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On "special" embeddings in complex and projective algebraic geometry

Thesis submitted for the degree of Doctor Philosophiae

Supervisor: *Ch.mo Prof. Ph. Ellia*

Candidate: Alessandro Arsie

ACADEMIC YEAR: 2000/2001

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To my Parents, for their Love, Patience and Support

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Contents

Chapter 1

Introduction

This thesis deals with some aspects of the geometry of "special" embeddings in Calabi-Yau manifolds and in projective spaces and it focuses on the main original results I have obtained during the last four years.

By special embeddings we just mean embeddings with some additional properties/structures or data. Throughout the thesis, we will see very different structures associated to embeddings: mainly Lagrangian embeddings in Calabi-Yau manifolds, small codimension embeddings and linearly normal embeddings in projective spaces. In all these cases, we investigate some consequences due to these extra structures and we try to get insight on how these additional requirements/data force the geometry of the subvariety, in each case.

Besides this introduction, the thesis is devided into four chapters, whose content we are going to briefly describe. Each chapter contains original results, which have been already published in the scientific literature or appear in preprint form.

In the next chapter, we describe Lagrangian embeddings (and also special Lagrangian embeddings) in Calabi-Yau manifolds. We give a consistent definition of the Maslov class for such submanifolds and we prove that it is possible the represent such a cohomological class via the contraction of the mean curvature vector field with the symplectic form. Essentially, this representation theorem extends a previous result (due indipendently to Morvan and Fomenko), which shows that for a Lagrangian embedding in a Euclidean symplectic vector space, the Maslov class can be represented as the contraction of the mean curvature vector field with the symplectic form. The main problem here comes from the fact that, in general, we can not use in this environment (Calabi-Yau manifolds) a trivial Levi-Civita connection as in the case of a Euclidean symplectic vector space. However, we succeed in proving our result, exploiting the fact that the holonomy of the Levi-Civita connection associated to a Calabi-Yau metric is sufficiently small. Finally, we comment on some properties of special Lagrangian embeddings in irreducible symplectic manifolds. The original results of this chapter have been published in ([58]).

In the third chapter, we leave the environment of Calabi-Yau manifolds and we focus our attention on projective embeddings. Here we deals with small codimension embeddings. We review the famous Hartshorne's conjecture and also some outstanding (but unfortunately partial) results concerning this field, just to put our result in a wider framework. The general philosophy operating behind Hartshorne's conjecture is that smooth subvarieties of small codimension (compared to their dimension) in a projective space, have to be complete intersections. This conjecture is extremely difficult and unfortunately is far from being solved (either positively or negatively).

We also review the construction of the Serre correspondence which have been essential for our work on codimension two subvarieties: this correspondence relates a subcanonical local complete intersection subvariety of codimension 2 in $\mathbb{P}^n$ to an algebraic vector bundle of rank 2 over $\mathbb{P}^n$, as soon as $n \geq 3$. Our contribution in this field concerns a scheme-theoretic criterion for recognizing complete intersections in codimension two. Briefly put, it is known that any locally complete intersection (l.c.i.) subvariety in $\mathbb{P}^n$ can be scheme-theoretically defined by $n+1$ equations. We prove that any codimension two subcanonical l.c.i subvariety in $\mathbb{P}^n$, scheme-theoretically defined by $p \leq n-1$ equations, is indeed a complete intersections (the proof works for varieties defined on an algebraically closed field of any characteristic). This result extends two previous theorems of Faltings (see [26]) and of Netsevetaev (see [34]).

As an application we answer (partially) to a question proposed recently by Franco, Kleiman and Lascu in [27]. Finally, we give an elementary proof of a weaker criterion on codimension two complete intersection subvarieties. These results are going to appear in [59].

The fourth chapter deals with linearly normal projective embeddings of a peculiar class of singular surfaces. These surfaces represent boundary points in the Alexeev's compactification of $\mathcal{A}_g$ (for $g = 2$); the Alexeev's contruction is particularly meaningful in that it selects among all possible compactifications of $\mathcal{A}_g$ the one which has good functorial properties for moduli problems. Thus these singular surfaces are degenerations of principally polarized abelian surfaces and are all equipped with an ample line bundle $\mathcal{L}$. Alexeev and Nakamura proved that for such surfaces $\mathcal{L}^{\otimes 5}$ is very ample and this corresponds to embed such surfaces in $\mathbb{P}^{24}$. (In fact the results of Alexeev and Nakamura are much wider in the sense that they hold for all degenerations of principally polarized abelian varieties, not just for surfaces, corresponding to the boundary points of their compactification of $\mathcal{A}_g$). Here we prove by elementary methods that $\mathcal{L}^{\otimes 3}$ is already very ample. In general, it is a difficult task to prove directly the very ampleness of a complete linear system, since it requires a deep knowledge of the intrinsic geometry of the variety we are going to embed: indeed, our proof is based on studying very concretely the corresponding linear systems on the various degeneration models. This permits to embed these surfaces in $\mathbb{P}^8$ and to study "by hand" the corresponding singular loci, since we have a clear description of the intrinsic geometry of these degenerations. Unfortunately, this underlines the fact that it is hopeless to solve (either positively or negatively) Hartshorne's conjecture by studying explicitly very ample complete linear systems of small dimension on 3-folds or 4-folds. The results of this chapter appear in a preprint which have been submitted to "Annali della Scuola Normale Superiore di Pisa" (see ([60])).

Finally, the last chapter is devoted to a study of some deformation properties of projective curves: for any integral smooth curve $C \subset \mathbb{P}^r$, we study the functorial map (not everywhere defined) $\phi : Hilb(d,g,r) \dashrightarrow \mathcal{M}_g$, which associates to each point $p(C)$ in $Hilb(d,g,r)$, representing the curve C, the corresponding isomorphism class $[C] \in \mathcal{M}_g$. In particular, we study in which cases the image of ϕ has positive dimension (i.e. the corresponding family is not isotrivial). In this study, a key role is played by linearly normal curves, since they tend to be less rigid than other classes of embedded curves, and via them, we can prove the non-isotriviality also for other curves, obtained by projecting down a given linearly normal curve. First of all, we focus on first order deformations, and then, under some additional assumptions we get some positive conclusions for finite deformations. The last paragraph deals with this kind of problems for some special classes of curves in $\mathbb{P}^3$ (e.g. curves of maximal rank), exploiting the minimal free resolution of their ideal sheaf. So, in this chapter we see how an embedding with some additional properties (linear normality, in this specific case) can help to study a seemingly totally unrelated problem. The results of this chapter appears in a preprint which has been developed in collaboration with Stefano Brangani (see ([61])).

Chapter 2

Lagrangian embeddings in Calabi-Yau manifolds

2.1 Introduction

In this chapter, we study some aspects of Lagrangian embeddings (and to some extent also of special Lagrangian embeddings) in a particular class of complex symplectic manifolds, called Calabi-Yau manifolds. We focus our attention to the proper definition of a topological invariant of these embeddings called Maslov class. Moreover, we prove that it is possible to define this class via the mean curvature vector field, analogously to what is known for Lagrangian embeddings in Euclidean vector spaces. Finally, at the end of the chapter, we give a brief description of Special Lagrangian embeddings and how they relate to complex submanifolds under some specific circumstances, following Bruzzo and his collaborators.

The Maslov class $[\mu_\Lambda]$ of a Lagrangian embedding $j : \Lambda \hookrightarrow V$ in the standard Euclidean symplectic vector space V has been constructed by Maslov in the study of global patching problem for asymptotic solutions of some PDEs (see [55] for further details on this point of view). Subsequently, this cohomological class has found applications in the analysis of several quantization procedure, starting from [1] up to recent aspects on its relations with asymptotic, semiclassical and geometric quantization, for which we refer to [13]. In spite of this, there are several problems in the very definition of the Maslov class for Lagrangian submanifolds of generic symplectic manifolds.

In [16] it has been proved that, for a Lagrangian embedding $j : \Lambda \hookrightarrow V$ in a Euclidean symplectic vector space (V, ω), the Maslov form μ_Λ can be represented by $\mu_\Lambda = i_H\omega$, that is by the contraction of the symplectic form with the mean curvature vector field H of the embedding j. Unfortunately, the very definition of Maslov form (and related class) as exposed in [1], [2] and [55], depends on the fact that the Lagrangian submanifold Λ is embedded in a symplectic vector space, in which we have chosen a projection $\pi : V \to \Lambda_0$ over a *fixed* Lagrangian subspace Λ_0; then the Maslov class $[\mu_\Lambda] \in H^1(\Lambda, \mathbb{R})$ can be defined as the Poincare' dual to the singular locus $Z(\Lambda) \hookrightarrow \Lambda$, where $Z(\Lambda) := \{\lambda \in \Lambda | \mathrm{rk}(\pi_*(\lambda)) < \max\} \cap H_{n-1}(\Lambda, \mathbb{Z})$. In the classical literature it is proved that if one changes projection π, that is if one changes the reference Lagrangian subspace Λ_0, then the *Maslov class* μ_Λ does not change, while its representative changes. This is achieved using the so called universal Maslov class construction on the Lagrangian Grassmannian $GrL(V)$, (the homogeneous space which parametrizes Lagrangian subspaces of (V, ω), see [1], [2] and [13]). These formulations depend heavily on the linear structure of the ambient manifold V; in particular it is assumed that V is endowed with the trivial connection. Therefore, it seems difficult even to define the Maslov class for Lagrangian submanifolds of symplectic manifolds, which are not vector spaces. For instance, it is possible to define the Maslov class of a Lagrangian embedding via the so called generating functions, or their generalization (Morse families), for which we refer to [55], and particularly [57]. In this way, one obtains a notion of Maslov class for Lagrangian submanifolds embedded in *any* cotangent bundle T^*M over a Riemannian manifold M, constructing a $\mathbb{Z}$-valued Čech cocycle, starting from the signature of the Hessian of a Morse family; however this construction depends strongly on the choice of a "base manifold" (M in the case of the cotangent bundle) and does not seem to be generalizable to Lagrangian embedding in any symplectic manifold. (See [57] for more details on this kind of construction).

Recently (see [26]), Fukaya has shown how to define a Maslov index for closed loops on Lagrangian submanifolds of a quite general class of symplectic manifolds, the so called pseudo-Einstein symplectic manifolds. The construction is developed using non trivial assumptions on the structure of the ambient manifold and is carried on only for a particular subclass of Lagrangian submanifolds; moreover, there is no explicit reference to the corresponding Maslov class.

In this chapter we show that, whenever the ambient manifold is Calabi-Yau, it is possible to give a consistent definition of Maslov class for its Lagrangian submanifolds, generalizing the approach of Arnol'd with the so called universal Maslov class. In this framework, we show that it is possible to generalize the result of Morvan and then we comment on various consequences of our construction, in particular on the possible definition of Maslov class for Lagrangian embedding in any symplectic manifold.

2.2 The Maslov class for Lagrangian embedding in Calabi-Yau

Let us briefly recall the standard construction of the Maslov class μ_Λ, for a Lagrangian submanifold Λ, embedded in a symplectic vector space (V, ω), of real dimension $2n$: first of all, one considers the tangent spaces to Λ as (affine) subspaces of V. Then, using the trivial parallel displacement one transports every tangent plane in a fixed point P of V, (for example the origin). Now, one has to consider the Lagrangian Grassmannian $GrL(T_P V)$, which by definition parametrizes all Lagrangian subspaces of $T_P V$. Using the trivial connection, we have thus obtained a map:

$$G : \Lambda \longrightarrow GrL(T_P V).$$

It is easy to see ([1], [2]),that $GrL(T_P V)$ has the natural structure of the homogeneous space $\dfrac{U(n)}{O(n)}$; then by the standard tool of the exact homotopy sequence for a fibration (see[7]), it is proved that $\pi_1(GrL(T_P V)) \cong \mathbb{Z}$. In fact, having fixed a Lagrangian plane Λ_0 in $T_P V$, all other Lagrangian planes are obtained via a unitary automorphism $A \in U(n)$. Obviously, we have a fibration:

$$SU(n) \longrightarrow U(n) \xrightarrow{det} S^1,$$

but this does not descend to $GrL(T_P S)$, since we have to quotient out the possible orthogonal automorphisms. However, since the square of the determinant of an orthogonal automorphism is always 1, we have a well defined map:

$$det^2 : GrL(T_P S) \longrightarrow S^1,$$

which sits in the following commutative diagram of fibrations:

$$\begin{array}{ccccc} SO(n) & \longrightarrow & O(n) & \xrightarrow{det} & S^0 \\ \downarrow & & \downarrow & & \downarrow \\ SU(n) & \longrightarrow & U(n) & \xrightarrow{det} & S^1 \\ \downarrow & & \downarrow & & \downarrow \; z^2 \\ GrSL(\mathbb{C}^n) & \longrightarrow & GrL(\mathbb{C}^n) & \xrightarrow{det^2} & S^1 \end{array}$$

In this diagram the space $GrSL(\mathbb{C}^n)$ denotes the Grassmannian of special Lagrangian planes in $\mathbb{C}^n$, that is the Grassmannian of Lagrangian planes which are *calibrated* by the top holomorphic form of $\mathbb{C}^n$; the corresponding Lagrangian submanifolds are called special

Lagrangian (see [11] for more details or the last section of this chapter). Notice that this space is always simply connected.

Finally, using Hurewicz isomorphism and taking a generator belonging to $H^1(GrL(T_PV),\mathbb{Z})$, which is thought as the pull-back via det^2 of the generator $[\alpha] \in H^1(S^1,\mathbb{Z})$, one defines the Maslov class $[\mu_\Lambda] := G^*(det^2)^*[\alpha]$. Obviously, this construction is indipendent on the choice of the point P, since if another point is chosen it is possible to construct a homotopy in such a way to prove the invariance of $[\mu_\Lambda]$. It is clear that, in this framework, the existence of the trivial connection is an (almost!) essential requirement for the construction to work. In fact, we will see in this section that, to have a consistent definition of Maslov class it is not necessary that the ambient manifold is endowed with the trivial connection, but is sufficient that the global holonomy of the symplectic manifold is "small" in a suitable sense.

From now on we restrict our attention to Lagrangian submanifolds of Calabi-Yau manifolds. Recall that Calabi-Yau manifolds can be defined as compact Kaehler manifolds with vanishing first Chern class; recall also that a celebrated theorem by Yau (proving a previous conjecture by Calabi) implies that for every choice of the Kaehler class on a Calabi-Yau, there exists a unique Ricci-flat Kaehler metric. Moreover, while the holonomy of a Kaehler manifold is contained in U(n), if g is the Ricci-flat metric of an n-dimensional Calabi-Yau, then the corresponding holonomy group is contained in SU(n). Finally, let us recall that, on every Kaehler manifold (X,g,J) (where g is a Kaehler metric and J the integrable almost complex structure) the corresponding symplect or Kaehler form ω is related to g via:

$$\omega(X,Y) := g(X,JY) \quad \forall X,Y \in \Gamma(TX), \tag{2.1}$$

and that the almost complex structure tensor J is covariantly constant with respect to the Levi-Civita connection induced by g. Considering a Kaehler metric g on a Calabi-Yau, we will always mean the Ricci-flat metric. Typical examples of Calabi-Yau are given by the zero locus of a homogeneous polynomial of degree $n+1$ in $\mathbb{P}^n(\mathbb{C})$ (whenever this locus is smooth); however it is by no means true that all Calabi-Yau are algebraic. For further details on this class of manifolds see for example [5] and [20].

The construction of Fukaya for defining the Maslov index of closed loops goes as follow (see [26] for details and motivations). He considers symplectic manifolds (X,ω) which are "pseudo-Einstein" in the sense that there exists an integer N such that $N\omega = c_1(X)$. By this relation, the line bundle $det(TX)$ is flat when restricted to every Lagrangian submanifold Λ of X, but Fukaya restricts further the class of Lagrangian submanifolds considering only the so called Bohr-Sommerfeld orbit Λ (BS-orbit for short), which are defined as the Lagrangian submanifolds for which the restriction of $det(TX)$ is not only flat, but even trivial. This implies that if we consider a closed loop $h : S^1 \longrightarrow \Lambda$ (Λ is a

BS-orbit), then the monodromy M of the tangent bundle TX along $h(S^1)$ is contained in $SU(n)$. Then the idea is to take a path in $SU(n)$ joining M with the identity, in order to get an induced trivialization of $h^*(TX_{|h(S^1)}) \cong S^1 \times \mathbb{C}^n$. In this trivial bundle there is a family of Lagrangian vector subspaces $T_{h(t)}\Lambda$ and in this way we get a loop in $GrL(\mathbb{C}^n)$ and hence a well-defined integer (the Maslov index) $m(h)$. Obviously $m(h)$ is independent of the choice of the path in $SU(n)$ which joins M to the unit, since $\pi_1(SU(n)) \cong 1$.

Now we come to our construction. Consider embedded Lagrangian submanifolds Λ of a Calabi-Yau (X, ω, g, J), where ω, g, J are related by (2.1). Define the *Lagrangian Grassmannization* $GrL(X)$ of TX as the fibre bundle over X obtained substituting T_xX with $GrL(T_xX)$, thus:

$$GrL(X) := \coprod_{x \in X} GrL(T_xX)$$

and in particular:

$$GrL(X)_\Lambda := \coprod_{x \in \Lambda} GrL(T_xX).$$

Let $G(j)$ be the Gauss map, which takes $x \in \Lambda$ in $T_x\Lambda$ thought as a Lagrangian subspace of T_xX. Via $G(j)$, the embedding $j : \Lambda \hookrightarrow X$ lifts to a section $G(j) : \Lambda \to GrL(X)_\Lambda$. We would like to define the Maslov class of Λ via a map $\mathcal{M} : \Lambda \longrightarrow S^1$ in the following way: to every point $x \in \Lambda$, we consider $G(j)(x)$ and then through the isomorphism $GrL(T_xX) \cong \dfrac{U(n)}{O(n)}$, taking the map det^2 we get a point in S^1. However, as we have seen, to establish an isomorphism to every space $GrL(T_xX)$ $(x \in \Lambda)$ with $\dfrac{U(n)}{O(n)}$ we need a reference Lagrangian plane in $GrL(T_xX)$ $\forall x \in \Lambda$, that is we need *another section* of $GrL(X)_\Lambda$, besides $G(j)(\Lambda)$.

To this aim, fix a point $p \in \Lambda$, consider $T_p\Lambda$ and use the parallel displacement, induced by the Levi Civita connection of g, along a system γ of paths on Λ starting from p, to construct a reference distribution of Lagrangian planes $\mathcal{D}_\gamma$ over Λ, that is another section of $GrL(X)_\Lambda$. This is indeed possible, since the holonomy is contained in $U(n)$, the parallel displacement is an isometry for g and J is covariantly constant: these facts, combined with the relation (2.1) imply that parallel transport sends Lagrangian planes in Lagrangian planes. Obviously this distribution $\mathcal{D}_\gamma$ is not uniquely determined, since it depends on the choice of the system of paths γ starting from p. In spite of this, due to the fact that the holonomy of a Calabi-Yau metric is very constrained, this dependence does not prevent us to reach our goal. Indeed, consider $q \in \Lambda$ and compare the two Lagrangian planes $(\mathcal{D}_\gamma)_q$ and $(\mathcal{D}_\delta)_q$ obtained by parallel transport of $T_p\Lambda$ along two different paths γ and δ.

By the holonomy property of a Calabi-Yau metric we have:

$$(\mathcal{D}_\gamma)_q = M(\mathcal{D}_\delta)_q \quad M \in SU(n).$$

Thus, if $A \in U(n)$ is such that $T_q\Lambda = A(\mathcal{D}_\gamma)_q$, then $T_q\Lambda = AM(\mathcal{D}_\delta)_q$; so to every $q \in \Lambda$ we can associate A_q such that $G(j)(q) = T_q\Lambda = A_q(\mathcal{D}_\gamma)_q$, where A_q is determined up to multiplication by a matrix $M \in SU(n)$. At this point the key observation is that $det^2(A_q) \in S^1$ is a well defined point, which is not affected by the ambiguity of A_q. In this way we have a well-defined map, the *Maslov map*:

$$\begin{array}{rccc} \mathcal{M}: & \Lambda & \longrightarrow & S^1 \\ & q & \mapsto & det^2(A_q) \end{array}$$

Take the generator $[\alpha]$ of $H^1(S^1, \mathbb{Z})$ represented by the form $\alpha := \frac{1}{2\pi}d\theta$. Observe that the target space of the Maslov map, is not only topologically a circle, but even a Lie group, the group $U(1)$: this implies that the choice of the form $\frac{1}{2\pi}d\theta$ is compulsory, since it is the unique normalized invariant 1-form. Now we can give the following:

Definition: Using the previous notations, we define the *Maslov form* of the Lagrangian embedding $j : \Lambda \hookrightarrow X$ as $\mu_\Lambda := \mathcal{M}^*\alpha$ and the corresponding *Maslov class* as $[\mu_\Lambda] = \mathcal{M}^*[\alpha] \in H^1(\Lambda, \mathbb{Z})$.

Remark 1 : The Maslov map $\mathcal{M}$ has been built up fixing a reference point p, from which we constructed $\mathcal{D}_\gamma$; in this way the map $\mathcal{M}$ associates $1 \in S^1$ to p. It is clear that if one takes a different reference point p', then the map $\mathcal{M}$ changes (this time p' goes to 1), but the Maslov class and the Maslov form do not change, as it is immediate to see. In particular, the invariance of the Maslov form is due to the invariance of α under the action of the Lie group $U(1)$.

Remark 2 : In [19], Trofimov costructed a generalized Maslov class, as a cohomological class defined on the space of paths $[X, \Lambda]$; these paths start from a fix point x_0 in a symplectic manifold X and end to a fixed Lagrangian submanifold Λ of X. We argue that the the Maslov class we have just defined can be obtained as a finite dimensional reduction of the class built up in [19], when one uses the Levi-Civita connection induced by the Calabi-Yau metric. In fact, Trofimov did not use metric connections, but instead affine torsion free connections, preserving the symplectic structure, which are generally not induced by a metric.

2.3 Representation of the Maslov class via the mean curvature vector field

In this section, generalizing what has been proved by Morvan in [16] for Lagrangian embeddings in Euclidean symplectic vector space, we prove the following:

Theorem: *Let $j : \Lambda \hookrightarrow X$ be a Lagrangian embedding in a Calabi-Yau X and let $H \in \Gamma(N\Lambda)$ be the mean curvature vector field of the embedding j (with repect to the Calabi-Yau metric), then:*

$$\mu_\Lambda = \frac{1}{\pi} i_H \omega,$$

where ω is the Kaehler form constructed from the Calabi-Yau metric g, and *μ_Λ is the Maslov form previously defined.*

Before proving the theorem we need various preliminary results, which we are going to state and prove, and we need also to decompose into simpler pieces the action of $\mathcal{M}^*$ on $[\alpha]$.

Recall that given an embedding j, the associated second fundamental form $\sigma : T\Lambda \times T\Lambda \to N\Lambda$ is a symmetric tensor defined by:

$$\sigma(X,Y) := \nabla^g_X Y - \nabla^{j^*g}_X Y, \quad \forall X, Y \in \Gamma(T\Lambda),$$

where ∇^g is the Levi-Civita connection in the ambient manifold, while ∇^{j^*g} is the connection induced on Λ via the pulled-back metric. If σ is identically vanishing, then the submanifold is called totally geodesic. Taking the trace of σ we get a field of normal vectors, that is the *mean curvature vector field* H of the embedding j. Those embeddings for which H is identically vanishing are called minimal.

First of all we need to understand the local structure of $TGrL(T_xX)$. Fix a point $q \in \Lambda$ and set $V := T_qX$ for short. We can prove the following:

Lemma 1: *The space $T_\pi GrL(V)$ over a Lagrangian n-plane π of V can be identified with the subspace of linear maps $\psi : \pi \to \pi^\perp$ ($\pi^\perp$ denotes the orthogonal subspace in V with respect to the metric g in q) such that:*

$$g(\psi(X), JY) = g(\psi(Y), JX), \quad \forall X, Y \in \pi.$$

Proof: First of all, we have $T_\pi GrL(V) \equiv S(\pi)$, where $S(\pi)$ is the space of all symmetric bilinear forms on π. In fact every $v \in T_\pi GrL(V)$ can be represented as $\frac{d}{dt}B(t)\pi_{|t=0}$, where $B(t)$ is a path of linear symplectic transformation of V, with the condition $B(0) = id_V$. To $v \in T_\pi GrL(V)$ we can associate a form S_v given by:

$$S_v(X,Y) := \omega(\frac{d}{dt}B(t)X_{|t=0}, Y).$$

This form is clearly bilinear and is symmetric:

$$S_v(X,Y) = \omega(\frac{d}{dt}B(t)X_{|t=0}, B(t)Y_{|t=0}) =$$

$$= \frac{d}{dt}\omega(B(t)X, B(t)Y)_{|t=0} - \omega(B(t)X_{|t=0}, \frac{d}{dt}B(t)Y_{|t=0}) = 0 - \omega(X, \frac{d}{dt}B(t)Y_{|t=0}) =$$

$$= \omega(\frac{d}{dt}B(t)Y_{|t=0}, X) = S_v(Y, X),$$

by the fact that $B(t)$ is a symplectic linear transformation of V and by skewsymmetry of ω. It is easy to verify that the corresponding map $T_\pi GrL(V) \to S(\pi)$ is an isomorphism. Moreover we have:

$$S_v(X, Y) = \omega(\frac{d}{dt}B(t)X_{|t=0}, Y) \overset{(2.1)}{=} g(\frac{d}{dt}B(t)X_{|t=0}, JY)$$

and thus, identifying $\psi : \pi \to \pi^\perp$ with $\frac{d}{dt}B(t)\pi_{|t=0}$ we get the result. □

By Lemma 1 it is clear that J itself, restricted to q, can be considered not only as an element of $T_\pi GrL(V)$ but even as an invariant vector field on $GrL(V)$, that is $J_q \in \Gamma(TGrL(V))$. Let $e_1, \ldots, e_n$ be an orthonormal basis of π and $f^1, \ldots, f^n$ the corresponding dual basis, in such a way that $Je_1, \ldots, Je_n$ is a basis of $\pi^\perp$ and $-Jf^1, \ldots, -Jf^n$ the associated dual basis. Then J as a vector belonging to $T_\pi GrL(V)$, can be represented as a section of $\pi^* \otimes \pi^\perp$, that is $J = f^i \otimes Je_i$ (Einstein summation convention is intended). From J in this representation one can construct a 1-form $\tilde{J} \in \omega^1(GrL(V))$ using the paring induced by the metric, that is $\tilde{J} = e_i \otimes -Jf^i$. This 1-form has a quite outstanding role:

Lemma 2: *Fix an arbitrary Lagrangian plane in V in order to have a map $det^2 : GrL(V) \to S^1$. Then:*

$$(det^2)^*(\alpha) = \frac{1}{\pi}\tilde{J},$$

so that $\tilde{J}$ defines a closed form on $GrL(V)$.

Proof: It is sufficient to prove that for every $X \in T_\pi GrL(V)$ one has $(det^2)^*(\alpha)(X) = \frac{1}{\pi}\tilde{J}(X)$. Indeed:

$$(det^2)^*(\alpha)(X) = (\alpha)(det^2_*(X)),$$

so we are led to compute the tangent map to det^2. Assume for simplicity that π is the reference Lagrangian plane in the isomorphism $GrL(V) \cong \frac{U(n)}{O(n)}$, so that it is represented by the identity matrix. Then, since $T_\pi GrL(V) \cong \frac{u(n)}{o(n)}$, consider a path $\gamma : (-\epsilon, \epsilon) \to u(n)$, such that $\gamma(0) = O$ and such that its image in $u(n)$ has empty intersection with $o(n)$ (except for the zero matrix). The exponential mapping determines in this way a path in

$GrL(V)$ through π. Now, we have:

$$\frac{d}{dt}det^2(e^{\gamma(t)})_{|t=0} = \frac{d}{dt}det(e^{2\gamma(t)})_{|t=0} = \frac{d}{dt}(e^{2Tr(\gamma(t))})_{|t=0} =$$
$$2Tr(\dot{\gamma}(0)) = 2Tr(X),$$

where $\dot{\gamma}(0)$ is identified with the tangent vector X in $T_\pi GrL(V)$. Hence one gets:

$$(det^2)^*(\alpha)(X) = (\alpha)(det^2_*(X)) = (\alpha)(2Tr(X)) = \frac{1}{\pi}Tr(X).$$

On the other hand, $X \in \Gamma(\pi^* \otimes \pi^\perp)$, so that it can be represented as $X = X^l_k f^k \otimes Je_l$; thus one gets:

$$\tilde{J}(X) = (e_i \otimes -Jf^i)(X^l_k f^k \otimes Je_l) = X^i_i = Tr(X).$$

□

Till now we have worked only locally, having fixed a point $q \in \Lambda$. To proceed we need to globalize the properties stated in lemma 1 and 2. Let us define the *vertical tangent bundle* $VT(GrL(X)_\Lambda)$ ($VT(GrL)$ for short) over $GrL(X)_\Lambda$ as:

$$VT(GrL(X)_\Lambda) := \coprod_{x\in\Lambda} TGrL(T_xX);$$

notice that this is not the tangent bundle of $GrL(X)_\Lambda$, since it is obtained taking the tangent bundle of the fibre only (thus the name vertical). Analogously, one can define the *vertical cotangent bundle* over $GrL(X)_\Lambda$ as:

$$VT^*(GrL(X)_\Lambda) := \coprod_{x\in\Lambda} T^*GrL(T_xX),$$

(from now on denoted as $VT^*(GrL)$ for short).

Now, by the previous reasoning and since J is covariantly constant on a Kaehler manifold X, we have that J defines a section of $VT(GrL)$ and analogously $\tilde{J}$ induces a section of $VT^*(GrL)$. In order to globalize the result of lemma 2, observe that the section $\mathcal{D}_\gamma$ of $GrL(X)_\Lambda$ over Λ, defined in the previous section, enables one to give a well-defined map $Det^2 : GrL(X)_\Lambda \to S^1$ (one takes as a reference Lagrangian plane in $GrL(T_xX)$ the subspace $(\mathcal{D}_\gamma)_x$). It is clear that one gets immediately the following:

Corollary 1: *Under the previous notations and considering the fibration $Det^2 : GrL(X)_\Lambda \to S^1$ induced by the reference distribution $\mathcal{D}_\gamma$ one has:*

$$(Det^2)^*(\alpha) = \frac{1}{\pi}\tilde{J}$$

where $\tilde{J}$ is viewed as a section of $VT^*(GrL)$.

Via the Gauss map we can pull-back $VT(GrL)$ to Λ:

$$\begin{array}{ccc} G(j)^*(VT(GrL)) & & VT(GrL) \\ \downarrow & & \downarrow pr_{VT} \\ \Lambda & \rightarrow & GrL(X)_\Lambda \end{array}$$

Lemma 3: *The bundle $G(j)^*VT(GrL)$ can be identified with the subspace of $T^*\Lambda \otimes N\Lambda$ consisting of those sections $\psi \in \Gamma(T^*\Lambda \otimes N\Lambda)$ (that is $N\Lambda$-valued 1-forms on Λ) such that:*

$$g(\psi(X), JY) = g(\psi(Y), JX), \quad \forall X, Y \in \Gamma(T\Lambda).$$

Proof: By the very definition of pulled-back bundle, we have that:

$$G(j)^*VT(GrL) \cong \{(x; x', \pi, X) \in \Lambda \times VT(GrL) : \quad (x, T_x\Lambda) = G(j)(x) =$$

$$= pr_{VT}(x', \pi, X) = (x', \pi)\},$$

which clearly implies the constraint $x = x'$ and $T_x\Lambda = \pi$ so that:

$$G(j)^*VT(GrL) \cong \coprod_{x \in \Lambda} T_{\pi = T_x\Lambda} GrL(T_xX).$$

On the other hand, by lemma 1:

$$T_{\pi=T_x\Lambda}GrL(T_xX) \cong \{\psi \in \Gamma(T_x^*\Lambda \otimes N_x\Lambda) \quad \text{such} \quad \text{that} :$$

$$g(\psi(X), JY) = g(\psi(Y), JX), \quad \forall X, Y \in T_x\Lambda\},$$

so one gets immediately the thesis. □

The tangent application to the Gauss map is related to the second fundamental form as shown in the following:

Lemma 4: *The tangent map to $G(j)$ in a point $x \in \Lambda$ can be identified with the second fundamental form σ, thought of as an application with values in $T^*\Lambda \otimes N\Lambda$; more exactly σ takes values in the subspace $G(j)^*(VT(GrL))$ of $T^*\Lambda \otimes N\Lambda$, in the sense that it satisfies $g(\sigma(X,Y), JZ) = g(\sigma(X,Z), JY)$.*

Proof: First of all, the identity $g(\sigma(X,Y), JZ) = g(\sigma(X,Z), JY)$ is a consequence of the fact that Lagrangian submanifolds of Kähler manifolds are always anti-invariant (also called totally real) submanifolds of top dimension (see [22] page 35). Hence, always by result of [22], page 43, we have the desired relation. Finally, the fact that the tangent map to the Gauss map can be identified with the second fundamental form, via the action of the almost complex structure J and the metric g, is a classically known result which can be found, for example in [6], page 196. $\square$

Observe that by lemma 3 and 4, the second fundamental form $\sigma(X,.)$, considered as a map taking values in $T^*\Lambda \otimes N\Lambda$ is an element of $G(j)^*(VT(GrL))$. Let us summarize the situation in the following diagram:

$$\begin{array}{ccccc} T\Lambda & \overset{G(j)_*}{\rightarrow} & G(j)^*(VT(GrL)) & \subset T^*\Lambda \otimes N\Lambda & VT(GrL) \\ \downarrow & & & & \downarrow \\ \Lambda & \overset{G(j)}{\rightarrow} & G(j)(\Lambda) & \hookrightarrow & GrL(X)_\Lambda \end{array}$$

Denote again with $\tilde{J}$ the restriction of $\tilde{J}$ to the bundle $G(j)^*(VT^*(GrL))$. By the previous diagram we can pull-back $\tilde{J}$ to a closed 1-form on Λ via $G(j)^*$:

$$(G(j)^*(\tilde{J}))(X) = \tilde{J}(G(j)_*(X)) = \tilde{J}(\sigma(X,.)) \quad \forall X \in \Gamma(T\Lambda), \tag{2.2}$$

where the last equality in equation (2.2) is due to lemma 4 and the pairing between $\tilde{J}$ and $\sigma(X,.)$ is induced by the natural pairing between $G(j)^*(VT^*(GrL))$ and $G(j)^*(VT(GrL))$, respectively.

Proof of the theorem: First of all, notice that the Maslov map $\mathcal{M} : \Lambda \to S^1$ can be decomposed as $\mathcal{M} = Det^2 \circ G(j)$, as is immediate to see. Then $\mu_\Lambda := \mathcal{M}^*(\alpha) = G(j)^* \circ (Det^2)^*(\alpha)$ and so $\mu_\Lambda = \frac{1}{\pi}G(j)^*(\tilde{J})$, by lemma 2. Now $\tilde{J} = e_l \otimes -Jf^l$ and $\sigma(X,.)$ can be represented as $\Gamma(T^*\Lambda \otimes N\Lambda) \ni \sigma(X,.) = \sigma_i^k(X) f^i \otimes Je_k$. In this way we have that for all $X \in \Gamma(T\Lambda)$:

$$(G(j)^*(\tilde{J}))(X) = (e_l \otimes -Jf^l)(\sigma_i^k(X) f^i \otimes Je_k) = \sigma_i^i(X) =$$
$$= \sum_i g(\sigma(X, e_i), Je_i) = (\text{by} \quad \text{lemma} \quad 4) = \sum_i g(\sigma(e_i, e_i), JX) =$$
$$= g(H, JX) = \omega(H, X) = i_H\omega(X).$$

Hence, one gets the result:

$$\mu_\Lambda = G(j)^*(\frac{1}{\pi}\tilde{J}) = \frac{1}{\pi} i_H\omega \quad \in H^1(\Lambda, \mathbb{Z}). \tag{2.3}$$

$\square$

By the result of the theorem, one can give the following:

Definition: Let $\Lambda \hookrightarrow X$ a Lagrangian embedding in a Calabi-Yau X; then the Maslov index m of a closed loop γ on Λ is given by:

$$m(\gamma) := \frac{1}{\pi} \int_\gamma i_H\omega \quad \in \mathbb{Z}.$$

2.4 Some comments and a conjecture

Calabi-Yau manifolds have received great attention as target spaces for superstring compactifications. Moreover their Lagrangian and special Lagrangian submanifolds are now considered as the cornerstones for understanding the mirror symmetry phenomenon between pairs of Calabi-Yau spaces, both from a categorical point of view ([14]), and from a physical-geometrical standpoint ([18]). Let us recall that special Lagrangian submanifolds Λ of a Calabi-Yau X are exactly what are called BPS states or supersymmetric cycles in the physical literature; on the other hand, it is known that special Lagrangian submanifolds are nothing else that *minimal* Lagrangian submanifolds (compare [11] page 96, where this is proved for special Lagrangian submanifolds of $\mathbb{C}^n$). From our result it turns out that the Maslov class of special Lagrangian submanifolds is identically vanishing; on the other hand, this can be seen just by considering the Grassmannian of special Lagrangian planes, which turns out to be diffeomorphic to $\dfrac{SU(n)}{SO(n)}$, hence simply connected (notice that the Grassmannian of special Lagrangian planes is isomorphic to the fibre in the fibration $det^2 : GrL(\mathbb{C}^n) \to S^1$). It is then clear that the Maslov index is identically vanishing for all special Lagrangian submanifolds Λ of a Calabi-Yau X. We believe that this simple observation can enhance our understanding of the structure of the A^∞-Fukaya category, whenever its objects are restricted to minimal Lagrangian submanifolds (see [26] for a definition of A^∞ category, and [14] for its application in the study of mirror symmetry). Indeed, this is a key point for the proof of homological mirror symmetry for K3 surfaces, for which we refer to [3].

The Maslov class so far constructed does not depend on the choice of a canonical projection, from which one could determine the singular locus (as usually happens when one considers Lagrangian embedding in cotangent bundles over an arbitrary Riemannian manifold). However, it is still possible to determine, rather then the singular locus, the *homology class* $[Z] \in H_{n-1}(\Lambda, \mathbb{Z})$ of a "singular locus", just considering the Poincare' dual to $[\mu_\Lambda]$, and setting $[Z] := Pd([\mu_\Lambda])$ (Pd stands for Poncare' duality). We have said "a singular locus", because Z is not determined at all uniquely, but only up to its homology class; in spite of this one could take as singular locus any representative of $[Z]$. So it makes sense to speak of a singular locus, even if there is no projection to which to refer it.

It is clear that it is not possible to extend our definition of Maslov class for Lagrangian embedding in arbitrary symplectic manifolds; even the construction of Fukaya (which is specifically designed for Maslov index of closed loops only on BS orbits) needs several assumption such that the ambient manifold admits a "prequantum bundle" and so on. We are thus tempted to suggest the following alternative description: we would like to define the Maslov class for a Lagrangian embedding in *any* symplectic manifold (X, ω), via the mean curvature representation $i_H\omega$. Two problems arise following this approach. First of all, to define the mean curvature vector field H it is necessary to fix a Riemannian metric on X; as it is well known, on any symplectic manifold one has lots of Riemannian metrics $g_J(X,Y) := \omega(X, JY)$, constructed using the given symplectic form ω and choosing an ω-compatible almost complex structure J; (recall that the set of ω-compatible almost complex structures on a given symplectic manifold is always non empty and contractible, see [52]). What is the "right" choice for g_J?

Once we have fixed the right metric, the second problem is related to the closure of the 1-form $i_H\omega$, considered as a form on Λ; indeed there is no reason, a priori, for which $i_H\omega$ has to be closed. We are thus led to the following:

Conjecture: Having fixed the Lagrangian embedding $j : \Lambda \hookrightarrow X$, on any symplectic manifold (X, ω) there exists at least one Riemannian metric g_J built up from an ω-compatible almost complex structure J, such that the 1-form $i_H\omega$, considered as a form on Λ is closed. Multiplying the corresponding cohomological class $[i_H\omega]$ for a suitable constant in such a way that it is integer valued, we call this class the *Maslov-Morvan class* of the Lagrangian submanifold Λ.

It does not seem possible to give an interpretation of this conjectured Maslov-Morvan class via the universal Maslov class, as we have done for Calabi-Yau manifolds, since, in general, we have no control on the holonomy of g_J.

2.5 Special Lagrangian embeddings

First of all, we recall from ([11]) the following:

Definition-Proposition: *Let X be a Calabi-Yau n-fold, with Kaehler form ω and holomorphic nowhere vanishing n-form Ω. A (real) n-dimensional submanifold $j : \Lambda \hookrightarrow X$ of X is called special Lagrangian if the following two conditions are setiesfied:*

1. *Λ is Lagrangian with respect to ω, i.e. $j^*\omega = 0$;*
2. *there exists a multiple Ω' of Ω such that $j^*Im(\Omega') = 0$.*

One can prove that both conditions are equivalent to $j^*Re(\Omega') = Vol_g(\Lambda)$.

This last condition means that the real part of Ω' restricts to the volume form of Λ, induced by the Calabi-Yau Riemannian metric g. In this way, special Lagrangian submanifolds are considered as a type of calibrated submanifolds (see [11] for further details on this point).

Despite their importance, it is difficult to construct examples of special Lagrangian submanifolds, especially in Calabi-Yau 3-folds, where there is no a priori evident relation between the Kaehler form and the holomorphic form which trivializes the canonical bundle. However, there is a subclass of Calabi-Yau n-folds, for which it easier construct such examples. These are the so called irreducible symplectic n-folds:

Definition: *A complex manifold X is called irreducible symplectic if the following three conditions are satisfied:*

1. *X is compact and Kaehler,*
2. *X is symply connected,*
3. *$H^0(X, \Omega_X^2)$ is spanned by an everywhere nondegenerate 2-form ω.*

In particular, irreducible symplectic manifolds are special cases of Calabi-Yau manifolds (the top holomorphic form which trivializes the canonical line bundle is given by a suitable power of the holomorphic 2-form ω) and moreover they are all hyperkaehler. Observe that in dimension 2, K3 surfaces are the only irreducible symplectic manifolds, and indeed irreducible symplectic manifolds appear as higher-dimensional analogues of K3 surfaces, as strongly suggested in ([12]). Unfortunately, up to now there are also few explicit examples of irreducible symplectic manifolds. Indeed, almost all known examples turn out to be birational to two standard series: Hilbert schemes of points on K3 surfaces and generalized Kummer variety (both series were first studied in ([4]) , but quite recently O'Grady has constructed irreducible symplectic manifolds which are not birational to any of the elements of these two groups (see ([17])).

Let us fix a Kaehler class $[\omega]$ in the Kaehler cone of X, a irreducible symplectic n-folds. (In this case, n is alwaus even, so we put n=2m). By Yau's theorem, this detemines a unique hyperkaehler metric g. Choose a hyperkaehler structure (I, J, K) compatible with this metric (notice that the triple (I, J, K) is not uniquely determined) and consider the associated symplectic the associated symplectic structures $\omega_I(.,.) = g(I.,.)$, $\omega_J(.,.) = g(J.,.)$ and $\omega_K(.,.) = g(K.,.)$. Fixing the complex structure K on X, the holomorphic n-form which trivializes the canonical bundle can be expressed as:

$$\Omega_K := \frac{1}{m!}(\omega_I + i\omega_J)^m. \tag{2.4}$$

Then we have the following:

Proposition (Bruzzo and others): *Let X as above. Let Λ be a submanifold (compact and without border) of X, which is Lagrangian with respect to the symplectic form ω_J and complex in the complex structure I. Then Λ is a special Lagrangian submanifold of X in the complex structure K.*

Proof: By assumption, Λ is Lagrangian in the symplectic structure ω_J and so $\omega_{J|\Lambda} = 0$; moreover Λ is a complex submanifold of X in the complex structure I so that by

Wirtinger's theorem $Vol_g(\Lambda) = \frac{1}{m!}\int(\omega_I)^m$. Now, since $\Omega_K = \frac{1}{m!}(\omega_I + i\omega_J)^m$, it is immediate that $Re(\Omega_K)_{|\Lambda} = Vol_g(\Lambda)$, so that Λ is a special Lagrangian submanifold of X in the structure K. □

Let us remark that this proposition is even stronger in dimension 2, i.e. for K3 surfaces. Indeed, in this case, it is just sufficient to require that Λ be a complex submanifold of X in the structure I: then it is immediate to see (again by Wirtinger's theorem) that Λ is calibrated by the real part of $\Omega_K = \omega_I + i\omega_J$ and thus it is special Lagrangian in the structure K. This means that on a K3 surface, special Lagrangian geometry is equivalent to the geometry of complex submanifolds of dimension 1 (curves).

As we have alredy observed, K3 surfaces or more generally Calabi-Yau manifolds need not to be algebraic. Starting from the next chapter, we leave the big realm of complex geometry we will focus our attention to projective algebraic varieties, and specifically to their projective embeddings.

Chapter 3

Small codimension embeddings in projective spaces

3.1 Introduction

With this chapter, we start to study "special" embeddings in projective spaces, thus we abandon the realm of Kaehler geometry and we focus on some aspects of (projective) algebraic geometry.

More specifically, in this chapter we address the problem of studying and possibly classifying (smooth) subvarieties of small codimension in some $\mathbb{P}^n$ (where small is intended with respect to n) and the ultimate goal would be to understand which properties of the geometry of subvarieties are forced by the small codimension condition. Indeed, for any subvariety X of $\mathbb{P}^n$, we have the following exact sequence:

$$0 \to TX \to T\mathbb{P}^n_{|X} \to N_{X/\mathbb{P}^n} \to 0.$$

Thus, (restricting to the case of X smooth) we obtain for the corresponding total Chern classes the relation $c(TX)c(N_{X/\mathbb{P}^n}) = c(T\mathbb{P}^n_{|X})$; this implies immediately that the Chern classes of TX can be computed using those of $N_{X/\mathbb{P}^n}$. On the other hand, if the codimension of X is small compared to its dimension, then all the Chern classes of TX depend on a small number of variables (i.e. the Chern classes of $N_{X/\mathbb{P}^n}$). This suggests somehow that, in general, we expect to have only very few small codimension subvarieties.

In a very famous paper (see [32]), Hartshorne formulated a key conjecture regarding smooth projective subvarieties of small codimension. Briefly, let us recall a fundamental definition. Let k be an algebraically closed field and let $\mathbb{P}^n$ be the n-dimesional projective space over k. Let $X \subset \mathbb{P}^n$ be a smooth irreducible subvariety of dimension r. Then we say that X is a *complete intersection* in $\mathbb{P}^n$, if one can find $n - r$ hypersurfaces $H_1, \ldots, H_{n-r}$, such that $X = H_1 \cap \ldots \cap H_{n-r}$, and such that this intersection is transversal, i.e. the hypersurfaces H_i are smooth at all points of X, and their tangent hyperplanes intersect properly at each point of X. In algebraic terms, X is a complete intersection iff its homogeneous prime ideal $I(X) \subset k[x_0, \ldots, x_n]$ can be generated by exactly $n - r$ homogeneous polynomials.

Fundamental Conjecture (Hartshorne): If X is a smooth subvariety of dimension r in $\mathbb{P}^n$, and $r > \frac{2}{3}n$, then X is a complete intersection.

Despite the fact that this conjecture can be stated in elementary form, it is still completely open, after almost thirty years! (In " Geometric Invariant Theory", Mumford wrote that this is the most important and most difficult conjecture in projective algebraic geometry which is still open).

3.2 Evidences for the conjecture

To put the previous conjecture in a wider framework, let us consider a more general question. Let X be any smooth algebraic variety, and let Y be a (possibly singular) subvariety. Then we can consider the cohomology class of Y, in any suitable cohomology theory on X. For simplicity, let us assume that X is a projective variety defined over $\mathbb{C}$. Then we can consider X as a compact complex manifold. If Y is a subvariety, it defines a homology class on X, which, by Poincaré duality, gives a cohomology class $[Y] \in H^{2q}(X, \mathbb{Z})$, where q is the complex codimension of Y in X. Now, we can ask: if Y is a smooth subvariety of codimension q in X, to what extent are the properties of Y determined by its cohomology class $[Y]$? Indeed, the conjecture can be viewed as a special case of this question. The general philosophy which operates here is that a nonsingular subvariety of *small codimension* of a fixed variety X must be subject to stringent conditions. More specifically, in the case of a subvariety X of $\mathbb{P}^n$, if we know that it is a complete intersection of hypersurfaces of degrees $d_1, \ldots, d_{n-r}$, then we essentially know all about it. (For simplicity, let us restrict to varieties over $\mathbb{C}$). Indeed, we have the following remarkable:

Theorem (Lefschetz): *Let X be a smooth subvariety of dimension r of $\mathbb{P}^n$, which is a complete intersection. Then:*

1. *The restriction map $H^i(\mathbb{P}^n, \mathbb{Z}) \to H^i(X, \mathbb{Z})$ is an isomorphism for $i < r$, and injective for $i = r$.*

2. *$\pi_1(X) = 1$, if $r \geq 2$.*

3. *$Pic(X) \cong \mathbb{Z}$, generated by $\mathcal{O}_X(1)$, if $r \geq 3$.*

On the other hand, the outstanding results of Barth and others show that many of the same properties hold for smooth subvarieties of small codimension. This supports the conjecture, in the sense that it shows that subvarieties of small codimension "look like" complete intersections, at least from a cohomological point of view.

Theorem (Barth, Larsen): *Let X be a smooth subvariety of dimension r in $\mathbb{P}^n$ (which is not necessarily a complete intersection). Then:*

1. *The restriction map $H^i(\mathbb{P}^n, \mathbb{C}) \to H^i(X, \mathbb{C})$ is an isomorphism for $i \leq 2r - n$.*

2. *The same is true for cohomology with $\mathbb{Z}$ coefficients.*

3. $\pi_1(X) = 1$ *if* $r \geq \frac{1}{2}(n+2)$.

4. *$Pic(X) \cong \mathbb{Z}$, generated by $\mathcal{O}_X(1)$ if $r \geq \frac{1}{2}(n+2)$.*

Note in the statements of the theorem that the restrictions on i and r operate so that if $r < \frac{1}{2}n$, then there is no conclusion. On the other hand, as the codimension of X becomes small with respect to its dimension, then one gets stronger and stronger conclusions.

Following ([32]), we want to show that part 1 of the Barth-Larsen's theorem is an easy consequence of the strong Lefschetz theorem. First of all, recall:

Theorem (Strong Lefschetz Theorem): *Let X be a smooth projective algebraic variety of dimension n, and let $h \in H^2(X, \mathbb{C})$ be the class of a hyperplane section. Then the cup-product map $H^i(X, \mathbb{C}) \xrightarrow{h^{n-i}} H^{2n-i}(X, \mathbb{C})$ is an isomorphism, for each $i = 0, \ldots, n$.*

This theorem is proved using Hodge's theory of harmonic integrals.

We have the following:

Corollary: *Within the hypotheses of the theorem, the cup-product map $H^i(X, \mathbb{C}) \xrightarrow{h^j} H^{i+2j}(X, \mathbb{C})$ is injective whenever $j \leq n - i$.*

Proof (part 1 of Barth-Larsen's theorem): Let X be a smooth subvariety of dimension r of $\mathbb{P}^n$, and let $j : X \hookrightarrow \mathbb{P}^n$ be the embedding map. Let $j^* : H^i(\mathbb{P}^n) \to H^i(X)$ be the restriction map on cohomology (always with $\mathbb{C}$ coefficients), and let $j_* : H^i(X) \to H^{i+2n-2r}(\mathbb{P}^n)$ be the covariant map induced by Poincaré duality, from the covariant map j_* on homology. Let $[X] \in H^{2n-2r}(\mathbb{P}^n)$ be the cohomology class of X. Then for any $x \in H^i(\mathbb{P}^n)$, by projection formula we get $j_*j^*(x) = x \cup [X] \in H^{i+2n-2r}(\mathbb{P}^n)$. On the other hand, for any $y \in H^i(X)$, by the Thom isomorphism theorem on a tubular neighbourhood of X, we have $j^*j_*(y) = y \cup j^*([X]) \in H^{i+2n-2r}(X)$.

Now suppose that X is a variety of degree d. Then $[X] = dh^{n-r}$, where $h \in H^2(\mathbb{P}^n)$ is the class of a hyperplane. Applying the previous corollary to X, we find that the map $j^*j_* : H^i(X) \to H^{i+2n-2r}(\mathbb{P}^n)$, which is the cup-product with $j^*([X]) = dj^*(h)^{n-r}$ is injective, provided that $n - r \leq r - i$, i.e., $i \leq 2r - n$. If this is so, then $j_* : H^i(X) \to$

$H^{i+2n-2r}(\mathbb{P}^n)$ must also be injective. On the other hand, $j^*j_* : H^i(\mathbb{P}^n) \to H^{i+2n-2r}(\mathbb{P}^n)$ is an isomorphism. Indeed, these groups are either both 0, if i is odd, or both $\mathbb{C}$, if i is even, and they are generated by the appropriate powers of h, so the cup-product with h^{n-r} is an isomorphism. Putting these facts together, we find that $j^* : H^i(\mathbb{P}^n) \to H^i(X)$ is an isomorphism, provided $i \leq 2r - n$. □

Observe, that the same argument can be applied to other ambient varieties, besides projective spaces. For example, suppose X is a smooth complete intersection variety of dimension n, and Y is a smooth subvariety of X of dimension r. By Lefschetz' Theorem we know the cohomology of X and by the same reasoning, we find that $j^* : H^i(X) \to H^i(Y)$ is an isomorphism for $i < 2r - n$, and injective for $i = 2r - n$.

3.2.1 Subvarieties of small degree

It is possible to approach the conjecture, studying varieties according to their degree, instead to their dimension. By so doing, it has been recognized that subvarieties of small codimension and small degree are indeed complete intersection. A variety of degree 1 is a linear variety, which is itself a projective space, hence a complete intersection. If X is a variety of degree 2 and dimension r in $\mathbb{P}^n$, then it is contained in some $\mathbb{P}^{r+1}$. Hence it is a hypersurfaces in $\mathbb{P}^{r+1}$, and as such it is a complete intersection. More generally, one can show that any variety of dimension r and degree d in some $\mathbb{P}^n$ is always contained in a linear subspace $\mathbb{P}^{r+d-1}$. Indeed, if $V = \mathbb{P}^{n-r}$ is a generic subspace, then $X \cap V$ is finite set of d points. This set spans at most a linear subspace of dimension $d - 1$ and one proves immediately by induction that X is contained in $\mathbb{P}^{r+d-1}$.

For varieties of degree 3, we have the following result, the proof of which is completely elementary:

Theorem: *Let X be a smooth variety of dimension r and degree 3 in $\mathbb{P}^n$, and assume that X is non degenerate (i.e. not contained in any $\mathbb{P}^{n-1}$). Then either $r = n - 1$ i.e. X is a hypersurface, or $r = n - 2$, $n = 3, n = 4$ or $n = 5$ and Y is obtained by linear automorphism of $\mathbb{P}^n$ from one of the following three varieties:*

1. *the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^5$,*
2. *its general hyperplane section, which is a rational ruled surface in $\mathbb{P}^4$,*
3. *the twisted cubic curve in $\mathbb{P}^3$.*

In particular, we see that if X is a smooth variety of degree 3, which is not a complete intersection, then it is contained in a projective space of dimension $n \leq 5$.

There is a similar analysis for varieties of degree 4:

Theorem (Swinnerton-Dyer): *Let X be a smooth variety of dimension r and degree 4 in $\mathbb{P}^n$, which is nondegenerate. Then either: $r = n-1$, so X is a hypersurface, or $r = n-2$, and Y is a complete intersection of two quadric hypersurfaces, or $n \leq 7$, and X is one of the following (up to an isomorphism of $\mathbb{P}^n$):*

1. *the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^3$ in $\mathbb{P}^7$,*
2. *the Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$,*
3. *a variety obtained from the two previous ones, by a succession of sections by a hyperplane and/or projections from a point into a lower-dimensional projective space.*

As in the case of degree 3, we see that those varieties of degree 4 which are not complete intersections are contained in a projective space of bounded dimension, in this case $n \leq 7$, and that there is a finite list of possibilities. Unfotunately, this method of classification is hopeless to extend to the general case.

However, using different tools, one can show the existence of such a bound for any d:

Theorem (Barth-Hartshorne-Van de Ven): *For any $d > 0$, there exists an $n_0(d) > 0$, such that if X is a smooth projective variety of degree d, defined over $\mathbb{C}$, which is not a complete intersection, then X is contained in some $\mathbb{P}^n$, with $n \leq n_0(d)$. Furthemore, there is only a finite number of continuous families of such varieties.*

3.3 Embedding varieties in projective spaces

One could try to approach the conjecture, studying which very ample complete linear systems exist on a given variety and selecting those of small dimension (compared to the dimension of the given variety). This is however an impossible task, since in this problem we have to deal with varieties of big dimension and it is very difficult, in general, even to study complete linear systems on a surface. Thus, we can follow another way and embed a projective algebraic variety X of dimension r in a big projective space $\mathbb{P}^n$; then we can define a mapping of X into $\mathbb{P}^{n-1}$, by projecting from a point of $\mathbb{P}^n$ to a hyperplane. One can easily show that if X is smooth, and if $n > 2r+1$, where $r = dim(X)$, then the projection can be chosen so that the image is still nonsingular, and hence we obtain an embedding of X in $\mathbb{P}^{n-1}$. Indeed, the only thing to check is that the center P of the projection map does not lie on any chord, or any tangent line of X. By counting parameters, we see immediately that the *chord variety* of X, which is the locus of all points on chords and tangents of X, has dimension $\leq 2r+1$. So, if $n > 2r+1$, we can find a point in $\mathbb{P}^n$, not on the chord variety.

Hence, by successive projection, we show that smooth projective variety of dimension r can always be embedded in $\mathbb{P}^{2r+1}$. In general, if we try to project further X into smaller projective spaces, the image will acquire singularities.

However, in special cases, X may already lie in a smaller projective space, or the projection of X into a smaller projective space may remain smooth. Let us investigate further on this point and consider some examples.

If X is a smooth subvariety of $\mathbb{P}^n$, then X can be realized as the projection of a variety in $\mathbb{P}^{n+1}$, not lying in any $\mathbb{P}^n$, iff the linear system of hyperplane sections (in $\mathbb{P}^n$) is not complete. On the other hand, if the linear system of hyperplane sections is complete, we will say that X is *linearly normal* for the given embedding (hence it is not obtained by projection!). In terms of sheaf cohomology, this means that the map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(X, \mathcal{O}_X(1))$ is surjective (and this happens, for example, if $H^1(\mathcal{I}_X(1)) = 0$). Being linearly normal, is weaker then the condition of being *projectively normal*, which means that for all integers $k \in \mathbb{Z}$, $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) \to H^0(X, \mathcal{O}_X(k))$ is surjective (and this happens if for example $H^1(\mathcal{I}_Y(k)) = 0$). X is projectively normal iff the vertex of the affine cone over X in $\mathbb{A}^{n+1}$ is a normal point (i.e. the correponding local ring at the vertex is integrally closed in its quotient field). As far as complete itersections is concerned, one knows that any smooth complete intersection variety is projectively normal, hence it is linearly normal and so cannot be realized as a projection of a variety sitting effectively in a higher dimensional projective space. (For a smooth codimension two subvariety X, in a projective space of dimension greater or equal to six, even more is true: indeed, in this case it is known that X is a complete intersection iff it is projectively normal (Hartshorne, Ogus, Szpiro)). Thus, for instance, any non singular hypersurface in $\mathbb{P}^n$ is a smooth complete intersection, hence it is linearly normal and so cannot be obtained by projection. This in particular applies for smooth curves in $\mathbb{P}^2$, smooth surfaces in $\mathbb{P}^3$, smooth threefolds in $\mathbb{P}^4$ and so on.

Now, let us look at some more examples. A smooth surface in $\mathbb{P}^3$ is projectivaly normal, hence it cannot be the projection of a surface in any higher $\mathbb{P}^n$. But, a surface in $\mathbb{P}^4$ might be a projection of a surface in $\mathbb{P}^5$. Indeed, there is a classical theorem of Severi, which tells us when the projection is smooth:

Theorem (Severi): *Let X be a nondegenerate surface in $\mathbb{P}^5$, whose generic projection into $\mathbb{P}^4$ is smooth. Then, up to an isomorphism of $\mathbb{P}^5$, X is the Veronese surface, which is the embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$ induced by the complete linear system of conics.*

In higher dimensions, we know a few interesting examples of varieties which can be projected down into smaller projective spaces than one would expect. Indeed, if X in $\mathbb{P}^n$ is an irreducible nondegenerate r dimensional variety, whose chordal variety $S(X)$ has dimension strictly less than $min(2r+1, n)$, we say that X has *deficient secant variety*; in these circumstances, we define the deficiency $\delta(X)$ to be: $\delta(X) := 2r + 1 - dim(S(X))$.

Here we have the following outstanding examples:

Proposition: *Consider the embedding of* $\mathbb{P}^r$ *into* $\mathbb{P}^N$ $(N = \frac{1}{2}r(r+3))$, *by the complete linear system of hyperquadrics. Let* X *be the projection into* $\mathbb{P}^{2r+1}$. *Then, in this case* X *can be projected into* $\mathbb{P}^{2r}$ *and remains smooth, i.e.* $\delta(X) = 1$.

Proof: First of all X is smooth. We claim that the chordal variety $S(X)$ has only dimension $2r$. To see this, suppose $q \in \mathbb{P}^{2r+1}$ is a general point lying on a secant line to X; we may write the secant line as $x_1\bar{x}_2$, for some pair of points $x_1, x_2 \in \mathbb{P}^{2r+1}$ and belonging to X. Now, the points x_1, x_2 correspond to two points q_1, q_2 in $\mathbb{P}^r$. The line $L := q_1\bar{q}_2 \subset \mathbb{P}^r$ is carried, under the Veronese embedding and the subsequent smooth projection down to $\mathbb{P}^{2r+1}$, to a plane conic curve $C \subset X \subset \mathbb{P}^{2r+1}$, and since $q \in x_1\bar{x}_2$, q will lie on the plane λ spanned by C. But then every line through q in λ, will be a secant line to C, and hence to X. In particular, it follows that a general point lying on a secant line to X lies also on a one-dimensional family of secant lines to X. We may deduce from this that the dimension of $S(X)$ is at most $2r$. Since it is clear on elementary grounds that $S(X)$ cannot have dimension less than $2r$, (for example all the cones $p\bar{X}$ over X with vertex $p \in X$ would have to coincide), we conclude that $dim(S(X)) = 2r$. □

The following two propositions, whose proof can be given using analogous arguments, give two other kind of examples of deficiency:

Proposition: *Consider the embedding of* $\mathbb{P}^r \times \mathbb{P}^s$ *into* $\mathbb{P}^N$ $(N = rs + r + s)$, *given by the Segre embedding. Assume moreover* $s \geq 3, r \geq 3$ *or* $s = 2, r \geq 4$ *(otherwise we cannot project down, since* N *is too small compared to* $r + s$*). Let* X *be the corresponding projection into* $\mathbb{P}^{2(r+s)+1}$. *Then, in this case* X *can be projected down to* $\mathbb{P}^{2(r+s)-1}$ *without acquiring singularities, i.e.* $\delta(X) = 2$.

Proposition: *Consider the natural embedding* X *of* $G(1, n)$ *(Grassmannian of lines in* $\mathbb{P}^n$*) into* $\mathbb{P}^{\frac{n(n+1)}{2}-1}$. *Assume* $n \geq 5$, *then in this case we have* $\delta(X) = 4$.

Since all smooth complete intersections are linearly normal, the validity of Hartshorne's conjecture on smooth varieties of small codimension would imply that all such varieties are linearly normal. Indeed, this was first proved by F. Zak in 1979, in the following outstanding:

Theorem (Zak): *Let* X *be a smooth r-dimensional variety in* $\mathbb{P}^n$. *If* $3r > 2(n-1)$, *then* X *is linearly normal.*

Zak classified also all varieties at the boundary of the inequality $3r > 2(n-1)$ and proved that, up to projective equivalence, there are exactly four such varieties (called Severi varieties). A clear exposition of such topics can be found in ([33]).

3.4 The Serre correspondence

In the next sections we will deal mostly with subvarieties of codimension 2. In this case, we can get stronger results due to the fact that it is possible to associate vector bundles of rank 2 over $\mathbb{P}^n$ to such subvarieties; this is the so called Serre correspondence which we are going to review in this section, following closely ([56]).

Let E be an algebraic vector bundle over $\mathbb{P}^n$ of rank 2, with a section s whose zero scheme $Z(s)$ is a locally complete intersection subscheme Y of codimension 2. Let $U \subset \mathbb{P}^n$ be open and such that $E_{|U}$ is trivial; let $s_1, s_2 \in H^0(U, E_{|U})$ be a local basis for E over U. Then $s_{|U} = f_1 s_1 + f_2 s_2$ for suitable regular functions $f_1, f_2 \in H^0(U, \mathcal{O})$. One obtains a global sheaf of ideals $\mathcal{I}_Y \subset \mathcal{O}_{\mathbb{P}^n}$ with $\mathcal{I}_Y|_U = (f_1, f_2)\mathcal{O}_{\mathbb{P}^n|U}$. We have $Supp(\mathcal{O}_{\mathbb{P}^n}/\mathcal{I}_Y) = Y$, it follows that $Y = (Y, \mathcal{O}_{\mathbb{P}^n}/\mathcal{I}_Y)$ is a codimension 2 locally complete intersection in $\mathbb{P}^n$, as we already observed. Notice that Y may not be reduced.

$\mathcal{I}_Y/\mathcal{I}_Y^2$ is in natural way a sheaf of $\mathcal{O}_{\mathbb{P}^n}/\mathcal{I}_Y$-modules and as such it is locally free of rank 2 (over Y), for if $s_{|U} = f_1 s_1 + f_2 s_2$, then the germs $f_{1,x}, f_{2,x}$, for $x \in U \cap Y$ form a regular sequence and represent a $\mathcal{O}_{Y,x}$-module basis of $\mathcal{I}_Y/\mathcal{I}_Y^2$. Usually, $\mathcal{I}_Y/\mathcal{I}_Y^2$ is called the *conormal bundle* of Y in $\mathbb{P}^n$, so that the normal bundle $N_{Y/\mathbb{P}^n} = (\mathcal{I}_Y/\mathcal{I}_Y^2)^*$.

Over U the ideal sheaf $\mathcal{I}_Y$ has the free resolution:

$$0 \to \mathcal{O}_U \xrightarrow{\alpha} \mathcal{O}_U \oplus \mathcal{O}_U \xrightarrow{\beta} \mathcal{I}_Y|_U \to 0,$$

with $\alpha(g) = (-f_{2,x}g, f_{1,x}g)$, and $\beta(g,h) = f_{1,x}g + f_{2,x}h$, for $x \in U$, $g, h \in \mathcal{O}_{U,x}$, $f_{i,x} \in \mathcal{O}_{U,x}$. The sequence is exact because $f_{1,x}, f_{2,x} \in \mathcal{O}_{U,x}$ is a regular sequence (in the sense of commutative algebra), for any point $x \in U$.

These local sequences yield a global resolution:

$$0 \to det(E^*) \xrightarrow{\alpha} E^* \xrightarrow{\beta} \mathcal{I}_Y \to 0, \tag{3.1}$$

with $\alpha(\phi_1 \wedge \phi_2) = \phi_1(s_x)\phi_2 - \phi_2(s_X)\phi_1$ and $\beta(\phi) = \phi(s_x)$, for $x \in \mathbb{P}^n$, $\phi_1, \phi_2, \phi \in E_x^* = Hom(E_x, \mathcal{O}_{\mathbb{P}^n,x})$. (As always, s_x denotes the germ of s at the point x).

The sequence (3.1) is the *Koszul complex* for s, and as we have just seen, it is a locally free resolution of $\mathcal{I}_Y$, if s has a codimension 2 zero set. If one restricts the Koszul complex to Y, one gets the exact sequence:

$$det(E^*) \otimes \mathcal{O}_Y \xrightarrow{\alpha \otimes 1_{\mathcal{O}_Y}} E^*_{|Y} \to \mathcal{I}_Y/\mathcal{I}_Y^2 \to 0.$$

$\alpha \otimes 1_{\mathcal{O}_Y}$ is the zero homomorphism as one sees from the local description. Hence we get an isomorphism:

$$E^*_{|Y} \cong \mathcal{I}_Y/\mathcal{I}_Y^2.$$

Thus E is an extension to all of $\mathbb{P}^n$ of the normal bundle $N_{Y/\mathbb{P}^n}$.

The Serre correspondence is essentially a successful attempt to reverse this construction. Thus, for a given locally complete intersection $(Y, \mathcal{O}_Y)$, we seek an extension of the normal bundle $N_{Y/\mathbb{P}^n}$ such that a regular section of this extension has as zero set precisely Y, with structure sheaf $\mathcal{O}_Y$. An obvious necessary condition for the extendability of the normal bundle $N_{Y/\mathbb{P}^n}$ is the extendability of the determinant bundle $det(N_{Y/\mathbb{P}^n})$. Indeed, if the determinant bundle is extendable, then E^*, if it exists, is an extension (in the sense of homological algebra) of the sheaf of ideals $\mathcal{I}_Y$, by an extension over $\mathbb{P}^n$ of $det(N^*_{Y/\mathbb{P}^n})$, as the sequence (3.1) shows. The following theorem shows that the extendability of $det(N_{Y/\mathbb{P}^n})$ is also sufficient for the extendability of $N_{Y/\mathbb{P}^n}$.

Theorem (Serre's correspondence): *Let Y be a locally complete intersection of codimension 2 in $\mathbb{P}^n$, $n \geq 3$, with sheaf of ideals $\mathcal{I}_Y \subset \mathcal{O}_Y$. Let the determinant bundle of the normal bundle be extendable: $det(N_{Y/\mathbb{P}^n}) = \mathcal{O}_{\mathbb{P}^n}(k)_{|Y}$ ($k \in \mathbb{Z}$). Then there is an algebraic 2-bundle E over $\mathbb{P}^n$ with a section s which has precisely $(Y, \mathcal{O}_Y)$ as zero set. Moreover s induces the exact sequence:*

$$0 \to \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\cdot s} E \to \mathcal{I}_Y(k) \to 0. \tag{3.2}$$

The Chern classes of E are given by: $c_1(E) = k$, $c_2(E) = deg(Y)$ (if Y is smooth).

Proof: If there is a bundle E as claimed, then $det(N^*_{Y/\mathbb{P}^n}) = det(E^*_{|Y})$, so we would choose $det(E^*) = \mathcal{O}_{\mathbb{P}^n}(-k)$. Thus E^* is an extension:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-k) \to E^* \to \mathcal{I}_Y \to 0.$$

Thus we investigate the extension of $\mathcal{I}_Y$ by $\mathcal{O}_{\mathbb{P}^n}(k)$. These are classified by global Ext-group (see [31], page 725) $Ext^1(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k))$. For the calculation of this group, we employ the lower term sequence of the spectral sequence (see [52], page 706):

$$E_2^{p,q} = H^p(\mathbb{P}^n, \mathcal{E}xt^q(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(k))) \Rightarrow E^{p+q} = Ext^{p+q}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)).$$

The induced exact sequence is the following:

$$0 \to H^1(\mathbb{P}^n, \mathcal{H}om(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k))) \to Ext^1(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) \to$$

$$\to H^0(\mathbb{P}^n, \mathcal{E}xt^1(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k))) \to H^2(\mathbb{P}^n, \mathcal{H}om(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)))$$

On the other hand, the sequence $0 \to \mathcal{I}_Y \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_Y \to 0$ gives rise to the long exact sequence:

$$0 \to \mathcal{H}om(\mathcal{O}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) \to \mathcal{H}om(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(-k)) \to$$

$$\to \mathcal{H}om(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) \to \mathcal{E}xt^1(\mathcal{O}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) \to \dots$$

Since Y is a locally complete intersection of codimension 2, we get (see [31], page 690) $\mathcal{E}xt^i(\mathcal{O}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) = 0$ for $i = 0, 1$, and thus $\mathcal{H}om(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) \cong \mathcal{H}om(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(-k)) = \mathcal{O}_{\mathbb{P}^n}(-k)$.

If we put these results into the lower term sequence induced by the spectral sequence, we get the exact sequence:

$$0 \to H^1(\mathbb{P}^n, \mathcal{O}(-k)) \to Ext^1(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) \to$$

$$\to H^0(\mathbb{P}^n, \mathcal{E}xt^1(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k))) \to H^2(\mathbb{P}^n, \mathcal{O}(-k)) \to \dots$$

In particular, for $n \geq 3$, we have $Ext^1(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) \cong H^0(\mathbb{P}^n, \mathcal{E}xt^1(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)))$. For $n = 2$, the group $H^2(\mathbb{P}^n, \mathcal{O}(-k))$ is zero only for $k < 3$, so that the previous isomorphism also holds in the case $n = 2, k < 3$.

Now, we calculate $\mathcal{E}xt^1(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k))$: from the $\mathcal{E}xt$-sequence associated to $0 \to \mathcal{I}_Y \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_Y \to 0$, we get the isomorphism $\mathcal{E}xt^1(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) \cong \mathcal{E}xt^2(\mathcal{O}_Y, \mathcal{O}_{\mathbb{P}^n}(-k))$. Since Y is a codimension 2 locally complete intersection, we have (see for instance [31], page 690), the local fundamental isomorphism (LFI):

$$\mathcal{E}xt^2(\mathcal{O}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) \cong \mathcal{H}om(det(\mathcal{I}_Y/\mathcal{I}_Y^2), \mathcal{O}_Y(-k)),$$

where $\mathcal{O}_Y(-k) = \mathcal{O}_{\mathbb{P}^n}(-k) \otimes \mathcal{O}_Y$.

However, by assumption $det(\mathcal{I}_Y/\mathcal{I}_Y^2) = \mathcal{O}_Y(-k)$, so $\mathcal{H}om(det(\mathcal{I}_Y/\mathcal{I}_Y^2), \mathcal{O}_Y(-k)) \cong \mathcal{O}_Y$. Thus, altogether, we have a canonical isomorphism of sheaves $\mathcal{E}xt^1(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) \cong \mathcal{O}_Y$, which implies that $Ext^1(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) \cong H^0(Y, \mathcal{O}_Y)$.

Now we consider the extension represented by $1 \in H^0(Y, \mathcal{O}_Y)$:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-k) \to F \to \mathcal{I}_Y \to 0,$$

with F a coherent sheaf over $\mathbb{P}^n$.

Claim: *F is a locally free sheaf over $\mathbb{P}^n$*

Proof of the claim: Let $x \in \mathbb{P}^n$. Due to what we have so far proved, the germ 1_x of 1 in the point x is an element in $(\mathcal{E}xt^1(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)))_x = Ext^1_{\mathcal{O},x}(\mathcal{I}_{Y,x}, \mathcal{O}_{\mathbb{P}^n,x}(-k))$ and defines the extension:

$$0 \to \mathcal{O}_{\mathbb{P}^n,x}(-k) \to F_x \to \mathcal{I}_{Y,x} \to 0.$$

Since 1_x naturally generates the $\mathcal{O}_{\mathbb{P}^n,x}$-module $Ext^1_{\mathbb{P},x}(\mathcal{I}_{Y,x}, \mathcal{O}_{\mathbb{P}^n,x}(-k)) \cong \mathcal{O}_{Y,x}$, it turns out that F_x is a free $\mathcal{O}_{\mathbb{P}^n,x}$-module and so F is locally free, according to the following lemma of Serre:

Lemma (Serre): *Let A be a noetherian local ring, $I \subset A$ an ideal with a free resolution of lenght 1:*

$$0 \to A^p \to A^q \to I \to 0.$$

(e.g. the Koszul complex $0 \to A \to A^{\oplus 2} \to I \to 0$, if I is generated by a regular sequence (f_1, f_2)). Let $e \in Ext^1_A(I, A)$ be represented by the extension:

$$0 \to A \to M \to I \to 0.$$

Then M is a free A-module iff e generates the A module $Ext^1_A(I, A)$.

Proof of the Lemma: Assume that M is a free A-module. Then, the Ext-sequence associated to $0 \to A \to M \to I \to 0$ gives:

$$\ldots \to Hom_A(A, A) \xrightarrow{\delta} Ext^1_A(I, A) \to Ext^1(M, A) \to Ext^1_A(A, A).$$

On the other hand, M and A are free A-modules so that $Ext^1_A(M, A) = Ext^1_A(A, A) = 0$ In this case δ is surjective and since $\delta(id_A) = e$, this happens precisely when e generates the A-module $Ext^1_A(I, A)$

On the other hand, since $Ext^1_A(A, A)$ is always zero, $Ext^1_A(M, A) = 0$ precisely when δ is surjective (and this in turns happens when e geerates $Ext^1_A(I, A)$). Thus, it remains to show that $Ext^1_A(M, A) = 0 \Rightarrow M$ is free.

To this aim, we construct out of the exact sequences:

$$0 \to A^p \to A^q \xrightarrow{\phi} I \to 0$$

$$0 \to A \xrightarrow{\alpha} M \xrightarrow{\beta} I \to 0$$

a free resolution of length 1 for M: let $\varphi : A^q \to M$ be a lifting of ϕ to M and $\psi : A \oplus A^q \to M$ be defined by $\psi(x, y) = \alpha(x) + \varphi(y)$. Then we have a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & A & \to & A \oplus A^q & \to & A^q & \to & 0 \\ & & \downarrow = & & \downarrow \psi & \swarrow \varphi & \downarrow \phi & & \\ 0 & \to & A & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & I & \to & 0 \end{array}$$

By snake lemma, it follows that $ker(\psi) \cong ker(\phi) = A^p$ and $coker(\psi) = 0$. Thus we have an exact sequence:

$$0 \to A^p \to A^r (= A \oplus A^q) \to M \to 0.$$

Since $Ext^1_A(M, A) = 0$, this sequence splits. Hence is a direct sum in A^r, hence projective and thus free. □

Coming back to the proof of the theorem (Serre correspondence), we have an extension:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-k) \to F \xrightarrow{\beta} \mathcal{I}_Y \to 0, \tag{3.3}$$

with a 2-bundle F over $\mathbb{P}^n$. $E = F^*$ is the bundle we are searching for, and (3.3) is just the Koszul complex of a section $s \in H^0(\mathbb{P}^n, E)$. Multiplication with s, $.s : \mathcal{O}_{\mathbb{P}^n} \to E$, is dual to the composition:

$$E^* \xrightarrow{\beta}> \mathcal{I}_Y \hookrightarrow \mathcal{O}_{\mathbb{P}^n}.$$

On the other hand, let $\beta' : E \to> \mathcal{I}_Y(k)$, be the composition:

$$E \cong E^* \otimes det(E) = E^*(k) \xrightarrow{\beta(k)} \mathcal{I}_Y(k),$$

where $E \cong E^* \otimes det(E) = \mathcal{H}om_{\mathbb{P}^n}(E, \wedge^2 E)$ is the canonical isomorphism given by $s \mapsto (t \mapsto s \wedge t)$. From this we get the exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^n} \xrightarrow{.s} E \xrightarrow{\beta'} \mathcal{I}_Y(k) \to 0,$$

and this concludes the proof of the theorem. □

The relation between codimension 2 locally complete intersections in $\mathbb{P}^n$ and rank 2 vector bundles is even closer, as displayed by the following:

Proposition: *Let E be a 2-bundle which is associated to a locally complete intersection $Y \subset \mathbb{P}^n$, $n \geq 3$. Then E splits iff Y is a* global *complete intersection.*

Proof: Let Y be defined by the section $s \in H^0(\mathbb{P}^n, E)$. If E splits, let us say $E = \mathcal{O}(a) \oplus \mathcal{O}(b)$, then $s = (s_1, s_2)$, with $s_1 \in H^0(\mathbb{P}^n, \mathcal{O}(a))$, $s_2 \in H^0(\mathbb{P}^n, \mathcal{O}(b))$, and $Y = \{s = 0\}$ is the intersection of the hypersurfaces given by $\{s_1 = 0\}$ and $\{s_2 = 0\}$.

Conversely, let Y be the intersection of two hypersurfaces V_a, V_b of degree a respectively b : $V_a = \{s_1 = 0\}$, $V_b = \{s_2 = 0\}$, with $s_1 \in H^0(\mathbb{P}^n, \mathcal{O}(a))$, $s_2 \in H^0(\mathbb{P}^n, \mathcal{O}(b))$. The Koszul complex of the section $s = (s_1, s_2)$ in $\mathcal{O}(a) \oplus \mathcal{O}(b)$ gives the extension:

$$0 \to \mathcal{O}_{\mathbb{P}^n}((-(a+b))) \to \mathcal{O}_{\mathbb{P}^n}(-a) \oplus \mathcal{O}_{\mathbb{P}^n}(-b) \to \mathcal{I}_Y \to 0.$$

This extension defines a non-zero element in $Ext^1(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-(a+b))) \cong H^0(Y, \mathcal{O}_Y)$. Thus, if we can show that $H^0(Y, \mathcal{O}_Y)$ is 1-dimensional, then every other non-trivial extension of $\mathcal{I}_Y$ by $\mathcal{O}_{\mathbb{P}^n}(-(a+b))$ must give the split bundle. Now we prove that $h^0(Y, \mathcal{O}_Y) = 1$ for every global complete intersection Y of codimension 2 in $\mathbb{P}^n$, $n \geq 3$. (In particular Y is connected). Indeed, from the cohomology sequence associated to:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-(a+b)) \to \mathcal{O}_{\mathbb{P}^n}(-a) \oplus \mathcal{O}_{\mathbb{P}^n}(-b) \to \mathcal{I}_Y \to 0,$$

we get immediately ($a, b > 0$, $n \geq 3$): $h^0(\mathbb{P}^n, \mathcal{I}_Y) = h^1(\mathbb{P}^n, \mathcal{I}_Y) = 0$. Thus $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) \cong H^0(Y, \mathcal{O}_Y)$. □

Due to this kind of correspondence, it is possible (and useful !) to relate the problem of recognizing complete intersection in codimension 2, with the problem of proving that a certain rank 2 vector bundle is split.

3.5 A scheme-theoretic criterion

As we have seen in the previous sections, the problem of detecting (global) complete intersections is a key question in projective algebraic geometry and commutative algebra. Up to now, this problem is far from being solved and a complete answer is known only in trivial cases, such that of hypersurfaces in projective spaces or in Grassmannians. Moreover, the outstanding conjecture of Hartshorne is still not proved. Besides the various "evidences" supporting the conjecture, in the last 25 years, there have been some partial results in this direction, particularly in the case of codimension 2. Essentially, the results obtained in the case X is a smooth codimension 2 subvariety of $\mathbb{P}^n$, $n \geq 6$ can be grouped into two kinds of criteria.

The first one says that if X is contained in a hypersurface V, such that $deg(V) \leq n-2$, then X is a complete intersection (see [35] or the recent improvement in [24], where it is shown that the bound on the degree of V can be increased to $n-1$, in the case of codimension 2 subvarieties of $\mathbb{P}^6$); using this kind of criterion one can give also a bound on the degree of X, so that to assure that X is a complete intersection.

The second kind of criterion is based on giving a bound on the number p of generators, not for the homogeneous ideal $I(X)$, but for an ideal $I_{sch}(X)$ which coincide with $I(X)$ only in high degree, that is $[I_{sch}(X)]_d = [I(X)]_d$, for $d \gg 0$. We call $I_{sch}(X)$ the "schematic" ideal of X, in that its generators define X scheme-theoretically. Following this approach, Faltings proved in [26] that if $p \leq n-2$ and $n \geq 8$ and X is a (possibly singular) subcanonical local complete intersection, then it is a complete intersection (in any characteristic). Some years later, this result was improved in [34], proving in characteristic zero that if $p \leq n-1$, $n \geq 8$, then X is a complete intersection (but assuming X smooth).

The aim of this section is twofold: on one hand we would like to give a different (in that we use Serre's correspondence) and simpler proof of the result announced in [34], hoping to give a "crystal clear" version of some obscure (to our opinion) arguments. Moreover, assuming only that X is a (possibly singular) subcanonical l.c.i., we prove in any characteristic that if $n \geq 3$ and $p \leq n-1$, then X is a complete intersection and we give some more results, assuming that the normal bundle of X extends to a numerically split bundle E on $\mathbb{P}^n$ (i.e. the Chern classes of E are those of a split bundle), $n \geq 3$, $p \leq n$. On the other hand, as an application of our result we answer to a question posed recently by Franco, Kleiman and Lascu in [27], (neglecting the case of space curves). Unfortunately, our result shows that the characterization given by Faltings *is not peculiar* of two codimension embeddings in high dimensional projective spaces.

Our proof is based in constructing and exploiting an exact sequence of locally free sheaves, (sequence (3.6)), which relates the rank 2 vector bundle E appearing in Serre's correspondence with the generators of the "scheme-theoretic" ideal of X.

Untill the end of this section, X will denote a codimension 2 *subcanonical* l.c.i. (possibly singular) closed subscheme of a projective space $\mathbb{P}^n$ over an algebraically closed field k of any characteristic, where, as usual $\mathbb{P}^n = Proj(k[x_0, \dots, x_n])$. Then, under these assumptions, we prove the following result:

Theorem A: *If $X \subset \mathbb{P}^n$, (X as above, $n \geq 3$) is a scheme-theoretic intersection of $p \leq n-1$ hypersurfaces, then X is a complete intersection.*

Proof: Since X is assumed to be subcanonical (i.e. its dualizing sheaf ω_X, which is locally free, is of the form $\mathcal{O}_X(e)$) , by Serre's correspondence there exists an algebraic vector bundle E of rank 2 over $\mathbb{P}^n$ and a section $s \in H^0(\mathbb{P}, E)$ such that X is identified with the scheme of zeroes of s, $Z(s)$. The Koszul complex for this section gives a projective resolution of the ideal sheaf of $Z(s)$, hence of the ideal sheaf of X:

$$0 \longrightarrow \bigwedge^2 E^* \longrightarrow E^* \longrightarrow \mathcal{I}_X \longrightarrow 0. \tag{3.4}$$

Since $\mathcal{I}_X$ is not itself projective, by (3.4) it turns out that the projective dimension of $\mathcal{I}_X$ is 1. On the other hand, if X is schematically cut out by $p \leq n-1$ hypersurfaces of degrees $d_1, \dots, d_p$, we have an exact sequence:

$$0 \longrightarrow Ker(f) \longrightarrow \bigoplus \mathcal{O}(-d_i) \xrightarrow{f} \mathcal{I}_X \longrightarrow 0. \tag{3.5}$$

Since pd($\mathcal{I}_X$)=1, then the first syzygy $Ker(f)$ is also projective (see for example [57]), hence it corresponds to a locally free sheaf. Certainly, we can construct a morphism

$h \in Hom(\oplus\mathcal{O}(-d_i) \oplus \mathcal{O}(-c_1), \mathcal{I}_X)$ which is given first by projecting to $\oplus\mathcal{O}(-d_i)$ and then composing with f, (c_1 is the first Chern class of E). Moreover, since $Ext^1(\oplus\mathcal{O}(-d_i) \oplus \mathcal{O}(-c_1), \mathcal{O}(-c_1)) = 0$, then h comes from an element g in $Hom(\oplus\mathcal{O}(-d_i) \oplus \mathcal{O}(-c_1), E^*)$; indeed, due to the fact that $\bigwedge^2 E^* \equiv \mathcal{O}(-c_1)$, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(-c_1) & \longrightarrow & \bigoplus \mathcal{O}(-d_i) \oplus \mathcal{O}(-c_1) & \longrightarrow & \bigoplus \mathcal{O}(-d_i) & \longrightarrow & 0 \\ & & \downarrow\equiv & & \downarrow g & & \downarrow f & & \\ 0 & \longrightarrow & \bigwedge^2 E^* & \longrightarrow & E^* & \longrightarrow & \mathcal{I}_X & \longrightarrow & 0 \end{array}$$

Applying the snake lemma to the previous commutative diagram, we see that g is surjectve and that $Ker(g) \equiv Ker(f)$, so that dualizing the sequence $0 \longrightarrow Ker(g) \longrightarrow \bigoplus \mathcal{O}(-d_i) \oplus \mathcal{O}(-c_1) \xrightarrow{g} E^* \longrightarrow 0$, we get a short exact sequence of *locally free* sheaves:

$$0 \longrightarrow E \xrightarrow{\alpha_1} \bigoplus \mathcal{O}(d_i) \oplus \mathcal{O}(c_1) \xrightarrow{\alpha_2} C \longrightarrow 0, \tag{3.6}$$

where C is just the cokernel sheaf. Since C is locally free, it can be identified with a vector bundle of rank equal to $p-1$ ($p \leq n-1$). Let in_j denote the canonical injection of $\mathcal{O}(d_j)$ into $\bigoplus \mathcal{O}(d_i) \oplus \mathcal{O}(c_1)$, and pr_j the corresponding projection from $\bigoplus \mathcal{O}(d_i) \oplus \mathcal{O}(c_1)$ to $\mathcal{O}(d_j)$. Considering the maps $f_j := pr_j \circ \alpha_1$ and $g_j := \alpha_2 \circ in_j$ we have a diagram like the following:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \xrightarrow{\alpha_1} & \bigoplus \mathcal{O}(d_i) \oplus \mathcal{O}(c_1) & \xrightarrow{\alpha_2} & C & \longrightarrow & 0 \\ & & & \overset{f_j}{\searrow} & pr_j \downarrow \ \uparrow in_j & \overset{g_j}{\nearrow} & & & \\ & & & & \mathcal{O}(d_j) & & & & \end{array}$$

Now consider the morphisms $E \xrightarrow{f_1} \mathcal{O}(d_1)$ and $\mathcal{O}(d_1) \xrightarrow{g_1} C$ and denote by $Z(f_1)$ and $Z(g_1)$, their respective degeneracy loci. Since in general we have that $codim(Z(f_j)) \leq 2$ and $codim(Z(g_j)) \leq p-1$, it turns out that if $p \leq n-1$, then $Z(f_1) \cap Z(g_1) \neq \emptyset$. On the other hand, by exactness of (3.6), it is clear that $Z(f_1) \cap Z(g_1) = \emptyset$. Indeed, if it exists $x \in Z(f_1) \cap Z(g_1)$, then $Ker(f_1)_x = E_x$, but since the morphism α_1 can not degenerate at any point (due to the fact that the cokernel C is locally free), we have that $Im(\alpha_1)_x \subset \oplus_{i\geq 2}\mathcal{O}_x(d_i) \oplus \mathcal{O}_x(C_1)$. On the other hand, $Ker(g_1)_x = \mathcal{O}_x(d_1)$, but since in_1 is always injective, we have that $Ker(\alpha_2)_x = in_1(\mathcal{O}(d_1))_x$. Hence, if exists $x \in Z(f_1) \cap Z(g_1)$, then $Ker(\alpha_2)_x \cap Im(\alpha_1)_x = \emptyset$ and so the sequence (3.6) can not be exact in the middle, at x. Absurd.

Hence $Z(f_1) = \emptyset$ or $Z(g_1) = \emptyset$.

If $Z(f_1) = \emptyset$, then f_1 is never degenerate, so dualizing $E \xrightarrow{f_1} \mathcal{O}(d_1) \longrightarrow 0$ we get $0 \longrightarrow \mathcal{O}(d_1) \xrightarrow{(f_1)^T} E^*$, where the map $(f_1)^T$ is never degenerate; hence E^* splits, so E splits and X is a complete intersection.

If instead $Z(g_1) = \emptyset$, we build up the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & \mathcal{O}(d_1) & \xrightarrow{\equiv} & \mathcal{O}(d_1) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow in_1 & & \downarrow g_1 & & \\
0 & \longrightarrow & E & \longrightarrow & \bigoplus \mathcal{O}(d_i) \oplus \mathcal{O}(c_1) & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Ker(\psi) & \longrightarrow & \bigoplus_{i\geq 2} \mathcal{O}(d_i) \oplus \mathcal{O}(c_1) & \xrightarrow{\psi} & C' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & &
\end{array}$$

By applying the snake lemma to the two central rows, we see that $Ker(\psi) \equiv E$, C' is locally free since $Z(g_1) = \emptyset$, and we obtain a short exact sequence of *locally free* sheaves:

$$0 \longrightarrow E \longrightarrow \bigoplus_{i\geq 2} \mathcal{O}(d_i) \oplus \mathcal{O}(c_1) \longrightarrow C' \longrightarrow 0. \tag{3.7}$$

Repeating the previous reasoning, we can consider the morphisms $E \xrightarrow{f_2} \mathcal{O}(d_2)$ and $\mathcal{O}(d_2) \xrightarrow{g_2} C'$ and as before $Z(f_2) = \emptyset$ or $Z(g_2) = \emptyset$. If $Z(f_2) = \emptyset$, then E splits and X is a complete intersection. On the other hand, if $Z(g_2) = \emptyset$, arguing as before, we obtain a short exact sequence of locally free sheaves:

$$0 \longrightarrow E \longrightarrow \bigoplus_{i\geq 3} \mathcal{O}(d_i) \oplus \mathcal{O}(c_1) \longrightarrow C'' \longrightarrow 0.$$

In this way we obtain a sequence $Z(f_1), \dots, Z(f_{p-1})$. If one of these is empty, we are done; otherwise, if all are not empty, then, necessarily, $Z(g_{p-1}) = \emptyset$ and as before we obtain:

$$0 \longrightarrow E \longrightarrow \mathcal{O}(d_p) \oplus \mathcal{O}(c_1) \longrightarrow 0,$$

and we are done. □

We can obtain a similar result, for the case $p \leq n$, under the additional hypotheses of the following:

Theorem B: *Let X (as above) be scheme-theoretically defined by $p \leq n$ hypersurfaces $V_1, \dots, V_n$ of degrees $d_1, \dots, d_p$, respectively. If the normal bundle of X can be extended to a rank 2 vector bundle E on $\mathbb{P}^n$ which is numerically split (i.e. $c_1(E) = a + b$ and $c_2(E) = ab$, $a, b \in \mathbb{Z}$) and a* or *b is in $(d_1, \dots, d_p)$, then X is a complete intersection.*

Proof: In our assumption X is cut out schematically by n hypersurfaces $V_1, \dots, V_n$ of degrees $d_1, \dots, d_n$ and we have that $c_1(E) := a + b = d_k + b$ and $c_2(E) := ab = d_k b$ for some $k \in (1, \dots, n)$. (It is not restrictive to assume that $a \in (d_1, \dots, d_n)$). Reordering the hypersurfaces, we can assume that $c_1(E) = d_1 + b$ and $c_2(E) = d_1 b$. From the exact sequence (3.6), it is clear that the rank of C is $n-1$, so that the morphism $\mathcal{O}(d_1) \xrightarrow{g_1} C$ degenerates at most in codimension $n-1$. On the other hand, the morphism $E \xrightarrow{f_1} \mathcal{O}(d_1)$ can not degenerate in codimension 2, otherwise, if $Z(f_1)$ is its degeneracy locus, the corresponding class in the Chow ring $[Z(f_1)] \in A^2(\mathbb{P}^n)$ would represent $c_2(E^* \otimes \mathcal{O}(d_1)) = c_1(E^*)d_1 + c_2(E^*) + d_1^2 = 0$, so that either the morphism f_1 does not degenerate at all, and in this case we are done as before, or it degenerates in codimension 1.

So if the morphism f_1 degenerates in codimension one, we have that $Z(f_1) \cap Z(g_1) \neq \emptyset$, provided that $Z(g_1) \neq \emptyset$. On the other hand, by exactness of (3.6), we must have $Z(f_1) \cap Z(g_1) = \emptyset$, so that we conclude that $Z(g_1) = \emptyset$. Thus, arguing as in part A, we obtain the following short exact sequence:

$$0 \longrightarrow E \longrightarrow \bigoplus_{i \geq 2}^{n} \mathcal{O}(d_i) \oplus \mathcal{O}(c_1) \longrightarrow C' \longrightarrow 0,$$

and, from this, we conclude as in Theorem A, since the rank of C' is equal to $n-2$. □

The result of Theorem B can be interpreted as a relation between degree $deg(X)$ and subcanonicity e, recalling the well-known fact that if E is the vector bundle associated to X via Serre's correspondence, then $deg(X) = c_2(E)$, while $e + n + 1 = c_1(E)$.

Corollary A: *Let $X \subset \mathbb{P}^n$ (X as in the hypotheses of Theorem B), $n \geq 3$ be scheme-theoretically defined by n hypersurfaces of degrees $d_1, \ldots, d_n$, and let l be an integer in the set $(d_1, \ldots, d_n)$. If the following relation is satisfied:*

$$deg(X) + l^2 - (e + n + 1)l = 0, \tag{3.8}$$

then X is a complete intersection.

Proof: Arguing as in part B, it is clear that to show that E splits is sufficient to show that $c_2(E^* \otimes \mathcal{O}(l)) = 0$ for some l as above. But the vanishing of the second Chern class of $E^* \otimes \mathcal{O}(l)$ is exactly the relation (3.8), as an easy computation can show. □

Remark 1: The approach of giving bounds on the degree of a subvariety to detect a complete intersection is particularly "effective", but it is obviously hopeless if one pretend to solve Hartshorne's conjecture. On the other hand, since any closed subscheme (irreducible or not) of $\mathbb{P}^n$ which is a local complete intersection is always scheme-theoretically defined by $n+1$ hypersurfaces, as proved in [28], the approach of recognizing a complete intersection via the number of generators of its scheme-theoretic ideal, could be in principle useful to solve the conjecture. Unfortunately, the cases $p = n$ and in particular $p = n + 1$ (the generic case) appear completely intractable, at least up to now, since it is very difficult to relate the algebro-geometric properties of a small codimension embedding, with those of its scheme-theoretic ideal.

Remark 2: In the light of the previous remark and of Theorem B, it would be nice to know when a subvariety can be scheme-theoretically defined by n equations. To get a sufficient condition, we can use the theory of excess and residual intersections as developed in [28]. For example, let us consider 4 hypersurfaces $\{V_1, \ldots, V_4\}$ in $\mathbb{P}^4$, such that $\cap V_i = X \cup \{p_1, \ldots, p_k\}$, where X is a smooth subcanonical surface and $\{p_1, \ldots, p_k\}$ are (possibly non reduced) points, i.e. the four hypersurfaces define scheme-theoretically the union of X and a bunch of points outside X. The theory of residual intersections enables us to predict the (weighted) number of residual points as a function of the degrees of the hypersurfaces, of the degree of X and of the degrees of the Chern classes of T_X, the tangent sheaf of X. Imposing that the number of residual points is zero, gives us a sufficient condition for a surface (a subvariety in general) to be scheme-theoretically defined by n equations. (The basic idea is not new, and it appears, perhaps for the first time, in the work of George Salmon ([29]), where it is exploited particularly for the case of space curves; then it appears again in the work of several classical geometers, such as Vahlen, Enriques ([25]), Severi, up to Gherardelli ([30])). In this way, combining Proposition 9.12 (page 154) of [28] with Example 9.1.5. we get (for a surface in $\mathbb{P}^4$):

$$deg(c_2(T_X)) + (\sigma_1(g_i) - 5)deg(c_1(T_X)) +$$

$$+(\sigma_2(g_i) - 5\sigma_1(g_i) + 15)deg(X) + W(p_1, \ldots, p_k) = \sigma_4(g_i), \tag{3.9}$$

where $W(p_1, \ldots, p_k)$ is the weighted number of residual points and $\sigma_j(g_i)$ is the j-th elementary symmetric polynomial in the degrees g_i of the hypersurfaces V_i. Assuming X subcanonical, from $c_1(T_X) = -K_X$, we get $c_1(T_X) = -eH$, where H is the class of a hyperplane section; moreover, from the exact sequence:

$$0 \longrightarrow T_X \longrightarrow T_{\mathbb{P}^4} \otimes \mathcal{O}_X \longrightarrow N_{X/\mathbb{P}^4} \longrightarrow 0,$$

we get $c_2(T_X) = 10H^2 + 5HK + K^2 - c_2(N_{X/\mathbb{P}^4})$, and by $c_1(T_X) = -eH$, we have $c_2(T_X) = (10 - 5e + e^2)H^2 - c_2(N_{X/\mathbb{P}^4})$. Since $deg(c_2(N_{X/\mathbb{P}^4})) = d^2$, $deg(H^2) = d$ and $deg(H) = d$, (where d is $deg(X)$), substituting in (3.9), taking degrees and imposing $W(p_1, \ldots, p_k) = 0$ reads:

$$[25 + \sigma_2(g_i) - (5 + e)\sigma_1(g_i) + e^2 - d]d = \sigma_4(g_i). \tag{3.10}$$

Thus, the relation (3.10) gives a sufficient condition for a subcanonical surface in $\mathbb{P}^4$ to be scheme-theoretically defined by 4 equations.

3.5.1 An application: the linkage criterion

As usual, if X and Y are l.c.i. of codimension 2 in $\mathbb{P}^n$, we say that X is (directly) linked to Y if there exists a complete intersection (F_1, F_2), such that Y is the residual scheme of X in the intersection $F_1 \cap F_2$, and viceversa. In [23], working in characteristic zero and assuming X smooth and subcanonical $dim(X) \geq 1$, Beorchia and Ellia proved that X is a complete intersection iff it is self-linked, i.e. iff there exists a complete intersection (F_1, F_2) such that $F_1 \cap F_2 = 2X$ (F_1 and F_2 define on X a double structure which is a complete intersection). They also asked if the same criterion holds also for possibly singular l.c.i.. Recently, in [27], Franco, Kleiman and Lascu have given a positive answer to this question proving that the same criterion holds avoiding smoothness: X can be reducible and nonreduced. Their proof works only in characteristic zero (unless $dim(X) \geq 4$, where it holds over any algebraically closed field, due to a previous result of Faltings), so they ask if the same holds in positive characteristic, for lower dimensional X. Using our Theorem A we prove the following:

Proposition A: *Let X be a subcanonical (possibly singular) l.c.i. subscheme of codimension 2 in $\mathbb{P}^n$, $n \geq 4$, defined over an algebraically closed field of any characteristic. Then X is a complete intersection iff it is self-linked.*

Proof: According to the "Gherardelli linkage theorem", which holds over any algebraically closed field (see [27] for its proof) we know that $X \subset F_1 \cap F_2$ is subcanonical iff its residual scheme Y (in the complete intersection $F_1 \cap F_2$) is scheme-theoretically defined by the intersection of F_1 and F_2 with a third hypersurface F_3. On the other hand, if X is self-linked by F_1 and F_2, then, by definition X is equal to its own residual scheme in the complete intersection of F_1 and F_2, and since X is assumed subcanonical, by the Gherardelli theorem it is scheme theoretically defined by F_1, F_2 and F_3; hence, by Theorem A, it is a complete intersection as soon as $dim(X) \geq 2$. Viceversa, if X is a complete intersection, it is immediate to see that it is self-linked (just consider the intersection of F_1 and $2F_2$ if $X = F_1 \cap F_2$). □

There is an immediate generalization of the previous proposition, which is the following:

Proposition B: *Let X as in Proposition A. Then X is a complete intersection iff it can be (directly) linked to Y, where Y is any subcanonical (possibly singular) l.c.i. subscheme.*

Proof: It is sufficient to use again the Gherardelli linkage and Theorem A. □

Remark 3: The X's as in Proposition B are self-linked iff they are scheme-theoretically defined by three hypersurfaces. Indeed, if X is self-linked, then by Gherardelli it is "schematically" defined by 3 equations; viceversa, if X is defined by 3 equations it is a complete intersection by Theorem A and then it is self-linked by Proposition B.

Remark 4: The most difficult case, in order to characterize complete intersections via self-linking is that of curves in $\mathbb{P}^3$. Beorchia and Ellia proved their criterion (in characteristic zero) also for curves, assuming that they are smooth, while Franco, Kleiman and Lascu extended this result to l.c.i. curves (always working in characteristic zero). In positive characteristic (characteristic 2), however, there is certainly a counterexample for this criterion to hold in the case of curves, due to Migliore (see the discussion at the end of [27]). So our extension of this criterion over a field of any characteristic is the best possible for low dimensional subvarieties: that is surfaces are the lowest dimensional (possibly singular) l.c.i. subvarieties where this criterion holds without exceptions. Unfortunately, up to now, there is no positive result for the case of space curves in characteristic grater than zero.

3.5.2 An elementary obstruction result

In this subsection, we give an obstruction result which forces a smooth codimension two subvariety to be defined scheme-theoretically by a number of equations which is not too small (unless the subvariety itself is a complete intersection). In fact, the number of equations depends on the dimension of the singular loci of the hypersurfaces which define scheme-theoretically the subvariety.

Proposition C: *Let X be a smooth subvariety of codimension two in $\mathbb{P}^n$, $n \geq 4$ and let X be defined scheme theoretically by the equations $\{f_1, \ldots, f_m\}$ corresponding to the hypersurfaces $\{Z_1, \ldots, Z_m\}$. Let also $dim(X \cap Sing(Z_i)) = n - \delta_i$. Then if there exists i such that $\delta_i > 4$ then X is a complete intersection. Moreover, in the other case, setting $\sigma_i := n - dim(Sing(Z_i))$ we have that $n+1 \geq m > f(n, \sigma_i)$, where $f(n,\sigma_i) := \frac{n}{\bar{\sigma}_i}$, and $\bar{\sigma}_i$ is the arithmetic mean of the $\{\sigma_i\}_{i=1,\ldots,m}$.*

Proof: If there exists i such that $\delta_i > 4$, this implies $dim(X \cap Sing(Z_i)) < n-4$, then cutting with a general $\mathbb{P}^4$ we will get a smooth surface S, lying on a hypersurface Z_i' in $\mathbb{P}^4$, such that $S \cap Sing(Z_i') = \emptyset$. But then , by Severi-Lefschetz theorem it would follow that S is a complete intersection and so also X itself is a complete intersection.

In the other case we have that $\delta_i \leq 4$ for all i. Now since $dim(X \cap Sing(Z_i)) = n - \delta_i \geq n-4$ for all i, this in particular implies that $n-1 \geq Sing(Z_i) \geq n-4$. Let us set $\sigma_i := n - dim(Sing(Z_i))$. Since X is smooth, hence a l.c.i., for any point $p \in X$ we have to find two hypersurfaces Z_i, Z_j which are smooth and meet transversally at p. On the other hand, if $dim(\cap_{i=1}^m Sing(Z_i) \geq 0$, then we certainly have that X can not be scheme-teoretically defined by these hypersurfaces (otherwise it would not be smooth). Since we are working in a projective space, we have that $dim(\cap_{i=1}^m Sing(Z_i) \geq n - \sum_{i=1}^m \sigma_i$, so that a necessary condition for X to be defined scheme-theoretically by these hypersurfaces is that $n - \sum_{i=1}^{m} \sigma_i < 0$, which can immediately be written as $m > f(n, \sigma_i)$ with f as in the statement of Proposition C. $\square$

For example, with the previous notation, observe that $1 \leq \sigma_i \leq 4$, so that if for example $\sigma_i = 4$ for all i then X can not be defined by m equations, for $m \leq \frac{n}{4}$, or if $\sigma_i = 1$ for all i, then X can not be defined by m equations with $m \leq n$.

Chapter 4

Embeddings of some singular surfaces

4.1 Introduction

In this chapter, we study quite concretely projective embeddings of singular surfaces which correspond to degenerations of principally polarized abelian surfaces. These specific degenerations realize boundary points of the (functorial) compactification of $\mathcal{A}_g$ recently constructed by Alexeev (here we deal only with surfaces, so set $g = 2$). For more details on this outstanding construction see ahead in this introduction and in the bibliography. By a theorem of Alexeev and Nakamura these singular surfaces are equipped with an ample line bundle $\mathcal{L}$ and they prove that $\mathcal{L}^{\otimes 5}$ is indeed very ample (this embeds these surfaces in $\mathbb{P}^{24}$). By elementary methods and analyzing the corresponding linear systems, we prove "by hand" that already $\mathcal{L}^{\otimes 3}$ is very ample (and this embeds these surfaces in $\mathbb{P}^8$). Hence, these are certainly not small codimension embeddings, in the sense of the previous chapter. On the other hand, it is clear from the constructions developed in this chapter that it is very difficult to study projective embeddings directly on a given variety X, unless one knows a lot of its intrinsic geometry. Unfortunately with regard to Hartshorne's conjecture, we have no such a deep understanding of the intrinsic geometry of 3-folds and 4-folds, not to say n-folds!

Now we give a very short review of the various attempts to construct compactifications of $\mathcal{A}_g$.

In the past, there have been many methods to construct suitable compactifications of the (coarse) moduli space of abelian varieties, both in the principally polarized and in the non-principally polarized cases (see [43] for a detailed review). Let us restrict our attention to the principally polarized case. In this case, the first solution was given by Satake (see [45]) who constructed a projective normal variety $\bar{\mathcal{A}}_g$, which is highly singular along the boundary (the boundary $\partial\bar{\mathcal{A}}_g$ of $\mathcal{A}_g$ is not a divisor in this case and it is set-theoretically the union of the moduli $\mathcal{A}_i$ for $i \leq g-1$). Subsequently, by blowing up along the boundary, Igusa constructed a partial desingularization of Satake's compactification: in his compactification the boundary has codimension 1 (see [44]). These ideas were the starting point for Mumford's theory of toroidal compactifications of quotients of bounded symmetric domains (see [39] for a detailed description of this theory). Namikawa proved that Igusa's compactification is a toroidal compactification in Mumford' s sense. Unfortunately, toroidal compactifications are not unique, since they depend on the choice of cone decompositions. Ideally, one would like to construct a compactification which is meaningful for moduli, so as to abtain a space which represents a functor (at least as a stack), described in terms of abelian varieties and well-understood degenerations. Of course, the model is the Deligne-Mumford compactification of $\mathcal{M}_g$. The fact that toroidal compactifications are not unique has made very difficult to select the right compactification (if there is one). In spite of this, quite recently, Alexeev and Nakamura (see [38] and [37]), building up on previous works of Nakamura and Namikawa, have shown that the toroidal compactification $\mathcal{A}_g^{Vor}$, associated to the second Voronoi decomposition represents a good functor (as a stack). This means that $\mathcal{A}_g^{Vor}$ represents the canonical compactification of the moduli space $\mathcal{A}_g$ of principally polarized Abelian varieties, as the Deligne-Mumford compactification represents the canonical compactification for the moduli space of curves (this point is investigated in [37]).

More specifically, $\mathcal{A}_g^{Vor}$ represents the functor of stable semi-abelic varieties (SSAV). Let us recall that a semi-abelian variety G is just an extension $1 \to T \to G \to A \to 1$, where A is an abelian variety and T is an algebraic torus $T = (\mathbb{C}^*)^r$, for some r. Then, in Alexeev's construction a SSAV Y is a "good" degeneration of an abelian variety (corresponding to a boundary point in $\mathcal{A}_g^{Vor}$) and on it there is an action of a semi-abelian variety G in such a way that there are finitely many orbits. We will not recall the whole construction (see [38]), we want just to remark that any SSAV Y is a projective, semi-normal variety (i.e. Y is isomorphic to its semi-normalization Y' in Y^{nor}: $Y^{nor} \to Y' \xrightarrow{\pi} Y$ and Y' is maximal such that for each $x \in Y$ there is a unique $x' \in Y'$ with $\pi(x') = x$ and $\mathbb{C}(x) \cong \mathbb{C}(x')$), equipped with an ample line bundle $\mathcal{O}_Y(1)$.

In this chapter, we study very ampleness of line bundles coming from multiples of principal polarization on degenerate abelian surface (over $\mathbb{C}$), corresponding to boundary components of $\mathcal{A}_2^{Vor}$, that is on SSAV's. A well-known theorem of Lefschetz states that if A is a *smooth* abelian variety of dimension g and $\mathcal{O}_A(1)$ is a principal polarization, then $\mathcal{O}_A(3)$ is very ample, (in fact, the theorem of Lefschetz is true for all polarizations, not just for principal polarizations). We want to understand how far is this statement if we replace

A, with a SSAV Y (restricting to the case of surfaces). Indeed, in [38] it is proved that for a SSAV Y of genus g, $\mathcal{O}_Y(n)$ is very ample as soon as $n \geq 2g+1$, that is, in the case of surfaces, as soon as $n \geq 5$. We improve this bound, showing that already $\mathcal{O}_Y(3)$ is very ample (that is Lefschetz's theorem still holds for a SSAV of dimension 2, which deforms to a principally polarized abelian variety). The proof of this result is elementary in spirit and it is based on proving the analogous statement for each degeneration type, providing a careful description of $\mathcal{O}_Y(3)$ and of its sections. Indeed, in the principally polarized case, there are only three types of degenerations for surfaces. Two degenerations where there is no remaining abelian part (which in the following sections are called degenerations of second and third type), corresponding to the two standard Delaunay decompositions of $\mathbb{Z}^2$ (lattice of rank 2) and one degeneration (of the first type) where there is still an abelian part surviving (an elliptic curve) and which corresponds to the unique Delaunay decomposition of a lattice of rank one.

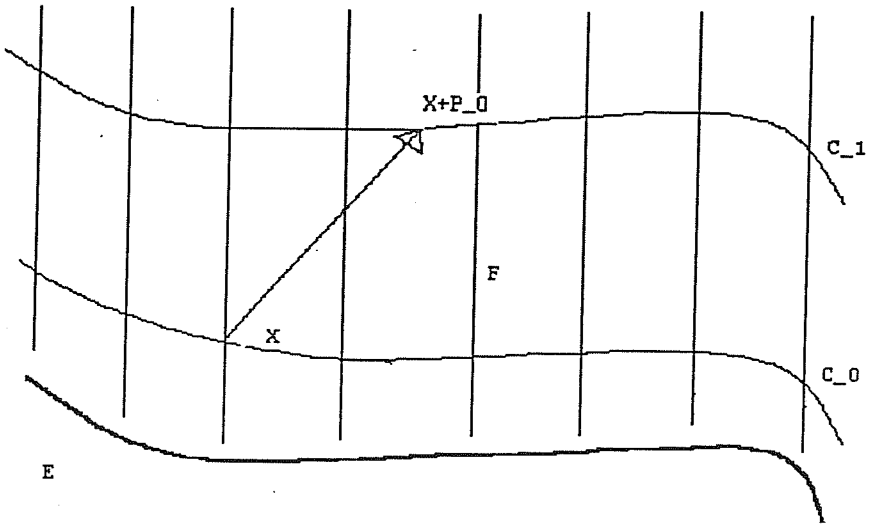
The case of the $\mathbb{P}^1$-bundle over an elliptic curve (degeneration of the first type) is the most general, since it depends on two moduli, i.e. the moduli of the elliptic curve and the glueing parameter. The second degeneration type depends on 1 moduli, namely the glueing parameter, while the third degeneration type depends on no moduli at all.

4.2 Very ampleness on the first type of degeneration

The first type of degeneration for smooth principally polarized abelian surfaces can be constructed from a $\mathbb{P}^1$-bundle X, with two sections, over an elliptic curve E, $\pi : X \to E$. In this case, in the degenerate surface, there is still an abelian part surviving and the smooth model is $X := \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(e-o))$ (e is a point on E and o is the zero of the group law on E); it has two sections C_0 and C_1, corresponding to the fact that the ruled surface X comes from a split rank two vector bundle on E. We can identify the two sections by saying what is their normal bundle, so we define C_0 so that $\mathcal{N}_{C_0/X} = \mathcal{O}_{C_0}(C_0) = \mathcal{O}_E(e-o)$, and correspondingly C_1 so that $\mathcal{N}_{C_1/X} = \mathcal{O}_{C_1}(C_1) = \mathcal{O}_E$.

It is well known that $Pic(X) = \pi^* Pic(E) \oplus \mathbb{Z}C_0$, while the Néron-Severi group of X is $NS(X) = \mathbb{Z}F \oplus \mathbb{Z}C_0$, where F is any fibre of X over E and the intersection pairing is $C_i^2 = 0$, $F^2 = 0$ and $C_i.F = 1$ (see for instance [42]).

The degenerate abelian surface Y is obtained by identifying each point $x \in C_0$ with the point $x + p_0 \in C_1$ for some parameter $p_0 \in E$, as displayed in the following picture:



Let ν be the desingularization map of Y: $\nu : X \to Y$. Let $\mathcal{L} \in Pic(Y)$ and $\mathcal{L}' := \nu^*\mathcal{L}$. Then $\mathcal{L}'$ is numerically equivalent to $aC_0 + bF$, for some a and b. We want $\mathcal{L}$ to represent a principal polarization: since on an abelian surface a principal polarization has self-intersection 2, and the self-intersection does not change in a flat family, we require $\mathcal{L}^2 = 2$, which pulling back to the normalization, implies $\mathcal{L}'^2 = 2$. Thus we get $(aC_0 + bF)^2 = 2$, which implies $a = b = 1$. This means that $\mathcal{L}' = \mathcal{O}_X(C_0 + F)$, for some $F = \pi^{-1}(p)$, $p \in E$, or equivalently $\mathcal{L}' = \mathcal{O}_X(C_0) \otimes \pi^*\mathcal{M}$, $\mathcal{M} \in Pic^1(E)$, $\mathcal{M} = \mathcal{O}_E(p)$. Now we ask under which conditions on the glueing process, such an $\mathcal{L}'$ descends to Y, or equivalently, when there exists $\mathcal{L} \in Pic(Y)$ such that $\nu^*\mathcal{L} = \mathcal{L}'$. The answer is given by the following:

Lemma 1: *Let $\mathcal{L}'$, $X = \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(e - o))$ and Y as above; then $\mathcal{L}'$ descend to Y if and only if $e + p_0 \sim o$ (where $\sim$ stands for linear equivalence and p_0 is the glueing parameter described above).*

Proof: Let $\varphi : C_0 \overset{\approx}{\to} C_1$ be the isomorphism between the two sections, given by $x \mapsto x + p_0$. Then it is clear that $\mathcal{L}'$ descends to Y iff:

$$\varphi^*\mathcal{L}'_{|C_1} = \mathcal{L}'_{|C_0}. \tag{4.1}$$

$\mathcal{L}'_{|C_0} = \mathcal{O}_X(C_0 + F) \otimes \mathcal{O}_{C_0}$ and this is equal to $\mathcal{N}_{C_0/X}(p)$, where $p = C_0 \cap F$. Since $C_0 \cong E \cong C_1$, and $\mathcal{N}_{C_0/X} = \mathcal{O}_{C_0}(C_0) = \mathcal{O}_E(e - o)$, we get that $\mathcal{L}'_{|C_0} = \mathcal{O}_E(e - o + p)$, where we identify C_0 with E and the point $p = C_0 \cap F$ with its projection to E. Analogously, $\mathcal{L}'_{|C_1} = \mathcal{O}_X(C_0 + F) \otimes \mathcal{O}_{C_1}$ which is equal to $\mathcal{O}_{C_1}(p')$ for $p' = C_1 \cap F$ since $\mathcal{O}_{C_1}(C_0) = \mathcal{O}_{C_1}$ (due to the fact that $C_1 \cap C_0 = \emptyset$). Then $\mathcal{O}_{C_1}(p') \cong \mathcal{O}_E(p)$ (since if π is the projection to E, then $\pi(p) = \pi(p')$) so that we can translate condition (4.1) as a relation on line bundles on E: $\mathcal{O}_E(p - p_0) = \mathcal{O}_E(e - o + p)$, which is equivalent to $e + p_0 \sim 0$. □

Fixed e, that is fixed X, from now on we assume that the parameter p_0 has been chosen in order to satisfy the condition $e+p_0 \sim o$. Under this assumption, not only $\mathcal{L}'$, but also all line bundles of the form $\mathcal{O}_X(nC_0+nF)$ descend to Y. Let us call $\mathcal{O}_Y(1)$ the line bundle $\mathcal{L}$ on Y such that $\nu^*\mathcal{L} = \mathcal{O}_X(C_0+F)$ so that $\nu^*\mathcal{O}_Y(n) = \mathcal{O}_X(nC_0+nF)$. Then, by the results of Alexeev and Nakamura ([38], Theorem 4.7) it turns out that the map on Y associated to the line bundle $\mathcal{O}_Y(5)$ gives an embedding. We will prove (Theorem 2) that the map associated to $\mathcal{O}_Y(3)$ gives also an embedding (this is again an analogue of Lefschetz theorem). Before doing this, we compute $h^0(Y, \mathcal{O}_Y(n))$:

Proposition 1: *Let Y and $\mathcal{O}_Y(n)$ as above. If $n \geq 0$ and $k \geq 0$, then $h^0(X, \mathcal{O}_X(nF+kC_0)) = (k+1)n$ and $h^0(Y, \mathcal{O}_Y(n)) = n^2$.*

Proof: Since $\nu^*\mathcal{O}_Y(n) = \mathcal{O}_X(nC_0+nF)$, first of all we compute $h^0(X, \mathcal{O}_X(nC_0+nF))$. From the exact sequence:

$$?0 \to \mathcal{O}_X(-C_0) \to \mathcal{O}_X \to \mathcal{O}_{C_0} \to 0,$$

twisting by $\mathcal{O}_X(C_0+nF)$, we obtain the exact sequence:

$$0 \to \mathcal{O}_X(nF) \to \mathcal{O}_X(nF+C_0) \to \mathcal{O}_{C_0}(C_0+nF) \to 0. \tag{4.2}$$

To go on, let us recall that if $\mathcal{S} \in Pic^d(E)$ ($d \geq 1$), where E is a curve of genus 1, then $h^0(E,\mathcal{S}) = d$ and $h^i(E,\mathcal{S}) = 0$ for $i \geq 1$ (this is just an immediate application of Riemann-Roch theorem and Serre duality); moreover, by Lemma 2.4 of Chapter V of ([42]), we have that $H^i(X, \mathcal{O}_X(D)) \cong H^i(E, \pi_*\mathcal{O}_X(D))$ for $i \geq 0$ and for any divisor D on X such that $D.F \geq 0$, and if $D.F = k$ then $\pi_*\mathcal{O}_X(D)$ is locally free of rank $k+1$.

Since $H^1(X, \mathcal{O}_X(nF)) = H^1(E, \pi_*(\mathcal{O}_X(nF))) = 0$, because $\pi_*\mathcal{O}_X(nF) = \mathcal{O}_E(np)$. Indeed, we have $\pi_*\mathcal{O}_X(nF) = \pi_*(\mathcal{O}_X \otimes \pi^*\mathcal{O}_E(np))$, which is equal by projection formula to $\pi_*\mathcal{O}_X \otimes \mathcal{O}_E(np) = \mathcal{O}_E \otimes \mathcal{O}_E(np) = \mathcal{O}_E(np)$.

Therefore, taking the long exact cohomology sequence from (4.2), we obtain that $h^0(X, \mathcal{O}_X(nF+C_0)) = h^0(X, \mathcal{O}_X(nF)) + h^0(C_0, \mathcal{O}_{C_0}(nF+C_0))$. Now, $\pi_*\mathcal{O}_X(nF)$ is a line bundle on E of degree n, so that $h^0(X, \mathcal{O}_X(nF)) = n$; moreover $\mathcal{O}_{C_0}(nF+C_0) \cong \mathcal{O}_E(e-o+np)$, under the identification $C_0 \cong E$, and $\mathcal{O}_{C_0}(C_0) = \mathcal{N}_{C_0/X} = \mathcal{O}_E(e-o)$. This implies that $h^0(C_0, \mathcal{O}_{C_0}(C_0+nF)) = n$, so that $h^0(X, \mathcal{O}_X(nF+C_0)) = 2n$.

Now we prove by induction that $h^0(X, \mathcal{O}_X(nF+kC_0)) = (k+1)n$. This is true for $k=1$; assume, by inductive hypothesis that $h^0(X, \mathcal{O}_X(nF+(k-1)C_0)) = kn$. We have the following exact sequence:

$$0 \to \mathcal{O}_X(nF+(k-1)C_0) \to \mathcal{O}_X(nF+kC_0) \to \mathcal{O}_{C_0}(nF+kC_0) \to 0. \tag{4.3}$$

Again, since $\mathcal{O}_{C_0}(nF+kC_0) \cong \mathcal{O}_E(ke-ko+np)$, we have that $h^0(C_0, \mathcal{O}_{C_0}(nF+kC_0)) = n$. Now we prove that $H^1(\mathcal{O}_X(nF+(k-1)C_0)) = 0$; first of all this is equal to

$H^1(E, \pi_*(\mathcal{O}_X(nF + (k-1)C_0)))$ and $\pi_*\mathcal{O}_X(nF + (k-1)C_0) = \pi_*(\mathcal{O}_X((k-1)C_0) \otimes \pi^*\mathcal{O}_E(np))$, which is, by projection formula $\pi_*(\mathcal{O}_X((k-1)C_0)) \otimes \mathcal{O}_E(np)$.

We have that $\pi_*(\mathcal{O}_X((k-1)C_0)) = Sym^{(k-1)}(\mathcal{O}_E \oplus \mathcal{O}_E(e-o))$, since $\pi_*(\mathcal{O}_X(dC_0)) = \pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(d)) = Sym^d(\mathcal{O}_E \oplus \mathcal{O}_E(e-o))$, where Sym^d is the d-th symmetric power and $\mathcal{E}$ is the decomposable rank two vector bundle $\mathcal{O}_E \oplus \mathcal{O}_E(e-o)$. (The equality $\mathcal{O}_X(C_0) = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ comes from the fact that we have chosen $\mathcal{O}_X(C_0)$ as one of the generators of $Pic(X)$ (see Proposition 2.3, page 370 of [42]) and from the fact that $Pic(X) = \pi^*Pic(E) \oplus \mathbb{Z}h$, where h is the class of the tautological sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ (see [40], Proposition 3.18, page 36)).

Thus, $\pi_*\mathcal{O}_X(nF + (k-1)C_0)$ is a direct sum of line bundles of positive degree on E and this implies that $H^1(E, \pi_*\mathcal{O}_X(nF + (k-1)C_0)) = 0$.

This proves our claim on $h^0(X, \mathcal{O}_X(nF+kC_0))$; in particular we have that $h^0(X, \mathcal{O}_X(nF+nC_0)) = n^2 + n$. To prove that $h^0(\mathcal{O}_Y(n)) = n^2$, consider the following restriction morphism (which is just the direct sum of the restriction morphisms to C_0 and C_1):

$$\begin{array}{ccc} H^0(\mathcal{O}_X(nC_0 + nF)) & \to & H^0(\mathcal{O}_{C_0}(nC_0 + nF)) \oplus H^0(\mathcal{O}_{C_1}(nC_0 + nF)) \\ s & \mapsto & (s_{|C_0}, s_{|C_1}) \end{array}$$

Now, the section s descends to a section of $\mathcal{O}_Y(n)$ iff the glueing conditions $\varphi^*(s_{|C_1}) = s_{|C_0}$ are satisfied, where, as above, φ is the isomorphism between C_1 and C_0, induced by translation. Since $H^0(\mathcal{O}_{C_0}(nC_0 + nF))$ and $H^0(\mathcal{O}_{C_1}(nC_0 + nF))$ are both n-dimensional vector spaces, these glueing conditions determine n equations in $H^0(\mathcal{O}_{C_0}(nC_0 + nF))$.

Thus, it remains to prove that these n relations are independent, but this is clear, because there is certainly at least one non-trivial relation and hence, due to Heisenberg invariance, there are n independent equations, since the action of the Heisenberg group is irreducible (recall that the divisor $nC_0 + nF$ cuts out on C_0 a divisor of degree n). This implies that $h^0(\mathcal{O}_Y(n)) = n^2$.

□

Collecting the results of Alexeev and Nakamura ([38], Theorem 4.7), and Proposition 1, we have that $|\mathcal{O}_Y(5)|$ gives an embedding of Y as a linearly normal surface in $\mathbb{P}^{24}$. In fact, we can do better, embedding Y as a linearly normal surface in $\mathbb{P}^8$ as proved by the following:

Theorem 1: *Let Y and $\mathcal{O}_Y(n)$ as above. Then the complete linear system $|\mathcal{O}_Y(3)|$ is base-point free and the associate morphism $\phi_{|\mathcal{O}_Y(3)|} : Y \hookrightarrow \mathbb{P}^8$ is an embedding.*

Proof: First of all we prove that $|\mathcal{O}_Y(3)|$ has no fixed component. Assume the contrary, and let K be an irreducible component of the 1-dimensional fixed locus. Now observe that any curve on the smooth model which is not equal to C_1, intersects C_0. Then K

corresponds to a locus on the smooth model, which will intersect C_0 or C_1. Thus, there exists always an $x \in K \cap C_0$ (or C_1 which is identified to C_0 on Y).

Now the restriction morphism $H^0(Y, \mathcal{O}_Y(3)) \to H^0(C_0, \mathcal{O}_{C_0}(3o))$ is clearly surjective by construction, so that there exists a $t \in H^0(C_0, \mathcal{O}_{C_0}(3o))$ such that $t(x) \neq 0$ (since the complete linear system $|\mathcal{O}_{C_0}(3o)|$ embeds C_0 in $\mathbb{P}^2$ and we can always find a hyperplane section of this embedded curve which does not hit the point x) ; this implies that there exists $s \in H^0(Y, \mathcal{O}_Y(3))$, such that $s(x) \neq 0$ so that x is not a point in the fixed component: this is a contradiction. Thus $|\mathcal{O}_Y(3)|$ has no fixed component and $Bs|\mathcal{O}_Y(3)|$ is at most a *finite set* of points.

Let $s \in H^0(Y, \mathcal{O}_Y(1))$ and consider $D := (s)_0$, the zero scheme of s (clearly $Bs|\mathcal{O}_Y(3)| \subset D$). If $e \neq o$ D is irreducible and its pull-back to X is numerically equivalent to $C_0 + F$ (if $e = o$ the proof that $\phi_{|\mathcal{O}_Y(3)|}$ is an embedding is completely trivial, see remark 1 at the end of the proof). Now $s^{\otimes 3} \in H^0(Y, \mathcal{O}_Y(3))$ and $red[(s^{\otimes 3})_0] = D$ (where *red* is the reduced scheme structure on the zero scheme $(s^{\otimes 3})_0$). Since $|\mathcal{O}_Y(3)|$ has no fixed-component, we can always choose $s' \in H^0(Y, \mathcal{O}_Y(3))$ such that $s'_{|D} \neq 0$.

Let $D' = (s')_0$, then the pull-back of D' to X is numerically equivalent to $3C_0 + 3F$, so that $D.D' = (C_0 + F).(3C_0 + 3F) = 6$. By this computation we have that $Bs|\mathcal{O}_Y(3)| \subset (s')_0.(s^{\otimes 3})_0$, which consists of (at most) 6 *distinct* points, possibly with multiplicities. On the other hand, let us consider the group $E^{(3)} := \{x \in E; 3x = o\}$ of 3-torsion points of E, which is (not canonically) isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. This is a finite group of order 9 and for any $\lambda \in E^{(3)}$, if $t_\lambda : x \mapsto x + \lambda$, then $t_\lambda^* \mathcal{O}_X(3C_0 + 3F) = \mathcal{O}_X(3C_0 + 3F)$ (since $F = F_p$ for some $p \in E$ and $3F_p \sim 3F_{p \pm \lambda}$, being $3\lambda = o$). This action of $E^{(3)}$ clearly descends to $\mathcal{O}_Y(3)$, since the two sections C_0 and C_1 are just glued together by a rigid translation.

In view of this action, it turns out that if $|\mathcal{O}_Y(3)|$ has one fixed point (possibly with multiplicities), then it has to have at least 9 *distinct* base-points (possibly with multiplicities), while by the previous computation $Bs|\mathcal{O}_Y(3)|$ has at most 6 distinct points. Contradiction. Thus $|\mathcal{O}_Y(3)|$ is base-point free and defines a morphism of Y to $\mathbb{P}^8$.

Now we have to prove that $|\mathcal{O}_Y(3)|$ separates points. We have to deal with different cases.

First case: the inverse images P_1 and P_2 of the two distinct points $x, y \in Y$ on the smooth model X belong to distinct fibers and $x, y \notin Sing(Y)$; then consider the sections $s \in H^0(X, \mathcal{O}_X(3C_0 + 3F_o))$ such that $s_{|C_0} = s_{|C_1} = 0$, which certainly descend onto Y. These sections are in one to one correspondence with the sections of $\mathcal{O}_X(3C_0 + 3F_o - C_0 - C_1)$. Since we have $C_1 \sim C_0 + (F_o - F_e)$, we have that $\mathcal{O}_X(3C_0 + 3F_o - C_0 - C_1) = \mathcal{O}_X(C_0) \otimes \pi^* \mathcal{O}_E(B)$, where B is a divisor of degree 3 on E. Since we can always find on E a divisor of degree 3, containing $\pi(P_1)$, but not $\pi(P_2)$ (recall that P_1 and P_2 belong to different fibres), it is always possible to find out a section of $\mathcal{O}_X(3C_0 + 3F_o - C_0 - C_1)$, which vanishes on P_1, but not on P_2 and consequently a section of $\mathcal{O}_X(3C_0+3F_o)$ which descends to Y and which separates x, y.

Second case: the two distinct points $x, y \in Y$ are on the glued sections C_0, C_1. In this case, since the restriction morphism $H^0(Y, \mathcal{O}_Y(3)) \to H^0(C_0, \mathcal{O}_{C_0}(3))$ is surjective by construction, and the complete linear system $|\mathcal{O}_{C_0}(3)|$ gives an embedding of C_0 into $\mathbb{P}^2$, we are done immediately.

Third case: the point $x \in C_0$, $y \notin C_0$. In this case we can do as in the first case or just observe that there are certainly sections which vanish on x (those sections which on the smooth model vanish on C_0 and C_1), but not on y.

Fourth case: the inverse images P_1 and P_2 of the two distinct points $x, y \in Y$ on the smooth model X belong to the same fiber F_p and are outside the two sections C_0 and C_1. In this case, consider the sections of $\mathcal{O}_X(C_0 + 2F_o + F_e)$, which are in one to one correspondence with the sections of $\mathcal{O}_X(3C_0+3F_o)$ which vanish on C_0 and C_1, that is those sections which descend automatically onto Y. We have the exact sequence:

$$0 \to \mathcal{O}_X(C_0+2F_o+F_e-F_p) \to \mathcal{O}_X(C_0+2F_o+F_e) \to \mathcal{O}_{F_p}(1) \to 0 \tag{4.4}$$

twisting the defining sequence of F_p, since $(C_0 + 2F_o + F_e).F_p = 1$ and $F_p \cong \mathbb{P}^1$. Now, $H^1(X, \mathcal{O}_X(C_0 + 2F_o + F_e - F_p)) = H^1(E, \pi_*(\mathcal{O}_X(C_0 + 2F_o + F_e - F_p))) = 0$, as in the proof of Proposition 1, so that taking the long exact cohomology sequence induced from (4.4), we get that the restriction morphism $H^0(X, \mathcal{O}_X(C_0+2F_o+F_e)) \to H^0(F_p, \mathcal{O}_{F_p}(1))$ is surjective. Then we can always find a section s of $\mathcal{O}_{F_p}(1)$ which vanishes on P_1, but not on P_2; we lift s to a section of $\mathcal{O}_X(C_0 + 2F_o + F_e)$, which corresponds to a section t of $\mathcal{O}_X(3C_0 + 3F_o)$ vanishing on C_0 and C_1; this section descends to a section of $\mathcal{O}_Y(3)$ and vanishes on x, but not on y. Thus, the linear system $|\mathcal{O}_Y(3)|$ separates points also in this case. So we have proved that the map $\phi_{|\mathcal{O}_Y(3)|} : Y \to \mathbb{P}^8$ is injective, since there are clearly no other cases for the relative position of the points x and y.

$|\mathcal{O}_Y(3)|$ separates tangent directions: to prove this we distinguish two different cases: $p \in Y$ is a smooth point (first case), or $p \in Y$ belongs to the the image of C_0 and so it is singular (second case).

First case: let $v \in T_pY \cong \mathbb{A}^2$. To prove that $|\mathcal{O}_Y(3)|$ separates tangent directions it is sufficient to find out a curve $C' \in |\mathcal{O}_Y(3)|$, passing through p and smooth at p such that $T_pC' \neq v$. If $v \neq T_pF$, then we consider any smooth curve C'' on X, inside the linear equivalence class of $2C_0 - C_1 + F_1 + F_2 + F_3$, where $p \in F_1$, $p \notin F_2$, $p \notin F_3$ and such that the fibres F_1, F_2 and F_3 are arranged so that $2C_0 - C_1 + F_1 + F_2 + F_3 \sim 2C_0 - C_1 + 3F_o$. In this case, $|2C_0 - C_1 + F_1 + F_2 + F_3|$ can be viewed as a subsystem of $|\mathcal{O}_X(3)|$, corresponding to sections vanishing on C_0 and C_1. All these sections clearly descend to Y and correspondingly any curve $C'' \in |2C_0 - C_1 + F_1 + F_2 + F_3|$. Thus it is sufficient to set $C' := \nu(C'')$, where $\nu : X \to Y$ is the desingularization map.

If instead, $v = T_pF$, it is sufficient to prove that the morphism $H^0(Y, \mathcal{O}_Y(3)) \to H^0(F, \mathcal{O}_F(3))$ is surjective, since the complete linear system $|\mathcal{O}_F(3)|$ defines an embedding of $\mathbb{P}^1$ as a twisted cubic. Assume that this is not the case; then, since we already know that the map $\phi_{|\mathcal{O}_Y(3)|}$ is injective, it means that the image in $\mathbb{P}^8$ of F is a plane rational curve, having at most a cusp as a singularity. Since the image is a plane curve, then there exists a unique plane $V \cong \mathbb{P}^2$ containing it. Now consider all hyperplanes of $\mathbb{P}^8$ containing this V: by a standard argument, they are parametrized by a $\mathbb{P}^5$. Observe that the rational curve intersects the image Z of the singular locus of Y (an elliptic curve) in two distinct points (because $p_0 \neq o$), which are obviously contained in V.

Let x and y be these two points and fix another point z on the image of the elliptic curve Z in $\mathbb{P}^8$ so that $x + y + z$ is not linearly equivalent to a hyperplane section of Z. Then, all hyperplanes containing V and z, have also to contain Z; these hyperplanes are parametrized by a $\mathbb{P}^4$, but their pull-back to X cut out a divisor D such that $3C_0+3F-D$ is numerically equivalent to C_0+2F. Now, by Proposition 1, $h^0(\mathcal{O}_X(C_0+2F)) = 4$, which implies that these hyperplanes should form a $\mathbb{P}^3$, not a $\mathbb{P}^4$. Contradiction. This happens because we have assumed that the image of F is a *plane* rational curve. Thus $\phi_{|\mathcal{O}_Y(3)|}$ separates tangent directions also in this case.

Second case: the point p belongs to the singular locus $Sing(Y)$ of Y and clearly $T_pY \cong \mathbb{A}^3$. In this case, to prove that $|\mathcal{O}_Y(3)|$ separates tangent directions it is sufficient to prove that the image of T_pY in $\mathbb{P}^8$ is 3-dimensional. Assume that the image is not 3-dimensional; then its image in $\mathbb{P}^8$ is at most a $V \cong \mathbb{P}^2$. Then look at the hyperplanes of $\mathbb{P}^8$ containing this V; by a standard argument, they are parametrized by a $\mathbb{P}^5$. The pull-back of any of these hyperplanes to the smooth model X determines a divisor on X having multiplicity 2 at the point x_0 and x_1, where $x_0 \in C_0$ and $x_1 \in C_1$ (the two points x and y are just the preimages in X of the point $p \in Sing(Y)$). Then choose an other point q on the image of the singular locus Z of Y (the image of $Sing(Y)$ in $\mathbb{P}^8$ is just a plane elliptic curve, since $H^0(Y, \mathcal{O}_Y(3)) \to H^0(C_0, \mathcal{O}_{C_0}(3))$ is surjective). Choose q such that $2p + q$ is not linearly equivalent to a hyperplane section of Z, the image of $Sing(Y)$ in $\mathbb{P}^8$.

Then we obtain a $\mathbb{P}^4$ of hyperplanes, containig V and q. Since we have chosen q in this way, it turns out these hyperplanes have to contain Z, hence on the smooth model they cut out a divisor containg C_0 and C_1 and the points x_0 and x_1 with multiplicity 2. Finally, choose on the smooth model X two other points: y_0 on the fiber passing through x_0 and y_1 on that passing through x_1. On the hyperplanes of $\mathbb{P}^8$ satisfying the previous conditions, impose also to pass through the images of y_0 and y_1: in this way we get a $\mathbb{P}^2$ of these hyperplanes. The pull-back of any of these hyperplanes to the smooth model X cut out C_0, C_1 and two fibers F_1 and F_2 and the remaining divisor in $3C_0 + 3F$ is numerically equivalent to $C_0 + F$. On the other hand, by Proposition 1 $h^0(X, \mathcal{O}_X(C_0 + F)) = 2$ and this is true if replace $C_0 + F$ with any other divisor numerically equivalent to it. This is a contradiction, because we have a $\mathbb{P}^2$ of these hyperplanes, while $|\mathcal{O}_X(C_0+F)| = \mathbb{P}^1$. The contradiction arises from the fact that we have assumed that the image of T_pY in $\mathbb{P}^8$ is at most 2-dimensional.

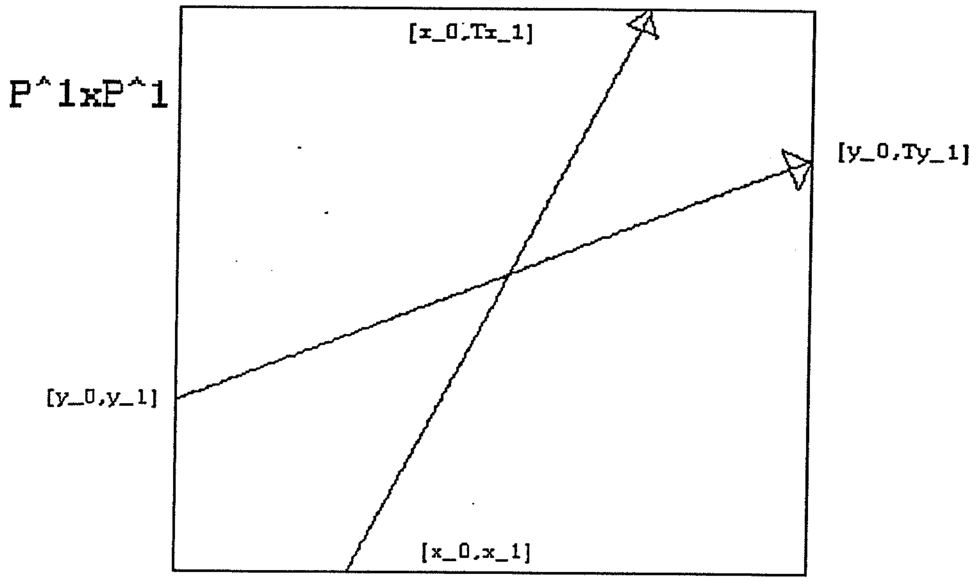
Thus, we have proved that $\phi_{|\mathcal{O}_Y(3)|} : Y \hookrightarrow \mathbb{P}^8$ is an embedding. □

Remark 1: In the proof of Theorem 1, we have assumed that $e \neq o$ (and consequently $p_0 \neq o$). Indeed, if $e = o$, by Lemma 1 we take $p_0 = o$ and in this case $Y = E \times C$, where E is an elliptic curve and C is a nodal cubic curve. Then we get $E \times C \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$, where the last embedding is a Segre map. Since $h^0(Y, \mathcal{O}_Y(3))$ is independent of the parameter e (hence p_0), it turns out that for *generic* e the map $\phi_{|\mathcal{O}_Y(3)|}$ gives an embedding. However, to prove this for any e we have to do as above.

Remark 2: The image of Y in $\mathbb{P}^8$ is a linerly normal surface. Assume $e \neq o$, then its singular locus is a smooth plane elliptic curve Z and through each point of this elliptic curve there are two twisted cubic curve, intersecting transversaly each other and also transversally with Z. The degree is 18, since $(3C_0 + 3F)^2 = 18$.

4.3 Very ampleness on the second type of degeneration

The second type of degeneration of smooth principally polarized abelian surfaces we are going to consider is obtained by a smooth quadric $X = \mathbb{P}^1 \times \mathbb{P}^1$ (with homogeneous coordinates $[x_0, x_1] \times [y_0, y_1]$), identifying the points of coordinates $[x_0, x_1] \times [1, 0]$ with those of coordinates $[x_0, x_1 T] \times [0, T]$ and the points of coordinates $[1, 0] \times [y_0, y_1]$ with those of coordinates $[0, T] \times [y_0, y_1 T]$ for some parameter $T \in \mathbb{C}^*$; in particular the points corresponding to coordinates $[1, 0] \times [1, 0]$, $[1, 0] \times [0, T]$, $[0, T] \times [0, T]$ and $[0, T] \times [1, 0]$ are all identified. This is symbolically displayed in the following picture:



Observe that this type of degeneration depends on 1 moduli, namely the glueing parameter T.

Let us call Y (strictly speaking Y_T, since it depends on the parameter T) the image of X under these identifications, $\pi : X \to Y$. Y is one of the degeneration type of smooth principally polarized abelian surfaces represented by a point in the boundary of $\mathcal{A}_2^{Vor}$.

Recall that $Pic(X) = \mathbb{Z}L_1 \oplus \mathbb{Z}L_2$, where L_1 and L_2 generate the two rulings on X, while the intersection pairing is symply $L_i^2 = 0$, $L_1.L_2 = 1$. Recall also that the self-intersection of a principal polarization on a smooth abelian surface is 2 and that self-intersection does not change in a flat family. Having recalled this, it is natural to consider as a degenerate principal polarization on Y a line bundle $\mathcal{L}$ such that $\pi^*\mathcal{L} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)$ (simply because the corresponding divisor class is of the form $L_1 + L_2$ and $(L_1 + L_2)^2 = 2$). In this light, proving a sort of Lefschetz theorem for this type of degeneration is equivalent to prove the following:

Theorem 2: *Let $\mathcal{L}$ be a line bundle on Y such that $\pi^*\mathcal{L} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3,3)$. Then the complete linear system $|\mathcal{L}|$ is base-point free and the corresponding map $\phi_{|\mathcal{L}|} : Y \hookrightarrow \mathbb{P}^8$ defines an embedding of the singular model Y.*

Proof: First of all, we have to exhibit a basis of $H^0(Y, \mathcal{L})$, that is we have to understand which sections of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3,3)$ descend to sections of $\mathcal{L}$.

Since $H^0(X, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3,3)) = H^0(\mathcal{O}_{\mathbb{P}^1}(3)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(3))$ any section σ of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3,3)$ can be written as:

$$\sigma = a_1 x_0^3 y_0^3 + a_2 x_0^3 y_0^2 y_1 + a_3 x_0^3 y_0 y_1^2 + a_4 x_0^3 y_1^3 +$$

$$+ a_5 x_0^2 x_1 y_0^3 + a_6 x_0^2 x_1 y_0^2 y_1 + a_7 x_0^2 x_1 y_0 y_1^2 + a_8 x_0^2 x_1 y_1^3 +$$

$$+a_9x_0x_1^2y_0^3 + a_{10}x_0x_1^2y_0^2y_1 + a_{11}x_0x_1^2y_0y_1^2 + a_{12}x_0x_1^2y_1^3 +$$
$$a_{13}x_1^3y_0^3 + a_{14}x_1^3y_0^2y_1 + a_{15}x_1^3y_0y_1^2 + a_{16}x_1^3y_1^3.$$

The necessary and sufficient condition for a section to descend is that it satisfies some compatibility conditions under the glueing process described above. In particular, it obvious that the sections represented by $x_0^2x_1y_0^2y_1$, $x_0^2x_1y_0y_1^2$, $x_0x_1^2y_0^2y_1$ and $x_0x_1^2y_0y_1^2$ always descend since thay are identically zero on the points which are going to be identified (hence $h^0(Y, \mathcal{L}) \geq 4$).

The compatibility conditions for σ can be expressed as

$$\sigma|_{[1,0]\times[y_0,y_1]} = \lambda\sigma|_{[0,1]\times[y_0,Ty_1]}, \tag{4.5}$$

and

$$\sigma|_{[x_0,x_1]\times[1,0]} = \lambda'\sigma|_{[x_0,Tx_1]\times[0,1]}, \tag{4.6}$$

for *some* $\lambda, \lambda' \in \mathbb{C}^*$. Since we have not fixed any λ, but we just say that there is some λ such that (4.5) holds, the equation (4.5) is equivalent to the vanishing of three 2×2 minors of the matrix:

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_{13} & Ta_{14} & T^2a_{15} & T^3a_{16} \end{pmatrix}.$$

This imposes 3 conditions on the coefficients a_i, while the equation (4.6) is equivalent (always because we have not fixed any λ') to the vanishing of three 2×2 minors of the matrix:

$$\begin{pmatrix} a_1 & a_5 & a_9 & a_{13} \\ a_4 & Ta_8 & T^2a_{12} & T^3a_{16} \end{pmatrix},$$

and this imposes three other conditions on the coefficients a_i, which are not all independent of the previous ones; it is immediate to check that only 2 of these conditions are independent of the previous ones, so that we get a total of 5 conditions. To these 5 conditions we have to add 2 other independent conditions determined by the glueing parameters λ, λ', so that we get $h^0(Y, \mathcal{L}) = h^0(X, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(3,3)) - (5+2) = 16 - 7 = 9$. Now we determine a basis for the 9-dimensional vector space $H^0(Y, \mathcal{L})$. From the condition (4.5), exchanging λ^{-1} with λ, we have:

$$\lambda a_1 = a_{13} \quad \lambda a_2 = Ta_{14} \quad \lambda a_3 = T^2a_{15} \quad \lambda a_4 = T^3a_{16},$$

while from condition (4.6), exchanging λ'^{-1} with λ', we get:

$$\lambda' a_1 = a_4 \quad \lambda' a_5 = Ta_8 \quad \lambda' a_9 = T^2a_{12} \quad \lambda' a_{13} = T^3a_{16},$$

so that we can choose $(a_1, a_2, a_3, a_5, a_6, a_7, a_9, a_{10}, a_{11})$ as coordinates on $H^0(Y, \mathcal{L})$, since, by the relations just described above, we have:

$$a_{16} = \frac{\lambda\lambda'}{T^3}a_1 \quad a_{15} = \frac{\lambda}{T^2}a_3 \quad a_{14} = \frac{\lambda}{T}a_2 \quad a_{13} = \lambda a_1$$

$$a_{12} = \frac{\lambda'}{T^2}a_9 \quad a_8 = \frac{\lambda'}{T}a_5 \quad a_4 = \lambda' a_1.$$

To write down an explicit basis of $H^0(Y, \mathcal{L})$ it is sufficient to substitute iteratively $(a_1, a_2, a_3, a_5, a_6, a_7, a_9, a_{10}, a_{11})$ equal to $(1, 0, \ldots, 0)$, $(0, 1, \ldots, 0)$,..., $(0, \ldots, 1)$ in the expression of the general section σ, taking into account the relations which define a_{16}, a_{15}, a_{14}, a_{13}, a_{12}, a_8 and a_4 in terms of the other a_i' s. Completing the computation, we obtain as a basis of $H^0(Y, \mathcal{L})$:

$$x_0^3y_0^3 + \lambda' x_0^3y_1^3 + \lambda x_1^3y_0^3 + \frac{\lambda\lambda'}{T^3}x_1^3y_1^3, \quad (x_0^3 + \frac{\lambda}{T}x_1^3)y_0^2y_1, \quad (x_0^3 + \frac{\lambda}{T^2}x_1^3)y_0y_1^2,$$

$$x_0^2x_1(y_0^3 + \frac{\lambda'}{T}y_1^3), \quad x_0^2x_1y_0^2y_1, \quad x_0^2x_1y_0y_1^2, \quad x_0x_1^2(y_0^3 + \frac{\lambda'}{T^2}y_1^3),$$

$$x_0x_1^2y_0^2y_1, \quad x_0x_1^2y_0y_1^2.$$

Different choices of λ and λ' do not lead to the same line bundle. Indeed, these are the two parameters in $Pic^0(Y) \cong (\mathbb{C}^*)^2$ and different choices leads to different line bundles. On the other hand, if we map x to λx and y to $\lambda' y$, this defines an automorphism of the singular variety (recall that the torus $T = (\mathbb{C}^*)^2$ acts on Y and this is exactly that action). Now pulling back via this automorphism, identifies the line bundle given by λ, λ', with that given by $1, 1$. Thus, by acting with this automorphism, we may indeed assume $\lambda = \lambda' = 1$. So from now on, we fix $\lambda = \lambda' = 1$.

Hence we get a rational map $\phi_{|\mathcal{L}|} : Y - -- > \mathbb{P}^8$; now we prove that $|\mathcal{L}|$ is base-point free, so that $\phi_{|\mathcal{L}|}$ is actually a morphism. To this aim, suppose that we have a point P of Y such that its image under the complete linear system $\mathcal{L}$ corresponds to the origin $(0, \ldots, 0)$ of $H^0(Y, \mathcal{L})$ (which is not a point of the corresponding projective space). In particular this implies that $x_0^2x_1y_0^2y_1 = 0$, that is at least one of the coordinates is zero. For instance, let us assume that $y_0 = 0$; then substituting in the fixed basis for $H^0(Y, \mathcal{L})$, we obtain: $x_0^2x_1y_1^3 = 0$, $x_0x_1^2y_1^3 = 0$ and $x_0^3y_1^3 + \frac{1}{T^3}x_1^3y_1^3 = 0$. From the first two equalities, we get that either $y_1 = 0$ (but this is not possible, since $[y_0, y_1] = [0, 0]$ is not a point of the projective line), or $x_0 = 0$ or $x_1 = 0$. If $x_0 = 0$, from the last equality we have $x_1 = 0$ (but this again impossible since $[x_0, x_1] = [0, 0]$ is not a point of the projective line), while if $x_1 = 0$, then $x_0 = 0$ or $y_1 = 0$, and we conclude as before. Hence we have a *morphism* $\phi_{|\mathcal{L}|} : Y \to \mathbb{P}^8$.

To conclude Theorem 2, we have to prove that $\phi_{|\mathcal{L}|}$ separates points and tangent lines. On the singular model Y we can distinguish three types of points: those inside the square of the second figure, which correspond to smooth points of Y and which have coordinates of the form $[1,\alpha] \times [1,\beta]$ for $\alpha, \beta \neq 0$ (first type); those for which only one of the coordinates $[x_0, x_1] \times [y_0, y_1]$ is zero (second type): these correspond to points on the edges on the square, but not on the vertices; and finally those points for which two of the coordinates $[x_0, x_1] \times [y_0, y_1]$ are zero (third type), the vertices of the square (actually, on Y there is just one point of the third type). Let us consider the image under $\phi_{|\mathcal{L}|}$ of a point P of the first type of the form $[1,\alpha] \times [1,\beta]$. The sections $x_0^2 x_1 y_0^2 y_1$ and $x_0 x_1^2 y_0^2 y_1$ are never vanishing on the points of this form; moreover, from their ratio one gets immediately the homogeneous coordinates $[x_0, x_1]$; a completely analogous reasoning, using now the sections $x_0^2 x_1 y_0 y_1^2$ and $x_0^2 x_1 y_0^2 y_1$ gives the homogeneous coordinates $[y_0, y_1]$. This means that $\phi^{-1}_{|\mathcal{L}|}(\phi_{|\mathcal{L}|}(P)) = P$, for a point of the first type.

Consider now a point P of the second type, for instance of the form $[1,\alpha] \times [1,0]$. Its image under $\phi_{|\mathcal{L}|}$ is given by $[1+\alpha^3, 0, 0, \alpha, 0, 0, \alpha^2, 0, 0] = Q \in \mathbb{P}^8$ and we have to prove that $\phi^{-1}_{|\mathcal{L}|}(Q)$ consists of two points on the smooth model X which are on the edges and which are going to be identified to a unique point on the singular model Y. Since $\alpha \neq 0$, from the expression of the coordiantes of Q we get that either $y_0 = 0$ or $y_1 = 0$. If $y_0 = 0$, from the expression $x_0^2 x_1 \frac{1}{T} y_1^3 = \alpha$, $x_0 x_1^2 \frac{1}{T^2} y_1^3 = \alpha^2$, taking their ratio we have $\frac{x_1}{x_0} = \alpha T$, so that we obtain the point $P_2 = [1, \alpha T] \times [0, 1]$. If instead $y_1 = 0$, from the relations $x_0^2 x_1 y_0^3 = \alpha$, $x_0 x_1^2 y_0^3 = \alpha^2$ we are led to the point $P_1 = P = [1,\alpha] \times [1,0]$. Now the points P_1 and P_2 are distinct on the model X, but they are identified under the glueing process, so that $\phi^{-1}_{|\mathcal{L}|}(\phi_{|\mathcal{L}|}(P)) = P$, also for a point P of the second type.

Finally, as for the points of third type, we can consider $P = P_1 = [1,0] \times [1,0]$ and its image $Q = [1, 0, \ldots, 0]$ under $\phi_{|\mathcal{L}|}$. Now, $\phi^{-1}_{|\mathcal{L}|}(Q)$ can be computed immediately, since from the expression of the cooordinates of Q we obtain the following four possibilities ($y_0 = 0$, $x_0 = 0$), ($y_0 = 0$, $x_1 = 0$), ($y_1 = 0$, $x_0 = 0$), and ($y_1 = 0$, $x_1 = 0$), which correspond to the four vertices of the square: again these four points are distinct on X, but are identified to a unique point on Y. This proves that $\phi_{|\mathcal{L}|}$ separates points on Y.

Now we prove that $\phi_{|\mathcal{L}|}$ separates tangent directions. First of all we prove that $d\phi_{|\mathcal{L}|_P} : T_P Y \to T_P \mathbb{P}^8$ is injective for any point P of the first type. So $P = [1,\alpha] \times [1,\beta]$; since the question about the injectivity of $d\phi_{|\mathcal{L}|}$ is local, we can substitute $x_0 = 1$, $y_0 = 1$ in the expression of the sections we have fixed as a basis of $H^0(Y, \mathcal{L})$ (this is equivalent to consider x_1, y_1 al local affine coordinates such that $(x_1, y_1)(P) = (\alpha, \beta)$). The fifth basis vector has then the form $x_1 y_1$ and is never vanishing for a point of the first type since $\alpha, \beta \neq 0$. Thus we can divide all the other sections by $x_1 y_1$ obtaining a map to $\mathbb{A}^8$:

$\rho := \phi_{|\mathcal{L}|}|_U : U \to \mathbb{A}^8$ (where U is an open neighbourhood of P), given by:

$$(x_1, y_1) \mapsto \left(\frac{1 + y_1^3 + x_1^3 + \frac{1}{T^3} x_1^3 y_1^3}{x_1 y_1}, \quad \frac{1}{x_1}(1 + \frac{1}{T} x_1^3), \quad \frac{y_1}{x_1}(1 + \frac{1}{T^2} x_1^3), \right.$$

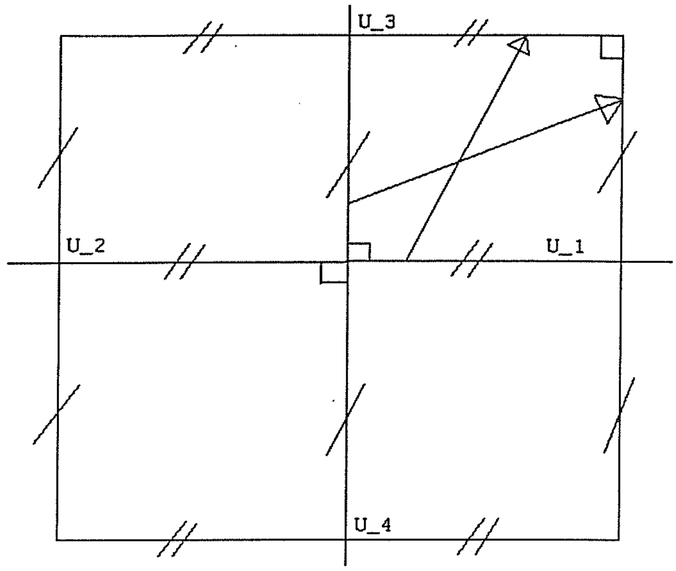
$$\left. \frac{1}{y_1}(1 + \frac{1}{T} y_1^3), \quad y_1, \quad \frac{x_1}{y_1}(1 + \frac{1}{T^2} y_1^3), \quad x_1, \quad x_1 y_1 \right).$$

To check injectivity of $d\phi_{|\mathcal{L}|}$ at the points of the first type, it is sufficient to compute the Jacobian matrix of the map ρ (with respect to x_1 and y_1) and evaluate it at (α, β) proving that it has rank 2. To prove that it has rank 2, just look at the form of the Jacobian matrix (without fulfilling the computation!):

$$\begin{pmatrix} - & - \\ - & - \\ - & - \\ - & - \\ 0 & 1 \\ - & - \\ 1 & 0 \\ - & - \end{pmatrix}.$$

The first column compute the derivative with respect to x_1 of the map ρ, the second column compute the derivative with respect to y_1, and the sign $-$ means that we have skipped the computation; however, it is clear that the rank of $d\phi_{|\mathcal{L}|}$ is 2, so that $\phi_{|\mathcal{L}|}$ separates tangent directions for the points of the first type.

To check the injectivity of the differential for points of the second and third type, we have to understand the singularities of Y on these kinds of points. From the toric description of Y and the corresponding Delaunay decomposition, we have that $T_P Y \cong \mathbb{A}^3$ if P is a point of second type, while $T_P Y \cong \mathbb{A}^4$ if P is of the third type. Indeed, the Delaunay decomposition from which Y arises is displayed in the following picture:



Let us study for example the singularity of P, the point of third type. Since the lattice points u_1, u_2, u_3 and u_4 are not cell-mates, from the associated toric construction it turns out that we have in $\mathbb{C}[u_1, u_2, u_3, u_4]$ the relations $u_1u_2 = 0$, $u_3u_4 = 0$; these correspond to four 2-planes meeting in one point (the point P). These four 2-planes are $\pi_1 = \{(u_1 = 0, u_3 = 0)\}$, $\pi_2 = \{(u_1 = 0, u_4 = 0)\}$, $\pi_3 = \{(u_2 = 0, u_3 = 0\}$ and $\pi_4 = \{(u_2 = 0, u_4 = 0)\}$. Clearly the intersection of all these four planes is just the origin (the point P) and there are pairs of these planes which intersect along a line, as π_1 and π_2. Moreover, T_PY can be spanned just by a pair of planes, which intersect each other just in P, such as (π_1, π_4). These two planes are represented as two small boxes near P in the above picture. This indeed proves that $T_PY \cong \mathbb{A}^4$. A completely analogous reasoning proves that $T_PY \cong \mathbb{A}^3$ for a point P of the second type.

To prove injectivity of $d\phi_{|\mathcal{L}|}$ for a point P of the second type, we prove that the vector space spanned by the image of the differential at the two points P_1 and P_2 (these two points are on the edges of the square and are identified to the unique point P on Y) is at least 3-dimensional.

Let us consider as $P_1 = [1, \alpha] \times [1, 0]$ and $P_2 = [1, T\alpha] \times [0, T]$ which correspond to the unique (singular) point P on Y. As before, on the sections forming a basis of $H^0(Y, \mathcal{L})$, we substitute $x_0 = 1$, $y_0 = 1$, and we divide by the fourth basis section, which is not vanishing on P. In this way we get a map ρ' to $\mathbb{A}^8$ (centered at $(\alpha, 0)$) given by:

$$(x_1, y_1) \mapsto \left(\frac{1 + y_1^3 + x_1^3 + \frac{1}{T^3}x_1^3y_1^3}{x_1(1 + \frac{1}{T}y_1^3)}, \ \frac{(1 + \frac{1}{T}x_1^3)y_1}{x_1(1 + \frac{1}{T}y_1^3)}, \ \frac{(1 + \frac{1}{T^2}x_1^3)y_1^2}{x_1(1 + \frac{1}{T}y_1^3)}, \right.$$

$$\frac{y_1}{1+\frac{1}{T}y_1^3}, \quad \frac{y_1^2}{1+\frac{1}{T}y_1^3}, \quad \frac{x_1(1+\frac{1}{T^2}y_1^3)}{1+\frac{1}{T}y_1^3}, \quad \frac{x_1y_1}{1+\frac{1}{T}y_1^3}, \quad \frac{x_1y_1^2}{1+\frac{1}{T}y_1^3}\Bigg).$$

Computing the Jacobian matrix of the map ρ' and evaluating at $(\alpha, 0)$ we have:

$$\begin{pmatrix} - & 0 \\ 0 & - \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & - \\ 0 & 0 \end{pmatrix}.$$

Now consider the point $P_2 = [1, \alpha T] \times [0, T]$ and substitute $x_1 = \alpha T$ and $y_1 = T$ in the expression of the sections forming a basis of $H^0(Y, \mathcal{L})$. We then divide all sections again by the fourth section (which is not vanishing on P_2) and we obtain a map to $\mathbb{A}^8$ (this time centered at $(1, 0)$) given by:

$$(x_0, y_0) \mapsto \Bigg(\frac{x_0^3y_0^3 + T^3x_0^3 + \alpha^3T^3y_0^3 + \alpha^3T^3}{x_0^2\alpha T(y_0^3+T^2)}, \quad \frac{y_0^2T(x_0^3+\alpha^3T^2)}{x_0^2\alpha T(y_0^3+T^2)}, \quad \frac{y_0T^2(x_0^3+\alpha^3T)}{x_0^2\alpha T(y_0^3+T^2)},$$

$$\frac{y_0^2T}{y_0^3+T^2}, \quad \frac{y_0\alpha T^3}{\alpha T(y_0^3+T^2)}, \quad \frac{\alpha^2T^2(y_0^3+T)}{x_0\alpha T(y_0^3+T^2)}, \quad \frac{y_0^2\alpha T^2}{x_0(y_0^3+T^2)}, \quad \frac{y_0\alpha T^3}{x_0(y_0^3+T^2)}\Bigg).$$

Now we just take the partials of this map with respect to y_0 and evaluate at $(1, 0)$ obtaining:

$$\begin{pmatrix} - \\ - \\ - \\ - \\ 1 \\ - \\ - \\ - \end{pmatrix}.$$

Comparing with the image of the differential at P_1, we see that the rank of $d\phi_{|\mathcal{L}|}$ at the point of the second type is at least 3, and this is sufficient to conclude that $\phi_{|\mathcal{L}|}$ separates tangent directions for the points of second type.

As for the points of the third type (actually there is just *one* point P of the third type on Y which corresponds to the four vertices of the square), we have to check that the rank of $d\phi_{|\mathcal{L}|}$ is 4. From the toric description of the singularity around P it turns out that this can be seen as the intersection of 4 2-planes, meeting in one point (the point P) and along some other lines (which corresponds to the points of the second type). In particular there are 2 2-planes just meeting in P which span $T_P Y \cong \mathbb{A}^4$. So it is sufficient to check that the image of these two planes under $d\phi_{|\mathcal{L}|}$ spans again an $\mathbb{A}^4$. To this aim, we compute $d\phi_{|\mathcal{L}|}$ at $[1,0]\times[1,0]$ and at $[0,T]\times[0,T]$.

As before, for the point $P_1 = [1,0]\times[1,0]$, we substitute $x_0 = 1$, $y_0 = 1$ in the sections forming a basis of $H^0(Y,\mathcal{L})$, and we divide by the first basis section, which is not vanishing on P_1. In this way, we get a map (centered in $(0,0)$) to $\mathbb{A}^8$, given by:

$$(x_1,y_1)\mapsto\Bigg(\frac{y_1(1+\frac{1}{T}x_1^3)}{1+y_1^3+x_1^3+\frac{1}{T^3}x_1^3y_1^3},\ \frac{y_1^2(1+\frac{1}{T^2}x_1^3)}{1+y_1^3+x_1^3+\frac{1}{T^3}x_1^3y_1^3},\ \frac{x_1(1+\frac{1}{T}y_1^3)}{1+y_1^3+x_1^3+\frac{1}{T^3}x_1^3y_1^3},$$
$$\frac{x_1y_1}{1+y_1^3+x_1^3+\frac{1}{T^3}x_1^3y_1^3},\ \frac{x_1y_1^2}{1+y_1^3+x_1^3+\frac{1}{T^3}x_1^3y_1^3},\ \frac{x_1^2(1+\frac{1}{T^2}y_1^3)}{1+y_1^3+x_1^3+\frac{1}{T^3}x_1^3y_1^3},$$
$$\frac{x_1^2y_1}{1+y_1^3+x_1^3+\frac{1}{T^3}x_1^3y_1^3},\ \frac{x_1^2y_1^2}{1+y_1^3+x_1^3+\frac{1}{T^3}x_1^3y_1^3}\Bigg).$$

Taking the Jacobian matrix of this map and evaluating at $(0,0)$ we have:

$$\begin{pmatrix} 0 & 1\\ 0 & 0\\ 1 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0 \end{pmatrix}.$$

Finally, to construct the corresponding map to $\mathbb{A}^8$ centered at $P_2 = [0,T]\times[0,T]$, we substitute $x_1 = T$ and $y_1 = T$ in the sections forming a basis of $H^0(Y,\mathcal{L})$ and we divide again by the first basis section (as we have done above), which is not vanishing on P_2. By so doing we get the map:

$$(x_0,y_0)\mapsto\Bigg(\frac{y_0^2T(x_0^3+T^2)}{x_0^3y_0^3+x_0^3T^3+y_0^3T^3+T^3},\ \frac{y_0T^2(x_0^3+T)}{x_0^3y_0^3+x_0^3T^3+y_0^3T^3+T^3},$$
$$\frac{x_0^2T(y_0^3+T^2)}{x_0^3y_0^3+x_0^3T^3+y_0^3T^3+T^3},\ \frac{x_0^2y_0^2T^2}{x_0^3y_0^3+x_0^3T^3+y_0^3T^3+T^3},\ \frac{x_0^2y_0T^3}{x_0^3y_0^3+x_0^3T^3+y_0^3T^3+T^3},$$

$$\frac{x_0T^2(y_0^3+T)}{x_0^3y_0^3+x_0^3T^3+y_0^3T^3+T^3}, \quad \frac{x_0y_0^2T^3}{x_0^3y_0^3+x_0^3T^3+y_0^3T^3+T^3}, \quad \frac{x_0y_0T^4}{x_0^3y_0^3+x_0^3T^3+y_0^3T^3+T^3}\Bigg).$$

Again, taking the Jacobian matrix and evaluating at $(0,0)$ we get:

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

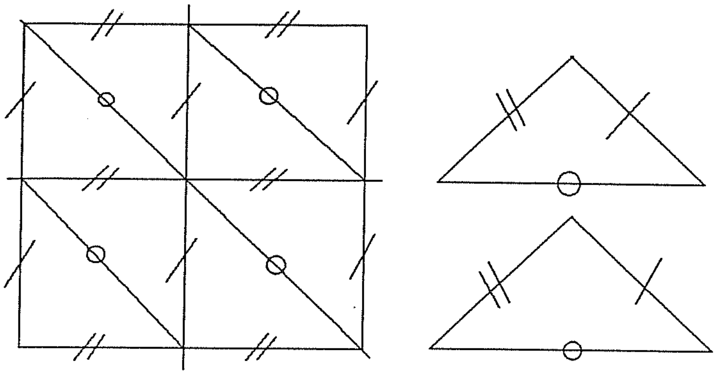
Thus, we see that the two 2-planes spanned by the image of $d\phi_{|\mathcal{L}|}$ at P_1 and P_2 are independent (in $\mathbb{A}^8$), so that the rank of $d\phi_{|\mathcal{L}|}$ at the point P of the third type on Y is 4. Then $\phi_{|\mathcal{L}|}$ separates tangent directions.

We thus have an embedding $\phi_{|\mathcal{L}|} : Y \hookrightarrow \mathbb{P}^8$. □

Remark 3: For the value $T = 1$, the singular model Y is just the product of two nodal curves C and C'; each of these curves is embedded into $\mathbb{P}^2$ and then via a Segre map into $\mathbb{P}^8$: $C \times C' \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$. Since the dimension of $H^0(Y_T, \mathcal{L})$ is independent of T, this automatically implies that for *generic* T $\phi_{|\mathcal{L}|}$ is an embedding; however to check this for any $T \in \mathbb{C}^*$ we have to give a proof as above.

4.4 Very ampleness on the third degeneration type and conclusion

The third degeneration type Y for smooth principally polarized abelian surfaces is constructed via a glueing of two disjoint copies of $\mathbb{P}^2$. It does not depend on any moduli, i.e. it is rigid. Indeed, from the toric construction associated to the following Delaunay decomposition:



it turns out that Y is obtained by glueing two disjoint $\mathbb{P}^2$' s along the following pairs of lines ($[x_0, x_1, x_2]$ denote homogeneous coordinates on the first $\mathbb{P}^2$, $[y_0, y_1, y_2]$ on the second): $\{x_0 = 0\}$ and $\{y_0 = 0\}$, $\{x_1 = 0\}$ and $\{y_1 = 0\}$, $\{x_2 = 0\}$ and $\{y_2 = 0\}$ and identifying moreover the coordinate points to a unique point. The desingularization of Y clearly consists of $X := \mathbb{P}^2 \amalg \mathbb{P}^2$, $\pi : X \to Y$. A line bundle on X is just the union of two line bundles, one on each copy of $\mathbb{P}^2$. Obviously, the divisor class group of X is generated by L_1 and L_2 (each of which is a line in $\mathbb{P}^2$, such that $L_1.L_2 = 0$). Since the self-intersection of a principal polarization on a smooth abelian surface is 2, and this does not change in a flat family, it turns out that we can consider as a degenerate principal polarization on the smooth model X the line bundle given by $\mathcal{O}_{\mathbb{P}^2}(1)$ on each $\mathbb{P}^2$. For simplicity, let us call this bundle $\mathcal{O}_X(1;1)$, and observe that $h^0(X, \mathcal{O}_X(1;1)) = 6$. If $\mathcal{L}$ is a line bundle on Y such that $\pi^*\mathcal{L} = \mathcal{O}_X(n;n)$, then we denote $\mathcal{L}$ as $\mathcal{O}_Y(n)$. By the results of Alexeev and Nakamura, it turns out that the complete linear system $|\mathcal{O}_Y(5)|$, gives an embedding of Y into some $\mathbb{P}^N$. This is concretely realized embedding each disjoint copy of $\mathbb{P}^2$, via $|\mathcal{O}_{\mathbb{P}^2}(5)|$, in such a way that they are glued along the prescribed lines and points. Indeed, the embedding of Y is induced by determining which sections of $\mathcal{O}_X(5;5)$ descend to Y:

$$\begin{array}{ccccc} \mathbb{P}^2 \amalg \mathbb{P}^2 & & \rightrightarrows & & \mathbb{P}^N \\ \pi & \searrow & & \nearrow & |\mathcal{O}_Y(5)| \\ & & Y & & \end{array}$$

Also in this case, to prove an analogue of Lefschetz theorem is equivalent to prove the following:

Theorem 3: *With the notations as above, the complete linear system $|\mathcal{O}_Y(3)|$ is base-point free and the map $\phi_{|\mathcal{O}_Y(3)|} : Y \hookrightarrow \mathbb{P}^8$ is an embedding.*

Proof: As previously noticed, the map $\phi_{|\mathcal{O}_Y(3)|}$ is induced by the sections of the line bundles $\mathcal{O}_{\mathbb{P}^2}(3)$ on each $\mathbb{P}^2$, imposing the glueing conditions. Let us call x_0, x_1, x_2 the homogeneous coordinates on the first $\mathbb{P}^2$ and y_0, y_1, y_2 those on the second. Then the general section $\sigma_x \in H^0(\mathcal{O}_{\mathbb{P}^2}(3))$ can be written as:

$$\sigma_x = a_0x_0^3 + a_1x_1^3 + a_2x_2^3 + a_3x_0^2x_1 + a_4x_0^2x_2 + a_5x_1^2x_0 + a_6x_1^2x_2 + a_7x_2^2x_0 + a_8x_2^2x_1 + a_9x_0x_1x_2,$$

and analogously for:

$$\sigma_y = b_0y_0^3 + b_1y_1^3 + b_2y_2^3 + b_3y_0^2y_1 + b_4y_0^2y_2 + b_5y_1^2y_0 + b_6y_1^2y_2 + b_7y_2^2y_0 + b_8y_2^2y_1 + b_9y_0y_1y_2.$$

In order to determine which sections descend onto Y, we express the glueing conditions as:

$$\begin{aligned} \sigma_x|_{\{x_0=0\}} &= \lambda\sigma_y|_{\{y_0=0\}}, && (4.7)\\ \sigma_x|_{\{x_1=0\}} &= \lambda'\sigma_y|_{\{y_1=0\}}, && (4.8)\\ \sigma_x|_{\{x_2=0\}} &= \lambda''\sigma_y|_{\{y_2=0\}}, && (4.9)\\ \sigma_x|_{[1,0,0]} &= \mu\sigma_x|_{[0,1,0]}, && (4.10)\\ \sigma_x|_{[1,0,0]} &= \nu\sigma_x|_{[0,0,1]}, && (4.11) \end{aligned}$$

for *some* (not fixed!) parameters $\lambda, \lambda', \lambda'', \mu, \nu \in \mathbb{C}^*$. The equations (4.7), (4.8) and (4.9), express the glueing conditions for the coordinate lines, while the remaining equations express the fact that the coordinate points have to be identified to a unique point.

From the first three equations we get $\sigma_x|_{[1,0,0]} = \lambda'\sigma_y|_{[1,0,0]} = \lambda''\sigma_y|_{[1,0,0]}$ and $\sigma_x|_{[0,1,0]} = \lambda\sigma_y|_{[0,1,0]} = \lambda''\sigma_y|_{[0,1,0]}$; from these relations we get $\lambda = \lambda' = \lambda''$ and $b_0 = \lambda a_0$, $b_1 = \lambda a_1$ and $b_2 = \lambda a_2$. On the other hand, from (4.10) and (4.11), we obtain $a_2 = \nu^{-1}a_0$ and $a_1 = \mu^{-1}a_0$. Combining these relations, we see that all coordinates a_2, a_3, b_0, b_1, b_2 are multiples of a_0. Now notice that the equations (4.7), (4.8) and (4.9) can be written as (just assuming that there are *some*, not fixed parameters $\lambda, \lambda', \lambda''$):

$$rk\begin{pmatrix} a_1 & a_2 & a_6 & a_8 \\ b_1 & b_2 & b_6 & b_8 \end{pmatrix} \leq 1, \tag{4.12}$$

$$rk\begin{pmatrix} a_0 & a_2 & a_4 & a_7 \\ b_0 & b_2 & b_4 & b_7 \end{pmatrix} \leq 1, \tag{4.13}$$

$$rk\begin{pmatrix} a_0 & a_1 & a_3 & a_5 \\ b_0 & b_1 & b_3 & b_5 \end{pmatrix} \leq 1. \tag{4.14}$$

From these and from the previous relations, we find immediately all the compatibility conditions:

$$b_0 = \lambda a_0, \quad b_1 = \lambda\mu^{-1}a_0, \quad b_2 = \lambda\nu^{-1}a_0, \quad a_2 = \nu^{-1}a_0, \quad a_1 = \mu^{-1}a_0,$$

$$b_3 = \lambda a_3, \quad b_4 = \lambda a_4, \quad b_5 = \lambda a_5, \quad b_6 = \lambda a_6, \quad b_7 = \lambda a_7, \quad b_8 = \lambda a_8.$$

Then we can choose as coordinates for determining a basis for the sections of $\mathcal{O}_Y(3)$, the coefficients $(a_0, a_3, a_4, a_5, a_6, a_7, a_8, b_9)$. Now observe that changing the parameter λ, does not change the line bundle, since the two planes are different and one can make a corresponding choice of coordinates on one of the planes, so as to cancel out the effect of changing λ. Thus we can set $\lambda = 1$. On the other hand, different choices of μ and ν lead to different line bundles; indeed, $Pic^0(Y) \cong (\mathbb{C}^*)^2$ and the parameters μ and ν are coordinates on $Pic^0(Y)$. But again modulo the action of the torus $T = (\mathbb{C}^*)^2$, all line bundles are the same, so one can choose one of them, e.g. by setting $\mu = \nu = 1$. Thus, from now on, we set $\lambda = \mu = \nu = 1$.

In the light of this analysis, the map $\phi_{|\mathcal{O}_Y(3)|}$ can be precisely described via a pair of maps $(\phi_x, \phi_y) : \mathbb{P}^2 \amalg \mathbb{P}^2 \rightrightarrows \mathbb{P}^8$, which is explicitly given by:

$$\Big[(x_0^3 + x_1^3 + x_2^3; y_0^3 + y_1^3 + y_2^3), \quad (x_0^2 x_1; y_0^2 y_1), \quad (x_1^2 x_0; y_1^2 y_0), \quad (x_0^2 x_2; y_0^2 y_2),$$

$$(x_1^2 x_2; y_1^2 y_2), \quad (x_2^2 x_0; y_2^2 y_0), \quad (x_2^2 x_1; y_2^2 y_1), \quad (x_0 x_1 x_2; 0), \quad (0; y_0 y_1 y_2)\Big].$$

If $[z_0, z_1, \dots, z_8]$ denotes homogeneous coordinates in $\mathbb{P}^8$, then the image of the first $\mathbb{P}^2$ is contained in the hyperplane $\{z_8 = 0\}$, while that of the second is contained in $\{z_7 = 0\}$. Now we prove that $\phi_{|\mathcal{O}_Y(3)|}$ is an embedding, checking the suitable properties on the pair of maps (ϕ_x, ϕ_y).

First of all, we check that $|\mathcal{O}_Y(3)|$ has no base points: just take into account one of the maps of the pair, for instance ϕ_x. If there is a point on Y such that all sections of $\mathcal{O}_Y(3)$ vanish, then for the correponding point(s) in X, we have $x_0 x_1 x_2 = 0$, $x_1^2 x_0 = 0$, $x_0^2 x_2 = 0$ and $x_1^2 x_2 = 0$. Then at least two of the x_i's is zero, but since also $x_0^3 + x_1^3 + x_2^3 = 0$, then all x_i' s are zero. This is clearly impossible since this is not a point of $\mathbb{P}^2$. Thus $\phi_{|\mathcal{O}_Y(3)|} : Y \to \mathbb{P}^8$ is a morphism.

Let us distinguish, also for this type of degeneration, three kinds of points: smooth points (first type), singular points obtained by glueing pairs of points on the edges of the triangles, which are not vertices (second type), and the unique point which comes from the glueing of the vertices (third type). Now we prove that $\phi_{|\mathcal{O}_Y(3)|}$ separates points. For the points of the first type, let us consider a point in the first copy of $\mathbb{P}^2$, of homogeneous coordinates $[1, \alpha, \beta]$, where $\alpha, \beta \in \mathbb{C}^*$. Then the image of this point under ϕ_x is given by: $[1 + \alpha^3 + \beta^3, \alpha, \alpha^2, \beta, \alpha^2\beta, \beta^2, \beta^2\alpha, \alpha\beta, 0]$. From this expression, since $\alpha\beta \neq 0$, then this point in $\mathbb{P}^8$ is never the image of a point of the second $\mathbb{P}^2$, and it is clear that one can recover the homogeneous coordinates $[1, \alpha, \beta]$, just taking ratios, so that the map $\phi_{|\mathcal{O}_Y(3)|}$ separates points of the first type. Now consider a point of the second type, which can be represented on the smooth model X, by a pair of points of the form (for example)

$[\alpha, \beta, 0], [\alpha', \beta', 0]$ such that $\alpha, \alpha', \beta, \beta' \in \mathbb{C}^*$ and $\beta/\alpha = \beta'/\alpha'$. Rescaling the homogeneous coordinates, one can represent these as $[1, \gamma, 0], [1, \gamma, 0]$. These two points have the same image under the two maps (ϕ_x, ϕ_y): $[1 + \gamma^3, \gamma, \gamma^2, 0, 0, 0, 0, 0, 0]$. Again taking ratios one sees that this point corresponds exactly to the pair of points, which are going to be identified on Y to a point of the second type (indeed, looking at the zero entries, the point $[1 + \gamma^3, \gamma, \gamma^2, 0, 0, 0, 0, 0, 0]$ can not be the image of a smooth point or of a point on a different singular line). Finally, the unique point of the third type is represented by three pairs of points on X. Any of these pairs has image in $\mathbb{P}^8$: $[1, 0, 0, 0, 0, 0, 0, 0, 0] = P$. Then it is clear that $(\phi_x, \phi_y)^{-1}(P)$ consists exactly of these three pairs, which correspond to the unique point of the third type on Y. Thus $\phi_{|\mathcal{O}_Y(3)|}$ is injective on Y.

To conclude the proof it remains to show that this map separates tangent directions. If Q is a point of the first type on Y, then it corresponds to a unique point on one of the two copies of $\mathbb{P}^2$. It is not restrictive to assume that the homogeneous coordinates of this point belong to the first $\mathbb{P}^2$ and are of the form $[1, \alpha, \beta]$ $(\alpha, \beta \in \mathbb{C}^*)$. Since $T_Q Y \cong \mathbb{A}^2$ for this type of points, it is sufficient to prove that the rank of $d\phi_x$ is 2 in a neighbourhood of $\pi^{-1}(Q) = [1, \alpha, \beta]$. Substituting $x_0 = 1$ and dividing by $x_0 x_1 x_2$ the entries of the map ϕ_x we obtain a map from a neighbourhood U of $\pi^{-1}(Q)$ to $\mathbb{A}^8$: $\phi_x' : U \to \mathbb{A}^8$, centered at (α, β), which is explicitly given by:

$$\left(\frac{1 + x_1^3 + x_2^3}{x_1 x_2}, \quad \frac{1}{x_2}, \quad \frac{x_1}{x_2}, \quad \frac{1}{x_1}, \quad x_1, \quad \frac{x_2}{x_1}, \quad x_2, \quad 0 \right).$$

It is immediate to check that the differential of this map computed at (α, β) has rank 2.

If Q is a point of the second type on Y, then since it is obtained by glueing two $\mathbb{P}^2$'s along lines, it is clear that $T_Q Y \cong \mathbb{A}^3$. Moreover, $\pi^{-1}(Q) = (P_1, P_2)$ where each P_i belongs to a $\mathbb{P}^2$. Without loosing generality, we can assume that $P_1 = [1, \alpha, 0]$ (x-coordinates) and $P_2 = [1, \alpha, 0]$ (y-coordinates). Then it is sufficient to prove that $d\phi_x | T_{P_1 \mathbb{P}^2}$ and $d\phi_y | T_{P_2 \mathbb{P}^2}$ span a vector space of dimension at least 3. To compute $d\phi_x$ we substitute $x_0 = 1$ and divide all entries by $x_0^2 x_1$, obtaining an explicit map ϕ_x' to $\mathbb{A}^8$ of the form:

$$\left(\frac{1 + x_1^3 + x_2^3}{x_1}, \quad x_1, \quad \frac{x_2}{x_1}, \quad x_2 x_1, \quad \frac{x_2^2}{x_1}, \quad x_2^2, \quad x_2, \quad 0 \right).$$

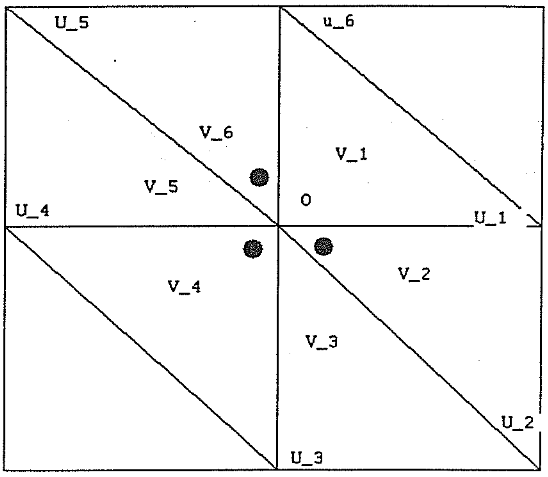
An analogous reasoning for the y-coordinates, gives an explicit map ϕ_y' to $\mathbb{A}^8$ given by:

$$\left(\frac{1 + y_1^3 + y_2^3}{y_1}, \quad y_1, \quad \frac{y_2}{y_1}, \quad y_2 y_1, \quad \frac{y_2^2}{y_1}, \quad y_2^2, \quad 0, \quad y_2 \right).$$

Computing the Jacobian of the two maps at the point $(\alpha, 0)$ (i.e. $(x_1, x_2) = (\alpha, 0)$ for the x-coordinates and $(y_1, y_2) = (\alpha, 0)$ for the y-coordinates) we get the two matrices:

$$\begin{pmatrix} - & - \\ 1 & 0 \\ - & - \\ - & - \\ - & - \\ - & - \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} - & - \\ 1 & 0 \\ - & - \\ - & - \\ - & - \\ - & - \\ 0 & 0 \\ 0 & 1 \end{pmatrix},$$

which span together at least a 3-dimensional vector space in $\mathbb{A}^8$. Finally, we have to prove that $\phi_{|\mathcal{O}_Y(3)|}$ separates tangent directions for the unique point Q of the third type. This point corresponds to the vertices of the triangles, which are all identified. It can be represented by the origin O of the following picture:



where there are 6 triangles meeting. Then a neighbourhood of Q can be described as 6 copies of $\mathbb{A}^2$, meeting along lines according to the pattern in the previous picture. We call this copies of $\mathbb{A}^2$ as $V_1, V_2, V_3, V_4, V_5, V_6$. Now $V_1 \cap V_2$ is a line so that they span together a 3-dimensional vector space W_1. Then $W_1 \cap V_3$ is again a line, so that the span of W_1 and V_3 is 4-dimensional vector space W_2. Again $W_2 \cap V_4$ is a line and they together span W_3 which is 5-dimensional and finally $W_3 \cap V_5$ is a line and they span W_4 which is 6-dimensional. Then observe that $V_6 \subset W_4$ since they have in common 2 lines. This implies that $T_Q Y = \mathbb{A}^6$, showing that Q is an extremely nasty singularity.

We can give a cleaner proof of the fact that the dimension of the tangent space is actually equal to 6, via toric geometry. Indeed, considering the points $u_1, u_2, u_3, u_4, u_5, u_6$ in the previous picture, since they are not cell-mates, we get in $\mathbb{C}[u_1, \ldots, u_6]$, the following exhaustive set of relations: $u_1u_3 = 0$, $u_1u_4 = 0$, $u_1u_5 = 0$, $u_2u_4 = 0$, $u_2u_5 = 0$, $u_2u_6 = 0$, $u_3u_6 = 0$, $u_3u_5 = 0$, $u_4u_6 = 0$. The ideal generated by these relations in $\mathbb{C}[u_1, \ldots, u_6]$, corresponds to six 2-planes: $\pi_1 = \{u_1 = u_2 = u_3 = u_4 = 0\}$, $\pi_2 = \{u_1 = u_2 = u_3 = u_6 = 0\}$, $\pi_3 = \{u_3 = u_4 = u_5 = u_6 = 0\}$, $\pi_4 = \{u_2 = u_3 = u_4 = u_5 = 0\}$, $\pi_5 = \{u_1 = u_2 = u_5 = u_6 = 0\}$ and $\pi_6 = \{u_1 = u_4 = u_5 = u_6 = 0\}$. All these six planes intersect just in the origin and the span of three of them, such as π_1, π_3 and π_5 is $\mathbb{A}^6$. This just proves that $T_QY \cong \mathbb{A}^6$.

Then, to conclude it is enough to show that the images of $d\phi_x$ at the three coordinate points span altogether a 6-dimensional vector space. Indeed, to span T_QY it is sufficient to take three copies of $\mathbb{A}^2$ around O, which meet only in O, as those selected with dots in the previous picture (they correspond to the 2-planes π_1, π_3 and π_5). These copies correpond to the three vertices of just one copy of $\mathbb{P}^2$, let us say the x-copy. Then we have just to compute $d\phi_x$ at the points $P_1 = [1,0,0]$, $P_2 = [0,1,0]$ and $P_3 = [0,0,1]$. To compute $d\phi_x$ at P_1 we set $x_0 = 1$ and divide by the entry $x_0^3 + x_1^3 + x_2^3$ all other entries, getting the map ϕ_{x,P_1} as follows:

$$\left(\frac{x_1}{1+x_1^3+x_2^3}, \frac{x_1^2}{1+x_1^3+x_2^3}, \frac{x_2}{1+x_1^3+x_2^3}, \frac{x_1^2x_2}{1+x_1^3+x_2^3}, \frac{x_2^2}{1+x_1^3+x_2^3}, \frac{x_2^2x_1}{1+x_1^3+x_2^3}, \frac{x_1x_2}{1+x_1^3+x_2^3}, 0\right).$$

Computing the differential of ϕ_{x,P_1} at $(0,0)$ we get:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Repeating the procedure with the point P_2 (this time setting $x_1 = 1$ but *always* dividing by $x_0^3 + x_1^3 + x_2^3$) we obtain the map ϕ_{x,P_2}:

$$\left(\frac{x_0^2}{1+x_0^3+x_2^3}, \frac{x_0}{1+x_0^3+x_2^3}, \frac{x_0^2x_2}{1+x_0^3+x_2^3}, \frac{x_2}{1+x_0^3+x_2^3},\right.$$

$$\frac{x_2^2x_0}{1+x_0^3+x_2^3}, \quad \frac{x_2^2}{1+x_0^3+x_2^3}, \quad \frac{x_0x_2}{1+x_0^3+x_2^3}, \quad 0\Bigg),$$

and also we get the differential at $(0,0)$:

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Finally, considering the point P_3, we have the map ϕ_{x,P_3}:

$$\Bigg(\frac{x_0^2x_1}{1+x_0^3+x_1^3}, \quad \frac{x_1^2x_0}{1+x_0^3+x_1^3}, \quad \frac{x_0^2}{1+x_0^3+x_1^3}, \quad \frac{x_1^2}{1+x_0^3+x_1^3},$$

$$\frac{x_0}{1+x_0^3+x_1^3}, \quad \frac{x_1}{1+x_0^3+x_1^3}, \quad \frac{x_0x_1}{1+x_0^3+x_1^3}, \quad 0\Bigg),$$

the differential of which at $(0,0)$ is given by:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This implies that $d\phi_{|\mathcal{O}_Y(3)|}$ is always injective on Y, also for the point of third type, since the rank of $d\phi_x$ is 6. Thus $\phi_{|\mathcal{O}_Y(3)|} : Y \hookrightarrow \mathbb{P}^8$ is an embedding. □

Recalling all the results of the previous sections, we get immediately the following main result:

Theorem 4: *Let Y be a SSAV of dimension 2, which is a degeneration of a principally polarized abelian surface and let $\mathcal{O}_Y(1)$ the associated ample line bundle. Then $\mathcal{O}_Y(3)$ is already very ample.*

Proof: Immediate in the light of Theorems 1, 2 and 3, since any SSAV Y of dimension 2, coming from a principally polarized abelian surface belongs to one of the three degeneration types studied in the previous sections. □

Remark 4: Theorem 4 gives an improvement (in the case of surfaces) with respect to the result proved in ([38], Theorem 4.7); indeed there they proved that $\mathcal{O}_Y(5)$ is very ample. In general, we expect that the bound given in ([38]) for very ampleness of $\mathcal{O}_Y(n)$ is not sharp, at least for some classes of SSAV' s. This is also suggested by the results of ([41]): there it is proved (among other things) that if C is an irreducible curve, having only nodes as singularities, then on the compactified Jacobian (considered as the moduli scheme parametrizing torsion-free, rank 1 sheaves of Euler characteristic 0 on C), there is a line bundle $\mathcal{L}$ representing a principal polarization, such that $\mathcal{L}^{\otimes 3}$ is already very ample.

Chapter 5

Deformation properties and linear normality of curves

5.1 Introduction

Let k be any algebraically closed field of characteristic zero and, as usual, let $\mathbb{P}^r := Proj(k[x_0, \dots, x_r])$ be the associated projective space. Inside the Hilbert scheme $H(d, g, r)$, parametrizing closed subschemes of dimension 1, arithmetic genus g, degree d in $\mathbb{P}^r$, let us consider the so called *restricted* Hilbert scheme $Hilb(d, g, r)$, which is the subscheme of $H(d, g, r)$, consisting of those points $p(C)$, such that every component K of $H(d, g, r)$ on which $p(C)$ lies has smooth, non degenerate and irreducible general element (see Definition 1.31 of [54]).

In this chapter we present some results concerning the behaviour of the rational functorial map $\phi : Hilb(d, g, r) \to \mathcal{M}_g$, which associates to each point $p(C)$ in $Hilb(d, g, r)$ representing a smooth non degenerate irreducible curve C the corresponding isomorphism class $[C] \in \mathcal{M}_g$. In particular, we study in which cases the image of ϕ has positive dimension (i.e. the corresponding family of curves is not isotrivial). In this study, a key role is played by linearly normal curves, since they tend to be less rigid than other class of embedded curves. So, in this chapter we see how an embedding with some additional properties can help to study a seemingly totally unrelated problem.

Observe that any non degenerate smooth integral subscheme C of dimension 1 in $\mathbb{P}^r$ determines a point $p(C) \in Hilb(d, g, r)$. We give the following:

Definition 1: *The projective curve $C \subset \mathbb{P}^r$ admits non-trivial first order deformations if the image of the map $D\phi : T_{p(C)}Hilb(d, g, r) \to T_{[C]}\mathcal{M}_g$ has positive dimension (or equivalently if $D\phi \neq 0$). In this case we say that the corresponding curve is non-rigid at* *the first order, for the given embedding.*

Definition 2: *The projective curve $C \subset \mathbb{P}^r$ admits non-trivial deformations if there exists at least a curve $\gamma \subset Hilb(d, g, r)$, through $p(C)$, which is not contracted to a point via ϕ. Equivalently, if there exists an irreducible component of $Hilb(d, g, r)$ containing $p(C)$, such that its image in $\mathcal{M}_g$ through ϕ has positive dimension. In this case we say that the curve is non-rigid for the given embedding.*

We can somehow get rid of the fixed embedding in some projective space taking into account *all* possible nondegenerate embedding, as in the following:

Definition 3: *The (abstract) smooth curve C is non-rigid at the first order, as a smooth non degenerate projective curve, if for* any *non degenerate projective embedding $j : C? \hookrightarrow \mathbb{P}^r$, the corresponding map $D\phi : T_{p(C)}Hilb(d, g, r) \to T_{[C]}\mathcal{M}_g$ is non zero.*

Analogously, one has the following:

Definition 4: *The (abstract) smooth curve C is non-rigid as a smooth non degenerate projective curve if, for* any *non degenerate projective embedding $j : C \hookrightarrow \mathbb{P}^r$, there exists an irreducible component of the associated $Hilb(d, g, r)$ containing $p(C)$, such that its image in $\mathcal{M}_g$ through ϕ has positive dimension.*

In this chapter, we prove that there exists a dense open subset $U \subset \mathcal{M}_g$ $(g \geq 1)$, such that any C, with $[C] \in U$, is non rigid at the first order as a smooth non degenerate projective curve in the sense of Definition 3; moreover, we prove that these curves are non-rigid (not only at the first order) under the additional assumption that $p(C)$ is a smooth point of $Hilb(d, g, r)_{red}$ (the restricted Hilbert scheme with reduced scheme structure) or at worst it is a reducible singularity of $Hilb(d, g, r)_{red}$ (see Definition 5 in section 3).

5.2 First order deformations

First of all we deal with the case of smooth projective curves of genus $g \geq 2$ in $\mathbb{P}^r$. We will prove that there exists a dense open subset $U^0_{BN} \subset \mathcal{M}_g$ such that for any $[C] \in U^0_{BN}$ and for *any* non degenerate smooth embedding of C in $\mathbb{P}^r$ the corresponding projective curve is non-rigid at the first order (in the sense of Definition 3).

From the fundamental exact sequence:

$$0 \to TC \to T\mathbb{P}^r_{|C} \to N_{C/\mathbb{P}^r} \to 0, \tag{5.1}$$

taking the associated long exact cohomology sequence, since $H^0(TC) = H^0(K_C^{-1}) = 0$ (genus $g \geq 2$), we get:

$$0 \to H^0(T\mathbb{P}^r_{|C}) \to H^0(N_{C/\mathbb{P}^r}) \xrightarrow{D\phi} H^1(TC) \to H^1(T\mathbb{P}^r_{|C}) \to H^1(N_{C/\mathbb{P}^r}) \to 0. \tag{5.2}$$

In sequence (5.2), as usual, we identify $H^0(N_{C/\mathbb{P}^r})$ with the tangent space $T_{p(C)}Hilb(d,g,r)$ to the Hilbert scheme at the point $p(C)$ representing C, and $H^1(TC)$ with $T_{[C]}\mathcal{M}_g$. Thus the coboundary map $D\phi$ represents the differential of the map $\phi : Hilb(d,g,r) \to \mathcal{M}_g$ we are interested in. If $D\phi = 0$ (i.e. the corresponding curve is rigid also at the first order) the sequence above splits and in particular $h^0(T\mathbb{P}^r_{|C}) = h^0(N_{C/\mathbb{P}^r})$; thus imposing $h^0(T\mathbb{P}^r_{|C}) < h^0(N_{C/\mathbb{P}^r})$ and estimating the dimension of the cohomology groups, we get a relation involving d, g, r, which, if it is fulfilled implies that the corresponding curve is not rigid (at least at the first order). This is the meaning of the following:

Proposition 1: *Let $C \subset \mathbb{P}^r$ a smooth non-degenerate curve of genus $g \geq 2$ and degree d. Then if $d > \frac{2}{r+1}[g(r-2)+3]$, or $\mathcal{O}_C(1)$ is non special (this holds if $d > 2g-2$), then $D\phi \neq 0$. Furthemore, if $C \subset \mathbb{P}^r$ is linearly normal, then $D\phi \neq 0$ provided that*

$$d > \frac{(r-2)g + r(r+1) + 3}{r+1}. \tag{5.3}$$

Proof: It is clear from the exactness of (5.2) that if $h^0(N_{C/\mathbb{P}^r}) > h^0(T\mathbb{P}^R_{|C})$, then $D\phi \neq 0$. On the other hand, $h^0(N_{C/\mathbb{P}^r}) = dim(T_{p(C)}Hilb(d,g,r)) \geq dim(Hilb(d,g,r))$ and $dim(Hilb(d,g,r)) \geq (r+1)d - (r-3)(g-1)$, where the last inequality always holds at points of $Hilb(d,g,r)$ parametrizing locally complete intersection curves (in particular smooth curves), see for example ([54]). Thus $h^0(N_{C/\mathbb{P}^r}) \geq (r+1)d - (r-3)(g-1)$. Now, applying Riemann-Roch to the vector bundle $T\mathbb{P}^r$ on C, we get $h^0(T\mathbb{P}^r_{|C}) = (r+1)d - r(g-1) + h^1(T\mathbb{P}^r_{|C})$. On the other hand, from the Euler sequence (twisted with $\mathcal{O}_C$):

$$0 \quad \to \quad \mathcal{O}_C \quad \to \quad (r+1)\mathcal{O}_C(1) \quad \to \quad T\mathbb{P}^r_{|C} \quad \to \quad 0, \tag{5.4}$$

we get immediately $h^1(T\mathbb{P}^r_{|C}) \leq (r+1)h^1(\mathcal{O}_C(1))$ and by Riemann-Roch the latter is equal to $(r+1)(h^0(\mathcal{O}_C(1)) - d + g - 1)$. Now, if $\mathcal{O}_C(1)$ is non special (i.e. if $d > 2g-2$), then $h^1(T\mathbb{P}^r_{|C}) = 0$, so that, imposing $h^0(N_{C/\mathbb{P}^r}) > h^0(T\mathbb{P}^R_{|C})$, we get $3(g-1) > 0$, which is always satisfied (if $g \geq 2$). This means that a smooth curve of genus $g \geq 2$, which is embedded via a non special linear system, is always non-rigid at least at the first order.

If instead $\mathcal{O}_C(1)$ is special, by Clifford's theorem we have $h^0(\mathcal{O}_C(1)) \leq d/2 + 1$, so that $h^1(T\mathbb{P}^r_{|C}) \leq (r+1)(g - d/2)$. Imposing again $h^0(N_{C/\mathbb{P}^r}) > h^0(T\mathbb{P}^r_{|C})$, that is $(r+1)d - (r-3)(g-1) > (r+1)d - r(g-1) + (r+1)(g-d/2)$, we get the relation $d > \frac{2}{r+1}[g(r-2)+3]$.

Finally, if $C \subset \mathbb{P}^r$ is linearly normal and non degenerate, then $h^0(\mathcal{O}_C(1)) = r+1$. Substituting in $h^1(T\mathbb{P}^r_{|C}) \leq (r+1)h^1(\mathcal{O}_C(1)) = (r+1)(h^0(\mathcal{O}_C(1)) - d + g - 1)$ and imposing the fundamental inequality $h^0(N_{C/\mathbb{P}^r}) > h^0(T\mathbb{P}^r_{|C})$, we get $d > \frac{(r-2)g+r(r+1)+3}{r+1}$. □

Since the bound (5.3) is particularly good, but it holds only for linearly normal curves and since any curve can be obtained via a series of (generic) projections from a linearly normal curve, we are going to study what is the relation among first order deformations of a linearly normal curve and the first order deformations of its projections. This is the aim of the following:

Proposition 2: *Let $C \subset \mathbb{P}^r$ a smooth curve of genus $g \geq 2$ which is non-rigid at the first order. Then any of its smooth projections $C' := \pi_q(C) \subset \mathbb{P}^{r-1}$ from a point $q \in \mathbb{P}^r$ is non-rigid at the first order.*

Proof: First of all, let us remark that the proposition states that in the following diagram:

$$\begin{array}{ccc} T_{p(C)}Hilb(d,g,r) & \stackrel{D\phi}{\to} & T_{[C]}\mathcal{M}_g \\ \downarrow & \nearrow D\phi' & \\ T_{p(C')}Hilb(d,g,r-1) & & \end{array}$$

if $Im(D\phi) \neq 0$, then $Im(D\phi') \neq 0$. Now consider the following commutative diagram:

$$\begin{array}{ccccccccc} & & 0 & \to & ker(a) & \to & ker(b) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & TC & \to & T\mathbb{P}^r_{|C} & \to & N_{C/\mathbb{P}^r} & \to & 0 \\ & & \downarrow\cong & & \downarrow a & & \downarrow b & & \\ 0 & \to & TC' & \to & T\mathbb{P}^{r-1}_{|C'} & \to & N_{C'/\mathbb{P}^r} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & \to & 0 & \to & 0 & & \end{array}$$

where the morphisms a and b are induced by the projection of C to C'. Clearly $TC \cong TC'$, because C and C' are isomorphic curves ($q \notin Sec(C)$ and thus the projection is an isomorphism) and moreover a and b are surjective by construction. Applying the snake lemma to the previous diagram, we see that $ker(a) \cong ker(b)$ and since a and b are surjective morphisms of vector bundles, it turns out that $ker(a) = ker(b) = \mathcal{L}$, where $\mathcal{L}$ is a line bundle on C. Restricting the attention to the last column of the previous diagram, it is clear from a geometric reasoning that the line bundle $\mathcal{L}$ can be identified with the ruling of the projective cone, with vertex q through which we project. Indeed, it is sufficient to look at the projection induced map b at a point $x \in C$: $b : N_{C/\mathbb{P}^r, x} \to N_{C'/\mathbb{P}^r, \pi(x)}$; the kernel is always the line on the cone with vertex q going trough x and this is never a subspace of TC, because $q \notin Sec(C)$. Clearly, we can identify the projective cone with vertex q through which we project, with the line bundle $\mathcal{L}$, since we can consider instead of just $\mathbb{P}^r$, the blowing-up $Bl_q(\mathbb{P}^r)$ in q in such a way to separate the ruling of the cone (this however does not affect our reasoning since we are dealing with line bundles over C and $q \notin C$).

Applying the cohomology functor to the previous commutative diagram and recalling that $h^0(TC) = 0$ since $g \geq 2$ we get the following diagram:

$$\begin{array}{ccccccccc} & & H^0(\mathcal{L}) & \cong & H^0(\mathcal{L}) & \rightarrow & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & H^0(T\mathbb{P}^r_{|C}) & \rightarrow & H^0(N_{C/\mathbb{P}^r}) & \overset{D\phi}{\rightarrow} & H^1(TC) & \rightarrow & \dots \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \rightarrow & H^0(T\mathbb{P}^{r-1}_{|C'}) & \rightarrow & H^0(N_{C'/\mathbb{P}^{r-1}}) & \overset{D\phi'}{\rightarrow} & H^1(TC') & \rightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & coker(\alpha) & \rightarrow & coker(\beta) & \rightarrow & 0 & \rightarrow & 0 \end{array}$$

Now, $Im(D\phi) \subset H^1(TC)$ and via the isomorphism γ it is mapped inside $H^1(TC')$. On the other hand, by commutativity of the square having as edges the maps $\beta, \gamma, D\phi$ and $D\phi'$ it is clear that $Im(D\phi) \subseteq Im(D\phi')$ so that if $D\phi \neq 0$, then a fortiori $D\phi' \neq 0$. □

The following corollary gives two simple sufficient conditions for having $Im(D\phi) \cong Im(D\phi')$.

Corollary 1: *Let C, C', $D\phi$ and $D\phi'$ as in Proposition 2. Then if $\mathcal{O}_C(1)$ is non special or if $h^1(T\mathbb{P}^r_{|C}) = 0$, then $Im(D\phi) \cong Im(D\phi')$.*

Proof: Rewrite the previous diagram as:

$$\begin{array}{ccccccccc} & & H^0(\mathcal{L}) & \cong & H^0(\mathcal{L}) & \rightarrow & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & H^0(T\mathbb{P}^r_{|C}) & \rightarrow & H^0(N_{C/\mathbb{P}^r}) & \rightarrow & Im(D\phi) & \rightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow & & \\ 0 & \rightarrow & H^0(T\mathbb{P}^{r-1}_{|C'}) & \rightarrow & H^0(N_{C'/\mathbb{P}^{r-1}}) & \rightarrow & Im(D\phi') & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & coker(\alpha) & \rightarrow & coker(\beta) & \rightarrow & coker(\alpha)/coker(\beta) & \rightarrow & 0 \end{array}$$

Observe that $coker(\alpha) \subseteq H^1(\mathcal{L})$ and the same is true for $coker(\beta)$. So if $H^1(\mathcal{L}) = 0$, then $Im(D\phi) = Im(D\phi')$. On the other hand, from the exact sequence $0 \rightarrow \mathcal{L} \rightarrow T\mathbb{P}^r_{|C} \rightarrow T\mathbb{P}^{r-1}_{|C'} \rightarrow 0$, taking Chern polynomials, we get that $\mathcal{L}$ is a line bundle of degree d (and one can identify $\mathcal{L}$ with $\mathcal{O}_C(1) \otimes \mathcal{L}'$ for some $\mathcal{L}' \in Pic^0(C)$). Thus, if $\mathcal{O}_C(1)$ is non special we conclude. If instead $h^1(T\mathbb{P}^r_{|C}) = 0$, then $coker(\alpha) = H^1(\mathcal{L})$ and $coker(\alpha) \subseteq coker(\beta) \subseteq H^1(\mathcal{L})$ so that $coker(\alpha) = coker(\beta)$ and we conclude again. □

Now we deal with the much simpler case of curves of genus $g = 1$.

Proposition 3: *For any smooth curve $[C] \in \mathcal{M}_1$ and for any non degenerate projective embedding of $C \hookrightarrow \mathbb{P}^r$, the corresponding projective curve is non-rigid at the first order.*

Proof: From the fundamental exact sequence:

$$?0 \rightarrow T_C \rightarrow T\mathbb{P}^r_{|C} \rightarrow N_{C/\mathbb{P}^r} \rightarrow 0,$$

since $T_C \cong \mathcal{O}_C$ $(g = 1)$, we obtain the long exact cohomology sequence:

$$0 \rightarrow H^0(\mathcal{O}_C) \rightarrow H^0(T\mathbb{P}^r_{|C}) \rightarrow H^0(N_{C/\mathbb{P}^r}) \stackrel{D\phi}{\rightarrow} H^1(\mathcal{O}_C) \rightarrow H^1(T\mathbb{P}^r_{|C}) \dots \qquad (5.5)$$

Twisting the Euler sequence with $\mathcal{O}_C$ and taking cohomology, we have that $h^1(T\mathbb{P}^r_{|C}) \leq (r+1)h^1(\mathcal{O}_C(1))$, but $\mathcal{O}_C(1)$ is always non special for a curve of genus $g = 1$ since $d > 2g - 2 = 0$. Thus $H^1(T\mathbb{P}^r_{|C}) = 0$ and being $h^1(\mathcal{O}_C) \neq 0$, from (5.5) we have that $D\phi \neq 0$ and it is even always surjective. □

We conclude this section with the following theorem, which is the analogue of Proposition 3 for curves of genus $g \geq 2$ (in this case we do not work over all $\mathcal{M}_g$, but just on an open dense subset).

Theorem 1: *For any $g \geq 2$, there exists a dense open subset $U_{BN} \subset \mathcal{M}_g$ such that for any $[C] \in U_{BN}$ and for* any *non degenerate projective embedding of $C \hookrightarrow \mathbb{P}^r$, the corresponding projective curve is non-rigid at the first order.*

Proof: According to theorem 1.8, page 216 of [46], there exists a dense open subset $U_{BN} \subset \mathcal{M}_g$ such that any $[C] \in U_{BN}$ can be embedded in $\mathbb{P}^r$ as a smooth non degenerate curve of degree d if and only if $\rho \geq 0$, where $\rho(d, g, r) := g - (r+1)(g - d + r)$ is the Brill-Noether number. Now we consider a curve $[C] \in U_{BN}$ and we embed it as a linerly normal curve $\bar{C}$ of degree d in some $\mathbb{P}^r$. Since $[C] \in U_{BN}$, we have that $\rho \geq 0$; on the other hand, $\bar{C}$ is linearly normal and the fundamental inequality (5.3) is satisfied since $\rho \geq 0$ (indeed, it is just a computation to see that (5.3) is equivalent to $\rho \geq -\epsilon$ for some $\epsilon > 0$). Thus, by Proposition 1 $\bar{C}$ is non-rigid at the first order, and moreover by Propositon 2 all of its smooth projections are non-rigid at the first order. To conclude, observe that any smooth non degenerate projective curve C such that $[C] \in U_{BN}$ can be obtained via a series of smooth projections from a linearly normal projective curve $\bar{C}$ with corresponding $\rho \geq 0$ (since for the curves in U_{BN} the Brill-Noether condition is necessary and sufficient).
□

5.3 Finite deformations

Our problem is now to extend the first order deformations studied in the previous section to finite deformations. By Theorem 1, we know that, for the curves C such that $[C] \in U_{BN}$ $(g \geq 2)$, the corresponding $Im(D\phi) \neq 0$ and an even stronger result holds for curves of genus $g = 1$. We need to prove that there exists a vector $v \in T_{p(C)}Hilb(d,g,r)$, corresponding to a smooth curve $\gamma \subset Hilb(d,g,r)$ through $p(C)$ such that the image of the curve via ϕ has positive dimension. To this aim, observe that if $[C] \in U_{BN}$ is not a smooth point of $\mathcal{M}_g$, then there are $w \in T_{[C]}U_{BN}$ which are obstructed deformations, that is which do not correspond to any curve in U_{BN} through $[C]$. We can easily get rid of this problem, just by restricting further the open subset U_{BN}. Indeed, for $g \geq 1$, there is a dense open subset $U^0 \subset \mathcal{M}_g$ such that any $[C] \in U^0$ is a smooth point (see for example [54]). Thus, for curves of genus $g \geq 2$ we consider the dense open subset $U^0_{BN} := U_{BN} \cap U^0$ and for any $w \in T_{[C]}U^0_{BN}$, the corresponding first order deformations are unobstructed, while for curves of genus $g = 1$ we just restrict to the smooth part of $\mathcal{M}_1$, that we denote as U^0_1.

We can draw a first conclusion to this reasoning via the following:

Proposition 4: *Let $[C] \in U^0_{BN}$ or $[C] \in U^1_0$ and let $C \hookrightarrow \mathbb{P}^r$ any projective embedding such that the corresponding point $p(C) \in Hilb(d,g,r)$ is a smooth point of the restricted Hilbert scheme. Then the projective curve $C \subset \mathbb{P}^r$ is non rigid.*

Proof: By Theorem 1 or Proposition 3, the associated map $D\phi \neq 0$, so that there exists a $w \in T_{p(C)}Hilb(d,g,r)$ such that $D\phi(w) \neq 0$. Since $p(C)$ is a smooth point of $Hilb(d,g,r)$, the tangent vector w corresponds to a smooth curve $\gamma \subset Hilb(d,g,r)$, through $p(C)$, such that $T_{p(C)}\gamma = w$. Now consider the image Z of this curve in U^0_{BN} via ϕ. Since $\mathcal{M}_g$ exists as a quasi-projective variety, in particular we can represent a neighbourhood of $[C] \in \mathcal{M}_g$, as $Spec(B)$, for some finitely generated k-algebra B. This implies that the map ϕ can be viewed locally around $p(C)$ as a morphism of affine schemes. Thus the image of the curve γ (which is a reduced scheme) via the morphism of affine schemes ϕ is the subscheme Z in $Spec(B)$. Then either Z is positive dimensional and in this case we are done, or it is a zero dimensional subscheme, supported at the point $[C]$; observe that this zero dimensional subscheme Z can not be the reduced point $[C]$, otherwise we would certainly have $D\phi(w) = 0$. So let us consider the case in which Z is a zero dimensional subscheme, supported at the point $[C]$, with non-reduced scheme structure: this case is clearly impossible since the image Z of a reduced subscheme (the curve γ) via the morphism of affine schemes ϕ can not be a non-reduced subscheme. Indeed, if it were the case, consider the rectriction of ϕ to γ: ϕ_γ, $Z_{red} = [C]$; then $\phi_\gamma^{-1}([C])$ is a reduced subscheme, which coincides with γ, since γ is reduced. But this would imply that $\phi(\gamma) = [C]$ and $D\phi(w) = 0$.

Thus, it turns out that Z has necessarily positive dimension and we conclude.

□

The hypothesis of Proposition 4, according to which $p(C)$ is a smooth point of $Hilb(d,g,r)$ is extremely strong. Ideally, one would like to extend the result of Proposition 4 to *any* non degenerate projective embedding for curves $[C] \in U^0_{BN}$. Before giving a partial extension of Proposition 4 (Theorem 2), let us give the following:

Definiton 5: *A point $p(C) \in Hilb(d,g,r)_{red}$ is called a reducible singularity if it is in the intersection of two or more irreducible components of $Hilb(d,g,r)_{red}$, each of which is smooth in $p(C)$.*

Theorem 2: *Let $[C] \in U^0_{BN}$ or $[C] \in U^1_0$ and let $C \hookrightarrow \mathbb{P}^r$ any projective embedding such that the corresponding point $p(C) \in Hilb(d,g,r)$ is a smooth point of $Hilb(d,g,r)_{red}$ (restricted Hilbert scheme with reduced structure) or such that $p(C)$ is a reducible singularity of $Hilb(d,g,r)_{red}$. Then the projective curve $C \subset \mathbb{P}^r$ is non rigid.*

Proof: Let us consider the exact sequence:

$$0 \rightarrow H^0(T\mathbb{P}^r_{|C}) \rightarrow T_{p(C)}Hilb(d,g,r) \overset{D\phi}{\rightarrow} T_{[C]}\mathcal{M}_g \qquad (5.6)$$

from which $ker(D\phi) \cong H^0(T\mathbb{P}^r_{|C})$. Take the reduced scheme $Hilb(d,g,r)_{red}$ and consider the induced morphism of schemes $r : Hilb(d,g,r)_{red} \rightarrow Hilb(d,g,r)$ (see for example [42], exercise 2.3, page 79). If $p(C)$ is a smooth point of $Hilb(d,g,r)_{red}$, then we have that $dim(Hilb(d,g,r)) = dim(T_{p(C)}Hilb(d,g,r)_{red})$. On the other hand, to prove that there are first order deformations we have just imposed $h^0(T\mathbb{P}^r_{|C}) < dim(Hilb(d,g,r))$. Now, we want to prove that in the following diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & H^0(T\mathbb{P}^r_{|C}) & \rightarrow & T_{p(C)}Hilb(d,g,r) & \overset{D\phi}{\rightarrow} & T_{[C]}\mathcal{M}_g \\ & & & & \uparrow Dr & \nearrow & \\ & & & & T_{p(C)}Hilb(g,d,r)_{red} & & \end{array}$$

the map Dr is injective, so that since $h^0(T\mathbb{P}^r_{|C}) < dim(Hilb(d,g,r)) = dim(T_{p(C)}Hilb(d,g,r)_{red})$, we can find a $w \in T_{p(C)}Hilb(d,g,r)_{red}$ the image of which in $T_{[C]}\mathcal{M}_g$ is non zero and then we can argue as in the proof of Proposition 4. Setting $Hilb(d,g,r)_{red} = X_{red}$, $p(C) = x$ and $Hilb(d,g,r) = X$, we have to prove that given $r : X_{red} \rightarrow X$, the associated morphism on tangent spaces is injective $Dr : T_xX_{red} \rightarrow T_xX$. Since X_{red} is a scheme, we can always find an open affine subscheme U_{red} of X_{red} containing x such that $U_{red} = Spec(A_{red})$, where A_{red} is a finitely generated k-algebra without nilpotent elements and the closed point x corresponds to a maximal ideal m_x. Recall that, from the point of view of the functor of points, the closed point x corresponds to a morphism $\lambda : Spec(k) \rightarrow Spec(A_{red})$ (which is induced by $A_{red} \rightarrow A_{red,m_x} \rightarrow A_{red,m_x}/m_xA_{red,m_x} = k(x) = k$, where A_{red,m_x} is the localization of A_{red} at the maximal ideal m_x). Recall also that via the algebra map $k[\epsilon]/\epsilon^2 \rightarrow k$ and the corresponding inclusion of schemes $i : Spec(k) \rightarrow Spec(k[\epsilon]/\epsilon^2)$, T_xX_{red} can be identified with $\{u \in Hom(Spec(k[\epsilon]/\epsilon^2, Spec(A_{red})))$ such that $u \circ i = \lambda\}$. Clearly, an analogous description holds for X and T_xX, (we denote the corresponding neighbourhood of x in X as $Spec(A)$). From the description of T_xX_{red} just given, it turns out any $w \in T_xX_{red}$, $w \neq 0$, corresponds to a (non-zero) ring homomorphism $u^\natural : A_{red} \to k[\epsilon]/(\epsilon)^2$, such that the following diagram is commutative:

$$\begin{array}{ccc} A_{red} & \stackrel{u^\natural}{\to} & k[\epsilon]/\epsilon^2 \\ & \lambda^\natural \searrow & \downarrow i^\natural \\ & & k(x) = k \end{array}$$

On the other hand, saying that $Dr(w) \neq 0$ is equivalent to say that we can lift the non zero ring homomorphism $u^\natural : A_{red} \to k[\epsilon]/\epsilon^2$ to a non zero ring homomorphism $\tilde{u}^\natural : A \to k[\epsilon]/\epsilon^2$ such that the following diagram is commutative:

$$\begin{array}{ccc} A & \stackrel{\tilde{\lambda}^\natural}{\to} & k(x) = k \\ r^\natural \downarrow & \tilde{u}^\natural \searrow & \uparrow i^\natural \\ A_{red} & \stackrel{u^\natural}{\to} & k[\epsilon]/\epsilon^2 \\ & \lambda^\natural \searrow & \downarrow i^\natural \\ & & k(x) = k \end{array}$$

It is clear that we can always do such a lifting, since the homomorphisms $\tilde{u}^\natural$ and $\tilde{\lambda}^\natural$ are just given precomposing the corresponding homomorphisms from A_{red}, with $r^\natural$. Moreover, since $r^\natural$ is a non zero ring homorphism, it turns out that if $u^\natural \neq 0$, then also $\tilde{u}^\natural \neq 0$ and the previous diagram is commutative. This implies that $Dr(w) \neq 0$ and thus that $Dr : T_{p(C)}Hilb(d,g,r)_{red} \hookrightarrow T_{p(C)}Hilb(d,g,r)$ is injective. Reasoning as in the proof of Proposition 4, we can find a curve $\gamma \subset Hilb(d,g,r)_{red}$ through $p(C)$ in such a way that $D\phi \circ Dr(T_{p(C)}\gamma) \neq 0$. Thus the image of this curve via $\phi \circ r$ contains the point $[C]$ in U^0_{BN} and a tangent direction. On the other hand the image via $\phi \circ r$ of a reduced scheme can not be a non reduced point (always because we can represent a neighbourhood of $[C]$ in $\mathcal{M}_g$ as an affine scheme and consider $\phi \circ r$ locally as a morphism of affine schemes). Thus the image of γ through $\phi \circ r$ must have positive dimension and in this way we conclude if $p(C)$ is a smooth point of $Hilb(d,g,r)_{red}$.

Finally, if $p(C)$ is a reducible singularity of $Hilb(d,g,r)_{red}$, it will be sufficient to repeat the previous reasoning, substituting $T_{p(C)}Hilb(d,g,r)_{red}$, with $T_{p(C)}H$, where H is an irreducible component of $Hilb(d,g,r)_{red}$ through $p(C)$, smooth at $p(C)$ and of maximal dimension, so that $dim_{p(C)}H = dim_{p(C)}Hilb(d,g,r)_{red} = dim_{p(C)}Hilb(d,g,r)$. In the same way, one can find a smooth curve $\gamma \subset H$, through $p(C)$, such that its image in $\mathcal{M}_g$ is positive dimensional, arguing again as in the proof of Proposition 4 (the image of γ has to a be a reduced scheme, hence necessarily positive dimensional, in order to have $D(\phi) \neq 0$). □

Remark: If $p(C)$ is a reducible singularity of $Hilb(d,g,r)_{red}$, for the Theorem 2 to work, it is not necessary that *all* irreducible components of $Hilb(d,g,r)_{red}$ through $p(C)$ are smooth in a neighbourhood of $p(C)$. Indeed, from the proof of Theorem 2, it is clear that it sufficient that there exists an irreducible component of maximal dimension H of $Hilb(d,g,r)_{red}$, which is smooth in a neighbourhood of $p(C)$.

In the light of the previous theorem, let us dicuss Mumford's famous example of a component of the restricted Hilbert scheme which is non reduced (see [55]). He considered smooth curves C on smooth cubic surfaces S in $\mathbb{P}^3$, belonging to the complete linear system $|4H + 2L|$, where H is the divisor class of a hyperplane section of S and L is the class of a line on S. It is immediate to see that the degree of such a curve is $d = 14$ and that its genus is $g = 24$. Therefore we are working with $Hilb(14, 24, 3)$. In [55], it is proved that the sublocus J_3 of $Hilb(14, 24, 3)$ cut out by curves C of this type, is dense in a component of the Hilbert scheme. Moreover, it turns out that this component is non reduced. Indeed, Mumford showed that the dimension of $Hilb(14, 24, 3)$ at the point $p(C)$ representing a curve C of the type just described, is 56, while the dimension of the tangent space to $Hilb(14, 24, 3)$ at $p(C)$ is 57. On the other hand, in [48] it is proved that for the points of type $p(C)$ an infinitesimal deformation (i.e. a deformation over $Spec(k[\epsilon]/\epsilon^2)$) is either obstructed at the second order (i.e. you can not lift the deformation to $Spec(k[\epsilon]/\epsilon^3)$), or at no order at all. This implies that the corresponding component of $Hilb(14, 24, 3)_{red}$ is smooth. Since for curves of this type, we have that $d > \frac{g+3}{2}$, by Proposition 1 we know that $D\phi \neq 0$. If $[C] \in \mathcal{M}_g$ is a smooth point, then by Theorem 2, being $Hilb(14, 24, 3)_{red}$ smooth at $p(C) \in J_3$, we have that the curve $C \hookrightarrow \mathbb{P}^3$ is non rigid for the given embedding.

For other interesting examples of singularities of Hilbert schemes of curves and related constructions, see [50], [53], [47] and [57].

5.4 Some special classes of curves in $\mathbb{P}^3$

In this section, we take into account some special classes of curves and prove that they are non-rigid at the first order or even non-rigid for the given embedding. As a first example, let us consider a projectively normal curve C in $\mathbb{P}^3$, which does not sit on a quadric or on a cubic. We prove that the curves of this class are non-rigid at the first order. Their ideal sheaf has a resolution of the type (with $a_j \geq 4$ and consequently $b_j \geq 5$):

$$0 \to \oplus_{j=1}^{s}\mathcal{O}_{\mathbb{P}^3}(-b_j) \to \oplus_{j=1}^{s+1}\mathcal{O}_{\mathbb{P}^3}(-a_j) \to \mathcal{I}_C \to 0,$$

from which, twisting with $T\mathbb{P}^3$, we get:

$$0 \to \oplus_{j=1}^{s}T\mathbb{P}^3(-b_j) \to \oplus_{j=1}^{s+1}T\mathbb{P}^3(-a_j) \to T_{\mathbb{P}^3}\otimes\mathcal{I}_C \to 0. \quad (5.7)$$

On the other hand, from the Euler sequence (suitably twisted) we have that $h^0(T\mathbb{P}^3(-k)) = 0$ and $h^1(T\mathbb{P}^3(-k)) = 0$ for $k \geq 4$. Thus, from (5.7) it follows that $h^0(T\mathbb{P}^3 \otimes \mathcal{I}_C) = 0$. Moreover, $H^2(T\mathbb{P}^3(-b_j))$ is equal by Serre duality to $H^1(\Omega^1_{\mathbb{P}^3}(b_j - 4))^*$ and this is zero by Bott formulas (see for example [56]), since we assumed $b_j \geq 5$. Therefore, again from (5.7), it follows that $h^1(T\mathbb{P}^3 \otimes \mathcal{I}_C) = 0$. Finally, from the defining sequence of C, twisting by $T\mathbb{P}^3$, we get that $H^0(T\mathbb{P}^3) \cong H^0(T\mathbb{P}^3|_C)$. Now, $h^0(T\mathbb{P}^3) = 15$, so that $D\phi \neq 0$ as soon as $15 < 4d$ (recall that $h^0(N_{C/\mathbb{P}^3}) \geq 4d$), that this $D\phi \neq 0$ for $d \geq 4$. Now, recall the important fact that if C is a projectively normal curve, then $Hilb(\mathbb{P}^3)$ is smooth at the corresponding point $p(C)$ (see [51]) and this implies that the projectively normal curve is non rigid (Theorem 2) as soon as it does not sit on a quadric or a cubic surface.

Now we consider a projectively normal curve which sits on a smooth cubic surface S in $\mathbb{P}^3$ and prove that this curve is non-rigid at the first order and hence non-rigid always by Theorem 2 and by the result of [51]. From the exact sequence:

$$0 \to N_{C/S} \to N_{C/\mathbb{P}^3} \to N_S|_C \to 0, \quad (5.8)$$

since $N_S|_C \cong \mathcal{O}_C(3)$ and $N_{C/S} \cong \omega_C \otimes \omega_S^{-1} \cong \omega_C(1) \cong \mathcal{O}_C(C)$, we get $\chi(N_{C/\mathbb{P}^3}) = \chi(\omega_C(1)) + \chi(\mathcal{O}_C(3))$. By Riemann-Roch $\chi(\mathcal{O}_C(3)) = 3d - g + 1$ and by Serre duality $h^1(\omega_C(1)) = h^0(\mathcal{O}_C(1)) = 0$, so that $\chi(\omega_C(1)) = C^2 + 1 - g$ and $\chi(N_{C/\mathbb{P}^3}) = 3d - g + 1 + h^0(\omega_C(1)) = 3d - 2g + 2 + C^2$. Again from the sequence (5.8), taking cohomology, we have that $h^1(N_{C/\mathbb{P}^3}) = h^1(\mathcal{O}_C(3))$. On the other hand, from the exact sequence:

$$0 \to \mathcal{I}_C(3) \to \mathcal{O}_{\mathbb{P}^3}(3) \to \mathcal{O}_C(3) \to 0,$$

assuming that C is projectively normal and that it sits on a unique cubic, we have $1 - 20 + h^1(\mathcal{O}_C(3)) + 3d - g + 1 = 0$, so that $h^1(N_{C/\mathbb{P}^3}) = 18 - 3d + g$. Thus $h^0(N_{C/\mathbb{P}^3}) = \chi(N_{C/\mathbb{P}^3}) + h^1(N_{C/\mathbb{P}^3}) = 20 - g + C^2$. As a remark, notice that since $h^0(N_{C/\mathbb{P}^3}) \geq 4d$, we obtain the inequality $4d \leq 20 - g + C^2$ for curves of this type. To give an estimate of $h^0(T\mathbb{P}^3_{|C})$, we use as before the Riemann-Roch Theorem and the Euler sequence, so that $h^0(T\mathbb{P}^3_{|C}) \leq 4d + 3(1 - g) + 4h^1(\mathcal{O}_C(1))$. On the other hand, from the defining sequence of C twisted by $\mathcal{O}_{\mathbb{P}^3}(1)$, assuming C projectively normal and nondegenerate, we get $h^1(\mathcal{O}_C(1)) = g - d + 3$, so that $h^0(T\mathbb{P}^3_{|C}) \leq g + 15$. Thus $D\phi \neq 0$ as soon as $g + 15 < 20 - g + C^2$. Using adjunction formula, i.e. $C.(C + K_S) = 2g - 2$, we can rewrite this as $C.K_S < 3$. Now, since S is a smooth cubic $K_S \equiv -H$ where H is an effective divisor representing a hyperplane section. Moreover any C is linearly equivalent to $al - \sum b_i e_i$ and $h \equiv 3l - \sum e_i$ (we identify S with $\mathbb{P}^2$ blown-up in 6 different points), so that $D\phi \neq 0$ as soon as $3a - \sum b_i > 3$, but $3a - \sum b_i = d$, and so we get the condition $d \geq 4$.

Finally , as an example we consider the case of projectively normal curves on a smooth quadric Q, proving that these curves are non-rigid (indeed it is sufficient to assume that $h^1(\mathcal{I}_C(2)) = 0$). First of all, from the sequence:

$$0 \to N_{C/Q} \to N_{C/\mathbb{P}^3} \to N_Q|_C \to 0,$$

being $N_{C/Q} \cong \omega_C(2)$ and $N_Q|_C \equiv \mathcal{O}_C(2)$, we have that $h^1(N_{C/\mathbb{P}^3}) = h^1(\mathcal{O}_C(2))$; from the defining sequence $0 \to \mathcal{I}_C(2) \to \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_C(2) \to 0$, since we assumed $h^1(\mathcal{I}_C(2)) = 0$, we have $1 - 10 + h^1(\mathcal{O}_C(2)) + 2d - g + 1 = 0$. Moreover, by Serre duality and Kodaira vanishing $h^1(\omega_C(2)) = h^1(N_{C/Q}) = 0$ so that $h^0(N_{C/\mathbb{P}^3}) = \chi(\omega_C(2)) + \chi(\mathcal{O}_C(2)) + h^1(N_{C/\mathbb{P}^3})$ and this is equal to $10 - g + C^2$. The previous estimate for $h^0(T\mathbb{P}^3_{|C})$ works also in this case (we just used the fact that C is linearly normal and non degenerate), so that $D\phi \neq 0$ as soon as $g + 15 < 10 - g + C^2$. By adjunction $2g - 2 = C.(C + K_Q)$, and by the fact that $K_Q \equiv -2H$, the inequality $g + 15 < 10 - g + C^2$ can be rewritten as $2C.H > 7$, so that for $d \geq 4$ C is non rigid at the first order for the given embedding and so they are non-rigid (Theorem 2 and [51]).

Let us take into account the wider class of curves of maximal rank in $\mathbb{P}^3$. By definition a curve C is of maximal rank iff $h^0(\mathcal{I}_C(k))h^1(\mathcal{I}_C(k)) = 0$ for any $k \in \mathbb{Z}$. Since we have already dealt with projectively normal curves, from now on we assume that C is a smooth irreducible curve of maximal rank in $\mathbb{P}^3$, which is not projectively normal. As usual, let $s := min\{k/h^0(\mathcal{I}_C(k)) \neq 0\}$ be the postulation index of C. Observe that $h^1(\mathcal{I}_C(k)) = 0$ for any $k \geq s$, since C is of maximal rank. Thus, having set $c(C) := max\{k/h^1(\mathcal{I}_C(k)) \neq 0\}$, we have that $c(C) \leq s - 1$ ($c(C)$ is called the completeness index).

As a first case, let us consider $c = s - 2$ and assume $h^1(\mathcal{O}_C(s-2)) = 0$ (which is certainly satisfied if $d(2-s) + 2g - 2 < 0$ or equivalently $d > \frac{2g-2}{s-2}$, $s \geq 3$). Observe that in this case, C is s-regular, i.e. $h^i(\mathcal{I}_C(s-i)) = 0$ for any $i > 0$. Indeed, from the defining sequence of C, we have that $h^1(\mathcal{O}_C(k)) = h^2(\mathcal{I}_C(k))$ and since $h^1(\mathcal{O}_C(s-2)) = 0$, we are done. Set $u := h^0(\mathcal{I}_C(s))$. Then, if

$$0 \to \oplus\mathcal{O}_{\mathbb{P}^3}(-n_{3i}) \to \oplus\mathcal{O}_{\mathbb{P}^3}(-n_{2i}) \to \oplus\mathcal{O}_{\mathbb{P}^3}(-n_{1i}) \to \mathcal{I}_C \to 0$$

is the minimal free resolution of $\mathcal{I}_C$, setting $n_j^+ := max\{n_{ji}\}$ and $n_j^- := min\{n_{ji}\}$, it is easy to see that $n_3^+ = c+4 = s+2$. Moreover, we have $n_3^+ > n_2^+ > n_1^+$, $n_3^- > n_2^- > n_1^-$ and also $n_3^+ = s+2 > n_2^+ \geq n_2^- > n_1^- = s$. From these we get $n_2^+ = n_2^- = s+1$, that is $n_{2i} = s+1$ for any i. Analogously, one gets $n_{3i} = s+2$ for any i. Thus, in this case, the minimal free resolution is

$$0 \to y\mathcal{O}_{\mathbb{P}^3}(-s-2) \to x\mathcal{O}_{\mathbb{P}^3}(-s-1) \to u\mathcal{O}_{\mathbb{P}^3}(-s) \to \mathcal{I}_C \to 0 \tag{5.9}$$

(resolution of the first kind), where $y = h^1(\mathcal{I}_C(c)) = h^1(\mathcal{I}_C(s-2))$. If we have a resolution of the first kind, we can split it as follows:

$$0 \to y\mathcal{O}_{\mathbb{P}^3}(-s-2) \to x\mathcal{O}_{\mathbb{P}^3}(-s-1) \to E \to 0, \tag{5.10}$$

$$0 \to E \to u\mathcal{O}_{\mathbb{P}^3}(-s) \to \mathcal{I}_C \to 0 \tag{5.11}$$

where E is only a locally free sheaf (indeed, if it were free, than C would be projectively normal by (5.11)). Twisting (5.10) and (5.11) by $T\mathbb{P}^3$ and taking cohomology, we get:

$$0 \to uH^0(T\mathbb{P}^3(-s)) \to H^0(\mathcal{I}_C \otimes T\mathbb{P}^3) \to H^1(E \otimes T\mathbb{P}^3) \to \dots \tag{5.12}$$

$$\dots \to xH^1(T\mathbb{P}^3(-s-1)) \to H^1(E \otimes T\mathbb{P}^3) \to yH^2(T\mathbb{P}^3(-s-2)) \to \dots \tag{5.13}$$

On the other hand, in the sequence (5.13), $h^1(T\mathbb{P}^3(-s-1)) = h^2(\Omega^1_{\mathbb{P}^3}(s-3)) = 0$ by Serre duality and Bott formulas, while $h^2(T\mathbb{P}^3(-s-2)) = h^1(\Omega^1_{\mathbb{P}^3}(s-2)) = 0$, if $s \geq 3$. Thus, we get that if $s \geq 3$, then $H^1(E \otimes T\mathbb{P}^3) = 0$. Moreover, twisting the Euler sequence with $\mathcal{O}_{\mathbb{P}^3}(-s)$, we obtain that $h^0(T\mathbb{P}^3(-s)) = 0$ as soon as $s \geq 2$. Therefore, from the sequence (5.12), we have that $h^0(\mathcal{I}_C \otimes T\mathbb{P}^3) = 0$ as soon as $s \geq 3$.

Now, twisting the defining sequence of C by $T\mathbb{P}^3$ and taking cohomology, we get (assuming $s \geq 3$):

$$0 \to H^0(T\mathbb{P}^3) \to H^0(T\mathbb{P}^3_{|C}) \to H^1(\mathcal{I}_C \otimes T\mathbb{P}^3) \to 0. \tag{5.14}$$

We want to give an estimate to $h^1(\mathcal{I}_C \otimes T\mathbb{P}^3)$. Continuing the long exact cohomology sequence (5.12), using again Serre duality and Bott formulas and assuming $s \geq 5$, we get that $h^1(\mathcal{I}_C \otimes T\mathbb{P}^3) = h^2(E \otimes T\mathbb{P}^3)$. Moreover, going on with the sequence (5.13), applying Serre duality and Bott formulas ($s \geq 5$), we obtain $h^2(E \otimes T\mathbb{P}^3) \leq yh^3(T\mathbb{P}^3(-s-2)) = \frac{ys(s-1)(s-3)}{2}$. Hence $h^1(\mathcal{I}_C \otimes T\mathbb{P}^3) = \frac{ys(s-1)(s-3)}{2}$ and from (5.14) we get $h^0(T\mathbb{P}^3) \leq 15 + \frac{ys(s-1)(s-3)}{2}$, $s \geq 5$. Thus, if $4d > 15 + \frac{ys(s-1)(s-3)}{2}$, or equivalently $d \geq 4 + \frac{ys(s-1)(s-3)}{8}$, $s \geq 5$, then a curve C of maximal rank, with a resolution of the first kind and with $h^1(\mathcal{O}_C(s-2)) = 0$, is non rigid at the first order for the given embedding.

As a final example, let us consider a curve C of maximal rank, such that $h^0(\mathcal{I}_C(s)) \leq 2$ and $h^1(\mathcal{O}_C(s-3)) = h^1(\mathcal{O}_C(s-2)) = h^1(\mathcal{O}_C(s-1)) = h^1(\mathcal{O}_C(s)) = 0$ (this happens for example if $d > \frac{2g-2}{s}$ and assuming $s \geq 4$). In this case, we have $c(C) = s-1$. Indeed, if it were $c < s-2$, then C would be (s-1)-regular and this contradicts the fact that s is the postulation. Moreover, if it were $c = s-2$, then C would be s-regular and since $h^0(\mathcal{I}_C(s)) \leq 2$, C would be a complete intersection of type (s,s), and in particular it would be projectively normal.

Thus, $c(C) = s-1$ and from the given hypotheses, the fact that $h^2(\mathcal{I}_C(s-1)) = h^1(\mathcal{O}_C(s-1)) = 0$, and $h^1(\mathcal{I}_C(s)) = 0$ (since $c(C) = s-1$), it easy to see that C is (s+1)-regular. This implies that the homogeneous ideal $I(C)$ is generated in degree less or equal to $s+1$. With notations as above, we have $n_3^+ = c+4 = s+3 > n_2^+ > n_1^+ = s+1$, where the last equality holds since $I(C)$ is generated in degree less or equal to $s+1$. From this, we get $n_2^+ = s+2$ and moreover $n_2^- > n_1^- = s$ so that $n_2^- \geq s+1$. On the other hand, we can say more, because the map $H^0(\mathcal{I}_C(s)) \otimes H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \to H^0(\mathcal{I}_C(s+1))$ is injective; indeed, $h^0(\mathcal{I}_C(s)) \leq 2$ and from a relation of the form $H_1F_s = H_2F'_s$ between the two generators in degree s, we would have that $H_1|F'_s$ but this is clearly impossible. It turns out that we have no relations in degree (s+1) between the generators of $I(C)$. Thus $n_2^- > s+1$, $n_3^- > n_2^- \geq s+2$, so that $n_{3i} = s+3$ for any i and also $n_{2i} = s+2$ for any i.

Hence,in this case, the minimal free resolution of $\mathcal{I}_C$ is the following:

$$0 \to v\mathcal{O}_{\mathbb{P}^3}(-s-3) \to x\mathcal{O}_{\mathbb{P}^3}(-s-2) \to w\mathcal{O}_{\mathbb{P}^3}(-s-1) \oplus u\mathcal{O}_{\mathbb{P}^3}(-s) \to \mathcal{I}_C \to 0, \quad (5.15)$$

(resolution of the second kind), where $v = h^1(\mathcal{I}_C(c)) = h^1(\mathcal{I}_C(s-1))$. In this case, that is under the hypotheses that $h^1(\mathcal{O}_C(k)) = 0$ for $k = s, s-1, s-2, s-3$ (which is satisfied if for example $d > \frac{2g-2}{s}$, $s \geq 4$), $h^0(\mathcal{I}_C(s)) \leq 2$ ($c = s-1$), assuming $s \geq 5$ and arguing as in the previous case, starting from the sequence (5.15), we get that C is non rigid at the first order for the given embedding as soon as $d \geq 4 + \frac{vs(s-2)(s+1)}{8}$. We leave to the interested reader the details of this case, which is completely analogous to the previous one.

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