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Cosmological Consistency Relations

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Chapter 1

Introduction

In the last several decades we witnessed a tremendous progress in cosmology. From a theory with mainly qualitative ideas about the expanding universe originated from a state with high density and temperature (Hot Big Bang cosmological model), cosmology rapidly evolved to a science in which the composition and the evolution of the universe are known well enough to make very detailed predictions for a large number of observables on many different scales and different redshifts. Many of these predictions can be tested nowadays in observations to a percent accuracy, and planned future experiments will push the limits even further. This progress brought to us some of the most important and exciting discoveries in the history of science, helping us to understand in detail the history of our universe over the whole 13.7 billion years of its evolution.

It is fair to say that the era of precision cosmology started with the first detection of anisotropies in the Cosmic Microwave Background (CMB) by the COBE satellite [1]. A series of ground-based and balloon-borne experiments improved the measurements of the power spectrum of temperature fluctuations in the CMB in the following decade and lead to the discovery of acoustic peaks which provided an independent measurement of cosmological parameters (see for example [2, 3]). Observations strongly supported the existence of non-baryonic dark matter and confirmed the existence of dark energy discovered around the same time in the observations of supernovae at high redshift [4, 5]. The characteristic shape of the power spectrum of temperature fluctuations with acoustic peaks, measured flatness of the universe and scale invariance of the power spectrum of primordial fluctuations were all in agreement with the inflationary predictions, severely constraining alternative mechanisms for generation of the CMB anisotropies. At the same time, the estimates of cosmological parameters from the CMB were in a good agreement with local measurements (see for example [6]), indicating that we have a consistent picture of the universe. Although all these steps brought groundbreaking discoveries, the potential of the CMB as the leading probe of cosmology was still to be fully explored.

The new era started with new satellites. The first to fly was WMAP. Apart from significantly improving the precision of measurements of the composition of the universe, WMAP also brought for the first time interesting constraints on the initial conditions, ruling out some

inflationary models [7]. It delivered tight constraints on non-Gaussianities, that shrank the parameter space for inflation more than ever [8, 9, 10]. The latest results from measurements of the CMB temperature anisotropies came form Planck [11, 12, 13]. Probably the most important single result is confirmation of a small negative tilt of the power spectrum of initial curvature perturbations, predicted by the simplest inflationary models. The scale invariant power spectrum was ruled out for the first time in a single experiment with more than 5σ [12]. On the other hand, with new, much tighter constraints on non-Gaussianities we are entering in the interesting regime of parameters where we can make some general statements about the nature of inflation. All these results can still improve after the data release form polarisation measurements by the Planck collaboration.

Although for some observations (like temperature fluctuations in the CMB) we are reaching the cosmic variance limited precision, the CMB still remains a gold mine of data. In particular, there is a lot of space for improvement in measurements of polarisation. The most interesting observable are the so called B-modes of the CMB polarisation, because in the absence of vector modes they can be induced only by tensor perturbations [14, 15]. Therefore, a detection of B-modes (on top of the signal coming form CMB lensing) would be a detection of primordial gravitational waves and confirmation of the inflationary paradigm. The first encouraging steps were made recently. The first evidence for B-modes induced by CMB lensing was found in [16, 17]. After that, the BICEP collaboration announced the first direct detection of B-modes at a degree scale, with amplitude much higher than the leasing contribution [18]. Although there is an ongoing debate about whether this signal is of cosmological origin or not [19, 20], it is certain that a new era in observational cosmology is opened and we will have much more data coming in the following years.

In the meantime a big progress has been made in extracting the cosmological parameters from observations of Large Scale Structure (LSS). A series of galaxy surveys (for example [21, 22, 23]) allowed us to learn a lot about structure formation and late evolution of the universe. A lot of work has been done in order to understand how to use this growing amount of data to learn about the statistics of the initial conditions, constrain possible modifications of gravity, study the properties of dark energy, explore galaxy formation etc. Despite huge complexity due to non-linear gravitational collapse and complicated baryonic physics involved in galaxy formation, a big advantage of LSS is that it, in principle, contains much more information than the CMB. The reason for this is simply that the amount of modes available is much larger. At the moment, some of the parameters obtained form LSS surveys are already competitive with CMB results. With future experiments, the LSS will become the leading probe of cosmology (see for example [24]).

Combining all the data mentioned above allowed us to measure the cosmological parameters better than ever and establish ΛCDM as the standard model of cosmology. All observational evidence tell us that our universe is homogeneous and isotropic on large scales, spatially flat, with around 5% of baryons, around 27% of cold dark matter (CDM) and dominated by the cosmological constant Λ that causes the accelerated expansion. The initial curvature perturbations have a scale-invariant power spectrum, they are highly Gaussian and perturbations

are adiabatic, all in agreement with predictions of the simplest models of inflation. So far, in observations, we do not see any statistically significant disagreement with the predictions of ΛCDM model. Still, many questions remain without answers. For example, just to mention some of them, we still do not know what is the statistic of primordial perturbations, what is the nature of dark energy or whether or not general relativity (GR) is modified on cosmological scales. For some of these questions we may find the answers already in some of the next generation experiments.

It is important to stress that almost all this huge progress has been made measuring different correlation functions. Although the study of the background evolution was important to establish the hot Big Bang model, only with the development of perturbation theory and measurements of the correlation functions of cosmological perturbations it became possible to make a crucial improvement in our ability to learn about the universe. There are many different observables, from fluctuations in the CMB (including temperature and polarization), neutral hydrogen at high redshifts, Lyman-alpha forest, to galaxy surveys in our local cosmological environment. All these observables are sensitive to different scales at different epochs of the universe and to many different processes that are involved in the evolution of cosmological perturbations. Therefore, one of the most important goals of theoretical cosmology is to find reliable methods to calculate different correlation functions for a given cosmological model. Over the past several decades a lot of progress has been made, from calculations of inflationary predictions, linear perturbation theory and CMB physics, to some techniques that capture non-linear effects and gravitational collapse in the late universe—all in order to better understand the evolution of cosmological perturbations. Cosmological consistency relations, as general theorems about correlation functions, are an important part of this effort.

1.1 Correlation Functions and Symmetries

One of the most powerful tools that we have at hand in the study of cosmological correlation functions are symmetries. They were used to construct a very useful and general effective field theory approach to single-field [25] and multi-field inflation [26], some of the dark energy models [27, 28] and large scale structure [29, 30, 31, 32, 33]. They also play an important role in the study of non-Gaussianities, independently of that whether they are imposed on the background [34], perturbations [35] or non-linearly realised [25, 36], just to mention some examples. The consistency relations, that are the main subject of this thesis, also came from studying the symmetries of cosmological perturbations. In all these cases, like in QFT, the symmetries help us to construct the theories, find general statements, constrain solutions or access the non-perturbative regime.

As an illustration, let us take a look at the example of isometries of de Sitter space, which is a good approximation for the space-time during inflation. Let us also, for the time being, focus on the case of a test field φ in de Sitter[1]. For example, using only de Sitter isometries

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[1]This is relevant in some inflationary models where the curvature perturbation dominantly come form a

we can show that the power spectrum of φ is scale-invariant. Indeed, de Sitter space-time is described by the following metric

$$\mathrm{d}s^2 = \frac{1}{H^2\eta^2}\left(-\mathrm{d}\eta^2 + \mathrm{d}\vec{x}^2\right) , \tag{1.1}$$

with a constant Hubble parameter H (η is conformal time). Being a maximally symmetric space-time, de Sitter has 10 isometries. Six of them are spatial translations and rotations, and they constrain the correlation functions in a trivial way. One of the remaining isometries is the rescaling of coordinates $\eta \to \lambda\eta$ and $\vec{x} \to \lambda\vec{x}$. Given this symmetry in real space, the 2-point function of φ in momentum space has to satisfy the following property

$$\langle \varphi_{\vec{k}_1/\lambda}(\lambda\eta)\varphi_{\vec{k}_2/\lambda}(\lambda\eta)\rangle = \lambda^6 \langle \varphi_{\vec{k}_1}(\eta)\varphi_{\vec{k}_2}(\eta)\rangle . \tag{1.2}$$

This constraint, after imposing the translational invariance, immediately tells us that

$$\langle \varphi_{\vec{k}_1}(\eta)\varphi_{\vec{k}_2}(\eta)\rangle = (2\pi)^3 \delta(\vec{k}_1 + \vec{k}_2)\frac{1}{k_1^3}F(k_1\eta) , \tag{1.3}$$

where $F(x)$ is some unknown function. However, we are usually interested in the late-time limit $\eta \to 0$, in which the unknown part becomes just a constant $F(0)$. In this way, without specifying any details of the theory, we were able to show that the two-point function indeed must be scale invariant.

Inspired by this example, we can take a look at the remaining three isometries of de Sitter space. They are parametrized by an infinitesimal parameter $\vec{b}$

$$\eta \to \eta - 2\eta(\vec{b}\cdot\vec{x}) , \quad x^i \to x^i - b^i(-\eta^2 + \vec{x}^2) + 2x^i(\vec{b}\cdot\vec{x}) . \tag{1.4}$$

Do these isometries constrain the correlation functions in a similar way as dilations? For a test field in de Sitter this must be the case, because the correlation functions depend on de Sitter invariant distances and therefore must be invariant under (1.4). Indeed, this can be explicitly checked for several situations that are relevant for inflation (the consequences of these isometries were first studied in [39]). For example, the correlation functions of gravitons are expected to be invariant under (1.4), because gravitons behave just as a test field in the inflating background [40]. It is interesting to notice that in the late time limit the isometries (1.4) act on the spatial coordinates as the special conformal transformation [39, 40]. In this sense, we can say that the graviton three-point functions must be conformally invariant [40]. This is a very strong statement, that immediately tells us that the shape of the three-point function for gravitons is fixed up to a normalisation, independently on the interactions. The same applies to correlation functions of curvature perturbations, in inflationary models where the curvature perturbations are generated by some spectator field [41]. Also in this case the shape of the three-point function is fixed by conformal invariance and the scaling dimension of

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light test field like the curvaton [37, 38].

the field (which depends on its mass). In all these examples, for a massless field φ, invariance under (1.4) in momentum space implies

$$b^i \cdot \sum_{a=1}^{n} \left[6\partial_{k_{ai}} - k_{ai}\vec{\partial}_{k_a}^2 + 2(\vec{k}_a \cdot \vec{\partial}_{k_a})\partial_{k_{ai}} \right] \langle \varphi_{\vec{k}_1} \varphi_{\vec{k}_2} \ldots \varphi_{\vec{k}_n} \rangle' = 0 \ , \qquad (1.5)$$

where the prime on the correlation functions indicates that the momentum conserving delta function has been removed.

Naturally, one is immediately tempted to think that these conclusions can be easily generalised to any inflationary model, given that quantum fluctuations that eventually produce curvature perturbations live in an approximately de Sitter background during inflation. However, it is straightforward to check that although approximately scale-invariant, the correlation functions of curvature perturbations in single-field inflation are *not* invariant under conformal transformations. The resolution to this apparent paradox is very simple. Unlike in the case of test fields, in single-field inflation the curvature perturbations are generated from the fluctuations of the inflaton around some time-dependent background. This time-dependent background breaks the isometries[2] of de Sitter spontaneously, and the correlation functions of perturbations do not depend on de Sitter invariant distances.

Let us take a more detailed look at what happens. We can parametrize the background as we like, and let us say that $\phi = t + \pi$. Now the isometries of de Sitter are non-linearly realised on π. For dilation $t \to t + C$, and consequently $\pi \to \pi - C$. However, an approximate shift symmetry is typically assumed in inflation. It is for this reason that the correlation functions are scale invariant. Dilation is spontaneously broken, but in combination with shift symmetry it is recovered. Indeed, the amount of breaking is proportional to the amount of breaking of the shift symmetry, which is related to the slow-roll parameters. On the other hand, nothing similar happens for conformal transformations. They are broken and there is no symmetry that help them to be restored[3].

However, this does not mean that one cannot say anything about the consequences of de Sitter isometries. The symmetry is still there in the original Lagrangian—on the level of perturbations it is only non-linearly realised. The situation is the same as in QFT, where usually some internal symmetry is spontaneously broken. It is well known that in these kind of theories one can derive Ward identities that relate correlation functions of different order. Although the spontaneous breaking of space-time symmetries is somewhat different from internal symmetries[4], similar identities can be written for the inflationary correlation functions as well. They have the following form

$$\langle \zeta_{\vec{q}} \zeta_{\vec{k}_1} \ldots \zeta_{\vec{k}_n} \rangle'_{q \to 0} = -P(q) \left[3(n-1) + \sum_a \vec{k}_a \cdot \vec{\partial}_{k_a} + \frac{1}{2} q^i \delta^{(n)}_{\mathcal{K}^i} + \mathcal{O}(q/k)^2 \right] \langle \zeta_{\vec{k}_1} \ldots \zeta_{\vec{k}_n} \rangle' \ , \quad (1.6)$$

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[2]It breaks isometries that are related to time. Spatial translations and rotations remain linearly realised.

[3]For more details, see the discussion in Section 4.6.

[4]For example the number of Goldstone bosons is not equal to the number of broken generators, for more details see [42].

where $\delta_{\mathcal{K}^i}^{(n)}$ is a differential operator in (1.5). These equations that relate the squeezed limit of the correlation functions with lower order correlation functions are called consistency relations.

1.2 Consistency Relations

So far we used arguments based on de Sitter isometries to claim that there are some non-trivial relations between inflationary correlation functions of different order called consistency relations. However, during inflation the metric fluctuates as well. Including gravity is never easy, but it turns out that the consistency relations remain valid in all single-field models, even when the coupling with gravity is taken into account. In order to prove this stronger statement we cannot use the isometries of de Sitter anymore. Instead we need a different kind of insight.

The insight is that in single-field models[5] the long-wavelength mode is locally indistinguishable from a coordinate transformation [43, 44, 45]. Indeed, if we have only one dynamical degree of freedom during inflation, then all parts of the universe follow the same classical history and quantum fluctuations of the inflaton are locally equivalent to a small shift of this classical history in time. This shift in time is then equivalent to an additional expansion that is equivalent to a simple rescaling of coordinates. Of course, this is true once the long mode crosses the horizon, because after that it becomes frozen and remains constant. For short modes its effect is not physical because it is just a classical background whose effect can be removed just by a rescaling of coordinates. Notice that this argument is independent on the details of interactions, the form of the kinetic term, slow-roll assumptions or any other details of the theory [45]. We will make this discussion a bit more formal in Chapter 3.

The other way to see the same thing is to say that in single-field models one can connect two different solutions of Einstein equations, one with short modes and a long mode and one with the same short modes but *without* the long mode, just using the change of coordinates. This is the heart of the consistency relations, because it allows us to find a relation between the correlation functions of different order. On one side we have a correlation function with one long mode and n short modes. In momentum space this means that the momentum of the long mode is much smaller than all the others, and we get the left hand side of (1.6). On the other side is a variation of the n-point function under the change of coordinates which in momentum space becomes the right hand side of (1.6).

At zeroth order in $\vec{q}$ (the first term on the right hand side of (1.6)) the consistency relations were first shown to hold for the three-point functions in single-field slow-roll inflation [43]. It was later proven that they hold for any n-point function in any model of single-field inflation [45] (for a proof in the framework of the EFT of inflation see [46]). Let us immediately stress the most important consequence of the consistency relations. Given the approximate scale invariance of the two-point function, they imply that the local non-Gaussianities must be small in *any* single-field model of inflation. In other words, any detection of $f_{\mathrm{NL}}^{\mathrm{loc.}} \gg 1$ would

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[5]Here we assume that the inflaton is on the dynamical attractor and standard Bunch-Davies vacuum.

rule out *all* single-field models. For this reason Maldacena's consistency relation is one of the most important theorems about non-Gaussianities.

The behaviour of the three-point function in the squeezed limit was further investigated in [47] where it was shown that the corrections to the leading result linear in $\vec{q}$ vanish. This is not the case in general for the squeezed limit of an n-point function. The form of the term linear in $\vec{q}$ was first discussed in [48]. It is the second term on the right hand side in (1.6). Given that the long mode at this order in gradients is equivalent to a special conformal transformation of spatial coordinates, the result is dubbed conformal consistency relation. Since then there was a lot of progress in understanding the origin of the consistency relations in different ways. They were derived as Ward identities [49, 50, 51], using the wavefunciton of the universe [52], as Slavnov-Taylor identities for diffeomorphism invariance [53, 54], using operator product expansion techniques [55, 56] and the holographic description of inflation [57, 58]. We will discuss some of these approaches in Chapter 4. As a final comment let us say that, although we are not going to consider it in this theses, the same arguments can be applied even in models that are alternatives to inflation. One example of this kind is the conformal mechanism [59, 60, 61, 62]. The consistency relations for this kind of models were derived in [63].

Although the consistency relations were originally derived for inflation, it is possible to use the same arguments in the late universe. As long as the long adiabatic mode is outside the dynamical horizon (the sound horizon) it can be seen locally as a gauge transformation (assuming single-field inflation). For example, using this property one can calculate analytically the squeezed limit of the bispectrum of temperature fluctuations in the CMB [64]. This calculation connects the inflationary consistency relations with observables, and shows precisely in what sense the single-field models predict small local non-Gaussianities.

This is not the end of consistency relation because we can keep applying the same logic even at later times. After recombination the sound horizon practically shrinks to zero. Therefore, all the modes that enter the horizon afterwards are outside the sound horizon and the same arguments as in inflation apply to them too. Locally, these long adiabatic modes are just a coordinate transformations. This insight leads to the consistency relations of LSS that were a subject of an intense study recently. Using a different approach they were first derived in Newtonian limit in [65, 66] and then following the arguments described here generalised in [67] to the relativistic result and including baryons. An incomplete set of references that explored and extended these results include [68, 69, 70, 71, 72, 73, 74, 75]. As we will show in Chapter 5, the consistency relations of LSS have the same form as the consistency relations for inflation. Here we quote just the result in the non-relativistic limit [65, 66, 69]

$$\langle \delta_{\vec{q}}(\eta)\, \delta_{\vec{k}_1}(\eta_1) \cdots \delta_{\vec{k}_n}(\eta_n) \rangle'_{q \to 0} = -P_\delta(q, \eta) \sum_a \frac{D_\delta(\eta_a)}{D_\delta(\eta)} \frac{\vec{q} \cdot \vec{k}_a}{q^2} \langle \delta_{\vec{k}_1}(\eta_1) \cdots \delta_{\vec{k}_n}(\eta_n) \rangle' . \qquad (1.7)$$

where δ is the density contrast and $D_\delta(\eta)$ the growth function of perturbations.

What makes the consistency relations of LSS very interesting is that they are non-perturbative result and they hold independently of baryon physics and bias. Indeed, as

we will see in Chapter 5, we can write them down for correlation functions of galaxies directly in redshift space. Notice that, although we cannot calculate non-pertubatively any of these correlation functions, we can still prove that the relation (1.7) holds. It is also important to stress that they are valid even for the long modes that are inside the Hubble radius and therefore observable. However, for the observationally most interesting case of equal-time correlation functions, the right hand side of the consistency relation vanishes. The situation is somewhat similar to what happens in inflation, where due to approximate scale invariance the right hand side of Maldacena's consistency relation is unobservable. Still, we can use this to our advantage. Instead of trying to confirm them, we can search for violations of the consistency relations. Indeed, if any of the assumptions entering the derivation of the consistency relations is violated, then the right hand side of (1.7) will not be zero. Apart from the assumption of single-field inflation, the other crucial ingredient that we need for eq. (1.7) to hold is the equivalence principle (EP). Therefore, what we learn from the consistency relations of LSS is that *any* violation of the EP on cosmological scales would induce the $1/q$ divergence in the squeezed limit of the three-point function. This provides a new, model independent test of general relativity on cosmological scales. We will explore the ability of future surveys to constrain the EP violations in Chapter 5.

To conclude, given the importance and complexity of cosmological correlation functions described at the beginning of this introduction, the consistency relations are interesting as a rare example of non-perturbative and model-independent statements that one can make. They are derived form the fact that, in any single-field model and assuming GR, ever since the long mode leaves the horizon and remains outside the sound horizon its effect on the short modes is equivalent to a change of coordinates and therefore unphysical. The future probes will have a potential to measure the correlation functions with increasing precision. As we pointed out, if the consistency relations are falsified in future observations, it would be a sign of either a violation of the EP on cosmological scales or multifield inflation, or both.

Chapter 2

Theory of Cosmological Perturbations

The theory of cosmological perturbations is in the heart of modern cosmology. According to the standard ΛCDM model, inhomogeneities that we observe in the universe come from quantum fluctuations stretched by inflation. These fluctuations eventually come back inside the horizon and evolve during different epochs of the history of the universe. We can see them as small fluctuations of the temperature in the CMB on the one side and as fluctuations in the number density of galaxies on the other. The best way to characterise these perturbations is through their correlation functions. Given complicated interactions over a very long period of time, these correlation functions are very sensitive to cosmological parameters. It is for this reason that the study of cosmological perturbations is crucial for our ability to compare theories and observations and find the correct model for our universe.

The purpose of this Chapter is not to go through all details of the evolution of cosmological perturbations. We will just fix the notation and give an overview of the main results that we are going to use in the following chapters. In particular, our focus will be on calculations of correlation functions that are necessary for many checks of the consistency relations, both for inflation and LSS. Many details of these calculations that we are going to skip can be found in many excellent articles, reviews, lecture notes and textbooks and we will refer the reader to some of them when needed.

2.1 Relativistic Perturbation Theory

At early times and large distances compared to the curvature scale, relativistic effects cannot be neglected. Therefore, for a complete description of cosmological fluctuations one necessarily needs a perturbation theory that is compatible with general relativity. We focus on it in this Section. Our main goal is to introduce the perturbed Einstein equations in Newtonian gauge that we will need in Chapter 3. Many more details about perturbation theory can be found in textbooks on cosmology (see for example [76, 77, 78]).

2.1.1 Background Evolution

In order to introduce some notation and set the stage for the study of perturbations, we begin by a brief review of the background evolution of the Universe. All observational evidence that we have tells us that the Universe is homogeneous and isotropic on cosmological scales. This large amount of symmetry simplifies its description significantly. Averaged over space, all relevant quantities such as density, temperature, pressure and others can depend only on time.

Indeed, assuming that the spatial curvature is zero[1], the line element takes the following simple form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \mathrm{d}\vec{x}^2 \ . \tag{2.1}$$

This is the well known Friedmann-Robertson-Walker (FRW) metric. In this metric $a(t)$ is some time dependent function called scale factor. The scale factor contains information about the expansion history of the Universe and it can be determined once the composition and the initial conditions are known. Before we move to the dynamics, let us introduce conformal time η that we will use frequently. Conformal time is defined in the following way

$$\eta = \int \frac{\mathrm{d}t}{a(t)} \ , \tag{2.2}$$

such that the FRW metric in new coordinates is conformal to Minkowski space-time

$$\mathrm{d}s^2 = a^2(\eta) \left(-\mathrm{d}\eta^2 + \mathrm{d}\vec{x}^2 \right) \ . \tag{2.3}$$

These coordinates are particularly useful for the study of the causal structure of the FRW space-time. For the rest of the thesis, dots will denote the standard time derivative with respect to t, while primes will denote derivatives with respect to conformal time, $' \equiv \partial_\eta$.

In order to describe the dynamics of the expansion we have to use the Einstein equations. With ansatz (2.1) the Einstein equations give

$$\left(\frac{\dot{a}}{a} \right)^2 \equiv H^2 = \frac{8\pi G}{3} \rho \ , \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p) \ . \tag{2.4}$$

These are the famous Friedmann equations, that describe the evolution of the scale factor. The quantity $H \equiv \dot{a}/a$ is Hubble expansion rate. Written in terms of conformal time these equations become

$$\left(\frac{a'}{a} \right)^2 \equiv \mathcal{H}^2 = \frac{8\pi G}{3} a^2 \rho \ , \quad \frac{a''}{a} = \frac{4\pi G}{3} a^2 (\rho - 3p) \ . \tag{2.5}$$

Notice that on the right hand side of these equations we have the total energy density ρ and pressure p of a fluid that fills the universe. They come from the energy-momentum tensor

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[1]This is one of the inflationary predictions and it is in a perfect agreement with observations [11]. We will always assume zero spatial curvature throughout the thesis.

on the right hand side of Einstein equations. It is a good approximation to use the energy-momentum of a perfect fluid, which in a frame comoving with the fluid has the following simple form $T_\nu^\mu = \text{diag}(\rho, -p, -p, -p)$.

Notice that equations (2.4) can be combined to get

$$\dot\rho + 3H(\rho + p) = 0 \; . \tag{2.6}$$

This is the continuity equation and it can be obtained from the conservation of the energy-momentum tensor too. Finally, in order to solve Friedmann equations we need to specify the equation of state of the fluid. We can use the following parametrization $p = w\rho$ in which case, for constant w, the continuity equation gives

$$\rho \sim a^{-3(1+w)} \; . \tag{2.7}$$

Now it is straightforward to find the solution for the scale factor. It is given by

$$a(t) \sim t^{\frac{2}{3(1+w)}} \; , \quad w \neq -1 \; . \tag{2.8}$$

Using $w = 0$ for ordinary matter or $w = 1/3$ for radiation, we recover the well known results in cases of matter and radiation dominated universe. In the case $w = -1$ the energy density is constant and the universe expands exponentially $a \sim e^{Ht}$. This is for example the case during inflation.

2.1.2 Perturbed Einstein Equations

The universe is homogeneous and isotropic only on cosmological scales and the Friedmann equations can describe well only the background evolution where all quantities are averaged in space. In order to study small perturbations around this background we have to allow for space and time dependence and the Einstein equations become more complicated. Once the background is subtracted we are left with

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu} \; . \tag{2.9}$$

If the perturbations are small, we can do a perturbative expansion on both sides. For most of the purposes in cosmology it is enough to keep only the linear terms.

The main problem with relativistic perturbation theory is gauge redundancy in GR. The freedom to choose the coordinates means that there is no clear distinction between what we call background and what are perturbations. Of course, physical quantities cannot depend on this choice and this is why only gauge invariant observables have physical sense. The way one typically deals with these issues is to fix some gauge, do the calculation and then make the connection to the observables. This situation is familiar form classical electrodynamics, which is a gauge theory too.

Let us take a closer look on the left hand side of eq. (2.9). In general the metric fluctuations can be written in the following way

$$ds^2 = -(1 + 2\Phi)dt^2 + 2aB_i dx^i dt + a^2 \left[(1 - 2\Psi)\delta_{ij} + E_{ij}\right] dx^i dx^j \; . \tag{2.10}$$

The perturbations Φ and Ψ are scalars and B_i and E_{ij} can be decomposed into scalar, vector and tensor parts[2]. Notice that, as in any gauge theory, the number of components in the metric is always larger than the number of actual physical degrees of freedom. It is also important to stress that the scalar components of the metric transform under the change of coordinates and therefore they are not physical quantities.

On the right hand side of eq. (2.9) we can write the fluctuations in energy-momentum tensor like

$$T_0^0 = -(\bar{\rho} + \delta\rho) \,, \tag{2.11}$$

$$T_i^0 = (\bar{\rho} + \bar{p})v_i \,, \tag{2.12}$$

$$T_j^i = \delta_j^i(\bar{p} + \delta p) + \Pi_j^i \,, \tag{2.13}$$

where bar denotes background quantities and Π_j^i is the anisotropic stress.

Let us at the end mention two important gauge invariant quantities that we will use later in the thesis. One is the curvature perturbation on uniform-density hypersurfaces

$$\zeta \equiv -\Psi - \frac{H}{\dot{\bar{\rho}}}\delta\rho \,. \tag{2.14}$$

The other one is the comoving curvature perturbation

$$\mathcal{R} \equiv \Psi - \frac{H}{\bar{\rho} + \bar{p}}v \,, \tag{2.15}$$

where v is the velocity potential $v_i = \partial_i v$. One can explicitly check that these two quantities remain invariant under the change of coordinates.

2.1.3 Newtonian Gauge

As we said, the way to deal with gauge redundancy is to fix the gauge and then do the calculation. In this section we show how the perturbed Einstein equations look like in Newtonian gauge. Newtonian gauge is defined by $B = E = 0$, such that the metric (2.10) takes the following form

$$\mathrm{d}s^2 = a^2(\eta)\left[-(1 + 2\Phi)\mathrm{d}\eta^2 + (1 - 2\Psi)\mathrm{d}\vec{x}^2\right] \,. \tag{2.16}$$

Notice that this metric has a familiar form of a metric in the weak gravitational field. This gauge is very convenient for LSS calculations, and it is intuitive because in the non-relativistic limit Φ becomes the gravitational potential.

Once we fix the gauge, it is straightforward to calculate the variation of the Einstein tensor. The perturbed Einstein equations in Newtonian gauge give

$$\nabla^2\Psi - 2\mathcal{H}(\Psi' + \mathcal{H}\Phi) = 4\pi Ga^2\bar{\rho}\,\delta \,, \tag{2.17}$$

$$\partial_i(\Psi' + \mathcal{H}\Phi) = (\mathcal{H}^2 - \mathcal{H}')\partial_i v \,, \tag{2.18}$$

$$\Psi'' + \mathcal{H}(\Phi' + 2\Psi') + (2\mathcal{H}' + \mathcal{H}^2)\Phi - \frac{1}{3}\nabla^2(\Psi - \Phi) = 4\pi Ga^2\delta p \,, \tag{2.19}$$

$$\partial_i\partial_j(\Psi - \Phi) = 8\pi G\Pi_{ij} \,. \tag{2.20}$$

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[2]Vector modes are not produced during inflation and we will neglect them in the rest of the discussion.

In the absence of vector and tensor perturbation we can write the anisotropic stress as $\Pi_{ij} = \partial_i \partial_j \delta\sigma$. We are going to use these equations in Chapter 3.

2.2 Cosmological Perturbations from Inflation

Inflation is a leading paradigm for describing the early universe. Originally, inflation was introduced as a solution for three famous problems of the standard hot Big Bang cosmology: horizon problem, flatness problem and magnetic monopoles problem (for example see [79, 80, 81, 82, 83]). However, it quickly turned out that inflation gives a natural framework for the generation of primordial cosmological perturbations (some of the early works include [84, 85, 86, 87, 88, 89]). In that sense, inflation can be seen as a theory of the initial conditions for the standard hot Big Bang cosmology. In this Section we will give a short summary of inflationary theory, with a particular accent on calculations of correlation functions that we will need in the rest of the thesis. Most of the time we will follow [43], where the power spectrum and different three-point functions were calculated in a very clean way. For a very nice review of inflation and many more details of the calculations, we invite the reader to take a look at [90].

2.2.1 Inflation

Inflation can be defined as a period of accelerated expansion $\ddot{a} > 0$ in the early universe. This accelerated expansion is essential for solving the horizon and curvature problems and it naturally explains why we do not see possible heavy and stable relics form the early universe, such as magnetic monopoles [80]. The easiest way to make the universe expand with acceleration is to use a scalar field minimally coupled to gravity (we can use units in which $M_{\mathrm{Pl}} = 1$) [80, 81]

$$S = \frac{1}{2} \int \sqrt{-g} \Big[R - (\partial\phi)^2 - 2V(\phi) \Big] . \tag{2.21}$$

If the potential is sufficiently "flat" in the sense that we will explain below, then the vacuum energy that couples to gravity can make the universe expand quasi-exponentially sufficiently long to create a huge homogeneous and spatially flat patch that we are living in. Indeed, one can easily find from the energy momentum tensor of the scalar field that on the background level energy density and pressure are given by [90]

$$\bar{\rho} = \frac{1}{2}\dot{\phi}^2 + V(\phi) , \quad \bar{p} = \frac{1}{2}\dot{\phi}^2 - V(\phi) . \tag{2.22}$$

If the potential is dominant compared to kinetic energy of the field, we see that the effective equation of state corresponds to $w \approx -1$ and therefore the universe expands almost exponentially.

In order to be more quantitative about the conditions that we need in order to have an inflating universe we can look at the background equations of motion for the action (2.21).

These are

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) , \quad \ddot{\phi} + 3H\dot{\phi} + \frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi} = 0 . \tag{2.23}$$

The first equation is just the Friedmann equation in which we used the energy density from (2.22). The second equation is a generalisation of the Klein-Gordon equation to curved spacetime. In order to have a sufficiently long and stable period of inflation we have to impose some conditions on the potential. These conditions are

$$\epsilon \equiv \frac{1}{2}\left(\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}\phi}\right)^2 \ll 1 , \quad \eta \equiv \frac{1}{V}\frac{\mathrm{d}^2V}{\mathrm{d}\phi^2} \ll 1 . \tag{2.24}$$

The parameters ϵ and η are called slow-roll parameters. If $\epsilon, \eta \ll 1$ then the potential energy dominates the kinetic energy and the relative change in the Hubble parameter H and the value of the field ϕ in one Hubble time are small. Indeed, using $\epsilon, \eta \ll 1$, we can see form the first equation in (2.23) that the expansion of the universe is quasi-exponential. If the Hubble friction is large, in the second equation we can neglect the $\ddot{\phi}$ term and find that $\dot{\phi} \approx -V_{,\phi}/3H$ is very small. Given that the field is "slowly rolling" down the potential, these kind of models are called slow-roll models.

Since the beginning of the eighties many different inflationary models were invented. Single-field slow-roll inflation is just the simplest example. We will not enter the details of other models, because we will not need them for most of the conclusions that we want to make. Let us just say that single-field slow-roll inflation is still a perfectly valid inflationary candidate, even with a simple inflationary potential such as $V(\phi) = m^2\phi^2/2$ [12]. After this short recap of inflation at the level of the homogenous background, we turn to the study of small perturbations in the next sections.

2.2.2 The Power Spectrum

We will begin the study of inflationary perturbations with the simplest object—the two point function. In this Section we will stick to single-field slow-roll inflationary models and follow closely the presentation in [43] (some of the original works are [84, 85, 86, 87, 88, 89]). The starting point is the action of the inflaton minimally coupled to gravity (2.21). We already studied this action at the level of the background. To study the perturbations let us first write the metric using the ADM decomposition

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + h_{ij}(\mathrm{d}x^i + N^i\mathrm{d}t)(\mathrm{d}x^j + N^j\mathrm{d}t) , \tag{2.25}$$

where N is the so-called lapse and N^i the shift function. If we use this metric, the action (2.21) becomes

$$S = \frac{1}{2}\int\sqrt{h}\Big[NR^{(3)} - 2NV(\phi) + N^{-1}(E_{ij}E^{ij} - E^2)$$
$$+ N^{-1}(\dot{\phi} - N^i\partial_i\phi)^2 - Nh^{ij}\partial_i\phi\partial_j\phi\Big] , \tag{2.26}$$

where we defined

$$E_{ij} = \frac{1}{2} \left(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i \right) . \tag{2.27}$$

So far we just rewrote the starting action. If we want to study perturbations around the homogeneous solution, we should perturb the inflaton field and the metric. As we already pointed out, due to the diffeomorphism invariance, the number of degrees of freedom that we have in general is larger than the number of physical degrees of freedom. To solve this issue, we have to fix the gauge. In inflation, the convenient choice is the so called ζ-gauge. In this gauge the inflaton is unperturbed and all perturbations are in the metric

$$\delta\phi = 0 , \quad h_{ij} = a^2 e^{2\zeta} (e^\gamma)_{ij} , \, \partial_i \gamma_{ij} = 0 , \quad \gamma_{ii} = 0 . \tag{2.28}$$

As we expect, we have three physical degrees of freedom: one scalar mode ζ and two polarisations of a transverse and traceless tensor mode γ_{ij}. After we choose the gauge, our next task is to find the action for the physical degrees of freedom ζ and γ_{ij}. In order to do that, we first have to solve the equations of motion for N and N_i. In the ADM formalism those two functions are not dynamical variables and their equations of motion are the momentum and hamiltonian constraints (we set $\delta\phi = 0$)

$$\nabla_i \left(N^{-1}(E^i_j - \delta^i_j E) \right) = 0 , \tag{2.29}$$
$$R^{(3)} - 2V(\phi) - N^{-2}(E_{ij}E^{ij} - E^2) - N^{-2}\dot{\phi}^2 = 0 . \tag{2.30}$$

These equations can be solved perturbatively, in principle up to any order. However, for the purpose of finding the quadratic action, it is enough to solve for N and N_i up to first order in perturbations. For example, the second order solution of N would multiply the hamiltonian constraint that vanishes, because it is evaluated at zeroth order and the solution obeys the equations of motion. If we write

$$N = 1 + \delta N , \quad N^i = \partial_i \psi + N^i_T , \quad \partial_i N^i_T = 0 , \tag{2.31}$$

then from equations (2.29) and (2.30) we get the following solutions [43]

$$\delta N = \frac{\dot{\zeta}}{H} , \quad N^i_T = 0 , \quad \psi = -\frac{\zeta}{a^2 H} + \chi , \quad \partial^2 \chi = \frac{\dot{\phi}^2}{2H^2}\dot{\zeta} . \tag{2.32}$$

Now it is straightforward to find the quadratic actions for ζ and γ_{ij}. We have to replace these solutions into eq. (2.26) and expand the action up to second order in perturbations. After some integrations by parts, the quadratic action for ζ takes the following simple form [43]

$$S^{(2)}_\zeta = \frac{1}{2} \int \mathrm{d}t \mathrm{d}^3\vec{x}\, a^3 \frac{\dot{\phi}^2}{H^2} \left(\dot{\zeta}^2 - \frac{1}{a^2}(\partial_i\zeta)^2 \right) . \tag{2.33}$$

Similarly, the quadratic action for tensor modes is given by

$$S^{(2)}_\gamma = \frac{1}{8} \int \mathrm{d}t \mathrm{d}^3\vec{x}\, a^3 \left(\dot{\gamma}_{ij}\dot{\gamma}_{ij} - \frac{1}{a^2}(\partial_k\gamma_{ij})^2 \right) . \tag{2.34}$$

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So far we have been treating small perturbations ζ and γ_{ij} as classical. As in flat space, we can quantize these fields. In order to do so we first have to define canonically normalized fields and find the solutions of the classical equations of motion. For example, for ζ, the canonically normalized field is $\zeta^c = \frac{\dot{\phi}}{H}\zeta$. The equation of motion that it satisfies can be easily obtained from the action (2.33). If we use conformal time, in momentum space it reads

$$\zeta_{\vec{k}}^{c\prime\prime}(\eta) + 2\mathcal{H}\zeta_{\vec{k}}^{c\prime}(\eta) + \vec{k}^2\zeta_{\vec{k}}^{c}(\eta) = 0 \,. \tag{2.35}$$

Using that the inflationary background is quasi de-Sitter we can write $\mathcal{H} = -1/\eta$. Small deviations are captured by the time dependence of the factor $\frac{\dot{\phi}}{H}$ which is proportional to the slow-roll parameters. The general solution of this equation of motion will contain two modes, described by Hankel functions.

Given a classical solution ζ^c, we can write a quantum field as

$$\hat{\zeta}_{\vec{k}}^{c}(t) = \zeta_{\vec{k}}^{c}(t)\hat{a}_{\vec{k}}^{\dagger} + \zeta_{\vec{k}}^{c*}(t)\hat{a}_{-\vec{k}} \,, \tag{2.36}$$

where $\hat{a}_{\vec{k}}$ and $\hat{a}_{\vec{k}}^{\dagger}$ are annihilation and creation operators. In order to fix the vacuum we have to specify additional boundary conditions for the modes. For example, when the modes are deep inside the horizon, $k\eta \gg 1$, we can neglect the curvature effects and choose the standard Minkowski vacuum. This imposes the conditions on the solution of eq. (2.35) bringing it to the form

$$\zeta_{\vec{k}}^{c}(\eta) = \frac{H}{\sqrt{2k^3}}(1 - ik\eta)e^{ik\eta} \,. \tag{2.37}$$

This form of the modes is compatible with the choice of the standard Bunch-Davies vacuum in de Sitter space [43].

With all this in mind, we can finally write the two-point function for scalar perturbations ζ

$$\langle \zeta_{\vec{k}_1}\zeta_{\vec{k}_2} \rangle = (2\pi)^3\delta(\vec{k}_1 + \vec{k}_2)\frac{H^2}{\dot{\phi}^2}|\zeta_{\vec{k}_1}^{c}(\eta)|^2 \,. \tag{2.38}$$

In the late time limit the two-point function becomes

$$\langle \zeta_{\vec{k}_1}\zeta_{\vec{k}_2} \rangle = (2\pi)^3\delta(\vec{k}_1 + \vec{k}_2)P(k_1) \,, \tag{2.39}$$

where the power-spectrum $P(k)$ is given by

$$P(k) = \frac{H^4}{\dot{\phi}^2}\frac{1}{2k^3} = \frac{H^2}{2\epsilon M_{\rm Pl}^2}\frac{1}{2k^3} \,, \tag{2.40}$$

where in the last equality we explicitly reintroduced Planck mass. This is the result that we were looking for. A few comments are in order here. First, notice that the modes outside the horizon freeze and they do not evolve in time. This is the main advantage of ζ-gauge. In this gauge any correlation function of ζ will remain frozen once all modes are outside the horizon[3].

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[3]This can be proven not only on the tree-level but including loops to all orders in perturbation theory, see [91, 92, 93, 94, 95].

Moreover, ζ directly tells us how much the universe has expanded in addition to the classical expansion. This is easy to see form the form of the line element in ζ-gauge $\mathrm{d}\vec{l}^2 = a^2 e^{2\zeta}\mathrm{d}\vec{x}^2$, or just by calculating the spatial curvature of the universe that reads $R^{(3)} = 4\partial^2\zeta$. The other important point to stress is that the quantities H and $\dot{\phi}$ have to be evaluated once the mode with a given $\vec{k}$ crosses the horizon, $k \approx aH$. Given that the inflation potential changes during the slow-roll phase, we expect some small k dependence of the power spectrum (on top of the scale-invariant part k^{-3}). This small scale dependence can be parametrized like $P(k) \sim k^{-3+(n_s-1)}$, and we can explicitly calculate the tilt $n_s - 1$ of the power spectrum. Using $k \approx aH$ we get

$$n_s - 1 = k\frac{\mathrm{d}}{\mathrm{d}k}\log\frac{H^4}{\dot{\phi}^2} = \frac{1}{H}\frac{\mathrm{d}}{\mathrm{d}t}\log\frac{H^4}{\dot{\phi}^2} = 2\eta - 6\epsilon \ . \tag{2.41}$$

In the large-field models we discussed in the previous section, $\eta \sim \epsilon$ and we see that we expect to find a small negative tilt of the power spectrum. Indeed, this what we see in the observations. The current best measured value of the tilt is $n_s = 0.9603 \pm 0.0073$ [12], in agreement with the prediction of the simplest inflationary models.

We can repeat the same procedure and find the power spectrum for tensor modes. In this case, the expansion in Fourier modes is slightly more complicated because different polarisation states of gravitons have to be taken into account properly. We have

$$\gamma_{ij}(\vec{x}, t) = \int\frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\sum_{s=\pm}\epsilon^s_{ij}(\vec{k})\gamma^s_{\vec{k}}(t)e^{i\vec{k}\cdot\vec{x}} \ , \tag{2.42}$$

where $\epsilon^s_{ii}(\vec{k}) = 0$, $k^i\epsilon^s_{ij}(\vec{k}) = 0$ and $\epsilon^s_{ij}(\vec{k})\epsilon^{s'}_{ij}(\vec{k}) = 2\delta_{ss'}$. The two point function is given by [43]

$$\langle\gamma^s_{\vec{k}_1}\gamma^{s'}_{\vec{k}_2}\rangle = (2\pi)^3\delta(\vec{k}_1 + \vec{k}_2)P_\gamma(k_1)\delta_{ss'} \ , \tag{2.43}$$

where the power spectrum is

$$P_\gamma(k) = \frac{8H^2}{M^2_{\mathrm{Pl}}}\frac{1}{2k^3} \ . \tag{2.44}$$

Notice that in this expression the only variable is the Hubble scale during inflation H. Therefore, the detection of primordial gravitational waves would tell us about the energy scale at which inflation happens. If we compare the tensor and the scalar power spectrum we have

$$r \equiv \frac{P_\gamma(k)}{P(k)} = 16\epsilon. \tag{2.45}$$

This is the famous tensor-to-scalar ratio r. For large-field inflationary models, the tensor-to-scalar ration is of order $\mathcal{O}(0.1)$ which is close to the current upper bounds [12]. Recently the BICEP collaboration announced the first detection of CMB polarisation on a degree angular scales [18], that if proven to be of cosmological origin[4], would indicate that r is indeed $\mathcal{O}(0.1)$.

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[4]Similar effect can come form polarisation of galactic dust. The amplitude of this signal is currently unknown, but it might be compatible with what BICEP detected [19, 20].

In the same way as for scalars, we can also calculate the tilt of the power spectrum for tensors. It is simply given by

$$n_t = k\frac{\mathrm{d}}{\mathrm{d}k}\log\frac{H^2}{M_{\mathrm{Pl}}^2} = -2\epsilon \ . \tag{2.46}$$

Notice that in four observables related to the power spectra of scalars and tensors we have only three parameters: ϵ, η and H. In principle, this allows for a consistency check of the theory. In terms of observables we have

$$\frac{P_\gamma(k)}{P(k)} = -8n_t \ . \tag{2.47}$$

If the tensor-to-scalar ratio turns out to be large enough, one day we might be able to test this relation in observations.

2.2.3 Non-Gaussianities

In this section we turn to higher order correlation functions. They are very important because they carry information about interactions of the inflaton. Given that inflationary potential is expected to be very flat, the interactions are expected to be very small. Therefore, one can say that in inflation the generic prediction is that initial curvature perturbation field is highly Gaussian, which is indeed what we observe in the CMB. However, non-Gaussianities are a very important tool in constraining inflationary models because even the absence of any detection and increasingly tighter upper bounds can still tell us a lot about the nature of inflation.

In order to calculate higher order correlation functions one has to find the action to higher order in perturbations ζ or γ. This is not a straightforward task. To illustrate the procedure, in this Section we will focus only on the cubic action for ζ and the corresponding three-point function. For the three-point functions involving gravitons and higher order correlation functions in slow-roll single-field inflation and some other inflationary models, see Appendix A.

In order to find the cubic action for ζ we have to expand the action (2.26) up to third order. Luckily, it turns out that in order to do this we do not have to solve for N and N_i neither at second nor at third order in perturbations. As before, the third order term would multiply the momentum and hamiltonian constraints evaluated at zeroth order on the solution of the equations of motion. The second order term would multiply constraints evaluated using eq. (2.32), but it turns out that this contribution is zero too. In conclusion, the first order solution (2.32) is all we need.

With this in mind the expansion of the action is (2.26) is straightforward. The final result

is given by [43]

$$S^{(3)} = \int \mathrm{d}^3 x \mathrm{d}t \left[ae^\zeta \left(1 + \frac{\dot\zeta}{H}\right)\left(-2\partial^2\zeta - (\partial\zeta)^2\right) + \epsilon a^3 e^{3\zeta}\dot\zeta^2\left(1 - \frac{\dot\zeta}{H}\right) + \right.$$
$$\left. + a^3 e^{3\zeta}\left(\frac{1}{2}\left(\partial_i\partial_j\psi\partial_i\partial_j\psi - (\partial^2\psi)^2\right)\left(1 - \frac{\dot\zeta}{H}\right) - 2\partial_i\psi\partial_i\zeta\partial^2\psi\right)\right] , \tag{2.48}$$

where ψ is given by eq. (2.32).

Using field redefinitions this action can be further simplified, which makes the calculation of the three-point function much easier. We will not repeat this procedure here, because it is not essential to describe the main steps of the calculation (for details see [43]). The most important difference compared to the standard QFT calculation is that we are not interested in scattering amplitudes but correlation functions. Therefore, in the interaction picture, the expectation value of an operator $\mathcal{O}$ is given by [43, 96]

$$\langle\mathcal{O}\rangle = \langle 0|\bar{T}e^{i\int_{-\infty(1+i\epsilon)}^t H_{int}\mathrm{d}t'}\, \mathcal{O}\, Te^{-i\int_{-\infty(1-i\epsilon)}^t H_{int}\mathrm{d}t'}|0\rangle , \tag{2.49}$$

where H_{int} is an interaction Hamiltonian, instead of what we usually compute in QFT

$$\langle\mathcal{O}\rangle = \langle 0|T\, \mathcal{O}\, e^{-i\int_{-\infty(1-i\epsilon)}^{+\infty(1+i\epsilon)} H_{int}\mathrm{d}t'}|0\rangle . \tag{2.50}$$

This formalism is known as "in-in" formalism, because we are not calculating scattering amplitudes (that correspond to "in-out"). Indeed, we evolve the vacuum using the evolution operator until some moment of time t, insert the operator $\mathcal{O}$ and then we evolve backwards in time. Notice that in order to project to the vacuum of the theory we had to deform the lower boundary of the integral. Here we use the standard $i\epsilon$ prescription.

The expectation value in eq. (2.49) is calculated in the standard way using the expansion of ζ in creation and annihilation operators (2.36) and Wick's theorem. For example, using the action (2.48), the three-point function of ζ is at the end given by

$$\langle\zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\zeta_{\vec{k}_3}\rangle' = \frac{H^4}{4\epsilon^2 M_{\rm Pl}^4}\frac{1}{\prod(2k_i^3)}\left[(2\eta - 3\epsilon)\sum_i k_i^3 + \epsilon\sum_{i\neq j}k_i k_j^2 + \epsilon\frac{8}{k_t}\sum_{i>j}k_i^2 k_j^2\right], \tag{2.51}$$

where $k_t = k_1 + k_2 + k_3$ and prime on the correlation function means that we have removed $(2\pi)^3\delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3)$ form the expression. In the same way one can calculate the other three-point functions involving gravitons, or higher order correlation functions (see Appendix A). We are going to use some of these results in checks of the inflationary consistency relations.

So far we were focusing on single-field slow-roll inflation. However, as we said, there are many other models that deviate from this minimal scenario. One particularly interesting class of models are those in which we have derivative interactions for the inflaton. For example, we can have a Lagrangian of the form [97]

$$S = \frac{1}{2}\int \mathrm{d}^4 x\sqrt{-g}\left(M_{\rm Pl}^2 R + 2P(X, \phi)\right) , \tag{2.52}$$

where $X = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi$. In these kind of models, scalar perturbations generically propagate with speed $c_s < 1$. It turns out that the three-point function is inversely proportional to c_s^2 (see Appendix A). For this reason, in the limit $c_s \ll 1$, these models predict large non-Gaussianities and this is why they are phenomenologically very interesting.

Let us stress that there is one important difference between the two models we described. They predict very different "shapes" of non-Gaussianities [98]. Here by shape we mean the momentum dependence of the three-point function. Given that the interactions in slow-roll models are local they peak in squeezed configuration, when one of the momenta is much smaller than the others $k_1 \ll k_2, k_3$ [98]. This kind of non-Gaussianities are called local and their amplitude is parametrized by $f_{\rm NL}^{\rm loc.}$. On the other hand, for derivative interactions, the three-point function has a maximum in configuration where all momenta have similar magnitudes $k_1 \approx k_2 \approx k_3$ [99, 98]. For these reason this situation corresponds to equilateral non-Gaussianities parametrized by $f_{\rm NL}^{\rm equil.}$. The current constraint form Planck on these two parameters are [13]

$$f_{\rm NL}^{\rm loc.} = 2.7 \pm 5.8 \ , \quad f_{\rm NL}^{\rm equil.} = -42 \pm 75 \ . \tag{2.53}$$

It is important to stress that although in general non-Gaussianities can be large, local non-Gaussianities in *any* single-field model of inflation must be always very small. As we are going to see later, this is one of the main consequences of the consistency relations for inflation.

Let us close this Section saying that, although we used some particular models to introduce non-Gaussianities, there exists a general, model-independent approach to study small fluctuations of the inflaton and their correlation functions. It is based on the effective field theory approach to inflation. The Effective Field Theory of Inflation was first formulated for single-field models [25] and later generalized to multi-field inflation [26]. In the EFT of inflation one writes down all possible operators compatible with the symmetry. Different inflationary models then correspond to different parameters that enter the Lagrangian for perturbations. This description is very general and apart form models discussed so far includes even more exotic models like Ghost Inflation [100].

2.3   Perturbation Theory in the Late Universe

Let us finally turn to the late universe and see how to calculate the correlation functions for small fluctuations in density at scales small compared to the horizon. These are results that we will use for checks of the consistency relations of LSS.

In the regime when we are deep inside the horizon and at low redshift, we can take the non-relativistic limit of the perturbed Einstein equation. The description of perturbations therefore simplifies significantly. One does not have to worry about gauge symmetry anymore and, under some assumptions about the nature of dark matter, the equations that describe the evolution of perturbations become the familiar equations of a perfect pressureless fluid in an expanding universe.

These equations are nonlinear and there has been a lot of effort in the past to find approximate solutions for as wide a range of scales and redshifts as possible. The most straightforward approach is the so called standard perturbation theory (SPT) that we will shortly describe in the following sections (for a review see [101]). The improved version of SPT is known as renormalized perturbation theory (RPT) [102]. Recently the Effective Field Theory approach to perturbation theory of LSS attracted a lot of attention. The main results can be found in [29, 30, 31, 32, 33] and references inside. A similar approach in spirit, based on coarse graining of the short modes, was recently proposed in [103, 104]. All this work is very important, because an improvement in the analytical understanding of the power spectrum or bispectrum on mildly non-linear scales is crucial for a successful comparison of theory and the data and extracting cosmological parameters from future galaxy surveys.

Notice that most of the approaches mentioned above focus on dark matter only. Of course, eventually we are not interested in dark matter correlation functions but correlation functions of the number density of galaxies. The step from the underlying dark matter distribution to the galaxy distribution is highly nontrivial and it is the subject of very intense research. For example, even in the linear regime, we know that galaxies are biased tracers of dark matter. The simplest relation that holds on large scales that we can write is

$$\delta^{(g)} = b_1 \delta \; , \tag{2.54}$$

where b_1 is a parameter called bias (for the original work see [105]). When necessary, throughout the thesis, we will refer to some results about bias. In this Section we do not need any of those, because our main goal is to find the correlation functions of dark matter density contrast. We will closely follow [101] where many more details can be found.

2.3.1 Fluid Description of Dark Matter

If we just focus on dark matter and neglect baryons, then the full description of the evolution is given by Boltzmann equation for a collisionless fluid

$$\frac{\partial f}{\partial t} + \frac{\vec{p}}{ma} \cdot \vec{\nabla} f - am\vec{\nabla}\phi \cdot \frac{\partial f}{\partial \vec{p}} = 0 \; , \tag{2.55}$$

where $f(t, \vec{x}, \vec{p})$ is the distribution function is phase space. In order to completely specify the system we have to add Poisson's equation

$$\nabla^2 \phi = 4\pi G a^2(t) \rho \; . \tag{2.56}$$

Of course, these two equations are impossible to solve analytically. However, we can use some approximations to bring them to a more useful form. First of all, we want to describe how small fluctuations evolve around some given background. For later convenience let us use conformal time. The density contrast δ is defined like

$$\delta(\vec{x}, \eta) \equiv \frac{\rho(\vec{x}, \eta) - \bar{\rho}(\eta)}{\bar{\rho}(\eta)} \; , \tag{2.57}$$

where $\bar{\rho}(\eta)$ is the background density. Similarly, the velocity perturbation $\vec{v}$ is given by

$$\vec{v}(\vec{x}, \eta) \equiv \vec{V}(\vec{x}, \eta) - \mathcal{H}\vec{x} \ . \tag{2.58}$$

Notice that from the full velocity field $\vec{V}(\vec{x}, \eta)$ we have subtracted the Hubble flow. Finally, the perturbation of the gravitational potential $\Phi(\vec{x}, \eta)$, when the background is subtracted, is

$$\Phi(\vec{x}, \eta) \equiv \phi(\vec{x}, \eta) + \frac{1}{2}\frac{\partial \mathcal{H}}{\partial \eta}x^2 \ . \tag{2.59}$$

Using these quantities we can write down moments of the distribution $f(t, \vec{x}, \vec{p})$ and find their equations of motion that follow from eq. (2.55). In principle one gets an infinite hierarchy of equations. At that point, we can use another approximation. For the fluid of dark matter particles, at least on large scales and before virialization, we can assume that pressure and viscosity are negligible. Therefore, anisotropic stress and higher order moments can be all set to zero [101]. Under these assumptions the set of equations that govern the evolution of density and velocity perturbations simplifies to the well known continuity, Euler and Poisson equations

$$\delta' + \partial_i\left((1+\delta)v^i\right) = 0 \ , \tag{2.60}$$

$$v_i' + \mathcal{H}(\eta)v_i + v_j\partial^j v_i = -\partial_i\Phi \ , \tag{2.61}$$

$$\partial^2\Phi = \frac{3}{2}\Omega_m(\eta)\mathcal{H}^2(\eta)\delta \ . \tag{2.62}$$

Obviously, these equations are non-linear and they cannot be solved exactly. However, as long as perturbations are small, we can apply techniques of perturbation theory and find perturbative solutions. These solutions then can be used to calculate higher order correlation functions.

2.3.2 Linear Perturbation Theory

Let us first focus on linear perturbation theory. If the density contrast and velocity are small, linear perturbation theory is a very good approximation. For example, we know that this is the case for CMB physics. Even later, as long as gravitational collapse does not start to dominate the dynamics, linear perturbation theory is a useful tool to get some ideas about the evolution of density perturbations on large scales.

Given that we are interested only at first order in δ and $\vec{v}$, we can drop the higher order terms in the equations of motion. In this way we get the following simple set of equations

$$\delta' + \partial_i v^i = 0 \ , \quad v_i' + \mathcal{H}v_i = -\partial_i\Phi \ . \tag{2.63}$$

If we use the velocity divergence $\theta = \partial_i v^i$ as a variable and use the Poisson's equation (2.62), the system of equations becomes

$$\delta' + \theta = 0 \ , \quad \theta' + \mathcal{H}\theta + \frac{3}{2}\mathcal{H}^2\delta = 0 \ . \tag{2.64}$$

Using $\theta = -\delta'$ from the first equation and replacing it into the second one, we find the evolution equation for density contrast

$$\delta'' + \mathcal{H}\delta' - \frac{3}{2}\mathcal{H}^2\delta = 0 . \tag{2.65}$$

This equation will have two solutions that are associated to a growing and a decaying mode. In general, we can write

$$\delta(\vec{x}, t) = D_\delta^{(+)}(\eta)A(\vec{x}) + D_\delta^{(-)}(\eta)B(\vec{x}) , \tag{2.66}$$

where $A(\vec{x})$ and $B(\vec{x})$ are two arbitrary functions describing the initial density field and the functions $D_\delta^{(\pm)}(\eta)$ are growth(decay) functions for the density contrast. In a matter dominated universe $\mathcal{H} = 2/\eta$ and it is easy to find the time dependence of δ analytically

$$D_\delta^{(+)}(\eta) = \eta^2 , \quad D_\delta^{(-)}(\eta) = \eta^{-3} . \tag{2.67}$$

Given that the equations are linear, the solution for divergence of velocity θ can be easily found

$$\theta(\vec{x}, t) = D_\theta^{(+)}(\eta)A(\vec{x}) + D_\theta^{(-)}(\eta)B(\vec{x}) , \tag{2.68}$$

where the relation between the growth(decay) functions of the density contrast and velocity is given by

$$D_\delta^{(\pm)}(\eta) = -\int^\eta \mathrm{d}\tilde{\eta} D_\theta^{(\pm)}(\tilde{\eta}) . \tag{2.69}$$

Notice that this is a general result, although we found explicit expressions for $D_\delta^{(\pm)}(\eta)$ in a matter dominated universe. We will use these equations in Chapter 5 when we derive the consistency relations of LSS in the Newtonian limit.

2.3.3 Non-linear Perturbation Theory

In the previous Section we focused on linear perturbation theory and found solutions that are valid as long as the density contrast and velocity of perturbations are small. In this Section we will go beyond the linear regime and explain how one can use perturbation theory to find non-linear solutions.

In what follows, we will neglect the anisotropic stress and vorticity and focus on a matter dominated universe. With these simplifications the equations of motion written in momentum space are

$$\delta_{\vec{k}}' + \theta_{\vec{k}} = -\int \mathrm{d}^3\vec{k}_1 \mathrm{d}^3\vec{k}_2 \, \delta(\vec{k} - \vec{k}_1 - \vec{k}_2) \, \alpha(\vec{k}_1, \vec{k}_2) \, v_{\vec{k}_1}\delta_{\vec{k}_2} , \tag{2.70}$$

$$\theta_{\vec{k}}' + \mathcal{H}\theta_{\vec{k}} + \frac{3}{2}\mathcal{H}^2\delta_{\vec{k}} = -\int \mathrm{d}^3\vec{k}_1 \mathrm{d}^3\vec{k}_2 \, \delta(\vec{k} - \vec{k}_1 - \vec{k}_2) \, \beta(\vec{k}_1, \vec{k}_2) \, v_{\vec{k}_1} v_{\vec{k}_2} . \tag{2.71}$$

In these expressions α and β are functions of momenta defined as

$$\alpha(\vec{k}_1, \vec{k}_2) \equiv \frac{(\vec{k}_1 + \vec{k}_2) \cdot \vec{k}_1}{k_1^2} , \quad \beta(\vec{k}_1, \vec{k}_2) \equiv \frac{(\vec{k}_1 + \vec{k}_2)^2 \vec{k}_1 \cdot \vec{k}_2}{2k_1^2 k_2^2} , \tag{2.72}$$

and they encode the non-linearities of the theory. The assumption that we are going to make is that we can do a perturbative expansion of the fields δ and θ and solve for them order by order in perturbation theory. In matter dominance, taking initial conditions on the fastest growing mode, the time dependence at any order is fixed and it is given by an appropriate power of the scale factor [101]. Therefore, we can write the following ansatz

$$\delta_{\vec{k}}(\eta) = \sum_{n=1}^{\infty} a^n(\eta)\delta_{\vec{k}}^{(n)}, \quad \theta_{\vec{k}}(\eta) = -\mathcal{H}(\eta)\sum_{n=1}^{\infty} a^n(\eta)\theta_{\vec{k}}^{(n)}. \tag{2.73}$$

Of course, at linear level we recover the results given by linear perturbation theory.

Provided the linear solution, we can use it in the equations of motion to find the second order solution. Then we can repeat the same steps and iteratively find solutions at higher and higher orders. The result of this procedure can be written in the following way [101]

$$\delta_{\vec{k}}^{(n)} = \int d^3\vec{q}_1 \cdots d^3\vec{q}_n \delta(\vec{k} - \vec{q}_1 - \cdots - \vec{q}_n)\, F_n(\vec{q}_1, \ldots, \vec{q}_n)\, \delta_{\vec{q}_1}^{(1)} \cdots \delta_{\vec{q}_n}^{(1)}, \tag{2.74}$$

$$\theta_{\vec{k}}^{(n)} = \int d^3\vec{q}_1 \cdots d^3\vec{q}_n \delta(\vec{k} - \vec{q}_1 - \cdots - \vec{q}_n)\, G_n(\vec{q}_1, \ldots, \vec{q}_n)\, \delta_{\vec{q}_1}^{(1)} \cdots \delta_{\vec{q}_n}^{(1)}, \tag{2.75}$$

where the kernels F and G satisfy the following recursion relations

$$F_n(\vec{q}_1, \ldots, \vec{q}_n) = \sum_{m=1}^{n-1} \frac{G_n(\vec{q}_1, \ldots, \vec{q}_m)}{(2n+3)(n-1)}\Big[(2n+1)\alpha(\vec{k}_1, \vec{k}_2)F_{n-m}(\vec{q}_{m+1}, \ldots, \vec{q}_n)$$
$$+ 2\beta(\vec{k}_1, \vec{k}_2)G_{n-m}(\vec{q}_{m+1}, \ldots, \vec{q}_n)\Big], \tag{2.76}$$

$$G_n(\vec{q}_1, \ldots, \vec{q}_n) = \sum_{m=1}^{n-1} \frac{G_n(\vec{q}_1, \ldots, \vec{q}_m)}{(2n+3)(n-1)}\Big[3\alpha(\vec{k}_1, \vec{k}_2)F_{n-m}(\vec{q}_{m+1}, \ldots, \vec{q}_n)$$
$$+ 2n\beta(\vec{k}_1, \vec{k}_2)G_{n-m}(\vec{q}_{m+1}, \ldots, \vec{q}_n)\Big], \tag{2.77}$$

with $\vec{k}_1 = \vec{q}_1 + \cdots + \vec{q}_m$, $\vec{k}_2 = \vec{q}_{m+1} + \cdots + \vec{q}_n$ and $F_1 = G_1 = 1$. For example, for $n = 2$ one finds

$$F_2(\vec{q}_1, \vec{q}_2) = \frac{5}{7} + \frac{1}{2}\frac{\vec{q}_1 \cdot \vec{q}_2}{q_1 q_2}\left(\frac{q_1}{q_2} + \frac{q_2}{q_1}\right) + \frac{2}{7}\frac{(\vec{q}_1 \cdot \vec{q}_2)^2}{q_1^2 q_2^2}, \tag{2.78}$$

$$G_2(\vec{q}_1, \vec{q}_2) = \frac{3}{7} + \frac{1}{2}\frac{\vec{q}_1 \cdot \vec{q}_2}{q_1 q_2}\left(\frac{q_1}{q_2} + \frac{q_2}{q_1}\right) + \frac{4}{7}\frac{(\vec{q}_1 \cdot \vec{q}_2)^2}{q_1^2 q_2^2}, \tag{2.79}$$

and the second order solution for, let us say $\delta_{\vec{k}}$, reads

$$\delta_{\vec{k}}^{(2)} = \int d^3\vec{q}_1 d^3\vec{q}_2 \delta(\vec{k} - \vec{q}_1 - \vec{q}_2)\left[\frac{5}{7} + \frac{1}{2}\frac{\vec{q}_1 \cdot \vec{q}_2}{q_1 q_2}\left(\frac{q_1}{q_2} + \frac{q_2}{q_1}\right) + \frac{2}{7}\frac{(\vec{q}_1 \cdot \vec{q}_2)^2}{q_1^2 q_2^2}\right]\delta_{\vec{q}_1}^{(1)}\delta_{\vec{q}_2}^{(1)}. \tag{2.80}$$

Once the higher order solutions are known, one can use them to calculate higher order correlation functions. For example, if we are interested in the three-point function of the
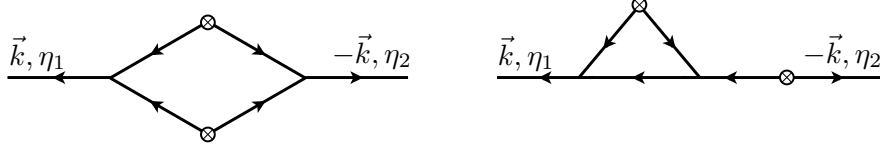
Figure 2.1: Two diagrams that contribute to the 1-loop power spectrum. Left: P_{22}. Right: P_{31}.

density contrast δ at lowest (nontrivial) order in perturbation theory, we can use $\delta^{(2)}_{\vec{k}}$ to calculate it. The contributions to the three-point function are

$$\langle \delta_{\vec{k}_1}(\eta)\delta_{\vec{k}_2}(\eta)\delta_{\vec{k}_3}(\eta)\rangle = a^3(\eta)\langle \delta^{(1)}_{\vec{k}_1}\delta^{(1)}_{\vec{k}_2}\delta^{(1)}_{\vec{k}_3}\rangle$$
$$+ a^4(\eta)\langle \delta^{(2)}_{\vec{k}_1}\delta^{(1)}_{\vec{k}_2}\delta^{(1)}_{\vec{k}_3}\rangle + a^4(\eta)\langle \delta^{(1)}_{\vec{k}_1}\delta^{(2)}_{\vec{k}_2}\delta^{(1)}_{\vec{k}_3}\rangle + a^4(\eta)\langle \delta^{(1)}_{\vec{k}_1}\delta^{(1)}_{\vec{k}_2}\delta^{(2)}_{\vec{k}_3}\rangle . \tag{2.81}$$

The first line takes into account just the linear evolution of the three-point function in the initial conditions. The second line contains non-linearities coming form the mode coupling.

In a similar way, using higher order solutions, one can calculate loop corrections to the correlation functions too. The simplest example is the one-loop power spectrum

$$\langle \delta_{\vec{k}_1}(\eta)\delta_{\vec{k}_2}(\eta)\rangle = a^2(\eta)\langle \delta^{(1)}_{\vec{k}_1}\delta^{(1)}_{\vec{k}_2}\rangle + a^4(\eta)\langle \delta^{(2)}_{\vec{k}_1}\delta^{(2)}_{\vec{k}_2}\rangle + a^4(\eta)\left(\langle \delta^{(1)}_{\vec{k}_1}\delta^{(3)}_{\vec{k}_2}\rangle + \langle \delta^{(3)}_{\vec{k}_1}\delta^{(1)}_{\vec{k}_2}\rangle\right) . \tag{2.82}$$

The first term on the right hand side just describes the linear evolution of the initial power spectrum. The second and third term are two different one-loop contributions. Typically they are denoted as P_{22} and P_{31} and they can be presented diagrammatically as in Fig. 2.1.

The explicit expressions for the three-point function or the one-loop power spectrum are quite complicated. Already the integral in eq. (2.80) is very hard to solve and, going to higher orders, expressions become even more complex. Fortunately, in most examples that we are going to use in the remaining chapters, we are going to be interested in a particular kinematical regime where some of the momenta are much smaller than the others. For these configurations, the calculations simplify significantly. The reason is that the kernels F and G become very simple if there is a separation of scales among momenta. For example, some useful identities are

$$\lim_{q_1 \ll q_2} F_2(\vec{q}_1, \vec{q}_2) = \frac{\vec{q}_1 \cdot \vec{q}_2}{2q_1^2} , \quad \lim_{q_1 \ll q_2} G_2(\vec{q}_1, \vec{q}_2) = \frac{\vec{q}_1 \cdot \vec{q}_2}{2q_1^2} . \tag{2.83}$$

Similarly, one can show that

$$\lim_{q_1, q_2 \ll q_3} F_3(\vec{q}_1, \vec{q}_2, \vec{q}_3) = \lim_{q_1, q_2 \ll q_3} G_3(\vec{q}_1, \vec{q}_2, \vec{q}_3) = \frac{(\vec{q}_1 \cdot \vec{q}_3)(\vec{q}_2 \cdot \vec{q}_3)}{6q_1^2 q_2^2} . \tag{2.84}$$

We are going to use some of these properties in Section 5.2 to check the consistency relations of LSS.

Chapter 3

Construction of Adiabatic Modes

We will start the discussion by reviewing the construction of adiabatic modes by Weinberg [44]. We will show that one can generate a long-wavelength adiabatic mode starting from an unperturbed FRW metric and just performing a change of coordinates. For this procedure to work, the details of the composition of the universe are not important. The construction of adiabatic modes is crucial for derivation of the consistency relations, because it allow us to trade modes with small wavenumbers $\vec{q} \to 0$ for a change of coordinates and in this way relate the squeezed limit of an $(n+1)$-point function to some variation of the corresponding n-point function.

The heart of Weinberg's argument is an observation that for modes in a given gauge in the limit $\vec{q} \to 0$ there is a residual gauge freedom, although for $\vec{q} \neq 0$ there is no remaining gauge symmetry. This can be used to construct solutions of the perturbed Einstein equations starting from a homogeneous background and just performing a change of coordinates. As we are going to see in what follows, this change of coordinates is not arbitrary. To get physical solutions, one has to impose that the generated perturbations satisfy the Einstein equations that vanish in the $\vec{q} \to 0$ limit [44]. Furthermore, this construction can be extended even to the case when the starting metric is already perturbed. In this way, one can always find the exact second order solution where one of the modes has a small momentum $\vec{q}$. This solution is related to another first order solution by an appropriate change of coordinates.

We will present Weinberg's construction first in Newtonian gauge which is more suitable for LSS and then in ζ-gauge which is used in inflation. Towards the end of the Chapter, we will give several checks that show that we are indeed able to generate the correct second order solutions starting from the solutions of linear perturbation theory. Most of this Chapter follows [44] and parts of [48, 67] related to the construction of the adiabatic modes including gradients, both in inflation and LSS.

3.1 Newtonian Gauge

We will first show how one can generate a long-wavelength adiabatic solution of Einstein equations working in Newtonian gauge. This gauge is particularly important because we are going to use it to derive the consistency relations of LSS and it has a nice and intuitive interpretation in the non-relativistic limit. We are first going to show how one constructs a homogenous mode starting from an unperturbed FRW metric, pointing out the key steps of Weinberg's construction. After that we are going to prove that a similar procedure is possible to generate solutions with a homogeneous gradient and generalise the construction to work also in the presence of short modes.

3.1.1 Weinberg's Construction

In order to illustrate the main steps in the construction of the adiabatic modes, we will focus on the simplest case in which we are looking at a homogeneous mode. In other words, we are interested in a solution of linear perturbation theory in the limit where we keep only the zeroth order terms in the small wavenumber limit, $\vec{q} \to 0$. Originally, the construction was done in [44].

The starting point is a universe filled with a fluid (or several fluids) with a given equation of state that we do not have to specify. On the background level, the solution of Einstein equations is given by the FRW metric

$$\mathrm{d}s^2 = a^2(\eta) \left(-\mathrm{d}\eta^2 + \mathrm{d}\vec{x}^2 \right) , \tag{3.1}$$

where the scale factor $a(\eta)$ depends on the composition of the universe. Apart from the background solution, we can also take a look at the perturbed Einstein equations and see how small perturbations around the given background evolve. In general, this is a complicated task because it is hard to solve the perturbation equations (2.17) to (2.20) analytically. However, there is a way to show that in the long-wavelength limit (or $\vec{q} \to 0$), the solution can be obtained just starting from the unperturbed FRW metric and doing a change of coordinates. Let us take a look at this procedure in more details.

In order to find the change of coordinates that generates perturbations in the metric extendable to physical solutions, we have to demand that the coordinate transformation leaves the metric in Newtonian gauge. In the exact $\vec{q} = 0$ limit the solution must be $\vec{x}$-independent. This means that the change of time must be x-independent and the change of $\vec{x}$ can be at most linear in $\vec{x}$. It is easy to check that the most general coordinate transformation that satisfies all these conditions is a time-dependent shift of conformal time and a constant rescaling of spatial coordinates

$$\eta \to \eta + \epsilon(\eta) , \quad \vec{x} \to \vec{x} + \lambda\vec{x} . \tag{3.2}$$

Any other space or time dependence of the parameters ϵ and λ would either generate a non-homogeneous solution or violate some of the gauge conditions. For example, a time

dependent rescaling of coordinates would generate a space-time component of the metric $g_{0i} = a^2(\eta)\lambda'(\eta)x_i$, which in Newtonian gauge must be zero.

Let us take a look at the result of the coordinate transformation (3.2). The metric (3.1) becomes

$$ds^2 = a^2(\eta)\left[-d\eta^2\left(1 + 2\epsilon' + 2\mathcal{H}\epsilon\right) + d\vec{x}^2\left(1 + 2\lambda + 2\mathcal{H}\epsilon\right)\right] . \qquad (3.3)$$

Thus, starting from an unperturbed FRW, we generate a metric that looks like a perturbed metric in Newtonian gauge with a homogeneous mode of the form

$$\Phi = \epsilon' + \mathcal{H}\epsilon , \qquad \Psi = -\lambda - \mathcal{H}\epsilon . \qquad (3.4)$$

Similarly, the background energy density and pressure perturbations are given by

$$\delta\rho = \bar{\rho}'\epsilon , \quad \delta p = \bar{p}'\epsilon . \qquad (3.5)$$

There is nothing special in what we did so far. The only thing we did is to rewrite the FRW metric in other coordinates, and of course, for arbitrary parameters ϵ and λ, the potentials in (3.4) are just gauge modes[1]. The key step in Weinberg's construction is to show that the potentials (3.4) can be smoothly extended to a physical mode for a particular choice of ϵ and λ. Let us see how this can be achieved.

If we want the gauge mode we generated to be a $\vec{q} \to 0$ limit of a physical solution of the perturbed Einstein equations, we have to make sure that for a small but finite q the generated mode satisfies the Einstein equations. Following Weinberg [44] we look for the Einstein equations that vanish in the $\vec{q} \to 0$ limit. These are eq. (2.18) and eq. (2.20) which we can write like

$$(\mathcal{H}' - \mathcal{H}^2)v = (\Psi' + \mathcal{H}\Phi) , \qquad (3.6)$$

$$\Phi = \Psi - 8\pi G\delta\sigma . \qquad (3.7)$$

As before, v is the velocity potential and $\delta\sigma$ is the anisotropic stress, defined by writing the spatial part of the stress-energy tensor as $T_{ij} = g_{ij}p + \partial_i\partial_j\delta\sigma$. From the second equation we can immediately see the relation between the physical mode and the parameters ϵ and λ. From eq. (3.4) and eq. (3.7) it is straightforward to get

$$\epsilon' + 2\mathcal{H}\epsilon = -\lambda - 8\pi G\delta\sigma . \qquad (3.8)$$

The solution to this equation is

$$\epsilon = -\frac{1}{a^2}\int^{\eta} a^2(\tilde{\eta})\left(\lambda + 8\pi G\delta\sigma(\tilde{\eta})\right)d\tilde{\eta} . \qquad (3.9)$$

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[1]Notice that the perturbations in the metric are homogeneous and that they do not fall off at spatial infinity. This is because we have performed the so called large diffeomorphism. Still, as we will see later, there are no issues with the boundary conditions, because these modes will be smoothly connected to the real physical solutions that have a regular behaviour at infinity.

In order to find the relation between the adiabatic mode and the parameters of the coordinate transformation, let us for simplicity (and without loss of generality) neglect the anisotropic stress[2]. Under this assumption we find

$$\Phi_{\vec{q}\to 0} = \Psi_{\vec{q}\to 0} = -\lambda \left(D_v' + \mathcal{H}D_v \right) \,, \tag{3.10}$$

where $D_v = \frac{1}{a^2} \int^\eta a^2(\tilde{\eta}) \mathrm{d}\tilde{\eta}$.

Let us now focus on the other Einstein equation (3.6). In the absence of anisotropic stress we find that

$$v = -\epsilon \,, \tag{3.11}$$

and that the comoving curvature perturbation $\mathcal{R} \equiv -\Psi + \mathcal{H}v$ becomes simply

$$\mathcal{R} = \lambda \,. \tag{3.12}$$

Given that we can combine these expressions to get

$$v = -\epsilon = \lambda \frac{1}{a^2} \int^\eta a^2(\tilde{\eta}) \mathrm{d}\tilde{\eta} = D_v \cdot \mathcal{R} \,, \tag{3.13}$$

we can see that D_v is nothing but the growth function of the velocity. As a particular example, we can take a look at a matter dominated universe. In this example $a \sim \eta^2$ and therefore $\mathcal{H} = 2/\eta$ and $D_v(\eta) = \eta/5$. If we use this in eq. (3.10) we recover a well known result[3]

$$\Phi = \Psi = -\frac{3}{5}\zeta \,, \tag{3.14}$$

which holds in a matter dominated universe without the anisotropic stress.

In conclusion, we were able to show that starting from an unperturbed FRW metric and performing a change of coordinates, one can find a solution of the perturbed Einstein equations in the limit when the wavenumber of the modes goes to zero. Notice that in order to do so we never had to specify any equations of state for the fluids that fill the universe or any other details that are typically needed in order to find the equations that govern the evolution of perturbations. For this reason the construction we described is very general.

So far we were working only with homogeneous modes. In momentum space, this means that the construction we described is valid at zeroth order in q. In the next Section, we are going to show that it is possible to extend Weinberg's construction to capture the homogeneous gradients as well (in other words, terms of order $\mathcal{O}(q)$).

3.1.2 Homogeneous Gradients

The construction of adiabatic modes from the previous Section can be extended to include first order in gradient expansion (or small q expansion). To show that a physical solution

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[2]This is anyhow a very good approximation in our universe.
[3]Up to order $\mathcal{O}(q^2)$ we have $\zeta = \mathcal{R}$.

at this order can be obtained just doing the change of coordinates, we can follow the same steps as before, with a difference that now we have to allow for a more general coordinate transformation. Starting from an FRW metric, we want to generate a perturbed metric in Newtonian gauge and show that it can be smoothly connected to physical solutions of the Einstein equations.

Let us first focus on the spatial part of the metric. We are looking for a change of coordinates that can generate an $\vec{x}$-dependent potential and leave the form of the spatial line element $\mathrm{d}\vec{x}^2$ unchanged. The most general transformations of this kind belong to the conformal group. Indeed, the change of spatial ordinates in the previous Section was just a dilation written in infinitesimal form. If we want to induce an x-dependent conformal factor, we have to use a special conformal transformation. In general, putting dilation and special conformal transformation together, the change of coordinates is

$$x^i \ \rightarrow \ x^i + \lambda x^i + 2(\vec{x} \cdot \vec{b})x^i - b^i \vec{x}^2 \ . \tag{3.15}$$

In principle, λ and $\vec{b}$ can be time dependent parameters, but it is easy to check that in order to satisfy other gauge conditions they must be constant. We already saw this for λ in the previous Section and a similar argument holds for $\vec{b}$ too. After performing the change of coordinates (3.15), the spatial line element becomes

$$\mathrm{d}\vec{x}^2 \ \rightarrow \ (1 + 2\lambda + 4\vec{b} \cdot \vec{x})\mathrm{d}\vec{x}^2 \ . \tag{3.16}$$

Notice that this is exactly what we wanted to get, because $\mathrm{d}\vec{x}^2$ is multiplied by a function linear in $\vec{x}$.

To be consistent we also have to generate a linear function of $\vec{x}$ in the g_{00} part of the metric. In order to do that we need a change of conformal time which is linear in $\vec{x}$ (in addition to a homogeneous shift we discussed before). In general

$$\eta \ \rightarrow \ \eta + \epsilon(\eta) + \vec{x} \cdot \vec{\xi}(\eta) \ , \tag{3.17}$$

which will transform $\mathrm{d}\eta^2$ in the following way

$$\mathrm{d}\eta^2 \ \rightarrow \ (1 + 2\epsilon'(\eta) + 2\vec{x} \cdot \vec{\xi}'(\eta))\mathrm{d}\eta^2 \ . \tag{3.18}$$

Again, we get a linear profile in front of $\mathrm{d}\eta^2$, as we wanted.

The only difficulty with the change of coordinates we found so far is that they induce the g_{0i} component of the metric as well. On the other hand, in Newtonian gauge, we have to demand $g_{0i} = 0$. To keep this gauge condition satisfied, the only choice we have is to introduce an additional time-dependent translation of the spatial coordinates. Notice that this induces only a space-time term in the metric and we can adjust it to insure that $g_{0i} = 0$. Using this condition, it is easy to find what this additional time translation has to be. Putting everything together, we get

$$\eta \ \rightarrow \ \eta + \epsilon(\eta) + \vec{x} \cdot \vec{\xi}(\eta) \ , \tag{3.19}$$

$$x^i \ \rightarrow \ x^i + \lambda x^i + 2(\vec{x} \cdot \vec{b})x^i - b^i \vec{x}^2 + \int^{\eta} \xi^i(\tilde{\eta})\mathrm{d}\tilde{\eta} \ . \tag{3.20}$$

Finally, we are ready to see how the full line element transforms. Taking into account that the scale factor will change because of the shift in conformal time and starting from an unperturbed FRW, after performing the change of coordinates (3.19) and (3.20), we can easily read off the long-wavelength mode we generate

$$\Phi = \epsilon' + \vec{x} \cdot \vec{\xi}' + \mathcal{H}(\epsilon + \vec{x} \cdot \vec{\xi}) , \qquad \Psi = -\lambda - 2\vec{x} \cdot \vec{b} - \mathcal{H}(\epsilon + \vec{x} \cdot \vec{\xi}) . \tag{3.21}$$

As before, to ensure that this mode is the long-wavelength limit of a physical solution, we have to consider the Einstein's equations that vanish in the limit $q \to 0$

$$(\mathcal{H}' - \mathcal{H}^2)v = (\Psi' + \mathcal{H}\Phi) , \tag{3.22}$$

$$\Phi = \Psi - 8\pi G \delta\sigma . \tag{3.23}$$

Using the equations for the potentials (3.21) and the first condition (3.22) one finds that the velocity potential is now given by

$$v = -(\epsilon + \vec{x} \cdot \vec{\xi}) , \tag{3.24}$$

and that the comoving curvature perturbation corresponds to

$$\mathcal{R} = \lambda + 2\vec{x} \cdot \vec{b} . \tag{3.25}$$

On the other hand, if we replace expressions (3.21) into eq. (3.23), we get equations that the functions $\epsilon(\eta)$ and $\vec{\xi}(\eta)$ have to satisfy

$$\epsilon' + 2\mathcal{H}\epsilon = -\lambda - 8\pi G \delta\sigma , \tag{3.26}$$

$$\vec{\xi}' + 2\mathcal{H}\vec{\xi} = -2\vec{b} - 8\pi G \vec{\nabla}\delta\sigma . \tag{3.27}$$

The solutions of these equations give us expressions for ϵ and $\vec{\xi}$ in terms of λ, $\vec{b}$ and $\delta\sigma$,

$$\epsilon = -\frac{1}{a^2}\int a^2(\lambda + 8\pi G \delta\sigma)\mathrm{d}\tilde{\eta} , \qquad \vec{\xi} = -\frac{1}{a^2}\int a^2(2\vec{b} + 8\pi G \vec{\nabla}\delta\sigma)\mathrm{d}\tilde{\eta} . \tag{3.28}$$

We can again, for simplicity, neglect the anisotropic stress. In this case the relation between Φ and ζ is

$$\Phi = \Psi = -(\lambda + 2\vec{b} \cdot \vec{x})(D_v' + \mathcal{H}D_v) = -\zeta(D_v' + \mathcal{H}D_v) . \tag{3.29}$$

As before, $D_v = \frac{1}{a^2}\int^\eta a^2\mathrm{d}\tilde{\eta}$ is the growth function of velocity and in the case of matter dominance we recover the simple relation between Φ and ζ. One important difference is that now it holds even including the first order in gradient expansion.

Let us at the end stress one important interpretation of the result we obtained. Taking the divergence of eqs. (3.24) and (3.25) one obtains, respectively,

$$\vec{\xi} = -\vec{v} \equiv -\vec{\nabla}v , \qquad \vec{b} = \frac{1}{2}\vec{\nabla}\zeta . \tag{3.30}$$

Therefore, one can check using eq. (3.23) and the definition of the comoving curvature perturbation that, in the absence of the anisotropic stress, eqs. (3.26) and (3.27) respectively reduce to the Euler equation for the bulk velocity potential $v_L = -\epsilon$ and bulk velocity $\vec{v}_L = -\vec{\xi}$,

$$v'_L + \mathcal{H}v_L = -\Phi_L , \qquad \vec{v}'_L + \mathcal{H}\vec{v}_L = -\vec{\nabla}\Phi_L . \tag{3.31}$$

This tells us that the coordinate transformation (3.20) corresponds to going into a free-falling elevator and the last term on its right hand side is simply the displacement field due to the large-scale velocity $\vec{\xi} = -\vec{\nabla}v$ [67]. We will come back to this interpretation when we will be talking about consistency relations of LSS and their relation with the equivalence principle.

3.1.3 Including Short Modes

So far we have we have discussed the procedure for generating physical modes in the limit $\vec{q} \to 0$ starting from the unperturbed FRW metric. In this Section we will show that a similar construction (up to some small technical details) can be also done in the more interesting and realistic case when the starting metric already contains some perturbations [67]. This will be the main result that we will need in order to derive the consistency relations of LSS.

Let us start from the first order metric in Newtonian gauge

$$\mathrm{d}s^2 = a^2(\eta)\left[-(1+2\Phi_S)\mathrm{d}\eta^2 + (1-2\Psi_S)\mathrm{d}\vec{x}^2\right] . \tag{3.32}$$

We will refer to these perturbations as "short". This is to make a difference with respect to the mode that we want to induce doing a change of coordinates that from now on will be referred as "long". The origin of this terminology is related to the fact that in momentum space we are always looking at configurations where momenta corresponding to the short modes $\vec{k}$ are much larger than the momentum of the long mode $\vec{q}$, $k \gg q$. The physical interpretation of this limit is that Φ_S and Ψ_S are metric perturbations in a region of space of some size R while the long mode has a wavelength much bigger than R. It therefore locally looks like a smooth background, and we can do the q/k expansion.

In writing the metric (3.32) we kept only first order terms in potentials. This is a good approximation because the gravitational potential is always small. This is true even in the late universe, although the density contrast δ can become large due to the gravitational collapse that leads to structure formation. Indeed, we allow for a general time and space dependence of short modes to allow for this possibility.

As before, we want to find a coordinate transformation that generates a long adiabatic mode starting from the metric above. We expect this change of coordinates to be similar to eq. (3.19) and eq. (3.20). However, the presence of the short modes introduces additional complications that are absent in the case of the homogeneous background. The g_{ij} and g_{00} components of the metric behave just as before, with additional transformation of the short modes. On the other hand, now we also have g_{0i} part of the metric[4] and we have to make

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[4]This part of the metric is zero in Newtonian gauge, in linear perturbation theory. However, in general, at higher orders in perturbation theory we expect it to be different from zero.

sure that it satisfies all gauge conditions. The Newtonian gauge generalises to Poisson gauge [106] and the g_{0i} has to satisfy $\partial_i g_{0i} = 0$. It is straightforward to check that the coordinate transformation (3.19) and (3.20) applied to the metric (3.32) generates

$$g_{0i} = -2a^2 \big(\Phi_S(\vec{x}, \eta) + \Psi_S(\vec{x}, \eta) \big) \xi_i(\eta) \ . \tag{3.33}$$

Obviously, $\partial_i g_{0i} = 0$ cannot be satisfied for generic Φ_S and Ψ_S.

In order to fix this issue, we have to extend the change of coordinates. It is enough to modify the transformation for η in the following way

$$\tilde{\eta} = \eta + \epsilon(\eta) + \vec{x} \cdot \vec{\xi}(\eta) + \alpha(\vec{x}, \eta) \ , \tag{3.34}$$

where α is of order Φ_S. The g_{0i} in this case becomes

$$g_{0i} = -a^2 \partial_i \alpha - 2a^2 \big(\Phi_S(\vec{x}, \eta) + \Psi_S(\vec{x}, \eta) \big) \xi_i(\eta) \ . \tag{3.35}$$

and the gauge condition $\partial_i g_{0i} = 0$ boils down to

$$\nabla^2 \alpha = -2 \partial_i \big[(\Phi_S(\vec{x}, \eta) + \Psi_S(\vec{x}, \eta)) \xi^i(\eta) \big] \ , \tag{3.36}$$

which can be always be satisfied with a suitable choice of $\alpha(\vec{x}, \eta)$. Therefore, the final change of coordinates is given by

$$\tilde{\eta} = \eta + \epsilon(\eta) + \vec{x} \cdot \vec{\xi}(\eta) + \alpha(\vec{x}, \eta) \ , \tag{3.37}$$

$$\tilde{x}^i = x^i + \lambda x^i + 2(\vec{x} \cdot \vec{b}) x^i - b^i \vec{x}^2 + \int^{\eta} \xi^i(\tilde{\eta}) \mathrm{d}\tilde{\eta} \ . \tag{3.38}$$

In the same way as before, the parameters of the transformation can be related to the curvature perturbation in the initial conditions.

There are a couple of things worth stressing here. The additional change of coordinates does not change anything we said in the previous Section. The reason is that α is proportional to Φ_S and any corrections but the one that appear in g_{0i} metric elements would have been second order in Φ_S. Notice also that in order to generate the long mode we have to know the configuration of the short modes. As we are going to see later this is not a problem, and if one is careful enough, one can still use this change of coordinates to derive the consistency relations of LSS.

It is very important to point out that the long mode leaves the small scale dynamics unaffected—so that its effect can always be removed by a suitable coordinate redefinition—only as long as it is much larger than the *sound* horizon. Indeed, in a Hubble time interactions cannot propagate to scales longer than the sound horizon scale. Therefore, the construction of the adiabatic modes holds as long as the long wavelength mode q satisfies $c_s q/(aH) \ll 1$, where c_s is the sound speed of fluctuations. In the case of dark matter the speed of sound is almost zero and our construction holds inside the Hubble radius. However, one has to keep in mind that we have to be able to remove the long mode from the *full* solution of the Einstein

equations, which involves time evolution as well. In other words, the long mode has to be outside the sound horizon during the entire history of the universe. We will comment more on this condition later when we derive the consistency relations of LSS.

Although we are are going to check the construction of the long mode using perturbation theory, let us stress that it works *non-perturbatively* in the short-scale modes: whatever the short-scale dynamics is, we can generate a new solution that includes a long mode by a coordinate transformation. This statement holds also when baryons are taken into account. Of course, as we already pointed out, we always assume that the metric perturbations are small, which is a very good approximation for all practical purposes.

3.2 ζ-Gauge

So far we have been discussing the construction adiabatic modes in Newtonian gauge. As we said, this gauge is useful for describing the perturbations in the late universe. However, one can also choose to work in some other gauge. For our purposes, the most interesting choice is ζ-gauge, that we already met in Section 2.2. This gauge is used in inflationary calculations and here we want to briefly repeat some steps of Weinberg's construction because we are going to need these results to derive inflationary consistency relations.

3.2.1 Scalar Modes

In this Section we want to show how to construct an adiabatic solution in ζ-gauge, which is correct not only in the homogeneous limit, but also including corrections linear in the gradients. An adiabatic mode is locally a coordinate redefinition of the unperturbed solution. We thus parallel what we did in Newtonian gauge and look for the most general transformation of the unperturbed FRW metric that leaves the metric in ζ-gauge. The inflaton is unperturbed in this gauge, the slicing is fixed and we cannot redefine the time variable. The only possibility is to change the spatial coordinates.

The spatial line element in ζ-gauge, in a flat universe and neglecting tensor modes, is given by

$$\mathrm{d}\vec{l}^2 = a^2(t)e^{2\zeta_S(x)}\mathrm{d}\vec{x}^2 \ . \tag{3.39}$$

Notice that we include short scalar modes from the beginning. It turns out that the change of coordinates we need to generate the long mode is the same as for the unperturbed FRW metric, and therefore we will immediately discuss this more general case. Since for the time being we are not interested in long tensor modes (we will consider them in the next Section), we want our change of coordinates to induce only a conformal rescaling of the spatial metric, in this way "exciting" only ζ_L. As we already said, the most general transformation that we can do is a dilation and a special conformal transformation of the spatial coordinates, in principle a different one at each time t

$$x^i \ \rightarrow \ x^i + \lambda(t)x^i + 2(\vec{b}(t) \cdot \vec{x})x^i - \vec{x}^2 b^i(t) \ . \tag{3.40}$$

If we perform this change of coordinates, the spatial line element becomes

$$\mathrm{d}\vec{l}^2 = a^2(t)e^{2\zeta_S(x)+2\lambda(t)+4\vec{b}(t)\cdot\vec{x}}\mathrm{d}\vec{x}^2 \ . \tag{3.41}$$

Obviously, the transformation (3.40) will turn on ζ_L, that will be given at each t by a constant piece and a term linear in $\vec{x}$

$$\zeta_L = \lambda(t) + 2\vec{b}(t)\cdot\vec{x} \ . \tag{3.42}$$

Of course, in general, this is just a gauge mode. As before, we have to check what are the additional constraints coming from the requirement that the solutions we found are the $q \to 0$ limit of physical solutions [44]. For standard slow-roll inflation the linear momentum constraint reads (see Section 2.2)

$$\partial_j(H\delta N - \dot{\zeta}) = 0 \ , \tag{3.43}$$

while the Hamiltonian constraint reads

$$(3H^2 + \dot{H})\delta N + H\partial_i N^i = -\frac{\nabla^2}{a^2}\zeta + 3H\dot{\zeta} \ . \tag{3.44}$$

The coordinate transformations we discussed do not induce δN (at linear order). Therefore, from the first equation, we have to require $\dot{\zeta} = 0$ up to order q^2 if we want our coordinate transformation to describe an adiabatic solution at order q. This implies that $\vec{b}$ and λ must be taken as time-independent

$$\zeta = 2\vec{b}\cdot\vec{x} + \lambda \ . \tag{3.45}$$

We expected this to be the case, because the long modes in single-field inflation do not evolve in time (up to order q^2) once they are outside the horizon.

Now we can focus on the Hamiltonian constraint. It implies that N^i is of order q. Its spatial dependence would be of order q^2 and can thus be neglected within our approximation. In other words, $N^i(t)$ just depends on time. We can always generate this $N^i(t)$ by a time-dependent translation of the spatial coordinates

$$\tilde{x}^i = x^i + \delta x^i(t) \quad \to \quad N^i(t) = -\frac{d}{dt}\delta x^i(t) \ . \tag{3.46}$$

This does not change the metric at constant t, i.e. it does not change ζ_L. Putting everything together, the total change of coordinates is given by

$$x^i \ \to \ x^i + \lambda x^i + 2(\vec{b}\cdot\vec{x})x^i - \vec{x}^2 b^i + \delta x^i(t) \ , \tag{3.47}$$

with the following identification $\zeta_L(x) = \lambda + 2\vec{b}\cdot\vec{x}$. In conclusion, it is always possible to write an adiabatic solution, valid including $\mathcal{O}(q)$ corrections, that is obtained by a (time-independent) dilation and a special conformal transformation of the spatial coordinates and a time-dependent translation. It is important to point out that this is only possible in single-field models. One may naively think that all we have to do in order to generate the long mode is to do a proper rescaling of coordinates in the limit $\eta \to 0$. However, this is not the case.

The long mode has to be equivalent to a coordinate redefinition during the entire history (in other words for the full solution of the Einstein equations involving time evolution), and this is true only for inflationary models with only one dynamical degree of freedom. We will come back to this discussion when we derive the inflationary consistency relations.

So far we were using the equations for the simplest single-field models with a standard kinetic term. However, the situation is very similar for a general model of single-field inflation. In this case the constraint equations are different, but from their general form (see for example [47]) one can still conclude that $\dot\zeta$ is of order q^2 and N^i of order q, which is all we need in the discussion above.

Let us finally stress one technical point. In solving the constraints (3.43) and (3.44) we have to choose boundary conditions. We choose them in a standard way, for example taking ζ to vanish at infinity. This does not clash with eq. (3.45), as this is only a local approximation of the solution which, once the constraints are solved, can be extended to a regular global solution that falls of at infinity.

3.2.2 Tensor Modes

In this Section we come back to the discussion of tensor modes and show that, similarly to what happens to scalars, a long physical solution $\gamma_{ij,L}$ is locally equivalent to a change of coordinates. We work in ζ-gauge, where the spatial line element in a presence of a long gravitational background is given by

$$\mathrm{d}\vec{l}^2 = a^2 e^{2\zeta_S} \left(e^{\gamma_L}\right)_{ij} \mathrm{d}x^i \mathrm{d}x^j \,, \tag{3.48}$$

and $\gamma_{ij,L}$ is transverse and traceless [43, 40]. Therefore, our task is to find a change of coordinates that induces a long gravitational wave in the presence of short scalar modes ζ_S, i.e. starting with a line element $a^2 e^{2\zeta_S} \mathrm{d}\vec{x}^2$. Notice that we do not include the short tensor modes, because in this case there are some difficulties in constructing the long tensor mode. We will come back to this issue later.

In order to have the long mode of γ_{ij} linear in $\vec{x}$, we have to consider a transformation of coordinates that is quadratic in $\vec{x}$. Given the anisotropy the gravitational wave induces, the change of coordinates is more general than a dilation and special conformal transformation. In general we can write

$$x_i \to x_i + A_{ij}x_j + B_{ijk}x_j x_k \,, \tag{3.49}$$

where B_{ijk} is symmetric in the last two indices $B_{ijk} = B_{ikj}$. With this transformation the spatial line element becomes

$$a^2 e^{2\zeta_S} \mathrm{d}\vec{x}^2 \to a^2 e^{2\zeta_S} \left(\delta_{ij} + 2A_{ij} + 4\frac{B_{ijk} + B_{jik}}{2}x^k\right) \mathrm{d}x^i \mathrm{d}x^j \,. \tag{3.50}$$

We can identify the terms in brackets with the gravitational wave. Imposing the symmetry properties of B_{ijk} we find that the parameters of the transformation are

$$A_{ij} = \frac{1}{2}\gamma_{ij} \,, \quad B_{ijk} = \frac{1}{4}(\partial_k\gamma_{ij} - \partial_i\gamma_{jk} + \partial_j\gamma_{ik}) \,. \tag{3.51}$$

Notice that we have chosen these parameters to be time-independent from the very beginning. This is what we expect (the long gravitational waves are frozen outside the horizon at order q), and it is easy to see that this is necessary to generate the physical solution and remain in ζ-gauge.

In our discussion so far we were focusing on the situation where the short modes are scalar perturbations ζ_S. It is interesting to see what happens when we have short tensor modes too. In this case we start from a line element of the form $a^2 e^{2\zeta_S} (e^{\gamma_S})_{ij} \, \mathrm{d}x^i \mathrm{d}x^j$. At zeroth order in gradients it is still possible to induce long scalar or tensor modes. However, transformations (3.47) and (3.49) fail to induce the proper long modes at order q.

It is easy to understand why this happens. For example, a special conformal transformation induces a conformal factor only if one starts from a line element proportional to $\mathrm{d}\vec{x}^2$, and not generic $(e^{\gamma_S})_{ij} \, \mathrm{d}x^i \mathrm{d}x^j$. It is thus not possible to induce a constant gradient of ζ just using transformation (3.47). The same arguments apply when we consider a long tensor mode. The transformation (3.49) does not work at first order in gradients in the presence of short tensor modes. One can explicitly check that in this case the part which is antisymmetric in ij and which drops out starting from $\mathrm{d}\vec{x}^2$, gives a contribution.

This difficulties can be resolved, as shown in [50]. We will come back to this discussion when we talk about consistency relations as Ward identities of non-linearly realised symmetries. However, for the time being, we will just focus on the configurations where the short tensor modes are not present.

3.3 Checks

In this Section we will check that the construction of the adiabatic modes we discussed so far indeed generates the proper second order solutions in the squeezed limit. The impatient reader can skip this Section and move directly to the following two Chapters where we derive the consistency relations for single-field inflation and LSS.

3.3.1 The Second Order Metric in a Matter Dominated universe

For our first check we are going to use the second order metric calculated for a matter dominated universe and show that, in the limit when one of the modes is much longer than the other, the metric can be obtain starting from a first order solution and performing the change of coordinates (3.37) and (3.38). As we already pointed out, the sound horizon in matter dominance is much smaller than the Hubble radius, and we expect our construction to hold even for a long-wavelength modes whose wavenumber q is much larger than H.

The second order metric in Poisson gauge has the following form [107, 108]

$$\mathrm{d}s^2 = \ a^2(\eta) \left[-\left(1 + 2\Phi_P\right) \mathrm{d}\eta^2 + \ 2\omega_{Pi}^{\perp} \mathrm{d}x^i \mathrm{d}\eta + \left(1 - 2\Psi_P\right) \mathrm{d}\vec{x}^2 \right] \ . \tag{3.52}$$

For simplicity we have neglected tensor modes. Following [108] we can write components of

the metric in the following way

$$\Phi_P = -\frac{3}{5}\zeta_0 + \frac{9}{25}\left[\zeta_0^2 + \partial^{-2}(\partial\zeta_0)^2 - 3\partial^{-4}\partial_i\partial_j(\partial_i\zeta_0\partial_j\zeta_0)\right]$$
$$+ \frac{6}{175\mathcal{H}^2}\partial^{-2}\left[2(\partial_i\partial_j\zeta_0)^2 + 5(\partial^2\zeta_0)^2 + 7\partial_i\zeta_0\partial_i\partial^2\zeta_0\right] , \qquad (3.53)$$

$$\Psi_P = -\frac{3}{5}\zeta_0 - \frac{9}{25}\left[\zeta_0^2 + \frac{2}{3}\partial^{-2}(\partial\zeta_0)^2 - 2\partial^{-4}\partial_i\partial_j(\partial_i\zeta_0\partial_j\zeta_0)\right]$$
$$+ \frac{6}{175\mathcal{H}^2}\partial^{-2}\left[2(\partial_i\partial_j\zeta_0)^2 + 5(\partial^2\zeta_0)^2 + 7\partial_i\zeta_0\partial_i\partial^2\zeta_0\right] , \qquad (3.54)$$

$$\omega_{Pi}^{\perp} = -\frac{24}{25\mathcal{H}}\partial^{-2}\left[\partial^2\zeta_0\partial_i\zeta_0 - \partial^{-2}\partial_i\partial_j(\partial^2\zeta_0\partial_j\zeta_0)\right] , \qquad (3.55)$$

where ζ_0 describes initial curvature perturbations. We are interested in the limit when one of the modes is much longer than the other. In this limit we can split ζ_0 into short- and long-wavelength parts $\zeta_0 = \zeta_S + \zeta_L$, and look at the components of the metric that are linear in ζ_L. We are interested only in zeroth and first orders in the gradient expansion, and we can neglect all terms where two or more derivatives hit ζ_L. It is straightforward to find that the relevant part of the metric in this limit becomes

$$\Phi_P = -\frac{3}{5}\zeta_L + \frac{9}{25}(2\zeta_L\zeta_S - 4\partial_i\zeta_L\partial^{-2}\partial_i\zeta_S) + \frac{6}{25a^2H^2}\partial_i\zeta_L\partial_i\zeta_S , \qquad (3.56)$$

$$\Psi_P = -\frac{3}{5}\zeta_L - \frac{9}{25}\left(2\zeta_L\zeta_S - \frac{8}{3}\partial_i\zeta_L\partial^{-2}\partial_i\zeta_S\right) , \qquad (3.57)$$

$$\omega_{Pi}^{\perp} = -\frac{24}{25aH}(\partial_i\zeta_L\zeta_S - \partial_j\zeta_L\partial^{-2}\partial_i\partial_j\zeta_S) . \qquad (3.58)$$

Notice that in these formulas ζ_L without derivatives is just a homogenous part of the long mode.

Now we want to show that one can get the same solution starting from the first order metric

$$ds^2 = a^2(\eta)\left[-\left(1 + 2\Phi_S^{(1)}\right)d\eta^2 + \left(1 - 2\Psi_S^{(1)}\right)d\vec{x}^2\right] , \qquad (3.59)$$

and performing the change of coordinates that generates the long mode. We assume the absence of any anisotropic stress and use results of the linear perturbation theory in matter dominance. With these assumptions $\Psi_S^{(1)} = \Phi_S^{(1)}$ and the gravitational potential is constant in time and can be related to the initial curvature perturbation by a simple linear relation $\Phi_S^{(1)} = -\frac{3}{5}\zeta_S$. The first order metric becomes

$$ds^2 = a^2(\eta)\left[-\left(1 - \frac{6}{5}\zeta_S\right)d\eta^2 + \left(1 + \frac{6}{5}\zeta_S\right)d\vec{x}^2\right] . \qquad (3.60)$$

The same linear relation $\Phi_L = -\frac{3}{5}\zeta_L$ holds for the long mode, as can be easily seen from eq. (3.29) using that the scale-factor and Hubble parameter in matter dominance are given by

$$a \propto \eta^2 , \quad \mathcal{H} = \frac{2}{\eta} . \qquad (3.61)$$

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In the absence of any anisotropic stress, $\Phi_L = \Psi_L$ and eq. (3.28) gives $\epsilon = -\eta\lambda/5$ and $\vec{\xi} = -2\eta\vec{b}/5$. The coordinate transformation (3.37) and (3.38) under all these assumptions become

$$\eta \;\rightarrow\; \eta - \frac{\eta}{5}\zeta_L - \frac{\eta}{5}\partial_i\zeta_L x^i - \frac{12}{25}\eta\,\partial_i\zeta_L\partial^{-2}\partial_i\zeta_S \;, \tag{3.62}$$

$$x^i \;\rightarrow\; x^i + x^i\zeta_L + \partial_j\zeta_L x^j x^i - \frac{1}{2}\vec{x}^2\partial^i\zeta_L - \frac{\eta^2}{10}\partial^i\zeta_L \;. \tag{3.63}$$

Notice that in these expressions ζ_L and $\partial_i\zeta_L$ do not depend on space and denote coefficients of the first two terms in the Taylor expansion of the long mode around the origin.

Let us see how the linear metric transforms. The scale factor changes as

$$a^2(\eta) \;\rightarrow\; a^2(\eta)\,(1 + 2\mathcal{H}\delta\eta) = a^2(\eta)\left(1 + \frac{4}{\eta}\delta\eta\right) \;, \tag{3.64}$$

where we have used that $2\mathcal{H} = 4/\eta$ in matter dominance. The transformation of the short modes is

$$\zeta_S \;\rightarrow\; \zeta_S + \partial_i\zeta_S\delta x^i \;, \tag{3.65}$$

while

$$\mathrm{d}\vec{x}^2 \;\rightarrow\; \mathrm{d}\vec{x}^2\left(1 + 2\zeta_L + 2\partial_i\zeta_L x^i\right) - \frac{2}{5}\eta\partial_i\zeta_L\mathrm{d}x^i\mathrm{d}\eta \;, \tag{3.66}$$

and

$$\mathrm{d}\eta^2 \;\rightarrow\; \mathrm{d}\eta^2\left(1 - \frac{2}{5}\zeta_L - \frac{2}{5}\partial_i\zeta_L x^i - \frac{24}{25}\partial_i\zeta_L\partial^{-2}\partial_i\zeta_S\right) - \frac{24}{25}\eta\,\partial_j\zeta_L\frac{\partial_j\partial_i}{\partial^2}\zeta_S\,\mathrm{d}x^i\mathrm{d}\eta \;. \tag{3.67}$$

Now we are ready to see how the first order metric transforms. Apart from terms linear in ζ_L and ζ_S, we generate the second order terms $\mathcal{O}(\zeta_S\zeta_L)$. This is the non-trivial part of the metric that we are interested in. It can be written in the form

$$\begin{aligned}
\mathrm{d}s^2 \supset a^2(\eta)\Bigg[&-\left(1 + 2\Phi_P - \frac{6}{5}\partial_i\zeta_S\left(x^i\zeta_L + \partial_j\zeta_L x^j x^i - \frac{1}{2}\vec{x}^2\partial^i\zeta_L\right)\right)\mathrm{d}\eta^2 \\
&+ 2\omega_{Pi}^{\perp}\mathrm{d}x^i\mathrm{d}\eta + \left(1 - 2\Psi_P + \frac{6}{5}\partial_i\zeta_S\left(x^i\zeta_L + \partial_j\zeta_L x^j x^i - \frac{1}{2}\vec{x}^2\partial^i\zeta_L\right)\right)\mathrm{d}\vec{x}^2\Bigg] \;,
\end{aligned} \tag{3.68}$$

where Φ_P and Ψ_P are exactly the second order solutions in the squeezed limit given by eq. (3.56).

We showed that the change of coordinates indeed generates the correct second-order solution, with some additional terms in eq. (3.68). These terms are nothing but a dilation and a special conformal transformation acting on the initial curvature perturbations. Indeed, we can expect these terms to appear, because the effect of the long mode does not show up only in the dynamics of the short modes, but also in the possible correlations in the initial conditions. We will come back to this issue later and see that this is the root of the connection between the consistency relations for inflation and LSS.

3.3.2 The Squeezed Limit of $\delta^{(2)}$ in a Matter Dominated Unverse

Another way to do this check is to use the density contrast δ calculated at second order in perturbation theory in a matter dominated universe. Being a second order result, $\delta_{\vec{k}}^{(2)}$ is proportional to a product of two initial conditions $\Phi_{\vec{q}}\Phi_{\vec{k}-\vec{q}}$. In the limit when one of these two modes is much longer than the other $q \ll k$, $\delta_{\vec{k}}^{(2)}$ can be obtained starting from a first order solution $\delta_{\vec{k}}^{(1)}$ and performing the change of coordinates (3.62) and (3.63). Although the underlying physics is the same as in the previous section, we show this check here because it shows some technical subtleties that will be relevant later for the derivation of the consistency relations. The check for a homogeneous long mode was first done in [109], and including gradients in [67].

The full result for second order $\delta^{(2)}$ is given by (see for example [109])

$$\delta_{\vec{k}}^{(2)} = 2\left[\left(\frac{k^4}{14} + \frac{3}{28}(q_1^2 + q_2^2)k^2 - \frac{5}{28}(q_1^2 - q_2^2)^2\right)\eta^4\right.$$
$$+ \left(\frac{59}{14}k^2 - \frac{125}{14}(q_1^2 + q_2^2) - \frac{9}{7}\frac{(q_1^2 - q_2^2)^2}{k^2}\right)\eta^2$$
$$\left.+ \left(90 + 36\frac{q_1^2 + q_2^2}{k^2} - 54\frac{(q_1^2 - q_2^2)^2}{k^4}\right)\right]\frac{\Phi_{\vec{q}_1}\Phi_{\vec{q}_2}}{36} . \tag{3.69}$$

To simplify the notation, here and in the following we assume a summation $\int \frac{d^3q_1\,d^3q_2}{(2\pi)^3}\delta(\vec{k} - \vec{q}_1 - \vec{q}_2)$ in all products of first order terms in Fourier space. In the limit $q_1 \ll q_2$, when one mode is much longer than the other, we get

$$\delta_{\vec{k}}^{(2)} = 2\left[72 - 6q_2^2\eta^2 + \frac{\vec{q}_1 \cdot \vec{q}_2}{q_2^2}\left(144 + 11\eta^2 q_2^2 + \frac{1}{2}\eta^4 q_2^4\right)\right]\frac{\Phi_{\vec{q}_1}\Phi_{\vec{q}_2}}{36}\left[1 + \mathcal{O}(q_1^2/q_2^2)\right] . \tag{3.70}$$

We are going to show that this solution is the same as the one that can be obtained starting from the linear $\delta^{(1)}$

$$\delta_{\vec{k}}^{(1)} = -(12 + k^2\eta^2)\frac{\Phi_{\vec{k}}}{6} , \tag{3.71}$$

and performing the change of coordinates (3.62) and (3.63).

Notice that δ does not transform linearly under the change of coordinates. Only the density ρ is a scalar. Therefore, the correct transformation for density contrast is

$$\delta \to \delta + \delta\eta\left(\frac{\bar{\rho}'}{\bar{\rho}}(1 + \delta) + \delta'\right) + \delta x^i \cdot \partial_i \delta . \tag{3.72}$$

with $\bar{\rho} \propto \eta^{-6}$ during matter dominance. In order to make the discussion more clear, we split the check in two parts. First we do the check at order $\mathcal{O}(q^0)$ (which corresponds to a constant Φ_L) and then we focus on order $\mathcal{O}(q^1)$ (for which the long mode is proportional to a constant gradient).

Constant Φ_L. In this case we can neglect terms with derivatives $\partial_i\Phi_L$ and the change of coordinates is simply given by

$$\eta \to \eta + \frac{\eta}{3}\Phi_L , \quad x^i \to x^i - \frac{5}{3}x^i\Phi_L . \tag{3.73}$$

Using eq. (3.72) we find that δ transforms as

$$\delta \;\to\; \delta - 2\Phi_L - 2\Phi_L\delta + \frac{\eta}{3}\Phi_L\delta' - \frac{5}{3}\Phi_L x^i\partial_i\delta \;. \tag{3.74}$$

which in Fourier space becomes

$$\delta_{\vec{k}} \to \delta_{\vec{k}} - 2\Phi_{\vec{q}}\delta_{\vec{k}} + \frac{1}{3}\Phi_{\vec{q}}\eta\delta'_{\vec{k}} + \frac{5}{3}\Phi_{\vec{q}}\partial_{k_i}(k_i\delta_{\vec{k}}) \;. \tag{3.75}$$

We can write $\delta_{\vec{k}}$ as $T_k(\eta)\Phi_{\vec{k}}$, where Φ describes the first-order (time-independent) potential in the matter dominated phase, whose value is obviously affected by the evolution during radiation dominance. $T_k(\eta)$ describes the linear relation between δ and Φ during MD and it is given by

$$T_k(\eta) \equiv -2 - \frac{k^2\eta^2}{6} \;. \tag{3.76}$$

Plugging this back in the transformation of δ, eq. (3.72), we get

$$\delta_{\vec{k}} \to \delta_{\vec{k}} + \Phi_{\vec{q}}\Phi_{\vec{k}}\left(-2T_k + \frac{1}{3}\eta T'_k + \frac{5}{3}k\partial_k T_k\right) + \frac{5}{3}\Phi_{\vec{q}}T_k\partial_{k_i}(k_i\Phi_{\vec{k}}) \;. \tag{3.77}$$

The last term describes the effect of a dilation of the initial conditions. We already met these terms before (see Section 3.3.1), and as we explained they come from the effect of the long mode on the initial conditions. We will neglect them in what follows, because they do not contribute to the dynamical evolution that we are trying to capture. With this in mind we have

$$\delta_{\vec{k}} \to \delta_{\vec{k}} + \Phi_{\vec{q}}\Phi_{\vec{k}}\left(4 - \frac{1}{3}k^2\eta^2\right) \;. \tag{3.78}$$

By replacing $\vec{q} = \vec{q}_1$ and $\vec{k} = \vec{q}_1 + \vec{q}_2$, this expression reproduces the first two terms in the eq. (3.70).

Gradient of Φ_L. When we add a constant gradient to the coordinate transformation (3.73) the situation becomes a bit more complicated. Let us look at it term by term by starting from the time redefinitions eq. (3.63). The last term on the right-hand side gives

$$\delta_{\vec{k}} \to \delta_{\vec{k}} + \frac{8\vec{q}\cdot\vec{k}}{k^2}\Phi_{\vec{q}}\Phi_{\vec{k}} \;, \tag{3.79}$$

while the second term, i.e. the space-dependent transformation, gives

$$\delta_{\vec{k}} \to \delta_{\vec{k}} + 2\Phi_{\vec{q}}q_i\partial_{k_i}\delta_{\vec{k}} - \frac{1}{3}\Phi_{\vec{q}}q_i\eta\partial_{k_i}\delta'_{\vec{k}} \;. \tag{3.80}$$

Replacing $\delta_{\vec{k}} = T_k(\eta)\Phi_{\vec{k}}$, this expression can be rewritten as

$$\delta_{\vec{k}} \to \delta_{\vec{k}} + \frac{\vec{q}\cdot\vec{k}}{k}\Phi_{\vec{q}}\left(2\partial_k T_k - \frac{1}{3}\eta\partial_k T'_k\right)\Phi_{\vec{k}} + \Phi_{\vec{q}}\left(2T_k - \frac{1}{3}\eta T'_k\right)\left(q_j\partial_{k_j}\Phi_{\vec{k}}\right) \;. \tag{3.81}$$

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The last term contains a derivative acting on the initial conditions that we drop for the same reasons as explained above.

Let us now discuss the spatial redefinitions, given by eq. (3.62). The last term on the right-hand side gives

$$\delta_{\vec{k}} \to \delta_{\vec{k}} - \frac{1}{6}\eta^2 \vec{q} \cdot \vec{k}\, \Phi_{\vec{q}} T_k \Phi_{\vec{k}} \; . \tag{3.82}$$

The two remaining terms in the transformation of x^i give

$$\delta_{\vec{k}} \to \delta_{\vec{k}} - \frac{5}{6}\Phi_{\vec{q}} q_j \left[2\partial_{k_i}\partial_{k_j}(k_i \delta_{\vec{k}}) - \partial_{k_i}\partial_{k_i}(k_i \delta_{\vec{k}}) \right] \; . \tag{3.83}$$

After replacing $\delta_{\vec{k}} = T_k(\eta)\Phi_{\vec{k}}$, this expression can be conveniently rewritten as

$$\delta_{\vec{k}} \to \delta_{\vec{k}} - \frac{5}{6}\Phi_{\vec{q}} \Bigg\{ \frac{\vec{q} \cdot \vec{k}}{k} \left(4\partial_k T_k + k\partial_k^2 T_k \right) \Phi_{\vec{k}} + 2q_j k\partial_k T_k\, \partial_{k_j}\Phi_{\vec{k}} $$
$$+ q_j T_k \left[2\partial_{k_i}\partial_{k_j}(k_i \Phi_{\vec{k}}) - \partial_{k_i}\partial_{k_i}(k_i \Phi_{\vec{k}}) \right] \Bigg\} \; . \tag{3.84}$$

Similarly to what happened in eq. (3.77), the last two terms of this expression—the ones proportional to T_k—contain derivatives acting only on the initial conditions and we neglect them. In the second term one derivative acts on T_k and the other on $\Phi_{\vec{k}}$. Adding this term to the last one on the right-hand side of eq. (3.81) gives

$$- \Phi_{\vec{q}} \cdot (q_j \partial_{k_j}\Phi_{\vec{k}}) \left(-2T_k + \frac{1}{3}\eta T_k' + \frac{5}{3}k_i \partial_{k_i} T_k \right) \; . \tag{3.85}$$

Since $\Phi_{\vec{k}-\vec{q}} \simeq \Phi_{\vec{k}} - (q_j \partial_{k_j}\Phi_{\vec{k}})$, this is just a shift $\vec{k} \to \vec{k} - \vec{q}$ in the momentum dependence of the initial condition in the dilation transformation, eq. (3.77).

Combining all the contributions, i.e. eqs. (3.77), (3.79), (3.81), (3.82) and (3.84), and dropping derivatives acting on the initial conditions which as already mentioned will be treated in the next section, we finally obtain

$$\delta_{\vec{k}} \to \delta_{\vec{k}} + \left(-2T_k + \frac{1}{3}\eta T_k' + \frac{5}{3}k_i \partial_{k_i} T_k \right) \Phi_{\vec{q}} \Phi_{\vec{k}-\vec{q}} $$
$$+ \frac{\vec{q} \cdot \vec{k}}{k^2} \left(8 - \frac{4}{3}k\partial_k T_k - \frac{5}{6}k^2 \partial_k^2 T_k - \frac{1}{3}\eta k\partial_k T_k' - \frac{1}{6}\eta^2 k^2 T_k \right) \Phi_{\vec{q}} \Phi_{\vec{k}} \; . \tag{3.86}$$

We can now plug in the expression for the transfer function, eq. (3.76), and obtain

$$\delta_{\vec{k}} \to \delta_{\vec{k}} + 12(12 - k^2\eta^2)\frac{\Phi_{\vec{q}}\Phi_{\vec{k}-\vec{q}}}{36} + 2\frac{\vec{q} \cdot \vec{k}}{k^2}\left(144 + 23\eta^2 k^2 + \frac{1}{2}\eta^4 k^4 \right)\frac{\Phi_{\vec{q}}\Phi_{\vec{k}}}{36} \; . \tag{3.87}$$

For $\vec{q} = \vec{q_1}$ and $\vec{k} = \vec{q_1} + \vec{q_2}$, this expression reproduces (after expanding the second term on the right-hand side) the squeezed limit $q_1 \ll q_2$ of the explicit result [109], i.e. eq. (3.70).

3.3.3 The Cubic Action for Single-field Inflation

One can do similar checks on the level of the metric in ζ-gauge. Here we are going to focus on the cubic action for curvature perturbation ζ in standard single-field slow-roll inflation. We will show that, in the limit in which one of the three modes is much longer than the other two, this action can be obtained with a coordinate transformation starting from the quadratic action (up to corrections quadratic in the long-wavelength momentum $\vec{q}$). The check at zeroth order in $\vec{q}$ was originally done in [47] and including gradients in [48]. The quadratic and cubic actions are given by [43] (see Section 2.2)

$$S^{(2)} = \int \mathrm{d}^3x\mathrm{d}t \; \epsilon a^3 \left[\dot{\zeta}^2 - \left(\frac{\partial \zeta}{a} \right)^2 \right] , \tag{3.88}$$

$$S^{(3)} = \int \mathrm{d}^3x\mathrm{d}t \left[ae^\zeta \left(1 + \frac{\dot{\zeta}}{H} \right) \left(-2\partial^2\zeta - (\partial\zeta)^2 \right) + \epsilon a^3 e^{3\zeta} \dot{\zeta}^2 \left(1 - \frac{\dot{\zeta}}{H} \right) + \right.$$
$$\left. + a^3 e^{3\zeta} \left(\frac{1}{2} \left(\partial_i\partial_j\psi\partial_i\partial_j\psi - (\partial^2\psi)^2 \right) \left(1 - \frac{\dot{\zeta}}{H} \right) - 2\partial_i\psi\partial_i\zeta\partial^2\psi \right) \right] , \tag{3.89}$$

where $\psi = -\frac{\zeta}{a^2 H} + \chi$ and $\partial^2\chi = \epsilon\dot{\zeta}$.

We first find the limit of the cubic action when one of the modes is much longer than the others. To do that, we separate long and short modes by writing $\zeta = \zeta_L + \zeta_S$. Notice that $\dot{\zeta}_L = \mathcal{O}(q^2)$ and whenever a time derivative acts on ζ we can replace it with ζ_S. Obviously, the same is true if two spatial derivatives act on ζ. The cubic action then becomes

$$S^{(3)}_{q\to 0} = \int \mathrm{d}^3x\mathrm{d}t \left[ae^\zeta \left(1 + \frac{\dot{\zeta}_S}{H} \right) \left(-2\partial^2\zeta_S - (\partial\zeta)^2 \right) + \epsilon a^3 e^{3\zeta} \dot{\zeta}_S^{\;2} \left(1 - \frac{\dot{\zeta}_S}{H} \right) + \right.$$
$$\left. + a^3 e^{3\zeta} \left(\frac{1}{2} \left(\partial_i\partial_j\psi_S\partial_i\partial_j\psi_S - (\partial^2\psi_S)^2 \right) \left(1 - \frac{\dot{\zeta}_S}{H} \right) - 2\partial_i\psi\partial_i\zeta\partial^2\psi_S \right) \right] , \tag{3.90}$$

We are interested in terms linear in ζ_L and quadratic in ζ_S. We can expand the action and try to simplify it using integration by parts. For example, the second term in $S^{(3)}_{q\to 0}$ gives

$$\epsilon a^3 e^{3\zeta} \dot{\zeta}_S^{\;2} \left(1 - \frac{\dot{\zeta}_S}{H} \right) \to 3\epsilon a^3 \zeta_L \dot{\zeta}_S^2 . \tag{3.91}$$

The other terms demand a bit more work. The first term in $S^{(3)}_{q\to 0}$ gives

$$ae^\zeta \left(1 + \frac{\dot{\zeta}_S}{H} \right) \left(-2\partial^2\zeta_S - (\partial\zeta)^2 \right) \to ae^{\zeta_L + \zeta_S} \left(1 + \frac{\dot{\zeta}_S}{H} \right) \left(-2\partial^2\zeta_S - (\partial\zeta_S)^2 - 2\partial_i\zeta_L\partial_i\zeta_S \right)$$
$$\to a\zeta_L(\partial\zeta_S)^2 - 2a\zeta_L \frac{\dot{\zeta}_S}{H}\partial^2\zeta_S - 2a\partial_i\zeta_L \frac{\dot{\zeta}_S}{H}\partial_i\zeta_S , \tag{3.92}$$

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where we have expanded the exponent $e^{\zeta_L + \zeta_S} = 1 + \zeta_L + \zeta_S + \zeta_L \zeta_S + \cdots$ and have done one integration by parts: $a\zeta_L \zeta_S \partial^2 \zeta_S \to -a\zeta_L(\partial \zeta_S) - a\partial_i \zeta_L \zeta_S \partial_i \zeta_S$. Now we can transform the second term in the previous expression

$$-2a\zeta_L \frac{\dot{\zeta}_S}{H} \partial^2 \zeta_S \;\to\; 2a\partial_i \zeta_L \frac{\dot{\zeta}_S}{H} \partial_i \zeta_S + \frac{a}{H} \zeta_L \frac{\mathrm{d}}{\mathrm{d}t}[(\partial \zeta_S)^2]$$
$$\to\; 2a\partial_i \zeta_L \frac{\dot{\zeta}_S}{H} \partial_i \zeta_S - a\zeta_L(\partial \zeta_S)^2 - a\epsilon \zeta_L(\partial \zeta_S)^2 \;, \tag{3.93}$$

such that the first term in $S_{q\to 0}^{(3)}$ finally simplifies to

$$ae^\zeta \left(1 + \frac{\dot{\zeta}_S}{H}\right) \left(-2\partial^2 \zeta_S - (\partial \zeta)^2\right) \;\to\; -a\epsilon \zeta_L (\partial \zeta_S)^2. \tag{3.94}$$

Now we can focus on the last term in $S_{q\to 0}^{(3)}$. After expanding the exponent, the relevant part is given by

$$\frac{3}{2} a^3 \zeta_L \left(\partial_i \partial_j \psi_S \partial_i \partial_j \psi_S - (\partial^2 \psi_S)^2\right) - 2a^3 \partial_i \zeta_L \partial_i \psi_S \partial^2 \psi_S - 2a^3 \partial_i \psi_L \partial_i \zeta_S \partial^2 \psi_S \;. \tag{3.95}$$

The first contribution can be rewritten as

$$\frac{3}{2} a^3 \zeta_L \partial_i \left(\partial_j \psi_S \partial_i \partial_j \psi_S - \partial_i \psi_S \partial^2 \psi_S\right) = \frac{3}{4} a^3 \zeta_L \partial^2 (\partial \psi_S)^2 - \frac{3}{2} a^3 \zeta_L \partial_i \left(\partial_i \psi_S \partial^2 \psi_S\right)$$
$$\to\; \frac{3}{4} a^3 (\partial^2 \zeta_L)(\partial \psi_S)^2 + \frac{3}{2} a^3 \partial_i \zeta_L \partial_i \psi_S \partial^2 \psi_S \;. \tag{3.96}$$

The first term on the left hand side has two spatial derivatives on ζ_L and it can be neglected while the second one has the same form as the second term in (3.95). If we rewrite it as

$$\partial_i \zeta_L \partial_i \psi_S \partial^2 \psi_S = \partial_i \zeta_L \partial_j (\partial_i \psi_S \partial_j \psi_S) - \frac{1}{2} \partial_i \zeta_L \partial_i (\partial \psi_S)^2 \;, \tag{3.97}$$

we can see that, after integration by parts, it is proportional to $\partial^2 \zeta_L$ too, and can be safely neglected. To summarise, only the last term in (3.95) gives a non-vanishing contribution to the cubic action in the limit we consider. It can be further simplified noticing that one is forced to replace $\partial^2 \psi_S$ with $\partial^2 \chi_S = \epsilon \dot{\zeta}_S$, because otherwise a term with two spatial derivatives on ζ_L is generated again. At leading order in slow-roll $\partial_i \psi_L = -\frac{1}{a^2 H} \partial_i \zeta_L$, and finally we get

$$-2a^3 \partial_i \psi_L \partial_i \zeta_S \partial^2 \psi_S \;\to\; \frac{2a\epsilon}{H} \partial_i \zeta_L \dot{\zeta}_S \partial_i \zeta_S \;. \tag{3.98}$$

Summing up all relevant contributions in the cubic action linear in ζ_L and quadratic in ζ_S we get

$$\delta S_{q\to 0}^{(3)} = \int \mathrm{d}^3 x \mathrm{d}t \left(3\epsilon a^3 \zeta_L \dot{\zeta}_S^2 - a\epsilon \zeta_L (\partial \zeta_S)^2 + \frac{2a\epsilon}{H} \dot{\zeta}_S \partial_i \zeta_S \partial_i \zeta_L\right) \;. \tag{3.99}$$

On the other hand, we can start from the quadratic action for the short modes and perform a change of coordinates

$$x^i \to \tilde{x}^i = x^i - b^i x^2 + 2x^i (b \cdot x) + \delta x^i(t) \tag{3.100}$$

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with

$$b_i = -\frac{1}{2}\partial_i\zeta_L \qquad \frac{d}{dt}\delta x^i(t) = -\frac{2b^i}{a^2 H} + \mathcal{O}(\epsilon) \ . \tag{3.101}$$

What we expect to find is exactly the cubic action (3.99) with one long and two short modes. Indeed, starting from the action (3.88) and performing the change of coordinates we get

$$\delta S^{(2)} = \int d^3x dt \left[-6a^3\epsilon(\vec{b}\cdot\vec{x})\dot{\zeta}^2 + 2a\epsilon(\vec{b}\cdot\vec{x})(\partial\zeta)^2 - 4a\epsilon\frac{\dot{\zeta}}{H}b_i\partial_i\zeta \right] \ . \tag{3.102}$$

This indeed coincides with the cubic action in the squeezed limit once we use $b_i = -\frac{1}{2}\partial_i\zeta_L$.

Notice that in this check we did an expansion in ζ_L. However, for the homogeneous ζ_L the check can be done exactly, without assuming that ζ_L is small. To see this, notice that the long mode appears in the action only in combination ae^{ζ}_L. Therefore, removing the long mode is equivalent to a finite resealing of the scale-factor [47]. The same does not hold for a large homogeneous gradient and, as we are going to see later, this has important consequences for studying multiple soft limits of the inflationary correlation functions.

A similar check can be done for a long tensor mode and two short scalar modes too. The cubic action in the single field slow-roll inflation in this case is given by

$$S = \int d^4x \ \frac{1}{2}\frac{\dot{\phi}^2}{H^2}a\gamma_{ij}\partial_i\zeta\partial_j\zeta \ . \tag{3.103}$$

From the form of the action it is immediately clear that the effect of γ is equivalent, including first gradient corrections, to the change of spatial coordinates discussed above (see eq. (3.49)). A similar analysis does not work for the vertices $\zeta_L\gamma\gamma$ and $\gamma_L\gamma\gamma$ when gradient corrections are included. As we already pointed out (see Section 3.2.2), this is related to the fact that in the presence of short tensor modes a naive change of coordinates does not induce a proper long mode and one has to be more careful (see [50]). We will come back to this issue when we discuss the consistency relations for single-field inflation involving short tensor modes.

Chapter 4

Inflationary Consistency Relations

In the previous Chapter we showed that the long-wavelength mode can be locally traded for a change of coordinates. In this Chapter we are going to use this as a starting point to derive the consistency relations for single-field inflation. These relations connect the squeezed limit of an $(n + 1)$-point function of curvature perturbations with a variation of the corresponding n-point function and have the following form

$$\langle \zeta_{\vec{q}} \zeta_{\vec{k}_1} \dots \zeta_{\vec{k}_n} \rangle'_{q \to 0} = - P(q) \left[\sum_a D_a + \mathcal{O}(q/k)^2 \right] \langle \zeta_{\vec{k}_1} \dots \zeta_{\vec{k}_n} \rangle' \, , \tag{4.1}$$

where D_a is some differential operator with derivatives acting on $\vec{k}_a$ which is linear in $\vec{q}$.

Given that the construction of the adiabatic modes is quite general, the consistency relations will be a very general result too. As we will show in the following sections they hold in any single-field model of inflation, independently on the details of inflationary potential and kind of interactions of the inflaton field. We will see that this is a very powerful statement that allows us to make model-independent observational tests that can rule out any single-field model of inflation.

Although eq. (4.1) involving soft scalars is best known, one can derive similar consistency relations for long tensor modes too. Furthermore, it is possible to use these results to investigate other kinematical regimes where there is a big separation in scales. We will focus on two examples, one where the sum of some of the external momenta goes to zero and the other where several external modes are soft. We are also going to discuss in what sense one can extend some of the arguments that go into the derivation of the consistency relations to capture also $\mathcal{O}(q^2)$ corrections in eq. (4.1). Towards the end of this Chapter, we will comment on the connection of the consistency relations and Ward identities for spontaneously broken space-time symmetries.

4.1 Consistency Relations for Scalars

In this Section we will focus on the squeezed limit of correlation functions involving only scalar modes ζ. We will derive Maldacena's consistency relation [43] and its generalisation when

gradients of the long mode are included—conformal consistency relation [48]. After giving some comments on the assumptions behind the derivation and importance of the consistency relations, we will give several non-trivial checks using explicit expressions for correlation functions calculated in several single-field models.

Let us begin with a physical argument that is behind the consistency relations [43, 45]. Given that we are interested in calculating the squeezed limit of an $(n + 1)$-point function, the configuration that we are considering in real space is the one where one mode is much longer than other short modes. It is obvious that during inflation the long mode goes outside the horizon first and by the time the short modes cross the horizon (which is the moment when the in-in integral picks up the most of the contribution), it can be considered as a classical background for the short modes. As we saw in the previous Chapter this background is equivalent to a simple change of coordinates, including first order in gradients. Therefore, performing a proper change of coordinates, we can go to a frame in which the long mode is removed. Essentially, this is what allows us to write down the consistency relations: the change of coordinates relates two solutions, with and without the long mode and the same short modes, and consequently the $(n + 1)$-point and n-point correlation functions.

Let us make this line of reasoning a bit more quantitative. The starting point is a statement in real space that, for the correlation functions of short modes, the presence of ζ_L can be identified with a change of coordinates. In other words

$$\langle \zeta(x_1) \cdots \zeta(x_n) | \zeta_L \rangle = \langle \zeta(\tilde{x}_1) \cdots \zeta(\tilde{x}_n) \rangle \, , \tag{4.2}$$

where the change of coordinates is such that going from x to $\tilde{x}$ corresponds to generating ζ_L. Our aim now is to rewrite this equality in terms of standard correlation functions. In order to do that, we should multiply both sides with ζ_L and average over it

$$\langle \zeta_L(x) \langle \zeta(x_1) \cdots \zeta(x_n) | \zeta_L \rangle \rangle = \langle \zeta_L(x) \langle \zeta(\tilde{x}_1) \cdots \zeta(\tilde{x}_n) \rangle \rangle \, . \tag{4.3}$$

Notice that the left hand side is just an $(n + 1)$-point function involving ζ_L and n short modes. The right hand side is a bit more complicated, because $\tilde{x}$ contains ζ_L and in general one cannot find the average over ζ_L easily. However, we can expand the n-point function in δx and take the average after that[1]. The first two terms in the expansion are

$$\langle \zeta(\tilde{x}_1) \cdots \zeta(\tilde{x}_n) \rangle = \langle \zeta(x_1) \cdots \zeta(x_n) \rangle + \sum_{a=1}^{n} \delta x_{ai} \cdot \partial_{ai} \langle \zeta(x_1) \cdots \zeta(x_n) \rangle + \cdots \, . \tag{4.4}$$

If we plug this expression into eq. (4.3) the first term is proportional to $\langle \zeta_L \rangle$ and averages to zero. Therefore, on the right hand side, the only relevant contribution comes from the variation of the n-point function

$$\delta \langle \zeta(x_1) \cdots \zeta(x_n) \rangle = \sum_{a=1}^{n} \delta x_{ai} \cdot \partial_{ai} \langle \zeta(x_1) \cdots \zeta(x_n) \rangle \, . \tag{4.5}$$

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[1]We will come back later to the case in which we keep the coordinate transformation finite when we talk about multiple soft limits.

Finally, what we can write down in real space is the following equality

$$\langle \zeta_L(x)\zeta(x_1)\cdots\zeta(x_n)\rangle = \left\langle \zeta_L(x) \cdot \sum_{a=1}^{n} \delta x_{ai} \cdot \partial_{ai}\langle\zeta(x_1)\cdots\zeta(x_n)\rangle\right\rangle . \tag{4.6}$$

This is the consistency relation written in real space. On the left hand side we have the $(n+1)$-point function. On the right hand side, the average $\langle \zeta_L \cdot \delta x \rangle$ gives the two point function of the long mode, multiplying a variation of the n-point function. To bring this result to the familiar form, one only has to rewrite this equation in momentum space.

Of course, in order to do that, one has to specify the change of coordinates δx. In the following two sections we are going to split the discussion. First, we are going to consider only the homogeneous part of the long mode that corresponds to a dilation of spatial coordinates. After that we will turn to the case where gradients are included as well and the change of coordinates contains both dilations and special conformal transformations.

Before moving on, let us introduce some notation that we are going to use here and in the following chapters. We define

$$\frac{d\vec{k}_1}{(2\pi)^3}\cdots\frac{d\vec{k}_n}{(2\pi)^3} \equiv \mathrm{dK}_n , \quad \langle\zeta_{\vec{k}_1}\cdots\zeta_{\vec{k}_n}\rangle = \mathcal{M}\cdot\delta(\vec{P}) , \quad \vec{P} = \vec{k}_1 + \cdots + \vec{k}_n , \tag{4.7}$$

and $\mathcal{M}$ contains all momentum dependence of the correlation function other than the delta function of momentum conservation.

4.1.1 Maldacena's Consistency Relation

At zeroth order in gradients, the result of the procedure described above is Maldacena's famous consistency relation. It was first proven to hold in standard single-field slow-roll inflation [43] and later it was shown that it is valid in all inflationary models with a single dynamical degree of freedom, independently on the slow-roll approximation and the form of the interactions [45, 46].

Let us follow the main steps in the derivation. The long mode at leading order in derivatives, as we saw in Section 3.2.1, is equivalent just to a simple time-independent rescaling of spatial coordinates

$$x^i \rightarrow x^i + \zeta_L(\vec{x}_+)x^i . \tag{4.8}$$

For the homogeneous long mode the point $\vec{x}_+$ is arbitrary, but physically it should be in the neighbourhood of the points $\vec{x}_1,\ldots,\vec{x}_n$. For concreteness we will make the following choice $\vec{x}_+ = (\vec{x}_1 + \cdots + \vec{x}_n)/n$.

We can start from eq. (4.5) and rewrite it in momentum space. Using $\delta\vec{x} = \zeta_L(\vec{x}_+)\vec{x}$, the variation of an n-point function is

$$\delta\langle\zeta(x_1)\cdots\zeta(x_n)\rangle = \int \frac{d\vec{q}_+}{(2\pi)^3}\mathrm{dK}_n\,\zeta_{\vec{q}_+}\sum_a \mathcal{M}\delta(\vec{P})k_{ai}\partial_{k_{ai}}e^{i\vec{q}_+\cdot\vec{x}_+ + i\vec{k}_b\cdot\vec{x}_b}$$

$$= -\int \frac{d\vec{q}_+}{(2\pi)^3}\mathrm{dK}_n\,\zeta_{\vec{q}_+}\sum_a \left(3\mathcal{M}\delta(\vec{P}) + k_{ai}\partial_{k_{ai}}\mathcal{M}\delta(\vec{P}) + k_{ai}\mathcal{M}\partial_{k_{ai}}\delta(\vec{P})\right)e^{i\vec{q}_+\cdot\vec{x}_+ + i\vec{k}_b\cdot\vec{x}_b} , \tag{4.9}$$

where we did one integration by parts to go to the second line. Notice that in the exponent we implicitly assume the summation over b which runs from 1 to n. The term that needs additional simplification is the one where derivatives act on the delta function. First, notice that we can write $\partial_{k_{ai}} \delta(\vec{P}) = \partial_{P_i} \delta(\vec{P})$. After this replacement it is easy to do the sum, and the last term becomes $\mathcal{M} \vec{P} \cdot \vec{\partial}_P \delta(\vec{P})$. We still have derivatives acting on the delta function and we have to do integration by parts again. What we gained in rewriting the last term in this form is that whenever the derivative hits anything else than $\vec{P}$, the result is proportional to the integral of $\vec{P} \delta(\vec{P})$ and therefore is zero. Having this in mind, we can finally write

$$\delta\langle \zeta(x_1) \cdots \zeta(x_n)\rangle = -\int \frac{d\vec{q}_+}{(2\pi)^3} dK_n \, \zeta_{\vec{q}_+} \left[\left(3(n-1) + \sum_a k_{ai}\partial_{k_{ai}} \right) \mathcal{M} \right] \delta(\vec{P}) e^{i\vec{q}_+ \cdot \vec{x}_+ + i\vec{k}_b \cdot \vec{x}_b},$$

(4.10)

where now derivatives act only on $\mathcal{M}$.

To complete the derivation we should only multiply this expression by the long mode $\zeta(x)$ and average over it. As we already said, on the left hand side we are going to get the $(n+1)$-point function while on the right hand side we get the two-point function of the long mode. After integrating one of the momenta $\vec{q}_+$, we are left with

$$\langle \zeta(x)\zeta(x_1) \cdots \zeta(x_n)\rangle = -\int \frac{d\vec{q}}{(2\pi)^3} dK_n \, P(q) \left(3(n-1) + \sum_a k_{ai}\partial_{k_{ai}} \right) \mathcal{M}\delta(\vec{P}) e^{i\vec{q}\cdot\vec{x} + i\vec{k}_b\cdot\vec{x}_b}.$$

(4.11)

On the other hand, the Fourier transform of the $(n+1)$-point correlation function is given by

$$\langle \zeta(x)\zeta(x_1) \cdots \zeta(x_n)\rangle = \int \frac{d\vec{q}}{(2\pi)^3} dK_n \langle \zeta_{\vec{q}}\zeta_{\vec{k}_1} \cdots \zeta_{\vec{k}_n}\rangle'(2\pi)^3 \delta(\vec{q} + \vec{P}) e^{i\vec{q}\cdot\vec{x} + i\vec{k}_b\cdot\vec{x}_b}$$

(4.12)

We can directly compare these two expressions to get Maldacena's consistency relation [43]

$$\langle \zeta_{\vec{q}}\zeta_{\vec{k}_1} \cdots \zeta_{\vec{k}_n}\rangle'_{\vec{q}\to 0} = -P(q) \left(3(n-1) + \sum_a k_{ai}\partial_{k_{ai}} + \mathcal{O}(q/k) \right) \langle \zeta_{\vec{k}_1} \cdots \zeta_{\vec{k}_n}\rangle'.$$

(4.13)

Notice that in the two integrals we have delta functions of momentum conservation with different arguments. At zeroth order in $\vec{q}$ this difference is irrelevant. Of course, when we include first order in gradients of the long mode, this subtlety will become important. We will come back to this issue in the following Section when we derive the conformal consistency relation for single-field inflation.

4.1.2 Conformal Consistency Relations

Maldacena's consistency relation describes the squeezed limit of an $(n+1)$-point function at leading order in gradients of the long mode. However, as we saw in Chapter 3.2.1, the long mode is equivalent to a change of coordinates including gradients. This implies that we can

use the same arguments as in previous sections to derive a consistency relation that captures $\mathcal{O}(q)$ corrections in the squeezed limit of an $(n+1)$-point function. Given that at this order the long mode is equivalent to a special conformal transformation of the spatial coordinates, the squeezed limit will be related to a conformal variation of the n-point function. For this reason we call these relations conformal consistency relations [48].

Before we move to the derivation, let us stress here one important difference compared to Maldacena's consistency relation. While the inflationary correlation functions are almost scale invariant, they are *not* invariant under special conformal transformations. Therefore, while in the limit of exact scale invariance the right hand side of Maldacena's consistency relation gives just zero, the conformal consistency relation will have a nontrivial contribution.

One easy way to understand where this difference is coming from is to consider the decoupling limit. In this limit we are interested in correlation functions of small fluctuations around the inflaton background $\phi(t)$. As we already explained in the Introduction, the time-dependent inflaton background breaks de Sitter isometries spontaneously. Two of these isometries, in the late time limit $\eta \to 0$, act as dilation and special conformal transformation of spatial coordinates, and one would expect that the correlation functions are not invariant under any of these two transformations. However, the approximate shift symmetry of the inflaton field $\phi \to \phi + C$ is what makes the correlation functions almost scale-invariant. On the other hand, the isometries corresponding to the spatial conformal transformation for $\eta \to 0$ remain broken. We will come back to this discussion in the following sections.

The derivation of the conformal consistency relations is essentially the same as for Maldacena's consistency relation. Again, we can start from eq. (4.5). The only difference now is that we have to use a more complicated change of coordinates that induces not only the homogeneous part of the long mode but also the homogeneous gradient. We already saw that this change of coordinates is

$$ x^i \;\to\; x^i + \zeta_L x^i + (\nabla\zeta_L \cdot \vec{x})x^i - \frac{1}{2}\vec{x}^2\nabla^i\zeta_L + \delta x^i(t) \;, \tag{4.14} $$

where the time-dependent translation is such that $N_L^i(t) = -\delta\dot{x}^i(t)$. The explicit form of this term is irrelevant, since all correlation functions are translationally invariant and in what follows we will drop it. Notice that in this expression ζ_L is just a homogenous part of the long mode while $\nabla\zeta_L$ is a homogeneous gradient: $\zeta_L(x) = \zeta_L + \nabla\zeta_L \cdot \vec{x}$. We also assume that ζ_L and $\nabla\zeta_L$ are evaluated at $\vec{x}_+$. As we will see, the choice of this point is irrelevant, but for concreteness we will use the midpoint $\vec{x}_+ \equiv (\vec{x}_1 + \ldots + \vec{x}_n)/n$.

Using the explicit form of the change of coordinates, the variation of the n-point function in momentum space is given by

$$ \delta\langle\zeta(x_1)\cdots\zeta(x_n)\rangle = \int \frac{d\vec{q}_+}{(2\pi)^3}dK_n \; \zeta_{\vec{q}_+} \sum_a \mathcal{M}\delta(\vec{P}) $$
$$ \times \left(k_{ai}\partial_{k_{ai}} + q_{+j}k_{ai}\partial_{k_{ai}}\partial_{k_{aj}} - \frac{1}{2}q_{+i}k_{ai}\partial_{k_{aj}}\partial_{k_{aj}} \right) e^{i\vec{q}_+\cdot\vec{x}_+ + i\vec{k}_b\cdot\vec{x}_b} \;. \tag{4.15} $$

The only thing that remains to be done is to do some integrations by parts and rewrite this expression such that all derivatives act on $\mathcal{M}$ and the delta function factors out. The first term in brackets comes from the rescaling of coordinates and we already saw how to deal with it. It will contribute to the final result as

$$\delta\langle\zeta(x_1)\cdots\zeta(x_n)\rangle \supset -\int\frac{\mathrm{d}\vec{q}_+}{(2\pi)^3}\mathrm{dK}_n\,\zeta_{\vec{q}_+}\left(3(n-1)+\sum_a k_{ai}\partial_{k_{ai}}\right)\mathcal{M}\delta(\vec{P})e^{i\vec{q}_+\cdot\vec{x}_++i\vec{k}_b\cdot\vec{x}_b}\ . \tag{4.16}$$

The nontrivial exercise is to rewrite the other two terms in eq. (4.15). If we do two integrations by parts we get

$$\delta\langle\zeta(x_1)\cdots\zeta(x_n)\rangle \supset \int\frac{\mathrm{d}\vec{q}_+}{(2\pi)^3}\mathrm{dK}_n\,\zeta_{\vec{q}_+}$$
$$\times\sum_a\left(q_{+j}\partial_{k_{ai}}\partial_{k_{aj}}(k_{ai}\mathcal{M}\delta(\vec{P}))-\frac{1}{2}q_{+i}\partial_{k_{aj}}\partial_{k_{aj}}(k_{ai}\mathcal{M}\delta(\vec{P}))\right)e^{i\vec{q}_+\cdot\vec{x}_++i\vec{k}_b\cdot\vec{x}_b}\ . \tag{4.17}$$

Let us look at the integrand in more detail. If we expand the derivatives we get

$$\partial_{k_{ai}}\partial_{k_{aj}}(k_{ai}\mathcal{M}\delta(\vec{P})) = 4\partial_{k_{aj}}\mathcal{M}\cdot\delta(\vec{P})+k_{ai}\partial_{k_{ai}}\partial_{k_{aj}}\mathcal{M}\cdot\delta(\vec{P})$$
$$+4\mathcal{M}\partial_{k_{aj}}\delta(\vec{P})+k_{ai}\partial_{k_{aj}}\mathcal{M}\partial_{k_{ai}}\delta(\vec{P})+k_{ai}\partial_{k_{ai}}\mathcal{M}\partial_{k_{aj}}\delta(\vec{P})$$
$$+k_{ai}\mathcal{M}\partial_{k_{ai}}\partial_{k_{aj}}\delta(\vec{P})\ . \tag{4.18}$$

Similarly, the other term is

$$-\frac{1}{2}\partial_{k_{aj}}\partial_{k_{aj}}(k_{ai}\mathcal{M}\delta(\vec{P})) = -\partial_{k_{ai}}\mathcal{M}\cdot\delta(\vec{P})-\frac{1}{2}k_{ai}\partial_{k_{aj}}\partial_{k_{aj}}\mathcal{M}\cdot\delta(\vec{P})$$
$$-\mathcal{M}\partial_{k_{ai}}\delta(\vec{P})-k_{ai}\partial_{k_{aj}}\mathcal{M}\partial_{k_{aj}}\delta(\vec{P})$$
$$-\frac{1}{2}k_{ai}\mathcal{M}\partial_{k_{aj}}\partial_{k_{aj}}\delta(\vec{P})\ . \tag{4.19}$$

All terms in the first lines of these two expressions are proportional to $\delta(\vec{P})$ and we do not have to do any additional work to bring them to the right form. They will contribute to the final result as

$$\delta\langle\zeta(x_1)\cdots\zeta(x_n)\rangle \supset \int\frac{\mathrm{d}\vec{q}_+}{(2\pi)^3}\mathrm{dK}_n\,\zeta_{\vec{q}_+}$$
$$\times\sum_a\frac{1}{2}q_{+i}\left(6\partial_{k_{ai}}+2k_{aj}\partial_{k_{aj}}\partial_{k_{ai}}-k_{ai}\partial_{k_{aj}}\partial_{k_{aj}}\right)\mathcal{M}\cdot\delta(\vec{P})e^{i\vec{q}_+\cdot\vec{x}_++i\vec{k}_b\cdot\vec{x}_b}\ . \tag{4.20}$$

Now we can focus on terms with one derivative on $\delta(\vec{P})$. They are in the second lines of equations (4.18) and (4.19). If we combine them we get two kind of contributions. One will be of the form

$$\sum_a q_i\partial_{P_j}\delta(\vec{P})\cdot\left(k_{aj}\partial_{k_{ai}}-k_{ai}\partial_{k_{aj}}\right)\mathcal{M}\ . \tag{4.21}$$

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This is just a rotation operator acting on $\mathcal{M}$, and since the correlation functions are rotationally invariant the action of this operator is zero. The remaing contribution is just

$$\delta\langle\zeta(x_1)\cdots\zeta(x_n)\rangle \supset \int\frac{\mathrm{d}\vec{q}_+}{(2\pi)^3}\mathrm{dK}_n\,\zeta_{\vec{q}_+}$$

$$\times\left(3n + \sum_a k_{ai}\partial_{k_{ai}}\right)\mathcal{M}\cdot q_{+i}\partial_{P_i}\delta(\vec{P})e^{i\vec{q}_+\cdot\vec{x}_+ + i\vec{k}_b\cdot\vec{x}_b}\,. \tag{4.22}$$

Notice that this is almost the form of the dilation operator. The missing part comes from the terms with two derivatives on the delta function. After we do two integrations by parts, they are schematically of the form

$$q_j\delta(\vec{P})\cdot\partial_{P_i}\partial_{P_j}(P_i\cdot X) = 4q_j\delta(\vec{P})\cdot\partial_{P_j}X + q_j\delta(\vec{P})P_i\cdot\partial_{P_i}\partial_{P_j}X\,. \tag{4.23}$$

and

$$-\frac{1}{2}q_j\delta(\vec{P})\cdot\partial_{P_i}\partial_{P_i}(P_j\cdot X) = -q_j\delta(\vec{P})\cdot\partial_{P_j}X - \frac{1}{2}q_j\delta(\vec{P})P_j\cdot\partial_{P_i}\partial_{P_i}X\,. \tag{4.24}$$

Terms proportional to $P_i\delta(\vec{P})$ under the integral give zero. Therefore, the only remaining contribution to the variation of the n-point function, after another integration by parts, is just given by

$$\delta\langle\zeta(x_1)\cdots\zeta(x_n)\rangle \supset \int\frac{\mathrm{d}\vec{q}_+}{(2\pi)^3}\mathrm{dK}_n\,\zeta_{\vec{q}_+}q_{+i}(-3)\,\mathcal{M}\cdot\partial_{P_i}\delta(\vec{P})e^{i\vec{q}_+\cdot\vec{x}_+ + i\vec{k}_b\cdot\vec{x}_b}\,. \tag{4.25}$$

Combining equations (4.16), (4.22) and (4.25), we get at the end

$$\delta\langle\zeta(x_1)\cdots\zeta(x_n)\rangle \supset -\int\frac{\mathrm{d}\vec{q}_+}{(2\pi)^3}\mathrm{dK}_n\,\zeta_{\vec{q}_+}$$

$$\times\left(3(n-1) + \sum_a k_{ai}\partial_{k_{ai}}\right)\mathcal{M}\cdot(1 - q_{+i}\partial_{P_i})\delta(\vec{P})e^{i\vec{q}_+\cdot\vec{x}_+ + i\vec{k}_b\cdot\vec{x}_b}\,. \tag{4.26}$$

Notice that all these complicated contributions at the end just give the dilation operator multiplying $(1 - q_{+i}\partial_{P_i})\delta(\vec{P})$. It was crucial to bring the variation of the n-point function in this form, because we can write

$$(1 - q_{+i}\partial_{P_i})\delta(\vec{P}) = \delta(\vec{P} - \vec{q}_+) \tag{4.27}$$

up to corrections of order $\mathcal{O}(q_+^2)$. After we average over the long mode $\zeta_{\vec{q}}$ and integrate in $\vec{q}_+$, this delta function will become $\delta(\vec{P} + \vec{q})$, which is exactly what we have on the left hand side when we write the Fourier transform of the $(n+1)$-point function. This kind of manipulation is not necessary for the expression (4.20) because it is already first order in $\vec{q}_+$.

After this long and rather technical procedure, summing up (4.20) and (4.26) and averaging over the long mode $\zeta_{\vec{q}}$, we get the final result: the consistency relation valid up to corrections of order $\mathcal{O}(q^2/k^2)$

$$\langle\zeta_{\vec{q}}\zeta_{\vec{k}_1}\ldots\zeta_{\vec{k}_n}\rangle'_{q\to 0} = -P(q)\left[3(n-1) + \sum_a \vec{k}_a\cdot\vec{\partial}_{k_a} + \frac{1}{2}q^i\delta^{(n)}_{\mathcal{K}^i} + \mathcal{O}(q/k)^2\right]\langle\zeta_{\vec{k}_1}\ldots\zeta_{\vec{k}_n}\rangle'\,, \tag{4.28}$$

where the operator $\delta_{\mathcal{K}i}^{(n)}$ is the operator of special conformal transformation acting in momentum space

$$q^i \delta_{\mathcal{K}i}^{(n)} \equiv \sum_{a=1}^{n} \left[6\vec{q} \cdot \vec{\partial}_{k_a} - \vec{q} \cdot \vec{k}_a \vec{\partial}_{k_a}^2 + 2\vec{k}_a \cdot \vec{\partial}_{k_a} (\vec{q} \cdot \vec{\partial}_{k_a}) \right] . \tag{4.29}$$

In the case where the n-point function depends only on the magnitudes of the momenta $|\vec{k}_a|$, this operator simply becomes [40]

$$\sum_{a=1}^{n} \vec{q} \cdot \vec{k}_a \left[\frac{4}{k_a} \frac{\partial}{\partial k_a} + \frac{\partial^2}{\partial k_a^2} \right] . \tag{4.30}$$

Notice that the procedure above implies that the *same* delta function of momentum conservation is removed from the correlation functions on both sides. For conformal consistency relation this is not relevant because the variation of the n-point function is explicitly proportional to q. However, this statement is not trivial for the scale variation. What we actually see is that the n-point function has to be evaluated for a configuration of momenta $\vec{k}_1 \ldots, \vec{k}_n$ that *do not* form a closed polygon, but $\vec{k}_1 + \cdots + \vec{k}_n = -\vec{q}$. Operationally, this means that in order to properly calculate $\mathcal{O}(q)$ terms in the squeezed limit, on has to apply the first operator in eq. (4.28) on the n-point function and after that expand it in q having in mind that $\sum \vec{k}_a$ is not zero. We will check this explicitly in some examples later.

The procedure we described might seem a bit confusing and ambiguous. The correlation functions that we calculate using the in-in formalism are always such that the sum of momenta is equal to zero. Therefore, it seems that there are many possible extensions to go to configurations where momenta do not form a closed polygon. In other words, we can always add to the n-point function a term of the form $\mathcal{M}_{add} = \vec{P} \cdot \vec{F}(\vec{k})$, where F is an arbitrary function. On the other hand, it so not obvious that this does not change the right hand side of the consistency relation.

In the short calculation that follows we are going to show that indeed this is exactly what happens and the consistency relation is unambiguous. To do that we will calculate the action of the operator

$$\left(3(n-1) + \sum_a k_{ai} \partial_{k_{ai}} + \frac{1}{2} q_i \delta_{\mathcal{K}i}^{(n)} \right) \tag{4.31}$$

on the additional amplitude $\mathcal{M}_{add}$. It is straightforward to find that, keeping only linear terms in $\vec{q}$, we get that the action of the operator is given by

$$3(n-1)P_i F_i + \sum_a k_{aj}(F_i + P_i \partial_{k_{aj}} F_i) + 3\sum_a q_i F_i - \sum_a q_j k_{aj} \partial_{k_{ai}} F_i$$
$$+ \sum_a q_j k_{ai} \partial_{k_{ai}} F_j + \sum_a q_j k_{ai} \partial_{k_{aj}} F_i . \tag{4.32}$$

After some trivial algebra, these terms can be rearranged in the following way

$$(P_i + q_i) \left[3(n-1) + \sum_a k_{al} \partial_{k_{al}} \right] F_i + 2q_i F_i + q_j \sum_a (k_{ai} \partial_{k_{aj}} - k_{aj} \partial_{k_{ai}}) F_i . \tag{4.33}$$

The first term is zero because $\vec{P} + \vec{q} = 0$. The other two terms cancel after we rewrite the function F in the following way

$$F_i = \sum_a k_{ai} A_a \,, \tag{4.34}$$

where A_a are scalar functions of the momenta and they are rotationally invariant. Indeed, this can be always done without loss of generality.

4.1.3 Comments

Before we move to checks of the consistency relations, let us pause for a bit and comment on some of their properties. First of all, we derived the consistency relations without specifying the details of the underlying physics driving inflation and creating the perturbations. In particular, although the consistency relations involve correlation functions, we did not have to specify any interactions. The only assumption we made is that the long mode always locally reduces to a diffeomorphism. This is always true in any single-field model of inflation if the inflaton is on the attractor[2]. Indeed, if we have only one dynamical degree of freedom during inflation, then all parts of the universe follow the same classical history and quantum fluctuations of the inflaton are locally equivalent to a small shift of this classical history in time. As a consequence, different regions of the universe expand by different amounts, but they all have the same history. Obviously, the perturbations created in this way are adiabatic.

Under these assumptions, once the long mode crosses the horizon, it becomes frozen and remains constant. For short modes its effect is not physical because it describes only a small perturbation in the expansion of the universe that can be removed just by a rescaling of coordinates. This is exactly what we used in the previous section to derive the consistency relations. To summarise, in any single-field model of inflation (assuming that the inflaton is on the dynamical attractor and choosing the Bunch-Davies vacuum), we expect the consistency relations to hold, independently on the details of interactions, the form of the kinetic term, slow-roll assumptions or any other details of the theory [45].

One immediate consequence of this statement is that in single-field models we expect small local non-Gaussianities. Indeed, one can check that the action of conformal operator on the two-point function vanish at order $\mathcal{O}(q)$. On the right hand side of the consistency relation remains only the dilation operator. Given the approximate scale invariance of the two-point function, in the squeezed limit of the three-point function we are left with

$$\langle \zeta_{\vec{q}} \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle'_{\vec{q} \to 0} = - \left[(n_s - 1) + \mathcal{O}(q^2/k^2) \right] P(q) P(k) \,. \tag{4.35}$$

From this equation we can immediately read that in any single-field model, the amplitude of local non-Gaussianities $f_{\rm NL}^{\rm loc.}$ is proportional to the tilt of the power spectrum. Of course, this makes the local non-Gaussianities unobservable in the near future. However, the main point of this result is that if one observes $f_{\rm NL}^{\rm loc.} \gg 1$, this would in one shot *rule out all single-field*

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[2]Notice that here we are assuming standard initial conditions—Bunch-Davies vacuum. We will comment on this assumption later.

models [45]. Therefore, phenomenologically it is much more interesting to look for violations of the consistency relations, than to try to check if they hold. The current constraints on non-Gaussianities from Planck are compatible with single-field models. Indeed, the limit is $f_{\rm NL}^{\rm loc.} = 2.7 \pm 5.8$ [13].

Notice that eq. (4.35) is written in ζ-gauge and the reader might be worried that going to observables in the CMB some of the conclusions can change. Although the long mode is always locally equivalent to an unobservable change of coordinates, the coordinate transformation can vary from one part of the sky to the other and this can induce some contribution to the three-point function of temperature fluctuations. Indeed, the squeezed limit of the CMB bispectrum can be calculated analytically, and the corresponding $f_{\rm NL}^{\rm loc.}$ is given by [64]

$$f_{\rm NL}^{\rm loc.} = -\frac{1}{6}(1 + 6\cos 2\theta)\left(1 - \frac{1}{2}\frac{{\rm d}\ln(l_S^2 C_{l_S})}{{\rm d}\ln l_S}\right) , \qquad (4.36)$$

where θ is the relative angle between the long and short modes and C_l the standard temperature angular power spectrum. From this expression we see that in single-field models local non-Gaussianities are always small, at most $f_{\rm NL}^{\rm loc.} \sim \mathcal{O}(1)$.

On a more theoretical side, being a non-perturbative and a model-independent result, the consistency relations are a powerful tool and have many applications in the study of single-field inflationary models. For example, using the consistency relations one can explore the infrared divergences in calculations of the correlation functions involving loops [92]. They are also an important ingredient in the proof that ζ is conserved outside the horizon, to all orders in perturbation theory [91, 92, 93, 94, 95].

Before moving to some more technical comments, let us briefly come back to the assumptions that enter the derivation of the consistency relation and see what happens if some of them are violated. One obvious choice is multi-field inflation. Although in models with more that one dynamical degree of freedom it might happen that accidentally the local non-Gaussianities are small too, this is not a generic case. Indeed, multifield models typically predict large $f_{\rm NL}^{\rm loc.}$. Some examples include the curvaton mechanism [37, 38] and inhomogeneous reheating [110] (for a review see [111]). Even in single-field models, if the inflaton is not on the attractor, large local non-Gaussianities can be still generated. This kind of violations of the consistency relations were recently studied in [112]. The same holds if we change the initial state. In a non-Bunch-Davies vacuum, explicit calculations show that the three-point function in the squeezed limit can be large [113]. For some recent discussion on this subject see [114, 115]. Let us at the end point out another interesting example of a single-field inflationary model in which the consistency relation is (in principle) violated—Khronon inflation [34]. The details of the model are briefly summarised in Appendix A. Let us here just point out that, due to a peculiar $1/a(t)$ decay of the decaying modes in Khronon inflation, it is possible to violate the consistency relations even though all other conditions are satisfied. Unfortunately, this violation turns out to be very small and suppressed by the small breaking of the time reparametrization symmetry.

Let us finally come back to eq. (4.28) and comment on some of the consequences for correlation functions and interactions in the Lagrangian. As we already pointed out, Maldacena's

consistency relation is somewhat trivial due to the scale invariance. On the other side, the conformal consistency relation carries more information, but only for higher order correlation functions. For example, the conformal variation of the two-point function vanishes (up to terms which are singular in Fourier space, see [40]). This is obvious from the fact that the two-point function does not specify any direction with which the vector $\vec{b}$ can be contracted. This implies that the conformal consistency relation just implies that the three-point function is suppressed, compared with a local shape, by (at least) $\mathcal{O}(q/k)^2$ [47].

Much less trivial is the implication of eq. (4.28) for the squeezed limit of the four-point function, as we are now going to discuss. First of all, at the level of the Lagrangian, it is well known that operators with a different number of fields are related due to the non-linear realization of the Lorentz symmetry [25]. The conformal consistency relations contain the same information at the level of observable correlation functions. For example, for $c_s = 1$, the operator $(g^{00}+1)^3$ in the language of the effective field theory of inflation [25], contributes both to the three-point function and to the four-point function. This is encoded in the conformal consistency relation, as the conformal variation of the three-point function must be matched by a corresponding (squeezed limit of the) four-point function.

Similarly, if an operator only contributes to the four-point function and not to the three-point function, then this four-point function must have a suppressed squeezed limit, i.e. the coefficient of the q^{-2} divergence must vanish. This is indeed what happens for the operator $(g^{00}+1)^4$. In [116] it was shown that it is technically natural to make this operator large and the four-point function arbitrarily large compared to the three-point function: in any case this happens for any n-point function, its q^{-2} divergence must vanish.

Finally, the conformal consistency relation eq. (4.28) states that the q^{-2} divergence of the four-point function is *always small*, of order $P_\zeta^{1/2}$ times the three-point function. This is not at all obvious at the level of the action. Indeed, the operator which reduces the speed of sound gives a four-point function of order $P_\zeta^{1/2} c_s^{-2}$ compared with the three-point function. Moreover, this operator is of the form $(\nabla \pi)^4$ so that it naively gives rise to a q^{-2} squeezed limit. It turns out that, in going to the Hamiltonian, the term $(\nabla \pi)^4$ gets a c_s^2 suppression while the leading source of the four-point function ($\propto c_s^{-4}$) is given by the exchange diagrams which, as discussed above, do not contribute to the q^{-2} divergence. The same kind of cancellation occurs for the operator $\lambda \dot{\pi}^2 (\nabla \pi)^2$ which, in the small c_s limit, would naively give a four-point function which is too large in the squeezed limit to satisfy eq. (4.28). Again, this term is cancelled in going to the Hamiltonian and gets suppressed by c_s^2, while the leading four-point function is given by the exchange diagrams. In the following section we will explicitly check some of these results.

4.1.4 Checks

Most of the correlation functions for different inflationary models are calculated in the limit of the exact scale-invariance. For this reason, most of the checks of Maldacena's consistency relation that we can do boil down to explicitly proving that at leading order the correlation

functions vanish in the limit $\vec{q} \to 0$. If deviations from the perfect scale-invariance are included in the calculation, then the checks become more nontrivial and one gets nonzero result on both sides of the consistency relation. For example, one can calculate the three-point function in models with reduced speed of sound including slow-roll corrections. Then, taking the squeezed limit, after a long calculation, one indeed recovers that the right hand side is simply proportional to the tilt of the power spectrum as expected (for the details see [117]).

$$\langle \zeta_{\vec{q}} \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle'_{\vec{q} \to 0} = -(n_s - 1) P(q) P(k) \ . \tag{4.37}$$

On the other hand, the conformal consistency relation gives a non-trivial check even in the limit of exact scale invariance. Unfortunately, as we pointed out, the action of the conformal operator on the two-point function is always zero. This means that the checks have to include higher order correlation functions that are more difficult to calculate. We are going to show a couple of examples involving three-point and four-point functions in different classes of single-field models.

Resonant non-Gaussianities. Let us first focus on example of models in which scale-invariance is broken and we can check that eq. (4.13) holds. We are going to consider models with a periodic modulation of the inflaton potential that have recently attracted some attention, as motivated by explicit string constructions of Monodromy Inflation [118]. These models represent an interesting example in which the n-point functions deviate sizeably from scale-invariance and in which many n-point functions can be analytically calculated at leading order in the parameter describing the potential modulation [119, 120].

The $(n+1)$-point function reads (see Appendix A for more details)

$$\langle \zeta_{\vec{q}} \zeta_{\vec{k}_1} \ldots \zeta_{\vec{k}_n} \rangle = (2\pi)^3 \delta(\vec{q} + \vec{k}_1 + \ldots + \vec{k}_n) \left(-\frac{H}{\dot{\phi}} \right)^{n+1} \frac{H^{2n-2}}{2q^3 \prod_{i=1}^{n} 2k_i^3} I_{n+1} \ , \tag{4.38}$$

where I_{n+1} is given by the following in-in integral:

$$I_{n+1} = -2 \operatorname{Im} \int_{-\infty - i\epsilon}^{0} \frac{\mathrm{d}\eta}{\eta^4} V^{(n+1)}(\phi(\eta))(1 - iq\eta)(1 - ik_1\eta) \ldots (1 - ik_n\eta) e^{ik_t\eta} \ . \tag{4.39}$$

Here $k_t = q + \sum_i k_i$ and $V^{(n+1)}$ is the $(n+1)$th derivative of the potential, evaluated on the unperturbed history $\phi(\eta)$. Now we can calculate the squeezed limit of this $(n+1)$-point function and check that it satisfies Maldacena's consistency relation. In the squeezed limit $q \to 0$, the expression above gives

$$\langle \zeta_{\vec{q}} \zeta_{\vec{k}_1} \ldots \zeta_{\vec{k}_n} \rangle'_{q \to 0} = -\left(-\frac{H}{\dot{\phi}} \right) \frac{H^2}{2q^3} \left(-\frac{H}{\dot{\phi}} \right)^{n} \frac{H^{2n-4}}{\prod_{i=1}^{n} 2k_i^3} 2 \operatorname{Im} \int_{-\infty}^{0} \frac{\mathrm{d}\eta}{\eta^4} V^{(n+1)} \prod_{i=1}^{n} (1 - ik_i\eta) e^{ik_i\eta} \ . \tag{4.40}$$

The trick that we are going to use is to convert the $(n+1)$th derivative of the potential using

$$\frac{d}{d\phi} V^{(n)}(\phi(\eta)) = -\frac{H}{\dot{\phi}} \left[\eta \frac{d}{d\eta} V^{(n)}(\phi(\eta)) \right] \tag{4.41}$$

and integrate by parts the derivative with respect to η. Given that in the expressions for the wavefunctions under the integral conformal time always appears in the combination $k_i\eta$, this derivative can be traded for derivatives with respect to k_i to give

$$-\left(\frac{H}{\dot\phi}\right)^2\frac{H^2}{2q^3}\left[3(n-1)+\sum_i k_i\frac{\partial}{\partial k_i}\right]\left(-\frac{H}{\dot\phi}\right)^n\frac{(-1)H^{2n-4}}{\prod_{i=1}^n 2k_i^3}2\,\mathrm{Im}\int_{-\infty}^0\frac{\mathrm{d}\eta}{\eta^4}V^{(n)}\prod_{i=1}^n(1-ik_i\eta)e^{ik_i\eta}$$

$$=-P(q)\left[3(n-1)+\sum_i k_i\frac{\partial}{\partial k_i}\right]\langle\zeta_{\vec k_1}\cdots\zeta_{\vec k_n}\rangle'\,. \tag{4.42}$$

In this way we proved that

$$\langle\zeta_{\vec q}\zeta_{\vec k_1}\cdots\zeta_{\vec k_n}\rangle'_{q\to 0}=-P(q)\left[3(n-1)+\sum_i k_i\frac{\partial}{\partial k_i}\right]\langle\zeta_{\vec k_1}\cdots\zeta_{\vec k_n}\rangle'\,, \tag{4.43}$$

which is nothing but Maldacena's consistency relation.

What about the conformal consistency relation? First, notice that the action of the operator $q^i\delta^{(n)}_{\mathcal{K}^i}$ on the n-point function vanishes. Given that the function only depends on the moduli of the wavevectors, the differential operator takes the simplified form (4.30)

$$\sum_{a=1}^n\vec q\cdot\vec k_a\left[\frac{4}{k_a}\frac{\partial}{\partial k_a}+\frac{\partial^2}{\partial k_a^2}\right]\langle\zeta_{\vec k_1}\cdots\zeta_{\vec k_n}\rangle'=0\,. \tag{4.44}$$

To see that this expression vanishes, it is enough to compute the action of the operator in brackets for one particular momentum k_a. The result does not depend on a (indeed the form of the amplitude remains the same with an additional $-\eta^2$ that appears in the integral). This means that the sum will be proportional to $\vec q\cdot\sum_{a=1}^n\vec k_a$, which is zero up to q^2 corrections.

This result seems to imply two things. First, the squeezed limit of any $(n+1)$-point function does not have $\mathcal{O}(q)$ corrections. Indeed, it can be checked explicitly that this is the case. The second thing is that it seems that all correlation functions in these models are invariant under special conformal transformations. This cannot be true because correlation functions are not scale-invariant, due to the features in the potential, and there is no such thing as invariance under special conformal transformations without scale-invariance (indeed, the commutator of a special conformal transformation with a translation contains a dilation). As we discussed above, there is an additional singular piece of the special conformal transformation, proportional to the derivative of the delta function and to the scale variation of the n-point function—which is not zero in this case—and this was used to derive eq. (4.28). The vanishing of the conformal operator in Fourier space tells us that in these models, where there is no operator with spatial derivatives that enters at order q in the consistency relation, the conformal transformation of the n-point function is just fixed by its dilation properties.

Models with reduced speed of sound. Following Appendix A, an n-point correlation function in these models can be written as

$$\langle\zeta_{\vec k_1}\cdots\zeta_{\vec k_n}\rangle=(2\pi)^3\delta(\vec k_1+\cdots+\vec k_n)P_\zeta^{n-1}\prod_{i=1}^n\frac{1}{k_i^3}\mathcal{M}^{(n)}(\vec k_1,...,\vec k_n)\,, \tag{4.45}$$

where P_ζ is defined as

$$P_\zeta = \frac{1}{2M_{\text{Pl}}^2} \frac{H^2}{c_s \epsilon} \, , \tag{4.46}$$

with $\epsilon \equiv -\dot{H}/H^2$. The amplitude of the three-point function is given by

$$\mathcal{M}^{(3)} = \left(\frac{1}{c_s^2} - 1 - \frac{2\lambda}{\Sigma} \right) \frac{3k_1^2 k_2^2 k_3^2}{2k_t^3} + \left(\frac{1}{c_s^2} - 1 \right) \left(-\frac{1}{k_t} \sum_{i>j} k_i^2 k_j^2 + \frac{1}{2k_t^2} \sum_{i \neq j} k_i^2 k_j^3 + \frac{1}{8} \sum_i k_i^3 \right) . \tag{4.47}$$

The speed of sound c_s and the parameters λ and Σ are determined by the function $P(X)$ (see Appendix A) and we will work under the assumption that they are constants.

The four-point function is a bit more complicated and contains two kind of contributions. One comes from diagrams with a quartic interaction while the other from diagrams in which one scalar mode is exchanged. It is easy to see that scalar-exchange diagrams do not contribute to the conformal consistency relation since they vanish as k_4^2 for $k_4 \to 0$. Let first write down the cubic Hamiltonian

$$\mathcal{H}^{(3)} = -\frac{a}{H^3} \Sigma(1 - c_s^2) \zeta'(\partial_i \zeta)^2 . \tag{4.48}$$

Form the form of the interaction it is obvious that when a time derivative acts on the soft external leg with momentum k_4, it gives a subleading term in k_4, as the time derivative of the wavefunction gives k_4^2. The same holds when a spatial derivative acts on the soft external leg. Indeed, at the relevant vertex we will have external momenta $\vec{k}_4$ and (say) $\vec{k}_3$ and an internal momentum $-\vec{k}_4 - \vec{k}_3$. Depending on which leg the second spatial derivative acts, we have two diagrams proportional to $\vec{k}_4 \cdot \vec{k}_3$ and $-\vec{k}_4 \cdot (\vec{k}_4 + \vec{k}_3)$. They cancel each other at leading order, leaving terms of order k_4^2. This behaviour can be checked in the explicit results of [121, 122].

Therefore, in order to check the conformal consistency relation, we can concentrate on the diagrams with a quartic interaction. Their contribution to the amplitude is given by [121]

$$\mathcal{M}_{cont}^{(4)} = \left[\frac{3}{2} \left(\frac{\mu}{\Sigma} - \frac{9\lambda^2}{\Sigma^2} \right) \frac{\prod_{i=1}^4 k_i^2}{k_t^5} - \frac{1}{8} \left(\frac{3\lambda}{\Sigma} - \frac{1}{c_s^2} + 1 \right) \frac{k_1^2 k_2^2 (\vec{k}_3 \cdot \vec{k}_4)}{k_t^3} \right.$$
$$\times \left(1 + \frac{3(k_3 + k_4)}{k_t} + \frac{12k_3 k_4}{k_t^2} \right) + \frac{1}{32} \left(\frac{1}{c_s^2} - 1 \right) \frac{(\vec{k}_1 \cdot \vec{k}_2)(\vec{k}_3 \cdot \vec{k}_4)}{k_t}$$
$$\left. \times \left(1 + \frac{\sum_{i<j} k_i k_j}{k_t^2} + \frac{3k_1 k_2 k_3 k_4}{k_t^3} \sum_{i=1}^4 \frac{1}{k_i} + \frac{12k_1 k_2 k_3 k_4}{k_t^4} \right) \right] + 23 \text{ perm.} , \tag{4.49}$$

where the parameter μ is defined through $P(X)$ (see Appendix A). We are interested in terms linear in k_4. It is quite straightforward to get

$$\mathcal{M}_{cont}^{(4)}|_{\vec{k}_4 \to 0} = \left[-\frac{1}{2} \left(\frac{3\lambda}{\Sigma} - \frac{1}{c_s^2} + 1 \right) \frac{k_1^2 k_2^2 (\vec{k}_3 \cdot \vec{k}_4)}{k_t^3} \left(1 + \frac{3k_3}{k_t} \right) \right.$$
$$\left. + \frac{1}{4} \left(\frac{1}{c_s^2} - 1 \right) \frac{(\vec{k}_1 \cdot \vec{k}_2)(\vec{k}_3 \cdot \vec{k}_4)}{k_t} \left(1 + \frac{k_1 k_2 + k_1 k_3 + k_2 k_3}{k_t^2} + \frac{3k_1 k_2 k_3}{k_t^3} \right) \right] + 2 \text{ terms} , \tag{4.50}$$

where the two additional terms have the same form with $\vec{k}_1 \leftrightarrow \vec{k}_3$ and $\vec{k}_2 \leftrightarrow \vec{k}_3$. Notice that, since in the squeezed limit $\vec{k}_1$, $\vec{k}_2$ and $\vec{k}_3$ form a triangle, we can make the following replacements $(\vec{k}_1 \cdot \vec{k}_2) \rightarrow \frac{1}{2}(k_3^2 - k_1^2 - k_2^2)$. Factoring out $(\vec{k}_3 \cdot \vec{k}_4)$, we finally see that the expression has the same structure as the right-hand side of the conformal consistency relation

$$\sum_{i=1}^{3} (\vec{k}_i \cdot \vec{k}_4) \left(\frac{4}{k_i} \frac{\partial}{\partial k_i} + \frac{\partial^2}{\partial k_i^2} \right) \langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \rangle . \tag{4.51}$$

Indeed it is straightforward to check that eq. (4.28) is satisfied. (To simplify the calculations one can choose, without loss of generality, $\vec{k}_4 = (1, 0, 0)$. Notice that one has to impose $\vec{k}_1 + \vec{k}_2 + \vec{k}_3 = 0$ at the end, after taking all the derivatives.)

Single-field slow-roll inflation. Let us verify the conformal consistency relation (4.28) for $n = 3$, for the case of slow-roll inflation with a standard kinetic term. The three-point function is the well-known Maldacena's result [43]

$$\langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \rangle' = \frac{H^4}{4\epsilon^2 M_P^4} \frac{1}{\prod(2k_i^3)} \left[(2\eta - 3\epsilon) \sum_i k_i^3 + \epsilon \sum_{i \neq j} k_i k_j^2 + \epsilon \frac{8}{k_t} \sum_{i > j} k_i^2 k_j^2 \right], \tag{4.52}$$

where $k_t = k_1 + k_2 + k_3$.

As in the case of models with small speed of sound, the four-point function is the sum of two comparable contributions, one coming from the quartic interactions of ζ [123] and the other from a graviton exchange[3] [124]. The contribution from the graviton exchange is irrelevant for the check of the conformal consistency relation as it decays too quickly in the squeezed limit $q \rightarrow 0$. This is easy to see from the cubic vertex that enters (twice) in the diagram:

$$S \sim \int d^4x \, \partial_i \zeta \partial_j \zeta \gamma_{ij} . \tag{4.53}$$

As the graviton is transverse, we can rewrite this vertex moving the two derivatives on the leg with momentum q. This implies that the graviton exchange gives a contribution suppressed by two powers of q in the squeezed limit compared with a local shape and this does not contribute to eq. (4.28).

The four-point function coming from the contact diagram has a very complicated momentum dependence given by [123]

$$\langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \zeta_{\vec{k}_4} \rangle'_{\text{cont}} = \frac{H^6}{4\epsilon^3 M_P^6} \frac{1}{\prod(2k_i^3)} \epsilon \sum_{perm} \mathcal{M}_4(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) , \tag{4.54}$$

with

$$\mathcal{M}_4(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) = -2 \frac{k_1^2 k_3^2}{k_{12}^2 k_{34}^2} \frac{W_{24}}{k_t} \left(\frac{\vec{Z}_{12} \cdot \vec{Z}_{34}}{k_{34}^2} + 2\vec{k}_2 \cdot \vec{Z}_{34} + \frac{3}{4} \sigma_{12} \sigma_{34} \right) -$$

$$- \frac{1}{2} \frac{k_3^2}{k_{34}^2} \sigma_{34} \left(\frac{\vec{k}_1 \cdot \vec{k}_2}{k_t} W_{124} + 2 \frac{k_1^2 k_2^2}{k_t^3} + 6 \frac{k_1^2 k_2^2 k_4}{k_t^4} \right) , \tag{4.55}$$

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[3]In principle, the scalar exchange contributes to the result too, but it is subdominant [123].

$$\sigma_{ab} = \vec{k}_a \cdot \vec{k}_b + k_b^2 \,, \quad \vec{Z}_{ab} = \sigma_{ab}\vec{k}_a - \sigma_{ba}\vec{k}_b \,, \quad W_{ab} = 1 + \frac{k_a + k_b}{k_t} + \frac{2k_a k_b}{k_t^2} \,, \tag{4.56}$$

$$W_{abc} = 1 + \frac{k_a + k_b + k_c}{k_t} + \frac{2(k_a k_b + k_b k_c + k_a k_c)}{k_t^2} + \frac{6k_a k_b k_c}{k_t^3} \,. \tag{4.57}$$

It is easy to verify that the first term in brackets in eq. (4.52) has vanishing conformal variation and thus does not contribute to eq. (4.28). This must be the case as the four-point function does not depend on η. The right-hand side of the conformal consistency relation therefore reads

$$\frac{H^6}{8\epsilon^3 M_P^6} \frac{1}{2q^3 \prod(2k_i^3)} \left(-\frac{\epsilon}{2}\right) \sum_{a=1}^{3} \vec{q} \cdot \vec{k}_a \left[\frac{4}{k_a} \frac{\partial}{\partial k_a} + \frac{\partial^2}{\partial k_a^2}\right] \left(\sum_{i \neq j} k_i k_j^2 + \frac{8}{k_t} \sum_{i > j} k_i^2 k_j^2\right) \,. \tag{4.58}$$

One can explicitly check that this indeed reproduces the $1/q^2$ behaviour of the four-point function (4.54).

4.2 Consistency Relations for Gravitational Waves

In the previous Sections we discussed the consistency relations involving long scalar modes. Given that we showed that a long gravitational wave is locally just a change of coordinates, we can derive similar relations involving long tensors too. At zeroth order in gradients the consistency relations for gravitational waves were first derived in [43], and including gradients in [48]. Here we will briefly repeat the derivation and show a check with the three-point function with one soft graviton and two scalar modes.

4.2.1 Derivation

The procedure to derive the consistency relations with one long tensor mode is essentially the same as for scalars. We only have to replace ζ_L with $\gamma_{ij,L}$. As before we can start from the observation that the presence of a long gravitational wave is equivalent to a change of coordinates

$$\langle \zeta(\vec{x}_1) \ldots \zeta(\vec{x}_n) \rangle_\gamma = \langle \zeta(\tilde{\vec{x}}_1) \ldots \zeta(\tilde{\vec{x}}_n) \rangle \,. \tag{4.59}$$

We only have to average this expression over the long mode and go to momentum space. The crucial object to calculate is again the variation of the n-point function

$$\delta \langle \zeta(\vec{x}_1) \cdots \zeta(\vec{x}_n) \rangle = \sum_{a=1}^{n} \delta x_{ai} \cdot \partial_{ai} \langle \zeta(\vec{x}_1) \cdots \zeta(\vec{x}_n) \rangle \,. \tag{4.60}$$

The coordinate transformation is in this case given by (see Section 3.2.2)

$$x_i \to x_i + A_{ij}x_j + B_{ijk}x_j x_k \,, \tag{4.61}$$

where the parameters of the transformation are related to the long tensor mode in the following way

$$A_{ij} = \frac{1}{2}\gamma_{ij,L} , \quad B_{ijk} = \frac{1}{4}(\partial_k\gamma_{ij,L} - \partial_i\gamma_{jk,L} + \partial_j\gamma_{ik,L}) . \quad (4.62)$$

Going to momentum space is rather long and technical, but it does not contain any additional subtleties compared to the case of scalars. After a lot of integrations by parts, one again finds that, up to corrections of order $\mathcal{O}(q^2)$, all terms with derivatives on delta functions rearrange such that the delta function is the same on both sides of the consistency relation. We will quote here just the final result, leaving the details of the calculation for the Appendix B. The consistency relation with a long gravitational wave, including gradient corrections, is given by

$$\langle \gamma_{\vec{q}}^s \zeta_{\vec{k}_1} \cdots \zeta_{\vec{k}_n} \rangle'_{q\to 0} = -\frac{1}{2}P_\gamma(q) \sum_a \epsilon_{ij}^s(\vec{q}) k_{ai}\partial_{k_{aj}} \langle \zeta_{\vec{k}_1} \cdots \zeta_{\vec{k}_n} \rangle'$$

$$-\frac{1}{4}P_\gamma(q) \sum_a \epsilon_{ij}^s(\vec{q}) \left(2k_{ai}(\vec{q}\cdot\vec{\partial}_{k_a}) - (\vec{q}\cdot\vec{k}_a)\partial_{k_{ai}} \right) \partial_{k_{aj}} \langle \zeta_{\vec{k}_1} \cdots \zeta_{\vec{k}_n} \rangle' . \quad (4.63)$$

As we already stressed, on the right hand side the momenta $\vec{k}_1, \ldots, \vec{k}_n$ do not sum up to zero, but to $-\vec{q}$.

It is important to stress one difference compared to the consistency relation for scalars. While a long scalar mode is equivalent to a dilation and a special conformal transformation that both annihilate a scale invariant two-point function, in the case of a long tensor mode the rescaling of coordinates is anisotropic and its action on the two-point function is not zero. Indeed, as we are going to see, one can check that the three-point function $\langle \gamma\zeta\zeta \rangle$ has a nontrivial squeezed limit both at zeroth and first order in $\vec{q}$.

4.2.2 Checks

The simplest way to check the consistency relation (4.63) is to look at the squeezed limit of the three-point function with one graviton and two scalars. In single-field slow-roll inflation, this three-point function is given by [43]

$$\langle \gamma_{\vec{q}}^s \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle' = \frac{H^4}{q^3 k_1^3 k_2^3} \frac{1}{4\epsilon} \epsilon_{ij}^s(\vec{q}) k_{1i} k_{2j} \left(-k_t + \frac{k_2 q + k_1 q + k_2 k_1}{k_t} + \frac{q k_1 k_2}{k_t^2} \right) . \quad (4.64)$$

In the limit $\vec{q} \to 0$ the expression above reduces to

$$\langle \gamma_{\vec{q}}^s \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle'_{q\to 0} = \frac{H^2}{4\epsilon k_1^3} \frac{H^2}{q^3} \epsilon_{ij}^s(\vec{q}) \frac{k_{1i} k_{1j}}{k_1^2} \frac{3}{2} \left(1 - \frac{5}{2} \frac{\vec{k}_1 \cdot \vec{q}}{k_1^2} \right) . \quad (4.65)$$

We can compare this result with the right hand side of the consistency relation (4.63). First, notice that the second line of eq. (4.63) vanishes when we act on the two-point function. We

are therefore left only with

$$\langle \gamma^s_{\vec{q}} \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle'_{q\to 0} = -\frac{1}{2} P_\gamma(q) \epsilon^s_{ij}(\vec{q}) \left(k_{1i}\partial_{k_{1j}} + k_{2i}\partial_{k_{2j}} \right) \langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle'$$

$$= P_\gamma(q) \epsilon^s_{ij}(\vec{q}) \frac{3}{4} \left(\frac{k_{1i}k_{1j}}{k_1^2} + \frac{k_{1i}k_{1j}}{k_1^2} \right) \langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle' , \qquad (4.66)$$

where we have used $\langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle' = H^2/(4\epsilon\sqrt{k_1^3 k_2^3})$. If we just naively replace $\vec{k}_2 = -\vec{k}_1$, we recover only the first term in eq. (4.65). Obviously, this is not the correct result.

We have to remember the derivation of the consistency relation imposes that on both sides we have $\vec{k}_1 + \vec{k}_2 + \vec{q} = 0$. If we use this, and expand the right hand side of eq. (4.66), we indeed find the correct squeezed limit eq. (4.65), and the consistency relation is satisfied.

4.3   Soft internal lines

So far we have considered squeezed limits of inflationary correlation functions in which one of the external momenta goes to zero. Another interesting kinematical regime is when the sum of m (out of n) of the external momenta goes to zero. In this case the n-point function is dominated by the exchange of a soft internal line. In this configuration the result factorizes in the product of the power spectrum of the soft internal line and the m-point and $(n-m)$-point functions, both calculated in the presence of the long mode [124, 119]. Therefore, we can still apply arguments we used for the external soft modes and use the consistency relations to calculate the correlation functions with soft internal lines.

The physical intuition behind these arguments is as follows. The soft mode—scalar or tensor—freezes much before the others and behaves as a classical background when short modes cross the horizon. Of course, this holds also taking into account a constant gradient of the long mode, i.e. the first correction in its momentum. What one has to calculate then is the n-point function in the presence of the long mode, in a particular configuration. In real space one should think about short modes localized around two different points $\vec{x}$ and $\vec{y}$ that interact through the common long background mode.

In general, we can take into account both long modes of scalars and gravitons. Once we go to momentum space, the result can be written in the following form

$$\langle \zeta_{\vec{k}_1} \cdots \zeta_{\vec{k}_n} \rangle'_{q\to 0} = P_\zeta(q) \langle \zeta_{-\vec{q}} \zeta_{\vec{k}_1} \cdots \zeta_{\vec{k}_m} \rangle^*_{q\to 0} \langle \zeta_{\vec{q}} \zeta_{\vec{k}_{m+1}} \cdots \zeta_{\vec{k}_n} \rangle^*_{q\to 0} +$$

$$+ P_\gamma(q) \sum_s \langle \gamma^s_{-\vec{q}} \zeta_{\vec{k}_1} \cdots \zeta_{\vec{k}_m} \rangle^*_{q\to 0} \langle \gamma^s_{\vec{q}} \zeta_{\vec{k}_{m+1}} \cdots \zeta_{\vec{k}_n} \rangle^*_{q\to 0} , \qquad (4.67)$$

where $1 < m < n$, $\vec{q} = \vec{k}_1 + \cdots + \vec{k}_m$ and $*$ on the squeezed-limit correlation functions means that both the delta function of momentum conservation and the long-mode power spectrum $P(q)$ are dropped. Notice that, as a consequence of momentum conservation, the vector $\vec{q}$ has opposite sign in the two terms of the product.

The last step is straightforward. Each of the terms on the right hand side of the equation above can be simply calculated using the standard consistency relations with long scalar or

tensor modes. In this way, in the limit when one internal line becomes soft, the consistency relation gives a connection between an n-point function and the corresponding m-point and $(n-m)$-point functions.

The simplest example in which we can study this configuration is the four-point function in the limit in which two of the external momenta become equal and opposite [124, 119]. In the case of scalar exchange we know that the three-point function $\langle \zeta_{\vec{q}} \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle$ does not have any linear corrections. This implies that, in the limit of exact scale-invariance, we have a q^2 suppression at each of the two vertices (that can be indeed verified by the explicit calculation [121, 122]) that completely kills the $q \to 0$ divergence from the power spectrum. For this reason the result is not dominated by the soft scalar exchange and the limit is therefore regular

$$\langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \zeta_{\vec{k}_4} \rangle_{\vec{q} \to 0}^{\prime SE} \sim \text{const.} \tag{4.68}$$

More interesting is the case of the four-point function of ζ with graviton exchange. As we already pointed out, the squeezed limit of the three-point function $\langle \gamma \zeta \zeta \rangle$ with a soft graviton does not vanish at leading order in q because the soft graviton induces an anisotropic rescaling of coordinates. For this reason, unlike in the case of a soft internal scalar mode, we expect to get some non-trivial contributions that can be captured using consistency relations.

Using results from the Appendix A, we can write the four-point function with graviton exchange in the collapsed limit ($\vec{q} = \vec{k}_1 + \vec{k}_2$, $\vec{k}_1 \approx -\vec{k}_2$ and $\vec{k}_3 \approx -\vec{k}_4$) in the following way

$$\langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \zeta_{\vec{k}_4} \rangle_{\vec{q} \to 0}^{\prime GE} = \frac{H^2}{q^3} \frac{H^2}{4\epsilon k_1^3} \frac{H^2}{4\epsilon k_3^3} \sum_s \epsilon_{ij}^{s*}(\vec{q}) \epsilon_{lm}^s(\vec{q}) \frac{9}{4} \frac{k_{1i} k_{1j}}{k_1^2} \frac{k_{3l} k_{3m}}{k_3^2} \left(1 + \frac{5}{2} \frac{\vec{k}_1 \cdot \vec{q}}{k_1^2} - \frac{5}{2} \frac{\vec{k}_3 \cdot \vec{q}}{k_3^2} \right). \tag{4.69}$$

On the other hand, this is exactly what one finds using expression (4.67) and the consistency relation for the soft gravitons (4.63).

Let us stress at the end that in the same way the standard consistency relations with soft external momenta are very interesting for observations, the same is true in the case of soft internal lines. For example, the power spectrum of the CMB μ distortions [125] is sensitive to the four-point function in the limit we discussed in this Section. Therefore, neglecting the effect of tensor modes, one can say in a model-independent way that the power spectrum of μ distortions is suppressed in all single-field models.

4.4   Going to Higher Order in q

The consistency relations derived so far are all based on the fact that the long mode, up to first order in gradients, locally reduces to a diffeomorphism. As such, they are very powerful, because they are model independent and hold under a very small number of assumptions. On the other hand, these relations are somewhat trivial, as they simply state that the long mode does not affect the short ones in a coordinate-independent way, and so they do not contain any dynamical information.

This changes if we want to go to higher orders in derivatives (or finite momentum q). For example, if we want to consider the second order terms in the gradient expansion, this means that we have two spatial derivatives acting on the long mode. These kind of terms are proportional to the spatial curvature induced by the long mode and therefore are *physical*. Indeed, at order $\mathcal{O}(q^2)$, the effect of the long mode on short modes cannot be removed just by a change of coordinates. Although this implies that one cannot write down general and model-independent consistency relations, we can still get some interesting results studying this order in the small q expansion.

In the rest of this Section we will focus on this leading "physical" effect, i.e. at order q^2, in the squeezed limit of single-field correlation functions following the discussion in [126]. We will show that at this order the long mode induces curvature and that, after we take a proper average over directions, its effect is the same as being in a curved FRW universe. For the three-point function this will imply

$$\langle \zeta_{\vec{q}} \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle_{q \to 0, \, \mathrm{avg}} = P(q) \, \frac{2}{3} \cdot q^2 \cdot \frac{\partial}{\partial \kappa} \langle \zeta_{\vec{k}_1} \zeta_{-\vec{k}_1} \rangle \,, \tag{4.70}$$

where P is the power spectrum, κ is the spatial curvature of the FRW and the subscript avg indicates that an angular average has been taken.

It is important to stress that the equation above is rather different from a standard consistency relation, which connects observable correlation functions in our universe, so that one can check whether it is experimentally violated. Here the right hand side of eq. (4.70) contains the power spectrum in a *curved* universe, which cannot be measured independently. Therefore one cannot check experimentally whether the relation holds or not. Its interest is mostly conceptual since, as we will shortly explain, it allows to understand in simple physical terms some of the properties of non-Gaussianities. Indeed, we will see that the equivalence between a long mode and a curved universe allows to learn something general about inflationary correlation functions.

4.4.1 A Heuristic Argument

Before we move on and show the equivalence of the long mode at order $\mathcal{O}(q^2)$ and a curved FRW universe, let us focus on a heuristic argument that allows us to understand all relevant physics involved in the problem.

The argument is very simple and it is illustrated in Figure 4.1. Let us consider a squeezed triangle with a given ratio between the long and the short modes $\varepsilon \equiv \frac{q}{k} \ll 1$. The power spectrum of short modes is imprinted at freezing and at this moment the long mode has momentum $\varepsilon \, k_{\mathrm{ph,f}}$. The effect of the long mode is the same as being in a curved FRW, with curvature $\kappa \sim q^2/a^2 \cdot \zeta_q \sim \varepsilon^2 k^2/a^2 \cdot \zeta_q$. Curvature will modify the spectrum of the short modes through its effect in the Friedmann equation and its consequent change of the inflaton speed $\dot{\phi}$. The relative correction introduced in the Friedmann equation, and thus in the power spectrum for the short modes, is obtained by comparing the spatial curvature with H^2 at the freezing time: $(q^2/a_f^2 \cdot \zeta_q)/H^2 \simeq \varepsilon^2 k_{\mathrm{ph,f}}^2 \zeta_q/H^2$. This implies that the three-point function
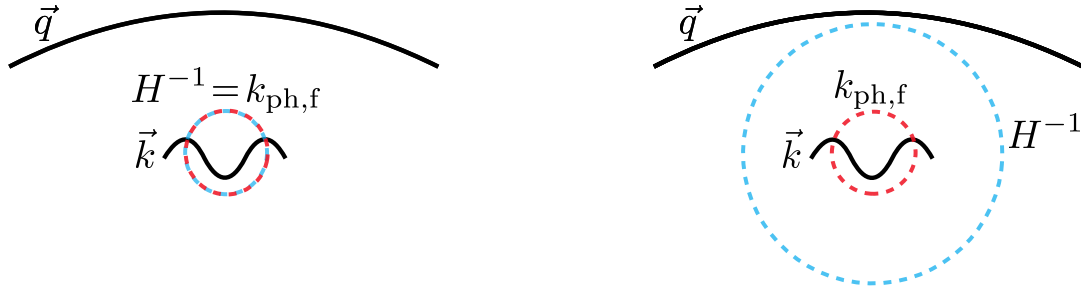
Figure 4.1: Parametric dependence of non-Gaussianity with the freezing scale. On the left the freezing scale is the Hubble scale. On the right the freezing scale is parametrically shorter than Hubble: in this case the effect of the curvature induced by the long mode is bigger, since it has to be compared with H^2 in the Friedmann equation.

behaves as $(k_{\mathrm{ph,f}}/H)^2(q/k)^2 P(q)P(k)$ in the squeezed limit, i.e. $f_{\mathrm{NL}} \sim (k_{\mathrm{ph,f}}/H)^2$. Although the derivation works only in the squeezed limit, we expect by continuity (since at this order we are capturing the physical effect of the long mode) that it gives the correct parametric dependence also when modes become comparable.

Notice that, strictly speaking, f_{NL} is defined as proportional to the value of the 3-point function in the equilateral configuration. In this regime, the derivative expansion in q/k clearly does not apply and we therefore have nothing to say for this configuration. Here for f_{NL} we simply mean a typical value of the 3-point function apart from points where it accidentally cancels, or equivalently the observational limit on f_{NL} that would be obtained by a dedicated analysis of the 3-point function in the data. Indeed, since at order q^2 we are capturing a true physical effect, it is hard to imagine that there are cancellations for all configurations so that the size of the non-Gaussianities is parametrically different than the one obtained by extrapolating our expression (4.71) to the equilateral limit. This expectation is indeed confirmed by explicit calculations of the 3-point function in all triangular configurations.

Therefore, a general conclusion one can draw is that the level of non-Gaussianity is always parametrically enhanced when modes freeze at a wavelength shorter than the Hubble scale. More precisely

$$f_{\mathrm{NL}} \sim \left(\frac{k_{\mathrm{ph,f}}}{H}\right)^2 , \tag{4.71}$$

where $k_{\mathrm{ph,f}}$ is the physical freezing scale. Notice that in order to get this result we never had to talk about interactions of the inflaton. The scaling (4.71) is obtained just by looking at the power spectrum.

The relation (4.71) obviously works for models with reduced speed of sound [97, 117, 25, 127]:

$$\text{Reduced } c_s : \qquad k_{\mathrm{ph,f}} = \frac{H}{c_s} \quad \Rightarrow \quad f_{\mathrm{NL}} \sim \left(\frac{k_{\mathrm{ph,f}}}{H}\right)^2 \sim \frac{1}{c_s^2} . \tag{4.72}$$

The same argument works when dissipative effects are present. In this class of models [128] a parameter $\gamma \gg H$ plays the role of an effective friction and the freezing happens when

$c_s^2 k^2 / a^2 \sim \gamma H$. This implies that Eq. (4.71) reads

$$\text{Dissipative:} \qquad k_{\mathrm{ph,f}} = \frac{\sqrt{\gamma H}}{c_s} \qquad \Rightarrow \qquad f_{\mathrm{NL}} \sim \left(\frac{k_{\mathrm{ph,f}}}{H} \right)^2 \sim \frac{\gamma}{H c_s^2} \; . \tag{4.73}$$

This matches the explicit calculations of [128]. The argument leading to Eq. (4.71) does not rely on a linear dispersion relation for the inflaton perturbations and indeed it gives the correct estimate for Ghost Inflation [100] too, where the dispersion relation is of the form $\omega = k^2/M$. In this case we have

$$\text{Ghost Inflation:} \qquad k_{\mathrm{ph,f}} = (HM)^{1/2} \qquad \Rightarrow \qquad f_{\mathrm{NL}} \sim \left(\frac{k_{\mathrm{ph,f}}}{H} \right)^2 \sim \frac{M}{H} \; . \tag{4.74}$$

A peculiar example is the one of Khronon Inflation [34]. In this case the power spectrum is not affected by the background curvature at order $1/c_s^2$ since in the quadratic action the scale factor does not appear at this order and the speed of the inflaton background is immaterial due to the field redefinition symmetry. Indeed, we check in Appendix A that $1/c_s^2$ terms of the three-point function at order q^2 vanish, when the proper angular average is taken. At the end, let us notice that the general argument above implies that one has completely non-Gaussian perturbations, i.e. strong coupling, when freezing occurs at a sufficiently short scale

$$\text{Strong coupling:} \qquad \left(\frac{k_{\mathrm{ph,f}}}{H} \right)^2 \sim P_\zeta^{-1/2} \; . \tag{4.75}$$

Some of these arguments can be straightforwardly extended to the trispectrum and higher correlation functions. For example one can apply twice the argument leading to Eq. (4.70) to conclude that the amplitude of the four-point function will behave as

$$\tau_{\mathrm{NL}} \sim \left(\frac{k_{\mathrm{ph,f}}}{H} \right)^4 \; . \tag{4.76}$$

This corresponds to $1/c_s^4$ for models with reduced speed of sound and $(M/H)^2$ for Ghost Inflation.

It is worth noticing that our arguments cannot be extended beyond the q^2 order. At order q^3 one *cannot* interpret the three-point function as a calculation of the two-point function on a long classical background. Indeed, at this order one cannot neglect the time-dependent phase of the long mode and therefore one cannot describe the long mode classically. Moreover, one cannot neglect anymore the probability that two short modes combine to give a long one. In this sense, eq. (4.70) is the last statement we can make treating the long mode as a classical background.

Let us at the end stress one important point. In the procedure we described above we are associating non-Gaussianities with the effect of the curvature, by setting the long mode fluctuation of the inflation to zero with a choice of coordinates. In this way, the interactions between the long mode and the short modes appear of gravitational origin. On the other hand,

we know that the three-point function can also be obtained (up to slow-roll corrections) in the decoupling limit. In particular, in models with $c_s \ll 1$, freezing happens at scales much shorter than the curvature and large non-Gaussianities can be understood in flat space. This simplification is somewhat obscured in the language we use in this Section.

4.4.2 From ζ-gauge to a Curved FRW Universe

In this Section our aim is to show that the long mode, at order $\mathcal{O}(q^2)$ is equivalent to a spatially curved universe. For simplicity, we will neglect the homogenous part and the homogeneous gradient as they can be always removed by a change of coordinates, and focus only on the second order in derivative expansion of the long mode

$$\zeta_L(\vec{x}, t) = \zeta_L(t) + \frac{1}{2}\partial_i\partial_j\zeta_L \, x^i x^j \ . \tag{4.77}$$

The time dependence of $\zeta_L(\vec{x}, t)$ is fixed by the equation of motion of ζ_L. Focusing on models with reduced speed of sound we have

$$\partial_t\left(\epsilon a_0^3\dot{\zeta}_L\right) - \epsilon a_0 c_s^2\,\partial^2\zeta_L = 0 \ , \tag{4.78}$$

where a_0 is the scale factor of the flat FRW universe. Notice that we cannot neglect the time dependence of ζ_L since it is of order q^2. Substituting eq. (4.77) and solving for $\zeta_L(t)$ at first order in the slow-roll parameters, we find that the long mode at quadratic order is

$$\zeta_L(\vec{x}, t) = -\frac{c_s^2\partial^2\zeta_L}{2a_0^2H_0^2}(1 - 2\epsilon) + \frac{1}{2}\partial_i\partial_j\zeta_L \, x^i x^j \ . \tag{4.79}$$

In order to show that at this order the long mode is equivalent to being in a spatially curved FRW, it is enough to consider the perturbed metric in ζ-gauge at linear order (see for example [117])

$$g_{00} = -1 - 2\frac{\dot{\zeta}_L}{H_0} \ ,$$

$$g_{0i} = -\frac{\partial_i\zeta_L}{H_0} + \frac{a_0^2\epsilon}{c_s^2}\frac{\partial_i\dot{\zeta}_L}{\partial^2} \ , \tag{4.80}$$

$$g_{ij} = a_0^2\left(1 + 2\zeta_L\right)\delta_{ij} \ ,$$

and make a gauge transformation to rewrite it in the usual FRW form (i.e. by setting $g_{00} = -1$ and $g_{0i} = 0$). Under the coordinate transformation $x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu(\tilde{x})$, the metric transforms as

$$\tilde{g}_{00} = g_{00} + 2\dot{\xi}^0 \ ,$$

$$\tilde{g}_{0i} = g_{0i} - a_0^2\dot{\xi}^i + \partial_i\xi^0 \ , \tag{4.81}$$

$$\tilde{g}_{ij} = g_{ij} - a_0^2\partial_i\xi^j - a_0^2\partial_j\xi^i - 2a_0^2H_0\xi^0\delta_{ij} \ ,$$

and we can set $\tilde{g}_{00} = -1$ and $\tilde{g}_{0i} = 0$ by choosing the transformation parameters to be

$$\xi^0 = \int^t dt' \frac{\dot{\zeta}_L}{H_0} = -\frac{c_s^2 \partial^2 \zeta_L}{2a_0^2 H_0^3}\left(1 - \frac{3}{2}\epsilon\right) \,,$$

$$\xi^i = -\int^t dt' \frac{\partial_i \zeta_L}{a_0^2 H_0}(1 - \epsilon) = \frac{\partial_i \partial_j \zeta_L \tilde{x}^j}{2a_0^2 H_0^2} \,. \tag{4.82}$$

The effect of the long mode is now entirely encoded in the spatial part of the metric

$$\tilde{g}_{ij} = a_0^2 \left(\delta_{ij} - \frac{\partial_i \partial_j \zeta_L}{a_0^2 H_0^2} + \epsilon \frac{c_s^2 \partial^2 \zeta_L}{2a_0^2 H_0^2}\delta_{ij}\right)\left(1 + \partial_a \partial_b \zeta_L \tilde{x}^a \tilde{x}^b\right) \,, \tag{4.83}$$

which, once averaged over angles, takes the familiar form

$$\tilde{g}_{ij,\,\mathrm{avg}} = a^2\left(1 - \frac{1}{2}\kappa \tilde{x}^2\right) \,. \tag{4.84}$$

In this expression the spatial curvature κ and the scale factor a in the curved universe are given by

$$\kappa = -\frac{2}{3}\partial^2 \zeta_L \,, \qquad a^2 = a_0^2\left(1 + \frac{\kappa}{2a_0^2 H_0^2} - \epsilon \frac{3c_s^2 \kappa}{4a_0^2 H_0^2}\right) \,. \tag{4.85}$$

In conclusion, there are two effects of the long mode at order $\mathcal{O}(q^2)$. One is to induce the spatial curvature. The other is to consistently change the scale factor compared to the starting flat FRW universe. This effect is dominant in the limit of small speed of sound, as we are going to check in the following Section. Intuitively, the reason is that as c_s goes to zero, the sound horizon where perturbations freeze also shrinks and geometrical effects can be neglected compared to the effect in the change in the expansion history.

It is important to stress that the change of coordinates (4.82) decays away at late times. This implies that, since we are interested in correlation functions at late times, that one can neglect the change of coordinates and just treat the long mode as adding spatial curvature to the FRW solution. This allows us to write the consistency relation at this order as

$$\langle \zeta_{\vec{q}} \zeta_{\vec{k}_1} \zeta_{\vec{k}_2}\rangle'_{q\to 0,\,\mathrm{avg}} = P_\zeta(q)\frac{2}{3}\cdot q^2 \cdot \frac{\partial}{\partial \kappa}\langle \zeta_{\vec{k}_1}\zeta_{-\vec{k}_1}\rangle'_\kappa \,, \tag{4.86}$$

where the subscript κ means that the two-point function is calculated in the curved universe. As before, the assumption used to derive this formula is the fact that the background solution is on the attractor, so that the time-dependence of the long mode ζ_L starts at order q^2. Notice that in deriving eq. (4.85) we used the slow-roll approximation, but we expect the relationship to hold at all orders in the slow-roll expansion. Our consistency condition should indeed hold for all FRW backgrounds that are accelerating so that modes freeze after horizon crossing. This implies that the effect of the long mode on the short ones is captured by a derivative expansion of the long mode at the time when the short modes cross the horizon.

One can generalize these arguments without taking the angular average and consider a curved anisotropic (but homogeneous) universe. In this case eq. (4.86) becomes

$$\langle \zeta_{\vec{q}} \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle'_{q\to 0} = P_\zeta(q) \, q_i q_j \frac{\partial}{\partial (q_i q_j \zeta_{\vec{q}})} \langle \zeta_{\vec{k}} \zeta_{-\vec{k}} \rangle'_{\text{local}} \;, \tag{4.87}$$

where $\langle \ldots \rangle_{\text{local}}$ means that the two-point function is calculated in a locally anisotropic curved universe.

4.4.3   Checks

As we already pointed out, eq. (4.86) is not the consistency relation in the usual sense, because it relates correlation function calculated in two different universes. One of them is flat and the other has a spatial curvature κ. Therefore, in order to check eq. (4.86), one has to calculate correlation functions in the curved universe. This is not so easy, in particular because of the geometrical effects (for example, one cannot expand the modes in plane waves). Furthermore, to do the comparison with the three-point function, the two-point function in the curved universe must be expressed in terms of the flat universe parameters (for example speed of sound c_s). Luckily, at least in some regimes, some of these difficulties are not present. For example, if we take a parametric limit $c_s \ll 1$, then the modes freeze at scales much smaller than the curvature scale, and one can neglect geometrical effects and work in the approximation of a flat space. The only way the curvature enters the problem is through the change of the expansion rate as described in (4.85).

Let us therefore focus on the models with small speed of sound and check eq. (4.86) at leading order in $1/c_s^2$. The simplest way to do the check is to use the action for K-inflation in the decoupling limit in a curved universe (see for example [97])

$$S = \int d^4x \sqrt{-g} P(X, \phi) \;, \tag{4.88}$$

where $X = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$. For the homogeneous background $\phi(t)$ this action boils down to

$$S = \frac{1}{2} \int d^4x \left(a^3 P_X \dot{\phi}^2 - a^3 V(\phi) \right) \;, \tag{4.89}$$

and the corresponding equation of motion is

$$\partial_t(a^3 P_X \dot{\phi}) - a^3 V' = 0 \;. \tag{4.90}$$

This equation implies that the effect of curvature is not only to change the scale factor. If we perturb $a(t)$, to be consistent we also have to change the background field $\phi(t)$. To find what change in the field compared to the flat universe comes from adding curvature, let us first write down the equation of motion in the following way

$$P_X \ddot{\phi} + 3H P_X \dot{\phi} + 2 P_{XX} X \ddot{\phi} = -V' \;. \tag{4.91}$$

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Now we can split the background in the part which corresponds to the evolution in a flat universe, and a small perturbation coming from curvature: $\dot{\phi} = \dot{\phi}_0 + \delta\dot{\phi}$. Using the background equation of motion for ϕ_0, we find that the small perturbation $\delta\dot{\phi}$ has to satisfy

$$2P_{XX}X\delta\ddot{\phi} + 6H_0\dot{\phi}_0^2 P_{XX}\delta\dot{\phi} + 3P_X\dot{\phi}_0\delta H = 0. \tag{4.92}$$

Using the change of the scale factor in the presence of curvature eq. (4.85), one can easily find the change of the Hubble constant δH. The equation of motion for $\delta\dot{\phi}$ becomes

$$\delta\ddot{\phi} + 3H_0\delta\dot{\phi} = \frac{3}{2}c_s^2 H_0\dot{\phi}_0\frac{R}{a_0^2 H_0^2}\,, \tag{4.93}$$

and the solution is simple to find

$$\delta\dot{\phi} = C\,a_0^{-3} + \frac{3}{2}c_s^2\dot{\phi}_0\frac{R}{a_0^2 H_0^2}\,. \tag{4.94}$$

The homogeneous part decays faster than the particular solution and we can neglect it.

To summarise, in models with the small speed of sound, the leading effect of the long mode is to change the background evolution. This change affects the inflation speed as well, that has to be consistently shifted by the amount we calculated above. The relative changes of the relevant quantities are

$$\frac{\delta a(\eta)}{a(\eta)} = \frac{\kappa}{6}\eta^2 \quad \text{and} \quad \frac{\delta X}{X} = 2\frac{\delta\dot{\phi}}{\dot{\phi}} = 3\frac{c_s^2\kappa}{H_0^2 a_0^2}\,. \tag{4.95}$$

Let us now turn back to the problem of calculating the power spectrum of perturbations in the presence of the curvature κ. We can start from the action for perturbations [97]

$$S = \frac{1}{2}\int d^4x\,\sqrt{-g}\Big((XP_X + 2X^2 P_{XX})\dot{\pi}^2 - XP_X g^{ij}\partial_i\pi\partial_j\pi + 3\kappa XP_X\pi^2\Big)\,, \tag{4.96}$$

and vary all relevant quantities. For example

$$P_X \to P_X\left(1 + \frac{XP_{XX}}{P_X}\frac{\delta X}{X}\right)\,, \quad P_{XX} \to P_{XX}\left(1 + \frac{XP_{XXX}}{P_{XX}}\frac{\delta X}{X}\right)\,. \tag{4.97}$$

Using the definition for the parameters c_s^2, λ and Σ of Ref. [117],

$$\frac{XP_{XX}}{P_X} = \frac{1}{2}\left(\frac{1}{c_s^2} - 1\right)\,, \quad \frac{2X^2 P_{XXX}}{P_X} = \frac{3\lambda}{c_s^2\Sigma} - \frac{3}{2}\left(\frac{1}{c_s^2} - 1\right)\,, \tag{4.98}$$

the action in Eq. (4.96) reads

$$S = \int d^4x\,H_0^2\epsilon a_0^2\left[\frac{1}{c_s^2}\left(1 + \frac{\kappa}{3a_0^2 H_0^2} + \frac{9c_s^2\kappa}{a_0^2 H_0^2}\frac{\lambda}{\Sigma}\right)\pi'^2 - \left(1 + \frac{11\kappa}{6a_0^2 H_0^2}\right)(\partial_i\pi)^2\right]\,. \tag{4.99}$$

Here we are considering only terms which contribute at order $1/c_s^2$. This greatly simplifies the calculation since the metric can be taken to be spatially flat and the field can be decomposed in Fourier modes. The equation of motion derived from this action is

$$\pi_{\vec{k}}'' - \frac{2}{\eta}\pi_{\vec{k}}' + c_s^2 k^2 \pi_{\vec{k}} + \frac{\kappa}{a_0^2 H_0^2}\left(\left(\frac{1}{3} + 9c_s^2\frac{\lambda}{\Sigma}\right)\pi_{\vec{k}}'' + \frac{11}{6}c_s^2 k^2 \pi_{\vec{k}}\right) = 0 , \qquad (4.100)$$

and can easily be solved perturbatively in κ. Once the solution has been properly normalised, the power spectrum for ζ can be calculated using the standard procedure. The final result is

$$\langle \zeta_{\vec{k}}\zeta_{-\vec{k}}\rangle_\kappa' = P_\zeta(k)\left(1 - \frac{19 + 18\frac{\lambda}{\Sigma}}{8c_s^2 k^2}\kappa\right) . \qquad (4.101)$$

The contribution to the squeezed limit of the three-point function is then obtained by averaging over the long mode and substituting $\kappa = \frac{2}{3}q^2\zeta_{\vec{q}}$

$$\langle \zeta_{\vec{q}}\zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\rangle_{q\to 0}' = -P_\zeta(q)P_\zeta(k)\frac{1}{c_s^2}\left(\frac{19}{12} + \frac{3}{2}\frac{\lambda}{\Sigma}\right)\frac{q^2}{k^2} . \qquad (4.102)$$

Indeed, this agrees with the squeezed limit of the three-point function at second order in q, once the angular average is taken and terms not enhanced by $1/c_s^2$ are neglected (see Appendix A). For a more detailed discussion, involving checks with anisotropic long mode and including geometrical effects, see [126].

4.5   Multiple Soft Limits

All consistency relations we discussed so far involve a correlation function with a single soft external momentum. It is possible to generalize the consistency relations to cases when multiple external momenta are taken to be soft. In this Section we are going to give a few most important results and highlight some of the problems one encounters trying to derive the consistency relations for the multiple squeezed limit. We are also going to focus on scalars. The full derivation of all results is beyond the scope of this thesis and more details can be found in [129].

The way to get the consistency relations with multiple soft external legs is essentially the same as for the single soft mode. The long mode ζ_L is locally equivalent to a change of coordinates and we can write

$$\langle \zeta(x_1)\cdots\zeta(x_n)|\zeta_L\rangle = \langle \zeta(\tilde{x}_1)\cdots\zeta(\tilde{x}_n)\rangle . \qquad (4.103)$$

To get the the correlation function with multiple soft modes we have to multiply this equality with $\zeta(y_1)\ldots\zeta(y_m)$ and average over the long modes. Notice that in order to be consistent, we also have to expand the right hand side up to order $\mathcal{O}(\delta x^m)$. This is an important difference compared to the case of a single soft mode. For multiple soft limit it is impossible to linearize the change of coordinates.

In the following sections we are going to sketch the derivation of the consistency relations with two soft modes at different orders in the gradient expansion. This simplified problem still captures all the subtleties of the derivation. At the end, we will present a check of these relations in models with small speed of sound.

Before we proceed, let us write down the exact (not infinitesimal) change of coordinates that corresponds to a dilation and a special conformal transformation (see, *e.g.*, [130])

$$x^i \;\rightarrow\; e^\lambda x^i \;, \qquad\qquad\qquad \text{(dilation)}$$

$$x^i \;\rightarrow\; \frac{x^i - b^i \vec{x}^2}{1 - 2\vec{x}\cdot\vec{b} + \vec{x}^2\vec{b}^2} \;. \qquad\qquad \text{(SCT)} \tag{4.104}$$

Under these transformations, the spatial part of the line element transforms as

$$\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \;\rightarrow\; \Omega^2(\vec{x})\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \;, \tag{4.105}$$

where the conformal pre-factor is given by

$$\Omega_{\mathrm{dil}}(\vec{x}) = e^\lambda \;,$$

$$\Omega_{\mathrm{SCT}}(\vec{x}) = (1 - 2\vec{x}\cdot\vec{b} + \vec{x}^2\vec{b}^2)^{-1} \;. \tag{4.106}$$

So far we were using only linearised versions of these equations, but in the following sections will need non-linearised expressions to derive consistency relation with multiple soft momenta.

4.5.1 Zeroth Order in Gradients

Let us first look at the simplest case of homogeneous long modes. As we said, for simplicity, we will consider two soft modes, which requires working at quadratic order in ζ_L. The nonlinear ζ_L is generated by a finite dilation of the spatial coordinates

$$x^i \;\rightarrow\; e^{\zeta_L(\vec{x}_+)}x^i \;. \tag{4.107}$$

Notice that this is a simple generalisation of eq. (4.8). This implies that the variation of coordinates, up to second order in ζ_L, is given by

$$\delta x^i = \tilde{x}^i - x^i \simeq \left(\zeta_L + \frac{1}{2}\zeta_L^2\right)x^i \;. \tag{4.108}$$

Given this setup, the derivation proceeds in the same way as in Section 4.1. After going to momentum space, commuting derivatives through delta functions and many integrations by parts [129], one finds that the variation of an n-point function is given by

$$\delta\langle\zeta_{\vec{k}_1}\cdots\zeta_{\vec{k}_n}\rangle' = \lim_{\vec{q}\to 0}\left(-\zeta_{\vec{q}}\delta_{\mathcal{D}}^{(n)} + \frac{1}{2}\int\mathrm{d}^3Q\,\zeta_{\vec{q}}\zeta_{\vec{Q}-\vec{q}}(\delta_{\mathcal{D}}^{(n)})^2\right)\langle\zeta_{\vec{k}_1}\cdots\zeta_{\vec{k}_n}\rangle' \;, \tag{4.109}$$

where $\delta_{\mathcal{D}}^{(n)}$ is just the dilation operator

$$\delta_{\mathcal{D}}^{(n)} = 3(n-1) + \sum_{a=1}^{n}k_{ai}\partial_{k_{ai}} \;. \tag{4.110}$$

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Notice that there are two different terms in the variation of the n-point function. They correspond to two different contributions to the double squeezed limit of the $(n + 2)$-point function. Indeed, if we average the previous relation over $\zeta_{\vec{q}_1}\zeta_{\vec{q}_2}$, we finally get

$$\lim_{\vec{q}_1, \vec{q}_2 \to 0} \langle \zeta_{\vec{q}_1} \zeta_{\vec{q}_2} \zeta_{\vec{k}_1} \cdots \zeta_{\vec{k}_n} \rangle' = -\langle \zeta_{\vec{q}_1} \zeta_{\vec{q}_2} \zeta_{-\vec{q}} \rangle' \delta_{\mathcal{D}}^{(n)} \langle \zeta_{\vec{k}_1} \cdots \zeta_{\vec{k}_n} \rangle' + P_\zeta(q_1) P_\zeta(q_2) (\delta_{\mathcal{D}}^{(n)})^2 \langle \zeta_{\vec{k}_1} \cdots \zeta_{\vec{k}_n} \rangle' . \tag{4.111}$$

The first term in (4.111) describes two soft modes combining and linking to the hard modes via a soft internal line. This causes the diagram to factorize into a three-point function with no hierarchy between the modes and an n-point function rescaled by the presence of a long internal line. The second term in (4.111) corresponds to the case where both soft lines come from a single vertex (the contact diagram), which induces a double dilation on the remaining hard modes. Equation (4.111) is a generalization of Maldacena's consistency relation for a single soft external leg. It was first considered in [131, 132] and derived in this form in [129][4].

It is important to stress that in some cases these two terms may be of different orders in slow-roll parameters, such that one is suppressed compared to the other. For example, in single-field models with reduced speed of sound, the three-point function is not slow-roll suppressed, while the second term is higher-order in slow-roll parameters. This is not a generic case. For instance, in standard slow-roll inflation both terms are expected to be equally relevant (second order in slow-roll parameters). One can also find examples where the second term is dominant. This happens in the case of resonant non-Gaussianities, where diagrams with more vertices are always subdominant [119].

The other way to see that one needs both terms in (4.111) is to look at the hierarchical limit $q_1 \ll q_2$. Physically, this corresponds to a situation where the mode with the wavenumber $\vec{q}_1$ goes outside the horizon much before all the others. After that, the other mode $\vec{q}_2$ becomes longer than the horizon and eventually short modes cross it too. Intuitively, in this kind of situation, to get the double soft limit of an $(n + 2)$-correlation function, we can apply Maldacena's consistency relation twice in the following way

$$\lim_{\vec{q}_1 \ll \vec{q}_2 \to 0} \langle \zeta_{\vec{q}_1} \zeta_{\vec{q}_2} \zeta_{\vec{k}_1} \cdots \zeta_{\vec{k}_n} \rangle' = P_\zeta(q_1) \delta_{\mathcal{D}}^{(n+1)} \left(P_\zeta(q_2) \delta_{\mathcal{D}}^{(n)} \langle \zeta_{k_1} \cdots \zeta_{k_n} \rangle' \right) . \tag{4.112}$$

Indeed, this is exactly what we find from eq. (4.111). In the limit $q_1 \ll q_2$ we have

$$\lim_{\vec{q}_1 \ll \vec{q}_2 \to 0} \langle \zeta_{\vec{q}_1} \zeta_{\vec{q}_2} \zeta_{\vec{k}_1} \cdots \zeta_{\vec{k}_n} \rangle' = P_\zeta(q_1) \delta_{\mathcal{D}}^{(2)} P_\zeta(q_2) \, \delta_{\mathcal{D}}^{(n)} \langle \zeta_{k_1} \cdots \zeta_{k_n} \rangle' + P_\zeta(q_1) P_\zeta(q_2) (\delta_{\mathcal{D}}^{(n)})^2 \langle \zeta_{k_1} \cdots \zeta_{k_n} \rangle'$$

$$= P_\zeta(q_1) \delta_{\mathcal{D}}^{(n+1)} \left(P_\zeta(q_2) \delta_{\mathcal{D}}^{(n)} \langle \zeta_{k_1} \cdots \zeta_{k_n} \rangle' \right) . \tag{4.113}$$

Notice that we have additional terms in the dilation operator when it acts on modes with momentum q_2.

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[4]Notice that in [131, 132] the consistency relation contained only the first term on the right-hand side of (4.111).

4.5.2 First Order in Gradients

In this Section we will extend the discussion about consistency relations with two soft modes beyond the homogeneous limit and keep the first order corrections in gradients. Up to second order in ζ_L and first order in the gradient expansion, the spatial line element is given by

$$\mathrm{d}\vec{l}^2 = a^2(t)e^{2\zeta_L(\vec{x})}\mathrm{d}\vec{x}^2$$

$$\simeq a^2(t)\left(1 + 2\zeta_L + 2\zeta_L^2 + 2\vec{x}\cdot\vec{\nabla}\zeta_L + 4\zeta_L\vec{x}\cdot\vec{\nabla}\zeta_L\right)\mathrm{d}\vec{x}^2 , \tag{4.114}$$

where we have expanded the long mode around the origin

$$\zeta_L(x) = \zeta_L + \vec{x}\cdot\vec{\nabla}\zeta_L + \cdots . \tag{4.115}$$

It is obvious that the change of coordinates that generates the long adiabatic mode at this order must involve one dilation and one special conformal transformation. However, it is well known that these two transformations do not commute. Therefore, one can be confused about the order in which these two transformations should be performed. For example, in the case of internal symmetries, taking multiple soft limits in different orders gives different answers. It turns out that different ordering of limits is related to the commutators of broken generators, allowing to probe the structure constants of the algebra of broken generators experimentally (see for example a discussion in [133]). On the other hand, in case of inflation, the only Goldstone boson is ζ and the correlation functions are symmetric, such that taking limits in different order gives always the same answer.

This is not a paradox, because it turns out that the order of transformations that induce the long mode is not arbitrary. It is fixed by demanding that the change of coordinates generates a physical solution, corresponding to the spatial line element (4.114). Indeed, using (4.106), we can see that doing a special conformal transformation first gives[5]

$$(\mathrm{D}\circ\mathrm{SCT})\mathrm{d}\vec{x}^2 = \left(1 + 2\lambda + 2\lambda^2 + 4\vec{b}\cdot\vec{x} + 12\lambda\vec{b}\cdot\vec{x} + \cdots\right)\mathrm{d}\vec{x}^2 , \tag{4.116}$$

while if we do the transformations in opposite order we get

$$(\mathrm{SCT}\circ\mathrm{D})\mathrm{d}\vec{x}^2 = \left(1 + 2\lambda + 2\lambda^2 + 4\vec{b}\cdot\vec{x} + 8\lambda\vec{b}\cdot\vec{x} + \cdots\right)\mathrm{d}\vec{x}^2 . \tag{4.117}$$

Comparing these expressions with (4.114), we see that only (4.117) can be consistently matched with the physical solution if we choose λ and $\vec{b}$ in the standard way

$$\lambda = \zeta_L , \qquad \vec{b} = \frac{1}{2}\vec{\nabla}\zeta_L . \tag{4.118}$$

Notice that the difference between eq. (4.116) and eq. (4.117) is just $4\lambda\vec{b}\cdot\vec{x}$. This reflects the fact that the commutator of generators for dilations and special conformal transformations is proportional to the generator of special conformal transformations.

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[5]Here we work at linear order in $\vec{b}$, but keep terms of order $\lambda\vec{b}$ and λ^2. The reason is that $\lambda \sim \zeta_L$ and $\vec{b} \sim \vec{\nabla}\zeta_L$, and we want to keep terms that are first order in gradients but second order in ζ_L.

The summary of this discussion is that, in order to generate the line element (4.114) corresponding to a physical solution, we must do a dilation first and then perform a special conformal transformation. The full change of coordinates (up to an irrelevant spatial translation) that is equivalent to a physical long mode up to second order in ζ_L and first order in derivatives is

$$\delta x^i = \left(\zeta_L + \frac{1}{2}\zeta_L^2 + (1 + \zeta_L)\,\vec{x} \cdot \vec{\nabla}\zeta_L\right) x^i - \frac{1}{2}\,(1 + \zeta_L)\,\vec{x}^2\nabla^i\zeta_L \ . \tag{4.119}$$

To get the consistency relation one can follow the same steps as before. Here we will just quote the final result. For more details and some other technical steps in the derivation, see [129]. After we consistently expand the variation of an n-point function to second order in ζ_L, go to momentum space, average over $\zeta_{\vec{q}_1}\zeta_{\vec{q}_2}$ and do a lot of integrations by parts, the final result is

$$\lim_{\vec{q}_1,\vec{q}_2 \to 0} \frac{\langle\zeta_{\vec{q}_1}\zeta_{\vec{q}_2}\zeta_{\vec{k}_1}\cdots\zeta_{\vec{k}_n}\rangle'}{P_\zeta(q_1)P_\zeta(q_2)} = -\frac{\langle\zeta_{\vec{q}_1}\zeta_{\vec{q}_2}\zeta_{-\vec{q}}\rangle'}{P_\zeta(q_1)P_\zeta(q_2)}\delta_{\mathcal{D}}^{(n)}\langle\zeta_{\vec{k}_1}\cdots\zeta_{\vec{k}_n}\rangle' + (\delta_{\mathcal{D}}^{(n)})^2\langle\zeta_{\vec{k}_1}\cdots\zeta_{\vec{k}_n}\rangle'$$
$$+ \frac{1}{2}q_1^i\left(\delta_{\mathcal{K}^i}^{(n)}\delta_{\mathcal{D}}^{(n)}\langle\zeta_{\vec{k}_1}\cdots\zeta_{\vec{k}_n}\rangle' - \frac{\langle\zeta_{\vec{q}_1}\zeta_{\vec{q}_2}\zeta_{-\vec{q}}\rangle'}{P_\zeta(q_1)P_\zeta(q_2)}\delta_{\mathcal{K}^i}^{(n)}\langle\zeta_{\vec{k}_1}\cdots\zeta_{\vec{k}_n}\rangle'\right) , \tag{4.120}$$

where, as before, $\delta_{\mathcal{K}^i}^{(n)}$ is the operator of the special conformal transformation in momentum space

$$\delta_{\mathcal{K}^i}^{(n)} = \sum_{a=1}^{n}\left(6\partial_{k_{ai}} - k_{ai}\partial_{k_a}^2 + 2\vec{k}_a \cdot \vec{\partial}_{k_a}\partial_{k_{ai}}\right) \ . \tag{4.121}$$

Notice that the first line matches the result (4.111) in the homogeneous limit, while the second line contains gradient corrections.

4.5.3 Second Order in Gradients

From what we described so far, it might seem that going to higher order in gradients is straightforward. One always has to make sure to expand the conformal factors (4.106) to a given order in q and ζ_L and to use the change of coordinates that matches the physical solution in ζ-gauge. However, already at second order in gradients, we meet a problem. We already saw that the effects of a *single* long mode at order $\mathcal{O}(q^2)$ are physical and cannot be removed by a change of coordinates. It turns out that the same happens if we want to consider two long modes, each at linear order in gradients, even though they individually locally act only as a coordinate transformation.

The intuitive reason for this somewhat surprising result is the following. Imagine that we want to calculate the squeezed limit of an $(n + 2)$-point function and keep the corrections of order $\mathcal{O}(q_1 q_2)$. Some contributions to the result will come from diagrams where the two long modes combine to a single soft internal line. As we already saw, this kind of diagrams factorise to the three-point function $\langle\zeta_{\vec{q}_1}\zeta_{\vec{q}_2}\zeta_{\vec{q}}\rangle'$ and the $(n+1)$-point function $\langle\zeta_{-\vec{q}}\zeta_{\vec{k}_1}\cdots\zeta_{\vec{k}_n}\rangle'$. Therefore, in order to capture corrections of order $\mathcal{O}(q_1 q_2)$, we unavoidably have to deal with

the effects of the soft internal mode on the remaining n hard modes that are of the order $(\vec{q}_1 + \vec{q}_2)^2$. But this is exactly the effect of the curvature of the long mode, and cannot be calculated using just the coordinate transformation.

The other way to see this problem is the following. Given that we are interested in the construction of the adiabatic mode at order $(\nabla \zeta_L)^2$, we expect that the change of coordinates we need to do is a special conformal transformation at second order in $\vec{b}^2$. However, one can explicitly check that the second order conformal transformation does not match the physical long mode. The reason is that at second order in q and $\zeta_{\vec{q}}$ one has to take into account the term $\zeta_L \nabla^2 \zeta_L$ that is related to the curvature of the long mode. Indeed, neglecting dilations for simplicity, the difference between the line element with a physical long mode and the conformal transformation is

$$e^{2\vec{x}\cdot\vec{\nabla}\zeta_L + x^i x^j \nabla_i \nabla_j \zeta_L} - \Omega_{\rm sct}^2\big|_{b_i = \frac{1}{2}\nabla_i \zeta_L} = x^i x^j \nabla_i \nabla_j \zeta_L + \left(\vec{x}\cdot\vec{\nabla}\zeta_L\right)^2 - \frac{1}{2}\vec{x}^2 \left(\vec{\nabla}\zeta_L\right)^2 \ . \tag{4.122}$$

This is not so surprising. One way to understand this difference is that for the physical long mode the spatial curvature at order we are interested in is proportional to $R^{(3)} \sim q^2 \zeta_{\vec{q}}^2$, while the special conformal transformation leaves the space flat, at any order in $\vec{b}$.

Despite these issues, it is still possible to write down a relation among the correlation functions that is correct at order $\mathcal{O}(q_1 q_2)$. The details of the calculation are rather technical, and we will not present them here (the interested reader can find it in [129]). Let us just quote the final result and comment on some of the terms. For two soft modes we get

$$\lim_{q_1,q_1 \to 0} \frac{1}{P_\zeta(q_1) P_\zeta(q_2)} \langle \zeta_{q_1} \zeta_{q_2} \zeta_{k_1} \cdots \zeta_{k_n}\rangle' = -\frac{\langle \zeta_{q_1} \zeta_{q_2} \zeta_{-q}\rangle'}{P_\zeta(q_1) P_\zeta(q_2)} \left(\delta_{\mathcal{D}}^{(n)} + \frac{1}{2}\vec{q}\cdot\delta_{\vec{\mathcal{K}}}^{(n)}\right) \langle \zeta_{k_1} \cdots \zeta_{k_n}\rangle'$$

$$+ \left((\delta_{\mathcal{D}}^{(n)})^2 + \frac{1}{2}\vec{q}\cdot\delta_{\vec{\mathcal{K}}}^{(n)}\delta_{\mathcal{D}}^{(n)} + \frac{1}{4}q_1^i q_2^j \delta_{\mathcal{K}^i}^{(n)}\delta_{\mathcal{K}^j}^{(n)}\right) \langle \zeta_{k_1} \cdots \zeta_{k_n}\rangle'$$

$$+ \lim_{q \to 0} \left[\frac{1}{2}(\vec{q}^2 \nabla_q^2 - 2q_i q_j \nabla_q^i \nabla_q^j)\langle \zeta_q \zeta_{k_1} \cdots \zeta_{k_n}\rangle'\right.$$

$$\left.+ \frac{\langle \zeta_{q_1} \zeta_{q_2} \zeta_{-q}\rangle'}{P_\zeta(q_1) P_\zeta(q_2)} q_i q_j \nabla_q^i \nabla_q^j \frac{1}{P(q)} \langle \zeta_q \zeta_{k_1} \cdots \zeta_{k_n}\rangle'\right] \ . \tag{4.123}$$

The first two lines in this expression contain what one would naively have guessed using the consistency relations with a single soft momentum. The first line describes the effect of a soft internal mode that comes from two long modes on the n-point function. The second line is a "square" of the sum of the dilation and the special conformal operator. On the other hand, the last two lines come from the effects of the curvature of the long mode. They are obtained using the arguments behind eq. (4.86). One can see that these terms correspond to the mismatches in eq. (4.122). The first line comes from the last two terms in eq. (4.122), while the last line comes from the first term.

The main importance of this result is to show that at higher orders in $\nabla \zeta_L$, even though each gradient is locally just a change of coordinates, when they combine their total effect has a physical part. This makes the extension to higher orders in gradients even harder.

Let us also stress one limitation of eq. (4.123). For example, in simple single-field slow-roll inflation, two scalar modes can combine to give a tensor mode. In principle, this can be also captured in the expression for the double squeezed limit. Therefore, our choice to neglect the contribution from tensor modes constrains the applicability of eq. (4.123) only to some situations, for example models with small speed of sound where we can work in the decoupling limit.

Let us finally notice that if we are interested only in homogeneous long modes, we can apply the procedure described above for an arbitrary number of soft external legs. As we will see later, the same is true in the context of the Newtonian limit of the consistency relations of LSS. We will come back to some of the issues presented here in Section 5.2.2.

4.5.4 Checks

In this Section we will perform a check of the consistency relations with two soft external modes using correlation functions calculated in models with small speed of sound. We must consider at least a four-point function on the left hand side. On the right hand side most of the terms vanish, because the two-point function is both scale and conformally invariant. The only nontrivial contributions are at order $\mathcal{O}(q_1 q_2)$. The consistency relation in this case boils down to

$$\lim_{\vec{q}_1,\vec{q}_2 \to 0} \nabla^j_{q_1} \nabla^i_{q_2} \left(\frac{\langle \zeta_{\vec{q}_1} \zeta_{\vec{q}_2} \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle'}{P_\zeta(q_1) P_\zeta(q_2)} \right) = \lim_{\vec{q} \to 0} \left[\frac{1}{2} \delta^{ij} \nabla^2_q + \left(\frac{\langle \zeta_{\vec{q}_1} \zeta_{\vec{q}_2} \zeta_{-\vec{q}} \rangle'}{P_\zeta(q_1) P_\zeta(q_2)} - 1 \right) \nabla^i_q \nabla^j_q \right] \frac{\langle \zeta_{\vec{q}} \zeta_{\vec{k}} \zeta_{-\vec{k}} \rangle'}{P_\zeta(q)} . \tag{4.124}$$

Let us calculate the contributions on the two sides of the consistency relation.

On the left hand side, there are two types of diagrams that contribute to the four-point function (see Appendix A). In the double-squeezed limit, the contact contribution gives

$$\lim_{\vec{q}_1,\vec{q}_2 \to 0} \langle \zeta_{\vec{q}_1} \zeta_{\vec{q}_2} \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle'_{\text{cont.}} = P_\zeta(q_1) P_\zeta(q_2) P_\zeta(k) \left(1 - \frac{1}{c_s^2} \right) \left(\frac{3}{4} \frac{\vec{q}_1 \cdot \vec{q}_2}{k^2} + \frac{5}{2} \frac{(\vec{k} \cdot \vec{q}_1)(\vec{k} \cdot \vec{q}_2)}{k^4} \right) . \tag{4.125}$$

Notice that for simplicity we set $\lambda = 0$ in all calculations. The exchange diagram is expected to factorize

$$\lim_{\vec{q}_1,\vec{q}_2 \to 0} \langle \zeta_{\vec{q}_1} \zeta_{\vec{q}_2} \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle'_{\text{exc.}} = \lim_{\vec{q} \to 0} \langle \zeta_{\vec{q}_1} \zeta_{\vec{q}_2} \zeta_{-\vec{q}} \rangle' \frac{\langle \zeta_{\vec{q}} \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle'}{P_\zeta(q)} . \tag{4.126}$$

Assuming scale invariance, the last factor vanishes up to $\mathcal{O}(q^2)$ by the usual consistency relations. At order q^2, however, we have the effects of the curvature of the long mode $\zeta_{\vec{q}}$

$$\lim_{\vec{q} \to 0} \langle \zeta_{\vec{q}} \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle' = P_\zeta(q) P_\zeta(k) \left(1 - \frac{1}{c_s^2} \right) \left(2 \frac{q^2}{k^2} - \frac{5}{4} \frac{(\vec{k} \cdot \vec{q})^2}{k^4} \right) . \tag{4.127}$$

Substituting into (4.126), the exchange contribution becomes

$$\lim_{\vec{q}_1,\vec{q}_2 \to 0} \langle \zeta_{\vec{q}_1} \zeta_{\vec{q}_2} \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle'_{\text{exc.}} = P_\zeta(k) \left(1 - \frac{1}{c_s^2} \right) \left(2 \frac{q^2}{k^2} - \frac{5}{4} \frac{(\vec{k} \cdot \vec{q})^2}{k^4} \right) \langle \zeta_{\vec{q}_1} \zeta_{\vec{q}_2} \zeta_{-\vec{q}} \rangle' . \tag{4.128}$$

Now we can calculate the left hand side of the consistency relation. Taking derivatives with respect to q_1 and q_2, we obtain[6]

$$\lim_{\vec{q}_1,\vec{q}_2\to 0} \nabla^j_{q_1}\nabla^i_{q_2} \frac{\langle \zeta_{\vec{q}_1}\zeta_{\vec{q}_2}\zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\rangle'}{P_\zeta(q_1)P_\zeta(q_2)} = P_\zeta(k)\left(1-\frac{1}{c_s^2}\right)\left[\left(\frac{3}{4}\frac{\delta^{ij}}{k^2}+\frac{5}{2}\frac{k^i k^j}{k^4}\right)\right.$$
$$\left. + \left(4\frac{\delta^{ij}}{k^2}-\frac{5}{2}\frac{k^i k^j}{k^4}\right)\frac{\langle \zeta_{\vec{q}_1}\zeta_{\vec{q}_2}\zeta_{-\vec{q}}\rangle'}{P_\zeta(q_1)P_\zeta(q_2)}\right]. \tag{4.129}$$

To compare this to the right hand side of (4.124), we need the three-point function with $\lambda = 0$ (see Appendix A)

$$\langle \zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\zeta_{\vec{k}_3}\rangle' = P_\zeta(k_2)P_\zeta(k_3)\frac{1}{2k_1^3}\left[\left(\frac{1}{c_s^2}-1\right)\frac{12 k_1^2 k_2^2 k_3^2}{k_t^3}\right.$$
$$\left. + \left(\frac{1}{c_s^2}-1\right)\left(-\frac{8}{k_t}\sum_{i>j}k_i^2 k_j^2 + \frac{4}{k_t^2}\sum_{i\ne j}k_i^2 k_j^3 + \sum_i k_i^3\right)\right], \tag{4.130}$$

where $k_t = k_1 + k_2 + k_3$. Using this, we obtain

$$\lim_{\vec{q}\to 0}\nabla^i_q\nabla^j_q \frac{\langle \zeta_{\vec{q}}\zeta_{\vec{k}}\zeta_{-\vec{k}}\rangle'}{P(q)} = P_\zeta(k)\left(1-\frac{1}{c_s^2}\right)\left(4\frac{\delta^{ij}}{k^2}-\frac{5}{2}\frac{k^i k^j}{k^4}\right). \tag{4.131}$$

This leads to the following expression for the right-hand side of (4.124)

$$\lim_{\vec{q}\to 0}\left[\frac{1}{2}\delta^{ij}\nabla_q^2 + \left(\frac{\langle \zeta_{\vec{q}_1}\zeta_{\vec{q}_2}\zeta_{-\vec{q}}\rangle'}{P_\zeta(q_1)P_\zeta(q_2)}-1\right)\nabla^i_q\nabla^j_q\right]\frac{\langle \zeta_{\vec{q}}\zeta_{\vec{k}}\zeta_{-\vec{k}}\rangle'}{P_\zeta(q)}$$
$$= P_\zeta(k)\left(1-\frac{1}{c_s^2}\right)\left[\left(\frac{3}{4}\frac{\delta^{ij}}{k^2}+\frac{5}{2}\frac{k^i k^j}{k^4}\right)+\left(4\frac{\delta^{ij}}{k^2}-\frac{5}{2}\frac{k^i k^j}{k^4}\right)\frac{\langle \zeta_{\vec{q}_1}\zeta_{\vec{q}_2}\zeta_{-\vec{q}}\rangle'}{P_\zeta(q_1)P_\zeta(q_2)}\right], \tag{4.132}$$

which precisely agrees with (4.129).

4.6 Consistency Relations as Ward Identities

In this Chapter we presented the derivation of the inflationary consistency relations based on the fact that the long mode (scalar or tensor) locally acts only as a rescaling of coordinates. There is another way to obtain the same result and it is based on Ward identities for spontaneously broken space-time symmetries. Indeed, in ζ-gauge, the co-moving curvature perturbation ζ can be thought of as the Goldstone boson of the spontaneously broken space-time symmetry

$$SO(4,1) \to \mathbb{R}^3 \times SO(3). \tag{4.133}$$

In this gauge there remain residual large gauge transformations under which ζ transforms non-linearly. These transformations are dilations and special conformal transformations [134]. The quantum-mechanical consequences of these symmetries are Ward identities which relate

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[6]The second term gets a factor of 2 from the fact that $q^2 \supset 2\vec{q}_1\cdot\vec{q}_2$.

correlation functions with a soft external line to a lower point function, leading exactly to the inflationary consistency relations [50, 51].

We will not enter the details of the derivation of the consistency relations using Ward identities. Given that this approach attracted some attention in the literature, in this Section our goal is just to very briefly highlight the main results and compare them with what we did so far. Before we move to the full theory including gravity, we will first focus on the simpler case of the decoupling limit.

4.6.1 Decoupling Limit

In the decoupling limit we can neglect the effects of gravity, which makes the discussion much simpler. Inflation takes place in an approximate de Sitter space

$$ds^2 = -dt^2 + e^{2Ht}d\vec{x}^2 \ , \tag{4.134}$$

a maximally symmetric space whose isometry group is $SO(4,1)$. Notice that this coincides with the group of conformal transformations in 3-dimensional Euclidean space. The time-evolving inflaton background is homogeneous and rotationally invariant, so that translations and rotations are exact symmetries of the whole system and show up in an obvious way at the level of correlation functions. This accounts for 6 out of the 10 generators of $SO(4,1)$.

The other 4 generators are related to the dilation isometry

$$t \to t - H^{-1}\log\lambda \ , \qquad \vec{x} \to \lambda\vec{x} \ , \tag{4.135}$$

and another isometry

$$t \to t - 2H^{-1}\vec{b}\cdot\vec{x} \ , \quad x^i \to x^i - b^i(-H^{-2}e^{-2Ht} + \vec{x}^2) + 2x^i(\vec{b}\cdot\vec{x}) \ , \tag{4.136}$$

parametrized by the infinitesimal vector $\vec{b}$, that at large times $t \to +\infty$, act on the spatial coordinates as special conformal transformations. None of these transformations, on general grounds, are symmetries of the inflaton background. Indeed the solution—which we can always assume to be of the form $\phi = t$ by a suitable field redefinition—is not invariant under a time shift.

However, in inflationary models it is quite common that there is an additional (approximate) symmetry, the shift symmetry $\phi \to \phi + c$. In this case the inflaton background is invariant under a combination of a dilation and a shift of the field[7]. As we saw in the Introduction, this unbroken symmetry is the origin of the approximate scale-invariance of the inflationary correlation functions [60]. The obvious question is whether a similar thing can happen for other isometries. One may wonder whether it is possible, like in the case of dilations, to impose an additional symmetry on the inflaton Lagrangian in such a way that some

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[7]One can also consider the case in which the system is only invariant under a discrete (and not continuous) shift symmetry, see [120].

combination of the isometries (4.136) and this internal symmetry is preserved. This would require that the inflaton action is invariant under

$$\phi \to \phi + \vec{b} \cdot \vec{x} \, , \tag{4.137}$$

so that it is possible to cancel the variation induced on ϕ by the isometries (4.136). Notice that this is a Galilean transformation [135] with a spatial transformation parameter. It is certainly possible to write an inflaton action which is Galilean invariant [136, 35], but it is well known that this symmetry is only well defined in the Minkowski limit and broken by a curved background [137]. This implies that correlation functions, which probe the theory at a scale comparable with the curvature, cannot be invariant under the isometries (4.136) [8].

The fact that some of the isometries are spontaneously broken must be reflected at the level of the correlation functions. Indeed, in QFT, one can always write down Ward identities for non-linearly realised symmetries. These are non-perturbative results that relate the soft limits of an $(n+1)$-point correlation function to an appropriate variation of the corresponding n-point function. It turns out that Ward identities for spontaneously broken dilations and special conformal transformations have exactly the form of the consistency relations.

Let us take a closer look at the derivation of the consistency relation in the decoupling limit. We will start by expanding the long-wavelength mode to linear order in $\vec{x}$

$$\pi_L(x) = \pi_L(0) + \partial_i \pi_L(0) x^i. \tag{4.138}$$

The constant piece $\pi_L(0)$ can be induced by a rescaling of coordinates which, in the presence of a shift symmetry, leaves the correlation functions invariant. We will focus on the linear piece $\partial_i \pi_L(0) x^i$. Let us write down the transformation of the perturbations under a special conformal transformation, eq. (4.136), parametrized by the infinitesimal parameter $\vec{b}$

$$\pi(x) \to \tilde{\pi}(\tilde{x}) = \pi(x) + 2H^{-1}\vec{b} \cdot \vec{x} \, . \tag{4.139}$$

According to this relation a constant gradient can be induced by a special conformal transformation on the coordinates with $\vec{b} = -(1/2)H\vec{\partial}\pi_L(0)$ [9]

$$\langle \pi(\vec{x}_1) \dots \pi(\vec{x}_n) \rangle_{\partial \pi_L} = \langle \pi(\vec{\tilde{x}}_1) \dots \pi(\vec{\tilde{x}}_n) \rangle \, , \tag{4.140}$$

where $\tilde{x}^i \equiv x^i - b^i \vec{x}^2 + 2x^i(\vec{b} \cdot \vec{x})$. This equation holds in the out-of-the-horizon limit, when one can drop the time-dependent piece in the transformation of the x^i in the isometry (4.136)

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[8] In exact de Sitter and in the decoupling limit, interactions of the form $(\Box\phi)^n$ are invariant under spatial Galilean transformations. However, it is easy to check that all these couplings are redundant, i.e. they can be set to zero by a proper field redefinition. This field redefinition is of the schematic form $\phi \to \phi + (\Box\phi)^{n-1}$ and thus vanishes on large scales, giving trivial cosmological correlation functions. Correlation functions are not trivial on short scales and are obviously $SO(4,1)$ invariant because they are produced by a de Sitter invariant field redefinition.

[9] The long wavelength mode is taken to be time-independent: we are implicitly assuming that its time-dependence arises at $\mathcal{O}(k^2)$ and can thus be neglected for our purposes.

and neglect the time-dependence of π. The variation of the correlation function under such a coordinate transformation is not zero and will therefore induce a relation between the variation of the n-point function and the squeezed limit of the $(n+1)$-point function. Let us proceed by computing this variation in Fourier space

$$\delta\langle\pi(\vec{x}_1)\ldots\pi(\vec{x}_n)\rangle = \int \mathrm{dK}_n \langle\pi_{\vec{k}_1}\ldots\pi_{\vec{k}_n}\rangle(-i)\sum_{a=1}^n k_a^i\left[b^i\vec{x}_a^2 - 2x_a^i(\vec{b}\cdot\vec{x}_a)\right]e^{i(\vec{k}_1\cdot\vec{x}_1+\cdots+\vec{k}_n\cdot\vec{x}_n)}$$

$$= \int \mathrm{dK}_n e^{i(\vec{k}_1\cdot\vec{x}_1+\cdots+\vec{k}_n\cdot\vec{x}_n)}(-i)\sum_{a=1}^n b^i\left(6\partial_{k_a^i} - k_a^i\vec{\partial}_{k_a}^2 + 2\vec{k}_a\cdot\vec{\partial}_{k_a}\partial_{k_a^i}\right)\langle\pi_{\vec{k}_1}\ldots\pi_{\vec{k}_n}\rangle. \quad (4.141)$$

We can thus compute the $(n+1)$-point correlation function where one of the modes is much longer than the other n to linear order in the gradient of the long mode

$$\langle\pi_{\vec{q}}\pi_{\vec{k}_1}\ldots\pi_{\vec{k}_n}\rangle_{\vec{q}\to 0} = \langle\pi_{\vec{q}}\langle\pi_{\vec{k}_1}\ldots\pi_{\vec{k}_n}\rangle_{\pi_L}\rangle$$

$$= (-i)\langle\pi_{\vec{q}}b^i\rangle'\sum_{a=1}^n\left(6\partial_{k_a^i} - k_a^i\vec{\partial}_{k_a}^2 + 2\vec{k}_a\cdot\vec{\partial}_{k_a}\partial_{k_a^i}\right)\langle\pi_{\vec{k}_1}\ldots\pi_{\vec{k}_n}\rangle$$

$$= \frac{H}{2}P_\pi(q)q^i\sum_{a=1}^n\left(6\partial_{k_a^i} - k_a^i\vec{\partial}_{k_a}^2 + 2\vec{k}_a\cdot\vec{\partial}_{k_a}\partial_{k_a^i}\right)\langle\pi_{\vec{k}_1}\ldots\pi_{\vec{k}_n}\rangle, \quad (4.142)$$

where in the second equality we used the fact that a constant mode leaves the correlation functions unaffected in the presence of the field shift symmetry. Since within the approximations used in this Section one can relate the curvature perturbation in comoving gauge ζ with the field perturbation by $\zeta = -H\pi$, we rewrite this last equation as[10]

$$\langle\zeta_{\vec{q}}\zeta_{\vec{k}_1}\ldots\zeta_{\vec{k}_n}\rangle'_{\vec{q}\to 0} = -\frac{1}{2}P(q)q^i\sum_{a=1}^n\left(6\partial_{k_a^i} - k_a^i\vec{\partial}_{k_a}^2 + 2\vec{k}_a\cdot\vec{\partial}_{k_a}\partial_{k_a^i}\right)\langle\zeta_{\vec{k}_1}\ldots\zeta_{\vec{k}_n}\rangle', \quad (4.143)$$

where P is the power-spectrum of ζ. This is exactly the consistency relation (4.28) in the limit of the exact scale invariance. However, here we derived it as a consequence of the spontaneously broken de Sitter isometries. One should not be surprised that we get the same result. At the end of the day, we are asking the same question about the behaviour of the squeezed limits of correlation functions in both approaches.

Let us at the end stress one technical point. Typically, when a symmetry is broken spontaneously, the number of Ward identities is equal to the number of broken generators. This is the case for the consistency relation too. If we count zeroth order in q and there identities proportional to q^i as independent, then indeed we see that we have four Ward identities for the four broken generators (one dilation and three special conformal transformations). However, notice that we have only *one* Goldstone boson π. Indeed, for spontaneously broken space-time symmetries, it is well known that the number of Goldstone bosons *is not* the same as number of broken generators (see for example [42]).

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[10]Here we used the fact that $\delta(\vec{q} + \vec{k}_1 + \cdots + \vec{k}_n) \approx \delta(\vec{k}_1 + \cdots + \vec{k}_n)$ up to subleading corrections in q, and that the transformation commutes with the Dirac delta up to vanishing terms proportional to the dilation of the n-point function.

4.6.2 Ward Identities

In the previous Section we saw that that, in the decoupling limit, consistency relations can be thought of as Ward identities of spontaneously broken de Sitter isometries. Remarkably, this can be extended to include gravity, as shown in a series of papers [49, 50, 51, 53, 54].

Let us here very briefly present the main logic of the derivation. The starting point is a realisation that once the comoving gauge is fixed, the co-moving curvature perturbation ζ non-linearly realises dilations and special conformal transformations [134]. More generally, even in the presence of short gravitons, the spatial metric in ζ-gauge

$$h_{ij} = a^2 e^{2\zeta} \left(e^\gamma\right)_{ij} \ , \tag{4.144}$$

is invariant under simultaneous particular change of spatial co-ordinates and shift in fields ζ and γ. These change of co-ordinates are residual diffeomorphisms that do not fall off at infinity and they leave the form of the metric (4.144) unchanged. The fact that they diverge at infinity is not necessarily a problem[11]. It turns out that one can generalize Weinbergs argument [44] to show that some linear combinations of the transformations can be smoothly extended to physical solutions that have a regular behaviour. As before, these transformations then correspond to adiabatic long modes. Finally, the last step is to derive Ward identities for this residual gauge symmetry.

Without entering the further details, let us just quote the main result of this approach. Schematically, one can write down (see for example [50])

$$\lim_{q \to 0} M_m \nabla_q^m \left(\frac{1}{P_\zeta(q)} \langle \zeta_q A(\vec{k}_1, \cdots, \vec{k}_n)\rangle + \frac{1}{P_\gamma(q)} \langle \gamma_q A(\vec{k}_1, \cdots, \vec{k}_n)\rangle \right) \sim M_m \nabla_k^m \langle A(\vec{k}_1, \cdots, \vec{k}_n)\rangle \ , \tag{4.145}$$

where $A(\vec{k}_1, \cdots, \vec{k}_n)$ is an arbitrary operator built out of tensor and scalar perturbations. Notice that eq. (4.145) seems to suggest that there are infinitely many identities, one at each order in q. On the other hand, from the discussion in Section 4.4, we know that at second order in gradient expansion the long mode has some physical effects on the short modes. The resolution to this apparent paradox is that the M_m are projection tensors, which only *partially* constrain the soft limit of the amplitude at each order $\mathcal{O}(q^m)$ for $m \geq 2$.([12]) The part that can be constrained at higher orders in q is related to long tensor modes. For $m = 0$ and $m = 1$, one recovers the standard Maldacena and conformal consistency relations.

Notice that one of the major advantages of using the Ward identities approach to consistency relations is that it automatically incorporates the situations in which one has short tensor modes in the correlation functions. As we discussed in Section 3.3.3 and Section 4.2, this is not automatic in the approach described in this thesis. Another important point is that Ward identities are exact to all orders in perturbation theory and including quantum mechanical effects [49, 50, 51]. On the other side, the derivation of the consistency relation using the equivalence of the long mode and the change of co-ordinates relies on treating the

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[11]We already discussed this issue in Chapter 3.

[12]For recent checks of these relations, see [138].

long mode as a classical background. Although we already claimed that the consistency relations must hold non-perturbatively, the agreement with Ward identities approach is a nice check.

Chapter 5

Consistency Relations of Large Scale Structure

During inflation vacuum fluctuations are stretched by the exponential expansion of space. In single-field models, once the modes are outside the horizon, they remain frozen and in the region of space much smaller than their wavelength they are indistinguishable from a rescaling of coordinates. Once inflation is over, the modes start to enter back into the Hubble radius and became observable. It is therefore interesting to ask whether the long mode can be seen locally as change of coordinates even when it is inside the horizon. The relevant quantity to answer this question, as we saw in Chapter 3, is the sound horizon, because it sets the length scale above which the interactions cannot propagate within a Hubble time. Before the decoupling of photons the sound horizon is almost the same as the Hubble scale, but after that it practically shrinks to zero. All modes that enter the Hubble horizon after the decoupling and that we can observe nowadays are indeed just a coordinate redefinition on scales sufficiently small compared to their wavelength.

In Chapter 3 we showed how one can add a long mode to a short-scale solution of the Einstein's equations just by performing a change of coordinates and generate in this way a new solution. We saw that this is possible both in inflation and in the late universe. Of course, if we want to use this trick to make contact with observations, we have to find a way to reformulate this statement on terms of correlation functions. In Chapter 4 we studied how the construction of the adiabatic modes allows us to derive consistency relations for single-field inflation. In this Chapter, we will turn to the late universe and show that for the correlation functions of the matter density contrast δ similar identities exist. These are consistency relations of LSS.

Schematically, the consistency relations of LSS have the following form

$$\langle \Phi_{\vec{q}}(\eta) \delta_{\vec{k}_1}(\eta_1) \cdots \delta_{\vec{k}_n}(\eta_n) \rangle'_{q \to 0} = P_\Phi(q) \sum_a \mathcal{O}_a \langle \delta_{\vec{k}_1}(\eta_1) \cdots \delta_{\vec{k}_n}(\eta_n) \rangle' \,, \tag{5.1}$$

where $\mathcal{O}_a$ are differential operators that contain derivatives with respect to $\vec{k}_i$ and η_i, and correspond to the various terms in the change of coordinates. $P_\Phi(q)$ is the power spectrum

of the gravitational potential Φ. In the Newtonian limit they were first derived in [65, 66], using symmetries of the non-relativistic equations of motion for dark matter. The derivation that we are going to present here is fully relativistic and uses only the equivalence principle [67]. It is therefore quite general, and independent on the details of the short scale physics.

Before we move on to the derivation, let us give a couple of general comments. The form of the consistency relations of LSS is similar to the ones for inflation, and indeed, they are related in a subtle way to each other. The reason for this is that correlation functions involve statistical averages, so we have to take into account not only the effect of the long mode on the short scale dynamics, but also its effect on the initial conditions of the short modes (indeed, this is exactly what we saw in Section 3.3.1). These initial conditions for the late universe are set by inflation. If we restrict to single-field models of inflation we know that the long mode acts simply as a coordinate transformation. Therefore, only with single-field models, we have a unified picture where the long mode *always* acts on the short scales as a coordinate transformation, from inflation until the matter and Λ dominated phases. Of course, this is true only if the long mode is always outside the sound horizon. We will discuss this in more details later.

It is worth stressing that in the same way the inflationary consistency relations are model-independent, the consistency relations of LSS are quite robust statements too. They are independent on the details of the short modes and they hold non-perturbatively even when the baryonic physics is taken into account. As such they are *exact* statements about correlation functions that we can directly observe in galaxy surveys. From this point of view, given the huge amount of data that future LSS surveys will deliver, they are possibly even more interesting than the inflationary consistency relations. If any violation is detected, it would mean that some of the basic assumptions that enter the derivation—single-field inflation and the equivalence principle—is wrong. The consistency relations of LSS are therefore a new model-independent test both of inflation and GR. We will say much more about this towards the end of this Chapter.

We will begin reviewing the full relativistic derivation and then study phenomenologically the most relevant Newtonian limit. After we show how to rewrite the consistency relations in redshift space, we will focus on estimating how well we can constrain the EP using these results. We will close the Chapter briefly discussing some other interesting applications of the consistency relations of LSS.

5.1 Relativistic Consistency Relations of LSS

We will begin our discussion with a derivation of the consistency relations of LSS using the full relativistic framework. The relativistic result is important for two reasons. The first one is that it naturally takes into account even the modes comparable to the scale of horizon where relativistic corrections cannot be neglected. Secondly, this kind of approach is necessary in order to make a connection with initial curvature perturbations and inflation.

5.1.1 Derivation

As in inflation, the starting point in the derivation of the consistency relations is a statement in real space that correlation functions of short-scale perturbations in the presence and the absence of a long mode are related just by a change of coordinates

$$\langle \delta(\vec{x}_1, \eta_1) \cdots \delta(\vec{x}_n, \eta_n) | \Phi_L \rangle = \langle \delta(\vec{\tilde{x}}_1, \tilde{\eta}_1) \cdots \delta(\vec{\tilde{x}}_n, \tilde{\eta}_n) \rangle \ . \tag{5.2}$$

Let us stress again that this statement is correct only if the long mode does not change the statistics of the initial conditions for the short-scale perturbations. This happens only in single-field models of inflation[1]. On the other hand, in multi-field models, the effect of the long mode on the initial conditions *is not* just a coordinate transformation. This would induce additional terms in the equation above, coming from local non-Gaussianities. For the rest of this Chapter we will assume that we have single-field inflation and that there is no statistical correlation between a long and short modes in the initial conditions.

To get the relation among correlation functions, the next step is to multiply both sides of eq. (5.2) by Φ_L and take the average over it. On the left-hand side we get the $(n + 1)$-point function with gravitational potential Φ_L. On the right-hand side we have to expand the correlation function in δx^μ. The density contrast δ is not a scalar and, as we saw in Chapter 3, transforms as

$$\delta \ \rightarrow \ \delta + \delta\eta_a \left(\frac{\bar{\rho}'}{\bar{\rho}}(1 + \delta) + \delta' \right) + \delta x_a^i \cdot \partial_i \delta \ . \tag{5.3}$$

Therefore, once averaged over the Φ_L, eq. (5.2) becomes

$$\langle \Phi_L(x)\delta(x_1) \cdots \delta(x_n) \rangle$$
$$= \left\langle \Phi_L(x) \sum_a \langle \delta(x_1) \cdots \left[\delta\eta_a \left(\frac{\bar{\rho}'}{\bar{\rho}}(1 + \delta) + \delta' \right) + \delta x_a^i \cdot \partial_i \delta \right]_{x_a} \cdots \delta(x_n) \rangle \right\rangle \ . \tag{5.4}$$

Notice that the zeroth order term in the δx^μ expansion is proportional to $\langle \Phi_L \rangle$ and averages to zero. The variation of coordinates δx^μ is proportional to Φ_L and the right hand side will become a product of the two-point function of the long gravitational potential and the n-point function of the density contrast δ. We can say that eq. (5.4) is the consistency relation of LSS written in real space. In the rest of this Section we will show how to bring it to a more familiar form in momentum space.

Before we move on let us recall that the change of coordinates that induces a long mode of the gravitational potential $\Phi_L(\eta)$ for an arbitrary background cosmology is

$$\eta \ \rightarrow \ \eta - D_v(\eta)\zeta_L - D_v(\eta)x^i\partial_i\zeta_L + 4D_v(\eta)\partial_i\zeta_L\partial^{-2}\partial_i\Phi_S \ , \tag{5.5}$$

$$x^i \ \rightarrow \ x^i + \zeta_L x^i + \partial_j\zeta_L x^j x^i - \frac{1}{2}\vec{x}^2\partial^i\zeta_L - \partial^i\zeta_L \int^\eta D_v(\tilde{\eta})\mathrm{d}\tilde{\eta} \ , \tag{5.6}$$

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[1]Notice that this implies small local non-Gaussianities but the amplitude of the three-point function in non-squeezed configurations can be still large.

where $D_v = \frac{1}{a^2} \int a^2 \mathrm{d}\eta$ is the growth function for the velocity. We decided to write the change of coordinates in terms of the initial curvature perturbation ζ_L such that all time dependence of the long mode is in $D_v(\eta)$. Of course, at the end, one can always go back to the gravitational potential using the linear relation between ζ_L and Φ_L eq. (3.29). Notice that for simplicity we have neglected the anisotropic stress, which is anyhow expected to give a very small contribution in the late universe. However, if needed, it can be included straightforwardly. All contributions on the right hand side of eq. (5.4) can be evaluated independently and then summed up at the end, and this is what we are going to do in what follows.

Transformation of spatial coordinates. Let us first focus on the transformation of spatial coordinates. The time independent part of eq. (5.6) describes a dilation and a special conformal transformation of spatial coordinates

$$\eta \rightarrow \eta , \quad x^i \rightarrow x^i + \zeta_L x^i + \partial_j \zeta_L x^j x^i - \frac{1}{2}\vec{x}^2 \partial^i \zeta_L . \tag{5.7}$$

We have already met the same change of coordinates in the derivation of the conformal consistency relation (see Section 4.1.2). Following the same algebra as before we get

$$\langle \zeta_{\vec{q}} \delta_{\vec{k}_1}(\eta_1) \cdots \delta_{\vec{k}_n}(\eta_n) \rangle'_{q \to 0} \supset -P_\zeta(q) \left(3(n-1) + \sum_a \vec{k}_a \cdot \vec{\partial}_{k_a} + \frac{1}{2}q^i \delta^{(n)}_{\mathcal{K}^i} \right) \langle \delta_{\vec{k}_1}(\eta_1) \cdots \delta_{\vec{k}_n}(\eta_n) \rangle' , \tag{5.8}$$

where $\delta^{(n)}_{\mathcal{K}^i}$ is the operator of the special conformal transformation written in momentum space

$$q^i \delta^{(n)}_{\mathcal{K}^i} \equiv \sum_{a=1}^{n} \left[6\vec{q} \cdot \vec{\partial}_{k_a} - \vec{q} \cdot \vec{k}_a \vec{\partial}^2_{k_a} + 2\vec{k}_a \cdot \vec{\partial}_{k_a}(\vec{q} \cdot \vec{\partial}_{k_a}) \right] . \tag{5.9}$$

Let us remind the reader that primes on correlation functions mean that the *same* momentum conservation delta function $(2\pi)^3 \delta(\vec{q} + \sum_i \vec{k}_i)$ is removed from the correlation functions on both sides. This is non-trivial because the correlation function on the right hand side does not contain q. Therefore, the momenta $\vec{k}_1, \ldots, \vec{k}_n$ do not form a closed polygon. As we are going to see (and as it was the case for inflation), this is important when checking the consistency relation using explicit results for the correlation functions.

It is important to notice that expression (5.8) is valid over the entire history of the universe, assuming single-field inflation and that the long mode is always outside the sound horizon. Indeed, in the limit $\eta_a \to 0$, eq. (5.8) becomes just the conformal consistency relation for the initial curvature perturbation. For this reason, the LSS consistency relations are valid only in models with a single dynamical degree of freedom during inflation. Later stages of evolution, such as radiation and matter dominance, are encoded in the time dependence of the short modes. As we will see later, eq. (5.8) is the only part of the consistency relations of LSS that involves initial conditions.

The time-dependent part of the coordinate transformation in eq. (5.6) is

$$\eta \rightarrow \eta , \quad x^i \rightarrow x^i - \partial^i \zeta_L \int^\eta D_v(\tilde{\eta})\mathrm{d}\tilde{\eta} . \tag{5.10}$$

This term is very important because it is the leading term in the non-relativistic limit. Indeed, for the short modes deep inside the horizon, $k_a \eta_a \to \infty$ and the last term in eq. (5.6) dominates the change of coordinates[2]. Under transformation (5.10) the variation of an n-point function is given by

$$\delta \langle \delta(x_1) \cdots \delta(x_n) \rangle = \sum_a \left(-\partial^i \zeta_L \int^{\eta_a} D_v(\tilde{\eta}) \mathrm{d}\tilde{\eta} \right) \partial_{ai} \langle \delta(x_1) \cdots \delta(x_n) \rangle \ . \tag{5.11}$$

In momentum space the right hand side becomes

$$\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \mathrm{d}K_n (2\pi)^3 \sum_a \left(\int^{\eta_a} D_v(\tilde{\eta}) \mathrm{d}\tilde{\eta} \right) \vec{p} \cdot \vec{k}_a \, \zeta_{\vec{p}} \, \langle \delta_{\vec{k}_1} \cdots \delta_{\vec{k}_n} \rangle e^{i\vec{p} \cdot \vec{x}_+ + i\sum_b \vec{k}_b \cdot \vec{x}_b} \ . \tag{5.12}$$

The last step is to multiply eq. (5.12) by $\zeta_{\vec{q}}$ and average over ζ. On the left hand side, we get the $(n+1)$-point correlation function. On the right hand side the average over $\zeta_{\vec{q}}$ gives the power spectrum of the long mode. Due to momentum conservation $\vec{p}$ becomes $-\vec{q}$. The contribution to the consistency relation is

$$\langle \zeta_{\vec{q}} \delta_{\vec{k}_1}(\eta_1) \cdots \delta_{\vec{k}_n}(\eta_n) \rangle'_{q\to0} \supset -P_\zeta(q) \sum_a \left(\int^{\eta_a} D_v(\eta) \mathrm{d}\eta \right) \vec{q} \cdot \vec{k}_a \langle \delta_{\vec{k}_1}(\eta_1) \cdots \delta_{\vec{k}_n}(\eta_n) \rangle' \ . \tag{5.13}$$

As we have already said, this term is the largest on short scales and it dominates the consistency relation when the short modes are deep inside the Hubble radius. In this limit, the construction of adiabatic modes and the consistency relation have a very intuitive interpretation based on the Equivalence Principle. We will discuss this in more details in the following Section.

Transformation of the time coordinate. Let us now focus on the part of the consistency relation that comes from the transformation of time

$$\delta\eta = -D_v(\eta)\zeta_L - D_v(\eta)x^i \partial_i \zeta_L + 4D_v(\eta)\partial_i \zeta_L \partial^{-2} \partial_i \Phi_S \ . \tag{5.14}$$

Given that we are interested in correlation functions of δ we should rewrite the short-wavelength gravitational potential in terms of δ_S. In a ΛCDM universe[3] the equation relating the density contrast and the gravitation potential reads

$$\partial^2 \Phi_S - 3\mathcal{H}^2 f_g(\eta) \Phi_S = \frac{3}{2} \mathcal{H}^2 \Omega_{\mathrm{m}} \delta \ , \tag{5.15}$$

where $f_g(\eta)$ is the linear growth rate of fluctuations and Ω_{m} is the time-dependent dark matter density fraction. Therefore, the change of coordinates becomes

$$\delta\eta = -D_v(\eta)\zeta_L - D_v(\eta)x^i \partial_i \zeta_L + 6D_v(\eta)\partial_i \zeta_L \frac{\mathcal{H}^2 \Omega_{\mathrm{m}}}{\partial^2(\partial^2 - 3\mathcal{H}^2 f_g)} \partial_i \delta_S \ . \tag{5.16}$$

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[2]For example in matter dominance this can be explicitly seen by taking the limit $k\eta \to \infty$.

[3]We assume $\Psi_S = \Phi_S$, because we are working at first order in Φ_S. This, however, does not mean that the result is also linear in δ_S, as we already explained when we discussed the construction of adiabatic modes in the late universe.

There are three different terms that have to be taken into account

$$\delta \;\rightarrow\; \delta + \delta\eta_a\left(\delta' + \frac{\bar{\rho}'}{\bar{\rho}} + \frac{\bar{\rho}'}{\bar{\rho}}\delta\right) . \tag{5.17}$$

We begin from the first one. The last part of eq. (5.14) would lead to a contribution that is second order in Φ_S and we can neglect it in this case. The relevant terms are

$$\eta \rightarrow \eta - D_v(\eta)\zeta_L - D_v(\eta)x^i\partial_i\zeta_L , \quad x^i \rightarrow x^i . \tag{5.18}$$

It is straightforward to find the variation of an n-point function

$$\delta\langle\delta(x_1)\cdots\delta(x_n)\rangle = -\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\mathrm{dK}_n\sum_a D_v(\eta_a)\zeta_{\vec{p}}\,\partial_{\eta_a}\langle\delta_{\vec{k}_1}\cdots\delta_{\vec{k}_n}\rangle(1+\vec{p}\cdot\vec{\partial}_{k_a})e^{i\vec{p}\cdot\vec{y}+i\sum_b\vec{k}_b\cdot\vec{x}_b}$$

$$= -\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\mathrm{dK}_n\sum_a\left(D_v(\eta_a)\zeta_{\vec{p}}\,\partial_{\eta_a}(1-\vec{p}\cdot\vec{\partial}_{k_a})\right)\langle\delta_{\vec{k}_1}\cdots\delta_{\vec{k}_n}\rangle e^{i\vec{p}\cdot\vec{y}+i\sum_b\vec{k}_b\cdot\vec{x}_b} . \tag{5.19}$$

Like in the case of special conformal transformations and dilations, when the derivatives act on the delta function of momentum conservation they combine with zeroth order term. In particular

$$(1-\vec{p}\cdot\vec{\partial}_{k_a})\delta(\vec{k}_1+\cdots+\vec{k}_n) = \delta(-\vec{p}+\vec{k}_1+\cdots+\vec{k}_n) , \tag{5.20}$$

which will lead to the *same* momentum conservation delta function on both sides of the consistency relation. Indeed, if we average over the long mode ζ_q, the contribution to the consistency relation is

$$\langle\zeta_{\vec{q}}\,\delta_{\vec{k}_1}(\eta_1)\cdots\delta_{\vec{k}_n}(\eta_n)\rangle'_{q\to0} \supset -P_\zeta(q)\sum_a D_v(\eta_a)(1+\vec{q}\cdot\vec{\partial}_{k_a})\partial_{\eta_a}\langle\delta_{\vec{k}_1}(\eta_1)\cdots\delta_{\vec{k}_n}(\eta_n)\rangle' . \tag{5.21}$$

The last term in eq. (5.17) will give a very similar contribution. Using $\bar{\rho}' = -3\mathcal{H}\bar{\rho}$ we get

$$\langle\zeta_{\vec{q}}\,\delta_{\vec{k}_1}(\eta_1)\cdots\delta_{\vec{k}_n}(\eta_n)\rangle'_{q\to0} \supset P_\zeta(q)\sum_a 3\mathcal{H}(\eta_a)D_v(\eta_a)(1+\vec{q}\cdot\vec{\partial}_{k_a})\langle\delta_{\vec{k}_1}(\eta_1)\cdots\delta_{\vec{k}_n}(\eta_n)\rangle' . \tag{5.22}$$

Finally, we can focus on the second term in eq. (5.17). This term is not proportional to δ_S and we have to use the last part of the coordinate transformation[4]. The relevant transformation of δ is

$$\delta \;\rightarrow\; \delta - 18\mathcal{H}\partial_i\zeta_L\frac{\mathcal{H}^2\Omega_\mathrm{m}D_v(\eta)}{\partial^2(\partial^2-3\mathcal{H}^2f_g)}\partial_i\delta , \tag{5.23}$$

where we have used $\bar{\rho}' = -3\mathcal{H}\bar{\rho}$. At the level of the consistency relation, it is easy to see that the contribution is

$$\langle\zeta_{\vec{q}}\,\delta_{\vec{k}_1}\cdots\delta_{\vec{k}_n}\rangle'_{q\to0} \supset -6P_\zeta(q)\sum_a\mathcal{H}D_v\Omega_\mathrm{m}\frac{\vec{q}\cdot\vec{k}_a}{k_a^2}\left(f_g+\frac{k_a^2}{3\mathcal{H}^2}\right)^{-1}\langle\delta_{\vec{k}_1}\cdots\delta_{\vec{k}_n}\rangle' . \tag{5.24}$$

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[4]The other parts just induce the long mode of δ but we are not interested in these terms.

The full result. Now we are ready to write down the final result. Summing up all terms derived above, we get the relativistic consistency relation for LSS

$$\langle \zeta_{\vec{q}} \delta_{\vec{k}_1}(\eta_1) \cdots \delta_{\vec{k}_n}(\eta_n) \rangle'_{q \to 0} = -P_\zeta(q) \left[3(n-1) + \sum_a \vec{k}_a \cdot \vec{\partial}_{k_a} + \sum_a D_v(\eta_a)(\partial_{\eta_a} - 3\mathcal{H}(\eta_a)) \right.$$

$$+ \sum_a \left(\int^{\eta_a} D_v(\eta)\mathrm{d}\eta \right) \vec{q} \cdot \vec{k}_a + \frac{1}{2} q^i \delta^{(n)}_{\mathcal{K}^i} + \sum_a D_v(\eta_a)(\partial_{\eta_a} - 3\mathcal{H}(\eta_a))\vec{q} \cdot \vec{\partial}_{k_a}$$

$$\left. + 6 \sum_a \Omega_{\mathrm{m}}(\eta_a) D_v(\eta_a) \mathcal{H}(\eta_a) \frac{\vec{q} \cdot \vec{k}_a}{k_a^2} \left(f_g(\eta_a) + \frac{k_a^2}{3\mathcal{H}^2(\eta_a)} \right)^{-1} \right] \langle \delta_{\vec{k}_1}(\eta_1) \cdots \delta_{\vec{k}_n}(\eta_n) \rangle' . \quad (5.25)$$

Let us comment on some of the terms on the right hand side. The first line contains terms of order $\mathcal{O}(q^0)$ which come from a homogeneous part of the long mode, while the contributions in the other two lines are of order $\mathcal{O}(q)$ and they come from gradients. As we already pointed out, the first term in the second line dominates in the non-relativistic limit, when all the modes are deep inside the Hubble radius. In this case, which is observationally most relevant, the consistency relation becomes very simple and we will come back to it in the following sections. Let us also stress that in the limit $\eta \to 0$ the only two terms that are non-zero are dilations and special conformal transformations. In this limit we have a relation that has to be satisfied by correlation functions of initial curvature perturbations. As we already pointed out, this is the case only in single-field inflation.

Before moving on and commenting on some of the properties of the consistency relation of LSS, let us write it here explicitly for an important case of a matter dominated universe. To do so we have to set

$$\Omega_{\mathrm{m}} = 1 , \quad \mathcal{H}(\eta) = \frac{2}{\eta} , \quad D_v(\eta) = \frac{\eta}{5} , \quad f_g(\eta) = 1 . \quad (5.26)$$

We can also write the final result in terms of the gravitational potential instead of curvature perturbations. For the long mode, in a matter dominated universe, the relation among the two is simply $\zeta_{\vec{q}} = -\frac{5}{3}\Phi_{\vec{q}}$. The consistency relation in this case simplifies to

$$\langle \Phi_{\vec{q}} \delta_{\vec{k}_1}(\eta_1) \cdots \delta_{\vec{k}_n}(\eta_n) \rangle'_{q \to 0} = P_\Phi(q) \left[5(n-1) + \frac{5}{3} \sum_a \vec{k}_a \cdot \vec{\partial}_{k_a} + \sum_a \frac{\eta_a}{3} \left(\partial_{\eta_a} - \frac{6}{\eta_a} \right) \right.$$

$$+ \frac{1}{6} \sum_a \eta_a^2 \vec{q} \cdot \vec{k}_a + \frac{5}{6} q^i \delta^{(n)}_{\mathcal{K}^i} + \sum_a \frac{\eta_a}{3} \left(\partial_{\eta_a} - \frac{6}{\eta_a} \right) \vec{q} \cdot \vec{\partial}_{k_a}$$

$$\left. + 4 \sum_a \frac{\vec{q} \cdot \vec{k}_a}{k_a^2} \left(1 + \frac{k_a^2 \eta_a^2}{12} \right)^{-1} \right] \langle \delta_{\vec{k}_1}(\eta_1) \cdots \delta_{\vec{k}_n}(\eta_n) \rangle' . \quad (5.27)$$

As we already pointed out, this equation is derived under the assumption that the anisotropic stress is zero but this can be easily relaxed.

Before we give some comments about the consistency relation, let us clarify one technical point about the final result. As we already stressed, the delta function of momentum conservation is the same on the two sides of the consistency relation. This can be a bit confusing,

because the correlation functions have different number of fields. On the left hand side there is no ambiguity, but on the right hand side the correlation function has to be evaluated for configuration of momenta that do not form a closed polygon. For this reason, the equation might seem ambiguous[5]. In principle, we can add to the correlation function on the right hand side any expression that is zero for $\sum \vec{k}_i = 0$, but it is far from obvious that the complicated differential operator will always give zero acting on this term. However, it is straightforward to show that this is indeed what happens. If we assume that the additional term is of the form $P_i \mathcal{M}_i(\vec{k}_1, \ldots, \vec{k}_n)$ and that it is translationally and rotationally invariant, then the following relation holds

$$\sum_a O_a [P_i \mathcal{M}_i(\vec{k}_1, \ldots, \vec{k}_n)] = \mathcal{O}(q^2) , \tag{5.28}$$

where $\vec{P} = \sum \vec{k}_i$, $\mathcal{M}$ is an arbitrary (translationally and rotationally invariant) function of the momenta and O_a is a differential operator on the right hand side of the consistency relation. In this way we explicitly checked that the consistency relation is indeed unambiguous.

5.1.2   Comments

Before moving on, it is important to summarize the conditions under which the consistency relations of LSS are derived and to highlight some of their properties that we are going to use in the following sections.

Let us begin the discussion by putting together all assumptions that we made in order to derive (5.25). They are related to the properties of the long mode, the initial conditions and the validity of general relativity (GR).

- The long mode must be longer than the sound horizon during the entire history of the universe (after inflation). This means that the long modes must be outside the horizon until decoupling. After that, the speed of sound practically becomes zero, and the sound horizon shrinks too. As we will see later, even if the modes enter the sound horizon before decoupling, as long as the separation of scales is large enough, the effects of the interactions are small at low redshift. The reason is that the interactions between long and short modes stop at decoupling and after that their effects decay with the expansion of the universe. The other condition on the long mode is that it has to be in the linear regime. Only then is our construction from Chapter 3 valid.

- The initial conditions for curvature perturbations must be compatible with single-field inflation. This is important because the validity of the consistency relation does not rely only on the fact that the long mode can be removed by a change of coordinates, but also that the correlation with the short modes in the initial conditions does not exists. This is the case only in single-field models. As we already discussed, the most explicit way to see this is to send $\eta \to 0$ in eq. (5.25). Notice that this condition implies that only local non-Gaussianities are negligible. The amplitude of the three-point function of

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[5]We have already seen how to resolve this problem for the consistency relation for single-field inflation.

initial curvature perturbations for other shapes, for example equilateral or orthogonal, can be still large.

- The construction of the adiabatic mode inside the Hubble radius in the late universe is valid only assuming GR. In the theories of modified gravity it might happen that different kind of objects couple to the gravitational potential in different ways (for example in a presence of a fifth force or due to different screening mechanisms). As a result, the effect of the long mode cannot be removed by a change of coordinates. In the non-relativistic limit, that will be the central topic in the following sections, this has a particularly nice and intuitive interpretation in terms of the EP. As we are going to see, if the consistency relations of LSS are violated, this indicates a violation of the EP on cosmological scales.

Apart form these basic assumptions, we do not have to make any other. This is particularly relevant because we do not have to assume anything about the dynamics of the short modes. The short modes can even be non-perturbative. Indeed, although the consistency relation is derived keeping only the terms up to second order in the metric perturbations, this doesn't mean that the result is perturbative in the density contrast δ. The gravitational potential for the short modes is always small, but δ can be arbitrarily large. The consistency relation holds always, and therefore also deep in the non-linear regime. Notice that in order to make this statement we do not have to assume any equations of motion for the short scales, no single stream or fluid approximations.

Even more, under some mild additional assumptions, the consistency relation holds also for the density contrast of galaxies[6]. This is a very important point. The construction of the adiabatic modes is insensitive to the short scale physics and the long mode is just a coordinate transformation even including the interactions of baryons. Therefore, we can directly replace δ_S with $\delta_S^{(g)}$ in the formulas above and they should still be valid. In this way we do not have to worry about bias, linear or non-linear, and all complicated details of galaxy formation. This is crucial if we want to connect the consistency relations with observations and use their non-perturbative nature. Of course, this does not hold for the long mode. In this case, we have to assume some model for bias. Given that the long mode is always in the linear regime, we can expect that Φ_L, δ_L and $\delta_L^{(g)}$ are related to each other through simple linear relations.

Let us at the end comment on the possibility to check the consistency relation in observations. It is important to notice that for equal-time correlation functions (that is what we observe) the right hand side of the consistency relation vanishes if we take the squeezed limit of the three-point function. As for the inflationary correlation functions, this is simply the consequence of the fact that the two-point function does not specify any direction and at order $\mathcal{O}(q)$ one cannot find an object to contract with $\vec{q}$. This does not happen for higher order correlation functions, but the terms that survive on the right hand side are all relativistic and

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[6]A modification to our formulas will come from the fact the number of haloes or galaxies is not conserved, but evolves in time, so that the effect of the time redefinition will be different.

hard to observe. The non-relativistic part vanishes for any equal-time n-point function (we will come back to this later).

We are therefore left with an option to use unequal-time correlation functions. However, let us stress that it will be difficult—if not impossible—to measure consistency relations at different times. To see the effect of the long mode, one would like to measure at quite different redshifts the short-scale correlation function at a spatial distance which is much smaller than Hubble. This is of course impossible since we can only observe objects on our past lightcone. This implies that, although one can check the consistency relations at different times in simulations, from real data we can only get the correlation functions at the same time.

This might seem disappointing, but we can turn this situation in our advantage noticing that any violation of the consistency relations would indicate that one of the assumptions in the derivation does not hold. In other words, measuring the equal-time correlation functions in the squeezed limit and finding non-vanishing $\mathcal{O}(q)$ terms on the right hand side, would represent a detection of either multi-field inflation or violation of the equivalence principle. In this sense, phenomenologically it is much more interesting to look for violations of the consistency relations. We will discuss this in much more details in last sections of this Chapter.

5.1.3   Checks

To check the consistency relation we can use the two-point and the three-point functions calculated for a matter dominated universe. The two-point function of δ reads

$$\langle \delta_{\vec{k}_1}(\eta_1)\delta_{\vec{k}_2}(\eta_2)\rangle' = \frac{1}{36}P_\Phi(k)(12+k_1^2\eta_1^2)(12+k_2^2\eta_2^2) \, , \tag{5.29}$$

where $k \equiv |\vec{k}_1 - \vec{k}_2|/2$ and we stress again that $P_\Phi(k)$ is the power spectrum of Φ in the matter dominated phase, whose scale dependence is affected by the evolution during radiation dominance. Plugging this expression on the right hand side of eq. (5.27) and expanding for small q using $\vec{k}_1 \equiv \vec{k} - \vec{q}/2$, $\vec{k}_2 \equiv -\vec{k} - \vec{q}/2$, we obtain

$$\langle \Phi_{\vec{q}}\delta_{\vec{k}_1}(\eta_1)\delta_{\vec{k}_2}(\eta_2)\rangle'_{q\to 0} = P_\Phi(q)P_\Phi(k)\left(\frac{-144+k^4\eta_1^2\eta_2^2}{9}\right.$$
$$\left. +\vec{q}\cdot\vec{k}(\eta_1^2-\eta_2^2)\frac{-168+k^4\eta_1^2\eta_2^2+12k^2(\eta_1^2+\eta_2^2)}{216}\right)$$
$$+\frac{5}{3}P_\Phi(q)\frac{1}{k^3}\frac{d(k^3 P_\Phi(k))}{d\ln k}\left[(12+k^2\eta_1^2)(12+k^2\eta_2^2)-12\vec{q}\cdot\vec{k}(\eta_1^2-\eta_2^2)\right] \, . \tag{5.30}$$

We can compare this result with the squeezed limit of the three-point function calculated explicitly using the perturbation theory. The first two lines on the right hand side of the previous equation contain the nonlinear evolution in matter dominance. One can verify this by computing the three-point function using the result of the exact second-order calculation from Ref. [46]

$$\langle \Phi_{\vec{q}}\delta_{\vec{k}_1}(\eta_1)\delta_{\vec{k}_2}(\eta_2)\rangle'_{q\to 0} = \langle \Phi_{\vec{q}}\delta_{\vec{k}_1}^{(2)}(\eta_1)\delta_{\vec{k}_2}^{(1)}(\eta_2)\rangle' + \langle \Phi_{\vec{q}}\delta_{\vec{k}_1}^{(1)}(\eta_1)\delta_{\vec{k}_2}^{(2)}(\eta_2)\rangle' \, . \tag{5.31}$$

Notice that we keep the long mode always in the perturbative regime. After we replace the explicit form of $\delta^{(2)}$ we indeed get the first two lines in eq. (5.30).

The last line in eq. (5.30) takes into account the initial conditions and directly follows from Maldacena's calculation of the primordial three-point correlation function of ζ [43], rewritten in terms of Φ in matter dominance and also taken in the squeezed limit

$$\langle \Phi_{\vec{q}}\, \Phi_{\vec{k}_1} \Phi_{\vec{k}_2} \rangle'_{q\to 0} = \frac{5}{3} P_\Phi(q) \left(3 + k\frac{\mathrm{d}}{\mathrm{d}k} \right) P_\Phi(k) \,. \tag{5.32}$$

Notice that we get only scale variation, because the conformal transformation of the initial conditions vanishes when acting on a two-point function.

5.2   Non-relativistic Limit and Resummation

So far we have been considering the full relativistic derivation and the consistency relations of LSS including relativistic corrections. On the other hand, observationally, the most interesting regime is the one when all the modes are deep inside the Hubble radius. In this case only one term on the right hand side of eq. (5.25) survives and the consistency relations simplify to [65, 66, 69]

$$\langle \zeta_{\vec{q}}\delta_{\vec{k}_1}(\eta_1)\cdots\delta_{\vec{k}_n}(\eta_n)\rangle'_{q\to 0} = -P_\zeta(q)\sum_a \left(\int^{\eta_a} D_v(\eta)\mathrm{d}\eta \right) \vec{q}\cdot\vec{k}_a \, \langle \delta_{\vec{k}_1}(\eta_1)\cdots\delta_{\vec{k}_n}(\eta_n)\rangle' \,. \tag{5.33}$$

We can rewrite this formula in a simpler way. Let us first define the growth function of the density field D_δ as $\delta_L = D_\delta(\eta)\nabla^2\zeta_L$. Then, from the continuity equation $\delta_L' + \vec{\nabla}\cdot\vec{v}_L = 0$, it follows that $D_\delta(\eta) = -\int^\eta D_v(\tilde{\eta})\mathrm{d}\tilde{\eta}$ and the above expression reduces to

$$\langle \delta_{\vec{q}}(\eta)\, \delta_{\vec{k}_1}(\eta_1)\cdots\delta_{\vec{k}_n}(\eta_n)\rangle'_{q\to 0} = -P_\delta(q,\eta)\sum_a \frac{D_\delta(\eta_a)}{D_\delta(\eta)}\frac{\vec{q}\cdot\vec{k}_a}{q^2}\langle \delta_{\vec{k}_1}(\eta_1)\cdots\delta_{\vec{k}_n}(\eta_n)\rangle' \,. \tag{5.34}$$

There is a very simple and intuitive interpretation of this result. Let us consider a flat unperturbed FRW universe and add to it a homogenous gradient of the Newtonian potential Φ_L.[7] Provided all species feel gravity the same way—namely, assuming the equivalence principle—we can can get rid of the effect of $\vec{\nabla}\Phi_L$ by going into a frame which is free falling in the constant gravitational field. The coordinate change to the free-falling frame is

$$\vec{x} \to \vec{x} + \delta\vec{x}(\eta)\,, \qquad \delta\vec{x}(\eta) \equiv -\int \vec{v}_L(\tilde{\eta})\,\mathrm{d}\tilde{\eta}\,, \tag{5.35}$$

while time is left untouched. The velocity $\vec{v}_L$ satisfies the Euler equation in the presence of the homogenous force, whose solution is

$$\vec{v}_L(\eta) = -\frac{1}{a(\eta)}\int a(\tilde{\eta})\vec{\nabla}\Phi_L(\tilde{\eta})\,\mathrm{d}\tilde{\eta}\,. \tag{5.36}$$

---

[7]Since we are interested in the non-relativistic limit, we do not consider a constant value of Φ_L, which is immaterial in this limit.

Replacing this result into eq. (5.35), we get a change of coordinates that leads to the consistency relation (5.34).

From this simple analysis we see that in the non-relativistic limit the physical argument behind the consistency relations is that at leading order in q a long mode gives rise to a homogeneous gravitational field $\vec{\nabla}\Phi$. The effect of this mode on the short-scale physics can be derived exactly using the equivalence principle and erasing the long mode with a suitable change of coordinates. As we already pointed out, this logic makes no assumptions about the physics at short scales, including the complications due to baryons. In the rest of this Chapter we will focus on this non-relativistic limit.

Before we move on, let us briefly comment on one of the conditions that the long mode in general has to satisfy. As already explained, for the validity of the consistency relations, the long mode has to be always out of the sound horizon since inflation. However, even if this condition is not satisfied, the corrections to the consistency relation are small. Let us look at what kind of corrections we expect.

One well-understood example where the consistency relations are not obeyed is the case of baryons and cold dark matter particles after decoupling. Before recombination, while dark matter follows geodesics, baryons are tightly coupled to photons through Thomson scattering and display acoustic oscillations. Later on, baryons and electrons recombine and decouple from photons. Thus, as their sound speed drops they start following geodesics, but with a larger velocity than that of the dark matter on comoving scales below the sound horizon at recombination. As discussed in [139], the long-wavelength relative velocity between baryons and CDM reduces the formation of early structures on small scales, through a genuinely nonlinear effect. The fact that baryons have a different initial large-scale velocity compared to dark matter implies, if the long mode is shorter than the comoving sound horizon at recombination, that the change of coordinates that erases the effect of the long mode is not the same for the two species. Thus the effect of the long mode does not cancel out in the equal-time correlators involving different species [140, 141]. In particular, the amplitude of the short-scale equal-time n-point functions becomes correlated with the long-wavelength isodensity mode, so that the $(n+1)$-point functions in the squeezed limit do not vanish at equal time. This effect, however, becomes rapidly negligible at low redshifts because the relative comoving velocity between baryons and dark matter decays as the scale factor, $|\vec{v}_{\rm b} - \vec{v}_{\rm CDM}| \propto 1/a$.[8] As a conclusion we can say that, while a deviation form the consistency relations can be sizeable at high redshifts, it can be actually neglected at low redshift (or deep inside the horizon). Therefore, for the rest of this Chapter, we will assume that in galaxy surveys the consistency relations apply also when the long mode is shorter than the comoving sound horizon at recombination.

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[8]The violation of the consistency relations decays as $(D_{\rm iso}/D)^2 \propto (a^2 H f D)^{-2} \sim (1+z)^{3/2}$ where $D_{\rm iso} \propto |\vec{v}_{\rm b} - \vec{v}_{\rm CDM}|/(aHf)$ is the growth function of the long-wavelength isodensity mode, D is the growth function of the long-wavelength adiabatic growing mode, f is the growth rate and H is the Hubble rate (see [140] for details); in the last approximate equality we have used matter dominance. Thus, the effect is already sub-percent at $z \sim 40$.

5.2.1 Resumming the long mode

The consistency relations we discussed so far were obtained using an infinitesimal transformations of coordinates. Physically, this corresponds to a small gradient of the long mode. While in the full relativistic setup there are some issues related to the generalisation of the results to the finite gradients (see Section 4.5), in the non-relativistic regime the situation is much simpler. Intuitively, no matter how large is the gradient of the long mode, using the EP it is always possible to go to a free falling reference frame in which its effect is removed.

Let us take a closer look to the expansion we did so far. The displacement due to a homogeneous gravitational potential scales with time as $\Delta \vec{x} \sim \vec{\nabla} \Phi_L \, t^2$, so that the effect on the short modes of momentum k goes as

$$\vec{k} \cdot \Delta \vec{x} \sim k \, q \, \Phi_L \, t^2 \sim \frac{k}{q} \delta_L \ . \tag{5.37}$$

In the derivation of the consistency relation we were expanding in $\vec{k} \cdot \Delta \vec{x}$, but this quantity is not necessarily small. The only thing we have to impose is that δ_L is small, such that we are always treating the long mode perturbatively. On the other hand, in the $q \ll k$ limit, the quantity $\frac{k}{q}\delta_L$ can become arbitrarily large. In this Section we are going to take this into account and derive the consistency relation without expanding in the displacement due to the long mode. Physically, this corresponds to a situation in which gradients of the long mode are large. As we are going to see, this will lead to a "resummed" consistency relation and it will allow us to study other interesting kinematical regimes of the correlation functions, such as multiple soft limits, soft internal lines or soft loops.

Our starting point is again real space. We will follow the same steps as before, with the only difference that we are not going to expand the change of coordinates. Following the argument above, any n-point correlation function[9] of short wavelength modes of $\delta^{(g)}$ in the presence of a slowly varying $\Phi_L(\vec{y})$ is equivalent to the same correlation function in displaced spatial coordinates. The change of coordinates in given by $\vec{\tilde{x}} \equiv \vec{x} + \delta \vec{x}(\vec{y}, \eta)$, where the displacement field $\delta \vec{x}(\vec{y}, \eta)$ is given by eq. (5.35) and $\vec{y}$ is an arbitrary point—e.g., the midpoint between $\vec{x}_1, \ldots, \vec{x}_n$—whose choice is irrelevant at order q/k. This statement can be formulated with the following relation,

$$\langle \delta^{(g)}(\vec{x}_1, \eta_1) \cdots \delta^{(g)}(\vec{x}_n, \eta_n) | \Phi_L(\vec{y}) \rangle \approx \langle \delta^{(g)}(\vec{\tilde{x}}_1, \eta_1) \cdots \delta^{(g)}(\vec{\tilde{x}}_n, \eta_n) \rangle_0$$
$$= \int \frac{\mathrm{d}^3 k_1}{(2\pi)^3} \cdots \frac{\mathrm{d}^3 k_n}{(2\pi)^3} \langle \delta^{(g)}_{\vec{k}_1}(\eta_1) \cdots \delta^{(g)}_{\vec{k}_n}(\eta_n) \rangle_0 \, e^{i \sum_a \vec{k}_a \cdot (\vec{x}_a + \delta \vec{x}(\vec{y}, \eta_a))} \ , \tag{5.38}$$

where in the last line we have simply taken the Fourier transform of the right-hand side of the first line. For the rest of this Section, by the subscript 0 after an expectation value we mean that the average is taken setting $\Phi_L = 0$ (and not averaging over it), while by $\approx$ we

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[9] Here, for definiteness, we denote by $\delta^{(g)}$ the density contrast of the galaxy distribution. However, the relations that we will derive are more general and hold for any species—halos, baryons, etc., irrespectively of their bias with respect to the underlying dark matter field.

mean an equality that holds in the limit in which there is a separation of scales between long and short modes. In momentum space this holds when the momenta of the soft modes is sent to zero. In other words, corrections to the right-hand side of $\approx$ are suppressed by $\mathcal{O}(q/k)$. As usual, we assume single-field models of inflation, that produce initial conditions with no statistical correlation between long and short modes.

As before, we can compute an $(n+1)$-point correlation function in the squeezed limit by multiplying the left-hand side of eq. (5.38) by δ_L and averaging over the long mode. Since the only dependence on Φ_L in eq. (5.38) is in the exponential of $i \sum_a \vec{k}_a \cdot \delta \vec{x}(\vec{y}, \eta_a)$, we obtain

$$
\begin{aligned}
&\langle \delta_L(\vec{x}, \eta) \langle \delta^{(g)}(\vec{x}_1, \eta_1) \cdots \delta^{(g)}(\vec{x}_n, \eta_n) | \Phi_L \rangle \rangle_{\Phi_L} \\
&\approx \int \frac{\mathrm{d}^3 k_1}{(2\pi)^3} \cdots \frac{\mathrm{d}^3 k_n}{(2\pi)^3} \langle \delta^{(g)}_{\vec{k}_1}(\eta_1) \cdots \delta^{(g)}_{\vec{k}_n}(\eta_n) \rangle_0 \, e^{i \sum_a \vec{k}_a \cdot \vec{x}_a} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} e^{i \vec{q} \cdot \vec{x}} \langle \delta_{\vec{q}}(\eta) e^{i \sum_a \vec{k}_a \cdot \delta \vec{x}(\vec{y}, \eta_a)} \rangle_{\Phi_L} .
\end{aligned}
$$

(5.39)

Notice that before we were expanding the exponential and taking the leading term that was giving us the power spectrum of the long modes. Now we want to keep the derivation exact and we have to calculate the average with the exponent. This can be done assuming Gaussian initial conditions. Notice that this assumption is stronger than assuming single-field inflation, because even single-field models can lead to large non-Gaussianities of, for example, equilateral type. However, with current limits from Planck [13], all corrections coming from non-Gaussianities are small and we will neglect them in what follows.

The exponential in the previous expression can be rewritten in the following form

$$
\exp \left[i \sum_a \vec{k}_a \cdot \delta \vec{x}(\vec{y}, \eta_a) \right] = \exp \left[\int^\Lambda \frac{\mathrm{d}^3 p}{(2\pi)^3} J(\vec{p}) \delta_0(\vec{p}) \right] ,
$$

(5.40)

where

$$
J(\vec{p}) \equiv \sum_a D(\eta_a) \frac{\vec{k}_a \cdot \vec{p}}{p^2} e^{i \vec{p} \cdot \vec{y}} .
$$

(5.41)

The integral is restricted to soft momenta, smaller than a UV cut-off Λ, which must be much smaller than the hard modes of momenta k_a. Averaging the right-hand side of eq. (5.40) over the long wavelength Gaussian random initial condition $\delta_0(\vec{p})$ yields

$$
\left\langle \exp \left[\int^\Lambda \frac{\mathrm{d}^3 p}{(2\pi)^3} J(\vec{p}) \delta_0(\vec{p}) \right] \right\rangle_{\Phi_L} = \exp \left[\frac{1}{2} \int^\Lambda \frac{\mathrm{d}^3 p}{(2\pi)^3} J(\vec{p}) J(-\vec{p}) P_0(p) \right] .
$$

(5.42)

We can use this relation to compute the expectation value of δ_L with the exponential. The only thing that we have to do is one functional derivative with respect to $J(\vec{q})$

$$
\begin{aligned}
\left\langle \delta_{\vec{q}}(\eta) \exp \left(i \sum_a \vec{k}_a \cdot \delta \vec{x}(\vec{y}, \eta_a) \right) \right\rangle_{\Phi_L} &= (2\pi)^3 D(\eta) \frac{\delta}{\delta J(\vec{q})} \left\langle \exp \left[\int^\Lambda \frac{\mathrm{d}^3 p}{(2\pi)^3} J(\vec{p}) \delta_0(\vec{p}) \right] \right\rangle_{\Phi_L} \\
&= P(q, \eta) \frac{J(-\vec{q})}{D(\eta)} \exp \left[\frac{1}{2} \int^\Lambda \frac{\mathrm{d}^3 p}{(2\pi)^3} J(\vec{p}) J(-\vec{p}) P_0(p) \right] ,
\end{aligned}
$$

(5.43)

where we have defined the power spectrum at time η: $P(q, \eta) \equiv D^2(\eta) P_0(q)$. Finally, rewriting eq. (5.39) in Fourier space using the above relation and the definition of J, eq. (5.41), we obtain the resummed consistency relations in the squeezed limit,

$$
\begin{aligned}
\langle \delta_{\vec{q}}(\eta) \delta^{(g)}_{\vec{k}_1}(\eta_1) \cdots \delta^{(g)}_{\vec{k}_n}(\eta_n) \rangle' \approx & - P(q, \eta) \sum_a \frac{D(\eta_a)}{D(\eta)} \frac{\vec{k}_a \cdot \vec{q}}{q^2} \langle \delta^{(g)}_{\vec{k}_1}(\eta_1) \cdots \delta^{(g)}_{\vec{k}_n}(\eta_n) \rangle'_0 \\
& \times \exp \left[-\frac{1}{2} \int^\Lambda \frac{\mathrm{d}^3 p}{(2\pi)^3} \left(\sum_a D(\eta_a) \frac{\vec{k}_a \cdot \vec{p}}{p^2} \right)^2 P_0(p) \right] .
\end{aligned}
\tag{5.44}
$$

However, what one observes in practice is not the expectation value $\langle \ldots \rangle_0$ with the long modes set artificially to zero: one wants to rewrite the right-hand side of eq. (5.44) in terms of an average over the long modes. Using eq. (5.42) it is easy to see that

$$
\begin{aligned}
\langle \langle \delta^{(g)}_{\vec{k}_1}(\eta_1) \cdots \delta^{(g)}_{\vec{k}_n}(\eta_n) | \Phi_L \rangle \rangle_{\Phi_L} \\
\approx \exp \left[-\frac{1}{2} \int^\Lambda \frac{\mathrm{d}^3 p}{(2\pi)^3} \left(\sum_a D(\eta_a) \frac{\vec{k}_a \cdot \vec{p}}{p^2} \right)^2 P_0(p) \right] \langle \delta^{(g)}_{\vec{k}_1}(\eta_1) \cdots \delta^{(g)}_{\vec{k}_n}(\eta_n) \rangle_0 .
\end{aligned}
\tag{5.45}
$$

Therefore, once written in terms of the observable quantities the consistency relation comes back to the simple form

$$
\langle \delta_{\vec{q}}(\eta) \delta^{(g)}_{\vec{k}_1}(\eta_1) \cdots \delta^{(g)}_{\vec{k}_n}(\eta_n) \rangle' \approx -P(q, \eta) \sum_a \frac{D(\eta_a)}{D(\eta)} \frac{\vec{k}_a \cdot \vec{q}}{q^2} \langle \delta^{(g)}_{\vec{k}_1}(\eta_1) \cdots \delta^{(g)}_{\vec{k}_n}(\eta_n) \rangle' .
\tag{5.46}
$$

This equation has the same form as the consistency relations obtained in Refs. [65, 66, 67], but now it *does not* rely on a linear expansion in the displacement field,

$$
\frac{|\delta \vec{x}|}{|\vec{x}|} \sim \frac{k}{q} \delta_L \ll 1 .
\tag{5.47}
$$

Indeed, to derive eq. (5.44) we have assumed that the long mode is in the linear regime, i.e. $\delta_L \ll 1$, but no assumption has been made on $(k/q)\delta_L$, which can be as large as one wishes.

5.2.2 Several soft legs

We saw in previous Section that the consistency relation written in terms of observable correlation functions remains unchanged even when we do not expand the change of coordinates. However, the method described above gives us a possibility to study additional momentum configurations of the correlation functions, that is inaccessible if we linearise in $\delta \vec{x}$. One example of this kind are multiple soft limits, in which we send several external momenta to zero. The generalisation of the consistency relation to this case relies on taking successive functional derivatives with respect to $J(\vec{q}_i)$ of eq. (5.42). As an example, let us explicitly

compute the consistency relations with two soft modes. In this case the $(n+2)$-point function reads

$$\langle \delta_L(\vec{y}_1, \tau_1) \delta_L(\vec{y}_2, \tau_2) \delta^{(g)}(\vec{x}_1, \eta_1) \cdots \delta^{(g)}(\vec{x}_n, \eta_n) \rangle \approx \int \frac{\mathrm{d}^3 k_1}{(2\pi)^3} \cdots \frac{\mathrm{d}^3 k_n}{(2\pi)^3} \langle \delta^{(g)}_{\vec{k}_1}(\eta_1) \cdots \delta^{(g)}_{\vec{k}_n}(\eta_n) \rangle_0 \, e^{i \sum_a \vec{k}_a \cdot \vec{x}_a}$$
$$\times \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3} \frac{\mathrm{d}^3 q_2}{(2\pi)^3} e^{i(\vec{q}_1 \cdot \vec{y}_1 + \vec{q}_2 \cdot \vec{y}_2)} \left\langle \delta_{\vec{q}_1}(\tau_1) \delta_{\vec{q}_2}(\tau_2) e^{\int^\Lambda \frac{\mathrm{d}^3 p}{(2\pi)^3} J(\vec{p}) \delta_0(\vec{p})} \right\rangle . \tag{5.48}$$

To compute the average over the long modes in the last line, it is enough to take two functional derivatives of eq. (5.42),

$$\left\langle \delta_{\vec{q}_1}(\tau_1) \delta_{\vec{q}_2}(\tau_2) e^{\int^\Lambda \frac{\mathrm{d}^3 p}{(2\pi)^3} J(\vec{p}) \delta_0(\vec{p})} \right\rangle = (2\pi)^6 D(\tau_1) D(\tau_2) \frac{\delta}{\delta J(\vec{q}_1)} \frac{\delta}{\delta J(\vec{q}_2)} \left\langle e^{\int^\Lambda \frac{\mathrm{d}^3 p}{(2\pi)^3} J(\vec{p}) \delta_0(\vec{p})} \right\rangle$$
$$= \frac{J(-\vec{q}_1)}{D(\tau_1)} \frac{J(-\vec{q}_2)}{D(\tau_2)} P(q_1, \tau_1) P(q_2, \tau_2) e^{\frac{1}{2} \int^\Lambda \frac{\mathrm{d}^3 p}{(2\pi)^3} J(\vec{p}) J(-\vec{p}) P_0(p)} , \tag{5.49}$$

where we have assumed $\vec{q}_1 + \vec{q}_2 \neq 0$ to get rid of unconnected contributions. In Fourier space, this yields

$$\langle \delta_{\vec{q}_1}(\tau_1) \delta_{\vec{q}_2}(\tau_2) \delta^{(g)}_{\vec{k}_1}(\eta_1) \cdots \delta^{(g)}_{\vec{k}_n}(\eta_n) \rangle' \approx P(q_1, \tau_1) P(q_2, \tau_2)$$
$$\times \sum_a \frac{D(\eta_a)}{D(\tau_1)} \frac{\vec{k}_a \cdot \vec{q}_1}{q_1^2} \sum_b \frac{D(\eta_b)}{D(\tau_2)} \frac{\vec{k}_b \cdot \vec{q}_2}{q_2^2} \langle \delta^{(g)}_{\vec{k}_1}(\eta_1) \cdots \delta^{(g)}_{\vec{k}_n}(\eta_n) \rangle' , \tag{5.50}$$

where again we have used eq. (5.45) to write the result in terms of correlation functions averaged over the long modes. This is the consistency relation for two soft external modes. The generalisation to arbitrary number of soft modes is straightforward.

We can check this result in a simple example of the four-point function for dark matter perturbations calculated in perturbation theory. The consistency relation with two soft modes in this case has the following form

$$\langle \delta_{\vec{q}_1}(\tau_1) \delta_{\vec{q}_2}(\tau_2) \delta_{\vec{k}_1}(\eta_1) \delta_{\vec{k}_2}(\eta_2) \rangle'$$
$$\approx \frac{(D(\eta_1) - D(\eta_2))^2}{D(\tau_1) D(\tau_2)} \frac{\vec{q}_1 \cdot \vec{k}_1}{q_1^2} \frac{\vec{q}_2 \cdot \vec{k}_1}{q_2^2} P(q_1, \tau_1) P(q_2, \tau_2) \langle \delta_{\vec{k}_1}(\eta_1) \delta_{\vec{k}_2}(\eta_2) \rangle'. \tag{5.51}$$

We can check that this expression correctly reproduces the tree-level trispectrum computed in perturbation theory in the double-squeezed limit. Although the full trispectrum is rather hard to calculate, it can be easily computed in the double squeezed limit by summing the two types of diagrams displayed in Fig. 5.1. The diagram on the left-hand side represents the case where the density perturbations of the short modes are both taken at second order, yielding

$$T_{1122} = D(\tau_1) D(\tau_2) D(\eta_1) D(\eta_2) P_0(q_1) P_0(q_2)$$
$$\times F_2(-\vec{q}_1, \vec{k}_1 + \vec{q}_1) F_2(-\vec{q}_2, \vec{k}_2 + \vec{q}_2) \langle \delta_{\vec{k}_1}(\eta_1) \delta_{\vec{k}_2}(\eta_2) \rangle' + \text{perms}$$
$$\approx -8 \frac{\vec{q}_1 \cdot \vec{k}_1}{2 q_1^2} \frac{\vec{q}_2 \cdot \vec{k}_1}{2 q_2^2} \frac{D(\eta_1) D(\eta_2)}{D(\tau_1) D(\tau_2)} P(q_1, \tau_1) P(q_2, \tau_2) \langle \delta_{\vec{k}_1}(\eta_1) \delta_{\vec{k}_2}(\eta_2) \rangle' , \tag{5.52}$$
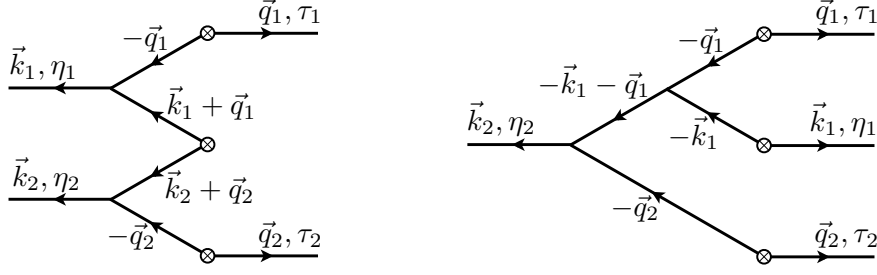
Figure 5.1: Two diagrams that contribute to the tree-level trispectrum. Left: T_{1122}. Right: T_{1113}.

where, on the right-hand side of the first line, $F_2(\vec{p}_1, \vec{p}_2)$ is the usual kernel of perturbation theory, which in the limit where $p_1 \ll p_2$ simply reduces to $\vec{p}_1 \cdot \vec{p}_2/(2p_1^2)$ [101]. The second type of diagram, displayed on the right-hand side of Fig. 5.1, is obtained when one of the short density perturbations is taken at third order; it gives

$$T_{1113} = D(\eta_2)^2 D(\tau_1) D(\tau_2) P_0(q_1) P_0(q_2) F_3(-\vec{q}_1, -\vec{q}_2, -\vec{k}_1)\langle \delta_{\vec{k}_1}(\eta_1)\delta_{\vec{k}_2}(\eta_2)\rangle' + \text{perms}$$

$$\approx 4\frac{\vec{q}_1 \cdot \vec{k}_1}{2q_1^2}\frac{\vec{q}_2 \cdot \vec{k}_1}{2q_2^2}\frac{D(\eta_2)^2}{D(\tau_1)D(\tau_2)}P(q_1, \tau_1)P(q_2, \tau_2)\langle \delta_{\vec{k}_1}(\eta_1)\delta_{\vec{k}_2}(\eta_2)\rangle' , \tag{5.53}$$

where, on the right-hand side of the first line, $F_3(\vec{p}_1, \vec{p}_2, \vec{p}_3)$ is the third-order perturbation theory kernel, which in the limit where $p_1, p_2 \ll p_3$ reduces to $(\vec{p}_1 \cdot \vec{p}_3)(\vec{p}_2 \cdot \vec{p}_3)/(6p_1^2 p_2^2)$ [101]. As expected, summing up all the contributions to the connected part of the trispectrum, i.e. $T_{1122} + T_{1131} + T_{1113}$, using eqs. (5.52) and (5.53) and $\vec{k}_2 \approx -\vec{k}_1$ one obtains eq. (5.51).

5.2.3 Soft Loops

So far we have derived consistency relations where the long modes appear explicitly as external legs. We now show that our arguments can also capture the effect on short-scale correlation functions of soft modes running in loop diagrams. We already did this in eq (5.45)

$$\left\langle\langle\delta_{\vec{k}_1}^{(g)}(\eta_1)\cdots\delta_{\vec{k}_n}^{(g)}(\eta_n)|\Phi_L\rangle\right\rangle_{\Phi_L}$$

$$\approx \exp\left[-\frac{1}{2}\int^\Lambda\frac{d^3p}{(2\pi)^3}\left(\sum_a D(\eta_a)\frac{\vec{k}_a \cdot \vec{p}}{p^2}\right)^2 P_0(p)\right]\langle\delta_{\vec{k}_1}^{(g)}(\eta_1)\cdots\delta_{\vec{k}_n}^{(g)}(\eta_n)\rangle_0 . \tag{5.54}$$

The exponential in this expression can be expanded at a given order, corresponding to the number of soft loops dressing the n-point correlation function. Each loop carries a contribution $\propto k^2 \int dp P_0(p)$ to the correlation function. However, this expression makes it very explicit that at all loop order these contributions have no effect on equal-time correlators, because in this case the exponential on the right-hand side is identically unity. This confirms previous analysis on this subject [142, 143, 140, 141, 144, 145]. It is important to notice again, however, that in our derivation this cancellation is more general and robust that in those references, as it takes place independently of the equations of motion for the short modes and is completely agnostic about the short-scale physics. It simply derives from the equivalence principle.
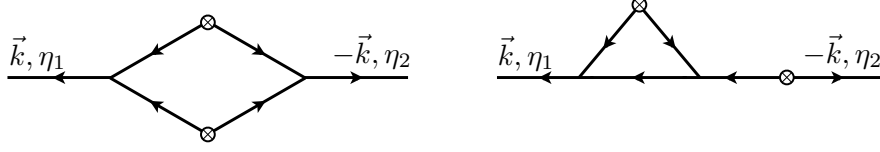
Figure 5.2: Two diagrams that contribute to the 1-loop power spectrum. Left: P_{22}. Right: P_{31}.

Nevertheless, soft loops contribute to unequal-time correlators. As a check of the expression above, one can compute the contribution of soft modes to the 1-loop unequal-time matter power spectrum, $\langle \delta_{\vec{k}_1}(\eta_1) \delta_{\vec{k}_2}(\eta_2) \rangle'$, and verify that this reproduces the standard perturbation theory result. Expanding at order $(\frac{k}{p}\delta)^2$ the exponential in eq. (5.54) for $n = 2$, one obtains the 1-loop contribution to the power spectrum,

$$\langle \delta_{\vec{k}}^{(g)}(\eta_1) \delta_{-\vec{k}}^{(g)}(\eta_2) \rangle'_{1-\text{soft loop}} \approx -\frac{1}{2} \left(D(\eta_1) - D(\eta_2) \right)^2 \int^\Lambda \frac{\mathrm{d}^3 p}{(2\pi)^3} \left(\frac{\vec{p} \cdot \vec{k}}{p^2} \right)^2 P_0(p) \langle \delta_{\vec{k}}^{(g)}(\eta_1) \delta_{-\vec{k}}^{(g)}(\eta_2) \rangle'_0 . \tag{5.55}$$

We can compute the analogous contribution in perturbation theory and check that the two expressions agree. There are two types of diagrams that are relevant. They are shown in Fig. 5.2. The one on the left, usually called P_{22}, yields

$$P_{22} \approx 4 D(\eta_1) D(\eta_2) \int^\Lambda \frac{\mathrm{d}^3 p}{(2\pi)^3} \left(\frac{\vec{p} \cdot \vec{k}}{2p^2} \right)^2 P_0(p) \langle \delta_{\vec{k}}(\eta_1) \delta_{-\vec{k}}(\eta_2) \rangle'_0 , \tag{5.56}$$

while the diagram on the right, P_{31}, gives

$$P_{31} \approx -2 D(\eta_1)^2 \int^\Lambda \frac{\mathrm{d}^3 p}{(2\pi)^3} \left(\frac{\vec{p} \cdot \vec{k}}{2p^2} \right)^2 P_0(p) \langle \delta_{\vec{k}}(\eta_1) \delta_{-\vec{k}}(\eta_2) \rangle'_0 . \tag{5.57}$$

In getting these expressions we were using the fact that the perturbation theory kernels become very simple in the limit $p \ll k$. Summing up all the different contributions, $P_{22} + P_{31} + P_{13}$, one obtains eq. (5.55).

5.2.4 Soft internal lines

Another kinematical regime in which the consistency relations can be applied is the limit in which the sum of some of the external momenta becomes very small, for instance $|\vec{k}_1 + \cdots + \vec{k}_m| \ll k_1, \ldots, k_m$. In this limit, the dominant contribution to the n-point function comes from the diagram where m external legs of momenta $\vec{k}_1, \ldots, \vec{k}_m$ exchange soft modes with momentum $\vec{q} = \vec{k}_1 + \cdots + \vec{k}_m$ with $n - m$ external legs with momenta $\vec{k}_{m+1}, \ldots, \vec{k}_n$. We have already discussed limits of soft internal lines in the case of inflation (see Section 4.3). Now we turn to the case of LSS and show how one can use the consistency relation to explore these kinematical regimes.

The contribution to the correlation function that we want to calculate, in the limit where some of the internal modes are soft, comes from averaging a product of m-point and $(n-m)$-point functions under the effect of long modes. In this case, the n-point function in real space can be written as

$$
\begin{aligned}
&\langle \delta^{(g)}(\vec{x}_1, \eta_1) \cdots \delta^{(g)}(\vec{x}_m, \eta_m)\, \delta^{(g)}(\vec{x}_{m+1}, \eta_{m+1}) \cdots \delta^{(g)}(\vec{x}_n, \eta_n) \rangle \\
&\approx \langle \langle \delta^{(g)}(\vec{x}_1, \eta_1) \cdots \delta^{(g)}(\vec{x}_m, \eta_m) | \Phi_L \rangle \langle \delta^{(g)}(\vec{x}_{m+1}, \eta_{m+1}) \cdots \delta^{(g)}(\vec{x}_n, \eta_n) | \Phi_L \rangle \rangle_{\Phi_L}\ ,
\end{aligned}
\tag{5.58}
$$

Now we can straightforwardly apply the equations from the previous sections. As before, the long mode can be traded for the change of coordinates. Rewriting the right-hand side in Fourier space we get

$$
\begin{aligned}
&\langle \delta^{(g)}(\vec{x}_1, \eta_1) \cdots \delta^{(g)}(\vec{x}_m, \eta_m)\, \delta^{(g)}(\vec{x}_{m+1}, \eta_{m+1}) \cdots \delta^{(g)}(\vec{x}_n, \eta_n) \rangle \\
&\approx \int \frac{\mathrm{d}^3 k_1}{(2\pi)^3} \cdots \frac{\mathrm{d}^3 k_n}{(2\pi)^3} \langle \delta^{(g)}_{\vec{k}_1}(\eta_1) \cdots \delta^{(g)}_{\vec{k}_m}(\eta_m) \rangle_0 \langle \delta^{(g)}_{\vec{k}_{m+1}}(\eta_{m+1}) \cdots \delta^{(g)}_{\vec{k}_n}(\eta_n) \rangle_0\, e^{i \sum_a \vec{k}_a \cdot \vec{x}_a} \\
&\times \left\langle \exp\left[i \sum_{a=1}^{m} \vec{k}_a \cdot \delta\vec{x}(\vec{y}_1, \eta_a) \right] \cdot \exp\left[i \sum_{a=m+1}^{n} \vec{k}_a \cdot \delta\vec{x}(\vec{y}_2, \eta_a) \right] \right\rangle_{\Phi_L}\ ,
\end{aligned}
\tag{5.59}
$$

where $\vec{y}_1$ and $\vec{y}_2$ are two different points respectively close to $(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m)$ and $(\vec{x}_{m+1}, \vec{x}_{m+2}, \ldots, \vec{x}_n)$. The average over the long mode can be rewritten as

$$
\left\langle \exp\left[\int^{\Lambda} \frac{\mathrm{d}^3 \vec{p}}{(2\pi)^3} \big(J_1(\vec{p}) + J_2(\vec{p}) \big) \delta_0(\vec{p}) \right] \right\rangle_{\Phi_L}
\tag{5.60}
$$

with

$$
J_1(\vec{p}) = \sum_{a=1}^{m} D(\eta_a) \frac{\vec{k}_a \cdot \vec{p}}{p^2} e^{i\vec{p}\cdot\vec{y}_1}\ , \qquad J_2(\vec{p}) = \sum_{a=m+1}^{n} D(\eta_a) \frac{\vec{k}_a \cdot \vec{p}}{p^2} e^{i\vec{p}\cdot\vec{y}_2}\ .
\tag{5.61}
$$

Taking the expectation value over the long mode using the expression for averaging the exponential of a Gaussian variable, i.e. eq. (5.42), eq. (5.59) can be written as

$$
\begin{aligned}
&\langle \delta^{(g)}(\vec{x}_1, \eta_1) \cdots \delta^{(g)}(\vec{x}_m, \eta_m)\, \delta^{(g)}(\vec{x}_{m+1}, \eta_{m+1}) \cdots \delta^{(g)}(\vec{x}_n, \eta_n) \rangle \\
&\approx \int \frac{\mathrm{d}^3 k_1}{(2\pi)^3} \cdots \frac{\mathrm{d}^3 k_n}{(2\pi)^3} \langle \delta^{(g)}_{\vec{k}_1}(\eta_1) \cdots \delta^{(g)}_{\vec{k}_m}(\eta_m) \rangle' \langle \delta^{(g)}_{\vec{k}_{m+1}}(\eta_{m+1}) \cdots \delta^{(g)}_{\vec{k}_n}(\eta_n) \rangle'\, e^{i \sum_a \vec{k}_a \cdot \vec{x}_a} \\
&\times \exp\left[-\int^{\Lambda} \frac{\mathrm{d}^3 p}{(2\pi)^3} J_1(\vec{p}) J_2(\vec{p}) P_0(\vec{p}) \right]\ .
\end{aligned}
\tag{5.62}
$$

We are interested in the soft internal lines, that come from the cross term, i.e. the last line of eq. (5.62). Notice that $J_1(\vec{p})$ and $J_2(\vec{p})$ are evaluated at different points $\vec{y}_1$ and $\vec{y}_2$ separated by a distance $\vec{x}$.[10] The last step is just to take the Fourier transform of this equation, which

[10]For definiteness, we can choose $\vec{y}_1 = \frac{1}{m}\sum_{a=1}^{m} \vec{x}_a$ and $\vec{y}_2 = \frac{1}{n-m}\sum_{a=m+1}^{n} \vec{x}_a$.

gives

$$
\begin{aligned}
\langle \delta^{(g)}_{\vec{k}_1}(\eta_1) \cdots \delta^{(g)}_{\vec{k}_m}(\eta_m) \delta^{(g)}_{\vec{k}_{m+1}}(\eta_{m+1}) \cdots \delta^{(g)}_{\vec{k}_n}(\eta_n) \rangle' \\
\approx \langle \delta^{(g)}_{\vec{k}_1}(\eta_1) \cdots \delta^{(g)}_{\vec{k}_m}(\eta_m) \rangle' \langle \delta^{(g)}_{\vec{k}_{m+1}}(\eta_{m+1}) \cdots \delta^{(g)}_{\vec{k}_n}(\eta_n) \rangle' \\
\times \int \mathrm{d}^3 x\, e^{-i \sum_{i=1}^m \vec{k}_i \cdot \vec{x}} \exp\left[-\int^\Lambda \frac{\mathrm{d}^3 p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \sum_{a=1}^m D(\eta_a) \frac{\vec{k}_a \cdot \vec{p}}{p^2} \sum_{a=m+1}^n D(\eta_a) \frac{\vec{k}_a \cdot \vec{p}}{p^2} P_0(p) \right].
\end{aligned}
$$
(5.63)

The last line encodes the effect of soft modes with total momentum $\vec{q} = \vec{k}_1 + \cdots + \vec{k}_m$ exchanged between m external legs of momenta $\vec{k}_1, \ldots, \vec{k}_m$ and $n - m$ external legs with momenta $\vec{k}_{m+1}, \ldots, \vec{k}_n$, in the limit $q/k_i \to 0$. Expanding the exponential at a given order in $P_0(p)$ counts the number of soft lines exchanged. The integral in $\mathrm{d}^3 x$ ensures that the sum of the internal momenta is $\vec{q}$.

As a concrete example, let us consider the case $m = 2, n = 4$, i.e. a 4-point function in the collapsed limit $|\vec{k}_1 + \vec{k}_2| \ll k_1, k_2$, and the exchange of a single soft line. In this case, expanding the exponential at first order in $P_0(p)$, the equation above gives

$$
\begin{aligned}
\langle \delta_{\vec{k}_1}(\eta_1) \delta_{\vec{k}_2}(\eta_2) \delta_{\vec{k}_3}(\eta_3) \delta_{\vec{k}_4}(\eta_4) \rangle'_c \\
\approx - \langle \delta(\vec{k}_1, \eta_1) \delta(\vec{k}_2, \eta_2) \rangle' \langle \delta(\vec{k}_3, \eta_3) \cdots \delta(\vec{k}_4, \eta_4) \rangle' \\
\times \int^\Lambda \mathrm{d}^3 p \big(D(\eta_1) - D(\eta_2) \big) \frac{\vec{k}_1 \cdot \vec{p}}{p^2} \big(D(\eta_3) - D(\eta_4) \big) \frac{\vec{k}_3 \cdot \vec{p}}{p^2} P_0(p) \delta_D(\vec{p} - \vec{k}_1 - \vec{k}_2) \,,
\end{aligned}
$$
(5.64)

where we have considered only the connected diagram and, for simplicity, we are neglecting soft loops attached to each lines. To compare with perturbation theory, we need to compute the tree-level exchange diagram. This can be easily done in the limit when the exchanged momentum is small. The contribution from taking $\vec{k}_1$ and $\vec{k}_3$ at second order is

$$
T_{2121} \approx -4 D(\eta_1) D(\eta_3) P_0(|\vec{k}_1 + \vec{k}_2|) \frac{\vec{k}_1 \cdot (\vec{k}_1 + \vec{k}_2)}{2|\vec{k}_1 + \vec{k}_2|^2} \frac{\vec{k}_3 \cdot (\vec{k}_1 + \vec{k}_2)}{2|\vec{k}_1 + \vec{k}_2|^2} \langle \delta_{\vec{k}_1}(\eta_1) \delta_{\vec{k}_2}(\eta_2) \rangle' \langle \delta_{\vec{k}_3}(\eta_3) \delta_{\vec{k}_4}(\eta_4) \rangle' \,.
$$
(5.65)

If we include the other permutations we finally get

$$
\begin{aligned}
\langle \delta_{\vec{k}_1}(\eta_1) \delta_{\vec{k}_2}(\eta_2) \delta_{\vec{k}_3}(\eta_3) \delta_{\vec{k}_4}(\eta_4) \rangle'_c \approx - \big(D(\eta_1) - D(\eta_2) \big) \big(D(\eta_3) - D(\eta_4) \big) P_0(|\vec{k}_1 + \vec{k}_2|) \\
\times \frac{\vec{k}_1 \cdot (\vec{k}_1 + \vec{k}_2)}{|\vec{k}_1 + \vec{k}_2|^2} \frac{\vec{k}_3 \cdot (\vec{k}_1 + \vec{k}_2)}{|\vec{k}_1 + \vec{k}_2|^2} \langle \delta_{\vec{k}_1}(\eta_1) \delta_{\vec{k}_2}(\eta_2) \rangle' \langle \delta_{\vec{k}_3}(\eta_3) \delta_{\vec{k}_4}(\eta_4) \rangle' \,,
\end{aligned}
$$
(5.66)

which is indeed the same as eq. (5.64).

5.3 Consistency Relations in Redshift Space

So far we were considering consistency relations in real space (and corresponding momentum space). As we saw, one of their main virtues is that they hold non-perturbatively, even when

baryonic physics is taken into account. For this reason they should hold directly at the level of the correlation functions of galaxy overdensities and checking if they hold in real data can help us to test the assumptions that enter in the derivation of the consistency relations.

However, galaxy surveys do not tell us about positions of galaxies in *real* space, but in *redshift* space. If one does not want to restrict only to large scales, the mapping between the two is very non-trivial and depends on the dynamics of the modes (in particular velocities) even in the non-perturbative regime. It is therefore impossible to exploit the full potential of the consistency relations at the level of observables, unless they can be rewritten in redshift space without using any approximations.

This is not a trivial task. One approach would be to start form the consistency relations in real space and try to rewrite them in redshift space. As we said, for this approach to work, one has to know the exact velocities of all particles, needed for the mapping between real and redshift space. Of course, this is impossible. One may hope that even without solving for the velocities, the effects of mapping will still somehow "resum" on both sides of the consistency relation, leading to a simple result. But even in the simplest case, if we use the fluid approximation[11], any n-point correlation function in real space is an infinite sum of all correlation functions in redshift space of order higher than n. This makes the task of going from real to redshift space very difficult to complete.

Luckily, there is a simple trick to derive the consistency relations directly in redshift space, without using the real space result as a starting point. In the following Section we present this derivation [69, 71].

5.3.1 Derivation

For simplicity, let us work in the plane-parallel approximation[12]. The mapping between real space $\vec{x}$ and redshift space $\vec{s}$ is given by

$$\vec{s} = \vec{x} + \frac{v_z}{\mathcal{H}}\hat{z} \ , \tag{5.67}$$

where $\hat{z}$ is the direction of the line of sight, $v_z \equiv \vec{v} \cdot \hat{z}$, and $\vec{v}$ is the peculiar velocity of a given object. The relation between z and η also receives corrections due to peculiar velocities. However, these corrections are small for sufficiently distant objects for which $v \ll Hx$. Notice that we do not assume that the peculiar velocity is a function of the position $\vec{x}$ since this holds only in the single-stream approximation, which breaks down for virialized objects on small scales [146, 147].

With this setup, the derivation of the consistency relations in redshift space follows closely what we did in real space. Intuitively, the long mode has two important effects. It again

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[11]In this case the system is fully described by density and velocity fields. Every point in space has only one velocity vector and the mapping between real and redshift space in "one-to-one". This is not the case when the fluid approximation breaks down.

[12]This approximation is valid on small angular scales only. For a large fraction of the sky, one has to rederive the consistency relations in terms of appropriate observables, which is beyond the scope of this thesis.

induces a time-dependent translation of real coordinates that we discussed so far. However, in redshift space one has to take into account the effects on velocities. Given that the long mode makes all objects moving with the same bulk velocity, there is an additional translation in redshift space, and this will lead to an additional term in the consistency relation. Let us see how this intuition can be made more formal.

The effect of the long mode on coordinates and velocities is

$$\vec{x} \to \vec{x} + D\,\vec{\nabla}\Phi_{0,L} \,, \tag{5.68}$$

$$\vec{v} \to \vec{v} + f\mathcal{H}D\,\vec{\nabla}\Phi_{0,L} \,. \tag{5.69}$$

As we already saw, in these equations $D(\eta)$ is the growth factor, $f(\eta) \equiv d\ln D/d\ln a$ is the growth rate and $\vec{\nabla}\Phi_{0,L}$ a homogenous gradient of the initial gravitational potential $\Phi_{0,L}$. Now we can use eq. (5.67) to find a corresponding translation in redshift space

$$\vec{s} = \vec{s} + D(\eta) \cdot (\vec{\nabla}\Phi_{0,L} + f\nabla_z\Phi_{0,L}\hat{z}) \,. \tag{5.70}$$

The first term is just a familiar translation in real space, while the second one captures the effects of the bulk velocity induced by the long mode.

Now the derivation of the consistency relation is straightforward. As in real space, we can start from the fact that the same redshift-space correlation functions in the presence or the absence of a long mode Φ_L are related only by a translation (5.70) in redshift space. Therefore, we can write

$$\langle \delta_s^{(g)}(\vec{s}_1, \eta_1) \cdots \delta_s^{(g)}(\vec{s}_n, \eta_n) | \Phi_L \rangle \approx \langle \delta_s^{(g)}(\vec{\tilde{s}}_1, \eta_1) \cdots \delta_s^{(g)}(\vec{\tilde{s}}_n, \eta_n) \rangle$$
$$= \sum_a \delta\vec{s}_a \langle \delta_s^{(g)}(\vec{s}_1, \eta_1) \cdots \vec{\nabla}_a\delta_s^{(g)}(\vec{s}_a, \eta_a) \cdots \delta_s^{(g)}(\vec{s}_n, \eta_n) \rangle \,, \tag{5.71}$$

where $\delta\vec{s}_a \equiv D(\eta_a)\,(\vec{\nabla}_a\Phi_{0,L} + f_a\nabla_{a,z}\Phi_{0,L}\hat{z})$. This statement is non-trivial because it involves a statistical average. To show that it is correct in a more rigorous way, we can notice that the density in redshift space can be written in terms of the real-space distribution function in the following way [146, 147]

$$\rho_s(\vec{s}) = ma^{-3} \int \mathrm{d}^3p\, f\left(\vec{s} - \frac{v_z}{\mathcal{H}}\hat{z}, \vec{p}\right) \,. \tag{5.72}$$

In this equation m is the mass of the particles and $\vec{p}$ is the physical momentum. In the presence of the long mode, the distribution function $\rho_s(\vec{s})$ can be written as

$$\rho_s(\vec{s})_{\Phi_L} = \frac{m}{a^3} \int \mathrm{d}^3p\, f\left(\vec{s} - \frac{v_z}{\mathcal{H}}\hat{z} + \delta\vec{x}, \vec{p} + am\delta\vec{v}\right)$$
$$= \frac{m}{a^3} \int \mathrm{d}^3p'\, f\left(\vec{s} - \frac{v_z - \delta v_z}{\mathcal{H}}\hat{z} + \delta\vec{x}, \vec{p}'\right) = \rho_s(\vec{s} + \delta\vec{s}), \tag{5.73}$$

where δx and $\delta\vec{v}$ are given by eqs. (5.68) and (5.69).

At this point what remains to be done is just to go to the Fourier space conjugate to redshift space coordiantes. It is straightforward to see that eq. (5.71) becomes

$$\langle \Phi_0(\vec{q})\delta^{(g)}_{s\,\vec{k}_1}(\eta_1)\cdots\delta^{(g)}_{s\,\vec{k}_n}(\eta_n)\rangle'$$
$$\approx P_\Phi(q)\sum_a D(\eta_a)\big[\vec{q}\cdot\vec{k}_a + f(\eta_a)q_z\,k_{a,z}\big]\langle\delta^{(g)}_{s\,\vec{k}_1}(\eta_1)\cdots\delta^{(g)}_{s\,\vec{k}_n}(\eta_n)\rangle' . \tag{5.74}$$

The most difficult task of dealing with the short modes is done in this way. If we want to write everything in terms of observables in redshift space, the last step is to express the long mode gravitational potential as a function of the density contrast $\delta^{(g)}_s$. This is not very difficult, since the long mode is in the linear regime. Assuming that a simple linear bias model works well on large scales we can write

$$\delta^{(g)}_s(\vec{q},\eta) = -(b_1 + f\mu^2_{\vec{q}})D(\eta)q^2\Phi_0(\vec{q}) , \tag{5.75}$$

where b_1 is a linear bias parameter between galaxies and dark matter and $\mu_{\vec{k}} \equiv \vec{k}\cdot\hat{z}/k$. With this approximations the consistency relation above takes its final form

$$\langle\delta^{(g)}_{s\,\vec{q}}(\eta)\delta^{(g)}_{s\,\vec{k}_1}(\eta_1)\cdots\delta^{(g)}_{s\,\vec{k}_n}(\eta_n)\rangle' \approx -\frac{P_{g,s}(q,\eta)}{b_1 + f\mu^2_{\vec{q}}}\sum_a\frac{D(\eta_a)}{D(\eta)}\frac{k_a}{q}\big[\hat{q}\cdot\hat{k}_a + f(\eta_a)\mu_{\vec{q}}\mu_{\vec{k}_a}\big]$$
$$\times\langle\delta^{(g)}_{s\,\vec{k}_1}(\eta_1)\cdots\delta^{(g)}_{s\,\vec{k}_n}(\eta_n)\rangle' . \tag{5.76}$$

As we expected, the only important difference compared to the real space result is the additional term related to the bulk velocity.

Everything we said so far about the non-relativistic consistency relation in real space holds also in redshift space. It is a non-perturbative statements that is exact even when the baryonic physics is taken into account. The expression above is non-perturbative in $\frac{k}{q}\delta_L$, and all results involving multiple soft limits, loops, internal soft legs and others can be straightforwardly generalized to redshift space.

One of the properties of the consistency relation in real space was that it vanishes (up to sub-leading corrections in q) for equal-time correlation functions. As we stressed, this can be used as a tool to look for deviations from the consistency relation and test the assumptions that enter its derivation. It is important to stress that the same thing holds even in redshift space. This is not surprising because, after all, the long mode is still just a translation that has slightly different form than before. Given that it does not change the distances between the points in redshift space and that everything is translationally invariant, the variation of the equal-time correlation functions is just zero. This observation is crucial for the tests of the Equivalence Principle that we are going to discuss in the following Section. Before that, let us do a small check of the equation (5.76).

5.3.2 Checks

As we did so far, we can check the consistency relations of LSS in redshift space using the correlation functions calculated in perturbation theory. Let us start with the three-point

function. The redshift space bispectrum is given by [101]

$$
\begin{aligned}
\langle \delta_s^{(g)}{}_{\vec{q}}(\eta) \delta_s^{(g)}{}_{\vec{k}_1}(\eta_1) \delta_s^{(g)}{}_{\vec{k}_2}(\eta_2) \rangle' = \\
2 Z_2(-\vec{q}, -\vec{k}_2; \eta_1) Z_1(\vec{q}; \eta) Z_1(\vec{k}_2; \eta_2) \langle \delta(\vec{q}, \eta) \delta(-\vec{q}, \eta_1) \rangle' \langle \delta(\vec{k}_1, \eta_1) \delta(\vec{k}_2, \eta_2) \rangle' + \text{cyclic} ,
\end{aligned}
\tag{5.77}
$$

where

$$
\begin{aligned}
Z_1(\vec{k}; \eta) &\equiv (b_1 + f \mu_{\vec{k}}^2) , \\
Z_2(\vec{k}_a, \vec{k}_b; \eta) &\equiv b_1 F_2(\vec{k}_a, \vec{k}_b) + f \mu_{\vec{k}}^2 G_2(\vec{k}_a, \vec{k}_b) \\
&\quad + \frac{f \mu_{\vec{k}} k}{2} \left[\frac{\mu_{\vec{k}_a}}{k_a}(b_1 + f \mu_{\vec{k}_b}^2) + \frac{\mu_{\vec{k}_b}}{k_b}(b_1 + f \mu_{\vec{k}_a}^2) \right] + \frac{b_2}{2} .
\end{aligned}
\tag{5.78}
$$

In these formulas b_1 and b_2 are the linear and non-linear bias parameters and F_2 and G_2 are the standard second-order perturbation kernels for density and velocity respectively [101]. In the limit $q \to 0$ the kernels simplify in the following way

$$
2 Z_2(-\vec{q}, -\vec{k}_2; \eta_1) \approx (b_1 + f_1 \mu_{\vec{k}_2}^2) \frac{\vec{q} \cdot \vec{k}_2}{q^2} + (b_1 + f_1 \mu_{\vec{k}_2}^2) f_1 \frac{k_2}{q} \mu_{\vec{q}} \mu_{\vec{k}_2} .
\tag{5.79}
$$

Using this result we can rewrite the redshift space three-point function in the squeezed limit in the following way

$$
\begin{aligned}
\langle \delta_s^{(g)}{}_{\vec{q}}(\eta) &\delta_s^{(g)}{}_{\vec{k}_1}(\eta_1) \delta_s^{(g)}{}_{\vec{k}_2}(\eta_2) \rangle' \\
&\approx \frac{P_{g,s}(q, \eta)}{b_1 + f \mu_{\vec{q}}^2} \frac{D(\eta_1)}{D(\eta)}(b_1 + f_1 \mu_{\vec{k}_2}^2) \left(\frac{\vec{q} \cdot \vec{k}_2}{q^2} + f_1 \frac{k_2}{q} \mu_{\vec{q}} \mu_{\vec{k}_2} \right) Z_1(\vec{k}_2; \eta_2) \langle \delta(\vec{k}_1, \eta_1) \delta(\vec{k}_2, \eta_2) \rangle' \\
&\approx -\frac{P_{g,s}(q, \eta)}{b_1 + f \mu_{\vec{q}}^2} \frac{D(\eta_1)}{D(\eta)} \frac{k_1}{q}(\hat{q} \cdot \hat{k}_1 + f_1 \mu_{\vec{q}} \mu_{\vec{k}_1}) \langle \delta(\vec{k}_1, \eta_1) \delta(\vec{k}_2, \eta_2) \rangle' + (1 \leftrightarrow 2) .
\end{aligned}
\tag{5.80}
$$

Notice that this is exactly the right hand side of the eq. (5.76) in the case $n = 2$. Therefore, we can conclude that the consistency relation is satisfied.

5.4 Violation of the Equivalence Principle

In previous sections we showed that one can derive a fully relativistic consistency relation of LSS, showed how it simplifies in the non-relativistic regime and proved that the relation remains almost unchanged when written in terms of observables in redshift space. For the case of equal-time (or equal-redshift) correlation functions, the consistency relation seems somewhat trivial[13]

$$
\lim_{q \to 0} \langle \delta_{\vec{q}}(\eta) \delta_{\vec{k}_1}(\eta) \delta_{\vec{k}_2}(\eta) \rangle' = \mathcal{O}\left[(q/k)^0 \right] .
\tag{5.81}
$$

---

[13]For the rest of this section we will focus only on the three-point function which is observationally the most relevant. We are also going to work with the real space expressions because they are simpler and the additional terms in the redshift space equations do not change any of the conclusions.

As we saw, the fact that the squeezed limit of the $(n + 1)$-point function vanishes at leading order in q is just a consequence of the translational invariance. The effect of the long mode is just to induce some bulk velocity and "translate" the short modes. Given the translational invariance, the variation of the n-point function on the right hand side is simply zero.

As we pointed out at the beginning of this Chapter, there are two important assumptions in this line of reasoning. The first one is that there is no statistical correlation between the long and short modes in the initial conditions. In other words we have to assume that local non-Gaussianities are very small. This happens in all single-field models of inflation, as we already discussed in the previous Chapter. The second assumption is that the long mode affects all objects in the same way. In other words we have to assume that the EP holds on cosmological scales.

Any violation of any of these two assumptions would lead to a violation of the consistency relation. For example, it is well-known that in the presence of the local non-Gaussianities the squeezed limit of the three-point function is given by (see for example [148])

$$\lim_{q \to 0} \langle \delta_{\vec{q}}(\eta) \delta_{\vec{k}_1}(\eta) \delta_{\vec{k}_2}(\eta) \rangle' \approx 6 f_{\mathrm{NL}}^{loc} \frac{\Omega_{m,0} H_0^2}{q^2 T(q)} P(q, \eta) P(k, \eta) \,. \tag{5.82}$$

This is a consequence of the fact that although dynamically it does not do anything, the long mode affects the statistics of the initial conditions for the short modes. As a consequence, this changes the evolution on the short scales and the correlation "survives" until late times. The other way to see the same effect is through the so-called scale-dependent bias. There is a lot of observational effort to use the scale-dependence of the bias or divergences in the squeezed limit of the three-point function to constrain the amplitude of the local non-Gaussianities f_{NL}^{loc}. Our best current constraints that come from LSS use these techniques.

In this Section we will focus on the other possible source of violation of the consistency relation — a violation of the EP on cosmological scales [70, 71, 68]. If the EP is violated, then the effect of the long mode even on equal-time correlation functions is not trivial. Different objects will fall in a different way in the homogeneous gravitational field induced by the long mode. The long mode is not just a change of coordinates anymore and although there is no way to calculate its effect non-perturbatively, one can claim that in general in the squeezed limit we expect

$$\lim_{q \to 0} \langle \delta_{\vec{q}}(\eta) \delta_{\vec{k}_1}^{(A)}(\eta) \delta_{\vec{k}_2}^{(B)}(\eta) \rangle' = \left(\epsilon \frac{\vec{k} \cdot \vec{q}}{q^2} + \mathcal{O}\big[(q/k)^0\big] \right) P(q, \eta) P_{AB}(k, \eta) \,, \tag{5.83}$$

where A and B are two different kind of objects that behave in a different way in the presence of a gravitational field induced by the long mode $\delta_{\vec{q}}$. We can see that in the absence of the EP ϵ is different form zero and the squeezed limit of the equal-time three-point function has a divergence that goes like k/q. Therefore, assuming single-field inflation, any detection of a divergence in the squeezed limit of the three-point function would be a signal for a violation of the EP.

Notice that the equations (5.82) and (5.83) have a very similar form. Apart form constants and different scaling with momenta, they are both divergent in $q \to 0$ limit. As we are going

to see, this means that we can use the same method for forecasting how well we can constrain $f_{\rm NL}^{\rm loc.}$ to constrain the violation of the EP on cosmological scales. The parameter that plays the role of $f_{\rm NL}^{\rm loc.}$ in this case is ϵ. However, we have to stress that there is one important difference between (5.82) and (5.83). While for constraining the local non-Gaussianities one can use the standard power spectra and bispectrum, in the case of a violation of the EP it is crucial to cross-correlate two objects of different kinds A and B. Therefore, for making the constraints tighter, is is crucial to be able to separate different kind of objects in galaxy surveys or other observations. We will come back to this issue later.

In the rest of this Section we are going to explore in more detail how well we can constrain the EP using the consistency relations. Before moving on, let us stress here that the statement we made holds directly on the level of observables — galaxy number densities in redshift space. This is the main advantage of using the consistency relations. Moreover, looking for a violation of the consistency relation, we do a model independent test of GR on cosmological scales because we do not have to assume any model of modified gravity. The only choice we have to make is what classes of objects A and B we want to cross-correlate.

5.4.1 Modifications of gravity and EP violation

Equation (5.83) is a general result that does not depend on the underlying modification of GR. In GR (and assuming single-field inflation) the right hand side vanishes. Therefore, if the EP is violated on cosmological scales, whatever is the mechanism, the squeezed limit of the three-point function must be given by eq. (5.83). In principle, this is enough to proceed and estimate how well we can constrain violations of the EP using LSS. However, in this section, we would like to give a very brief overview of some of the most popular modifications of GR that can induce violations of the EP on cosmological scales (for a review, see for example [149]). The list is certainly not complete, but we need it just to give a flavour of what kind of effects one can expect.

The main motivation for looking for modifications of GR is the accelerated expansion of the universe. The value of cosmological constant needed to explain the acceleration is unnaturally small, and it is interesting to explore whether it is possible to consistently modify GR such that gravity behaves differently on cosmological scales, while keeping the standard form on small scales where we have a lot of experimental evidence that GR works just fine. Over the years, several models of consistent infrared modifications of GR were proposed. All of them have the common property that they pass Solar System tests (typically the effects of modifying GR are unobservable due to some "screening" mechanism) and make significant modification only on the cosmological scales. Based on the screening mechanism there are several models and we will discuss some of them. We are also going to mention other possible sources of the EP violation.

For all of these possibilities there are already good constraints coming from precise measurements in Solar System and the study of structure formation. As we are going to see, some of them are stronger than what we can get using the consistency relation violations.

However, it is important to stress that constraints coming from the squeezed limit of the three-point function are model independent (up to the choice of objects to cross correlate). If the accelerated expansion of the universe really has to do with modifications of GR on large scales, it can still be that our current theories are the final word on the subject and any model independent tests are worth doing.

Let us now go through some of the situations in which the EP is violated.

Screening. The screening of extra forces needed to satisfy the Solar System constraints induces violations of the EP [150]. Depending on the screening mechanism, there are several relevant cases. We focus on some of them.

One possibility is *chameleon* [151] or *symmetron* [152, 153, 154] screening. The screening in these models depends on the typical value of the gravitational potential GM/r of the object and the EP violation can be of order unity between screened and unscreened objects. However, imposing the screening inside the Solar System makes the impact of the fifth force on cosmological scales very limited. One can show that there is a model-independent limit on the mass of the scalar [155, 156]

$$m^2 \gtrsim 10^6 \alpha^2 H^2 \, , \tag{5.84}$$

where α^2 measures a deviation from GR. This inequality is valid at low redshifts and limits the effect of the scalar on short scales $k/a \lesssim m$. We will show later how this bound compares with the constraints one can get form a violation of the consistency relations.

The other possibility is *Galileon* screening [135]. In this case, due to the nonlinearities of the scalar equations, the issue of the EP violation is quite subtle. On one hand, one can show that an object immersed in an external field which is constant over the size of the object will receive an acceleration proportional to the mass and independent of the possible Vainshtein screening of the object [150]. On the other hand, due to the nonlinearities, the value of the external field may not be the same before and after the object is put into place. For example, the Moon changes the solution of the Galileon around the Earth and the nonlinearity of the system is such that the acceleration the Moon experiences is different from the one of a test particle orbiting around the Earth [157, 158].

The complicated nonlinear behaviour in the Galileon screening and related violations of the EP are hard to control in general. This is why it is a bit surprising that one can actually prove that the Galileon models do *not* induce violations of the consistency relations. To see this let us focus on the non-relativistic limit of structure formation. Deep inside the horizon, structure formation in the presence of the Galileon π is governed by the following equations

$$\dot{\vec{v}} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla}\Phi - \alpha\vec{\nabla}\pi \, , \tag{5.85}$$

$$F(\partial_i \partial_j \pi) = \alpha 8\pi G \rho \, , \tag{5.86}$$

$$\nabla^2 \Phi = 4\pi G \rho \, , \tag{5.87}$$

where F is the equation of motion for the Galileon, which only depends on the second derivatives of π. The point is that one can apply the same argument as in the standard case. A long

mode of $\Phi + \alpha \pi$, up to first order in gradients, can be removed just by a change of coordinates. As before, this change of coordinates brings us to an accelerated frame. For this to happen the symmetry of Galileons is crucial, since it makes a homogeneous gradient of π drop out of eq. (5.86) [159]. However, it is important to notice that this works only deep inside the horizon. Only in the non-relativistic regime, when time-derivatives can be neglected, we can remove a homogeneous field with arbitrary time-dependence although this is not a symmetry of the full Galileon theory. This does not happen, for example, in the case of the chameleon and there we expect the consistency relation to be violated.

One important step in the argument above is to make sure that the long mode induced by a change of coordinates is a physical mode. This is indeed the case. The homogenous gradient can describe a long mode in the linear regime as shown in simulations [160, 161, 162, 163, 164]. This means that, assuming no primordial local non-Gaussianity, the effect of a long mode boils down to the change of coordinates. This is a time dependent translation which does not give any effect for equal-time correlation functions.

Another example of the screening mechanism is *K-mouflage* [165], where the screening depends on the first derivative of the gravitational potential. This happens in theories with a generic kinetic term of the form $P(X)$ with $X \equiv (\partial \phi)^2$. Although this case has not been thoroughly studied, there is no reason to expect that the consistency relations holds. In the absence of Galileon symmetry in the non-relativistic limit, the argument above does not go through.

Baryon content. If dark matter and baryons have a different coupling with a light scalar, one has a violation of the EP at the fundamental level. This causes different astrophysical objects, with a different baryon/dark matter ratio to fall at different rates in an external field. This scenario is however very constrained: Planck [166] limits this kind of couplings to be $\lesssim 10^{-4}$ smaller than gravity. This is far from what we can achieve with our method, since most astrophysical objects have a quite similar baryon content and this suppresses substantially the EP violation.

Gravitational potential. The no-hair theorem implies that black-holes do not couple with a scalar force. More generally, the mass due to self-gravity will violate the EP in the presence of a fifth force. Unfortunately, it seems impossible to observe isolated objects with a sizable component of gravitational mass. The mass of clusters only receives a contribution in the range $10^{-5} \div 10^{-4}$ from the gravitational potential and the correction is even smaller for less massive objects. Black holes, whose mass is entirely gravitational in origin, do not significantly contribute to the mass of the host galaxy.

Environment. Another possibility is to divide the objects depending not on some intrinsic feature but on their environment, for example comparing galaxies in a generic place against galaxies in voids [71]. The fifth force tends to be screened in a dense environment (blanket screening), while it is active in voids. Notice that this is not a test of the Galilean EP (different

objects fall at the same rate in the same external field), but it still checks whether the effect of the long mode can be reabsorbed completely by a change of coordinates. The arguments made above for the Galileon case work also here and we expect no violation of the consistency relation in this case. This effect will be present in the case of chameleon screening (with the same limitations on the Compton wavelength discussed above) and in K-mouflage.

In all these examples, as a consequence of a EP violation, there are different kinds of astrophysical objects (or they are in a different environment) that behave in different ways in an external gravitational field induced by the long mode. We can describe this generic situation using a toy model in which the universe is composed of two non-relativistic fluids A and B, with the latter coupled to a scalar field mediating a fifth force. For example, the two fluids could be baryons and dark matter, but the model can also describe two populations of astrophysical objects, like two different types of galaxies. If the scalar field φ has a negligible time evolution, the continuity equations of the two fluids are the same,

$$\delta'_X + \vec{\nabla} \cdot [(1 + \delta_X)\vec{v}_X] = 0 \, , \quad X = A, B \, , \tag{5.88}$$

where a prime denotes the derivative with respect to the conformal time η, $' \equiv \partial_\eta$. The Euler equation of B contains the fifth force, whose coupling is parameterized by α,

$$\vec{v}'_A + \mathcal{H}\vec{v}_A + (\vec{v}_A \cdot \vec{\nabla})\,\vec{v}_A = -\vec{\nabla}\Phi \, , \tag{5.89}$$

$$\vec{v}'_B + \mathcal{H}\vec{v}_B + (\vec{v}_B \cdot \vec{\nabla})\,\vec{v}_B = -\vec{\nabla}\Phi - \alpha\vec{\nabla}\varphi \, , \tag{5.90}$$

where $\mathcal{H} \equiv \partial_\eta a/a$ is the comoving Hubble parameter. To close this system of equations we need Poisson's equation and the evolution equation of the scalar field. Assuming that the scalar field stress-energy tensor is negligible, only matter appears as a source in Poisson's equation,

$$\nabla^2\Phi = 4\pi G\,\rho_{\rm m}\,\delta = 4\pi G\,\rho_{\rm m}\,(w_A\delta_A + w_B\delta_B) \, , \tag{5.91}$$

where $\rho_{\rm m}$ is the total matter density and $w_X \equiv \rho_X/\rho_{\rm m}$ is the density fraction of the X species. Moreover, in the non-relativistic approximation we can neglect time derivatives in comparison with spatial gradients and the equation for the scalar field reads

$$\nabla^2\varphi = \alpha \cdot 8\pi G\rho_{\rm m}w_B\delta_B \, , \tag{5.92}$$

where we have neglected the mass of the scalar field, assuming we are on scales much shorter than its Compton wavelength.

Using this setup one can proceed and using perturbation theory calculate explicitly the three-point functions of density perturbations. We leave the details of the calculation for Appendix C. The most interesting three-point function in the squeezed limit is given by

$$\lim_{q \to 0} \langle \delta_{\vec{q}}(\eta)\delta^{(A)}_{\vec{k}_1}(\eta)\delta^{(B)}_{\vec{k}_2}(\eta) \rangle' \simeq \frac{7}{5}w_B\,\alpha^2\,\frac{\vec{k} \cdot \vec{q}}{q^2}P(q,\eta)P_{AB}(k,\eta) \, . \tag{5.93}$$

As we expect, the result is proportional to k/q and it has the form of eq. (5.83).

The result that we obtained remains qualitatively the same if A or B represent extended objects, for example a fluid of galaxies of a given kind [167]. For instance, one can take A to represent galaxies that are not coupled to the fifth force because they are screened (see Sec. 5.4.1), while B represents the dark matter fluid. In this case one should start from initial conditions in which there is a bias between the galaxy and the dark matter overdensities: $u_a = (b_g, 1, b, b)$ (for the definition see Appendix C). This is equivalent to exciting decaying modes, given that asymptotically the galaxy bias becomes unity[14] (b_g is the initial galaxy bias). Consequently, the result (5.93) will be different. Still, it is straightforward to check that, as expected, there is no $1/q$ divergence if the EP is not violated, i.e. $\alpha = 0$. In the limit $w_A \ll 1$ (i.e. the screened galaxies contribute a subdominant component of the overall mass density) and keeping only the slowest decaying mode, one gets (in this case we take the long mode to be dark matter only)

$$\lim_{q \to 0} \langle \delta_{\vec{q}}^{(B)}(\eta) \delta_{\vec{k}_1}^{(A)}(\eta) \delta_{\vec{k}_2}^{(B)}(\eta) \rangle' \simeq \frac{7}{5} \alpha^2 \frac{\vec{k} \cdot \vec{q}}{q^2} \left(1 + \frac{10}{7}(b_g - 1)e^{-(y-y_0)} \right) P(q, \eta) P_{AB}(k, \eta). \quad (5.94)$$

Where y is the logarithm of the linear growth factor, $y \equiv \ln D_+$ (see Appendix C). Notice that y_0 here represents the initial value when the local galaxy bias b_g is set up.

Of course, this model is quite rough to capture all processes in a realistic process of structure formation. For example, one possible complication comes from the fact that objects become screened only at a certain stage of their evolution, so that the coupling of the fluid A with the scalar is time-dependent. However, this and other subtleties cannot change the form of the final result, but just modify the numerical value on the right-hand side of eq. (5.93) by a factor of order one. In any case, given the model-dependence of the result, in the following Section where we estimate the possible constraints on the EP violation, we will stick to eq. (5.93) as our benchmark model.

5.4.2   Signal to noise for the bispectrum

Let us finally focus on the question how well we can constrain the violation of the EP using large scale structure surveys. In order to address this question quantitatively, we have to calculate the signal to noise ratio for a given model that violates the EP. We will use our toy model described in the previous Section, but the constraints can be straightforwardly translated to other models of modified gravity. Given that equations (5.82) and (5.83) are quite similar, the calculation closely follows the standard approach used to make forecasts for constraining the primordial non-Gaussianities (see for example [168]).

Before moving to a more detailed calculation, we can try to roughly estimate what kind of constraints we can get just counting the number of modes available in the survey. The simple way to do it is as follows. The experimental constraint on $f_{\rm NL}^{\rm loc}$ goes like $\Delta f_{\rm NL}^{\rm loc} \sim 5\sqrt{10^6/N}$, where N is the number of available modes in the survey. This is consistent with the Planck limit $f_{\rm NL}^{\rm loc} \lesssim 5$, for $N \sim 10^6$ [13]. If we turn to the large scale structure and consider a survey

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[14]Here we are assuming conservation of galaxies.

with a volume $V \sim 1(h^{-1}\text{Gpc})^3$ and $k_{\max}$ of the order of the non-linear scale, then the number of modes is $N \sim (k_{\max}/k_{\min})^3 \sim 4 \cdot 10^3$ and so $\Delta f_{\text{NL}}^{\text{loc}} \sim 80$. This is in a good agreement with Fig. 3 of Ref. [168]. To get the constraints on the EP violation we only have to compare the equations (5.82) and (5.83) and see that $f_{\text{NL}}^{\text{loc}} \sim \alpha^2 \times q \, k/(\Omega_m H_0^2)$. Assuming $k/q \sim 10$, a bound on α^2 is of the order 10^{-3}.

To make this estimate more precise, we will do the full signal-to-noise calculation. Let us imagine that we have a survey of a given comoving volume V. This volume defines the fundamental scale in momentum space, $k_f = 2\pi/V^{1/3}$. This means that we cannot measure the modes with momentum smaller than k_f. In this setup, the bispectrum estimator is given by

$$B(k_1, k_2, k_3) = \frac{V_f}{V_{123}} \int_{k_1} \mathrm{d}^3\vec{q}_1 \int_{k_2} \mathrm{d}^3\vec{q}_2 \int_{k_3} \mathrm{d}^3\vec{q}_3 \, \delta(\vec{q}_1 + \vec{q}_2 + \vec{q}_3) \cdot \delta_{\vec{q}_1} \delta_{\vec{q}_2} \delta_{\vec{q}_3} \, , \tag{5.95}$$

where $V_f = (2\pi)^3/V$ is the volume of the fundamental cell and the integration is done over the spherical shells with bins $q_i \in (k_i - \delta k/2, k_i + \delta k/2)$. The volume V_{123} is given by

$$V_{123} \equiv \int_{k_1} \mathrm{d}^3\vec{q}_1 \int_{k_2} \mathrm{d}^3\vec{q}_2 \int_{k_3} \mathrm{d}^3\vec{q}_3 \, \delta(\vec{q}_1 + \vec{q}_2 + \vec{q}_3) \approx 8\pi^2 \, k_1 k_2 k_3 \, \delta k^3 \, . \tag{5.96}$$

In what follows we will assume no significant correlation among different triangular configurations. In other words, we assume that the bispectrum covariance matrix is diagonal and given by a Gaussian statistics. This is a good approximation for the modes in the linear regime that we are interested in. Under this assumption the variance is given by

$$\Delta B^2(k_1, k_2, k_3) = k_f^3 \frac{s_{123}}{V_{123}} P_{\text{tot}}(k_1) P_{\text{tot}}(k_2) P_{\text{tot}}(k_3) \, , \tag{5.97}$$

where $s_{123} = 6, 2, 1$ for equilateral, isosceles and general triangles, respectively. The power spectrum $P_{\text{tot}}(k)$ contains two parts

$$P_{\text{tot}}(k) = P(k) + \frac{1}{(2\pi)^3} \frac{1}{\bar{n}} \, . \tag{5.98}$$

The first term is the standard power spectrum, while the second accounts for the shot noise. For example, in a galaxy survey, $\bar{n}$ is the number density of galaxies. In what follows we will neglect the shot noise contribution. The first reason is that for our estimates we use only modes that are in the linear regime where the shot noise is expected to be negligible. Furthermore, neglecting the shot noise we can estimate the total amount of signal in principle available for a survey of a given volume. In this way we do not restrict our analysis specifically to galaxy surveys.

Given the definitions above, the signal-to-noise ratio is calculated as

$$\left(\frac{S}{N}\right)^2 = \sum_T \frac{(B_{\text{new physics}}(k_1, k_2, k_3) - B_{\text{standard}}(k_1, k_2, k_3))^2}{\Delta B^2(k_1, k_2, k_3)} \, , \tag{5.99}$$

where the sum runs over all possible triangles formed by $\vec{k}_1$, $\vec{k}_2$ and $\vec{k}_3$ given $k_{\min}$ and $k_{\max}$. It is possible to write the sum in such a way that the same triangles are not counted twice. The

symmetry factor s_{123} is then used to take into account some special configurations. In the case with two different species of particles that we study in our toy model, the bispectrum is not symmetric when momenta are exchanged and the sum in the signal-to-noise has to be written accordingly. The simplest way is to impose $s_{123} = 1$ for all configurations and the sum over triangles in the following way

$$\sum_T \equiv \sum_{k_1=k_{\min}}^{k_{\max}} \sum_{k_2=k_{\min}}^{k_{\max}} \sum_{k_3=k_{\min}^*}^{k_{\max}^*} , \tag{5.100}$$

where $k_{\min}^* \equiv \max(k_{\min}, |\vec{k}_1 - \vec{k}_2|)$, $k_{\max}^* \equiv \min(|\vec{k}_1 + \vec{k}_2|, k_{\max})$ and the discrete sum is done with $|\vec{k}_{\max} - \vec{k}_{\min}|/\delta k$ steps where we set δk to be the same as the fundamental scale k_f.

Let us now apply this machinery to the toy model with two fluids we discussed at the end of the previous Section. We expect that the signal-to-noise will get the largest contribution from the squeezed configurations. However, to be completely general, we will keep the full result for the bispectrum (calculated up to first order in α^2) in the analysis. The details of the calculation can be found in Appendix C. For the toy model, the signal-to-noise is given by

$$\left(\frac{S}{N}\right)^2 = \sum_T \frac{\left[B_{\alpha^2}^{(AB)}(k_1, k_2, k_3) - B_{\alpha^2=0}^{(AB)}(k_1, k_2, k_3)\right]^2}{\Delta[B^{(AB)}]^2(k_1, k_2, k_3)} , \tag{5.101}$$

where the bispectrum $B^{(AB)}(k_1, k_2, k_3)$ is defined by

$$\langle \delta_{\vec{k}_1}(\eta)\delta_{\vec{k}_2}^{(A)}(\eta)\delta_{\vec{k}_3}^{(B)}(\eta)\rangle = (2\pi)^3 \delta_D(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) B^{(AB)}(k_1, k_2, k_3) , \tag{5.102}$$

and the left hand side of this equation is computed at leading order in α^2. In order to do the calculation we use

$$\delta^{(A)}(k) \equiv \frac{1}{w_A + w_B b}\delta(k) , \qquad \delta^{(B)}(k) \equiv \frac{1}{w_A/b + w_B}\delta(k) , \tag{5.103}$$

where the transfer function of the total matter density contrast δ is computed with CAMB [169]. With the following definitions

$$\langle\delta_{\vec{k}}(\eta)\delta_{\vec{k}'}(\eta)\rangle = (2\pi)^3 \delta_D(\vec{k}+\vec{k}')P(k) , \qquad \langle\delta_{\vec{k}}^{(X)}(\eta)\delta_{\vec{k}'}^{(X)}(\eta)\rangle = (2\pi)^3 \delta_D(\vec{k}+\vec{k}')P^{(X)}(k) , \tag{5.104}$$

and using eq. (5.97), the variance of the bispectrum becomes

$$\Delta[B^{(AB)}]^2(k_1, k_2, k_3) = k_f^3 \frac{s_{123}}{V_{123}} P(k_1)P^{(A)}(k_2)P^{(B)}(k_3) . \tag{5.105}$$

Figure 5.3 shows the estimated error on α^2, $\sigma(\alpha^2)$, for three different surveys of volume $V = 1(h^{-1}\mathrm{Gpc})^3$ at redshift $z = 0$, $z = 0.5$ and $z = 1$, respectively. On the left panel this is shown as a function of $k_{\max}$ for the smallest possible $k_{\min}$, i.e. $k_{\min} = k_f = 2\pi/V^{1/3}$. The smallest measurable value of α^2 roughly scales as $k_{\max}^{-2.8}$, so that it crucially depends on our ability to capture the shortest scales.
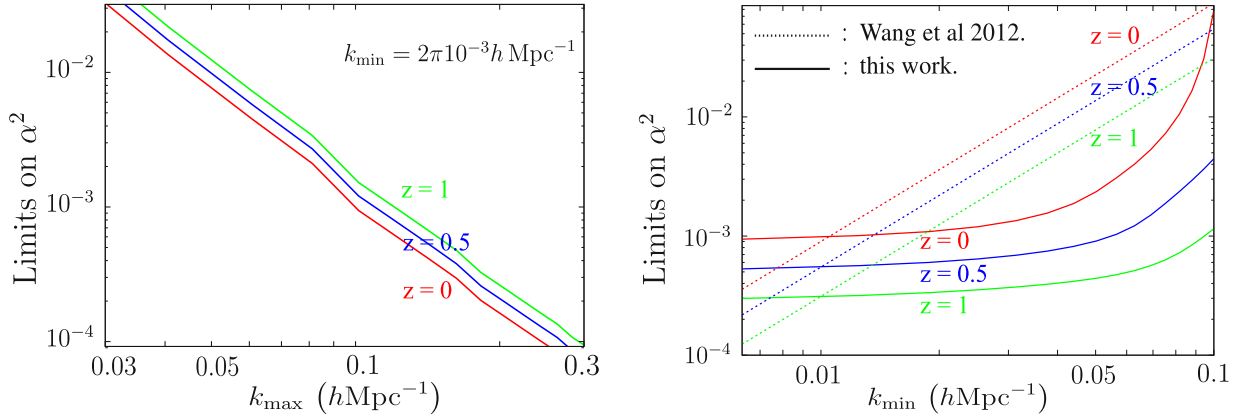
Figure 5.3: Expected error on α^2, $\sigma(\alpha^2)$, for a survey with volume $V = 1(\mathrm{Gpc}/h)^3$ at three different redshifts, $z = 0$, $z = 0.5$ and $z = 1$. Left: $\sigma(\alpha^2)$ is plotted as a function of $k_{\max}$. We have chosen $k_{\min} = 2\pi/V^{1/3}$ so that the violation of the EP extends to the whole survey. Right: $\sigma(\alpha^2)$ is plotted as a function of $k_{\min}$. $k_{\max}$ is given by 0.10, 0.14, 0.19 for $z = 0, 0.5, 1$ respectively. The dotted lines represent $\alpha^2 \lesssim 10^{-6}(m/H)^2$, i.e. the bound on α^2 from screening the Milky Way [155].

On the right panel, the estimated relative variance is shown as a function of $k_{\min}$. For each survey, we take $k_{\max}$ such that we are still in a quasi-linear regime where theoretical control in perturbation theory is possible. In particular, we fix $k_{\max} = \pi/(2R)$ where R is chosen in such a way that σ_R, the root mean squared linear density fluctuation of the matter field in a ball of radius R, is 0.5. This yields $k_{\max} = 0.10, 0.14, 0.19$ for $z = 0, 0.5, 1$ respectively. From Fig. 5.3 we see that the dependence on $k_{\min}$ is very mild when going to zero. This seems counterintuitive, because eq. (5.93) indicates that the bispectrum diverges as $1/q$ at small q, giving more signal. However, in that limit the power spectrum of matter fluctuations scales as q, $P(q) \propto q$, canceling the enhancement. This differs from the familiar case of local non-Gaussianity where the divergence scales as $1/q^2$, causing the known increase of precision on $f_{\mathrm{NL}}^{\mathrm{loc.}}$ when going to larger surveys. The improvement of the constraints at higher redshifts, discussed also in [71], is due to the fact that $k_{\max}$ increases and, assuming a fixed volume, we have access to more modes. Let us stress again that all these results are obtained assuming that the covariance matrix is diagonal and given by the Gaussian statistics. To make a realistic forecast, one has to take into account the geometry of the survey and induced covariance matrix between different triangles. However, for our choices of $k_{\min}$, the corrections are expected to be small and they should not change the estimated errors on α^2 significantly (see for example [168]).

Our constraints can be compared with that for chameleon models derived in Ref. [155] from requiring that the Milky Way must be screened. This yields

$$\alpha^2 \lesssim 10^{-6}(m/H)^2 \,, \tag{5.106}$$

where m is the Compton mass of the chameleon. In this case $k_{\min}$ can be identified with m^{-1}, the Compton wavelength of the chameleon and we show in Fig. 5.3 that for $m^{-1} \gtrsim 0.01$ our

constraints can in principle improve that of Ref. [155]. However, it is important to notice that the screening in these kind of models depends on the typical value of the gravitational potential GM/r of the object. Given that we know the Milky Way is screened, one should look for objects with a lower Φ to find unscreened objects. This looks challenging since in a survey one is typically sensitive to galaxies which are more luminous and therefore more massive than the Milky Way.

The constraints that we get in our analysis are expected to be correct up to an order of magnitude and more detailed analysis is necessary to get better predictions. There are several things that have to be taken into account and here we discuss some of them.

First, when looking for EP violation, a possible contaminant is the initial density or velocity bias between two different species. As we already said, even in single-field inflation we know that baryons and dark matter have different initial conditions on scales below the sound horizon at recombination, because at recombination baryons are tightly coupled to photons through Thomson scattering, while dark matter particles are free falling. As discussed in [139, 140, 141], the relative velocity between baryons and dark matter excites long wavelength isodensity modes that couple to small scales reducing the formation of early structures. However, one can check that this effect decays more rapidly than the one described by eq. (5.93). For instance, assuming no violation of the EP but an initial density and velocity bias between the two species A and B, $u_a = (b_A, b_A, b_B, b_B)$, one obtains

$$\lim_{q \to 0} \langle \delta_{\vec{q}}(\eta) \delta_{\vec{k}_1}^{(A)}(\eta) \delta_{\vec{k}_2}^{(B)}(\eta) \rangle' \simeq 4 \, (b_A - b_B) e^{-\frac{3}{2}(y-y_0)} \frac{\vec{k} \cdot \vec{q}}{q^2} P(q, \eta) P_{AB}(k, \eta) \,, \tag{5.107}$$

independently of w_A and w_B. Thus, the effect is still divergent as $1/q$ but rapidly decays, so that it is typically suppressed by a factor $\sim (1+z_0)^{-3/2}$ where z_0 represents the initial redshift. For the example discussed above of baryons and dark matter we can take $z_0 \simeq 1100$ and today this effect is thus suppressed by $\sim \mathcal{O}(10^{-5})$. In a particular example of the EP violation coming form an additional fifth force between dark matter particles, it will be further suppressed by the fact that the baryon-to-dark matter ratio is rather constant in different galaxies.

When using galaxies, one should also remember that their density field is a biased tracer and that in general we expect the bias to contain nonlinearities. Thus, other contributions are expected in eq. (5.93), for instance of $\mathcal{O}[(k/q)^0]$ if the nonlinear bias is scale independent. To compute the signal-to-noise ratio correctly taking into account this effect, one should include these nonlinear contributions and marginalise over the bias parameters, similarly to what done in the context of non-Gaussianity, for instance in Ref. [168]. However, due to its different scale and angular dependence, we do not expect the marginalization over nonlinear bias to dramatically change our estimates.

Before concluding, it is important to stress that our estimates so far assume that we exactly know which are the two classes of objects that violate the EP. In other words, our estimate is just cosmic variance limited for a given volume. In practice, one will have to classify objects either based on some intrinsic property (mass, luminosity, dark matter content) or some environmental property (like being inside an overdense or underdense region),

and astrophysical uncertainties in the selection of these objects may significantly suppress the signal. In particular, if the kind of objects we aim for is quite rare, the shot noise, which we have neglected so far, will be an important limitation. Indeed, this is the biggest limitation of using the consistency relations to test the EP. However, our ability to distinguish different kind of objects in LSS surveys should improve with better telescopes and more data, and this is something certainly worth exploring in the future.

Despite all these limitations, the absence of any signal when the EP holds is a very robust prediction. The long mode cannot give any $1/q$ effect, independently of the bias of the objects and of the selection strategy we use. Furthermore, as we saw, the same statement holds also in redshift space, which makes the connection with observations even more straightforward. In other words, all complications that enter when one wants to use the data to infer the underlying dark matter three-point function are not relevant here if we only want to show that the EP is violated. Of course, if a violation of the consistency relations is detected, it would be much more challenging to better characterize which mechanism is responsible for it.

5.5 Other Applications

In this final Section, let us briefly mention some other applications of the consistency relations of LSS that are worth exploring more in the future.

Velocity bias. One can read eq. (5.83), when the EP holds, i.e. $\epsilon = 0$, as the statement that there is no velocity bias between species A and B on large scales. In other words, the long mode induces exactly the same velocity for all objects. It is important to stress that this holds even considering *statistical* velocity bias. Objects do not form randomly, but in special places of the density field. Therefore, even if they locally fall together with the dark matter, there can be a velocity bias in a statistical sense [170, 171]. However, the arguments of [67, 69, 65] tell us that the long mode (at leading order in q) is equivalent to a change of coordinates. Apart from this change of coordinates the long mode does not affect neither the dynamics nor the *statistics* of short modes. Therefore, the EP implies that the statistical velocity bias disappears on large scales. Again, this statement is completely non-perturbative in the short scales and includes the effect of baryons. For the case of dark matter only, we know that the statistical velocity bias vanishes on large scales as $\sim q^2 R^2$, where R is a length scale of order the Lagrangian size of the object; this can be calculated by looking at the statistics of peaks [170, 171] and verified in numerical simulations [172]. One expects that the effect of statistical velocity bias is therefore subdominant with respect to the unknown corrections in eq. (5.83) since $k \lesssim R^{-1}$.

Gravitational waves. One interesting question is whether it is possible to write down consistency relations of LSS with soft tensor modes. As we saw in Chapter 4 this is certainly possible for inflationary correlation functions. The answer to this question is negative. It turns out that, using only the change of coordinates, one cannot generate a physical solution

for a long gravitational wave inside the horizon even at lowest order in gradients. In other words, the gravitational waves *do* have physical effect on short scales. For example, after they decay their imprint in the two-point function of matter fluctuations remains. For this reason these effects are called fossil effects and they were explicitly calculated in perturbation theory and studied in [173, 174].

Galilean Invariance. Another important question is about the connection of the consistency relations in the non-relativistic limit and Galilean invariance. In reference [66] the consistency relations were derived as Ward identities for non-linearly realised Galilean symmetry (for a similar approach see [75]). On the other hand, as we saw in this Chapter, the non-relativistic consistency relations can be thought of as a consequence of the EP. This is apparently a contradiction because it is not difficult to imagine theories that are Galilean invariant and where the EP is violated (this is for example the case in model described in Appendix C). In these kind of theories, do the consistency relations hold or not?

As we saw in explicit calculation in the previous Section, the consistency relations are indeed violated. In the absence of the EP different species fall in a different way in the homogeneous gravitational field of the long mode which eventually turns into k/q divergence in the squeezed limit of the three-point function. From the point of view of the derivation we did in this Chapter, there is nothing more that one can say. However, the starting theory is still Galilean invariant and on the level of perturbations this symmetry is non-linearly realized. Indeed, as it was shown in [68], this leads to a Ward identity that now has a more complicated form, even though the EP is violated. Notice that both approaches agree that in the case of a violation of the EP the three-point function goes like k/q in the squeezed limit. It would be certainly interesting to explore in more details what are the consequences of Galilean invariance for LSS and how do they relate to the derivation we were following in this thesis.

Spread of the BAO peak. The derivation of the consistency relation was based on the fact that the long mode is locally unobservable—its effect is just to induce a bulk velocity that can be always removed by a change of coordinates. However, there is a situation in which this bulk flows are important: they are responsible for the spread of the BAO peak. Indeed, if we imagine modes shorter than the BAO scale and longer than the width of the BAO peak, their effect on the equal time two-point function does not vanish (see for example [175, 176]). The reason is that the long mode is not coherent on the BAO scale, and it induces different bulk velocity at different points. As a consequence, the BAO peak is spread. Notice that this statement is valid beyond perturbation theory and including baryons. It would be certainly interesting to explore this in the future.

Chapter 6

Conclusions and Outlook

In this thesis we derived consistency relations for inflation and LSS. These are model independent and non-perturbative relations between an $(n+1)$-point function in the squeezed limit and a variation of the corresponding n-point function.

The derivation of the consistency relations is based on the fact that a long adiabatic mode can be locally generated starting from an unperturbed FRW metric just by a change of coordinates. We showed how this construction works in Chapter 3, including gradients, in Newtonian and ζ-gauge. We also proved that, with some minor modifications, the same procedure of generating the long adiabatic mode can be applied even when short scale perturbations are present in the metric. As we showed, this allows for generating the exact second order solution of the Einstein equations in the squeezed limit.

In Chapter 4 we used the construction of adiabatic modes to derive the consistency relations for inflation. We proved that they hold in any single-field model, assuming the standard vacuum and that the inflaton is on the attractor. At zeroth order in gradients we recovered the well-known Maldacena result, while at first order in gradients we got the conformal consistency relation. We also derived the consistency relation with soft tensor modes, explored different kinematical regimes where the consistency relations can be applied, such as multiple squeezed limits and limits in which the sum of some of the momenta goes to zero and investigated higher orders in gradient expansion where the effects of the long mode are physical, but some general statement about non-Gaussianities can still be made.

The construction of adiabatic modes is valid if the long mode is outside the sound horizon, such that within one Hubble time interactions with short modes can be neglected. After recombination the sound horizon becomes much smaller than the Hubble radius, and the construction is valid even for the modes that are observable. We used this to derive consistency relations of LSS in Chapter 5. These relations are non-perturbative in short modes and hold including baryonic physics. Although we obtained the fully relativistic result, observationally the most interesting is the Newtonian limit when all modes are deep inside the horizon, and we explored it in great detail. We further showed that one can derive the consistency relations directly in redshift space. In all these cases, the right hand side of the consistency relations vanish for equal-time correlation functions. Any deviation from this behavior would indicate

that some of the few assumptions in the derivation are wrong.

There are essentially two crucial assumptions we made in the derivation. These are single-field inflation and the equivalence principle. Therefore, detection of any violation of the consistency relations is a model-independent probe of these two assumptions. It is well known for a long time that a detection of $f_{\rm NL}^{\rm loc.} \gg 1$ would rule out all single-field models. In a similar way, appearance of a k/q divergence in the galaxy bispectrum in $q \to 0$ limit would indicate that the EP is violated on cosmological scales. Therefore, the consistency relations of LSS provide a new, model-independent test of GR. We explored how much one can constrain the violations of the EP using these technique in Chapter 5.

Although our presentation was focused on inflation and LSS separately, let us stress again that in both cases there is the same underlying principle that leads to the consistency relations. Indeed, as we already pointed out, taking the $\eta \to 0$ limit in the relativistic consistency relations of LSS, one recovers the inflationary result. We can say that the consistency relations of LSS are just a final piece in a unified picture valid during the entire history of the universe. Ever since the long mode leaves the horizon during inflation, at leading orders in derivatives, it is just a classical background for the short modes equivalent to a change of coordinates. If the long mode remains outside the sound horizon until recombination, its effects on short modes remain unphysical even if it enters the Hubble radius afterwards. The relativistic consistency relations of LSS capture all this evolution, from the end of inflation until today.

Still, a lot of work remains to be done. Let us just mention some of the major open questions. As we pointed out, the consistency relations can be derived as Ward identities for spontaneously broken symmetries. In the case of inflation these are residual diffeomorphisms in comoving gauge, while in the case of LSS Galilean invariance is non-linearly realized. It would be certainly worth to explore more carefully what are connections between Ward identities and the approach we described in this thesis. In particular, it would be nice to understand better this connection for LSS, where Galilean invariance seems to give non-trivial relation between correlation functions even when the EP is violated. On a more observational side one obvious step forward is to write down the consistency relations of LSS in terms of observables, including relativistic effects. This is particularly important for future galaxy surveys, and it is necessary in order to fully exploit them as a test of GR on cosmological scales.

Appendix A

Correlation Functions in Inflation

In this appendix we will give some results needed for checks of inflationary consistency relations. We will present four different single-field models and quote the results for some of the three-point and four-point correlation functions.

A.1   Single-field Slow-roll Inflation

We first focus on single-field slow-roll inflation. In these model the standard kinetic term is assumed and inflation is driven by the potential energy of the field. The potential satisfies slow-roll conditions.

A.1.1   Three-point Correlation Functions

We already saw the the three-point function for scalars is given by [43]

$$\langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \rangle' = \frac{H^4}{4\epsilon^2 M_P^4} \frac{1}{\prod(2k_i^3)} \left[(2\eta - 3\epsilon) \sum_i k_i^3 + \epsilon \sum_{i \neq j} k_i k_j^2 + \epsilon \frac{8}{k_t} \sum_{i > j} k_i^2 k_j^2 \right], \qquad (A.1)$$

where $k_t = k_1 + k_2 + k_3$.

Similarly, one can calculate the three-point function with one graviton and two scalars. The action that describes this interaction is [43]

$$S = \int d^4x \, \frac{1}{2} \frac{\dot{\phi}^2}{H^2} a \gamma_{ij} \partial_i \zeta \partial_j \zeta \, . \qquad (A.2)$$

Therefore, the in-in integral is easy to solve and the final result is given by

$$\langle \gamma_{\vec{q}}^s \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle' = \frac{H^4}{q^3 k_1^3 k_2^3} \frac{1}{4\epsilon} \epsilon_{ij}^s(\vec{q}) k_{1i} k_{2j} \left(-k_t + \frac{k_2 q + k_1 q + k_2 k_1}{k_t} + \frac{q k_1 k_2}{k_t^2} \right) \, . \qquad (A.3)$$

Notice that in the limit $\vec{q} \to 0$ we have

$$\langle \gamma_{\vec{q}}^s \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle'_{q \to 0} = \frac{H^2}{4\epsilon k^3} \frac{H^2}{q^3} \epsilon_{ij}(\vec{q}) \frac{k_i k_j}{k^2} \frac{3}{2} \left(1 - \frac{5}{2} \frac{\vec{k} \cdot \vec{q}}{k^2} \right) \, . \qquad (A.4)$$

A.1.2 Four-point Correlation Functions

Let also also quote the result for the four-point function of ζ [123, 124]. The four-point function is the sum of two comparable contributions, one coming from four-legs interactions of ζ [123] and one from a scalar or graviton exchange [124]. As explained in [124] scalar exchange gives a subdominant contribution and we can neglect it.

The contact term contributes to the four-point function by [123]

$$\langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \zeta_{\vec{k}_4} \rangle'_{\text{cont}} = \frac{H^6}{4\epsilon^3 M_P^6} \frac{1}{\prod(2k_i^3)} \epsilon \sum_{perm} \mathcal{M}_4(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) , \tag{A.5}$$

with:

$$\mathcal{M}_4(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) = -2 \frac{k_1^2 k_3^2}{k_{12}^2 k_{34}^2} \frac{W_{24}}{k_t} \left(\frac{\vec{Z}_{12} \cdot \vec{Z}_{34}}{k_{34}^2} + 2\vec{k}_2 \cdot \vec{Z}_{34} + \frac{3}{4}\sigma_{12}\sigma_{34} \right) -$$

$$- \frac{1}{2} \frac{k_3^2}{k_{34}^2} \sigma_{34} \left(\frac{\vec{k}_1 \cdot \vec{k}_2}{k_t} W_{124} + 2\frac{k_1^2 k_2^2}{k_t^3} + 6\frac{k_1^2 k_2^2 k_4}{k_t^4} \right) , \tag{A.6}$$

$$\sigma_{ab} = \vec{k}_a \cdot \vec{k}_b + k_b^2 , \tag{A.7}$$

$$\vec{Z}_{ab} = \sigma_{ab}\vec{k}_a - \sigma_{ba}\vec{k}_b , \tag{A.8}$$

$$W_{ab} = 1 + \frac{k_a + k_b}{k_t} + \frac{2k_a k_b}{k_t^2} , \tag{A.9}$$

$$W_{abc} = 1 + \frac{k_a + k_b + k_c}{k_t} + \frac{2(k_a k_b + k_b k_c + k_a k_c)}{k_t^2} + \frac{6k_a k_b k_c}{k_t^3} . \tag{A.10}$$

Notice that in order to find this result one has to solve constraint equations up to second order in perturbations. After finding the action at fourth order, one has to be careful with going to interaction Hamiltonian which at this order is not just equal to the negative interaction Lagrangian. All these details can be found in [123].

The graviton exchange diagram is even more complicated to calculate. The final result can be found in [124]. Let us just quote here what it becomes in the limit $\vec{k}_1 + \vec{k}_2 = \vec{q} \to 0$

$$\langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \zeta_{\vec{k}_4} \rangle'^{GE}_{\vec{q} \to 0} = \frac{H^2}{q^3} \frac{H^2}{4\epsilon k_1^3} \frac{H^2}{4\epsilon k_3^3} \sum_s \epsilon_{ij}^{s*}(\vec{q}) \epsilon_{lm}^s(\vec{q}) \frac{9}{4} \frac{k_{1i} k_{1j}}{k_1^2} \frac{k_{3l} k_{3m}}{k_3^2} \left(1 + \frac{5}{2} \frac{\vec{k}_1 \cdot \vec{q}}{k_1^2} - \frac{5}{2} \frac{\vec{k}_3 \cdot \vec{q}}{k_3^2} \right) . \tag{A.11}$$

This result is relevant for checking the consistency relation when one internal line is soft (see Section 4.3).

A.2 Models with Reduced Speed of Sound

These models are described by a Lagrangian of the form [97]

$$S = \frac{1}{2} \int \mathrm{d}^4 x \sqrt{-g} \left(M_{\text{Pl}}^2 R + 2P(X, \phi) \right) , \tag{A.12}$$

where $X = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi$. In order to compute the three-point and four-point correlation functions we have to expand the action up to fourth order, find the interaction Hamiltonian and use the standard in-in formalism. All these computations have been done in [121, 122] and here we follow the notation of the first paper. Any n-point function can be written as

$$\langle \zeta_{\vec{k}_1} \cdots \zeta_{\vec{k}_n}\rangle = (2\pi)^3\delta(\vec{k}_1 + \cdots + \vec{k}_n)P_\zeta^{n-1}\prod_{i=1}^{n}\frac{1}{k_i^3}\mathcal{M}^{(n)}(\vec{k}_1,...,\vec{k}_n) \tag{A.13}$$

and the two-point function is given by

$$\langle \zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\rangle = (2\pi)^3\delta(\vec{k}_1 + \vec{k}_2)P_\zeta\frac{1}{2k_1^3}, \quad P_\zeta = \frac{1}{2M_{\rm Pl}^2}\frac{H^2}{c_s\epsilon}, \tag{A.14}$$

with $\epsilon \equiv -\dot{H}/H^2$ and the speed of sound defined as

$$c_s^2 \equiv \frac{P_{,X}}{P_{,X} + 2XP_{,XX}} . \tag{A.15}$$

The amplitude of the three-point function is

$$\mathcal{M}^{(3)} = \left(\frac{1}{c_s^2} - 1 - \frac{2\lambda}{\Sigma}\right)\frac{3k_1^2k_2^2k_3^2}{2k_t^3} + \left(\frac{1}{c_s^2} - 1\right)\left(-\frac{1}{k_t}\sum_{i>j}k_i^2k_j^2 + \frac{1}{2k_t^2}\sum_{i\neq j}k_i^2k_j^3 + \frac{1}{8}\sum_i k_i^3\right), \tag{A.16}$$

where the parameters λ and Σ are related to derivatives of the Lagrangian with respect to X

$$\lambda = X^2P_{,XX} + \frac{2}{3}X^3P_{,XXX}$$
$$\Sigma = X^2P_{,X} + 2X^2P_{,XX} . \tag{A.17}$$

Notice that in the squeezed limit the three-point function becomes

$$\langle \zeta_{\vec{q}}\zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\rangle'_{\vec{q}\to 0} = P(k)P(q)\left[\left(-1 + \frac{1}{c_s^2}\right)\left(-2\frac{q^2}{k^2} + \frac{5}{4}\frac{(\vec{q}\cdot\vec{k})^2}{k^4}\right) - \frac{3}{2}\frac{q^2}{k^2}\frac{\lambda}{\Sigma}\right]. \tag{A.18}$$

As in single field inflation, the four-point function contains two kind of contributions. One comes from diagrams with a quartic interaction and the other from scalar-exchange diagrams. Here we focus only on the first one, more details about scalar exchange can be found in [121]. The diagrams with a quartic interaction contribute to the amplitude like [121]

$$\mathcal{M}_{cont}^{(4)} = \left[\frac{3}{2}\left(\frac{\mu}{\Sigma} - \frac{9\lambda^2}{\Sigma^2}\right)\frac{\prod_{i=1}^4 k_i^2}{k_t^5} - \frac{1}{8}\left(\frac{3\lambda}{\Sigma} - \frac{1}{c_s^2} + 1\right)\frac{k_1^2k_2^2(\vec{k}_3\cdot\vec{k}_4)}{k_t^3}\left(1 + \frac{3(k_3 + k_4)}{k_t} + \frac{12k_3k_4}{k_t^2}\right)\right.$$
$$\left. + \frac{1}{32}\left(\frac{1}{c_s^2} - 1\right)\frac{(\vec{k}_1\cdot\vec{k}_2)(\vec{k}_3\cdot\vec{k}_4)}{k_t}\left(1 + \frac{\sum_{i<j}k_ik_j}{k_t^2} + \frac{3k_1k_2k_3k_4}{k_t^3}\sum_{i=1}^4\frac{1}{k_i} + \frac{12k_1k_2k_3k_4}{k_t^4}\right)\right] + 23 \text{ perm.} \tag{A.19}$$

with μ defined as:

$$\mu = \frac{1}{2}X^2 P_{,XX} + 2X^3 P_{,XXX} + \frac{2}{3}X^4 P_{,XXXX} \,. \tag{A.20}$$

When one of the momenta is much smaller we find

$$\mathcal{M}_{cont}^{(4)}\big|_{\vec{k}_4 \to 0} = \left[-\frac{1}{2}\left(\frac{3\lambda}{\Sigma} - \frac{1}{c_s^2} + 1\right) \frac{k_1^2 k_2^2 (\vec{k}_3 \cdot \vec{k}_4)}{k_t^3}\left(1 + \frac{3k_3}{k_t}\right) \right.$$

$$\left. + \frac{1}{4}\left(\frac{1}{c_s^2} - 1\right) \frac{(\vec{k}_1 \cdot \vec{k}_2)(\vec{k}_3 \cdot \vec{k}_4)}{k_t}\left(1 + \frac{k_1 k_2 + k_1 k_3 + k_2 k_3}{k_t^2} + \frac{3 k_1 k_2 k_3}{k_t^3}\right)\right] + 2 \text{ terms }, \tag{A.21}$$

where the two additional terms have the same form with $\vec{k}_1 \leftrightarrow \vec{k}_3$ and $\vec{k}_2 \leftrightarrow \vec{k}_3$. Notice that since in the squeezed limit $\vec{k}_1$, $\vec{k}_2$ and $\vec{k}_3$ form a triangle, we can replace $(\vec{k}_1 \cdot \vec{k}_2) \to \frac{1}{2}(k_3^2 - k_1^2 - k_2^2)$. We used this result to check the conformal consistency relation.

A.3   Resonant non-Gaussianities

In these models the inflaton potential has periodic modulation and they are motivated by explicit string constructions of Monodromy Inflation [118]. It turns out that, at first order in the size of modulation, all n-point functions can be analytically calculated [119, 120].

The $(n + 1)$-point function reads [119, 120]

$$\langle \zeta_{\vec{q}} \zeta_{\vec{k}_1} \dots \zeta_{\vec{k}_n} \rangle = (2\pi)^3 \delta(\vec{q} + \vec{k}_1 + \dots + \vec{k}_n) \left(-\frac{H}{\dot{\phi}}\right)^{n+1} \frac{H^{2n-2}}{2q^3 \prod_{i=1}^{n} 2k_i^3} I_{n+1} \,, \tag{A.22}$$

where I_{n+1} is given by the following in-in integral:

$$I_{n+1} = -2\,\mathrm{Im} \int_{-\infty - i\epsilon}^{0} \frac{\mathrm{d}\eta}{\eta^4} V^{(n+1)}(\phi(\eta))(1 - iq\eta)(1 - ik_1\eta)\dots(1 - ik_n\eta)e^{ik_t\eta} \,. \tag{A.23}$$

Here $k_t = q + \sum_i k_i$ and $V^{(n+1)}$ is the $(n + 1)$th derivative of the potential, evaluated on the unperturbed history $\phi(\eta)$. In resonant models, one starts from a trigonometric function and therefore these derivatives are simply sines or cosines. Here we keep the discussion more general, without specifying the form of the potential.

Notice that this result holds in the decoupling limit and neglecting diagrams with more than one vertex. Corrections to the decoupling limit are suppressed by ϵ and they vanish in the limit $\epsilon \to 0$ keeping constant the power spectrum of ζ [120]. Terms with more than one vertex contain extra powers of the amplitude of the potential modulation. These corrections eventually become sizeable for n-point functions with large n [119]. However, the consistency relations must be satisfied independently by the single-vertex diagrams, as they are the only ones proportional to the first power of the amplitude of the modulation. The integral over η can be explicitly done in the saddle-point approximation, in the limit of many oscillations per Hubble time [119, 120]. Since the consistency relations must hold independently of this limit, and not only for oscillatory but for general potentials, here we prefer to keep the integral form.

A.4 Khronon Inflation

Let us finally turn to an interesting example of Khronon Inflation [34]. In Khronon Inflation the approximate shift symmetry of the inflaton

$$t \to \tilde{t} = t + \text{const} , \tag{A.24}$$

is promoted to the full time reparametrization invariance

$$t \to \tilde{t}(t) . \tag{A.25}$$

This symmetry has been studied in the context of Hořava gravity and its healthy extensions [177, 178, 179]. In these references the scalar mode describing the preferred foliation has been dubbed 'khronon'.

We want to write an inflaton action in which the usual (approximate) symmetry $\phi \to \phi + c$ is promoted to the full invariance under field redefinition $\phi \to \tilde{\phi}(\phi)$. We are going to assume an exact de Sitter metric and take the decoupling limit $M_{\text{Pl}} \to \infty$, in which the dynamics of the scalar perturbations can be studied without considering the mixing with gravity. The time dependent inflaton background defines a foliation and in the presence of ϕ reparameterization invariance, the only invariant object is the 4-vector perpendicular to the foliation [179]

$$u_\mu = \frac{\partial_\mu \phi}{\sqrt{-g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi}} , \tag{A.26}$$

which is indeed invariant under $\phi \to \tilde{\phi}(\phi)$. At low energy the operators with the smallest number of derivatives will dominate. One can show that the action to lowest order in derivative—and any order in u^μ—can thus be written as [34]

$$S = \frac{1}{2} \int \mathrm{d}^4 x \sqrt{-g} \left(M_{Pl}^2 R - 2\Lambda - M_\lambda^2 \left(\nabla_\mu u^\mu - 3H \right)^2 + M_\alpha^2 u^\mu u^\nu \nabla_\mu u_\rho \nabla_\nu u^\rho \right) , \tag{A.27}$$

where M_α and M_λ are the two parameters of the model, besides the vacuum energy Λ which is driving inflation.

A.4.1 The Power Spectrum

To calculate the power spectrum we expand the action (A.27) at second order. Using the field redefinition symmetry we can assume to perturb around $\phi_0 = t$, i.e. $\phi(\vec{x}, t) = t + \pi(\vec{x}, t)$, in an unperturbed de Sitter space, which is a good approximation in the decoupling limit

$$S_2 = \int \mathrm{d}^3 x \, \mathrm{d}\eta \left(\frac{M_\alpha^2}{2} (\partial \pi')^2 - \frac{M_\lambda^2}{2} (\partial^2 \pi)^2 \right) . \tag{A.28}$$

This result is pretty unconventional. First of all, compared with the usual free-field action, each term has two additional spatial derivatives. This is not worrisome as additional spatial derivatives do not introduce extra pathological degrees of freedom. Second, the action does

not contain any η dependence so that the field is not sensitive to the expansion of the Universe and behaves as in Minkowski space (though with a speed of sound which is, in general, different from the speed of light). Actually these two peculiarities in some sense cancel each other to give a scale-invariant spectrum. Indeed, we expect the mode functions to be of the Minkowski form, but with an additional factor of $1/k$ because of the presence of the additional spatial derivatives. It is easy to get the wavefunctions

$$\pi_k(\eta) = \frac{1}{\sqrt{2k^3}} \frac{1}{\sqrt{M_\alpha M_\lambda}} e^{\pm i \frac{M_\lambda}{M_\alpha} k\eta} , \tag{A.29}$$

which give a scale-invariant spectrum for π at late times $\eta \to 0$. The curvature perturbation ζ is given by $\zeta = -H\pi$ so that

$$\langle \zeta_{\vec{k}} \zeta_{\vec{k}'} \rangle = (2\pi)^3 \delta(\vec{k} + \vec{k}') \frac{1}{2k^3} \frac{H^2}{M_\alpha M_\lambda}. \tag{A.30}$$

Notice that the scale invariance of the power spectrum (and of higher-order correlation functions) can be justified by symmetry arguments [60], since we are in exact de Sitter and the action is shift symmetric. Of course, a small tilt is induced if the field redefinition symmetry is slightly broken.

Apart from other subtleties of this result (see the discussion in [34]), let us only notice here that above feature leaves an interesting signature in the correlation functions of the model. Indeed, the "decaying" mode decays much slower than in the conventional case (as $1/a$ instead of $1/a^3$). This has remarkable consequences for the squeezed limits of correlation functions: the standard single-field theorems hold, but only at first order in the momentum of the long mode. In principle, it is possible to get $1/q^2$ divergence in the three-point function in the squeezed limit. As it turns out in explicit calculations, this term is proportional to a small breaking of the reparametrization symmetry and therefore practically unobservable.

A.4.2 The 3-point function

The interaction Lagrangian can be computed by expanding the action, equation (A.27), to third order. We get, after several integrations by parts,

$$S_3 = \int \mathrm{d}^3 x \, \mathrm{d}\eta \, \frac{1}{a} \Big[M_\lambda^2 \big(2\partial_i \pi' \partial_i \pi \partial^2 \pi + \pi' \partial_i \partial_j \pi \partial_i \partial_j \pi \big) + M_\alpha^2 \big(\pi' \partial_i \pi'' \partial_i \pi - \partial_i \pi' \partial_j \pi \partial_i \partial_j \pi \big) \Big]. \tag{A.31}$$

In order to compute the three-point function for ζ we use the relation $\zeta = -H\pi$[1]

$$\langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \rangle' = \frac{1}{\prod k_i^3} P_\zeta^2 \left[-\frac{k_1}{k_t^2} (k_3^2 \vec{k}_1 \cdot \vec{k}_2 + k_2^2 \vec{k}_1 \cdot \vec{k}_3) - \frac{k_1^2}{k_t} \vec{k}_2 \cdot \vec{k}_3 - \frac{M_\alpha^2}{M_\lambda^2} \frac{k_1^3}{k_t^2} \vec{k}_2 \cdot \vec{k}_3 \right]$$
$$+ \text{ cyclic perms. }, \tag{A.32}$$

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[1]Additional non-linear terms in this relation either involve higher derivatives, which vanish outside of the horizon, or are suppressed by slow-roll factors, see [43, 46].

where $k_t \equiv k_1 + k_2 + k_3$ and $P_\zeta = H^2/(2M_\alpha M_\lambda)$ is the ζ power spectrum, eq. (A.30). All the contributions but the last cannot be large and give an $f_{\mathrm{NL}} \sim 1$. The contribution from the last term on the other hand is proportional to $M_\alpha^2/M_\lambda^2 \equiv 1/c_s^2$. It is therefore a good approximation to take the shape as equilateral. Its amplitude can be read from the expression (A.32) above

$$f_{\mathrm{NL}}^{\mathrm{eq}} = \frac{5}{108}\frac{1}{c_s^2} \,. \tag{A.33}$$

Notice that this is the opposite of what happens in more conventional models with reduced speed of sound (K-inflation), where the operator which reduces the speed of sound gives $f_{\mathrm{NL}}^{\mathrm{eq.}} \propto -1/c_s^2$.

In the squeezed limit, this three-point function gives

$$\langle \zeta_{\vec{q}} \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle'_{\vec{q}\to 0} = \frac{P_\zeta^2}{k^3 q^3}\frac{1}{4}\frac{M_\alpha^2}{M_\lambda^2}\left(\frac{q^2}{k^2} - 3\frac{(\vec{k}\cdot\vec{q})^2}{k^4} \right) \,. \tag{A.34}$$

We can explicitly see that, after we average over the angles, the squeezed limit at order $\mathcal{O}(q^2)$ vanishes.

Appendix B

Derivation of the Consistency Relation for Gravitational Waves

In this Appendix we are going to give some details on the derivation of the consistency relation for short scalars and a long gravitational wave. We want to compute the average over the long gravitational wave $\gamma_{mn}(x)$ of the variation of the scalar n-point function induced by the change of coordinates given in eq. (4.61). In order to simplify the notation let us define

$$C_{ij}(\vec{x}_+) = A_{ij}(\vec{x}_+) - 2B_{ijk}(\vec{x}_+)x_k^+, \quad \vec{P} = \sum_i \vec{k}_i, \quad \langle \zeta_{\vec{k}_1} \dots \zeta_{\vec{k}_n} \rangle = \delta(\vec{P})\mathcal{M} . \tag{B.1}$$

Under this change of coordinates, the transformation of the n-point function is given by

$$\delta\langle \zeta(\vec{x}_1) \dots \zeta(\vec{x}_n) \rangle = \sum_{a=1}^{n} \left(C_{ij}(\vec{x}_+)x_{aj}\partial_{ai} + B_{ijk}(\vec{x}_+)x_{aj}x_{ak}\partial_{ai} \right) \langle \zeta(\vec{x}_1) \dots \zeta(\vec{x}_n) \rangle . \tag{B.2}$$

We can compute it term by term. The contribution from dilations is given by

$$\delta\langle \zeta(\vec{x}_1) \dots \zeta(\vec{x}_n) \rangle \supset \sum_{a=1}^{n} C_{ij}(\vec{x}_+)x_{aj}\partial_{ai}\langle \zeta(\vec{x}_1) \dots \zeta(\vec{x}_n) \rangle$$

$$= \int \mathrm{dK}_n \sum_a C_{ij}(\vec{x}_+)\delta(\vec{P})\mathcal{M}k_{ai}\partial_{k_{aj}}e^{i\vec{k}_1 \cdot \vec{x}_1 + \dots + i\vec{k}_n \cdot \vec{x}_n}$$

$$= \int \mathrm{dK}_n \delta(\vec{P}) \left(-\sum_a C_{ij}(\vec{x}_+)k_{ai}\partial_{k_{aj}}\mathcal{M} \right) e^{i\vec{k}_1 \cdot \vec{x}_1 + \dots + i\vec{k}_n \cdot \vec{x}_n} . \tag{B.3}$$

Here we used the fact that the tensor C_{ij} is traceless and that a derivative on the delta function can be rewritten as $\partial_{P_j}\delta(\vec{P})$ leading to a term proportional to $P_i\delta(\vec{P})$. Let us now use the explicit expression for C_{ij}

$$C_{ij}(\vec{x}_+) = \frac{1}{2}(1 - x_k^+ \partial_k^+)\gamma_{ij}(\vec{x}_+) + \frac{1}{2}\left(x_k^+ \partial_i^+ \gamma_{jk}(\vec{x}_+) - x_k^+ \partial_j^+ \gamma_{ik}(\vec{x}_+) \right) \tag{B.4}$$

and focus on the first term. Averaging over the long background mode $\gamma_{mn}(x)$ we get

$$\langle \gamma_{mn}(x)\zeta(\vec{x}_1)\dots\zeta(\vec{x}_n)\rangle \supset \int \frac{d^3\vec{q}}{(2\pi)^3} dK_n \delta(\vec{P}) \left(-\sum_a k_{ai}\partial_{k_{aj}}\mathcal{M}\right)$$

$$\times \sum_s \epsilon^s_{mn}(\vec{q})\epsilon^s_{ij}(\vec{q})\frac{H^2}{q^3}\frac{1}{2}(1 - x^+_k\partial^+_k)e^{i\vec{k}_1\cdot\vec{x}_1+\dots+i\vec{k}_n\cdot\vec{x}_n}e^{i\vec{q}\cdot\vec{x}-i\vec{q}\cdot\vec{x}_+} . \tag{B.5}$$

Notice that $(1 - x^+_k\partial^+_k)e^{-i\vec{q}\cdot\vec{x}_+} = 1 + \mathcal{O}(q^2)$ which proves that there is no dependence of the result on a choice of the point $\vec{x}_+$ at first order in the small momentum $\vec{q}$. The second term of eq. (B.4) has an additional dependence on $\vec{x}_+$ but it gives no contribution. After the averaging over the long background mode $\gamma_{mn}(x)$ the result is

$$\langle \gamma_{mn}(x)\zeta(\vec{x}_1)\dots\zeta(\vec{x}_n)\rangle \supset \int \frac{d^3\vec{q}}{(2\pi)^3} dK_n \delta(\vec{P}) \left(-\sum_a k_{ai}\partial_{k_{aj}}\mathcal{M}\right)$$

$$\times \sum_s \epsilon^s_{mn}(\vec{q})\frac{H^2}{q^3}\frac{1}{2}\left(iq_j\epsilon^s_{ik}(\vec{q}) - iq_i\epsilon^s_{jk}(\vec{q})\right)x^+_k e^{i\vec{k}_1\cdot\vec{x}_1+\dots+i\vec{k}_n\cdot\vec{x}_n}e^{i\vec{q}\cdot\vec{x}-i\vec{q}\cdot\vec{x}_+} . \tag{B.6}$$

We can rewrite x^+_k in front of the exponent like $i\partial_{q_k}$ and do a partial integration. In the first line in the integral nothing depends on $\vec{q}$. The action on the second line will produce three terms respectively proportional to

$$\sim \sum_s \left(\partial_{q_k}\epsilon^s_{mn}(\vec{q})\right)\frac{1}{q^3}\left(q_j\epsilon^s_{ik}(\vec{q}) - q_i\epsilon^s_{jk}(\vec{q})\right) ,$$

$$\sim \sum_s \epsilon^s_{mn}(\vec{q})\frac{q_k}{q^5}\left(q_j\epsilon^s_{ik}(\vec{q}) - q_i\epsilon^s_{jk}(\vec{q})\right) , \tag{B.7}$$

$$\sim \sum_s \epsilon^s_{mn}(\vec{q})\frac{1}{q^3}\left(\epsilon^s_{ij}(\vec{q}) + q_j\partial_{q_k}\epsilon^s_{ik}(\vec{q}) - \epsilon^s_{ji}(\vec{q}) - q_i\partial_{q_k}\epsilon^s_{jk}(\vec{q})\right) .$$

The first term will vanish when we multiply both sides of the final equation by $\epsilon^{*s}_{mn}(\vec{q})$ in order to express everything in terms of the amplitudes. It is easy to see that $\epsilon^{*s}_{mn}(\vec{q})\partial_{q_k}\epsilon^s_{mn}(\vec{q}) = 0$. The second term is trivially zero because polarization tensors are transverse. To prove that the last one is zero let us first note that one can write the polarization tensor like

$$\epsilon^{s_a}_{ij}(\vec{k}_a) = \frac{1}{\sqrt{2}}(\epsilon^P_{ij}(\vec{k}_a) + is_a\epsilon^X_{ij}(\vec{k}_a)) , \quad \epsilon^P_{ij}(\vec{k}_a) = z_iz_j - u^a_iu^a_j, \quad \epsilon^X_{ij}(\vec{k}_a) = z_iu^a_j + z_ju^a_i , \tag{B.8}$$

and unit vectors $\vec{z}$ and $\vec{u}_a$ are orthogonal to $\vec{k}_a$ and each other [40]. Using these expressions one can derive the following formula for the derivative of the polarization tensor

$$\partial_{k_j}\epsilon^s_{lr}(\vec{k}) = -\frac{1}{k^2}\left(k_l\epsilon^s_{jr}(\vec{k}) + k_r\epsilon^s_{lj}(\vec{k})\right) , \tag{B.9}$$

which one can use to complete the proof.

We can focus now on the second part of the coordinate transformation. The variation of the n-point function is

$$\delta\langle\zeta(\vec{x}_1)\dots\zeta(\vec{x}_n)\rangle \supset \sum_{a=1}^{n} B_{ijk}(\vec{x}_+)x_{aj}x_{ak}\partial_{ai}\langle\zeta(\vec{x}_1)\dots\zeta(\vec{x}_n)\rangle$$

$$= \int dK_n \sum_a (-i)B_{ijk}(\vec{x}_+)\delta(\vec{P})\mathcal{M}k_{ai}\partial_{k_{aj}}\partial_{k_{ak}}e^{i\vec{k}_1\cdot\vec{x}_1+\dots+i\vec{k}_n\cdot\vec{x}_n}$$

$$= -\int dK_n \sum_a iB_{ijk}(\vec{x}_+)k_{ai}\left(2\partial_{k_{ak}}\delta(\vec{P}) + \delta(\vec{P})\partial_{k_{aj}}\right)\partial_{k_{ak}}\mathcal{M}e^{i\vec{k}_1\cdot\vec{x}_1+\dots+i\vec{k}_n\cdot\vec{x}_n}, \qquad \text{(B.10)}$$

where we have used similar tricks as before to get the final result. It is convenient to express B_{ijk} as a sum of two terms:

$$B_{ijk}(\vec{x}_+) = \frac{1}{4}\left(\partial_k^+\gamma_{ij}(\vec{x}_+)\right) - \frac{1}{4}\left(\partial_i^+\gamma_{jk}(\vec{x}_+) - \partial_j^+\gamma_{ik}(\vec{x}_+)\right). \qquad \text{(B.11)}$$

The term containing a derivative of the delta function in the variation of the n-point function together with the first term in B_{ijk} give:

$$\langle\gamma_{mn}(x)\zeta(\vec{x}_1)\dots\zeta(\vec{x}_n)\rangle \supset \int\frac{d^3\vec{q}}{(2\pi)^3}dK_n\left(-\sum_a k_{ai}\partial_{k_{aj}}\mathcal{M}\right)$$

$$\times \vec{q}\cdot\nabla\delta(\vec{P})\sum_s \epsilon_{mn}^s(\vec{q})\epsilon_{ij}^s(\vec{q})\frac{H^2}{q^3}\frac{1}{2}e^{i\vec{k}_1\cdot\vec{x}_1+\dots+i\vec{k}_n\cdot\vec{x}_n}e^{i\vec{q}\cdot\vec{x}-i\vec{q}\cdot\vec{x}_+}. \qquad \text{(B.12)}$$

The form of this contribution is the same as for dilations. When they sum up the final result will be the same as for dilations but with a modified delta function $\delta(\sum\vec{k}_i + \vec{q})$. This means that in all computations the n-point function has to be computed slightly off-shell. The two additional terms that contain a derivative of the delta function will give a total contribution proportional to $\sum_a\left(k_{ai}\partial_{k_{aj}} - k_{aj}\partial_{k_{ai}}\right)\mathcal{M}$, which is zero because the correlation functions are rotationally invariant. Finally, the last part of the variation is

$$\delta\langle\zeta(\vec{x}_1)\dots\zeta(\vec{x}_n)\rangle \supset -\int dK_n\delta(\vec{P})\sum_a iB_{ijk}(\vec{x}_+)k_{ai}\partial_{k_{aj}}\partial_{k_{ak}}\mathcal{M}e^{i\vec{k}_1\cdot\vec{x}_1+\dots+i\vec{k}_n\cdot\vec{x}_n}. \qquad \text{(B.13)}$$

After averaging over the long mode it gives the following contribution

$$\langle\gamma_{mn}(x)\zeta(\vec{x}_1)\dots\zeta(\vec{x}_n)\rangle \supset \int\frac{d^3\vec{q}}{(2\pi)^3}dK_n\delta(\vec{P})\left(-\sum_a k_{ai}\partial_{k_{aj}}\partial_{k_{ak}}\mathcal{M}\right)$$

$$\sum_s \epsilon_{mn}^s(\vec{q})\frac{H^2}{q^3}\frac{1}{4}\left(q_k\epsilon_{ij}^s(\vec{q}) - q_i\epsilon_{jk}^s(\vec{q}) + q_j\epsilon_{ik}^s(\vec{q})\right)e^{i\vec{q}\cdot\vec{x}+i\vec{k}_1\cdot\vec{x}_1+\dots+i\vec{k}_n\cdot\vec{x}_n}. \qquad \text{(B.14)}$$

Summing up all nonzero terms and going to momentum space on the left hand side, we get the consistency relation for a background tensor mode:

$$\langle\gamma_{\vec{q}}^s\zeta_{\vec{k}_1}\dots\zeta_{\vec{k}_n}\rangle'_{q\to0} = -\frac{1}{2}P_\gamma(q)\sum_a \epsilon_{ij}^s(\vec{q})\left(k_{ai} + k_{ai}(\vec{q}\cdot\vec{\partial}_{k_a}) - \frac{1}{2}(\vec{q}\cdot\vec{k}_a)\partial_{k_{ai}}\right)\partial_{k_{aj}}\langle\zeta_{\vec{k}_1}\dots\zeta_{\vec{k}_n}\rangle'.$$

$$\text{(B.15)}$$

Appendix C

An Example of the Equivalence Principle Violation

In this section we are going to study a toy model in which the Universe is composed of two non-relativistic fluids A and B, with the latter coupled to a scalar field mediating a fifth force. For example, the two fluids could be baryons and dark matter but, with some modifications that we will discuss below, the model can also describe two populations of astrophysical objects, say different types of galaxies. If the scalar field φ has a negligible time evolution, the continuity equations of the two fluids are the same,

$$\delta'_X + \vec{\nabla} \cdot [(1 + \delta_X)\vec{v}_X] = 0 \, , \quad X = A, B \, . \tag{C.1}$$

The Euler equation of B contains the fifth force, whose coupling is parameterized by α,

$$\vec{v}'_A + \mathcal{H}\vec{v}_A + (\vec{v}_A \cdot \vec{\nabla}) \, \vec{v}_A = -\vec{\nabla}\Phi \, , \tag{C.2}$$

$$\vec{v}'_B + \mathcal{H}\vec{v}_B + (\vec{v}_B \cdot \vec{\nabla}) \, \vec{v}_B = -\vec{\nabla}\Phi - \alpha\vec{\nabla}\varphi \, , \tag{C.3}$$

where $\mathcal{H} \equiv \partial_\eta a/a$ is the comoving Hubble parameter. To close this system of equations we need Poisson's equation and the evolution equation of the scalar field. Assuming that the scalar field stress-energy tensor is negligible, only matter appears as a source in the Poisson's equation,

$$\nabla^2\Phi = 4\pi G \, \rho_{\mathrm{m}} \, \delta = 4\pi G \, \rho_{\mathrm{m}} \left(w_A\delta_A + w_B\delta_B \right) \, , \tag{C.4}$$

where ρ_{m} is the total matter density and $w_X \equiv \rho_X/\rho_{\mathrm{m}}$ is the density fraction of the X species. Moreover, in the non-relativistic approximation we can neglect time derivatives in comparison with spatial gradients and the equation for the scalar field reads

$$\nabla^2\varphi = \alpha \cdot 8\pi G\rho_{\mathrm{m}}w_B\delta_B \, , \tag{C.5}$$

where we have neglected the mass of the scalar field, assuming we are on scales much shorter than its Compton wavelength.

Let us start with the linear theory and, following [180], look for two of the four independent solutions of the system in which the density and the velocity of the species B differ from those of the species A by a (possibly time-dependent) bias factor b,

$$\delta_{\vec{k}}^{(A)}(\eta) = D(\eta)\,\delta_0(\vec{k})\,, \tag{C.6}$$

$$\theta_{\vec{k}}^{(A)}(\eta) = -\mathcal{H}(\eta)f(\eta)\delta_{\vec{k}}^{(A)}(\eta)\,, \tag{C.7}$$

$$\delta_{\vec{k}}^{(B)}(\eta) = b(\eta)\delta_{\vec{k}}^{(A)}(\eta)\,, \tag{C.8}$$

$$\theta_{\vec{k}}^{(B)}(\eta) = -\mathcal{H}(\eta)f(\eta)\delta_{\vec{k}}^{(B)}(\eta)\,, \tag{C.9}$$

where we have defined $\theta^{(X)} \equiv \vec{\nabla}\cdot\vec{v}_X$ and $\delta_0(\vec{k})$ is a Gaussian random variable. Plugging this ansatz in eqs. (C.1)–(C.5) and using the background Friedmann equations for a flat universe, we find, at linear order,

$$f = \frac{\mathrm{d}\ln D}{\mathrm{d}\ln a}\,, \tag{C.10}$$

$$\frac{\mathrm{d}f}{\mathrm{d}\ln a} + f^2 + \left(2 - \frac{3}{2}\Omega_m\right)f - \frac{3}{2}\Omega_m(w_A + w_B b) = 0\,, \tag{C.11}$$

$$\frac{\mathrm{d}b}{\mathrm{d}\ln a} = 0\,, \tag{C.12}$$

$$w_B b + w_A\left(1 - \frac{1}{b}\right) - w_B(1 + 2\alpha^2) = 0\,. \tag{C.13}$$

Using eqs. (C.10) and (C.11), the linear growth factor D satisfies a second-order equation,

$$\frac{\mathrm{d}^2 D}{\mathrm{d}\ln a^2} + \left(2 - \frac{3}{2}\Omega_m\right)\frac{\mathrm{d}D}{\mathrm{d}\ln a} - \frac{3}{2}\Omega_m(w_A + w_B b)D = 0\,, \tag{C.14}$$

whose growing and decaying solutions are D_+ and D_-. Note that eq. (C.12) implies that the bias b is time independent. In the absence of EP violation ($\alpha = 0$) we get $b = 1$ (using $w_A + w_B = 1$) and we recover from eq. (C.14) the usual evolution of the growth of matter perturbations.

Following [102, 181], we introduce $y \equiv \ln D_+$ as the time variable. Defining the field multiplet

$$\Psi_a \equiv \begin{pmatrix} \delta^{(A)} \\ -\theta^{(A)}/\mathcal{H}f_+ \\ \delta^{(B)} \\ -\theta^{(B)}/\mathcal{H}f_+ \end{pmatrix}\,, \tag{C.15}$$

the equations of motion of the two fluids can be then written in a very compact form as

$$\partial_y \Psi_a(\vec{k}) + \Omega_{ab}\Psi_b(\vec{k}) = \gamma_{abc}\Psi_b(\vec{k}_1)\Psi_c(\vec{k}_2)\,, \tag{C.16}$$

where integration over $\vec{k}_1$ and $\vec{k}_2$ is implied on the right-hand side. The entries of γ_{abc} vanish

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except for

$$\gamma_{121} = \gamma_{343} = (2\pi)^3 \delta_D(\vec{k} - \vec{k}_1 - \vec{k}_2) \frac{\vec{k}_1 \cdot (\vec{k}_1 + \vec{k}_2)}{k_1^2} \,,$$

$$\gamma_{222} = \gamma_{444} = (2\pi)^3 \delta_D(\vec{k} - \vec{k}_1 - \vec{k}_2) \frac{\vec{k}_1 \cdot \vec{k}_2 (\vec{k}_1 + \vec{k}_2)^2}{2k_1^2 k_2^2} \,,$$

(C.17)

the matrix Ω_{ab} reads

$$\Omega_{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -\frac{3}{2}\frac{\Omega_m}{f_+^2}w_A & \frac{3}{2}\frac{\Omega_m}{f_+^2}(w_A + bw_B) - 1 & -\frac{3}{2}\frac{\Omega_m}{f_+^2}w_B & 0 \\ 0 & 0 & 0 & -1 \\ -\frac{3}{2}\frac{\Omega_m}{f_+^2}w_A & 0 & -\frac{3}{2}\frac{\Omega_m}{f_+^2}(w_B b + w_A(1 - \frac{1}{b})) & \frac{3}{2}\frac{\Omega_m}{f_+^2}(w_A + bw_B) - 1 \end{pmatrix} \,,$$

(C.18)

and we have employed eq. (C.13) to replace the dependence on α^2 by a dependence on the bias b. The solution of eq. (C.16) can be formally written as

$$\Psi_a(y) = g_{ab}(y)\phi_b + \int_0^y \mathrm{d}y' g_{ab}(y - y')\gamma_{bcd}\Psi_c(y')\Psi_d(y') \,,$$

(C.19)

where ϕ_b is the initial condition, $\phi_b = \Psi_b(y = 0)$, and $g_{ab}(y)$ is the linear propagator which is given by [102]

$$g_{ab}(y) = \frac{1}{2\pi i} \int_{\xi - i\infty}^{\xi + i\infty} \mathrm{d}\omega \, (\omega I + \Omega)_{ab}^{-1} e^{\omega y} \,,$$

(C.20)

where ξ is a real number larger than the real parts of the poles of $(\omega I + \Omega)^{-1}$.

In the following we consider small couplings to the fifth force, $\alpha^2 \ll 1$, which by virtue of eq. (C.13) implies $b \simeq 1$. In this case, it is reasonable to use the approximation $f_+^2 \simeq \Omega_m$, which for $b = 1$ is very good throughout the whole evolution [182]. We choose to use this approximation because it considerably simplifies the presentation but one can easily drop it and make an exact computation. The linear evolution is characterized by four modes. Expanding for small $b - 1$, apart from the "adiabatic" growing and decaying modes already introduced above, respectively going as $D_+ = e^y$ and $D_- = e^{-\frac{3}{2}[1 + w_B(b-1)]y}$, one finds two "isodensity" modes, one decaying as $D_\mathrm{i} = e^{-\frac{1}{2}[1 + 3(1 + w_A)(b-1)]y}$ and an almost constant one going as $D_\mathrm{c} = e^{3w_A(b-1)\alpha^2 y}$.[1]

We are interested in the equal-time 3-point function involving the two species. In particular, we compute

$$\langle \delta_{\vec{k}_3}(\eta) \delta_{\vec{k}_1}^{(A)}(\eta) \delta_{\vec{k}_2}^{(B)}(\eta) \rangle = w_A \langle \Psi_1(k_3, \eta)\Psi_1(k_1, \eta)\Psi_3(k_2, \eta) \rangle + w_B \langle \Psi_3(k_3, \eta)\Psi_1(k_1, \eta)\Psi_3(k_2, \eta) \rangle \,,$$

(C.21)

---

[1]With an abuse of language, we denote the modes $(+)$ and $(-)$ as adiabatic and (i) and (c) as isodensity even though, strictly speaking, they do not correspond to the usual notion of adiabatic and isocurvature. Indeed, $(+)$ and $(-)$ correspond to $\delta_A = \delta_B/b$ and not to $\delta_A = \delta_B$ as in the standard adiabatic case, while (i) and (c) yield $w_A\delta^{(A)} + bw_B\delta^{(B)} = 0$ instead of $w_A\delta^{(A)} + w_B\delta^{(B)} = 0$ which one finds in the standard isodensity case (see [140] for a discussion of adiabatic and isodensity modes in the standard case $b = 1$).

where $\delta \equiv w_A \delta^{(A)} + w_B \delta^{(B)}$. The calculation can be straightforwardly done at tree level by perturbatively expanding the solution (C.19) as $\Psi_a = \Psi_a^{(1)} + \Psi_a^{(2)} + \ldots$, which up to second order in δ_0 yields

$$
\begin{aligned}
\Psi_a^{(1)}(y) &= g_{ab}(y)\phi_b \,, \\
\Psi_a^{(2)}(y) &= \int_0^y \mathrm{d}y' g_{ab}(y-y')\gamma_{bcd}\Psi_c^{(1)}(y')\Psi_d^{(1)}(y') \,,
\end{aligned}
\tag{C.22}
$$

and by applying Wick's theorem over the Gaussian initial conditions. In the squeezed limit, the expression for (C.21) simplifies considerably. Assuming that the initial conditions are in the most growing mode, i.e. they are given by $\phi_a(\vec{k}) = u_a \delta_0(\vec{k})$ with $u_a = (1,1,b,b)$, at leading order in $b-1$ one finds

$$
\lim_{q\to 0}\langle \delta_{\vec{q}}(\eta)\delta_{\vec{k}_1}^{(A)}(\eta)\delta_{\vec{k}_2}^{(B)}(\eta)\rangle \simeq -(b-1)P(q,\eta)P(k,0)\frac{\vec{k}\cdot\vec{q}}{q^2}\int_0^y \mathrm{d}y' e^{2y'}\left[g_{11}+g_{12}-g_{31}-g_{32}\right](y-y') \,,
\tag{C.23}
$$

which shows that the long wavelength adiabatic evolution has no effect on the 3-point function[2] [140, 141]. Retaining the most growing contribution and using $b \simeq 1+2w_B\alpha^2$ one finally finds

$$
\lim_{q\to 0}\langle \delta_{\vec{q}}(\eta)\delta_{\vec{k}_1}^{(A)}(\eta)\delta_{\vec{k}_2}^{(B)}(\eta)\rangle' \simeq \frac{7}{5}w_B\,\alpha^2\,\frac{\vec{k}\cdot\vec{q}}{q^2}P(q,\eta)P_{AB}(k,\eta)\,.
\tag{C.24}
$$

---

[2]For $b=1$ one finds $g_{11}^{(+)} = g_{31}^{(+)}$, $g_{12}^{(+)} = g_{32}^{(+)}$, $g_{11}^{(-)} = g_{31}^{(-)}$ and $g_{12}^{(-)} = g_{32}^{(-)}$.

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