, upon lifting the solution to 6D, is in fact the scale $\rho$ of an $SU(2)$ Yang-Mills instanton. We will then find the new solution of the supergravity equations of motion up to second order in the space-time derivatives of $\rho$. We will find that demanding regularity of the deformed geometry forces to switch on a source for a scalar field. We will then compute the on-shell bulk action and verify that this reproduces the correct kinetic term boundary action for the massless scalar field $\tau = \log\rho$, with coefficient $\delta c = c_{UV} - c_{IR}$ [1]. The computation of the CFT effective action is done up to second order in derivative expansion. Namely, only the leading term in the full IR effective action is computed, and our procedure is similar to the one followed in [24] for the derivation of the equations of hydrodynamics from AdS/CFT. In section 2.3 we reconsider the problem from a 6D viewpoint [25]: the 6D description has the advantage of making more transparent the $10D$ origin of our geometry in terms of a configuration of $D1$ and $D5$ branes in type I string theory [2].

Here we take one step further: not only we determine the deformed background involving two derivatives of $\rho$ but also solve the linearized equations of motion around it to determine the on-shell fluctuations. This allows us to compute one-point functions of the boundary stress-energy tensor $< T_{\mu\nu} >$, from which we deduce that the boundary action for $\tau$ is precisely the 2D Wess-Zumino term of broken conformal invariance, i.e. a massless scalar coupled to the 2D curvature $R^{(2)}$ and overall coefficient $\delta c$. An obvious question is what the field $\tau$ and its action represent on the dual CFT. We will argue that the interpretation of the effective field theory for $\tau$ is a manifestation of the mechanism studied in [28], describing the separation of a $D1/D5$ sub-system from a given $D1/D5$ system from the viewpoint of the $(4,4)$ boundary CFT. There, from the Higgs branch, one obtains an action for the radial component of vector multiplet scalars which couple to the hypermultiplets, in the form of a 2D scalar field with background

[1] This new field $\tau$ should not be confused with the spurion fields discussed in sections 2.1 and 2.2. We hope not to confuse with this abuse of notation.

[2] As it will be clear in section 2.3, the background geometry involves to a superposition of $D5$ branes and a gauge 5-brane [26] supported by the $SU(2)$ instanton. The latter is interpreted as a $D5$ branes in the small instanton limit $\rho \to 0$ [27].

charge, such that its conformal anomaly compensates the variation of the central charge due to the emission of the sub-system. In our case we will see that in the limit $\rho \to \infty$, the gauge five-brane decouples, whereas in the limit $\rho \to 0$ it becomes a D5-brane: these two limits correspond in turn to the IR and UV regions of the RG flow, respectively. The effective action for $\tau = \log \rho$ accounts, in the limit of large charges, precisely for the $\delta c$ from the UV to the IR in the RG flow. We will give an interpretation of the action for $\tau$ in terms of the effective field theory of the D1-D5 system in presence of D9 branes in type I theory.

We stress that the above procedure, although, for technical reasons, implemented explicitly in the context of an $AdS_3/CFT_2$ example, we believe should produce the correct Wess-Zumino dilaton effective action even in the $D = 4$ case, had we an explicit, analytic and smooth RG flow triggered by a v.e.v. in the UV. Of course, in this case we should have pushed the study of equations of motion up to fourth order in the derivative expansion.

In the second part of this thesis we attempt to study distinctive features of black holes in the context of 3D higher spin theories. We start by trying to provide a hint for classification of charges in the case of $sl(3, \mathbb{R})$ Chern Simon formulation. Thereafter we proceed to present and study a class of solutions that we argue should be interpreted as black holes.

Higher spin (HS) theories [29, 30, 31, 32] in 3D, have been of great interest recently and specifically, the study of higher spin generalisations of black holes in the Chern-Simons formulation has been one of the most active lines of research [1, 2, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43].

The 3D Chern-Simons (CS) is a theory of pure gauge degrees of freedom. However, in backgrounds with conformal boundaries, like $AdS_3$, it is not a trivial theory. To have a well defined variational principle, boundary terms should be added to the original action. These boundary terms are designed to make the total action stationary under motion in a given region of the moduli space of flat connections. The selection of that region, a.k.a. imposition of boundary conditions, defines the domain of the moduli space to work with: the phase space. Motion outside of the phase space does not leave the action invariant and it is incompatible with the variational principle. The corresponding gauge transformations are dubbed non residual. Motion inside the phase space instead, leaves the total action invariant by construction, then it is admissible. The corresponding gauge transformations are called residual and they emerge as global symmetry transformations. It is very important to stress that throughout chapter 3 we will use the term phase space in the sense stated above, and not to denote all possible initial data in a given Cauchy surface, as it is usually done.

In the last few years some families of phase spaces have been argued to contain generalisations of the BTZ black hole [44]: They are called higher spin black holes. See [2, 33]. Each one of these families is labeled by a set of numbers $\mu, \bar{\mu}$ usually called chemical potentials. The name derives from the fact that they can be identified with the chemical potentials of conserved higher spin currents in a 2D CFT. Recently, attention has been drawn to the fixed time canonical symplectic structure of these families [1, 2] (studies for highest weight boundary conditions can be found at [32, 45, 46]). One main point of concern, regards whether the associated Asymptotic Symmetry Algebra (ASA) is or is not independent on the chemical potentials $(\mu, \bar{\mu})$. An important fact that calls for attention is that black holes are zero modes of a corresponding family, and so different phase spaces sharing one of them, will provide different descriptions of the given black hole [47]. In fact the initial gauge invariance guarantees the presence of a map between any two of such descriptions, the gauge transformation being of course non residual. However, as we shall show, not all non residual gauge transformations take to a new description of the phase space while preserving the form of the zero modes. In chapter 3 we will address issues related to these questions. In order to simplify the analysis we will do it in a perturbative framework and for the case in which the gauge algebra is $sl(3, \mathbb{R})$.

The outline of this chapter is as follows. In section 3 we start by showing how to identify the Regge-Teitelboim (RT) formalism [48] with the Dirac one, for a family $(\mu_3, \bar{\mu}_3)$ in $sl(3, \mathbb{R})$ CS presented in [1]. Even though, as already known [1], one can arrive at a fixed time $W_3$ symplectic structure by use of RT formalism, we will show that this procedure is equivalent to the implementation of a non residual gauge transformation to a new phase space, that does not include the $(\mu_3, \bar{\mu}_3)$ black hole as zero mode. Thereafter we compute the Dirac brackets at a fixed time and show that they can not be identified to the $W_3$ algebra. Finally, we compute the fixed time Dirac brackets in a different phase space that does include the $(\mu_3, \bar{\mu}_3)$ black hole as zero mode [2, 33], and show that their algebra is isomorphic to $W_3^{(2)}$[2].

A distinctive feature of Black Holes (BH), in both asymptotically flat or asymptotically $(A)dS$ space-times, is the existence of "Quasi Normal Modes" (QNM): if one perturbs a Black Hole one finds damped modes, i.e. modes whose frequencies are complex, signalling the fact that the corresponding field can decay by falling into the Black Hole. In the AdS case these modes have an interpretation in the dual CFT as describing the approach to equilibrium of the perturbed thermal state [49, 50, 51]. This phenomenon has been studied extensively, especially after the proposal of the AdS/CFT correspondence, in the ordinary (super)gravity context in various dimensions. In particular, for gravity coupled to various matters in $D = 3$, the case of the BTZ BH has been studied in detail.

In chapter 4 we will be interested in generalising the problem to the context of higher spin systems in $D = 3$. Such systems, with finite number of higher spins $\leq N$, can be formulated via Chern-Simons theories based on $sl(N)$ algebras, but, like ordinary 3D gravity, they do not contain propagating degrees of freedom and, moreover, they do not allow coupling to propagating matter. In order to introduce (scalar) matter coupled to the higher spin sector, one formulates the theory in terms of a flat connection $(\mathcal{A}, \bar{\mathcal{A}})$ for the infinite dimensional algebra $hs(\lambda) \times hs(\lambda)$ [29, 30, 31]. The matter fields are packaged in an algebra valued master field $C$, a section obeying the horizontality condition with respect to the flat connection, in a way that will be detailed below. It turns out that if one embeds the BTZ BH in this system, one can follow a "folding" procedure to reduce the equation of motion for $C$ in the BTZ background to an ordinary second order equation for the lowest, scalar, component of the field $C$, with a $\lambda$ dependent mass given by $m^2 = \lambda^2 - 1$. Therefore the corresponding QNM are the usual ones found for a massive scalar field coupled to BTZ in the ordinary gravity case.

However, the higher spin systems are expected to admit generalized BH's carrying different charges, other than the mass and angular momentum carried in the BTZ case [1, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 52]. The issue then arises to study matter fluctuations in their background and possibly identify the spectrum of the corresponding QNM. Unlike for the BTZ background, one expects in general the "folding" procedure to give rise to a differential equation of order higher than two for the matter scalar field and it is not to be expected to be able to solve it analytically.

In chapter 4 we present a class of flat connections, depending on parameters $(\mu, \bar{\mu})$, in such a way that when $\mu$ and $\bar{\mu}$ go to zero we recover the BTZ connection. We will argue that they correspond to BH configurations in 3D $hs(\lambda) \times hs(\lambda)$ higher spin gravity. In addition, we will be able to solve the equations for matter fluctuations analytically and therefore identify the presence of QNM.

As first discussed in [33], establishing whether a given geometry represents a BH in higher spin theories is a subtle issue, due to the presence of a higher spin gauge degeneracy that can, to mention an example, relate seemingly BH geometries to geometries without horizons. We will follow the criterion of [33] and impose the BTZ holonomy conditions on the connection around the euclidean time $S^1$. As a result spacetime tensor fields [32] will be shown to behave smoothly at the horizon.

But as remarked above, further evidence arises from the analysis of their interaction with matter, in particular from the existence of QNM and their dispersion relations. Another subtle and important issue, whose general aspects have been subject of recent investigations, with different conclusions, [1, 2, 39, 40, 53], concerns the precise determination of the charges carried by our backgrounds and, more generally, their asymptotic symmetry algebras. Perhaps, one could get a clue of the general answer by studying the truncations of $hs(\lambda)$ with integer $\lambda$, we hope to come back to this problem in the future. In this way one would be able to, first, properly define their charges and, second, identify whether they are of higher spin character or not.

As for the bulk to boundary 2-point function, even though the differential equations of motion that determine them are of order higher than two, they are described by combinations of pairs of solutions of a 2nd order PDE's. Only one of all these pairs is smoothly related to the solutions corresponding to a real scalar field with $m^2 = \lambda^2 - 1$ propagating in the BTZ black hole [30, 54], as $\mu, \bar{\mu} \to 0$.

The outline of chapter 4 is as follows. In the first section in 4 we introduce the ansätze mentioned above and show that they define smooth horizons by use of the relation between connections and metric-like fields proposed in [32]. Then, we identify our $(\mu,\bar{\mu})$ with the so called chemical potentials in the solutions introduced in [33] and [2], that from now on we denote as GK and BHPT2[1], respectively. We do it by identifying the gauge transformation relating our connection to those ansätze. Next, in section 4.1 we discuss the equations of motion for the effective scalar in the BH backgrounds and describe the strategy to solve them for a generic element in the class. We give the explicit solutions for a couple of particular cases. An important fact to stress on, is that even though connections and generic metric like fields do break the asymptotic of $AdS$, the equations for fluctuations do preserve the behaviour of scalars minimally coupled to AdS in all cases.

In section 4.2 we show how to obtain the QNM and bulk to boundary 2-point functions for a generic element in the class and discuss them in the same particular cases. As a last check, we transform our results to the GK and BHPT2 descriptions and verify that the result of the gauge transformation coincides with the perturbative solution of the equations of motion for linear fluctuations of matter, written in those ansätze, as it should.

[1]Strictly speaking its embedding into $hs(\lambda) \times hs(\lambda)$.

# Part I

# Holographic Weyl anomaly matching

# 2

# Holographic anomaly matching

In this chapter we discuss various issues related to the understanding of the conformal anomaly matching in CFT from the dual holographic viewpoint. First, we act with a PBH diffeomorphism on a generic 5D RG flow geometry and show that the corresponding on-shell bulk action reproduces the Wess-Zumino term for the dilaton of broken conformal symmetry, with the expected coefficient $a_{UV} - a_{IR}$. Then, we consider a specific 3D example of RG flow whose UV asymptotics is normalisable and admits a 6D lifting. We promote a modulus $\rho$ appearing in the geometry to a function of boundary coordinates. In a 6D description $\rho$ is the scale of an $SU(2)$ instanton. We determine the smooth deformed background up to second order in the space-time derivatives of $\rho$ and find that the 3D on-shell action reproduces a boundary kinetic term for the massless field $\tau = \log \rho$ with the correct coefficient $\delta c = c_{UV} - c_{IR}$. We further analyse the linearized fluctuations around the deformed background geometry and compute the one-point functions $< T_{\mu\nu} >$ and show that they are reproduced by a Liouville-type action for the massless scalar $\tau$, with background charge due to the coupling to the 2D curvature $R^{(2)}$. The resulting central charge matches $\delta c$. We give an interpretation of this action in terms of the $(4, 0)$ SCFT of the D1-D5 system in type I theory.

## 2.1 The holographic spurion

The aim of this section is to verify that the quantum effective action for the holographic spurion in 4D contains the Wess-Zumino term, a local term whose variation under Weyl shifts of the spurion field reproduces the conformal anomaly [1], with coefficient given by the difference of UV and IR $a$-central charges, in accordance with the anomaly matching argument. We start by characterizing a generic RG flow background and the action of $PBH$ diffeomorphisms on it. The action of a special class of $PBH$ diffeomorphisms introduces a dependence of the background on a boundary conformal mode which will play the role of the spurion. Indeed, we will verify that the corresponding on-shell Einstein-Hilbert action gives the correct Wess-Zumino term for the conformal mode introduced through $PBH$ diffeo's. We then study the case of a flow induced by a dimension $\Delta = 2$ CFT operator, and check that boundary contributions coming from the Gibbons-Hawking term and counter-terms do not affect the bulk result. A derivation of the Wess-Zumino action has appeared in [22], studying pure gravity in AdS in various dimensions: the spurion $\phi$ is introduced as deformation of the UV cut-off boundary surface from $z$ constant to $z = e^{\phi(x)}$, $z$ being the radial coordinate of AdS. In appendix A.1.5 we present a covariant approach to get the same result for the WZ term.

### 2.1.1 Holographic RG flows

We start by characterizing a generic RG flow geometry. For the sake of simplicity, we are going to work only with a single scalar minimally coupled to gravity. In the next section we will consider a specific example involving two scalar fields. The action comprises the Einstein-Hilbert term, the kinetic and potential terms for a scalar field $\phi$, and the Gibbons-Hawking extrinsic curvature term at the boundary of the space-time manifold $M$:

$$S = \int_M d^{d+1}x\sqrt{G}(\frac{1}{4}R + (\partial\phi)^2 - V(\phi)) - \int_{\partial M} d^d x\sqrt{\gamma}\frac{1}{4}2K, \tag{2.1}$$

where $K$ is the trace of the second fundamental form,

$$2K = \gamma^{\alpha\beta} L_n \gamma_{\alpha\beta}, \tag{2.2}$$

[1]This is a combination of a Weyl shift of the background metric with a compensating shift in the spurion field. In this way the remaining variation is independent of the spurion field. It depends only on the background metric.

and $\gamma$ is the induced metric on the boundary of $M$, $\partial M$. $L_n$ is the Lie derivative with respect to the unit vector field $n$ normal to $\partial M$.

The metric has the form:

$$ds^2 = \frac{l^2(y)}{4}\frac{dy^2}{y^2} + \frac{1}{y}g_{\mu\nu}(y)dx^\mu dx^\nu, \tag{2.3}$$

which is an $AdS_5$ metric for constant $l(y)$ and $g_{\mu\nu}(y)$ $(\mu, \nu = 0, 1, 2, 3.)$. A RG flow geometry is then characterized by the fact that the above geometry is asymptotic to $AdS_5$ both in the UV and IR limits, $y \to 0$ and $y \to \infty$, respectively.

We assume that the potential $V(\phi)$ has two $AdS_5$ critical points that we call $\phi_{UV(IR)}$ and the background involves a solitonic field configuration $\phi(y)$ interpolating monotonically between these two critical points:

$$\phi(y) \quad \sim \delta\phi(y) + \phi_{UV}, \text{ when } y \to \infty \tag{2.4}$$

$$\phi(y) \quad \sim \delta\phi(y) + \phi_{IR}, \text{ when } y \to 0 \tag{2.5}$$

Around each critical point there is an expansion:

$$V(\phi) \sim \Lambda_{UV(IR)} + m^2_{UV(IR)}\delta\phi(y)^2 + o(\delta\phi(y)^4), \tag{2.6}$$

where $\delta\phi(y) = \phi(y) - \phi_{UV(IR)}$. By using (2.6) in the asymptotic expansion of the equations of motion:

$$\frac{1}{4}R_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi + \frac{1}{3}V[\phi], \tag{2.7}$$

one sees that the constants $\Lambda_{UV(IR)}$ play the role of cosmological constants and fix also the radii of the two $AdS_5$'s.

We discuss here the possibility to work in a gauge that makes easier to appreciate how only the boundary data is determining the spurion effective action. Consider a RG flow geometry of the form (2.3). Poincaré invariance of the asymptotic value of the metric implies $g_{\mu\nu}(y) = g(y)\eta_{\mu\nu}$. This is going to be an important constraint later on. The scale length function $l^2(y)$ has the following asymptotic behaviour:

$$l^2(y_{UV}) \quad \sim L^2_{UV} + \delta l_{UV} y_{UV}^{n_{UV}}, \; L^2(y_{IR}) \sim L^2_{IR} + \tfrac{\delta l_{IR}}{y_{IR}^{n_{IR}}}. \tag{2.8}$$

Notice that there is still the gauge freedom:

$$(x, y) \to (x, h \times y),$$

where $h = h(y)$ is any smooth function with asymptotic values 1 in the UV/IR fixed points. This gauge freedom allows to choose positive integers $n_{UV}$ and $n_{IR}$ as large as desired. In particular it is always possible to choose $n_{UV} > 2$. This gauge choice does not change the final result for the effective action because this is a family of proper diffeomorphisms leaving invariant the Einstein-Hilbert action (we will comment on this fact later on). Its use is convenient in order to make clear how only leading behaviour in the background solution is relevant to our computation. At the same time it allows to get rid of any back-reaction of $\delta l_{UV}$ and $\delta l_{IR}$ in the leading UV/IR asymptotic expansion of the equations of motion. The metric $g_{\mu\nu}$ has the following UV expansion, for $y \to 0$:

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + g^{(2)}_{\mu\nu} y + y^2 \left( g^{(4)}_{\mu\nu} + h^{(4)}_{\mu\nu} \log(y) + \tilde{h}^{(4)}_{\mu\nu} \log^2(y) \right) + o(y^3), \tag{2.9}$$

and a bulk scalar field dual to a UV field theory operator of conformal dimension $\Delta = 2$ that we denote as $\mathbf{O}_{(2)}$, behaves like:

$$\delta\phi = \phi^{(0)} y + \widetilde{\phi}^{(0)} y \log(y) + ..., \tag{2.10}$$

where the ... stand for UV subleading terms. From the near to boundary expansion of the Klein-Gordon equations it comes out the useful relation between the conformal weight of $\mathbf{O}_{(2)}$ and the mass of $\phi$ on dimensional $AdS_{d+1}$:

$$\Delta_{UV} = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2 L^2_{UV}}. \tag{2.11}$$

In this critical case we have the standard relation between asymptotic values of bulk fields and v.e.v.'s or sources, for the dual CFT operators: namely $\phi_{(0)}$ is the v.e.v. and $\tilde{\phi}_{(0)}$ the source. We have chosen the case $\Delta = 2$ to take a particular example, but one can generalise the results to any other value of $\Delta \leq 4$. In the remaining of the section we refer only to relevant perturbations.

### 2.1.2 On the $PBH$ diffeomorphisms

The $PBH$ diffeomorphisms transform, by definition, the line element (2.3) into:

$$ds^2 = \frac{l^2(e^\tau y)}{4y^2}dy^2 + \frac{1}{y}\tilde{g}_{\mu\nu}(y)dx^\mu dx^\nu, \tag{2.12}$$

with $\tilde{g}_{\mu\nu}$ given by a UV asymptotic expansion of the form (2.9):

$$\tilde{g}_{\mu\nu} = e^{-\tau}g^{(0)}_{\mu\nu} + ... \tag{2.13}$$

and $h^{(4)}_{1,2}$ and $g^{(i)}$, with $i = 2, 4$, determined in terms of the boundary data by the near to boundary expansion of the equations of motion (A.10).

For the static RG flow geometry at hand, (2.3), a $PBH$ transformation has the following structure in terms of derivatives of $\tau$:

$$\begin{aligned} x^\mu \quad \rightarrow \quad x^{\tau\mu} &= x^\mu - a^{(1)}[e^\tau y]\partial^\mu\tau - a^{(2)}[e^\tau y]\partial_\mu\Box\tau - a^{(3)}[e^\tau y]\Box^{\mu\nu}\tau\partial_\nu\tau \\ &\quad -a^{(4)}[e^\tau y]\Box\tau\partial^\mu\tau - a^{(5)}[e^\tau y](\partial\tau)^2\partial^\mu\tau + O\left(\partial^5\right), \\ y \quad \rightarrow \quad y^\tau &= ye^\tau + b^{(1)}[e^\tau y](\Box\tau) + b^{(2)}[e^\tau y](\Box\Box\tau) + b^{(3)}[e^\tau y](\partial\tau)^2 \\ &\quad +b^{(4)}[e^\tau y](\Box\tau)(\partial\tau)^2 + b^{(5)}[e^\tau y](\partial\tau)^4 \\ &\quad +b^{(6)}[e^\tau y]\partial_\mu\tau\Box^{\mu\nu}\tau\partial_\nu\tau + b^{(7)}[e^\tau y](\Box\tau)^2 \\ &\quad +b^{(8)}[e^\tau y]\partial_\mu(\Box\tau)\partial^\mu\tau + O(\partial^6). \end{aligned} \tag{2.14}$$

Notice we have written the most general boundary covariant form and that this derivative expansion of the full transformation is valid along the full flow geometry up to the IR cut off, not only close to the boundary. The constraints implied by preserving the form (2.12) allow to determine the form factors $a^{(i)}$ and $b^{(i)}$ in terms of the scale length function $l$. To begin with, it is immediate to see that :

$$\partial_z a^{(1)} = \frac{l^2(z)}{4},$$

where $z = e^\tau y$, which can be readily integrated. Some of these form factors can be settled to zero without lost of generality, since they are solution of homogeneous differential equations. Let us study the following factor: $b^{(1)}$. We can look at second order in derivatives contribution of $\delta x^\mu \equiv x^{\tau\mu} - x^\mu$ to the $(y, y)$ component of the metric, which is $\sim (\partial\tau)^2$. The contributions coming from $\delta y \equiv y^\tau - y$ contains a linear order in $y$ term proportional to

$$\left(\left(-\frac{l}{z} + \partial_z l\right) b^{(1)} + l\partial_y b^{(1)}\right) \Box\tau,$$

that does not match any contribution from $\delta x^\mu$ and also a term proportional to $(\partial\tau)^2$. This implies $b^{(1)}$ has to be taken to vanish. Consequently $a^{(2)}$ would vanish. In the same fashion one can prove $b^{(2)}$ can be taken to vanish and $b^{(3)}$ can be found to obey the following inhomogeneous first order differential equation:

$$\left(\frac{\partial_z l}{l} - \frac{1}{z}\right) b^{(3)} + \partial_z b^{(3)} = -\frac{l^2}{8} z,$$

which can be solved asymptotically to give:

$$b^{(3)} \sim -\tfrac{L_{UV}^2}{8} z^2 + O\left(z^{n_{UV}+2}\right),\ b^{(3)} \sim -\tfrac{L_{IR}^2}{8} z^2 + O\left(z^{-n_{IR}+2}\right). \tag{2.15}$$

Notice that so far, we have always taken the trivial homogeneous solution. In fact we are going to see that this choice corresponds to the minimal description of the spurion. The choice of different $PBH$ representative [1] would translate in a local redefinition of the field theory spurion. In the same line of logic one can find that:

$$\partial_z a^{(5)} = \frac{l^2}{4} \partial_z b^{(3)},\ \partial_z a^{(3)} = \frac{l^2}{2} \frac{b^{(3)}}{z},\ \partial_z a^{(4)} = 0. \tag{2.16}$$

From these we can infer that $b^{(4)}$, $b^{(7)}$ and $b^{(8)}$ obey homogeneous differential equations provided $a^{(4)}$ is taken to vanish, so we set them to zero too. The following constraints:

$$\begin{aligned}\left(\left(\frac{\partial_z l}{l} - \frac{1}{z}\right) b^{(5)} + \partial_z b^{(5)}\right) &= -\left(\frac{(\partial_z l)^2 + l\partial_z^2 l}{2l^2} + \frac{3}{2}\frac{1}{z^2}\right)\left(b^{(3)}\right)^2 - \\ &\quad \left(2\left(\frac{\partial_z l}{l} - \frac{1}{z}\right) b^{(3)} + \frac{1}{2}\partial_z b^{(3)}\right)\partial_z b^{(3)} - \frac{l^2}{4} z\partial_z b^{(3)},\end{aligned} \tag{2.17}$$

$$\left(\left(\frac{\partial_z l}{l} - \frac{1}{z}\right) b^{(6)} + \partial_z b^{(6)}\right) = -\frac{l^2}{2} b^{(3)}, \tag{2.18}$$

[1] Namely, to pick up non trivial solutions of the homogeneous differential equations for the form factors.

give the UV/IR asymptotic expansions for the form factors:

$$b^{(5)} \sim -\frac{L_{UV}^4}{128} z^3 + ..., \; b^{(5)} \sim -\frac{L_{IR}^4}{128} z^3 + ..., \tag{2.19}$$
$$b^{(6)} \sim -\frac{L_{UV}^4}{32} z^3 + ..., \; b^{(6)} \sim -\frac{L_{IR}^4}{32} z^3 + ..., \tag{2.20}$$

where the ... stand for subleading contributions. In appendix (A.1.2) we extend these results to the case of non static geometries. We use those non static cases in section 2.2 to check out the general results of this section in a particular example.

Before closing the discussion, let us comment about a different kind of $PBH$ modes. To make the discussion simpler we restrict our analysis to the level of $PBH$ zero modes i.e. $\tau$ is taken to be a constant. Then, is easy to see that one can take the transformation

$$y \to y^\tau = e^{h\times\tau} y, \text{ with } h(y) \xrightarrow[y\to(0,\infty)]{} h_{(UV,IR)}.$$

This arbitrary function $h$ constitutes a huge freedom. In particular we notice that one can choose a $PBH$ which does not affect the UV boundary data at all, but does change the IR side, namely such that:

$$h \sim 0, \quad h \sim 1,$$

respectively, or vice versa. This kind of $PBH$'s are briefly considered in appendix A.1.2.

Besides acting on the metric the change of coordinates also changes the form of the scalars in our background. We focus on the UV asymptotic. So, for instance the case of the dual to a $\Delta = 2$ operator the transformation laws are:

$$\widetilde{\phi}^{(0)} \to e^\tau \widetilde{\phi}^{(0)}, \; \phi^{(0)} \to e^\tau \phi^{(0)} + \tau e^\tau \widetilde{\phi}^{(0)}. \tag{2.21}$$

Notice the source transforms covariantly, but not the v.e.v.. This asymptotic action will be useful later on when solving the near to boundary equations of motion.

As already mentioned, we assume smoothness of the scalar field configurations in the IR. It is interesting however to explore an extra source of IR divergencies.

The original 5D metric is assumed to be smooth and asymptotically AdS in the IR limit, $y \to \infty$:

$$ds^2_{IR} = \frac{L^2_{IR}}{4}\frac{dy^2}{y^2} + \frac{1}{y}g^{(0)}_{\mu\nu}dx^\mu dx^\nu. \tag{2.22}$$

This AdS limit assumption implies that $g^{(0)}_{\mu\nu} = \eta_{\mu\nu}$. Non trivial space time dependence for $g_{(0)}$ sources an infinite tower of extra contributions that break AdS limit in the IR. For instance, a Weyl shifted representative will alter the IR AdS behaviour. The change is given by:

$$g^{(0)}_{\mu\nu} \to e^{-\tau}g^{(0)}_{\mu\nu} + yg^{(2)}_{\mu\nu}[e^{-\tau}g^{(0)}] + y^2g^{(4)}_{\mu\nu}[e^{-\tau}g^{(0)}] + y^2(h^{(4)}_{\mu\nu}[e^{-\tau}g^{(0)}]\log(y)) + ...,$$

in (2.22). Clearly AdS IR behaviour, $y \to \infty$, is broken in this case. This is related with the fact that $PBH$ diffeomorphisms are singular changes of coordinates in the IR. These modes alter significantly the IR behaviour of the background metric.

Let us comment on a different approach that will be employed in the following to study the effect of $PBH$ diffeo's on specific background solutions. Clearly $PBH$ diffeo's map a solution of the EoM into another. By knowing the UV and IR leading behaviours, one could then use near to boundary equations of motion to reconstruct next to leading behaviour in both extrema of the flow. Namely we can find the factors $g^{(2)}$, $g^{(4)}$ and $h^{(4)}$'s in (2.9) in terms of the Weyl shift of the boundary metric $e^\tau g^{(0)}$. We can then evaluate the bulk and boundary GH terms of the action with this near to boundary series expansion. Some information will be unaccessible with this approach, concretely the finite part of the bulk term remains unknown after use of this method. In appendix A.1.4.1 we compute the divergent terms of the bulk term and find exact agreement with the results posted in the next subsection. We will use this procedure to evaluate the GH and counter-terms indeed.

### 2.1.3 Wess-Zumino Term

Given its indefinite $y$-integral $S[y]$, the bulk action can be written as:

$$S_{bulk} = S[y_{UV}] - S[y_{IR}].$$

The divergent parts of the bulk action come from the asymptotic expansions of the primitive $S$:

$$S \sim \int d^4x \left( \frac{a^{(0)}_{UV}}{y^2_{UV}} + \frac{a^{(2)}_{UV}}{y_{UV}} + a^{(4)}_{UV} \log(y_{UV}) + O(1) \right), \; S \sim S_{IR}. \tag{2.23}$$

For a generic static RG flow solution $a^{(0)}_{UV,IR} = \frac{1}{2L_{UV/IR}}$. The factors $a^{(2)}_{UV/IR}$ and $a^{(4)}_{UV/IR}$, will depend on the specific matter content of the bulk theory at hand. As for our particular choice of $\Delta$'s in the UV/IR, the $a^{(2)}_{UV/IR}$ coefficients are proportional to the 2D Ricci Scalar $R$ of the boundary metric $g^{(0)}$ and vanish for the static case $g^{(0)}_{\mu\nu} = \eta_{\mu\nu}$ (See equation (A.14) and (A.15)). However, a different choice of matter content could provide a non trivial $a^{(2)}_{UV/IR}[\eta_{\mu\nu}]$ dependence on the parameters of the flow, so in order to keep the discussion as general as possible until the very end of the section, we keep the static limit of both $a^{(2)}_{UV/IR}$ as arbitrary. As for the expansions of the primitive $S$ in a generic static case, one gets thence:

$$S[y_{UV}] \sim \int d^4x \left( \frac{1}{2L_{UV}} \frac{1}{y^2_{UV}} + \frac{a^{(2)}_{UV}[\eta_{\mu\nu}]}{y_{UV}} + a^{(4)}_{UV}[\eta_{\mu\nu}] \log(y_{UV}) + O\left(1\right) \right), \tag{2.24}$$

$$S[y_{IR}] \sim \int d^4x \left( \frac{1}{2L_{IR}} \frac{1}{y^2_{IR}} + \frac{a^{(2)}_{IR}[\eta_{\mu\nu}]}{y_{IR}} + a^{(4)}_{IR}[\eta_{\mu\nu}] \log(y_{IR}) + O\left(1\right) \right). \tag{2.25}$$

The terms $a^{(4)}_{uv,ir}[\eta_{\mu\nu}]$ are the contributions to the Weyl anomaly coming from the matter sector of the dual CFT, they must be proportional to the sources of the dual operators. The order one contribution is completely arbitrary in near to boundary analysis. Notice that we have freedom to add up an arbitrary, independent of y functional, $\int d^4x \; C$, in the expansions. The difference of both of these functionals carries all the physical meaning and it is undetermined by the near to boundary analysis. To determine its dependence on the parameters of the flow, full knowledge of the primitive $S$ is needed.

Next we aim to compute the change of the bulk action introduced before, under an active $PBH$ diffeomorphism. The full action is invariant under (passive) diffeomorphisms $x^\mu = f^\mu(x')$, under which, for example, the metric tensor changes as:

$$g'_{\mu\nu}(x') = \left(\frac{\partial x^\rho}{\partial x'^\mu}\right)\left(\frac{\partial x^\sigma}{\partial x'^\nu}\right) g_{\rho\sigma}(x), \tag{2.26}$$

and similarly for other tensor fields. Here the transformed tensors are evaluated at the new coordinate $x'$. On the other hand by an active diffeomorphism, the argument of a tensor field is kept fixed, i.e:

$$g(x) \to g'(x). \tag{2.27}$$

The infinitesimal version of this transformation above is given by the Lie-derivative acting on $g$. The difference between the two viewpoints becomes apparent on a manifold $M$ with boundaries. Let us take a manifold with two disconnected boundaries to be time-like hypersurfaces. An integration of a scalar density over this manifold is invariant in the following sense:

$$\begin{aligned} S[B_{UV}, B_{IR}, g] &= \int_{B_{IR}}^{B_{UV}} d^D x \sqrt{g(x)} L[g(x)] \\ &= \int_{f^{-1}(B_{IR})}^{f^{-1}(B_{UV})} d^D x' \sqrt{g'(x')} L[g'(x')] \\ &= S[f^{-1}(B_{UV}), f^{-1}(B_{IR}), g'], \end{aligned} \tag{2.28}$$

where the boundaries are denoted by $B_{UV(IR)}$. By $f^{-1}(B_{UV})$ we mean the shape of the boundaries in the new coordinates $x' = f(x)$. On the other hand, under an active transformation we have the change:

$$S[B_{UV}, B_{IR}, g] \to S[B_{UV}, B_{IR}, g']. \tag{2.29}$$

By using (2.28), the variation of the corresponding functional under an active diffeomorphism can be written as:

$$\begin{aligned} \Delta_f S &= S[B_{UV}, B_{IR}, g'] - S[B_{UV}, B_{IR}, g] \\ &= S[f(B_{UV}), f(B_{IR}), g] - S[B_{UV}, B_{IR}, g], \end{aligned} \tag{2.30}$$

where in the last step we have used the invariance under the passive diffeomorphism induced by the inverse map $f^{-1}$. Of course, if the maps $f$ or $f^{-1}$ leave invariant the boundary conditions then the functional $S$ is invariant even under the active transformation induced by them.

From now on in this section we specialize to $D = 5$ with $x^5 \equiv y$. We take as diffeomorphism the $PBH$ diffeomorphism discussed earlier. The aim is to compute the on-shell action of the $PBH$ mode $\tau$. From the last discussion it follows that all needed is the on-shell action in terms of the original background, namely the background before performing the $PBH$ transformation, and a choice of time-like boundary surfaces, which we take to be:

$$y = y_{UV}, \; y = y_{IR}. \tag{2.31}$$

Under a generic $PBH$ GCT this region transforms into:

$$-\infty < t, \; x < \infty, \qquad y^\tau_{IR} < y < y^\tau_{UV}, \tag{2.32}$$

with $y^\tau_{UV}$ and $y^\tau_{IR}$ given by the action (2.14) on $y_{UV}$ and $y_{IR}$ respectively. In virtue of (2.30) we compute the transformed bulk action:

$$\begin{aligned} S[y_{UV}] - S[y_{IR}] &= \int d^4x \int_{y_{IR}}^{y_{UV}} dy\sqrt{-g}L \\ &\to \int d^4x \int_{y^\tau_{IR}}^{y^\tau_{UV}} dy\sqrt{-g}L = S[y^\tau_{UV}] - S[y^\tau_{IR}], \end{aligned} \tag{2.33}$$

where $y^\tau$ is given in (2.14). Given the near to boundary expansion of the bulk action for boundary metric $g^{(0)} = \eta$:

$$S_{div} = \int d^4x \left( \frac{1}{2L_{UV}} \frac{1}{y^2_{UV}} + \frac{a^{(2)}_{UV}[\eta_{\mu\nu}]}{y_{UV}} + a^{(4)}_{UV}[\eta_{\mu\nu}] \log(y_{UV}) + ... \right),$$

with cut off surface at $y = y_{UV}$, we can then compute the leading terms in the $PBH$ transformed effective action by using (2.14) and (2.33):

$$\begin{aligned} \int d^4x \frac{1}{y^2_{UV}} \;\; &\to \int d^4x \left( \frac{1}{z^2_{UV}} - 2\left( \frac{b^{(3)}(\partial\tau)^2}{z^3_{UV}} \right) - \right. \\ &\left. 2\left( \frac{b^{(5)}(\partial\tau)^4 + b^{(6)}\partial_\mu\tau\Box^{\mu\nu}\tau\partial_\nu\tau}{z^3_{UV}} \right) + 3\left( \frac{\left(b^{(3)}\right)^2(\partial\tau)^4}{z^4_{UV}} \right) \right) \\ &\to \int d^4x \left( \frac{1}{z^2_{UV}} + \frac{L^2_{UV}}{4}\frac{(\partial\tau)^2}{z_{UV}} + \frac{L^4_{UV}}{32}\left( (\partial\tau)^4 + 2\partial_\mu\tau\partial_\nu\tau\Box^{\mu\nu}\tau \right) ... \right) \\ &\to \int d^4x \left( \frac{1}{z^2_{UV}} + \frac{L^2_{UV}}{4}\frac{(\partial\tau)^2}{z_{UV}} + \frac{L^4_{UV}}{32}\left( (\partial\tau)^4 - 4\Box\tau(\partial\tau)^2 \right) ... \right). \end{aligned} \tag{2.34}$$

Similar contribution comes from the IR part of the primitive $S$. Should we demand IR smoothness of every background field, the static coefficients $a_{IR}^{(2)}[\eta_{\mu\nu}]$ and $a_{IR}^{(4)}[\eta_{\mu\nu}]$ will vanish automatically (See last paragraph in appendix A.1.4). So finally, we get the following form for the regularized bulk action:

$$\begin{aligned} S_{bulk}^{reg} &= \int d^4x \Bigg( \frac{e^{-2\tau}}{2L_{UV}y_{UV}^2} + \frac{L_{UV}e^{-\tau}}{8y_{UV}}(\partial\tau)^2 + \\ &\qquad \frac{(L_{UV}^2 a_{UV}^{(2)}[\eta_{\mu\nu}] - L_{IR}^2 a_{IR}^{(2)}[\eta_{\mu\nu}])}{8}(\partial\tau)^2 + \\ &\qquad (a_{UV}^{(4)}[\eta_{\mu\nu}] - a_{IR}^{(4)}[\eta_{\mu\nu}])\tau + \frac{\Delta a}{8}\left((\partial\tau)^4 - 4\Box\tau(\partial\tau)^2\right)\Bigg) + \ldots, \end{aligned} \tag{2.35}$$

where $\Delta a = a_{UV} - a_{IR}$ with $a_{UV/IR} = \frac{L_{UV/IR}^3}{8}$. The $\ldots$ stand for logarithmic divergent terms that will be minimally subtracted. Notice that the gravitational Wess-Zumino term comes out with a universal coefficient $\Delta a$, independent of the interior properties of the flow geometry. Specific properties of the flow determine the normalization of the kinetic term and the Wess-Zumino term corresponding to the matter Weyl Anomaly. Next, we have to check whether this result still holds after adding the GH term and performing the holographic renormalization. So, from now on we restrict the discussion to the case of $\Delta = 2$. The finite Gibbons-Hawking contribution can be computed with the data given in appendix A.1.4.1. One verifies that the contributions of both boundaries are independent of derivatives of $\tau$. The difference $S_{GH}|_{IR}^{UV}$ gives a finite contribution proportional to $\int d^4x\phi^0\tilde{\phi}^{(0)}$ which after a $PBH$ tranformation reduces to a potential term for $\tau$.

Notice that this term vanishes for a v.e.v. driven flow, so in this case no finite contribution at all arises. We will crosscheck this in the particular example studied in the next sections. In the case of a source driven flow, the finite contribution $\int d^4x\phi^0\tilde{\phi}^{(0)}$ give a potential term which is not Weyl invariant, as one can notice from the transformation properties (2.21). In fact its infinitesimal Weyl transformation generates an anomalous variation proportional to the source square $\delta\tau\left(\frac{8\ L_{UV}^3}{3}(\tilde{\phi}^{(0)})^2\right)$. From the passive point of view, the GH term presents an anomaly contribution $\log(y_{UV})\left(\frac{8\ L_{UV}^3}{3}(\tilde{\phi}^{(0)})^2\right)$ that after the cut off redefinition originates a matter Wess-Zumino term $\int d^4x \left( \frac{8}{3} \frac{L_{UV}^3}{3} (\tilde{\phi}^{(0)})^2 \right) \tau$ (See equations (A.12) and (A.18)).

Next, we analyse the counter-terms that are needed in order to renormalize UV divergencies. Covariant counter-terms involve the boundary cosmological constant and curvatures for $g^{(0)}$ and the boundary values of the scalar field, namely v.e.v. and source:

$$\int d^4x\sqrt{\gamma} = \int d^4x\sqrt{g_{(0)}} \left( \frac{1}{y_{UV}^2} + \frac{1}{y_{UV}} \frac{R}{12} + \frac{2}{3}\tilde{\phi}_{(0)}^2 + \frac{4}{3}\phi_{(0)}^2 + ... \right), \tag{2.36}$$

$$\int d^4x\sqrt{\gamma}R[\gamma] = \int d^4x\sqrt{g_{(0)}} \left( \frac{R}{y_{UV}} + \frac{R^2}{12} \right), \tag{2.37}$$

$$\int d^4x\sqrt{\gamma}\Phi^2(x, y_{UV}) = \int d^4x\sqrt{g_{(0)}} \left( \phi_{(0)}^2 + \dots \right), \tag{2.38}$$

where $\dots$ stand for logarithmic dependences that at the very end will be minimally substracted. We take $g^{(0)}$ to be conformally flat and then use the Weyl transformation properties of the boundary invariants to compute the Weyl factor dependence of counter-terms. The "volume" counter-term (2.36) is used to renormalize the infinite volume term of an asymptotically $AdS_5$ space. One then needs to use the $R$-term to cancel the next to leading divergent term. In the process one remains with a finite potential contribution that even for a v.e.v. driven flow gives a non vanishing energy-momentum trace contribution. The usual procedure [15, 55] is then to use the finite covariant counter-term (2.38) to demand conformal invariance in the renormalized theory, when the source is switched off. The counter-term action satisfying this requirements is:

$$S_{CT} = \int d^4x\sqrt{\gamma} \left( \frac{3}{2} - \frac{1}{8}R[\gamma] - 2\Phi^2 \right) |_{UV}.$$

This action will provide an extra finite contribution to (2.35) proportional to:

$$\int d^4x e^{-2\tau} R^2[e^{-\tau}\eta] \sim \int d^4x \left( \Box\tau - (\partial\tau)^2 \right)^2.$$

Finally the renormalized action takes the form:

$$\begin{aligned} S_{ren}^{\Delta=2} &= S_{reg}^{\Delta=2} + S_{GH}^{\Delta=2} + S_{CT} \\ &= \int d^4x \left( \frac{16\ L_{UV}^3}{3} \tilde{\phi}_{(0)}^2 \tau + \frac{\Delta a}{8} \left( (\partial\tau)^4 - 4\Box\tau(\partial\tau)^2 \right) \right. \\ &\qquad \left. + \beta \left( \Box\tau - (\partial\tau)^2 \right)^2 + O(1) + O\left(\partial^6\right) \right) . \end{aligned}$$

We should notice that no second derivative term, $(\partial\tau)^2$, is present in this particular case, just as in the similar discussion of [22]. However, there is a source of higher derivative terms: due to the fact that the $PBH$ diffeomorphism used is singular in the IR. In fact, the higher orders in derivatives come with the higher order IR singularities. So, the higher derivative terms are counted by powers of the IR cut off. We do not address here the issue of renormalizing these terms. The main idea here was to show the presence of a Wess-Zumino term compensating the anomaly difference between fixed points. The term $O(1)$ stands for possible finite contributions (4D cosmological constants) in the static on shell action plus GH term and CT. As for the GH term this contribution vanishes for v.e.v. driven flows.

## 2.2 RG flow in $N = 4$ 3D gauged supergravity

In this section we consider a particular, explicit and analytic example of a Holographic RG flow in 3D gauged supergravity. The reason to analyse this particular example is twofold: first, it is relatively simple and analytic, and, second, it is completely smooth, even in the infrared region. Indeed smoothness will be our guiding principle in deforming the background geometry in the way we will detail in this section. We will promote some integration constants (moduli) present in the flow solution to space-time dependent fields and identify among them the one which corresponds to a specific field in the boundary CFT. To get still a solution of the equations of motion we will have to change the background to take into account the back reaction of space-time derivatives acting on the moduli fields. This will be done in a perturbative expansion in the number of space-time derivatives. The starting point is one of the explicit examples of RG flows studied in [23], where domain wall solutions in $N = 4$ 3D gauged supergravity were found. These solutions are obtained by analyzing first order BPS conditions and respect 1/2 of the bulk supersymmetry. They describe holographic RG flows between $(4, 0)$ dual SCFT's. It turns out that the solution we will be considering admits a consistent lift to 6D supergravity, which will be reviewed and used in the next section. In this section the analysis will be purely three-dimensional.

We start by writing the action and equations of motion for the three dimensional theory at hand. In this case the spectrum reduces to the metric $g$, and a pair of scalars $A$ and $\phi$, which are left over after truncating the original scalar manifold. The action is:

$$S^{bulk}_{scalars} = \int d^3x\sqrt{-g}\left(-\frac{R}{4} - \frac{3}{4}\frac{(\partial A)^2}{(1-A^2)^2} - \frac{1}{4}(\partial\phi)^2 - V(A,\phi)\right), \qquad (2.39)$$

with potential for the scalar fields given by:

$$V \;= \tfrac{1}{2}e^{-4\phi}\left(\frac{2e^{2\phi}\left(A^2\left(g_2A\left(g_2A\left(A^2-3\right)+4g_1\right)-3g_1^2\right)+g_1^2\right)}{\left(A^2-1\right)^3} + 4c_1^2\right). \qquad (2.40)$$

The corresponding set of equations of motion is then given by:

$$
\begin{aligned}
\frac{1}{2}\Box\phi - \partial_\phi V(A,\phi) &= 0, & (2.41)\\
\frac{3}{2}\frac{1}{\sqrt{-g}}\partial_\mu\left(\frac{1}{(1-A^2)^2}g^{\mu\nu}\partial_\nu A\right) - \partial_A V(A,\phi) &= 0, & (2.42)\\
-\frac{1}{4}R_{\mu\nu} - \frac{3}{4}\frac{\partial_\mu A\partial_\nu A}{(1-A^2)^2} - \frac{1}{4}\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}V(A,\phi) &= 0. & (2.43)
\end{aligned}
$$

### 2.2.1 The domain wall solution and its moduli

In this subsection we review the domain wall solution describing the RG flow on the dual CFT and identify its moduli. Let us choose coordinates $x^\nu = t, x, r$ and the 2D $(t,x)$-Poincaré invariant domain wall ansatz for the line element:

$$
ds^2 = dr^2 + e^{2f(r)}\eta_{\mu\nu}dx^\mu dx^\nu, \tag{2.44}
$$

and the scalar field profiles $A_B(r)$ and $\phi_B(r)$.

The equations of motion reduce then to the following set:

$$
\begin{aligned}
f'\phi_B' + \frac{\phi_B''}{2} - \partial_{\phi_B}V &= 0, & (2.45)\\
3A_B'' + 6A_B'f' + 6\frac{A_B A_B'^2}{(1-A_B^2)} - 2\left(1-A_B^2\right)^2\partial_{A_B}V &= 0, & (2.46)\\
\left(2f'' + 2f'^2 + \phi_B'^2 + \frac{3A_B'^2}{(A_B^2-1)^2} + 4V\right) &= 0, & (2.47)
\end{aligned}
$$

where the primes denote derivative with respect to $r$. It is then straightforward to show that the following field configuration:

$$
\begin{aligned}
e^{\phi_B(r)} &= \frac{2c_1\left(g_2^2 - \frac{g_1^2\rho^2}{(\rho+y(r))^2}\right)}{g_1 g_2^2\sqrt{1-A_B(r)^2}}, & (2.48)\\
A_B(r) &= \frac{g_1}{g_2}\frac{\rho}{(\rho+y(r))},\quad e^{2f(r)} = \frac{1}{2}e^{2s_p}y(r)\left(\frac{g_2^2(\rho+y(r)) - g_1^2\rho}{(\rho+y(r))}\right)^2, & (2.49)
\end{aligned}
$$

with $y(r) = e^{2g_1F(r)}$ is the most general solution of (2.45),(2.42), (2.41), provided:

$$
F'(r) = \frac{g_1 g_2^2(\rho+y(r))^2}{2c_1\left(g_2^2(\rho+y(r))^2 - g_1^2\rho^2\right)}. \tag{2.50}
$$

We can solve this equation explicitly for r(F):

$$r(F) = \frac{2c_1\left(F\left(g_2^2-g_1^2\right)-\frac{g_1\rho}{2\left(e^{2Fg_1}+\rho\right)}+\frac{1}{2}g_1\log\left(e^{2Fg_1}+\rho\right)\right)}{g_1 g_2^2}+\tau. \tag{2.51}$$

Notice the presence of three moduli $\tau$, $s_p$, $\rho$. The first one corresponds to a freedom in shifting the radial coordinate by a constant amount $\tau$, $r \to r+\tau$. This mode is a $PBH$ rigid diffeomorphism in the domain wall coordinates. As mentioned a rigid $PBH$ in domain wall coordinates becomes a warped one in the Fefferman-Graham coordinates. The second modulus $s_p$ can be identified with a rigid conformal transformation in the boundary coordinates $(t, x)$. The third modulus $\rho$ is an internal mode respecting the boundary conditions for the metric in both UV and IR limits but changing the scalar modes and it corresponds to a normalisable zero mode. In the next section we will see this mode is basically the instanton size modulus in the 6D description of the RG flow. But can be also thought of as a linear combination of a $PBH$ and $s_p$ mode. In order to have a flavor of the properties of the flow geometry it is useful to make a change of coordinates, from $(t, x, r)$ to $(t, x, y)$ with $y = e^{2g_1F(r)}$. In these coordinates the metric becomes:

$$ds^2 = \frac{\left(g_2^2(y+\rho)^2-g_1^2\rho^2\right)}{2(y+\rho)^2}\left(\frac{2c_1^2\left(g_2^2(y+\rho)^2-g_1^2\rho^2\right)}{g_1^4g_2^4y^2(y+\rho)^2}dy^2+e^{2s_p}y\left(dx^2-dt^2\right)\right). \tag{2.52}$$

This geometry approaches $AdS_3$ in both the UV($y \to \infty$) and the IR ($y \to 0$) limits, with corresponding radii:

$$\frac{L_{IR}^2}{4} = \frac{c_1^2\left(g_1^2-g_2^2\right)^2}{g_1^4g_2^4} \text{ and } \frac{L_{UV}^2}{4} = \frac{c_1^2}{g_1^4}. \tag{2.53}$$

These radii determine the central charges of the (4,0) CFT's at the fixed points, through the expression $c = 3L/2G_N$, $G_N$ being the 3D Newton's constant [1]. Additionally the limit:

$$g_2 \to \infty \text{ with } g_1 \text{ fixed}, \tag{2.54}$$

recovers $AdS_3$ space with radius $L$ given by $\frac{L^2}{4} = \frac{c_1^2}{g_1^4}$. An additional transformation in the boundary metric is needed to keep it finite in the limit, $\eta \to \frac{2}{g_2^2}\eta$. The

[1]In our conventions $G_N$=4.

scalar fields go in the UV and IR to different fixed points (extrema) of the potential $V(A,\phi)$. In particular, in the UV, $A \to 0$ and $\phi \to \log\left(\frac{2c_1}{g_1}\right)$. Expanding the potential (2.40) around the extremum we find out the masses of the bulk fields $A(r)$, $\phi(r)$ at the UV fixed point:

$$m_A^2 = 0 \text{ and } m_\phi^2 = \frac{h^2 g_1^4}{c_1^2} = \frac{8}{L_{UV}^2}. \tag{2.55}$$

The allowed conformal dimension of the corresponding dual boundary operators are:

$$\Delta_{A+} = 2,\ \Delta_{A-} = 0 \text{ and } \Delta_\phi = 4, \tag{2.56}$$

respectively. By looking at (2.48) we can read off their asymptotic expansions near the UV boundary ($y \to \infty$):

$$\delta A(y) \sim \frac{g_1}{g_2}\frac{\rho}{y} \text{ and } \delta\phi(y) \sim -\frac{g_1^2}{2g_2^2}\frac{\rho^2}{y^2}. \tag{2.57}$$

These are "normalisable" excitations, and in the standard quantization, which adopts $\Delta = \Delta_+$, they would correspond to a vacuum state in the dual CFT, where the dual operators $\mathcal{O}_A$ and $\mathcal{O}_\phi$ acquire a v.e.v.. This clashes with the fact that in $D = 2$ we cannot have spontaneous breaking of conformal invariance [1]. Notice that the problem arises also in the well known case of the D1-D5 system in IIB, when one deforms the $AdS_3 \times S^3$ background by going to multi-center geometries. Most probably this is a feature of the supergravity approximation, or dually, of the leading large-$N$ expansion on the CFT side. It would be interesting to see how the picture is modified in going beyond the supergravity approximation, as discussed, in a different context, in [57].

At the IR region,

$$\delta A(y) \sim -\frac{g_1}{g_2}\frac{y}{\rho} \text{ and } \delta\phi(y) \sim -\frac{g_1^2}{g_1^2 - g_2^2}\frac{y}{\rho}. \tag{2.58}$$

In particular, the background is completely smooth. Now we notice a property of the metric (2.52) : the UV/IR AdS limits of the geometry are independent

[1]On the other hand, the "alternate" quantization [56] would interpret this background as a source term for the $\Delta_- = 0$ operator $\mathcal{O}_A$. However, this interpretation clashes with the standard axioms of 2D CFT.

of $\rho$, and, as mentioned earlier, this modulus corresponds to a normalisable zero mode.

It is instructive to look at how can be represented a $PBH$ diffeomorphism zero-mode of the form $y \to e^{2\sigma_{PBH}} y$ in terms of the moduli appearing in the background geometry: it amounts to take the combined set of transformations $\rho \to \rho e^{2\sigma_{PBH}}$ and $s_p \to sp + \sigma_{PBH}$. Conversely, the $\rho$ modulus can be thought of as a combination of a $PBH$ mode mentioned before plus a suitable choice of $s_p$ such that the boundary metric remains unchanged. We should stress that the $PBH$ zero-modes $\tau$ and $\sigma_{PBH}$ aren't precisely the same. The difference will come about in the next subsection. But we can already say that there is a choice of $\tau$ and $s_p$ for fixed $\rho = 1$ that preserves normalisability. We can explore then two possibilities, either we analyse the combined pair of moduli $(\tau,\ s_p)_{\rho=1}$ or the single modulus $\rho$. In the next subsection we analyse both cases. We will also check the geometrical procedure discussed in section 2.1.3.

### 2.2.2 Fluctuations analysis

In this subsection we are going to analyse a deformation of the background geometry which arises when one gives a non trivial $(t, x)$ dependence to some of the moduli introduced in the previous subsection. Specifically, we will promote the integration constants $s_p$ and $\tau$ to functions of $t$ and $x$, $s_p(t, x)$ and $\tau(t, x)$. In doing so, of course, we have to take into account the back reaction due to the $(t, x)$ derivatives acting on these fields. The equations of motion will involve therefore inhomogeneous terms containing derivatives of $s_p(t, x)$ and $\tau(t, x)$. We will work in a perturbative expansion in the number of $t$ and $x$ derivatives. For that purpose it is convenient to introduce a counting parameter $q$, whose powers count the number of $t, x$ derivatives. As for the metric, we keep the axial gauge condition and therefore start with the expression:

$$ds^2 = dr^2 + (e^{2f}\eta_{\mu\nu} + q^2 g^{(2)}_{\mu\nu})dx^\mu dx^\nu, \tag{2.59}$$

where $x_0 = t$ and $x_1 = x$, and $\mu,\ \nu = 0, 1$.

For the background deformations, at second order in $(t, x)$ derivatives, we adopt the following ansatz for the scalar fields:

$$\begin{aligned} A &= A_B + q^2 A^{(2)}, \\ \phi &= \phi_B + q^2 \phi^{(2)}, \end{aligned} \tag{2.60}$$

whereas for the metric components:

$$g_{tt}^{(2)} = -e^{2f}(g^{(2)} + T),\ g_{xx}^{(2)} = e^{2f}(g^{(2)} - T), \tag{2.61}$$

and we redefine $g_{tx}^{(2)} \to e^{2f} g_{tx}^{(2)}$. The homogeneous part of the equations of motion will involve an ordinary linear differential operator in the $r$ variable acting on the fluctuations and this will be sourced by an inhomogeneous term involving two $t, x$ derivatives acting on $s_p$ and $\tau$, which represents the moduli back reaction to the original background. Now we have five unknown functions and eight equations, (2.41), (2.42), (2.43), so that we need to reduce the number of independent equations. It is a long but straightforward procedure to find out the general solutions to the system. We are going to sketch the procedure we followed to solve them. Details are given in appendices. Specifically the equations of motions at order $q^2$ are given in appendix A.2.1.

A change of coordinates is useful to render the system of partial differential equations simpler. We perform a change from the domain wall coordinates $(t, x, r)$ to the Poincaré like coordinates $(t, x, y)$ already introduced in the previous subsections:

$$y = e^{2g_1 F(\tau(t,x),r)}, \tag{2.62}$$

where,

$$\partial_r F - \frac{g_1 g_2^2 \left(e^{2g_1 F} + 1\right)^2}{2c_1 \left(g_2^2 \left(e^{2g_1 F} + 1\right)^2 - g_1^2\right)} = 0. \tag{2.63}$$

Notice that if we are using a non fluctuating cut off surface $r = r_{UV}$ in the original coordinates, in the new coordinates the same surface will be fluctuating at a pace dictated by $\tau(t, x)$. We can however use a different choice of coordinates:

$$\widetilde{y} = e^{2g_1 F(0,r)}. \tag{2.64}$$

It is then easy to show based on (2.51), that cut off shapes in the $y$-system and $\widetilde{y}$-system are related as follows:

$$y_{UV} \to e^{\frac{g_1^2}{c_1}\tau}\widetilde{y}_{UV}, \;\; y_{IR} \to e^{\frac{g_1^2 g_2^2}{c_1\left(g_2^2-g_1^2\right)}\tau}\widetilde{y}_{IR}. \tag{2.65}$$

The set of equations, (2.41), (2.42), (2.43) provides a system of second order differential equations for the fluctuations in terms of the inhomogeneities produced by derivatives acting on $s_p(t,x)$ and $\tau(t,x)$. We are going to denote the five Einstein equations (2.43) by $(t,t)$, $(x,x)$, $(t,x)$, $(r,r)$, $(t,r)$, $(x,r)$, with obvious meaning. Equations $(t,t)$, $(x,x)$ and $(r,r)$ form a set of second order equations in the $\eta$-trace part of the metric parametrized by $g^{(2)}(t,x,r)$ and the traceless part parametrized by $T(t,x,r)$, together with the scalar fluctuations, which only appear up to first order in radial derivatives. It turns out that the combination $(t,t)-(x,x)$ gives an equation for the trace part and scalar fluctuations, but the traceless part decouples in the combination $(t,t)+(x,x)$. Namely it gives the equation:

$$y\partial_y^2 T + 2\partial_y T + \frac{2e^{-2s_p}2g_1^2(g_1^2+3g_2^2(1+y^2))\left((\partial_t^2\tau)^2+(\partial_x^2\tau)^2\right)}{(g_1^2-g_2^2(y^2+1))^3} = 0, \tag{2.66}$$

whose general solution is:

$$T = C_3(t,x) - \frac{1}{y}C_2(t,x) + \frac{g_1^2}{g_2^2 y\left(g_2^2(y+1)^2-g_1^2\right)}e^{-2s_p}\left((\partial_t\tau)^2+(\partial_x\tau)^2\right), \tag{2.67}$$

where $C_3$ and $C_2$ are integration constants promoted to be arbitrary functions of $t$ and $x$. Let's focus then on the set of equations $(t,t)-(x,x)$ and $(r,r)$. This is a coupled system for the trace part and the scalars which can be solved in many different ways, here we present one. First of all $(r,r)$ can be integrated to get:

$$\partial_y g^{(2)} = R^{(1)}_{\partial_y g^{(2)}} A^{(2)} + R^{(2)}_{\partial_y g^{(2)}}\phi^{(2)} + \frac{1}{y^2}C_5, \tag{2.68}$$

where,

$$R^{(1)}_{\partial_y g^{(2)}} = -\frac{6g_1 g_2^3(y+1)^2}{\left(g_1^2-g_2^2(y+1)^2\right)^2}, \;\; R^{(2)}_{\partial_y g^{(2)}} = -\frac{2g_1^2}{(y+1)\left(g_2^2(y+1)^2-g_1^2\right)}, \tag{2.69}$$

with an integration constant $C_5$. Then, one can notice that Eq. $(t,t)-(x,x)$ only contains derivatives of the trace part of the metric fluctuations, so we can use (2.68) and its derivative to eliminate this function. The remaining equation will contain the scalar fluctuations up to first order in "radial" derivatives:

$$\begin{aligned}\partial_y\phi^{(2)} &= R^{(1)}_{\partial_y\phi^{(2)}}\partial_y A^{(2)} + R^{(2)}_{\partial_y\phi^{(2)}}\phi^{(2)} + R^{(3)}_{\partial_y\phi^{(2)}}A^{(2)} \\ &\qquad + R^{(4)}_{\partial_y\phi^{(2)}}(C_5 - \frac{2c_1}{g_1^2 g_2^2}\Box\tau) + R^{(5)}_{\partial_y\phi^{(2)}}(\partial\tau)^2 + R^{(6)}_{\partial_y\phi^{(2)}}e^{-2s_p}\Box s_p.\end{aligned} \tag{2.70}$$

Under the conditions already found the remaining equations (2.41), (2.42) reduce to the final algebraic equation for $\phi^{(2)}$ in terms of y-derivatives of $A^{(2)}$ up to second order. By solving it and plugging the result in (2.70) we obtain the third order differential equation:

$$\partial_y^{(3)}A^{(2)} + R^{(2)}_{A^{(2)}}\partial_y^2 A^{(2)} + R^{(1)}_{A^{(2)}}\partial_y A^{(2)} + R^{(0)}_{A^{(2)}}A^{(2)} = e^{-2s_p}F, \tag{2.71}$$

where the inhomogeneous part takes the form:

$$F = F^{(1)}C_5 + F^{(2)}\Box s_p + F^{(3)}\Box\tau + F^{(4)}(\partial\tau)^2. \tag{2.72}$$

The $R^{(i)}_{A^{(2)}}$ and $F^{(i)}$ are rational functions in the radial coordinate $y$ (they are given in the appendix A.2.2). We solve this equation by Green's function method (See appendix A.2.3).

The $(t,x)$ equation:

$$\partial_y^2 g^{(2)}_{tx} = -\frac{2}{y}\partial_y g^{(2)}_{tx} + \frac{4g_1^2 e^{-2s_p}\left(3g_2^2(y+1)^2 + g_1^2\right)}{y\left(g_2^2(y+1)^2 - g_1^2\right)^3}e^{-2s_p}\partial_t\tau\partial_x\tau, \tag{2.73}$$

can be solved to get:

$$g^{(2)}_{tx} = -\frac{C_6(t,x)}{y} + C_7(t,x) - \frac{2g_1^2}{g_2^2 y\left(g_2^2(y+1)^2 - g_1^2\right)}e^{-2s_p}\partial_t\tau\partial_x\tau. \tag{2.74}$$

As for the mixed equations, $(t,r)$ and $(x,r)$, they involve odd number of $(t,x)$ derivatives and one needs to go to third order, were in fact they reduce to differential constraints for the integration constants $C_2$, $C_5$ and $C_6$ sourced by second derivatives of the moduli $\tau$ and $s_p$. Before solving for these constraint equations it is convenient to analyse the constraints that IR regularity imposes on the modulus $C_5$.

At this point we should comment about an important issue. We have nine integration functions $C_i(t,x)$ and our general on shell fluctuations develop generically infrared singularities and/or UV non-normalisability, in the latter case representing source terms on the dual CFT. We have two ways to deal with possible IR divergencies in our deformed background geometry: we could allow infrared singularities of the geometry and setup a cut off at the IR side, or demand IR-smoothness. This latter will spoil full normalisability of all fluctuations, as we will see. This is something perhaps we could allow because at $q^0$ order the modulus which could be associated to the "dilaton" is still a normalisable bulk mode. The first option will guarantee full normalisability to order $q^2$, but will require the presence of an IR Gibbons Hawking (GH) term (2.93). In any case we will see that the GH term will give no contribution to the boundary effective action of the moduli. In this chapter we take the first point of view and demand full smoothness of the deformed geometry. By demanding regularity at the IR side for matter fluctuations $A^{(2)}$ and $\phi^{(2)}$ we get the following set of relations for the integration functions:

$$C_5(t,x) = -\frac{2c_1}{g_1^2 g_2^2} e^{-2s_p}\Box\tau + \frac{4c_1^2\left(g_1^2 - g_2^2\right)}{g_1^4 g_2^4} e^{-2s_p}\Box s_p, \tag{2.75}$$

$$C_{10}(t,x) = \frac{9g_1^5}{4g_2^9} e^{-2s_p}(\partial\tau)^2 - \frac{c_1^2\left(9g_1^4 - 17g_2^2 g_1^2 + 8g_2^4\right)}{2g_1 g_2^{11}} e^{-2s_p}\Box s_p. \tag{2.76}$$

At this point we could solve the $(t,r)$ and $(x,r)$ fluctuation equations for the moduli:

$$\begin{aligned} e^{2s_p}C_2(t,x) = {} & \frac{4c_1^2\left(g_1^2 - g_2^2\right)}{g_1^4 g_2^4}\left((\partial_t s_p)^2 - \partial_t^2 s_p\right) \\ & -\frac{4c_1}{g_1^2 g_2^2}\partial_t s_p \partial_t\tau - \frac{1}{g_2^2}(\partial_t\tau)^2 + \frac{2c_1}{g_1^2 g_2^2}\partial_t^2\tau + (\partial_t \to \partial_x)\,, \end{aligned} \tag{2.77}$$

$$\begin{aligned} e^{2s_p}C_6(t,x) = {} & -\frac{8c_1^2\left(g_1^2 - g_2^2\right)}{g_1^4 g_2^4}(\partial_t s_p \partial_x s_p - \partial_{tx}^2 s_p) \\ & +\frac{4c_1}{g_1^2 g_2^2}(\partial_t s_p \partial_x\tau + \partial_x s_p \partial_t\tau - \partial_{tx}^2\tau) + \frac{2}{g_2^2}(\partial_t\tau\partial_x\tau). \end{aligned} \tag{2.78}$$

According to the AdS/CFT dictionary, a state in the boundary CFT should correspond to a normalisable bulk mode, whereas non normalisable modes correspond to source deformations of the CFT. In our case, the UV boundary metric in the Fefferman-Graham gauge looks like $e^{2s_p+\frac{g_1^2}{c_1}\tau}\eta$. So, assuming the "standard" quantization, if we do not want to turn on sources for the trace of the boundary energy momentum tensor we need to take:

$$\tau = -\frac{2c_1}{g_1^2}s_p. \tag{2.79}$$

This is not the case in the IR boundary where the induced metric picks up a shifting factor that we can not avoid by staying in the axial gauge ($g_{rr} = 1$). By requiring not to turn on sources, even at second order in the derivative expansion for other components of the UV boundary CFT stress tensor, we see that:

$$C_3(t,x) = 0,\ C_4(t,x) = 0,\ C_7(t,x) = 0. \tag{2.80}$$

At this point of the nine integration constants at our disposal, after requiring regularity and normalisability of the metric fluctuations, two are left over, $C_8$ and $C_9$. Together with $\tau$ they determine the CFT sources inside the matter fluctuations $\phi^{(2)}$ and $A^{(2)}$. This remaining freedom can be used just to require normalisability of either $\phi^{(2)}$ or $A^{(2)}$, but not both of them. From here onwards we choose to make $\phi_{(2)}$ normalisable but for our purposes the two choices are equivalent. Finally we get:

$$\begin{aligned} C_9(t,x) &= 4C_8(t,x) + \frac{(-3g_1^7 + 13g_2^2g_1^5 - 4g_2^4g_1^3)}{g_2^7}e^{-2s_p}(\partial\tau)^2 \\ &+ \frac{2c_1^2\,(27g_1^8 - 144g_2^2g_1^6 + 139g_2^4g_1^4 + 23g_2^6g_1^2 - 12g_2^8)}{9g_1^3g_2^9}e^{-2s_p}\Box s_p. \end{aligned} \tag{2.81}$$

This choice turns on a source for the CFT operator dual to $A$. Indeed the UV expansion for $A$-fluctuation reads:

$$A^{(2)} \sim -\frac{2c_1^2}{3g_1^3g_2^3}e^{-2s_p}\left(\Box s_p\right). \tag{2.82}$$

To summarize, requiring IR regularity forces us to turn on a source term for one of the scalar fields. Notice that under the condition (2.79) the traceless and off-diagonal modes $T$ and $g^{(2)}$ are IR divergent. They go as $\frac{1}{y}$ in the IR limit. Nevertheless the IR limit of the metric is not divergent because of the extra warp factor, which is proportional to $y$. Notice that The AdS IR limit is in fact broken by $q^2$ order fluctuations, as already argued in section 2.1.

### 2.2.3 Evaluating the on-shell action

The regularized boundary Lagrangian coming from the bulk part is obtained by performing the integral over the radial coordinate with IR and UV cut-offs $y_{IR}$ , $y_{UV}$ respectively:

$$L_{2D}^{bulk} = \int_{y_{IR}}^{y_{UV}} dy\ L_{3D}. \tag{2.83}$$

First we present the result for the presence of both the moduli $s_p$ and $\tau$. We can write down the 3D lagrangian as:

$$\begin{aligned} L_{3D} &= l^{(0)} + l^{(1)}(\partial\tau)^2 + l^{(2)}\Box\tau + l^{(3)}\Box s_p \\ &+ \partial_y \left( l^{(4)}\partial_y g_{tt}^{(2)} + l^{(5)} g_{tt}^{(2)} + l^{(6)} A^{(2)} + l^{(7)}\phi^{(2)} \right). \end{aligned} \tag{2.84}$$

After integration and evaluation at the cut off surfaces we arrive to a boundary regularized action:

$$\begin{aligned} \int dtdx L_{2D}^{bulk} &= \int dtdx \left( \frac{g_1^2 g_2^2}{8c_1} e^{2s_p(t,x)} \left[y\right]_{IR}^{UV} \right. \\ &\quad - \frac{g_1^4}{8c_1\left(g_1^2 - g_2^2\right)}(\partial\tau)^2 + \left(\frac{1}{4}\Box\tau + \frac{c_1}{2g_1^2}\Box s_p\right)\log y_{UV} \\ &\quad \left. - \left(\frac{1}{4}\Box\tau + \frac{c_1(g_2^2 - g_1^2)}{2g_1^2 g_2^2}\Box s_p\right)\log y_{IR} + ... + \left[L_{hom}\right]_{IR}^{UV} \right), \end{aligned} \tag{2.85}$$

where the ... stand for infinitesimal contributions and a total derivative term

$$-\frac{c_1}{2g_2^2}\Box s_p + \log\left(1 - \frac{g_2^2}{g_1^2}\right)\Box\tau,$$

which is irrelevant for our conclusions. Notice that the logarithmic divergent part is a total derivative, as it should be. Moreover the coefficient in front of it is proportional to the difference of central charges at the UV and IR fixed points.

The contribution of the homogeneous part of the solutions to the on-shell bulk action can be written as:

$$L_{hom} = \left( l^{(4)} \partial_y g_{tt}^{(2)} + l^{(5)} g_{tt}^{(2)} + l^{(6)} A^{(2)} + l^{(7)} \phi^{(2)} \right) . \tag{2.86}$$

As we will show in a while, this contribution does not affect the finite value of the moduli $\tau$ and $s_p$ effective action at all! In next section we will see this will not be the case if we work in Fefferman-Graham gauge since the beginning. In that case, the solution of homogeneous equations do affect the final result but upon regularity conditions the contributions are total derivatives of the moduli and hence irrelevant. The explanation in this mismatch comes from the fact that the coordinate transformation from one gauge to the other is singular at $q^2$ order. After using (2.68) on (2.86) we get:

$$L_{hom} = \left( l^{(5)} g_{tt}^{(2)} + \left( l^{(6)} + l^{(4)} \times R^{(1)}_{\partial_y g_{tt}^{(2)}} \right) A^{(2)} + \left( l^{(7)} + l^{(4)} \times R^{(2)}_{\partial_y g_{tt}^{(2)}} \right) \phi^{(2)} \right) . \tag{2.87}$$

Now, we asymptotically expand $L_{hom}$. For this we need to use the most general form of the solutions to $g_{tt}^{(2)}$, $A^{(2)}$ and $\phi^{(2)}$. After a straightforward computation one gets:

$$L_{hom} \xrightarrow[y \to y_{UV}]{} \frac{g_1^2 g_2^2}{8c_1} e^{2s_p} C_5(t,x) + O\left(\frac{1}{y_{UV}}\right) , \tag{2.88}$$

$$L_{hom} \xrightarrow[y \to y_{IR}]{} \frac{g_1^2 g_2^2}{8c_1} e^{2s_p} C_5(t,x) + O\left(y_{IR}\right) . \tag{2.89}$$

The only integration constant entering the boundary data is given by $C_5(t,x)$. However $[L_{hom}]_{y_{IR}}^{y_{UV}}$ vanishes, and the boundary effective action for the moduli $s_p$ and $\tau$ coming from the bulk action is independent of all the integration constants, namely, any particular solution of the inhomogeneous system of differential equations gives the same final result, so far. We say so far, because still we have not commented about the GH and CT contributions. This is an interesting outcome, since the result holds independently of the IR regularity and normalisability conditions imposed on the fluctuations discussed earlier. The GH term will not affect this observation, but the CT contribution does it. In any case, we choose integration constants in order to satisfy our cardinal principle: IR regularity.

### 2.2.4 Gibbons-Hawking contribution

Let us discuss now the GH contribution. In the domain wall coordinates, (2.113), it reads:

$$\frac{1}{2}\int dtdx L_{2D}^{GH} = \frac{1}{2}\int dtdx\sqrt{g^{rr}}\partial_r\left(\sqrt{g^{rr}}\sqrt{-\det g}\right)|_{boundary}, \tag{2.90}$$

where so far $g_{rr} = 1$, but for later purposes it is convenient to write the most general form above. In the $(t, x, y)$ coordinates and after using (2.68) it is simple to show that:

$$L_{2D}^{GH} = \left(-\frac{g_1^2 g_2^2 y\left(g_2^2(y+1)^3 + g_1^2(y-1)\right)}{4c_1(y+1)\left(g_2^2(y+1)^2 - g_1^2\right)}e^{2s_p} - 2L_{hom}\right)|_{boundary}, \tag{2.91}$$

The UV and IR asymptotic expansions are thence given by:

$$L_{2D}^{GH} \xrightarrow[y\to y_{UV}]{} -\frac{g_1^2 g_2^2}{4c_1}e^{2s_p}y_{UV} + O\left(\frac{1}{y_{UV}}\right), \tag{2.92}$$

$$L_{2D}^{GH} \xrightarrow[y\to y_{IR}]{} \frac{g_1^2 g_2^2}{4c_1}e^{2s_p}y_{IR} + O\left(y_{IR}^2\right). \tag{2.93}$$

Even though we are not taking the approach of cutting off the geometry in the IR side, we present the IR behaviour of GH term just for completeness of analysis. An important point to stress on is that there is not finite contribution coming from them and again one should notice the independence of the final result on the integration constants, as previously mentioned.

**Regularized Action** At this point we can write down the regularized Lagrangian for the "normalisable" modulus $s_p$. We first make the change to the Fefferman-Graham gauge at $q^0$ order, $y \to \tilde{y}$, make use of the normalisability condition (2.79) and the final result becomes:

$$S_{reg}^{2D} = \int dtdx\left(\frac{g_1^2 g_2^2}{8c_1}\tilde{y}_{UV} - \frac{c_1}{2g_2^2}\Box s_p \log\tilde{y}_{IR} + \frac{1}{2}\frac{c_1}{(g_2^2 - g_1^2)}(\partial s_p)^2 + ...\right), \tag{2.94}$$

where the ... stand for subleading contributions in terms of the cutoffs and finite total derivative terms. Notice that there is no logarithmic divergence at the UV cutoff. This is because this modulus is not affecting the UV boundary metric. On the other hand the IR side does have a logarithmic divergent factor, which however is a total derivative.

Now, we discuss possible contributions coming from covariant counterterms. Let us start by gravitational countertems. In the asymptotically AdS$_3$ geometries the leading divergence in the on-shell action is renormalized by using the covariant term

$$\int d^2x\sqrt{-\det\gamma}|_{bdry} = \int d^2x \frac{g_2^2}{2}\tilde{y}_{UV} + \frac{c_1}{g_1^2}\left(\frac{2c_1}{g_1^2}\Box s_p + \Box\tau\right) + O\left(\frac{1}{\tilde{y}_{UV}}\right).$$

Other possible counterterms are:

$$\int dtdx\sqrt{-\gamma}R^{(2D)}[\gamma]|_{bdry} = \frac{1}{c_1}\left(\frac{2c_1}{g_1^2}\Box s_p + \Box\tau\right) + O\left(\frac{1}{\tilde{y}_{UV}}\right), \tag{2.95}$$

$$\int dtdx\sqrt{-\gamma}(\delta A)^2|_{bdry} = -\frac{2c_1^2}{3g_1^2 g_2^2}\Box s_p, \quad \int dtdx\sqrt{-\gamma}(\delta\phi)^2|_{bdry} = O\left(\frac{1}{\tilde{y}_{UV}^3}\right), \tag{2.96}$$

where $\delta A$, $\delta\phi$ denote the fluctuations around the UV stationary point of the potential. Notice that after imposing the normalisability condition (2.79) the finite contributions of this counterterm disappear except for the $\delta A$ fluctuation which is a total derivative contribution. The remaining IR logarithmic divergence is minimally subtracted. Finally the renormalized action takes the form:

$$S_{ren}^{2D} = \int dtdx\left(\frac{1}{2}\frac{c_1}{(g_2^2 - g_1^2)}(\partial s_p)^2 + O\left(\partial^4\right)\right). \tag{2.97}$$

The coefficient in front of this action is not the difference of central charges of the UV/IR fixed points. Although we can always rescale the field, this mismatch is unpleasant, because a rigid shifting in the spurion mode $\tau$ (not on $s_p$) rescales the CFT metric (UV side) in accordance with the normalization used in [5], and the mode $s_p$ only contributes through total derivatives to the boundary Lagrangian. So, the QFT side is saying that once fixed the proper normalization, the corresponding coefficient of the kinetic term of the spurion should coincide with the difference of central charges. This, points towards the conclusion that the modulus $\tau$ seems not to be the optimal description for the QFT spurion. In fact the $PBH$ modulus $\tau$ looks like a warped $PBH$ in the Fefferman-Graham gauge, see A.1.2, so the outcome of the 2D version of the computation done in section 2.1.3 will change. We will show the result in the next subsection.

The appropriate description of the spurion from the bulk side seems to be associated to a rigid $PBH$ in Fefferman-Graham gauge. As we already said the modulus $\rho$ could be seen as a combination of a $PBH$ of that kind and the mode $s_p$. So, following our line of reasoning $\rho$ seems to be the most natural bulk description of the dilaton. In fact in the 6D analysis to be discussed in section 4 this identification will become even more natural.

### 2.2.5 Checking the $PBH$ procedure.

There is an equivalent way to arrive to (2.97). We present it here because it gives a check of the procedure we used to compute the spurion effective action in a 4D RG flow. As was already noticed the modulus $\tau$ can be related to a family of diffeomorphisms. To check the procedure we take as starting point the bulk on-shell action of the modulus $s_p$ without turning on $\tau$:

$$\begin{aligned}\int dtdx L_{2D}^{bulk} &= \int dtdx \Bigg( \frac{g_1^2 g_2^2}{8c_1} e^{2s_p(t,x)} y\big|_{IR}^{UV} + \frac{c_1}{2g_1^2} \Box s_p \log y_{UV} \\ &\quad - \frac{c_1(g_2^2 - g_1^2)}{2g_1^2 g_2^2} \Box s_p \log y_{IR} + \ldots + L_{hom}^{\tau=0}\big|_{IR}^{UV} \Bigg), \end{aligned} \qquad (2.98)$$

and perform the UV and IR asymptotic expansions of the corresponding $PBH$ transformation (A.3) keeping only terms up to second order in derivatives. The result coincides with (2.85). Notice that the $PBH$ transformations do not affect the boundary conditions of the matter field (2.21), provided we take the restriction (2.79). So all the IR constraints and normalisability conditions we imposed before will still hold in this second approach provided they were imposed at $\tau = 0$.

Finally, after applying the same previous procedure to the GH term and to the counterterms, namely transforming the metric (2.113) at vanishing $\tau$-modulus, gives (2.92) and (2.95) respectively.

### 2.2.6 The $\rho$-branch analysis

We can repeat the same computations done before but using the $\rho$ modulus instead of the pair $(\tau, s_p)$. The trace and off-diagonal modes $T$ and $g_{tx}^{(2)}$ can be solved from the decoupled equations $(t,t)+(x,x)$ and $(t,x)$ to be:

$$T = C_3(t,x) - \frac{1}{y}C_2(t,x) + \frac{c_1^2}{g_1^2 g_2^2 y \left(g_2^2 (y+\rho)^2 - g_1^2\rho^2\right)} \left((\partial_t\rho)^2 + (\partial_x\rho)^2\right), \tag{2.99}$$

$$g_{tx}^{(2)} = -\frac{C_6(t,x)}{y} + C_7(t,x) - \frac{2c_1^2}{g_1^2 g_2^2 y \left(g_2^2(y+\rho)^2 - g_1^2\rho^2\right)}\partial_t\rho\partial_x\rho. \tag{2.100}$$

In the same manner, we can then solve for all fluctuations in terms of $A^{(2)}$ by integrating the $(t,t)-(x,x)$ and $(r,r)$ equations:

$$\partial_y g^{(2)} = R^{(1)}_{\partial_y g^{(2)}} A^{(2)} + R^{(2)}_{\partial_y g^{(2)}}\phi^{(2)} + \frac{1}{y^2}C_5, \tag{2.101}$$

$$\begin{aligned}\partial_y\phi^{(2)} &= R^{(1)}_{\partial_y\phi^{(2)}}\partial_y A^{(2)} + R^{(2)}_{\partial_y\phi^{(2)}}\phi^{(2)} + R^{(3)}_{\partial_y\phi^{(2)}}A^{(2)} + \\ &\quad R^{(4)}_{\partial_y\phi^{(2)}}C_5 + R^{(5)}_{\partial_y\phi^{(2)}}\Box\rho + R^{(6)}_{\partial_y\phi^{(2)}}(\partial\rho)^2,\end{aligned} \tag{2.102}$$

with:

$$R^{(1)}_{\partial_y g^{(2)}} = \frac{6g_1 g_2^3\rho(\rho+y)^2}{\left(g_2^2(y+\rho)^2 - g_1^2\rho^2\right)^2},\ R^{(2)}_{\partial_y g^{(2)}} = -\frac{2g_1^2\rho^2}{(\rho+y)\left(g_2^2(y+\rho)^2 - g_1^2\rho^2\right)}, \tag{2.103}$$

which is also found to obey a third order linear differential equation of the form:

$$\partial_y^{(3)}A^{(2)} + R^{(2)}_{A^{(2)}}\partial_y^2 A^{(2)} + R^{(1)}_{A^{(2)}}\partial_y A^{(2)} + R^{(0)}_{A^{(2)}}A^{(2)} = F_\rho, \tag{2.104}$$

where

$$F_\rho = F^1(y)\Box\rho + F^{(2)}(y)(\partial\rho)^2 + F^{(3)}(y)C_5(t,x). \tag{2.105}$$

The rational functions $F^{(1)}$, $F^{(2)}$ and $F^{(3)}$ are given in the second paragraph of appendix A.2.2. We solve this equation by the Green's function method (see second paragraph appendix A.2.3). As for the case before, we use the nine integration constants to demand IR regularity and as much normalisability as possible. In this case we are able to turn off UV sources except for one of the two corresponding to $\Delta = 2$ and $\Delta = 4$ CFT operators. We choose to allow a non vanishing source of the $A$ scalar field, namely at the UV boundary, $y = y_{UV}$:

$$A^{(2)} \sim \frac{c_1^2}{3g_1^3 g_2^3}\frac{(\partial\rho)^2 - \rho\Box\rho}{\rho^3}. \tag{2.106}$$

We compute then the full renormalized boundary action

$$S_{ren} = S_{bulk} + S_{GH} + S_{CT}.$$

The result up to total derivatives and without ambiguity in renormalization (as for the previous case) is:

$$S_{ren} = \int dtdx \left( \frac{c_1}{g_2^2} (\partial s)^2 + O\left(\partial^4\right) \right), \tag{2.107}$$

where $s = \log(\rho)$. Notice that the coefficient in front of this kinetic term is proportional to the difference of holographic central charges among the interpolating fixed points, which in 2D can be identified with the difference of $AdS_3$ radii $\Delta L = \frac{2c_1}{g_2^2}$. Notice that we have a freedom in normalization of $s$. We have chosen the normalization to agree with [4, 5]. Namely, the associated $PBH$ diffeo shifts the UV/IR metric from $\eta$ to $e^{-2\sigma_{PBH}}\eta$. As we mentioned the $\rho$ modulus is a combination of a $PBH$ mode with $s_p$. So we can again check the procedure used in section 2.1.3 via (2.107).

We can see the rigid $\rho$ modulus as a combination of a $PBH$ mode $y \to e^{2\sigma_{PBH}}y$ and the $s_p = -\sigma_{PBH}$ mode. This last constraint guarantees not to turn on sources for the CFT's energy momentum tensor (nor for the hypothetical IR one). To obtain the bulk contribution we perform the $PBH$ transformation (A.7)-(A.8), on the on-shell action with only $s_p$ turned on (2.98). Before performing the $PBH$ transformation, explicit solutions in terms of $s_p$ are demanded to be IR regular and as normalisable as possible. As usual, we choose to let on the source of the dimension $\Delta = 2$ CFT operator, which we can read from (2.82). As in previous cases. The GH and Counterterms (CT) contributions are evaluated by explicit use of the transformed metric and fields. The GH term does not contribute to the final result for the regularized action at all. As for the CT's, they contribute with total derivatives to the final result of the effective action which, under the identification $\sigma_{PBH} \equiv s$, coincides with (2.107).

A last comment about the relation between bulk normalisability and the identification of (2.107) as quantum effective action for $s$: Notice that demanding normalisability of the mode $s$ amounts to impose the on-shell condition

$$\Box s = 0,$$

in both equations (2.82) and (2.106). This is in agreement with holographic computations of hadron masses, where normalisability gives rise to the discreteness of the spectrum and indeed puts on-shell the states corresponding to the hadrons[1]. On the other hand, the on shell supergravity action, as already mentioned in the paragraph below (2.86), is independent of $A^{(2)}$. Also, as shown in (2.96), the contributions coming from counter terms which depend on $A^{(2)}$ give contributions that are linear in the source for the operator dual to $A$, at order $q^2$, but at the end, these contributions reduce to total derivatives in (2.107). Notice that no other sources, apart from the one corresponding to the operator dual to $A$ are turned on. Therefore (2.107) has no source dependence and can be interpreted as the (off-shell) effective action for the massless mode $s$.

## 2.3 6D Analysis

Six dimensional supergravity coupled to one anti-self dual tensor multiplet, an $SU(2)$ Yang-Mills vector multiplet and one hypermultiplet is a particular case of the general $N = 1$ 6D supergravity constructed in [25] and admits a supersymmetric action. The bosonic equations of motion for the graviton $g_{MN}$, third rank anti-symmetric tensor $G_{3MNP}$, the scalar $\theta$ and the $SU(2)$ gauge fields $A^I_M$ are:

$$
\begin{aligned}
R_{MN} - \frac{1}{2}g_{MN}R - \frac{1}{3}e^{2\theta}\big(3G_{3MPQ}G_{3N}{}^{PQ} - \frac{1}{2}g_{MN}G_{3PQR}G_3{}^{PQR}\big) & \\
-\partial_M\theta\partial^M\theta + \frac{1}{2}g_{MN}\partial_P\theta\partial^P\theta - e^{\theta}\big(2F^{IP}_M F^I_{NP} - \frac{1}{2}g_{MN}F^I_{PQ}F^{IPQ}\big) &= 0, \qquad (2.108)\\
e^{-1}\partial_M(eg^{MN}\partial_N\theta) - \frac{1}{2}e^{\theta}F^I_{MN}F^{IMN} - \frac{1}{3}e^{2\theta}G_{3MNP}G_3{}^{MNP} &= 0, \qquad (2.109)\\
\mathcal{D}_N(ee^{\theta}F^{IMN}) + ee^{2\theta}G^{MNP}F^I_{NP} &= 0, \qquad (2.110)\\
D_M(ee^{2\theta}G_3{}^{MNP}) &= 0. \qquad (2.111)
\end{aligned}
$$

[1] We would like to thank a referee from JHEP for pointing out this analogy.

The three-form $G_3$ is the field strength of the two form $B_2$ modified by the Chern-Simons three-form, $G_3 = dB_2 + tr(FA - \frac{2}{3}A^3)$, with the $SU(2)$ gauge field strength $F = dA + A^2$. As a result there is the modified Bianchi identity for the 3-form:

$$dG_3 = trF \wedge F. \tag{2.112}$$

We are going to consider all the fields depending on coordinates $u$, $v$ and $r$ where $u$ and $v$ are light-cone coordinates given by $u = t + x$, $v = t - x$, and $r$ is a radial coordinate. For the metric we take the following $SO(4)$ invariant ansatz:

$$ds_6^2 \;=\; e^{2f}(g_{uu}du^2 + g_{vv}dv^2 + 2g_{uv}dudv) + e^{-2f}\left(dr^2 + r^2 d\Omega^2\right), \tag{2.113}$$

where $d\Omega^2$ is the $SO(4)$ invariant metric on $S^3$:

$$d\Omega^2 = d\phi^2 + sin^2(\phi)\left(d\psi^2 + sin^2(\psi)d\chi^2\right), \tag{2.114}$$

and $f$, $g_{uu}$, $g_{uv}$, $g_{vv}$ are functions of $(u, v, r)$, from now on we will not show this dependence. As for the $SU(2)$ one-form $A$, we take it to be non trivial only along $S^3$, preserving a $SU(2)$ subgroup of $SO(4)$,

$$A = is\sum_{k=1}^{3} \sigma^k \omega^k, \tag{2.115}$$

where $\sigma^k$ are Pauli matrices and $\omega^k$ left-invariant one-forms on $S^3$, and $s$ is a function of $(u, v, r)$. For the three-form $G_3$, we take it to be non trivial only along $u, v, r$ and along $S^3$,

$$G_3 = G_3^{(1)} du \wedge dv \wedge dr + G_3^{(2)} \sin^2(\phi)\sin(\psi) d\phi \wedge d\psi \wedge d\chi, \tag{2.116}$$

where the functions $G_3^{(1,2)}$ only depend on $(u, v, r)$ . Finally we will have a non trivial scalar field $\theta(u, v, r)$.

### 2.3.1 Deforming the RG flow background

The aim of this section is to look for a solution of the above equations of motion which deforms the RG flow solution of [23], with the appropriate boundary conditions to be specified in due course (in order to demand IR regularity). To be more specific, this background is actually BPS. It preserves half of the 8 supercharges and interpolates between two $AdS_3 \times S^3$ geometries for $r \to \infty$, the UV region, and $r \to 0$, the IR region, with different $S^3$ and $AdS_3$ radii. It describes a naively speaking, v.e.v. driven RG flow between two $(4,0)$ SCFT's living at the corresponding $AdS$ boundaries parametrized by the coordinates $u, v$. The solution involves an $SU(2)$ instanton centered at the origin of the $R^4$ with coordinates $r$, $\phi$, $\psi$, $\chi$, corresponding to $s = \rho^2/(r^2+\rho^2)$. The scale modulus $\rho$ enters also in the other field configurations, as will be shown shortly. Our strategy here is to promote $\rho$ to a function of $u, v$, $\rho = \rho(u,v)$. So, the starting point will be given by the field configurations:

$$\begin{aligned} g^{(0)}_{uu} &= g^{(0)}_{vv} = 0,\ g^{(0)}_{uv} = -1/2, \\ s^{(0)} &= \rho^2/(r^2+\rho^2), \\ f^{(0)} &= -\frac{1}{4} log[\frac{c}{r^2}(\frac{d}{r^2} + \frac{1}{r^3}\partial_r(r^3\partial_r log(r^2+\rho^2))], \\ \theta^{(0)} &= 2f^{(0)} + log(c/r^2). \end{aligned} \tag{2.117}$$

Notice that $s^{(0)}$ goes like $\rho^2/r^2$ in the UV. As for the three-form, it turns out that the following expressions for $G_3^{(1)}$ and $G_3^{(2)}$ solve identically the Bianchi identity and equations of motion:

$$\begin{aligned} G_3^{(1)} &= e^{4f-2\theta}\sqrt{-detg}\ c/r^3, \\ G_3^{(2)} &= -\left(4+d+4s^2(-3+2s)\right), \end{aligned} \tag{2.118}$$

where $det(g) = -g_{uu}g_{vv} + g_{uv}^2$ and $f$, $\theta$ and $s$ are functions of $(u,v,r)$. As explained in [23, 58], the positive constants $c$ and $d$ are essentially electric and magnetic charges, respectively, of the dyonic strings of 6D supergravity. More precisely we have:

$$\begin{aligned} Q_1 &= \frac{1}{8\pi^2}\int_{S^3} e^{2\theta} * G = c/4, \\ Q_5 &= \frac{1}{8\pi^2}\int_{S^3} G = d/4+1, \end{aligned} \tag{2.119}$$

where we see that the instanton contributes to $Q_5$ with one unit as a consequence of the modified Bianchi identity (2.112). The constants $c$ and $d$ determine the central charges of the UV and IR CFT's, respectively: $c_{UV} = c(4+d)$, $c_{IR} = cd$ [23].

These fields solve the equations of motion only if $\rho$ is constant (apart from $G_3^{(1,2)}$ which solve them identically). We will then deform the above background to compensate for the back reaction due to the $u, v$ dependence of $\rho$. In this way one can set up a perturbative expansion in the number of $u, v$ derivatives. For the purpose of analyzing the equations of motion keeping track of the derivative expansion, again it is convenient to assign a counting parameter $q$ for each $u, v$ derivative. The first non-trivial corrections to the above background will involve two $u, v$-derivatives of $\rho(u, v)$. i.e. terms that are linear in two derivatives of $\rho(u, v)$ or quadratic in its first derivatives. From now on we will not write down the coordinate dependence of the modulus $\rho$. Therefore we start with the following ansatz for the deformed background:

$$\begin{aligned}
f_b(u,v,r) &= f^{(0)}(u,v,r) + q^2 f^{(2)}(u,v,r), \\
s_b(u,v,r) &= s^{(0)}(u,v,r) + q^2 s^{(2)}(u,v,r), \\
\theta_b(u,v,r) &= \theta^{(0)}(u,v,r) + q^2 \theta^{(2)}(u,v,r), \\
g_{buv}(u,v,r) &= -1/2 + q^2 g_{uv}^{(2)}(u,v,r), \\
g_{buu}(u,v,r) &= q^2 g_{uu}^{(2)}(u,v,r),\ g_{bvv}(u,v,r) = q^2 g_{vv}^{(2)}(u,v,r).
\end{aligned} \tag{2.120}$$

Our first task is to determine these deformations as functions of $\rho$ and its derivatives. The structure of the resulting, coupled differential equations for the deformations is clear: they will be ordinary, linear second order differential equations in the radial variable $r$ with inhomogeneous terms involving up to two derivatives of $\rho$. Due to the symmetry of the problem, there is only one independent equation for the gauge field, with free index along $S^3$, say $\phi$, and the non trivial Einstein's equations, $E_{MN}$, arise only when $M, N$ are of type $u, v, r$ and for $M = N$ along one of the three coordinates of $S^3$, e.g. $\phi$. The traceless part of the Einstein equations $E_{uu}$ and $E_{vv}$ involve only $g_{uu}^{(2)}$ and $g_{vv}^{(2)}$ respectively and these differential equations can be solved easily. The equations $E_{uv}$, $E_{\phi\phi}$, $E_{rr}$, the gauge field equation and the $\theta$ equation involve only $g_{uv}^{(2)}$, $s^{(2)}(u, v, r)$, $f^{(2)}$ and $\theta^{(2)}$. Since a constant scaling of $u$ and $v$ in the zeroth order background solution is equivalent to turning on a constant $g_{uv}^{(2)}$, the latter enters these equations only with derivatives with respect to $r$ at $q^2$ order. Therefore we can find three linear combinations of these equations that do not involve $g^{(2)}_{uv}$. To simplify these three equations further, it turns out that an algebraic constraint among the fields $f$, $\theta$ and $s$, dictated by consistency of the $S^3$ dimensional reduction of the 6D theory down to 3D, gives a hint about a convenient way to decouple the differential equations by redefining the field $\theta$ in the following way

$$e^{\theta} = \frac{r^2 e^{-2f} e^{\varphi}}{(4+d-s^2)}. \tag{2.121}$$

Note that for the reduction ansatz, $\varphi = 0$. In general the new field $\varphi$ will also have an expansion in $q$ of the form:

$$\varphi(u,v,r) = \varphi^{(0)}(u,v,r) + q^2\varphi^{(2)}(u,v,r). \tag{2.122}$$

For the zeroth order solution defined above one can see that $\varphi^{(0)} = 0$. The reduction ansatz indicates that at order $q^2$ one can find a combination of the linear second order differential equations which gives a decoupled homogeneous second order equation for $\varphi^{(2)}$. This equation can be solved for $\varphi^{(2)}$, which involves two integration constants denoted by $a_1$ and $a_2$ (that are functions of $u$ and $v$)

$$\begin{aligned}
\varphi^{(2)}_h &= a_1(u,v)\frac{48r^6(r^2+\rho^2)^2\log(\frac{r^2+\rho^2}{r^2}) - 48r^6\rho^2 - 24r^4\rho^4 + (12+d)r^2\rho^6 + d\rho^8}{r^4\rho^2((4+d)r^4 + 2(4+d)r^2\rho^2 + d\rho^4)} \\
&+ a_2(u,v)\frac{4r^2(r^2+\rho^2)}{\rho^2((4+d)r^4 + 2(4+d)r^2\rho^2 + d\rho^4)},
\end{aligned} \tag{2.123}$$

and after substituting this solution, we get two second order differential equations for $s^{(2)}$ and $f^{(2)}$. In general one can eliminate $f^{(2)}$ from these two equations and obtain a fourth order differential equation for $s^{(2)}$. However, it turns out that in these two equations $f^{(2)}/r^2$ appears only through $r$-derivatives [1] and this results in a third order decoupled differential equation for $s^{(2)}$

$$A_3(r)\partial_r^3 s^{(2)} + A_2(r)\partial_r^2 s^{(2)} + A_1(r)\partial_r s^{(2)} + A_0(r)s^{(2)} = B(r), \tag{2.124}$$

[1] This can be understood by observing that one can add a constant to the solutions for $e^{\theta-2f}$ and $e^{-\theta-2f}$ in equations (3.28) and (3.26). At the infinitesimal level this is equivalent to turning on a constant $f^{(2)}/r^2$.

where

$$\begin{aligned}
A_3(r) &= r^3(r^2+\rho^2)^6((4+d)r^4+2(4+d)r^2\rho^2+d\rho^4)^2, \\
A_2(r) &= r^2(r^2+\rho^2)^5(11(4+d)^2r^{10}+51(4+d)^2r^8\rho^2 \\
&\quad +2(4+d)(128+47d)r^6\rho^4 \\
&\quad +2(4+d)(24+43d)r^4\rho^6+d(80+39d)r^2\rho^8+7d^2\rho^{10}), \\
A_1(r) &= r(r^2+\rho^2)^4(21(4+d)^2r^{12}+130(4+d)^2r^{10}\rho^2 \\
&\quad +(4+d)(948+311d)r^8\rho^4 \\
&\quad +4(4+d)(100+91d)r^6\rho^6+(-192+456d+211d^2)r^4\rho^8 \\
&\quad +10d(-8+5d)r^2\rho^{10}+d^2\rho^{12}), \\
A_0(r) &= 16\rho^2(r^2+\rho^2)^3(4(4+d)^2r^{12}+(4+d)(72+19d)r^{10}\rho^2 \\
&\quad +(4+d)(72+35d)r^8\rho^4 \\
&\quad +2(16+54d+15d^2)r^6\rho^6+2d(6+5d)r^4\rho^8-d^2r^2\rho^{10}-d^2\rho^{12}), \\
B(r) &= 16c\rho(r^2+\rho^2)^2((4+d)r^4+2(4+d)r^2\rho^2+d\rho^4)^3\partial_u\partial_v\rho \\
&\quad +16c(r^4+2r^2\rho^2-3\rho^4)((4+d)r^4+2(4+d)r^2\rho^2+d\rho^4)^3\partial_u\rho\partial_v\rho.
\end{aligned} \tag{2.125}$$

The three independent solutions of the homogeneous part of the above equation are

$$\begin{aligned}
s_h^{(2)} &= a_3(u,v)\frac{3(4+d)r^8+24(4+d)r^6\log(r/\rho)\rho^2}{12r^4(r^2+\rho^2)^2}+ \\
&\quad a_3(u,v)\frac{-6(10+3d)r^4\rho^4-6(2+d)r^2\rho^6-d\rho^8}{12r^4(r^2+\rho^2)^2}+ \\
&\quad a_4(u,v)\frac{\rho^2(24r^6\log(1+\rho^2/r^2)-24r^4\rho^2+3(8+d)r^2\rho^4+2d\rho^6)}{144r^4(r^2+\rho^2)^2} \\
&\quad +a_5(u,v)\frac{r^2\rho^2}{(r^2+\rho^2)^2}.
\end{aligned} \tag{2.126}$$

Using the most general solution of the homogeneous equation one can construct the Green's function for the third order differential equation and obtain a particular solution of the full inhomogeneous equation

$$\begin{aligned} s_p^{(2)} &= \frac{c(3(4+d)r^6 - 6(4+d)r^4\rho^2 - 2(30+7d)r^2\rho^4 - 5d\rho^6)}{3r^4(r^2+\rho^2)^3}\partial_u\rho\partial_v\rho \\ &+\frac{c\rho(3(4+d)r^4 + 3(4+d)r^2\rho^2 + d\rho^4)}{3r^4(r^2+\rho^2)^2}\partial_u\partial_v\rho. \end{aligned} \tag{2.127}$$

Substituting the general solution for $s^{(2)}$ in the remaining equations one gets first order linear differential equations for $f^{(2)}$ and $g_{uv}^{(2)}$ which can be solved easily resulting in two more integration constants. Moreover, $E_{uu}$ and $E_{vv}$ give two decoupled second order differential equations for the traceless part of the metric $g_{uu}^{(2)}$ and $g_{uu}^{(2)}$ that can also be readily solved giving another four integration constants. In all there are eleven integration constants as compared to nine integration constants in the 3D case discussed in the previous sections. This is to be expected since the $S^3$ reduction ansatz from 6D to 3D sets $\varphi = 0$. Finally $E_{ru}$ and $E_{rv}$ at order $q^3$ give first order partial differential equations in $u$ and $v$ variables on the integration constants. The full homogeneous solution and a particular solution for the inhomogeneous equations are given in Appendix A.3.

Now we turn to the analysis of the IR and UV behaviour of the general solutions. The general solution for $s^{(2)}$ is a sum of the particular solution (2.127) and the homogeneous solution (2.126). Near $r = 0$ this solution has divergent $1/r^4$ and $1/r^2$ terms that can be set to zero by choosing:

$$a_3(u,v) = \frac{4c\partial_u\partial_v \log\rho}{3\rho^2}, \qquad a_4(u,v) = \frac{16c}{\rho^4}(7\partial_u\rho\partial_v\rho - \rho\partial_u\partial_v\rho). \tag{2.128}$$

Similarly analyzing the general solution for $\varphi^{(1)}$ one finds that it has also IR divergent $1/r^4$ and $1/r^2$ terms that can be set to zero by setting $a_1(u,v) = 0$. With these choices we have checked that Ricci scalar and Ricci square curvature invariants are non-singular at $r = 0$.

Finally, the Einstein equations $E_{ur}$ and $E_{vr}$ give certain partial differential equations with respect to $v$ and $u$ on the integration constants $b_1$ and $c_1$ respectively and these are solved by:

$$b_1 = \frac{4c\,(-2(\partial_u\rho)^2 + \rho\partial_u^2\rho)}{\rho^2}, \qquad c_1 = \frac{4c\,(-2(\partial_v\rho)^2 + \rho\partial_v^2\rho)}{\rho^2}. \tag{2.129}$$

With these conditions even the metric functions $g_{uu}$, $g_{vv}$ and $g_{uv}$ have no power like singularities in $r$ near $r \to 0$. Thus we have a smooth solution near IR up to $q^2$ order.

In the UV region, $r \to \infty$, the source terms behave as $O(r^2)$ for $\varphi$ and $f$, and $O(1)$ for the metric $g_{uv}$, $g_{uu}$ and $g_{vv}$. By making an asymptotic expansion of the homogeneous solutions one can see that $a_2$, $a_4$, $a_7$, $b_2$ and $c_2$ control these source terms. Since in our background we do not want to turn on any sources, we set these integration constants to zero.

Finally, the UV behaviour of the gauge field $s^{(2)}$ is:

$$\frac{c(4+d)\partial_u\partial_v \log\rho}{3\rho^2}(1-\frac{2\rho^2}{r^2}(4\log(\frac{\rho}{r}+1))+\frac{\rho^2}{r^2}a_5. \tag{2.130}$$

It turns out though that IR regularity forces us to allow a source term for the $s_b(u,v,r)$ field, this is a term of order $r^0$ for $r \to \infty$ and of order $q^2$:

$$s^{(2)} \to \frac{c(4+d)(\rho\partial_u\partial_v\rho - \partial_u\rho\partial_v\rho)}{3\rho^2} + \mathcal{O}(1/r), \tag{2.131}$$

as $r \to \infty$. Notice that here, like in the 3D case, discussed at the end of section 2.2.6, the source term for the operator dual to $s$ is proportional to the EoM for the massless scalar $\log\rho$, and therefore vanishes on-shell.

### 2.3.2 Finding linearised fluctuations around the deformed background

Having determined the background corrected by the leading terms involving two space-time derivatives of the modulus $\rho$, we could compute the regularized on shell action, as was done in the 3D case. We find it more convenient to compute directly one-point functions of dual operators (especially of the stress energy tensor). To this end we need to switch on corresponding sources and therefore to solve the linearized equations of motion of the various fields on the deformed background. This is done again in a derivative expansion starting with the following ansatz for the fields fluctuations:

$$\delta s = \delta^{(0)}s + q^2\delta^{(2)}s,\ \delta g_{uu} = \delta^{(0)}g_{uu} + q^2\delta^{(2)}g_{uu}, \delta g_{vv} = \delta^{(0)}g_{vv} + q^2\delta^{(2)}g_{vv}, \tag{2.132}$$

$$\delta g_{uv} = \delta^{(0)}g_{uv} + q^2\delta^{(1)}g_{uv},\ \delta f = \delta^{(0)}f + q^2\delta^{(2)}f,\ \delta\theta = \delta^{(0)}\theta + q^2\delta^{(2)}\theta, \tag{2.133}$$

where $\delta^{(0)}$ stands for the zeroth order in space-time derivatives, and $\delta^{(2)}$ stands for fluctuations coming at second order in space time derivatives and this is why is weighted by $q^2$. The general solution for $\delta^{(0)}$ is the homogeneous solution given in Appendix A.3. We fix the integration constants so that

$$\delta^{(0)}g_{uu} = h_{uu},\ \delta^{(0)}g_{vv} = h_{vv},\ \delta^{(0)}g_{uv} = h_{uv}, \tag{2.134}$$

$$\delta^{(0)}f = \frac{2\rho^4 r^2}{(\rho^2+r^2)\left(d\rho^4 + (4+d)\,r^4 + 2\,(4+d)\,\rho^2 r^2\right)}a_5(u,v), \tag{2.135}$$

$$\delta^{(0)}\theta = \frac{4\rho^4 r^2}{(\rho^2+r^2)\left(d\rho^4 + (4+d)\,r^4 + 2\,(4+d)\,\rho^2 r^2\right)}a_5(u,v), \tag{2.136}$$

$$\delta^{(0)}s = \frac{r^2\rho^2 a_5(u,v)}{(r^2+\rho^2)^2}, \tag{2.137}$$

where $h_{uu}$, $h_{vv}$ and $h_{uv}$ are the integration constants $b_2(u,v)$, $c_2(u,v)$ and $a_7(u,v)$ respectively. Consequently they are the sources for the boundary stress energy tensor components $T_{uu}$, $T_{vv}$, and $T_{uv}$. These $h$'s are small fluctuations around the flat boundary metric, $g^{(0)} = \eta + h$, and the corresponding linearized curvature is

$$R^{(2)}(g^{(0)}) = -2(\partial_v^2 h_{uu} - 2\partial_u\partial_v h_{uv} + \partial_u^2 h_{vv}). \tag{2.138}$$

We have also kept the integration constant $a_5$ for reasons that will become apparent later on.

The next step is to solve the equations of motion at order $q^2$ for the $\delta^{(2)}$ fields. The equations for $\delta^{(2)}$ fields contain also inhomogeneous terms that involve $\delta^{(0)}$ fields and their derivatives, up to second order with respect to $u$ and $v$. The procedure is the same as the one employed in solving for the corrected background. As the differential equations are inhomogeneous, the general solution will be the sum of the homogeneous solution and a particular solution of the inhomogeneous one, which can be obtained using Green's functions once we have the homogeneous solutions. The integration constants in the homogeneous part of solution can be partially fixed by requiring IR smoothness and absence of sources for $\delta^{(2)}\theta$ and $\delta^{(2)}f$. Moreover some sources can be reabsorbed in the already existing sources at zeroth order. Finally, the mixed $u, v$ and $r$ Einstein's equations result in differential constraints among the integration constants.

Concerning the IR behaviour, the metric components go, for $r \to 0$, as:

$$\begin{aligned}
\delta^{(2)} g_{uv} &\sim -\frac{cd}{4} R^{(2)}/r^2, \\
\partial_v \delta^{(2)} g_{uu} &\sim -\frac{cd}{4} \partial_u R^{(2)}/r^2, \\
\partial_u \delta^{(2)} g_{vv} &\sim -\frac{cd}{4} \partial_v R^{(2)}/r^2.
\end{aligned} \tag{2.139}$$

The apparent $1/r^2$ singularity is presumably a coordinate singularity: we have verified that both the 6D Ricci scalar and Ricci squared are finite both at the IR and UV. The other fields are manifestly regular at the IR. We have seen that there is a physical fluctuation for the operator $\mathcal{O}_s$ proportional to $\rho^2$ at order $q^0$ and that at order $q^2$ there is a source, $J_s$, which couples to it, proportional to $\Box \log(\rho)/\rho^2$. Therefore we expect that, at order $q^2$, the corresponding term $\mathcal{O}_s J_s$ in the boundary action will not give any contribution being a total derivative. So, this type of term will not contribute to the dilaton $\rho$ effective action if we were to compute it, as it was done in the 3D case, by evaluating the regularized bulk action on the background together with boundary GH and counter-terms. We close this subsection by writing down the full source term $J_s$ for the operator $\mathcal{O}_s$ dual to the bulk field $s$, i.e. the sum of the source in the background $s_b$ plus the one in the fluctuation $\delta s$:

$$\begin{aligned}
J_s = {} & \frac{c(4+d)}{12} \left( \frac{\Box_{g^{(0)}} \log(\rho) - \frac{1}{2} R^{(2)}[g^{(0)}]}{\rho^2} \right) \\
& + \frac{c(4+d)}{12} \frac{\Box \log(\rho)}{\rho^2} a_5 + \frac{c(4+d)}{24} \frac{1}{\rho^2} \Box a_5.
\end{aligned} \tag{2.140}$$

Next, we go to compute the contribution of the term $\int \sqrt{g^{(0)}} J_s \mathcal{O}_s$ to the 2D boundary action. While $J_s$ is the coefficient of $r^0$ in the UV expansion of $s$, $< \mathcal{O}_s >$ is proportional to the coefficient of $1/r^2$. We will determine this proportionality constant in the following by studying the dependence of the regularized bulk action on $a_5$. Note that $J_s$ is already of order $q^2$, therefore we need only $q^0$ term in the coefficient of $1/r^2$ in $s$, which can be seen from (2.117) and (2.134) to be[1]

$$< \mathcal{O} >_s \sim \rho^2(1 + a_5). \tag{2.141}$$

Using the fact that $\sqrt{g^{(0)}}$ at order $q^0$ is $1/2(1 - 2h_{uv})$, it can be shown that $\sqrt{g^{(0)}}J_s < \mathcal{O}_s >$ up to the order we are working at, is a total derivative and therefore the corresponding integral vanishes.

### 2.3.3 Boundary Action

Here, we will determine the boundary action in presence of sources for the dual stress energy tensor $T_{\mu\nu}$ , which will allow to compute its one-point functions. We will expand the bulk action around the determined background to linear order in the fluctuation fields, at order $q^2$. First of all, we need to point out a subtlety concerning the bulk action. Recall that the bosonic equations of motion of $(1, 0)$ 6D supergravity, (2.111), can be derived from the following action:

$$S^{bulk}_{6D} = \int d^6x\sqrt{-g_{6D}} \left( -\frac{1}{4}R + \frac{1}{4}e^{\theta}F^2 - \frac{1}{4}e^{2\theta}(G_3)^2 - \frac{1}{4}(\partial\theta)^2 \right), \tag{2.142}$$

where the equations of motion are obtained by varying with respect to all the fields, including the two form $B_{MN}$. The 6D equations of motion have been shown in [23] to reduce consistently to the 3D equations discussed earlier. In particular the 3D flow solution discussed before has a 6D uplift. For convenience, we give the map of the 6D fields and parameters in terms of 3D ones used in the previous sections:

$$\begin{aligned} &r^6e^{-8f}dr^2 \to dr^2, \quad r^3e^{-2f} \to e^f, \quad s \to 2A, \quad e^{4\theta} \to \frac{g_1^6 e^{2\phi}}{256 g_2^8 (1 - A^2)^3}, \\ &\varphi \to 0, \quad 4 + d \to \frac{4g_2^2}{g_1^2}, \quad c \to \frac{c_1}{2g_2^2}. \end{aligned} \tag{2.143}$$

[1] Of course, the same remarks about the CFT interpretation of the asymptotic data of bulk fields made in sub-section 2.2.1, implying spontaneous symmetry breaking of conformal invariance, apply here.

In the 6D action (2.142) above, $(G_3)^2$ equals $(G_3^{(1)})^2 + (G_3^{(2)})^2$. However the 3D gauged supergravity action is not the reduction of $S_{6D}^{bulk}$. The difference lies in the fact that in reducing to 3D, one eliminates $G_3$ by using its 6D solution in terms of the remaining fields. The 3D action is constructed by demanding that its variation gives the correct equations for the remaining fields. From the explicit solutions for $G_3^{(1)}$ and $G_3^{(2)}$ in (2.118), one can easily prove that the modified action $\tilde{S}_{6D}^{bulk}$, obtained by replacing $(G_3)^2 \to (G_3^{(1)})^2 - (G_3^{(2)})^2$ in $S_{6D}^{bulk}$, reproduces the correct equations of motion for all the remaining fields. From the AdS/CFT point of view, it seems reasonable to use $\tilde{S}_{6D}^{bulk}$, since the two-form potential in 3D is not a propagating degree of freedom and does not couple to boundary operators. We should point out that the boundary action that we will compute in the following is not the same for $S_{6D}^{bulk}$ and $\tilde{S}_{6D}^{bulk}$. Only the latter reproduces the results of the 3D analysis. The flow solution studied in this chapter can be described in the 3D gauged supergravity, however there are many solutions describing flows in 2D or 4D CFTs that cannot be described in 3D or 5D gauged supergravities. Instead one has to directly work in higher dimensions. In such cases, we think, that the bulk action that should be used in the holographic computations, is the one that reproduces the correct equations for the fields that couple to the boundary operators, after having eliminated 2-form and 4-form fields respectively.

As promised at the beginning of this subsection our goal will be to evaluate $S_{6D}^{bulk}$, with the modification just mentioned, on the field configurations which are sums of the background fields plus the $\delta$ fields, at first order in the latter and to order $q^2$. Since the background solves the equations of motion, the result will be a total derivative and there will be possible contributions from the UV and IR boundaries, i.e. $r \to \infty$ and $r \to 0$, respectively. It is simpler to give the sum, $S_1$, of the boundary term coming from the bulk action and the Gibbons-Hawking term, which in our case is $\int dudv \frac{\partial_r\left(e^{2f} det(g)\right)}{\sqrt{(-detg)}}$:

$$\begin{aligned} S_1 &= \int \frac{dudv}{\sqrt{-detg}}[-r^3(4g_{buv}^2\partial_r f_b - g_{bvv}\partial_r g_{buu} + 2g_{buv}\partial_r g_{buv} \\ &- g_{buu}(4g_{bvv}\partial_r f_b + \partial_r g_{bvv}))\delta f/2 \\ &- r^2(g_{buv}(-6+4r\partial_r f_b) - r\partial_r g_{buv})\delta g_{uv}/4 \\ &+ r^2(g_{bvv}(-6+4r\partial_r f_b) - r\partial_r g_{bvv})\delta g_{uu}/8 \\ &+ r^2(g_{buu}(-6+4r\partial_r f_b) - r\partial_r g_{buu})\delta g_{vv}/8 \\ &- r^3(-detg)\partial_r\theta_b\delta\theta \\ &- 6e^{2f_b+\theta_b}(-detg)\delta s]. \end{aligned} \tag{2.144}$$

By looking at the solutions for the various fields one can see that this expression has a quadratic divergence for $r \to \infty$ at order $q^0$, which can be renormalized by subtracting a counterterm proportional to the boundary cosmological constant:

$$S_{CT} = \frac{1}{2(c(4+d))^{1/4}} \int dudve^f \sqrt{-detg}. \tag{2.145}$$

The final term $S_f = S_1 - S_{CT}$, at order $q^2$, for $r \to \infty$ is obtained using the explicit solutions:

$$\begin{aligned} S_f &= \int dudv \frac{c}{8\rho^2}(2h_{uu}(9(\partial_v\rho)^2 - \rho\partial_v^2\rho) \\ &+ 2\partial_u\rho(8a_5\partial_v\rho - 16h_{uv}\partial_v\rho + 9h_{vv}\partial_u\rho \\ &+ 2\rho(7\partial_u\rho\partial_u h_{vv} - 8a_5\partial_u\partial_v\rho + 16h_{uv}\partial_u\partial_v\rho \\ &+ h_{vv}\partial_u^2\rho) + 7\rho^2(\partial_{vv}h_{uu} + \partial_u\partial_v a_5 - 2\partial_u\partial_v h_{uv} + \partial_u^2 h_{vv}))). \end{aligned} \tag{2.146}$$

For $r \to 0$ one can readily verify that there is no finite contribution left over. Before coming to the computation of $< T_{uu} >$, $< T_{vv} >$ and $< T_{uv} >$, let us analyse more precisely $\mathcal{O}_s$ . This can be obtained by comparing $J_s$ from (2.140), after setting to zero the sources of $T_{\mu\nu}$, with the corresponding term in $S_f$, which gives $\int \sqrt{g^{(0)}} < \mathcal{O}_s > J_s$. Setting the sources of $T_{\mu\nu}$ to zero, i.e. keeping only $a_5$, $S_f$ is $2c(\partial_u\rho\partial_v\rho - \rho\partial_u\partial_v\rho)/\rho^2 a_5$ which by the holographic map is equal to $\int \sqrt{g^{(0)}} < \mathcal{O}_s > J_s$. Using the expression for $J_s$ given in (2.140) one finds:

$$< \mathcal{O}_s >^{(0)} = \frac{6\rho^2}{4+d}. \tag{2.147}$$

Notice that using the fact that $< \mathcal{O}_s >$ is proportional to $\rho^2$ (2.141), the term proportional to $\Box a_5$ in $J_s$ is a total derivative. The above equation actually gives the proportionality constant in (2.141) so that including the first order fluctuation:

$$< \mathcal{O}_s >= \frac{6\rho^2}{4+d}(1+a_5). \tag{2.148}$$

### 2.3.4 One-point function of $T_{\mu\nu}$

The one-point functions of the stress energy tensor, $< T_{uu} >$, $< T_{vv} >$ and $< T_{uv} >$, are determined as the coefficients of $h_{vv}$, $h_{uu}$ and $h_{uv}$, respectively, in $S_f$. After performing a partial integration one obtains the result :

$$\begin{aligned} < T_{uu} > &= \frac{-2c(2(\partial_u\rho)^2 + \rho\partial_u^2\rho)}{\rho^2}, \\ < T_{vv} > &= \frac{-2c(2(\partial_v\rho)^2 + \rho\partial_v^2\rho)}{\rho^2}, \\ < T_{uv} > &= \frac{2c(-\partial_u\rho\partial_v\rho + \rho\partial_u\partial_v\rho)}{\rho^2}. \end{aligned} \tag{2.149}$$

This stress energy tensor can be derived from an effective action for the field $\rho$:

$$S_\rho = 2c\int dudv\sqrt{-g^{(0)}}[(\partial\log(\rho))^2 - R^{(2)}(g^{(0)})\log(\rho)]. \tag{2.150}$$

Note that the coefficient that appears in $S_\rho$ is $c$ which is proportional to $c_{UV} - c_{IR}$. Under the Weyl transformation

$$g^{(0)} \to e^{2\sigma}g^{(0)}, \quad \rho \to e^{-\sigma}\rho, \quad S_\rho \to S_\rho + 2c\int dudv\sqrt{-g^{(0)}}\sigma R^{(2)}(g^{(0)}), \tag{2.151}$$

and therefore $S_\rho$ precisely produces the anomalous term. Finally note that $J_s$ in (2.140) transforms, up to the linearized fluctuation that we have computed here, covariantly as $J_s \to e^{2\sigma}J_s$ under the Weyl transformation.

Finally, using (4.2), (2.140) and (2.148), we find that the conservation of stress tensor is modified by the source terms as:

$$\partial^i < T_{ij} >= J_s\partial_j < \mathcal{O}_s >, \tag{2.152}$$

which is the Ward identity for diffeomorphisms in the CFT in the presence of a source term $\int j_s\mathcal{O}_s$.

Now we would like to interpret (2.150) from the dual $(4,0)$ SCFT point of view. It is useful to recall some facts from the better understood type IIB $(4,4)$ SCFT describing bound states of $Q_1$ D1-branes and $Q_5$ D5 branes [28, 59]. If one wants to study the separation of, say, one D1 or D5 brane from the rest, one has to study the effective action for the scalars in the vector multiplets, $\vec{V}$, in the relevant branch of the 2D (4,4) gauge theory, which is the Higgs branch, where (semiclassically) the hypermultiplet scalars $H$ acquire v.e.v., whereas for the vector multiplet scalars, which carry dimension 1, $< V >= 0$. One can obtain an effective action for $V$ either by a probe supergravity approach [28] or by a field theory argument [59, 60, 61], i.e. by integrating out the hypermultiplets and observing that in the 2D field theory there is a coupling schematically of the form $\vec{V}^2 H^2$. This can be shown to produce for $\log|\vec{V}|$ a lagrangian of the form (2.150) with the correct background charge to produce a conformal anomaly which matches the full conformal anomaly, to leading order in the limit of large charges.

In our case, where we have a D1-D5 system in presence of D9 branes in type I theory, the role of the vector multiplet scalars is played by the field $\rho$, the instanton scale in the background geometry. The "separation" of one D-brane corresponds geometrically to the limit $\rho \to \infty$, where the gauge 5-brane decouples, making a reduction in the central charge from an amount proportional to $Q_1 Q_5$ in the UV to $Q_1(Q_5 - 1)$ in the IR, where, as shown earlier, $Q_1 = c/4$ and $Q_5 = d/4 + 1$. Therefore the variation of the central charge, $\delta c$, is proportional to $Q_1$. On the other hand, from the D-brane effective field theory point of view the instanton scale corresponds to a gauge invariant combination of the D5-D9 scalars, $h$, with $h^2 \sim \rho^2$. The $h$'s are in the bifundamental of $Sp(1) \times SO(3)$, $Sp(1)$ being the gauge group on the D5-brane and $SO(3)$ that on the D9- branes. The $h$'s couple to D1-D5 scalars $H$ which are in the bifundamental of $SO(Q_1) \times Sp(1)$ and belong to (4,4) hypermultiplets. In the Higgs branch, which gives the relevant dual CFT, again $H$'s can have v.e.v. semiclassically, while $< h >= 0$. In the 2D effective action there is a coupling of the form $H^2 h^2$ and upon 1-loop integration of $H$'s one gets a term $(\partial h)^2/h^2$[60], with coefficient proportional to $Q_1$. The presence of the background charge term can be justified along the lines of [28, 59] and it guarranties the matching of Weyl anomalies along the flow.

# Part II

# On black holes in 3D higher spin theories

# 3

# The phase space of $sl(3,\mathbb{R})$ black holes.

In this chapter we address some issues of recent interest, related to the asymptotic symmetry algebra of higher spin black holes in $sl(3,\mathbb{R}) \times sl(3,\mathbb{R})$ Chern Simons (CS) formulation. In our analysis we resort to both, Regge-Teitelboim and Dirac bracket methods and when possible identify them. We compute explicitly the Dirac bracket algebra on the phase space, in both, diagonal and principal embeddings. The result for principal embedding is shown to be isomorphic to $W_3^{(2)} \times W_3^{(2)}$.

## 3.1 The Regge-Teitelboim formalism

We start this section by reviewing the Regge-Teitelboim (RT) formalism in the context of Chern Simons theory in a 3D space with boundaries. Firstly, we provide some tips that the reader should keep in mind during this section.

- Along our discussion we will use the $\lambda = 3$ truncation of $hs(\lambda)$ to $sl(3,\mathbb{R})$. However many of the procedures to be reviewed in the next section do generalise straightforwardly to any of the truncations gotten for positive integer $\lambda$.

- The super index (0) in a given quantity $X$ stands for its restriction to the Cauchy surface $X^{(0)}$. Or equivalently to its initial condition under a given flow equation.

- The symbol $\delta$ stands for an arbitrary functional variation whereas $\delta_\Lambda$ stands for a variation due to a residual gauge transformation $\Lambda$.

Let us denote by $(\mathcal{A},\bar{\mathcal{A}})$ the left and right $sl(3,\mathbb{R})$-valued connections of interest. Let us focus on the sector $\mathcal{A}$ and let us denote the space-time coordinates by $(\rho, x_1, x_2)$. The Chern Simons action supplemented by a boundary term is

$$S_{CS} = \int tr\left(\mathcal{A}d\mathcal{A} + \frac{2}{3}\mathcal{A}^3\right) + I_{bdry}. \tag{3.1}$$

Part of the $hs(\lambda)$ [1] gauge freedom is fixed by the choice

$$\mathcal{A}_\rho = V_0^2,\ \left(\bar{\mathcal{A}}_\rho = -V_0^2\right). \tag{3.2}$$

The $(1,\rho)$ and $(2,\rho)$ components of the equations of motion $d\mathcal{A}+\mathcal{A}^2=0$ impose the form

$$\mathcal{A}_a = bA_ab^{-1},\ \ b = e^{-\rho V_0^2}\ \left(\bar{\mathcal{A}}_a = \bar{b}A_a\bar{b}^{-1},\ \ \bar{b} = e^{\rho V_0^2}\right), \tag{3.3}$$

with $a = 1, 2$[2]. The remaining $(1,2)$ components read

$$dA + A^2 = 0,\ d \equiv dx^a\partial_a. \tag{3.4}$$

Up to this point we have twice as many variables than equations. Equation (3.4) can be thought of as:

- $x_2$ evolution equation for $A_1$ (I). $(\partial_2 A_1 + \ldots = 0)$.

where the ... define quantities that do not involve derivatives with respect to $x_2$.

From this point of view $A_2$ is an arbitrary source and the Cauchy surface initial condition is $A_1|_{x_2=fixed}$. The arbitrariness of the source $A_2$ represents an extra gauge freedom that tunes the $x_2$ evolution of a Cauchy data surface $A_1|_{x_2=fixed}$. Should we make the choice $A_2 = 0$, evolution is trivial and all Cauchy surfaces

[1] See appendix B.1 for notations, conventions and definitions concerning the $hs(\lambda)$ algebra.

[2] From now on we will focus on the unbarred sector $\mathcal{A}$. The results for the barred sector $\bar{\mathcal{A}}$ can be obtained in the same way.

have the same data $A_1(x_1)$. Data $A_1(x_1)$ and $A_1(x_1)+\delta_\Lambda A_1(x_1)$ are physically inequivalent as the gauge degeneracy has been already fixed.

However, notice that one can map $\delta_\Lambda A_1(x_1)$ to a "improper" $hs(\lambda)$ gauge transformation with parameter $\Lambda(x_1)$[1]. In this way the gauge choice $A_2=0$ is preserved and

$$\delta_\Lambda A_1(x_1) \equiv \partial_1 \Lambda(x_1) + [A_1, \Lambda]. \tag{3.5}$$

The gauge parameters $\Lambda$ carry thence some physical meaning, they will define global charges $Q(\Lambda)$ whose Poisson bracket with the initial data $A_1(x_1)$ will generate the changes $\delta A_1(x_1)$. In fact, in virtue of what was said, it results that

$$Q(\Lambda) = G|_{\mathcal{A}_\rho = V_0^2, \mathcal{A}_1 = bA_1 b^{-1}}(b\Lambda(x_1)b^{-1}). \tag{3.6}$$

Where $G$ is the generator of gauge transformations in a given Cauchy surface before imposing any second class constraint. Even though we did not make it explicit in (3.6), we have also imposed $\mathcal{A}_2 = 0$.

Before defining $G$ let us stress that in the following paragraph we do not impose neither (3.2) nor (3.3) which are not compatible (second class) with the $x_2 = fixed$ Poisson bracket algebra

$$\{\mathcal{A}_1, \mathcal{A}_\rho\}_{PB} = -\{\mathcal{A}_\rho, \mathcal{A}_1\}_{PB} = V_0^1 \delta^{(2)}. \tag{3.7}$$

Where by $V_0^1$ we mean the identity operator in the $hs(\lambda)$ algebra (See appendix B.1). However we are free to take $\mathcal{A}_2 = 0$ as it is compatible (first class) with (3.7). The quantity

$$G(\Gamma) \equiv \int dx_1 tr(\Gamma \mathcal{A}_1)|_{\rho=\infty} + \int dx_1 d\rho\ tr(\Gamma \mathcal{F}_{1\rho}), \tag{3.8}$$

is defined over each $x_2 = fixed$ Cauchy surface and obeys the following properties

$$\begin{aligned} \{G(\Gamma), \mathcal{A}_{1,\rho}\}_{PB} &= D_{1,\rho}\Gamma \equiv \delta_\Gamma \mathcal{A}_{1,\rho}, \\ \delta_{\mathcal{A}_1} G(\Gamma) &= -\int dx_1 d\rho\ tr\left(D_\rho \Gamma \delta \mathcal{A}_1\right), \end{aligned} \tag{3.9}$$

[1] In terms of the $\mathcal{A}$ components the parameter is $b\Lambda(x_1)b^{-1}$, in such a way that it preserves the $hs(\lambda)$ gauge choice $\mathcal{A}_\rho = V_0^2$. The gauge transformation $\Lambda$ is usually called "improper" as it changes the specified boundary conditions. In a manner that will be explicitly shown below these transformations define global symmetries.

under the brackets (3.7). Namely, it generates the gauge transformations on a given Cauchy surface under (3.7), and it is properly differentiable under off-shell variations $\delta\mathcal{A}_1$. By computing the gauge variation of (3.8) and regrouping some terms one arrives to the algebra

$$\{G(\Gamma_1), G(\Gamma_2)\}_{PB} \equiv \delta_{\Gamma_1} G(\Gamma_2) = G([\Gamma_1, \Gamma_2]) - \int dx_1 \; tr(\Gamma_1 \partial_1 \Gamma_2), \qquad (3.10)$$

which is inherited through (3.6) by the $Q(\Lambda)$'s.

In fact, after plugging (3.8) into (3.6) one gets

$$Q(\Lambda) = \int dx_1 tr(\Lambda A_1). \qquad (3.11)$$

From the first line in (3.9) and after imposing the second class constraints (3.2) and (3.3) we arrive to

$$\{Q(\Lambda), A_1\}_{PB} = D_1 \Lambda \equiv \delta_\Lambda A_1(x_1), \qquad (3.12)$$

which after taking $\Lambda = \delta^2 \tau_a$, $A_1 = A_1^b \tau_b$ reduces to the Kac-Moody algebra

$$\{A_1^a(x_1), A_1^b(y_1)\}_{PB} = f^{ab}{}_c A_1^c \delta(x_1 - y_1) - g^{ab} \partial_{x_1} \delta(x_1 - y_1), \qquad (3.13)$$

where $g^{ab}$ is the inverse of the Killing metric, $g_{ab} = tr\,(\tau_a \tau_b)$, that is also used to raise indices. To lower indices we use the Killing metric $g_{ab}$ itself. For instance $f_c^{ab} = g^{a\bar{a}} g^{b\bar{b}} g_{c\bar{c}} f_{\bar{a}\bar{b}}{}^{\bar{c}}$. Where $[\tau_a, \tau_b] = f_{ab}{}^c \tau_c$. Notice that the same result (3.13) can be deduced from (3.10) and the definition (3.6).

It is worth to notice that in the previous definition of $G$, the gauge parameter $\Gamma$ was supposed to be field independent. Should this not be the case, then (3.8) should be replaced by

$$G(\Gamma) \equiv B(\Gamma, \mathcal{A}) + \int dx_1 d\rho \; tr(\Gamma \mathcal{F}_{1\rho}), \qquad (3.14)$$

where the boundary term $B$ is such that

$$\delta_{\mathcal{A}_1} B(\Gamma) = \int dx_1 tr(\Gamma \delta \mathcal{A}_1)|_{\rho=\infty}. \qquad (3.15)$$

Is easy to check that (3.14) still obeys the properties (3.9), but in a weak sense, namely up to terms that vanish when one imposes the equations of motion, $\mathcal{F}_{1\rho} =$

0. Clearly when $\Gamma$ is field independent both definitions (3.8) and (3.14) are equivalent. But (3.14) is more general. So we will stick to (3.14).

For later use we impose (3.2), (3.3), and $\Gamma = b\Lambda b^{-1}$, onto (3.15) and rewrite it as

$$\delta Q(\Lambda) = \int dx_1 tr(\Lambda \delta A_1). \tag{3.16}$$

Where now we note that the $\rho$ dependence has disappeared, and the non linearity of $\Gamma$ is inherited by $\Lambda$. The integration of (3.16), $Q(\Lambda)$ generates the residual gauge transformations that preserve any further constraint, with $\Lambda$ being the corresponding residual gauge parameter. From (3.12) we have then a way to find out the Poisson brackets on a further reduced phase space.

A shortcut to find out the algebra without integrating (3.15) is at hand. After use of the equivalence relation in (3.10) inherited by the $Q$, together with (3.16) one gets

$$\{Q(\Lambda_1), Q(\Lambda_2)\}_{PB} \equiv \delta_{\Lambda_1} Q(\Lambda_2) = -\int dx_1 \; tr(\Lambda_1 D_1 \Lambda_2). \tag{3.17}$$

In this way we just need to use $A_1$ and the residual gauge parameter $\Lambda$ to evaluate the RHS [1]. We will not resort to this way.

Notice also, that in the process we have been neglecting total derivative terms with respect to $x_1$ under integration. To take care of them, one imposes boundary conditions on the field and gauge parameters, like for instance periodicity under $x_1 \to x_1 + 2\pi$. In the next section we will study a case in which such a periodicity is lost due to the use of perturbation theory.

## 3.2 Regge-Teitelboim method in the principal embedding

In this section we impose extra constraints (boundary conditions) on the phase space of the theory with Lie algebra $sl(3,\mathbb{R})$. We will explicitly set up the RT method in order to make it equivalent to Dirac formalism. In the process we will show, as already known, that it is also possible to set up the RT formalism in order to define a $W_3$ algebra at fixed time slices [1]. We will show explicitly that this choice can be thought of as the realisation of a non residual gauge transformation that resets the initial constraints in favour of the usual highest weight one. Next section we will show that such $W_3$ is not isomorphic to the fixed time Dirac bracket algebra.

Let us relax the condition $A_2 = 0$ used in the previous section. Besides (3.2) and (3.3), we impose the following constraints

$$\begin{aligned} A_1 &= V_1^2 + \mathcal{L}V_{-1}^2 + \mathcal{W}V_{-2}^3, \\ A_2 &= \mu_3\left(V_2^3 + \text{ lower components }\right), \end{aligned} \tag{3.18}$$

where the highest weight elements $(\mathcal{L}, \mathcal{W}, \ldots)$ are arbitrary functions of $(x_1, x_2)$. From now on to save some notation we denote the set of all of them $(\mathcal{L}, \mathcal{W}, \ldots)$ as $\mathcal{M}$. The flatness conditions along the generators $V^s_{m_s\geq -s+1}$ provide algebraic equations for the "lower components" in terms of $(\mathcal{M}, \partial_2\mathcal{M})$.

$$A_2 = \mu_3\left(V_2^3 + 2\mathcal{L}V_0^3 - \frac{2}{3}\partial_1\mathcal{L}V_{-1}^3 + \left(\mathcal{L}^2 + \frac{1}{6}\partial_1^2\mathcal{L}\right)V_{-2}^3 - 2\mathcal{W}V_{-1}^2\right). \tag{3.19}$$

The remaining ones provide the $x_2$-flow equations

$$\partial_2\mathcal{L} = -2\mu_3\partial_1\mathcal{W}, \quad \partial_2\mathcal{W} = \mu_3\left(\frac{8}{3}\mathcal{L}\partial_1\mathcal{L} + \frac{1}{6}\partial_1^3\mathcal{L}\right), \tag{3.20}$$

which determine the $\mathcal{M}$ out of the initial conditions $\mathcal{M}(x_1, 0)$. Solutions can be found in terms of perturbations of the chemical potential $\mu_3$ and will have the generic form

$$\mathcal{M} = \mathcal{M}^{(0)} + \mu_3\left(x_2\mathcal{M}^{(1)} + \mathcal{M}_1^{(0)}\right) + O(\mu_3^2), \tag{3.21}$$

where $\mathcal{M}^{(1)}$, are local functionals of the initial conditions $\mathcal{M}^{(0)}$, $\mathcal{M}_1^{(0)}$. Notice that the integration constants $\mathcal{M}_1^{(0)}$ are just shifts in $\mathcal{M}^{(0)}$. In general we will take $\mathcal{M}_1^{(0)}$ as the most general functional of $x_1$ and $\mathcal{M}^{(0)}$ consistent with dimensional analysis. The explicit dependence in $x_1$ will play an important role.

We ask now for the set of linear gauge transformations preserving the boundary conditions (3.18)

$$\delta A_a = \partial_{x_a}\Lambda + [A_a, \Lambda], \tag{3.22}$$

$$\Lambda = \epsilon V_1^2 + \eta V_2^3 + \text{ higher components}, \tag{3.23}$$

[1] where the lowest components $\{\epsilon, \eta\}$ are arbitrary functions of $(x_1, x_2)$. We will denote the set of lowest components $\{\epsilon, \eta\}$ by $\Theta$. The projection along the generators $V^s_{m_s>-s+1}$ of the $x_1$ equation in (3.22) solves algebraically for the highest components in terms of the lowest ones $\Theta$:

$$\Lambda(\epsilon,\eta) = \epsilon V_1^2 - \partial_1\epsilon V_0^2 + \left(\mathcal{L}\epsilon - 2\mathcal{W}\eta + \frac{1}{2}\partial_1^2\epsilon\right) V_{-1}^2 + \eta V_2^3 - \partial_1\eta V_1^3 + \left(2\mathcal{L}\eta + \frac{1}{2}\partial_1^2\eta\right) V_0^3 - \left(\frac{2}{3}\partial_1\mathcal{L}\eta + \frac{5}{3}\mathcal{L}\partial_1\eta + \frac{1}{6}\partial_1^3\eta\right) V_{-1}^3 + \left(\mathcal{W}\epsilon + \mathcal{L}^2\eta + \frac{7}{12}\partial_1\mathcal{L}\partial_1\eta + \frac{1}{6}\partial_1^2\mathcal{L}\eta + \frac{2}{3}\mathcal{L}\partial_1^2\eta + \frac{1}{4}\partial_1^4\eta\right) V_{-2}^3. \quad (3.24)$$

Notice that the $A_2$ component (3.19) can be viewed as a residual gauge parameter $\Lambda(0, \mu_3)$. This is of course a reminiscence of its spurious character.

The remaining $x_1$ equations provide variations of the gauge field parameters $\mathcal{M}(x_1, x_2)$

$$\begin{aligned}
\delta_\Lambda\mathcal{L} &= \partial_1\mathcal{L}\epsilon + 2\mathcal{L}\partial_1\epsilon - 2\partial_1\mathcal{W}\eta - 3\mathcal{W}\partial_1\eta + \frac{1}{2}\partial_1^3\epsilon, \\
\delta_\Lambda\mathcal{W} &= \partial_1\mathcal{W}\epsilon + 3\mathcal{W}\partial_1\epsilon + \frac{1}{6}\left(16\mathcal{L}\partial_1\mathcal{L} + \partial_1^3\mathcal{L}\right)\eta + \\
&\quad \frac{1}{12}\left(9\partial_1^2\mathcal{L} + 32\mathcal{L}^2\right)\partial_1\eta + \frac{5}{4}\partial_1\mathcal{L}\partial_1^2\eta + \frac{5}{6}\mathcal{L}\partial_1^3\eta + \frac{1}{24}\partial_1^5\eta,
\end{aligned} \quad (3.25)$$

From flatness conditions and the Dirichlet boundary condition to impose, it is clear that any other component variation of the gauge fields can be deduced from these ones. Demanding the lowest weight components $(V_1^2, V_2^3)$ of the final $A_2$ connection to be fixed, determines the $x_2$-flow equations

$$\partial_2\epsilon = -\mu_3\left(\frac{8}{3}\mathcal{L}\partial_1\eta + \frac{1}{6}\partial_1^3\eta\right), \ \partial_2\eta = 2\mu_3\partial_1\epsilon, \quad (3.26)$$

which allow to solve for the gauge parameter $\Theta(x_1, x_2)$ in terms of the initial conditions $\Theta(x_1, 0)$. Again, solutions can be found in perturbations of the chemical potential $\mu_3$

$$\Theta = \Theta^{(0)} + \mu_3\left(x_2\Theta^{(1)} + \Theta_1^{(0)}\right) + O(\mu_3^2), \quad (3.27)$$

[1]Notice that in (3.22) we have used $\delta$ and not $\delta_\Lambda$. In fact we use $\delta_\Lambda A$ to denote the solution of the condition (3.22), meanwhile $\delta$ stands for an arbitrary functional variation.

where the $\Theta^{(1)}$, are local functionals of the initial conditions $\Theta^{(0)}$. The $\Theta_1^{(0)}$ are shifts of $\Theta^{(0)}$ and we will define them as general functionals of $x_1$, $\mathcal{M}^{(0)}$ and $\Theta^{(0)}$ consistent with dimensional analysis, and linear in the $\Theta^{(0)}$.

Let us define our coordinates $x_1 = \frac{1}{2}(t_0 + \phi)$, $x_2 = \frac{1}{2}(-t_0 + \phi)$ and consider time evolution. This choice of coordinates identify (3.18) with the first two lines in equation (3.1) of [1] under our conventions [1].

The Cauchy data at a fixed time slice and the corresponding residual gauge transformations are

$$A d\tilde{\phi} = 2A_\phi d\tilde{\phi} = A_1 dx_1 + A_2 dx_2, \quad \delta_\Lambda A = 2\delta_\Lambda A_\phi = \delta_\Lambda A_1 + \delta_\Lambda A_2, \tag{3.28}$$

where the effective angular variable is $\tilde{\phi} = \frac{1}{2}\phi$. By the following redefinition

$$\begin{aligned}
\mathcal{L}_1^{(0)} &= 2\mathcal{W}^{(0)} + 2x_1\partial_1\mathcal{W}^{(0)}, \\
\mathcal{W}_1^{(0)} &= -\mathcal{L}^{(0)2} - \frac{1}{6}\partial_1^2\mathcal{L}^{(0)} - x_1\frac{1}{6}\left(16\mathcal{L}^{(0)}\partial_1\mathcal{L}^{(0)} + \partial_1^3\mathcal{L}^{(0)}\right), \\
\epsilon_1^{(0)} &= x_1\left(\frac{8}{3}\mathcal{L}^{(0)}\partial_1\eta^{(0)} + \frac{1}{6}\partial_1^3\eta^{(0)}\right), \\
\eta_1^{(0)} &= -2x_1\partial_1\epsilon^{(0)},
\end{aligned} \tag{3.29}$$

we get rid of all terms in the connection $A$ and residual gauge transformation $\delta_\Lambda A$ that break periodicity under $\phi \to \phi + 2\pi$. The periodic terms however are chosen by convenience[2]. The $V_{-1}^2$ and $V_{-2}^3$ components of $A$ become $\mathcal{L}^{(0)} + \frac{1}{2}\mu_3 t_0 \mathcal{L}^{(1)} + O(\mu_3^2)$ and $\mathcal{W}^{(0)} + \frac{1}{2}\mu_3 t_0 \mathcal{W}^{(1)} + O(\mu_3^2)$ respectively. The $(\mathcal{L}^{(1)}, \mathcal{W}^{(1)})$ are determined by the equations of motion (3.20) to be

$$\begin{aligned}
\mathcal{L}^{(1)} &= 2\partial_1\mathcal{W}^{(0)}, \\
\mathcal{W}^{(1)} &= -\frac{1}{6}\left(16\mathcal{L}^{(0)}\partial_1\mathcal{L}^{(0)} + \partial_1^3\mathcal{L}^{(0)}\right).
\end{aligned} \tag{3.30}$$

Notice that explicit dependence in the Cauchy surface position $t_0$ remains in both $A$ and $\delta_\Lambda A$. The contribution of this explicit dependence in $t_0$ to the charge $Q$ is a

[1]Should we have chosen $x_1 = \phi$ and $x_2 = t$ the fixed time Dirac bracket algebra of (3.18) is seen to be $W_3$ [2].

[2]Later on we will compare the result for the ASA with the choice 3.29 with the Dirac bracket algebra. (3.29) is the consistent choice for that case.

total derivative whose integration vanishes upon imposing our periodic boundary conditions. The integrated charge out of (3.16), for any $t_0$

$$Q(t_0)=\int_0^\pi d\tilde{\phi}\left(\epsilon^{(0)}\mathcal{L}^{(0)}-\eta^{(0)}\left(\mathcal{W}^{(0)}+\mu_3\left(\frac{1}{3}\partial_1^2\mathcal{L}^{(0)}+\frac{1}{3}\mathcal{L}^{(0)^2}\right)\right)\right)+O(\mu_3^2), \tag{3.31}$$

and the variations

$$\begin{aligned}
\delta_\Lambda\mathcal{L}^{(0)} &= \ldots+\mu_3\left(2\partial_1\mathcal{W}^{(0)}\epsilon^{(0)}+4\mathcal{W}^{(0)}\partial_1\epsilon^{(0)}+4\mathcal{L}^{(0)}\partial_1\mathcal{L}^{(0)}\eta^{(0)}\right.\\
&\qquad +3\mathcal{L}^{(0)^2}\partial_1\eta^{(0)}+3\partial_1\eta^{(0)}\partial_1^2\mathcal{L}^{(0)}+\frac{11}{2}\partial_1\mathcal{L}^{(0)}\partial_1^2\eta^{(0)}\\
&\quad\left.+\frac{1}{3}\partial_1^3\mathcal{L}^{(0)}\eta^{(0)}+\frac{8}{3}\mathcal{L}^{(0)}\partial_1^3\eta^{(0)}+\frac{1}{6}\partial_1^5\eta\right)+O(\mu_3^2), \qquad (3.32)\\
\delta_\Lambda\mathcal{W}^{(0)} &= \ldots+\mu_3\left(-\frac{8}{3}\mathcal{L}^{(0)}\partial_1\mathcal{L}^{(0)}\epsilon^{(0)}-\frac{13}{3}\mathcal{L}^{(0)^2}\partial_1\epsilon^{(0)}-\frac{4}{3}\partial_1^2\mathcal{L}^{(0)}\partial_1\epsilon^{(0)}\right.\\
&\qquad -\frac{25}{6}\partial_1\mathcal{L}^{(0)}\partial_1^2\epsilon^{(0)}-\frac{1}{6}\partial_1^3\mathcal{L}^{(0)}\epsilon^{(0)}-\frac{11}{3}\mathcal{L}^{(0)}\partial_1^3\epsilon^{(0)}-\frac{1}{3}\partial_1^5\epsilon^{(0)}\\
&\qquad +\frac{16}{3}\mathcal{W}^{(0)}\partial_1\mathcal{L}^{(0)}\eta^{(0)}+\frac{20}{3}\mathcal{L}^{(0)}\partial_1\mathcal{W}^{(0)}\eta^{(0)}+\frac{38}{3}\mathcal{L}^{(0)}\mathcal{W}^{(0)}\partial_1\eta^{(0)}\\
&\qquad \left.\frac{10}{3}\partial_1^2\mathcal{W}^{(0)}\partial_1\eta^{(0)}+\frac{11}{3}\partial_1\mathcal{W}^{(0)}\partial_1^2\eta^{(0)}+\frac{5}{3}\mathcal{W}^{(0)}\partial_1^3\eta^{(0)}+\partial_1^3\mathcal{W}^{(0)}\eta^{(0)}\right)\\
&\qquad +O(\mu_3^2),\\
\delta_\Lambda\mathcal{L}^{(1)} &= \left(\delta\mathcal{L}^{(1)}\right)|_{\delta\to\delta_\Lambda},\\
\delta_\Lambda\mathcal{W}^{(1)} &= \left(\delta\mathcal{W}^{(1)}\right)|_{\delta\to\delta_\Lambda}, \qquad (3.33)
\end{aligned}$$

determine, after long but straightforward computation, the Poisson bracket algebra (3.55) by means of (3.12)[1]. The … in (3.32) stand for the zeroeth order in $\mu_3$ contribution, which is given by the right hand side of (3.25) after substituting $(\mathcal{L},\mathcal{W},\epsilon,\eta)$ by $(\mathcal{L}^{(0)},\mathcal{W}^{(0)},\epsilon^{(0)},\eta^{(0)})$ respectively. Remember that $\delta$ stands for arbitrary functional differential and so by $(\delta\ldots)|_{\delta\to\delta_\Lambda}$ we mean to take the functional differential of … in terms of $(\delta\mathcal{L}^{(0)},\delta\mathcal{W}^{(0)})$ and after substitute $\delta$ by $\delta_\Lambda$. We will prove that the ASA on a fixed time $t_0$ slice that is obtained by imposition of (3.29) upon the Regge-Teitelboim bracket definition (3.17), namely (3.55), coincides with the fixed time $t_0$ Dirac bracket algebra in the space of flat connections (3.18). We will check that the $\mu_3$ deformation of (3.55) can not be

[1]… with the substitution $(x_1,\partial_1)\to(\frac{t_0}{2}+\tilde{\phi},\partial_{\tilde{\phi}})$ always implicitly intended.

absorbed by a field redefinition. In other words the ASA (3.55) is not isomorphic to $W_3$.

However, there is a way to associate a $W_3$ algebra to (3.18). In fact the choice

$$\begin{aligned} \mathcal{L}_1^{(0)} &= \ldots + \mathcal{W}^{(0)}, \quad \mathcal{W}_1^{(0)} = \ldots - \frac{5}{3}\mathcal{L}^{(0)2} - \frac{7}{12}\partial_1^2\mathcal{L}^{(0)}, \\ \epsilon_1^{(0)} &= \ldots - \left(\frac{8}{3}\eta^{(0)}\mathcal{L}^{(0)} + \frac{1}{4}\partial_1^2\eta^{(0)}\right), \quad \eta_1^{(0)} = \ldots + \epsilon^{(0)}, \end{aligned} \tag{3.34}$$

with the $\ldots$ denoting the rhs of the previous choice (3.29), defines the integrated charge

$$Q(t_0) = \int_0^\pi d\tilde{\phi}\left(\epsilon^{(0)}\mathcal{L}^{(0)} - \eta^{(0)}\mathcal{W}^{(0)}\right) + O(\mu_3^2), \tag{3.35}$$

with variations $(\delta_\Lambda\mathcal{L}^{(0)}, \delta_\Lambda\mathcal{W}^{(0)})$ given precisely as in (3.25) with $(\mathcal{L}, \mathcal{W}, \epsilon, \eta)$ substituted by the initial conditions $(\mathcal{L}^{(0)}, \mathcal{W}^{(0)}, \epsilon^{(0)}, \eta^{(0)})$.

The variations $(\delta_\Lambda\mathcal{L}^{(1)}, \delta_\Lambda\mathcal{W}^{(1)})$ are given in terms of $(\delta_\Lambda\mathcal{L}^{(0)}, \delta_\Lambda\mathcal{W}^{(0)})$, as presented in the last two lines in (3.33). Thence from (3.12) one derives (3.52) which is $W_3$. As already stated this Poisson structure is not equivalent to the Dirac structure (3.55) mentioned before. The technical reason being the presence of the field dependent redefinition of gauge parameters (3.34) that is not equivalent to a redefinition of $(\mathcal{L}^{(0)}, \mathcal{W}^{(0)})$. As we will show this procedure is somehow violating the Dirichlet boundary conditions of (3.18).

But before going on let us write down the expression for the original $(V_{-1}^2, V_{-2}^3)$ components of the projection $A_1$ of $A$ and the corresponding residual gauge parameters, $(\mathcal{L}, \mathcal{W}, \epsilon, \eta)$, in terms of the $(\mathcal{L}^{(0)}, \mathcal{W}^{(0)}, \epsilon^{(0)}, \eta^{(0)})$ for the choice (3.34)

$$\begin{aligned} \mathcal{L} &= \mathcal{L}^{(0)} + 3\mu_3\mathcal{W}^{(0)} + \mu_3 t_0\partial_1\mathcal{W}^{(0)} + O(\mu_3^2), \\ \mathcal{W} &= \mathcal{W}^{(0)} - \mu_3\left(\frac{8}{3}\mathcal{L}^{(0)2} + \frac{3}{4}\partial_{x_1}^2\mathcal{L}^{(0)}\right) - \frac{1}{12}\mu_3 t_0\left(16\mathcal{L}^{(0)}\partial_1\mathcal{L}^{(0)} + \partial_1^3\mathcal{L}^{(0)}\right) + O(\mu_3^2), \\ \epsilon &= \epsilon^{(0)} - \mu_3\left(\frac{8}{3}\eta^{(0)}\mathcal{L}^{(0)} + \frac{1}{4}\partial_{x_1}^2\eta^{(0)}\right) + \frac{1}{12}\mu_3 t_0\left(16\mathcal{L}^{(0)}\partial_1\eta^{(0)} + \partial_1^3\eta^{(0)}\right) + O(\mu_3^2), \\ \eta &= \eta^{(0)} + \mu_3\epsilon^{(0)} - \mu_3 t_0\partial_1\epsilon^{(0)} + O(\mu_3^2). \end{aligned} \tag{3.36}$$

The $(V_{-1}^2, V_{-2}^3)$ components of $A$ are recovered by dropping the terms linear in $\mu_3$ without $t_0$ dependence in the first two lines in (3.36).

## 3.3 The RT reduction to $W_3$ as a non residual gauge transformation

As promised, we will show that the process that follows the choice (3.34) in defining a $W_3$ algebra for (3.18), is equivalent to the process of performing a gauge transformation that maps the phase space (3.18) with $\mu_3 \neq 0$ to the one with $\mu_3 = 0$. Namely to perform a gauge transformation that changes the original boundary conditions. First let us collect useful information. Let $A$ be the space of flat connections with residual gauge transformation condition $\delta A = D_A \Lambda_A$.

Let $g$ be an arbitrary field dependent gauge group element which is not a residual transformation of $A$. By performing the similarity transformation by $g$ on both sides of $(\delta A) = D_A \Lambda_A$ we get

$$
\begin{aligned}
g\delta A g^{-1} &= \delta A_g - D_{A_g}(g\delta g^{-1}), \\
g D_A(\Lambda_A) g^{-1} &= D_{A_g}(g\Lambda_A g^{-1}),
\end{aligned} \tag{3.37}
$$

where $A_g \equiv gAg^{-1} + g\partial g^{-1}$. From (3.37) we read out the transformation law for the residual gauge parameter $\Lambda$

$$
\Lambda_{A_g} = g\Lambda_A g^{-1} + g\delta g^{-1}, \tag{3.38}
$$

where at this point, we are free to substitute the arbitrary differential $\delta$ by $\delta_{\Lambda_A}$, the initial residual gauge transformation.

Now we notice that equations (3.20) and (3.26) are integrable at any order in $\mu_3$ as it follows from gauge invariance [1, 38]. One way to solve them is to express the solution in terms of a gauge group element $g = g(\tilde{\mathcal{L}}, \tilde{\mathcal{W}}, \mu_3 x_2)$ that takes the highest weight connection

$$
\tilde{A}_1 = V_1^2 + \tilde{\mathcal{L}} V_{-1}^2 + \tilde{\mathcal{W}} V_{-2}^3, \quad \tilde{A}_2 = 0, \tag{3.39}
$$

to (3.18), via the usual transformation law $\tilde{A} \to \tilde{A}_g \equiv A$. The element $g$ that transforms (3.39) into (3.18) is generated at the first order in $\mu_3$ and linear order in the algebra element by:

$$
\begin{aligned}
\Lambda_g &= \Lambda(\tilde{\epsilon}_g, \tilde{\eta}_g) - x_2 A_2 + O(\mu_3^2) \\
&= \Lambda(\tilde{\epsilon}_g, \tilde{\eta}_g) + \Lambda(0, -\mu_3 x_2) + O(\mu_3^2),
\end{aligned} \tag{3.40}
$$

with $\Lambda$, as a function of $(\tilde{\epsilon},\tilde{\eta})$, given by (3.24) with background fields $(\tilde{\mathcal{L}},\tilde{\mathcal{W}})$ instead of $(\mathcal{L},\mathcal{W})$. From the second line in (3.40) it follows that $\Lambda_g$ generates transformations of the kind (3.25) on the $(\tilde{\mathcal{L}},\tilde{\mathcal{W}})$ and relate them with the new parameters $(\mathcal{L},\mathcal{W})$ by

$$\mathcal{L}=\tilde{\mathcal{L}}-2\mu_3 x_2\partial_1\tilde{\mathcal{W}}+O(\mu_3^2),\ \mathcal{W}=\tilde{\mathcal{W}}+\mu_3 x_2\left(\frac{8}{3}\tilde{\mathcal{L}}^2+\frac{1}{6}\partial_1^2\tilde{\mathcal{L}}\right)+O(\mu_3^2),\ (3.41)$$

where we have hidden the arbitrariness $\Lambda(\tilde{\epsilon}_g,\tilde{\eta}_g)$ in (3.40), inside of the $(\tilde{\mathcal{L}},\tilde{\mathcal{W}})$. From the $x_2$ flow equations (3.20) and (3.41) one is able to identify the parameters $(\tilde{\mathcal{L}},\tilde{\mathcal{W}})$ with the initial conditions

$$\tilde{\mathcal{L}}\equiv\mathcal{L}^{(0)}+\mu_3\mathcal{L}_1^{(0)}+O(\mu_3^2),\ \tilde{\mathcal{W}}\equiv\mathcal{W}^{(0)}+\mu_3\mathcal{W}_1^{(0)}+O(\mu_3^2). \qquad (3.42)$$

The map induced by $\mathcal{H}_g$ is then identified with the Hamiltonian evolution along $x_2$ that recovers $(\mathcal{L},\mathcal{W})$ out of the initial conditions (3.42).

Now we can apply (3.38) to this specific case

$$\begin{aligned}\Lambda &= g\tilde{\Lambda}g^{-1}+g\delta g^{-1}\\ &= \tilde{\Lambda}+x_2\left(\delta A_2-[A_2,\Lambda]\right)+O(\mu_3^2)=\tilde{\Lambda}+x_2\partial_2\Lambda|_{x_2=0}+O(\mu_3^2)\\ &= \tilde{\Lambda}+x_2\left(-\mu_3\left(\frac{8}{3}\tilde{\mathcal{L}}\partial_1\tilde{\eta}+\frac{1}{6}\partial_1^3\tilde{\eta}\right)V_1^2+2\mu_3\partial_1\tilde{\epsilon}V_2^3+\ldots\right)+O(\mu_3^2).\end{aligned} \qquad (3.43)$$

Where by $\delta$ we mean the analog of the variations (3.25), and again we have hidden the arbitrariness $\Lambda(\tilde{\epsilon}_g,\tilde{\eta}_g)$ inside the parameters $\tilde{\Lambda}\equiv\Lambda(\tilde{\epsilon},\tilde{\eta})$. The last line in (3.43), together with the $x_2$ flow equations (3.26), allows us to identify the parameters $(\tilde{\epsilon},\tilde{\eta})$ with the initial conditions $(\epsilon^{(0)}+\mu_3\epsilon_1^{(0)}+O(\mu_3^2),\eta^{(0)}+\mu_3\eta_1^{(0)}+O(\mu_3^2))$. For later reference

$$\tilde{\epsilon}\equiv\epsilon^{(0)}+\mu_3\epsilon_1^{(0)}+O(\mu_3^2),\ \tilde{\eta}\equiv\eta^{(0)}+\mu_3\eta_1^{(0)}+O(\mu_3^2). \qquad (3.44)$$

After imposing (3.34), the explicit form of $\Lambda$ (3.24), (3.42), (3.44) on (3.41) and (3.43), one finds the same relations gotten from the previous procedure, (3.36). This was expected a priori, since the latter approach is simply a way to encode the $x_2$ evolution in the element $g$. Additionally, it provides an alternative perspective to understand the significance of the choice (3.34).

From (3.38) it follows that the differential of charge $\delta Q \equiv \int_0^\pi d\tilde{\phi}\; tr(\tilde{\Lambda}\delta A)$ is not invariant under a gauge transformations $g$. In particular, the differential of charge for (3.39) previous to the gauge transformation $g$ encoding the $x_2$ evolution, is:

$$\delta Q(\tilde{\epsilon},\tilde{\eta}) \equiv \int_0^\pi d\tilde{\phi}\; tr(\tilde{\Lambda}\delta\tilde{A}_1) = \int_0^\pi d\tilde{\phi}\left(\tilde{\epsilon}\delta\tilde{\mathcal{L}} - \tilde{\eta}\delta\tilde{\mathcal{W}}\right), \tag{3.45}$$

and picks up an extra $\mu_3$ dependence after the gauge transformation $g$ is performed. The choice (3.34) is the one that cancels, up to trivial integrations of total derivatives, this extra $\mu_3$ dependence contribution to the final differential of charge. The final result for the transformed charge, after functional integration is performed, coincides with (3.35).

Notice however that the non residual gauge transformation $g$ takes to a phase space (3.39) that does not include the $(\mu_3,\bar{\mu}_3)$ GK ansätze [33].

## 3.4 Dirac bracket in the principal embedding

In this section we compute the Dirac bracket on the phase space (3.18), on a Cauchy surface at fixed $t_0$. From there, we will check that they define an algebra which is not isomorphic to $W_3$. To make things easier we start by computing them on a Cauchy surface at fixed $x_2$. In this case the phase space is given by a generic $sl(3,\mathbb{R})$ valued function of $x_1$

$$\begin{aligned} a(x_1) &= A^s_{m_s}V^s_{m_s} = A^a V_a, \\ V_a &= \left(V^2_1, V^2_0, V^2_{-1}, V^3_2, V^3_1, V^3_0, V^3_{-1}, V^3_{-2}\right), \end{aligned} \tag{3.46}$$

We start from the Kac-Moody algebra (3.13) and proceed to impose the following 6 second class constraints

$$C^i = \Bigg(A^2_1 - 1, A^2_0, A^3_2, A^3_1, A^3_0, A^3_{-1}\Bigg), \tag{3.47}$$

onto it, but first we choose the integration constants to be

$$\begin{aligned} \mathcal{L}^{(0)}_1 &= 2\mathcal{W}^{(0)} + 2x_1\partial_1\mathcal{W}^{(0)}, \\ \mathcal{W}^{(0)}_1 &= -\mathcal{L}^{(0)^2} - \frac{1}{6}\partial_1^2\mathcal{L}^{(0)} - x_1\frac{1}{6}\left(16\mathcal{L}^{(0)}\partial_1\mathcal{L}^{(0)} + \partial_1^3\mathcal{L}^{(0)}\right), \end{aligned} \tag{3.48}$$

precisely as in (3.29). From now on, to save space we will not write down the explicit $t_0$ dependence but the reader should keep in mind that the full result is recovered by making the substitutions

$$\begin{aligned} \mathcal{L}^{(0)} &\rightarrow \mathcal{L}^{(0)} + \mu_3 t_0 \mathcal{W}^{(0)} + O(\mu_3^2), \\ \mathcal{W}^{(0)} &\rightarrow \mathcal{W}^{(0)} + \mu_3 t_0 \frac{1}{12}\left(16\mathcal{L}^{(0)}\partial_1\mathcal{L}^{(0)} + \partial_1^3\mathcal{L}^{(0)}\right) + O(\mu_3^2), \end{aligned} \tag{3.49}$$

at the end. The constraints (3.47) define the Dirac bracket

$$\{A^a(x_1), A^b(y_1)\}_D = \{A^a(x_1), A^b(y_1)\}_{PB} - \left(\{A^a, C^i\}_{PB} M_{ij} \{C^j, A^b\}_{PB}\right)(x_1, y_1), \tag{3.50}$$

in the reduced phase space with configurations $A^a = (\mathcal{L}^{(0)}, \mathcal{W}^{(0)})$. The object $M_{ij}(x_1, y_1)$ is the inverse operator of $\{C^i(x_1), C^j(x_2)\}_{PB}$, whose non trivial components are computed to be

$$\begin{gathered} M_{12} = \frac{1}{2}\delta_{x_1y_1},\ M_{21} = -M_{12},\ M_{22} = \frac{1}{2}\partial_{x_1}\delta_{x_1y_1},\ M_{36} = -\frac{1}{4}\delta_{x_1y_1}, \\ M_{45} = \frac{1}{12}\delta_{x_1y_1},\ M_{46} = -\frac{1}{12}\partial_{x_1}\delta_{x_1y_1},\ M_{54} = -M_{45},\ M_{55} = \frac{1}{24}\partial_{x_1}\delta_{x_1y_1}, \\ M_{56} = -\frac{1}{4}(\mathcal{L}^{(0)}\delta_{x_1y_1} + \frac{1}{6}\partial_{x_1}^2\delta_{x_1y_1}),\ M_{63} = -M_{36},\ M_{64} = M_{46},\ M_{65} = -M_{56}, \\ M_{66} = -\frac{1}{4}\left(\partial_{x_1}\mathcal{L}^{(0)}\delta_{x_1y_1} + 2\mathcal{L}^{(0)}\partial_{x_1}\delta_{x_1y_1} + \frac{1}{6}\partial_{x_1}^3\delta_{x_1y_1}\right). \end{gathered} \tag{3.51}$$

It is easy to check that $M_{ij}(x_1, y_1) = -M_{ji}(y_1, x_1)$ as it should be. After some algebra (3.50) takes the explicit form

$$\begin{aligned} \{\mathcal{L}^{(0)}(y_1), \mathcal{L}^{(0)}(x_1)\}_D &= \partial_{x_1}\mathcal{L}^{(0)}\delta_{x_1y_1} + 2\mathcal{L}^{(0)}\partial_{x_1}\delta_{x_1y_1} + \frac{1}{2}\partial_{x_1}^3\delta_{x_1y_1}, \\ \{\mathcal{L}^{(0)}(y_1), \mathcal{W}^{(0)}(x_1)\}_D &= 2\partial_{x_1}\mathcal{W}^{(0)}\delta_{x_1y_1} + 3\mathcal{W}^{(0)}\partial_{x_1}\delta_{x_1y_1}, \\ \{\mathcal{W}^{(0)}(y_1), \mathcal{W}^{(0)}(x_1)\}_D &= -\frac{1}{6}\left(16\mathcal{L}^{(0)}\partial_{x_1}\mathcal{L}^{(0)} + \partial_{x_1}^3\mathcal{L}^{(0)}\right)\delta_{x_1y_1} - \\ &\quad \frac{1}{12}\left(9\partial_{x_1}^2\mathcal{L}^{(0)} + 32{\mathcal{L}^{(0)}}^2\right)\partial_{x_1}\delta_{x_1y_1} - \frac{5}{4}\partial_{x_1}\mathcal{L}^{(0)}\partial_{x_1}^2\delta_{x_1y_1} - \\ &\quad \frac{5}{6}\mathcal{L}^{(0)}\partial_{x_1}^3\delta_{x_1y_1} - \frac{1}{24}\partial_{x_1}^5\delta_{x_1y_1}, \end{aligned} \tag{3.52}$$

where all the $\mathcal{L}^{(0)}$ and $\mathcal{W}^{(0)}$ in the right hand side are evaluated on $x_1$. The brackets (3.52), define a $W_3$ algebra at fixed light cone coordinate $x_2$ slices[1] for

[1] This is, when evolution along $x_2$ is considered.

the phase space (3.18) [53]. Notice that in this case, the $\mu_3$ dependence is implicit in the fields through the redefinitions (3.49).

Now we go a step forward to compute the Dirac bracket on a Cauchy surface at fixed time $t_0$. This time the constraints will look like

$$C^i = \left(A_1^2 - 1, A_0^2, A_2^3 - \mu_3, A_1^3, A_0^3 - 2\mu_3\mathcal{L}, A_{-1}^3 + \frac{2}{3}\mu_3\partial_1\mathcal{L}\right), \tag{3.53}$$

and the corresponding first order in $\mu_3$ corrections to (3.51) are

$$\begin{gathered}
M_{14}^1 = \frac{1}{6}\delta_{x_1y_1},\ M_{15}^1 = -\frac{1}{6}\partial_{x_1}\delta_{x_1y_1},\ M_{16}^1 = \delta_{x_1y_1}\mathcal{L}^{(0)} + \frac{1}{4}\partial_{x_1}^2\delta_{x_1y_1}, \\
M_{23}^1 = -\frac{1}{2}\delta_{x_1y_1},\ M_{24}^1 = \frac{1}{3}\partial_{x_1}\delta_{x_1y_1},\ M_{25}^1 = -\frac{2}{3}\delta_{x_1y_1}\mathcal{L}^{(0)} - \frac{1}{4}\partial_{x_1}^2\delta_{x_1y_1}, \\
M_{26}^1 = \frac{5}{3}\delta_{x_1y_1}\partial_{x_1}\mathcal{L}^{(0)} + \frac{7}{3}\partial_{x_1}\delta_{x_1y_1}\mathcal{L}^{(0)} + \frac{1}{3}\partial_{x_1}^3\delta_{x_1y_1},\ M_{32}^1 = -M_{23}^1, M_{41}^1 = -M_{14}^1, \\
M_{42}^1 = M_{24}^1,\ M_{51}^1 = M_{15}^1,\ M_{52}^1 = -M_{25}^1,\ M_{56}^1 = -\frac{1}{6}\delta_{x_1y_1}\mathcal{W}^{(0)},\ M_{61}^1 = -M_{16}^1, \\
M_{62}^1 = \frac{2}{3}\delta_{x_1y_1}\partial_{x_1}\mathcal{L}^{(0)} + \frac{7}{3}\partial_{x_1}\delta_{x_1y_1}\mathcal{L}^{(0)} + \frac{1}{3}\partial_{x_1}^3\delta_{x_1y_1},\ M_{65}^1 = -M_{56}^1, \\
M_{66}^1 = -\frac{1}{3}\delta_{x_1y_1}\partial_{x_1}\mathcal{W}^{(0)} - \frac{2}{3}\partial_{x_1}\delta_{x_1y_1}\mathcal{W}^{(0)}.
\end{gathered} \tag{3.54}$$

Again it is easy to check that $M_{ij}^1(x_1,y_1) = -M_{ji}^1(y_1,x_1)$. From (3.50), (3.51) and (3.54) we compute the corresponding Dirac bracket. They can be checked to obey the compatibility property $\{C^i,\ldots\}_D = 0$.

The corrections to (3.52) are given by

$$\begin{aligned}
\{\mathcal{L}^{(0)}(y_1),\mathcal{L}^{(0)}(x_1)\}_D &= \ldots + 2\mu_3\partial_{x_1}\mathcal{W}^{(0)}\delta_{x_1y_1} + 4\mu_3\mathcal{W}^{(0)}\partial_{x_1}\delta_{x_1y_1}, \\
\{\mathcal{L}^{(0)}(y_1),\mathcal{W}^{(0)}(x_1)\}_D &= \ldots - \mu_3\left(\frac{8}{3}\mathcal{L}^{(0)}\partial_{x_1}\mathcal{L}^{(0)}\delta_{x_1y_1} + \frac{1}{6}\partial_{x_1}^3\mathcal{L}^{(0)}\delta_{x_1y_1} + \right. \\
&\qquad \frac{13}{3}\mathcal{L}^2\partial_{x_1}\delta_{x_1y_1} + \frac{4}{3}\partial_{x_1}^2\mathcal{L}^{(0)}\partial_{x_1}\delta_{x_1y_1} + \\
&\qquad \left.\frac{25}{6}\partial_{x_1}\mathcal{L}^{(0)}\partial_{x_1}^2\delta_{x_1y_1} + \frac{11}{3}\mathcal{L}^{(0)}\partial_{x_1}^3\delta_{x_1y_1} + \frac{1}{3}\partial_{x_1}^5\delta_{x_1y_1}\right), \\
\{\mathcal{W}^{(0)}(y_1),\mathcal{W}^{(0)}(x_1)\}_D &= \ldots - \mu_3\left(\frac{22}{3}\partial_{x_1}(\mathcal{W}^{(0)}\mathcal{L}^{(0)})\delta_{x_1y_1} + \frac{44}{3}\mathcal{L}^{(0)}\mathcal{W}^{(0)}\partial_{x_1}\delta_{x_1y_1} + \right. \\
&\qquad \partial_{x_1}^3\mathcal{W}^{(0)}\delta_{x_1y_1} + \frac{10}{3}\partial_{x_1}^2\mathcal{W}^{(0)}\partial_{x_1}\delta_{x_1y_1} + \\
&\qquad \left.4\partial_{x_1}\mathcal{W}^{(0)}\partial_{x_1}^2\delta_{x_1y_1} + \frac{8}{3}\mathcal{W}^{(0)}\partial_{x_1}^3\delta_{x_1y_1}\right),
\end{aligned} \tag{3.55}$$

and cannot be reabsorbed by a general analytical redefinition at first order in $\mu_3$

$$\mathcal{L} \to \mathcal{L} + \mu_3 \mathcal{L}^0_{1hom}, \;\; \mathcal{W} \to \mathcal{W} + \mu_3 \mathcal{W}^0_{1hom}, \tag{3.56}$$

where the $(\mathcal{L}^{(0)}_{1\;hom}, \mathcal{W}^{(0)}_{1\;hom})$ are given in the first line of (B.9). So the fixed time Dirac bracket algebra (3.55) on the phase space (3.18) is not isomorphic to $W_3$. However as we will see (3.18) can be embedded in a larger phase space whose constrained algebra at fixed time slices is isomorphic to $W_3^{(2)}$.

## 3.5 Dirac bracket in the diagonal embedding

As promised, in this section we first review how to embed the phase space (3.18) into a larger phase space with gravitational $sl(2,\mathbb{R})$ diagonally embedded into $sl(3,\mathbb{R})$. Thereafter we compute the corresponding fixed time Dirac bracket algebra and show that it is isomorphic to $W_3^{(2)}$.

First we redefine our generators as

$$\begin{gathered} J_0 = \frac{1}{2}V^2_0, \;\; J_\pm = \pm\frac{1}{2}V^3_{\pm 2}, \;\; \Phi_0 = V^3_0, \\ G^{(\pm)}_{\frac{1}{2}} = \frac{1}{\sqrt{8}}\left(V^2_1 \mp 2V^3_1\right), \quad G^{(\pm)}_{-\frac{1}{2}} = -\frac{1}{\sqrt{8}}\left(V^2_{-1} \pm 2V^3_{-1}\right), \end{gathered} \tag{3.57}$$

with the non trivial commutation relations being:

$$\begin{gathered} [J_i, J_j] = (i-j)J_{i+j}, \quad [J_i, \Phi_0] = 0, \quad [J_i, G^{(a)}_m] = (\frac{i}{2} - m)G^{(a)}_{i+m}, \\ [\Phi_0, G^{(a)}_m] = aG^{(a)}_m, \quad [G^{(+)}_m, G^{(-)}_n] = J_{m+n} - \frac{3}{2}(m-n)\Phi_0, \end{gathered} \tag{3.58}$$

with $i = -1, 0, 1$, $m = -\frac{1}{2}, \frac{1}{2}$ and $a = \pm$. The $J$'s denoting the $sl(2,\mathbb{R})$ generators in the diagonal embedding. After the shift $\rho \to \rho - \frac{1}{2}\log(\mu_3)$, the space of flat connections (3.18) can be embedded into

$$\begin{aligned} A_1 &= \nu_3 \left( \sqrt{2}\left(G^{(+)}_{\frac{1}{2}} + G^{(-)}_{\frac{1}{2}}\right) - \frac{1}{\sqrt{2}}\left(\mathcal{G}^+ + \mathcal{G}^-\right) J_- - \sqrt{3}\mathcal{J}\left(G^{(+)}_{-\frac{1}{2}} + G^{(-)}_{-\frac{1}{2}}\right)\right), \\ A_2 &= 2J_+ + 2\mathcal{G}^+ G^{(+)}_{-\frac{1}{2}} + 2\mathcal{G}^- G^{(-)}_{-\frac{1}{2}} + \sqrt{6}\mathcal{J}\Phi_0 + 2\mathcal{T}' J_-, \end{aligned} \tag{3.59}$$

where $\nu_3 \equiv \mu_3^{-\frac{1}{2}}$ and

$$\begin{aligned} \mathcal{G}^+ &= \frac{\sqrt{2}}{6}\mu_3^{\frac{3}{2}}\left(\partial_1\mathcal{L} + 6\mathcal{W}\right), \; \mathcal{G}^- = -\frac{\sqrt{2}}{6}\mu_3^{\frac{3}{2}}\left(\partial_1\mathcal{L} - 6\mathcal{W}\right), \\ \mathcal{J} &= \sqrt{\frac{2}{3}}\mu_3\mathcal{L}, \; \mathcal{T}' = -\frac{1}{6}\mu_3^2(\partial_1^2\mathcal{L} + 6\mathcal{L}^2). \end{aligned} \tag{3.60}$$

To obtain the previous phase space (3.18) out of (3.59), one must impose restrictions on the latter. This is, relations (3.60) imply the constraints

$$\mathcal{G}^+ - \mathcal{G}^- - \frac{1}{\sqrt{3}\,\nu_3}\partial_1\mathcal{J} = 0, \; \mathcal{T}' + \frac{1}{2\sqrt{6}\,\nu_3^2}\left(\partial_1^2\mathcal{J} + \nu_3^2\sqrt{\frac{3}{2}}\mathcal{J}^2\right) = 0, \tag{3.61}$$

which are not compatible with the equations of motion

$$\begin{aligned} \partial_1\mathcal{G}^\pm &= \mp\frac{\nu_3}{2\sqrt{2}}\left(6\mathcal{J}^2 \pm \sqrt{6}\partial_2\mathcal{J} + 4\mathcal{T}'\right), \; \partial_1\mathcal{J} = \sqrt{3}\nu_3\left(\mathcal{G}^+ - \mathcal{G}^-\right), \\ \partial_1\mathcal{T}' &= -\nu_3\left(\sqrt{3}\left(\mathcal{G}^- - \mathcal{G}^+\right)\mathcal{J} + \frac{1}{2\sqrt{2}}\left(\partial_2\mathcal{G}^- + \partial_2\mathcal{G}^+\right)\right), \end{aligned} \tag{3.62}$$

and hence they define second class constraints on the corresponding phase space of solutions. We will not impose them, in fact they are non perturbative in $\nu_3$.

The gauge parameter of residual gauge transformations for (3.59)

$$\begin{aligned} \Lambda &= 2\Lambda_{J_+}J_+ + 2\Lambda_{G^+_{\frac{1}{2}}}G^+_{\frac{1}{2}} + 2\Lambda_{G^-_{\frac{1}{2}}}G^-_{\frac{1}{2}} + \sqrt{6}\Lambda_{\Phi_0}\Phi_0 \\ &+ \left(-\frac{1}{2}\partial_2\Lambda_{J_+}\right)J_0 + \left(-\mathcal{G}^+\Lambda_{G^{(-)}_{\frac{1}{2}}} - \mathcal{G}^-\Lambda_{G^{(+)}_{\frac{1}{2}}} + 2\mathcal{T}'\Lambda_{J_+} + \frac{1}{4}\partial_2^2\Lambda_{J_+}\right)J_- \\ &+ \left(-\sqrt{6}\mathcal{J}\Lambda_{G^+_{\frac{1}{2}}} + 2\mathcal{G}^{(+)}\Lambda_{J_+} - \partial_2\Lambda_{G^{(+)}_{\frac{1}{2}}}\right)G^+_{-\frac{1}{2}} \\ &+ \left(-\sqrt{6}\mathcal{J}\Lambda_{G^-_{\frac{1}{2}}} + 2\mathcal{G}^{(-)}\Lambda_{J_+} + \partial_2\Lambda_{G^{(-)}_{\frac{1}{2}}}\right)G^-_{-\frac{1}{2}}, \end{aligned} \tag{3.63}$$

define the variations

$$\begin{aligned} \delta_{\Lambda_{J_+}}\mathcal{T}' &= \Lambda_{J_+}\partial_2\mathcal{T}' + 2\partial_2\Lambda_{J_+}\mathcal{T}' + \frac{1}{8}\partial_2^3\Lambda_{J_+}, \\ \delta_{\Lambda_{\Phi_0}}\mathcal{J} &= \partial_2\Lambda_{\Phi_0}, \;\; \delta_{\Lambda_{G^{(+)}_{\frac{1}{2}}}}\mathcal{J} = -\sqrt{6}\Lambda_{G^{(+)}_{\frac{1}{2}}}\mathcal{G}^-, \;\; \delta_{\Lambda_{G^{(-)}_{\frac{1}{2}}}}\mathcal{J} = \sqrt{6}\Lambda_{G^{(-)}_{\frac{1}{2}}}\mathcal{G}^+, \\ \delta_{\Lambda_{J_+}}\mathcal{G}^{(\pm)} &= \partial_2\Lambda_{J_+}G + \frac{3}{2}\Lambda_{J_+}\partial_2\mathcal{G}^{(\pm)} \pm \sqrt{6}\Lambda_{J_+}\mathcal{J}\mathcal{G}^\pm, \\ \delta_{\Lambda_{G^+_{\frac{1}{2}}}}\mathcal{G}^- &= \left(2\mathcal{T}' + 3\mathcal{J}^2 - \sqrt{\frac{3}{2}}\partial_2\mathcal{J}\right)\Lambda_{G^+_{\frac{1}{2}}} - \sqrt{6}\mathcal{J}\partial_2\Lambda_{G^+_{\frac{1}{2}}} + \frac{1}{2}\partial_2\Lambda_{G^+_{\frac{1}{2}}}, \end{aligned} \tag{3.64}$$

and the following differential of charge in the case of $x_1$ evolution

$$\delta Q = \int dx_2 tr\left(\Lambda \delta A_2\right) = \int dx_2 \left(\Lambda_{J_+} d\mathcal{T} - \Lambda_{\Phi_0} d\mathcal{J} - \Lambda_{G^{(-)}_{\frac{1}{2}}} d\mathcal{G}^+ - \Lambda_{G^{(+)}_{\frac{1}{2}}} d\mathcal{G}^-\right). \tag{3.65}$$

We could now repeat the Regge-Teitelboim analysis done for the case of the principal embedding to this case, but instead we choose to work out the Dirac bracket algebra.

For the seek of brevity we will work at $t_0 = 0$, but the conclusion of this computation remains unchanged at any other fixed time slice. The difference being that the charges will carry an explicit $t_0$ dependence as in the previous case. At $t_0 = 0$ the Cauchy data at first order in $\nu_3$ can be written in the form

$$\begin{aligned} A &= 2A_\phi d\tilde{\phi} = \left(A_{x_1} dx_1 + A_{x_2} dx_2\right) \\ &= \left(2J_+ + \sqrt{2}\nu_3\left(G^{(+)}_{-\frac{1}{2}} + G^{(-)}_{-\frac{1}{2}}\right) + 2\tilde{\mathcal{G}}^{+(0)} G^{(+)}_{-\frac{1}{2}} + 2\tilde{\mathcal{G}}^{-(0)} G^{(-)}_{-\frac{1}{2}} \right. \\ &\qquad \left. + \sqrt{6}\mathcal{J}^{(0)}\Phi_0 + 2\tilde{\mathcal{T}}'^{(0)} J_-\right) d\tilde{\phi} + O(\nu_3^2), \end{aligned} \tag{3.66}$$

by a choice of integration constants. Where

$$\tilde{\mathcal{G}}^{\pm(0)} = \mathcal{G}^{\pm(0)} - \frac{\sqrt{3}}{2}\nu_3 \mathcal{J}^{(0)}, \; \tilde{\mathcal{T}}'^{(0)} = \mathcal{T}'^{(0)} - \frac{1}{2\sqrt{2}}\nu_3\left(\mathcal{G}^{+(0)} + \mathcal{G}^{-(0)}\right). \tag{3.67}$$

Again, we remind that by super index (0) we refer to the initial conditions of the system of $x_1$ evolution equations (3.62). Some comments on notation are in order. Let the components of $A$ in the $W_3^{(2)}$ basis (3.57), be denoted again by $A_a$ with $a = 1, \ldots, 8$ and the ordering corresponding to

$$\left(J_0, J_+, J_-, \Phi_0, G^{(+)}_{-\frac{1}{2}}, G^{(-)}_{-\frac{1}{2}}, G^{(-)}_{-\frac{1}{2}}, G^{(+)}_{-\frac{1}{2}}\right). \tag{3.68}$$

At this point, we impose the four second class constraints

$$C^i = \left(A_1, A_2 - 2, A_7 - \sqrt{2}\nu_3, A_8 - \sqrt{2}\nu_3\right), \tag{3.69}$$

on the phase space (3.66) endowed with the algebra (3.13) written in the basis (3.68). Notice that we shall not impose at this point the second class constraints coming from (3.61). As already mentioned they are non perturbative in $\nu_3$.

Next, is straightforward to compute the Dirac bracket (3.50). For completeness we write down the non vanishing elements of $M_{ij}$ in this case

$$M_{11} = \frac{1}{8}\partial_{x_2}\delta_{x_2y_2},\ M_{12} = -M_{21} = -\frac{1}{2\sqrt{2}}\delta_{x_2y_2},\ M_{34} = -M_{43} = \frac{1}{2}\delta_{x_2y_2},$$
$$M_{13} = -M_{31} = M_{41} = -M_{14} = \frac{\nu_3}{4\sqrt{2}}\delta_{x_2y_2}, \tag{3.70}$$

from where we can check explicitly by using (3.50) that $\{C^i,\ldots\}_D = 0$.

The algebra in the reduced phase space will depend on $\nu_3$ explicitly, but after implementing the change

$$\mathcal{G}^{\pm(0)}_{\nu_3} = \tilde{\mathcal{G}}^{\pm(0)} - \frac{\sqrt{3}}{2}\nu_3\mathcal{J}^{(0)},\ \mathcal{T}'_{\nu_3} = \tilde{\mathcal{T}}' - \frac{1}{\sqrt{2}}\nu_3(\tilde{\mathcal{G}}^{+(0)} + \tilde{\mathcal{G}}^{-(0)}), \tag{3.71}$$

we obtain the undeformed $W_3^{(2)}$ algebra:

$$\begin{aligned}
\{\mathcal{T}'^{(0)}_{\nu_3}(y_2),\mathcal{T}'^{(0)}_{\nu_3}(x_2)\}_D &= \mathcal{T}'^{(0)}_{\nu_3}\delta_{x_2y_2} + 2\partial_{x_2}\mathcal{T}'^{(0)}_{\nu_3}\delta_{x_2y_2} + \frac{1}{8}\partial_{x_2}\delta_{x_2y_2},\\
\{\mathcal{J}^{(0)}_{\nu_3}(y_2),\mathcal{J}^{(0)}_{\nu_3}(x_2)\}_D &= \delta_{x_2y_2},\\
\{\mathcal{J}^{(0)}_{\nu_3}(y_2),\mathcal{G}^{\pm(0)}_{\nu_3}(x_2)\}_D &= \pm\sqrt{6}\mathcal{G}^{\pm(0)}_{\nu_3}\delta_{x_2y_2},\\
\{\mathcal{T}'^{(0)}_{\nu_3}(y_2),\mathcal{G}^{\pm(0)}_{\nu_3}(x_2)\}_D &= \partial_{x_2}\mathcal{G}^{\pm(0)}_{\nu_3}\delta_{x_2y_2} + \frac{3}{2}\mathcal{G}^{\pm(0)}_{\nu_3}\partial_{x_2}\delta_{x_2y_2} \pm \sqrt{6}\mathcal{J}^{(0)}_{\nu_3}\mathcal{G}^{\pm(0)}_{\nu_3}\delta_{x_2y_2},\\
\{\mathcal{G}^{+(0)}_{\nu_3}(y_2),\mathcal{G}^{-(0)}_{\nu_3}(x_2)\}_D &= -\left(2\mathcal{T}'^{0}_{\nu_3} + 3\mathcal{J}^{(0)^2}_{\nu_3} - \sqrt{\frac{3}{2}}\partial_{x_2}\mathcal{J}^{(0)}_{\nu_3}\right)\delta_{x_2y_2}\\
&\quad +\sqrt{6}\mathcal{J}^{(0)}_{\nu_3}\partial_{x_2}\delta_{x_2y_2} - \partial^2_{x_2}\delta_{x_2y_2},
\end{aligned} \tag{3.72}$$

that agrees precisely with the signature of charges in (3.65) and the transformation laws (3.64). The most canonical form can be achieved by the usual redefinition of energy momentum tensor $\mathcal{T}'^{(0)}_{\nu_3} \to \mathcal{T}'^{(0)}_{\nu_3} + \frac{1}{2}\mathcal{J}^{(0)^2}_{\nu_3}$ that makes $G^{\pm(0)}_{\nu_3}$ and $\mathcal{J}^{(0)}_{\nu_3}$ primaries of weight $\frac{3}{2}$ and 1 respectively. It is then proven that the fixed time asymptotic symmetry algebra of the space of solutions (3.59) is $W_3^{(2)}$ at first order in the parameter $\nu_3$. However, it would be strange would this not be the case at any order in $\nu_3$.

Notice that (3.59) does contain the $(\mu_3,\bar{\mu}_3)$ higher spin black hole solutions [33] (of course, after performing the shift $\rho \to \rho - \frac{1}{2}\log(\mu_3)$ on them), as zero modes.

Thence, both families (3.18) and (3.59) can be used to define the charges of these black holes. However, the two possibilities are not equivalent as we have already shown that (3.59) is larger than (3.18) and thence the corresponding algebras are not isomorphic. The family (3.59) is the preferred one, as for (3.18) it is impossible to define a basis of primary operators for the corresponding algebra [1].

We make a last comment before concluding. Notice that should we have worked with the following coordinates

$$x_1 = \frac{t+\phi}{2},\ x_2 = \frac{\phi}{2}, \tag{3.73}$$

all previously done remains valid, up to dependence on $t_0$. This dependence only affects implicitly the $W_3^{(2)}$ algebra through field redefinitions. The $hs(\lambda)$ ansätze to be introduced in the next chapter [62], belong to (3.59) under (3.73) for the truncation to $sl(3,\mathbb{R})$ via the limit $\lambda = 3$[2]. Thenceforth, in this case, the corresponding charges are not of higher spin character.

In this study we will not attempt to meddle with the issue of asymptotic symmetry algebras coming from generalised boundary conditions in the context of $hs(\lambda)$. We hope to report on that point in the near future.

[1]One can define a quasi-primary field of dimension 2, as a Virasoro subalgebra can be identified in (3.55), but the remaining generator can not be redefined in order to form a primary with respect to the Virasoro one.

[2]However one should keep in mind the extra shift in the coordinate $\rho \to \rho - \frac{1}{2}\log(\mu_3)$.

# 4

# A class of black holes in the $hs(\lambda) \times hs(\lambda)$ theory

In this chapter we argue that a given class of $hs(\lambda) \times hs(\lambda)$ flat connections do have a space time interpretation as black holes. As a first argument, we resort to the usual relation between connections and metric like tensor fields discussed in the finite dimensional case in [32].

We start by writing down the generic form for the flat connections of interest:

$$\begin{aligned} \mathcal{A}_\rho = V_0^2, \quad \bar{\mathcal{A}}_\rho = -V_0^2, \\ \mathcal{A}_{t,\phi} = bA_{t,\phi}b^{-1}, \quad \bar{\mathcal{A}}_{t,\phi} = \bar{b}\bar{A}_{t,\phi}\bar{b}^{-1}, \end{aligned} \tag{4.1}$$

with $b = e^{-\rho V_0^2}$, $\bar{b} = e^{\rho V_0^2}$. The generators and structure constants for $hs(\lambda)$ algebra are listed in appendix B.1. Let us denote our space-time coordinates as $(\rho, t, \phi)$ and restrict our analysis to connections that obey the gauge choice (4.1) with $A$ independent of $x_a = (t, \phi)$.

The relation between the connection and the space time tensor fields is:

$$g^{(n)} = -\frac{1}{2}tr(e^n), \qquad e = \mathcal{A} - \bar{\mathcal{A}}, \tag{4.2}$$

with $e$ being the dreibein. As a starting point we remind the condition:

$$e_t|_{\rho=0} = 0, \tag{4.3}$$

required in order to have a smooth horizon at $\rho = 0$ in the spacetime tensor field $g^{(n)}$. Under (4.3) each $t$ component in $g^{(n)}$ will have a zero at $\rho = 0$ with the appropriate order. By appropriate orders we mean those that make the corresponding reparameterisation invariant quantities smooth at $\rho = 0$. For instance, $g_t^{(n)} \sim \rho^n$, and thence, it will be smooth after transforming to a regular coordinate system about the horizon. In virtue of (4.1) we can rewrite (4.3) as:

$$\bar{A}_t = A_t. \tag{4.4}$$

From the flatness condition the $\phi$ components are constrained to be of the form:

$$A_\phi = P\left(A_t\right), \, \bar{A}_\phi = \bar{P}\left(A_t\right)^{1}, \tag{4.5}$$

where we take $P$ and $\bar{P}$ to be polynomials in $A_t$ and $\bar{A}_t$ respectively. The condition:

$$g^{(n)}(\rho) = g^{(n)}(-\rho), \tag{4.6}$$

guarantees that all the components of $g^{(n)}$ will be $C^\infty$ in the Cartesian coordinates in the plane $(\rho, t)$, with $\rho$ thought as the radial coordinate. Condition (4.6) ensures smoothness for the $g^{(n)}$ at $\rho = 0$. As far as Euclidean conical singularity is concerned, it will be automatically excluded by requiring fulfilment of the BTZ holonomy condition [33]. See the paragraph before (4.85) for more details.

Let us identify a sufficient condition on the connections $(A, \bar{A})$ for (4.6) to hold. Consider the generic connections:

$$A_a = \sum_{(s,m_s)} c^s_{m_s} V^s_{m_s}, \, \bar{A}_a = \sum_{(s,m_s)} \bar{c}^s_{m_s} V^s_{m_s}. \tag{4.7}$$

Notice that the change $\rho$ to $-\rho$ in (4.7) is equivalent to the change $V^s_{m_s} \to V^s_{-m_s}$ [2].

By inserting (4.7) in (4.2), and using the properties of the $\star$-product, we can notice that $tr(e_a^n)$ is invariant under the combined action of $\rho \to -\rho$ and any of the following pair of $\mathbb{Z}_2$ transformations:

$$\mathrm{I} : c^s_{m_s}\left(\bar{c}^s_{m_s}\right) \to c^s_{-m_s}\left(\bar{c}^s_{-m_s}\right) \quad \text{AND/OR} \quad \mathrm{I} \times \mathrm{II}, \tag{4.8}$$

[1] It could be the case that $A_t = P(A_\phi)$ and not the other way around, but for our purposes we stick to the case written above. In fact the most general case is $A_\phi = P_\phi(A)$ and $A_t = P_t(A)$ with a generic $A \in hs(\lambda)$.

[2] Here we consider $s = 1, \dots \infty$, $m_s = -2s+1, \dots, 2s-1$. So that under summation the indices $s$ and $m_s$ are mute and can be renamed without lack of rigor.

with the $\mathbb{Z}_2$ II given by

$$\text{II} \ : \bar{c}^s_{-m_s} \to -c^s_{m_s}. \tag{4.9}$$

Transformation I together with $V^s_{m_s} \to V^s_{-m_s}$ leaves the dreibein $e_a = \mathcal{A}_a - \bar{\mathcal{A}}_a$ invariant and therefore the trace of powers of $e_a$. The transformation II leaves $tr(e^n_a)$ invariant but generically not the dreibein $e_a$.

A trivial (even) representation of $(A, \bar{A})_a$ under (4.8) is sufficient condition for (4.6). Should some components in $(A, \bar{A})$ not remain invariant under the $\mathbb{Z}_2$ I or I×II, but carry a non trivial (odd) representation under any of them, then the corresponding component of the dreibein $e$ will carry a non trivial (odd) representation too. Condition (4.6) will thus imply that traces involving an odd number of such components must vanish.

Let us analyze the particular case of the BTZ connection

$$A_t = \bar{A}_t = \frac{1}{2}a, \quad A_\phi = -\bar{A}_\phi = \frac{1}{2}a, \tag{4.10}$$

where

$$a = V^2_1 + MV^2_{-1}. \tag{4.11}$$

From now on, for simplicity, we will choose the value $M = -1$, which locates the horizon at $\rho = 0$. For later use we define $a_{\pm\rho} = bab^{-1}$.

The $\phi$ component of the pair $(A, \bar{A})$ remains invariant under the transformation II whereas the $t$ component is odd. However the $t$ component is also odd under I and so even under the composition I× II. Finally, the following symmetries of the corresponding $t$ and $\phi$ components of the dreibeins

$$\begin{aligned} e_t &= \frac{1}{2}(a_\rho - a_{-\rho}) \equiv a_{\text{ I} \times \text{II} -even}, \\ e_\phi &= \frac{1}{2}(a_\rho + a_{-\rho}) \equiv a_{\text{ II}-even}, \end{aligned} \tag{4.12}$$

imply that (4.6) holds for the connection (4.10). We can still get further information from symmetries. As $e_t$ and $e_\phi$ are odd under I, any tensor field component with an odd number of $t$ plus $\phi$ directions, vanishes. As $e_t$ and $e_\phi$ are odd and even respectively, under II, any tensor component with an odd number of $t$ components vanish. Finally, what said before implies that any tensor component with and odd number of $\phi$ directions vanish too.

Much of what was used for the BTZ case before, holds also for generic connections. Specifically:

- Any pair of connections $(A, \bar{A})$ that carries a trivial representation under I or I $\times$ II, will define metric-like fields obeying (4.6).

Additionally, one can argue also for a necessary condition for (4.6) to hold. Let us suppose that a pair $(A, \bar{A})$ contains a part $(A^{rep}, \bar{A}^{rep})$ that satisfies the conditions above, and a part $(\delta A, \delta\bar{A})$ that does not, but still defines metric like fields which are even under $\rho$ to $-\rho$. In that case the term $(\delta\mathcal{A} - \delta\bar{\mathcal{A}})$ should be orthogonal to itself[1], its powers, and powers of the generators in $(\mathcal{A}^{rep} - \bar{\mathcal{A}}^{rep})$ (This is possible to find, for example $V_2^3$ is orthogonal with itself and its powers). Should this not be the case, the term $(\delta\mathcal{A} - \delta\bar{\mathcal{A}})$ would give contributions which are not even in $\rho$ (based on the invariance property of the trace mentioned above). However, if $(\mathcal{A}^{rep} - \bar{\mathcal{A}}^{rep})$ contains all of the $sl(2, \mathbb{R})$ elements, $V^2_{0,\pm 1}$, it is impossible to find a set of generators in $hs(\lambda)$ that is orthogonal to every power of them. In that case, symmetry under any of the $\mathbb{Z}_2$ transformations in the maximal set, out of the (4.8), (I, I $\times$ II) for any $(s, m_s)$[2] is also a necessary condition for (4.6).

At this point we specify our class of connections:

$$\begin{aligned} A_t &= \bar{A}_t = P_t\left(a\right), \\ A_\phi = \frac{1}{2}a + P_\phi\left(a\right), &\quad \bar{A}_\phi = -\frac{1}{2}a + \bar{P}_\phi\left(a\right), \end{aligned} \tag{4.13}$$

with $P_t$, $P_\phi$ and $\bar{P}_\phi$ being arbitrary traceless polynomials of the form

$$\begin{aligned} P_t &= \sum_{i=0}^{\infty} \nu_i \left(a^{2i+1} - \text{ trace }\right), \\ P_\phi = \sum_{i=0}^{\infty} \mu_{i+3}\left(a^{2i+2} - \text{ trace }\right), &\quad \bar{P}_\phi = \sum_{i=0}^{\infty} \bar{\mu}_{i+3}\left(a^{2i+2} - \text{ trace }\right). \end{aligned} \tag{4.14}$$

Notice that (4.14) obeys (4.3) and that $P_t$ and $P_\phi$ are selected in such a way that $g_{t\phi} = 0$. We also choose the components $g_{\rho t}$ and $g_{\rho\phi}$ to vanish. In particular (4.14)

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[1] The orthogonality is meant with respect to the trace operation in $hs(\lambda)$.

[2] Notice that there are many possible $\mathbb{Z}_2$'s. The number grows exponentially with the number of generators in $(\mathcal{A} - \bar{\mathcal{A}})$. The calligraphic letters indicate the full connection, $\rho$ component and $\rho$ dependence included.

reduce to the non rotating BTZ$_{M=-1}$ connection in the limit $\nu_0 = \frac{1}{2}$, $\nu_{i>0} = 0$, and vanishing $\mu_i$, $\bar{\mu}_i$. Now:

- The transformations of $a$, the corresponding deformation polynomials ($P_\phi(a)$, $\bar{P}_\phi(a)$, $P_t(a)$) and the $\rho$ components $\pm V_0^2$ under I in (4.8), are odd, even, odd and even respectively.
- In virtue of properties of the $\star$-product, the traces with odd numbers of $a$ and $P_t(a)$ with any number of insertions of $V_0^2$ and $(P_\phi(a), \bar{P}_\phi(a))$, vanish, and so all non vanishing traces are even under I and henceforth even under $\rho \to -\rho$.

We conclude that the ansätze (4.14) give rise to spacetime tensor fields that obey (4.6). In fact we explicitly checked (4.6) to hold up to arbitrary higher order in $n$ and the order of the polynomials $P$ and $\bar{P}$.

In the near horizon expansion, $g^{(2)}$, the line element defined by (4.2), will look like:

$$d\rho^2 - \frac{4}{T^2}\rho^2 dt^2 + \ldots = \rho^* dv^2 + \frac{1}{2} d\rho^* dv + \ldots, \tag{4.15}$$

with $v = t - \frac{T}{2} log(\rho) + \ldots$ and $\rho^* = \frac{4}{T^2}\rho^2 + \ldots$ being coordinate redefinitions that are going to be useful later on when analyzing fluctuations. The $\ldots$ denoting higher orders corrections in $\rho$. The temperature:

$$T\left(P_t\right) \equiv \frac{1}{\sqrt{\frac{1}{2} tr\left([P_t(a), V_0^2]^2\right)}},^{1} \tag{4.16}$$

defines the thermal periodicity under $t \to t + \pi T i$.

We will focus our study in the cases $\nu_0 = \frac{1}{2}$, $\nu_{i>0} = 0$. These are solutions that obey the usual BTZ holonomy-smoothness condition as the temporal component of the connection coincides with the BTZ one with $M = -1$. This implies that not only the eigenvalues of the time component of connection are the same as BTZ$_{M=-1}$, but also that the holonomy around the contractible euclidean time

[1]From the positiveness of the traces $tr(V^{2s}_{2m_s+1} V^{2s}_{-2m_s-1})$, see (B.4), in the interval $0 < \lambda < 1$ and the fact we have chosen odd powers of $a$ in $P_t$ it follows that the quantity inside the roots in (4.17) and (4.19) is a sum of positive defined quantities and hence positive defined. We stress that we restrict our study to the interval $0 < \lambda < 1$.

cycle coincides with the BTZ case, since the euclidean periodicity, determined by the temperature $T\left(\frac{1}{2}a\right) = 2$, is the same as for the $\mathrm{BTZ}_{M=-1}$.

However before going on, let us comment on the possibility of arbitrary $\nu_i$. The euclidean smoothness condition is:

$$e^{\pi i T(P_t) P_t(a)} \sim V_0^1. \tag{4.17}$$

To solve for (4.17) we use the fact that $\pi i P(a)$, with $P(a)$ an arbitrary polynomial in $a$ with arbitrary integer coefficients, are known to exponentiate to $V_0^1$ in the region $0 < \lambda < 1$, see [63].

Then relations (4.17) reduce to find out the $\nu_i$ such that $\nu_i T(P_t)$ are integers. To study this quantization conditions it is useful to write down $P_t$ in the basis

$$a_\perp^{s-1} \equiv \frac{1}{N_s} \sum_{t=0}^{s-1} (-1)^t \begin{pmatrix} s-1 \\ t \end{pmatrix} V_{s-1-2t}^s \sim (a^{s-1})\big|_{V_{m_t}^{t<s} \to 0}, \tag{4.18}$$

where $N_s$ is a normalization factor, chosen in such a way that: $((a_\perp^{s-1})^2) = 1$. We get thus

$$P_t(a) = \sum_{s=0}^{\infty} \nu_{\perp s} \frac{a_\perp^{2s+1}}{\sqrt{\frac{1}{2} tr([a_\perp^{s-1}, V_0^2]^2)}}, \qquad \nu_\perp^s = M^{si} \nu_i, \tag{4.19}$$

where the linear transformation matrix $M$ is upper triangular. In the appendix C.1 we present the explicit form for $M$, (C.1), for the case $\mu_{2i+1} \neq 0$, with $i = 0, \ldots, 4$. An important property to use is that the eigenvalues (the diagonal elements) of $M$ can be checked to be larger or equal than 1 in the range $0 < \lambda < 1$ until arbitrary large $i$.

The desired quantization conditions can be written as:

$$\nu_i T(P_t) = (M^{-1})_{is} \cos\theta^s = n_i, \tag{4.20}$$

with $\cos\theta^s \equiv \frac{\nu_\perp^s}{\sqrt{\sum_s (\nu_\perp^s)^2}}$ and $n_i$ an arbitrary integer. The condition for the quantization relation (4.20) to admit solutions is:

$$\sum_{s=1}^{\infty} (M \centerdot n)^{s2} = 1. \tag{4.21}$$

In appendix C.1 we show that the property of the eigenvalue of $M$ mentioned above excludes the presence of other solutions to the consistency condition (4.21) in the region $0 < \lambda < 1$, apart from the trivial one, $n_0 = 1$ ($\nu_0 = \frac{1}{2},\ \nu_{i>0} = 0$). Here we just continue with the cases that are continuously linked to the BTZ connection in the limit $\mu_i$, $\bar{\mu}_i$ to zero. Namely $\nu_0 = \frac{1}{2},\ \nu_{i>0} = 0$. The requirement of the BTZ holonomy condition will guarantee the absence of any possible conical singularity in the tensor like fields as the dreibein itself is thermal periodic.

Generically, (4.14) will define asymptotically Lifshitz metrics with critical exponent $z < 1$, except for the cases in which the contributions out of the deformation parameters $\mu_i$, $\bar{\mu}_i$ will not provide $\rho$ dependence. An example being when $\bar{\mu}_i = 0$ (or $\mu_i = 0$) in which case the only contribution to $g_{\phi\phi}$ comes at quadratic order in $\mu_i$(or $\bar{\mu}_i$) but it is independent of $\rho$ due to the cyclic property of the trace. In those cases the metric becomes asymptotically AdS.

To summarize, (4.14) will define metrics of two classes:

- Generically Lifshitz metric with $z < 1$.
- AdS metrics when $\mu_{2i} = 0$ (or $\bar{\mu}_{2i} = 0$).

This classification relies on the definition (4.2). For instance the line elements coming from (4.2) for the cases $\mu_3 \neq 0$, $\bar{\mu}_3 = -\mu_3 \neq 0$ and $\bar{\mu}_3 = \mu_3 \neq 0$ look like :

$$\begin{aligned}
ds^2_{(\mu_3,\ 0)} &= d\rho^2 - \sinh^2\rho\ dt^2 + \left(\cosh^2\rho + \frac{16(\lambda^2-4)}{15}\mu_3^2\right) d\phi^2,\\
ds^2_{(\mu_3,-\mu^3)} &= d\rho^2 - \sinh^2\rho\ dt^2 + \frac{1}{30}\left(12\left(\lambda^2-4\right)\mu_3^2\cosh(4\rho)\right.\\
&\qquad \left.+5\left(4\left(\lambda^2-4\right)\mu_3^2 + 3\cosh(2\rho)+3\right)\right) d\phi^2,\\
ds^2_{(\mu_3,\ \mu_3)} &= d\rho^2 - \sinh^2\rho\ dt^2 + \frac{1}{5}\cosh^2(\rho)\left(-8\left(\lambda^2-4\right)\mu_3^2\cosh(2\rho)\right.\\
&\qquad \left.+8\left(\lambda^2-4\right)\mu_3^2+5\right) d\phi^2.
\end{aligned} \tag{4.22}$$

The first line element in (4.22) behaves asymptotically as $\mathrm{AdS}_3$ and shows a smooth horizon at $\rho = 0$, while the last two cases are Lifshitz metrics with dynamical critical exponent $z = \frac{1}{2} < 1$. Should we have turned on a higher spin $\mu$ deformation, the parameter $z$ would have decreased like $z = \frac{1}{4}, \frac{1}{8} \ldots$.

The bulk of the present chapter, section 4.1, will be devoted to the study of matter fluctuations around the connections (4.13), which are not just gravitational but involve also higher spin tensor fields turned on. This further analysis will confirm the expectation that these backgrounds truly describe black holes, through the "dissipative" nature of matter fluctuations we will find.

Before closing this section, we make contact (perturbatively in $\mu_3$) with other relevant backgrounds studied in the literature recently. More precisely, we look for static gauge parameters $(\Lambda, \bar{\Lambda})$ (independent of $x_{1,2}$), that transform (4.14) to the GK [33] and BHPT2 [2, 64] backgrounds. Notice that these gauge transformations will not change the eigenvalues of the components $(A_{1,2}, \bar{A}_{\bar{1},\bar{2}})$ of the connections because they are just similarity transformations. The two classes of backgrounds we want to relate ours, are described by the following connections:

$$
\begin{aligned}
A_1 &= V_1^2 + \mathcal{L} V_{-1}^2 + \mathcal{W} V_{-2}^3 + \mathcal{Z} V_{-3}^4 + \ldots, \quad A_2 = \sum_{i=0}^{\infty} \mu_{i+3} \left( A_1^{i+2} - \text{ traces} \right), \\
\bar{A}_{\bar{1}} &= V_{-1}^2 + \bar{\mathcal{L}} V_1^2 + \bar{\mathcal{W}} V_2^3 + \bar{\mathcal{Z}} V_3^4 + \ldots, \quad \bar{A}_{\bar{2}} = \sum_{i=0}^{\infty} \bar{\mu}_{i+3} \left( \bar{A}_{\bar{1}}^{i+2} - \text{ traces } \right).
\end{aligned}
\tag{4.23}
$$

Our parameters $(\mu_i, \bar{\mu}_i)$ will be identified precisely with the chemical potentials in (4.23). In our approach the charge-chemical potential relations [33, 65] are determined a priori by the condition $\nu_0 = \frac{1}{2}$, $\nu_{i>0} = 0$. Namely, after applying the gauge transformations $(\Lambda, \bar{\Lambda})$ the charges $\mathcal{L}$, $\mathcal{W}$ and $\mathcal{Z}$ will be already written in terms of the chemical potentials $(\mu_i, \bar{\mu}_i)$. In this way one can generate GK, and BHPT2 ansätze with more than one $(\mu_i, \bar{\mu}_i)$ turned on, and with the holonomy conditions already satisfied. However, with the choice $\nu_0 = \frac{1}{2}$, $\nu_{i>0} = 0$ one can only reach branches that are smoothly related to the $\text{BTZ}_{M=-1}$.
Taking $x_1 = x_{\bar{2}} = x_+$ and $x_2 = x_{\bar{1}} = x_-$, we recover the GK background, whereas for $x_1 = x_{\bar{1}} = \phi$ and $x_2 = x_{\bar{2}} = t$ we get BHPT2.

For later use, we write down the particular gauge transformations that takes the representative with non vanishing $\mu_3 = -\bar{\mu}_3$ into the wormhole ansatz for

GK's case. They read, respectively, to leading order in $\mu_3 = -\bar{\mu}_3$:

$$\begin{aligned} \Lambda_{GK} &= \mu_3 \left( -\frac{5}{3} e^{-\rho} V^3_{-1} + e^{\rho} V^3_1 \right) + \text{ commutant of } a_\rho + O(\mu_3^2), \\ \bar{\Lambda}_{GK} &= \mu_3 \left( e^{\rho} V^3_{-1} - \frac{5}{3} e^{-\rho} V^3_1 \right) + \text{ commutant of } a_{-\rho} + O(\mu_3^2). \end{aligned} \tag{4.24}$$

The holonomy conditions are satisfied a priori and so the corresponding charge-chemical potential relations are as follows:

$$\mathcal{L} = \bar{\mathcal{L}} = -1 + O(\mu_3^2),\ \mathcal{W} = -\bar{\mathcal{W}} = \frac{8}{3}\mu_3 + O(\mu_3^3),\ \mathcal{Z} = \bar{\mathcal{Z}} = O(\mu_3^2),\ \ldots \tag{4.25}$$

For BHPT2, namely when the chemical potentials are turned on along the $t$ direction and the asymptotic symmetry algebra is the undeformed $W_\lambda \times W_\lambda$ [64, 66], they are given by:

$$\begin{aligned} \Lambda_{BHPT2} &= 2\Lambda_{GK} + O(\mu_3^2), \\ \bar{\Lambda}_{BHPT2} &= 2\bar{\Lambda}_{GK} + O(\mu_3^2). \end{aligned} \tag{4.26}$$

In this case the relations charge-chemical potential are:

$$\mathcal{L} = \bar{\mathcal{L}} = -1 + O(\mu_3^2),\ \mathcal{W} = -\bar{\mathcal{W}} = \frac{16}{3}\mu_3 + O(\mu_3^3),\ \mathcal{Z} = \bar{\mathcal{Z}} = O(\mu_3^2). \tag{4.27}$$

Later on, we will apply these transformations to the matter fluctuations in the $\bar{\mu}_3 = -\mu_3 \neq 0$ background in (4.14).

## 4.1 Equations for fluctuations

In this subsection we show how to obtain the differential equations for the scalar fluctuations over the backgrounds (4.14). Firstly, we review how this works for the BTZ$_{M=-1}$ case. This will allow us to identify a strategy for the cases (4.14).

As mentioned in the introduction, the equation of motion of the master field $C$ in generic background connections $(A, \bar{A})$ is simply the horizontality condition:

$$\tilde{\nabla} C \equiv dC + \mathcal{A} \star C - C \star \overline{\mathcal{A}} = 0 \ \text{ with } \ C = \sum C^s_{m_s} V^s_{m_s}, \tag{4.28}$$

whose formal solution and its corresponding transformation law under left multiplication $(g,\bar{g})\to(e^{\Lambda}g,e^{\bar{\Lambda}}\bar{g})$, are, respectively:

$$C = g\,\mathcal{C}\,\bar{g}^{-1} \text{ and } C_{(\Lambda,\bar{\Lambda})} = e^{\Lambda}Ce^{-\bar{\Lambda}}, \tag{4.29}$$

where $d\mathcal{C}=0$ and $\mathcal{C}=\sum \mathcal{C}^s_m V^s_m$.

The trace part of the master field $C$ and its transformation law are also:

$$C^1_0 = (C)\big|_{V^1_0} \text{ and } C^1_{0\,(\Lambda,\bar{\Lambda})} = \left(e^{(\Lambda-\bar{\Lambda})}C\right)\big|_{V^1_0}. \tag{4.30}$$

The integration constant $\mathcal{C}$ is evaluated in the limit $C\big|_{g\to 1}$. In our cases (4.14) $g$ goes to 1 at the points $(\rho,x_a)=0$. However notice that these points are located at the horizon $\rho=0$ of (4.14) and, as we shall see, many of the components of the master field $C$ will diverge there.

Our aim is to "fold" (4.28) for our ansätze (4.14) with $\nu_0=\frac{1}{2}$, $\nu_{i>0}=0$. By "folding" we mean the process of expressing every $C^s_{m_s}$ in terms of $C^1_0$ and its derivatives, and finally to obtain a differential equation for $C^1_0$. For such a purpose we start by reviewing how this process works for the simplest case, BTZ$_{M=-1}$, and in doing so we will discover how to fold the matter fluctuations in the case of the backgrounds (4.14).

We start by proving that for BTZ$_{M=-1}$ every higher spin component $C^s_{m_s}$, can be expressed in terms of $\partial_\pm$ derivatives of $C^1_0$ and $C^2_0$. Using the explicit forms for $g$ and $\bar{g}$ in this case:

$$C = e^{-a_\rho x_+}\mathcal{C}(\rho)e^{-a_{-\rho}x_-}. \tag{4.31}$$

It is easy to see that:

$$\partial_\pm C^1_0 = -(a_{\pm\rho}C)\big|_{V^1_0} \sim -(e^{\pm\rho}C^2_1 - e^{\mp\rho}C^2_{-1}), \tag{4.32}$$

from where (C.15) of the Appendix C.3 is immediate. By $\left(\dots\right)\big|_{V^1_0}$ we denote the coefficient of $V^1_0$ in $\left(\dots\right)$.

Now we can repeat the procedure at second order in $\pm$ derivatives of $C^1_0$. At this stage we can write down three combinations:

$$\partial^2_+,\ \partial^2_-,\ \partial^2_{+-},$$

which would generate the following quadratic relations inside the trace element:

$$\begin{aligned} a_\rho^2 &= \tilde{V}_0^1 + e^{2\rho} V_2^3 - 2V_0^3 + e^{-2\rho} V_{-2}^3, && (4.33)\\ a_{-\rho}^2 &= \tilde{V}_0^1 + e^{-2\rho} V_2^3 - 2V_0^3 + e^{2\rho} V_{-2}^3, && (4.34)\\ a_\rho a_{-\rho} &= \cosh 2\rho(\tilde{V}_0^1 - 2V_0^3) - 2\sinh 2\rho V_0^2 + V_2^3 + V_{-2}^3, && (4.35) \end{aligned}$$

where $\tilde{V}_0^1 = \frac{(\lambda^2-1)}{3} V_0^1$.

Equations (4.33), (4.34) and (4.35), allow to write down $C_{-2}^3$, $C_0^3$ and $C_2^3$ in terms of

$$\left(\partial_+^2 C_0^1,\ \partial_-^2 C_0^1,\ \partial_{+-}^2 C_0^1, C_0^2\right),$$

so that one arrives to the relations (C.17) and (C.19).

Proceeding this way, we see that at the level $s = 3$ we can still use first derivatives acting on $C_0^2$:

$$\partial_+ C_0^2 = -(V_0^2 a_\rho C)\big|_{V_0^1} \quad \text{and} \quad \partial_- C_0^2 = -(a_{-\rho} V_0^2 C)\big|_{V_0^1}. \qquad (4.36)$$

Then, if we use:

$$\begin{aligned} V_0^2 a_\rho &= -\frac{1}{2}(e^\rho V_1^2 + e^{-\rho} V_{-1}^2) - e^{-\rho} V_{-1}^3 + e^\rho V_1^3, && (4.37)\\ a_{-\rho} V_0^2 &= \frac{1}{2}\left(e^{-\rho} V_1^2 + e^\rho V_{-1}^2\right) - e^\rho V_{-1}^3 + e^{-\rho} V_1^3, && (4.38) \end{aligned}$$

on both equations in (4.36), together with (4.32), we get the spin three components $C_{\pm 1}^3$ in terms of:

$$\left(\partial_+ C_0^1,\ \partial_- C_0^1,\ \partial_+ C_0^2,\ \partial_- C_0^2\right),$$

as shown in (C.18).

Now we show how this process of reduction works at any spin level $s$. First we remind some useful properties of the lonestar product. Let us start by the generic product

$$V_{m_1}^{s_1} \star V_{m_2}^{s_2},$$

that will reduce to a combination of the form:

$$V_{m_1+m_2}^{s_1+s_2-1} + \ldots + V_{m_1+m_2}^{s_1+s_2-1-j} + \ldots + V_{m_1+m_2}^{|m_1+m_2|+1}, \qquad (4.39)$$

where we are not paying attention to the specific coefficients, which will be used in due time. The index $j$ goes from 0 to $s_1 + s_2 - 2 - |m_1 + m_2|$. From (4.39) it follows that the products: $V^{s_1}_{m_1} \star a$ and $a \star V^{s_1}_{m_1}$, with $a = V^2_1 - V^2_{-1}$, will contain combinations of the form:

$$V^{s_1+1}_{m_1+1} + V^{s_1+1}_{m_1-1} + \dots, \tag{4.40}$$

where the $\dots$ stand for lower total spin $s$ contributions. For our purposes only the highest total spin generators are relevant.

Furthermore, for any chain of $2s-1$ generators with even spin $2s$ and even projections, $\sum\limits_{m=-s+1}^{s-1} V^{2s}_{2m} + \dots$, further left or right multiplication by $a$ will change it into a chain of $2s$ generators $\sum\limits_{m=-s}^{s-1} V^{2s+1}_{2m+1} + \dots$ at the next spin level $2s+1$. As a consequence, arbitrary powers of $a$ look like:

$$a^{2s} = \sum_{m=-s}^{s} V^{2s+1}_{2m} + \dots \quad \text{and} \quad a^{2s+1} = \sum_{m=-s-1}^{s} V^{2s+2}_{2m+1} + \dots. \tag{4.41}$$

From (4.30) and (4.31), it follows that each $\partial_\pm$ derivative acting on $C^1_0$ is equivalent to a left or right multiplication by $-a_{\pm\rho}$ inside the trace. In particular, taking $2s$ of these derivatives on $C^1_0$ is equivalent to take $2s$ powers of $\pm a_{\pm\rho}$ inside the trace.

The number of different derivatives of order $2s$ denoted by: $\partial^{2s}_\pm$ is $2s+1$. This number coincides precisely with the number of components with total spin=$2s+1$ in the first power of (4.41). So one can use the $2s+1$ relations:

$$\partial^{2s}_\pm C^1_0 = (a^{2s}_{\pm\rho} C)\big|_{V^1_0}, \tag{4.42}$$

to solve for $2s+1$ components of $C$:

$$[C^{2s+1}_{2m}] \text{ with } m = -s, \dots, s, \tag{4.43}$$

in terms of components with lower total spin and their $\pm$ derivatives.

One can always solve equations (4.42) in terms of (4.43) because the set of symmetrized powers of $a^{2s}_{\pm\rho}$ (more precisely, their components with the highest total spin) will generate a basis for the $2s+1$ dimensional space generated by:

$$[V^{2s+1}_{2m}] \text{ with } m = -s-1, \dots, s.$$

In order to prove this statement, we take the large $\rho$ limit. In this limit a given symmetric product $a_{\pm}^{2s}$ with $2m_+$ plus signs and $2m_- = 2\,(s - m_+)$ minus signs reduces to a single basis element $V^{2s}_{2(m_+-m_-)}$. So, the set of all possible symmetric products $a_{\pm}^{2s}$ span an $2s+1$-dimensional vector space. Consequently the system of equations (4.42) is non-degenerate.

Similarly, increasing the spin by one, one can solve the $2s+2$ relations:

$$\partial_{\pm}^{2s+1} C_0^1 = -(a_{\pm\rho}^{2s+1} C)\big|_{V_0^1}, \tag{4.44}$$

for the $2s+2$ components

$$[C^{2s+2}_{2m+1}] \text{ with } m = -s-1, \dots,\ s, \tag{4.45}$$

in terms of lower spin components and their $\pm$ derivatives.

Summarizing, what we have done is to use the identities:

$$\partial_+ = -a_\rho \star_L, \qquad \partial_- = -a_{-\rho} \star_R, \tag{4.46}$$

with left $\star_L$ and right $\star_R$ multiplication inside any trace. Notice that in Fourier space $(-i\partial_t, -i\partial_\phi) = (w, k)$ the master field (4.31) is an eigenstate of the operators on the right hand side of (4.46). This will turn out to be a crucial observation, and it will be useful for later purposes, but for now we just use (4.46) to solve for every component of $C^s_{m_s}$ with $(s, m_s)$ being points in a "semi-lattice" with origin $(1, 0)$ and generated by positive integral combinations of basis vectors $(2, 1)$ and $(2, -1)$. From now on we will refer to this particular "semi-lattice" as $I$ and to the corresponding set of components of the master field $C$ in it as $C^I$.

In exactly the same manner one can show how the set of powers

$$a_\rho^{s_+} V_0^2 a_{-\rho}^{s_-}, \tag{4.47}$$

with $s = s_+ + s_- + 1$ spans the complementary "semi-lattice" of spin $s+1$ and projection $m_s = -s+1, -s+3, \dots, s-3, s-1$ generators. Namely the "semi-lattice" with origin at $(2, 0)$ and positive integral combinations of $(2, 1)$ and $(2, -1)$. We refer to it as $II$, and the corresponding components of the master field $C$, $C^{II}$. More in detail, this means that we can solve the $s$ relations:

$$\partial_+^{s_+} \partial_-^{s_-} C_0^2 = (-1)^{s_+ + s_-} \left(a_{-\rho}^{s_-} V_0^2 a_\rho^{s_+} C\right)\big|_{V_0^1}, \tag{4.48}$$

for the set of components in $C^{II}$ with highest spin$= s+1$ and projections $m_s = -s+1, -s+3, \ldots, s-3, s-1$.

- In conclusion, equations (4.42)-(4.45) and (4.48) allow to solve for every components of $C^I$ and $C^{II}$ in terms of $C^1_0$ and $C^2_0$ and their derivatives along $\pm$ directions.

Finally, the $V^1_0$-$d\rho$ component of (4.28) gives $C^2_0 \sim \partial_\rho C^1_0$ and the $V^2_0$-$d\rho$ component of (4.28) will determine the differential equation $D_2 C^1_0 = 0$ with

$$D_2 = \Box - \left(\lambda^2 - 1\right), \tag{4.49}$$

being the Klein Gordon operator in the $\mathrm{BTZ}_{M=-1}$ background, for a scalar field with mass squared $\lambda^2 - 1$.

Now we go back to our case $\nu_0 = \frac{1}{2}$ $\nu_{i>0} = 0$. Here the $t$ component of (4.28) is the same as for the $\mathrm{BTZ}_{M=-1}$ case and so we use it as before

$$\partial_t C^{s-1}_{m_s+1} = C^s_{m_s} + C^s_{m_s+2} + \ldots, \tag{4.50}$$

to solve for the highest spin, with the lowest spin projection components $(s, m_s)$. The dots refer to components with lower total spin and we have omitted precise factors. That is, we solve for all components in $C^I$ and $C^{II}$ in terms of the line of highest weight and its contiguous next-to-highest weight components, namely:

$$C^{s+1}_s \text{ and } C^{s+2}_s \text{ with } s = 0, \ldots, \infty. \tag{4.51}$$

Next, $\partial_\phi \sim a^{1+\tilde{s}_{Max}} +$ lower powers, and therefore from (4.41) one can prove that the use of the $d\phi$ component of the equations (4.28) reduces the set of independent elements in (4.51) to:

$$C^{s+1}_s \text{ and } C^{s+2}_s \text{ with } 0 \leq s \leq s_{max}, \tag{4.52}$$

with $s_{max} + 1$ being at most $\tilde{s}_{max} + 1$, the maximum value of the power in the polynomials $(P(a), \bar{P}(a))$, that determines the $\phi$ component of the connections $(A_\phi, \bar{A}_\phi)$. Notice that for some configurations in (4.14) there are degeneracies and the number of independent components decreases in those cases. In fact

$s_{max}$ determines the degree of the differential equation for $C_0^1$ (or equivalently the number of $\rho$-components one has to use to close the system) to be given by $2\,(s_{max}+1)$, after the $\rho$ components of the equations of motion are imposed.

### 4.1.1 Solving the matter equations of motion

In this subsection we show how to proceed for the simplest cases, and later on we prove in general that the equations of motion for scalars in (4.14), can be expressed in terms of simpler building blocks. Let us start by explicitly exhibiting the solutions for matter fluctuations in the case of the backgrounds with $\mu_3 \neq 0$. Firstly, we determine the differential equation for $C_0^1$ by using the procedure outlined in the last paragraph of the previous section. In this case $s_{max} = 1$ and we get a differential equation for $C_0^1$ with degree $2(s_{max}+1) = 4$ in $\rho$. It is convenient to Fourier transform from $(\phi, t)$ to $(k, \omega)$ for the fileds $C_m^s$ :

$$C_m^s[\rho, t, \phi] = e^{i\omega t} e^{ik\phi} C_m^s[\rho]. \tag{4.53}$$

The final form of the equation for $C_0^1$ is given in (C.13), here we will be somewhat schematic. After the change of coordinates $\rho = \tanh^{-1}(\sqrt{z})$[1] and the following redefinition of the dependent variable $C[z] = z^{\frac{-i\omega}{2}}(1-z)^{\frac{1-\lambda}{2}} G[z]$ one gets a new form for the original differential equation:

$$D_4 G[z] = 0. \tag{4.54}$$

The differential operator $D_4$, whose precise form is given in (C.13), has three regular singularities at 0,1 and $\infty$ with the following $4 \times 3 = 12$ characteristic exponents:

$$\begin{array}{ccc} \alpha_0^I = (0, i\omega) & \alpha_1^I = (0, \lambda) & \alpha_\infty = (\delta_-^+, \delta_+^+) \\ \alpha_0^{II} = (1, 1+i\omega) & \alpha_1^{II} = (1, 1+\lambda) & \widetilde{\alpha}_\infty = (\delta_-^-, \delta_+^-), \end{array}$$

where:

$$\begin{aligned} \delta_+^+ &= \tfrac{1-\lambda}{2} + \delta_0^+(\mu_3), \quad \delta_-^+ = \tfrac{1-2i\omega-\lambda}{2} - \delta_0^+(\mu_3), \\ \delta_+^- &= \tfrac{1-\lambda}{2} + \delta_0^-(\mu_3), \quad \delta_-^- = \tfrac{1-2i\omega-\lambda}{2} - \delta_0^-(\mu_3), \end{aligned} \tag{4.55}$$

[1]Notice that this implies that $z$ lies in the positive real axis.

and:

$$\delta_0^{\pm}(\mu_3) \quad = \quad \tfrac{-3 \pm \sqrt{9 - 36 i \mu_3(\omega + k) + 12 \mu_3^2(\lambda^2 - 1)}}{12 \mu_3}. \tag{4.56}$$

Notice that $\delta_0^+$ is regular in the limit of vanishing $\mu_3$ whereas $\delta_0^-$ is not.

For a Fuchsian differential equation of order $n$ with $m$ regular singular points the sum of characteristic exponents is always $(m-2) \times \frac{n(n-1)}{2}$ [67]. It is easy to check that in our case $n = 4$, $m = 3$ the sum of characteristic exponents is indeed 6. An interesting case is when $n = 2$ and $m = 3$ in that case one has $m \times n = 6$ characteristic exponents whose sum equals 1. Conversely, it is a theorem that any set of 6 numbers adding up to 1 defines a unique Fuchsian operator of order $n = 2$ with $m = 3$ regular singular points. It is also a theorem that such a sextuple of roots defines a subspace of solutions that carry an irreducible representation of the monodromy group of $D_n$ and hence a factor $D_2$ [67]. Namely:

$$D_n = D_{n-2}^L D_2^R, \tag{4.57}$$

and $D_{n-2}^L$ is also Fuchsian and the $L$ and $R$ denote the left and right operator, respectively, in the factorisation.

Before proceeding, let us review some facts that will be used in the following [67, 68]. The most general form of a Fuchsian differential operator $D_2$ once the position of the regular singular points are fixed at $0, 1, \infty$ and a pair of characteristic exponents is fixed to zero, is:

$$D_2 \equiv y(y-1)\frac{d^2}{dy^2} + ((a+b+1)y - c)\,\frac{d}{dy} + ab. \tag{4.58}$$

The characteristic exponents are:

$$\alpha_0 = (0, 1-c),\ \ \alpha_1 = (0, c-a-b),\ \ \alpha_\infty = (a, b). \tag{4.59}$$

The kernel of $D_2$ is generated by the linearly independent functions:

$$\begin{aligned} u_1(a,b,c|z) \quad &\equiv \quad {}_2F_1(a,b,c|z), \\ z^{1-c} u_2(a,b,c|z) \quad &\equiv \quad z^{1-c}\,{}_2F_1(a+1-c, b+1-c, 2-c|z), \end{aligned} \tag{4.60}$$

which are eigenstates of the monodromy action at $z = 0$. The second solution is independent only when $c$ is not in $\mathbb{Z}$. The monodromy eigenstates at $z = 1$ are:

$$\begin{aligned} \tilde{u}_1(a, b, c|z) &\equiv {}_2F_1(a, b, 1 + a + b - c|1 - z), \\ (1 - z)^{c-a-b}\tilde{u}_2(a, b, c|z) &\equiv (1 - z)^{c-a-b}\, {}_2F_1(c - a, c - b, 1 + c - a - b|1 - z), \\ &. \end{aligned} \tag{4.61}$$

when $c - a - b$ is not in $\mathbb{Z}$. In a while we will see that $c - a - b = \lambda$.

Our operator $D_4$ does have the properties mentioned in the paragraph before (4.57). In fact each one of the set of characteristic exponents:

$$\begin{aligned} &\left(\alpha_0^I,\, \alpha_1^I,\, \alpha_\infty\right), \\ &\left(\alpha_0^I,\, \alpha_1^I,\, \widetilde{\alpha}_\infty\right), \end{aligned} \tag{4.62}$$

adds up to 1, and hence defines the second order Fuchsian operators:

$$\begin{aligned} D_2^R &: \;\; a = \delta_+^+(\mu_3), \;\; b = \delta_-^+(\mu_3), \;\; c = 1 - i\omega, \\ \widetilde{D}_2^R &: \;\; a = \delta_+^-(\mu_3), \;\; b = \delta_-^-(\mu_3), \;\; c = 1 - i\omega. \end{aligned} \tag{4.63}$$

As a result $D_4$ has two independent factorizations:

$$D_4 = D_2^L D_2^R \text{ and } D_4 = \widetilde{D}_2^L \widetilde{D}_2^R, \tag{4.64}$$

as one can check explicitly. Consequently we have:

$$ker D_4 = ker D_2^R \bigoplus ker \widetilde{D}_2^R, \tag{4.65}$$

where $ker D_2^R$ is given by the hypergeometric functions $u_1$ and $u_2$ given in (4.60), with the parameters $a$, $b$ and $c$ defined in (4.63). This proves that the fluctuation equation in the background $\mu_3 \neq 0$ is solved in terms of four linearly independent hypergeometric functions, which, from now on we refer to as "building blocks".

One can explicitly verify this factorization pattern for the next background, with $\mu_3, \mu_5 \neq 0$. In this case $s_{Max} = 3$ and the corresponding differential operator $D_8$, has order 8, and is again Fuchsian with 3 regular singularities in the $z$ coordinate system previously defined (we always place them at 0, 1 and $\infty$). The characteristic exponents are:

$$\begin{array}{lll}
\alpha_0^I = (0, i\omega) & \alpha_1^I = (0, \lambda) & \alpha_\infty^I = (\delta_-^{++}, \delta_+^{++}) \\
\alpha_0^{II} = (1, 1 + i\omega) & \alpha_1^{II} = (1, 1 + \lambda) & \alpha_\infty^{II} = (\delta_-^{+-}, \delta_+^{+-}) \\
\alpha_0^{III} = (2, 2 + i\omega) & \alpha_1^{III} = (2, 2 + \lambda) & \alpha_\infty^{III} = (\delta_-^{-+}, \delta_+^{-+}) \\
\alpha_0^{IV} = (3, 3 + i\omega) & \alpha_1^{IV} = (3, 3 + \lambda) & \alpha_\infty^{IV} = (\delta_-^{--}, \delta_+^{--}),
\end{array}$$

where for each of the couples of exponents $\alpha_\infty$ the following property holds: $\delta_+^{\pm\pm}(\mu_3, \mu_5) + \delta_-^{\pm\pm}(\mu_3, \mu_5) = 1 - i\omega - \lambda$. As a consequence there are four triads of characteristic exponents whose sums equal 1 :

$$\begin{aligned}
&\left(\alpha_0^I,\, \alpha_1^I,\, \alpha_\infty^I\right), \quad \left(\alpha_0^I,\, \alpha_1^I,\, \alpha_\infty^{II}\right), \\
&\left(\alpha_0^I,\, \alpha_1^I,\, \alpha_\infty^{III}\right), \quad \left(\alpha_0^I,\, \alpha_1^I,\, \alpha_\infty^{IV}\right).
\end{aligned} \tag{4.66}$$

Each of them defines a second order "Hypergeometric operator" as in (4.63):

$$D_2^{I\,R}, D_2^{II\,R},\, D_2^{III\,R} \text{ and } D_2^{IV\,R}$$

such that

$$ker D_8 = ker D_2^{I\,R} \bigoplus ker D_2^{II\,R} \bigoplus ker D_2^{III\,R} \bigoplus ker D_2^{IV\,R}.$$

In fact there is a simple way to prove that the above pattern generalizes, showing that the solutions of our higher order differential equations can be expressed in terms of ordinary hypergeometric functions, for all of the representatives in (4.14). The point is to use the fact that the Fourier components $C(\omega, k)$ of the full master field $C(t, x)$ defined by the arbitrary polynomial $P_\phi$ and $\bar{P}_\phi$, are eigenstates of the operators in the right hand side of:

$$\begin{aligned}
\partial_t &= \frac{-a_\rho \star_L + a_{-\rho} \star_R}{2}, \\
\partial_\phi &= -\left(\frac{a_\rho}{2} + P_\phi(a_\rho)\right) \star_L - \left(\frac{a_{-\rho}}{2} - \bar{P}_\phi(a_{-\rho})\right) \star_R,
\end{aligned} \tag{4.67}$$

with eigenvalues $(i\omega, ik)$ respectively. The same can be said of the trace component $C_0^1(\omega, k)$ but in this case, the left and right multiplication are equivalent by cyclic property of the trace. As the operators on the right hand side of (4.67) are polynomials in $a_{\pm\rho}$, they share eigenvectors with the latter. But as we pointed out around (4.46):

$$\begin{aligned} i(\omega' + k')C_{BTZ}(\omega', k') &= -a_\rho \star_L C_{BTZ}(\omega', k'), \\ i(k' - \omega')C_{BTZ}(\omega', k') &= -a_{-\rho} \star_R C_{BTZ}(\omega', k'), \end{aligned} \tag{4.68}$$

where $C_{BTZ}$ is the master field for the $\mathrm{BTZ}_{M=-1}$ connection. So from (4.67) and (4.68) it follows that:

$$C_0^1(\omega, k) = C_{0BTZ}^1(\omega', k'), \tag{4.69}$$

where $(\omega', k')$ are any of the roots of the algebraic equations:

$$\begin{aligned} i\omega &= i\omega', \\ ik &= ik' - \left(P_\phi(-i(\omega' + k')) - \bar{P}_\phi(-i(k' - \omega'))\right). \end{aligned} \tag{4.70}$$

Relations (4.69) imply that the differential equation for $C_0^1$ in the class of ansätze (4.14) is always integrable in terms of hypergeometric functions ${}_2F_1$. The number of linearly independent modes being given by twice the order of the algebraic equations (4.70), which can be checked to be, $2(s_{Max}+1)$. Here $s_{Max}+1$ coincides with the order of the polynomial equation (4.70) for $k'$ in terms of $(\omega, k)$.

Summarising, the most general solution for fluctuations in (4.14) is:

$$\begin{aligned} C_0^1(\omega, k) = \sum_r e^{i(\omega t + k\phi)}(1-z)^{\frac{1-\lambda}{2}} \Big( & c_r^{in} z^{-\frac{i\omega}{2}} u_1(a_r, b_r, 1 - i\omega, z) \\ & + c_r^{out} z^{\frac{i\omega}{2}} u_2(a_r, b_r, 1 - i\omega, z) \Big), \\ a_r \equiv \frac{i(k'_r - \omega) + 1 - \lambda}{2}, \ & b_r \equiv \frac{-i(k'_r + \omega) + 1 - \lambda}{2}, \end{aligned} \tag{4.71}$$

where $k'_r$ are the roots of (4.70) and $r = 1, \ldots, 2(s_{Max} + 1)$.

For later reference we write down (4.71) in terms of monodromy eigenstates at the boundary $z = 1$:

$$\begin{aligned} C_0^1(\omega, k) &= \sum_r e^{i(\omega t + k\phi)} z^{\frac{-i\omega}{2}} (1-z)^{\frac{1-\lambda}{2}} \left(\tilde{c}_r^1 \tilde{u}_1(a_r, b_r, 1 - i\omega; z)\right. \\ &\quad \left. + \tilde{c}_r^2 (1-z)^\lambda \tilde{u}_2(a_r, b_r, 1 - i\omega; z)\right). \end{aligned} \tag{4.72}$$

As a check, let us reproduce the first result of this section by using this method. For the case $\mu_3 \neq 0$ the equation for $k'_r$ are:

$$ik = ik'_r - \mu_3 \left( -(\omega + k'_r)^2 + \frac{1-\lambda^2}{3} \right), \tag{4.73}$$

whose solutions are :

$$ik'_\pm = -i\omega - \delta_0^\mp(\mu_3). \tag{4.74}$$

This coincides with the solution one obtains from (4.63), as can be seen using the definitions in the second line of (4.71). We note that only $k'_+$ is smooth in the BTZ limit $\mu_3$ to zero.

As an interesting observation, we would like to draw the attention of the reader to the fact that the boundary conditions for the most general fluctuation (4.71) at the horizon and boundary, $z = 0$ and $z = 1$, respectively, are not affected by the fact that connections (4.14) and the corresponding background tensor fields $g^{(n)}$, defined as (4.2), do break the original $BTZ_{M=-1}$ boundary conditions!

## 4.2 QNM and bulk to boundary 2-point functions

As anticipated, in this subsection we will further argue that the connections (4.14) describe a class of black hole configurations. We will do so by showing the presence of Quasi Normal Modes(QNM). We will compute their spectrum for any representative in (4.14) and, in particular, more explicitly for the simplest cases discussed in the previous section.

We start by recalling the conditions for QNM for AdS Black Holes [49]: they behave like ingoing waves at the horizon, $z = 0$ and as subleading modes at the boundary $z = 1$. In the language employed before, the QNM conditions reduce to ask for solutions with indicial roots $\alpha_0 = 0$ at the horizon $z = 0$, and $\alpha_1 = \lambda$ at the boundary $z = 1$. In this section we are considering the region $0 < \lambda < 1$ so that $(1-z)^{\frac{(1-\lambda)}{2}}$ is the leading behaviour near the boundary. In terms of the most general solution (4.71), the ingoing wave condition reads: $c_r^{out} = 0$. The subleading behaviour requirement implies the quantisation conditions[1].

$$\omega \pm k'_r + i(1+2n+\lambda) = 0, \quad r = 0, \dots 2(s_{Max}+1), \tag{4.75}$$

where $n$ is an arbitrary and positive integer.

We should elaborate about the smoothness of the QNM at the horizon. In the Eddington-Finkelstein coordinates $v = t - \frac{T}{2} log(\rho) + \dots$ and $\rho^* = \frac{4}{T^2}\rho^2 + \dots$, see (4.15) the incoming waves, namely the $c_r^{in}$ modes , behave as plane waves $e^{Iwv}$, at leading order in the near-horizon expansion. In contrast, the $c_r^{out}$ modes are not $C^\infty$ as they look like $e^{i\omega v}\left(\rho^{*i\omega}\right)$. In other words, the requirement of incoming waves at the horizon amounts to have a smooth solution at the horizon [49].

In our example $\mu_3 \neq 0$, $s_{Max} = 1$, there are $2 \times 2$ branches in the quantisation conditions (4.75). The associated branches of QNM being:

$$\begin{aligned} \omega_n^0 &= -k - i\left(1+2n+\lambda - \tfrac{2\mu_3}{3}\left(1+(1+2\lambda)(1+\lambda)+6n(1+\lambda)+6n^2\right)\right), \\ \omega_n^\pm &= -\tfrac{1}{2}i(1+2n+\lambda) + \delta^\pm(n,\mu_3), \end{aligned} \tag{4.76}$$

where:

$$\delta^\pm(n,\mu_3) = \frac{-i \pm \sqrt{-1+8(1+2ik+2n+\lambda)\mu_3 - \frac{16(\lambda^2-1)\mu_3^2}{3}}}{8\mu_3}. \tag{4.77}$$

Before going on, let us briefly mention some relevant issues about the stability of the branches (4.76). It is not hard to see that for large enough values of $k \in \mathbb{R}$ at least one of the branches $\omega_n^\pm$ will exhibit a finite number of undamped modes, namely modes with positive imaginary parts. However for a fixed value of $k$ and $\mu_3$ the UV modes ($n \gg 1, k, \mu_3$) will go like $\omega_n^\pm \sim -in$ and hence will be stable. The branch $\omega_n^0$ is stable for $\mu_3 < 0$. Finally notice also that $(\omega_n^0, \omega_n^+)$ become the left and right moving branches of the BTZ$_{M=-1}$ case, in the limit of vanishing $\mu_3$, whereas $\omega_n^-$ is not analytic in that limit.

[1] We have the identity ${}_2F_1[a,b,c,z] = \frac{\Gamma[c]\Gamma[a+b-c]}{\Gamma[c-b]\Gamma[c-a]}\,{}_2F_1[a,b,a+b-c+1,1-z] + (1-z)^{c-a-b}\frac{\Gamma[c]\Gamma[c-a-b]}{\Gamma[b]\Gamma[a]}\,{}_2F_1[c-a,c-b,c-a-b+1,1-z]$ [68]. The quantisation condition (4.75) is equivalent to $c-a=-n$ and $c-b=-n$ respectively. These choices guarantees that the first term on the rhs of the previous identity vanishes. Indeed, this is the term that carries the leading behaviour of the field at the boundary.

We have $2 \times 2(s_{Max} + 1)$ independent solutions $(c^{in}, c^{out})_r$ in (4.71). Each block $r$ represents an independent degree of freedom and a general fluctuation in the background (4.14) can be re-constructed as a combination of them. So, for the moment we restrict our analysis to a given sector, let us say the block $r$.

In order to define the bulk to boundary 2-point function we set $\tilde{c}_r^2 = 0$ in (4.72), corresponding to the solution with the leading behaviour $(1-z)^{\frac{1-\lambda}{2}}$ at the boundary. We will further fix $\tilde{c}_r^1 = 1$, to guarantee independence on $\omega$ and $k$ of the leading term in the expansion of the solution near the boundary, in such a way that its Fourier transform becomes proportional to $\delta^{(2)}(t, \phi)$ at the boundary, which is the usual UV boundary condition in coordinate space. As a result, in Fourier space, the bulk to boundary 2-point function of the block of solutions $r$ is given by:

$$G_r^{(2)}(\omega, k, z) \equiv \tilde{u}_1(a_r, b_r, 1 - i\omega; 1 - z). \tag{4.78}$$

After Fourier transforming back in $(t, \phi)$ space and using the $\rho$ coordinate one gets preliminary:

$$G_r^{(2)}(t, \phi, \rho) = J_r(-i\partial_t, -i\partial_\phi) \left( G_{BTZ}^{(2)}(t, \phi; \rho) + \delta G_r^{(2)}(t, \phi, \rho) \right). \tag{4.79}$$

We stress that (4.79) obeys the boundary condition:

$$G_r^{(2)}(t, \phi, \rho) \to \delta^{(2)}(t, \phi), \text{ when } \rho \to \infty. \tag{4.80}$$

The quantity:

$$J_r(\omega, k) \equiv \frac{1}{\frac{\partial k_r'(\omega,k)}{\partial k}} e^{i\left(k - k_r'(\omega,k)\right)\phi},$$

is the product of the Jacobian from the change of variables from $k$ to $k_r'$ times an exponential contribution. For our specific case:

$$J_r(\omega, k) = \left(1 + 2i\mu_3 \delta_0^{\pm}(\omega, k)\right) e^{i\left(k - k_r'(\omega,k)\right)\phi}. \tag{4.81}$$

The quantity:

$$G_{BTZ}^{(2)}(t, \phi, \rho) = -\frac{\lambda}{\pi} \left( \frac{e^{-\rho}}{e^{-2\rho} \cosh x_+ \cosh x_- + \sinh x_+ \sinh x_-} \right)^{1-\lambda}, \tag{4.82}$$

is the bulk to boundary 2-point function for $\mathrm{BTZ}_{M=-1}$. Notice that (4.82) is smooth in the near-horizon expansion as its leading contribution is independent of $t$. We note that the contributions coming from $G^{(2)}_{BTZ}$ to (4.79) are also smooth at the horizon provided the Taylor expansion of $J_r(w,k)$ around $(\omega,k)=0$ starts with a constant or an integer power of $k$. This is always the case, as one can infer from (4.70) that $J_r = 1 + O(\mu_3)$, as in the particular case (4.81).

Finally $\delta G^{(2)}_r$ is a contribution that comes from the deformation of the countour of integration that follows from the change $k \to k'_r$ . The change of variable from $k$ to $k'_r(\omega,k)$ deforms the real line $\mathbb{R}$ to a contour $C_{r,\omega} \equiv k'_r(\mathbb{R},\omega)$. Integration over the contours $k'_r \in \mathbb{R}$ and $k'_r \in C_{r,\omega}$ (followed by integration over $\omega \in \mathbb{R}$) of the integrand

$$e^{ik'_r\phi+i\omega t}\tilde{u}_1(a_r,b_r,1-i\omega;1-z),$$

differ by the quantity $\delta G^{(2)}_r(t,\phi,z)$. This quantity can be obtained imposing the condition (4.80). In Fourier space $(\omega,k'_r)$ It reads:

$$\delta G^{(2)}_r(\omega,k'_r,z) = \left(\frac{\partial k'_r}{\partial k} - 1\right)\tilde{u}_1(a_r,b_r,1-i\omega;1-z).^1 \tag{4.83}$$

Finally, (4.79) takes the form:

$$G^{(2)}_r(t,\phi,\rho) = e^{-\left(ik'_r\left(-i\partial_t,-i\partial_\phi\right)-\partial_\phi\right)\phi}G^{(2)}_{BTZ}(t,\phi,\rho).^2 \tag{4.84}$$

For the same reasons explained before (4.84) is smooth at the horizon, namely its leading behaviour is independent on $t$.

Notice that periodicity under $t \to t+2\pi i$ is preserved by all building blocks (4.84). The preservation of thermal periodicity comes after imposing the BTZ holonomy condition on (4.14). It is a global statement in the sense that is determined by the exponentiation properties of the algebra. Namely the gauge group elements generating the family (4.14) with $\nu_0=\frac{1}{2}$, $\nu_{i>0}=0$:

$$\begin{aligned} g &= e^{-\rho V_0^2}e^{-\frac{a}{2}t-\left(\frac{a}{2}+P_\phi(a)\right)\phi}, \\ \bar{g} &= e^{\rho V_0^2}e^{-\frac{a}{2}t+\left(\frac{a}{2}-\bar{P}_\phi(a)\right)\phi}, \end{aligned} \tag{4.85}$$

are thermal periodic due to the fact $i\pi a$ exponentiates to the center of the group whose Lie algebra is $hs(\lambda)$ [63].

[1] Notice that the quantity $\delta G^{(2)}_r(\omega,k'_r,z)$ ( as $G^{(2)}_{BTZ}(\omega,k'_r,z)$) is in the kernel of the BTZ Klein-Gordon operator $D_2(\omega,k'_r,z)$.

[2] We note that the $\phi$ in the exponential (4.84) is located to the right of the derivatives.

### 4.2.1 Making contact with other relevant backgrounds

In this section we perform the gauge transformations (4.24) and (4.26) taking our backgrounds to the GK (BHPT2) ones. As already said, the backgrounds to be transformed have critical exponent $z < 1$. Here we will focus in performing gauge transformations (4.24) and (4.26) on the scalar fluctuations for $\bar{\mu}_3 = -\mu_3 \neq 0$ and we will explicitly verify that they solve the equation of motion for matter fluctuations in the GK (BHPT2) backgrounds. The analysis will be done perturbatively, to first order in a $\mu_3$ expansion.

To this purpose we introduce the series expansion:

$$C = \sum_{i=0}^{\infty} \mu_3^i \overset{(i)}{C}, \tag{4.86}$$

for the master field in equations (4.28) with the connections $(A, \bar{A})$ given by (4.23), (4.25) and (4.27). Taking the $\mu_3^i$ component of (4.28):

$$(d + \overset{(0)}{\mathcal{A}} \star_L - \overset{(0)}{\bar{\mathcal{A}}} \star_R) \overset{(i)}{C} = - \sum_{j=1}^{i} (\overset{(j)}{\mathcal{A}} \star_L - \overset{(j)}{\bar{\mathcal{A}}} \star_R) \overset{(i-j)}{C}, \; i = 0, \ldots, \infty, \tag{4.87}$$

where $\overset{(j)}{\mathcal{A}}$ is the coefficient of $\mu_3^j$ in the Taylor expansion of $\mathcal{A}$ about $\mu_3 = 0$. Notice that if $\overset{(i)}{C}$ is a particular solution of (4.87), then $\overset{(i)}{C} +$ constant $\overset{(0)}{C}$ is also a solution. This is in fact the maximal freedom in defining $\overset{(i)}{C}$ and it constraints the form of the "folded" version of (4.87) to be of the form:

$$\begin{aligned} D_2 \overset{(0)}{C_0^1} &= 0, \; i = 0, \\ D_2 \overset{(i)}{C_0^1} &= \overset{(i)}{D} \left( \overset{(0)}{C_0^1}, \ldots, \overset{(i-1)}{C_0^1} \right), \; i = 1, \ldots \infty, \end{aligned} \tag{4.88}$$

where the differential operator $D_2$ is the BTZ Klein-Gordon operator (4.49) and $\overset{(i)}{D}$ is a linear differential operator in $\rho$ that we shall find out explicitly when analysing up to first order in $\mu_3$.

Let us write down the connections (4.14) with $\mu_3 = -\bar{\mu}_3 \neq 0$ as:

$$\mathcal{A}_{ours} = \overset{(0)}{\mathcal{A}} + \mu_3 \overset{(1)}{\mathcal{A}}_{ours}, \; \mathcal{A}_{ours} = \overset{(0)}{\mathcal{A}} + \mu_3 \overset{(1)}{\mathcal{A}}_{ours}. \tag{4.89}$$

The full answer $C^1_{0\,ours}$ is defined as the building block $r$ in (4.71) with $k'_r$, given by the root (C.5) of equation (C.4) which is the analytic solution in the limit $\mu_3$ to zero. By using the folding method one can check until arbitrary order in $i$ that (4.88) works for the expansion coefficients $\overset{(i)}{C}_{ours}$. Here we restrict to the $i=1$:

$$D_2 \overset{(1)}{C^1_{0\,ours}} = \overset{(1)}{D}_{ours} \overset{(0)}{C^1_0}, \tag{4.90}$$

where:

$$\overset{(1)}{D}_{ours} = \frac{16ike^{2\rho}\left(\frac{1}{3}(\lambda^2-1)+k^2+w^2\right)}{\left(e^{2\rho}+1\right)^2}. \tag{4.91}$$

Let us solve (4.90). We can expand in series the solution for $C^1_{0\,ours}$ (4.71), but we will use gauge covariance instead. From the use of the transformation laws:

$$\begin{aligned} \mathcal{A}_{ours} &= e^{\Lambda_{ours}}\mathcal{A}e^{-\Lambda_{ours}} + e^{\Lambda_{ours}}d\, e^{-\Lambda_{ours}}, \\ \bar{\mathcal{A}}_{ours} &= e^{\bar{\Lambda}_{ours}}\bar{\mathcal{A}}e^{-\bar{\Lambda}_{ours}} + e^{\bar{\Lambda}_{ours}}d\, e^{-\bar{\Lambda}_{ours}}, \end{aligned} \tag{4.92}$$

at linear order, with:

$$\Lambda_{ours} = -\phi P_\phi(a_\rho),\ \bar{\Lambda}_{ours} = -\phi \bar{P}_\phi(a_{-\rho}), \tag{4.93}$$

and $C^1_{0\,ours} = \left((e^{\Lambda_{ours}-\bar{\Lambda}_{ours}})\overset{(0)}{C^1_0}\right)\Big|_{V^1_0}$, for the case $\mu_3 = -\mu_3 \neq 0$ in Fourier space, it follows that:

$$\begin{aligned} \overset{(1)}{C^1_{0\,ours}} &= -i\partial_k\left((a^2_\rho + a^2_{-\rho} - \text{ trace})\overset{(0)}{C}\right)\Big|_{V^1_0} \\ &= -i\left(\frac{2}{3}(1-\lambda^2) - 2(k^2+w^2)\right)\partial_k \overset{(0)}{C^1_0} + \ldots, \end{aligned} \tag{4.94}$$

where the ... in (4.94) stand for terms that are proportional to $\overset{(0)}{C^1_0}$ and hence are in the kernel of $D_2$.

To check that (4.94) is solution of (4.90) it is enough to check that:

$$\left[i\left(\frac{2}{3}(1-\lambda^2) - 2(k^2+w^2)\right)\partial_k,\ D_2\right] = \overset{(1)}{D}_{ours}, \tag{4.95}$$

by using (C.12) or to notice that (4.94) coincides with the first order coefficient in the Taylor expansion around $\mu_3=0$ of the corresponding solution $C^1_{0\,ours}$ which is given by $(\frac{\partial k'}{\partial\mu_3}\partial_{k'}C^1_{0\,ours})|_{\mu_3=0}=\frac{\partial k'}{\partial\mu_3}|_{\mu_3=0}\partial_k\overset{(0)}{C^1_0}$.

Next, we truncate the GK background at first order in $\mu_3$ and after following the procedure we can explicitly show again that the form (4.88) holds until $i=1$ [1]. Here we just present the $i=1$ equation:

$$D_2\overset{(1)}{C^1_{0\,GK}}=\overset{(1)}{D}_{GK}\overset{(0)}{C^1_0}. \tag{4.96}$$

The expression for $\overset{(1)}{D}_{GK}$ is given in (C.14). We should stress again that (4.96) refers only to fluctuations over the GK ansatz that are analytic when $\mu_3$ goes to zero. Finally we check explicitly that the transformed fluctuation:

$$\begin{aligned}\overset{(1)}{C^1_{0\,GK}} &= \overset{(1)}{C^1_{0\,ours}}+\left.\left((\overset{(1)}{\Lambda}_{GK}-\overset{(1)}{\bar{\Lambda}}_{GK})\overset{(0)}{C}\right)\right|_{V^1_0}\\ &= \overset{(1)}{C^1_{0\,ours}}-\frac{ik\,(3e^{2\rho}+5)}{3\,(e^{2\rho}+1)^2}\left(\left(e^{2\rho}-1\right)\overset{(0)}{C^1_0}-\left(e^{2\rho}+1\right)\partial_\rho\overset{(0)}{C^1_0}\right),\end{aligned} \tag{4.97}$$

solves (4.96), after using (4.90) and the $i=0$ equation in (4.88). We have then reproduced the result of [35, 69], by starting from our ansatz.

[1] We checked it up to $i=2$, when the GK background is truncated at second order in $\mu_3$.

# 5

# Conclusions

This thesis is divided in two parts that were organised in three chapters. Chapter 2 consisted of two parts. In the first part, section 2.1, we have shown how Weyl anomaly matching and the correspondig Wess-Zumino action for the conformal "spurion" is reproduced holographically, from kinematical arguments on the bulk gravity side: there, its universality comes from the fact that only the leading boundary behaviour of bulk fields enters the discussion. The $PBH$ diffeomorphisms affect the boundary data and consequently the gravity action depends on them, in particular on the field $\tau$. The regulated effective action is completely fixed by the kinematical procedure detailed in section 2.1. For a specific representative in the family of diffeomorphisms the Wess Zumino term takes the minimal form reported in literature. In appendix A.1.5 we present a different way to approach the same result (We do it for an arbitrary background metric).

We then moved on in sections 2.2 and 2.3 to analyze an explicit 3D holographic RG flow solution, which has a "normalisable" behaviour in the UV. In section 2.2 we studied the problem in the context of 3D gauged supergravity. We started by identifying the possible moduli of the background geometry: out of the zero modes $(\tau, s_p, \rho)$, there come out two independent normalisable combinations. We promoted these integration constants to functions of the boundary coordinates $(t, x)$ and solve the EoM up to second order in a derivative expansion. In a first approach we used a combination of $(\tau, s_p)$ dictated by normalisability, in a second approach we used $\rho$. In both cases we find a boundary action for a free scalar field with the expected normalisation. As argued in section 2.2.6, agreement with

QFT arguments in [5] points towards $\rho$ as the right description for the would-be-dilaton scalar field. For possible extensions to higher dimensional computations, could be helpful to keep on mind that this mode $\rho$ can be seen as the normalisable combination of a rigid $PBH$ in Fefferman-Graham gauge and the mode $s_p$.

Then we moved in section 2.3 to elucidate the QFT interpretation of this normalisable mode by lifting the 3D theory to the 6D one: we promoted the modulus $\rho$, the $SU(2)$ instanton scale, to a boundary field, $\rho(u,v)$, and solved the EoM in a derivative expansion both for the background geometry and the linearized fluctuations around it, up to second order. This allowed us to compute $< T_{\mu\nu} >$ and determine the boundary action for $\log\rho$: this is the action of a free scalar with background charge and its conformal anomaly is $c_{UV} - c_{IR}$, therefore matching the full $c$. We identified $\tau = log\rho$ with a D5-D9 mode in the $(4,0)$ effective field theory of the D1-D5 system in the presence of D9 branes in type I theory.

Finally, as an open problem, it would be interesting to apply the procedure followed in sections 2.2 and 2.3 to a v.e.v. driven RG flow in a 5D example, where we would give spacetime dependence to the moduli associated, say, to the Coulomb branch of a 4D gauge theory: in this case no subtleties related to spontaneous symmetry breaking arise and we should be able to obtain a genuine dilaton effective action.

The second part of the thesis aimed to study higher spin generalisations of black holes in 3D. In chapter 3, we started by analysing the symplectic structure on the phase space $sl(3,\mathbb{R})$ higher spin black holes in principal embedding, (3.18), with $x_1 = \frac{t+\phi}{2}$ and $x_2 = \frac{-t+\phi}{2}$. We were able to identify the conditions that match the Regge-Teitelboim (RT) and Dirac procedures. The fixed time Dirac brackets algebra is not isomorphic to $W_3$. However a $W_3$ structure can be defined by use of Regge-Teitelboim [1]. The phase space of connections associated to this construction does not contain the zero modes that are identified with higher spin black holes but a highest weight description of them. Upon analysis in diagonal embedding we computed the Dirac brackets algebra and as expected [1, 2] it turned out to be isomorphic to $W_3^{(2)}$. Our results complement the viewpoints in [1, 2][1].

[1]When $x_1 = \phi$ and $x_2 = t$ the fixed time Dirac brackets algebra, (3.52), is $W_3$ [2, 39].

It would be necessary to address similar questions for a generic value of the deformation parameter $\lambda$. For that, analysis in perturbations of the generalised boundary conditions in the corresponding embeddings, like $(\mu, \bar{\mu})$ in the principal, or $(\nu, \bar{\nu})$ in the diagonal of the $\lambda = 3$ truncation, could result helpful. Presumably the map between zero modes in different embeddings could be identified at any order in the chemical potentials. Related maps have been studied for the usual conical defects [70, 71, 72]. Nevertheless we believe that an alternative and more general path to follow can be developed.

Finally, in chapter 4, we have presented a family of connections constructed out of arbitrary polynomial combinations of the $\mathrm{BTZ}_{M=-1}$ connection in $hs(\lambda) \times hs(\lambda)$ 3D CS theory. Their space time tensor fields present smooth horizons. The system of higher order differential equations of motion for matter fluctuations can be solved in terms of hypergeometric functions related to the solutions in the BTZ background. This allows to solve explicitly for Quasi Normal Modes and 2-point functions. As a check, we have made contact with other backgrounds studied in the literature. Among the open problems that our work leaves unanswered, we mention the following ones. The first regards the understanding of which (higher spin ?) charges are carried by these backgrounds, or, more generally what is the asymptotic symmetry algebra associated to them. Recent progress on this problem for BH backgrounds in the $sl(3)$ CS theory, as argued at the end of chapter 3, may allow to get an answer for the cases presented here. Secondly, one would like to use the results found here for the matter fluctuations, to solve for more general backgrounds by using appropriate gauge transformations (either "proper" or "improper" ) carrying our backgrounds to these. Unfortunately, a perturbative analysis along the lines discussed in chapter 4 seems to be unavoidably beset by singularities at the horizon $\rho = 0$. It would be interesting to know whether this is an artifact of the perturbative expansion and if a full non perturbative analysis would be free of such singularities. This would allow to study QNM virtually for any BH background.

# Appendix A

## A.1 $PBH$ diffeomorphisms

### A.1.1 Conventions

We use the mostly positive convention for the metric, namely signature $(-,+,+,+)$ in 4D and $(-,+)$ in 2D. The Riemann tensor we define as:

$$R_{\mu\nu\alpha}{}^{\beta} = 2\partial_{[\mu}\Gamma^{\beta}_{\nu]\alpha} + 2\Gamma^{\beta}_{[\mu\lambda}\Gamma^{\lambda}_{\nu]\alpha},$$

with the Christoffel symbols:

$$\Gamma^{\beta}_{\nu\alpha} = \frac{1}{2}g^{\beta\eta}\left(\partial_\nu g_{\eta\alpha} + \partial_\alpha g_{\eta\nu} - \partial_\eta g_{\nu\alpha}\right).$$

The 4D Euler density and Weyl tensors are defined as:

$$E_4 = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2,\ C = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2. \tag{A.1}$$

### A.1.2 Non Static domain wall ansatz

Let the domain wall form for the metric be:

$$ds^2 = dr^2 + e^{2f(r,x)}g_{\mu\nu}(x,r)dx^\mu dx^\nu. \tag{A.2}$$

The $PBH$ diffeomorphism until second order in derivatives of $\tau$, can be written by symmetry arguments as:

$$x^\mu \rightarrow x^\mu - a^{(1)}[r+\tau,x]\partial^\mu\tau + O\left(\partial^3\right), \tag{A.3}$$

$$r \rightarrow r + \tau + b^{(3)}[r+\tau,x](\partial\tau)^2 + O\left(\partial^4\tau\right), \tag{A.4}$$

where index contractions and raising of covariant indices are made by using the metric $g^{\mu\nu}(r, x^\mu)$. The gauge preserving conditions on the form factors arenormalisation

$$\partial_z a^{(1)}[z,x] = e^{-2f},\ \partial_z b^{(3)}[z,x] = \frac{e^{2f}}{2}(\partial_z a^{(1)})^2, \tag{A.5}$$

where $z = r + \tau$. Notice that if we go to the Fefferman-Graham gauge this mode will look like a "warped" diffeomorphism. Namely, the induced $y$-transformation at zeroth order in derivatives of $\tau$ will look like:

$$y \to y e^{h(y)\tau},$$

with $h$ some function of $y$ interpolating between constant values. This is the technical cause behind the fact that the coefficient in the kinetic term (2.97) does not coincides with the difference of holographic central charges. Namely, if we choose the right normalised in the UV $h(\infty) = 1$, thence $h(0) \neq 1$, and so the IR kinetic contribution is not properly normalised to the IR central charge.

### A.1.3 Non Static Fefferman Graham gauge

Let us suppose we are in the Fefferman-Graham gauge, namely:

$$ds^2 = g_{yy}(y)\frac{dy^2}{y^2} + y\left(g_{\mu\nu}dx^\mu dx^\mu\right), \tag{A.6}$$

where $g_{yy}$ and $g_{\mu\nu}$ go as a constant and a Weyl factor times $\eta_{\mu\nu}$ respectively, in both UV and IR limits. Next, we can ask for the 3D diffeomorphisms preserving this form above. We write it as

$$x^\mu \to x^\mu - a^{(1)}[e^{2s}y, x]\partial^\mu\tau + O\left(\partial^3\right), \tag{A.7}$$

$$y \to e^{2s}y + b^{(3)}[e^{2s}y, x](\partial\tau)^2 + O\left(\partial^4\right), \tag{A.8}$$

where the covariant form factors obey the following constraints

$$\partial_z a^{(1)}[z,x] = 2\frac{g_{yy}(z)}{z^2},\quad \partial_z b^{(1)} + \left(\frac{\partial_z g_{yy}}{2g_{yy}} - \frac{1}{z}\right)b^{(1)} + \frac{z^3}{2g_{yy}}(\partial_z a^{(1)})^2 = 0, \tag{A.9}$$

which can be solved easily for a given RG flow metric in this gauge.

### A.1.4 Near To Boundary Analysis

We use the near to boundary analysis to reproduce the results for the bulk action in presence of a $PBH$ mode and to compute the GH and counterterm contribution. We start by writing the near to boundary expansion of the equations of motion. We then evaluate the onshell bulk contribution and finally the onshell contributions from GH and counterterm.

#### A.1.4.1 Near to boundary expansion of the EoM

The near to boundary expansion of the equations of motion in the Fefferman-Graham gauge choice (2.3) comes from:

$$\begin{aligned} y[2g''_{ij} - 2(g'g^{-1}g')_{ij} + Tr(g^{-1}g')g_{ij}] \\ +R_{ij} - 2g_{ij} - Tr(g^{-1}g')g_{ij} &= \frac{4}{3}\frac{g_{ij}}{y}(V[\phi] - V_{fp}) \\ Tr(g^{-1}g'') - \frac{1}{2}Tr(g^{-1}g'g^{-1}g') &= \frac{8}{3}g_{yy}(V[\phi] - V_{fp}) + 8(\phi')^2, \end{aligned}$$

where the primes denote derivative with respect to the flow variable $y$ and $V_{fp}$ is the potential at the corresponding fixed point. In the boundary $V_{fp} = V[0]$.

Another useful relation that is going to be helpful in computing the spurion effective action is the following form for the on-shell action:

$$S^{os}_{bulk} = \frac{L_{UV}}{2}\int d^4x \int dy\sqrt{g}\Big(-\frac{2}{3}V[\phi]\Big). \tag{A.10}$$

**Solutions** We can solve the equations of motions for a generic potential of the form (2.6). Let us start by the UV side.

**The UV side** We can check now the result (2.35) (with exception of the finite part) for the bulk action, after a $\tau$:$PBH$ is performed. We just need to use near to boundary analysis. As said before, we take the near to boundary expansion of the scalar field to be:

$$\phi \sim y\phi^{(0)}(x) + y\log(y)\widetilde{\phi}^{(0)}(x),$$

where the $\widetilde{\phi}^{(0)}$ and $\phi^{(0)}$ are identified with the source and vev of a dimension $\Delta = 2$ CFT operator, respectively. The terms in the near to boundary expansion

(2.9) of the metric, are obtained as:

$$g_{ij}^{(2)} = \frac{1}{2}\left(R_{ij}[g^{(0)}] - \frac{1}{6}g_{ij}^{(0)}R\right),$$

$$\begin{aligned} Tr(h_1^{(4)}) &= \frac{16}{3}\phi_{(0)}\widetilde{\phi}_{(0)}, & \text{(A.11)} \\ Tr(h_2^{(4)}) &= \frac{8}{3}\widetilde{\phi}^2_{(0)}, & \text{(A.12)} \\ Tr(g^{(4)}) &= \frac{1}{4}tr(g^2_{(2)}) - \frac{3}{2}tr(h_1^{(4)}) - tr(h_2^{(4)}) + \frac{8}{3}\phi^2_{(0)} + 4\widetilde{\phi}^2_{(0)} + 8\phi_{(0)}\widetilde{\phi}_{(0)} & \\ &= \frac{1}{4}tr(g^2_{(2)}) + \frac{8}{3}\phi^2_{(0)} + \frac{16}{3}\widetilde{\phi}^2_{(0)}. & \text{(A.13)} \end{aligned}$$

The volume measure expansion:

$$\begin{aligned} \sqrt{g} &= \sqrt{g_{(0)}}\Bigg(1 + \frac{1}{2}Tr(g_{(2)})y + \Bigg(\frac{1}{2}Tr(g_{(4)}) + \frac{1}{8}Tr(g_{(2)})^2 - \frac{1}{4}Tr(g^2_{(2)}) \\ &\quad + \frac{1}{2}Tr(h^1_{(4)})\log(y) + \frac{1}{2}Tr(h^2_{(4)})\log^2(y)\Bigg)y^2\Bigg), \qquad \text{(A.14)} \end{aligned}$$

is used to evaluate the near to boundary expansion of bulk lagrangian in (A.10). The result for the UV expansion of the onshell action (2.23), is evaluated by use of the following result for a conformally flat metric $g_{(0)} = e^{-\tau}\eta$

$$\begin{aligned} a^{(0)}_{UV} &= \frac{1}{2L_{UV}}\int d^4x\sqrt{g_{(0)}} = \frac{1}{2L_{UV}}\int d^4x e^{-2\tau}, \\ a^{(2)}_{UV} &= \frac{L_{UV}}{2}\int d^4x\sqrt{g_{(0)}}Tr(g_{(2)}) = \frac{L_{UV}}{8}\int d^4x e^{-\tau}(\partial\tau)^2, \qquad \text{(A.15)} \\ a^{(4)}_{UV} &= L^3_{UV}\int d^4x\sqrt{g_{(0)}}\left(\frac{1}{2}Tr(g_{(4)}) + \frac{1}{8}Tr(g_{(2)})^2 - \frac{1}{4}Tr(g^2_{(2)}) - \frac{4}{3}\phi^2_{(0)}\right) \\ &= L^3_{UV}\int d^4x\frac{8}{3}\tilde{\phi}^2_{(0)}. \end{aligned}$$

The Weyl transformation properties of the Ricci scalar in 4D was used in getting this result.

**GH term contribution** In the UV side we can expand the Gibbons Hawking term in a near to boundary series:

$$\begin{aligned}\frac{1}{4}\int d^4x\sqrt{\gamma}2K|_{UV} &= \frac{1}{L_{UV}}\int d^4x\frac{1}{y_{UV}^2}(-2\sqrt{g}+y\partial_y\sqrt{g})\\ &= \int d^4x\left(\frac{b^{(0)}}{y_{UV}^2}+\frac{b^{(2)}}{y_{UV}}+b^{(4)}\log(y_{UV})+b_{finite}\right),\end{aligned} \tag{A.16}$$

where,

$$b^{(0)} = -\frac{2}{L_{UV}}\int d^4x\sqrt{g_{(0)}},\ b^{(2)} = -\frac{L_{UV}}{2}\int d^4x\sqrt{g_{(0)}}Tr(g_{(2)}), \tag{A.17}$$

$$b^{(4)} = L_{UV}^3\int d^4x\sqrt{g_{(0)}}Tr(h_2^{(4)}),\ b_{finite} = \frac{L_{UV}^3}{2}\int d^4x\sqrt{g_{(0)}}Tr(h_1^{(4)}). \tag{A.18}$$

The finite contribution $b_{finite}$ is proportional to $\int d^4x\sqrt{g_{(0)}}Tr(h_1^{(4)})$ which by (A.11) is proportional to the product of the v.e.v. and the source $\phi^{(0)}$ and $\tilde{\phi}^{(0)}$ respectively. Namely, for a v.e.v. driven flow the GH term does not contribute at all to the finite part of the regularized onshell action. In the case of a source driven flow, the finite contribution gives a potential term which is not Weyl invariant, as one can notice from the transformation properties (2.21). In fact its infinitesimal Weyl transformation generates an anomalous variation proportional to the source square $\delta\tau(\tilde{\phi}^{(0)})^2$. This fact can be noticed by simple eye inspection, one just needs to analyse the transformation properties (2.21) for the static case.

**The IR side** In this case we can do the same. As already said, we assume IR regularity in the corresponding background, namely,

$$\phi \sim \phi_{IR} + \frac{1}{\rho^m}\phi^{(0)} + ...,\ m > 0.$$

We start by writing the IR asymptotic expansion of the GH term in the IR:

$$\frac{1}{4}\int d^4x\sqrt{\gamma}2K|_{IR} \sim \int d^4x\left(\frac{b_{IR}^{(0)}}{y_{IR}^2}+\frac{b_{IR}^{(2)}}{y_{IR}}+b_{IR}^{(4)}\log y+b_{finite}+\sum_{n=1}^{\infty}y_{IR}^n b_{IR}^{(n)}\right).$$

We compute the factors $b$ in terms of the components of the near to IR expansion of the metric:

$$b_{IR}^{(0)} = \frac{1}{2l_{IR}} \int d^4x \sqrt{g_{(0)}},\ b_{IR}^{(2)} = \frac{L_{IR}}{2} \int d^4x \sqrt{g_{(0)}} e^{-2\tau} Tr\left(g_{(2)}\right), \tag{A.19}$$

$$b_{IR}^{(4)} = L_{IR}^3 \int d^4x \sqrt{g_{(0)}} e^{-2\tau} \left(\frac{1}{2} Tr\left(g_{(4)}\right) + \frac{1}{8} Tr\left(g_{(2)}\right)^2 - \frac{1}{4} Tr\left(g_{(2)}^2\right)\right), \tag{A.20}$$

$$b_{finite} = L_{IR}^3 \int d^4x \sqrt{g_{(0)}} e^{-2\tau} Tr\left(h_1^{(4)}\right). \tag{A.21}$$

By using the near to IR expansion of the equations of motions (A.10) at second order we get:

$$g_{ij}^{(2)} = \frac{1}{2}\left(R_{ij}[g^{(0)}] - \frac{1}{6} g_{ij}^{(0)} R\right),$$

and additionally:

$$Tr(h_1^{(4)}) = 0,\ Tr(h_2^{(4)}) = 0,$$
$$Tr(g^{(4)}) = \frac{1}{4} tr(g_{(2)}^2) - \frac{3}{2} tr(h_1^{(4)}) - tr(h_2^{(4)}) = \frac{1}{4} tr(g_{(2)}^2).$$

It is then easy to see how the IR GH term does not contribute to the finite part of the regularized action! provided the background solutions are smooth in the IR.

### A.1.5 Anomaly matching from $PBH$ transformations

In this appendix we present an alternative way to compute the gravitational WZ term. The approach is covariant in the sense that it works with an arbitrary boundary background metric $g^{(0)}$ and shows how the 4D anomaly matching argument of [4, 6] is linked to the 5D $PBH$ transformation properties.

The relevant terms in the cut off expansion of the bulk action are:

$$\begin{aligned} S[\tau] &= \int d^4x \sqrt{\hat{g}^0} \Bigg( \frac{1}{y_{UV}^2} - \frac{1}{y_{IR}^2} + \frac{a_{UV}^{(2)}[\hat{g}^{(0)}, \hat{\phi}^{(0)}]}{y_{UV}} - \frac{a_{IR}^{(2)}[\hat{g}^{(0)}, \hat{\phi}^{(0)}]}{y_{IR}} + \\ &+ a_{UV}^{(4)}[\hat{g}^{(0)}, \hat{\phi}^{(0)}] \log(y_{UV}) - a_{IR}^{(4)}[\hat{g}^{(0)}, \hat{\phi}^{(0)}] \log(y_{IR}) \Bigg) + S_{finite}[\tau] + \ldots, \end{aligned} \tag{A.22}$$

after a finite $PBH$ transformation parameterized by $\tau$ is performed. The $S_{finite}[\tau]$ stands for the cut off independent contribution to the bulk action and $\hat{g}^{(0)} =$

$e^{-\tau}g^{(0)}$ and $\hat{\phi}^{(0)}$ stand for the $PBH$ transformed boundary data. The leading "matter" boundary data $\hat{\phi}^{(0)}$ (UV/IR need not be the same), does not transform covariantly, unlike the background boundary metric $g^{(0)}$.

Next, one can perform a second infinitesimal $PBH$, $\delta\tau_1$, and think about it in two different ways:

- Keep the cut-off fixed and transform the fields (I).
- Keep the fields fixed and transform the cut-offs (II).

In approach $I$, in virtue of additivity of $PBH$ transformations:

$$\delta S_{finite} = \delta\tau_1 \left(\frac{\delta S_{finite}[\tau]}{\delta\tau}\right) . \tag{A.23}$$

In approach $II$, one needs the generalization of (2.14) for a linear parameter $\delta\tau_1$ and arbitrary boundary metric $g^{(0)}$. An important point is that (2.14) is not a near to boundary expansion, but rather an IR expansion valid along the full flow geometry. Notice also that, in principle, some contribution proportional to $\Box\delta\tau_1$, $\Box\Box\delta\tau_1$, .., could come out of the cut off powers in (A.22). As discussed for (A.22), these terms can be completely gauged away. Then approach $II$ gives:

$$\delta S_{finite} = \int d^4x\sqrt{\hat{g}^0}\ \delta\tau_1 \left(a^{(4)}_{UV}[\hat{g}^{(0)},\hat{\phi}^{(0)}] - a^{(4)}_{IR}[\hat{g}^{(0)},\hat{\phi}^{(0)}]\right) . \tag{A.24}$$

Equating (A.23) and (A.24) we get:

$$\frac{\delta S_{finite}[\tau]}{\delta\tau} = \int d^4x\sqrt{\hat{g}^0}\ \left(a^{(4)}_{UV}[\hat{g}^{(0)},\hat{\phi}^{(0)}] - a^{(4)}_{IR}[\hat{g}^{(0)},\hat{\phi}^{(0)}]\right) . \tag{A.25}$$

Now we can expand the gravitational contribution to $a^{(4)}_{UV}[\hat{g}^{(0)}] - a^{(4)}_{IR}[\hat{g}^{(0)}]$:

$$\frac{(L^3_{UV} - L^3_{IR})}{64}\left(E_{(4)}[\hat{g}^{(0)}] - W[\hat{g}^{(0)}]^2\right) ,$$

by using the Weyl expansions:

$$\begin{aligned}
\hat{W}^2 &= e^{2\tau}W^2, \\
\hat{E}_{(4)} &= e^{2\tau}\left(E_{(4)} + 4\left(R^{\mu\nu} - \frac{1}{2}g^{(0)\mu\nu}R\right)\nabla_\mu\partial_\nu\tau\right) \\
&+ e^{2\tau}\left(2\left((\Box\tau)^2 - \Box^{\mu\nu}\tau\Box_{\mu\nu}\tau\right) - \left((\Box\tau)(\partial\tau)^2 + 2\ \partial_\mu\tau\Box^{\mu\nu}\tau\partial_\nu\tau\right)\right) .
\end{aligned} \tag{A.26}$$

Hence, from (A.25) and (A.26) one can integrate out the gavitational contribution to $S_{finite}$:

$$\int d^4x\sqrt{g^0}\left(\Delta a\ E_{(4)}\frac{\tau}{2}-\Delta a\left(R^{\mu\nu}-\frac{1}{2}g^{(0)\mu\nu}R\right)\partial_\mu\tau\partial_\nu\tau+\right.$$
$$\left.+\Delta a\frac{1}{8}\left((\partial\tau)^4-4\,\Box\tau(\partial\tau)^2\right)-\Delta c\ W^2\frac{\tau}{2}\right), \qquad (A.27)$$

where in the case we are considering $c=a$. Notice that in the above derivation, we implicitly assumed the group property of the $PBH$ transformations on fields, that is:

$$L_{\tau_1}\circ L_{\tau_2}=L_{\tau_1+\tau_2},$$

were $L$ represents the transformation thought of as an operator acting on the fields (boundary data). As for the case of matter contributions, a problem arises when a v.e.v. or source transforms non covariantly

$$\phi^{(0)}\to e^\tau\phi^{(0)}+\tau e^\tau\tilde{\phi}^{(0)}.$$

So, it is not clear to us how to use this procedure to compute "matter" contributions to the Weyl anomaly. An efficient procedure to compute anomalies for generic backgrounds (in a spirit similar to the approach presented here), had appeared in [73] (section 3.1).

## A.2 3D N=4 SUGRA example

### A.2.1 Equations of motion for the background fluctuations

We start by writing down the gravitational side of the set of equations of motion for the background fluctuations $g^{(2)}$, $T^{(2)}$, $g^{(2)}_{tx}$, $A^{(2)}$ and $\phi^{(2)}$, at second order in time $t$ and space $x$ derivatives. We use here the notation used through out the main text, namely denoting the equations as the space time components they descend from. So the equations $(r,r)$, $(t,t)-(x,x)$, $(t,t)+(x,x)$ and $(t,x)$, read off respectively:

$$\partial_r^2 g^{(2)} + 2\partial_r f_B \partial_r g^{(2)} + 4\partial_{\phi_B} V \phi^{(2)} + 2\partial_r \phi_B \partial_r \phi^{(2)} \\ + 4\left(\partial_{A_B} V + \frac{3A_B}{(1-A_B^2)^3}(\partial_r A_B)^2\right) A^{(2)} + \frac{6}{(1-A_B^2)^2}\partial_r A_B \partial_r A^{(2)} = 0, \quad \text{(A.28)}$$

$$\partial_r^2 g^{(2)} + 4\partial_r f_B \partial_r g^{(2)} + 2\left(4V + 2(\partial_r f_B)^2 + \partial_r^2 f_B\right) g^{(2)} \\ + 8\partial_{\phi_B} V \phi^{(2)} + 8\partial_{A_B} V A^{(2)} + e^{-2f_B}\left(\frac{3}{(1-A_B^2)^2}(\partial A_B)^2 + (\partial\phi_B)^2 + 2\Box f_B\right) = 0, \quad \text{(A.29)}$$

$$\partial_r^2 T + 2\partial_r f_B \partial_r T + 2\left(4V + 2(\partial_r f_B)^2 + \partial_r^2 f_B\right) T \\ - e^{-2f_B}\left(\frac{3}{(1-A_B^2)^2}(\partial A_B)^2 + (\partial\phi_B)^2\right) = 0, \quad \text{(A.30)}$$

$$\partial_r^2 g_{tx}^{(2)} + 2\partial_r f_B \partial_r g_{tx}^{(2)} + 2\left(4V + 2(\partial_r f_B)^2 + \partial_r^2 f_B\right) g_{tx}^{(2)} \\ + 2e^{-2f_B}\left(\frac{3}{(1-A_B^2)^2}(\partial A_B)^2 + (\partial\phi_B)^2\right) = 0, \quad \text{(A.31)}$$

where for a $Y \equiv A_B, \phi_B, f_B$, we use the notation $(\partial Y)^2 \equiv (\partial_x Y)^2 - (\partial_t Y)^2$ and $\Box Y = (\partial_x^2 Y - \partial_t^2 Y)$. We also used the equations $(t,r)$ and $(x,r)$ respectively:

$$\left(\partial_{tr}^2 T + 2\partial_t f_B \partial_r T\right) + \\ \left(\partial_{xr}^2 g_{tx}^{(2)} + 2\partial_x f_B \partial_r g_{tx}^{(2)}\right) - \partial_{tr}^2 g^{(2)} - 2\left(\partial_t \phi_B \partial_r \phi^{(2)} + \partial_r \phi_B \partial_t \phi^{(2)}\right) - \\ \frac{6}{(1-A_B^2)^2}\left(\partial_t A_B \partial_r A^{(2)} + \partial_r A_B \partial_t A^{(2)}\right) - \frac{24}{(1-A_B^2)^3}(A_B \partial_r A_B \partial_t A_B) A^{(2)} = 0,$$

$$-\left(\partial_{xr}^2 T + 2\partial_x f_B \partial_r T\right) - \\ \left(\partial_{tr}^2 g_{tx}^{(2)} + 2\partial_t f_B \partial_r g_{tx}^{(2)}\right) - \partial_{xr}^2 g^{(2)} - 2\left(\partial_x \phi_B \partial_r \phi^{(2)} + \partial_r \phi_B \partial_x \phi^{(2)}\right) - \\ \frac{6}{(1-A_B^2)^2}\left(\partial_x A_B \partial_r A^{(2)} + \partial_r A_B \partial_x A^{(2)}\right) - \frac{24}{(1-A_B^2)^3}(A_B \partial_r A_B \partial_x A_B) A^{(2)} = 0.$$

These equations reduce to constraints for the integration constants that appear.

The Klein-Gordon equations for the scalar fields $\phi$ and $A$ give the following couple of equations for the fluctuations respectively:

$$\begin{aligned}&\partial_r^2\phi^{(2)}+2\partial_r f_B\partial_r\phi^{(2)}-2\partial^2_{\phi_B}V\phi^{(2)}\\&\qquad -2\partial^2_{A_B,\phi_B}VA^{(2)}+\partial_r\phi_B\partial_r g^{(2)}+e^{-2f_B}\Box\phi_B=0,\end{aligned}\tag{A.32}$$

$$\begin{aligned}&\partial_r^2A^{(2)}+\left(2\partial_r f_B+\frac{4A_B}{(1-A_B^2)^2}\partial_r A_B\right)\partial_r A^{(2)}\\&\quad+\frac{2}{3}\left(-(1-A_B^2)^2\partial^2_{A_B}V+3\frac{(1-5A_B^2)}{(1-A_B^2)^2}(\partial_r A_B)^2\right.\\&\qquad\left.+\frac{6A_B}{(1-A_B^2)}\left(2\partial_r A_B\partial_r f_B+\partial_r^2A_B\right)\right)A^{(2)}+\partial_r A_B\partial_r g^{(2)}\\&\qquad+\frac{2}{3}(1-A_B^2)^2\partial^2_{A_B,\phi_B}V\phi^{(2)}+e^{-2f_B}\left(\frac{A_B}{1-A_B^2}(\partial A_B)^2+\Box A_B\right)=0.\end{aligned}\tag{A.33}$$

### A.2.2 Rational functions for the pair $(s_p, \tau)$

In this subsection we write down the rational functions appearing in the equations in section 2.2.

$$\begin{aligned}R^{(1)}_{\partial_y\phi^{(2)}}&=-\frac{3g_2^3(y+1)^3}{g_1(g_1^2-g_2^2(y+1)^2)},\quad R^{(2)}_{\partial_y\phi^{(2)}}=-\frac{(g_1^2(2y+1)+g_2^2(y+1)^2(2y-1))}{y(y+1)\left(g_2^2(y+1)^2-g_1^2\right)},\\R^{(3)}_{\partial_y\phi^{(2)}}&=\frac{3g_2^3(y+1)^3\left(g_2^2\left(y^2-1\right)+g_1^2\right)}{g_1y\left(g_1^2-g_2^2(y+1)^2\right)^2},\quad R^{(4)}_{\partial_y\phi^{(2)}}=\frac{(g_2^2(y+1)^3+g_1^2(y-1))}{2g_1^2y^3},\\R^{(5)}_{\partial_y\phi^{(2)}}&=\frac{(y+1)(3g_2^2(y+1)^2+g_1^2)}{y(g_1^2-g_2^2(y+1)^2)^2},R^{(6)}_{\partial_y\phi^{(2)}}=-\frac{2c_1^2\left(g_1^2-g_2^2(y+1)^2\right)^2}{g_1^6g_2^4y^3(y+1)},\end{aligned}\tag{A.34}$$

$$\begin{aligned}
F^{(1)} &= -\frac{2\left(g_1^4\left(5y^2+6y+2\right)+2g_2^4(y+1)^5-4g_1^2g_2^2(y+1)^4\right)}{g_1g_2y^4(y+1)^3\left(g_1^2(3y+2)-2g_2^2(y+1)^2\right)}e^{2s_p},\\
F^{(2)} &= \frac{4c_1^2\left(g_1^2(3y+2)-2g_2^2(y+1)^2\right)\left(g_1^2-g_2^2(y+1)^2\right)}{g_1^5g_2^5y^4(y+1)^4},\\
F^{(3)} &= -\frac{4c_1\left(g_1^4\left(5y^2+6y+2\right)+2g_2^4(y+1)^5-4g_1^2g_2^2(y+1)^4\right)}{g_1^3g_2^3y^4(y+1)^3\left(g_1^2(3y+2)-2g_2^2(y+1)^2\right)},\\
F^{(4)} &= \frac{2g_1\left(g_1^6\left(12y^2+13y+4\right)-4g_2^6(y+1)^8\right)}{g_2y^3(y+1)^2\left(g_2^2(y+1)^2-g_1^2\right)^3\left(2g_2^2(y+1)^2-g_1^2(3y+2)\right)}\\
&\quad+\frac{2g_1^2g_2^4(y+1)^4\left(9y^3+32y^2+29y+12\right)}{g_2y^3(y+1)^2\left(g_2^2(y+1)^2-g_1^2\right)^3\left(2g_2^2(y+1)^2-g_1^2(3y+2)\right)},\\
&\quad-\frac{2g_1^4g_2^2(y+1)^2\left(21y^3+40y^2+34y+12\right)}{g_2y^3(y+1)^2\left(g_2^2(y+1)^2-g_1^2\right)^3\left(2g_2^2(y+1)^2-g_1^2(3y+2)\right)},
\end{aligned}\tag{A.35}$$

$$\begin{aligned}
R^{(0)}_{A^{(2)}} &= \frac{8g_2^8(y+1)^8-2g_1^2g_2^6(y+1)^5(y(y(8y+27)+29)+16)}{y^3(y+1)^2\left(g_2^2(y+1)^2-g_1^2\right)^3\left(2g_2^2(y+1)^2-g_1^2(3y+2)\right)}\\
&\quad+\frac{2\left(g_1^8(y(12y+13)+4)+g_2^4g_1^4(y+1)^3(y(y(2y(9y+32)+79)+63)+24)\right)}{y^3(y+1)^2\left(g_2^2(y+1)^2-g_1^2\right)^3\left(2g_2^2(y+1)^2-g_1^2(3y+2)\right)}\\
&\quad-\frac{2g_1^6g_2^2(y+1)\left(y\left(y\left(42y^2+68y+77\right)+55\right)+16\right)}{y^3(y+1)^2\left(g_2^2(y+1)^2-g_1^2\right)^3\left(2g_2^2(y+1)^2-g_1^2(3y+2)\right)},
\end{aligned}\tag{A.36}$$

$$\begin{aligned}
R^{(1)}_{A^{(2)}} &= \frac{4g_1^4g_2^2(y+1)^2(y(y+1)(15y-13)-6)}{y^2(y+1)^2\left(g_1^2-g_2^2(y+1)^2\right)^2\left(2g_2^2(y+1)^2-g_1^2(3y+2)\right)}\\
&\quad-\frac{2g_1^2g_2^4(y+1)^4(y(2y(9y+13)-13)-12)}{y^2(y+1)^2\left(g_1^2-g_2^2(y+1)^2\right)^2\left(2g_2^2(y+1)^2-g_1^2(3y+2)\right)}\\
&\quad+\frac{2\left(2g_2^6(y+1)^6\left(5y^2-2\right)+g_1^6(y(12y+13)+4)\right)}{y^2(y+1)^2\left(g_1^2-g_2^2(y+1)^2\right)^2\left(2g_2^2(y+1)^2-g_1^2(3y+2)\right)},
\end{aligned}\tag{A.37}$$

$$R^{(2)}_{A^{(2)}} = \frac{g_2}{g_2y-g_1+g_2}+\frac{g_2}{g_2y+g_1+g_2}+\frac{4g_2^2(y+1)-3g_1^2}{g_1^2(3y+2)-2g_2^2(y+1)^2}+\frac{2}{y}+\frac{6}{y+1},$$

$$
\begin{aligned}
l^{(0)} &= \frac{g_1^2 g_2^2 y\left(g_2^2(y+1)^3+g_1^2(y-1)\right)}{8c_1(y+1)\left(g_2^2(y+1)^2-g_1^2\right)}e^{2s_p},\ l^{(1)}=\frac{g_1^4 g_2^2 y\left(3g_2^2(y+1)^2+g_1^2\right)}{4c_1\left(g_1^2-g_2^2(y+1)^2\right)^3},\\
l^{(2)} &= \frac{g_2^2(y+1)^3+g_1^2(y-1)}{4y(y+1)\left(g_2^2(y+1)^2-g_1^2\right)},\ l^{(3)}=-\frac{c_1\left(g_1^2-g_2^2(y+1)^2\right)}{2g_1^2 g_2^2 y(y+1)^2},\\
l^{(4)} &= \frac{g_1^2 g_2^2 y^2}{4c_1}e^{2s_p},\ l^{(5)}=\frac{g_1^2 g_2^2 y\left(g_2^2 y^3+3g_2^2 y^2+\left(g_1^2+3g_2^2\right)y-g_1^2+g_2^2\right)}{8c_1(y+1)\left(g_2^2 y^2+2g_2^2 y-g_1^2+g_2^2\right)}e^{2s_p},\\
l^{(6)} &= -\frac{3g_1^3 g_2^5 e^{2s_p}y^2(y+1)^2}{4c_1\left(g_2^2 y^2+2g_2^2 y-g_1^2+g_2^2\right)^2},\ l^{(7)}=\frac{g_1^4 g_2^2 e^{2s_p}y^2}{4c_1(y+1)\left(g_2^2 y^2+2g_2^2 y-g_1^2+g_2^2\right)}.
\end{aligned}
$$

**Case of the modulus $\rho$**

$$
\begin{aligned}
R^{(1)}_{\partial_y\phi^{(2)}} &= -\frac{3g_2^3(y+\rho)^3}{\rho g_1(g_1^2-g_2^2(y+\rho)^2)},\ R^{(2)}_{\partial_y\phi^{(2)}}=-\frac{g_1^2\rho^2(\rho+2y)+g_2^2(2y-\rho)(\rho+y)^2}{y(\rho+y)\left(g_2^2(\rho+y)^2-g_1^2\rho^2\right)},\\
R^{(3)}_{\partial_y\phi^{(2)}} &= \frac{3g_2^3(\rho+y)^3\left(g_1^2\rho^2+g_2^2\left(y^2-\rho^2\right)\right)}{g_1\rho y\left(g_1^2\rho^2-g_2^2(\rho+y)^2\right)^2},R^{(4)}_{\partial_y\phi^{(2)}}=\frac{g_1^2\rho^2(y-\rho)+g_2^2(\rho+y)^3}{2g_1^2\rho^2 y^3},\\
R^{(5)}_{\partial_y\phi^{(2)}} &= \frac{c_1^2\left(g_1^2\rho^2-g_2^2\left(\rho^2+3y^2+4\rho y\right)\right)}{g_1^4 g_2^4\rho y^3(\rho+y)},
\end{aligned}
\tag{A.38}
$$

$$
\begin{aligned}
R^{(6)}_{\partial_y\phi^{(2)}} &= \frac{c_1^2\left(g_1^6\rho^5-g_2^4 g_1^2\rho(\rho+y)^2\left(3\rho^2+8y^2+10\rho y\right)\right)}{g_1^4 g_2^4\rho y^3(\rho+y)\left(g_1^2\rho^2-g_2^2(\rho+y)^2\right)^2}\\
&\quad+\frac{c_1^2\left(g_2^2 g_1^4\rho^3\left(3\rho^2+5y^2+8\rho y\right)+g_2^6(\rho+y)^4(\rho+4y)\right)}{g_1^4 g_2^4\rho y^3(\rho+y)\left(g_1^2\rho^2-g_2^2(\rho+y)^2\right)^2},
\end{aligned}
\tag{A.39}
$$

$$
\begin{aligned}
F^{(1)} &= \frac{2c_1^2\left(-g_1^2 g_2^2\rho\left(8\rho^4+9y^4+32\rho y^3+47\rho^2y^2+32\rho^3 y\right)\right)}{g_1^3 g_2^5 y^4(\rho+y)^4\left(2g_2^2(\rho+y)^2-g_1^2\rho(2\rho+3y)\right)}\\
&\quad+\frac{2c_1^2\left(g_1^4\rho^3(2\rho+3y)^2+4g_2^4(\rho+y)^5\right)}{g_1^3 g_2^5 y^4(\rho+y)^4\left(2g_2^2(\rho+y)^2-g_1^2\rho(2\rho+3y)\right)},
\end{aligned}
\tag{A.40}
$$

$$F^{(2)} =$$
$$-\frac{2c_1^2\left(-2g_2^8g_1^2\rho(\rho+y)^6\left(10\rho^3+6y^3+27\rho y^2+28\rho^2y\right)\right)}{g_1^3g_2^5y^4(\rho+y)^4\left(g_2(\rho+y)-g_1\rho\right)^3\left(g_1\rho+g_2(\rho+y)\right)^3\left(2g_2^2(\rho+y)^2-g_1^2\rho(2\rho+3y)\right)}$$
$$-\frac{2c_1^2\left(g_2^4g_1^6\rho^4(\rho+y)^2\left(40\rho^4+54y^4+178\rho y^3+251\rho^2y^2+164\rho^3y\right)\right)}{g_1^3g_2^5y^4(\rho+y)^4\left(g_2(\rho+y)-g_1\rho\right)^3\left(g_1\rho+g_2(\rho+y)\right)^3\left(2g_2^2(\rho+y)^2-g_1^2\rho(2\rho+3y)\right)}$$
$$+\frac{2c_1^2\left(g_2^6g_1^4\rho^2(\rho+y)^4\left(40\rho^4+15y^4+98\rho y^3+180\rho^2y^2+140\rho^3y\right)\right)}{g_1^3g_2^5y^4(\rho+y)^4\left(g_2(\rho+y)-g_1\rho\right)^3\left(g_1\rho+g_2(\rho+y)\right)^3\left(2g_2^2(\rho+y)^2-g_1^2\rho(2\rho+3y)\right)}$$
$$+\frac{2c_1^2\left(2g_2^2g_1^8\rho^6\left(10\rho^4+18y^4+61\rho y^3+79\rho^2y^2+46\rho^3y\right)\right)}{g_1^3g_2^5y^4(\rho+y)^4\left(g_2(\rho+y)-g_1\rho\right)^3\left(g_1\rho+g_2(\rho+y)\right)^3\left(2g_2^2(\rho+y)^2-g_1^2\rho(2\rho+3y)\right)}$$
$$-\frac{2c_1^2\left(g_1^{10}\rho^8(2\rho+3y)^2+4g_2^{10}(\rho+y)^{10}\right)}{g_1^3g_2^5y^4(\rho+y)^4\left(g_2(\rho+y)-g_1\rho\right)^3\left(g_1\rho+g_2(\rho+y)\right)^3\left(2g_2^2(\rho+y)^2-g_1^2\rho(2\rho+3y)\right)},$$

$$F^{(3)} = -\frac{2\left(g_1^4\rho^3\left(2\rho^2+5y^2+6\rho y\right)+2g_2^4(\rho+y)^5-4g_1^2g_2^2\rho(\rho+y)^4\right)}{g_1g_2y^4(\rho+y)^3\left(2g_2^2(\rho+y)^2-g_1^2\rho(2\rho+3y)\right)}, \qquad \text{(A.41)}$$

$$\begin{aligned} R^{(0)}_{A^{(2)}} &= \frac{2g_2^2\rho^2y\left(g_2^2g_1^4\rho\left(510\rho^3+18y^3+118\rho y^2+325\rho^2y\right)\right)}{(\rho+y)^2\left(g_2^2(\rho+y)^2-g_1^2\rho^2\right)^3\left(2g_2^2(\rho+y)^2-g_1^2\rho(2\rho+3y)\right)} \\ &+\frac{2g_2^2\rho^2y\left(+4g_2^6\left(70\rho^4+y^4+8\rho y^3+28\rho^2y^2+56\rho^3y\right)-2g_1^6\rho^3(55\rho+21y)\right)}{(\rho+y)^2\left(g_2^2(\rho+y)^2-g_1^2\rho^2\right)^3\left(2g_2^2(\rho+y)^2-g_1^2\rho(2\rho+3y)\right)} \\ &+\frac{2g_2^2\rho^2y\left(-g_2^4g_1^2\left(680\rho^4+8y^4+67\rho y^3+244\rho^2y^2+511\rho^3y\right)\right)}{(\rho+y)^2\left(g_2^2(\rho+y)^2-g_1^2\rho^2\right)^3\left(2g_2^2(\rho+y)^2-g_1^2\rho(2\rho+3y)\right)}, \end{aligned} \qquad \text{(A.42)}$$

$$\begin{aligned} R^{(1)}_{A^{(2)}} &= \frac{2\left(2g_2^6(\rho+y)^6\left(5y^2-2\rho^2\right)+g_1^6\rho^6\left(4\rho^2+12y^2+13\rho y\right)\right)}{y^2(\rho+y)^2\left(g_1^2\rho^2-g_2^2(\rho+y)^2\right)^2\left(2g_2^2(\rho+y)^2-g_1^2\rho(2\rho+3y)\right)} \\ &+\frac{2\left(+2g_2^2g_1^4\rho^3(\rho+y)^2\left(-6\rho^3+15y^3+2\rho y^2-13\rho^2y\right)\right)}{y^2(\rho+y)^2\left(g_1^2\rho^2-g_2^2(\rho+y)^2\right)^2\left(2g_2^2(\rho+y)^2-g_1^2\rho(2\rho+3y)\right)} \\ &+\frac{2\left(g_2^4g_1^2\rho(\rho+y)^4\left(12\rho^3-18y^3-26\rho y^2+13\rho^2y\right)\right)}{y^2(\rho+y)^2\left(g_1^2\rho^2-g_2^2(\rho+y)^2\right)^2\left(2g_2^2(\rho+y)^2-g_1^2\rho(2\rho+3y)\right)}, \end{aligned}$$

$$R^{(2)}_{A^{(2)}} = \frac{g_1^4\rho^3\left(4\rho^2+21y^2+19\rho y\right)-g_2^2g_1^2\rho(\rho+y)^3(8\rho+27y)+4g_2^4(\rho+y)^4(\rho+4y)}{y(\rho+y)\left(g_1^4\rho^3(2\rho+3y)-g_2^2g_1^2\rho(\rho+y)^2(4\rho+3y)+2g_2^4(\rho+y)^4\right)}.$$

### A.2.3 Solving the third order differential equation for $A^{(2)}$

In this subsection we find the solutions of the homogeneous equation corresponding to (2.71):

$$A^{(2)}_{h1} = a^{(2)}_{h1}(y)C_8(t,x),\ A^{(2)}_{h2} = a^{(2)}_{h2}(y)C_9(t,x) \text{ and } A^{(2)}_{h3} = a^{(2)}_{h1}(y)C_{10}(t,x), \qquad \text{(A.43)}$$

where:

$$a_{h1}^{(2)}(y) = \frac{y\left(g_1^2(9-7y)+4g_2^2(y+1)(4y-5)\right)}{4\left(g_1^2-4g_2^2\right)(y+1)\left(g_1^2-g_2^2(y+1)^2\right)}, \tag{A.44}$$

$$a_{h2}^{(2)}(y) = \frac{(y-1)y\left(g_1^2-2g_2^2(y+1)\right)}{2\left(g_1^2-4g_2^2\right)(y+1)\left(g_1^2-g_2^2(y+1)^2\right)}, \tag{A.45}$$

$$\begin{aligned} a_{h3}^{(2)}(y) &= \frac{g_2^2g_1^2(y+1)\left(y\left(12y^3-5y+2\right)-2\right)+g_1^4\left(2y\left(6y^2+3y-1\right)+1\right)}{6y^2(y+1)^2\left(g_2y-g_1+g_2\right)\left(g_2y+g_1+g_2\right)} \\ &+\frac{6g_1^2y^3(y+1)\left(g_2^2\left(2y^2+y-1\right)+2g_1^2\right)\log(\frac{y}{y+1})+g_2^4(y+1)^2}{6y^2(y+1)^2\left(g_2y-g_1+g_2\right)\left(g_2y+g_1+g_2\right)}. \end{aligned} \tag{A.46}$$

With this at hand we define the Green function:

$$G(z,y) = u_{h1}(z)a_{h1}^{(2)}(y)+u_{h2}(z)a_{h2}^{(2)}(y)+u_{h3}(z)a_{h3}^{(2)}(y), \tag{A.47}$$

where

$$u_{h3}(z) = \frac{z^4(z+1)^4}{g_1^2(3z+2)-2g_2^2(z+1)^2}, \tag{A.48}$$

$$\begin{aligned} u_{h1}(z) &= \frac{(z+1)^2\left(\left(g_1^2+g_2^2\right)g_1^2\left(12(z+1)^2z^4\log\left(\frac{z}{z+1}\right)\right)\right)}{g_1^2(9z+6)-6g_2^2(z+1)^2} \\ &+\frac{(z+1)^2\left(\left(g_1^2+g_2^2\right)g_1^2\left((z(2z+1)(6z(z+1)-1)+4)z-3\right)\right)}{g_1^2(9z+6)-6g_2^2(z+1)^2} \\ &+\frac{(z+1)^2\left(g_2^2g_1^2(z+1)\left(\left(2z\left(6z^2+3z-1\right)-11\right)z+9\right)\right)}{g_1^2(9z+6)-6g_2^2(z+1)^2} \\ &+\frac{(z+1)^2\left(g_2^2g_1^2(z+1)\left(2g_2^4(z+1)^2(4z-3)\right)\right)}{g_1^2(9z+6)-6g_2^2(z+1)^2}, \end{aligned} \tag{A.49}$$

$$\begin{aligned} u_{h2}(z) &= \frac{(z+1)^2\left(\left(7g_1^2+12g_2^2\right)g_1^2\left(12(z+1)^2z^4\log\left(\frac{z}{z+1}\right)\right)\right)}{6\left(g_1^2(3z+2)-2g_2^2(z+1)^2\right)} \\ &+\frac{(z+1)^2\left(3g_2^2g_1^2(z+1)\left(\left(8z\left(6z^2+3z-1\right)-27\right)z+29\right)\right)}{6\left(g_1^2(3z+2)-2g_2^2(z+1)^2\right)} \\ &+\frac{(z+1)^2\left(\left(7(z(2z+1)(6z(z+1)-1)+4)z-27\right)\right)}{6\left(g_1^2(3z+2)-2g_2^2(z+1)^2\right)} \\ &+\frac{(z+1)^2\left(4g_2^4(z+1)^2(16z-15)\right)}{6\left(g_1^2(3z+2)-2g_2^2(z+1)^2\right)}. \end{aligned} \tag{A.50}$$

With this at hand we can compute a particular solution

$$A_p^{(2)} = -\int dw\; G(y,w)e^{-2s_p}F(s_p,\tau,w), \tag{A.51}$$

where $e^{-2s_p}F$ is the RHS inhomogeneity in (2.71). After integration we get the final expression for $A^{(2)}$. We do not post the result but the computation is straightforward. The remaining background fluctuations, $g^{(2)}$ and $\phi^{(2)}$ are evaluated by use of (2.68) and (2.70) once $A^{(2)}$ is known.

**The case of the modulus $\rho$** In this paragraph we present the results towards the derivation of the Green function of the very last third order differential equation in case only the modulus $\rho$ is turned on. In this case we get the homogeneous solutions of (2.104) from:

$$\begin{aligned} a_{h1}^{(2)}(y) \;\; &= \;\; -\frac{y\left(g_2^2(\rho+1)\left(-2\rho(2\rho+3)+(3\rho+5)y^2+\left(3\rho^2+\rho-6\right)y\right)\right)}{(\rho+1)^2\left(g_1^2\rho^2-g_2^2(\rho+1)^2\right)(\rho+y)\left(g_1^2\rho^2-g_2^2(\rho+y)^2\right)} \\ &\quad -\frac{y\left(g_1^2\rho^2(4\rho-(3\rho+4)y+5)\right)}{(\rho+1)^2\left(g_1^2\rho^2-g_2^2(\rho+1)^2\right)(\rho+y)\left(g_1^2\rho^2-g_2^2(\rho+y)^2\right)}, \end{aligned} \tag{A.52}$$

$$a_{h2}^{(2)}(y) = \frac{(y-1)y\left(g_2^2(\rho+1)(\rho+y)-g_1^2\rho^2\right)}{(\rho+1)\left(g_1^2\rho^2-g_2^2(\rho+1)^2\right)(\rho+y)\left(g_1^2\rho^2-g_2^2(\rho+y)^2\right)}, \tag{A.53}$$

$$\begin{aligned} a_{h3}^{(2)}(y) \;\; &= \;\; -\frac{g_1^4\rho^2\left(12y^3(\rho+y)\log\left(\frac{y}{\rho+y}\right)+\rho\left(\rho^3+12y^3+6\rho y^2-2\rho^2 y\right)\right)}{6\rho^4y^2(\rho+y)^2\left(g_2^2(\rho+y)^2-g_1^2\rho^2\right)} \\ &\quad +\frac{g_2^2g_1^2(\rho+y)\left(6y^3\left(-\rho^2+2y^2+\rho y\right)\log\left(\frac{y}{\rho+y}\right)+2\rho^4 y\right)}{6\rho^4y^2(\rho+y)^2\left(g_2^2(\rho+y)^2-g_1^2\rho^2\right)} \\ &\quad +\frac{g_2^2g_1^2(\rho+y)\left(-2\rho^5+12\rho y^4-5\rho^3y^2\right)+g_2^4\rho^4(\rho+y)^2}{6\rho^4y^2(\rho+y)^2\left(g_2^2(\rho+y)^2-g_1^2\rho^2\right)}. \end{aligned} \tag{A.54}$$

To compute the particular solution we obtain :

$$u_{h3}(z) = \frac{z^4(\rho+z)^4}{2g_2^2(\rho+z)^2-g_1^2\rho(2\rho+3z)}, \tag{A.55}$$

$$
\begin{aligned}
u_{h1}(z) = {} & \frac{(\rho+1)(\rho+z)^2 \left(g_2^2 g_1^2(\rho+z)\left(6\left(\rho^2-\rho-2\right)z^4(\rho+z)\log\left(\frac{z}{\rho+z}\right)\right)\right)}{6\rho^4\left(g_1^2\rho(2\rho+3z)-2g_2^2(\rho+z)^2\right)} \\
& - \frac{(\rho+1)(\rho+z)^2\left(g_2^2 g_1^2(\rho+z)\left(\rho\left(-3\rho^4(2\rho+1)+6\left(\rho^2-\rho-2\right)z^4\right)\right)\right)}{6\rho^4\left(g_1^2\rho(2\rho+3z)-2g_2^2(\rho+z)^2\right)} \\
& - \frac{(\rho+1)(\rho+z)^2\left(g_2^2 g_1^2(\rho+z)\left(\rho\left(+3\rho\left(\rho^2-\rho-2\right)z^3\right)\right)\right)}{6\rho^4\left(g_1^2\rho(2\rho+3z)-2g_2^2(\rho+z)^2\right)} \\
& - \frac{(\rho+1)(\rho+z)^2\left(g_2^2 g_1^2(\rho+z)\left(\rho\left(\rho^2\left(-\rho^2+\rho+2\right)z^2+\rho^3\left(8\rho^2+4\rho-1\right)z\right)\right)\right)}{6\rho^4\left(g_1^2\rho(2\rho+3z)-2g_2^2(\rho+z)^2\right)} \\
& + \frac{(\rho+1)(\rho+z)^2\left(g_1^4\rho^2\left(12z^4(\rho+z)^2\log\left(\frac{z}{\rho+z}\right)\right)+g_2^4\rho^4(\rho+1)(4z-3)(\rho+z)^2\right)}{6\rho^4\left(g_1^2\rho(2\rho+3z)-2g_2^2(\rho+z)^2\right)} \\
& + \frac{(\rho+1)(\rho+z)^2\left(g_1^4\rho^2\left(\rho\left(-3\rho^4+12z^5+18\rho z^4+4\rho^2 z^3-\rho^3 z^2+4\rho^4 z\right)\right)\right)}{6\rho^4\left(g_1^2\rho(2\rho+3z)-2g_2^2(\rho+z)^2\right)},
\end{aligned}
\tag{A.56}
$$

$$
\begin{aligned}
u_{h2}(z) = {} & -\frac{(\rho+z)^2\left(-3g_2^2 g_1^2(\rho+z)\left(6\left(\rho^3-5\rho-4\right)z^4(\rho+z)\log\left(\frac{z}{\rho+z}\right)\right)\right)}{} \\
& - \frac{(\rho+z)^2\left(-3g_2^2 g_1^2(\rho+z)\left(\rho\left(-\rho^4\left(8\rho^2+15\rho+6\right)+6\left(\rho^3-5\rho-4\right)z^4\right)\right)\right)}{} \\
& - \frac{(\rho+z)^2\left(-3g_2^2 g_1^2(\rho+z)\left(\rho\left(3\rho\left(\rho^3-5\rho-4\right)z^3\right)\right)\right)}{6\rho^4\left(g_1^2\rho(2\rho+3z)-2g_2^2(\rho+z)^2\right)} \\
& - \frac{(\rho+z)^2\left(-3g_2^2 g_1^2(\rho+z)\left(\rho\left(\rho^2\left(-\rho^3+5\rho+4\right)z^2+\rho^3\left(8\rho^3+16\rho^2+5\rho-2\right)z\right)\right)\right)}{} \\
& - \frac{(\rho+z)^2\left(g_1^4\rho^2\left(12(3\rho+4)z^4(\rho+z)^2\log\left(\frac{z}{\rho+z}\right)+\rho\left(-3\rho^4(4\rho+5)\right)\right)\right)}{} \\
& - \frac{(\rho+z)^2\left(g_1^4\rho^2\left(\rho\left(12(3\rho+4)z^5+18\rho(3\rho+4)z^4+4\rho^2(3\rho+4)z^3-\rho^3(3\rho+4)z^2\right)\right)\right)}{} \\
& - \frac{(\rho+z)^2\left(g_1^4\rho^2\left(\rho\left(4\rho^4(3\rho+4)z\right)\right)+2g_2^4\rho^4(\rho+1)(\rho+z)^2(-6\rho+2(3\rho+5)z-9)\right)}{6\rho^4\left(g_1^2\rho(2\rho+3z)-2g_2^2(\rho+z)^2\right)},
\end{aligned}
\tag{A.57}
$$

that allow us to compute the corresponding Green function from (A.47). Then we calculate the particular solution by the convolution:

$$
A_p^{(2)} = -\int dw \; G(y,w) F_\rho(w). \tag{A.58}
$$

The remaining background fluctuations $g^{(2)}$ and $\phi^{(2)}$ are obtained by use of (2.68) and (2.70).

## A.3 6D solutions

**Homogeneous solutions** In the text we have already given the solutions to the homogeneous differential equations for $\varphi_1$ and $s_1$. For completeness we give here the solutions to the homogeneous differential equations for the remaining fields:

$$\begin{aligned}
g^{(2)}_{uv\,h} &= -a_3\frac{2r^2\rho^2+\rho^4}{2r^4}+a_1\frac{3\rho^4}{4r^4}- \\
&\quad a_4(u,v)\big(\frac{1}{4}\log(F/r^2)-\frac{d\rho^4}{32r^4}-\frac{\rho^2}{4F}-\frac{(3r^2+\rho^2)\rho^4}{12r^4F}\big)+a_7, \\
g^{(2)}_{uu\,h} &= -b_1\frac{1}{2r^2}+b_2, \qquad g^{(1)}_{vv\,h}=-c_1\frac{1}{2r^2}+c_2, \\
f^{(2)}_h &= \frac{\log(F/r^2)}{6\rho^2FG}\big(72r^2F^3a_1+r^2(FG+2\rho^6)a_4\big)+\frac{\log(r/\rho)}{FG}4(4+d)r^2\rho^4a_3 \\
&\quad+\rho^2\frac{3(4+d)r^8-5(4+d)r^6\rho^2-3(32+11d)r^4\rho^4+(8-3d)r^2\rho^6+2d\rho^8}{12FGr^4}a_3 \\
&\quad-\frac{48r^8+72r^6\rho^2+(20+d)r^4\rho^4+2(2+d)r^2\rho^6+d\rho^8}{4Gr^4}a_1 \\
&\quad-\frac{48F^3G-120F^2G\rho^2+(100+3d)FG\rho^4}{288FGr^4}a_4 \\
&\quad-\frac{-12(-4+d)F^2\rho^6-12(24+d)F\rho^8+4(60+d)\rho^{10}}{288FGr^4}a_4 \\
&\quad+\frac{2r^2\rho^4}{FG}a_5+\frac{r^2F^2}{\rho^2G}a_2-\frac{r^2}{4\rho^2}a_6,
\end{aligned} \tag{A.59}$$

where $F=r^2+\rho^2$ and $G=((4+d)r^4+2(4+d)r^2\rho^2+d\rho^4)$ and $a$, $b$ and $c$ are integration constants that depend only on $u$ and $v$.

**Particular solutions** The particular solution for $s^{(1)}$ is given in the text. The particular solution for the remaining fields is:

$$
\begin{aligned}
\varphi_p^{(2)} &= 0, \\
g_{uv\,p}^{(2)} &= -\log(F/r^2)\frac{8c\,(-5\partial_u\rho\partial_v\rho+\rho\partial_u\partial_v\rho)}{\rho^4} \\
&\quad -c\partial_u\rho\partial_v\rho\frac{(80r^2+7d\rho^2)F^2-\rho^2(12r^4-20\rho^4)}{2r^4\rho^2F^2} \\
&\quad +c\partial_u\partial_v\rho\frac{16r^4+(12+d)r^2\rho^2+(4+d)\rho^4}{2r^4\rho F}, \\
g_{uu\,p}^{(2)} &= -\log(F/r^2)\frac{8c\,(-3(\partial_u\rho)^2+\rho\partial_u^2\rho)}{\rho^4} \\
&\quad +\partial_u^2\rho\frac{4c(2r^2+\rho^2)}{r^2\rho F}-(\partial_u\rho)^2\frac{4c\,(6r^4+9r^2\rho^2+2\rho^4)}{r^2\rho^2F^2}, \\
g_{vv\,p}^{(2)} &= -\log(F/r^2)\frac{8c\,(-3(\partial_v\rho)^2+\rho\partial_v^2\rho)}{\rho^4}+\partial_v^2\rho\frac{4c(2r^2+\rho^2)}{r^2\rho F} \\
&\quad -(\partial_v\rho)^2\frac{4c\,(6r^4+9r^2\rho^2+2\rho^4)}{r^2\rho^2F^2}, \\
f_p^{(2)} &= \log(F/r^2)\frac{2cr^2\,(-9\partial_u\rho\partial_v\rho+\rho\partial_u\partial_v\rho)}{\rho^6}+ \\
c &\quad \partial_u\rho\partial_v\rho\frac{7G^2\rho^6+28dF\rho^{12}+144r^2\rho^{12}}{6r^4\rho^6F^2G}+ \\
c &\quad \partial_u\rho\partial_v\rho\frac{G\,(169F^5-393F^4\rho^2+216F^3\rho^4+56F^2\rho^6-27F\rho^8-29\rho^{10})}{6r^4\rho^6F^2G} \\
- &\quad c\partial_u\partial_v\rho\frac{(4+d)\rho^8\,(dF^2-4r^4+4r^2\rho^2)}{6r^4\rho^5FG}. \\
- &\quad c\partial_u\partial_v\rho\frac{G\,(28r^8+40r^6\rho^2+6r^4\rho^4+(2+d)r^2\rho^6)}{6r^4\rho^5FG}.
\end{aligned}
\tag{A.60}
$$

# Appendix B

## B.1 Conventions

The construction of the $hs(\lambda)$ algebra can be seen for example in [74]. The algebra is spanned by the set of generators $V_t^s$ with $s = 0, \ldots, \infty$ and $1 - s \le t \le s - 1$. To define the algebra we use the $\star$-product representation constructed in [75]:

$$V_m^s \star V_n^t = \frac{1}{2} \sum_{i=1,2,3,\ldots}^{s+t-Max[|m+n|,|s-t|]-1} g_i^{st}(m,n;\lambda) V_{m+n}^{s+n-i}. \tag{B.1}$$

With the constants:

$$g_i^{st}(m,n;\lambda) \equiv \frac{q^{i-2}}{2(i-1)!}\, {}_4F_3 \left[ \begin{array}{cccc|c} \frac{1}{2}+\lambda & \frac{1}{2}-\lambda & \frac{2-i}{2} & \frac{1-i}{2} & \\ \frac{3}{2}-s & \frac{3}{2}-t & \frac{1}{2}+s+t-i & & 1 \end{array} \right] N_i^{st}(m,n), \tag{B.2}$$

$q = \frac{1}{4}$ and:

$$N_i^{st}(m,n) = \textstyle\sum_{k=0}^{i-1} (-1)^k \binom{i-1}{k} \left(s-1+m+1\right)_{k-i+1} \left(s-1-m+1\right)_{-k} \left(t-1+n+1\right)_{-k} \left(t-1-n+1\right)_{k-i+1}. \tag{B.3}$$

Where the $(n)_k$ are the ascending Pochhammer symbols. The generators $V_0^2, V_{\pm 1}^2$ can be checked to form a $sl(2,\mathbb{R})$ sub algebra.

Let our definition of trace be

$$tr\left(V_{m_s}^s V_{-m_s}^s\right) \equiv \frac{6}{1-\lambda^2} \frac{(-1)^{m_s} 2^{3-2s} \Gamma(s+m_s)\Gamma(s-m_s)}{(2s-1)!!(2s-3)!!} \prod_{\sigma=1}^{s-1} \left(\lambda^2 - \sigma^2\right) \tag{B.4}$$

In chapter 3 we used $\lambda = 3$ and remain with the ideal part, $2 \leq s \leq 3$.

The Killing metric on the principal embedding for the ordering given in (3.46)

$$g_{ab} = tr(V_a V_b) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{B.5}$$

The Killing metric in diagonal embedding for the ordering given in (3.68)

$$g_{ab} = tr(V_a V_b) = \begin{pmatrix} -\frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 \end{pmatrix} \tag{B.6}$$

**Useful results** Here we report some results that were useful during the computations in section 3.2. In particular, the solution to the conditions

$$\begin{aligned} (\delta \mathcal{L}_1^{(0)})_{\delta \to (\delta_\Lambda)|_{\mu_3 \to 0}} &= (\delta_\Lambda \mathcal{L})\Big|_{\text{At } \mu_3 \text{ \& } x_2 \to 0}, \\ (\delta \mathcal{W}_1^{(0)})_{\delta \to (\delta_\Lambda)|_{\mu_3 \to 0}} &= (\delta_\Lambda \mathcal{W})\Big|_{\text{At } \mu_3 \text{ \& } x_2 \to 0}, \end{aligned} \tag{B.7}$$

where we remind the reader that by $(\delta \ldots)|_{\delta \to \delta_\Lambda}$ we mean:

- Take the functional differential of ... in terms of $(\delta\mathcal{L}^{(0)}, \delta\mathcal{W}^{(0)})$ and after substitute $\delta$ by $\delta_\Lambda$. The expressions for $(\delta_\Lambda\mathcal{L}^{(0)}, \delta_\Lambda\mathcal{W}^{(0)})$ are reported in (3.32). The expressions for $(\delta_\Lambda\mathcal{L}, \delta_\Lambda\mathcal{W})$ are reported in (3.25).

The most general solution to (B.7) reads out

$$
\begin{aligned}
\mathcal{L}_1^{(0)} &= 3c_1\mathcal{W}^{(0)} + c_2\partial_1\mathcal{L}^{(0)} + 2c_1x_1\partial_1\mathcal{W}^{(0)},\\
\mathcal{W}_1^{(0)} &= -c_1\left(\frac{8}{3}\mathcal{L}^{(0)2} + \frac{3}{4}\partial_1^2\mathcal{L}^{(0)}\right) + c_2\partial_1\mathcal{W}^{(0)} - c_1x_1\left(\frac{8}{3}\partial_1\mathcal{L}^{(0)} + \frac{1}{6}\partial_1^3\mathcal{L}^{(0)}\right),\\
\epsilon_1^{(0)} &= -c_1\left(\frac{8}{3}\eta^{(0)}\mathcal{L}^{(0)} + \frac{1}{4}\partial_1^2\eta^{(0)}\right) + c_2\partial_1\epsilon^{(0)} + c_1x_1\left(\frac{8}{3}\partial_1\eta^{(0)}\mathcal{L}^{(0)} + \frac{1}{6}\partial_1^3\eta^{(0)}\right),\\
\eta_1^{(0)} &= c_1\epsilon^{(0)} + c_2\partial_1\eta^{(0)} - 2c_1x_1\partial_1\epsilon^{(0)}.
\end{aligned}
\tag{B.8}
$$

It is straightforward to check that (B.8) coincides with (3.34) for $c_1 = 1$ and $c_2 = 0$. In fact this is the unique choice out of (B.8) that allows to integrate the differential of charge to (3.35).

It is also useful to write down the most general choice of $(\mathcal{L}_1^{(0)}, \mathcal{W}_1^{(0)}, \epsilon_1^{(0)}, \eta_1^{(0)})$ that is consistent without explicit dependence on $\phi$ and dimensional analysis. It is given by

$$
\begin{aligned}
{\mathcal{L}^{(0)}}_{1hom} &= c_3\mathcal{W} + c_4\partial_1\mathcal{L}, \qquad {\mathcal{W}^{(0)}}_{1hom} = c_5\mathcal{L}^2 + c_6\partial_1^2\mathcal{L} + c_7\partial_1\mathcal{W},\\
{\epsilon_1^{(0)}}_{hom} &= c_8\partial_1\epsilon + c_9\partial_1^2\eta + 2c_{10}\mathcal{L}\eta, \qquad {\eta_1^{(0)}}_{hom} = c_{11}\epsilon + c_{12}\partial_1\eta.
\end{aligned}
\tag{B.9}
$$

We use (B.9) to show that (3.55) is not isomorphic to $W_3$.

# Appendix C

## C.1 Uniqueness of the choice $\nu_0 = \frac{1}{2}$, $\nu_{i>0} = 0$ for $0 < \lambda < 1$

Here we show how the only solution to the integrability condition (4.21) in the region $0 < \lambda < 1$ is the trivial one $n_0 = 1$. First we write down the first $4 \times 4$ block of the upper triangular matrix $M$

$$\begin{pmatrix} 1 & \frac{4(\lambda^2-4)}{15} & \frac{4(\lambda^2-4)\left(11\lambda^2-71\right)}{315} & \frac{4(\lambda^2-4)\left(107\lambda^4-1630\lambda^2+6563\right)}{4725} \\ 0 & \frac{12\prod_{\sigma=2}^{3}\sqrt{(\lambda^2-\sigma^2)}}{5\sqrt{14}} & \frac{4\left(7\lambda^2-67\right)\prod_{\sigma=2}^{3}\sqrt{(\lambda^2-\sigma^2)}}{15\sqrt{14}} & \frac{4\prod_{\sigma=2}^{3}\sqrt{(\lambda^2-\sigma^2)}\left(893\lambda^4-19090\lambda^2+113957\right)}{2475\sqrt{14}} \\ 0 & 0 & \frac{8\sqrt{\frac{5}{11}}\prod_{\sigma=2}^{5}\sqrt{(\lambda^2-\sigma^2)}}{21} & \frac{80\sqrt{\frac{5}{11}}\prod_{\sigma=2}^{5}\sqrt{(\lambda^2-\sigma^2)}\left(5\lambda^2-89\right)}{819} \\ 0 & 0 & 0 & \frac{32\sqrt{\frac{7}{5}}\prod_{\sigma=2}^{7}\sqrt{(\lambda^2-\sigma^2)}}{429} \end{pmatrix}. \tag{C.1}$$

The eigenvalues can be checked to be greater or equal than 1, for $0 < \lambda < 1$. In fact they grow as the diagonal index $i$ grows. Next we show this excludes the presence of any other solution. Let the following definition and couple of properties be

$$n_O{}^i \equiv O^i_j n^j, \quad O M^T M O^T = Diag((M^{ii})^2), \quad O^T O = 1. \tag{C.2}$$

As $(M^{ii})^2 \geq 1$ it is clear that

$$\sum_{i=1}^{\infty} \left((M \,.\, n)^i\right)^2 = \sum_{i=1}^{\infty} \left(M^{ii}\right)^2 (n_{Oi})^2 \geq \sum_{i=1}^{\infty} n_{O}{}_i^2 = \sum_{i=1}^{\infty} n_i^2 \geq 1. \tag{C.3}$$

The saturation in (C.3) comes when one of the integers $n_i$ is $\pm 1$. As $(M^{ii})^2 = 1$ only if $i = 1$ thence the only solution to (4.21) is the trivial one. Notice however that our conclusions do breakdown when we are out of the region $0 < \lambda < 1$. This is, to define a new solution we just need to tune up $\lambda$ in such a way that for a given $i$, $M^{ii} = \pm 1$.

## C.2 Solutions with $z < 1$

Here we study the fluctuations for a specific background $z < 1$. We take as a toy example the case $\bar{\mu}_3 = -\mu_3 \neq 0$. The secular polynomial reads out

$$ik = ik'_r - 2\mu_3 \left( \omega^2 + k'^2_r + \frac{\lambda^2 - 1}{3} \right), \tag{C.4}$$

whose roots are

$$k'_{\pm} = \frac{-i + \sqrt{-1 + 8ik\mu_3 - \frac{16}{3}\left(\lambda^2 + 3\omega^2 - 1\right)\mu_3^2}}{4\mu_3}. \tag{C.5}$$

From the quantisation condition (4.75)

$$\begin{aligned} w^{\pm}_{1-n} &= -i\frac{1}{2}\left(1 + 2n + \lambda\right) + \delta^{\pm}_{1\ z<1}, \\ w^{\pm}_{2-n} &= -i\frac{1}{2}\left(1 + 2n + \lambda\right) + \delta^{\pm}_{2\ z<1}, \end{aligned} \tag{C.6}$$

where the $\pm$ refer to the $\pm$ in (C.5) and the $(1, 2)$ refer to the $(+, -)$ in (4.75) respectively, and

$$\begin{aligned} \delta^{\pm}_{1\ z<1} &= \frac{3i \mp \sqrt{-1+8(-1+2ik-2n-\lambda)\mu_3 + \frac{16}{3}(5+12n^2+6\lambda+\lambda^2+12n(1+\lambda))\mu_3^2}}{8\mu_3}, \\ \delta^{\pm}_{2\ z<1} &= \frac{-3i \pm \sqrt{-1+8(1+2ik+2n+\lambda)\mu_3 + \frac{16}{3}(5+12n^2+6\lambda+\lambda^2+12n(1+\lambda))\mu_3^2}}{8\mu_3}. \end{aligned} \tag{C.7}$$

We can also study the case $\bar{\mu}_3 = \mu_3$, we get in this case from (4.70):

$$k' = \frac{k + 4ik\omega\mu_3}{1 + 16\omega^2\mu_3^2}. \tag{C.8}$$

We get just one root, which means that after the folding process of section 4.1, the final equation obtained is of second order, as can be explicitly checked. The QNM in this case are given by:

$$\begin{aligned} \omega_{1\pm} &= \frac{-i-4i(1+2n+\lambda)\mu_3 \mp \sqrt{-1+8(1-2ik+2n+\lambda)\mu_3 - 16(1+2n+\lambda)^2\mu_3^2}}{8\mu_3}, \\ \omega_{2\pm} &= \frac{-i-4i(1+2n+\lambda)\mu_3 \mp \sqrt{-1+8(1+2ik+2n+\lambda)\mu_3 - 16(1+2n+\lambda)^2\mu_3^2}}{8\mu_3}. \end{aligned} \tag{C.9}$$

In section 4 we have given the metric for these solutions (4.22). Propagation in Lifshitz metrics with $z < 1$ is typically associated with the presence of superluminal excitations in the dual field theory, see for instance [76, 77]. For each one of our blocks $r$ we can make use of the AdS/CFT dictionary. The dispersion relations for the corresponding physical excitation, $n$, is given by the condition for a pole in the retarded 2-point function (4.75) and the expression for the auxiliary momentum $k'_r$ of the given block in terms of $k$ and $w$ are given in (C.5) and (C.8) respectively. The wavefront velocity $v_f = \lim_{\omega\to\infty} \frac{\omega}{k_R(\omega,n)}$, [78], can be computed to be

$$v_{f1} = \lim_{\omega\to\infty} \frac{\omega}{-\omega + 4\omega\mu_3 + 8n\omega\mu_3 + 4\lambda\omega\mu_3} = \frac{1}{-1 + 4\mu_3 + 8n\mu_3 + 4\lambda\mu_3}, \quad \text{(C.10)}$$

$$v_{f2} = \lim_{\omega\to\infty} \frac{\omega}{\omega + 4\omega\mu_3 + 8n\omega\mu_3 + 4\lambda\omega\mu_3} = \frac{1}{1 + 4\mu_3 + 8n\mu_3 + 4\lambda\mu_3}. \quad \text{(C.11)}$$

We end up by noticing that for $|\mu_3| \geq \frac{1}{2(1+\lambda)}$ there are no superluminal modes ($|v_f| \leq 1$) in these examples. But for other values there is a finite number of them. However the tale of large $n$ excitations have all $|v_f| \leq 1$.

## C.3 Differential operators and $C_{BTZ}$

We present some differential operators that were referenced in the main body of the text. The Klein Gordon operator in $\rho$ coordinates:

$$D_2 \equiv \frac{d^2}{d\rho^2} \quad + \frac{2(e^{4\rho}+1)}{(e^{4\rho}-1)}\frac{d}{d\rho} + \frac{(1-\lambda^2)(e^{8\rho}-1)}{(e^{4\rho}-1)^2} - \frac{2\Big(2(k^2-\omega^2)(e^{2\rho}+e^{6\rho})+\lambda^2-1-e^{4\rho}(4k^2+4\omega^2+\lambda^2-1)\Big)}{(e^{4\rho}-1)^2}. \quad \text{(C.12)}$$

The operator $D_4$ for the background $\mu_3 \neq 0$

$$\begin{aligned} D_4(z) \quad &\equiv \partial_z^4 - \frac{2iw(z-1)+2(\lambda-4)z+4}{(z-1)z}\partial_z^3 + \Bigg(\frac{-3(z-1)z+6i\mu_3(z-1)z(k+2w)}{12\mu_3^2(z-1)^2z^2} \\ &\quad - \frac{3w^2(z-1)^2-9iw(z-1)((\lambda-3)z+1)+z((\lambda-18)\lambda-(\lambda-4)(4\lambda-11)z+44)-6}{3(z-1)^2z^2}\Bigg)\partial_z^2 \\ &\quad + \frac{(w(z-1)-i((\lambda-2)z+1))(6k\mu_3+4\mu_3(3w+(\lambda-2)\mu_3(3w-i(\lambda-4)))+3i)}{12\mu_3^2(z-1)^2z^2}\partial_z \\ &\quad - \frac{(-i(\lambda-1)(2(\lambda-2)\mu_3+3)+3k+3w)\Big(-i(\lambda-1)(2(\lambda-2)\mu_3-3)+3k+12i\mu_3w^2+3w(4(\lambda-1)\mu_3-1)\Big)}{144\mu_3^2(z-1)^2z^2}. \end{aligned} \quad \text{(C.13)}$$

The differential operator $\overset{(1)}{D}_{GK}$ that we make reference to in section (4.2.1)

$$
\begin{aligned}
D^{(1)}_{GK} &= \frac{64ie^{2\rho}(3e^{2\rho}-1)k}{(e^{2\rho}-1)^2(1+e^{2\rho})^3(\lambda^2-1)}\frac{d}{d\rho} \\
&+ \frac{8k\left(\frac{1-11k^2-\omega^2-\lambda^2+e^{6\rho}(-7k^2+3\omega^2-5\lambda^2-11)}{(e^{2\rho}-1)^3}+e^{4\rho}(3k^2+9\omega^2+\lambda^2-1)\right)}{-ie^{-2\rho}(1+e^{2\rho})^4(\lambda^2-1)} \\
&+ \frac{8k\left(\frac{e^{8\rho}(42\omega^2+6k^2+2\lambda^2-2)+e^{4\rho}(29-15k^2+59\omega^2+3\lambda^2)+e^{2\rho}(27k^2+25\omega^2+\lambda^2-17)}{(e^{2\rho}-1)^3}\right)}{-ie^{-2\rho}(1+e^{2\rho})^4(\lambda^2-1)}. \qquad \text{(C.14)}
\end{aligned}
$$

Finally, we give the master field $C$ for the BTZ$_{M=-1}$ background up to spin 4. We have used the Fourier basis (4.53) and redefined $C^1_0 \equiv C$:

$$C^2_{\pm 1} = \frac{6ie^{\rho}\left(\mp(e^{2\rho}-1)k+(e^{2\rho}+1)\omega\right)C[\rho]}{(e^{2\rho}-1)(e^{2\rho}+1)(\lambda^2-1)}, \tag{C.15}$$

$$C^2_0 = -\frac{6C'[\rho]}{\lambda^2-1}, \tag{C.16}$$

$$\begin{aligned} C^3_0 &= \frac{30\left(\frac{6(k^2-\omega^2)(e^{2\rho}+e^{6\rho})}{\lambda^2-1}+1+e^{8\rho}-2e^{4\rho}(\frac{6k^2+6\omega^2}{\lambda^2-1}+1)\right)C[\rho]}{(e^{4\rho}-1)^2(\lambda^2-4)} \\ &- \frac{90(e^{8\rho}-1)C'[\rho]}{(e^{4\rho}-1)^2(4-5\lambda^2+\lambda^4)}, \end{aligned} \tag{C.17}$$

$$C^3_{\pm 1} = \frac{\left(\frac{\mp(e^{3\rho}-e^{\rho})}{(1+e^{2\rho})^2}k+\omega\frac{(e^{3\rho}+e^{\rho})}{(e^{2\rho}-1)^2}\right)C[\rho]+\left(\frac{\pm e^{\rho}}{(1+e^{2\rho})}k-\frac{e^{\rho}}{(e^{2\rho}-1)}\omega\right)C'[\rho]}{\frac{(4-5\lambda^2+\lambda^4)}{60i}}, \tag{C.18}$$

$$C^3_{\pm 2} = -\frac{30\left(\frac{\mp e^{\rho}}{(e^{2\rho}+1)}k+\frac{e^{\rho}}{(e^{2\rho}-1)}\omega\right)^2C[\rho]+\frac{30e^{2\rho}}{(e^{4\rho}-1)}C'[\rho]}{(4-5\lambda^2+\lambda^4)} \tag{C.19}$$

$$\begin{aligned} C^4_0 &= \frac{\left((e^{2\rho}+4e^{6\rho}+e^{10\rho})\frac{(k^2-\omega^2)}{\lambda^2-1}+\left(\frac{1+e^{12\rho}}{8}-(e^{4\rho}+e^{8\rho})(\frac{3k^2+3\omega^2}{\lambda^2-1}+\frac{1}{8})\right)\right)C[\rho]}{\frac{(e^{4\rho}-1)^3(\lambda^2-9)(\lambda^2-4)}{5600}} \\ &- \frac{\left((e^{2\rho}+e^{6\rho})(k^2-\omega^2)+\frac{(1+e^{8\rho})(11+\lambda^2)}{10}-2e^{4\rho}(k^2+\omega^2+\frac{\lambda^2-29}{10}\right)C'[\rho]}{\frac{(e^{4\rho}-1)^2(\lambda^2-9)(\lambda^2-4)(\lambda^2-1)}{42000}}, \end{aligned} \tag{C.20}$$

$$\begin{aligned} C^4_{\pm 1} &= \left(\frac{\pm k\left(\frac{(1+\lambda^2)(1+e^{8\rho})}{5}-(e^{2\rho}+e^{6\rho})(2+\omega^2)-2e^{4\rho}(\omega^2+\frac{\lambda^2-9}{5})\right)}{\frac{ie^{-\rho}(e^{2\rho}-1)^2(e^{2\rho}+1)^3(\lambda^2-9)(\lambda^2-4)(\lambda^2-1)}{2100}}\right. \\ &+ \left.\frac{\pm e^{2\rho}k^3-e^{2\rho}\frac{(e^{2\rho}+1)}{(e^{2\rho}-1)}k^2\omega-\frac{(e^{2\rho}+1)^3}{(e^{2\rho}-1)^3}\omega\left(\frac{(1+\lambda^2)(1+e^{4\rho})}{5}+\frac{e^{2\rho}(8-5\omega^2-2\lambda^2)}{5}\right)}{\frac{ie^{-\rho}(e^{2\rho}+1)^3(\lambda^2-9)(\lambda^2-4)(\lambda^2-1)}{2100}}\right)C[\rho] \\ &- \frac{2(e^{2\rho}-1)\left(\pm(e^{2\rho}-e^{4\rho}+\frac{e^{6\rho}-1}{2})k-(e^{2\rho}+e^{4\rho}+\frac{e^{6\rho}+1}{2})\omega\right)C'[\rho]}{\frac{ie^{-\rho}(e^{2\rho}+1)^2(\lambda^2-9)(\lambda^2-4)(\lambda^2-1)}{2100}}, \end{aligned} \tag{C.21}$$

$$\begin{aligned} C^4_{\pm 2} &= -420e^{2\rho}\left(\frac{\pm 8k\omega+(1-\lambda^2\mp 4k\omega+4\omega^2)(1+e^{8\rho})+2e^{4\rho}(1+20\omega^2)}{(e^{4\rho}-1)^3(\lambda^2-9)(\lambda^2-4)(\lambda^2-1)}\right. \\ &+ \left.\frac{(20e^{4\rho}-12(e^{2\rho}+e^{6\rho})+2(1+e^{8\rho}))(k^2-\omega^2)}{(e^{4\rho}-1)^3(\lambda^2-9)(\lambda^2-4)(\lambda^2-1)}\right)C[\rho] \\ &+ 420e^{2\rho}\frac{(\pm 4k\omega-2e^{2\rho}(k^2-\omega^2)+(1+e^{4\rho})(k^2\mp 2k\omega+\omega^2-4))C'[\rho]}{(e^{4\rho}-1)^2(\lambda^2-9)(\lambda^2-4)(\lambda^2-1)}, \end{aligned} \tag{C.22}$$

(C.23)

$$
\begin{aligned}
C^4_{\pm 3} &= \frac{\left(\frac{\pm k\left(3\omega^2+e^{4\rho}(3\omega^2-2)+e^{2\rho}(4+6\omega^2)-2\right)}{(e^{2\rho}-1)^2} \pm k^3 - \frac{3(1+e^{2\rho})k^2\omega}{e^{2\rho}-1} - \frac{(1+e^{2\rho})^3\omega(\omega^2-2)}{(e^{2\rho}-1)^3}\right) C[\rho]}{\frac{-ie^{-3\rho}(e^{2\rho}+1)^3(\lambda^2-9)(\lambda^2-4)(\lambda^2-1)}{140}} \\
&+ \frac{(\pm(e^{2\rho}-1)k-(1+e^{2\rho})\omega)\, C'[\rho]}{\frac{-ie^{3\rho}(e^{4\rho}-1)^3(\lambda^2-9)(\lambda^2-4)(\lambda^2-1)}{420}}. \qquad \text{(C.24)}
\end{aligned}
$$

The primes stand for derivative along $\rho$, and one can recover the result in coordinate space $(t,\phi)$ by replacing $k \to -i\partial_\phi$ and $\omega \to -i\partial_t$. Notice that all these higher spin components are generically singular at the horizon.

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