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Applications of the functional renormalization group in curved spacetime

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Foreword

Quantum field theory is the underlying framework of most of our progress in modern particle physics and has been succesfully applied also to statistical mechanics and cosmology. A basic concept of quantum field theory is the renormalization group which describes how physics changes according to the energy at which we probe the system. The functional renormalization group (fRG) for the effective average action (EAA) describes the Wilsonian integration of high momentum modes without expanding in any small parameter. As such this is a non–perturbative framework and can be used to obtain non–perturbative insights, even though some other approximations are necessary. However we are not assured that quantum field theory is the correct framework to describe physics up to arbitrary high energies. This may happen if the theory approaches an ultra violet fixed point so that all physical quantities remain finite. In this case predictivity requires a finite number of relevant directions in such a way that only a finite number of parameters needs to be fixed by the experiments.

In this thesis we consider the fRG to address several problems. In chapter 1 we briefly review the fRG for the EAA deriving its flow equation and describing how theories with local symmetries can be handled and possible strategies of computation. In chapter 2 we describe how this framework can be used to investigate whether a quantum theory of gravity can be consistently built within the framework of standard quantum field theory. In particular we consider a new approximation of the flow equation for the EAA where the difference between the anomalous dimension of the fluctuating metric and the Newton's constant is taken into account. In chapter 3 we show that Weyl invariance can be maintained along the flow if a dilaton is present and if a judicious choice of the cutoff is made. This seems to contradict the standard lore according to which the renormalization group breaks Weyl invariance introducing a mass scale which is the origin of the so called trace anomaly. We analyze this in detail and show that standard results can be reobtained in a specific choice of gauge. Finally in chapter 4 we discuss a global feature of the renormalization group in two dimensions: the c–theorem. This is a global feature of the RG since it regards the whole RG trajectory from the UV to the IR. In particular we derive an exact equation for the c–function and, with some approximations, compute it explicitly in some examples. This also leads to some insights about a generic form of a truncation for the EAA. Some background material and technical details are confined to several appendices at the end of the thesis.

Contents

CHAPTER 1

Functional Renormalization Group Equations

The renormalization group (RG) is a key concept in statistical mechanics and quantum field theory. The functional Renormalization Group (fRG) is a convenient framework which we can use to construct and define a quantum field theory. In particular we consider the Effective Average Action (EAA) functional whose dependence on a scale k is known to satisfy an exact equation [1, 2]. This equation is often called Exact Renormalization Group Equation (ERGE) which, being non–perturbative in nature, can be used to address even non–perturbative problems. Nevertheless in actual computations some approximations must be implemented choosing a particular ansatz for the EAA. In this chapter we introduce the basis of this formalism: we first derive the scale dependence of the EAA and then discuss how this framework can be extended to handle theories with local symmetries. Finally we review some types of approximations such as the vertex expansion and the loop expansion. In appendix A we describe the generic features of the Wilsonian renormalization group and derive two other exact equations for different functionals.

1.1 Functional Renormalization Group Equations

In this section we consider the Wilsonian RG for quantum field theories. The basic idea is to implement the coarse graining directly at the level of the action adding a suitable term which takes care of restricting the integration to the high momentum modes. Looking at the integration of an infinitesimal momentum shell one can deduce different functional equations according to the type of functional one considers. First we derive an exact equation for a scale dependent generalization of the effective action, called Effective Average Action (EAA). This functional will be the central object that we shall study. In appendix A we consider an alternative derivation of the same equation and review the equation coming from the generating functional of the

connected Green's functions.

As we will see these equations are exact since no approximations are made in deriving them. As such these equations are non–perturbative and lead to non–perturbative predictions even if some approximations may be necessary in practical computations. The derivation of all these equations is based on the following reasoning: let us suppose we are given a Hamiltonian H_Λ. To perform an RG transformation a la Wilson we need to integrate out a shell of modes (from Λ to $\Lambda - \Delta\Lambda$); one can do this for an infinitesimal difference and consider the result of $H_\Lambda - H_{\Lambda-\delta\Lambda}$ in the limit $\delta\Lambda \to 0$. This expression gives an exact integro-differential equation for H_Λ.

The study of the flows of these equations have found application ranging from statistical mechanics to quantum gravity and the main applications and formal aspects can be found in many reviews [3–9].

1.2 The Effective Average Action and its Exact Renormalization Group Equation

In this thesis we use a scale dependent generalization of the effective action, called effective average action (EAA) and denoted Γ_k [1,2]. From the computational point of view this functional has some practical advantages with respect to those used in other exact equations (see section A.3). Moreover the scheme is very intuitive and it is very easy to retrieve a 1-loop calculation (and more: we will see that the EAA contains the loop expansion, cfr. section 1.4.2). An important property that the EAA satisfies is that it interpolates between the bare action S and the effective action Γ in the following way: $\Gamma_{k=\Lambda} = S$ and $\Gamma_{k=0} = \Gamma$.

Scale dependence of the Effective Average Action

Let us introduce the EAA formalism for a scalar field in order not to deal with gauge-fixing and other complications for the time being. The idea is to modify the generating functional of connected Green's functions in such a way that momentum modes higher than the scale k are integrated without any suppression while the others contribute with a reduced weight depending on the implementation used. The new functional $W_k[J]$ is defined:

$$e^{W_k[J]} \equiv \int \mathcal{D}\chi \exp\left\{-S[\chi] - \Delta_k S[\chi] + \int d^dx \sqrt{g}\chi J\right\} \tag{1.1}$$

where the factor $\Delta_k S$ has the purpose to suppress low momentum modes and is quadratic in the field:

$$\Delta_k S[\chi] = \frac{1}{2}\int d^dx \sqrt{g}\chi R_k(\Delta)\chi. \tag{1.2}$$

The shape of R_k is arbitrary except is overall behaviour:

$$R_k(\Delta) \approx \begin{cases} k^2 & \Delta < k^2 \\ 0 & \Delta > k^2 \end{cases} . \tag{1.3}$$

Moreover the function R_k has to be monotonically decreasing in Δ. Note that the modes lower than k are suppressed giving them a mass. In full analogy with the usual generating functionals in quantum field theory we consider the Legendre transform $\tilde{\Gamma}_k$ of W_k, let $\varphi = \langle\chi\rangle$:

$$\tilde{\Gamma}_k[\varphi] \equiv \Gamma_k[\varphi] + \Delta S_k[\varphi] = \int d^dx\sqrt{g}J(\varphi)\varphi - W_k[J(\varphi)]. \tag{1.4}$$

Note that $\tilde{\Gamma}_k$ and W_k satisfy the same relations of the similar generating functionals in standard QFT. In particular we recall that:[1]

$$e^{-\Gamma[\varphi]} = \int \mathcal{D}\chi e^{-S[\varphi+\chi]+\int \frac{\delta\Gamma}{\delta\varphi}\chi}, \qquad \langle\chi\rangle = 0. \tag{1.5}$$

Thus we can write:

$$e^{-\Gamma_k[\varphi]} = \int \mathcal{D}\chi \exp\left\{-S[\varphi+\chi] - \Delta S_k[\varphi+\chi] + \int dx\sqrt{g}\left(\frac{\delta\Gamma_k[\varphi]}{\delta\varphi} + \frac{\delta\Delta S_k[\varphi]}{\delta\varphi}\right)\chi + \Delta S_k[\varphi]\right\},$$

which has to be considered together with the condition $\langle\chi\rangle = 0$. We have:

$$-\Delta S_k[\varphi+\chi] + \int dx\sqrt{g}\frac{\delta\Delta S_k[\varphi]}{\delta\varphi}\chi + \Delta S_k[\varphi] = -\Delta S_k[\chi]$$

and inserting this into the previous expression we finally get:

$$e^{-\Gamma_k[\varphi]} = \int \mathcal{D}\chi \exp\left\{-S[\varphi+\chi] - \Delta S_k[\chi] + \int dx\sqrt{g}\frac{\delta\Gamma_k[\varphi]}{\delta\varphi}\chi\right\}. \tag{1.6}$$

Since ΔS_k vanishes for $k = 0$ we have

$$\lim_{k\to 0}\Gamma_k[\varphi] = \Gamma[\varphi] \tag{1.7}$$

and recover the standard definition of effective action in QFT. The opposite limit is related to the bare action

$$\lim_{k\to\infty}\Gamma_k[\varphi] = S[\varphi]. \tag{1.8}$$

A simple argument goes as follows: the cutoff action for $k \to \infty$ is such that $\Delta S_k \sim k^2\chi^2$. If we redefine the fluctuation via $\chi \to (k_0/k)\chi$ we have

$$\begin{aligned} \lim_{k\to\infty} e^{-\Gamma_k[\varphi]} &= \lim_{k\to\infty}\int \mathcal{D}\chi \exp\left\{-S\left[\varphi + \frac{k_0}{k}\chi\right] - \frac{k_0^2}{k^2}\Delta S_k[\chi] + \frac{k_0}{k}\int dx\sqrt{g}\frac{\delta\Gamma_k[\varphi]}{\delta\varphi}\chi\right\} \\ &= Ce^{-S[\varphi]}. \end{aligned}$$

[1] To see this consider $e^{W_J} = \int \mathcal{D}\chi\exp(-S[\chi] + \int J\chi) = \int \mathcal{D}\chi\exp(-S[\chi] + \int\frac{\delta\Gamma}{\delta\varphi}\chi)$. Moreover $-\Gamma[\varphi] = -(\int J\varphi - W) = \log\left[\int \mathcal{D}\chi\exp(-S[\chi] + \int\frac{\delta\Gamma}{\delta\varphi}(\chi-\varphi))\right]$ and using the invariance of the measure under translations we can shift $\chi \to \chi + \varphi$ we have $\exp(-\Gamma[\varphi]) = \int \mathcal{D}\chi\exp(-S[\varphi+\chi] + \int\frac{\delta\Gamma}{\delta\varphi}\chi)$

In the second line we use the fact that χ disappears from every term in the exponent except the second one which is proportional to χ^2 and can be integrated out to give a numerical constant. For further details see [10]. We will often refer to $\Gamma_{k=\Lambda}$ as the bare action.

Now the most important aspect of this functional is that it satisfies an exact equation which describe the EAA dependence on the cutoff scale k. Let us differentiate (1.6) with respect to $t = \log(k/\mu)$, we have:

$$-e^{-\Gamma_k}\partial_t\Gamma_k = \int \mathcal{D}\chi e^{\left\{-S[\varphi+\chi]-\Delta S_k[\chi]+\int dx\sqrt{g}\frac{\delta\Gamma_k[\varphi]}{\delta\varphi}\chi\right\}}\left[-\partial_t\Delta S_k[\chi]+\int dx\sqrt{g}\partial_t\frac{\delta\Gamma_k[\varphi]}{\delta\varphi}\chi\right], \quad (1.9)$$

which gives:

$$\begin{aligned}\partial_t\Gamma_k[\varphi] &= \langle\partial_t\Delta S_k[\varphi]\rangle - \int dx\sqrt{g}\partial_t\frac{\delta\Gamma_k[\varphi]}{\delta\varphi}\langle\chi\rangle \\ &= \frac{1}{2}\int dx\sqrt{g}\langle\chi\chi\rangle\partial_t R_k,\end{aligned}$$

where we used the fact that $\langle\chi\rangle = 0$. Moreover since $\langle\chi\rangle = 0$ we can consider the two–point function above as the connected one. Thus we can relate the two point function above to functional derivatives of W_k and so to the $\tilde{\Gamma}_k = \Gamma_k + \Delta S_k$ in the following way:

$$\langle\chi\chi\rangle = \left(\frac{\delta^2(\Gamma_k+\Delta S_k)}{\delta\varphi\delta\varphi}\right)^{-1} = \left(\frac{\delta^2\Gamma_k}{\delta\varphi\delta\varphi}+R_k\right)^{-1}.$$

Inserting this into the equation for $\partial_t\Gamma_k$ we finally have:

$$\partial_t\Gamma_k[\varphi] = \frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2\Gamma_k}{\delta\varphi\delta\varphi}+R_k\right)^{-1}\partial_t R_k. \quad (1.10)$$

The above equation is exact since no approximation has been made deriving it. The crucial ingredient to achieve this is the fact that the cutoff action is chosen to be quadratic in the fields. If it was not so further terms would be present and we would not have anymore the one loop structure of (1.10). It is indeed very easy to retrieve a standard one loop calculation from the above equation. To see this consider the one loop EAA:

$$\Gamma_k^{1loop} = S + \Delta S_k + \frac{1}{2}\mathrm{tr}\log\frac{\delta^2(S+\Delta S_k)}{\delta\varphi\delta\varphi} - \Delta S_k = S + \frac{1}{2}\mathrm{tr}\log\frac{\delta^2(S+\Delta S_k)}{\delta\varphi\delta\varphi}.$$

Taking the derivative of the above expression with respect to t we have:

$$\partial_t\Gamma_k^{1loop} = \frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2 S}{\delta\varphi\delta\varphi}+R_k\right)^{-1}\partial_t R_k.$$

The one loop flow equation has the same structure of (1.10) with the only difference that in the r.h.s. there is the bare action instead of the EAA itself. Therefore computing the r.h.s. of (1.10) requires the same effort of a one loop computation with the caveat that one should be as general as possible in considering the terms appearing in Γ_k. In the derivation the step of rescaling

the momenta to their initial values is implemented (via dimensional analysis) expressing all the dimensionful quantities in terms of k itself.

Finally let us stress that the FRGE (1.10) is independent of the bare action S and that when we integrate down to $k = 0$ we obtain the EA which is the object that we actually would like to know. The FRGE is a *functional* equation which in principle can be solved for and used to find the form of the EAA. This is generally impossible and one has to resort to some approximations that will be discussed in section 1.4.

1.3 The Effective Average Action for theories with local symmetries

We are interested in applying the above exact functional equation also when a theory possesses local symmetries. This immediately brings two problems: the first one is that the EA is not a gauge invariant functional of its argument: only physical quantities like S-matrix elements are. The second one is how one can implement a cutoff in a gauge invariant fashion.

The first issue is solved using the background field method which is briefly reviewed in appendix B. For what concerns the coarse graining procedure if we applied the cutoff directly to the Laplacian $-\partial^2$ we would incur the following unpleasant situation: for a slowly varying gauge field $A(x)$ there is also a fast varying gauge-equivalent one related via a fast varying gauge parameter $\omega(x)$. Thus the cutoff procedure would lose somehow its intuitive meaning. In order to avoid this one organizes the modes according to the covariant Laplacian (or a similar operator which transforms nicely with respect to background gauge transformations). In the EAA formalism the requirement of a gauge covariant implementation of the cutoff boils down in being able to write a (background) gauge invariant cutoff action ΔS_k. This can be naturally accomplished extending the background field formalism not only to the gauge-fixing part but also to the cutoff action [11]. In this way the EAA becomes a gauge invariant functional of its arguments. Nevertheless the EAA still depends on the choice of background gauge condition. In order to have a completely gauge independent functional one should work with the Vilkovisky-De Witt formalism [12–14] which has been discussed in relation to the functional RG in [15, 16]. An alternative is to construct a gauge invariant functional RG equation as done in [17–19].

As we already said we will adopt the background field method to work in a gauge covariant manner. Since in this thesis we are mainly interested in gravity theories we will specialise the discussion to the metric field; nevertheless the same reasoning holds for any theory with local symmetries. In the background field formalism the metric field $g_{\mu\nu}$ is split into a fluctuating (quantum) part $h_{\mu\nu}$ and a background one $\bar{g}_{\mu\nu}$. Even if the fields $\bar{g}_{\mu\nu}$ and $h_{\mu\nu}$ appear in the combination $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ in the classical action, the gauge-fixing term and the gauge covariant

cutoff are quadratic in the fluctuations and break the "split symmetry"

$$\bar{g}_{\mu\nu} \rightarrow \bar{g}_{\mu\nu} + m_{\mu\nu} \tag{1.11}$$

$$h_{\mu\nu} \rightarrow h_{\mu\nu} - m_{\mu\nu} \tag{1.12}$$

which is present in the combination $g_{\mu\nu}$. Thus the introduction of gauge-fixing and cutoff terms implies that the EAA is a generic functional $\Gamma_k[g, \bar{g}]$ or, equivalently, $\Gamma_k[h; \bar{g}]$. In general we can define

$$\Gamma_k[g, \bar{g}] = \bar{\Gamma}_k[g] + \hat{\Gamma}_k[g, \bar{g}]$$

where

$$\hat{\Gamma}_k[g, \bar{g}] \equiv \Gamma_k[g, \bar{g}] - \Gamma_k[g, g] .$$

We can also rewrite Γ_k as

$$\Gamma_k[h; \bar{g}] = \bar{\Gamma}_k[\bar{g} + h] + \hat{\Gamma}_k[h; \bar{g}] .$$

The functional $\hat{\Gamma}_k$ vanishes for $h = 0$ by construction and contains also the gauge–fixing term.

When we integrate down to $k = 0$ we must recover the split symmetry since the cutoff action disappears. In the case of gauge theories the situation is more complicated because the gauge-fixing part of the ansatz is also split symmetry breaking. What one should recover at $k = 0$ is instead the standard BRS symmetry, which for $k \neq 0$ is also broken by the cutoff action. Let us write explicitly the quantum gauge transformation as well as the background gauge transformations in the case of a metric theory of gravity. The gauge transformation is

$$\delta\bar{g}_{\mu\nu} = 0 , \qquad \delta h_{\mu\nu} = \mathcal{L}_\xi (\bar{g}_{\mu\nu} + h_{\mu\nu})$$

while the background gauge transformation is

$$\delta\bar{g}_{\mu\nu} = \mathcal{L}_\xi (\bar{g}_{\mu\nu}) , \qquad \delta h_{\mu\nu} = \mathcal{L}_\xi (h_{\mu\nu}) .$$

The functional $\bar{\Gamma}_k$ is called gauge invariant EAA (gEAA) since it is invariant under true gauge transformation besides the background ones. This follows from the fact that a functional which is invariant both under background gauge transformation and split symmetry is invariant under the quantum gauge transformations. We will refer to the functional $\hat{\Gamma}_k$ as the "remainder" term in the EAA (rEAA).

It is clear that the EAA is invariant under the second transformation and not under the first one. Of course to control the split symmetry by means of modified Ward identities provides a non-perturbative check of the reliability of the computations. Imposing the restoration of the

split symmetry at $k = 0$ impose severe constraints on the initial conditions of the couplings belonging to $\hat{\Gamma}_k$. For a recent discussion on the role of the modified Ward identity associated with the split symmetry we refer the reader to [20].

At the level of ansatzs one distinguishes between the "single metric" truncations and the "bimetric" truncations. The bimetric truncations are the one in which the running of $\hat{\Gamma}_k$ is retained in some form. In the single metric case one takes a non–running $\hat{\Gamma}_k$ made only of the gauge–fixing terms. The discussion above tells us that RG flow constructed via the background field metric is inherently bimetric.

1.4 Approximation schemes for the Effective Average Action

The flow equation (1.10) is very difficult to solve in general. Indeed one typically has to choose an ansatz in order to do computations. There are various strategies and criteria that might be used to write down a sensible ansatz. Possible methods of organizing an expansion for the EAA are:

- truncation which takes into account a finite number of operators in the EAA.
- vertex expansion described in section 1.4.1.
- the derivative expansion: one expands the ansatz in derivatives and suppose that the higher is the order of the derivatives the less significative is the operator. Thus a derivative expansion looks like the following ansatz

$$\Gamma_k = \int \left[V_k(\varphi) + Z_k(\varphi) (\partial\varphi)^2 + W_k(\varphi) (\partial\varphi)^4 + \cdots \right].$$

- the loop expansion. Such expansion is analogous to the usual loop expansion in QFT and is reviewed in section 1.4.2.
- the curvature expansion. One expands assuming that the Riemann tensor is small.

1.4.1 Flow equation for the vertices

In this section we describe in some detail the vertex expansion which we will use later on. The EAA is expanded in functional derivatives (similarly to the EA) as:

$$\Gamma_k = \sum_{n=0}^{\infty} \int_{x_1 \cdots x_n} \frac{1}{n!} \Gamma^{(n)}_{k, x_1 \cdots x_n} [\bar{\varphi}] [\varphi(x_1) - \bar{\varphi}(x_1)] \cdots [\varphi(x_n) - \bar{\varphi}(x_n)]. \tag{1.13}$$

The flow equation for the n–point functions can be derived from the flow equation (1.10) taking functional derivatives of the arguments of the EAA. As can be observed from the computation

of the running two–point function below, the flow equation of a vertex is not closed since higher vertices are required. One typically stops the hierarchy at some order making some approximation for the vertices.

In this thesis we will never really consider a vertex expansion but we will write down some ansatz and eventually compute the running of the two–point function. As we have seen in section 1.3 the EAA is a generic functional of the background and the fluctuating metric. For this reason the vertices coming from functional derivatives with respect to $\bar{g}_{\mu\nu}$ are very different from those obtained from functional derivatives with respect to $h_{\mu\nu}$. Nevertheless the most delicate technical point is the evaluation of the functional derivative of the cutoff kernel. In the case of gauge theories this has been achieved using the non–local heat kernel expansion [21]. Let us write down explicitly the equations for the running of the various two–point functions:

$$\partial_t \Gamma_k^{(2;0)} = \mathrm{Tr}\, G_k\, \Gamma_k^{(3;0)} G_k\, \Gamma_k^{(3;0)} G_k \partial_t R_k - \frac{1}{2}\mathrm{Tr}\, G_k\, \Gamma_k^{(4;0)} G_k \partial_t R_k \tag{1.14}$$

$$\begin{aligned}\partial_t \Gamma_k^{(1;1)} = {} & \mathrm{Tr}\, G_k \left(\Gamma_k^{(2;1)} + R_k^{(1)}\right) G_k\, \Gamma_k^{(3;0)} G_k \partial_t R_k \\ & + \mathrm{Tr}\, G_k\, \Gamma_k^{(3;0)} G_k \left(\Gamma_k^{(2;1)} + R_k^{(1)}\right) G_k \partial_t R_k \\ & - \frac{1}{2}\mathrm{Tr}\, G_k \Gamma_k^{(3;1)} G_k \partial_t R_k - \frac{1}{2}\mathrm{Tr}\, G_k\, \Gamma_k^{(3;0)} G_k \partial_t R_k^{(1)}\end{aligned} \tag{1.15}$$

$$\begin{aligned}\partial_t \Gamma_k^{(0;2)} = {} & \mathrm{Tr}\, G_k \left(\Gamma_k^{(2;1)} + R_k^{(1)}\right) G_k \left(\Gamma_k^{(2;1)} + R_k^{(1)}\right) G_k \partial_t R_k \\ & - \frac{1}{2}\mathrm{Tr}\, G_k \left(\Gamma_k^{(2;2)} + R_k^{(2)}\right) G_k \partial_t R_k \\ & - \mathrm{Tr}\, G_k \left(\Gamma_k^{(2;1)} + R_k^{(1)}\right) G_k \partial_t R_k^{(1)} + \frac{1}{2}\mathrm{Tr}\, G_k \partial_t R_k^{(2)}\end{aligned} \tag{1.16}$$

where $\Gamma_k^{(n,m)}$ stands for n functional derivatives with respect to the fluctuations and m functional derivatives with respect to the background. It is typically convenient to set to zero the fluctuations after taking the functional derivatives, in this case we shall denote $\gamma_k^{(n,m)}$. We can represent diagramatically the equations (1.14,1.15,1.16) as in figure 1.1.

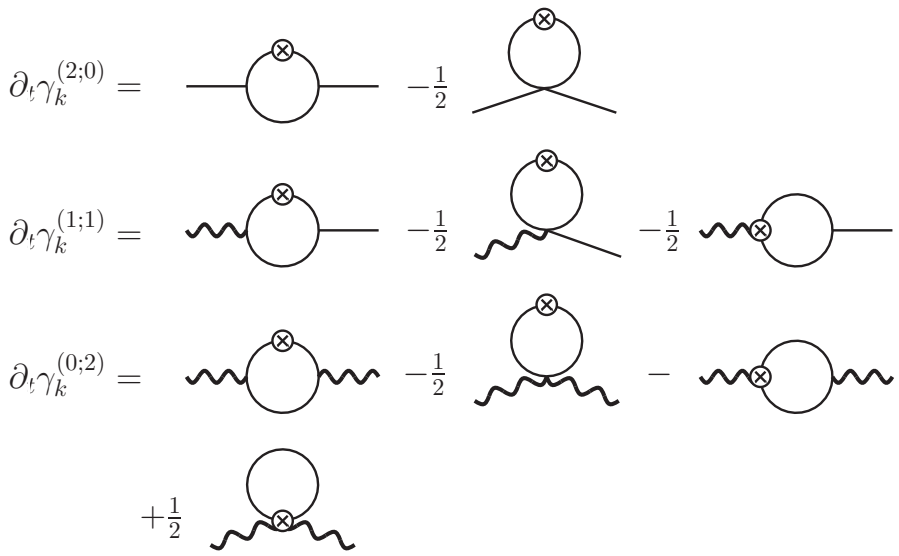


Figure 1.1: Diagrammatic representation of the flow equations for the vertices $\partial_t \gamma_k^{(2;0)}$, $\partial_t \gamma_k^{(1;1)}$ and $\partial_t \gamma_k^{(0;2)}$ as in equation (1.14). The lines in bold represent background "legs" while the others the fluctation "legs".

The expansion (1.13) in the context of gravity, where $\varphi - \bar{\varphi}$ is the fluctuating metric $h_{\mu\nu}$, is sometimes referred to as level expansion [22]. In this case the couplings proportional to the n–th power of the fluctuation are called level–n couplings. To stress the difference between single metric and bimetric computations let us expand a single monomial of the truncation in the fluctuation. For instance, limiting ourselves to terms with two derivatives, we can consider (in the following $G_k^{(n)}$ is a coupling and should not be confused with the regularized propagator G_k):

$$\begin{aligned}\Gamma\left[h,\bar{g}\right] \quad \sim \quad & \frac{1}{16\pi G_k^{(0)}}\int \sqrt{\bar{g}}R\left(\bar{g}\right) + \frac{1}{16\pi G_k^{(1)}}\int E^{\mu\nu}\left[\bar{g}_{\mu\nu}\right]h_{\mu\nu} \\ & + \frac{1}{16\pi G_k^{(2)}}\int \frac{1}{2}E^{\mu\nu\rho\sigma}\left[\bar{g}_{\mu\nu}\right]h_{\mu\nu}h_{\rho\sigma} + O(h^3).\end{aligned}$$

The couplings $G_k^{(0)}$ is referred to as level–0 coupling, $G_k^{(1)}$ as level–1 coupling and so on. In the single metric case one considers only the running of the gEAA so

$$\begin{aligned}\bar{\Gamma}_k\left[g\right] \quad \sim \quad & \frac{1}{16\pi G_k}\int \sqrt{g}R\left(g\right) = \frac{1}{16\pi G_k}\int \sqrt{\bar{g}}R\left(\bar{g}\right) + \frac{1}{16\pi G_k}\int \frac{\delta}{\delta g_{\mu\nu}}\left[\sqrt{\bar{g}}R\right]_{g_{\mu\nu}=\bar{g}_{\mu\nu}}h_{\mu\nu} \\ & + \frac{1}{16\pi G_k}\int \frac{1}{2}\frac{\delta^2}{\delta g_{\mu\nu}\delta g_{\rho\sigma}}\left[\sqrt{\bar{g}}R\right]_{g_{\mu\nu}=\bar{g}_{\mu\nu}}h_{\mu\nu}h_{\rho\sigma} + O(h^3).\end{aligned}$$

From the above expansion it is clear that in the single metric case all levels have the same running couplings. Due to the split–symmetry breaking this is not true and all couplings of different levels run independently.

1.4.2 Loop expansion from the Effective Average Action

A natural question is how the standard pertubative renormalization procedure can be understood in this framework. First let us recall that the standard EA yields a loop expansion (see [23] for a nice connection with the heat kernel formalism). The EAA also yields a loop expansion since the exact flow equation satisfied by the EAA can be solved iteratively via a loop expansion. If we choose as seed for the iteration the bare action, then the iteration procedure reproduces the renormalized loop expansion [24, 25]. In [25] an explicit connection between the exact flow scheme and the MS scheme has been given.

To see this let us introduce $\hbar$ as a loop counting parameter and expand the EAA:

$$\Gamma_k = S_\Lambda + \sum_{L=1}^{\infty} \hbar^L \Gamma_{L,k} \,. \tag{1.17}$$

One starts with $\Gamma_{0,k} \equiv S_\Lambda$, where S_Λ is the UV or bare action, and sets up an iterative solution (the subscript 0 indicates the order of the iteration, Λ is the UV cutoff and k is the RG scale) by plugging $\Gamma_{0,k}$ into the r.h.s. of the flow equation. Then one integrates the resulting differential equation with the boundary condition $\Gamma_{1,\Lambda} = S_\Lambda$. The solution $\Gamma_{k,1}$ is then plugged back into the r.h.s. of the flow equation and the procedure is repeated. The bare action is k–independent $\partial_t S_\Lambda = 0$. The exact flow equation (1.10) now takes the form:

$$\hbar\, \partial_t \Gamma_{1,k}\left[\varphi\right] + \hbar^2 \partial_t \Gamma_{2,k}\left[\varphi\right] + \ldots = \frac{\hbar}{2} \mathrm{Tr} \left[S_\Lambda^{(2)}\left[\varphi\right] + R_k + \hbar\, \Gamma_{1,k}^{(2)}\left[\varphi\right] + \hbar^2 \Gamma_{2,k}^{(2)}\left[\varphi\right] + \ldots \right]^{-1} \partial_t R_k \,. \tag{1.18}$$

The original flow equation (1.10) is finite both in the UV and IR: to maintain these properties the bare action S_Λ has to contain counterterms to cancel the divergences that may appear in the $\Gamma_{L,k}$. Thus we define:

$$S_\Lambda = S_0 + \sum_{L=1}^{\infty} \hbar^L \Delta S_{L,\Lambda} \,, \tag{1.19}$$

where each counterterm $\Delta S_{L,\Lambda}$ is chosen to cancel the divergent part of $\Gamma_{L,0}$. Since the divergent part of $\Gamma_{L,0}$ is the same as the divergent part of $\Gamma_{L,k}$ (we refer to [25] for more details on this point), this choice renders the denominator of (1.18) finite. Here S_0 is the renormalized action, i.e. the bare action with renormalized fields, masses and couplings. From (1.18) we can read off the flow of the L–th loop contribution:

$$\partial_t \Gamma_{L,k}\left[\varphi\right] = \frac{1}{(L-1)!} \frac{\partial^{L-1}}{\partial \hbar^{L-1}} \left. \frac{\partial_t \Gamma_k\left[\varphi\right]}{\hbar} \right|_{\hbar \to 0} \,. \tag{1.20}$$

The one–loop equation is straightforward:

$$\partial_t \Gamma_{1,k}\left[\varphi\right] = \frac{1}{2} \mathrm{Tr}\, G_k\left[\varphi\right] \partial_t R_k \,, \tag{1.21}$$

where the k–dependent renormalized propagator,

$$G_k[\varphi] = \frac{1}{S_0^{(2)}[\varphi] + R_k}\,, \tag{1.22}$$

depends on k only through the cutoff R_k. Thus, within the loop expansion, the operator

$$\tilde{\partial}_t \equiv (\partial_t R_k - \eta_k R_k)\frac{\partial}{\partial R_k}$$

is equivalent to ∂_t.

We can integrate the one–loop flow equation (1.21) between the UV and IR scales. We choose the UV initial condition $\Gamma_{L,\Lambda} = 0$ for $L > 0$ since the UV action is just the bare action. We find:

$$\begin{aligned}\Gamma_{1,k} &= -\int_k^\Lambda \frac{dk'}{k'}\partial_{t'}\Gamma_{1,k'} = -\frac{1}{2}\int_k^\Lambda \frac{dk'}{k'}\mathrm{Tr}\, G_{k'}\partial_{t'}R_{k'} \\ &= \frac{1}{2}\int_k^\Lambda dk'\mathrm{Tr}\,\partial_{k'}\log G_{k'} = \frac{1}{2}\mathrm{Tr}\log G_k\Big|_k^\Lambda .\end{aligned}$$

Note that in the second line we have exchanged the order of the trace and the derivative. This has been possible since we inserted an additional UV regulator Λ (one can also use dimensional regularization [25]). In the following all manipulations are intended with an implicit UV cutoff Λ.

We now choose $\Delta S_{L,\Lambda} = -[\Gamma_{L,0}]_{div}$ and define the renormalized one–loop contribution:

$$[\Gamma_{1,0}]_{ren} \equiv \lim_{\Lambda\to\infty}(\Gamma_{1,k} + \Delta S_{1,\Lambda}) = \frac{1}{2}[\mathrm{Tr}\log G_k]_{ren}\,.$$

Obviously, this limit is finite only if the theory is perturbatively renormalizable.

Now let us consider the two-loop contribution:

$$\partial_t\Gamma_{2,k} = \frac{\partial}{\partial\hbar}\frac{\partial_t\Gamma_k}{\hbar}\Big|_{\hbar\to 0} = -\frac{1}{2}\mathrm{Tr}\, G_k[\Gamma^{(2)}_{1,k}]_{ren}G_k\partial_t R_k = \frac{1}{2}\mathrm{Tr}[\Gamma^{(2)}_{1,k}]_{ren}\partial_t G_k.$$

We can plug in the one–loop result previously found. To do that we need to compute the Hessian $\Gamma^{(2)}_{1,k}$:

$$\Gamma^{(2)}_{1,k} = -\frac{1}{2}G_k S_0^{(3)}G_k S_0^{(3)} + \frac{1}{2}S_0^{(4)}G_k\,,$$

where we suppressed all indices. Using the above equation and

$$\tilde{\partial}_t G_k[\tau] = -G_k[\tau]\partial_t R_k[\tau]G_k[\tau] \qquad \tilde{\partial}_t\log G_k[\tau] = G_k^{-1}[\tau]\tilde{\partial}_t G_k[\tau] = -G_k[\tau]\partial_t R_k[\tau]\,. \tag{1.23}$$

we get:

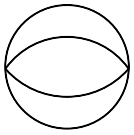
$$\begin{aligned}\partial_t\Gamma_{2,k} &= \frac{1}{2}\left[-\frac{1}{2}G_k S_0^{(3)}G_k S_0^{(3)} + \frac{1}{2}S_0^{(4)}G_k\right]^{ab}_{ren}[\partial_t G_k]^{ba} \\ &= \frac{1}{2}\partial_t\left[-\frac{1}{3\cdot 2}G_k^{cd}S_0^{(3)ade}G_k^{ef}S_0^{(3)bfc}G_k^{ab} + \frac{1}{2\cdot 2}S_0^{(4)abcd}G_k^{cd}G_k^{ab}\right]_{ren}\,,\end{aligned} \tag{1.24}$$

Integrating and renormalizing as before gives:

$$\Gamma_{2,k} = \left[-\frac{1}{12} G_k^{cd} S_0^{(3)ade} G_k^{ef} S_0^{(3)bfc} G_k^{ab} + \frac{1}{8} S_0^{(4)abcd} G_k^{cd} G_k^{ab} \right]_{ren} . \tag{1.25}$$

In the limit $k \to 0$ we recovered the usual two–loop result with the correct coefficients and in (nested) renormalized form. We can represent diagrammatically these contributions by adopting the same rules of section 1.4.1 with the difference that a continuous line represents a renormalized regularized propagator and vertices are constructed from the renormalized action $S_0^{(m)}$. To each loop we associate an integration $\int d^d x$ in coordinate space or $\int \frac{d^d q}{(2\pi)^d}$ in momentum space and we act overall with ∂_t. Proceeding along these lines all the standard loop expansion can be recovered at any loop order. From now on, for notational simplicity we will omit to explicitly report renormalized quantities with bracket, since these can be understood from the context.

Starting at three–loop order there are many different contributions. Here we show how to compute the following diagram,



that we will be interested in when we will compute the c-function via a loop expansion in chapter 4. We start from the following three–loop term flow:

$$\partial_t \Gamma_{3,k} = \frac{1}{2} \left(G_k^{ab} \Gamma_{1,k}^{(2)bc} G_k^{cd} \Gamma_{1,k}^{(2)de} G_k^{eg} - G_k^{ab} \Gamma_{2,k}^{(2)bc} G_k^{cg} \right) \partial_t R_k^{ga} . \tag{1.26}$$

We need the Hessian of the two-loop renormalized contribution, considering that we are interested only in the three-loop contribution in which there are two vertices $S_0^{(4)}$. Therefore we select:

$$\begin{aligned} \Gamma_{2,k}^{(2)mn} &= \frac{1}{12} \left[G_k^{cd} S_0^{(4)adem} G_k^{ef} S_0^{(4)bfcn} G_k^{ab} + G_k^{cd} S_0^{(4)aden} G_k^{ef} S_0^{(4)bfcm} G_k^{ab} \right] \\ &- \frac{1}{8} \left[\left(-G_k^{aa_1} S_0^{(4)a_1 a_2 mn} G_{k,a_2 b} \right) S_0^{(4)abcd} G_k^{cd} + G_k^{ab} S_0^{(4)abcd} \left(-G_k^{ca_1} S_0^{(4)a_1 a_2 mn} G_k^{a_2 d} \right) \right] . \end{aligned}$$

So we find:

$$\begin{aligned} \partial_t \Gamma_{3,k} &= \frac{1}{2} \left[G_k^{ab} \left(-\frac{1}{2} S_\Lambda^{(4)bca_1 a_2} G_k^{a_1 a_2} \right) G_k^{cd} \left(-\frac{1}{2} S_\Lambda^{(4)dea_3 a_4} G_k^{a_3 a_4} \right) G_k^{eg} \right. \\ &\left. - G_k^{ab} \Gamma_{2,k}^{(2)bc} G_k^{cg} \right] \partial_t R_k^{ga} ; \end{aligned}$$

recalling $-\partial_t G_k G_k^{(-1)} = G_k \partial_t R_k$ we pick up the contribution of the diagram we are interested

in:

$$\begin{aligned}
\partial_t \Gamma_{3,k} &= -\frac{1}{2} G_k^{qm} \left(\frac{1}{6} G_k^{cd} S_0^{(4)adem} G_k^{ef} S_0^{(4)bfcn} G_k^{ab} \right) (-\partial_t G_k^{nr}) G_k^{(-1)rq} + \cdots \\
&= -\frac{1}{2} \left(\frac{1}{6} G_k^{cd} S_0^{(4)adeq} G_k^{ef} S_0^{(4)bfcn} G_k^{ab} \right) (-\partial_t G_k^{nq}) + \cdots \\
&= \partial_t \left[\left(-\frac{1}{4 \cdot 12} \right) G_k^{qm} G_k^{cd} S_0^{(4)adem} G_k^{ef} S_0^{(4)bfcn} G_k^{ab} G_k^{nq} \right] + \cdots , \qquad (1.27)
\end{aligned}$$

where we used the cyclicity of the trace. Note that the symmetry factor of the three–loop contribution to the effective action is automatically recovered. Similarly one can easily obtain all the higher loop diagrams of this form.

CHAPTER 2

Improved closure of flow equation in Quantum Gravity

2.1 The Asymptotic Safety scenario for Quantum Gravity

Quantum Gravity (QG) is one of the most elusive open problem in physics. Indeed on one hand we lack some experimental hints which may guide us in the right direction and on the other hand the computations are generally rather involved. Nevertheless it is possible to accommodate quantum mechanics and general relativity in a consistent way in the effective field theory (EFT) framework [26]. This leads to consistent predictions such as the quantum corrections to the Newtonian potential [27].

The EFT quantization of gravity is achieved quantizing the metric which is the suitable field to describe the dynamics at low energy. Remaining in the framework of QFTs one may try to quantize gravity in a perturbative manner, namely around the gaussian fixed–point. Unfortunately this program is not successful since either the theory is not pertubatively renormalizable (Einstein-Hilbert action at two loops [28]) or it lacks unitarity (as in the case of higher derivative theories [29]). To overcome these problems one must resort to some new framework different from QFT or must go beyond perturbative techniques. In the former category we find discretized approaches like Causal Dynamical Triangulation and Loop Quantum Gravity as well as String Theory. In the latter approach instead one is somehow more conservative and attempts to properly define a gravitational path integral integrating the RG flow starting from a non–Gaussian UV fixed point. In order to have predictivity one requires a finite number of relevant directions which will correspond to the number of experiments to be performed to completely fix the theory. This is the approach that we shall be interested in and is usually referred to as "Asymptotic Safety" (AS). This scenario has been first proposed by Weinberg [30] and has found a more concrete framework when a non–perturbative flow equation for QG had been constructed [31].

The final aim of the AS program is to construct a well defined path integral for QG out

of which one can compute observable quantities. This is achieved integrating the flow of EAA down to $k = 0$. Clearly the piecemeal integral that we perform lowering the scale k requires some operator to which we can compare the scale k. In QFT this is typically offered by the Laplacian. Nevertheless, in the case of GR, we would like that our theory is background independent. This means that spacetime and in particular the metric must be a dynamical prediction of the theory and no metric should play any distinguished role a priori. The question is then which are the modes we integrate out in the RG flow? The Laplacian is built via some peculiar metric? In AS one solves the puzzle by employing the background field method: one chooses a background metric $\bar{g}_{\mu\nu}$ at the intermediate steps of the calculation and verifies that no physical predictions depend on the chosen metric (similarly to what happens in gauge theories with the gauge parameter). Therefore one divides the integration according to the eigenmodes of a covariant Laplacian built via this background metric. As we discussed in section 1.3 the price to pay is the bimetric character of the flow. In this chapter we discuss a possible way of taking this into account.

The AS scenario has now gained credit via many investigations which consistently find an UV attractive FP with a finite number of relevant directions. The flow of the EAA has been projected on subspaces of increasing complexity with a large number of operators. Evidence has been found for the presence of three relevant directions in polynomial truncations containing the Ricci scalar [32, 33] and higher derivative theories [34, 35]. Polynomials in the Ricci scalar have been pushed to higher order reinforcing this picture [33, 36]. Nevertheless the Weyl cube term has not been investigated so far. This is due to the technical difficulties but it is of utmost importance since the Goroff-Sagnotti divergence is proportional to this term [28]. So far the AS findings hold also in presence of a suitable number of matter fields [37, 38].

One of the biggest challenges of the AS program is to take into account infinitely many couplings. Indeed assume that we have found a fixed point with finitely many relevant directions in a truncation. When we add a new operator we may destroy the fixed point or change the number of relevant directions. To test the quality of the predictions one may rely on "consistency checks" like the stability of the critical exponents and the (approximate) modified Ward identities. Nevertheless one may consider ansatz with infinitely many couplings and it is clear that finding a FP solution of this type (scaling solution) would be a more convincing result. This has been achieved in statistical systems [39, 40]. In the case of gravity this has been addressed within the $f_k(R)$ approximation. Here one retains an infinite number of terms but neglects more complicated tensor structures like Riemann squared terms. The final analysis of this type of approximations seems not to be concluded [41–44] but there is hope that the scaling solution exists and has a finite number of relevant direction [44]. Of course a generalization to other tensor structure would be a very important result.

It is clear that there are two types of problems that have to be considered: on the one hand there is the bimetric character of the flow and on the other hand the problem of testing infinitely many couplings. The latter will not be addressed here but can hopefully be tackled exploiting the functional character of the flow equation. Instead we will consider in detail a possible way to deal with bimetric nature of the EAA.

First of all we want a formalism which naturally connects with the standard QFT approach. This is our main reason to adopt the parametrization of the EAA in terms of background and fluctuating metric, $\Gamma_k[h_{\mu\nu};\bar{g}_{\mu\nu}]$, instead that the one based on the background and the full metric $g_{\mu\nu}$: $\Gamma_k[g_{\mu\nu},\bar{g}_{\mu\nu}]$. In the usual application of the background field method (reviewed in appendix B) one sets the averaged fluctuation to zero and ends up in a gauge invariant functional which gives the correct scattering amplitudes. As we have seen in section 1.3 the EAA can be written as the sum of two parts:

$$\Gamma_k[h_{\mu\nu};\bar{g}_{\mu\nu}] = \bar{\Gamma}_k[\bar{g}_{\mu\nu}+h_{\mu\nu}] + \hat{\Gamma}_k[h_{\mu\nu};\bar{g}_{\mu\nu}].$$

If we set $h_{\mu\nu}=0$ we have that $\hat{\Gamma}_k=0$ and we are left with $\bar{\Gamma}_k$, the gEAA, from which we can generate scattering amplidutes. Therefore we are lead to assume that the "interesting physical couplings" are those contained in the gEAA while the other couplings are needed to accomodate the flow into a bigger space which is required by background independence and by the need of a covariant procedure. This definition of "physical couplings" coincides with the one proposed in [45–47]. Of course it may be natural to consider different ansatzs according to which fields are used to parametrize the EAA: $\Gamma_k[h_{\mu\nu};\bar{g}_{\mu\nu}]$ or $\Gamma_k[g_{\mu\nu},\bar{g}_{\mu\nu}]$. This has been further analysed recently in [22]. In this thesis we express the EAA as a functional of the background and of the fluctuation: $\Gamma_k[h_{\mu\nu};\bar{g}_{\mu\nu}]$. Moreover we consider the renormalized fields taking explicitly into account the wave function renormalization. As we said we are mainly interested in the couplings appearing in the gEAA whose running is found considering the flow of the EAA for $h_{\mu\nu}=0$:

$$\partial_t\bar{\Gamma}_k[0,\bar{g}_{\mu\nu}] = \frac{1}{2}\mathrm{Tr}\left(\Gamma_k^{(2,0)}[0,\bar{g}_{\mu\nu}]+R_k\right)^{-1}\partial_t R_k \tag{2.1}$$

where we used the notation introduced in section 1.4.1. Unfortunately this equation is not closed because the r.h.s. depends on the Hessian of the EAA which contains $\hat{\Gamma}_k^{(2,0)}[0,\bar{g}_{\mu\nu}]$. This term is not zero even in the simplest truncation where $\hat{\Gamma}_k$ is build just via the gauge-fixing.

A further problem in the AS scenario for QG is the check of unitarity. At present the question of whether unitarity is present or not is unanswered. Even though it is overwhelmingly likely that a UV FP solution contains higher derivatives this does not necessarily imply the presence of ghosts. First of all in this non–perturbative setting unitarity should be checked once the limit $k\to 0$ has been performed so that the full quantum theory is considered. The functional derivatives of the EA and its relation with the observables and the physical Hilbert space of the

theory tell us about the unitarity/positivity of the theory (for further details see [48]). There are three possible way-outs to the unitarity "problem":

- Non–trivial vacuum of the theory. The appearence of ghosts might be due to quantizing around the wrong vacuum. See e.g.: [49].
- The fixed-point action will presumably contain infinitely many non-zero couplings. As we said, to inspect if any ghost pole is present one has to consider the full propagator given by the EA. It may well be that a finite truncation of the effective action shows fictitious poles in the propagator, that are not present in the complete propagator.
- Running mass. Due to the running of its mass the ghost never really propagates [50] (see also [51]).

In this chapter we will consider the so called Einstein-Hilbert truncation in which we retain only two field monomials: a cosmological constant term and a term proportional to the Ricci scalar. Differently from what has been done in the literature we will not identify the anomalous dimension of the graviton $h_{\mu\nu}$ via $\eta_h = -\partial_t G_k/G_k$ but we will evaluate its running via an independent computation. This allows to close the equation without any ad hoc approximation such as the one we just mentioned. We will also evaluate the influence of the anomalous dimension of the ghosts on the beta functions of the gravitational couplings (also analyzed in [52, 53]). Indeed quantum fluctuations are responsible for the anomalous scaling of the fields; this fact is accounted for by introducing scale dependent wave–function renormalization constants for all the fluctuating fields present in the theory; in our case we redefine the fluctuating metric and the ghost fields according to:

$$h_{\mu\nu} \to Z_{h,k}^{1/2} h_{\mu\nu} \quad \bar{C}_\mu \to Z_{C,k}^{1/2} \bar{C}_\mu \quad C^\nu \to Z_{C,k}^{1/2} C^\nu$$

and we define the fluctuating metric and ghost anomalous dimensions:

$$\eta_{h,k} = -\partial_t \log Z_{h,k} \qquad \eta_{C,k} = -\partial_t \log Z_{C,k}\,.$$

As it will be clear from the computation this quantity enters in the flow equation and the choice $\eta_h = -\partial_t G_k/G_k$ correspond to limit ourselves to the so called single metric truncation.

2.2 The Einstein-Hilbert truncation

We will limit ourselves to the Einstein-Hilbert truncation. To be more precise our ansatz for the gEAA is given by

$$\bar{\Gamma}_k[g] = \frac{1}{16\pi G_k} \int d^d x \sqrt{g}\,(2\Lambda_k - R) \tag{2.2}$$

where

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \sqrt{32\pi G_k}\, h_{\mu\nu}\,. \tag{2.3}$$

We have re–scaled the fluctuating metric so that the combination $\sqrt{32\pi G_k}$ acts as the gravitational coupling, G_k being the scale dependent Newton's constant. In this way a gravitational vertex with n–legs is accompanied by a factor $(\sqrt{32\pi G_k})^{n-2}$. As we said the fluctuation field will also be re–scaled by the wave–function renormalization factor. As for the remainder $\hat{\Gamma}_k$ we consider an expansion to second power in $h_{\mu\nu}$ and first in $\bar{C}_\mu, C^\mu$:

$$\begin{aligned}\hat{\Gamma}_k[Z_{h,k}^{1/2}h, Z_{C,k}^{1/2}\bar{C}, Z_{C,k}^{1/2}C; \bar{g}] \;=\; & \frac{1}{2}Z_{h,k}\int d^dx\sqrt{\bar{g}}\left(h_{\mu\nu}h^{\mu\nu} - h^2\right)m_{h,k}^2 \\ & + \frac{1}{2\alpha_k}Z_{h,k}\int d^dx\sqrt{\bar{g}}\left(\bar{\nabla}^\alpha h_{\alpha\mu} - \frac{\beta_k}{2}\bar{\nabla}_\mu h\right)^2 \\ & - \frac{Z_{C,k}}{\sqrt{\alpha_k}}\int d^dx\sqrt{\bar{g}}\,\bar{C}_\mu\left[\bar{g}^{\mu\rho}\bar{g}^{\sigma\lambda}\bar{\nabla}_\lambda\left(g_{\rho\nu}\nabla_\sigma + g_{\sigma\nu}\nabla_\rho\right) - \beta_k\bar{g}^{\rho\sigma}\bar{g}^{\mu\lambda}\bar{\nabla}_\lambda\left(g_{\sigma\nu}\nabla_\rho\right)\right]C^\nu\end{aligned} \tag{2.4}$$

where the running of the gauge-fixing parameters α_k and β_k is made explicit. The action (2.4) amounts to an RG improvement of the usual gauge-fixing and ghost actions where the parameters are not allowed to run. Even if it is possible to evaluate the running of the gauge parameters, we will limit ourselves to consider the case where the gauge-fixing parameters are fixed to the values $\alpha_k = \beta_k = 1$. This is of course a further approximation. For the time being we also consider a sort of Pauli-Fierz mass $m_{h,k}$ which we will set to zero at some point of the calculation [54]. We stress that this is not a mass for the graviton, instead is a mass parameter which is there to accomodate the flow in the larger space required by the bimetric nature of EAA. This parameter should be endowed with the boundary condition of being zero at vanishing k to recover the split symmetry. Note that in (2.4) the ghost action involves both covariant derivatives in the full quantum metric ∇_μ and in the background metric $\bar{\nabla}_\mu$. This is because the ghost action comes from the (genuine) gauge variation of the gauge–fixing and such a variation is expressed in terms of derivatives of the full metric $g_{\mu\nu}$.

2.3 Beta function for the Cosmological and Newton's constants

In this section we derive the beta function of the Cosmological and Newton's constants. To read off the running of these couplings we simply consider the flow equation for the gEAA (2.2). Indeed plugging in our ansatz $h_{\mu\nu} = 0$ gives

$$\bar{\Gamma}_k[\bar{g}] = \frac{1}{16\pi G_k}\left(2\Lambda_k I_0\left(\bar{g}\right) - I_1\left(\bar{g}\right)\right) \tag{2.5}$$

where

$$I_0\left(\bar{g}\right) \equiv \int\sqrt{\bar{g}}, \qquad I_1\left(\bar{g}\right) \equiv \int\sqrt{\bar{g}}R\left(\bar{g}\right).$$

To extract the k-dependence from the r.h.s. of the flow eqution we will use standard heat kernel (HK) techniques which are reviewed in appendix C. Our calculation follows the one presented in [33]. Differentiating (2.5) with respect to the RG time gives:

$$\partial_t \bar{\Gamma}_k[\bar{g}] = \partial_t \left(\frac{\Lambda_k}{8\pi G_k} \right) I_0(\bar{g}) - \partial_t \left(\frac{1}{16\pi G_k} \right) I_1(\bar{g}) \ . \tag{2.6}$$

From (2.6) we see that we can extract the beta functions of the cosmological constant and of Newton's constant from those terms proportional to the invariants $I_0[g] = \int \sqrt{g}$ and $I_1[g] = \int \sqrt{g}R$ stemming from the expansion of functional traces on the rhs of the flow equation

$$\begin{aligned} \partial_t \bar{\Gamma}_k[\bar{g}] \quad = \quad & \frac{1}{2} \mathrm{Tr} \left(\Gamma_k^{(2,0,0,0)}[0,0,0;\bar{g}]^{\mu\nu}_{\alpha\beta} + R_k[\bar{g}]^{\mu\nu}_{\alpha\beta} \right)^{-1} \partial_t R_k[\bar{g}]^{\alpha\beta}_{\mu\nu} + \\ & - \mathrm{Tr} \left(\Gamma_k^{(0,1,1,0)}[0,0,0;\bar{g}]^{\mu}_{\nu} + R_k[\bar{g}]^{\mu}_{\nu} \right)^{-1} \partial_t R_k[\bar{g}]^{\mu}_{\nu} \end{aligned} \tag{2.7}$$

for the gEAA. Note that in (2.7) the cutoff kernels in the graviton and ghost sectors are distinguished by the indices. From here on we will consider only the gauge $\alpha = \beta = 1$ that simplifies the analysis since the Hessian operator $\Gamma_k^{(2)}$ is diagonalized and proportional to the Laplacian. Because of this we can avoid to decompose the field via the York decomposition or the use of the off-diagonal HK techniques. These important tools and their combined use are described in [55]. The Hessian of the EAA takes the following form:

$$\Gamma_k^{(2,0,0;0)}[h,\bar{C},C;\bar{g}]^{\mu\nu}_{\alpha\beta} = \bar{\Gamma}_k^{(2)}[\bar{g}+h]^{\mu\nu}_{\alpha\beta} + \hat{\Gamma}_k^{(2,0,0;0)}[h,\bar{C},C;\bar{g}]^{\mu\nu}_{\alpha\beta} \tag{2.8}$$

and

$$\Gamma_k^{(0,1,1;0)}[h,\bar{C},C;\bar{g}]^{\mu\nu} = Z_C S_{gh}^{(0,1,1;0)}[h,\bar{C},C;\bar{g}]^{\mu\nu} \ . \tag{2.9}$$

In (2.9) we used our ansatz for the rEAA given in equation (2.4). To calculate the gravitational Hessian needed in equation (2.7), we can extract the quadratic part in the fluctuation metric of the action (2.2) using equation (D.45) of appendix D:

$$\begin{aligned} \frac{1}{2} \int d^dx \sqrt{\bar{g}} h_{\mu\nu} \Gamma_k^{(2,0,0;0)}[h,\bar{C},C;\bar{g}]^{\mu\nu}_{\alpha\beta} h^{\alpha\beta} = \frac{1}{2} Z_h \int d^dx \sqrt{\bar{g}} \Big[& \frac{1}{2} h^{\mu\nu} \bar{\Delta} h_{\mu\nu} - \frac{1}{4} h \bar{\Delta} h \\ & + m_h^2 \left(h^{\alpha\beta} h_{\alpha\beta} - h^2 \right) - h^{\mu\nu} h^{\alpha}_{\mu} \bar{R}_{\nu\alpha} - h^{\mu\nu} h^{\alpha\beta} \bar{R}_{\alpha\mu\beta\nu} \\ & + h \bar{R}^{\mu\nu} h_{\mu\nu} + \left(\frac{1}{4} h^2 - \frac{1}{2} h^{\alpha\beta} h_{\alpha\beta} \right) (2\Lambda - \bar{R}) \Big] \ . \end{aligned} \tag{2.10}$$

The gravitational Hessian can now be easily extracted from (2.10) and reads:

$$\Gamma_k^{(2,0,0;0)}[0,0,0;g]^{\mu\nu}_{\rho\sigma} = \frac{Z_h}{2} \left[\delta^{\mu\nu}_{\alpha\beta} - \frac{1}{2} g^{\mu\nu} g_{\alpha\beta} \right] \left[\delta^{\alpha\beta}_{\rho\sigma} (\Delta + m_h^2 - 2\Lambda) + \frac{m_h^2}{d-2} g^{\alpha\beta} g_{\rho\sigma} + U^{\alpha\beta}_{\rho\sigma} \right] , \tag{2.11}$$

where we have introduced the symmetric spin two tensor identity $\delta^{\mu\nu}_{\rho\sigma} = \frac{1}{2}\left(\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} + \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho}\right)$ and the trace projector $P^{\mu\nu}_{\rho\sigma} = \frac{1}{d}g^{\mu\nu}g_{\rho\sigma}$ and we have defined the following tensor:

$$\begin{aligned} U^{\alpha\beta}_{\rho\sigma} &= \left(\delta^{\alpha\beta}_{\rho\sigma} - \frac{1}{2}g^{\alpha\beta}g_{\rho\sigma}\right)R + g^{\alpha\beta}R_{\rho\sigma} + R^{\alpha\beta}g_{\rho\sigma} + \\ &\quad -\frac{1}{2}\left(\delta^{\alpha}_{\rho}R^{\beta}_{\sigma} + \delta^{\alpha}_{\sigma}R^{\beta}_{\rho} + R^{\alpha}_{\rho}\delta^{\beta}_{\sigma} + R^{\alpha}_{\sigma}\delta^{\beta}_{\rho}\right) - \left(R^{\beta\ \ \alpha}_{\ \rho\ \ \sigma} + R^{\beta\ \ \alpha}_{\ \sigma\ \ \rho}\right) + \\ &\quad -\frac{d-4}{2(d-2)}\left(-R\,g^{\alpha\beta}g_{\rho\sigma} + g^{\alpha\beta}R_{\rho\sigma} + R^{\alpha\beta}g_{\rho\sigma}\right). \end{aligned} \tag{2.12}$$

We will sometimes suppress indices for notational clarity and we will use boldface symbols to indicate linear operators in the space of symmetric tensors. For example, the operators just defined will be indicated as $\mathbf{1}$, $\mathbf{P}$ and $\mathbf{U}$. Note that $\mathbf{1}-\mathbf{P}$ and $\mathbf{P}$ are orthogonal projectors into the trace and trace free subspaces in the space of symmetric tensors. With this notation we can rewrite (2.11) in the following way:

$$\mathbf{\Gamma}^{(2,0,0;0)}_{k}[0,0,0;g] = \frac{1}{2}Z_h\left[(\mathbf{1}-\mathbf{P}) - \frac{d-2}{2}\mathbf{P}\right]\left[\mathbf{1}(\Delta + m_h^2 - 2\Lambda) + m_h^2\frac{d}{d-2}\mathbf{P} + \mathbf{U}\right]. \tag{2.13}$$

The ghost action (2.4) when evaluated at zero fluctuation metric becomes:

$$S_{gh}[0,\bar{C},C;\bar{g}] = \int d^dx\sqrt{\bar{g}}\,\bar{C}^{\mu}\left[\bar{\Delta}\bar{g}_{\mu\nu} - (1-\beta)\bar{\nabla}_{\mu}\bar{\nabla}_{\nu} - \bar{R}_{\mu\nu}\right]C^{\nu}. \tag{2.14}$$

If we then set $\beta = 1$ in (2.14) we find the following ghost Hessian:

$$\Gamma^{(0,1,1;0)}_{k}[0,0,0;g]^{\mu}_{\nu} = Z_C\left(\Delta\delta^{\mu}_{\nu} - R^{\mu}_{\nu}\right). \tag{2.15}$$

For later use we report here the following traces of the tensors defined in (2.12) and before:

$$\mathrm{tr}\,\mathbf{1} = \frac{d(d+1)}{2} \qquad \mathrm{tr}\,\mathbf{P} = 1 \qquad \mathrm{tr}\,\mathbf{U} = \frac{d(d-1)}{2}R. \tag{2.16}$$

We have now to choose the cutoff operator that is used to separate the slow modes from the fast modes in the functional integral. We use the nomenclature of [33] and define: a type I cutoff as a kernel R_k function of the Laplacian $\Delta = -\nabla^2\mathbb{I}$, a type II cutoff as a kernel R_k function of $\Delta\mathbb{I} + \mathbb{E}$ (where $\mathbb{E}$ is an endomorphism) and type III cutoff as a kernel R_k function of the the full propagator $\Gamma^{(2)}_k$. In the following we shall consider the computations of the beta functions for the type I and II cutoffs and we refer to [33] for a complete analysis. For what concerns the function of the cutoff kernel we shall use the Litim's cutoff [56] which is particularly suitable for analytic calculations. The type II cutoff is considered only in this section for generality. Afterwards in this chapter we will always employ the type I cutoff.

Type I

We start to consider type I cutoff. We define the graviton cutoff kernel as:

$$\mathbf{R}_k(\Delta) = Z_h\left[(\mathbf{1}-\mathbf{P}) - \frac{d-2}{2}\mathbf{P}\right]R_k(\Delta), \tag{2.17}$$

while for the ghost cutoff kernel we take:

$$R_k(\Delta)^\mu_\nu = Z_C \delta^\mu_\nu R_k(\Delta)\,. \tag{2.18}$$

Remembering that the anomalous dimension of the fluctuation metric is defined by $\eta_h = -\partial_t \log Z_h$, we see that the flow equation (2.7) for the gEAA becomes:

$$\partial_t \bar{\Gamma}_k[g] = \frac{1}{2}\mathrm{Tr}\frac{\partial_t R_k(\Delta) - \eta_h R_k(\Delta)}{\mathbf{1}\left(\Delta + m_h^2 - 2\Lambda\right) + m_h^2 \frac{d}{d-2}\mathbf{P} + \mathbf{U}} - \mathrm{Tr}\frac{\partial_t R_k(\Delta) - \eta_C R_k(\Delta)}{\Delta\delta^{\mu\nu} - R^{\mu\nu}}\,. \tag{2.19}$$

Note that the wave-function renormalization factors in (2.19) have deleted each other leaving terms proportional to the anomalous dimension of the fluctuation metric and of the ghost fields.

Now we can use the freedom of choosing the background to isolate the monomials of interest. A convenient choice of the background metric is that of a d-dimensional sphere. [1] On the sphere the Riemann and Ricci tensors are proportional to the Ricci scalar:

$$R_{\mu\nu} = \frac{R}{d} g_{\mu\nu} \qquad\qquad R_{\mu\nu\rho\sigma} = \frac{R}{d(d-1)}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)\,. \tag{2.20}$$

Considering (2.20), the $\mathbf{U}$ tensor in (2.12) becomes simply:

$$\mathbf{U} = (\mathbf{1} - \mathbf{P})\frac{d^2 - 3d + 4}{d(d-1)}R + \mathbf{P}\frac{d-4}{d}R\,. \tag{2.21}$$

Using the fact that now (2.21) is decomposed in the orthogonal basis of the trace and trace–free projectors, we can easily re-express the Hessian (2.13) in the following way:

$$\begin{aligned}\mathbf{\Gamma}_k^{(2,0,0;0)}[0,0,0;g] \;=\; & \frac{1}{2}Z_h\left[(\mathbf{1}-\mathbf{P})\left(\Delta + m_h^2 - 2\Lambda + \frac{d^2-3d+4}{d(d-1)}R\right)\right.\\ & \left. -\frac{d-2}{2}\mathbf{P}\left(\Delta + 2\frac{d-1}{d-2}m_h^2 - 2\Lambda + \frac{d-4}{d}R\right)\right]\,.\end{aligned} \tag{2.22}$$

It is easy now to write down explicitly the full regularized graviton propagator:

$$\begin{aligned}&\left[\mathbf{1}\left(\Delta + m_h^2 - 2\Lambda + R_k(\Delta)\right) + m_h^2\frac{d}{d-2}\mathbf{P} + \mathbf{U}\right]^{-1} =\\ &\qquad = (\mathbf{1}-\mathbf{P})\frac{1}{\Delta + R_k(\Delta) + m_h^2 - 2\Lambda + \frac{d^2-3d+4}{d(d-1)}R}\\ &\qquad -\frac{2}{d-2}\mathbf{P}\frac{1}{\Delta + R_k(\Delta) + 2\frac{d-1}{d-2}m_h^2 - 2\Lambda + \frac{d-4}{d}R}\,.\end{aligned} \tag{2.23}$$

Equation (2.23) expresses the full regularized graviton propagator steming from our truncation (2.3) and (2.4) in the gauge $\alpha = \beta = 1$ when the background metric is a metric on the

[1] This choice allows an exact inversion of the Hessian. The same beta functions that we are going to derive can also be found expanding in a generic background with small curvature as shown in [33]. This holds up to the first order in the curvatures: for higher orders the spherical background is not sufficient to distinguish all the terms.

d-dimensional sphere. Note also, that there is a kinematical singularity in the regularized propagator (2.23) for $d = 2$. We can now define the trace and trace-free parts of the regularized graviton propagator on the d-dimensional sphere as follows:

$$\begin{aligned} G_{TF,k}(z) &= \frac{1}{z + R_k(z) + m_h^2 - 2\Lambda_k + \frac{d^2-3d+4}{d(d-1)}R} \\ G_{T,k}(z) &= \frac{1}{z + R_k(z) + 2\frac{d-1}{d-2}m_h^2 - 2\Lambda_k + \frac{d-4}{d}R}\,. \end{aligned} \tag{2.24}$$

Note that due to the presence of the Pauli–Fierz mass term, the trace and trace-free regularized propagators in (2.24) are different even at $R = 0$. The ghost regularized propagator on the d-dimensional sphere becomes simply:

$$G_{C,k} = \frac{1}{z + R_k(z) - \frac{R}{d}}\,. \tag{2.25}$$

The flow equation can thus be rewritten as

$$\begin{aligned} \partial_t \bar{\Gamma}_k[g] &= \frac{1}{2}\mathrm{Tr}(\mathbf{1}-\mathbf{P})(\partial_t R_k - \eta_h R_k)G_{TF,k} + \frac{1}{2}\mathrm{Tr}\mathbf{P}(\partial_t R_k - \eta_h R_k)G_{T,k} \\ &\quad -\mathrm{Tr}\delta^\mu_\nu(\partial_t R_k - \eta_C R_k)G_{C,k} \\ &= \frac{d^2+d-2}{4}\mathrm{Tr}_x(\partial_t R_k - \eta_h R_k)G_{TF,k} + \frac{1}{2}\mathrm{Tr}_x(\partial_t R_k - \eta_h R_k)G_{T,k} \\ &\quad -d\,\mathrm{Tr}_x(\partial_t R_k - \eta_C R_k)G_{C,k}\,. \end{aligned} \tag{2.26}$$

We evaluated the traces in (2.26) with the help of (2.16).

Collecting all terms of zeroth and first order in the scalar curvature that are present on the rhs of (2.26), stemming from the expansion of $G_{TF,k}$, $G_{T,k}$ and from the heat kernel expansion, we find :

$$\begin{aligned} \partial_t\left(\frac{\Lambda_k}{8\pi G_k}\right) &= \frac{1}{(4\pi)^{d/2}}\left\{\frac{d^2+d-2}{4}Q_{\frac{d}{2}}\left[(\partial_t R_k - \eta_h R_k)G_{TF,k}\right] + \frac{1}{2}Q_{\frac{d}{2}}\left[(\partial_t R_k - \eta_h R_k)G_{T,k}\right]\right. \\ &\quad \left. -dQ_{\frac{d}{2}}\left[(\partial_t R_k - \eta_C R_k)G_{C,k}\right]\right\} \end{aligned}$$

$$\begin{aligned} \partial_t\left(\frac{1}{16\pi G_k}\right) &= \frac{1}{(4\pi)^{d/2}}\left\{\frac{d^2+d-2}{24}Q_{\frac{d}{2}-1}\left[(\partial_t R_k - \eta_h R_k)G_{TF,k}\right]\right. \\ &\quad +\frac{1}{12}Q_{\frac{d}{2}-1}\left[(\partial_t R_k - \eta_h R_k)G_{T,k}\right] - \frac{d}{6}Q_{\frac{d}{2}-1}\left[(\partial_t R_k - \eta_C R_k)G_{C,k}\right] \\ &\quad -\frac{d^2-3d+4}{d(d-1)}\frac{d^2+d-2}{4}Q_{\frac{d}{2}}\left[(\partial_t R_k - \eta_h R_k)G^2_{TF,k}\right] \\ &\quad \left. -\frac{d-4}{2d}Q_{\frac{d}{2}}\left[(\partial_t R_k - \eta_h R_k)G^2_{T,k}\right] - Q_{\frac{d}{2}}\left[(\partial_t R_k - \eta_C R_k)G^2_{C,k}\right]\right\}\,. \end{aligned} \tag{2.27}$$

We can evaluate the beta functions (2.27) using the optimized cutoff shape function. In terms of the dimensionless couplings $\tilde{\Lambda} = k^{-2}\Lambda$, $\tilde{G} = k^{d-2}G$ and $\tilde{m}_h^2 = k^{-2}m_h$ we find the following

forms:

$$\begin{aligned}\partial_t\tilde{\Lambda} &= -2\tilde{\Lambda}+\frac{8\pi}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\left\{-4+\frac{d-1}{d}\frac{d+2-\eta_h}{1-2\tilde{\Lambda}+\tilde{m}_h^2}-\right.\\ &\quad\frac{2\left(d^2-3d+4\right)\tilde{\Lambda}}{d^2}\frac{2+d-\eta_h}{\left(1-2\tilde{\Lambda}+\tilde{m}_h^2\right)^2}+\frac{2}{d(d+2)}\frac{2+d-\eta_h}{1-2\tilde{\Lambda}+2\frac{d-1}{d-2}\tilde{m}_h^2}\\ &\quad\left.-\frac{4\left(d-4\right)\tilde{\Lambda}}{d^2(d+2)}\frac{2+d-\eta_h}{\left(1-2\tilde{\Lambda}+2\frac{d-1}{d-2}\tilde{m}_h^2\right)^2}-\frac{8\tilde{\Lambda}}{d(d+2)}\left(d+2-\eta_C\right)+\frac{4}{2+d}\eta_C\right\}\tilde{G}\\ &\quad+\frac{4\pi}{3(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\left\{-\frac{d^2+d-2}{d}\frac{d-\eta_h}{1-2\tilde{\Lambda}-\tilde{m}_h^2}\right.\\ &\quad\left.+\frac{2}{d}\frac{d-\eta_h}{1-2\tilde{\Lambda}+2\frac{d-1}{d-2}\tilde{m}_h^2}-4\left(d-\eta_C\right)\right\}\tilde{\Lambda}\tilde{G}\end{aligned}\tag{2.28}$$

and

$$\begin{aligned}\partial_t\tilde{G} &= (d-2)\tilde{G}+\frac{16\pi}{(4\pi)^{d/2}d^2\Gamma\left(\frac{d}{2}\right)}\left\{-\frac{4d}{d+2}\left(d+2-\eta_C\right)\right.\\ &\quad\left.-\left(d^2-3d+4\right)\frac{d+2-\eta_h}{\left(1-2\Lambda+\tilde{m}_h^2\right)^2}-\frac{2\left(d-4\right)}{d+2}\frac{d+2-\eta_h}{\left(1-2\tilde{\Lambda}+2\frac{d-1}{d-2}\tilde{m}_h^2\right)^2}\right\}\tilde{G}^2\\ &\quad+\frac{4\pi}{3(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\left\{\frac{d^2+d-2}{d}\frac{d-\eta_h}{1-2\tilde{\Lambda}+\tilde{m}_h^2}\right.\\ &\quad\left.+\frac{2}{d}\frac{d-\eta_h}{1-2\tilde{\Lambda}+2\frac{d-1}{d-2}\tilde{m}_h^2}-4\left(d-\eta_C\right)\right\}\tilde{G}^2\,.\end{aligned}\tag{2.29}$$

These beta functions represent the generalization of the beta functions for the dimensionless cosmological and Newton's constant in presence of a non–zero Pauli–Fierz mass.

Type II

We now turn to consider type II cutoff where we take as cutoff operators $\mathbf{\Delta}_2=\Delta\mathbf{1}+\mathbf{U}$ for the gravitons and $(\Delta_1)^\mu_\nu=\Delta\delta^\mu_\nu-R^\mu_\nu$ for the ghosts. The flow equation for the gEAA, at $m_h=0$, becomes now simply the following:

$$\partial_t\bar{\Gamma}_k[g]=\frac{1}{2}\mathrm{Tr}_{xL}\mathbf{G}_k(\mathbf{\Delta}_2)\partial_t\mathbf{R}_k(\mathbf{\Delta}_2)-\mathrm{Tr}_{xL}G_k(\Delta_1)^\mu_\nu\partial_t R_k(\Delta_1)^\nu_\mu\,.\tag{2.30}$$

It is now easy to evaluate the traces in (2.30) using the local heat kernel expansion. Using the following heat kernel coefficients for the cutoff operators we are considering

$$\mathrm{tr}\,b_2(\mathbf{\Delta}_2)=\mathrm{tr}\left[\mathbf{1}\frac{R}{6}-\mathbf{U}\right]=-\frac{d(5d-7)}{12}R\qquad\qquad\mathrm{tr}\,b_2(\Delta_1)=\mathrm{tr}\left[\delta^\mu_\nu\frac{R}{6}+R^\mu_\nu\right]=\frac{d+6}{d}R\,,$$

we find, to linear order in the curvature, the following expansion:

$$\begin{aligned}\partial_t\bar{\Gamma}_k[g] &= \frac{1}{(4\pi)^{d/2}}\int d^dx\sqrt{g}\left\{\frac{d(d+1)}{4}Q_{\frac{d}{2}}\left[(\partial_tR_k-\eta_hR_k)G_k\right]\right.\\ &\quad -d\,Q_{\frac{d}{2}}\left[(\partial_tR_k-\eta_CR_k)G_{C,k}\right]-\left[\frac{d(5d-7)}{24}Q_{\frac{d}{2}-1}\left[(\partial_tR_k-\eta_hR_k)G_k\right]\right.\\ &\quad \left.\left.\frac{d+6}{6}Q_{\frac{d}{2}-1}\left[(\partial_tR_k-\eta_CR_k)G_{C,k}\right]R\right\}+O\left(\mathcal{R}^2\right)\right. .\end{aligned} \tag{2.31}$$

From (2.31) we can extract the following relations that determine the beta functions of Λ and G:

$$\begin{aligned}\partial_t\left(\frac{\Lambda_k}{8\pi G_k}\right) &= \frac{1}{(4\pi)^{d/2}}\left\{\frac{d(d+1)}{4}Q_{\frac{d}{2}}\left[(\partial_tR_k-\eta_hR_k)G_k\right]\right.\\ &\quad \left.-d\,Q_{\frac{d}{2}}\left[(\partial_tR_k-\eta_CR_k)G_{C,k}\right]\right\}\\ \partial_t\left(\frac{1}{16\pi G_k}\right) &= \frac{1}{(4\pi)^{d/2}}\left\{\frac{d(5d-7)}{24}Q_{\frac{d}{2}-1}\left[(\partial_tR_k-\eta_hR_k)G_k\right]\right.\\ &\quad \left.+\frac{d+6}{6}Q_{\frac{d}{2}-1}\left[(\partial_tR_k-\eta_CR_k)G_{C,k}\right]\right\} .\end{aligned} \tag{2.32}$$

The same calculation goes through in the case of a non-zero mass. In this case we find:

$$\begin{aligned}\partial_t\left(\frac{\Lambda_k}{8\pi G_k}\right) &= \frac{1}{2(4\pi)^{d/2}}\left[\frac{(d+2)(d-1)}{2}Q_{d/2}\left[(\partial_tR_k-\eta_hR_k)\,G_{TF,k}\right]+Q_{d/2}\left[(\partial_tR_k-\eta_hR_k)\,G_{T,k}\right]\right.\\ &\quad \left.-d\,Q_{\frac{d}{2}}\left[(\partial_tR_k-\eta_CR_k)G_{C,k}\right]\right]\end{aligned} \tag{2.33}$$

$$\begin{aligned}\partial_t\left(-\frac{1}{16\pi G_k}\right) &= \frac{1}{2(4\pi)^{d/2}}\left[-\frac{(d+2)\left[d(5d-7)+24\right]}{12d}Q_{d/2-1}\left[(\partial_tR_k-\eta_hR_k)\,G_{TF,k}\right]\right.\\ &\quad \left.+\left(\frac{-5d+24}{6}\right)Q_{d/2-1}\left[(\partial_tR_k-\eta_hR_k)\,G_{T,k}\right]+\frac{d+6}{6}Q_{\frac{d}{2}-1}\left[(\partial_tR_k-\eta_CR_k)G_{C,k}\right]\right]\end{aligned} \tag{2.34}$$

2.4 Closure of the flow equations

The flow equations for the dimensionless Cosmological and Newton's constants depend on the anomalous dimensions of the graviton and the ghost fields and the graviton mass besides $\tilde{\Lambda}_k$ and $\tilde{G}_k$ themselves. From now on we will set the mass parameter to zero $m_{h,k}=0$ for simplicity but it will be clear how this can be easily retained in our formalism. This means that the equations of the couplings of the gEAA are not closed as expected from general considerations. In this section we analyze three possible ways of closing the equation:

- 1-loop beta functions: simply set: $\eta_{h,C}=0$.
- "standard closure": set $\eta_h=\partial_tG_k/G_k$ and $\eta_C=0$. This type of closure is the one mostly used so far.

- "improved closure": independent computation of η_h and η_C.

In the following we analyze these three possibilities considering the beta functions found via the cutoff of type I.

1-loop beta functions

As we said the one-loop beta functions can be retrieved setting the anomalous dimension to zero and expanding the denominator. We solve these equations for $\frac{d\tilde{\Lambda}}{dt}$ and $\frac{d\tilde{G}}{dt}$, obtaining

$$\begin{aligned}\frac{d\tilde{\Lambda}}{dt} &= -2\tilde{\Lambda} + \frac{1}{2}\frac{16\,\pi\,(d-3)}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}\tilde{G} + \frac{-4\pi(-d^3+15d^2-12d+48)}{3(4\pi)^{\frac{d}{2}}\,d\,\Gamma(\frac{d}{2})}\tilde{G}\tilde{\Lambda}\,,\\ \frac{d\tilde{G}}{dt} &= (d-2)\tilde{G} + \frac{-4\pi(-d^3+15d^2-12d+48)}{3(4\pi)^{\frac{d}{2}}\,d\,\Gamma(\frac{d}{2})}\tilde{G}^2\,,\end{aligned}$$

A detail comparison of these results with those we will see in the next subsection is done in [33]. The fact to be noticed is that the 1–loop computation already shows remarkably similar features to that of the "standard closure result" in the sense that a UV attractive fixed–point is found which is numerically close to the one obtained using the standard closure. For further detail we refer to [33].

Standard closure of the flow equations

Here we will close the flow equation setting:

$$\eta_{h,k} = \frac{\partial_t G_k}{G_k} = 2 - d + \frac{\partial_t \tilde{G}_k}{\tilde{G}_k} \qquad \eta_{C,k} = 0\,.$$

The above identification implies a non–trivial, but difficult to interpret, RG improvement of the beta functions. An important aspect of this closure is that via this definition the anomalous dimension at a non–trivial fixed point is always $2-d$ independently of the fixed–point [57].

Moreover this closure is equivalent to a single metric truncation. Indeed suppose that we take our Einstein–Hilbert truncation and expand this ansatz in $h_{\mu\nu}$, we would have schematically:

$$\Gamma_k\,[g] = \frac{1}{G_k}\gamma\,[\bar{g}] + \frac{1}{G_k}\gamma^{(1)}\,[\bar{g}]\,h + \frac{1}{G_k}\gamma^{(2)}\,[\bar{g}]\,h^2 + \cdots$$

where we wrote $\Gamma_k = G_k^{-1}\gamma$. This means that we are assuming that all the terms in the expansion, in particular the zero order and the second order ones, have in front the same running coefficient. Of course it is not so due to the breaking of the split–symmetry and all orders have a different running coefficient. This has been analyzed in gravity in [22, 45–47] and in gauge theories, see for instance [11, 58].

Making this improvement we obtain, for type I cutoff, the following beta functions from (2.28,2.29):

$$\begin{aligned}\partial_t\tilde{\Lambda}_k &= -2\tilde{\Lambda}_k + \frac{1}{6\pi}\frac{(3-4\tilde{\Lambda}_k-12\tilde{\Lambda}_k^2-56\tilde{\Lambda}_k^3)\tilde{G}_k+\frac{1}{12\pi}(107-20\tilde{\Lambda}_k)\tilde{G}_k^2}{(1-2\tilde{\Lambda}_k)^2-\frac{1}{12\pi}(1+10\tilde{\Lambda}_k)\tilde{G}_k} \\ \partial_t\tilde{G}_k &= 2\tilde{G}_k - \frac{1}{3\pi}\frac{(11-18\tilde{\Lambda}_k+28\tilde{\Lambda}_k^2)\tilde{G}_k^2}{(1-2\tilde{\Lambda}_k)^2-\frac{1}{12\pi}(1+10\tilde{\Lambda}_k)\tilde{G}_k}\,. \end{aligned} \tag{2.35}$$

Now we specilize to four dimensions. The system of beta functions (2.35) has two following fixed points: the Gaussian one $\left(\tilde{\Lambda}_k^*=0,\tilde{G}_k^*=0\right)$ and a non-Gaussian one $\left(\tilde{\Lambda}_k^*=0.1932,\tilde{G}_k^*=0.7073\right)$. For the AS scenario the interesting FP is the non–Gaussian one which has a pair of complex conjugate critical exponents $\theta'\pm i\theta''=-1.475\pm 3.043i$. As we can see this non-Gaussian FP satisfies the AS picture since it has a negative real part which, in our conventions, correspond to a relevant direction. The fact that the critical exponents are complex can be seen in figure 2.4 from the spiralling behaviour around the non-Gaussian FP.

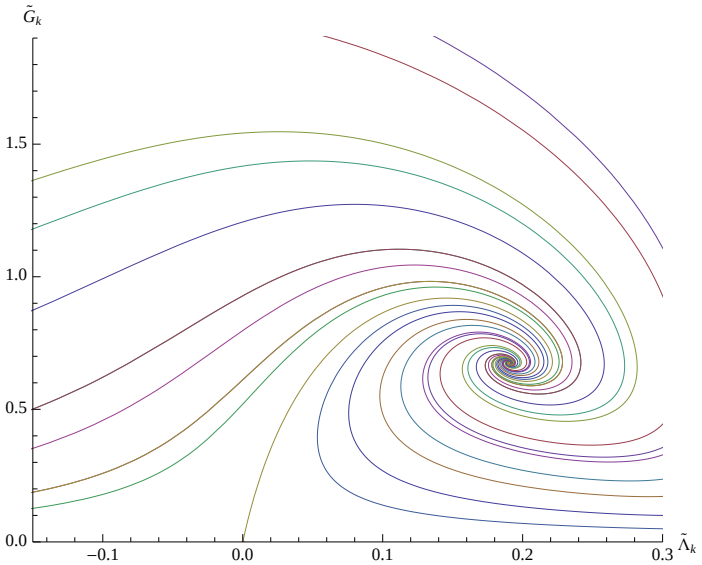


Figure 2.1: RG flow in the space $(\tilde{\Lambda}_k,\tilde{G}_k)$. The spiralling behaviour is due to the complex nature of the critical exponents.

The main features of the flow are left unchanged by varying the cutoff type and the cutoff kernel. Namely we always find a fixed–point with a pair of complex conjugated exponents. As we said in the introduction of this chapter enlarging the truncation one find a fixed point with one further relevant direction but so far no other relevant directions have been found [33].

Improved closure of the flow equations

In this section we close the beta functions (2.28) and (2.29) via an independent computations of the anomalous dimensions of the graviton and the ghost fields. The results of this section can also be found in [59]. This is achieved studying the running of the two–point function of the graviton and of the ghost. In particular we consider the running of the two–point function of the graviton at zero field, i.e.: $h_{\mu\nu} = 0$, and in flat spacetime via the techniques introduced in section 1.4.1.

First of all let us study the l.h.s. of the flow equation for the two point functions via the projectors introduced in appendix D. The two–point function of the graviton is thus expressed in momentum space as follows

$$\begin{aligned}\gamma_k^{(2,0,0;0)}(p,-p) \;=\; & Z_h\left\{\frac{1}{2}\left(p^2+2m_h^2-2\Lambda\right)\mathbf{P}_2+\left(\frac{1}{2\alpha}p^2+m_h^2-\Lambda\right)\mathbf{P}_1\right.\\ & +\left[-\left(\frac{d-2}{2}-\frac{(d-1)\beta^2}{4\alpha}\right)p^2+\frac{\Lambda}{2}\right]\mathbf{P}_S+\left(\frac{(2-\beta)^2}{4\alpha}p^2-\frac{\Lambda}{2}\right)\mathbf{P}_\sigma\\ & \left.+\frac{\sqrt{d-1}}{2}\left[\frac{\beta(\beta-2)}{2\alpha}p^2+2m_h^2+\Lambda\right]\mathbf{P}_{S\sigma}\right\}\,. \end{aligned} \tag{2.36}$$

It is interesting to note that the gauge–fixing parameters do not enter in the coefficient proportional to $\mathbf{P}_2$ which indeed represents the transverse traceless graviton mode (see appendix D). It is clear that from the computation of the two–point function one can extract much informations such as the running of the anomalous dimension Z_h, of the mass m_h and of the gauge–fixing parameters. It is also clear that we are not considering a single metric truncation as we did in the previous subsection since now the level zero and level 2 of the $h_{\mu\nu}$ expansion have different running coefficients.

In the following we set $m_h = 0$ and $\alpha = \beta = 1$ and focus on the running of the wave–functions Z_h and Z_C. The r.h.s. is computed via the diagrammatic approach of section 1.4.1. In this case we have to consider the diagrams in the figures 2.2 and 2.3 where the curly line represents the graviton and the dashed line represents the ghost.

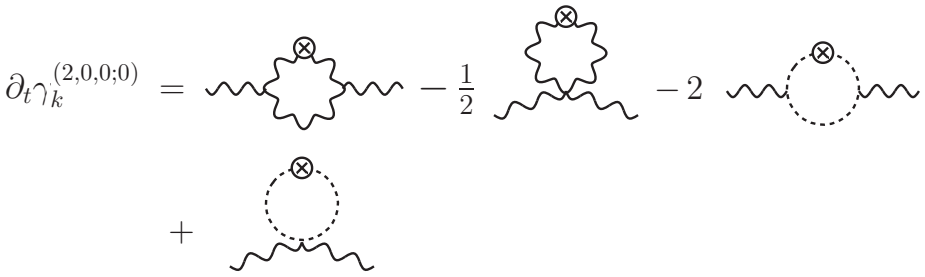


Figure 2.2: Diagrammatic representation of the RG flow equations for anomalous dimensions of the fluctuating metric. The cross–cap stands for a cutoff insertion.

and

Figure 2.3: Diagrammatic representation of the RG flow equations for anomalous dimensions of the ghost fields. The cross–cap stands for a cutoff insertion.

Due to our particular choice of gauge–fixing parameters we can express the graviton propagator in a simple manner via:

$$\mathbf{G}_k[0;\delta] = (\mathbf{1}-\mathbf{P})\, G_{TF,k}(p^2) - \frac{2}{d-2}\mathbf{P}\, G_{T,k}(p^2)\,, \tag{2.37}$$

where $\mathbf{P}$ is the trace projector (D.61) and the arguments in the bracket of $\mathbf{G}_k[0;\delta]$ indicates that the fluctuation $h_{\mu\nu}$ is set to zero while the background metric to the flat one $\bar{g}_{\mu\nu} = \delta_{\mu\nu}$. In (2.37) $G_{TF,k}$ and $G_{T,k}$ are, respectively, the trace-free and trace parts of the regularized graviton propagator and are defined in equation (2.24). The cutoff kernel, when written in terms of $\mathbf{P}$ using (D.61) and (D.62) reads as follows:

$$\mathbf{R}_k[\delta] = Z_h \left[\mathbf{1} - \mathbf{P} - \frac{d-2}{2}\mathbf{P}\right] R_k(p^2)\,. \tag{2.38}$$

In order to extract the running of Z_h, or similarly Z_C, we observe that the former appear as an overall multiplicative factor in (2.36). However the terms of order p^0 are also mixed with other quantities such as the mass parameter and the cosmological constant. Therefore to unambiguously select the wave function we consider just terms proportional to p^2. Moreover, even if we chose the condition $\alpha = \beta = 1$, the gauge parameters run and such running mixes with that of Z_h in the terms of order p^2. To avoid this we will contract the two–point function with the projector $\mathbf{P}_2$ since this picks up the first term in (2.36) which does not depend on the gauge parameters. Of course other choices are possible, for instance in [38] the authors used the complete tensor structure in front of the Laplacian instead of the projector adopted here. For the ghost we can simply consider the p^2-terms contracted with the identity. The evaluation of the two–point function immediately leads to the anomalous dimension. Let us recall that each external leg carries a factor $Z_{h,C}^{1/2}$, this means that the r.h.s., represented by the diagrams in figures (2.2,2.3), has an overall factor $Z_{h,C}$. Since the l.h.s. is simply proportional to $\partial_t Z_{h,C}$ we can divide the flow equation by $Z_{h,C}$ and immediately read–off $\eta_{h,C} = -\partial_t Z_{h,C}/Z_{h,C}$.

Of course the r.h.s. of the flow equation has a much more complicated momentum dependence than that of the l.h.s. shown in (2.36). Indeed we Taylor expand the r.h.s. and consider the terms of order p^2. A more complicated momentum structure has been considered in [60] where

the wave–function is momentum dependent. After expanding in the momentum one is left with a flat spacetime integral which can be solved using the following formula:

$$\int_q \to \frac{S_{d-2}}{(2\pi)^d} \int_0^\infty dq\, q^{d-1} \int_{-1}^1 dx \left(1-x^2\right)^{\frac{d-3}{2}} ,$$

where $S_d = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})}$ is the volume of the d-dimensional sphere. We also change variable $z = q^2$ in the radial integral so that:

$$\int_0^\infty dq\, q^{d-1} \to \frac{1}{2} \int_0^\infty dz\, z^{\frac{d}{2}-1} .$$

The integrals which arise can thus be converted in Q-functional (which are reviewed in appendix C) and we obtain

$$
\begin{aligned}
\eta_h \;=\; & \frac{\kappa^2}{2(4\pi)^{d/2}}\frac{(d+1)}{16\,(d^2-d-2)}\big[8\Lambda Q_{\frac{d}{2}+1}\left[(\partial_t R_k-\eta_h R_k)\,G_T^2 G'_{\mathrm{TF}}\right](d-4)^2 \\
& -\frac{8\Lambda^2 Q_{\frac{d}{2}+1}\left[(\partial_t R_k-\eta_h R_k)\,G_{\mathrm{TF}}^2 G''_T\right](d-4)^2}{d} \\
& -\frac{8\Lambda^2 Q_{\frac{d}{2}+1}\left[(\partial_t R_k-\eta_h R_k)\,G_T^2 G''_{\mathrm{TF}}\right](d-4)^2}{d} \\
& -\frac{8\Lambda^2 Q_{\frac{d}{2}}\left[(\partial_t R_k-\eta_h R_k)\,G_{\mathrm{TF}}^2 G'_T\right](d-4)^2}{d} \\
& -\frac{8\Lambda^2 Q_{\frac{d}{2}}\left[(\partial_t R_k-\eta_h R_k)\,G_T^2 G'_{\mathrm{TF}}\right](d-4)^2}{d} \\
& +8(d-2)\Lambda Q_{\frac{d}{2}+1}\left[(\partial_t R_k-\eta_h R_k)\,G_{\mathrm{TF}}^2 G'_T\right](d-4) \\
& +8(d-3)\Lambda Q_{\frac{d}{2}+2}\left[(\partial_t R_k-\eta_h R_k)\,G_{\mathrm{TF}}^2 G''_T\right](d-4) \\
& +8(d-3)\Lambda Q_{\frac{d}{2}+2}\left[(\partial_t R_k-\eta_h R_k)\,G_T^2 G''_{\mathrm{TF}}\right](d-4) \\
& +\frac{8(d-6)\Lambda Q_{\frac{d}{2}}\left[(\partial_t R_k-\eta_h R_k)\,G_T^2 G_{\mathrm{TF}}\right](d-4)}{d} \\
& +\frac{16(d-3)\Lambda Q_{\frac{d}{2}}\left[(\partial_t R_k-\eta_h R_k)\,G_T G_{\mathrm{TF}}^2\right](d-4)}{d} \\
& +\frac{(d((41-4d)d-116)+76)Q_{\frac{d}{2}+1}\left[(\partial_t R_k-\eta_h R_k)\,G_T^2 G_{\mathrm{TF}}\right]}{d} \\
& -\frac{(d-2)(d(8d-37)+50)Q_{\frac{d}{2}+1}\left[(\partial_t R_k-\eta_h R_k)\,G_T G_{\mathrm{TF}}^2\right]}{d} \\
& +\frac{4(d-2)^2(d(d+2)-11)Q_{\frac{d}{2}+1}\left[(\partial_t R_k-\eta_h R_k)\,G_{\mathrm{TF}}^3\right]}{d} \\
& -8(d-3)(d-2)(d+4)\Lambda Q_{\frac{d}{2}+1}\left[(\partial_t R_k-\eta_h R_k)\,G_{\mathrm{TF}}^2 G'_{\mathrm{TF}}\right] \\
& +\frac{16(d-2)^2(d+4)\Lambda^2 Q_{\frac{d}{2}+1}\left[(\partial_t R_k-\eta_h R_k)\,G_{\mathrm{TF}}^2 G''_{\mathrm{TF}}\right]}{d} \\
& +\frac{8(d-3)^2(d-2)Q_{\frac{d}{2}+2}\left[(\partial_t R_k-\eta_h R_k)\,G_T^2 G'_T\right]}{d^2} \\
& -\frac{2(d-3)(d-2)(d+3)(d(d+2)-4)Q_{\frac{d}{2}+2}\left[(\partial_t R_k-\eta_h R_k)\,G_{\mathrm{TF}}^2 G'_T\right]}{d^2}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{2\left(d\left((d-5)(d+1)d^2+96\right)-72\right)Q_{\frac{d}{2}+2}\left[\left(\partial_t R_k-\eta_h R_k\right)G_T^2 G_{\mathrm{TF}}'\right]}{d^2}\\
&+\frac{(d-3)(d-2)(d(d(d+3)(d+8)+8)-24)Q_{\frac{d}{2}+2}\left[\left(\partial_t R_k-\eta_h R_k\right)G_{\mathrm{TF}}^2 G_{\mathrm{TF}}'\right]}{d^2}\\
&-8(d-3)(d-2)(d+4)\Lambda Q_{\frac{d}{2}+2}\left[\left(\partial_t R_k-\eta_h R_k\right)G_{\mathrm{TF}}^2 G_{\mathrm{TF}}''\right]\\
&-8(d-2)(6Q_{\frac{d}{2}+1}\left[\left(\partial_t R_k-\eta_C R_k\right)G_C^3\right]\\
&+(d+2)(d+4)\left(Q_{\frac{d}{2}+2}\left[\left(\partial_t R_k-\eta_C R_k\right)G_C^2 G_C'\right]+Q_{\frac{d}{2}+3}\left[\left(\partial_t R_k-\eta_C R_k\right)G_C^2 G_C''\right]\right)\\
&+\frac{8(d-3)^2(d-2)Q_{\frac{d}{2}+3}\left[\left(\partial_t R_k-\eta_h R_k\right)G_T^2 G_T''\right]}{d^2}\\
&-\frac{2(d-3)^2\left(d(d+2)^2-8\right)Q_{\frac{d}{2}+3}\left[\left(\partial_t R_k-\eta_h R_k\right)G_{\mathrm{TF}}^2 G_T''\right]}{d^2}\\
&-\frac{2(d-3)^2\left(d(d+2)^2-8\right)Q_{\frac{d}{2}+3}\left[\left(\partial_t R_k-\eta_h R_k\right)G_T^2 G_{\mathrm{TF}}''\right]}{d^2}\\
&+\frac{(d-3)(d-2)(d(d(d+3)(d+8)+8)-24)Q_{\frac{d}{2}+3}\left[\left(\partial_t R_k-\eta_h R_k\right)G_{\mathrm{TF}}^2 G_{\mathrm{TF}}''\right]}{d^2}\\
&+\frac{2(d-8)(d-6)Q_{\frac{d}{2}}\left(G_T^2\right)+(d-2)(d+2)((d-13)d+24)Q_{\frac{d}{2}}\left[\left(\partial_t R_k-\eta_h R_k\right)G_{\mathrm{TF}}^2\right]}{d}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{16(d-3)(d-2)(d+4)\Lambda Q_{\frac{d}{2}}\left[\left(\partial_t R_k-\eta_h R_k\right)G_{\mathrm{TF}}^3\right]}{d}\\
&+\frac{16(d-2)^2(d+4)\Lambda^2 Q_{\frac{d}{2}}\left[\left(\partial_t R_k-\eta_h R_k\right)G_{\mathrm{TF}}^2 G_{\mathrm{TF}}'\right]}{d}\,. \qquad (2.39)
\end{aligned}
$$

For the ghosts we have:

$$
\begin{aligned}
\eta_C \;=\; &\frac{\kappa^2}{(4\pi)^{d/2}}[\frac{\left(d^2-4\right)}{4(d-2)d^2}(dQ_{\frac{d}{2}+2}\left[\left(\partial_t R_k-\eta_h R_k\right)G_T^2 G_C'\right]\\
&+d(d-1)Q_{\frac{d}{2}+2}\left[\left(\partial_t R_k-\eta_h R_k\right)G_{\mathrm{TF}}^2 G_C'\right]\\
&+(d-1)(4d-5)Q_{\frac{d}{2}+1}\left[\left(\partial_t R_k-\eta_h R_k\right)G_C G_{\mathrm{TF}}^2\right])+\\
&\left(d^2-4\right)((d-1)(3d-5)Q_{\frac{d}{2}+1}\left[\left(\partial_t R_k-\eta_C R_k\right)G_C^2 G_{\mathrm{TF}}\right]\\
&-dQ_{\frac{d}{2}+2}\left[\left(\partial_t R_k-\eta_C R_k\right)G_C^2 G_T'\right]\\
&-d(d-1)Q_{\frac{d}{2}+2}\left[\left(\partial_t R_k-\eta_C R_k\right)G_C^2 G_{\mathrm{TF}}'\right])\\
&+(20-d((d-7)d+28))Q_{\frac{d}{2}+1}\left[\left(\partial_t R_k-\eta_C R_k\right)G_C^2 G_T\right]\\
&+(d(7d-32)+20)Q_{\frac{d}{2}+1}\left[\left(\partial_t R_k-\eta_h R_k\right)G_C G_T^2\right]]\,. \qquad (2.40)
\end{aligned}
$$

Evaluating the anomalous dimension using the optimized cutoff in $d=4$ we have:

$$
\begin{aligned}
\eta_h \;&=\; -\frac{G\left(G\left(19968\Lambda^4-59136\Lambda^3+56256\Lambda^2-22096\Lambda+2699\right)-96\pi\left(384\Lambda^3-544\Lambda^2+322\Lambda-73\right)\right)}{(2\Lambda-1)\left(G^2\left(1200\Lambda^2-1200\Lambda+157\right)+48\pi G\left(104\Lambda^3-156\Lambda^2+34\Lambda+9\right)+4608\pi^2(1-2\Lambda)^4\right)}\\
\eta_C \;&=\; \frac{G\left(G\left(224\Lambda^2+4576\Lambda-1519\right)+768\pi(13\Lambda-19)(1-2\Lambda)^4\right)}{(1-2\Lambda)^2\left(G^2\left(1200\Lambda^2-1200\Lambda+157\right)+48\pi G\left(104\Lambda^3-156\Lambda^2+34\Lambda+9\right)+4608\pi^2(1-2\Lambda)^4\right)}
\end{aligned}
$$

	$\tilde{\Lambda}_*$	$\tilde{G}_*$	$\theta' \pm i\theta''$	$\tilde{\Lambda}_*\tilde{G}_*$	$\eta_{h,*}$	$\eta_{C,*}$
One–loop	0.121	1.172	$-1.868 \pm 1.398i$	0.142	0	0
[31]	0.193	0.707	$-1.475 \pm 3.043i$	0.137	-2	0
[52]	0.135	0.859	$-1.774 \pm 1.935i$	0.116	-2	-1.8
This calculation	-0.062	1.617	$-4.119, -1.338$	-0.100	0.686	-1.356

Table 2.1: Fixed–points and critical exponents for the various closures of the beta functions of Λ_k and G_k, in $d = 4$.

Now we can close the equations for the running Cosmological and Newton's constants (2.28) and (2.29). Given the complexity of the anomalous dimensions, it is clear that the modifications to the beta functions are by no means trivial and it is not guaranteed at all to find a fixed point with the usual features. We specify to $d = 4$ and, solving numerically the beta functions, we find a UV attractive fixed point suitable for the AS scenario. The values found in this calculation are reported in table 2.1.

The critical exponents turn out to be real contrary to most of the findings so far (real critical exponents were found in [61] and [38]). This can be appreciated in figure 2.4.

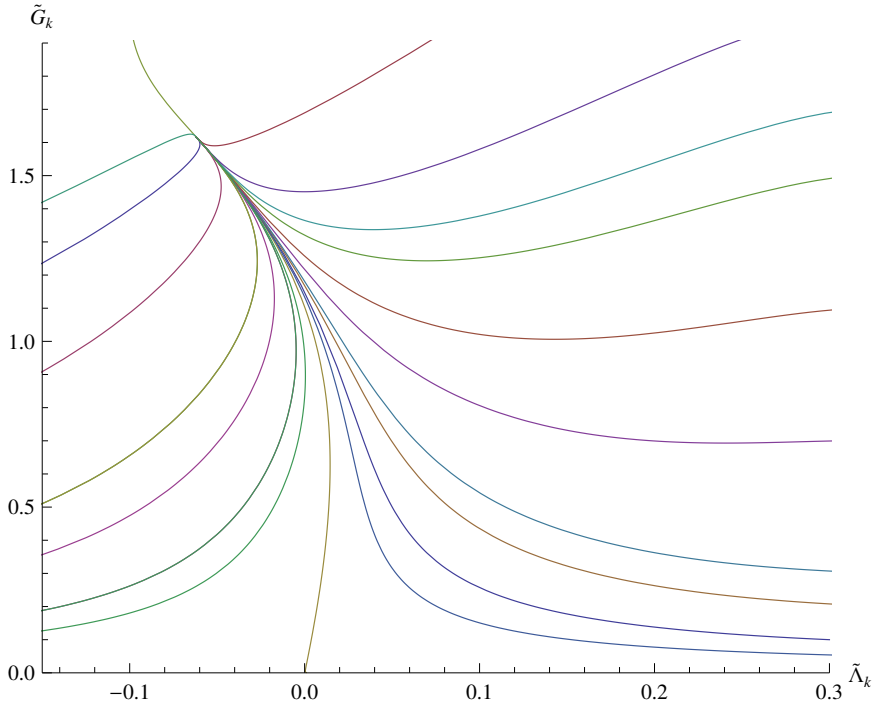


Figure 2.4: RG flow in the space $(\tilde{\Lambda}_k, \tilde{G}_k)$. The spiralling behaviour present in the case of the standard improvement is not present anymore since the critical exponents are real.

This analysis highlights the importance of taking consistently into account the bimetric nature

of the EAA. Therefore our result can be interpreted as a non–trivial confirmation of the AS picture. In order to make manifest the non–trivial modifications with respect to the case of the standard closure we plot the anomalous dimension in the two cases. The value of $\tilde{\Lambda}$ in each case is fixed to the corresponding fixed point value. As we can see in figure (2.5) there is qualitative difference in the two cases.

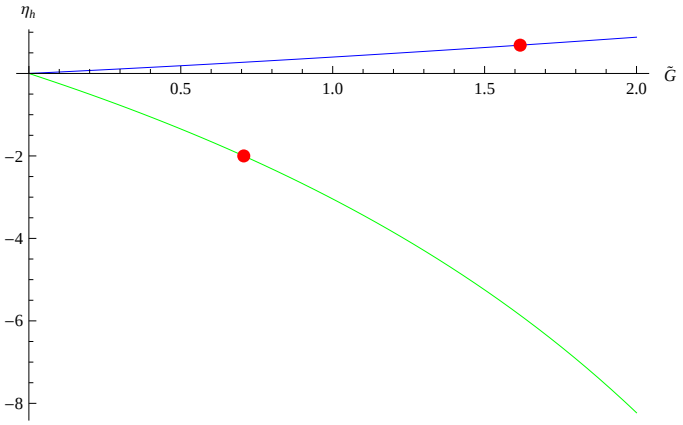


Figure 2.5: η_h as a function of $\tilde{G}$ with $\tilde{\Lambda}$ fixed at the fixed point value. The blue curve shows η_h in the case of the "improved closure" while the green one in the case of the "standard closure". The red dot denotes the value of η_h at the fixed point in each case.

Due to these remarkable differences it seems clear that the bimetric character of the flow has to be taken into account. In this chapter we did this for the Einstein–Hilbert truncation by computing the running of the wave function from the two point function instead of imposing the relation $\eta_h = G^{-1}\partial_t G$. Nevertheless we still find an UV fixed point with a finite number of relevant direction. This analysis is clearly not complete as the truncation needs to be enlarged. For instance $\bar{\Gamma}_k$ should contain also higher derivative terms which are known to play a role from the single metric computations. Moreover also the truncation for $\hat{\Gamma}_k$ should be enlarged to include the mass and further terms (as done in [22, 47]). Nevertheless the techniques employed for this computation are general and can be applied to larger truncations as well.

CHAPTER 3

Weyl invariance and the functional Renormalization Group

A characteristic feature of quantum field theories and Renormalization Group flow is the introduction of a mass scale. This is necessary to regularize the path integral even when the bare action is Weyl invariant and does not contain any dimensionful parameter. As a consequence the quantum effective action in general is not Weyl invariant and the introduction of such a scale is the origin of the trace anomaly [62, 63]. In this chapter we describe a procedure capable of maintaing Weyl invariance along the RG flow. As we will see to achieve this it is necessary to introduce a dilaton, or more generally, a Weyl vector. This fact has been noted several times in the literature. In [64], under the assumption of spontaneous breaking of the conformal symmetry (i.e.: when a dilaton is present), it was first observed that the conformal symmetry can be maintained at the quantum level. The use of the dilaton in this sense has also appeared in [65–67]. In [68] a suitable non–local function of the metric was used in place of the dilaton.

To embed this type of construction in a geometric framework suitable to QFTs in curved spacetime we shall employ Weyl geometry, which is described in section 3.1. Weyl's initial aim was to include the electromagnetic interactions in a geometric framework similar to that of General Relativity (GR). In GR if a vector undergoes parallel displacement along a loop its direction changes but not its length. Weyl takes this a step further and allows also the length to change. Thus one must set up an arbitrary standard of length at each point. This requires a new connection $\hat{\nabla}$ since, for a vector v^μ with the measure of length $l^2 = g_{\mu\nu}v^\mu v^\nu$, this length changes under parallel displacement as:

$$\hat{\nabla}_\mu l = 2b_\mu l, \tag{3.1}$$

which in turn implies:

$$\hat{\nabla}_\mu g_{\alpha\beta} = 2b_\mu g_{\alpha\beta}.$$

This means that the connection $\hat{\Gamma}_\mu{}^\rho{}_\nu$, which is assumed to be symmetric, is non–metric and can

be written as [69, 70]:

$$\hat{\Gamma}_{\mu}{}^{\rho}{}_{\nu} = \Gamma_{\mu}{}^{\rho}{}_{\nu} + \left(-\delta^{\lambda}_{\mu} b_{\nu} - \delta^{\lambda}_{\nu} b_{\mu} + g_{\mu\nu} b^{\lambda}\right)$$

we shall refer to b_μ as the Weyl vector. We require invariance of the action under $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$ where Ω^2 is an arbitrary function of spacetime. Indeed the unit of measure is arbitrary and the factor of Ω^2 represents the rescaling of all lengths. In flat space, scale transformations are usually interpreted as the map $x \to \Omega x$. As such, they form a particular subgroup of diffeomorphisms. Alternatively, one can think of rescaling the metric

$$g_{\mu\nu} \to \Omega^2 g_{\mu\nu}.$$

The two points of view are completely equivalent, since lengths are given by integrating the line element $ds = \sqrt{g_{\mu\nu} dx^\mu dx^\nu}$. For our purposes it will be convenient to adopt the second point of view. The connection $\hat{\Gamma}$ is invariant under a rescaling of the metric if we let the one–form b_μ transform as

$$b_\mu \mapsto b_\mu + \Omega^{-1} \partial_\mu \Omega$$

which can be rewritten as $b_\mu \mapsto b_\mu + \partial_\mu \sigma$ with $\Omega = e^\sigma$. This expression is clearly similar to the law of transformation of a gauge field for electromagnetic interactions, i.e.: the photon. This is the reason which lead Weyl to consider his theory as a possible unification of gravitational and electromagnetic forces. By now we know that the gauge field b_μ cannot be identified with the photon, for instance the coupling of b_μ with fermions is different from the one experimentally measured between fermions and photons which is determined by the value of the electric charge. Let us denote the curvature tensors built out of the connection $\hat{\Gamma}_{\mu}{}^{\rho}{}_{\nu}$ with an hat such as $\hat{R}_{\mu\nu\rho\sigma}$. One can note that in order to write down a theory which is invariant under Weyl transformations only four invariants are allowed in $4d$:

$$\hat{R}^2, \quad \hat{R}_{\mu\nu}\hat{R}^{\mu\nu}, \quad \hat{R}_{\mu\nu\rho\sigma}\hat{R}^{\mu\nu\rho\sigma}, \quad F_{\mu\nu}F^{\mu\nu}$$

where $F_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu$. Other invariants are either a combination of the above terms or are not allowed by Weyl invariance. For instance a term in the action which is third power in the curvature has a dimensionful coupling which breaks Weyl invariance.

We will describe a procedure via which any theory can be made Weyl invariant. To achieve this we will introduce a further degree of freedom: χ which we call the dilaton. The dilaton is a scalar field which under Weyl rescaling transforms as $\chi \mapsto \Omega^{-1}\chi$. We will refer to the power of Ω as the Weyl weight of the field, for instance the dilaton has Weyl weight -1. The dilaton also offers a possible unit of measure which varies in spacetime: we can parametrize any dimensionful quantity via a dimensionless coefficient times a suitable power of the dilaton. We will consider two possible realizations of the Weyl symmetry which are known as Weyl integrable and non–integrable theories. The integrable theory is the one in which the Weyl vector is a pure gauge

field and can be written as $b_\mu = -\chi^{-1}\partial_\mu\chi$ (and so $F_{\mu\nu} = 0$) while the non–integrable theory has a generic Weyl vector b_μ. The most striking difference between these two types of theories is that in the non–integrable case the parallel displacement around a loop produces no effects to the length while in the integrable case the change is proportional to $F_{\mu\nu}$.

As we said this geometric framework will be applied to QFTs to maintain Weyl invariance at the quantum level. In standard QFTs this invariance is broken even starting with a Weyl invariant bare action by the trace anomalies whose main features are reviewed in section 3.2. In order to make the discussion self–contained we review Weyl geometry in section 3.1. In section 3.3 we describe generically how to construct a Weyl invariant effective action and explain its relation with the standard one. In section 3.4 we consider the coupling to matter fields explicitly while in section 3.5 we make gravity dynamical. Finally we summarise our results in section 3.6.

3.1 Weyl geometry

In this section we introduce Weyl geometry which is our tool to keep track of Weyl invariance. Moreover we introduce a recipe which can be used to make any action Weyl invariant. In the introduction of this chapter we already defined the connection:

$$\hat{\Gamma}_\mu{}^\rho{}_\nu = \Gamma_\mu{}^\rho{}_\nu + \left(-\delta^\lambda_\mu b_\nu - \delta^\lambda_\nu b_\mu + g_{\mu\nu}b^\lambda\right). \tag{3.2}$$

This connection have the property to be invariant under Weyl rescalings $\left(g_{\mu\nu} \mapsto \Omega^2 g_{\mu\nu}, b_\mu \mapsto b_\mu + \Omega^{-1}\partial_\mu\Omega\right)$. Nevertheless, in order to be able to make any action Weyl invariant, this is not sufficient. Indeed if we act with $\hat{\nabla}$ on a tensor of weight w we do not obtain a Weyl covariant quantity. To achieve this we need a new connection D_μ which is defined for any tensor t by

$$D_\mu t = \hat{\nabla}_\mu t - w b_\mu t \ , \tag{3.3}$$

which was called by Dirac a "co-covariant" derivative [71].

We denote the curvature tensor of D_μ as $\mathcal{R}_{\mu\nu}{}^\rho{}_\sigma$; this depends on the "Weyl charge" of the field, w. If $w = 0$ we have $\mathcal{R}_{\mu\nu}{}^\rho{}_\sigma = \hat{R}_{\mu\nu}{}^\rho{}_\sigma$, and we can further express $\hat{R}_{\mu\nu}{}^\rho{}_\sigma$ in terms of the Riemann tensor $R_{\mu\nu}{}^\rho{}_\sigma$ (the curvature of the Levi-Civita connection) as

$$\begin{aligned}\hat{R}_{\mu\nu\rho\sigma} \;=\; & R_{\mu\nu\rho\sigma} - F_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}\left(\nabla_\nu b_\sigma + b_\nu b_\sigma\right) - g_{\mu\sigma}\left(\nabla_\nu b_\rho + b_\nu b_\rho\right) \\ & - g_{\nu\rho}\left(\nabla_\mu b_\sigma + b_\mu b_\sigma\right) + g_{\nu\sigma}\left(\nabla_\mu b_\rho + b_\mu b_\rho\right) - \left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)b^2 \ ,\end{aligned} \tag{3.4}$$

where $F_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu$ is the curvature of the Weyl gauge field b_μ. Since $\hat{\nabla}$ is not metric, its curvature is not anti–symmetric in the second pair of indices:

$$\hat{R}_{\mu\nu\rho\sigma} + \hat{R}_{\mu\nu\sigma\rho} = -2F_{\mu\nu}g_{\rho\sigma} \ . \tag{3.5}$$

There are thus two independent "Ricci tensors", obtained contracting the first index of the curvature with the third or the fourth. We will only need one of these definitions, and we observe that the trace of this "Ricci tensor" is unique:

$$\hat{R}_{\mu\nu} \equiv \hat{R}_{\rho\mu}{}^{\rho}{}_{\nu} = R_{\mu\nu} + F_{\mu\nu} + (d-2)(\nabla_\mu b_\nu + b_\mu b_\nu) + \nabla^\rho b_\rho g_{\mu\nu} - (d-2)b^2 g_{\mu\nu} \ , \quad (3.6)$$

$$\hat{R} = R + 2(d-1)\nabla^\mu b_\mu - (d-1)(d-2)b^2 \ . \quad (3.7)$$

The curvature of the connection D_μ acting on a vector of weight w is

$$\mathcal{R}_{\mu\nu}{}^{\rho}{}_{\sigma} = \hat{R}_{\mu\nu}{}^{\rho}{}_{\sigma} - wF_{\mu\nu}\,\delta^\rho_\sigma \ . \quad (3.8)$$

Clearly all this applies equally well in the integrable case where one has simply to substitute $b_\mu \to -\chi^{-1}\nabla_\mu\chi$. The simplest diffeomorphism– and Weyl–invariant actions constructed only with the metric and b_μ are of the form $c_1\mathcal{R}^2 + c_2\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + c_3\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} + c_4F_{\mu\nu}F^{\mu\nu}$. We observe that changing the value of w, the first three terms generate further contributions of the type of the fourth term. In order to establish a basis of independent field monomials we thus have to fix the value of w. In the following we will use $w = 0$, which seems the most natural choice. In this case the curvatures $\mathcal{R}_{\mu\nu\rho\sigma}$ coincide with $\hat{R}_{\mu\nu\rho\sigma}$.

It is important to note that a Weyl invariant theory does not necessarily contain a dilaton or a Weyl vector. In such theories the terms generated by a Weyl transformation that contain the derivatives of the transformation parameter are compensated by terms generated by variations of Ricci tensors. Since Weyl–invariance can be viewed as a gauged version of global scale invariance, this has been called "Ricci gauging" in [72]. It was also shown that such Ricci–gauged theories correspond (under mild additional assumptions) to theories that are conformal–invariant, as opposed to merely scale–invariant, in flat space. A simple example is the following action for a scalar field φ of weight -1 in $d = 4$:

$$\int \sqrt{g}\frac{1}{2}\varphi\left(\Delta + \frac{R}{6}\right)\varphi.$$

It is clear that the above action could not be made Weyl invariant via "Ricci gauging" if we added a mass term. Achieving Weyl invariance for this action (as well as for a generic one) requires the presence of the dilaton χ, a scalar field of weight -1. In this case the mass can be expressed as a dimensionless coefficient times a suitable power of the dilaton. We will see that there are actually two ways in which this can be made realized: the Weyl gauging in which we employ the Weyl vector b_μ (with b_μ eventually set to $-\chi^{-1}\nabla_\mu\chi$) and the Stückelberg trick where all the action is re-expressed via dimensionless quantities defined using suitable powers of χ.

Before embarking into the discussion of the recipes to build a Weyl invariant action let us define the scaling dimension of a quantity. Consider a theory with fields ψ_a, parameters g_i (which include masses, couplings, wave function renormalizations etc.) and action $S(g_{\mu\nu}, \psi_a, g_i)$. There

is a unique choice of numbers w_a (one per field) and w_i (one per parameter) such that S is invariant:

$$S(g_{\mu\nu}, \psi_a, g_i) = S(\Omega^2 g_{\mu\nu}, \Omega^{w_a}\psi_a, \Omega^{w_i} g_i) \ . \tag{3.9}$$

Here Ω is constant and the invariance is a consequence of the fact that S is dimensionless (it does not matter here whether the metric is fixed or dynamical). The numbers w_a, w_i are called the scaling dimensions, or the weights, of ψ_a and g_i. We shall adopt the point of view in which spacetime coordinates are dimensionless and we use natural units where $c = 1$, $\hbar = 1$. Then, the scaling dimensions are equal to the ordinary length dimensions of ψ_a and g_i in the sense of dimensional analysis. Since in particle physics it is customary to use mass dimensions, when we talk of "dimensions" without further specification we will refer to the mass dimensions $d_a = -w_a$ and $d_i = -w_i$. In d spacetime dimensions, the dimensions of scalar, spinor and vector fields are $(d-2)/2$, $(d-1)/2$ and $(d-4)/2$, respectively. One can easily convince oneself that the dimensions of all parameters in the Lagrangian, such as masses and couplings, are the same as in the more familiar case when coordinates have dimension of length.

Changing couplings is usually interpreted as changing theory, so in general the transformations (3.9) are not symmetries of a theory but rather maps from one theory to another. In the case when all the w_i are equal to zero, we have

$$S(g_{\mu\nu}, \psi_a, g_i) = S(\Omega^2 g_{\mu\nu}, \Omega^{w_a}\psi_a, g_i) \ . \tag{3.10}$$

Since these are transformations that map a theory to itself, a theory of this type is said to be globally scale invariant.

Scale transformations with Ω a positive real function of x are called Weyl transformations. They act on the metric and the fields exactly as in (3.9). What about the parameters? They are supposed to be x-independent, so transformation $g_i \to \Omega(x)^{w_i} g_i$ would not make much sense. One can overcome this difficulty by promoting the dimensionful parameters to fields. One can then meaningfully ask whether (3.9) holds. In general the answer will be negative, but there is a simple procedure that allows one to make a scale invariant theory also Weyl–invariant. We pick a mass parameter of the theory, let's call it μ and we promote it to a function that we shall denote χ. We can write

$$\chi(x) = \mu e^{\sigma(x)} \ , \tag{3.11}$$

where μ is constant. The function χ, or sometimes σ, is called the dilaton. Notice that unlike an ordinary scalar field, it has dimension one independently of the spacetime dimensionality. Now we can take any other dimensionful coupling of the theory and write

$$g_i = \chi^{-w_i}\hat{g}_i = \chi^{d_i}\hat{g}_i \ , \tag{3.12}$$

where $\hat{g}_i$ is dimensionless (and therefore Weyl–invariant). In general, a caret over a symbol denotes the same quantity measured in units of the dilaton. In principle one could promote more

than one dimensionful parameter, or even all dimensionful parameters, to independent dilatons. This may have interesting applications, but for the sake of simplicity we shall restrict ourselves to the case when there is a single dilaton.

Now start from a generic action for matter and gravity of the form $S(g_{\mu\nu}, \psi_a, g_i)$. Express every parameter g_i as in (3.12). Replace all covariant derivatives ∇ by Weyl covariant derivatives D and all curvatures R by the Weyl covariant curvatures $\mathcal{R}$ where $b_\mu = -\chi^{-1}\nabla_\mu\chi$ for the time being. Now all the terms appearing in the action are products of Weyl covariant objects, and local Weyl invariance just follows from the fact that the action is dimensionless. In this way we have defined an action $\hat{S}(g_{\mu\nu}, \chi, \psi_a, \hat{g}_i)$. It contains only dimensionless couplings $\hat{g}_i$, and is Weyl invariant by construction. One can choose a gauge where $\chi = \mu$ is constant (equivalently, $\sigma = 0$), and in this gauge the action $\hat{S}(g_{\mu\nu}, \chi, \psi_a, \hat{g}_i)$ reduces to the original one. This procedure is called Weyl gauging as we replaced standard covariant derivative with Weyl covariant ones and so on. One can use both the Weyl integrable and non–integrable geometries, the dilaton is always needed to make the couplings dimensionless.

If we are interested only in using the dilaton, and not a generic Weyl vector, there is also another way of defining a Weyl–invariant action from a non–invariant one, namely to replace all the arguments in S by the corresponding dimensionless quantities $\hat{g}_{\mu\nu} = \chi^2 g_{\mu\nu}$, $\hat{\psi}_a = \chi^{w_a}\psi_a$ and $\hat{g}_i = \chi^{w_i} g_i$ and subsequently reexpress the action in terms of the original fields

$$\hat{S}(g_{\mu\nu}, \chi, \psi_a, \hat{g}_i) = S(\hat{g}_{\mu\nu}, \hat{\psi}_a, \hat{g}_i) \ . \tag{3.13}$$

We will refer to this procedure as Stückelberg trick and it is easy to see that this construction gives the same result as the preceding one. This follows from the fact that (3.2) are the Christoffel symbols of $\hat{g}_{\mu\nu}$, that $\hat{\nabla}_\mu\hat{\psi}_a = \chi^{w_a} D_\mu\psi_a$ and that the curvature tensor of $\hat{\Gamma}$ is $\mathcal{R}_{\mu\nu\rho\sigma}$. [1]

3.2 Standard effective action and trace anomaly

In this section we review some known results concerning Weyl invariance in standard QFT coupled to external gravity. In particular we show how conformal anomalies arise in free theories computing the variation of the one–loop effective action. Moreover we discuss generic aspects of anomalies such as consistency condition and the role that anomalies play in effective actions.

3.2.1 The standard measure

In this section we review the evaluation of the effective action for free, massless matter fields conformally coupled to a metric. This will provide the basis for different quantization procedures

[1] If we call $\hat{R}_{\mu\nu\rho\sigma}$ the Riemann tensor of $\hat{g}_{\mu\nu}$, we have $\hat{R}^\mu{}_{\nu\rho\sigma} = \mathcal{R}^\mu{}_{\nu\rho\sigma}$ and $\hat{R}_{\mu\nu\rho\sigma} = \chi^2\mathcal{R}_{\mu\nu\rho\sigma}$.

to be described in the following. Much of the discussion can be carried out in arbitrary even dimension d.

For definiteness let us consider first a single conformally coupled scalar field, with equation of motion $\Delta^{(0)}\varphi = 0$, where $\Delta^{(0)} = -\nabla^2 + \frac{d-2}{4(d-1)}R$. Functional integration over φ in the presence of a source J leads to a generating functional $W(g_{\mu\nu}, j)$, whose Legendre transform $-\Gamma(g_{\mu\nu}, \varphi) = W(g_{\mu\nu}, j) - \int J\varphi$ is the effective action. For the definition of the functional integral one needs a metric (more precisely an inner product) in the space of the fields. We choose

$$\mathcal{G}(\varphi, \varphi') = \mu^2 \int dx \sqrt{g}\, \varphi\, \varphi' \ , \tag{3.14}$$

where μ is an arbitrary mass that has to be introduced for dimensional reasons. The action can be written as

$$S_S(g_{\mu\nu}, \varphi) = \frac{1}{2}\int dx \sqrt{g}\, \varphi \Delta^{(0)} \varphi = \frac{1}{2}\mathcal{G}\left(\varphi, \frac{\Delta^{(0)}}{\mu^2}\varphi\right) = \frac{1}{2}\sum_n a_n^2 \lambda_n/\mu^2 \ , \tag{3.15}$$

where λ_n are the eigenvalues of $\Delta^{(0)}$, φ_n the corresponding eigenfunctions and a_n are the (dimensionless) coefficients of the expansion of φ on the basis of the eigenfunctions:

$$\Delta^{(0)}\varphi_n = \lambda_n \varphi_n \ ; \qquad \mathcal{G}(\varphi_n, \varphi_m) = \delta_{nm} \ ; \qquad \varphi = \sum_n a_n \varphi_n \ ; \qquad a_n = \mathcal{G}(\varphi, \varphi_n) \ . \tag{3.16}$$

(For simplicity we assume that the manifold is compact and without boundary, so that the spectrum of the Laplacian is discrete.) Weyl–covariance means that under a Weyl transformation the operator $\Delta^{(0)}$ transforms as

$$\Delta^{(0)}_{\Omega^2 g} = \Omega^{-1-\frac{d}{2}} \Delta^{(0)}_g \Omega^{\frac{d}{2}-1} \ , \tag{3.17}$$

where we have made the dependence of the metric explicit. For an infinitesimal transformation $\Omega = 1 + \omega$,

$$\delta_\omega \Delta^{(0)} = -2\omega\Delta^{(0)} + \left(\frac{d}{2} - 1\right)[\Delta^{(0)}_g, \omega] \ . \tag{3.18}$$

The functional measure is $(d\varphi) = \prod_n da_n$, so the Gaussian integral can be evaluated as

$$e^{W(g_{\mu\nu}, J)} = \prod_n \left(\int da_n e^{-\frac{1}{2}a_n^2\lambda_n/\mu^2 + a_n J^n}\right) = \det\left(\frac{\Delta^{(0)}}{\mu^2}\right)^{-1/2} e^{\frac{1}{2}\int J\Delta^{-1}J} \tag{3.19}$$

up to a field–independent multiplicative constant. From here one gets (using the same notation for the VEV as for the field) $\varphi = -\left(\Delta^{(0)}\right)^{-1} J$, so finally the Legendre transform gives

$$\Gamma(\varphi, g_{\mu\nu}) = S_S(\varphi, g_{\mu\nu}) + \frac{1}{2}\mathrm{Tr}\log\left(\frac{\Delta^{(0)}}{\mu^2}\right) \ . \tag{3.20}$$

An UV regularization is needed to define this trace properly. We see that the scale μ, which has been introduced in the definition of the measure, has made its way into the functional determinant.

Things work much in the same way for the fermion field, which contributes to the effective action a term

$$S_D(\bar{\psi}, \psi, g_{\mu\nu}) - \frac{1}{2}\mathrm{Tr}\log\left(\frac{\Delta^{(1/2)}}{\mu^2}\right) , \tag{3.21}$$

where S_D is the classical action and $\Delta^{(1/2)} = -\nabla^2 + \frac{R}{4}$ is the square of the Dirac operator.

The Maxwell action is Weyl–invariant only in $d = 4$. With our conventions the field A_μ is dimensionless and the Weyl–invariant inner product in field space is:

$$\mathcal{G}(A, A') = \mu^2 \int d^4x\sqrt{g}\, g^{\mu\nu} A_\mu A_\nu \ . \tag{3.22}$$

Using the standard Faddeev-Popov procedure, we add gauge fixing and ghost actions

$$S_{GF} = \frac{1}{2\alpha}\int d^4x\sqrt{g}\,(\nabla_\mu A^\mu)^2 \ ; \qquad S_{gh} = \int d^4x\sqrt{g}\,\bar{C}\Delta^{(gh)}C \ , \tag{3.23}$$

with $\Delta^{(gh)} = -\nabla^2$. Then, in the gauge $\alpha = 1$, the gauge–fixed action becomes

$$S_M + S_{GF} = \frac{1}{2}\int d^4x\sqrt{g}\, A^\mu \Delta^{(1)\nu}_\mu A_\nu = \frac{1}{2}\mathcal{G}\left(A, \frac{\Delta^{(1)}}{\mu^2}A\right) \ , \tag{3.24}$$

where $\Delta^{(1)\nu}_\mu = -\nabla^2\delta^\nu_\mu + R^\nu_\mu$ is the Laplacian on one–forms. Following the same steps as for the scalar field, we obtain a contribution to the effective action equal to

$$S_M(A_\mu, g_{\mu\nu}) + \frac{1}{2}\mathrm{Tr}\log\left(\frac{\Delta^{(1)}}{\mu^2}\right) - \mathrm{Tr}\log\left(\frac{\Delta^{(gh)}}{\mu^2}\right) \ . \tag{3.25}$$

Note that even though the Maxwell action S_M is Weyl–invariant, the gauge fixing action is not, nor is the ghost action. As a result the operators $\Delta^{(1)}$ and $\Delta^{(gh)}$ are not Weyl–covariant. Instead of an equation like (3.18), they satisfy (in four dimensions)

$$\delta_\omega \Delta^{(gh)} = -2\omega\Delta^{(h)} - 2\nabla^\nu\omega\nabla_\nu \ ; \tag{3.26}$$

$$\delta_\omega \Delta^{(1)\nu}_\mu = -2\omega\Delta^{(1)\nu}_\mu + 2\nabla_\mu\omega\nabla^\nu - 2\nabla^\nu\omega\nabla_\mu - 2\nabla_\mu\nabla^\nu\omega \ . \tag{3.27}$$

We shall see in the next section how these non–invariances compensate each other in the effective action, so that the breaking of Weyl–invariance is only due to the presence of the scale μ which was introduced in the inner product.

In general, the need for an inner product in field space can also be seen in a more geometrical way as follows. The classical action, being quadratic in the fields, has the form $\mathcal{H}(\varphi, \varphi)$, where $\mathcal{H} = \frac{\delta^2 S}{\delta\varphi\delta\varphi}$ can be viewed as a covariant symmetric tensor in field space: when contracted with

a field (a vector in field space) it produces a one–form in field space. Now, the determinant of a covariant symmetric tensor is not a basis-independent quantity. One can only define in a basis-independent way the determinant of an operator mapping a space into itself, i.e. a mixed tensor. One can transform the covariant tensor $\mathcal{H}$ to a mixed tensor $\mathcal{O}$ by "raising an index" with a metric: [2]

$$\mathcal{H}(\varphi,\varphi') = \mathcal{G}(\varphi,\mathcal{O}\varphi') \ . \tag{3.28}$$

It is the determinant of the operator $\mathcal{O}$ that appears in the effective action. Again we see that the scale μ appears through the metric $\mathcal{G}$, which is needed to define the determinant. Notice that since $\mathcal{O}\varphi$ is another field of the same type as φ, $\mathcal{O}$ must necessarily be dimensionless, and this is guaranteed by the factors of μ contained in $\mathcal{G}$. For example, in the scalar case, $\mathcal{O} = \frac{1}{\mu^2}\Delta^{(0)}$.

3.2.2 The trace anomaly

The phenomenon of conformal (Weyl) anomalies is a generic one in QFT. It refers to the fact that even starting from an action which, at the classical level, possesses invariance under conformal transformations this symmetry is broken due to quantum effects. The intuitive reason for this is that in order to define a QFT we need to introduce some regularization method and a mass scale associated to the renormalization of our theory. The introduction of a dimensionlful parameter is the source of the breaking of the symmetry which manifests itself in a non-zero expectation value of the trace of the stress–energy tensor. In flat space this trace is zero but it is possible to observe the effects of the anomaly in the correlators of two (or more) stress–energy tensors. We will typically work with curved background and we will refer to the set of local transformation $\left(g_{\mu\nu} \to e^{2\sigma(x)}g_{\mu\nu}, \psi \to e^{w\sigma(x)}\psi\right)$ as a conformal transformation while we will refer to the global set $\left(g_{\mu\nu} \to e^{2\sigma}g_{\mu\nu}, \psi \to e^{w\sigma}\psi\right)$ as a scale transformation (here w denotes the Weyl weight of the field). Note that in principle the constraints on the trace of the stress–energy tensor derived from these symmetries are different. In the case of the conformal transformation the trace of the stress-energy tensor must be zero in flat spacetime while in the case of the scale symmetry the trace can be a total derivative. Nevertheless we will assume that in the cases of our interests scale invariance implies conformal invariance.[3]

A convenient method to compute trace anomalies is given by the heat kernel (HK) which is reviewed in appendix C. So let us briefly show how to connect these anomalies with some HK coefficients. Under an infinitesimal Weyl transformation the variation of the effective action is

$$\delta_\omega\Gamma \ = \ \int dx \, \frac{\delta\Gamma}{\delta g_{\mu\nu}} 2\omega g_{\mu\nu} = -\int dx \sqrt{g}\, \omega \langle T^\mu_\mu \rangle \ .$$

[2] In de Witt's condensed notation, where an index i stands both for a point x in spacetime and whatever tensor or spinor indices the field may be carrying, this equation reads $\mathcal{O}_i{}^j = \mathcal{H}_{ik}\mathcal{G}^{kj}$.

[3] For unitary theories in two dimension the equivalence has been established long time ago [73]. Very recently a proof for the four dimensional case also appeared [74].

The trace of the energy–momentum tensor vanishes for a Weyl–invariant action, so the appearance of a nonzero trace is the physical manifestation of the anomaly.

For non–interacting fields the one–loop effective action is exact. Let us denote the Hessian of the bare action Δ and use the proper time representation for the EA

$$\Gamma \;=\; S - \frac{1}{2}\int_{\varepsilon/\mu^2}^{\infty}\frac{dt}{t}\mathrm{Tr}e^{-t\Delta}\;, \tag{3.29}$$

where ε is a dimensionless UV regulator. As we have seen in the previous section, since the bare action is Weyl invariant, the Hessian is Weyl covariant. Therefore, for a field of weight w, we have:

$$\delta_\omega\Delta = -2\omega\Delta + w[\Delta,\omega]\;. \tag{3.30}$$

Varying (3.29) and using that the commutator cancels under the trace, one finds

$$\delta_\omega\Gamma = \frac{1}{2}\int_{\varepsilon/\mu^2}^{\infty}dt\,\mathrm{Tr}\,\delta_\omega\Delta e^{-t\Delta} = -\int_{\varepsilon/\mu^2}^{\infty}dt\,\mathrm{Tr}(\omega\Delta e^{-t\Delta}) = \int_{\varepsilon/\mu^2}^{\infty}dt\frac{d}{dt}\mathrm{Tr}(\omega e^{-t\Delta}) = -\mathrm{Tr}\left[\omega e^{-\varepsilon\Delta/\mu^2}\right]\;.$$

For $\varepsilon\to 0$ one has from the asymptotic expansion of the heat kernel:

$$\mathrm{Tr}\left[\omega e^{-\varepsilon\Delta/\mu^2}\right] = \frac{1}{(4\pi)^{d/2}}\int d^dx\sqrt{g}\,\omega\Big[\frac{\mu^d}{\varepsilon^{d/2}}b_0(\Delta) + \frac{\mu^{d-2}}{\varepsilon^{d/2-1}}b_2(\Delta) + \ldots + b_d(\Delta) + \ldots\Big]\;, \tag{3.31}$$

where b_i are scalars constructed with i derivatives of the metric. All terms b_i with $i > d$ tend to zero in the limit, so assuming that the power divergences (for $i < d$) are removed by renormalization, there remains a universal, finite limit

$$\delta_\omega\Gamma = -\frac{1}{(4\pi)^{d/2}}\int dx\sqrt{g}\,\omega\,b_d(\Delta)\;. \tag{3.32}$$

which implies that

$$\langle T^\mu_\mu\rangle = \frac{1}{(4\pi)^{d/2}}b_d(\Delta)\;. \tag{3.33}$$

We note that this can also be seen as a direct manifestation of the dependence of the result on the scale μ. In fact one has, formally [75]

$$\mu\frac{d}{d\mu}\frac{1}{2}\mathrm{Tr}\log\frac{\Delta}{\mu^2} = -\mathrm{Tr}\mathbf{1} = -\frac{1}{(4\pi)^{d/2}}\int dx\sqrt{g}\,b_d(\Delta) = -\int dx\sqrt{g}\,\langle T^\mu_\mu\rangle\;, \tag{3.34}$$

where in the second step we have used zeta function regularization [75].

Aside from the different prefactor the calculation follows the same steps in the case of massless spinors. The Maxwell field, however, requires some additional considerations, because the operators $\Delta^{(1)}$ and $\Delta^{(gh)}$ that appear in (3.25) are not covariant. (We restrict ourselves now to $d = 4$). The first two steps of the preceding calculation give:

$$\begin{aligned}\delta_\omega\Gamma \;&=\; \frac{1}{2}\int_{\varepsilon/\mu^2}^{\infty}dt\,\mathrm{Tr}\,\delta_\omega\Delta^{(1)}e^{-t\Delta^{(1)}} - \int_{\varepsilon/\mu^2}^{\infty}dt\,\mathrm{Tr}\,\delta_\omega\Delta^{(gh)}e^{-t\Delta^{(gh)}}\\ &=\; \frac{1}{2}\int_{\varepsilon/\mu^2}^{\infty}dt\,\mathrm{Tr}\,(-2\omega\Delta^{(1)}+\rho^{(1)})e^{-t\Delta^{(1)}} - \int_{\varepsilon/\mu^2}^{\infty}dt\,\mathrm{Tr}\,(-2\omega\Delta^{(gh)}+\rho^{(gh)})e^{-t\Delta^{(gh)}}\end{aligned} \tag{3.35}$$

where the violation of Weyl covariance is due to

$$\rho^{(gh)} = -2\nabla^\nu \omega \nabla_\nu \; ; \qquad \rho_\mu^{(1)\nu} = 2\nabla_\mu \omega \nabla^\nu - 2\nabla^\nu \omega \nabla_\mu - 2\nabla_\mu \nabla^\nu \omega \; . \tag{3.36}$$

Since $\Delta^{(1)}$ maps longitudinal fields to longitudinal fields and transverse fields to transverse fields, $\rho^{(1)} e^{-t\Delta^{(1)}}$ has vanishing matrix elements between transverse gauge fields. Therefore the trace containing $\rho^{(1)}$ can be restricted to the subspace of longitudinal gauge potentials. Let φ_n be a basis of eigenfunctions of $\Delta^{(gh)}$ satisfying an orthonormality condition as in (3.16). Then a basis in the space of longitudinal potentials satisfying a similar orthonormality condition with respect to the inner product (3.22) is given by the fields $A^L_{n\mu} = \frac{1}{\sqrt{\lambda_n}} \nabla_\mu \varphi_n$. The traces of the terms violating Weyl–covariance are therefore:

$$\frac{1}{2} \mathrm{Tr} \rho^{(1)} e^{-t\Delta^{(1)}} - \mathrm{Tr} \rho^{(gh)} e^{-t\Delta^{(gh)}} = \frac{1}{2} \sum_n \mathcal{G}\left(A^L_n, \rho^{(1)} e^{-t\Delta^{(1)}} A^L_n \right) - \sum_n \mathcal{G}\left(\varphi_n, \rho^{(gh)} e^{-t\Delta^{(gh)}} \varphi_n \right) \; . \tag{3.37}$$

Noting that

$$\Delta^{(1)} A^L_n = \frac{1}{\sqrt{\lambda_n}} \Delta^{(1)} \nabla_\mu \varphi_n = \frac{1}{\sqrt{\lambda_n}} \nabla_\mu \Delta^{(gh)} \varphi_n = \lambda_n A^L_n \; ,$$

we can evaluate the matrix elements:

$$\mathcal{G}\left(A^L_n, \rho^{(1)} e^{-t\Delta^{(1)}} A^L_n \right) = -4 e^{-t\lambda_n} \mathcal{G}\left(\varphi_n, \nabla^\nu \omega \nabla_\nu \varphi_n \right) \; ,$$

whereas in the ghost trace we have

$$\mathcal{G}\left(\varphi_n, \rho^{(gh)} e^{-t\Delta^{(gh)}} \varphi_n \right) = -2 e^{-t\lambda_n} \mathcal{G}\left(\varphi_n, \nabla^\nu \omega \nabla_\nu \varphi_n \right) \; .$$

We see that the sums in (3.37) cancel mode by mode. As a result only the first term remains in each of the traces in (3.35). From this point onwards the calculation proceeds as in the case of the scalar and finally gives

$$\delta_\omega \Gamma = \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} \left[b_4(\Delta^{(1)}) - 2 b_4(\Delta^{(gh)}) \right] \; . \tag{3.38}$$

The coefficients of the expansion of the heat kernel for Laplace-type operators are well-known. If there are n_S scalar, n_D spinors, one has in two dimensions

$$\langle T^\mu{}_\mu \rangle = -\frac{c}{24\pi} R \tag{3.39}$$

with

$$c = n_S + n_D \tag{3.40}$$

whereas in four dimensions (assuming also the existence of n_M Maxwell fields)

$$\langle T^\mu{}_\mu \rangle = c\, C^2 - a E \tag{3.41}$$

where $E = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is the integrand of the Euler invariant, $C^2 = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ is the square of the Weyl tensor and the anomaly coefficients are [4]

$$a = \frac{1}{360(4\pi)^2}\left(n_S + 11n_D + 62n_M\right) \ ; \quad c = \frac{1}{120(4\pi)^2}\left(n_S + 6n_D + 12n_M\right) \ . \tag{3.42}$$

Let us focus on the four dimensional case. Note that no R^2 term is present in the four dimensional anomaly. This fact is a direct consequence of the WZ consistency condition which is nothing but the requirement that the anomaly satisfy the group structure of the symmetry transformation. In the Weyl case we must therefore have

$$\delta_{\sigma_2(y)}\delta_{\sigma_1(x)}\Gamma - \delta_{\sigma_1(x)}\delta_{\sigma_2(y)}\Gamma \quad = \quad 0.$$

Moreover let us notice that the anomaly proportional to ΔR can be modified by adjusting the coefficient of the (local) R^2 term present in the EA. As a consequence this term is not viewed as a genuine anomaly as it can be changed according to the chosen scheme. Thus the genuine anomalies in $4d$ are the $a-$ and $c-$anomalies.

In [76] anomalies are classified into two different types: the so called type A and type B. The distinction comes from the fact that type A anomalies comes from finite parts of the EA which has no scale dependence. Type B anomalies come from terms in the EA which need to be regularized and therefore depend on a scale μ. The simplest example of type A anomaly can be found in two dimension where the EA of a massless gaussian scalar field can be integrated exactly giving the Polyakov action [77]:

$$\Gamma\left[\varphi = 0, g_{\mu\nu}\right] \quad = \quad -\frac{1}{96\pi}\int \sqrt{g}R\frac{1}{\Delta}R.$$

We will discuss this type of action in more detail later on. In $4d$ type A anomalies have a unique term proportional to the topological Euler term of the dimension. Types B anomalies are characterized from the fact that the WZ consistency condition is trivially satisfied since $\delta_{\sigma_2}\delta_{\sigma_1}\Gamma = 0$. For instance, in the four dimensional case, the $c-$anomaly belongs to the type B class since $\delta_{\sigma_2}\int C^2 = 0$ and a–anomaly is of type A. Moreover these anomalies comes from terms in the EA which need to be regularized, e.g.: in the case of the four dimensional $c-$anomaly the relevant term is $C_{\mu\nu\rho\sigma}\log\left(\Delta/\mu^2\right)C^{\mu\nu\rho\sigma}$. We will come back on this issue in section 3.4.1.

Finally let us stress that we are not considering the possibility of parity violating matter field (e.g.: a chiral fermion). Indeed in this case we would have another possible invariant of dimension four to consider: the Pontryagin density $\tilde{R}R \equiv \varepsilon^{\mu\nu\rho\sigma}R_{\mu\nu\alpha\beta}R_{\rho\sigma}{}^{\alpha\beta}$. The investigation of these anomalies is still matter of research but it seems that the presence of such an anomaly would imply the breaking of unitarity since an imaginary coefficient appears in the correlator of stress-energy tensors [78, 79].

[4] the coefficients c and a were called b and $-b'$ in [62].

At this point it is convenient to introduce the so called Wess-Zumino (WZ) action. Working with the metric as an external background we define [80]

$$\Gamma_{WZ}\left[g_{\mu\nu},\sigma\right] \equiv \Gamma\left[e^{2\sigma}g_{\mu\nu}\right]-\Gamma\left[g_{\mu\nu}\right]. \tag{3.43}$$

It has been noticed that the WZ action is finite given that the regulating counterterms in the EA have poles which are cancelled by the Weyl variation of the same term [80]. In $d=4$ the anomaly comes from the cancellation of the pole $(d-4)^{-1}$ with the coefficient of the Weyl variation of the (dimensionally regularized) EA which is proportional to $(d-4)$. This yields a finite WZ action.

From a more formal point of view conformal anomalies can be understood from a cohomological setting. This was first noticed in [81] where the authors consider the generators of Weyl transformation modified via a grassmanian odd parameter ξ. For instance for the metric field the generator (analogous to the Ward operators we will introduce in appendix E) reads:

$$\Xi \equiv \int_x 2\xi g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}$$

and it is easy to check that $\Xi^2=0$ implementing the WZ consistency condition coming from the fact that the Weyl transformation are abelian. One considers $\Xi\Gamma=\sum_i \mathcal{A}_i$ and check which anomalies $\mathcal{A}_i$ can be expressed via $\mathcal{A}_i=\Xi\gamma_i$. These anomalies are not considered as genuine anomalies as their values can be modified by adjusting a local functionals of the EA. Therefore genuine anomalies can be found modding out such functional and considering the non-trivial cohomology classes build via the operator Ξ. The interested reader can find further details in [81]. To highlight the relation between the cohomological approach to conformal anomalies and the form of the functionals which can be found in the EA we can consider the different approach developed by Mottola [80]. The coboundary operator is defined as follows:

$$\Delta_\sigma \circ F\left(\bar{g}\right) \equiv F\left(e^{2\sigma}\bar{g}\right)-F\left(\bar{g}\right).$$

In full analogy with the exterior derivatives of differential forms the further application of the coboundary operator is defined

$$\Delta_{\sigma_{k+1}}\circ F^{(k)}\left(\bar{g},\sigma_1,\cdots,\sigma_k\right) \equiv \sum_{i=1}^{k+1}(-1)^{i+1}\left(\Delta F\right)_{\sigma_i}\left(\bar{g},\sigma_1,\cdots,\hat{\sigma}_i,\cdots,\sigma_{k+1}\right).$$

Let us observe that the application of the operator Δ_{σ_1} to $\Gamma\left[\bar{g}\right]$ gives back the WZ action and the further application of Δ_{σ_2}together with the property $\Delta^2=0$ corresponds precisely to the WZ consistency condition. Another form of the WZ consistency condition can be found starting from the definition of WZ action [82]:

$$\begin{aligned}\Gamma_{WZ}\left[g,\sigma\right] &= \Gamma\left[ge^{2\sigma}\right]-\Gamma\left[g\right]\\ &= \Gamma\left[ge^{2\omega}e^{2\sigma-2\omega}\right]-\Gamma\left[ge^{2\omega}\right]+\Gamma\left[ge^{2\omega}\right]-\Gamma\left[g\right]\\ &= \Gamma_{WZ}\left[ge^{2\omega},\sigma-\omega\right]+\Gamma_{WZ}\left[g,\omega\right].\end{aligned} \tag{3.44}$$

In [80] starting from these considerations and using dimensional regularization it is argued that the consistent functional which reproduces the anomaly is given by the so called Riegert action [83]:

$$W(g_{\mu\nu}) = \int dx\,\sqrt{g}\frac{1}{8}\left(E-\frac{2}{3}\Box R\right)\Delta_4^{-1}\left[2c\,C^2 - a\left(E-\frac{2}{3}\Box R\right)\right] + \frac{a}{18}R^2 \tag{3.45}$$

where

$$\Delta_4 = \nabla^4 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu + \frac{1}{3}\nabla^\mu R\nabla_\nu - \frac{2}{3}R\nabla^2$$

We will discuss the role of the Riegert action in detail in section 3.4.

3.3 Effective action and the dilaton

We discuss the role of the dilaton employed to preserve Weyl invariance at the quantum level. In this section we shall limit ourselves to free fields and to some general considerations about the relation between the standard quantization procedure and the one which makes use of the dilaton. In particular we shall see that the resulting effective actions are related by a Wess–Zumino action.

3.3.1 The Weyl–invariant measure

Let us assume that the theory contains also a dilaton χ. For the purposes of this section it will be considered as part of the gravitational sector and treated as an external field. For notational simplicity we will discuss the case $d = 4$ but it is easy to generalize to arbitrary even dimensions.

The crucial observation is that we can now construct Weyl invariant metrics in the spaces of scalar, Dirac and Maxwell fields, replacing the fixed scale μ by the dilaton:

$$\mathcal{G}_S(\varphi,\varphi') = \int d^4x\sqrt{g}\,\chi^2\varphi\varphi'\ , \tag{3.46}$$

$$\mathcal{G}_D(\bar\psi,\psi') = \int d^4x\sqrt{g}\,\frac{1}{2}\chi[\bar\psi\psi' + \bar\psi'\psi]\ , \tag{3.47}$$

$$\mathcal{G}_M(A,A') = \int d^4x\sqrt{g}\,\chi^2 A_\mu g^{\mu\nu}A'_\nu\ . \tag{3.48}$$

One can follow step by step the calculation in section 3.2, the only change being the replacement of μ by χ. The final result for the one–loop contribution to the effective action can be written as

$$\frac{n_S}{2}\mathrm{Tr}\log\mathcal{O}_S - \frac{n_D}{2}\mathrm{Tr}\log\mathcal{O}_D + \frac{n_M}{2}\mathrm{Tr}\log\mathcal{O}_M - n_M\mathrm{Tr}\log\mathcal{O}_{gh}\ , \tag{3.49}$$

where now

$$\{\mathcal{O}_S,\mathcal{O}_D,\mathcal{O}_{M\mu}{}^\nu,\mathcal{O}_{gh}\} = \chi^{-2}\left\{\Delta^{(0)},\Delta^{(1/2)},g_{\mu\sigma}\left(\Delta^{(1)}\right)^{\sigma\nu},\Delta^{(gh)}\right\} \tag{3.50}$$

One can then verify that

$$\mathcal{O}_S^\Omega(\Omega^{-1}\varphi) = \Omega^{-1}\mathcal{O}_S\varphi \tag{3.51}$$

$$\mathcal{O}_D^\Omega(\Omega^{-3/2}\psi) = \Omega^{-3/2}\mathcal{O}_D\psi \tag{3.52}$$

$$\mathcal{O}_{M\mu}^{\Omega\ \nu}A_\nu = \mathcal{O}_{M\mu}{}^\nu A_\nu \tag{3.53}$$

$$\mathcal{O}_{gh}^\Omega(\Omega^{-1}c) = \Omega^{-1}\mathcal{O}_{gh}c. \tag{3.54}$$

where the notation $\mathcal{O}^\Omega$ stands for the operator $\mathcal{O}$ constructed with the transformed metric $g^\Omega = \Omega^2 g$ and dilaton $\chi^\Omega = \Omega^{-1}\chi$. These operators map fields into fields transforming in the same way. (As observed earlier, they are dimensionless.) This implies that the eigenvalues of the operators $\mathcal{O}$ are Weyl–invariant and therefore also their determinants are invariant. We conclude that in the presence of a dilaton there exists a quantization procedure for noninteracting matter fields that respects Weyl invariance.

3.3.2 The Wess–Zumino action

We have seen that in the presence of a dilaton one has a choice between different quantization procedures, which can be understood as different functional measures: one of them breaks Weyl–invariance while the other maintains it. Let us denote $\Gamma^{\rm I}$ the effective action obtained with the standard measure and $\Gamma^{\rm II}$ the one obtained with the Weyl–invariant measure. The first is anomalous:

$$\delta_\omega\Gamma^{\rm I} = \int dx\, 2\omega\frac{\delta\Gamma^{\rm I}}{\delta g_{\mu\nu}}g_{\mu\nu} = -\int dx\sqrt{g}\,\omega\langle T^\mu{}_\mu\rangle^I \neq 0 \tag{3.55}$$

whereas the second is Weyl invariant: $\Gamma^{\rm II}(g^\Omega,\chi^\Omega) = \Gamma^{\rm II}(g,\chi)$, or in infinitesimal form

$$0 = \delta_\omega\Gamma^{\rm II} = \int dx\sqrt{g}\,\omega\left(2\frac{\delta\Gamma^{\rm II}}{\delta g_{\mu\nu}}g_{\mu\nu} - \frac{\delta\Gamma^{\rm II}}{\delta\chi}\chi\right)\ . \tag{3.56}$$

The Weyl invariant measure differs from the standard one simply by the replacement of the fixed mass μ by the dilaton χ, therefore we have

$$\Gamma^{\rm II}(g_{\mu\nu},\mu) = \Gamma^{\rm I}(g_{\mu\nu})\ . \tag{3.57}$$

We see that $\Gamma^{\rm II}$ can be obtained from $\Gamma^{\rm I}$ by applying the Stückelberg trick *after* quantization, *i.e.* to the mass parameter μ that has been introduced by the functional measure.

Another useful point of view is the following. Noting that $\Omega = \chi/\mu$ can be interpreted as the parameter of a Weyl transformation, the variation of $\Gamma^{\rm I}$ under a finite Weyl transformation gives functional $\Gamma_{WZ}(g,\chi)$: [5]

$$\Gamma^{\rm I}(g^\Omega) - \Gamma^{\rm I}(g) = \Gamma_{WZ}(g,\mu\Omega)\ . \tag{3.58}$$

[5]Here we view the Wess-Zumino action as a functional of a metric and a dilaton, two dimensionful fields. Sometimes one may prefer to think of it as as a functional of a metric and a Weyl transformation, the latter being a dimensionless function. The two points of view are related by some factors of μ.

In order to make contact with $\Gamma^{\rm II}$ we recall $\Gamma^{\rm II}[g,\chi]=\Gamma^{\rm II}\left[g^\Omega,\chi^\Omega\right]$. We can choose $\Omega=\chi/\mu$ so that $\chi^\Omega=\Omega^{-1}\chi=\mu$. With this choice we have $\Gamma^{\rm II}[g,\chi]=\Gamma^{\rm II}(g^\Omega_{\mu\nu},\mu)=\Gamma^{\rm I}(g^\Omega_{\mu\nu})$ where the last step we use (3.57). Substituing in (3.58) we find:

$$\Gamma^{\rm II}(g,\chi)=\Gamma^{\rm I}(g)+\Gamma_{WZ}(g,\chi)\ . \tag{3.59}$$

The Weyl invariance of $\Gamma^{\rm II}$ can be checked using the relations (3.43) and (3.44). In the next section these statements will be verified by direct calculation in $d=2$, where all these functionals can be written explicitly. We can think of the Weyl–invariant effective action as the ordinary effective action to which a Wess-Zumino term has been added, with the effect of canceling the Weyl anomaly. [6]

In the case of non–interacting, massless, conformal matter fields the WZ actions can be computed explicitly by integrating the trace anomaly. Let Ω_t be a one-parameter family of Weyl transformations with $\Omega_0=1$ and $\Omega_1=\Omega$, and let $g(t)_{\mu\nu}=g^{\Omega(t)}_{\mu\nu}$.

$$\Gamma_{WZ}(g_{\mu\nu},\Omega)=\int_0^1 dt\int dx\,\frac{\delta\Gamma}{\delta g_{\mu\nu}}\bigg|_{g(t)}\delta g(t)_{\mu\nu}=-\int_0^1 dt\int dx\,\sqrt{g(t)}\langle T^\mu_\mu\rangle_k\Omega(t)^{-1}\frac{d\Omega}{dt}\ . \tag{3.60}$$

In two dimensions, integrating the anomaly (3.39) and using the parametrization (3.11), one finds [80]

$$\Gamma_{WZ}(g_{\mu\nu},\mu e^\sigma)=-\frac{c}{24\pi}\int d^2x\sqrt{g}\left(R\sigma-\sigma\nabla^2\sigma\right)\ . \tag{3.61}$$

A similar procedure in four dimensions using (3.41) leads to [80]

$$\Gamma_{WZ}(g_{\mu\nu},\mu e^\sigma)=-\int dx\,\sqrt{g}\left\{c\,C^2\sigma-a\left[\left(E-\frac{2}{3}\Box R\right)\sigma+2\sigma\Delta_4\sigma\right]\right\}\ , \tag{3.62}$$

where

$$\Delta_4=\Box^2+2R^{\mu\nu}\nabla_\mu\nabla_\nu-\frac{2}{3}R\Box+\frac{1}{3}\nabla^\mu R\nabla_\mu\ . \tag{3.63}$$

At this point the reader will wonder whether the two procedures described above lead to different physical predictions or not. If the metric and dilaton are treated as classical external fields, but we allow them to be transformed, the two quantization procedures yield equivalent physics. In the Weyl–invariant procedure one has the freedom of choosing a gauge where $\chi=\mu$ and in this gauge all the results reduce to those of the standard procedure. In particular we observe that the trace of the energy-momentum tensor derived from the two actions $\Gamma^{\rm I}$ and $\Gamma^{\rm II}$ are the same. Indeed conformal invariance generically holds because of the dilaton and the trace of the stress–energy tensor is related to the one–point function of the dilaton via (3.56). This encodes exactly the same information of the trace anomaly. Moreover we observe that one obtains the same results taking functional derivatives with respect to $g_{\mu\nu}$ from $\Gamma^{\rm II}[g,\chi]$ and then

[6] This is completely analogous to what happens with gauge invariance in chiral theories [84–86].

setting $\chi = \mu$ or taking the derivative directly from $\Gamma^{\rm I}[g, \mu]$ because of (3.57). On the other hand if we assume that the metric (and dilaton) are going to be quantized too, the answer hinges on the choice of *their* functional measure. We defer a discussion of this point to section 3.6.

3.4 The Effective Average Action of matter fields coupled to an external gravitational field

So far we have made some general considerations on how Weyl invariance can be preserved under quantization via a careful choice of the measure. We also found a connection between the two procedures via the relation between $\Gamma^{\rm II}$ and $\Gamma^{\rm I}$. In this section we will be more concrete and show explicitly how Weyl invariance can be maintained along the RG flow. This is achieved studying the flow of the EAA. In section 3.4.1 we will consider the flow of the EAA induced by free matter fields conformally coupled to gravity while in section 3.4.2 we generalize the discussion to interacting matter fields. We shall perform the piecemeal functional integral computing the EAA via the integration of the flow equation. This will allows us to discuss some general aspects of the EA induced by conformally invariant matter actions. Gravity will be quantized later in section 3.5.

3.4.1 The EAA and its flow at one loop

We consider matter fields conformally coupled to gravity. Due to the conformal coupling one can observe that implementation of the procedure described in section 3.1 to make any theory Weyl invariant is not useful since even implementing this procedure all the extra terms cancel each other. In this case, as we will see, the dilaton appears only in the cutoff. Since for free fields the 1–loop effective action is complete we will (re)derive the flow equation so that the differences with the standard procedure are highlighted.

The definition of the EAA follows the same steps of the definition of the ordinary effective action, except that one modifies the bare action by adding to it a cutoff term $\Delta S_k(\varphi)$ that is quadratic in the fields and therefore modifies the propagator without affecting the interactions. Using the notation of (3.15), the cutoff term is:

$$\Delta S_k(g_{\mu\nu}, \varphi) = \frac{1}{2}\,\mathcal{G}\left(\varphi, \frac{R_k(\Delta)}{\mu^2}\varphi\right) = \frac{1}{2}\frac{k^2}{\mu^2}\sum_n a_n^2 r\left(\frac{\lambda_n}{k^2}\right) \ , \tag{3.64}$$

where we have written the cutoff (which has dimension of mass squared) as $R_k(z) = k^2 r(z/k^2)$. The generic features that this kernel has to satisfy has been described in section 1.2.

We define a k-dependent generating functional W_k by

$$e^{W_k(g_{\mu\nu}, J)} = \int D\varphi\, \exp\left(-S(g_{\mu\nu}, \varphi) - \Delta S_k(g_{\mu\nu}, \varphi) + \int dx\, J\varphi\right) \ . \tag{3.65}$$

The EAA is obtained by Legendre transforming, and then subtracting the cutoff:

$$\Gamma_k(g_{\mu\nu},\varphi) = -W_k(g_{\mu\nu},J) + \int dx\, J\varphi - \Delta S_k(g_{\mu\nu},\varphi)\,. \tag{3.66}$$

The evaluation of the EAA for Gaussian matter fields, conformally coupled to a metric, follows the same steps that led to (3.20). The only differences are the replacement of S by $S+\Delta S_k$ and hence of the "inverse propagator" Δ by the regularized inverse propagator" $P_k(\Delta)=\Delta+R_k(\Delta)$, and in the end the subtraction of ΔS_k. The result is

$$\Gamma_k^{\rm I}(g_{\mu\nu},\varphi) = S(g_{\mu\nu},\varphi) + \frac{1}{2}{\rm Tr}\log\left(\frac{P_k(\Delta)}{\mu^2}\right)\,. \tag{3.67}$$

We used here the superscript I to denote that this EAA has been obtained by using the standard measure and reduces to $\Gamma^{\rm I}$ for $k=0$. We would like now to define a Weyl–invariant form of EAA, to be called $\Gamma_k^{\rm II}$ in analogy to the effective action $\Gamma^{\rm II}$ discussed previously.

The first step is to clarify the meaning of the cutoff k in this context. In the usual treatment k is a constant with dimension of mass. In the present context these two properties are contradictory, it makes no sense to consider a dimensionful quantity constant in Weyl geometry as lengths at different points cannot be compared. A quantity that has a nonzero dimension cannot generally be a constant: it can only be constant in some special gauge. This means that the cutoff must be allowed to be a generic non-negative function of spacetime and, as such, it transforms under Weyl rescaling. Now we must give a meaning to the notion of couplings depending on such a cutoff. In a Weyl–invariant theory all couplings are dimensionless, and the only way they can depend on k is via the dimensionless combination $u=k/\chi$. This is general indeed we can think of the dilaton as a unit of measure and we parametrize our cutoff in term of the dilaton via $k=u\chi$. Note that by definition the dilaton cannot vanish anywhere, whereas the cutoff should be allowed to go to zero. So u is a non-negative dimensionless function on spacetime which we use to parametrize the RG flow for $\Gamma^{\rm II}$. This raises the question of the meaning of a running coupling whose argument is itself a function on spacetime. This would also induce further divergent terms which depend on the derivative of the couplings as considered for instance in [87]. In order to avoid such issues we will restrict ourselves to the case when u is a constant, in other words the cutoff and the dilaton are proportional.

With this point understood, the evaluation of the EAA with the Weyl–invariant measure is very simple: as in section 3.3 we just have to replace μ by χ

$$\Gamma_u^{\rm II}(g_{\mu\nu},\varphi) = S(g_{\mu\nu},\varphi) + \frac{1}{2}{\rm Tr}\log\left(\frac{\Delta+R_k(\Delta)}{\chi^2}\right) \tag{3.68}$$

$$= S(g_{\mu\nu},\varphi) + \frac{1}{2}{\rm Tr}\log\left(\mathcal{O}+u^2 r(u^2\mathcal{O})\right)\,. \tag{3.69}$$

In the second line we have reexpressed the EAA as a function of the Weyl–covariant operator $\mathcal{O}=\chi^{-2}\Delta$, the Weyl–invariant cutoff parameter u and the dimensionless function $r(z/k^2)=R_k(z)/k^2$. It is manifest that all dependence on k is via u and that $\Gamma_u^{\rm II}$ is Weyl–invariant.

Now we want to compute the functional integral in (3.67) and (3.68). These expressions need to be regularized so, as already anticipated, we will compute them starting from the 1–loop ERGE which can easily be derived taking a k–derivative of (3.67) and the u–derivative (3.68). First let us recall the general form of the flow equation: [7]

$$k\frac{d\Gamma_k}{dk} = \frac{1}{2}\mathrm{Tr}\left[\frac{\delta^2(\Gamma_k + \Delta S_k)}{\delta\varphi\delta\varphi}\right]^{-1} k\frac{d}{dk}\frac{\delta^2\Delta S_k}{\delta\varphi\delta\varphi} \ . \tag{3.70}$$

If we take the derivative of (3.67) with respect to k, using the definition $R_k(\Delta) = k^2 r(\Delta/k^2)$, we obtain

$$\begin{aligned} k\frac{d\Gamma^{\mathrm{I}}_k}{dk} &= \frac{1}{2}\mathrm{Tr}\left(\frac{1}{\Delta + R_k(\Delta)} k\frac{dR_k(\Delta)}{dk}\right) \\ &= \mathrm{Tr}\frac{r(\Delta/k^2) - (\Delta/k^2)r'(\Delta/k^2)}{(\Delta/k^2) + r(\Delta/k^2)} \ . \end{aligned} \tag{3.71}$$

It is easy to see, especially using the form in the first line, that this is a special case (i.e.: for free conformally coupled fields) of the ERGE (3.70).

One can repeat this argument in the case of the Weyl–invariant EAA with little changes, and the flow equation reads

$$u\frac{d\Gamma^{\mathrm{II}}_u}{du} = \mathrm{Tr}\frac{r(\mathcal{O}/u^2) - (\mathcal{O}/u^2)r'(\mathcal{O}/u^2)}{(\mathcal{O}/u^2) + r(\mathcal{O}/u^2)} \ . \tag{3.72}$$

In this form the r.h.s. of the ERGE is manifestly Weyl–invariant, since u is Weyl–invariant and one has the trace of a function of a Weyl–covariant operator. [8]

The EAAs Γ^{I}_k and Γ^{II}_u are not well–defined functionals, but their derivatives are well–defined. As explained above, one can integrate the ERGE and obtain, in the IR limit, the ordinary effective action. If one starts from a given Weyl–invariant classical matter action at scale Λ and integrates the flow of $k\frac{d\Gamma^{\mathrm{I}}_k}{dk}$, respectively $u\frac{d\Gamma^{\mathrm{II}}_u}{du}$, down to $(k,u) = 0$ one obtains exactly the effective action Γ^{I}, respectively Γ^{II}. Furthermore, at each u, Γ^{II}_u is obtained from Γ^{I}_k by the Stückelberg trick as we have seen in section 3.3. It is instructive to explicitly illustrate these statements in the case of $d = 2$ and, for the c–anomaly, also in the case $d = 4$.

[7] Note that the structure of (3.70) in field space is the trace of a contravariant two tensor times a covariant two–tensor (in de Witt notation, $((\Gamma^{(2)}_k + \Delta S^{(2)}_k)^{-1})^{ij}(\partial_t \Delta S^{(2)}_k)_{ji}$, where a superscript (2) denotes second functional derivative and $t = \log k$) and is therefore an invariant expression. In passing from (3.70) to (3.71) one uses the field space metric $\mathcal{G}$ to raise and lower indices and transform the covariant and contravariant tensors into mixed tensors, each of which can be seen as a function of Δ. In practice this amounts to canceling all factors of $\sqrt{g}$ and μ.

[8] Note that $\Delta/k^2 = \mathcal{O}/u^2$ so the r.h.s. of (3.71) and (3.72) are identical. The reason for the lack of invariance of the EAA Γ^{I} (and its derivative) is the measure which contains the absolute mass scale μ. If one allowed μ to be transformed, in the same way as we allow the cutoff k to be transformed, the two actions would be seen to be the same.

$d = 2$: the Polyakov action

In this section we consider the effective action of a free massless scalar field, such effective action is known exactly and has been computed in [77] integrating the conformal anomaly. It has been derived by integrating the ERGE in [88]. The main tool in this derivation is the non–local expansion of the heat kernel in powers of curvature which is reviewed in appendix C. Keeping terms up to two curvatures one has

$$\mathrm{Tr} e^{-s\Delta} = \frac{1}{4\pi s}\int d^2x\sqrt{g}\left[1 + s\frac{R}{6} + s^2 R f_{R,2d}(s\Delta)R + \dots\right] , \tag{3.73}$$

where

$$f_{R,2d}(x) = \frac{1}{32}f(x) + \frac{1}{8x}f(x) - \frac{1}{16x} + \frac{3}{8x^2}f(x) - \frac{3}{8x^2} ; \qquad f(x) = \int_0^1 d\xi e^{-x\xi(1-\xi)} .$$

Note that $f_{R,2d}$ is the sum of different form factors presented in appendix C which merge together once one sets

$$R_{\mu\nu} = \frac{1}{2}Rg_{\mu\nu}.$$

The r.h.s. of (3.71) can be written, after some manipulations,

$$k\frac{d\Gamma_k^{\mathrm{I}}}{dk} = \int ds\,\tilde{h}(s)\mathrm{Tr}\, e^{-s\Delta} ; \qquad h(z) = \int_0^\infty ds\,\tilde{h}(s)e^{-s\,z} ,$$

where $\tilde{h}(s)$ is the Laplace anti-transform of $h(z) = \frac{\partial_t R_k(z)}{z+R_k(z)}$. Using the explicit cutoff $R_k(z) = (k^2 - z)\theta(k^2 - z)$, we have simply $h(z) = 2\theta(k^2 - z)$ and the integrals give

$$\begin{aligned} k\frac{d\Gamma_k^{\mathrm{I}}}{dk} &= \int d^2x\sqrt{g}\left[\frac{k^2}{4\pi} + \frac{1}{24\pi}R\right. \\ &\left. + \frac{1}{64\pi}R\frac{1}{\Delta}\left(\sqrt{\frac{\widetilde{\Delta}}{\widetilde{\Delta}-4}} - \frac{\widetilde{\Delta}+4}{\widetilde{\Delta}}\sqrt{\frac{\widetilde{\Delta}-4}{\widetilde{\Delta}}}\right)\theta(\widetilde{\Delta}-4)R\right] + O(R^3) \end{aligned} \tag{3.74}$$

with $\widetilde{\Delta} = \Delta/k^2$. On the other hand, keeping terms at most quadratic in curvature, the EAA can be written in the form

$$\Gamma_k^{\mathrm{I}} = \int d^2x\sqrt{g}\left[a_k + b_k R + R\,c_k(\Delta)R\right] + O\left(R^3\right) \tag{3.75}$$

where $c_k(\Delta)$ is a nonlocal form-factor which, for dimensional reasons, can be written in the form $c_k(\Delta) = \frac{1}{\Delta}c(\widetilde{\Delta})$. The beta functions of a_k, b_k and c_k are then

$$\partial_t a_k = \frac{k^2}{4\pi} ; \quad \partial_t b_k = \frac{1}{24\pi} ; \quad \partial_t c = \frac{1}{64\pi}\left(\sqrt{\frac{\widetilde{\Delta}}{\widetilde{\Delta}-4}} - \frac{\widetilde{\Delta}+4}{\widetilde{\Delta}}\sqrt{\frac{\widetilde{\Delta}-4}{\widetilde{\Delta}}}\right)\theta(\widetilde{\Delta}-4) \tag{3.76}$$

In order to obtain the effective action, one integrates this flow from some UV scale Λ, that can later be sent to infinity, down to $k = 0$. Setting $a_\Lambda = \frac{\Lambda^2}{4\pi}$, one has $a_k = \frac{k^2}{4\pi}$ and therefore the

renormalized cosmological term vanishes in the IR limit. The Hilbert term has a logarithmically running coefficient $b_k = b_\Lambda - \frac{1}{24\pi}\log\frac{\Lambda}{k}$. We will not consider this term in the following because it is topological. We assume that c_k vanishes at $k \to \infty$, since the UV action only contains the matter terms. The integral over k is finite even in the limit $\Lambda \to \infty$, and one finds

$$c(\widetilde{\Delta}) = -\frac{1}{96\pi}\frac{\sqrt{\widetilde{\Delta}-4}(\widetilde{\Delta}+2)}{\widetilde{\Delta}^{3/2}}\theta(\widetilde{\Delta}-4)\ . \tag{3.77}$$

The explicit form of c_k can be found also employing the mass cutoff $R_k(z) = k^2$, in which case the computation can also be done analytically, giving

$$c(\tilde{\Delta}) = -\frac{1}{16\pi}\left[\frac{1}{6} - \frac{1}{\tilde{\Delta}} + \frac{4\mathrm{Arctanh}\left(\sqrt{\frac{\tilde{\Delta}}{\tilde{\Delta}+4}}\right)}{\tilde{\Delta}^{3/2}\sqrt{\tilde{\Delta}+4}}\right] \tag{3.78}$$

and with the exponential cutoff $R_k(z) = \frac{z}{exp(\frac{z}{k^2})-1}$, in which case it is computed numerically. All three give the same qualitative running, as depicted in figure 3.1. In the limit $k \to 0$ one obtains, in all cases, the Polyakov action: [9]

$$\Gamma^{\mathrm{I}}(g_{\mu\nu}) = -\frac{1}{96\pi}\int d^2x\sqrt{g}R\frac{1}{\Delta}R\ . \tag{3.79}$$

Note that the non–local behaviour of equation (3.79) is obtained only when $k \to 0$. Indeed, for $k > 0$, the interpolating EAA Γ_k should always allow a derivative expansion (namely an expansion in ∇^μ/k) if the starting point of the flow, roughly the bare action, is local and if the cutoff action is a smooth function. This requirement comes from the fact that at any RG step the variables are affected only in a localised patch and that long range interactions appear only after infinitely many steps (this guarantees that there are no infrared singularities for $k > 0$), see [17, 18]. We note that in the case of the optimized cutoff, which is not smooth, many terms in the (3.77) do not possess a well defined derivative expansion. The full function (3.77) has a trivial derivative expansion with zero coefficients to all orders. These pathologies are due to the nature of the optimized cutoff as shown in [89]. In the case of the mass cutoff it is possible to expand in $\tilde{\Delta}$ and check that (3.78) has a proper Taylor expansion.

The function c_k admits a series expansion $c_k(\Delta) = \frac{1}{k^2}\sum_{n=1}^{\infty} c_n\frac{k^{2n}}{\Delta^n}$. Then, one can explicitly perform the variation with respect to the metric and obtain the energy–momentum tensor. In particular, conformal variation of Γ_k^{I} gives the k–dependent trace anomaly:

$$\langle T^\mu_\mu\rangle^I_k = -\frac{2}{\sqrt{g}}g_{\mu\nu}\frac{\delta\Gamma^{\mathrm{I}}_u}{\delta g_{\mu\nu}} = -4c(\tilde{\Delta})R - \frac{2}{k^2}\sum_{n=0}^{\infty}\sum_{k=1}^{n-1}c_n\left(\frac{1}{\tilde{\Delta}^k}R\right)\left(\frac{1}{\tilde{\Delta}^{n-k}}R\right)\ . \tag{3.80}$$

[9] Using this action in (3.58) one recovers the WZ action (3.61). Conversely, the Polyakov action can be obtained from the WZ action by using the equation of motion for σ.

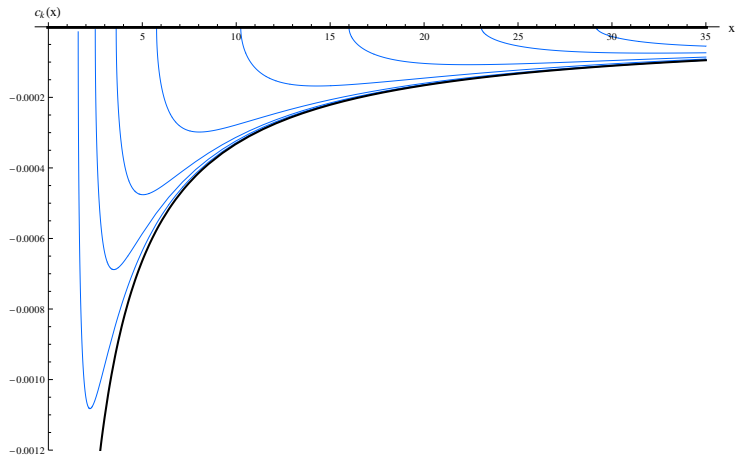


Figure 3.1: Shape of the form factor $c_k(x)$ of eq. (3.77) as a function of $x = \Delta$ for different values of k. The thick line shows the case $k = 0$ (Polyakov action).

We observe that the integrated trace anomaly can be written more explicitly

$$\int dx\sqrt{g}\,\langle T^{\mu}_{\mu}\rangle_k = \int dx\sqrt{g}\,\left(-4c(\tilde{\Delta})R + \frac{2}{k^2}R\,c'(\tilde{\Delta})R\right)\ . \tag{3.81}$$

For a fixed momentum Δ the linear term of the trace anomaly grows monotonically as k decreases, from zero at infinity to its canonical value at $k = 0$. The second term shows a nontrivial flow for $k \neq 0$, going to zero both in the UV and IR.

Let us now come to the effective action $\Gamma^{\rm II}$. Using the Weyl–invariant measure, the effective action is given by the determinant of the dimensionless operator $\mathcal{O} = \hat{\Delta} = \frac{1}{\chi^2}\Delta$, which can be identified with $\Delta_{\hat{g}}$, the operator constructed with the dimensionless, Weyl–invariant metric $\hat{g}_{\mu\nu} = \chi^2 g_{\mu\nu}$. We have to generalize this for finite $k \neq 0$. As discussed above, we assume that the cutoff is a constant multiple of the dilaton: $k = u\chi$. Neglecting the a– and b–terms, the effective average action can then be written in the manifestly Weyl–invariant form

$$\Gamma^{\rm II}_u(g_{\mu\nu},\chi) = \int d^2x\sqrt{g}\,\mathcal{R}\,\frac{1}{\chi^2\mathcal{O}}c\left(\frac{\mathcal{O}}{u^2}\right)\mathcal{R}\ , \tag{3.82}$$

with the same function c given in (3.77). In particular the Weyl–invariant version of the Polyakov action is obtained in the limit $u \to 0$:

$$\Gamma^{\rm II}(g_{\mu\nu},\chi) = -\frac{1}{96\pi}\int d^2x\sqrt{g}\,\mathcal{R}\,\frac{1}{\chi^2\mathcal{O}}\mathcal{R}\ . \tag{3.83}$$

It is now easy to check explicitly equation (3.59): for $c = 1$, using $\mathcal{R} = R + 2\Delta\sigma$, one finds

$$\Gamma^{\rm II} = -\frac{1}{96\pi}\int d^2x\sqrt{g}\,R\,\frac{1}{\Delta}R - \frac{1}{24\pi}\int d^2x\sqrt{g}\,\sigma(\Delta\sigma + R) = \Gamma^{\rm I} + \Gamma_{WZ}\ .$$

We have claimed in the end of section 3.3 that the trace of the energy–momentum tensors computed from $\Gamma^{\rm II}$ and $\Gamma^{\rm I}$ coincide in the gauge $\chi = \mu$. This statement actually holds also for

$k \neq 0$. A direct calculation yields

$$\langle T^{\mu}_{\mu} \rangle^{\rm II}_{u} = -\frac{2}{\sqrt{g}} g_{\mu\nu} \frac{\delta \Gamma^{\rm II}_{u}}{\delta g_{\mu\nu}} = -4c \left(\frac{\mathcal{O}}{u^2}\right) \mathcal{R} - \frac{2}{u^2 \chi^2} \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} c_n \left(\frac{u^{2k}}{\mathcal{O}^k} \mathcal{R}\right) \left(\frac{u^{2(n-k)}}{\mathcal{O}^{n-k}} \mathcal{R}\right) \tag{3.84}$$

One can verify that this is also equal to $\frac{1}{\sqrt{g}} \chi \frac{\delta \Gamma^{\rm II}_{u}}{\delta \chi}$, thereby obtaining an explicit check of the general statement (3.56). It is also interesting to observe that if we think of $\Gamma^{\rm II}_{u}$ as a function of k, χ and $g_{\mu\nu}$, and vary each keeping the other two fixed, the metric variation is again given by equation (3.84), the χ variation gives the first term in the r.h.s. of (3.84) and the k variation gives the second term. We also note that the "beta functional" can be written in general as [90]

$$u \frac{d\Gamma^{\rm II}_{u}}{du} = -\int dx \sqrt{g} \, \frac{2}{u^2 \chi^2} \mathcal{R} \, c' \left(\frac{\mathcal{O}}{u^2}\right) \mathcal{R} \, . \tag{3.85}$$

$d = 4$: the c–anomaly action

One would like to repeat the analysis of the previous section in $d = 4$ as much as possible. The main difference is that while in $d = 2$ the Polyakov action is the full effective action, in $d = 4$ there are terms with higher powers of curvature. We limit our analysis to second order in the curvatures. The EAA $\Gamma^{\rm I}_{k}$ has been computed in [91] and when $k \to 0$ the result can be compared to those found in [92] after changing the basis expansion from powers of $(R, C_{\mu\nu\rho\sigma})$ and their derivatives to powers of $(R, R_{\mu\nu})$ and their derivatives.

The computation shows that the first term of the EA has the form suggested by Deser and Schwimmer [76] as the source of the c–anomaly, namely the terms proportional to $C^2_{\mu\nu\rho\sigma}$ in (3.41). This action (in contrast to the Riegert action discussed below) also produces the correct flat spacetime limit for the correlation functions of the energy momentum tensor $\langle T_{\mu\nu} T_{\rho\sigma} \rangle$ [93,94].

In the basis of the tensors $(R, R_{\mu\nu})$ the terms cubic in curvature are known explicitly [95]. When the Riemann squared term in the anomaly is expanded in an infinite series in $(R, R_{\mu\nu})$, the action of [95] correctly reproduces the first terms of this expansion [96]. In order to reproduce the full anomaly (both c– and a–terms) one would need also terms in the effective action of order higher than three.

It is possible to write closed form actions that generate the full anomaly. A functional that generates both c– and a–anomaly is the Riegert action [83]

$$W(g_{\mu\nu}) = \int dx \, \sqrt{g} \frac{1}{8} \left(E - \frac{2}{3} \Box R\right) \Delta_4^{-1} \left[2c \, C^2 - a \left(E - \frac{2}{3} \Box R\right)\right] + \frac{a}{18} R^2 \, . \tag{3.86}$$

It has the drawback that it gives zero for the flat spacetime limit of the correlator of two energy–momentum tensors. This does not mean, however, that one cannot write the full effective action as the sum of the Riegert action and Weyl–invariant terms, because one can write the Deser–Schwimmer action as the Riegert action plus Weyl–invariant terms. In this case the energy–momentum correlator would come from the Weyl–invariant terms, as we shall see below.

The relation between the Wess–Zumino term (3.62) and the Riegert action (3.86) is very similar to the one between the two–dimensional Wess–Zumino action (3.61) and the Polyakov action (3.79): using the Riegert action in (3.58) one recovers the WZ action (3.62). Unlike the two–dimensional case, however, the converse procedure is not unique. The general idea is to replace the dilaton $\chi = \mu e^{\sigma}$, which in the WZ action is treated as an independent variable, by a functional of the metric $g_{\mu\nu}$ having the right transformation properties. One choice, which has been proposed in [68, 97] is

$$\sigma(g_{\mu\nu}) = \log\left(1 - \frac{1}{\Delta + R/6}\frac{R}{6}\right). \tag{3.87}$$

Another possibility is

$$\sigma(g_{\mu\nu}) = -\frac{1}{4}\frac{1}{\Delta_4}\left(E + \frac{2}{3}\Delta R + b\,C^2\right), \tag{3.88}$$

where b is an arbitrary constant. In both cases $\sigma(g_{\mu\nu}) \mapsto \sigma(g_{\mu\nu}) - \log\Omega$ under a Weyl transformation. Note that (3.88), for $b = c$ is the equation of motion for the dilaton coming from the WZ action (3.62), while for $b = 0$ it is the equation of motion coming from the a–term of the WZ action. The latter choice exactly reproduces (3.86); other choices of b give the Riegert action plus Weyl–invariant terms, while (3.87) gives another form of the anomaly functional.

In $d = 2$, knowing the explicit form of the effective action, we were able to explicitly check equation (3.59). In $d = 4$ we have only limited knowledge of the effective action. Instead of trying to check equation (3.59) we can use it to obtain some additional information on the effective action Γ^{I}. In particular we want to understand if it is possible to write Γ^{I} in such a way that the conformally anomalous term is written in a closed form as it happens for the Riegert action. We have

$$\Gamma^{\mathrm{I}}(g_{\mu\nu}) = \Gamma^{\mathrm{II}}(g_{\mu\nu}, \chi) - \Gamma_{WZ}(g_{\mu\nu}, \chi)\ , \tag{3.89}$$

where the first term in the r.h.s. is Weyl–invariant by construction and the anomaly comes entirely from the second term. For example if we use (3.88) with $b = 0$, the second term exactly reproduces the Riegert action and the correlator of two energy–momentum tensors must come from the first term. One finds

$$\Gamma^{\mathrm{II}}(g_{\mu\nu}, \mu e^{\sigma(g_{\mu\nu})}) = -\frac{1}{2}\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\,\frac{N_0 + 6N_{1/2} + 12N_1}{120}\,\mathcal{C}_{\mu\nu\rho\sigma}\log\left(\frac{\mathcal{O}}{u_0^2}\right)\mathcal{C}^{\mu\nu\rho\sigma} + \dots \tag{3.90}$$

where $\mathcal{C}_{\mu\nu\rho\sigma}$ is the Weyl tensor constructed with the metric $e^{2\sigma(g)}g_{\mu\nu}$. Expanding this to second order in the curvature of $g_{\mu\nu}$ one reobtains as a leading term the one proposed by Deser. The lack of Weyl–invariance of this leading term is compensated by higher terms in the expansion. This shows that there is no contradicton between the presence of the Riegert and the Deser–Schwimmer terms in the effective action Γ^{I}, and the flat space limit of energy–momentum tensor correlators. Thus there is also no disagreement with [80] and with [97].

Finally, following [91], we can write the explicit form of the interpolating EAA $\Gamma_u^{\rm II}$. For a scalar field we have

$$\begin{aligned}\Gamma_u^{\rm II}(g_{\mu\nu},\chi) &= -\frac{1}{2}\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\,\mathcal{C}_{\mu\nu\rho\sigma}\Bigg\{\frac{1}{120}\log u^2 \\ &+ \theta\left(\frac{\mathcal{O}}{u^2}-4\right)\Bigg[-\frac{1}{120}\log u^2 - \frac{4u^6\sqrt{\frac{\mathcal{O}}{u^2}-4}\sqrt{\frac{\mathcal{O}}{u^2}}}{75\mathcal{O}^3} + \frac{11u^4\sqrt{\frac{\mathcal{O}}{u^2}-4}\sqrt{\frac{\mathcal{O}}{u^2}}}{225\mathcal{O}^2} \\ &- \frac{23}{900}\sqrt{1-\frac{4u^2}{\mathcal{O}}} + \frac{1}{120}\log\left(\frac{\mathcal{O}}{2}\left(\sqrt{1-\frac{4u^2}{\mathcal{O}}}+1\right)-u^2\right)\Bigg]\Bigg\}\mathcal{C}^{\mu\nu\rho\sigma} + \dots \end{aligned} \tag{3.91}$$

The same computation can be repeated in the case of fermions and vectors and a different interpolating function can be found. When $u \to 0$ we get back equation (3.90).

3.4.2 Interacting matter fields

In the preceding sections we have shown that there exists a quantization procedure such that the effective action which is obtained by integrating out free (Gaussian) matter fields remains Weyl invariant. The proof was simple because the integration over matter was Gaussian. Here we generalize the result to the case when there are matter interactions.

As in the preceding section, we begin by considering the case when the initial matter action is Weyl invariant even without invoking a coupling to the dilaton. This is the case for massless, renormalizable quantum field theories such as φ^4, Yang-Mills theory and fermions with Yukawa couplings in $d = 4$. The interactions are of the form $S_{int}(g_{\mu\nu}, \Psi_a) = \lambda \int dx\sqrt{g}\,\mathcal{L}_{int}$ where $\mathcal{L}_{int}$ is a dimension d operator and λ is dimensionless. Interactions generate new anomalous terms over and above those that we have already considered for Gaussian matter. The trace anomaly of free matter vanishes in the limit of flat space, but this is not true for interacting fields: the trace is then proportional to the beta function. For the interaction term given above one has in flat space

$$\int dx\,\omega\,\langle T^{\mu}{}_{\mu}\rangle = -\delta_\omega S_{int} = \int dx\,\omega\,\beta_\lambda\,\mathcal{L}_{int} \tag{3.92}$$

where $\beta_\lambda = k\frac{d\lambda}{dk}$.

We want to study the effective action of this theory, which is obtained by integrating out the matter fields. In order to be able to make non–perturbative statements we will use the ERGE as a machine for calculating the effective action, as discussed in the introduction and exemplified by the calculations in the previous section. The general idea is to begin with some Weyl–invariant bare action at some scale and to integrate the RG flow. If the "beta functional" is itself Weyl–invariant, the action at each scale will be Weyl–invariant. The effective action, which is obtained by letting $k \to 0$, will also be Weyl–invariant.

This statement is seemingly in contrast with (3.92), which implies that Weyl invariance can only be achieved when all beta functions are zero. How can one maintain Weyl–invariance along a flow? The trick is to consider the flow as dependence of λ on the dimensionless parameter $u = k/\chi$. We assume u to be constant to avoid issues related to the interpretation of a coupling depending on a function. [10] Since u is Weyl–invariant, also $\lambda(u)$ is. This is very much in the spirit of Weyl's geometry, where the dilaton is interpreted physically as the unit of mass and u is the cutoff measured in the chosen units.

We now see that with this definition of RG, the running of couplings does not in itself break Weyl invariance. In the spirit of Weyl's theory the dilaton is taken as a reference scale and the couplings are functions of u. Since u is Weyl–invariant,

$$\delta_\omega S_{int} = 0 \ , \tag{3.93}$$

even when the beta function $\beta_\lambda = u\frac{d\lambda}{du}$ is not zero. It is important to stress that this should not be interpreted as vanishing trace of the energy–momentum tensor. We argued in section 3.3 that the energy–momentum tensor is the same whether one uses the standard or the Weyl–invariant measure. That argument is not restricted to non–interacting matter and applies here too. So, as in the case of free matter fields discussed at the end of the preceding section, the physical content of the Weyl–invariant theory is exactly the same as in the usual formalism. The recovery of Weyl–invariance is due to additional terms that involve the variation of the action with respect to the dilaton.

Let us return now to the issue of the Weyl–invariance of the flow. In full analogy with the discussion of conformally coupled free matter fields, in order to have Weyl invariance in the presence of the cutoff k, the latter must be transformed as a field of dimension one. Then, one can construct a Weyl–invariant cutoff action. Since the cutoff action is always quadratic in the quantum fields, one can use exactly the same procedure that we followed in the case of free fields. The r.h.s. of the ERGE is given in (3.70) as a trace of a function involving the Hessian and the k–derivative of the cutoff. If the field φ has weight w, the two terms in (3.70) have the transformation properties:

$$\frac{\delta^2(\Gamma_k + \Delta S_k)}{\delta\varphi\delta\varphi} \ \mapsto \ \Omega^{-w}\frac{\delta^2(\Gamma_k + \Delta S_k)}{\delta\varphi\delta\varphi}\Omega^{-w} \ , \tag{3.94}$$

$$k\frac{d}{dk}\frac{\delta^2\Delta S_k}{\delta\varphi\delta\varphi} \ \mapsto \ \Omega^{-w}k\frac{d}{dk}\frac{\delta^2\Delta S_k}{\delta\varphi\delta\varphi}\Omega^{-w} \ . \tag{3.95}$$

As a consequence, the trace in the r.h.s. of (3.70) is invariant. Since the beta functional is Weyl–invariant, if we start from some initial condition that is Weyl invariant we will remain within the

[10] Note that in this way the couplings will remain constant in spacetime. In this sense our approach differs from those in [98–100], where the couplings are allowed to become functions on spacetime.

subspace of theories that are Weyl–invariant. The effective action Γ^{II}, which is obtained as the limit of the flow for $k \to 0$, will also be Weyl invariant.

The advantage of the calculation based on the ERGE is that it extends easily to arbitrary theories. We can extend our results also to theories which are not Weyl invariant to begin with and relax all constraints on the functional form of the action $S(g_{\mu\nu}, \psi_a, g_i)$. Let us suppose that we know the form of the action S at some (constant) cutoff $k = u\mu$. It gives rise, via its flow, to an effective action Γ. Using the Stückelberg trick, as discussed in the end of section 3.1, we can construct an action $\hat{S}(g_{\mu\nu}, \chi, \psi_a, \hat{g}_i)$ and take it as initial point of the flow at cutoff $k = u\chi$ (which could now be some function of position). Flowing towards the IR from this starting point leads to an effective action Γ^{II} that is still Weyl invariant. When Γ^{II} is evaluated at constant $\chi = \mu$ it agrees with the effective action Γ^{I} evaluated in the Weyl–non–invariant flow. In other words, Γ^{II} could be obtained from Γ^{I} using the Stückelberg trick. We thus see that *quantization commutes with the Stückelberg trick.* [11]

As mentioned earlier, renormalizability is not required for these arguments, because the ERGE is UV finite. Divergences manifest themselves when one tries to solve for the flow towards large k. The question whether this theory has a sensible UV limit can be answered by studying the flow for increasing u. If the trajectory tends to an UV fixed point it is called a "renormalizable" or "asymptotically safe" trajectory. If instead the trajectory diverges in the UV, it describes an effective field theory with an UV cutoff scale. We will address in section 3.5 the meaning of a fixed point in a theory space consisting entirely of Weyl–invariant actions.

3.5 Dynamical gravity: the non–integrable case

Until now we discussed the compatibility of Weyl invariance when matter fields are quantized in curved spacetime. One may wonder if something "goes wrong" trying to go through the same steps when quantizing gravity. As we will see this is not the case. We will quantize gravity in the sense of an effective field theory employing the techniques explained in section 1.3. We will compute the one–loop beta functions and search for fixed points of this theory (in the case of Asymptotic Safety we have seen that the one–loop computation gives qualitatively correct information).

We consider the most general class of Weyl-invariant actions for $g_{\mu\nu}$, b_μ and χ that contain at most two derivatives, see equation (3.96). It defines a four-dimensional theory space. In this theory Weyl invariance is "higgsed": in the "unitary" gauge the kinetic term of χ becomes a mass term for b_μ. However, there is a three-dimensional subspace of theories where b_μ is massless

[11] The relation between Γ^{I} and Γ^{II} will always be as in (3.59), but in the general case Γ^{I}, and consequently also the Wess–Zumino action, will contain infinitely many Weyl–non–invariant terms.

and an additional abelian gauge invariance appears. In addition to the issue of preservation of Weyl invariance, there is therefore the issue of preservation of this additional gauge invariance. This theory space thus offers an interesting opportunity to study the RG flow in theory spaces admitting subspaces with special properties. Somewhat similar issues appear in topologically massive gravity and in three-dimensional higher derivative gravity. The RG flows studied in [101, 102] and [35] did not preserve the special subspaces, in those cases. In the case studied here we can construct flows that either preserve or do not preserve the special subspace. Here we will mainly consider the non–integrable case, the integrable case goes through with obvious modifications, see [90, 103].

3.5.1 The classical action

If we restrict ourselves to actions that contain at most two derivatives of the fields, we have the following four–parameter family of actions [71]:

$$S = \int d^4x\sqrt{g}\left[\frac{g_1}{2}D_\mu\chi D^\mu\chi + g_2\chi^4 + \frac{g_3}{4}F_{\mu\nu}F^{\mu\nu} - g_4\chi^2\mathcal{R}\right] . \tag{3.96}$$

Every term in the above action is separately Weyl invariant. The equations of motion that follow from this action, written in explicitly Weyl–covariant form, are

$$0 = -g_1D^2\chi + 4g_2\chi^3 - 2g_4\chi\mathcal{R} \tag{3.97}$$

$$0 = -g_3D_\nu F^{\nu\mu} + (g_1 + 12g_4)\chi D^\mu\chi \tag{3.98}$$

$$\begin{aligned} 0 = {} & g_4\chi^2\left(\mathcal{R}^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\mathcal{R}\right) - \frac{g_3}{2}\left(F^{\mu\rho}F^{\nu}{}_{\rho} - \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) \\ & -\frac{g_1}{2}\left(D^\mu\chi D^\nu\chi - \frac{1}{2}g^{\mu\nu}D_\rho\chi D^\rho\chi\right) + \frac{1}{2}g_2g^{\mu\nu}\chi^4 + g_4\left(g^{\mu\nu}D^2\chi^2 - D^{(\mu}D^{\nu)}\chi^2\right) . \end{aligned} \tag{3.99}$$

In the special case of integrable Weyl geometry the Weyl connection is flat: $F_{\mu\nu} = 0$. This case can be obtained as follows. With the dilaton one constructs a "pure gauge" Weyl vector

$$s_\mu = -\chi^{-1}\partial_\mu\chi . \tag{3.100}$$

One can use this gauge field to construct a covariant derivative $D^{(s)}$ and a curvature $\mathcal{R}^{(s)}$, as in equations (3.3,3.8). When ambiguities can arise we will denote the previously defined covariant derivative and curvature of b_μ by $D^{(b)}$ and $\mathcal{R}^{(b)}$. Moreover one has

$$D^{(s)}_\mu\chi = 0 . \tag{3.101}$$

Note that at the classical level this can be seen as a special solution of the equations of motion: from (3.98) one sees that if $g_1 + 12g_4 \neq 0$, $F_{\mu\nu} = 0$ implies $D_\mu\chi = 0$, which in turn is solved by $b_\mu = s_\mu$. If we use this condition in the action, it reduces to:

$$\int\sqrt{g}\left[g_2\chi^4 - g_4\chi^2\mathcal{R}^{(s)}\right] = \int\sqrt{g}\left[g_2\chi^4 - g_4\chi^2\left(R - 6\chi^{-1}\nabla^2\chi\right)\right] . \tag{3.102}$$

As already observed in [104], the kinetic term of χ has the wrong sign (note that here we are writing the Euclidean action). This action is Weyl–invariant even without the Weyl gauge field. It is said to be obtained from that of a massless scalar by "Ricci gauging" [72].

This theory is just ordinary general relativity, with cosmological constant, rewritten in Weyl–invariant form by use of a compensator field. In fact, from the assumption that $\chi > 0$ everywhere and from the transformation property $\chi \to \Omega^{-1}\chi$ one deduces the existence of a gauge where χ is constant. We can set

$$g_4\chi^2 = \frac{1}{16\pi G} \; ; \qquad g_2\chi^2 = 2g_4\Lambda \; . \tag{3.103}$$

Then the action (3.102) becomes just

$$S(g) = \frac{1}{16\pi G}\int d^4x\sqrt{g}(2\Lambda - R) \; . \tag{3.104}$$

Now let us observe that

$$\mathcal{R}^{(b)} = \mathcal{R}^{(s)} + 6\chi^{-1}D^2\chi \; . \tag{3.105}$$

Using this and the rule for integration by parts (E.15) one finds that (3.96) can be rewritten in the form:

$$S = \int d^4x\sqrt{g}\left[\frac{g_1 + 12g_4}{2}D_\mu\chi D^\mu\chi + g_2\chi^4 + \frac{g_3}{4}F_{\mu\nu}F^{\mu\nu} - g_4\chi^2\mathcal{R}^{(s)}\right] \; . \tag{3.106}$$

This form makes it clear that a Higgs phenomenon is at work in this theory. Going to the gauge (3.103) the action reads

$$S(g) = \int d^4x\sqrt{g}\left[\frac{1}{16\pi G}(2\Lambda - R) + \frac{g_3}{4}F_{\mu\nu}F^{\mu\nu} + \frac{g_1 + 12g_4}{32\pi G g_4}b_\mu b^\mu\right] \; , \tag{3.107}$$

describing gravity coupled to a massive vector field. In the special case when $g_1 + 12g_4 = 0$, the Weyl gauge field is massless and we are left with

$$\int \sqrt{g}\left[\frac{g_3}{4}F_{\mu\nu}F^{\mu\nu} + g_2\chi^4 - g_4\left(\chi^2 R + 6(\nabla\chi)^2\right)\right] \; . \tag{3.108}$$

This is the same as (3.102), plus the action of an abelian vector field that is decoupled from χ. As a result, while the general action (3.96) is only invariant under the Weyl transformation

$$g'_{\mu\nu} = \Omega^2 g_{\mu\nu} \; , \qquad b'_\mu = b_\mu + \Omega^{-1}\partial_\mu\Omega \; , \qquad \chi' = \Omega^{-1}\chi \; , \tag{3.109}$$

the action (3.108) is additionally invariant under the "modified Weyl transformation" where b_μ is inert:

$$g'_{\mu\nu} = \Omega^2 g_{\mu\nu} \; , \qquad b'_\mu = b_\mu \; , \qquad \chi' = \Omega^{-1}\chi \; , \tag{3.110}$$

This additional invariance is a consequence of the fact that the Maxwell action in four dimensions is invariant under Weyl transformations when the gauge field is treated as a field of Weyl weight

zero. One can reparametrize these two gauge invariances as modified Weyl transformations and ordinary abelian gauge transformations

$$g'_{\mu\nu} = g_{\mu\nu} \ , \qquad b'_\mu = b_\mu + g^{-1}\partial_\mu g \ , \qquad \chi' = \chi \ , \tag{3.111}$$

where g is a gauge transformation parameter. Thus (3.108) can be interpreted as the action of conformal gravity (or equivalently the action of a Ricci–gauged scalar) coupled to an abelian gauge field which has nothing to do with Weyl transformations.

In the following we will refer to the subspace defined by the equation $g_1 + 12g_4 = 0$ as the "massless subspace". One of the goals of this section is to understand how Weyl invariance can be maintained under quantization in the non–integrable Weyl theory and in particular whether the massless subspace is preserved by the renormalization group flow.

3.5.2 The quadratic action

In this section we give the second variation of the action, which is required for the quantization of the theory. We will use the background field method. For each field we choose generic background values, henceforth denoted $g_{\mu\nu}$, b_μ and χ and expand:

$$g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu} \ ; \qquad b_\mu \to b_\mu + w_\mu \ ; \qquad \chi \to \chi + \eta \ . \tag{3.112}$$

To second order in $h_{\mu\nu}$, w_μ and η, the action (3.96) becomes

$$\begin{aligned}
\frac{1}{2} \quad & \int dx\,\sqrt{g}\,\Big\{ g_4\chi^2\Big(-\frac{1}{2}h^{\mu\nu}D^2h_{\mu\nu} + h^{\mu\nu}D_\mu D_\rho h^\rho{}_\nu - hD_\mu D_\nu h^{\mu\nu} + \frac{1}{2}hD^2h \\
& + \mathcal{R}^{\mu\nu}h\,h_{\mu\nu} - \mathcal{R}^{\mu\nu}h_{\mu\rho}h_\nu{}^\rho - \mathcal{R}_{\alpha\mu\beta\nu}h^{\mu\nu}h^{\alpha\beta}\Big) - g_4D^\rho\chi^2\,(2hD_\sigma h^\sigma{}_\rho + h_{\rho\nu}D_\sigma h^{\sigma\nu}) \\
& + g_4D^2\chi^2\left(\frac{1}{4}h^2 - \frac{3}{4}h^{\mu\nu}h_{\mu\nu}\right) + g_4D^\mu D^\nu\chi^2(h_\mu{}^\rho h_{\nu\rho} - 2h\,h_{\mu\nu}) \\
& + \Big[\frac{g_1}{2}(D\chi)^2 + g_2\chi^4 + \frac{g_3}{4}F^2 - g_4\chi^2\mathcal{R}\Big]\left(\frac{1}{4}h^2 - \frac{1}{2}h^{\mu\nu}h_{\mu\nu}\right) \\
& + g_1D_\mu\chi D_\nu\chi\left[h^{\mu\alpha}h^\nu_\alpha - \frac{1}{2}hh^{\mu\nu}\right] + \frac{g_3}{2}\left[F_{\mu\nu}F_\rho{}^\nu(2h^{\mu\alpha}h^\rho{}_\alpha - hh^{\mu\rho}) + F_{\mu\nu}F_{\rho\sigma}h^{\mu\rho}h^{\nu\sigma}\right] \\
& + g_3w_\mu\left(-g^{\mu\nu}D^2 + D^\mu D^\nu + \mathcal{R}^{\mu\nu}\right)w_\nu + (g_1 + 12g_4)\chi^2w_\mu w^\mu \\
& + \eta\left(-g_1D^2 + 12g_2\chi^2 - 2g_4\mathcal{R}\right)\eta \\
& + g_3F^{\mu\nu}hD_\mu w_\nu - 2g_3F_\rho{}^\nu h^{\mu\rho}\left(D_\mu w_\nu - D_\nu w_\mu\right) + (g_1 + 12g_4)\chi D^\mu\chi\left(hw_\mu - 2h_{\mu\nu}w^\nu\right) \\
& + g_1D^\mu\chi\left(hD_\mu\eta - 2h_{\mu\nu}D^\nu\eta\right) + 4g_4\chi\left(D^2h - D^\mu D^\nu h_{\mu\nu} + \mathcal{R}_{\mu\nu}h^{\mu\nu} - \frac{1}{2}\mathcal{R}h\right)\eta \\
& + 4g_2\chi^3h\,\eta - 2(g_1 + 12g_4)\chi\eta D_\mu w^\mu\Big\} \ .
\end{aligned} \tag{3.113}$$

We have chosen to collect first the terms quadratic in h, w, η and then the mixed terms h-w, h-η and w-η. The origin of each term can be easily traced by looking at the coefficients g_1, g_2, g_3 and g_4.

The gauge fixing

The quadratic action has zero modes corresponding to infinitesimal diffeomorphisms ξ and infinitesimal Weyl transformations ω:

$$h_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} \; ; \qquad w_\mu = \mathcal{L}_\xi b_\mu \; ; \qquad \eta = \mathcal{L}_\xi \chi \; , \tag{3.114}$$

$$h_{\mu\nu} = 2\omega g_{\mu\nu} \; ; \qquad w_\mu = \partial_\mu \omega \; ; \qquad \eta = -\omega\chi \; . \tag{3.115}$$

Quantization requires a nondegenerate operator, which is achieved by adding a suitable gauge fixing condition. In the background field method, the gauge fixing is designed so as to preserve the "background transformations"

$$\delta^{(D)}_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} \; ; \qquad \delta^{(D)}_\xi b_\mu = \mathcal{L}_\xi b_\mu \; ; \qquad \delta^{(D)}_\xi \chi = \mathcal{L}_\xi \chi \; ; \tag{3.116}$$

$$\delta^{(D)}_\xi h_{\mu\nu} = \mathcal{L}_\xi h_{\mu\nu} \; ; \qquad \delta^{(D)}_\xi w_\mu = \mathcal{L}_\xi w_\mu \; ; \qquad \delta^{(D)}_\xi \eta = \mathcal{L}_\xi \eta \; , \tag{3.117}$$

$$\delta^{(W)}_\omega g_{\mu\nu} = 2\omega g_{\mu\nu} \; ; \qquad \delta^{(W)}_\omega b_\mu = \partial_\mu \omega \; ; \qquad \delta^{(W)}_\omega \chi = -\omega\chi \; , \tag{3.118}$$

$$\delta^{(W)}_\omega h_{\mu\nu} = 2\omega h_{\mu\nu} \; ; \qquad \delta^{(W)}_\omega w_\mu = 0 \; ; \qquad \delta^{(W)}_\omega \eta = -\omega\eta \; , \tag{3.119}$$

For the sake of defining a Weyl-covariant ghost operator it is convenient to define modified diffeomorphism generators [105] (see also appendix E for further details)

$$\tilde{\delta}^{(D)}_\xi = \delta^{(D)}_\xi + \delta^{(W)}_{-\xi^\mu b_\mu} \; .$$

A gauge fixing term for the modified diffeomorphisms that manifestly preserves the background gauge transformations is

$$S_{GF} = \frac{g_4}{2\alpha} \int d^4x \sqrt{g}\, \chi^2 F_\mu \bar{g}^{\mu\nu} F_\nu \; , \tag{3.120}$$

where

$$F_\nu = D_\mu h^\mu{}_\nu - \frac{1}{2} D_\nu h \; . \tag{3.121}$$

The ghost action corresponding to the gauge (3.121) is given by

$$S_{gh} = -\int d^4x \sqrt{g}\, \bar{C}^\mu \left(\delta^\nu_\mu D^2 + \mathcal{R}_\mu{}^\nu\right) C_\rho \; , \tag{3.122}$$

where $\bar{C}$ and C are anticommuting vector fields. To gauge–fix Weyl invariance we impose that $\eta = 0$, a condition that does not lead to ghosts. With this condition we can simply delete from the Hessian the rows and columns that involve the η field and we remain with a Hessian that is a quadratic form in the space of the covariant symmetric tensors $h_{\mu\nu}$.

In the following we will choose the Feynman–de Donder gauge $\alpha = 1/g_4$, which simplifies the kinetic operators. With these choices the gauge fixing can be expanded as

$$\begin{aligned} S_{GF} &= \frac{g_4}{2}\int d^4x\sqrt{g}\Big[\chi^2\left(-h_{\mu\nu}D^\mu D_\rho h^{\rho\nu} + hD^\mu D^\nu h_{\mu\nu} - \frac{1}{4}hD^2h\right) \\ &\qquad + D_\mu\chi^2\left(-h^{\mu\nu}D^\rho h_{\rho\nu} + hD_\nu h^{\mu\nu} - \frac{1}{4}hD^\mu h\right)\Big] \ . \end{aligned} \tag{3.123}$$

In [103] the gauge fixing had the same form, but with b_μ replaced by s_μ. Because of (3.101), the second line vanished.

When the gauge fixing is taken into account, the total quadratic action takes the form

$$\begin{aligned} \frac{1}{2} & \int dx\,\sqrt{g}\Big\{g_4\chi^2\left(-\frac{1}{2}h^{\mu\nu}D^2h_{\mu\nu} + \frac{1}{2}hD^2h + \mathcal{R}^{\mu\nu}h\,h_{\mu\nu} - \mathcal{R}^{\mu\nu}h_{\mu\rho}h_\nu{}^\rho - \mathcal{R}_{\alpha\mu\beta\nu}h^{\mu\nu}h^{\alpha\beta}\right) \\ & - g_4D^\rho\chi^2\left(hD_\sigma h^\sigma{}_\rho + 2h_{\rho\nu}D_\sigma h^{\sigma\nu}\right) \\ & + g_4D^2\chi^2\left(\frac{3}{8}h^2 - \frac{3}{4}h^{\mu\nu}h_{\mu\nu}\right) + g_4D^\mu D^\nu\chi^2(h_\mu{}^\rho h_{\nu\rho} - 2h\,h_{\mu\nu}) \\ & + \left[\frac{g_1}{2}(D\chi)^2 + g_2\chi^4 + \frac{g_3}{4}F^2 - g_4\chi^2\mathcal{R}\right]\left(\frac{1}{4}h^2 - \frac{1}{2}h^{\mu\nu}h_{\mu\nu}\right) \\ & + g_1D_\mu\chi D_\nu\chi\left[h^{\mu\alpha}h_\alpha^\nu - \frac{1}{2}hh^{\mu\nu}\right] + \frac{g_3}{2}\left[F_{\mu\nu}F_\rho{}^\nu(2h^{\mu\alpha}h^\rho{}_\alpha - hh^{\mu\rho}) + F_{\mu\nu}F_{\rho\sigma}h^{\mu\rho}h^{\nu\sigma}\right] \\ & + g_3w_\mu\left(-g^{\mu\nu}D^2 + D^\mu D^\nu + \mathcal{R}^{\mu\nu}\right)w_\nu + (g_1+12g_4)\chi^2w_\mu w^\mu \\ & + g_3F^{\mu\nu}hD_\mu w_\nu - 2g_3F_\rho{}^\nu h^{\mu\rho}\left(D_\mu w_\nu - D_\nu w_\mu\right) + (g_1+12g_4)\chi D^\mu\chi\left(hw_\mu - 2h_{\mu\nu}w^\nu\right)\Big\} \ . \end{aligned} \tag{3.124}$$

Note in particular that the last term in the second last line is a mass term for w_μ proportional to $g_1 + 12g_4$, in accordance with the previous statement that a Higgs phenomenon is occurring in this theory.

For technical reasons it proves convenient to decompose the field w_μ into its transverse and longitudinal components. We refer to appendix E.3 for some details.

The cutoff

We use the formalism of the effective average action, which is an effective action Γ_k calculated in the presence of an infrared cutoff of the form

$$\Delta S_k = \frac{1}{2}\int dx\sqrt{g}\Psi\mathcal{R}_k(-D^2)\Psi \ . \tag{3.125}$$

Here $\Psi = (h_{\mu\nu}, w_\mu)$ is the multiplet formed by the fluctuation fields and $\mathcal{R}$ is a matrix in field space containing the couplings g_i, times a cutoff profile function R_k which we choose to be $R_k(z) = (k^2 - z)\theta(k^2 - z)$ [56]. We neglect the k-derivatives of the couplings in the cutoff (one loop approximation).

The k-derivative of the effective average action satisfies the Wetterich equation [1,2]

$$k\frac{d\Gamma_k}{dk} = \frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2\Gamma_k}{\delta\Psi\delta\Psi} + \mathcal{R}_k\right)^{-1} k\frac{d\mathcal{R}_k}{dk} \ .$$

As we have seen in the first chapter the r.h.s. of this equation is the "beta functional" of the theory, the generating functional of all beta functions (in the sense that the coefficient of some field monomial is the beta function of the corresponding coupling). Of course the effective average action will generally contain infinitely many terms but here we concentrate our attention just on the ones of the form appearing in the action (3.96).

In order to maintain Weyl invariance along the flow, the computation is carried out along the same lines of section 3.4 since, for all purposes, we can now treat the graviton as a matter field in a curved spacetime with (background) metric $\bar{g}_{\mu\nu}$. The following procedure is used. First, as already indicated in (3.125), the cutoff is chosen to be a function of the Weyl–covariant operator $-D^2$. Then, instead of thinking of $\log k$ as the independent variable of the flow, we assume that the cutoff k is proportional to χ and we take the Weyl–invariant, dimensionless, constant quantity $u = k/\chi$ as independent variable. Thus, the couplings will be functions of u. The cutoff can be rewritten

$$\Delta S_k(-D^2) = \frac{1}{2}\int dx\sqrt{g}\,\chi^2\Psi(u^2-\mathcal{O})\theta(u^2-\mathcal{O})\Psi \tag{3.126}$$

where $\mathcal{O} = -(1/\chi^2)D^2$.

Since the r.h.s. of the Wetterich equation is the trace of a function of a Weyl–covariant operator, it is Weyl–invariant. Using heat kernel methods, it can be expanded as a sum of monomials constructed with the background fields, their derivatives, and the curvatures. By isolating terms of the form (3.96) one reads the beta functions of the couplings g_i.

3.5.3 The RG flow

Derivation of the beta functions

In order to project out the beta functions of the various couplings one has to calculate the relevant terms of the functional trace on the r.h.s. of the Wetterich equation. In order to simplify the calculations, we take advantage of the independence of the results from the choice of background, and we choose for each coupling/beta function the simplest background that makes the corresponding field monomial nonzero. As long as the same gauge condition and cutoff are used in all calculations, the result is the same as computing the functional trace with a general background.

To calculate the beta function of g_1 we choose backgrounds with $\mathcal{R}_{\mu\nu\rho\sigma} = 0$, $F_{\mu\nu} = 0$ but $D_\mu\chi \neq 0$. To extract the terms proportional to $(D\chi)^2$, we note that the full kinetic operator

$\mathcal{O} = \frac{\delta^2\Gamma_k}{\delta\Psi\delta\Psi}$, which can be read off (3.124), can be expanded as

$$\mathcal{O} = \mathcal{O}_0 + P^{(1)} + P^{(2)} \tag{3.127}$$

where $\mathcal{O}_0 = \mathcal{O}|_{D\chi\to 0}$, $P^{(1)}$ are the terms of order $D\chi^2$ and $P^{(2)}$ are the terms of order $(D\chi)^2$ or $D^2\chi^2$. We treat $P^{(1)}$ and $P^{(2)}$ as perturbations and expand

$$\frac{1}{\mathcal{O} + \mathcal{R}_k(-D^2)} = G - GP^{(1)}G - GP^{(2)}G + GP^{(1)}GP^{(1)}G + \dots \tag{3.128}$$

where $G = \frac{1}{\mathcal{O}_0+\mathcal{R}_k(-D^2)}$. One then has to evaluate a trace of a function of $-D^2$ with some insertions of powers of D_μ. Such traces can be evaluated using the "universal RG machine" developed in [55].

To calculate the beta function of g_3 we need a background with $F_{\mu\nu} \neq 0$. Since $F_{\mu\nu}$ appears in the kinetic operator, we can proceed as in the preceding case, assuming that F^2 is small and expanding as in (3.127,3.128). The h-h part of the second variation contains a term of order F^2 while the non-diagonal terms are of order F. There are thus contributions linear in $P^{(2)}$ and quadratic in $P^{(1)}$, multiplied by the the heat kernel coefficient $B_0(-D^2)$. There is another potential source of F^2 terms: it consists of terms of order zero in the perturbations proportional to the heat kernel coefficient $B_4(-D^2)$. Indeed the latter contains terms quadratic in curvature, which themselves contain F. As noted in section 3.1, these terms depend on the choice of basis of invariant operators. Furthermore, we do not currently have the formula for the b_4 coefficient of $-D^2$ (in appendix E we have evaluated only the coefficient b_2). For this reason we shall leave this contribution in the form of an undetermined coefficient K in the beta function of g_3 (see equation (3.132) below). We observe that this contribution is easily distinguishable from the remaining ones, which are proportional to g_3, whereas the one coming from b_4 is purely numerical.

As manifested in (3.103), the couplings g_2 and g_4 are related to the cosmological constant and Newton's constant respectively, so to calculate their beta functions one needs some curved background. The simplest possibility is to choose the background $b_\mu = 0$, χ constant, since in this way our quadratic action reduces to the usual Einstein-Hilbert action plus a minimally coupled massive vector field w_μ. The calculation is similar to the one in [103], except for the presence of the Weyl vector. Of course in this way one does not see explicitly that Weyl invariance is preserved by the beta functions: one has to appeal to the Weyl invariance of the general construction. As an additional check, in appendix E we show that this particular choice is not necessary and that Weyl invariance emerges explicitly.

The flow equation for the E-H terms is:

$$\begin{aligned}\partial_t\Gamma_k &= \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\Bigg\{5Q_2\left(\frac{\partial_t R_k}{P_k-\frac{g_2}{g_4}\chi^2}\right)-4Q_2\left(\frac{\partial_t R_k}{P_k}\right)+\mathcal{R}\left[\frac{5}{6}Q_1\left(\frac{\partial_t R_k}{P_k-\frac{g_2}{g_4}\chi^2}\right)\right.\\ &- \left.\frac{2}{3}Q_1\left(\frac{\partial_t R_k}{P_k}\right)-3Q_2\left(\frac{\partial_t R_k}{(P_k-\frac{g_2}{g_4}\chi^2)^2}\right)-Q_2\left(\frac{\partial_t R_k}{P_k^2}\right)\right] \qquad (3.129)\\ &+ \frac{3}{2}Q_2\left(\frac{\partial_t R_k}{P_k+\frac{g_1+12g_4}{g_3}\chi^2}\right)+R\left[\frac{1}{8}Q_1\left(\frac{\partial_t R_k}{P_k+\frac{g_1+12g_4}{g_3}\chi^2}\right)-\frac{3}{8}Q_2\left(\frac{\partial_t R_k}{\left(P_k+\frac{g_1+12g_4}{g_3}\chi^2\right)^2}\right)\right]\Bigg\}\end{aligned}$$

The first two lines contain the contribution of the graviton and ghost. The last line gives the contribution of the transverse part of the gauge field w_μ. The contribution of the longitudinal part of w_μ is cancelled by that of the scalar field which takes the jacobian of the decomposition into account.

Finally we collect here all the beta functions. Denoting $\beta_i = u\frac{dg_i}{du}$, we find

$$\begin{aligned}\beta_1 &= \frac{1}{16\pi^2}\left[\frac{3(g_1+12g_4)^2}{g_3g_4\left(1-\frac{g_2}{g_4u^2}\right)^2\left(1+\frac{g_1+12g_4}{g_3u^2}\right)}+\frac{3(g_1+12g_4)^2}{g_3g_4\left(1-\frac{g_2}{g_4u^2}\right)\left(1+\frac{g_1+12g_4}{g_3u^2}\right)^2}-8u^2\right. \qquad (3.130)\\ &\left.-\frac{2u^2(g_1+12g_4)}{3g_4\left(1-\frac{g_2}{g_4u^2}\right)}-\frac{2u^2(g_1+12g_4)}{3g_4\left(1-\frac{g_2}{g_4u^2}\right)^2}+\frac{12u^2}{\left(1-\frac{g_2}{g_4u^2}\right)^3}+\frac{3(g_1+12g_4)}{g_3\left(1+\frac{g_1+12g_4}{g_3u^2}\right)}+\frac{3(g_1+12g_4)}{g_3\left(1+\frac{g_1+12g_4}{g_3u^2}\right)^2}\right]\\ \beta_2 &= \frac{u^4}{16\pi^2}\left[-4+\frac{3}{2\left(1+\frac{g_1+12g_4}{g_3u^2}\right)}+\frac{5}{1-\frac{g_2}{g_4u^2}}\right] \qquad (3.131)\\ \beta_3 &= \frac{1}{16\pi^2}\left[\frac{K}{1-\frac{g_2}{g_4u^2}}-\frac{3g_3u^2}{g_4\left(1-\frac{g_2}{g_4u^2}\right)^2}+\frac{2g_3u^2}{g_4}\frac{\left(1-\frac{g_2}{g_4u^2}\right)+\left(1+\frac{g_1+12g_4}{g_3u^2}\right)}{\left(1-\frac{g_2}{g_4u^2}\right)^2\left(1+\frac{g_1+12g_4}{g_3u^2}\right)^2}\right] \qquad (3.132)\\ \beta_4 &= \frac{u^2}{16\pi^2}\left[\frac{7}{3}-\frac{1}{4\left(1+\frac{g_1+12g_4}{g_3u^2}\right)}+\frac{3}{8\left(1+\frac{g_1+12g_4}{g_3u^2}\right)^2}-\frac{5}{3\left(1-\frac{g_2}{g_4u^2}\right)}+\frac{3}{\left(1-\frac{g_2}{g_4u^2}\right)^2}\right] \qquad (3.133)\end{aligned}$$

Fixed points

In the standard Wilsonian approach to the renormalization group one uses the cutoff k as independent variable and also measures all dimensionful couplings in units of k. This leads to flow equations that are autonomous, meaning that the independent variable does not appear explicitly, but only as argument of the running couplings. In this context the definition of fixed point is very simple: it is just a zero of the beta functions. To find the fixed points one needs not solve the flow equations, which are differential equations: it is enough to solve a system of algebraic equations.

The price we have to pay for manifest Weyl–invariance is that the beta functions contain u explicitly: the flow is not autonomous. In this situation it is generally not obvious how fixed points can be defined, since any zeroes of the beta functions will in general move as functions of the renormalization group time $t = \log u$. It looks like any analysis of the flow will require solving differential equations. Fortunately one can again reduce the flow to autonomous equations: if one performs the redefinitions

$$g_1 = f_1 u^2 \; ; \qquad g_2 = f_2 u^4 \; ; \qquad g_3 = f_3 \; ; \qquad g_4 = f_4 u^2 \; , \tag{3.134}$$

in the beta functions u factors, leaving only overall powers that can be cancelled between the left and right hand sides of the flow equations. Then one can find fixed points for f_1, f_2, f_3 and f_4 in the usual way.

It is easy to see why this procedure should work. From (3.96), note that the powers of u in (3.134) are equal to the power of χ in the corresponding field monomial. Also recall that it is possible to go to the gauge where χ is constant. Then one can absorb the powers of χ in the coupling constants and the powers of χ are the mass dimensions of these dimensionful couplings. But then one sees that the couplings f_i are just the usual Wilsonian couplings made dimensionless by dividing them by powers of k, and we know that such couplings satisfy autonomous flow equations.

Solving numerically the fixed point equations for the couplings f_i, one finds several real solutions. Recalling that g_3 can be seen as the inverse of the QED coupling e^2, one expects a fixed point at $e^2 = 0$, which obviously is not visible in the original parameterization. If we rewrite the RG equations in terms of e^2 one indeed finds a fixed point at

	f_{1*}	f_{2*}	e^2_*	f_{4*}	$\tilde{\Lambda}_*$	$\tilde{G}_*$
FP_1	0.0161	0.008585	0.0000	0.02327	0.1845	0.8549

For the sake of comparison with the literature we have given here also the fixed point values of

$$\tilde{\Lambda} = \frac{f_2}{2f_4}; \qquad \tilde{G} = \frac{1}{16\pi f_4} \; .$$

(These relations follow from (3.103) and (3.134).) The following table gives the eigenvalues of the linearized flow, ordered from the most to the least relevant

	λ_1	λ_2	λ_3	λ_4
FP_1	-2.74294	$-2.28003 + 1.96824i$	$-2.28003 - 1.96824i$	-0.179136

The corresponding eigenvectors are f_1, complex mixtures of f_1, f_2, f_4 and a mixture mostly along f_3, respectively. There is also the true Gaussian fixed point with $G = 0$, which would require a further change of variable. The properties of these two fixed points are independent of the value of the undetermined constant K. In addition there are three real fixed points with $e^2 \neq 0$, whose

properties depend to some extent on K. This dependence is not very strong, however, and we have checked that their qualitative properties would be the same for a wide range of values of K.

	f_{1*}	f_{2*}	f_{3*}	f_{4*}	$\tilde{\Lambda}_*$	$\tilde{G}_*$
FP_2	0.47503	0.004118	0.0000	0.01698	0.1213	1.1718
FP_3	0.0382	0.008357	4.8524	0.02291	0.1823	0.8681
FP_4	−0.1493	−0.005328	−0.0488	0.01531	−0.1493	1.5387

and the eigenvalues of the linearized flow.

	λ_1	λ_2	λ_3	λ_4
FP_2	−1.95041	$-1.86782 + 1.39828i$	$-1.86782 - 1.39828i$	−0.811311
FP_3	−2.36814	$-2.25983 + 1.99667i$	$-2.25983 + 1.99667i$	0.175235
FP_4	−2.78268	−2.12422	−1.54627	15.3501

We do not list the eigenvectors but we note the following: at FP_2 there is a clean separation between the eigenvector of λ_1, which is a mixture of g_1 and g_3, the eigenvector of λ_4 which is exactly g_3 and the complex eigenvalues, which have no component on g_3; at FP_3 the eigenvectors have a very similar structure, but they all have some component on all couplings; at FP_4 all eigenvectors have a strong component only along g_1. This, together with the unphysical values of g_1 and g_3 make this an uninteresting, probably spurious fixed point and we shall not consider it further.

We have explored the properties of these fixed points for $-10 < K < 10$. All the listed parameters of FP_2 and FP_3 change only on the second or third significant digit for K in this range. The value of f_{3*} for FP_2 has the same sign as K and ranges between ± 0.04.

From these tables, knowing that in the gauge where χ is constant the theory reduces to Einstein-Hilbert gravity coupled to a massive vector field, and comparing with results from the literature, one may venture to say that FP_1 and one between FP_2 and FP_3 probably correspond to known fixed points and may have some physical relevance whereas the other two are most likely artifacts of the truncation.

3.5.4 An alternative cutoff

Now we consider in greater detail the subspace of theory space where the Weyl field is massless. As we have seen in section 3.5.1, if we set $g_1 + 12g_4 = 0$ we recover the Weyl integrable theory with a massless, minimally coupled abelian gauge field. The action reduces to the form (3.108), which, aside from the presence of the abelian gauge field, has been discussed also in [103]. It is natural to ask whether the RG flow preserves this subspace. To this effect, one has to compute the beta function of $g_1 + 12g_4$ and check whether it is zero when one sets $g_1 + 12g_4 = 0$. From (3.130,3.133) one sees that this is not the case.

The reason for this is not hard to understand. The massless subspace $g_1+12g_4=0$ is characterized by the enlarged symmetry (3.110). The operator $-D^{(b)2}$ which was used in the definition of the cutoff is not covariant under the transformations (3.110), where b_μ is inert. Thus the beta functions do not preserve the enlarged symmetry. This immediately suggests an alternative cutoff procedure: to define the cutoff using the operator $-D^{(s)2}$, which, being independent of b_μ, is covariant both under ordinary and modified Weyl transformations. In this section we discuss the calculation of the beta functions obtained from this alternative regularization procedure.

The modified beta functions

The calculation of the beta functions of g_2 and g_4, with the background $b_\mu=0$ and χ constant, is exactly as in section 3.5.3. Thus β_2 and β_4 remain as in (3.131,3.133). The calculation of the beta function of g_1 also proceeds along the same lines as before but now there are some differences: in the second variation of the action (3.106) the terms containing derivatives of χ (second and third line in (3.124)) are now zero because of (3.101). This removes several contributions to β_1. In the case of g_3 the term proportional to $b_4(-D^2)$ is now absent, because now D is $D^{(s)}$ and the fields strength of s_μ is zero. Thus if we choose a basis for operators containing powers of $\hat{R}^{(s)}_{\mu\nu\rho\sigma}$ (namely the curvature given in equation (3.6), with b_μ replaced by s_μ), there is no contribution to β_3 from $b_4(-D^2)$, in other words we can set the parameter $K=0$.

With these modifications, one arrives at the following beta functions:

$$\begin{aligned}\beta_1 \;=\; & \frac{1}{16\pi^2}\left[-28u^2+\frac{3(g_1+12g_4)^2}{g_3g_4\left(1-\frac{g_2}{g_4u^2}\right)^2\left(1+\frac{g_1+12g_4}{g_3u^2}\right)}+\frac{3(g_1+12g_4)^2}{g_3g_4\left(1-\frac{g_2}{g_4u^2}\right)\left(1+\frac{g_1+12g_4}{g_3u^2}\right)^2}\right.\\ & \left.-\frac{2u^2(g_1-18g_4)}{3g_4\left(1-\frac{g_2}{g_4u^2}\right)}-\frac{2u^2(g_1+66g_4)}{3g_4\left(1-\frac{g_2}{g_4u^2}\right)^2}+\frac{3u^2}{\left(1+\frac{g_1+12g_4}{g_3u^2}\right)}-\frac{9u^2}{2\left(1+\frac{g_1+12g_4}{g_3u^2}\right)^2}\right] \qquad (3.135)\end{aligned}$$

$$\beta_3 \;=\; \frac{u^2}{16\pi^2}\frac{g_3}{g_4}\left[\frac{2}{\left(1-\frac{g_2}{g_4u^2}\right)^2\left(1+\frac{g_1+12g_4}{g_3u^2}\right)}+\frac{2}{\left(1-\frac{g_2}{g_4u^2}\right)\left(1+\frac{g_1+12g_4}{g_3u^2}\right)^2}-\frac{3}{\left(1-\frac{g_2}{g_4u^2}\right)^2}\right] \qquad (3.136)$$

while the other two have remained as in (3.131,3.133). We see that that for $g_1+12g_4=0$, $\beta_1+12\beta_4=0$, so the massless subspace is indeed invariant.

Inessential couplings and fixed points

Let us consider the action, written in the form (3.106), choose the gauge where χ is constant and use equations (3.134) and the relation $u=k/\chi$ to write

$$S=\int d^4x\sqrt{g}\left[\frac{f_1+12f_4}{2}k^2b^2+f_2k^4+\frac{f_3}{4}F_{\mu\nu}F^{\mu\nu}-f_4k^2\mathcal{R}^{(s)}\right]\,. \qquad (3.137)$$

In this gauge f_3 can be seen as the coefficient of the kinetic term for the vector while the mass is given by the combination f_1+12f_4. It is clear that via a suitable rescaling of b one can eliminate either f_3 or f_1+12f_4. If we redefine the couplings as

$$f_1+12f_4=Z_b\kappa_1\ ;\qquad f_2=\kappa_2\ ;\qquad f_3=Z_b\ ;\qquad f_4=\kappa_4\ , \tag{3.138}$$

then Z_b can be eliminated by a redefinition of b_μ: it is an inessential coupling. [12] (We consider the alternative choice in appendix E.6.) This interpretation is confirmed by the explicit form of the beta functions:

$$\begin{aligned}
\beta_{\kappa_1} &= -2\kappa_1+\eta_b\kappa_1+\frac{(1-2\kappa_1)(2-\kappa_1)\kappa_2-(4-10\kappa_1-5\kappa_1^2)\kappa_4}{48\pi^2\,(\kappa_1+1)^{\,2}\,(\kappa_2-\kappa_4)^{\,2}}\kappa_1\\
\beta_{\kappa_2} &= -4\kappa_2-\frac{8\kappa_2\kappa_1+5\kappa_2+2\kappa_4\kappa_1+5\kappa_4}{32\pi^2(1+\kappa_1)(\kappa_2-\kappa_4)}\\
\beta_{\kappa_4} &= -2\kappa_4+\frac{(56\kappa_1^2+106\kappa_1+59)\kappa_2^2-6(12\kappa_1^2+22\kappa_1+13)\kappa_2\kappa_4+(88\kappa_1^2+170\kappa_1+91)\kappa_4^2}{384\pi^2(1+\kappa_1)^2(\kappa_2-\kappa_4)^2}
\end{aligned} \tag{3.139}$$

and the anomalous dimension

$$\eta_b=-\frac{\beta_Z}{Z}=\frac{2\kappa_2+\kappa_4\left(3\kappa_1^2+4\kappa_1-1\right)}{16\pi^2(1+\kappa_1)^2(\kappa_2-\kappa_4)^2}\ , \tag{3.140}$$

which only depend on the essential couplings κ_i.

The system of three equations $\beta_{\kappa_i}=0$ admits three real fixed points with κ_1 finite or zero, and one with $1/\kappa_1=0$:

	η_{b*}	$1/\kappa_{1*}$	κ_{2*}	κ_{4*}	$\tilde\Lambda_*$	$\tilde G_*$
FP_2	1.9504	0	0.00411798	0.0169775	0.1213	1.1718

	η_{b*}	κ_{1*}	κ_{2*}	κ_{4*}	$\tilde\Lambda_*$	$\tilde G_*$
FP_1	−0.179136	0	0.00858496	0.0232715	0.184452	0.854881
FP_3	1.27047	1.28633	0.00628253	0.0198675	0.158111	1.00135
FP_4	1.76077	−2.5551	0.000150384	0.0127745	0.00588611	1.55735

The inverted numbering of the first two fixed points is deliberate: it is such that the values of κ_2 and κ_4 are equal to the values of f_2 and f_4 for the fixed point by the same name in section 3.5.3. This suggests that perhaps these fixed points can be identified. This observation is strengthened by the results for the eigenvalues:

	λ_1	λ_2	λ_3
FP_2	$-1.86782+1.39828i$	$-1.86782-1.39828i$	−1.1391
FP_1	−2.92208	$-2.28003+1.96824i$	$-2.28003-1.96824i$
FP_3	$-2.02559+1.87941i$	$-2.02559-1.87941i$	0.923836
FP_4	−3.13639	−1.40315	3.36778

[12] This would no longer be true if b_μ was coupled to some matter field.

The eigenvectors at FP_2 are complex mixtures of κ_2 and κ_4, and a mixture mostly along $1/\kappa_1$, respectively. The eigenvectors at FP_1 are complex mixtures of κ_2 and κ_4, and a mixture mostly along κ_1, respectively. It is interesting to note that FP_1 lies in the massless subspace, since $\kappa_1 = 0$ means $g_1 + 12g_4 = 0$. The linearized flow tells us that this choice is attractive in the UV.

We observe that the eigenvalues λ_2 and λ_3 of FP_1 coincide with λ_2 and λ_3 of the fixed point FP_1 in section 3.5.3. Furthermore, the anomalous dimension is equal to $-\lambda_1$ of FP_1 in section 3.5.3. Similar identifications can be made for FP_2, suggesting that these four fixed points can be identified pairwise. (This was the motivation for the names in the first place.) The identification of FP_3 and FP_4 with the other two fixed points of section 3.5.3 is also relatively obvious, but in these two cases the values do not coincide numerically.

Inside the massless subspace

In the preceding section we have considered a set of beta functions in the full theory space that preserve the massless subspace. The correct beta functions inside the massless subspace are however different, since they must take into account the enlarged gauge symmetry that is present there. As already noted, in the massless subspace the theory is equivalent to gravity coupled to a Maxwell field. Therefore, one has to add a gauge fixing and a ghost term for the new abelian gauge symmetry (the abelian ghost is decoupled in flat space but it contributes to the beta functions of g_2 and g_4 because it is coupled to gravity). There is no need to add these terms outside the massless subspace. Here we discuss the modifications that follow.

We choose a standard Lorentz gauge condition, such that the gauge fixing and ghost terms are

$$S_{\mathrm{gf}} + S_{\mathrm{gh}} = \int d^4x \left[\frac{1}{2\alpha} (D_\mu w^\mu)^2 + \frac{1}{\sqrt{\alpha}} \bar{c}(-D^2)c \right] . \tag{3.141}$$

These have to be added to the quadratic action. As in the preceding sections, we decompose the field w^μ into its transverse and longitudinal components $w_\mu = w_\mu^T + D_\mu(-D^2)^{-1/2}\tilde{\varphi}$. This transformation has a trivial Jacobian so the new terms in the action amount to

$$S_{\mathrm{gf}} + S_{\mathrm{gh}} = \int d^4x \left[\frac{1}{2\alpha} \tilde{\varphi}(-D^2)\tilde{\varphi} + \frac{1}{\sqrt{\alpha}} \bar{c}(-D^2)c \right] \tag{3.142}$$

which contributes to the beta functional

$$\frac{1}{2}\mathrm{Tr}\left[\frac{\partial_t R_k(-D^2)}{-D^2 + R_k(-D^2)} \right] - \mathrm{Tr}\left[\frac{\partial_t R_k(-D^2)}{-D^2 + R_k(-D^2)} \right] . \tag{3.143}$$

The additional contribution is therefore equivalent to that of an anticommuting real scalar. [13]

[13] If we had chosen the gauge $\alpha = 0$, which amounts to imposing the gauge condition strongly, one would not have the contributions from $\tilde{\varphi}$ and the ghosts but instead there would be the contribution from the Jacobian of the decomposition, which is again equivalent to an anticommuting real scalar.

The calculation of the beta functions of g_2 and g_4, if we choose the background $b_\mu = 0$ and χ constant, is exactly the same as in section 3.5.3. This only changes the contribution of the abelian vector field given in the last line of equation (3.129), which now becomes

$$\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\left\{Q_2\left(\frac{\partial_t R_k}{P_k}\right)+R\left[\frac{1}{24}Q_1\left(\frac{\partial_t R_k}{P_k}\right)-\frac{3}{8}Q_2\left(\frac{\partial_t R_k}{P_k^2}\right)\right]\right\}\ . \tag{3.144}$$

It is interesting to compare this to the result given in equation (23) of [33]:

$$\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\left\{Q_2\left(\frac{\partial_t R_k}{P_k}\right)+R\left[\frac{1}{6}Q_1\left(\frac{\partial_t R_k}{P_k}\right)-\frac{1}{2}Q_2\left(\frac{\partial_t R_k}{P_k^2}\right)\right]\right\}\ . \tag{3.145}$$

In both cases one is using a cutoff "of type I", but the difference lies in the fact that here we decompose the vector field into its transverse and longitudinal parts, and impose cutoffs separately, whereas in [33] no such decomposition was used. Numerically, when one uses the optimized cutoff, the coefficient of R turns out to be $-7/24$ in the first case and $-4/24$ in the second.

The new terms bring only small changes to the beta functions of the "massive" case: the beta function β_1 is as in (3.135) except that -28 is replaced by -30; the beta function β_2 is as in (3.131) except that -4 is replaced by $-9/2$; the beta function β_3 remains as in (3.136); the beta function β_4 is as in (3.133) except that $7/3$ is replaced by $5/2$. The beta functions written in this way are extensions of ones valid in the massless subspace to the whole theory space. The flow they describe is very similar to the one described in the "massive" case, aside from minor numerical corrections which are anyway within the theorietical uncertainties of this type of calculation. There is however no reason to gauge fix outside the massless subspace, so these beta functions are strictly speaking not correct there. They are correct in the massless subspace $g_1 + 12g_4 = 0$, where they reduce to the following simple beta functions:

$$\beta_2 = \frac{u^4}{16\pi^2}\left[-3+\frac{5}{1-\frac{g_2}{g_4u^2}}\right] \tag{3.146}$$

$$\beta_3 = \frac{u^2}{16\pi^2}\frac{g_3}{g_4}\left[\frac{2}{\left(1-\frac{g_2}{g_4u^2}\right)}-\frac{1}{\left(1-\frac{g_2}{g_4u^2}\right)^2}\right] \tag{3.147}$$

$$\beta_4 = \frac{u^2}{16\pi^2}\left[\frac{21}{8}-\frac{5}{3\left(1-\frac{g_2}{g_4u^2}\right)}+\frac{3}{\left(1-\frac{g_2}{g_4u^2}\right)^2}\right] \tag{3.148}$$

We have not written β_1 since it is still true that $\beta_1 = -12\beta_4$. Furthermore, note that g_1 does not appear in any of the other beta functions at all.

There are now only two fixed points: one with $g_3 = 0$ and one with $1/g_3 \equiv e^2 = 0$. We list here their properties:

	f_{2*}	f_{3*}	f_{4*}	$\tilde{\Lambda}_*$	$\tilde{G}_*$
FP_2	0.007013	0.0000	0.0214442	0.163518	0.927726
	f_{2*}	e^2_*	f_{4*}	$\tilde{\Lambda}_*$	$\tilde{G}_*$
FP_1	0.007013	0.0000	0.0214442	0.163518	0.927726

The eigenvalues are as follows:

	λ_1	λ_2	λ_3
FP_2	$-2.14278 + 1.75252i$	$-2.14278 - 1.75252i$	0.225565
FP_1	$-2.14278 + 1.75252i$	$-2.14278 - 1.75252i$	-0.225565

with the complex eigenvalues referring to a mixture of g_2 and g_4, while the real eigenvalue is for g_3.

If one neglects the threshold effects represented by the nontrivial denominators, and uses the definitions (3.103), (3.147) becomes just

$$\beta_3 = \frac{1}{\pi} g_3 \tilde{G} \ . \tag{3.149}$$

Without the coupling to gravity the field b_μ would be just a free vector field and its beta function would vanish. The beta function (3.149) is entirely due to the effect of the gravitational coupling. This effect has been the subject of some interest in recent years [106–109]. One should not attach to these beta functions the same physical meaning of the usual perturbative beta functions [110]. The calculation we have done here is very similar to the one in [111] and finds a nonvanishing, positive coefficient. We note that if b_μ was coupled to some charged fields, for example as in QED, there would be an additional constant contribution $-C$ to (3.149). This would then translate into a beta function for e^2 of the form

$$Ce^4 - \frac{1}{\pi} \tilde{G} e^2 \ , \tag{3.150}$$

which is indeed of the form found in [111]. If $C > 0$, as is the case in QED, this, together with the beta functions for $\tilde{\Lambda}$ and $\tilde{G}$ admits, in addition to FP_1 and FP_2 also a third fixed point with finite, nonzero e^2 and e^2 irrelevant.

3.6 Summary

In discussions about conformal invariance, misunderstandings frequently arise due to the different physical interpretation of the transformations that are used by different authors. In particle physics language, a theory that contains dimensionful parameters is obviously not conformal. Thus conformal invariance is a property of a very restricted class of theories. In particular, in quantum field theory the definition of the path integral generally requires the use of dimensionful

parameters (cutoffs, renormalization points) which break conformality even if it was present in the original classical theory. True conformality is only achieved at a fixed point of the renormalization group. Let us call this the point of view I.

On the other hand in Weyl's geometry and its subsequent ramifications, conformal (Weyl) transformations are usually interpreted as relating different local choices of units. Since the choice of units is arbitrary and cannot affect the physics, it follows that essentially any physical theory can be formulated in a Weyl–invariant way. This point of view is more common among relativists. Let us call it the point of view II.

The way in which a generic theory containing dimensionful parameters can be made Weyl–invariant is by allowing those parameters to become functions on spacetime, *i.e.* to become fields. This is the step that the adherents of the interpretation I are generally unwilling to make, since then one would have to ask whether these fields have a dynamics of their own or not, and, in the quantum case, whether they have to be functionally integrated over or not. It can be unnatural to have fields in the theory that do not obey some specific dynamical equation, and it is clear that in general, if one allows all the dimensionful couplings to become dynamical fields, the theory is physically distinct from the original one. There is however one way in which Weyl–invariance can be introduced in any theory without altering its physical content, and that is to introduce *a single* scalar field, which we called a dilaton (sometimes also called a "Stückelberg" or "Weyl compensator" or "spurion" field) and to assume that all dimensionful parameters are proportional to it. This field carries a nonlinear realization of the Weyl group, since it is not allowed to become zero anywhere. Even though the new field obeys dynamical equations, it does not modify the physical content of the theory because it is exactly neutralized by the enlarged gauge invariance. In practice, it can be eliminated by choosing the Weyl gauge such that it becomes constant.

All this is well–known in the classical case. It had already been observed both in a perturbative and nonperturbative context that the above considerations can be generalized to the context of quantum field theory by treating the cutoff or the renormalization point in the same way as the mass or dimensionful parameters that are present in the action. Here we have discussed in particular the formulation of the renormalization group using the point of view II. It has proven convenient to adopt a non–perturbative definition of the renormalization group, where one considers the dependence of the effective action on an externally prescribed smooth cutoff k. The advantage of this procedure is that the resulting "beta functional" is both UV and IR finite and one can use it to define a first order differential equation whose solution, for $k \to 0$, is the effective action. It can therefore be viewed as a non–perturbative way of defining (and calculating) the effective action. Using this method we have shown in complete generality that one can define a flow of Weyl–invariant actions whose IR endpoint is a Weyl–invariant effective action. It is important to emphasize that this holds also when the dilaton and metric field are

quantized, at least in the background field method. This is our main result.

This provides an answer to the following question. Suppose we start from a theory that contains dimensionful parameters, and recast it in a Weyl–invariant form by the Stückelberg trick of introducing a dilaton field. If we quantize this Weyl–invariant theory, is the result equivalent to the one we would have obtained by quantizing the original theory? The answer is affirmative, if we use *for all fields* the Weyl–invariant measure. [14] Thus, there is a quantization procedure that commutes with the Stückelberg trick.

It is important to understand that although Weyl invariance is not anomalous, there is still a trace anomaly, in the sense that the trace of the energy–momentum, which is classically zero, is not zero in the quantum theory. This can be easily understood from the fact that in the Weyl–invariant quantization one obtains an effective action that depends not only on the metric but also on the dilaton. Weyl–invariance of the effective action is compatible with a nonvanishing trace, because the latter cancels out against the variation of the dilaton. (We have provided fully explicit examples of this phenomenon in section 3.4).

In order not to modify the dynamics we have used the "only one dilaton" prescription, with the consequence that the cutoff and the dilaton are proportional. The proportionality factor, which we have called u, is the RG parameter in this formulation. It is dimensionless (since it expresses the cutoff relative to the unit of mass), Weyl–invariant and constant on spacetime. Thus, running couplings are functions $g(u)$. In this we differ from other approaches to dilaton dynamics where the couplings depend on a mass scale k that is allowed to be a function on spacetime. In practice the difference is not important, because the variation of $g(u)$ with respect to k, keeping χ constant, is the same that one would obtain if one assumed that g is a function of k.

Given that in this formalism all theories are conformally invariant, one can also ask what is special about conformal field theories (in the standard sense of quantum field theory), and in particular about fixed points of the renormalization group. The answer is that for generic theories, conformal invariance is only achieved at the price of having a dilaton in the effective action. True conformal field theories are conformal even without the dilaton, so one must expect that as the RG flow approaches a fixed point, the dilaton must decouple.

[14] By contrast, suppose that after having quantized the matter fields we also quantize the metric and dilaton, using the standard, Weyl–non–invariant measure I. (One does not need to have a full quantum gravity for this argument, it is enough to think of a one loop calculation in the context of an effective field theory). The integration over metric and dilaton will now proceed with total actions $S_G + \Gamma^{\rm I}$ and $S_G + \Gamma^{\rm II}$, depending on whether we used for the matter the measures I or II. Clearly the resulting theories are physically inequivalent: In the first case the action is not Weyl invariant, so the dilaton field is physical, in the second case the action is Weyl invariant and the dilaton can be gauged away. So, *all else being equal*, quantizing matter with measures I and II leads to physically different theories.

Finally we discussed the quantization of gravity in the general case of Weyl non–integrable geometry. This formalism is not equivalent to the standard Einstein–Hilbert truncation since the equivalence holds only using the one–dilaton prescription. Nevertheless we were able to show that Weyl invariance can be maintained along the flow also in this case. Moreover the theory analyzed in section 3.5 has some subspaces of the theory space with special properties which one can either preserve or not according to the choice of the cutoff. In the calculation of the beta functions for the couplings g_2 and g_4 (related to the cosmological constant and Newton's contant) we have used a maximally symmetric background metric, with background $b_\mu = 0$ and χ constant. This is sufficient to determine the beta functions but Weyl invariance of the flow is not manifest. We have shown in appendix E that the relevant heat kernel coefficients are actually Weyl invariant. The calculation could thus have been done on an arbitrary background. This answers a minor issue that has remained lingering for some time. The beta function of Newton's coupling had been computed in [112] in the so-called "CREH" approximation, where only the conformal factor of the metric is dynamical. In this approximation the Einstein-Hilbert action has the form (3.102), where R is the curvature of a fixed reference metric. One can then read the beta function of Newton's coupling (or equivalently the "anomalous dimension" $\eta = \partial_t G/G$) either from the second or from the third term of (3.102). Since the second term can be viewed as a potential for the conformal factor χ the result was denoted $\eta^{(\mathrm{pot})}$, and since the third term is the ordinary kinetic term of χ the result was called $\eta^{(\mathrm{kin})}$. The two calculations in [112] gave $\eta^{(\mathrm{kin})} \neq \eta^{(\mathrm{pot})}$, so the question remained whether a quantization exists for which $\eta^{(\mathrm{kin})} = \eta^{(\mathrm{pot})}$. We have shown that the answer is positive.

CHAPTER 4

The functional Renormalization Group and the c-function

4.1 Introduction

In this thesis we have dealt with the renormalization of quantum field theories in curved spacetime where the graviton is eventually quantized. We focused on the ultraviolet behaviour of such theories and discussed the specific results at hand. In this chapter we shall not attempt to quantize gravity but rather we will use curved spacetime as a bookkeeping device to obtain relevant information about RG flows of quantum field theories in flat spacetime. In particular we will discuss some global features of the RG flow, i.e.: features that regards the entire RG trajectory from the UV to the IR. Of course the control of the whole RG flow is an incredibly difficult challenge and much of our understanding comes from the study of the two dimensional theories. Indeed in two dimensions, if we limit ourselves to unitary and scale invariant field theories, we have that fixed point theories are conformal field theories [73]. The classification of all possible conformal field theories (CFT) has been achieved via algebraic methods, exploiting the properties of the associated Virasoro algebra [113, 114]. A fixed point theory can then be deformed by adding weakly coupled operators that trigger a nontrivial flow out of the fixed point.

The study of a theory in the neighborhood of a fixed point is the main idea of conformal perturbation theory and it is generally hard to make rigorous statements beyond this level. However some important information is encoded in Zamolodchikov's c–theorem [115], which states that in every unitary Poincaré invariant theory there exists a function of the coupling constants, the c–function, that decreases from UV to IR, and that is stationary at the endpoints of the flow, where its value equals the central charge of the corresponding CFT (or equivalently the coefficient of the conformal anomaly). Note that the difference between the two central charges is an intrinsic quantity (intrinsic meaning independent of arbitrary choices in the renormalization procedure), so the content of the theorem is highly nontrivial.

In this case a complete analysis of the RG requires the ability to follow the flow arbitrarily far away from a fixed point. Unless the two fixed points are sufficiently close to each other we cannot rely on perturbative schemes. A natural framework to address such kind of problems is offered by the functional Renormalization Group (fRG) of the effective average action described in chapter 1. A first application of exact RG equations to the c–function has been explored in [116, 117], however here we will present a different construction. The c–function was also studied via spectral representation methods in [118] and in terms of the entanglement entropy in [119]. As far as the application of entanglement entropy is concerned the non–perturbative tool which has been mostly used is holography [120]. This approach has also being extended in higher dimensions [121, 122].

In this chapter we move the first steps necessary in order to give a bridge between these two general results: the c–theorem or, more generally, the computation of universal quantities related to the integrated flow between fixed points (that is, to global properties of theory space) and the fRG formalism based on the exact flow for the EAA. Our approach will be mainly a constructive one. We will give a general recipe to construct a c–function compatible with Zamolodchikov's theorem within the fRG framework. After identifying a natural candidate for a scale dependent c–function, c_k, we will be able to write an exact non-perturbative flow equation for it. Of course, there are only few cases in which the exactness of the flow equation can be used and one usually needs to resort to approximations. However, we will see that already for a simple truncation as the local potential approximation the flow equation gives results compatible with the c–theorem.

Our formalism will be based upon a curved space construction since functional derivatives of the metric allow to compute the n–point functions of stress–energy tensor which, as we shall see, are the crucial quantities that we will consider. Moreover we have seen that the fRG can be implemented in curved spacetime very efficiently, this will allow to have an explicitly anomalous term in the effective action (in flat spacetime the anomaly appears in the correlator of two stress energy tensors). Such term is given by the Polyakov action and we shall define its running coefficient c_k as our c–function. As we will see this definition and the results we will find are in agreement with the c–theorem. Furthermore we shall see that to achieve this it is necessary to consistently take into account other generic features of QFTs such as the scale anomaly. This will tell us something about the general form of the EAA. Nevertheless further study is needed to understand the mapping between our approach and the one based on local RG with spacetime dependent couplings [87].

The chapter will be organized as follows. In section 4.2 we will construct a Weyl–invariant functional measure and discuss the form of a CFT on curved background. This will lead us to a re–derivation of the trace anomaly matching condition, from which the "integrated" c–theorem follows from known results [100]. We will then move on to discuss the scale dependent c–function,

and obtain our flow equation for it, in section 4.3. This construction uses the EAA as the main tool and in section 4.4 we will investigate its general form. In section 4.5 we discuss various applications of our formalism while in section 4.6 we put forward a simple relation between the beta function of Newton's constant and the running c–function. In section 4.7 we summarize our findings and discuss possible generalizations to higher dimensions.

4.2 The integrated c–theorem

In this section we give the path–integral proof of the "integrated" c–theorem which says that the change of the central charges satisfies

$$\Delta c = c_{UV} - c_{IR} > 0.$$

This does not provide any information about the monotonicity and the shape of the c–function. We will work in curved space where the central charge, or equivalently the conformal anomaly, can be seen as the coefficient of the Polyakov term in the effective action. We will specify the background metric to be of the specific form $g_{\mu\nu} = e^{2\tau}\delta_{\mu\nu}$, where τ will be called "dilaton". Unlike in chapter 3 we shall refer to the dilaton as a specific choice of background. We will not use the dilaton to convert dimensionful couplings to dimensionless ones via the Stückelberg trick. Δc becomes the coefficient of the operator $\int \tau \Delta \tau$ and can be easily extracted. Before doing this we need to briefly discuss functional measures in curved space, Weyl–invariant quantization and the form of the effective action for a CFT on a curved background. The Weyl invariant quantization that we discuss below is different from the one in chapter 3 since here we will maintain Weyl invariance only at the UV fixed point and not along all the RG trajectory.

Finally let us recall that the EAA is a functional that interpolates between the UV microscopic action and the IR effective action. Because of this we will eventually denote the UV action as Γ_{UV} and the IR action as Γ_{IR}.

4.2.1 Weyl–invariant quantization and functional measures

The standard diffeomorphism invariant path integral measure in curved space [123], denoted here $\mathcal{D}^I_g$, is Weyl–anomalous [124]: under a Weyl transformation of the background metric $g_{\mu\nu} \to e^{2\tau} g_{\mu\nu}$ and of the fields $\varphi \to e^{w\tau}\varphi$, where w is the conformal weight of the field[1], one encounters the conformal anomaly:

$$\mathcal{D}^I_{e^{2\tau}g}\left(e^{w\tau}\varphi\right) = \mathcal{D}^I_g \varphi \, e^{-\Gamma_{WZ}[\tau,g]}\,, \tag{4.1}$$

[1] For a scalar field $w_\varphi = -\left(\frac{d}{2} - 1 + \frac{\eta_\varphi}{2}\right)$, while for a fermion field $w_\psi = -\left(\frac{d}{2} - \frac{1}{2} + \frac{\eta_\psi}{2}\right)$. The conformal weight of the metric is $w_g = 2$ in every dimension.

where c is the central charge of the CFT, which we want to use as UV action in the path integral, and $\Gamma_{WZ}[\tau, g]$ is the Wess–Zumino action:

$$\Gamma_{WZ}[\tau, g] = -\frac{c}{24\pi}\int d^2x\sqrt{g}\left[\tau\Delta\tau + \tau R\right] , \tag{4.2}$$

where $\Delta \equiv -\nabla_\mu\nabla^\mu$ is the Laplacian.

The Wess–Zumino action can be integrated to give the related Polyakov action,

$$S_P[g] = -\frac{c}{96\pi}\int d^2x\sqrt{g}R\frac{1}{\Delta}R\,, \tag{4.3}$$

which, upon Weyl variation, gives back (4.1):

$$S_P[e^{2\tau}g] - S_P[g] = \Gamma_{WZ}[\tau, g]\,. \tag{4.4}$$

The Polyakov action generates the following quantum energy–momentum tensor,

$$\begin{aligned}\langle T^{\mu\nu}\rangle \;=\; \frac{c}{48\pi}\Bigg[&-2\nabla^\mu\nabla^\nu\frac{1}{\Delta}R - \left(\nabla^\mu\frac{1}{\Delta}R\right)\left(\nabla^\nu\frac{1}{\Delta}R\right) + \\ &-2g^{\mu\nu}R + \frac{1}{2}g^{\mu\nu}\left(\nabla^\alpha\frac{1}{\Delta}R\right)\left(\nabla_\alpha\frac{1}{\Delta}R\right)\Bigg] ,\end{aligned}$$

which is anomalous:

$$\left\langle T^\mu_\mu\right\rangle = -\frac{c}{24\pi}R\,. \tag{4.5}$$

This is the conformal anomaly, in the two dimensional case. In curved space, where it can be written in terms of curvature invariants, the conformal anomaly manifests itself already in the one–point function (4.5), while in flat space it is seen only starting from the two–point function. For example, in flat space the two point function of the energy–momentum tensor obtained from the Polyakov action, when written in complex coordinates, reproduces the standard CFT result [113]:

$$4\frac{\delta}{\delta g_{\mu\nu}\delta g_{\rho\sigma}}S_P = \langle T_{zz}T_{ww}\rangle = \frac{1}{(2\pi)^2}\frac{c/2}{(z-w)^4}\,. \tag{4.6}$$

This relation shows the equivalence between the central charge and anomaly coefficient.

We can use the Polyakov action to define, formally, a new measure in the following way:

$$\mathcal{D}^{II}_g\varphi \equiv \mathcal{D}^I_g\varphi\, e^{S_P[g]}\,. \tag{4.7}$$

This is not the standard procedure. [2] Now using (4.1) and (4.4) one can show that indeed (4.7) is Weyl–invariant:

$$\begin{aligned}\mathcal{D}^{II}_{e^{2\tau}g}\left(e^{w\tau}\varphi\right) &= \mathcal{D}^I_{e^{2\tau}g}\left(e^{w\tau}\varphi\right)\, e^{S_P[e^{2\tau}g]} \\ &= \mathcal{D}^I_g\varphi\, e^{-\Gamma_{WZ}[\tau,g]}e^{S_P[g]+\Gamma_{WZ}[\tau,g]} \\ &= \mathcal{D}^{II}_g\varphi\,.\end{aligned} \tag{4.8}$$

[2] We note that the addition of a WZ term to cancel an anomaly has been discussed for the chiral anomalies in [86, 125–127]. In these cases the WZ term was expressed via a functional integral over an auxiliary field. What we do here is much similar in spirit but the non–locality of the Polyakov action cast doubts about this use. We will come back to the issue of non–locality in section 4.2.2 and at the end of section 4.4.2.

With these definitions, we now look at the effective action. First we define the standard Weyl non–invariant effective action [3]:

$$e^{-\Gamma_I[\varphi,g]} = \int_{1PI} \mathcal{D}_g^I \chi \, e^{-S[\varphi+\chi,g]} \,. \tag{4.9}$$

Even if the bare or UV action is conformally invariant $S[e^{w\tau}\varphi, e^{2\tau}g] = S[\varphi, g]$, it is not so for the effective action, which instead satisfies

$$\Gamma_I[e^{w\tau}\varphi, e^{2\tau}g] - \Gamma_I[\varphi, g] = \Gamma_{WZ}[\tau, g] \,, \tag{4.10}$$

where Γ_{WZ} is the Wess–Zumino action. Using instead the Weyl–invariant measure defined in (4.7) to define the effective action (without adding any relevant operator),

$$e^{-\Gamma_{II}[\varphi,g]} = \int_{1PI} \mathcal{D}_g^{II} \chi \, e^{-S[\varphi+\chi,g]} \,, \tag{4.11}$$

gives rise to a Weyl–invariant effective action. Using (3.43) and (4.1) we have:

$$\Gamma_{II}[e^{w\tau}\varphi, e^{2\tau}g] = \Gamma_{II}[\varphi, g] \,. \tag{4.12}$$

Equation (4.12) holds because we used a Weyl invariant measure and we did not add any relevant directions which would break the conformal symmetry.

Note that equation (4.12) is valid only when $\Gamma_{II}[\varphi, g] = S[\varphi, g]$. The fact that the effective action has the same form of the microscopic action can be understood as follows: computing the EA, i.e.: performing the functional integral above, is equivalent to consider the flow down to $k = 0$ (see chapter 1 for more details). Since we did not add any relevant directions the flow is trivial and we end up at the same point in the theory space when we lower k to zero. Moreover since we use a Weyl invariant measure we are not breaking the conformal symmetry when performing the functional integral. Thus the (bare) UV action and the (effective) IR action are the same in this case. Said in other words, the path integration amounts to the substitution of the quantum field with the average field. A purely Gaussian theory provides an example where one can check explicitly the validity of equation (4.12): if we integrate a free scalar field action with the measure $\mathcal{D}_g^I$ we generate a Polyakov term (see section 3.4.1 for the explicit computation). Instead, if we use the measure $\mathcal{D}_g^{II}$, we generate a Polyakov term which cancel against the one present in the measure and leaves just the bare action as a function of the averaged fields. Here the bare action is Weyl invariant, in the following we will assume this since, as explained in the next section, a generic CFT in curved spacetime possesses an anomalous term.

[3] We define $\int_{1PI} \equiv \int e^{\int \sqrt{g}\,\Gamma^{(1,0)}[\varphi,g]\chi}$ where $\Gamma^{(1,0)}[\varphi, g] \equiv \frac{\delta\Gamma[\varphi,g]}{\delta\varphi}$.

4.2.2 CFT action on curved background

In the previous section we have seen how to define, at least formally, a Weyl–invariant effective action starting from a Weyl invariant UV action via the functional measure (4.7), which has to be understood as the measure we will use from now on. Nevertheless on a curved background the effective action of a CFT is not Weyl–invariant since every CFT with $c \neq 0$ is anomalous and its action must contain a Polyakov term. Still, in absence of relevant perturbations, quantization will just give the IR effective action equal to the UV action.

These considerations, which hold both in the UV and in the IR, lead to the following "split" form for the effective action of a general CFT in presence of a background metric:

$$\Gamma_{UV,IR}[\varphi, g] = S^{(CFT)}_{UV,IR}[\varphi, g] + c_{UV,IR} S_P[g]\,, \tag{4.13}$$

where the first term is explicitly Weyl invariant, $S_P[g]$ is the Polyakov action and c its central charge. Note that this form of the EA is totally analogous to the one argued in sections 3.3 and 3.4. Other possible Weyl–invariant terms depending on the metric alone are not present in $d = 2$, but appear in higher dimensions as we have seen in section 3.4. Unfortunately very few CFT actions can be written in local form, these are the Gaussian, the Ising model (in the fermion representation) and the Wess–Zumino–Witten AKM actions [114].

The reader may wonder if it is legitimate to insert a Polyakov action term in a UV action since this is non–local. As we will see in the following we will always perform computations with $g_{\mu\nu} = e^{2\tau}\delta_{\mu\nu}$ where this non–locality does not appear. Nevertheless we are would like to gain some insights on the covariant form of the ansatz and therefore we now give an argument in favour of the presence of such a term in the action. Suppose we are given a local action in the UV and start an RG flow which ends in Γ_{IR}. It is natural to suppose that Γ_{IR} contains some non–local terms and that the Polyakov action is present to reproduce the conformal anomaly. If the flow ends in a non–trivial IR fixed point we have an IR fixed point action which is non–local. Now we note that an IR fixed point action can also be seen as a UV action once one deforms it in a suitable manner. This implies that this non–local IR action can also be used as a starting point for an RG flow and thus it may be generally admissible to consider non–local action to begin with.

We now give an explicit example of our construction. The Gaussian theory has $c = 1$ and is the simplest example of a CFT:

$$S^{c=1}_{CFT}[\varphi, g] = \frac{1}{2}\int \sqrt{g}\varphi\Delta\varphi \tag{4.14}$$

to which we add $S_P[g]$ to define $\Gamma_{UV}[\varphi, g] = S^{c=1}_{CFT}[\varphi, g] + S_P[g]$. Using the one–loop trace–log formula starting from the Gaussian UV action Γ_{UV} we find:

$$\Gamma_{IR}[\varphi, g] = \Gamma_{UV}[\varphi, g] + \frac{1}{2}\mathrm{Tr}\log\Delta - S_P[g] = \Gamma_{UV}[\varphi, g]\,, \tag{4.15}$$

where the second term is due to the integration of the fluctuations, while the Polyakov term with the minus sign comes from the Weyl–invariant measure (4.7). The two cancel since the $\frac{1}{2}\mathrm{Tr}\log\Delta = S_P[g]$. In order to have $\Gamma_{UV} \neq \Gamma_{IR}$ one needs to add a relevant perturbation triggering the RG flow.

4.2.3 Anomaly matching from the path-integral

Starting from $\Gamma_{UV}[\varphi, g] = S_{UV}[\varphi, g] + c_{UV} S_P[g]$ plus relevant operators, we can consider the IR effective action obtained by integrating out fluctuations:

$$\begin{aligned} e^{-\Gamma_{IR}[\varphi,g]} &= \int_{1PI} \mathcal{D}_g\chi\, e^{-S_{UV}[\varphi+\chi,g]-c_{UV}S_P[g]+relevant} \\ &= e^{-c_{UV}S_P[g]} \int_{1PI} \mathcal{D}_g\chi\, e^{-S_{UV}[\varphi+\chi,g]+relevant}\,. \end{aligned} \tag{4.16}$$

Since the metric is non–dynamical we passed the Polyakov term through the path integral. Here by *relevant* we mean, depending on the case, massive deformations or marginally relevant ones. An example of the first are mass terms like $\frac{m^2}{2}\varphi^2$ or $m\bar{\psi}\psi$, while Yang–Mills theory is an example of the second case.

If we now flow to an IR fixed point, by virtue of the splitting property (4.13), we must have $\Gamma_{IR}[\varphi, g] = S_{IR}[\varphi, g] + c_{IR} S_P[g]$. Choosing a dilaton background of the form $g_{\mu\nu} = e^{2\tau}\delta_{\mu\nu}$, we are left with:

$$e^{-S_{IR}[\varphi,e^{2\tau}\delta]} e^{(c_{UV}-c_{IR})\Gamma_{WZ}[\tau,\delta]} = \int_{1PI} \mathcal{D}_{e^{2\tau}\delta}\chi\, e^{-S_{UV}[\varphi+\chi,e^{2\tau}\delta]+relevant}\,, \tag{4.17}$$

where we used (4.4) on flat space $\Gamma_{WZ}[\tau, \delta] = S_P[e^{2\tau}\delta]$. In order to recover the flat space measure we first shift $\chi \to e^{w\tau}\chi$ and $\varphi \to e^{w\tau}\varphi$ and then use the invariance (4.7):

$$e^{-S_{IR}[e^{w\tau}\varphi,e^{2\tau}\delta]} e^{(c_{UV}-c_{IR})\Gamma_{WZ}[\tau,\delta]} = \int_{1PI} \mathcal{D}_\delta\chi\, e^{-S_{UV}[e^{w\tau}(\varphi+\chi),e^{2\tau}\delta]+relevant}\,. \tag{4.18}$$

Then we use the conformal invariance properties of the actions, i.e. we substitute $S_{UV}[e^{w\tau}\varphi, e^{2\tau}\delta] = S_{UV}[\varphi]$ and $S_{IR}[e^{w\tau}\varphi, e^{2\tau}\delta] = S_{IR}[\varphi]$ since both actions are Weyl–invariant and obtain:

$$e^{-S_{IR}[\varphi]} e^{(c_{UV}-c_{IR})\Gamma_{WZ}[\tau,\delta]} = \int_{1PI} \mathcal{D}_\delta\chi\, e^{-S_{UV}[\chi+\varphi]+relevant}\,. \tag{4.19}$$

Note that $\mathcal{D}_\delta\chi \equiv \mathcal{D}\chi$ is the flat space measure. The only remaining dependence on τ is due to the relevant terms, which make the path integral non–trivial. Equation (4.19) tells us that the dilaton effective action (generated by matter loops) compensates exactly the difference between the anomalies in the UV and IR. This is precisely the anomaly matching condition considered in [100].

4.2.4 Proof of the integrated c–theorem

We can now prove the integrated c–theorem following [100]. From equation (4.19),

$$e^{-S_{IR}[\varphi]} e^{-\frac{c_{UV}-c_{IR}}{24\pi}\int \tau\Delta\tau} = \int_{1PI} \mathcal{D}\chi\, e^{-S_{UV}[\varphi+\chi]+relevant}\,, \tag{4.20}$$

we can read off Δc from the terms of the dilaton two–point function quadratic in momenta. The *relevant* terms can be expanded in powers of τ:

$$relevant = \int d^2x\, \tau\, \Theta + O(\tau^2)\,, \tag{4.21}$$

where $\Theta \equiv T^\mu_\mu$ and we omitted all terms of order τ^2 or greater since they do not contribute to $\int \tau\Delta\tau$ [100]. We are thus interested in the following expectation:

$$\left\langle e^{\int \tau\,\Theta} \right\rangle\Big|_{\tau^2} = \frac{1}{2}\int d^2x \int d^2y\, \tau_x\tau_y \left\langle \Theta_x\Theta_y \right\rangle\,. \tag{4.22}$$

We now only have to expand τ_y around τ_x:

$$\tau_y = \tau_x + (y-x)^\mu\, \partial_\mu\tau_x + \frac{1}{2}(y-x)^\mu\,(y-x)^\nu\, \partial_\mu\partial_\nu\tau_x + \dots\,, \tag{4.23}$$

use translation invariance and compare with the coefficient of $\int \tau\Delta\tau$ to find:

$$\Delta c = 3\pi \int d^2x\, x^2\, \left\langle \Theta_x\Theta_0 \right\rangle_{IR}\,, \tag{4.24}$$

which is the integrated version of the c–theorem. From here one notices that the integral is positive assuming reflection positivity and concludes that $\Delta c \geq 0$ [114, 115]. The above equation has also been found with different techniques and expressed as sum rule [118].

4.3 Flow equation for the c–function

The c–theorem states [115] that for a two–dimensional unitary quantum field theory, invariant under rotations and whose energy–momentum tensor is conserved, there exists a function c of the coupling constants which is monotonic along the RG flow and, at a fixed point, is stationary and equal to the central charge of the corresponding CFT. This function c is such that $\partial_t c < 0$ (where the "RG time" is given by the logarithm of the radius $t = \log r$, so the flow is towards the infrared for $r \to \infty$, hence the minus sign). The differential equation for c can be integrated from $r = 0$ to $r = \infty$ and gives back (4.24). A natural trial definition for an interpolating c–function is given by taking (4.24) with the integral which has been cut off at some scale μ (see for instance [128]):

$$\Delta c\,(\mu) \equiv c_{UV} - c\,(\mu) = 3\pi \int_0^{2\pi} d\varphi \int_0^{\mu^{-1}} dr\, r^3\, \left\langle \Theta(r)\Theta(0) \right\rangle\,. \tag{4.25}$$

We will follow a different approach. Instead of cutting off directly in real space we will cutoff in momentum space. This will allow us to naturally connect with the framework of the fRG and to derive an exact RG flow equation for the c–function. In the following we derive an equation for the c–function following the steps used to derive the FRGE for the EAA in chapter 1 and then we prove once again the integrated c–theorem

4.3.1 The fRG flow equation for the c–function

To construct the c–function we consider the Wilsonian RG prescription. A way to perform the momentum shell integration in a smooth way, is to introduce a suppressing factor in the path integral via $\mathcal{D}_g\chi \to \mathcal{D}_g\chi\, e^{-\Delta S_k[\chi,g]}$. The role of the cutoff action $\Delta S_k[\chi,g]$ is to restrict the integration to modes above the IR scale k. Here we take our cutoff action to be Weyl invariant via a suitable construction. In particular we will always employ a modified mass cutoff of the form $R_k = e^{-2\sigma_g}k^2$ where $\sigma_g = (2\Delta)^{-1}R$. The cutoff action

$$\Delta S_k = \frac{1}{2}\int \sqrt{g}\chi\left(e^{-2\sigma_g}k^2\right)\chi$$

is Weyl invariant since the Weyl variation of R_k cancels against the one coming from $\sqrt{g}$. This feature is also encountered in the derivation of the c–theorem via "anomaly matching" [100]. We consider a scale dependent effective action $\Gamma_k[\varphi,g]$, which, using (4.13), can be decomposed as:

$$\Gamma_k[\varphi,g] = S_k[\varphi,g] + c_k S_P[g] + \text{gravitational terms}\,. \tag{4.26}$$

where $S_k[\varphi,g]$ is defined by $S_k[0,g]=0$ and c_k is the scale dependent c–function. By "gravitational terms" we mean the purely geometrical terms depending on the metric alone, like $\int\sqrt{g}$ or $\int\sqrt{g}R$, generated by fluctuations. In order to pick up the coefficient of the Polyakov action we will always consider the metric $g_{\mu\nu} = e^{2\tau}\delta_{\mu\nu}$ and perform a derivative expansion so that it is possible to select terms of the form $\tau\Delta\tau$, this allows us to distinguish the Polyakov term from the other "gravitational terms". We will always work using $g_{\mu\nu} = e^{2\tau}\delta_{\mu\nu}$ where the non–locality does not appear. We come back to the issue of non–locality at the end of section 4.4.2 where we put forward the entire ansatz for Γ_k.

The collection of the $\Gamma_k[\varphi,g]$ for all k constitute the RG trajectory connecting $\Gamma_{UV}[\varphi,g]$ to $\Gamma_{IR}[\varphi,g]$; a cartoon of this shown in figure 4.1. If we now repeat the steps leading to equation (4.20), but with the cutoff term added, we arrive at:

$$e^{-S_k[\varphi,e^{2\tau}\delta]}e^{-\frac{c_{UV}-c_k}{24\pi}\int\tau\Delta\tau} = \int_{1PI}\mathcal{D}\chi\, e^{-S_{UV}[\varphi+\chi]+relevant}e^{-\Delta S_k[\chi,\delta]}\,. \tag{4.27}$$

Note that the choice of a Weyl invariant cutoff gives us the possibility of coupling τ only to the trace of the stress–energy of Γ_k and to avoid "spurious" contribution from the cutoff action. Now

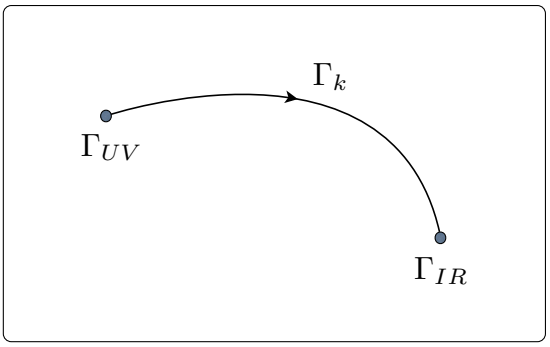


Figure 4.1: Cartoon depicting the flow in theory space: Γ_{UV} represents the bare action which, after turning on a relevant operator, flows to Γ_{IR} which, is the EA.

a derivative of (4.27) with respect to the "RG time" $t = \log(k/\mu)$ gives the RG flow of the central charge:

$$\partial_t c_k = -24\pi \left\langle \partial_t \Delta S_k[\chi, \delta] \right\rangle \Big|_{\int \tau \Delta \tau} , \tag{4.28}$$

in which the expectation value is calculated within the regularized path integral and the subscript indicates that one has to select the coefficient in front of the monomial $\tau\Delta\tau$. The running of c_k is related to the coarse–grained dilaton two–point function. Clearly to handle this equation it is convenient to consider the EAA since its flow equation is derived in much the same way.

Given the splitting property of our microscopic action we can write:

$$e^{-\Gamma_k[\varphi,g]} = \int_{1PI} D\chi \, e^{-S_{UV}[\varphi+\chi,g] - c_{UV} S_P[g] - \Delta S_k[\chi,g]} , \tag{4.29}$$

whose scale dependence is given by:

$$\partial_t \Gamma_k[\varphi, g] = \left\langle \partial_t \Delta S_k[\chi, g] \right\rangle = \frac{1}{2} \mathrm{Tr} \left\{ \left\langle \chi_A \chi_B \right\rangle \partial_t R_k^{AB}[g] \right\} , \tag{4.30}$$

$$= \frac{1}{2} \mathrm{Tr} \left(\frac{\delta^2 \Gamma_k[\varphi, g]}{\delta\varphi \delta\varphi} + R_k[g] \right)^{-1} \partial_t R_k[g] . \tag{4.31}$$

From the exact flow equation for the EAA we obtain a corresponding equation for the c–function. In particular, we can express the r.h.s. of (4.28) using (4.31):

$$\partial_t c_k = -24\pi \, \partial_t \Gamma_k[e^{w\tau}\varphi, e^{2\tau}\delta] \Big|_{\int \tau \Delta \tau} . \tag{4.32}$$

Equation (4.32) is the exact flow equation for the c–function in the fRG framework. Using (4.31) in the r.h.s. leads to the following explicit form:

$$\partial_t c_k = -12\pi \, \mathrm{Tr} \left(\frac{\partial_t R_k}{\Gamma_k^{(2,0)}[\varphi, \tau] + R_k} \right) \Bigg|_{\int \tau \Delta \tau} , \tag{4.33}$$

Figure 4.2: Diagrammatic representation of the two terms in the r.h.s. of the flow equation (4.35) for the c–function.

where we defined $\Gamma_k[\varphi,\tau] \equiv \Gamma_k[e^{w\tau}\varphi, e^{2\tau}\delta]$. The exact RG flow equation for the c–function is the main result of this section.

To write more explicitly the flow equation for the c–function we define the regularized propagator $G_k[\tau] \equiv (\Gamma_k^{(2,0)}[\varphi,\tau] + R_k)^{-1}$, perform two functional derivatives of (4.32) with respect to the dilaton, set $\tau = 0$ and extract the term proportional to Δ:

$$\begin{aligned} \partial_t c_k &= -24\pi \Big\{ \mathrm{Tr}\, G_k \left(\Gamma_k^{(2,1)}\right) G_k \left(\Gamma_k^{(2,1)}\right) G_k \partial_t R_k \\ &\quad -\frac{1}{2}\mathrm{Tr}\, G_k \left(\Gamma_k^{(2,2)}\right) G_k \partial_t R_k \Big\}\Big|_\Delta , \end{aligned} \tag{4.34}$$

where all quantities are evaluated at $\varphi = \tau = 0$.

The flow equation in the form (4.34) is a bit cumbersome so we introduce a compact notation to rewrite it in a simpler way. If we introduce the formal operator $\tilde{\partial}_t = \partial_t R_k \frac{\partial}{\partial R_k}$, we can rewrite the flow equation (4.32) for the c–function using the following simple relations:

$$\tilde{\partial}_t G_k[\tau] = -G_k[\tau]\partial_t R_k G_k[\tau] \qquad \tilde{\partial}_t \log G_k[\tau] = G_k^{-1}[\tau]\tilde{\partial}_t G_k[\tau] = G_k[\tau]\partial_t R_k \,.$$

Now we can rewrite the flow equation (4.34) in the following compact form:

$$\begin{aligned} \partial_t c_k &= 12\pi\, \mathrm{Tr}\, \tilde{\partial}_t \left\{ \left(\Gamma_k^{(2,1)}\right) G_k \left(\Gamma_k^{(2,1)}\right) G_k \right\} \\ &\quad -12\pi\, \mathrm{Tr}\, \tilde{\partial}_t \left\{ \left(\Gamma_k^{(2,2)}\right) G_k \right\}\Big|_\Delta , \end{aligned} \tag{4.35}$$

where again all quantities are evaluated at $\varphi = \tau = 0$. This is the form that we will use in applications in section 4.5. Finally, we can represent diagrammatically the two terms on the r.h.s. of (4.35) as in figure 4.2 and switch to momentum space to evaluate the diagrams by employing the techniques presented in [21]. In particular, continuous lines represent matter regularized propagators $G_k[0]$, while vertices with m–external wavy lines are the matter–dilaton vertices $\Gamma_k^{(2,m)}[\varphi,\tau]$. Finally, each loop represents a $\int d^2x\, \tilde{\partial}_t$ or a $\int \frac{d^2q}{(2\pi)^2}\, \tilde{\partial}_t$ trace.

In order to make explicit computations out of the flow equation for the c–function we need to make some statements about the response of the EAA to a Weyl variation away from a fixed point. Indeed, away from criticality, the effect of a Weyl variation is not only described by a Wess–Zumino action like (4.10). This relation needs to be generalized in such a way that it goes

back to (4.10) once we reach a fixed point. This will be done in detail in the next section. For the time being let us note that this simple information tells us that the new terms vanish at a fixed point and so they must be proportional to the (dimensionless) beta functions. We can thus make the following ansatz:

$$\Gamma_k[e^{w\tau}\varphi, e^{2\tau}g] - \Gamma_k[\varphi, g] = \Gamma_{WZ}[\tau, g] + \beta\text{–terms}\,, \tag{4.36}$$

where the dependence on c_k is inside $\Gamma_{WZ}[\tau, g]$. This relation can be read as a generalized running Wess–Zumino action. The β–terms indicate terms proportional to (at least one) dimensionless beta function which vanish at the CFTs and are generated along the flow by the fact that we are moving away from criticality.

Finally let us note that another way to see that the flow of the c–function is given by (4.32) is to recognize that c_k is nothing more than the coupling constant of the Polyakov action. As we said, when working on curved backgrounds one should always add the Polyakov term to a truncation. Thus the Wess–Zumino action on the r.h.s of (4.36) derives from the presence of the Polyakov action, with coefficient c_k, in the EAAs on the l.h.s of the same equation. Then, as just seen in the previous paragraph, a t–derivative relates $\partial_t c_k$ to the two–point function of the dilaton. In principle one can obtain the flow of c_k directly as the coefficient of $\int \sqrt{g} R \frac{1}{\Delta} R$ but this is more laborious. Finally, note that the inclusion of the Polyakov action with running central charge makes the truncation consistent with the conformal anomaly both in the UV and in the IR. To understand the β–terms we will consider, in the next section, the scale anomaly.

4.4 General form of the effective average action

Now we put forward some requirements which an ansatz for the EAA should satisfy. These requirements are motivated from the fact that the EAA should reproduce some generic features of QFTs, namely the scale and the conformal anomaly. In particular we will try to shed light on the nature of the β–terms introduced in equation (4.36).

4.4.1 The local ansatz and its limitations

When studying truncations of the EAA, one typically starts expanding the functional in terms of local operators compatible with the symmetries of the system:

$$\Gamma_k[\varphi, g] = \sum_i g_{i,k} \int d^2x \sqrt{g}\, \mathcal{O}_i[\varphi, g]\,. \tag{4.37}$$

This equation defines the running coupling constants $g_{i,k}$, which become the coordinates that parametrize theory space in the given operator basis.

A class of operators, which is not complete, but allows many computations to be performed analytically, is the one composed of powers of the field, i.e. $\mathcal{O}_i[\varphi, g] = \varphi^{2i}$ and $g_{i,k} = \frac{\lambda_{2i,k}}{(2i)!}$. In this approximation, one usually re–sums the field powers into a running effective potential $V_k(\varphi)$ and considers the following ansatz for the EAA:

$$\Gamma_k[\varphi, g] = \int d^2x\sqrt{g}\left[\frac{1}{2}\varphi\Delta\varphi + V_k(\varphi)\right], \tag{4.38}$$

known as local potential approximation (LPA). Within this truncation the exact flow equation (4.31) becomes a partial differential equation:

$$\partial_t V_k(\varphi) = c_d \frac{k^d}{1 + V_k''(\varphi)/k^2}, \tag{4.39}$$

with $c_d^{-1} = (4\pi)^{d/2}\Gamma(d/2+1)$. Even such a simple truncation is able to manifest qualitatively all the critical information relative to the theory space of scalar theories and in particular the fixed point structure [40, 129].

However, the effective action usually contains also quasilocal terms. Some of these quasilocal terms are directly related to the finite part of the effective action, which generally has a complicated form encoding all the information contained in the correlation functions or amplitudes. These terms are not present in the LPA which can be seen as the limit where we discard all the momentum structure of the vertices.

Nevertheless there are other semilocal terms that are non–zero only away from a fixed point: these are the β–terms introduced in equation (4.36). As we will explain in this section these terms are needed to recover known results and will play a central role in our computations. If we limit ourselves to the local truncation ansatz (4.37), then one finds that the flow equation for the c–function is driven only by the classical non Weyl–invariant terms, which is not correct. This is not due to the fact that the flow equation (4.32) is wrong, rather, it is the truncation ansatz (4.37) that is insufficient. Fluctuations induce the β–terms of equation (4.36) and we will see that these are crucial in driving the flow of the c–function. We will argue that these nonlocal terms have a precise form. We will do this requiring the EAA to reproduce the scale anomaly.

4.4.2 Nonlocal ansatz and the scale anomaly

It is easy to understand the origin of the terms on the r.h.s. of (4.36) which are linear in τ: they are related to the scale anomaly. To see this let us rescale the fields and expand the EAA in powers of the dilaton:

$$\Gamma_k[\varphi, \tau] = \Gamma_k[\varphi, 0] + \int d^2x\, \tau\, \langle\Theta\rangle_k + O(\tau^2), \tag{4.40}$$

where:

$$\langle\Theta\rangle_k = \frac{\delta}{\delta\tau}\Gamma_k[\varphi, \tau]\Big|_{\tau\to 0}, \tag{4.41}$$

defines the scale dependent energy–momentum tensor trace. In the IR the EAA reduces to the standard effective action, which generally is scale anomalous. If we start with some UV action deformed by terms of the form $\sum_j g_j \int d^2x \sqrt{g}\, \mathcal{O}_i$, the corresponding scale anomaly in flat space reads:

$$\int d^2x \sqrt{g}\, \langle \Theta \rangle = -\sum_i (\beta_i - d_i g_i) \int d^2x\, \mathcal{O}_i[\varphi, \delta]\,, \tag{4.42}$$

where d_i are the dimensions of the coupling constants. The expression in brackets is nothing but the beta function of the dimensionless coupling:

$$k^{d_i} \tilde{\beta}_i = \beta_i - d_i g_i\,. \tag{4.43}$$

This is a standard result known from both ordinary and conformal perturbation theories [114].

Now we consider again the β–terms on the r.h.s. of (4.36). They come from the conformal variation of the EAA which should include also the terms due to the scale anomaly. Therefore it is natural to generalize the above equation for a generic k:

$$\langle \Theta_x \rangle_k = -\sum_i k^{d_i} \tilde{\beta}_i \int d^2x\, \mathcal{O}_i[\varphi, \delta]\,. \tag{4.44}$$

If we insert this into (4.40) we find:

$$\Gamma_k[\varphi, \tau] = \Gamma_k[\varphi, 0] - \sum_i k^{d_i} \tilde{\beta}_i \int d^2x\, \tau \mathcal{O}_i[\varphi, \delta] + O(\tau^2)\,. \tag{4.45}$$

This expression gives a non trivial contribution to the flow of the c–function since we now have the vertex

$$\Gamma_k^{(2,1)}[\varphi, \tau]\Big|_{\varphi=\tau=0} = -\sum_i k^{d_i} \tilde{\beta}_i \int \mathcal{O}_i^{(2,0)}[0, 0] \tag{4.46}$$

to insert in the r.h.s. of the flow equation for c_k.

We now propose a covariant form for (4.45) using the following properties:

$$g_{\mu\nu} \to e^{2\tau} g_{\mu\nu} \qquad \to \qquad \frac{1}{2\Delta} R \to \frac{1}{2\Delta} R + \tau\,. \tag{4.47}$$

With this and $\mathcal{O}_i \to e^{w_i \tau} \mathcal{O}_i$, it is easy to verify that the action

$$\Gamma_k[\varphi, g] = \sum_i g_{i,k} \int \sqrt{g}\, \mathcal{O}_i[\varphi, g] - \frac{1}{2} \sum_i \beta_i \int \sqrt{g}\, \mathcal{O}_i[\varphi, g] \frac{1}{\Delta} R + \cdots\,, \tag{4.48}$$

reproduces (4.45) to linear order in τ. In order to get an ansatz consistent also with the conformal anomaly we need to add to (4.48) the Polyakov term with the running central charge c_k as coefficient:

$$\Gamma_k[\varphi, g] = \sum_i g_{i,k} \int \sqrt{g}\, \mathcal{O}_i[\varphi, g] - \frac{1}{2} \sum_i \beta_i \int \sqrt{g}\, \mathcal{O}_i[\varphi, g] \frac{1}{\Delta} R - \frac{c_k}{96\pi} \int \sqrt{g}\, R \frac{1}{\Delta} R\,. \tag{4.49}$$

The form (4.49) represents a parametrization of the EAA consistent with (4.36) to linear order in the beta functions. For the time being we will not improve further our ansatz, since we will see in the next section, that the understanding of the linear terms in the beta functions is already sufficient to build the c–function in non–trivial cases.

The covariant ansatz (4.49) is explicitly non–local. This casts doubts on its validity since we know that the Γ_k should be quasilocal if the flow starts from a local action [17, 18]. Non–locality should appear only in the limit of $k \to 0$ as we have checked in section 3.4.1. In our case one may consider the fact that we allowed a non–local action as a starting point of the flow (see section 4.2.2). Nevertheless it would be certainly very useful to have a recipe which promotes the τ terms to covariant terms that does not immediately imply non–locality, i.e.: quasilocal terms. So far this has not been achieved. Finally one could argue that our non–local terms can be seen as the integration over an auxiliary field but in such case one should perform the RG also for this field and not leave them out. Finally let us note that the Polyakov action has already been used in an ansatz for the EAA in [130] to describe membranes reproducing known results.

Finally we hope to come back to the issue of higher order terms in τ, which may play a role in making a bridge between the fRG perspective adopted here and the ideas related to the local RG [87].

4.5 Applications

4.5.1 Checking exact results

Here we provide two examples where the c–function and the difference $c_{UV} - c_{IR}$ are computed and can be compared to known exact results. We will consider a free scalar field and a free (Majorana) fermionic field whose fixed point actions are perturbed by a mass term, so they flow to $c_{IR} = 0$.

Massive deformation of the Gaussian fixed point

We consider a scalar field with Gaussian action and $c_{UV} = 1$ perturbed by a mass term. Since the beta function of the mass is zero (there are no interactions), our general ansatz (4.49) for the EAA reads:

$$\Gamma_k[\varphi, g] = \frac{1}{2}\int d^2x\sqrt{g}\,\varphi(\Delta + m^2)\varphi - \frac{c_k}{96\pi}\int \sqrt{g}R\frac{1}{\Delta}R\,, \tag{4.50}$$

which implies

$$\Gamma_k[\varphi, e^{2\tau}\delta] = \frac{1}{2}\int d^2x\,\varphi\left(\Delta + e^{2\tau}m^2\right)\varphi - \frac{c_k}{24\pi}\int \tau\Delta\tau\,. \tag{4.51}$$

It's clear that the only interaction between φ and τ is the one induced by the dimension of the mass. We use a mass cutoff and introduce the parameter a to check the cutoff independence of

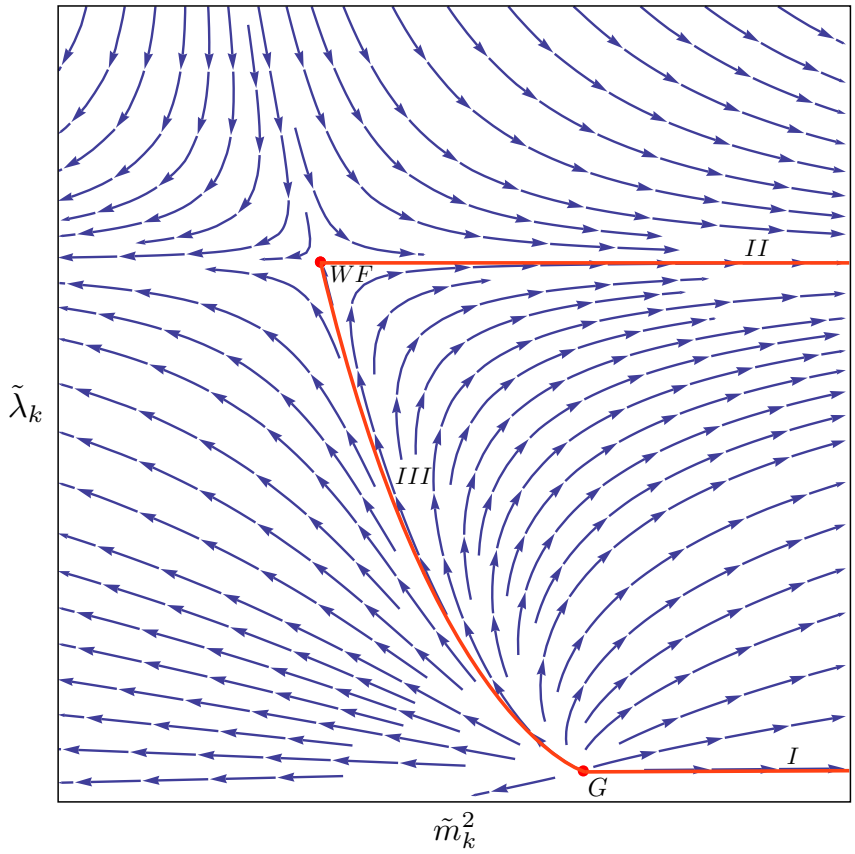


Figure 4.3: The flow in the $(\tilde{m}_k^2, \tilde{\lambda}_k)$ plane showing the Gaussian (G) and Ising (WF) fixed points. The flow induced by the massive deformation of the Gaussian fixed point is represented by trajectory–I, the flow induced by the massive deformation of the Ising fixed point is represented by trajectory–II while the flow between the two fixed points happens along trajectory–III.

the result. After a short computation[4] we find the following flow:

$$\partial_t c_k = \frac{4ak^2m^4}{(ak^2+m^2)^3}\,, \tag{4.52}$$

where m is the dimensionful mass. This RG flow is similar to trajectory–I of figure 4.3, which we will meet later.

Integrating the above differential equation, with the initial condition $c_\infty = 1$ (the central charge of the Gaussian fixed point) we find:

$$c_k = 1 - \frac{m^4}{(ak^2+m^2)^2}\,. \tag{4.53}$$

[4]We need to evaluate the first diagram of figure 4.2, for more details see section 4.5.2.

In the $k \to 0$ limit this gives $c_0 = 0$ which implies $\Delta c = 1$ independently of the cutoff parameter a. As expected a massive deformation of the Gaussian fixed point leads in the IR to a theory with zero central charge.

Massive deformation of the Ising fixed point

In this example we consider a massive deformation of the Ising fixed point. The critical Ising model is described by a free Majorana fermion and a massive deformation of this correspond to consider $T > T_c$ [114]. According to our general ansatz (4.49) and considering that, as before, the mass beta function is zero, the EAA reads:

$$\Gamma_k[\bar{\psi}, \psi, g] = \int d^2x \sqrt{g}\, \bar{\psi} \left(\not{\nabla} + m \right) \psi - \frac{c_k}{96\pi} \int \sqrt{g} R \frac{1}{\Delta} R \,, \tag{4.54}$$

which gives:

$$\Gamma_k[e^{\tau/2}\bar{\psi}, e^{\tau/2}\psi, e^{2\tau}\delta] = \int d^2x\, \bar{\psi} \left(\not{\nabla} + e^{\tau} m \right) \psi - \frac{c_k}{24\pi} \int \tau \Delta \tau \,. \tag{4.55}$$

The computation proceeds along the lines of the scalar case. Once again we use the mass cutoff $R_k = ak$ and we find:

$$\partial_t c_k = \frac{akm^2}{(ak+m)^3}.$$

This RG flow occurs is similar to trajectory–II of figure 4.3, which we will consider in the next section.

Integrating this equation with boundary condition $c_\infty = \frac{1}{2}$ (the central charge of the Ising model) leads to

$$c_k = \frac{1}{2} - \frac{m^2}{2\,(ak+m)^2} \,, \tag{4.56}$$

which gives $c_0 = 0$ and $\Delta c = \frac{1}{2}$ as expected independent of a.

4.5.2 The c–function in the local potential approximation

The local potential approximation (LPA), introduced in section 4.4.1, is characterized by the action (4.38); in our case it generalizes to:

$$\Gamma_k[\varphi, g] = \int d^2x \sqrt{g} \left[\frac{1}{2} \varphi \Delta \varphi + V_k(\varphi) - \frac{1}{2} \partial_t V_k(\varphi) \frac{1}{\Delta} R - \frac{c_k}{96\pi} R \frac{1}{\Delta} R \right] . \tag{4.57}$$

This implies:

$$\Gamma_k[\varphi, e^{2\tau}\delta] = \int d^2x \left[\frac{1}{2} \varphi \Delta \varphi + e^{2\tau} V_k(\varphi) - \partial_t V_k(\varphi)\, \tau - \frac{c_k}{24\pi} \tau \Delta \tau \right] . \tag{4.58}$$

If we now pass to dimensionless variables, $\varphi = k^{-w}\tilde{\varphi}$ and $V_k(\varphi) = k^2\tilde{V}_k(\tilde{\varphi})$, then the second and third terms in the above equation, to linear order in τ, become $-k^2\partial_t\tilde{V}_k(\tilde{\varphi})\tau$, so that the scalar–dilaton interaction is proportional to the dimensionless scale derivative of the potential.

To obtain the flow equation for the c–function we use (4.35) and the mass cutoff $R_k(z) = k^2$. Only the first diagram of figure 4.2 contributes terms of order p^2 in the external momenta, more specifically we need to evaluate the integral:

$$\partial_t c_k = -12\pi(\partial_t\tilde{V}_k''(\varphi_0))^2 k^4 \int \frac{d^2q}{(2\pi)^d}\, G_k^2(q^2) G_k\left((p+q)^2\right)\partial_t R_k(q^2)\Big|_{p^2}\,, \tag{4.59}$$

with the following regularized propagator:

$$G_k(q^2) = \frac{1}{q^2 + V_k''(\varphi_0) + R_k(q^2)}\,. \tag{4.60}$$

Here φ_0 is the minimum of the running effective potential, i.e. the solution of $V_k'(\varphi) = 0$. With the mass cutoff one finds the following result:

$$\int \frac{d^2q}{(2\pi)^d}\, G_k^2(q^2) G_k\left((p+q)^2\right)\partial_t R_k(q^2)\Big|_{p^2} = -\frac{1}{12\pi k^4(1+\tilde{V}_k''(\varphi_0))^3}\,, \tag{4.61}$$

provided that $\tilde{V}_k''(\varphi_0) > -1$, since otherwise the momentum integral does not converge. Inserting this back in (4.59) finally gives:

$$\partial_t c_k = \frac{(\partial_t\tilde{V}_k''(\varphi_0))^2}{(1+\tilde{V}_k''(\varphi_0))^3}\,, \tag{4.62}$$

which is the flow equation for the c–function in the LPA with a mass cutoff. This the main result of this section. Note that since (4.62) is valid only under the condition $\tilde{V}_k''(\varphi_0) > -1$, the c–theorem $\partial_t c_k \geq 0$ is indeed satisfied within the LPA.

Flow between the Gaussian and Ising fixed points

We now consider the simple case where there are just two running couplings parametrizing theory space, i.e. we expand the running effective potential in a Taylor series:

$$V_k(\varphi) = \frac{1}{2!}m_k^2\varphi^2 + \frac{1}{4!}\lambda_k\varphi^4 + \ldots \tag{4.63}$$

where m_k^2 is the mass and λ_k the quartic self–interaction. Inserting (4.63) in the flow equation for the effective potential (4.39) and projecting out the flow of the two couplings gives, after passing to dimensionless variables $m_k^2 = k^2\tilde{m}_k^2$ and $\lambda_k^2 = k^2\tilde{\lambda}_k$, the following system of beta functions:

$$\partial_t\tilde{m}_k^2 = -2\tilde{m}_k^2 - \frac{1}{4\pi}\frac{\tilde{\lambda}_k}{(1+\tilde{m}_k^2)^2} \qquad \partial_t\tilde{\lambda}_k = -2\tilde{\lambda}_k + \frac{3}{2\pi}\frac{\tilde{\lambda}_k^2}{(1+\tilde{m}_k^2)^3}\,. \tag{4.64}$$

This system has two fixed points: the Gaussian $(\tilde{m}_k^2, \tilde{\lambda}_k) = (0,0)$ and the Ising $(\tilde{m}_k^2, \tilde{\lambda}_k) = (-\frac{1}{4}, \frac{3\pi}{2})$. The Gaussian fixed point has two IR repulsive directions, while the Ising fixed point

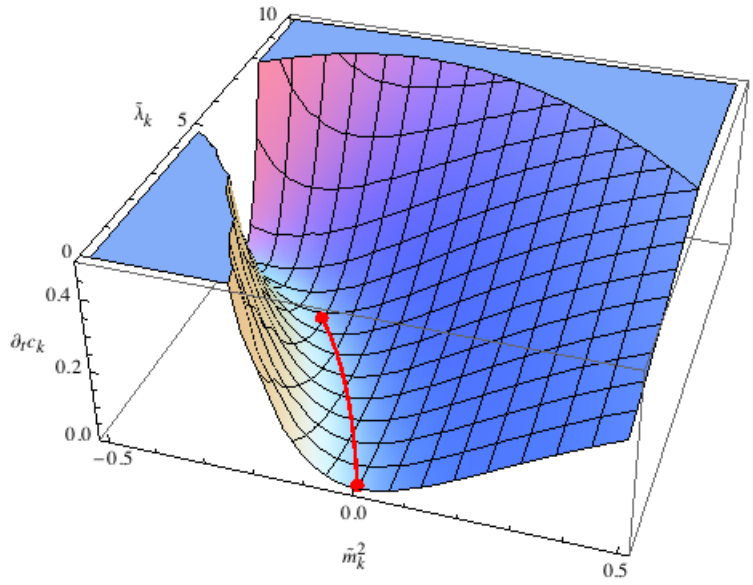


Figure 4.4: $\partial_t c_k$ in the $(\tilde{m}_k^2, \lambda_k)$ plane. We marked with a red dot the position of the Gaussian and Ising fixed points.

has one IR repulsive and one IR attractive direction. The trajectories starting along these directions are shown in figure 4.3, in particular trajectory–III connects the two fixed points.

We can now use (4.62) to evaluate the c–function in this truncation. This turns out to be simply related to the square of the dimensionless mass beta function:

$$\partial_t c_k = \frac{1}{(1+\tilde{m}_k^2)^3} \left(\partial_t \tilde{m}_k^2\right)^2 = \frac{1}{(1+\tilde{m}_k^2)^3} \left(2\tilde{m}_k^2 + \frac{1}{4\pi} \frac{\tilde{\lambda}_k}{(1+\tilde{m}_k^2)^2}\right)^2 . \tag{4.65}$$

As for (4.62), the result is only valid for $\tilde{m}_k^2 > -1$, so in this range we do have $\partial_t c_k \geq 0$, which is consistent with the c–theorem. Equation (4.65) is the first non–trivial example of explicit flow equation for the c–function obtained using the procedure presented in this work. In figure 4.4 we plot $\partial_t c_k$ in the plane $(\tilde{m}_k^2, \tilde{\lambda}_k)$: one can see that the magnitude of $\partial_t c_k$ is smaller along a "valley" containing the two fixed points. Along this valley lies the trajectory connecting them, trajectory–III of figure 4.3.

We would like to compute Δc by integrating the flow of the central charge along the path connecting the Gaussian and Ising fixed points and find an approximate result $\Delta c \approx 1/2$. In this simple truncation we do not have quantitatively good results and $\Delta c \approx 0.03$. To improve we need to consider a more refined truncation ansatz for the running effective potential. We leave these studies to future work.

Sine–Gordon model

We now consider the Sine–Gordon model which, in the continuum limit, is described by the following action [114]:

$$S_{SG}[\varphi] = \int d^2x \left[\frac{1}{2}\varphi\Delta\varphi - \frac{m^2}{\beta^2}\left(\cos\left(\beta\varphi\right) - 1\right)\right] , \tag{4.66}$$

where m is the mass and β is a coupling constant. This theory can be seen as a massive deformation of the Gaussian fixed point action (with $c_{UV} = 1$) and indeed we will find $c_{IR} = 0$.

The Sine-Gordon model can be described by an LPA with effective potential:

$$V_k(\varphi) = -\frac{m_k^2}{\beta_k^2}\left(\cos\left(\beta_k\varphi\right) - 1\right) . \tag{4.67}$$

We find the following form for the beta functions of m_k and β_k:

$$\begin{aligned} \partial_t\tilde{m}_k^2 &= \frac{\tilde{m}_k^2\left(\beta_k^2 - 8\pi\left(1+\tilde{m}_k^2\right)\right)}{4\pi\left(1+\tilde{m}_k^2\right)} \\ \partial_t\beta_k &= -\frac{3\tilde{m}_k^2\beta_k^3}{8\pi\left(1+\tilde{m}_k^2\right)^2} , \end{aligned}$$

where $\tilde{m}_k^2 = m_k^2/k^2$ is the dimensionless mass. Inserting the Sine–Gordon running potential (4.67) into the flow equation (4.62) now gives:

$$\partial_t c_k = \frac{\tilde{m}_k^4\left(\beta_k^2 - 8\pi\left(1+\tilde{m}_k^2\right)\right)^2}{16\pi^2\left(1+\tilde{m}_k^2\right)^5} . \tag{4.68}$$

We solved the system of equations numerically imposing $c_{UV} = 1$ finding $\Delta c \simeq 0.998$, in satisfactory agreement with the exact result $\Delta c = 1$. In figure 4.5 we plot the running of c_k as well as its beta function.

4.5.3 The c–function in the loop expansion

The last approximation we will consider is the loop expansion for the EAA which we introduced in section 1.4.2. In the first part of this section we will look at the various contributions diagrammatically, while in the second part we will explicitly evaluate one subclass of these.

Zamolodchikov's metric: diagrammatics

Using relation (1.20) we can compute the running of the EAA at each order in the loop expansion. The running of the L–th term $\partial_t\Gamma_{L,k}$, say, will contain a contribution to the running of c_k that we will call $\partial_t c_{L,k}$. The term $c_{L,k}$ arises only from diagrams with L matter loops and two dilaton external lines. In this way we can build a loop expansion for the c–function.

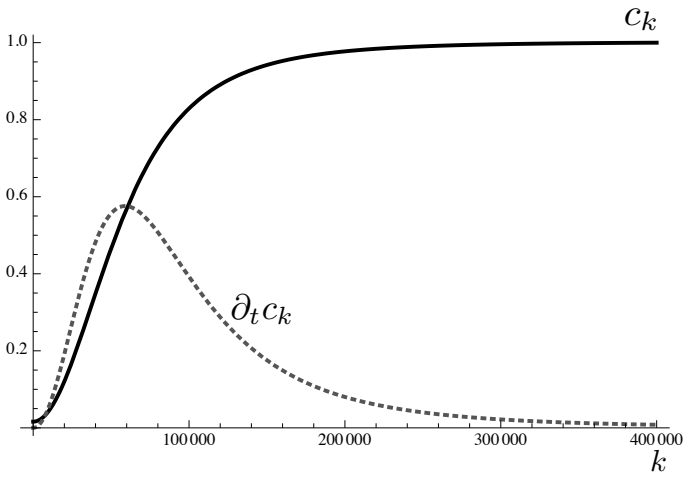


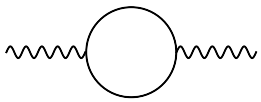
Figure 4.5: Flow of the Sine-Gordon model: the continuous line shows the running of the c-function and the dotted line has a bell shape meaning that the beta function of c_k is zero at the endpoints of the flow.

We can start by applying this construction step by step so to make clear how everything works. We will work with a $\mathbb{Z}_2$–symmetric scalar theory, so that the part linear in the dilaton of our general ansatz (4.49) takes the form:

$$\sum_n \frac{1}{(2n)!}\tilde{\beta}_{2n}\varphi^{2n}\tau\,, \tag{4.69}$$

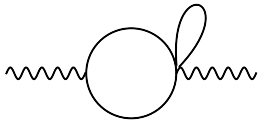
where $\tilde{\beta}_2$ is the mass beta function, $\tilde{\beta}_4$ is the φ^4 coupling beta function, and so on.

At one loop, we have only the following diagram, obtained from (1.21) by functional derivation with respect to the dilaton,

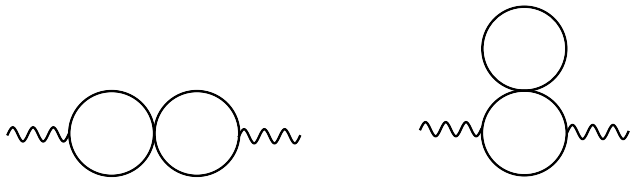


Here we adopt the same diagrammatic rules of section 4.3.1 where the continuous line represents the regularized propagator (in this case given in equation (1.22)), while the wavy line represents the dilaton. On every diagram the operator $\tilde{\partial}_t$ acts, but in this case it is equivalent to ∂_t. In this diagram the vertex, derived from (4.69), is the mass beta function, so this contribution goes like $\tilde{\beta}_2^2$ and we recover the LPA result (4.62) as one would expect.

From the flow of the two–loop contribution (1.24) we obtain different terms. We get the "non–diagonal" contribution (we will make this jargon clear in a second):

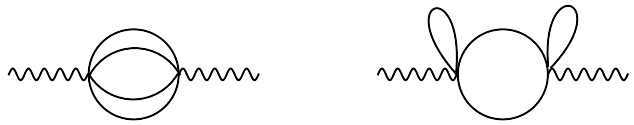


proportional to $\tilde{\beta}_2\,\tilde{\beta}_4$. Together with this, we also have the following 2–loop diagonal contributions:

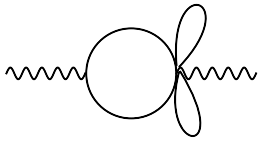


which are proportional to $\lambda_2\,\tilde{\beta}_2^2$. These represent a diagonal but coupling–dependent contribution, in the sense that couplings do not only appear through the beta functions. When going to 3–loops, 4–loops and so on, corresponding diagrams must be considered for all the diagonal contributions.

At three loops (remember we are considering a $\mathbb{Z}_2$–symmetric theory, so there are no scalar odd power interactions) we get again the "diagonal" contributions:



both proportional to $\tilde{\beta}_4^2$, as well as a nondiagonal one:



proportional to $\tilde{\beta}_2\,\tilde{\beta}_6$. From these first diagrams we clearly see from the structure of the loop expansion that we only get terms quadratic in the beta functions.

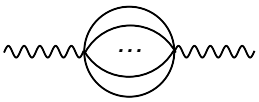
We can indeed follow Zamolodchikov and define the "metric" g_{ij} through:

$$\partial_t c_k = g_{ij}\tilde{\beta}^i\tilde{\beta}^j\,. \tag{4.70}$$

Our construction gives a diagrammatic representation of it within the loop expansion. It is also clear now what we meant by diagonal or nondiagonal contributions: they refer to the entries of this metric. In principle one can evaluate all these diagrams for a generic cutoff $R_k(z)$ but this turns out to be a difficult analytical task. In the next section we will be able to evaluate analytically one particular class of diagonal entries.

Diagonal contributions

At L–loop order, the simplest coupling–independent diagonal contribution comes from the following diagram:



corresponding to the expression:

$$\partial_t \Gamma_{L,k} = -\frac{1}{2(L+1)!}\,\tilde{\beta}^2_{L+1}\,k^4 \int d^2x \int d^2y\,\tau_x\tau_y\,\tilde{\partial}_t\left[G_k\left(x-y\right)\right]^{L+1} \tag{4.71}$$

(which generalizes equation (1.27)). In the above equation the $2(L+1)!$ comes from the symmetry factor of the diagram, and the minus sign from the fact that we are acting with an overall $\tilde{\partial}_t$. To recover the contribution to $\partial_t c_k$ is simple: expand τ_y around x as in equation (4.23), and isolate the proper term according to equation (4.32). In spirit this procedure is the practical implementation of the steps which we did in the general proof of section 4.2.4.

To see more explicitly the form that the Zamolodchikov's metric takes, we need some preliminary results. Using a mass cutoff $R_k = k^2$, the zero mass running renormalized propagator (1.22) will be the same as the standard massive one, only with k^2 in place of the mass m^2. In real space the propagator reads:

$$G_k\left(x-y\right) = \frac{1}{2\pi} K_0\left(\left|x-y\right|\sqrt{ak^2}\right)\,, \tag{4.72}$$

where K_0 is the Bessel K–function of order zero. We introduced the parameter a, eventually to be sent to 1, since in this way we have the simpler formula

$$\tilde{\partial}_t f\left[R_k\right] = 2\partial_a\, f\left[ak^2\right]\big|_{a\to 1}\,. \tag{4.73}$$

The different contributions are then calculated after expanding τ_y around x using (4.23). We find:

$$\partial_t \Gamma_{L,k} = \frac{k^4}{(L+1)!}\,\tilde{\beta}^2_{L+1} \int d^2x\,\tau_x \Delta\tau_x \int d^2y \frac{y^2}{2(2\pi)^{L+1}} \partial_a \left[K_0\left(\left|y\right|\sqrt{ak^2}\right)\right]^{L+1}\Bigg|_{a\to 1}\,. \tag{4.74}$$

These diagonal terms can be written to all orders, they give a contribution to the flow equation for c_k of the form:

$$\partial_t c_{L,k} = \mathcal{A}_L\,\tilde{\beta}^2_{L+1}\,, \tag{4.75}$$

in which we defined the quantity

$$\mathcal{A}_L \equiv \frac{3}{2^L \pi^{L-1} L!} \int_0^\infty dx\, x^4 \left[K_0\left(x\right)\right]^L K_1\left(x\right)\,. \tag{4.76}$$

We observe that contributions at loop order L are proportional to the square of the beta function of the coupling $\tilde{\lambda}_{L+1,k}$. Thus the flow of c_k receives contributions from all loops but a higher power in the interaction starts to contribute only at a higher loop order. All the $\mathcal{A}_L$ can be evaluated numerically and they turn out to be positive. The numerical values of the first $\mathcal{A}_L$ are shown in table 4.1. Note the fast decrease relative to the one–loop value.

$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	$\mathcal{A}_6$	$\mathcal{A}_7$
1	0.0182	4.778×10^{-4}	1.485×10^{-5}	5.066×10^{-7}	1.825×10^{-8}	6.8×10^{-10}

Table 4.1: First few numerical values of $\mathcal{A}_L$.

We can now write down the contribution of this class of diagrams to the running of the c–function at all loops in the $\mathbb{Z}_2$–symmetric case:

$$\partial_t c_k^{(diagonal)} = \sum_{i=1}^{\infty} \mathcal{A}_{2i-1}\, \tilde{\beta}_{2i}^2 \,, \tag{4.77}$$

which also gives the explicit form for the diagonal entries of the Zamolodchikov metric. Since this sum is manifestly positive, we can say that the c–theorem is satisfied to all loops by the diagonal terms considered.

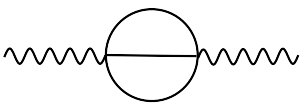
As we have seen previously, the entries of Zamolodchikov's metric contain a coupling–independent piece, plus further pieces proportional to increasing powers of the coupling constants, as we increase the loop order. The positivity properties of the metric are far from trivial when all these terms are involved. However, when the couplings are sufficiently small, the positivity will be determined solely by the coupling independent terms.

Non–unitary theories

Finally we make a comment on when the c–theorem is not satisfied, i.e. the case when $\partial_t c_k < 0$. We know that the c–theorem does not hold without the unitarity assumption [115]. This can indeed be checked explicitly. For example it is easy to see that when one considers interactions with complex couplings then the coefficients in the loop expansion turn negative. For instance, one notable example is the Lee-Yang model [114], in which one introduces the non–unitary complex interaction:

$$S_{LY}[\varphi] = \int d^2x \left[\frac{1}{2}\varphi\Delta\varphi + ig\varphi^3\right] . \tag{4.78}$$

A simple analysis reveals that this interaction contributes to the running of c_k through the following diagram:



which turns out to have the wrong sign to be consistent with the c–theorem:

$$\partial_t c_k = -\mathcal{A}_2\, \tilde{\beta}_3^2 < 0 \,, \tag{4.79}$$

since $\mathcal{A}_2 > 0$, as reported in table 4.1.

4.6 The c–function and Newton's constant

In this section we derive an interesting relation between the c–function and the matter–induced beta function of Newton's constant.[5] This can then be used to obtain another form of the flow of the central charge $\partial_t c_k$.

4.6.1 Relation between c_k and β_{G_k}

To obtain this relation we need to consider what happens when in equation (4.48) we set $\mathcal{O} = R$. Since the coupling constant of the invariant $\int \sqrt{g}R$ is $-\frac{1}{16\pi G_k}$, where G_k is the running Newton's constant, one finds, for the gravitational part of the EAA, the following form[6]:

$$\Gamma_k[0,g] = \int d^2x\sqrt{g}\left[-\frac{1}{16\pi G_k}R - \frac{1}{4}\partial_t\left(-\frac{1}{16\pi G_k}\right)R\frac{1}{\Delta}R + ...\right] . \tag{4.80}$$

We recognize that the Polyakov term above is the same that we included in our general anstaz for the EAA (4.49). Thus we infer that there is a relation between the beta function of Newton's constant and the running c–function:

$$\partial_t\left(-\frac{1}{16\pi G_k}\right) = \frac{c_k}{24\pi} . \tag{4.81}$$

This is a nontrivial statement by itself. It tells us that the running c–function for a certain matter field type can also be computed from the contributions of that kind of matter to the beta function of Newton's constant. In fact a derivative of (4.81) with respect to the RG scale gives

$$\partial_t c_k = \frac{3}{2G_k^2}\left(\partial_t\beta_{G_k} - 2\frac{\beta_{G_k}^2}{G_k}\right) , \tag{4.82}$$

where $\beta_{G_k} \equiv \partial_t G_k$ is the beta function of the Newton's constant. We will check the consistency of relation (4.81) in the case of a minimally coupled and self–interacting scalar.

Note that a relation similar to the one we are proposing has been discovered in the application of sigma models to string theory [131]. In that case we refer to the dilaton as the massless scalar field $\Phi(X^\mu)$ which is a function of the coordinates of the target space $X^\mu(\sigma,\tau)$, which are scalar fields living in two dimensions. In string theory one considers the following action

$$S = \int\left\{\frac{1}{4\pi\alpha'}\sqrt{\gamma}\gamma^{ab}\partial_a X^\mu\partial_b X^\nu G_{\mu\nu}(X) + \frac{1}{4\pi\alpha'}\varepsilon^{ab}\partial_a X^\mu\partial_b X^\nu B_{\mu\nu}(X) + \frac{1}{8\pi}\sqrt{\gamma}R_{(2d)}\Phi(X)\right\} .$$

The computation of the conformal anomaly and of the beta functions of $\Phi(X)$ shows that:

$$2\alpha'\int T^\mu_\mu = \int\left(-\frac{\alpha'}{4}\right)R_{(2d)}\left[\gamma(\Phi) + \theta(G)\right] + \cdots$$

where the combination $\gamma + \theta$ is exactly the beta function of the coupling Φ thus confirming our claim [131].

[5] In what follows we identify the Newton's constant as the coupling in front of the Ricci scalar.

[6] We need here 1/4 instead of 1/2 because of the the further symmetry we have in exchanging the two Rs

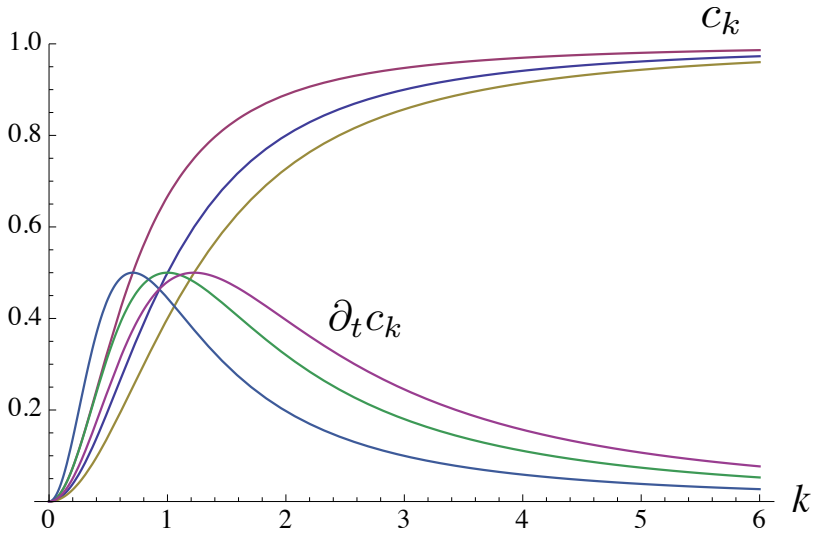


Figure 4.6: c_k and $\partial_t c_k$ as a function of k for a massive deformation of a minimally coupled scalar. Mass ($a = 1$), optimized ($a = 1$) and exponential ($a = 1$, $b = 1$) cutoffs (upper curves), exponential ($a = 1$, $b = \frac{1}{2}$) cutoff (middle curves), exponential ($a = 1$, $b = \frac{3}{2}$) cutoff (lower curves). In all cases we set $m^2 = 1$.

4.6.2 Scalar minimally coupled to gravity

Consider a minimally coupled scalar describing a massive deformation of the Gaussian fixed point as discussed in section 4.5.1. The action is given in (4.50) and the exact flow equation (4.31) for this case reads:

$$\partial_t \Gamma_k[\varphi, g] = \frac{1}{2}\mathrm{Tr}\frac{\partial_t R_k(\Delta)}{\Delta + m^2 + R_k(\Delta)}\,. \tag{4.83}$$

Given the discussion of the previous section we will not try to identify the c–function via the running of the Polyakov action. Instead, to find c_k using (4.81) we need to extract the terms in the trace on the r.h.s. of (4.83) that are proportional to the invariant $\int \sqrt{g}R$. As in chapter 2 this can be done using the heat kernel expansion. Defining $h_k(z) = \frac{\partial_t R_k(z)}{z+m^2+R_k(z)}$, one finds:

$$\frac{1}{2}\mathrm{Tr}\, h_k(\Delta)\Big|_{\int\sqrt{g}R} = \frac{1}{8\pi}\frac{1}{6}h_k(0)\int d^2x\sqrt{g}R\,, \tag{4.84}$$

which, when compared with the scale derivative of $-\frac{1}{16\pi G_k}\int\sqrt{g}R$ on the l.h.s. of (4.83), gives:

$$\partial_t\left(-\frac{1}{16\pi G_k}\right) = \frac{1}{8\pi}\frac{1}{6}h_k(0)\,. \tag{4.85}$$

Thus our formula (4.81) leads to

$$c_k = \frac{1}{2}h_k(0)\,. \tag{4.86}$$

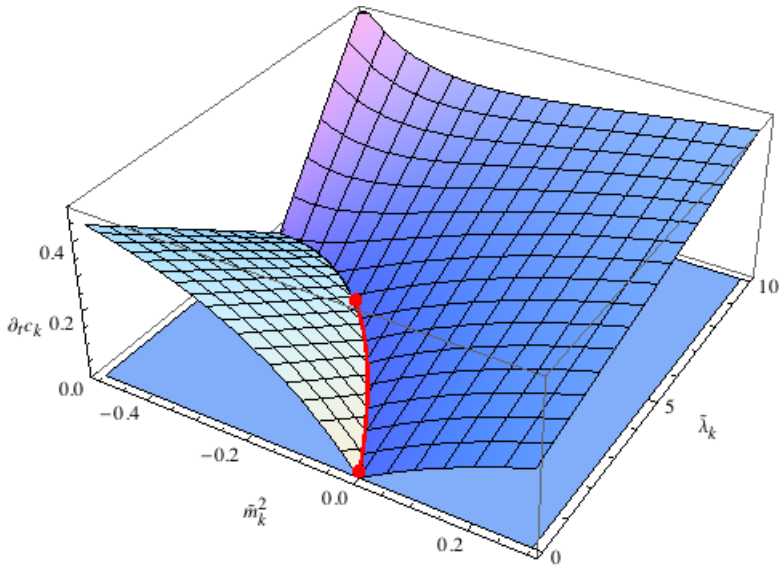


Figure 4.7: $\partial_t c_k$ in the $(\tilde{m}_k^2, \lambda_k)$ plane according to (4.92) for $a = 1$ and $b = \frac{1}{2}$. We marked with a red dot the position of the Gaussian and Ising fixed points and the trajectory–III connecting them. One can note that the trajectory connecting the two fixed points lies along a "valley".

Note that this relation is easily implemented for arbitrary cutoff function $R_k(z)$. For both the mass cutoff $R_k(z) = ak^2$ and the optimized cutoff $R_k(z) = a(k^2 - z)\theta(k^2 - z)$ we find the following form:

$$c_k = \frac{ak^2}{ak^2 + m^2} \,. \tag{4.87}$$

For the exponential cutoff $R_k(z) = \frac{az}{e^{bz/k^2}-1}$, with parameters a and b, we find:

$$c_k = \frac{ak^2}{ak^2 + bm^2} \,. \tag{4.88}$$

In all cases and for all values of the parameters a and b we find that $c_{UV} = 1$ and $c_{IR} = 0$ as expected. A derivative of (4.88) gives the flow of the c–function:

$$\partial_t c_k = \frac{2abk^2m^2}{(ak^2 + bm^2)^2} \,. \tag{4.89}$$

The interpolating c_k of equation (4.88) and the flow of the last equation are shown in figure 4.6. We clearly see that the flow is scheme dependent, but the integral of it along a trajectory, giving Δc, is universal.

4.6.3 Self–interacting scalar

We consider now an interacting scalar, i.e. the LPA action (4.58) of section 4.5.2. We can obtain c_k directly from equation (4.88) by just making the replacement $m^2 \to V_k''(\varphi_0)$:

$$c_k = \frac{ak^2}{ak^2 + bV_k''(\varphi_0)} \,. \tag{4.90}$$

A scale derivative now gives:

$$\partial_t c_k = -\frac{abk^2\left(\partial_t V_k''(\varphi_0) - 2V_k''(\varphi_0)\right)}{\left(ak^2 + bV_k''(\varphi_0)\right)^2}\,. \tag{4.91}$$

We need to decide the value of φ_0 where to evaluate this expression. In this case it is important to distinguish the ordered from the broken phase. If the running effective potential has the polynomial form (4.63), then we have $\varphi_0 = 0$ in the ordered phase and $\varphi_0 = \pm\sqrt{6m_k^2/\lambda_k}$ in the broken phase, the two phases being separated by trajectory–III and its continuation. Inserting these expressions in (4.91) gives the following form for the flow of c_k:

$$\partial_t c_k = \begin{cases} -\frac{ab\,\partial_t \tilde{m}_k^2}{(a+b\,\tilde{m}_k^2)^2} & \text{ordered phase} \\ \\ \frac{2ab\,\partial_t \tilde{m}_k^2}{(a-2b\,\tilde{m}_k^2)^2} & \text{broken phase}\,. \end{cases} \tag{4.92}$$

As shown in figure 4.7, the flow (4.92), even if not proportional to the square of the dimensionless beta function, is positive $\partial_t c_k \geq 0$ in the $(\tilde{m}_k^2, \lambda_k)$ plane. This calculation represents a non–trivial check of relation (4.81) and shows how this relation can be used explicitly to compute c_k in a given truncation by means of heat kernel techniques.

4.7 Summary

In this chapter we have explored a new method to study the flow of the c–function within the framework of the functional RG based on the effective average action (EAA). This function interpolates between the UV and IR central charges of the corresponding CFTs and is thus a global feature of the flow, related to the integration of it along a trajectory connecting two fixed points, independent of scheme choices.

Our main result is an RG exact equation for the running c–function based on the identification of it with the coefficient of the running Polyakov action. This equation relates the flow of the central charge to the exact flow of the EAA. To solve the equation for non–trivial cases we built a suitable ansatz requiring the EAA to reproduce generic features of QFTs, namely the scale and the conformal anomalies. This is an interesting result in its own right since it teaches us that a consistent ansatz for the EAA off criticality should include some terms proportional to beta functions. These terms are a direct consequence of the generalized WZ action (4.36). We put forward an ansatz which takes this into account in a covariant manner but is non–local and this casts doubts on its validity and further study is needed in this direction. Nevertheless when this ansatz is expressed via the metric $g_{\mu\nu} = e^{2\tau}\delta_{\mu\nu}$ we have a local ansatz for τ which is consistent with (4.36) and allows to compute an approximate $c-$function. Of course we do not claim full generality for this ansatz, but we found that it is sufficiently accurate to trigger the flow of the

c–function in non–trivial cases. Explicit computations, within the local potential approximation and the loop expansion, have been presented in section 4.5 showing the compatibility of our framework with the c–theorem.

Moreover we have put forward a relation between the beta function of Newton's constant and the running conformal anomaly. This relation comes from internal consistency of the generic ansatz for the EAA we proposed and allows us to use heat kernel techniques to compute the RG running of the c–function. We also checked this other relation in explicit cases finding consistency. Nevertheless we point out that our analysis is not complete. The works [87, 128] highlight that there are some subtleties related to the definition of the c–function. A complete mapping between the local RG approach and the fRG is still lacking and further study is needed in this direction. Another issue, which has not been touched at all, is the generalization of these ideas to the higher dimensional case, in particular $d = 4$ where one can consider similar constructions for the a–function [87, 132, 133]. Also in this case it is important to understand the connection between the approach pursued in this chapter and the results found with the local RG and how this enters in a consistent ansatz for the EAA. In the four dimensional case one should look to the four point function of the dilaton τ which enters in the Komargodski-Schwimmer proof [99] and it has been connected to the local RG framework [134].

APPENDIX A

The Wilsonian renormalization group

A.1 Brief review of the Wilsonian Renormalization Group

The renormalization group (RG) is a key concept in statistical mechanics and quantum field theory. In general the renormalization group is a set of ideas which has to be adapted to the problem at hand ranging from field theories and spin systems to partial differential equation. All the applications of the RG have the common feature of re-expressing the parameters of the problem in terms of some others keeping the physics of interest unchanged (e.g.: long distance behaviour and low energy physics). This is, of course, a very important conceptual step. For instance if we think about statistical mechanics we are used to consider all the degrees of freedom at once, e.g. computing the partition function. The RG changes that and instructs us that it is convenient to deal with degrees of freedom via consecutive steps. The generality of these ideas have found implementations in quantum field theory, statistical mechanics, out of equilibrium phenomena, chaos and so on. Equilibrium critical phenomena are a paradigm of the whole approach and we shall describe the ideas behind the RG using the typical example of a spin system on a lattice such as the Ising model. It is important to note that in such systems there might be no small parameter about which we can expand. Because of this it is difficult (but not impossible) to control if the results are quantitatively correct. Nevertheless the qualitative predictions such as universality and scaling relations arise as general properties and many times, despite the rough approximations, good quantitative results are obtained. We will first discuss the idea of blocking, first put forward by Kadanoff [135] and then we will see how this fits into the Wilsonian framework [136–138].

A.1.1 Kadanoff blocking

Let us consider a spin system on the lattice whose distance from site to site is a and whose interaction is only with the nearest neighbor. To simplify the problem one can imagine a coarse-graining procedure which averages the spin contained in a block of size $l \cdot a$. This averaged block

is called block spin. This block contains l^d original spins and if we began with N spins we end up in Nl^{-d} block spins. We define the block spin S_I by [139]:

$$S_I \equiv \frac{1}{|\bar{m}_l|}\frac{1}{l^d}\sum_{i\in I} S_i$$

where

$$\bar{m}_l \equiv \frac{1}{l^d}\sum_{i\in I}\langle S_i\rangle.$$

Note that with this normalization S_I has value in ± 1. The above definition is just an example of a possible coarse graining procedure. This procedure is not possible above the critical temperature since $\langle S_i\rangle = 0$ and one should adopt some other recipes, e.g.: decimation. Doing the coarse graining implies that we see the system somehow out of focus meaning that we can no longer see the microscopic details. The first assumption which Kadanoff did is the following: since the original spins interacts with the nearest neighbor we assume that also the block spins interacts only with the nearest neighbor. The coupling in the Hamiltonian are changed of course and thus after a block spin transformation the Hamiltonian is changed from $\mathcal{H}$ to $\mathcal{H}_l$. Here $\mathcal{H}$ is expressed in units of block spacing a and $\mathcal{H}_l$ in units of block spacing $l \cdot a$. The long distance physics must be preserved meaning that the correlation length measured in $\mathring{A}$ is the same if computed via $\mathcal{H}$ or $\mathcal{H}_l$. Nevertheless the correlation length measured in units of $l \cdot a$ of the block spin ξ_l is smaller than the one measured in units of lattice spacing a between the original spins which we denote ξ_1. Recall indeed that the $\mathcal{H}_l$ does not know about the size of the previous blocks and we have [139]:

$$\xi = \xi_l \cdot (l \cdot a) = \xi_1 \cdot (a)$$

and so

$$\xi_l = \frac{\xi_1}{l}$$

which tells us that implementing a block spin transformation we are going away from criticality (which occurs when $\xi = \infty$).

One is typically interested in understanding the scaling behaviour near the critical region and makes the further assumption that one can express quantities via a power law, for instance the reduced temperature is taken $t_l = t \cdot l^{y_t}$. If one works out the consequences of such assumptions one can show that the critical exponents $(\alpha, \Delta, \nu, \cdots)$ are connected to y_t and other similar exponents. Moreover one discovers some non-trivial relations relating the various exponents which turn out to explain the scaling relations found by Widom [139].

All this is very good but unfortunately is somehow not practical since we do not have any recipe to actually compute y_t or the scaling functions (such as the free energy density). Moreover

the assumption that after a coarse graining transformation the interaction between block spins is still between nearest neighbor is too strong. Indeed also other interactions are generated. Therefore we turn now to discuss the Wilsonian RG which successfully takes into account this fact and gives a framework thanks to which it is possible to compute the quantities of interest. Indeed Wilsonian RG allows us to write down differential equation for the transformations of the parameters turning the machinery just described into a much more convenient setup.

A.1.2 Wilsonian Renormalization Group

To overcome the difficulties encountered in the Kadanoff's approach we implement two steps: first we perform a block spin transformation (coarse graining) via which the block spins are separated by a distance $l \cdot a$. After this we rescale lengths in such a way that the block spins are now separated by a distance a, as the original spins were. Now the systems looks like before but with a different Hamiltonian. Repeating these steps yields a system of Hamiltonians which describe statistical mechanics further and further away from criticality (i.e.: the correlation function diminishes by the argument of the previous section). Basically we start from $\mathcal{H}$ and arrive to $\mathcal{H}_l$ via a coarse graining transformation, a further coarse graining transformation would lead to $\mathcal{H}_{l^2}$. This is conveniently achieved by rescaling all the lengths in $\mathcal{H}_l$ in such a way that $\mathcal{H}_l$ "looks like" $\mathcal{H}$ (with different couplings) so that the procedure can be repeated at will. In introducing the rescaled variables one eliminates any reference to the new lattice spacing and the Hamiltonians of the sequence differ only because the coupling differ.

As we have said an RG transformation changes the interactions terms and creates new ones (for instance in the Ising model new interactions besides the nearest neighbor are created). In general what is preserved under the RG transformation are some general characteristics of the system like dimensionality, symmetries and field content. These features define the arena were the RG transformation takes place: the theory space. The theory space can be parametrized via a set of couplings $g = \{g_i\}$ which is generally infinite. Let us denote the renormalization group transformation, acting on the couplings $\{g_i\}$, R_l. We observe that

$$\begin{aligned} g' &= R_{l_1}[g] \\ g'' &= R_{l_2}\left[g'\right] = R_{l_2} \cdot R_{l_1}[g] \end{aligned}$$

and thus

$$g'' = R_{l_1 l_2}[g], \quad \Rightarrow R_{l_1 l_2}[g] = R_{l_2} R_{l_1}[g]$$

so that R_l form a semigroup.

Of course there is some freedom in choosing a specific coarse-graining procedure but the long distance physics remains untouched. Let us make our statements more formal. The effective

Hamiltonian is

$$e^{\mathcal{H}'_{N'}[S,g']} = \mathrm{Tr}'_{\{s\}} e^{\mathcal{H}[s,g]} = \mathrm{Tr}_{\{s\}} P(s,S)\, e^{\mathcal{H}[s,g]}$$

where P is a projector which allows to do an unrestricted trace over the degrees of freedom and is constructed so that S has the same range of values as s. This projector must satisfy

- $P(s,S) \geq 0$ so that $e^{\mathcal{H}'_{N'}[S,g']} \geq 0$.
- $P(s,S) \geq 0$ respects the symmetries of the system. Otherwise one may end up in a different universality class. The impossibility of achieving this leads to anomalies.
- $\sum_s P(s,S) = 1$. This implies the invariance of the partition function:

$$\begin{aligned} Z_{N'}\left[g'\right] &= \mathrm{Tr}_{\{S\}} e^{\mathcal{H}'_{N'}[S,g']} = \mathrm{Tr}_{\{S\}} \mathrm{Tr}_{\{s\}} P(s,S)\, e^{\mathcal{H}[s,g]} \\ &= \mathrm{Tr}_{\{s\}} e^{\mathcal{H}[s,g]} = Z_N[g]\,. \end{aligned}$$

  For the free energy we have

$$\begin{aligned} \frac{1}{N}\log Z_N[g] &= \frac{1}{N}\log Z_{N'}\left[g'\right] = \frac{l^d}{N\cdot l^d}\log Z_{N'}\left[g'\right] \\ F[g] &= l^{-d} F\left[g'\right]. \end{aligned}$$

The above equation does not only imply that the partition function is left invariant by the RG transformation but also that the probability distributions of quantities that depend on $s', s'', \cdots$ are left invariant:

$$\begin{aligned} \mathrm{Tr}_{\{S\}} e^{\mathcal{H}'_{N'}[S,g']} f(S) &= \mathrm{Tr}_{\{S\}} \mathrm{Tr}_{\{s\}} P(s,S)\, e^{\mathcal{H}[s,g]} f(S = B(s)) \\ &= \mathrm{Tr}_{\{s\}} e^{\mathcal{H}[s,g]} f(S = B(s))\,. \end{aligned}$$

In particular this means that all the long wavelength degrees of freedom can be considered via the transformed Hamiltonian $\mathcal{H}'$. The iteration of this procedure yields a collection of Hamiltonians which are parametrized by the couplings g[1].

We define a fixed point (FP) as a point in the coupling space such that $g^* = R_l(g^*)$. Now recall that an RG step drives us away from criticality since:

$$\xi\left(g'\right) = \frac{\xi(g)}{l}.$$

[1] In the case of the Ising model the rescaling step is not manifest. This is due to the fact that the coarse grained Hamiltonian does not know about the length of the block $a, l\cdot a, l^2\cdot a, \cdots$. Therefore we are immediately able to read off the recursion relations. This has a further implication: suppose we have two couplings K_l and h_l, since the Hamiltonian does not know about $l, l^2, \cdots$ the coarse grained couplings depend only on those of the previous step. Making l continuous we have that $\frac{dK_l}{dl} = \frac{1}{l} f(K_l, h_l)$ which tells us that the beta functions do not depend on l separately.

Therefore for a FP the following relation holds

$$\xi\left(g^*\right) = \frac{\xi\left(g^*\right)}{l}.$$

This relation is true only if $\xi = 0$ or $\xi = \infty$. Critical behaviour is identified by the fact that the correlation length diverges and the respective FPs is called critical fixed points. Given the importance of the FPs let us study the behaviour of the flow near them. In order to do so we linearize the RG transformation near the FP:

$$\begin{aligned} g_n'\left(g^* + \delta g\right) &= g_n^* + \left.\frac{\partial g_n'}{\partial g_m}\right|_{g=g^*} \delta g_m + \cdots \\ &= g_n^* + M_{nm}\delta g_m + \cdots . \end{aligned}$$

Now consider the (right) eigenvectors e_a of the matrix M_{mn} (called stability matrix)

$$M_{nm}e_m = \lambda_n e_n.$$

The above consideration holds for a generic RG transformation R_l which depends on the length parameter l which we make explicit below. For consistency we have:

$$\begin{aligned} M_{ab}^{(l_1)}M_{bc}^{(l_2)} &= M_{ac}^{(l_1 l_2)} \\ \lambda_n^{(l_1)}\lambda_n^{(l_2)} &= \lambda_n^{(l_1 l_2)}. \end{aligned}$$

The solution to the above equation has the form [139]

$$\lambda_n^{(l)} = l^{y_n}.$$

Expanding the linearized RG transformation in the basis of eigenvector of the stability matrix we have the following behaviour near the FP (we use boldface symbols for vector and matrices):

$$\begin{aligned} \mathbf{g}'\left(g^* + \delta g\right) &\approx \mathbf{M}\cdot\mathbf{g} = \sum_a c^{(a)}\lambda_a \mathbf{e}^{(a)} \\ &= \sum_a c^{(a)} l^{y_a}\mathbf{e}^{(a)} \end{aligned}$$

We are now able to distinguish which components of δg grow and which shrink under an infinitesimal RG transformation. We can distinguish three cases:

- $y_a > 0$: the RG along the direction $\mathbf{e}^{(a)}$ flows away from the FP. This combination is said to be relevant.
- $y_a < 0$: the RG along the direction $\mathbf{e}^{(a)}$ flows to zero. This combination is said to be irrelevant.

- $y_a = 0$: the linearized analysis is insufficient to determine whether the couplings are driven away or towards the FP. In these cases one refers to marginally relevant or irrelevant couplings.

Suppose that we have n relevant couplings and $n' - n$ irrelevant ones. If we think of the space of the couplings as a manifold we refer to the $(n' - n)$-dimensional space of irrelevant couplings as critical surface or stable manifold. In the vicinity of the FP this space is spanned by the eigenvectors associated to irrelevant couplings. The long distance property of the system are controlled by the FP so all this surface is characterized from the fact of having infinite correlation length. The set of points reached from a trajectory emanating from the FP form the unstable manifold and any such trajectory is called renormalized trajectory. Individual flow lines starting outside the stable manifold will typically point towards the FP and then will be repelled from it (effect of the irrelevant operator). Via an ideal fine tuning we can start the flow inside the stable manifold so that the couplings can approach the FP. In this case the irrelevant couplings will move inside the stable manifold going into the FP while the relevant ones will emerge from the FP. This latter possibility is indeed an example of renormalized trajectory. To such trajectories one associates the so called "perfect actions", in the sense that the effect of the cutoff on observables is completely erased, even when the couplings are not close to their fixed point values [48]. The renormalized trajectory given by backtracing perfect actions gives a notion of non–perturbative renormalizability. Indeed we did not mention any small parameters. Note that in order to have a predictive continuum limit we need a finite number of relevant directions in such a way that only a finite number of experiments is required to fix the renormalization conditions of the relevant couplings. The irrelevant couplings become computable functions of the relevant ones [48].

So far what we have said regards a particular coarse graining procedure R_l. One may consider a different coarse graining procedure $\tilde{R}_l$, this will change the location of the fixed-points. In statistical mechanics the eigenvalues of the relevant couplings can be shown to be related to the critical exponent of the system [139]. The critical exponent are physical quantities and therefore cannot depend on the coarse graining procedure. Indeed when one considers a sufficient approximation of the theory space it can be observed that the values of the critical exponent is stable both changing R_l and the truncation. This is a first possible test of the reliability of computations in other contexts, for instance quantum gravity. The rate of approach to the FP, i.e.: the eigenvalues of the linearized RG flow, is a physical quantity which is independent of the choice of R_l. In this picture the irrelevant couplings determine some corrections to the scaling behaviour which typically can be neglected [140]. When one considers the corrections of the scaling due to irrelevant couplings one typically expands the scaling function in these couplings. If the limit in which the couplings are set to zero is not well defined we talk about dangerous irrelevant couplings and more information is needed to reconstruct the form of the

scaling function [140].

So far the discussion has been based on the example of a spin system on a lattice. Nevertheless the same logic applies to any theory as well since the RG is a conceptual framework for theories of any kind. Despite being intuitive, the analytic computations of RG flows in cases like the Ising model become soon very cumbersome. In the next section we will present a functional approach which implements the Wilsonian RG for QFTs.

Finally let us note that the notion of perturbative renormalizability is neither sufficient (e.g.: φ^4 theory in $4d$) nor necessary (e.g.: Gross-Neveu model in $2d$ [141] and $3d$ [142]). Being non perturbative, the Wilsonian RG is a suitable framework to investigate if a theory can be a fundamental theory.

A.2 Scale dependence of the Effective Average Action II

In this section we provide a further derivation of the flow equations. As before we first consider:

$$e^{W_k[J]} = \int \mathcal{D}\chi e^{-S[\chi]-\Delta S_k[\chi]+J\cdot\chi}$$

and take the k-derivative:

$$\begin{aligned}
\partial_k e^{W_k[J]} &= -\frac{1}{2}\frac{\delta}{\delta J_a}\partial_k R_{k,ab}\frac{\delta}{\delta J_b}\int \mathcal{D}\chi e^{-S[\chi]-\Delta S_k[\chi]+J\cdot\chi} = -\frac{1}{2}\frac{\delta}{\delta J_a}\partial_k R_{k,ab}\frac{\delta}{\delta J_b}e^{W_k[J]} \\
e^{W_k[J]}\partial_k W_k[J] &= -\frac{1}{2}\frac{\delta}{\delta J_a}\partial_k R_{k,ab}\left[e^{W_k[J]}\frac{\delta W_k[J]}{\delta J_b}\right] \\
&= -\frac{1}{2}\partial_k R_{k,ab}\left[e^{W_k[J]}\frac{\delta W_k[J]}{\delta J_a}\frac{\delta W_k[J]}{\delta J_b} + e^{W_k[J]}\frac{\delta^2 W_k[J]}{\delta J_a \delta J_b}\right] \\
\partial_k W_k[J] &= -\frac{1}{2}\partial_k R_{k,ab}\left[\frac{\delta W_k[J]}{\delta J_a}\frac{\delta W_k[J]}{\delta J_b} + \frac{\delta^2 W_k[J]}{\delta J_a \delta J_b}\right].
\end{aligned}$$

At this point it is worth to notice that the above equation is in form analog to the Polchinski's equation (see section A.3). The difference is in the fact that the above equation involve W_k which is a functional of the current, rather than the field.

To make conctact with the flow equation for the EAA we recall

$$\tilde{\Gamma}_k[\varphi] = J\cdot\varphi - W_k[J]$$

and given that J is a function of the mean field we have

$$\begin{aligned}
\frac{\delta\tilde{\Gamma}_k[\varphi]}{\delta\varphi} &= J + \frac{\delta J}{\delta\varphi}\cdot\varphi - \frac{\delta W}{\delta\varphi} = J + \frac{\delta J}{\delta\varphi}\cdot\varphi - \frac{\delta W}{\delta J}\cdot\frac{\delta J}{\delta\varphi} \\
&= J + \frac{\delta J}{\delta\varphi}\cdot\varphi - \varphi\cdot\frac{\delta J}{\delta\varphi} = J.
\end{aligned}$$

Note that by definition φ is k-depedent if we do not put any k-depedence in the source. Due to this fact the k-derivative acting on the EAA can be written

$$\frac{d}{dk} = \partial_k^{\varphi} + \partial_k \varphi \cdot \frac{\delta}{\delta \varphi}$$

where ∂_k^{φ} is the derivative at constant field. We observe:

$$\Gamma_k[\varphi] = \tilde{\Gamma}_k[\varphi] - \Delta S_k[\varphi] = J \cdot \varphi - W[J] - \Delta S_k[\varphi].$$

Now we consider the k derivative of the terms in the l.h.s. and r.h.s. of the above equation. The l.h.s. reads:

$$\begin{aligned}\frac{d}{dk}\Gamma_k[\varphi] &= \partial_k^{\varphi}\Gamma_k[\varphi] + \frac{\delta \Gamma_k[\varphi]}{\delta \varphi}\partial_k \varphi = \\ &= \partial_k^{\varphi}\Gamma_k[\varphi] + \frac{\delta \tilde{\Gamma}_k[\varphi]}{\delta \varphi}\partial_k \varphi - \varphi R_k \partial_k \varphi\end{aligned}$$

while the r.h.s. is

$$\begin{aligned}\text{r.h.s.} &= J\partial_k\varphi + \frac{1}{2}\partial_k R_{k,ab}\left[\frac{\delta W_k[J]}{\delta J_a}\frac{\delta W_k[J]}{\delta J_b} + \frac{\delta^2 W_k[J]}{\delta J_a \delta J_b}\right] - \frac{1}{2}\varphi\left(\partial_k R_k\right)\varphi - \varphi R_k \partial_k \varphi \\ &= \frac{\delta \tilde{\Gamma}_k[\varphi]}{\delta \varphi}\partial_k \varphi + \frac{1}{2}\partial_k R_{k,ab}\left[\frac{\delta^2 W_k[J]}{\delta J_a \delta J_b}\right] - \varphi R_k \partial_k \varphi\end{aligned}$$

Matching the l.h.s. and r.h.s. we have

$$\partial_k^{\varphi}\Gamma_k[\varphi] = \frac{1}{2}\partial_k R_{k,ab}\left[\frac{\delta^2 W_k[J]}{\delta J_a \delta J_b}\right].$$

This is exactly the expression we already found. Note that we can write $\partial_k^{\varphi} = \partial_k$ since from now on all the derivatives will be performed at fixed φ.

A.3 Other exact renormalization group equations

In this section we review two others equations which have been used in the literature: the Wegner-Houghton equation [143] and the Polchinski equation [144].

A.3.1 The Wegner-Houghton equation

The Wegner-Houghton equation describes the flow of the Wilsonian effective action and has been derived at the same time of the Wilson's equation [143]; we will follow the derivation presented in [145]. Let Λ be the UV cutoff of the theory and S_k be the Wilsonian effective action whose boundary condition in the UV is $S_\Lambda = S_0$ where S_0 is the bare action defined at this scale. We decompose the field $\varphi(p)$ in the following way:

$$\varphi(p) = \begin{cases} \varphi_<(p) = \varphi(p), & 0 \leq |p| \leq k - \Delta k \\ \varphi_s(p) = \varphi(p), & k - \Delta k \quad |p| < k \end{cases} \tag{A.1}$$

For definiteness, let us consider an action which can be split into a free part S^{kin} and an interacting one S^{int} between shell modes φ_s and the lower ones $\varphi_<$:

$$\begin{aligned} Z &= \int \mathcal{D}\varphi e^{-S_k} = \int \mathcal{D}\varphi_{\leq}\mathcal{D}\varphi_s e^{-S_k(\varphi_{\leq},\varphi_s)} \\ &= \int \mathcal{D}\varphi_{\leq} e^{-S_k^{kin}(\varphi_{\leq})} \int \mathcal{D}\varphi_s e^{-S_k^{kin}(\varphi_s)} e^{-S_k^{int}(\varphi_{\leq},\varphi_s)} \\ &= \int \mathcal{D}\varphi_{\leq} e^{-S_k^{kin}(\varphi_{\leq})} \langle e^{-S_k^{int}(\varphi_{\leq},\varphi_s)} \rangle_{\varphi_s}, \qquad \langle \cdots \rangle_{\varphi_s} \equiv \int \mathcal{D}\varphi_s e^{-S_k^{kin}(\varphi_s)} (\cdots) \\ &= \int \mathcal{D}\varphi_{\leq} e^{-S_k^{kin}(\varphi_{\leq}) - \Delta S_k(\varphi_{\leq})}, \qquad \Delta S_k \equiv= -\log \langle e^{-S_k^{int}(\varphi_{\leq},\varphi_s)} \rangle_{\varphi_s} \\ &\equiv \int \mathcal{D} e^{-S_{k-\Delta k}(\varphi_s)} \end{aligned} \tag{A.2}$$

This procedure must be done step by step, namely one goes from an action S_k to $S_{k'}$ and so on. Then we consider the following limit:

$$\lim_{\Delta k \to 0} \frac{S_k - S_{k-\Delta k}}{\Delta k} = \lim_{\Delta k \to 0} \frac{1}{\Delta k} \log \langle e^{-S_k^{int}(\varphi_{\leq},\varphi_s)} \rangle_{\varphi_s}. \tag{A.3}$$

Since we are interested in terms of order Δk we can use an approximate Gaussian integration which leads to an *exact* result for our purposes. Moreover we recall:

$$\int dx e^{-\frac{S_2}{2}x^2 - S_1 x - S_0} \sim \frac{e^{\frac{S_1^2}{2S_2} - S_0}}{\sqrt{S_2}}. \tag{A.4}$$

Inserting the result of the integration in the numerator of our expression and considering only a running effective potential we have:

$$k\frac{\partial S_k}{\partial k} = \frac{k}{2 \cdot \delta k} \int_{shell} \left[-\log \frac{\delta^2 S_k}{\delta\varphi_p \delta\varphi_{-p}} + \frac{\delta S_k}{\delta\varphi_p} \left(\frac{\delta^2 S_k}{\delta\varphi_p \delta\varphi_q} \right)^{-1} \frac{\delta S_k}{\delta\varphi_q} \right]. \tag{A.5}$$

A.3.2 The Polchinski equation

The Polchinski equation is widely used in addressing many issues in QFTs, for review see [7, 8]. We will follow the derivation presented in [2] where both conceptual and practical aspects are discussed. Let Δ be the propagator and S_{Λ_0} the bare interaction. The action we start with has a cutoff at Λ_0 which is introduced directly at the level of the propagator:

$$\Delta = \Delta_> + \Delta_<$$

where

$$\begin{aligned} \Delta_> &= [\theta_\varepsilon(p,\Lambda) - \theta_\varepsilon(p,\Lambda_0)]\,\Delta \\ \Delta_< &= [1 - \theta_\varepsilon(p,\Lambda)]\,\Delta. \end{aligned}$$

Here $\theta_\varepsilon(p,\Lambda)$ is a smooth cutoff function such that $\theta_\varepsilon(p,\Lambda)\approx 0$ for $p<\Lambda-\varepsilon$ and $\theta_\varepsilon(p,\Lambda)\approx 1$ for $p<\Lambda+\varepsilon$. As a consequence for $p>\Lambda_0$ the propagator is zero implementing the UV cutoff. Let us denote $\varphi=\varphi_<+\varphi_>$ in which we separate the fields in slow and fast modes and note that

$$\begin{aligned}\varphi\Delta\varphi &= (\varphi_>+\varphi_<)[\Delta_>+\Delta_<](\varphi_>+\varphi_<)\\ &= \varphi_>\Delta_>\varphi_>+\varphi_<\Delta_<\varphi_<.\end{aligned}$$

The action can thus be rewritten

$$\begin{aligned}Z[J] &\equiv \int\mathcal{D}\varphi\exp\left\{-\int\frac{1}{2}\varphi\Delta^{-1}\varphi-S_{\Lambda_0}+J\cdot\varphi\right\}\\ &= \int\mathcal{D}\varphi\exp\left\{-\int\frac{1}{2}\varphi_>\Delta_>^{-1}\varphi_>+\frac{1}{2}\varphi_<\Delta_<^{-1}\varphi_<-S_{\Lambda_0}+J\cdot\varphi\right\}.\end{aligned}$$

To make progress let us observe

$$\begin{aligned}Z[J] &\equiv \int\mathcal{D}\varphi\exp\left\{-\int\frac{1}{2}\varphi_>\Delta_>^{-1}\varphi_>+\frac{1}{2}\varphi_<\Delta_<^{-1}\varphi_<-S_{\Lambda_0}(\varphi)+J\cdot\varphi\right\}\\ &= \int\mathcal{D}\varphi_<\exp\left\{-\int\frac{1}{2}\varphi_<\Delta_<^{-1}\varphi_<\right\}\int\mathcal{D}\varphi_>\exp\left\{-\int\frac{1}{2}\varphi_>\Delta_>^{-1}\varphi_>-S_{\Lambda_0}(\varphi_>+\varphi_<)+J\cdot(\varphi_>+\varphi_<)\right\}\\ &= \cdots\int\mathcal{D}\tilde{\varphi}\exp\left\{-\int\frac{1}{2}\varphi_<\Delta_>^{-1}\varphi_<\right\}\times\\ &\quad\times\exp\left\{-\int\frac{1}{2}\tilde{\varphi}\Delta_>^{-1}\tilde{\varphi}-S_{\Lambda_0}(\tilde{\varphi})+J\cdot(\tilde{\varphi})+\tilde{\varphi}\cdot\Delta_>^{-1}\cdot\varphi_<\right\},\ \varphi_>\equiv\tilde{\varphi}-\varphi_<\\ &= \int\mathcal{D}\varphi_<\exp\left\{-\int\frac{1}{2}\varphi_<\Delta_<^{-1}\varphi_<\right\}\exp\left[\frac{1}{2}J\Delta_>J+J\varphi_<\right]\exp\left[-S_\Lambda(J\Delta_>+\varphi_<)\right].\end{aligned}$$

We denote:

$$\begin{aligned}Z_\Lambda[\varphi_<,J] &\equiv \int\mathcal{D}\varphi_>\exp\left\{-\int\frac{1}{2}\varphi_>\Delta_>^{-1}\varphi_>-S_{\Lambda_0}(\varphi_>+\varphi_<)+J\cdot(\varphi_>+\varphi_<)\right\}\\ &= \exp\left[\frac{1}{2}J\Delta_>J+J\varphi_<\right]\exp\left[-S_\Lambda(J\Delta_>+\varphi_<)\right]\\ Z[J] &= \int\mathcal{D}\varphi_<\exp\left\{-\int\frac{1}{2}\varphi_<\Delta_<^{-1}\varphi_<\right\}Z_\Lambda[\varphi_<,J].\end{aligned}$$

Note that if we suppose that J has support only for momenta $p<\Lambda$ as originally did by Polchinski we find

$$Z[J] = \int\mathcal{D}\varphi_<\exp\left\{-\int\frac{1}{2}\varphi_<\Delta_<^{-1}\varphi_<\right\}\exp[J\varphi_<]\exp\left[-S_\Lambda(\varphi_<)\right]$$

which is the expression for the Wilsonian effective action. We can also establish a relation between the generating functionals of connected Green's functions for the bare and Wilsonian

action [7]:

$$\begin{aligned}
e^{W[J]} &= \int \mathcal{D}\varphi_< \exp\left\{-\int \frac{1}{2}\varphi_< \Delta_<^{-1}\varphi_<\right\} \exp\left[\frac{1}{2}J\Delta_> J + J\varphi_<\right] \exp\left[-S_\Lambda\left(J\Delta_> + \varphi_<\right)\right] \\
&= \int \mathcal{D}\varphi \exp\left[-\int \frac{1}{2}\varphi \Delta_<^{-1}\varphi - S_\Lambda(\varphi)\right] \exp\left[J\varphi\left(1+\Delta_<^{-1}\Delta_>\right)\right] \times \\
&\quad \times \exp\left\{-\int\left[\frac{1}{2}\frac{\Delta_>}{\Delta_<}\left(\Delta_> + \Delta_<\right)\right] J^2\right\}, \quad \varphi = \varphi_< + \Delta_> J \\
&= e^{W\left[\left(1+\Delta_<^{-1}\Delta_>\right)J\right] - \int\left[\frac{1}{2}\frac{\Delta_>}{\Delta_<}\left(\Delta_>+\Delta_<\right)\right]J^2}.
\end{aligned}$$

This relation implies that the correlation functions of the bare theory can also be computed from the Wilson action. To find the flow equation for the interacting part of the Wilsonian action we consider:

$$\begin{aligned}
e^{-S_\Lambda^{(int)}} &= \int \mathcal{D}\varphi_> \exp\left\{-\int \frac{1}{2}\varphi_> \Delta_>^{-1}\varphi_> - S_{\Lambda_0}\left(\varphi_> + \varphi_<\right)\right\} \\
&= \int \mathcal{D}\varphi \exp\left\{-\int \frac{1}{2}\left(\varphi - \varphi_<\right)\Delta_>^{-1}\left(\varphi - \varphi_<\right) - S_{\Lambda_0}(\varphi)\right\} \\
-\Lambda\frac{d}{d\Lambda}e^{-S_\Lambda^{(int)}} &= \int \mathcal{D}\varphi \left[\int \frac{1}{2}\left(\varphi-\varphi_<\right)\left(-\frac{1}{\left[\theta_\varepsilon(p,\Lambda) - \theta_\varepsilon(p,\Lambda_0)\right]^2 \Delta(p)}\Lambda\frac{d}{d\Lambda}\theta_\varepsilon(p,\Lambda)\right)\left(\varphi-\varphi_<\right)\right] \times \\
&\quad \times \exp\left\{-\int \frac{1}{2}\left(\varphi-\varphi_<\right)\Delta_>^{-1}\left(\varphi-\varphi_<\right) - S_{\Lambda_0}(\varphi)\right\}.
\end{aligned}$$

Furthermore let us consider

$$\begin{aligned}
\frac{\delta}{\delta\varphi_<}e^{-S_\Lambda^{(int)}} &= \int \mathcal{D}\varphi \left[\int \Delta_>^{-1}\left(\varphi-\varphi_<\right)\right] \exp\left\{-\int \frac{1}{2}\left(\varphi-\varphi_<\right)\Delta_>^{-1}\left(\varphi-\varphi_<\right) - S_{\Lambda_0}(\varphi)\right\} \\
\frac{\delta^2}{\delta\varphi_<\delta\varphi_<}e^{-S_\Lambda^{(int)}} &= \int \mathcal{D}\varphi \left\{\left[\int \Delta_>^{-1}\left(\varphi-\varphi_<\right)\right]\left[\int \Delta_>^{-1}\left(\varphi-\varphi_<\right)\right]\right\} \times \\
&\quad \times \exp\left\{-\int \frac{1}{2}\left(\varphi-\varphi_<\right)\Delta_>^{-1}\left(\varphi-\varphi_<\right) - S_{\Lambda_0}(\varphi)\right\}.
\end{aligned}$$

where in the last line we dropped a term in the brackets since it is field independent and, as can be seen below, it enters as a field independent term also in the flow equation:

$$\begin{aligned}
-\Lambda\frac{d}{d\Lambda}e^{-S_\Lambda^{(int)}} &= \int \mathcal{D}\varphi \left[\int \frac{1}{2}\left(\varphi-\varphi_<\right)\left(-\frac{1}{\Delta_>^2}\Lambda\frac{d}{d\Lambda}\theta_\varepsilon(p,\Lambda)\Delta(p)\right)\left(\varphi-\varphi_<\right)\right] \times \\
&\quad \times \exp\left\{-\int \frac{1}{2}\left(\varphi-\varphi_<\right)\Delta_>^{-1}\left(\varphi-\varphi_<\right) - S_{\Lambda_0}(\varphi)\right\} \\
&= \int \mathcal{D}\varphi \left(-\Lambda\frac{d}{d\Lambda}\theta_\varepsilon(p,\Lambda)\Delta(p)\right)\frac{\delta^2}{\delta\varphi_<\delta\varphi_<}e^{-S_\Lambda^{(int)}} \\
-\Lambda\frac{d}{d\Lambda}S_\Lambda^{(int)} &= \int \mathcal{D}\varphi \left(-\Lambda\frac{d}{d\Lambda}\theta_\varepsilon(p,\Lambda)\Delta(p)\right)\left\{-\frac{\delta^2 S_\Lambda^{(int)}}{\delta\varphi_<\delta\varphi_<} + \frac{\delta S_\Lambda^{(int)}}{\delta\varphi_<}\frac{\delta S_\Lambda^{(int)}}{\delta\varphi_<}\right\} \\
&= \mathrm{Tr}\left[\frac{1}{p^2+m^2}\left(-\Lambda\frac{d}{d\Lambda}\theta_\varepsilon(p,\Lambda)\right)\left(\frac{\delta S_\Lambda^{(int)}}{\delta\varphi_<}\frac{\delta S_\Lambda^{(int)}}{\delta\varphi_<} - \frac{\delta^2 S_\Lambda^{(int)}}{\delta\varphi_<\delta\varphi_<}\right)\right].
\end{aligned}$$

The flow equation for the full Wilsonian action reads [6]:

$$-k\frac{\partial S_k}{\partial k} = \int_q \frac{\Delta(q/k)}{q^2+m^2}\left[\frac{q^2+m^2}{K(q/k)}\varphi(q)\frac{\delta S_k}{\delta\varphi(q)} + \frac{1}{2}\left(\frac{\delta S_k}{\delta\varphi(q)}\frac{\delta S_k}{\delta\varphi(-q)} + \frac{\delta^2 S_k}{\delta\varphi(q)\delta\varphi(-q)}\right)\right]$$

where

$$K(p) = \begin{cases} 1 & p^2 \leq 1 \\ 0 & p^2 > 1 \end{cases}, \qquad \Delta(p) \equiv -2p^2\frac{dK(p^2)}{dp^2}.$$

APPENDIX B

Brief review of background field method

In order to use the EAA and its exact equation we would like to use a method which allows to quantize theories with local symmetries preserving a form of gauge invariance. Such a method is the background field method and here we provide a brief reminder of this formalism taken from [146,147]. Typically in gauge theory one has a gauge invariant Lagrangian to start with and in order to quantize the system a gauge must be chosen thus breaking gauge invariance explicitly. As a consequence in the conventional formulation the Green's functions obey complicated Taylor identities. In the background field approch everything is arranged in such a way that even when gauge-fixing and ghost terms are considered there is explicit gauge invariance. As a result Green's functions obey the naive Ward identities due to gauge invariance and counterterms appear in a gauge invariant form.

Following the notation in [146, 147] consider:

$$Z[J] = \int \mathcal{D}Q \exp i\{S[Q] + JQ\}.$$

Now let us consider the following modified functional:

$$\tilde{Z}[J,\varphi] = \int \mathcal{D}Q \exp i\{S[Q+\varphi] + JQ\},$$

where we introduced the background field φ. In full analogy with the usual functional of quantum field theory we define the functionals $\tilde{W}$ and $\tilde{\Gamma}$:

$$\tilde{W}[J,\varphi] = -i \log \tilde{Z}[J,\varphi], \qquad \tilde{\Gamma}[\tilde{Q},\varphi] = \tilde{W}[J,\varphi] - J\tilde{Q}$$

where $\tilde{Q} = \delta_J W$. In order to relate the usual functionals with the background ones let us shift the integration variable $Q \to Q - \varphi$, we have:

$$\tilde{Z}[J,\varphi] = Z[J]e^{-iJ\varphi}, \qquad \tilde{W}[J,\varphi] = W[J] - J\varphi.$$

These relations impliy that $\tilde{Q} = \bar{Q} - \varphi$ and

$$\tilde{\Gamma}[\tilde{Q},\varphi] = W[J] - J\varphi - J\bar{Q} + J\varphi = \Gamma[\bar{Q}] = \Gamma[\tilde{Q}+\varphi]. \tag{B.1}$$

As a special case of the above relation we have:

$$\tilde{\Gamma}[0,\varphi] = \Gamma[\varphi]. \tag{B.2}$$

Let us observe that $\tilde{\Gamma}[\tilde{Q},\varphi]$ generetes 1PI Green's functions in the presence of the background field φ. If we are interested in the EA we can just consider $\tilde{\Gamma}[0,\varphi]$ which has no dependence on $\tilde{Q}$ so there is no graph with external lines. This means that it is possible to compute the EA as the sum of all 1PI vacuum diagrams in the presence of the background φ. The calculation of $\tilde{\Gamma}[0,\varphi]$ has two main approaches. The first is to specify explicitly a background φ and keep it into account making no approximation. The second one is to treat φ perturbatively and use it also to generate external lines, for further details see [146, 147].

As we already said this method turns out to be particularly useful in the case of gauge theories. Let Q^a_μ be the fluctuation field and A^a_μ the background one. The construction of the background EA follows exactly the same steps as before. The crucial point is that there exist a choice of the gauge-fixing term $\tilde{G}^a$ for which the background EA $\tilde{\Gamma}[0,A]$ is a gauge invariant functional of A. This gauge choice is

$$\tilde{G}^a = \partial_\mu Q^a_\mu + g f^{abc} A^b_\mu Q^c_\mu. \tag{B.3}$$

With this choice the background field EA is invariant under:

$$\begin{aligned} \delta A^a_\mu &= -f^{abc}\omega^b A^c_\mu + \frac{1}{g}\partial_\mu \omega^a, & \text{(B.4)}\\ \delta J^a_\mu &= -f^{abc}\omega^b Q^c_\mu, & \text{(B.5)} \end{aligned}$$

To see this it is necessary to perform the following change of variable in the functional integral $Q^a_\mu \to Q^a_\mu - f^{abc}\omega^b Q^c_\mu$. The complete field transforms as

$$\delta(Q^a_\mu + A^a_\mu) = -f^{abc}\omega^b(Q^c_\mu + A^c_\mu) + \frac{1}{g}\partial_\mu\omega^a. \tag{B.6}$$

The functional $\tilde{Z}$ is thus invariant under the above transformations. $\tilde{Q}$ is the conjugate variable of J and finally $\tilde{\Gamma}[\tilde{Q},A]$ is invariant under

$$\begin{aligned} \delta A^a_\mu &= -f^{abc}\omega^b A^c_\mu + \frac{1}{g}\partial_\mu \omega^a, & \text{(B.7)}\\ \delta \tilde{Q}^a_\mu &= -f^{abc}\omega^b \tilde{Q}^c_\mu. & \text{(B.8)} \end{aligned}$$

Finally it is clear that $\tilde{\Gamma}[\tilde{Q}=0,A]$ is a gauge invariant functional of A. This is a great advantage since by background gauge invariance the RG flow is non-trivially constrained. For a dedicated treatment of the background field method in the case of gravity we address the interested reader to [48].

Finally let us list some interesting properties of the background EA:

- $\tilde{\Gamma}[\tilde{Q}, A]$ generates the 1PI Green's functions via functional derivatives of $\tilde{Q}$ where A is an fixed external background. Nevertheless the on-shell Green's functions (which are related to scattering observable amplitudes) are independent on the background chosen (while the off-shell ones do depend on it). Remarkably the same on-shell Green's function can be obtained by differentiating the functional $\Gamma[A] = \tilde{\Gamma}[0, A]$. In this context on-shell means that the background satisfies $\delta\Gamma/\delta A = 0$.

- In the case of non-gauge theories one can establish a "spilitting Ward identity" which greatly simplifies the RG flow. Indeed it is possible to prove the the wave function renormalization of the background and the fluctuation fields are the same (see appendix B in [48] and references therein). In the case of gauge theories such Ward identities are ruined since the "linear split symmetry" between the fluctuation and the background is ruined by the gauge-fixing term and from the cutoff action.

APPENDIX C

Trace technology

In this thesis we employ covariant methods for the computation of the effective action and the beta functions. This is often achieved by means of the heat kernel (HK) which we review in this section. We also review some integrals that often appears when using the HK.

C.0.3 Local heat kernel

In this section we present the local HK expansion and a simple recipe to compute the coefficients of this expansion.[1] The HK expansion is defined via the following differential equation [149]

$$\begin{cases} \left(\partial_s + \Delta_x\right) K^s_{x,y} & = 0 \\ K^{s=0}_{x,y} & = \varphi \end{cases}$$

where φ is the initial "temperature" distribution which at later times is $e^{-s\Delta}\varphi$. If a boundary on the manifold is present a further boundary condition is required: $\mathcal{B}K^s = 0$. In this thesis we do not need such heat kernels and we address the interested reader to [149]. The above solution has an asymptotic expansion for $s \to 0$:

$$\mathrm{Tr}\left[e^{-s\Delta}\right] \quad = \quad \frac{1}{(4\pi s)^{d/2}} \mathrm{tr} \int \sqrt{g} \left\{ b_0 + s b_2 + s^2 b_4 + \cdots \right\}$$

and the coefficients b_i are called heat kernel coefficients. Let us first review some properties of the HK. Suppose the the manifold $\mathcal{M}$ is the product of $\mathcal{M}_1 \times \mathcal{M}_2$ and that the operator Δ can be written as $\Delta_1 \times \mathbf{1}_2 + \mathbf{1}_1 \times \Delta_2$ then we have [149]:

$$b_n \quad = \quad \sum_{n=n_1+n_2} b_{n_1} b_{n_2}.$$

The following two formulae derive from the study of the conformal variation of the HK. More precisely we will consider local scale transformation where the metric transforms as $g_{\mu\nu} \to e^{2\varepsilon\sigma} g_{\mu\nu}$

[1] The presentation in this section is due to work in collaboration with Alessandro Codello [148].

Spin s	Eigenvalues $\lambda_l(d,s)$	Multiplicity $D_l(d,s)$
0	$\frac{l(l+d-1)}{d(d-1)}R;\ l=0,1\dots$	$\frac{(2l+d-1)(l+d-2)!}{l!(d-1)!}$
1	$\frac{l(l+d-1)-1}{d(d-1)}R;\ l=1,2\dots$	$\frac{l(l+d-1)(2l+d-1)(l+d-3)!}{(d-2)!(l+1)!}$
2	$\frac{l(l+d-1)-2}{d(d-1)}R;\ l=2,3\dots$	$\frac{(d+1)(d-2)(l+d)(l-1)(2l+d-1)(l+d-3)!}{2(d-1)!(l+1)!}$

Table C.1: Eigenvalues and their multiplicities of the Laplacian on the d-sphere.

but the Laplacian is supposed to transform covariantly, i.e.: $\Delta \to e^{2\varepsilon\sigma}\Delta$. Let d be the dimension of the manifold, one has [149, 150]:

$$\begin{aligned} \frac{d}{d\varepsilon} b_k\left(1, e^{-2\varepsilon f}\Delta\right)\Big|_{\varepsilon=0} &= (d-k)\, b_k\left(f, e^{-2\varepsilon f}\Delta\right) \\ \frac{d}{d\varepsilon} b_{d-2}\left(e^{-2\varepsilon f}F, e^{-2\varepsilon f}\Delta\right)\Big|_{\varepsilon=0} &= 0 \end{aligned} \tag{C.1}$$

where f is a function which multiply all the HK coefficients. The latter of the above formulas is often used in the literature as it states that the conformal variation of the HK coefficient b_{d-2} vanishes and this provides relations among the coefficient which can be used once some of these are known.

Nevertheless one first needs to do some actual computations. An easy way to achieve this is to consider some specific manifold on which we know explicitly the spectrum of the Laplacian. In particular we will work with the sphere S_d. We exploit the fact that the HK coefficient of the Laplacian on unconstrained fields do not depend on the dimension of the manifold (note that this is not the case for differentially constrained fields and other types of operators) [149]. Therefore we will compute the expansion of the HK in various dimensions and equate this with the HK coefficient in such a way to end up with an algebraic systems of equations. In the following table we report eigenvalues and multiplicities of the Laplacian of the sphere on differentially constrained fields of different spin [33]. Nevertheless we need the trace $\mathrm{Tr}\left[e^{-s\Delta}\right]$ on unconstrained fields. In order to do this we consider the relation between the two spectra [151]:

$$\mathrm{Tr}_{(1)}\left[e^{-s\left(-D^2-q\,R\right)}\right] = \mathrm{Tr}_{(1T)}\left[e^{-s\left(-D^2-q\,R\right)}\right] + \mathrm{Tr}_{(0)}\left[e^{-s\left(-D^2-\frac{dq+1}{d}R\right)}\right] - e^{s\,\frac{dq+1}{d}R}$$

and

$$\begin{aligned} \mathrm{Tr}_{(2S)}\left[e^{-s(-D^2-qR)}\right] &= \mathrm{Tr}_{(2ST^2)}\left[e^{-s(-D^2-qR)}\right] + \mathrm{Tr}_{(1T)}\left[e^{-s\left(-D^2-\left(\frac{d+1}{d(d-1)}+q\right)R\right)}\right] \\ &\quad + \mathrm{Tr}_{(0)}\left[e^{-s\left(-D^2-\left(\frac{2}{d-1}+q\right)R\right)}\right] + \mathrm{Tr}_{(0)}\left[e^{-s(-D^2-qR)}\right] - e^{-s\left(-\frac{2}{d-1}-q\right)R} \\ &\quad - (d+1)\, e^{-s\left(-\frac{1}{d-1}-q\right)R} - \frac{d(d+1)}{2}\, e^{-s\left(-\frac{2}{d(d-1)}-q\right)R}. \end{aligned}$$

On the sphere $R = \frac{d(d-1)}{r^2}$ is the Ricci scalar. The heat kernel trace can be writen as a sum:

$$\mathrm{Tr}\, e^{-s\Delta} = \sum_{n=0}^{\infty} m_n\, e^{-s\lambda_n}\,.$$

On symmetry grounds we know that:

$$\mathrm{Tr}\, e^{-s\Delta} = \frac{1}{(4\pi s)^{d/2}} \int d^d x \sqrt{g}\, [a + sbR + ...]\,, \tag{C.2}$$

to determine the coefficients a and b we use the knowledge of the spectrum and combine it with the Euler-Maclauren expansion. Since the heat kernel coefficients are independent of the dimension we can work in $d = 2$ where:

$$\mathrm{Tr}\, e^{-s\Delta} = \sum_{n=0}^{\infty} (2n+1) e^{-\frac{sR}{2} n(n+1)}\,. \tag{C.3}$$

The Euler-Maclauren expansion teaches that:

$$\begin{aligned} \sum_{i=m}^{n} f\,(i) &= \int_m^n dx\, f\,(x) + \frac{f\,(n) + f\,(m)}{2} + \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}\,(n) - f^{(2k-1)}\,(m) \right) + Rem, \\ Rem &= \int_m^n dx\, f^{(2p+1)}\,(x)\, \frac{P_{2p}\,(x)}{(2p+1)!} \end{aligned}$$

which can also be rewritten:

$$\sum_{n=0}^{\infty} f(n) = \int_0^{\infty} dx\, f(x) + \frac{1}{2} f(0) - \frac{1}{12} f'(0) + \frac{1}{720} f'''(0) + \cdots\,.$$

Note that the larger is p the higher will be the power of the Ricci scalar R coming from the remainder. Thus choosing p high enough the remainder can be disregarded. Now using $f(x) = (2x+1)\, e^{-\frac{sR}{2} x(x+1)}$ in this last expression we have:

$$\mathrm{Tr}\, e^{-s\Delta} = \frac{2}{sR} + \frac{1}{3} + O(s)$$

which can be compared with the r.h.s. of (C.2) evaluated on a two dimensional sphere:

$$\begin{aligned} \mathrm{Tr}\, e^{-s\Delta} &= \frac{1}{4\pi s} \frac{8\pi}{R} \left[a + sbR + O(s^2) \right] \\ &= \frac{2}{sR} a + 2b + O(s)\,, \end{aligned}$$

which implies $a = 1$ and $b = \frac{1}{6}$. From this we learn the form of the heat kernel expansion to linear order in the curvature:

$$\mathrm{Tr}\, e^{-s\Delta} = \frac{1}{(4\pi s)^{d/2}} \int d^d x \sqrt{g} \left[1 + s\frac{R}{6} + ... \right]\,. \tag{C.4}$$

The second order terms in the curvatures can be computed as well. We simply have to repeat the above calculation for several dimensions and build a system of equations. Nevertheless choosing

a background also brings its own difficulties. In particular in the case of the sphere we have that some terms in the HK expansion "collapse", i.e.: $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \sim R^2$. As we said at the beginning this problem is avoided considering several equation obtained using different dimensions. A much more severe problem comes from the fact that some terms, such as $\nabla^2 R$ are zero and are not present at all in our computation. It is possible to compute them employing the variational relations (C.1). The final result for the operator $\Delta = -\nabla^2 + \mathbf{E}$ is:

$$\begin{aligned} b_0 &= \mathbf{1} \\ b_2 &= \mathbf{1}\frac{R}{6} - \mathbf{E} \\ b_4 &= \mathbf{1}\frac{1}{180}\left(R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} - R^{\mu\nu}R_{\mu\nu} + \frac{5}{2}R^2 + 6\nabla^2 R\right) \\ &\quad + \frac{1}{12}\mathbf{\Omega}_{\mu\nu}\mathbf{\Omega}^{\mu\nu} - \frac{1}{6}R\mathbf{E} + \frac{1}{2}\mathbf{E}^2 - \frac{1}{6}\nabla^2\mathbf{E}. \end{aligned}$$

The reader may wonder if this technique works also for higher HK coefficients. The answer is that it works of course but, so far, it is difficult to solve for all the coefficients appearing for instance in b_6. We can easily get the HK coefficients on the sphere by simple means of the Euler-Maclaurin expansion but to disentangle all of them appears difficult. Also the use of the Gauss-Bonnet theorem does not provide any new results.[2] The Gauss-Bonnet theorem is related to the HK via the following formula [149]:

$$\frac{1}{(4\pi)^{d/2}}\sum_{q=0}^{d}(-1)^q B_n\left(\Delta_q\right) = \begin{cases} 0 & \text{if } n \neq d \\ \chi\left(\mathcal{M}\right) & \text{if } n = d \end{cases}$$

where Δ_q is the Laplacian acting on the q-forms and B_n represents the trace of the HK coefficient: $B_n = \mathrm{tr} b_n$. By means of the Poincare duality we can relate $B_d\left(\Delta_q\right) = B_d\left(\Delta_{d-q}\right)$. The Laplacian on a k-form is defined as

$$\begin{aligned} \Delta\omega &= \left(\delta d + d\delta\right)\omega_{\mu_1\cdots\mu_k} \\ &= -\left(k+1\right)\nabla^\lambda\nabla_{[\lambda}\omega_{\mu_1\cdots\mu_k]} - k\nabla_{[\mu_1}\nabla^\lambda\omega_{|\lambda|\mu_2\cdots\mu_k]}. \end{aligned}$$

This equation provides non-trivial constraints among the coefficients but they can already be found via the application of the Laplacian spectrum on the sphere. A possible way out would be to be able to run our recipe on some other manifolds and use the product formula for product manifolds as $S^2 \times S^2$.

For completeness we also mention the relation between HK coefficients and pole of the zeta functions [152]. If d is even, we find simple poles at $s = d/2 - k$ for $0 \leq k \leq d/2 - 1$, with

$$\mathrm{Res}_{s=\frac{d}{2}-k}\zeta(s,x) = \frac{b_k(x)}{(4\pi)^{d/2}\Gamma(d/2-k)} \qquad 0 \leq k \leq d/2 - 1\,.$$

[2] This has somehow to be expected since, being a topological relation, it cannot depend on the metric and indeed depends only on the endomorphism part of the Laplacian.

Moreover, the would-be poles at zero or negative integer values of s are canceled by the poles of the Γ function and we find

$$\zeta(-k,x) = (-1)^k k! \frac{b_{d/2+k}(x)}{(4\pi)^{d/2}} \, k \geq 0 \,, \qquad d \text{ even.}$$

If d is odd, there are simples poles at $s = d/2 - k$ for all $k \geq 0$. The pole structure of the Γ function also yields in this case

$$\zeta(-k,x) = 0 \,, \quad k \geq 0 \,, \qquad d \text{ odd.}$$

C.0.4 Non-local heat kernel

In this section we review the calculation method for the non-local heat kernel set up in [153]. The non-local HK is an expansion in curvatures of the solution of the heat equation. The difference with the local HK of the previous section is that now infinitely many terms are resummed into form factors, i.e.: functions of the Laplacian, [92]. Indeed it is possible to check via a suitable expansion that some terms appearing in the local HK expansion are included in the form factors [92].

We consider an operator of the form

$$\Delta = -D^2 + U$$

where the covariant derivative is built via the Levi-Civita connection plus a vector bundle connection while U is an endomorphism. We look for a kernel K^s_{xy} which satisfies

$$\begin{cases} (\partial_s + \Delta_x) \, K^s_{x,y} & = 0 \\ K^{s=0}_{x,y} & = \delta \, (x-y) \, . \end{cases}$$

Note that K^s is now a bi-tensor of weight 1 (it would be a scalar if we had considered $\delta(x-y)/\sqrt{g}$ on the r.h.s.). The above boundary condition implies that the the Laplacian which has to be considered is the one acting on tensors ψ of weight $w = 1/2$. In this way the scalar product $\int_x \psi_1(x)\psi_2(x)$ is invariant under coordinate transformations. It is important to note that when the Laplacian acts on tensor of different weight its representation changes as follows:

$$\Delta_{w=1/2}\psi = g^{1/4}\Delta_{w=0}g^{-1/4}\psi = g^{1/4}\Delta_{w=0}\varphi_{w=0}.$$

It is easy to check that $\Delta_{w=1/2}$ and $\Delta_{w=0}$ share the same spectrum.

The formal solution to the HK equation is

$$K^s_{xy} = e^{-s\Delta_x}\delta_{xy}.$$

We choose a basis of eigenfunction ψ_n of weight $w = 1/2$:

$$\Delta_x \psi_n = \lambda_n \psi_n, \qquad \int_x \psi_n(x)\psi_m(x) = \delta_{nm} \quad \sum_n \psi_n(x)\psi_n(y) = \delta_{xy}.$$

In this way we meet the initial condition since, using the spectral representation of K^s, we have

$$K^s_{xy} = \sum_n e^{-s\lambda_n}\varphi_n(x)\varphi_n(y), \quad K^0_{xy} = \sum_n \varphi_n(x)\varphi_n(y) = \delta_{xy}.$$

Finally the HK trace reads:

$$\mathrm{Tr}K^s_{xy} = \mathrm{tr}\int_x K^s_{xx} = \int_x \sum_n e^{-s\lambda_n}\varphi_n(x)\varphi_n(x) = \sum_n e^{-s\lambda_n}$$

where in the first integral there is no $\sqrt{g}$ factor due to the choice of weight of our basis.

To compute the HK expansion we will consider an expansion around flat spacetime solution which we can compute exactly. We adopt the following convetions and notations: $\delta(x-y) = \delta_{xy}$, $\int_x = \int d^d x$ and $\int_q = \int \frac{d^d q}{(2\pi)^d}$. We have

$$K^s_{0,xy} = \frac{1}{(4\pi s)^{d/2}} e^{-\frac{(x-y)^2}{4s}}$$

which satisfy

$$K^{s_1+s_2}_{0,xy} = \int_z K^{s_1}_{0,xz} K^{s_2}_{0,zy}.$$

Note that

$$\Delta = -\partial^2 + V$$

where V contains all possible terms coming from the covariant derivative and the endomorphism. It is convenient to define the operator $U^s_{xy} \equiv \int_z K^{-s}_{0,xz} K^s_{zy}$ which satisfies

$$\partial_s U^s = -\int_{zw} K^{-s}_{0,xz} V_z K^s_{0,zw} U^s_{wy}$$

and is solved using a Dyson's series. The solution can be formally written

$$U^s = T\exp\left[-\int_0^s dt \int_z K^{-t}_{0,xz} V_z K^t_{0,zy}\right]$$

where the exponential is time-ordered with respect to the parameter s. Finally

$$K^s_{xy} = \int_z K^s_{0,xz} T\exp\left\{-\int_0^s dt \int_w K^{-t}_{0,zw} V_w K^t_{0,wy}\right\}$$

and rescaling the integration variable $t \to t/s$ we obtain the final formula for the perturbative expansion of the un-traced HK

$$K^s_{xy} \;=\; K^s_{0,xy} - s\int_0^1 dt \int_z K^{s(1-t)}_{0,xz} V_z K^{st}_{0,zy} + s^2 \int_0^1 dt_1 \int_0^{t_1} dt_2 \int_{zw} K^{s(1-t_1)}_{0,xz} V_z K^{s(t_1-t_2)}_{0,zw} V_w K^{st_2}_{0,wy} + O(V^3).$$

To obtain the traced HK expansion one has simply to set $y = x$.

It is convenient to introduce the so called "Laplacian action" [153] $L[\varphi, \Phi]$ which is defined in order to satisfy the following property: its Hessian with respect to the auxiliary field φ gives the Laplacian $\Delta[\Phi]$ we are interested in (φ has density weight zero and value in the internal space). The vertices needed for the perturbative computation of the HK expansion can also be found starting from this action. We consider:

$$\begin{aligned} L[\varphi,\Phi] &= \int d^dx \frac{1}{2}\left(\varphi_{w=0} g^{1/4}\right) \Delta[\Phi] \left(g^{-1/4}\varphi_{w=0}\right) \\ &= \int d^dx \sqrt{g}\frac{1}{2}\left[g^{\mu\nu} D_\mu\left(\varphi_{w=0}g^{-1/4}\right) D_\nu\left(g^{-1/4}\varphi_{w=0}\right) + \left(\varphi_{w=0}g^{-1/4}\right)\mathbf{U}\left(g^{-1/4}\varphi_{w=0}\right)\right]. \end{aligned}$$

Now we can functionally expand the Laplacian

$$\begin{aligned} g^{1/4}\Delta g^{-1/4} &= g_x^{1/4}\left(-\partial^2 + V\right)_x g_y^{-1/4} = L^{(2,0)}[\varphi,\Phi] \\ &= L^{(2,0)}_{xy}[0,0] + L^{(2,1)}_{xyz}[0,0]\,\Phi_z + \frac{1}{2}L^{(2,2)}_{xyzw}[0,0]\,\Phi_z\Phi_w + \cdots \\ V_x\delta_{xy} &= \int_z L^{(2,1)}_{xyz}[0,0]\,\Phi_z + \frac{1}{2}\int_{zw} L^{(2,2)}_{xyzw}[0,0]\,\Phi_z\Phi_w + \cdots \end{aligned}$$

where typically the field Φ is the set $h_{\mu\nu}, A_\mu, U$.

The most general expansion of the HK trace up to second order in the curvatures is [92]:

$$\begin{aligned} \mathrm{Tr}K^s &= \frac{1}{(4\pi s)^{d/2}}\int d^dx\sqrt{g}\,\mathrm{tr}\Big\{\mathbf{1} - s\mathbf{U} + s\mathbf{1}\frac{R}{6} + s^2\Big[\mathbf{1}R_{\mu\nu}f_{Ric}(s\Box)R^{\mu\nu} + \mathbf{1}R\,f_R(s\Box)R \\ &\quad + R\,f_{RU}(s\Box)\mathbf{U} + \mathbf{U}f_U(s\Box)\mathbf{U} + \Omega_{\mu\nu}f_\Omega(s\Box)\Omega^{\mu\nu}\Big] + O(\mathcal{R}^3)\Big\}, \end{aligned} \tag{C.5}$$

where $\mathbf{U}$ represents the endomorphism or any of the curvatures and $\Box = -D^2$. The form factors f_i can be computed comparing functional derivatives of the above ansatz with functional derivatives of the expansion in V which we have seen before [153]. The results is

$$\begin{aligned} f_{Ric}(x) &= \frac{1}{6x} + \frac{1}{x^2}\left[f(x) - 1\right] \\ f_R(x) &= \frac{1}{32}f(x) + \frac{1}{8x}f(x) - \frac{7}{48x} - \frac{1}{8x^2}\left[f(x) - 1\right] \\ f_{RU}(x) &= -\frac{1}{4}f(x) - \frac{1}{2x}\left[f(x) - 1\right] \\ f_U(x) &= \frac{1}{2}f(x) \\ f_\Omega(x) &= -\frac{1}{2x}\left[f(x) - 1\right], \end{aligned} \tag{C.6}$$

and all depend on the basic heat kernel form factor $f(x)$ that is defined in terms of a parameter integral

$$f(x) = \int_0^1 d\xi\, e^{-x\xi(1-\xi)}. \tag{C.7}$$

The full computation can be found in [153]. A different approach has been used in [92,95] where the traced non-local HK expansion has been computed up to third order in the curvatures. The advantage of the approach we have seen is that it should be possible to easily extend it to more general operators beside the Laplacian.

C.0.5 Q-functionals

The ERGE is a functional equation which typically involves very complicated functions of an operator (for instance the Laplacian). The technical difficulty of solving this equation, even when a local ansatz is specified, is to compute the trace on the r.h.s. of the ERGE. It is often convenient to express this trace as an integral over a parameter s which involves the heat kernel (HK) trace. This is achieved introducing the inverse Laplace transform $\widetilde{W}$ of the functional at hand in the following way:

$$\mathrm{Tr}W\left(\Delta\right) \quad = \quad \int_0^\infty ds\widetilde{W}\left(s\right)\mathrm{Tr}e^{-s\Delta}$$

and

$$W\left(z\right) \quad = \quad \mathcal{L}^{-1}\left[W\right]\left(s\right).$$

It is clear that once expanded the HK will produce the following type of integrals:

$$Q_n\left[W\right] \quad = \quad \int_0^\infty dss^{-n}\widetilde{W}\left(s\right)$$

which we call Q-functionals. This type of integral appears in all the computations and in the following we will list many useful properties.

For $n>0$ one has

$$Q_n\left[W\right] \quad = \quad \frac{1}{\Gamma\left[n\right]}\int_0^\infty dzz^{n-1}W\left(z\right)$$

which can be verified using the integral representation of the Gamma function.

$$\begin{aligned}
Q_n\left[W\right] &= \int_0^\infty dss^{-n}\widetilde{W}\left(s\right) = \frac{1}{\Gamma\left[n\right]}\Gamma\left[n\right]\int_0^\infty dss^{-n}\widetilde{W}\left(s\right) \\
&= \frac{1}{\Gamma\left[n\right]}\left(\int_0^\infty dtt^{n-1}e^{-t}\right)\int_0^\infty dss^{-n}\widetilde{W}\left(s\right) = \frac{1}{\Gamma\left[n\right]}\int_0^\infty dt\int_0^\infty dst^{n-1}e^{-t}s^{-n}\widetilde{W}\left(s\right) \\
&= \frac{1}{\Gamma\left[n\right]}\int_0^\infty ds\int_0^\infty dy\, s\left(ys\right)^{n-1}e^{-ys}s^{-n}\widetilde{W}\left(s\right) = \frac{1}{\Gamma\left[n\right]}\int_0^\infty dy\, y^{n-1}\int_0^\infty dse^{-ys}\widetilde{W}\left(s\right) \\
&= \frac{1}{\Gamma\left[n\right]}\int_0^\infty dy\, y^{n-1}W\left(y\right).
\end{aligned}$$

If $n=0$ from the definition of Laplace transform we have

$$Q_0\left[W\right] \quad = \quad W\left(0\right).$$

Finally for $n<0$ the Gamma function has a pole and we cannot use the above trick. Nevertheless we can exploit the following property of the Laplace transform

$$\frac{d^n}{ds^n}\mathcal{L}\left(f\right)\left(s\right) = \mathcal{L}\left(\left(-1\right)^n t^n f(t)\right)$$

therefore

$$\begin{aligned}\frac{d^n}{ds^n}\mathcal{L}\left(\tilde{h}\right) &= \mathcal{L}\left(\left(-1\right)^n t^n \tilde{h}\right)\\ \partial_s^n h(s=0) &= \int_0^\infty dt\, e^{-0t}\left(-1\right)^n t^n \tilde{h}(t)\\ \int_0^\infty dt\, t^n \tilde{h}(t) &= \left(-1\right)^n \partial_s^n h(s=0).\end{aligned}$$

Finally another property which we will use is the following one:

$$Q_n\left[W\left(z+\alpha\right)\right] = \int_0^\infty ds s^{-n}\widetilde{W}\left(s\right)e^{-s\alpha} = \int_0^\infty ds s^{-n}\widetilde{W}\left(s+a\right).$$

These formulas are very general and hold for any cutoff function R_k. Interestingly these integrals can be computed analytically if we employ the optimized cutoff $R_k\left(z\right)=\left(k^2-z\right)\theta\left(k^2-z\right)$ [56]. Let us define $P_k\equiv\Delta+R_k$ and $\tilde{q}\equiv qk^{-2}$, we have:

$$Q_n\left(\frac{\partial_t R_k}{\left(P_k+q\right)^\ell}\right) = \frac{2}{n!}\frac{1}{\left(1+\tilde{q}\right)^\ell}k^{2(n-\ell+1)}$$

and

$$Q_n\left(\frac{\partial_t R_k}{\left(P_k+q\right)^\ell}\right) = 0 \quad \text{for } n<0\ .$$

When using the non local HK as in chapter 3 we also use the formulae below. Let $h_k(z)=\frac{\partial_t R_k(z)}{z+R_k(z)}$, we have (see also [154]):

$$\int_0^1 \mathrm{d}\xi\, Q_{-1}\left[h_k\left(z+x\xi(1-\xi)\right)\right] = \frac{4}{\Delta}\sqrt{1+\frac{4}{u}}\theta(u-4),$$

$$\int_0^1 \mathrm{d}\xi\, Q_{0}\left[h_k\left(z+x\xi(1-\xi)\right)\right] = 2\left[1-\sqrt{1-\frac{4}{u}}\theta(u-4)\right],$$

$$\int_0^1 \mathrm{d}\xi\, Q_{1}\left[h_k\left(z+x\xi(1-\xi)\right)\right] = 2k^2\left[1-\frac{u}{6}+\frac{u}{6}\left(1-\frac{4}{u}\right)^{\frac{3}{2}}\theta(u-4)\right],$$

$$\int_0^1 \mathrm{d}\xi\, Q_{2}\left[h_k\left(z+x\xi(1-\xi)\right)\right] = 2k^4\left[\frac{1}{2}-\frac{u}{6}+\frac{u^2}{60}-\frac{u^2}{60}\left(1-\frac{4}{u}\right)^{\frac{5}{2}}\theta(u-4)\right].$$

where $u=x/k^2$, $x\equiv\Delta$.

APPENDIX D

Variations

One of the crucial ingredient of the FRGE is the Hessian of the EAA. This is typically a trivial step when dealing with statitistical system decribed by scalar field but it might be an involved work when we consider complicated ansatzs for gravity. In this section we derive some basic formulae which can be seen as building blocks for generic variations.

D.0.6 Variations and functional derivatives

We start introducing the basic curvature invariants. The invariants of the Einstein-Hilbert action are the volume and the integral of the Ricci scalar:

$$I_0[g] = \int d^d x \sqrt{g} \qquad\qquad I_1[g] = \int d^d x \sqrt{g} R \,. \tag{D.1}$$

Note that in $d = 2$ the integrand of the Ricci scalar is proportional to the Euler characteristic for a two dimensional manifold:

$$\chi(\mathcal{M}) = \frac{1}{4\pi} \int_{\mathcal{M}} d^2 x \sqrt{g}\, R \,. \tag{D.2}$$

Up to two curvatures, or four derivatives, the invariants we can construct are:

$$I_{2,1}[g] = \int d^d x \sqrt{g} R^2 \qquad\qquad I_{2,2}[g] = \int d^d x \sqrt{g} R_{\mu\nu} R^{\mu\nu}$$

$$I_{2,3}[g] = \int d^d x \sqrt{g} R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} \qquad\qquad I_{2,4}[g] = \int d^d x \sqrt{g} \Box R \,. \tag{D.3}$$

The last invariant in (D.3) is a total derivative and is usually dropped. In $d = 4$ the three curvature square invariants are not independent since the linear combination

$$E = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \,, \tag{D.4}$$

is the integrand of the the Euler characteristic for a four dimensional manifold:

$$\chi(\mathcal{M}) = \frac{1}{32\pi^2} \int_{\mathcal{M}} d^4 x \sqrt{g}\, E \,. \tag{D.5}$$

Relation (D.2) and (D.5) can be proven using heat kernel methods [149]. We define the invariant:

$$I_E[g] = \int d^d x \sqrt{g}\, E\,. \tag{D.6}$$

There is another interesting combination of the four derivatives invariants, this defines the Weyl tensor, the square of which is:

$$C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - \frac{4}{d-2}R_{\mu\nu}R^{\mu\nu} + \frac{2}{(d-1)(d-2)}R^2\,. \tag{D.7}$$

The Weyl tensor is completely traceless and the action

$$I_C[g] = \int d^d x \sqrt{g}\, C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}\,, \tag{D.8}$$

is invariant under local conformal transformations, i.e. $I_C[e^{2\sigma}g] = I_C[g]$ for any $\sigma(x)$.

We now calculate the variations of the basic invariants just defined. We define $h_{\mu\nu} = \delta g_{\mu\nu}$ to be the first variation of the metric tensor. The first variations of the inverse metric can be deduced from the following relations, valid for any invertible matrix M,

$$M^{-1}M = 1 \qquad \Rightarrow \qquad \delta M^{-1}M + M^{-1}\delta M = 0 \qquad \Rightarrow \qquad \delta M^{-1} = -M^{-1}\delta M\, M^{-1}\,. \tag{D.9}$$

Setting $M_{\mu\nu} = g_{\mu\nu}$ and $\delta M_{\mu\nu} = h_{\mu\nu}$ in (D.9) gives:

$$\delta g^{\alpha\beta} = -g^{\alpha\mu}g^{\beta\nu}\delta g_{\mu\nu} = -h^{\alpha\beta}\,. \tag{D.10}$$

The second variation can be calculated iterating (D.10):

$$\begin{aligned}
\delta^2 g^{\alpha\beta} &= -\delta g^{\alpha\mu}g^{\beta\nu}h_{\mu\nu} - g^{\alpha\mu}\delta g^{\beta\nu}h_{\mu\nu} \\
&= g^{\alpha\lambda}g^{\mu\rho}g^{\beta\nu}h_{\lambda\rho}h_{\mu\nu} + g^{\alpha\mu}g^{\beta\lambda}g^{\nu\rho}h_{\lambda\rho}h_{\mu\nu} \\
&= 2h^{\alpha\lambda}h^{\beta}_{\lambda}\,.
\end{aligned} \tag{D.11}$$

The third variation is similarly found to be:

$$\delta^3 g^{\alpha\beta} = -3! h^{\alpha}_{\rho}h^{\rho}_{\sigma}h^{\sigma\beta}\,. \tag{D.12}$$

Combining (D.10), (D.11) and (D.12) gives the following expansion for the inverse metric around the background metric $\bar{g}_{\mu\nu}$:

$$\begin{aligned}
g^{\alpha\beta} &= \bar{g}^{\alpha\beta} + \delta g^{\alpha\beta} + \frac{1}{2}\delta^2 g^{\alpha\beta} + \frac{1}{3!}\delta^3 g^{\alpha\beta} + O\left(h^4\right) \\
&= \bar{g}^{\alpha\beta} - h^{\alpha\beta} + h^{\alpha\lambda}h^{\beta}_{\lambda} - h^{\alpha}_{\rho}h^{\rho}_{\sigma}h^{\sigma\beta} + O\left(h^4\right)\,.
\end{aligned} \tag{D.13}$$

It is not difficult to write the general n-th variation of the inverse metric tensor, it can be proven by induction that: [1]

$$\delta^n g^{\alpha\beta} = (-1)^n n! h^{\alpha}_{\lambda_1}h^{\lambda_1}_{\lambda_2}\cdots h^{\lambda_{n-2}}_{\lambda_{n-1}}h^{\lambda_{n-1}\beta}\,. \tag{D.14}$$

[1] A simpler way to derive this equation is the following. First note that $(g+h)^{-1} = (1+g^{-1}h)^{-1}g^{-1}$. Then expanding binomially the bracket in the r.h.s. we have: $(g+h)^{-1} = (\sum_n (-1)^n (g^{-1}h))g^{-1}$. This is our result, the $n!$ comes from the definition of the Taylor expansion.

The variations of the determinant of the metric tensor can be easily found using the following relation, valid again for any invertible matrix M,

$$\log\det M = \operatorname{tr}\log M\,. \tag{D.15}$$

A variation of equation (D.15) gives:

$$\delta\det M = \delta e^{\log\det M} = \det M\,\delta\operatorname{tr}\log M = \det M\operatorname{tr}\left(M^{-1}\delta M\right)\,. \tag{D.16}$$

Inserting in (D.16) $M_{\mu\nu} = g_{\mu\nu}$ and $\delta M_{\mu\nu} = h_{\mu\nu}$ brings to

$$\delta\sqrt{g} = \frac{1}{2}\sqrt{g}g^{\alpha\beta}\delta g_{\alpha\beta} = \frac{1}{2}\sqrt{g}h\,. \tag{D.17}$$

The second variation follows easily:

$$\delta^2\sqrt{g} = \frac{1}{4}\sqrt{g}\delta g^\alpha_\alpha\delta g^\beta_\beta - \frac{1}{2}\sqrt{g}\delta g^{\alpha\beta}\delta g_{\alpha\beta} = \sqrt{g}\left(\frac{1}{4}h^2 - \frac{1}{2}h^{\alpha\beta}h_{\alpha\beta}\right)\,. \tag{D.18}$$

For completeness the third variation of the metric determinant is found to be:

$$\delta^3\sqrt{g} = \sqrt{g}\left(\frac{1}{8}h^3 - \frac{3}{4}hh_{\mu\nu}h^{\mu\nu} + h_{\mu\nu}h^{\nu\alpha}h^\mu_\alpha\right)\,. \tag{D.19}$$

We do not have a closed formula for the n-th variation of the square root of the determinant of the metric, but for any given n these can be easily determined. We find now the variations of the Christoffel symbols, defined:

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}\right)\,. \tag{D.20}$$

Using geodesic coordinates, it can be proven that the first variation of the Christoffel symbols is:

$$\delta\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(\nabla_\mu h_{\nu\beta} + \nabla_\nu h_{\mu\beta} - \nabla_\beta h_{\mu\nu}\right)\,. \tag{D.21}$$

More generally we have the fundamental relation, that can again be proven by induction on n, for the n-th variation of the Christoffel symbols:

$$\delta^n\Gamma^\alpha_{\mu\nu} = \frac{n}{2}\left(\delta^{n-1}g^{\alpha\beta}\right)\left(\nabla_\mu h_{\nu\beta} + \nabla_\nu h_{\mu\beta} - \nabla_\beta h_{\mu\nu}\right)\,. \tag{D.22}$$

All the non-linearities of the Christoffel symbols are due the inverse metric of which we know exactly the n-variation (D.14). Introducing the tensor:

$$G_{\mu\nu\alpha} = \frac{1}{2}\left(\nabla_\mu h_{\nu\alpha} + \nabla_\nu h_{\mu\alpha} - \nabla_\alpha h_{\mu\nu}\right)\,, \tag{D.23}$$

we can rewrite the n-th variation of the Christoffel symbols simply as:

$$\delta^n\Gamma^\alpha_{\mu\nu} = n\delta^{n-1}g^{\alpha\beta}\,G_{\mu\nu\beta}\,. \tag{D.24}$$

Proof. We compute the n^{th} variation of the inverse metric and the Christofell symbol. It is easy to check by induction:

$$\delta^n g^{\mu\nu} = (-1)^n n! ghghg...ghg$$

where

$$ghg = g^{\mu\alpha} h_{\alpha\beta} g^{\beta\nu}.$$

We recall that

$$\delta\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\rho} G_{\mu\nu\rho}$$

where

$$G_{\mu\nu\rho} = \left(\nabla_\mu h_{\rho\nu} + \nabla_\nu h_{\mu\rho} - \nabla_\rho h_{\mu\nu}\right).$$

We want to compute the n^{th} variation of the tensor G, which in terms of connection is

$$G_{\mu\nu\rho} = \left(-\partial_\rho h_{\mu\nu} + \partial_\mu h_{\rho\nu} + \partial_\nu h_{\mu\rho} - 2\Gamma^{\alpha}_{\mu\nu} h_{\alpha\rho}\right).$$

Now we observe that:

$$\delta G_{\mu\nu\rho} = -h_{\rho\alpha} g^{\alpha\beta} G_{\mu\nu\beta} = -G_{\mu\nu\alpha} g^{\alpha\beta} h_{\beta\rho} \equiv -Ggh.$$

Always by induction we see:

$$\delta^n G_{\mu\nu\rho} = (-1)^n n! Ggh...ghgh.$$

Finally let us consider:

$$\begin{aligned}
\delta^n \left(g^{\alpha\rho} G_{\mu\nu\rho}\right) &= \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) \delta^{n-m} g \delta^m G \\
&= \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) \left((-1)^{n-m}(n-m)! ghghg...ghg\right) \left((-1)^m m! Gghghghg...gh\right) \\
&= \sum_{m=0}^{n} (-1)^n \, n! ghghg...ghgGhgh...gh = (-1)^n (n+1) n! ghghg...hgG \\
&= (-1)^n (n+1)! ghgh...hgG.
\end{aligned}$$

Therefore

$$\delta^n \Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} (-1)^{n-1} \, n! ghghgh..hgG = \frac{n}{2} \cdot \delta^{n-1} \left(g^{\alpha\rho}\right) \cdot G_{\mu\nu\rho}$$

for $n \geq 2$ and where $ghg...hg$ contains $(n-1)$ times $h_{\alpha\beta}$. QED.

Note that the tensor (D.23) is symmetric in the first two indices $G_{\mu\nu\alpha} = G_{\nu\mu\alpha}$. In particular we have the useful contractions:

$$G^{\alpha\ \ \mu}_{\ \alpha} = \nabla^\alpha h^\mu_\alpha - \frac{1}{2}\nabla^\mu h \qquad\qquad G^{\alpha}_{\mu\alpha} = \frac{1}{2}\nabla_\mu h\,. \tag{D.25}$$

We turn now the variations of the fundamental building block of all gravitational invariants: the Riemann tensor. This is defined:

$$R_{\mu\nu}{}^{\alpha}{}_{\beta} = \partial_\mu \Gamma^\alpha_{\nu\beta} - \partial_\nu \Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\lambda}\Gamma^\lambda_{\nu\beta} - \Gamma^\alpha_{\nu\lambda}\Gamma^\lambda_{\mu\beta}\,. \tag{D.26}$$

The Ricci tensor and the Ricci scalar are defined by the following contractions:

$$R_{\beta\nu} = R^\alpha_{\beta\alpha\nu} \qquad\qquad R = g^{\beta\nu}R_{\beta\nu}\,. \tag{D.27}$$

The n-th variation of the Riemann tensor is found directly from the definition (D.26) and using the binomial theorem for the variation of a product:

$$\delta^n R^\alpha_{\beta\mu\nu} = \nabla_\mu \delta^n \Gamma^\alpha_{\beta\nu} - \nabla_\nu \delta^n \Gamma^\alpha_{\beta\mu} + \sum_{i=1}^{n-1} \begin{pmatrix} n \\ i \end{pmatrix} \left(\delta^{n-i}\Gamma^\alpha_{\mu\lambda}\delta^i\Gamma^\lambda_{\beta\nu} - \delta^{n-i}\Gamma^\alpha_{\nu\lambda}\delta^i\Gamma^\lambda_{\beta\mu}\right)\,. \tag{D.28}$$

This relation together with equation (D.22) or (D.24) and (D.14) gives us, in a closed form, all possible variations of the Riemann tensor. This is a fundamental result.

Proof. Now we consider the Riemann tensor variations. First let us consider the first variation, one can check that:

$$\delta R^\alpha{}_{\beta\mu\nu} = \nabla_\mu \delta\Gamma^\alpha_{\beta\nu} - \nabla_\nu \delta\Gamma^\alpha_{\beta\mu}$$

Then we vary the above expression n times. Let $\delta^n\Gamma \equiv \Gamma^{(n)}$, if we vary the above expression once we have:

$$\delta^2 R^\alpha{}_{\beta\mu\nu} = \nabla_\mu \Gamma^{(2)\alpha}_{\beta\nu} - \nabla_\nu \Gamma^{(2)\alpha}_{\beta\mu} + 2\left[\Gamma^{(1)\alpha}_{\mu\rho}\Gamma^{(1)\rho}_{\beta\nu} - \Gamma^{(1)\alpha}_{\nu\rho}\Gamma^{(1)\rho}_{\beta\mu}\right].$$

We observe that the variation of the first term in the r.h.s. always produce the type of contribution:

$$\nabla_\mu \Gamma^{(n+1)\alpha}_{\beta\nu} - \nabla_\nu \Gamma^{(n+1)\alpha}_{\beta\mu} + \Gamma^{(1)\alpha}_{\mu\rho}\Gamma^{(n)\alpha}_{\beta\nu} - \Gamma^{(1)\rho}_{\beta\mu}\Gamma^{(n)\alpha}_{\mu\rho} - \Gamma^{(1)\alpha}_{\rho\nu}\Gamma^{(n)\rho}_{\beta\mu} + \Gamma^{(1)\rho}_{\beta\nu}\Gamma^{(n)\alpha}_{\mu\rho}.$$

The contributions in which the covariant derivative does not appear explictly are there every time one makes a variation. Clearly they have to be taken into account in the next variation. Keeping track of these terms and the using binomial formulas after some manipulations we get:

$$\begin{aligned}
\delta^{n+1} R^\alpha{}_{\beta\mu\nu} &= \nabla_\mu \Gamma^{(n+1)\alpha}_{\beta\nu} - \nabla_\nu \Gamma^{(n+1)\alpha}_{\beta\mu} \\
&+ \sum_{i=0}^{n-1}\sum_{m=1}^{n-1} \begin{pmatrix} m \\ i \end{pmatrix} \left[\Gamma^{(1+i)\alpha}_{\mu\rho}\Gamma^{(n-i)\rho}_{\beta\nu} - \Gamma^{(1+i)\rho}_{\mu\beta}\Gamma^{(n-i)\alpha}_{\rho\nu} - \Gamma^{(1+i)\alpha}_{\rho\nu}\Gamma^{(n-i)\rho}_{\mu\beta} + \Gamma^{(1+i)\rho}_{\beta\nu}\Gamma^{(n-i)\alpha}_{\mu\rho}\right] \\
&= \nabla_\mu \Gamma^{(n+1)\alpha}_{\beta\nu} - \nabla_\nu \Gamma^{(n+1)\alpha}_{\beta\mu} \\
&+ \sum_{i=0}^{n-1} \left[\frac{n-i}{i+1}\begin{pmatrix} n \\ i \end{pmatrix} - \frac{i}{i+1}\begin{pmatrix} 0 \\ i \end{pmatrix}\right] \left[\Gamma^{(1+i)\alpha}_{\mu\rho}\Gamma^{(n-i)\rho}_{\beta\nu} - \Gamma^{(1+i)\rho}_{\mu\beta}\Gamma^{(n-i)\alpha}_{\rho\nu} - \Gamma^{(1+i)\alpha}_{\rho\nu}\Gamma^{(n-i)\rho}_{\mu\beta} + \Gamma^{(1+i)\rho}_{\beta\nu}\Gamma
\end{aligned}$$

So:

$$\begin{aligned}
\delta^n R^{\alpha}{}_{\beta\mu\nu} &= \partial_\mu \delta^n \Gamma^\alpha_{\beta\nu} - \partial_\nu \delta^n \Gamma^\alpha_{\beta\mu} + \delta^n \left(\Gamma^\alpha_{\rho\nu} \Gamma^\rho_{\beta\mu} - \Gamma^\rho_{\beta\nu} \Gamma^\alpha_{\rho\mu} \right) \\
&= \partial_\mu \delta^n \Gamma^\alpha_{\beta\nu} - \partial_\nu \delta^n \Gamma^\alpha_{\beta\mu} + \Gamma^\rho_{\beta\mu} \delta^n \Gamma^\alpha_{\rho\nu} - \Gamma^\alpha_{\rho\mu} \delta^n \Gamma^\rho_{\beta\nu} + \Gamma^\alpha_{\rho\nu} \delta^n \Gamma^\rho_{\beta\mu} - \Gamma^\rho_{\beta\nu} \delta^n \Gamma^\alpha_{\rho\mu} + \\
&\quad + \sum_{i=1}^{n-1} \left(\begin{array}{c} n \\ i \end{array} \right) \left[\delta^{n-i} \Gamma^\alpha_{\rho\nu} \delta^i \Gamma^\rho_{\beta\mu} - \delta^{n-i} \Gamma^\rho_{\beta\nu} \delta^i \Gamma^\alpha_{\rho\mu} \right] \\
&= \text{(D.28)}
\end{aligned}$$

where we chose a coordinate system (e.g. normal coordinates in a local patch) in order to make $\delta\Gamma$ transform as a tensor, and we used

$$\delta^n (AB) = \sum_{i=0}^{n} \left(\begin{array}{c} n \\ i \end{array} \right) \left(\delta^{n-i} A \right) \left(\delta^i B \right) .$$

QED.

The n-th variations of the Ricci tensor (D.27) are obtained straightforwardly from (D.28) by contraction:

$$\delta^n R_{\beta\nu} = \delta^n R^\alpha_{\beta\alpha\nu} . \tag{D.29}$$

The n-variation of the Ricci scalar follows from (D.27) and is:

$$\delta^n R = \sum_{i=1}^{n} \left(\begin{array}{c} n \\ i \end{array} \right) \delta^{n-i} g^{\beta\nu} \delta^i R_{\beta\nu} . \tag{D.30}$$

We can now study some particular examples. From the fundamental relation (D.28), for $i = 1$, we find

$$\delta R^\alpha_{\beta\mu\nu} = \nabla_\mu G^\alpha_{\beta\nu} - \nabla_\nu G^\alpha_{\beta\mu} .$$

Using (D.29) and the second relation in (D.25) gives the first variation of the Ricci tensor[2]:

$$\begin{aligned}
\delta R_{\mu\nu} &= \nabla_\alpha G^\alpha_{\mu\nu} - \nabla_\nu G^\alpha_{\mu\alpha} \\
&= \frac{1}{2} \left[\nabla_\alpha \left(\nabla_\mu h^\alpha_\nu + \nabla_\nu h^\alpha_\mu - \nabla^\alpha h_{\mu\nu} \right) - \nabla_\nu \nabla_\mu h \right] \\
&= \frac{1}{2} \left(-\nabla^2 h_{\mu\nu} - \nabla_\nu \nabla_\mu h + \nabla_\alpha \nabla_\mu h^\alpha_\nu + \nabla_\alpha \nabla_\nu h^\alpha_\mu \right) .
\end{aligned} \tag{D.31}$$

Combining (D.31) with (D.30) gives the first variation of the Ricci scalar:

$$\begin{aligned}
\delta R &= g^{\mu\nu} \delta R_{\mu\nu} + \delta g^{\mu\nu} R_{\mu\nu} \\
&= -\nabla^2 h + \nabla^\mu \nabla^\nu h_{\mu\nu} - h_{\mu\nu} R^{\mu\nu} .
\end{aligned} \tag{D.32}$$

[2] $\nabla_\nu \nabla_\mu h = \nabla_\mu \nabla_\nu h$

From (D.28) with $n = 2$ we get the second variation of the Riemann tensor

$$\delta^2 R^{\alpha}_{\beta\mu\nu} = -2\nabla_\mu \left(h^{\alpha\gamma} G_{\gamma\beta\nu}\right) + 2\nabla_\nu \left(h^{\alpha\gamma} G_{\gamma\beta\mu}\right) + 2\left(G^{\alpha}_{\mu\gamma} G^{\gamma}_{\beta\nu} - G^{\alpha}_{\nu\gamma} G^{\gamma}_{\beta\mu}\right) , \tag{D.33}$$

while the second variation of the Ricci tensor is again just the contraction of (D.33):

$$\delta^2 R_{\mu\nu} = -2\nabla_\alpha \left(h^{\alpha\beta} G_{\beta\mu\nu}\right) + 2\nabla_\nu \left(h^{\alpha\beta} G_{\beta\mu\alpha}\right) + 2\left(G^{\alpha}_{\alpha\beta} G^{\beta}_{\mu\nu} - G^{\alpha}_{\nu\beta} G^{\beta}_{\mu\alpha}\right) . \tag{D.34}$$

The second variation of the Ricci scalar is given in terms of (D.10), (D.11), (D.31) and (D.34):

$$\delta^2 R = \delta^2 g^{\mu\nu} R_{\mu\nu} + 2\delta g^{\mu\nu} \delta R_{\mu\nu} + g^{\mu\nu} \delta^2 R_{\mu\nu} . \tag{D.35}$$

We can now find the variations of the curvature invariants $I_i[g]$. Using (D.17) and (D.18) we find:

$$\delta I_0[g] = \frac{1}{2} \int d^d x \sqrt{g} h \qquad \delta^2 I_0[g] = \int d^d x \sqrt{g} \left(\frac{1}{4} h^2 - \frac{1}{2} h^{\alpha\beta} h_{\alpha\beta}\right) . \tag{D.36}$$

Using (D.17) and (D.32) we find:

$$\delta I_1[g] = \int d^d x \left(\delta\sqrt{g} R + \sqrt{g}\delta R\right) = \int d^d x \sqrt{g} \left(-\nabla^2 h + \nabla^\mu \nabla^\nu h_{\mu\nu} - h_{\mu\nu} R^{\mu\nu} + \frac{1}{2} h R\right) . \tag{D.37}$$

For the second variation we have:

$$\delta^2 I_1[g] = \int d^d x \left(\delta^2\sqrt{g} R + 2\delta\sqrt{g}\delta R + \sqrt{g}\delta^2 R\right) , \tag{D.38}$$

the first two terms in (D.38) are rapidly evaluated using (D.17), (D.18) and (D.32). The last term in (D.38) can be expanded as:

$$\int d^d x \sqrt{g} \delta^2 R = \int d^d x \sqrt{g} \left(\delta^2 g^{\mu\nu} R_{\mu\nu} + 2\delta g^{\mu\nu} \delta R_{\mu\nu} + g^{\mu\nu} \delta^2 R_{\mu\nu}\right) . \tag{D.39}$$

Again, the first two terms in (D.39) need just the relations (D.10), (D.11) and (D.31), the last can be written employing (D.34). Modulo a total derivative, we have:

$$\int d^d x \sqrt{g} g^{\mu\nu} \delta^2 R_{\mu\nu} = 2 \int d^d x \sqrt{g} \left(G^{\alpha}_{\alpha\beta} G^{\beta\gamma}_{\ \ \gamma} - G^{\alpha\gamma}_{\ \ \beta} G^{\beta}_{\gamma\alpha}\right) , \tag{D.40}$$

using in (D.40) the relations (D.25) and the product

$$G^{\alpha\gamma}_{\ \ \beta} G^{\beta}_{\gamma\alpha} = \frac{1}{4} \left(-\nabla^\gamma h_{\alpha\beta} \nabla_\gamma h^{\alpha\beta} + 2\nabla^\gamma h^{\alpha\beta} \nabla_\alpha h_{\beta\gamma}\right) ,$$

we find

$$\begin{aligned} \int d^d x \sqrt{g} g^{\mu\nu} \delta^2 R_{\mu\nu} &= 2 \int d^d x \sqrt{g} \left(G^{\alpha}_{\alpha\beta} G^{\beta\gamma}_{\ \ \gamma} - G^{\alpha\gamma}_{\ \ \beta} G^{\beta}_{\gamma\alpha}\right) , \\ &= \int d^d x \sqrt{g} \left(\nabla_\mu h^{\mu\nu} \nabla_\nu h - \frac{1}{2} \nabla_\mu h \nabla^\mu h \right. \\ &\quad \left. + \frac{1}{2} \nabla^\alpha h_{\mu\nu} \nabla_\alpha h^{\mu\nu} - \nabla^\alpha h^{\mu\nu} \nabla_\mu h_{\nu\alpha}\right) . \end{aligned} \tag{D.41}$$

Inserting in (D.38) the variation (D.39) and (D.41) finally gives:

$$\begin{aligned}\delta^2 I_1[g] &= \int d^dx\sqrt{g}\left[-\frac{1}{2}h\nabla^2 h+\frac{1}{2}h^{\mu\nu}\nabla^2 h_{\mu\nu}-h^{\mu\nu}\nabla_\alpha\nabla_\mu h^\alpha_\nu+h\nabla^\mu\nabla^\nu h_{\mu\nu}+\right.\\ &\left.+2h^{\mu\nu}h^\alpha_\mu R_{\nu\alpha}-hR^{\mu\nu}h_{\mu\nu}+\left(\frac{1}{4}h^2-\frac{1}{2}h^{\alpha\beta}h_{\alpha\beta}\right)R\right]. \qquad \text{(D.42)}\end{aligned}$$

Commuting covariant derivatives in the third term of (D.42) as

$$\nabla_\alpha\nabla_\mu h^\alpha_\nu=\nabla_\mu\nabla_\alpha h^\alpha_\nu+R_{\mu\alpha}h^\alpha_\nu-R_{\alpha\mu\beta\nu}h^{\alpha\beta},$$

we can recast (D.42) to the form:

$$\begin{aligned}\delta^2 I_1[g] &= \int d^dx\sqrt{g}\left[-\frac{1}{2}h^{\mu\nu}\Delta h_{\mu\nu}+\frac{1}{2}h\Delta h-h^{\mu\nu}\nabla_\nu\nabla_\alpha h^\alpha_\mu+h\nabla^\mu\nabla^\nu h_{\mu\nu}\right.\\ &\left.+h^{\mu\nu}h^\alpha_\mu R_{\nu\alpha}+h^{\mu\nu}h^{\alpha\beta}R_{\alpha\mu\beta\nu}-hR^{\mu\nu}h_{\mu\nu}+\left(\frac{1}{4}h^2-\frac{1}{2}h^{\alpha\beta}h_{\alpha\beta}\right)R\right], \qquad \text{(D.43)}\end{aligned}$$

which can be later combined with the gauge-fixing action. It is straightforward now to calculate higher order variations of both the actions $I_0[g]$ and $I_1[g]$, since their variations can always be reduced to combinations of variations of the inverse metric, of the metric determinant and of the Christoffel symbols, which are all known exactly. In the same way, we can easily calculate the variations of the higher curvature invariants (D.3). We will not do this here since, in this thesis, we will concentrate to truncations where only variations of $I_0[g]$ and $I_1[g]$ are needed.

The background gauge fixing action (2.4) is already quadratic in the metric fluctuation, when expanded reads:

$$S_{gf}[h;g]=\frac{1}{2\alpha}\int d^dx\sqrt{g}\left(-h^{\mu\nu}\nabla_\nu\nabla_\alpha h^\alpha_\mu+\beta h\nabla^\mu\nabla^\nu h_{\mu\nu}+\frac{\beta^2}{4}h\Delta h\right). \qquad \text{(D.44)}$$

Combining (D.44) with (D.43) gives:

$$\begin{aligned}-\frac{1}{2}\delta I_1[g]+S_{gf}[h;g] &= \frac{1}{2}\int d^dx\sqrt{g}\left[\frac{1}{2}h^{\mu\nu}\Delta h_{\mu\nu}-\frac{1}{2}\left(1-\frac{\beta^2}{2\alpha}\right)h\Delta h\right.\\ &+\left(1-\frac{1}{\alpha}\right)h^{\mu\nu}\nabla_\nu\nabla_\alpha h^\alpha_\mu-\left(1-\frac{\beta}{\alpha}\right)h\nabla^\mu\nabla^\nu h_{\mu\nu}\\ &-h^{\mu\nu}h^\alpha_\mu R_{\nu\alpha}-h^{\mu\nu}h^{\rho\alpha}R_{\rho\nu\alpha\mu}+hR^{\mu\nu}h_{\mu\nu}\\ &\left.-\left(\frac{1}{4}h^2-\frac{1}{2}h^{\alpha\beta}h_{\alpha\beta}\right)R\right]. \qquad \text{(D.45)}\end{aligned}$$

We will use (D.45) in section 2.2 to construct the Hessian's needed in the flow equation for the bEAA. Note that the gauge choice $\alpha=\beta=1$ diagonalizes the Hessian (D.45).

From the variations just obtained we can calculate all the functional derivatives of the previous defined invariants by employing the following relation between variations and functional

derivatives[3]:

$$\delta^{(n)}(...)(x) = \frac{1}{n!}\int_{x_1...x_n}\Big[(...)^{(n)}(x)\Big]^{\mu_1\nu_1...\mu_n\nu_n}(x_1,...,x_n)h_{\mu_1\nu_1}(x_1)...h_{\mu_n\nu_n}(x_n). \tag{D.46}$$

Using (D.46) we can derive all the gravitational vertices needed in the flow equations for the zero-field proper-vertices used in section 2.4.

D.0.7 Decomposition and projectors

In this section we study the different degrees of freedom that are contained in the fluctuation metric, we understand which degrees of freedom are physical and which are pure gauge. We use this knowledge to construct the projector basis that we will use in the next section to construct the regularized graviton propagator.

We start decomposing the metric fluctuation in transverse $h^T_{\mu\nu}$ and longitudinal $h^L_{\mu\nu}$ components:

$$h_{\mu\nu} = h^T_{\mu\nu} + h^L_{\mu\nu}\,, \tag{D.47}$$

with the following transversality condition $\nabla^\mu h^T_{\mu\nu} = 0$. The longitudinal part can be written in terms of the vector ξ_μ as:

$$h^L_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = \nabla_\mu\xi^T_\nu + \nabla_\nu\xi^T_\mu + 2\nabla_\mu\nabla_\nu\sigma\,. \tag{D.48}$$

In (D.48) we decomposed the vector into a transverse ξ^T_μ vector and the gradient of the scalar σ as $\xi_\mu = \xi^T_\mu + \nabla_\mu\sigma$, with the transversality condition $\nabla^\mu\xi^T_\mu = 0$. We can extract the trace of the fluctuation metric

$$h = g^{\mu\nu}h_{\mu\nu} = g^{\mu\nu}h^T_{\mu\nu} - 2\Delta\sigma\,, \tag{D.49}$$

writing the transverse component of $h_{\mu\nu}$ in the following way:

$$h^T_{\mu\nu} = h^{TT}_{\mu\nu} + \frac{1}{d}g_{\mu\nu}(h + 2\Delta\sigma)\,, \tag{D.50}$$

with $h^{TT}_{\mu\nu}$ the transverse-traceless metric satisfying $g^{\mu\nu}h^{TT}_{\mu\nu} = 0$. Inserting (D.48) and (D.50) in (D.47) gives:

$$h_{\mu\nu} = h^{TT}_{\mu\nu} + \nabla_\mu\xi^T_\nu + \nabla_\nu\xi^T_\mu + 2\nabla_\mu\nabla_\nu\sigma + \frac{1}{d}g_{\mu\nu}(h + 2\Delta\sigma)\,. \tag{D.51}$$

In (D.51) the metric fluctuation is decomposed into a transverse-traceless symmetric tensor, a transverse vector and two scalar degrees of freedom, the trace and the longitudinal component of the vector. To see which of these degrees of freedom are physical and which are pure gauge we can insert in (D.51) the gauge transformation of the metric fluctuation parametrized by the vector χ_μ:

$$\delta h_{\mu\nu} = \nabla_\mu\chi_\nu + \nabla_\nu\chi_\mu = \nabla_\mu\chi^T_\nu + \nabla_\nu\chi^T_\mu + 2\nabla_\mu\nabla_\nu\chi\,. \tag{D.52}$$

[3] We use the convention $\int_x \equiv \int d^dx\sqrt{g_x}$.

In (D.52) with decomposed the gauge transformation vector as $\chi_\mu = \chi^T_\mu + \nabla_\mu \chi$ with as usual $\nabla^\mu \chi^T_\mu = 0$. Matching (D.52) to

$$\delta h_{\mu\nu} = \delta h^{TT}_{\mu\nu} + \nabla_\mu \delta\xi^T_\nu + \nabla_\nu \delta\xi^T_\mu + 2\nabla_\mu\nabla_\nu \delta\sigma + \frac{1}{d} g_{\mu\nu}(\delta h + 2\Delta\delta\sigma)\,,$$

we find:

$$\delta h^{TT}_{\mu\nu} = 0 \qquad \delta\xi^T_\mu = \chi^T_\mu \qquad \delta\sigma = \chi \qquad \delta h = -2\Delta\chi\,. \tag{D.53}$$

These are the gauge transformation properties of the metric fluctuation components. We see that the transverse-traceless symmetric tensor is a physical degree of freedom, which can be associated with the graviton. Also the following combination of the two scalar degrees of freedom

$$S = h + 2\Delta\sigma\,, \tag{D.54}$$

is gauge invariant $\delta S = 0$ and is as well physical. It correspond to the conformal mode that in the path integral formulation of gravity is dynamical as the graviton. Instead, the transverse vector ξ^T_μ and the scalar field σ are pure gauge fields.

Using the properties of these projectors we can then easily obtain the regularized gravitational propagator $\mathbf{G}_k[0;\delta] = \left(\gamma_k^{(2,0,0;0)} + \mathbf{R}_k[\delta]\right)^{-1}$. The basic longitudinal projector is defined by $P^{\mu\nu} = \partial^\mu\partial^\nu/\partial^2$ and projects out the longitudinal component of a vector field, $\delta^{\mu\nu} - P^{\mu\nu}$ instead projects out the transverse component of a vector field. The graviton is the transverse part of the traceless component of the metric, in flat space we can define it as follows:

$$\begin{aligned} h^{TT}_{\mu\nu} &= \left[\frac{1}{2}\left(\delta^\alpha_\mu - P^\alpha_\mu\right)\left(\delta^\beta_\nu - P^\beta_\nu\right) + \frac{1}{2}\left(\delta^\alpha_\mu - P^\alpha_\mu\right)\left(\delta^\beta_\nu - P^\beta_\nu\right) + \right. \\ &\quad \left. -\frac{1}{d-1}\left(g_{\mu\nu} - P_{\mu\nu}\right)\left(g^{\alpha\beta} - P^{\alpha\beta}\right)\right] h_{\alpha\beta} \\ &= \left[\tilde{\delta}^{\alpha\beta}_{\mu\nu} - \frac{1}{d-1}\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\right] h_{\alpha\beta}\,, \end{aligned} \tag{D.55}$$

where we defined $\tilde{g}^{\mu\nu} = g^{\mu\nu} - P^{\mu\nu}$. We also have the following relations for the scalar degrees of freedom:

$$S = \frac{d}{d-1}\tilde{g}^{kl} h_{kl} \qquad \Box\sigma = \frac{d}{d-1}\left(P^{kl} - \frac{1}{d} g^{kl}\right) h_{kl}\,. \tag{D.56}$$

Inspired by (D.55) and (D.56) we define the following projectors:

$$\begin{aligned} P_2^{\mu\nu,\alpha\beta} &= \tilde{\delta}^{\mu\nu,\alpha\beta} - \frac{1}{d-1}\tilde{g}^{\mu\nu}\tilde{g}^{\alpha\beta} \\ P_1^{\mu\nu,\alpha\beta} &= \frac{1}{2}\left(\tilde{g}^{\mu\alpha}P^{\nu\beta} + \tilde{g}^{\mu\beta}P^{\nu\alpha} + \tilde{g}^{\nu\alpha}P^{\mu\beta} + \tilde{g}^{\nu\beta}P^{\mu\alpha}\right) \\ P_S^{\mu\nu,\alpha\beta} &= \frac{1}{d-1}\tilde{g}^{\mu\nu}\tilde{g}^{\alpha\beta} \\ P_\sigma^{\mu\nu,\alpha\beta} &= P^{\mu\nu}P^{\alpha\beta} \\ P_{S\sigma}^{\mu\nu,\alpha\beta} &= \frac{1}{\sqrt{d-1}}\left(\tilde{g}^{\mu\nu}P^{\alpha\beta} + P^{\mu\nu}\tilde{g}^{\alpha\beta}\right)\,. \end{aligned} \tag{D.57}$$

It is useful to note that these projectors can be rewritten as:

$$\begin{aligned}
P_{2;\mu\nu,\alpha\beta} &= \frac{1}{2}\left(P^T_{\mu\alpha}P^T_{\nu\beta}+P^T_{\mu\beta}P^T_{\nu\alpha}\right)-\frac{1}{d-1}P^T_{\mu\nu}P^T_{\alpha\beta}\\
P_{1;\mu\nu,\alpha\beta} &= \frac{1}{2}\left[P^T_{\mu\alpha}P^L_{\nu\beta}+P^T_{\mu\beta}P^L_{\nu\alpha}+P^T_{\nu\alpha}P^L_{\mu\beta}+P^T_{\nu\beta}P^L_{\mu\alpha}\right]\\
P_{S;\mu\nu,\alpha\beta} &= \frac{1}{d-1}P^T_{\mu\nu}P^T_{\alpha\beta}\\
P_{S\sigma;\mu\nu,\alpha\beta} &= \frac{1}{\sqrt{d-1}}P^T_{\mu\nu}P^L_{\alpha\beta}\\
P_{\sigma S;\mu\nu,\alpha\beta} &= \frac{1}{\sqrt{d-1}}P^L_{\mu\nu}P^T_{\alpha\beta}\\
P_{\sigma;\mu\nu,\alpha\beta} &= P^L_{\mu\nu}P^L_{\alpha\beta}
\end{aligned}$$

where

$$P^{\mu\nu}_T \equiv \delta^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\,,\qquad\qquad P^{\mu\nu}_L \equiv \frac{p^\mu p^\nu}{p^2}\,.$$

The projectors in (D.57) have the following traces (where we use the notation $A=\mu\nu$ and $B=\alpha\beta$ and hats mean contractions):

$$\begin{aligned}
P_2^{\hat{AB}} &= \frac{d^2-d-2}{2} & P_2^{\hat{A}\hat{B}} &= 0\\
P_1^{\hat{AB}} &= d-1 & P_1^{\hat{A}\hat{B}} &= 0\\
P_S^{\hat{AB}} &= 1 & P_S^{\hat{A}\hat{B}} &= d-1\\
P_\sigma^{\hat{AB}} &= 1 & P_\sigma^{\hat{A}\hat{B}} &= 1\\
P_{S\sigma}^{\hat{AB}} &= 0 & P_{S\sigma}^{\hat{A}\hat{B}} &= 2\sqrt{d-1}
\end{aligned}\tag{D.58}$$

and satisfy the following relations:

$$\begin{aligned}
[P_2+P_1+P_S+P_\sigma]^{\mu\nu,\alpha\beta} &= \delta^{\mu\nu,\alpha\beta}\\
\left[(d-1)P_S+P_\sigma+\sqrt{d-1}P_{S\sigma}\right]^{\mu\nu,\alpha\beta} &= g^{\mu\nu}g^{\alpha\beta}\\
\left[2P_\sigma+\sqrt{d-1}P_{S\sigma}\right]^{\mu\nu,\alpha\beta} &= g^{\mu\nu}P^{\alpha\beta}+P^{\mu\nu}g^{\alpha\beta}\\
[P_1+2P_\sigma]^{\mu\nu,\alpha\beta} &= \frac{1}{2}\left(g^{\mu\alpha}P^{\nu\beta}+g^{\mu\beta}P^{\nu\alpha}+g^{\nu\alpha}P^{\mu\beta}+g^{\nu\beta}P^{\mu\alpha}\right)\\
P_\sigma^{\mu\nu,\alpha\beta} &= P^{\mu\nu}P^{\alpha\beta}\,.
\end{aligned}\tag{D.59}$$

We can also introduce the trace projection operator as follows:

$$P^{\mu\nu,\alpha\beta}=\frac{1}{d}g^{\mu\nu}g^{\alpha\beta}\,,\tag{D.60}$$

and from (D.59) this can be expressed in terms of the other projection operators as[4]:

$$\mathbf{P}=\frac{d-1}{d}\mathbf{P}_S+\frac{1}{d}\mathbf{P}_\sigma+\frac{\sqrt{d-1}}{d}\mathbf{P}_{S\sigma}\,,\tag{D.61}$$

[4]We will sometimes suppress indices for notation clarity and we will use boldface symbols to indicate linear operators in the space of symmetric tensors.

so that

$$\mathbf{1} - \mathbf{P} = \mathbf{P}_2 + \mathbf{P}_1 + \frac{1}{d}\mathbf{P}_S + \frac{d-1}{d}\mathbf{P}_\sigma - \frac{\sqrt{d-1}}{d}\mathbf{P}_{S\sigma}\,. \tag{D.62}$$

The non-zero products between these projection operators are:

$$\mathbf{P}_S\mathbf{P}_{S\sigma} + \mathbf{P}_\sigma\mathbf{P}_{S\sigma} = \mathbf{P}_{S\sigma} \qquad \mathbf{P}_S\mathbf{P}_{S\sigma} = \mathbf{P}_{S\sigma}\mathbf{P}_\sigma$$

$$\mathbf{P}_{S\sigma}\mathbf{P}_S = \mathbf{P}_\sigma\mathbf{P}_{S\sigma} \qquad \mathbf{P}_{S\sigma}\mathbf{P}_{S\sigma} = \mathbf{P}_S + \mathbf{P}_\sigma\,. \tag{D.63}$$

The general structure of the inverse propagator that we will encounter in the next section is as follows:

$$\mathbf{M} = \lambda_2\mathbf{P}_2 + \lambda_1\mathbf{P}_1 + \lambda_S\mathbf{P}_S + \lambda_\sigma\mathbf{P}_\sigma + \lambda_{S\sigma}\mathbf{P}_{S\sigma}\,, \tag{D.64}$$

We can invert (D.64) to obtain:

$$\mathbf{M}^{-1} = \frac{1}{\lambda_2}\mathbf{P}_2 + \frac{1}{\lambda_1}\mathbf{P}_1 + \frac{\lambda_\sigma}{\lambda_S\lambda_\sigma - \lambda_{S\sigma}^2}\mathbf{P}_S + \frac{\lambda_S}{\lambda_S\lambda_\sigma - \lambda_{S\sigma}^2}\mathbf{P}_\sigma - \frac{\lambda_{S\sigma}}{\lambda_S\lambda_\sigma - \lambda_{S\sigma}^2}\mathbf{P}_{S\sigma}\,. \tag{D.65}$$

APPENDIX E

Weyl geometry: some details

Here we report some details concerning various technical aspects of the application of Weyl geometry to QFTs.

E.1 Modified diffeomorphism

In this section we will define some suitably generalized diffeomorphism transformation. The need for this generalization has been pointed out in a geometrical framework in [155] and in terms of operators in [105] and references therein.

Let us suppose that our theory is invariant both under diffeomorphisms and another local symmetry (in our case Weyl transformations). The crucial point is that the diffeomorphisms are not a normal subgroup of the full symmetry group while the gauge group is a normal subgroup. This can also be understood from the fact that the Ward operators (i.e.: the operators which implement the transformations, see below) of the diffeomorphisms and of the gauge symmetry do not commute. This implies that a diffeomorphism does not map Weyl covariant tensors into Weyl covariant tensors. We will see that it is possible to define a modified diffeomorphism which on the contrary maps covariant tensors into covariant tensors. The so called Ward operator (e.g.: for diffeomorphisms) is defined as follows:

$$W_D \equiv -\int dx \left[\delta_D g_{\mu\nu}(x)\frac{\delta}{\delta g_{\mu\nu}(x)} + \delta_D b^\mu(x)\frac{\delta}{\delta b^\mu(x)} + \cdots\right]$$

where the dots stand for all the other fields present in the theory.

In order to promote W_D to an operator which maps Weyl covariant tensors into Weyl convariant tensor we add to W_D a particular Weyl transformation, implemented via W_W, which depend on the Weyl field itself. This is completely analogous to what as been done in [105] in the case of $SU(N)$ gauge theories. As shown in section E.1.1 this is accomplished via the following Ward operator:

$$\widetilde{W}_D(\varepsilon) \equiv W_D(\varepsilon) + W_W(-\varepsilon \cdot b) \tag{E.1}$$

where the first piece implements a standard diffeomorphism transformation while the latter a Weyl transformation with parameter $\sigma = -\varepsilon^\rho b_\rho$. The transformation of the fields implemented by $\widetilde{W}$ are given in section E.1.1. To see how this works let us consider the simple case of a scalar field of weight -1. A standard and a modified diffeomorphism give respectively:

$$\begin{aligned}\delta_D\psi &= \varepsilon^\rho\nabla_\rho\psi \\ \tilde{\delta}_D\psi &= \varepsilon^\rho\nabla_\rho\psi - (-\varepsilon^\rho b_\rho)\psi = \varepsilon^\rho D_\rho\psi.\end{aligned}$$

We note that the modified diffeomorphism maps a scalar of weight -1 into a scalar of the same weight while Weyl covariance is broken by the standard diffeomorphism transformation. Let us also check the fact that standard diffeomorphisms do not commute with Weyl transformations while the modified ones do:

$$\begin{aligned}\delta_W\delta_D\psi - \delta_D\delta_W\psi &= \delta_W(\varepsilon^\rho\nabla_\rho\psi) - \delta_D(-\sigma\psi) = \varepsilon^\rho\nabla_\rho(-\sigma\psi) - (-\sigma\varepsilon^\rho\nabla_\rho\psi) = -\varepsilon^\rho\nabla_\rho\sigma\cdot\psi \\ \delta_W\tilde{\delta}_D\psi - \tilde{\delta}_D\delta_W\psi &= \delta_W(\varepsilon^\rho D_\rho\psi) - \tilde{\delta}_D(-\sigma\psi) = -\sigma\varepsilon^\rho D_\rho\psi - (-\sigma\varepsilon^\rho D_\rho\psi) = 0.\end{aligned}$$

E.1.1 Diffeomorphism infinitesimal transformation

Standard diffeomorphisms*

For the metric we have:

$$\delta_D g_{\mu\nu} = h_{\mu\nu} = \mathcal{L}_\varepsilon g_{\mu\nu} = \varepsilon^\rho\partial_\rho g_{\mu\nu} + g_{\rho\mu}\partial_\nu\varepsilon^\rho + g_{\rho\nu}\partial_\mu\varepsilon^\rho = \nabla_\mu\varepsilon_\nu + \nabla_\nu\varepsilon_\mu \tag{E.2}$$

where ∇ is the covariant derivative of the Christoffell symbols.

For a vector we have:

$$\delta_D b^\mu = \mathcal{L}_\varepsilon b^\mu = \mathcal{L}_\varepsilon b^\mu = [\varepsilon, b] = \varepsilon^\rho\partial_\rho b^\mu - b^\rho\partial_\rho\varepsilon^\mu = \varepsilon^\rho\nabla_\rho b^\mu - b^\rho\nabla_\rho\varepsilon^\mu. \tag{E.3}$$

And

$$\delta_D b_\mu = \delta_D g_{\mu\rho}\cdot b^\rho + g_{\mu\rho}\cdot\delta_D b^\rho = (\nabla_\mu\varepsilon_\rho + \nabla_\rho\varepsilon_\mu)b^\rho + g_{\mu\rho}\left(\varepsilon^\alpha\nabla_\alpha b^\mu - b^\alpha\nabla_\alpha\varepsilon^\mu\right). \tag{E.4}$$

Finally for a scalar we have:

$$\delta_D\chi = \mathcal{L}_\varepsilon\chi = \varepsilon^\rho\partial_\rho\chi. \tag{E.5}$$

If b is tought of as a 1-form we use the following relation:

$$\delta_D b = \mathcal{L}_\varepsilon b = i_\varepsilon db + d\left(i_\varepsilon b\right) = \varepsilon^\rho B_{\rho\mu} + \partial_\mu\left(\varepsilon^\rho b_\rho\right) \tag{E.6}$$

where $B_{\mu\nu} \equiv \partial_\mu b_\nu - \partial_\nu b_\mu$

Modified diffeomorphisms*

In this subsection we list the modified diffemorphisms of scalar, vectors and the metric. We recall that a modified diffeomorphism $\tilde{W}_D(\varepsilon) = W_D(\varepsilon) + W_W(-\varepsilon \cdot b)$. We have:

$$\begin{aligned}
\tilde{\delta}_D\chi &= \varepsilon^\rho D_\rho \chi \\
\tilde{\delta}_D v^\mu &= \varepsilon^\rho D_\rho v^\mu - v^\rho D_\rho \varepsilon^\mu \\
\tilde{\delta}_D g_{\mu\nu} &= D_\mu \varepsilon_\nu + D_\nu \varepsilon_\mu \\
\tilde{\delta}_D b_\mu &= \varepsilon^\rho B_{\rho\mu}.
\end{aligned}$$

As one can note all the transformation are now covariant with respect to Weyl transformations.

E.2 Weyl covariant derivatives acting on $h_{\mu\nu}$: a list

We report here the explicit expression of some terms with Weyl covariant derivatives and curvatures that enter in the second variation of the action.

$$\begin{aligned}
h^{\mu\nu} D^2 h_{\mu\nu} &= h^{\mu\nu}\nabla^2 h_{\mu\nu} + 4h^{\mu\nu} b_\mu \nabla^\rho h_{\rho\nu} - 4h^{\mu\nu} b^\rho \nabla_\mu h_{\rho\nu} - 2h^{\mu\nu} b^\rho \nabla_\rho h_{\mu\nu} - 8h^{\mu\nu} b_\mu b^\rho h_{\rho\nu} - 2h^{\mu\nu} h_{\mu\nu} b^2 \\
&+ 4hh^{\mu\nu} b_\mu b_\nu \\
h_{\mu\nu} D^\mu D_\rho h^{\rho\nu} &= h_{\mu\nu}\Big[\nabla^\mu \nabla_\rho h^{\rho\nu} - 4\nabla^\mu b_\rho \cdot h^{\rho\nu} - 4 b_\rho \nabla^\mu h^{\rho\nu} + \nabla^\mu b^\nu \cdot h + b^\nu \nabla^\mu h - g^{\mu\nu} b_\alpha \nabla_\beta h^{\alpha\beta} \\
&+ \left(-g^{\mu\nu} b^2 + 2b^\mu b^\nu\right) h + 4 g^{\mu\nu} b_\alpha b_\beta h^{\alpha\beta} + 2b^\mu \nabla_\rho h^{\rho\nu} - 8 b^\mu b_\rho h^{\rho\nu}\Big] \\
(h D^\mu D^\nu h_{\mu\nu}) &= h\left[\nabla^\mu \nabla^\nu h_{\mu\nu} + \nabla^\mu b_\mu \cdot h + b^\mu \nabla_\mu h - 4\nabla^\mu b^\nu \cdot h_{\mu\nu} - 6 b^\mu \nabla^\nu h_{\mu\nu} - 2b^2 h + 8 b^\mu b^\nu h_{\mu\nu}\right] \\
h D^2 h &= h\left[\nabla^2 - 2b^\mu \nabla_\mu\right] h
\end{aligned} \tag{E.7}$$

and:

$$\begin{aligned}
h^{\mu\nu} h^\alpha_\mu \mathcal{R}_{\nu\alpha} &= h^{\mu\nu} h^\alpha_\mu \left\{R_{\nu\alpha} + 2\nabla_\nu b_\alpha + g_{\nu\alpha}\nabla^\sigma b_\sigma + 2 b_\nu b_\alpha - 2 g_{\nu\alpha} b^2\right\} \\
h^{\mu\nu} h^{\alpha\beta} \mathcal{R}_{\alpha\mu\beta\nu} &= h^{\mu\nu} h^{\rho\sigma}\left\{R_{\rho\mu\sigma\nu} + g_{\rho\sigma}(\nabla_\mu b_\nu + b_\mu b_\nu) - g_{\rho\nu}(\nabla_\mu b_\sigma + b_\mu b_\sigma) - g_{\mu\sigma}(\nabla_\rho b_\nu + b_\rho b_\nu)\right. \\
&+ \left. g_{\mu\nu}(\nabla_\rho b_\sigma + b_\rho b_\sigma) - (g_{\rho\sigma} g_{\mu\nu} - g_{\rho\nu} g_{\mu\sigma}) b^2\right\} \\
h h^{\mu\nu} \mathcal{R}_{\mu\nu} &= h h^{\mu\nu}\left\{R_{\mu\nu} + 2\nabla_\mu b_\nu + g_{\mu\nu}\nabla^\sigma b_\sigma + 2 b_\mu b_\nu - 2 g_{\mu\nu} b^2\right\}
\end{aligned} \tag{E.8}$$

Finally:

$$\begin{aligned}
-h_{\mu\nu} D^\mu D_\rho h^{\rho\nu} &= (-1) h_{\mu\nu}\Big[\nabla^\mu \nabla_\rho h^{\rho\nu} - 4\nabla^\mu b_\rho \cdot h^{\rho\nu} - 4 b_\rho \nabla^\mu h^{\rho\nu} + \nabla^\mu b^\nu \cdot h + b^\nu \nabla^\mu h - g^{\mu\nu} b_\alpha \nabla_\beta h^{\alpha\beta} \\
&+ \left(-g^{\mu\nu} b^2 + 2b^\mu b^\nu\right) h + 4 g^{\mu\nu} b_\alpha b_\beta h^{\alpha\beta} + 2b^\mu \nabla_\rho h^{\rho\nu} - 8 b^\mu b_\rho h^{\rho\nu}\Big] \\
\frac{1+\beta}{4}(h D^\mu D^\nu h_{\mu\nu}) &= \frac{1+\beta}{4} h\left[\nabla^\mu \nabla^\nu h_{\mu\nu} + \nabla^\mu b_\mu \cdot h + b^\mu \nabla_\mu h - 4\nabla^\mu b^\nu \cdot h_{\mu\nu} - 6 b^\mu \nabla^\nu h_{\mu\nu} - 2b^2 h + 8 b^\mu b^\nu h_{\mu\nu}\right] \\
\frac{1+\beta}{4}(h_{\mu\nu} D^\mu D^\nu h) &= \frac{1+\beta}{4}\left[h_{\mu\nu}\nabla^\mu \nabla^\nu h - h b^\mu \nabla_\mu h + h_{\mu\nu}\left(b^\mu \nabla^\nu + b^\nu \nabla^\mu\right) h\right] \\
-\frac{(1+\beta)^2}{16} h D^2 h &= -\frac{(1+\beta)^2}{16} h\left[\nabla^2 - 2b^\mu \nabla_\mu\right] h
\end{aligned} \tag{E.9}$$

E.3 Weyl covariant decomposition

Let us recall that when a vector A_μ in a functional integral is decomposed into its transverse and longitudinal parts $A^T_\mu + \nabla_\mu \varphi$, there arises a Jacobian [123]:

$$1 = \int \mathcal{D}A_\mu e^{-\frac{1}{2}\int dx \sqrt{g} A^\mu A_\mu} = J \int \mathcal{D}A^{(T)}_\mu \mathcal{D}\varphi e^{-\frac{1}{2}\int dx \sqrt{g} A^{(T)\mu} A^{(T)}_\mu + \varphi(-\nabla^2)\varphi} = J \int \mathcal{D}\varphi e^{-\frac{1}{2}\int dx \sqrt{g}\varphi(-\nabla^2)\varphi}. \tag{E.10}$$

The above gaussian normalized measure is not Weyl invariant. Since we want to use a Weyl invariant measure, the above steps are modified as follows. Let $A_\mu = A^{(T)}_\mu + D_\mu \varphi$ with $D^\mu A^{(T)}_\mu = 0$, then:

$$1 = \int \mathcal{D}A_\mu e^{-\frac{1}{2}\int dx \sqrt{g} \chi^2 A^\mu A_\mu} = J \int \mathcal{D}A^{(T)}_\mu \mathcal{D}\varphi e^{-\frac{1}{2}\int dx \sqrt{g} \chi^2 A^{(T)\mu} A^{(T)}_\mu + \varphi(-D^2)\varphi} = J \int \mathcal{D}\varphi e^{-\frac{1}{2}\int dx \sqrt{g}\chi^2\varphi(-D^2)\varphi}. \tag{E.11}$$

Note that the above derivation hold if the background is such that $D\chi = 0$. If this is not the case we have:

$$\begin{aligned} 1 &= \int \mathcal{D}A_\mu e^{-\frac{1}{2}\int dx \sqrt{g} \chi^2 A^\mu A_\mu} = J \int \mathcal{D}A^{(T)}_\mu \mathcal{D}\varphi e^{-\frac{1}{2}\int dx \sqrt{g}\chi^2 \left[A^{(T)\mu} A^{(T)}_\mu + \varphi(-D^2)\varphi - \frac{D^\mu \chi^2}{\chi^2} A^{(T)}_\mu \varphi - \varphi \frac{D^\mu \chi^2}{\chi^2} D^\mu \varphi\right]} \\ J^{-1} &= \int \mathcal{D}A^{(T)}_\mu \mathcal{D}\varphi e^{-\frac{1}{2}\int \sqrt{g}\psi M \psi} \end{aligned} \tag{E.12}$$

where $\psi = (A^{(T)}_\mu, \varphi)$ and M is the following matrix:

$$M = \frac{\chi^2}{2} \begin{pmatrix} g^\mu_\nu & -\frac{D_\nu \chi^2}{\chi^2} \\ -\frac{D^\mu \chi^2}{\chi^2} & -D^2 - \frac{D^\rho \chi^2}{\chi^2} D_\rho \end{pmatrix} \tag{E.13}$$

Since the above field are bosonic we need to evaluate $\det(M)$ which can be done introducing two auxiliary (grassmaniann odd) fields (ξ_μ, τ). In order to be able to exponentiate in the action this determinant we also perform the redefinition $\xi_\mu \to \frac{\sqrt{-D^2}}{\chi}\xi_\mu$. This further redefinition gives a jacobian which also has to be taken into account via another auxiliary field (v_μ).

E.4 Rule for integration by parts

We discuss here the integration by parts with the Weyl-covariant derivative (3.3). As an illustration it will be sufficient to consider an integral of the form

$$\int d^4x \sqrt{g}\, A D_\mu B^\mu \ , \tag{E.14}$$

where A is a scalar and B is a vector. The case when A and B have additional contracted indices works in the same way. The important assumption that we have to make is that the integral is not only invariant under diffeomorphisms, as is already clear, but also under Weyl

transformations. This is equivalent to saying that it is dimensionless. Not all integrals need to be dimensionless, but this is the case for the action, and this is the only case we are interested in chapter 3. Since $\sqrt{g}$ has weight 4, if A has weight w_A, B must have $-4-w_A$. Assuming that surface terms can be discarded we then find

$$\begin{aligned}\int d^4x\sqrt{g}\, AD_\mu B^\mu &= \int d^4x\sqrt{g}\, A(\partial_\mu B^\mu + \hat{\Gamma}_\mu{}^\mu{}_\rho B^\rho + (4+w_A)b_\rho B^\rho) \\ &= \int d^4x\sqrt{g}\left[-B^\mu\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}A) + A\Gamma_\mu{}^\mu{}_\rho B^\rho + w_A A b_\rho B^\rho)\right] \\ &= \int d^4x\sqrt{g}\,[-B^\mu\partial_\mu A + w_A A b_\rho B^\rho] \\ &= -\int d^4x\sqrt{g}B^\mu D_\mu A\ . \end{aligned} \tag{E.15}$$

We see that Weyl covariant derivatives can be integrated by parts provided the integral is dimensionless.

E.5 Some results on the heat kernel

In section 3.5 we have calculated the beta functions using simple backgrounds that are just sufficient to make the relevant invariants nonzero, for example a spherical metric with $b_\mu = 0$ and χ constant. Gauge invariance was taken from the general construction of the RG flow and was not checked explicitly. Here we point out that gauge invariance follows from properties of the heat kernel of $-D^2$. More precisely we have

$$b_0 = \frac{1}{16\pi^2}\mathrm{tr}1\ ; \qquad b_2 = \frac{1}{16\pi^2}\frac{1}{6}\mathrm{tr}\mathcal{R}\mathbf{1}\ . \tag{E.16}$$

Since the connection $\hat{\Gamma}$ is non-metric we cannot apply directly known results, so we express $-D^2$ in terms of ∇, the Levi-Civita connection, and b_μ. For a scalar we can use the known property that for an operator $\Delta = -\nabla^2 + A^\mu\nabla_\mu + X$ the coefficients b_2 reads [156]:

$$b_2(\Delta) = \frac{1}{(4\pi)^{d/2}}\left[\frac{R}{6} - X + \frac{1}{2}\nabla_\mu A^\mu - \frac{1}{4}A_\mu A^\mu\right]$$

For a scalar of weight -1:

$$-D^2 \;=\; -\nabla^2\varphi + b^\mu b_\mu\varphi - \nabla^\mu b_\mu\cdot\varphi. \tag{E.17}$$

Inserting in the above equation one obtains (E.16). For the graviton the situation is more complicated since $-D^2$ contains terms which are of the form $s^\mu\nabla_\alpha h_{\mu\beta}$. To overcome this problem we expand the non-minimal terms in $e^{-s(-D^2)}$ and employ the off-diagonal HK coefficients [55, 157]. In this way one arrives again at (E.16).

E.6 Alternative definition of the essential couplings

In section 3.5.4 the redefinition

$$g_1 + 12g_4 = Z_b u^2 \; ; \qquad g_2 = \kappa_2 u^4 \; ; \qquad g_3 = Z_b \kappa_3 \; ; \qquad g_4 = \kappa_4 u^2 \; , \tag{E.18}$$

provides an alternative division of the couplings into inessential and essential ones. In this parametrization the beta functions are:

$$\begin{aligned}
\beta_{\kappa_2} &= \frac{8\kappa_2 + 2\kappa_4 + 5(\kappa_2+\kappa_4)\kappa_3}{32\pi^2(1+\kappa_3)(\kappa_4-\kappa_2)} - 4\kappa_2 \\
\beta_{\kappa_3} &= -\frac{\kappa_3\left(3\kappa_4 + 4\kappa_3\kappa_4 + \kappa_3^2(2\kappa_2-\kappa_4)\right)}{16\pi^2(1+\kappa_3)^2(\kappa_4-\kappa_2)^2} + \eta_b\kappa_3 \\
\beta_{\kappa_4} &= \frac{8(7\kappa_2^2 - 9\kappa_2\kappa_4 + 11\kappa_4^2) + 2\kappa_3(53\kappa_2^2 - 66\kappa_2\kappa_4 + 85\kappa_4^2) + \kappa_3^2(59\kappa_2^2 - 78\kappa_2\kappa_4 + 91\kappa_4^2)}{384\pi^2(1+\kappa_3)^2(\kappa_4-\kappa_2)^2} - 2\kappa_4
\end{aligned} \tag{E.19}$$

and the anomalous dimension is

$$\eta_b = -\frac{\beta_Z}{Z} = 2 - \frac{2\kappa_2 + 5\kappa_4 - 5\kappa_3(\kappa_2 - 2\kappa_4) + 2\kappa_3^2(\kappa_2 - 2\kappa_4)}{48\pi^2(1+\kappa_3)^2(\kappa_4-\kappa_2)^2} \; . \tag{E.20}$$

The system of three equations $\beta_{\kappa_i} = 0$ admits four real fixed points, one of which occurs at $\kappa_3 = \infty$ or $e^2 = 0$:

	η_{b*}	κ_{2*}	e^2_*	κ_{4*}	$\tilde{\Lambda}_*$	$\tilde{G}_*$
FP_1	2.7272	0.008585	0.0000	0.02327	0.1237	1.1338

	η_{b*}	κ_{2*}	κ_{3*}	κ_{4*}	$\tilde{\Lambda}_*$	$\tilde{G}_*$
FP_2	0.8113	0.004118	0.0000	0.01698	0.1213	1.1718
FP_3	1.2705	0.006282	0.7774	0.01987	0.1581	1.0013
FP_4	1.7608	0.000150	−0.3914	0.01277	0.0059	1.5573

The eigenvalues read:

	λ_1	λ_2	λ_3
FP_1	−2.92208	$-2.28003 + 1.96824i$	$-2.28003 - 1.96824i$
FP_2	$-1.86782 + 1.39828i$	$-1.86782 - 1.39828i$	−1.1391
FP_3	$-2.02559 + 1.87941i$	$-2.02559 - 1.87941i$	0.923836
FP_4	−3.13639	−1.40315	3.36778

At FP_1 the complex eigenvectors are mixture of κ_2, κ_4 and the real, least relevant, eigenvalue is almost entirely κ_3.

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