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Elliptic variational problems with critical exponent

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To Chiara and Giulia

I am grateful to my supervisor Prof. Ambrosetti, for proposing me the study of these problems and for all the time, patience, and attention he spent to follow my work. I also would like to thank Boumediene Abdellaoui, Mohameden Ould Ahmedou, Frédéric Robert, Matthias Schneider, and Francesco Uguzzoni for their stimulating and pleasant collaboration.

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Part III H-bubbles

Notation

We collect here a list of notation commonly used in this thesis.

$\mathbb{R}$	the set of real numbers.
$\mathbb{N}$	the set of natural numbers including 0.
$\mathbb{R}^N$	N-fold cartesian product of $\mathbb{R}$ with itself.
$B(x_0, R)$	the ball $\{x \in \mathbb{R}^N : \|x - x_0\| < R\}$.
Ω	open set in $\mathbb{R}^N$.
$\wedge$	the exterior product in $\mathbb{R}^3$.
$\mathbb{S}^N$	the unit N-sphere $\{x = (x_1, \dots, x_{N+1}) \in \mathbb{R}^{N+1} : \sum_i x_i^2 = 1\}$.
Δ	the Laplace operator defined by $\Delta u = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} u$ for any regular function $u : \mathbb{R}^N \to \mathbb{R}$.
$C_C(\mathbb{R}^N)$	the set of continous functions on $\mathbb{R}^N$ with compact support.
$C_0(\mathbb{R}^N)$	the closure of $C_C(\mathbb{R}^N)$ with respect to the uniform norm.
$\mathcal{C}_0^\infty(\mathbb{R}^N)$	the space of smooth functions from $\mathbb{R}^N \to \mathbb{R}$ with compact support.
$\mathcal{D}^{1,2}(\mathbb{R}^N)$	the closure space of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with respect to the norm $$\|\|u\|\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} \|\nabla u\|^2 dx \right)^{1/2}.$$ By the Sobolev inequality we can see that $\mathcal{D}^{1,2}(\mathbb{R}^N)$ can be equivalently defined as the class of functions in $L^{2^*}(\mathbb{R}^N)$ the distributional gradient of which satisfies $\int_{\mathbb{R}^N} \|\nabla u\|^2 dx < \infty$.
2^*	the critical Sobolev exponent $2N/(N-2)$.

$D_a^{1,2}(\mathbb{R}^N)$ the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\left[\int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \, dx\right]^{1/2}.$$

$H_a^1(\Omega)$ the closure of $C^\infty(\bar{\Omega})$ with respect to the norm

$$\|u\|^2_{H_a^1(\Omega)} := \int_\Omega |x|^{-2a} \left(|\nabla u|^2 + |u|^2\right).$$

$\mathcal{D}'(R^N)$ the space of distributions on $\mathbb{R}^N$.

$\mathcal{M}(\mathbb{R}^N)$ the space of finite measures on $\mathbb{R}^N$.

Part I

Preliminaries

1 Introduction

In physics and geometry non-compact group actions arise naturally from scale or gauge invariance. These phenomena are mathematically reflected in a loss of compactness which expresses as the failure of the Palais-Smale condition (PS) at certain levels. Given a differentiable functional $J : X \to \mathbb{R}$ on a Banach space X, we say that J satisfies the Palais-Smale (PS) condition at level $c \in \mathbb{R}$ if any sequence $\{u_n\}_{n\in\mathbb{N}}$ in X satisfying

$$J(u_n) \to c \quad \text{and} \quad DJ(u_n) \to 0$$

in the dual space X' has a convergent subsequence, where DJ denotes the Fréchet derivative of J. In several situations (PS) provides the good compactness property to solve nonlinear elliptic equations by finding critical point of an associate functional, as in the case in which such a functional has a mountain pass geometry, as described in the following theorem due to Ambrosetti and Rabinowitz [18].

Theorem 1.1 (Mountain pass). *Let J be a C^1 functional on a Banach space X. Suppose*

(*i*) *there exist a neighborhood U of 0 in X and a constant ρ such that $J(u) \geq \rho$ for every u on the boundary of U,*

(*ii*) *$J(0) < \rho$ and $J(v) < \rho$ for some $v \notin U$.*

Set

$$c = \inf_{\gamma\in\Gamma} \max_{w\in\gamma} J(w) \geq \rho,$$

where Γ denotes the class of continuous paths joining 0 to v. Then there exists a sequence $\{u_j\}$ in X such that $J(u_j) \to c$ and $DJ(u_j) \to 0$ in the dual space X'. If J satisfies $(PS)_c$, then c is a critical level for J.

Important examples of failure of (PS) can be found in situations involving the critical exponent in the Sobolev embedding theorems or in the cases of the action of a noncompact group, as in the case of the action of the conformal group acting on Dirichlet's integral for minimal surfaces (see Chapter 7).

For $p \in [1, \infty]$ and Ω open set in $\mathbb{R}^N$, let $W^{m,p}(\Omega)$ denote the Sobolev space

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) : \partial^\alpha u \in L^p(\Omega),\ |\alpha| \leq m\}$$

where, for a multi-index $\alpha \in \mathbb{N}^k$, we denote by $|\alpha| = \alpha_1 + \cdots + \alpha_k$ its length and $\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_k}}{\partial x_1^{\alpha_k}}$. We endow $W^{m,p}(\Omega)$ with the norm

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha|\le m} \|\partial^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}.$$

For $p = 2$, the space $W^{m,2}(\Omega)$ is a Hilbert space denoted by $H^m(\Omega)$. If $\Omega \subset \mathbb{R}^N$ is a sufficiently smooth domain, from the Sobolev Embedding Theorem we know that the following embedding is continuous

$$W^{j+m,p}(\Omega) \subset W^{j,q}(\Omega) \quad \text{when } p \le q \le \frac{Np}{N-mp}.$$

Moreover if Ω is bounded and $p \le q < \frac{Np}{N-mp}$ then the embedding is compact. On the other hand the embedding of $H^1(\mathbb{R}^N)$ into $L^{2N/(N-2)}(\mathbb{R}^N)$ is not compact. The reasons why $H^1(\mathbb{R}^N) \subset L^{2N/(N-2)}(\mathbb{R}^N)$ fails to be compact are essentially twofold: the unboundedness of the domain and the presence of the exponent $2N/(N-2)$ which is the critical threshold for compactness of the Sobolev immersion. This phenomenon can be clearly illustrated by the following two examples.

Example 1.1. Let $\varphi : \mathbb{R}^N \to \mathbb{R}$ be a smooth function with compact support, $\varphi \not\equiv 0$. If $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of points in $\mathbb{R}^N$ such that $|x_n| \to +\infty$ as $n \to \infty$, then the sequence of functions $u_n(x) := \varphi(x + x_n)$ obtained by translating φ (see fig. 1.1) is bounded in $H^1(\mathbb{R}^N)$ and converges to 0 almost everywhere. On the other hand it is clear that it does not contain any subsequence converging in $L^p(\mathbb{R}^N)$ for any $1 \le p \le \frac{2N}{N-2}$.

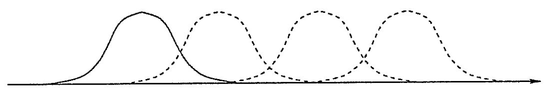

Fig. 1.1. $\varphi(x + x_n)$

Example 1.2. Let φ be as in Example 1.1 and consider the sequence of dilated functions $u_n(x) := n^{(N-2)/2}\varphi(n(x - x_0))$. It is easy to verify that $\{v_n\}_n$ is bounded in $H^1(\mathbb{R}^N)$, converges to 0 almost everywhere, and does not contain any subsequence converging in $L^{2N/(N-2)}(\mathbb{R}^N)$ (see fig. 1.2).

The critical threshold $2^* = \frac{2N}{N-2}$ is related to the Pohozaev identity: if u is a smooth solution of

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in a C^1 domain $\Omega \subset \mathbb{R}^N$, then

$$N \int_\Omega F(u(x))\,dx + \frac{2-N}{2} \int_\Omega u(x) f(u(x))\,dx = \frac{1}{2} \int_{\partial\Omega} (x \cdot \nu_x) \left| \frac{\partial u}{\partial \nu} \right|^2 d\sigma,$$

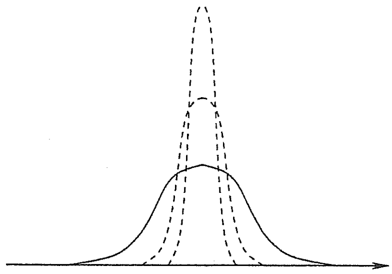

Fig. 1.2. $n^{(N-2)/2}\varphi(n(x-x_0))$.

where F is the primitive of f, i.e. $F(s) := \int_0^s f(t)\,dt$, and ν_x denotes the outward normal unit vector to Ω. As a consequence, if u is a solutions to the problem

$$\begin{cases} -\Delta u = |u|^{q-1}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a star-shaped domain with respect to some point $x^* \in \Omega$ (for simplicity let us assume $x^* = 0$ so that star-shape of the domain means $x \cdot \nu_x \geq 0$ on $\partial\Omega$), there holds

$$\left(\frac{N}{q+1} - \frac{N-2}{2}\right) \int_\Omega |u|^q \geq 0.$$

Hence (1.1) has only the trivial solution $u \equiv 0$, whenever $q \geq \frac{N+2}{N-2} = 2^* - 1$.

The critical exponent 2^* appears also in the Sobolev inequality in $\mathbb{R}^N$. Let $\mathcal{D}^{1,2}(\mathbb{R}^N)$ be the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^{1/2}.$$

By Sobolev embedding, $\mathcal{D}^{1,2}(\mathbb{R}^N) \subset L^{2^*}(\mathbb{R}^N)$ continuously and there exists a constant $S = S(N)$ such that

$$S\|u\|_{L^{2^*}}^2 \leq \|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}^2 \tag{1.2}$$

for all $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$. The best constant S in the Sobolev inequality is given by

$$S = \inf_{\substack{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \\ \|u\|_{L^{2^*}}=1}} \int_{\mathbb{R}^N} |\nabla u|^2\,dx. \tag{1.3}$$

Note that the above minimization problem presents lack of compactness in the sense that it is invariant under translations and dilations; indeed if

$$v^{\lambda,y}(x) = \lambda^{\frac{N-2}{2}} v(\lambda x + y)$$

then

$$\|v^{\lambda,y}\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = \|v\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \quad \text{and} \quad \|v^{\lambda,y}\|_{L^{2^*}} = \|v\|_{L^{2^*}}.$$

Let us recall that the best constant S in (1.3) is attained by the istanton (Aubin, Talenti 1976)

$$U(x) = \frac{(N(N-2))^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}}.$$

The phenomenon of lack of compactness arising from the criticality of the limiting Sobolev exponent was studied by Brezis and Nirenberg [27], who considered the following critical elliptic equation in a bounded domain $\Omega \subset \mathbb{R}^N$

$$\begin{cases} -\Delta u + \lambda u = u^{2^*-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega, \quad u \in H_0^1(\Omega) \end{cases} \tag{1.4}$$

where λ is a real parameter and $N \geq 3$. In the case of a subcritical nonlinearity, the problem is completely solvable. Indeed if $2 < p < 2^*$, then the problem

$$\begin{cases} -\Delta u + \lambda u = u^{p-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega, \quad u \in H_0^1(\Omega) \end{cases} \tag{1.5}$$

has a nontrivial solution if and only if $\lambda > -\lambda_1(\Omega)$, being $\lambda_1(\Omega)$ the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ (for the proof we refer to [83]). The main result in [27] is that for $N \geq 4$ problem (1.4) has a nontrivial solution provided $-\lambda_1(\Omega) < \lambda < 0$.

In this thesis we are going to study some non-compact elliptic problems on $\mathbb{R}^N$ in which the lack of compactness originates not only from the invariance by translations due to unboundedness of the domain but also from two other kinds of invariances:

- invariance under the action of a non-compact group of dilations due to the presence of a nonlinear term which is critical with respect the Caffarelli-Kohn-Nirenberg inequality (see Section 4.1), which can be seen as a generalization of Sobolev inequality;
- invariance under the action of the class of conformal diffeomorphisms of the sphere $\mathbb{S}^2$ (see Chapter 7).

In order to treat this kind of noncompact problems, we will essentially follow three methods. The first one is the finite dimensional reduction method introduced by Ambrosetti and Badiale [7, 8] and successfully used in a perturbative setting to treat noncompact elliptic problems arising in Nonlinear Analysis, such as the Yamabe and scalar curvature problems and nonlinear Schödinger equations, by several authors: Ambrosetti-Garcia Azorero-Peral [11], Ambrosetti-Li-Malchiodi [14], Ambrosetti-Malchiodi [15], Berti-Malchiodi [22], Ambrosetti-Badiale-Cingolani [9], Ambrosetti-Malchiodi-Secchi [17], Ambrosetti-Malchiodi-Ni [16] etc. This method allows to find critical points of a perturbed functional $f_\varepsilon = f_0 - \varepsilon G$ by constructing a natural constraint Z_ε as a perturbation of the manifold of solutions of the unperturbed problem, i.e. of the problem with $\varepsilon = 0$. In this way we are reduced to study the functional restricted to Z_ε, namely to study a finite dimensional functional. An introduction to this abstract method is given in Chapter 3 whereas applications to the study of degenerate critical elliptic equations related to the Caffarelli-Kohn-Nirenberg inequalities and H-systems can be found respectively in Chapters 4 and 7. An application of such a method was also used by the author to treat the problem of prescribing a fourth order conformal invariant related to the Paneitz-Branson operator [44] and by F. Uguzzoni and the author to study the Webster scalar curvature problem on the Heisenberg group [51].

The second method is a fine blow-up analysis which will be the key ingredient in Chapter 5, where the aim is to prove a-priori bounds for the solutions to degenerate critical elliptic equations in suitable weighted spaces in order to exploit the homotopy invariance of the Leray-Schauder degree and thus to obtain existence also in the nonperturbative case. To do this we will mainly follow the scheme of [69]; see also [46] and [47] for an application of such blow-up analysis to the proof of compactness results in deformations of Riemannian metrics on compact manifolds with boundary.

The third method, mainly used in Chapter 6, is the Concentration-Compactness Principle by P.L. Lions [70, 71]; in order to make this thesis the most self-contained as possible, we will illustrate it in Chapter 2.

Contents of the thesis

The thesis is organized as follows. After presenting in Chapter 2 the Concentration-Compactness argument by P.L. Lions [70, 71] and in Chapter 3 the abstract perturbation method introduced by Ambrosetti and Badiale in [7, 8], we will mainly treat two problems:

1. Degenerate critical elliptic equations related to Caffarelli-Kohn-Nirenberg inequality (Chapters 4, 5, and 6). In Chapter 4 we use perturbation methods to study a class of degenerate elliptic equations related to the Caffarelli-Kohn-Nirenberg inequality, which present lack of compactness due to invariance of the problem under the action of a noncompact group of dilations. Moreover, the study of nondegeneracy properties of the unperturbed problem which are needed to apply the method gives rise to some precise information about the so called symmetry breaking phenomenon. We present the results obtained by the author in collaboration with M. Schneider in [48], where the following elliptic equation is considered in $\mathbb{R}^N$, $N \geq 3$

$$- \operatorname{div}(|x|^{-2a}\nabla u) - \frac{\lambda}{|x|^{2(1+a)}}\, u = K(x)\frac{u^{p-1}}{|x|^{bp}}, \quad x \in \mathbb{R}^N \setminus \{0\} \tag{1.6}$$

$$p = p(a,b) = \frac{2N}{N-2(1+a-b)}$$

where $-\infty < a < \frac{N-2}{2}$, $-\infty < \lambda < \left(\frac{N-2a-2}{2}\right)^2$, and $a \leq b < a+1$. We start by discussing in Chapter 4 the perturbative case $K(x) = 1 + \varepsilon k(x)$, where $k \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$. Our approach is based on an abstract perturbative variational method discussed in Chapter 3. The unperturbed problem has a one dimensional manifold of radial solutions. If this manifold is non-degenerate in an appropriate sense, a one dimensional reduction of the perturbed variational problem is possible, so that we are reduced to look for critical points of a Melnikov-type functional defined on the real line. The exact knowledge of the critical manifold enables us to clarify the question of non-degeneracy, i.e. to find for which values of the parameters a, b, λ the tangent to the unperturbed manifold coincides with the space of the solutions to the linearized problem. This nondegeneracy result fairly highlights the symmetry breaking phenomenon of the unperturbed problem observed in [34], i.e. the existence of non-radial minimizers of the associated minimization problem. Following the abstract scheme presented in Chapter 3, we prove existence of a solution under suitable assumptions on k for the parameters satisfying the nondegeneracy condition and for all $|\varepsilon|$ sufficiently small.

In Chapter 5 we use blow-up analysis techniques to prove an a-priori estimate in a weighted space of continuous functions. From this compactness result, we prove existence in the non perturbative case by exploiting the homotopy invariance of the Leray-Schauder degree. In particular the computation of the degree of the solutions can be reduced to the computation of the degree of the finite dimensional Melnikov-type function introduced in Chapter 4; such degree turns out to have explicit expression, which allows us to find sufficient conditions on K for existence of solutions to equation (1.6).

In Chapter 6 we present some results obtained in collaboration with B. Abdellaoui and I. Peral [2] concerning equation (1.6) in the case $a = b = 0$, also considering the case in which, instead of λ, we have some function satisfying suitable assumptions. We prove existence by Concentration-Compactness arguments and multiplicity using techniques that previously had been introduced to study related problems by Cao-Chabrowsky in [32] and the Lusternik-Schnirelman category.

2. Existence of $\mathbb{S}^2$-type surfaces with prescribed mean curvature (Chapter 7). In Chapter 7 we discuss the problem of existence of surfaces in $\mathbb{R}^3$ parametrized on the unit sphere $\mathbb{S}^2$ with prescribed mean curvature H (H-bubble), which can be formulated as follows: given a function $H \in C^1(\mathbb{R}^3)$, find a smooth nonconstant function $\omega : \mathbb{R}^2 \to \mathbb{R}^3$ which is conformal as a map on $\mathbb{S}^2$ and solves the problem

$$\begin{cases} \Delta\omega = 2H(\omega)\,\omega_x \wedge \omega_y, & \text{in } \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} |\nabla\omega|^2 < +\infty. \end{cases}$$

After presenting a short description of the models of capillarity phenomena motivating the study of this problem (see Section 7.1), we will present the existence results obtained by the author in [45], where, using the perturbative method discussed in Chapter 3, the above equation is studied in the case $H = H_0 + \varepsilon H_1$, for some $H_0 \in \mathbb{R} \setminus \{0\}$ and $H_1 \in C^2(\mathbb{R}^3)$. Under the assumptions

$$\lim_{|p|\to\infty} H_1(p) = 0 \quad \text{and} \quad \nabla H_1 \in L^\infty(\mathbb{R}^3, \mathbb{R}^3)$$

the existence of a smooth H-bubble is proved provided $|\varepsilon|$ sufficiently small. Moreover we prove the existence of two or three solutions under some extra assumption on the perturbation H_1.

2 The Concentration-Compactness principle

In this chapter we discuss the P. L. Lions Concentration-Compactness principle, see [70, 71]. Let $C_C(\mathbb{R}^N)$ denote the set of continuous functions on $\mathbb{R}^N$ with compact support and let $C_0(\mathbb{R}^N)$ be its closure with respect to the uniform norm. A finite measure μ on $\mathbb{R}^N$ is a continuous linear functional on $C_0(\mathbb{R}^N)$ and its norm is defined by

$$||\mu||_{\mathcal{M}(\mathbb{R}^N)} := \sup_{\substack{u\in C_0(\mathbb{R}^N)\\ ||u||_{L^\infty(\mathbb{R}^N)}=1}} \left| \int_{\mathbb{R}^N} u\, d\mu \right|.$$

We denote by $\mathcal{M}(\mathbb{R}^N)$ the space of finite measures and will say that a sequence $\{\mu_n\}_n$ converges weakly in the sense of measures to μ if for any $u \in C_0(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} u\, d\mu_n \longrightarrow \int_{\mathbb{R}^N} u\, d\mu.$$

In this case we will use the notation $\mu_n \rightharpoonup \mu$.

Theorem 2.1. (Concentration-compactness) *Let $\{u_n\}$ be a sequence weakly converging to u in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Then, up to subsequences,*

(i) *$|\nabla u_n|^2$ weakly converges in $\mathcal{M}(\mathbb{R}^N)$ to a nonnegative measure μ,*

(ii) *$|u_n|^{2^*}$ weakly converges in $\mathcal{M}(\mathbb{R}^N)$ to a nonnegative measure ν,*

and there exist an at most countable index set J, a family $\{x_j : j \in J\}$ of distinct points of $\mathbb{R}^N$, and families $\{\nu_j : j \in J\}$, $\{\mu_j : j \in J\}$ of positive numbers such that

$$\nu = |u|^{2^*} dx + \sum_{j\in J} \nu_j \delta_{x_j}$$
$$\mu \geq |\nabla u|^2 dx + \sum_{j\in J} \mu_j \delta_{x_j}$$

and for all $j \in J$

$$S(\nu_j)^{2/2^*} \leq \mu_j,$$

where S is the best Sobolev constant given in (1.3) and δ_{x_j} is the Dirac measure at point x_j. In particular $\sum_{j\in J}(\nu_j)^{2/2^*} < \infty$. Moreover if $u \equiv 0$ and

$$\int_{\mathbb{R}^N} d\mu \leq S\Big(\int_{\mathbb{R}^N} d\nu\Big)^{\frac{2}{2^*}} \tag{2.1}$$

then J is a singleton and $\nu = \gamma\delta_{x_0} = S^{-1}\gamma^{\frac{2}{N}}\mu$ for some $\gamma \geq 0$.

Proof. Let us sketch the proof. For details we refer to [70, Lemma I.1]. Let $u_n \rightharpoonup u$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Then

$$\big\||\nabla u_n|^2\,dx\big\|_{\mathcal{M}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\nabla u_n|^2\,dx \leq \text{const}$$

and

$$\big\||u_n|^{2^*}\,dx\big\|_{\mathcal{M}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |u_n|^{2^*}\,dx \leq S^{-\frac{2^*}{2}}\Big(\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx\Big)^{2^*/2} \leq \text{const}.$$

Therefore, up to a subsequence,

$$\begin{aligned} |\nabla u_n|^2\,dx &\rightharpoonup \mu \geq 0 \\ |u_n|^{2^*}\,dx &\rightharpoonup \nu \geq 0. \end{aligned}$$

Set $v_n := u_n - u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$. We have that $v_n \rightharpoonup 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. From the Brezis-Lieb Lemma (see e.g. [83], Lemma 1.32)

$$\omega_n := |u_n|^{2^*}\,dx - |u|^{2^*}\,dx = |v_n|^{2^*}\,dx + o(1). \tag{2.2}$$

Set $\lambda_n := |\nabla v_n|^2\,dx$. There exist $\lambda, \omega \in \mathcal{M}(\mathbb{R}^N)$ such that

$$\lambda_n \rightharpoonup \lambda \geq 0 \quad \text{and} \quad \omega_n \rightharpoonup \omega \geq 0.$$

From (2.2) we have that $\omega = \nu - |u|^{2^*}$. For any $\xi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned} \int_{\mathbb{R}^N} |\xi|^{2^*}\,d\omega &= \lim_n \int_{\mathbb{R}^N} |\xi|^{2^*}\,d\omega_n = \lim_n \int_{\mathbb{R}^N} |\xi\, v_n|^{2^*}\,dx \\ &\leq S^{-\frac{2^*}{2}} \liminf_n \Big(\int_{\mathbb{R}^N} |\nabla(v_n\,\xi)|^2\Big)^{\frac{2^*}{2}}. \end{aligned} \tag{2.3}$$

Since $\big|\|f+g\|_{L^2} - \|f\|_{L^2}\big| \leq \|g\|_{L^2}$ we have

$$\Big|\Big(\int_{\mathbb{R}^N} |\nabla(v_n\,\xi)|^2\Big)^{1/2} - \Big(\int_{\mathbb{R}^N} |\xi|^2|\nabla v_n|^2\Big)^{1/2}\Big| \leq \Big(\int_{\mathbb{R}^N} |\nabla\xi|^2|v_n|^2\Big)^{1/2}; \tag{2.4}$$

moreover, for some compact set $\mathcal{K}$, up to a subsequence, $v_n \to 0$ in $L^2(\mathcal{K})$ and hence

$$\int_{\mathbb{R}^N} |\nabla\xi|^2|v_n|^2 \leq \text{const}\int_{\mathcal{K}} |v_n|^2 \longrightarrow 0. \tag{2.5}$$

From (2.3), (2.4), and (2.5), we obtain

$$S\Big(\int_{\mathbb{R}^N}|\xi|^{2^*}\,d\omega\Big)^{\frac{2}{2^*}} \le \int_{\mathbb{R}^N}|\xi|^2\,d\lambda \tag{2.6}$$

for any $\xi \in C_0^\infty(\mathbb{R}^N)$. Let $\mathcal{A} = \{x_j,\ j\in J\} \subseteq \mathbb{R}^N$ be the set of atoms of ω and ω_0 the atom less part of ω, i.e.

$$\omega = \omega_0 + \sum_{j\in J}\nu^j\,\delta_{x_j}.$$

Since $\omega(\mathbb{R}^N) < +\infty$, J is at most countable. Moreover, for any Borel set A

$$\omega_0(A) = \omega_0(A\cap\mathcal{A}) + \omega_0(A\setminus\mathcal{A}) = \omega(A\setminus\mathcal{A}) \ge 0$$

and hence $\omega_0 \ge 0$. From (2.6) it follows

$$\lambda \ge S(\nu^j)^{2/2^*}\delta_{x_j} \quad \forall j\in J.$$

Furthermore for any $\xi \in C_0^\infty(\mathbb{R}^N)$

$$\begin{aligned}\int|\nabla v_n|^2\,\xi\,dx &= \int\xi|\nabla u|^2\,dx + \int\xi|\nabla u_n|^2\,dx - 2\int\xi\nabla u\cdot\nabla u_n\,dx\\ &\longrightarrow \int\xi\,d\mu - \int\xi|\nabla u|^2\,dx.\end{aligned}$$

Hence $\lambda = \mu - |\nabla u|^2\,dx$ which yields

$$\mu \ge \lambda \ge S(\nu^j)^{2/2^*}\delta_{x_j} \quad\text{and}\quad \mu \ge |\nabla u|^2\,dx.$$

For any $\varphi \in C_0(\mathbb{R}^N)$, $\varphi \ge 0$

$$\begin{aligned}\int_{\mathbb{R}^N}\varphi\,d\mu &= \int_{\mathbb{R}^N\setminus\mathcal{A}}\varphi\,d\mu + \sum_j\int_{\{x_j\}}\varphi\,d\mu\\ &\ge \int_{\mathbb{R}^N}\varphi\,|\nabla u|^2\,dx + \sum_j S(\nu^j)^{\frac{2}{2^*}}\varphi(x_j)\end{aligned}$$

hence $\mu \ge |\nabla u|^2 + \sum_j S(\nu^j)^{\frac{2}{2^*}}\delta_{x_j}$. From (2.6) it follows that ω_0 is λ-absolutely continuous. From the Radon-Nikodym Theorem there exists $f \in L^1(\mathbb{R}^N,\lambda)$ such that $\omega_0 = f\lambda$ and

$$f(x) = \lim_{\rho\to 0^+}\left(\frac{\int_{B_\rho(x)}d\omega_0}{\int_{B_\rho(x)}d\lambda}\right).$$

If x is not an atom of λ

$$S\,f(x)^{\frac{2}{2^*}} = \lim_{\rho\to0^+}\frac{S\big(\int_{B_\rho(x)}d\omega_0\big)^{\frac{2}{2^*}}}{\big(\int_{B_\rho(x)}d\lambda\big)^{\frac{2}{2^*}}} \le \lim_{\rho\to0^+}\int_{B_\rho(x)}d\lambda = 0$$

which yields $\omega_0 \equiv 0$. Hence

$$|\nabla u_n|^2 \, dx \rightharpoonup \mu \geq |\nabla u|^2 + \sum_j S(\nu^j)^{\frac{2}{2^*}} \delta_{x_j}$$

$$|u_n|^{2^*} \, dx \rightharpoonup \nu = |u|^{2^*} + \sum_{j \in J} \nu^j \delta_{x_j}.$$

Assume now that $u \equiv 0$ and $\int_{\mathbb{R}^N} d\mu \leq S\big(\int_{\mathbb{R}^N} d\nu\big)^{\frac{2}{2^*}}$. Hence $\omega = \nu$ and $\lambda = \mu$. From (2.6) it follows that $S\nu(\mathbb{R}^N)^{\frac{2}{2^*}} \leq \mu(\mathbb{R}^N)$ so that (2.1) yields

$$\mu(\mathbb{R}^N) = S\nu(\mathbb{R}^N)^{\frac{2}{2^*}}. \tag{2.7}$$

From (2.6) and Hölder inequality we have that

$$S\left(\int_{\mathbb{R}^N} |\xi|^{2^*} \, d\nu\right)^{\frac{2}{2^*}} \leq \mu(\mathbb{R}^N)^{\frac{2}{N}} \left(\int_{\mathbb{R}^N} |\xi|^{2^*} \, d\mu\right)^{\frac{2}{2^*}}$$

for any $\xi \in C_0^\infty(\mathbb{R}^N)$ and hence

$$\nu \leq S^{-\frac{2^*}{2}} \mu(\mathbb{R}^N)^{\frac{2}{N-2}} \mu. \tag{2.8}$$

From (2.7) and (2.8), we deduce that $\nu = S^{-\frac{2^*}{2}} \mu(\mathbb{R}^N)^{\frac{2}{N-2}} \mu$ and hence (2.6) yields

$$\left(\int_{\mathbb{R}^N} |\xi|^{2^*} \, d\nu\right)^{\frac{1}{2^*}} \leq \nu(\mathbb{R}^N)^{-\frac{1}{N}} \left(\int_{\mathbb{R}^N} |\xi|^2 \, d\nu\right)^{\frac{1}{2}}$$

for any $\xi \in C_0^\infty(\mathbb{R}^N)$. Hence by approximation we get $\nu(E)^{\frac{1}{2^*}} \nu(\mathbb{R}^N)^{\frac{1}{N}} \leq \nu(E)^{\frac{1}{2}}$ for any Borel set E, which is possible only if J is a singleton and $\nu = \gamma\delta_{x_0} = S^{-1}\gamma^{\frac{2}{N}}\mu$ for some $\gamma \geq 0$. The proof is now complete. □

The following theorem provides information about possible loss of mass at infinity.

Theorem 2.2. (Loss of mass at infinity) *Let $\{u_n\}$ be a sequence weakly converging to u in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and define*

$$\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |u_n|^{2^*} \, dx,$$

$$\mu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |\nabla u_n|^2 \, dx.$$

Then the quantities ν_∞ and μ_∞ are well defined and satisfy

$$\text{i)} \quad \limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2^*} \, dx = \int_{\mathbb{R}^N} d\nu + \nu_\infty,$$

$$\text{ii)} \quad \limsup_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx = \int_{\mathbb{R}^N} d\mu + \mu_\infty,$$

$$\text{iii)} \quad S\nu_\infty^{\frac{2}{2^*}} \leq \mu_\infty,$$

where μ and ν are as in Theorem 2.1.

Proof. For any $R > 0$, let ϕ_R be a smooth function on $\mathbb{R}^N$ satisfying

$$\begin{aligned} &\phi_R(x) = 0 \quad \text{if } |x| < R, \\ &\phi_R(x) = 1 \quad \text{if } |x| > 2R, \\ &0 \le \phi_R \le 1 \quad \text{and} \quad |\nabla \phi_R| \le 2/R. \end{aligned}$$

From Sobolev inequality (1.2) we have that

$$S\bigg(\int_{\mathbb{R}^N} |u_n\phi_R|^{2^*}\,dx\bigg)^{\frac{2}{2^*}} \le \int_{\mathbb{R}^N} |\nabla(\phi_R u_n)|^2\,dx. \tag{2.9}$$

Since

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N} |u_n|^{2^*}\phi_R^{2^*} = \nu_\infty, \quad \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N} |\nabla u_n|^2\phi_R^2 = \mu_\infty,$$

$$\begin{aligned} \bigg|\int_{\mathbb{R}^N} u_n\nabla u_n\phi_R\nabla\phi_R\bigg| &\le \bigg(\int_{\mathbb{R}^N} u_n^2|\nabla\phi_R|^2\bigg)^{\frac{1}{2}}\bigg(\int_{\mathbb{R}^N} |\nabla u_n|^2\phi_R^2\bigg)^{\frac{1}{2}} \\ &\le \text{const}\bigg(\int_{\mathbb{R}^N} u_n^2|\nabla\phi_R|^2\bigg)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \limsup_{n\to\infty}\int_{\mathbb{R}^N} |u_n|^2|\nabla\phi_R|^2 = \int_{\mathbb{R}^N} |u|^2|\nabla\phi_R|^2 &\le \bigg(\int_{|x|>R} |u|^{2^*}\,dx\bigg)^{\frac{2}{2^*}}\bigg(\int_{\mathbb{R}^N} |\nabla\phi_R|^N\bigg)^{\frac{2}{N}} \\ &\le \text{const}\bigg(\int_{|x|>R} |u|^{2^*}\,dx\bigg)^{\frac{2}{2^*}} \underset{R\to\infty}{\longrightarrow} 0 \end{aligned}$$

from (2.9) we obtain iii). Observing that

$$\begin{aligned} \limsup_{n\to\infty}\int_{\mathbb{R}^N} |u_n|^{2^*} &= \limsup_{n\to\infty}\bigg(\int_{|x|<R} |u_n|^{2^*} + \int_{|x|>R} |u_n|^{2^*}\bigg) \\ &= \nu(|x| < R) + \limsup_{n\to\infty}\int_{|x|>R} |u_n|^{2^*} \end{aligned}$$

and letting $R \to \infty$ we obtain i). The proof of ii) is analogous. □

An an example of the use of the Concentration-Compactness principle, we present here the proof that the best constant in the Sobolev inequality (1.2) is attained.

Theorem 2.3. *Every minimizing sequence $\{u_n\}_n$ of (1.3) is relatively compact in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ up to a translation and a dilation. More precisely for any $\alpha \in (0,1)$ and every $y \in \mathbb{R}^N$ there exists a subsequence of $\{u_n\}_n$, still denoted by $\{u_n\}_n$, and a sequence $\{\sigma_n\}_n \subset (0,+\infty)$ such that*

$$\alpha = \int_{B(0,1)} \bigg|u_n\bigg(\frac{x+\sigma_n y}{\sigma_n}\bigg)\bigg|^{2^*}\sigma_n^{-N}\,dx \tag{2.10}$$

and

$$u_n\left(\frac{\cdot + \sigma_n y}{\sigma_n}\right)\sigma_n^{-\frac{N}{2^*}} \to u \text{ as } n \to \infty \text{ in } \mathcal{D}^{1,2}(\mathbb{R}^N),$$

where

$$S = \int_{\mathbb{R}^N} |\nabla u|^2\, dx \quad \text{and} \quad \int_{\mathbb{R}^N} |u(x)|^{2^*}\, dx = 1.$$

Proof. Since by a change of variable

$$\int_{B(0,1)} \left| u_n\left(\frac{x+\sigma_n y}{\sigma_n}\right)\right|^{2^*} \sigma_n^{-N}\, dx = \int_{B(y,1/\sigma)} |u_n(x)|^{2^*}\, dx,$$

for each n we can choose σ_n such that (2.10) holds. The sequence w_n defined by

$$w_n(x) = \sigma_n^{-\frac{N}{2^*}} u_n\left(\frac{x+\sigma_n y}{\sigma_n}\right)$$

is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and hence weakly converges to some $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ up to a subsequence. Let us prove that $u \not\equiv 0$. By contradiction, let us assume that $u \equiv 0$. Let $F : \mathcal{D}^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ be defined by

$$F(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*}\, dx.$$

The sequence $v_n = S^{\frac{1}{2^*-2}} w_n$ satisfies $F(v_n) \to \left(\frac{1}{2} - \frac{1}{2^*}\right)S^{\frac{N}{2}}$ and $F'(v_n) \to 0$. Since

$$\lim_{n\to\infty} \langle F'(v_n), v_n \phi_R\rangle = 0$$

uniformly in $R \geq 1$ (where ϕ_R is the cut-off function defined at p. 17), there holds

$$\mu_\infty = S\nu_\infty$$

where μ_∞ and ν_∞ are as in Theorem 2.2. From Theorem 2.2 it follows that

$$1 = \int_{\mathbb{R}^N} d\nu + \nu_\infty \quad \text{and} \quad S = \int_{\mathbb{R}^N} d\mu + \mu_\infty \tag{2.11}$$

and hence

$$\int_{\mathbb{R}^N} d\mu = S(1-\nu_\infty) = S\int_{\mathbb{R}^N} d\nu \leq S\left(\int_{\mathbb{R}^N} d\nu\right)^{\frac{2}{2^*}}.$$

From Theorem 2.1 ν must be a singleton of the form

$$\nu = \gamma\delta_{x_0} = S^{-1}\gamma^{\frac{2}{N}}\mu \tag{2.12}$$

for some $\gamma > 0$ and $x_0 \in \mathbb{R}^N$. From (2.11)-(2.12) it follows that

$$\gamma = 1 - \nu_\infty = \int_{\mathbb{R}^N} d\nu$$

and therefore we have

$$1-\nu_\infty = S^{-1}\gamma^{\frac{2}{N}} S(1-\nu_\infty) = (1-\nu_\infty)^{1+\frac{2}{N}}.$$

Hence $\nu_\infty = 0$ and $\gamma = 1$. On the other hand

$$\alpha = \int_{B(0,1)} |w_n|^{2^*}\,dx \to \nu(B(0,1)) = \delta_{x_0}(B(0,1)) = 1$$

which is not possible since $\alpha \in (0,1)$. Hence we have proved that $u \not\equiv 0$. Let us set $\rho = \int_{\mathbb{R}^N} |u|^{2^*} \in (0,1]$. If $\rho < 1$ then from Theorems 2.1 and 2.2 we have

$$\rho = 1 - \sum_{j\in J} \nu_j - \nu_\infty$$

and

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u|^2\,dx &\le S - \sum_{j\in J}\mu_j - \mu_\infty \le S\Big(1 - \sum_{j\in J}\nu_j^{\frac{2}{2^*}} - \nu_\infty^{\frac{2}{2^*}}\Big)\\ &< S\Big(1 - \sum_{j\in J}\nu_j - \nu_\infty\Big)^{\frac{2}{2^*}} = S\rho^{\frac{2}{2^*}}.\end{aligned}$$

On the other hand Sobolev inequality yields

$$S\rho^{\frac{2}{2^*}} \le \int_{\mathbb{R}^N} |\nabla u|^2\,dx$$

giving rise to a contradiction. Hence $\rho = 1$, which implies that $\nu_j = 0$ for $j \in J$ and $\nu_\infty = 0$. Thus neither concentration nor loss of mass at infinity may occur and the theorem follows. □

3 The perturbation technique

In this chapter we describe the perturbation method which was developed by Ambrosetti and Badiale ([7, 8]) to deal with problems which present lack of compactness, see also [11, Section 2]. This method allows us to find existence and multiplicity results for problems in which a small perturbation parameter appears and applies successfully also to some situations in which the concentration-compactness arguments fail or lead to involved calculations. The key ingredient is a kind of finite dimensional reduction which permits to exploit some precise knowledge of solutions to *unperturbed equations* to find one or more solutions of a *perturbed equation*. For the sake of completeness, we present below this technique in some detail.

Let E be a Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let us consider the family of perturbed functionals

$$f_\varepsilon = f_0 - \varepsilon G$$

where $f_0, G \in C^2(E,\mathbb{R})$. Let $D^2 f_0(u) \in \mathcal{L}(E,E')$ denote the second Fréchet derivative of f_0 at u. Through the Riesz Representation Theorem, we can identify $D^2 f_0(u)$ with $f_0''(u) \in \mathcal{L}(E,E)$ given by $f_0''(u)v = \mathcal{K}(D^2 f_0(u)v)$ where $\mathcal{K} : E' \to E$ satisfies $\left(\mathcal{K}(\varphi),\psi\right)_E = {}_{E'}\langle\varphi,\psi\rangle_E$, for any $\varphi \in E', \psi \in E$. Suppose that f_0 satisfies

$$f_0 \text{ has a finite dimensional manifold of critical points } Z; \tag{3.1}$$

$$\text{for all } z \in Z,\ f_0''(z) \text{ is a Fredholm operator of index } 0; \tag{3.2}$$

$$\text{for all } z \in Z, \text{ there results } T_z Z = \ker f_0''(z). \tag{3.3}$$

Let us recall that a linear continuous operator A is said to be a Fredholm map if its kernel $N(A)$ is finite-dimensional and its range $R(A)$ is closed and with finite codimension. We set $\operatorname{Index} A = \dim N(A) - \operatorname{codim} R(A)$. If (3.1)-(3.3) hold, we will say that Z is a *non degenerate manifold.*

Condition (3.3) is in fact a nondegeneracy condition which is needed to apply the Implicit Function Theorem. The inclusion $T_z Z \subseteq \ker f_0''(z)$ always holds due to the criticality of Z, so that to prove (3.3) it is enough to show that $\ker f_0''(z) \subseteq T_z Z$, which means that every solution of the linearized equation for the unperturbed problem belongs to the tangent space.

Consider the perturbed functional $f_\varepsilon(u) = f_0(u) - \varepsilon G(u)$, and denote by Γ the functional $G\big|_Z$. Due to assumptions (3.1), (3.2), and (3.3), it is possible to prove (see

Lemma 3.1) that there exists, for $|\varepsilon|$ small, a smooth function $w_\varepsilon(z) : Z \to (T_z Z)^\perp$ such that any critical point $\bar{z} \in Z$ of the functional

$$\Phi_\varepsilon : Z \longrightarrow \mathbb{R}, \qquad \Phi_\varepsilon(z) = f_\varepsilon(z + w_\varepsilon(z))$$

gives rise to a critical point $u_\varepsilon = \bar{z} + w_\varepsilon(\bar{z})$ of f_ε; in other words, the perturbed manifold $Z_\varepsilon = \{z + w_\varepsilon(z) : z \in Z\}$ is a *natural constraint* for f_ε.

We will assume that $Z = \zeta(\mathbb{R}^d)$ with $\zeta \in C^2(\mathbb{R}^d, E)$. Denote $Z^R = \zeta(B_R)$, where $B_R = \{x \in \mathbb{R}^d : \|x\| < R\}$.

Lemma 3.1. *Given $R > 0$, there exist $\varepsilon_0 > 0$ and a smooth function $w = w(\varepsilon, z)$ defined for $|\varepsilon| < \varepsilon_0$ and $z \in Z^R$, $w(\varepsilon, z) \in E$, such that*

$$\begin{aligned}
&\text{(i)} \quad w(0, z) = 0 \quad \forall\, z \in Z^R;\\
&\text{(ii)} \quad w(\varepsilon, z) \perp T_z Z \quad \forall\, z \in Z,\ \varepsilon \in (-\varepsilon_0, \varepsilon_0);\\
&\text{(iii)} \quad f_\varepsilon'(z + w(\varepsilon, z)) \in T_z Z \quad \forall\, z \in Z,\ \varepsilon \in (-\varepsilon_0, \varepsilon_0);\\
&\text{(iv)} \quad w(z, \varepsilon) = O(\varepsilon) \ \textit{as } \varepsilon \to 0 \textit{ uniformly in } z \in Z^R;\\
&\text{(v)} \quad \tfrac{\partial w}{\partial z}(z, \varepsilon) \to 0 \ \textit{as } \varepsilon \to 0 \textit{ uniformly in } z \in Z^R.
\end{aligned}$$

Proof. For any $z \in Z^R$, let $q = q(z) = (q_1, \ldots, q_d)$ denote an orthonormal basis for $T_z Z$. Consider the map (ε_0 to be determined later)

$$\begin{aligned}
\mathcal{H} :\quad ((-\varepsilon_0, \varepsilon_0) \times Z^R) \times E \times \mathbb{R}^d &\longrightarrow E \times \mathbb{R}^d\\
((\varepsilon, z), w, \alpha) &\longmapsto \begin{pmatrix} f_\varepsilon'(z + w) - \alpha q \\ (w|q) \end{pmatrix} = \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{pmatrix},
\end{aligned}$$

where $(w|q) = \big((w|q_1), (w|q_2), \ldots, (w|q_d)\big)$. Solving

$$\mathcal{H}(\varepsilon, z, w, \alpha) = 0 \tag{3.4}$$

we mean to find w satisfying (i-iii). Since

$$\mathcal{H}(0, z, 0, 0) = \begin{pmatrix} f_0'(z) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and $\mathcal{H}$ is of class C^1 by definition, we can try to solve equation (3.4) by means of the Implicit Function Theorem. In order to do that, we have to prove that

$$L = \frac{\partial \mathcal{H}}{\partial(w, \alpha)}\bigg|_{(0,z,0,0)}$$

is invertible. We get

$$\begin{aligned}
\frac{\partial \mathcal{H}_1}{\partial(w, \alpha)}[v, \beta] &= f_0''(z)v - \beta q,\\
\frac{\partial \mathcal{H}_2}{\partial(w, \alpha)}[v, \beta] &= (v|q).
\end{aligned}$$

Since $f_0''(z)$ is a Fredholm map, it is enough to prove injectivity. In order to do that, let us assume $L(z)[\beta, v] = (0,0)$. Then

$$\begin{cases} f_0''(z)v - \beta q = 0 \\ (v|q) = 0. \end{cases}$$

Hence, since $q \in \ker f_0''(z)$,

$$0 = (f_0''(z)q|v) = (f_0''(z)v|q) = \beta|q|^2 \quad \Longrightarrow \quad \beta = 0,$$

so that we have

$$\begin{cases} f_0''(z)v = 0 \\ (v|q) = 0 \end{cases} \quad \Longrightarrow \quad v \in \ker f_0''(z),$$

and then, by assumption (3.3), $v = \lambda q$. Since

$$\lambda(q|q) = (v|q) = 0$$

we deduce $\lambda = 0$ and therefore $v = 0$. We can apply the Implicit Function Theorem to find w, whose regularity is the same of $\mathcal{H}$ and of f', namely w is of class C^1. Since $w(z,0) = 0$ for all $z \in Z^R$, it follows that $w(\varepsilon, z)$ tends to 0 as $\varepsilon \to 0$ uniformly in Z^R. Let us now prove (iv). Setting $\tilde{w}_\varepsilon(z) = \varepsilon^{-1}w(\varepsilon, z)$, (iii) yields

$$Pf_0''(z)\tilde{w}_\varepsilon(z) - PG'(z) - PG''(z)w(\varepsilon, z) + \|\tilde{w}_\varepsilon\|o(1) = 0.$$

Since $w(\varepsilon, z)$ tends to 0 as $\varepsilon \to 0$, we find that $\|\tilde{w}_\varepsilon\| \le$ const, for $|\varepsilon|$ sufficiently small, thus proving (iv). Finally (v) comes easily from the Implicit Function Theorem. □

Let us define

$$Z_\varepsilon := \{z + w(\varepsilon, z) : \ (\varepsilon, z) \in (-\varepsilon_0, \varepsilon_0) \times Z^R\}$$

and note that Z_ε is locally diffeomorphic to Z.

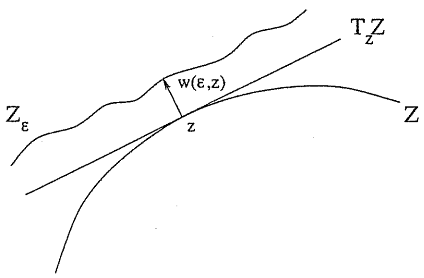

Fig. 3.1. The perturbed manifold.

Lemma 3.2. *Z_ε is a natural constraint for f_ε', namely: if $z_\varepsilon = z + w(\varepsilon, z)$ is a critical point of $\Phi_\varepsilon := f_\varepsilon\big|_{Z_\varepsilon}$, then $f_\varepsilon'(z_\varepsilon) = 0$.*

Proof. By assumption, one has

$$f'_\varepsilon(z_\varepsilon) \perp T_{z_\varepsilon} Z_\varepsilon$$

i.e.

$$(f'_\varepsilon(z + w_\varepsilon)|q + \dot{w}_\varepsilon) = 0,$$

where $\dot{w}_\varepsilon$ stands for the derivative with respect to z. (iii) implies that

$$f'_\varepsilon(z + w_\varepsilon) = \alpha_\varepsilon q$$

for some α_ε, hence

$$0 = (\alpha_\varepsilon q|q + \dot{w}_\varepsilon) = \alpha_\varepsilon |q|^2 + \alpha_\varepsilon (q|\dot{w}_\varepsilon). \tag{3.5}$$

From (ii) we get $(w_\varepsilon|q) = 0$ and, after a derivation, $(\dot{w}_\varepsilon|q) + (w_\varepsilon|\dot{q}) = 0$. Since $w_\varepsilon \to 0$ as $\varepsilon \to 0$, $(\dot{w}_\varepsilon|q) = -(w_\varepsilon|\dot{q})$ goes to 0 as $\varepsilon \to 0$. From (3.5) (remember $|q| = 1$) we get

$$0 = \alpha_\varepsilon + \alpha_\varepsilon o(1), \qquad \varepsilon \to 0$$

which implies $\alpha_\varepsilon = 0$ so that $f'_\varepsilon(z_\varepsilon) = 0$. □

The previous lemma says that critical points of f_ε constrained on Z_ε give rise to critical points of f_ε (free critical points). This means that in order to find critical points of f_ε it is sufficient to study the critical points of a finite dimensional functional Φ_ε defined on $Z \approx \mathbb{R}^d$. Therefore we can say that the previous lemma is a sort of Lyapunov-Schmidt reduction, in the sense that an infinite dimensional problem is reduced to a finite one.

Remark 3.1. If Z is compact, then Φ_ε must have either a maximum point or a minimum point and hence Lemma 3.2 allows to conclude the existence of a critical point of f_ε. Actually in [10] it is proved that if Z is compact and nondegenerate, then f_ε has at least $\mathrm{cat}(Z)$ critical points provided $|\varepsilon|$ is small enough, where $\mathrm{cat}(Z)$ denotes the Lusternik-Schnirelman category of Z, i.e. the least integer k such that $Z \subset U_1 \cup \ldots \cup U_k$, being U_i, $i = 1, \ldots, k$, closed subsets of Z contractible to a point in Z.

In the case in which Z is not compact, in order to find critical points of Φ_ε, the following Lemma, which provides an expansion for Φ_ε, can be useful.

Lemma 3.3. *Uniformly in $z \in Z^R$, there holds*

$$\Phi_\varepsilon(z) = b - \varepsilon\Gamma(z) + o(\varepsilon) \quad as\ \varepsilon \to 0 \tag{3.6}$$

where $f_0 \equiv b$ on Z and $\Gamma = G\big|_Z$.

Proof. The result can be proved simply by writing a Taylor expansion for Φ_ε

$$\begin{aligned}\Phi_\varepsilon(z) &= f_\varepsilon(z + w(\varepsilon, z)) = f_0(z + w(\varepsilon, z)) - \varepsilon G(z + w(\varepsilon, z)) \\ &= b + (f'_0(z)|w(\varepsilon, z)) - \varepsilon\left[G(z) + (G'(z)|w(\varepsilon, z))\right] + o(\varepsilon) \\ &= b - \varepsilon\Gamma(z) + o(\varepsilon).\end{aligned}$$

□

Since Γ is the leading term in the expansion, we will find results making assumptions on Γ. Such a Γ is the *Poincaré function* and its derivative is the *Melnikov function.* From Lemma 3.3 and Lemma 3.2, the following theorem follows.

Theorem 3.1. *Let f_0 satisfy* (3.1), (3.2), *and* (3.3) *and assume that Γ has a critical point $\bar{z} \in Z$ satisfying one of the following conditions:*

(a) *$\bar{z}$ is nondegenerated;*

(b) *$\bar{z}$ is a proper local maximum or minimum point;*

(c) *$\bar{z}$ is isolated and the local topological degree of Γ' at $\bar{z}$ is different from zero.*

Then for $|\varepsilon|$ small enough, the functional f_ε has a critical point u_ε such that $u_\varepsilon \to \bar{z}$ as $\varepsilon \to 0$.

Proof. Let us briefly sketch the proof of (b). If z_0 is a proper minimum point of Γ, then there exists an open neighborhood $A \subset Z$ of z_0 such that $\Gamma(z_0) < \inf_{\partial A} \Gamma$. From the expansion (3.6) it is easy to get, for ε small, that

$$f_\varepsilon(z_0 + w(\varepsilon, z_0)) > \sup_{z \in \partial A} f_\varepsilon(z + w(\varepsilon, z)).$$

Since Z_ε is finite dimensional, f_ε must have a maximum point of the form $z + w(\varepsilon, z)$ for $z \in A$. In the same way if z_0 is a proper maximum point of Γ, we can find maximum points of f_ε. Hence the theorem is proved under assumption (b). In order to treat case (a), we expand Φ'_ε as

$$\Phi'_\varepsilon(z) = -\varepsilon\Gamma'(z) + O(\varepsilon^2) = -\varepsilon(\Gamma''(\bar{z}) + o(1))(z - \bar{z}) + O(\varepsilon^2) \quad \text{as } z \to \bar{z}.$$

From the invertibility of $\Gamma''(\bar{z})$ we can find for ε small some z_ε near $\bar{z}$ such that

$$z_\varepsilon = \bar{z} + (\Gamma''(\bar{z}) + o(1))^{-1} O(\varepsilon) \quad \text{and} \quad \Phi'_\varepsilon(z_\varepsilon) = 0$$

hence also (a) is proved. (c) can be handled with the same kind of arguments. □

Remark 3.2. If $Z_0 = \{z \in Z : \Gamma(z) = \min_Z \Gamma\}$ is compact, it is still possible to prove that f_ε has a critical point near Z_0. The set Z_0 can also consist of local minimum points; the same holds for maximum points. In statement (c) we can allow that Γ as an isolated set of critical points C such that $\deg(\Gamma', \Omega, 0) \neq 0$ in an open neighborhood Ω of C.

Example. Let us consider the problem of finding homoclinics of

$$\ddot{x} - x + x^3 = \varepsilon h(t).$$

It is easy to see that $z_0(t) = \frac{\sqrt{2}}{\cosh t}$ is a solution of the unperturbed problem (see fig. 3.2). Let us take $E = H^1(\mathbb{R})$,

$$Z = \{z_\tau(t) = z_0(t - \tau) : \tau \in \mathbb{R}\},$$

$$G(u) = \int_{\mathbb{R}} h(t)u(t)\, dt,$$

and introduce the Poincaré functional

$$\Gamma(\tau) = G(z_\tau) = \int_{\mathbb{R}} h(t) z_0(t-\tau)\, dt = \int_{\mathbb{R}} h(t+\tau) z_0(t)\, dt.$$

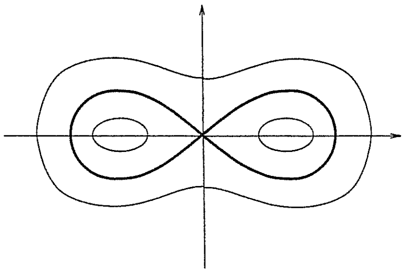

Fig. 3.2. The homoclinic solution to the unperturbed problem.

Z satisfies assumptions (3.1)-(3.3). Suppose that $h \in L^2(\mathbb{R}^N)$, so that the integral makes sense. Γ is a one dimension function and its proper maximum or minimum points give rise to homoclinic solutions to the problem.

Remark 3.3. More general perturbations are treated in [7], where functionals of the form

$$f_\varepsilon(u) = f_0(u) - G(\varepsilon, u) \tag{3.7}$$

are considered.

Part II

Degenerate critical elliptic problems

4 Caffarelli-Kohn-Nirenberg type equations

In this part, we study the problem of existence of solutions to some degenerate critical elliptic equations related to the Caffarelli-Kohn-Nirenberg inequality with a singular Hardy-type potential. In particular in the present chapter we discuss some existence results obtained in [48] in a perturbative setting whereas in the next chapter we treat the nonperturbative case through the blow-up analysis and the Leray-Schauder degree arguments developed in [49]. In chapter 6 some existence and multiplicity results obtained in [2] for a less general class of equations involving Hardy inequality and critical Sobolev exponent via concentration-compactness arguments are presented. Let us mention that on bounded domains related degenerate equations were studied in [4, 5, 39], while some results concerning elliptic equations with Hardy-type potentials can be found in [3, 52, 74]. Concerning parabolic equations related to the Caffarelli-Kohn-Nirenberg inequality one can see [1, 40].

Let us start by recalling Hardy and Caffarelli-Kohn-Nirenberg inequalities.

4.1 Hardy and Caffarelli-Kohn-Nirenberg inequalities

Let us recall the following inequality due to Hardy (see [59]). For the proof we refer to [21] and [54].

Lemma 4.1. (Hardy inequality) *Let $N \geq 3$ and assume that $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, then $\dfrac{u}{|x|} \in L^2(\mathbb{R}^N)$ and*

$$\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx \leq C_N \int_{\mathbb{R}^N} |\nabla u|^2 dx,$$

where $C_N = \left(\frac{2}{N-2}\right)^2$ is optimal and not attained.

The above inequality actually says that the embedding of $\mathcal{D}^{1,2}(\mathbb{R}^N)$ in L^2 with respect to the weight $|x|^{-2}$ is continuous. On the other hand it is possible to see that such an inclusion is not compact.

In [28] Caffarelli, Kohn, and Nirenberg established the following inequalities which can be considered as a generalization of Hardy and Sobolev inequalities (see (1.2))

$$\left(\int_{\mathbb{R}^N} |x|^{-bp}|u|^p \, dx\right)^{2/p} \leq \mathcal{C}_{a,b} \int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2 \, dx \qquad \forall u \in C_0^\infty(\mathbb{R}^N) \tag{4.1}$$

where for $N \geq 3$

$$\begin{aligned} & -\infty < a < \frac{N-2}{2}, \quad a \leq b < a+1, \\ & p = p(a,b) = \frac{2N}{N-2(1+a-b)}. \end{aligned} \tag{4.2}$$

Let us note that the above inequalities contain the classical Sobolev inequality (1.2) ($a = b = 0$) and the Hardy inequality ($a = 0$, $b = 1$). The problem of sharp constants and extremal functions was faced by Catrina and Wang in [34] (see also [33]). Let $D_a^{1,2}(\mathbb{R}^N)$ be defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\left[\int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \, dx\right]^{1/2}, \tag{4.3}$$

so (4.1) holds for $u \in D_a^{1,2}(\mathbb{R}^N)$. The best constant in (4.1) is given by

$$S(a,b) = {C_{a,b}}^{-1} := \inf_{u \in D_a^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int |x|^{-2a} |\nabla u|^2}{\left(\int |x|^{-bp} |u|^p\right)^{\frac{2}{p}}}. \tag{4.4}$$

The extremal functions for $S(a,b)$ are solutions of the Euler equation

$$-\operatorname{div}\left(|x|^{-2a} \nabla u\right) = \frac{u^{p-1}}{|x|^{bp}}, \quad u \geq 0, \quad x \in \mathbb{R}^N \tag{4.5}$$

which is the prototype of more general nonlinear degenerate elliptic equations describing anisotropic physical phenomena. In [34] it is proved that

1. for $b = a+1$, $S(a, a+1) = \left(\frac{N-2-2a}{2}\right)^2$ and it is not achieved;
2. for $a = b < 0$, $S(a,a)$ is equal to the best Sobolev constant and is not achieved;
3. for $a < b < a+1$, $S(a,b)$ is always achieved.

In addition, in [34] the symmetry breaking phenomenon, i.e. the existence of non-radial minimizers of (4.4), is studied; in particular the following result is proved.

Theorem 4.1. [34, Theorem 1.3] *There exist an open subset $H \subset \mathbb{R}^2$ containing $\{(a,a) \mid a < 0\}$, a real number $a_0 \leq 0$ and a function $h :]-\infty, a_0] \to \mathbb{R}$ satisfying $h(a_0) = a_0$ and $a < h(a) < a+1$ for all $a < a_0$, such that for every $(a,b) \in H \cup \{(a,b) \in \mathbb{R}^2 \mid a < a_0,\ a < b < h(a)\}$ the minimizer in (4.4) is non-radial.*

The exponent $p(a,b)$ defined in (4.2) is *critical* for the Caffarelli-Kohn-Nirenberg inequality; indeed equation (4.5) is invariant under the action of the noncompact group of dilations, in the sense that if u is a solution of (4.5) then for any positive μ the dilated function

$$\mu^{-\frac{N-2-2a}{2}} u(x/\mu)$$

is also a solution with the same norm in $D_a^{1,2}(\mathbb{R}^N)$.

4.2 Caffarelli-Kohn-Nirenberg equations in a perturbative setting

Motivated by [34], in [48] M. Schneider and the author studied the following elliptic equation in $\mathbb{R}^N$ in dimension $N \geq 3$

$$-\mathrm{div}\,(|x|^{-2a}\nabla u) - \frac{\lambda}{|x|^{2(1+a)}}\,u = K(x)\frac{u^{p-1}}{|x|^{bp}}, \quad x \in \mathbb{R}^N \setminus \{0\} \tag{4.6}$$

where a, b, and p satisfy (4.2) and

$$-\infty < \lambda < \left(\frac{N-2a-2}{2}\right)^2. \tag{4.7}$$

The above equation is a prototype of more general nonlinear degenerate elliptic equations describing anisotropic physical phenomena. For $\lambda = 0$ equation (4.6) is related to the family of inequalities (4.1) discussed in Section 4.1. We will deal with the perturbative case $K(x) = 1 + \varepsilon k(x)$, namely with the problem

$$\begin{cases} -\mathrm{div}\,(|x|^{-2a}\nabla u) - \dfrac{\lambda}{|x|^{2(1+a)}}\,u = \big(1+\varepsilon k(x)\big)\dfrac{u^{p-1}}{|x|^{bp}} \\ u \in D_a^{1,2}(\mathbb{R}^N), \quad u > 0 \text{ in } \mathbb{R}^N\backslash\{0\}. \end{cases} \tag{$\mathcal{P}_{a,b,\lambda}$}$$

Concerning the perturbation k we assume

$$k \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N). \tag{4.8}$$

Our approach is based on the abstract perturbative variational method discussed in Chapter 3, which splits our procedure in three main steps. First we consider the unperturbed problem, i.e. $\varepsilon = 0$, and find a one dimensional manifold of radial solutions. If this manifold is non-degenerate (see Theorem 4.2 below) a one dimensional reduction of the perturbed variational problem in $D_a^{1,2}(\mathbb{R}^N)$ is possible. Finally we have to find a critical point of a functional defined on the real line. Solutions of $(\mathcal{P}_{a,b,\lambda})$ are critical points in $D_a^{1,2}(\mathbb{R}^N)$ of

$$f_\varepsilon(u) := \frac{1}{2}\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2\,dx - \frac{\lambda}{2}\int_{\mathbb{R}^N}\frac{u^2}{|x|^{2(1+a)}}\,dx - \frac{1}{p}\int_{\mathbb{R}^N}\big(1+\varepsilon k(x)\big)\frac{u_+^p}{|x|^{bp}}\,dx,$$

where $u_+ := \max\{u, 0\}$. For $\varepsilon = 0$ we show that f_0 has a one dimensional manifold of critical points

$$Z_{a,b,\lambda} := \left\{ z_\mu^{a,b,\lambda} := \mu^{-\frac{N-2-2a}{2}} z_1^{a,b,\lambda}\left(\frac{x}{\mu}\right) \mid \mu > 0 \right\},$$

where

$$z_1^{a,b,\lambda}(x) = \left[\frac{N(N-2-2a)\sqrt{(N-2-2a)^2-4\lambda}}{N-2(1+a-b)}\right]^{\frac{N-2(1+a-b)}{4(1+a-b)}} \cdot$$

$$\cdot \left[|x|^{\left(1-\frac{\sqrt{(N-2-2a)^2-4\lambda}}{N-2-2a}\right)\frac{(N-2-2a)(1+a-b)}{N-2(1+a-b)}}\left[1+|x|^{\frac{2(1+a-b)\sqrt{(N-2-2a)^2-4\lambda}}{N-2(1+a-b)}}\right]\right]^{-\frac{N-2(1+a-b)}{2(1+a-b)}}.$$

These radial solutions were computed for $\lambda = 0$ in [34], the case $a = b = 0$ and $-\infty < \lambda < (N-2)^2/4$ was done by [82]. The exact knowledge of the critical manifold enables us to clarify the question of non-degeneracy.

Theorem 4.2. *Suppose a, b, λ, p satisfy (4.2) and* (4.7). *Then the critical manifold $Z_{a,b,\lambda}$ is non-degenerate, i.e.*

$$T_z Z_{a,b,\lambda} = \ker D^2 f_0(z) \quad \forall\, z \in Z_{a,b,\lambda}, \tag{4.9}$$

if and only if for any $j \in \mathbb{N} \setminus \{0\}$

$$b \neq h_j(a,\lambda) := \frac{N}{2}\left[1 + \frac{4j(N+j-2)}{(N-2-2a)^2-4\lambda}\right]^{-\frac{1}{2}} - \frac{N-2-2a}{2}. \tag{4.10}$$

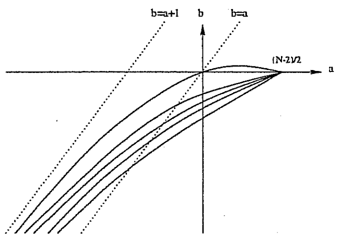

Fig. 4.1. $\lambda = 0$ and $h_j(\cdot, 0)$ for $j = 1 \dots 5$

The above theorem is explicit and fairly highlights the symmetry breaking phenomenon of the unperturbed problem observed in [34], see Theorem 4.1. In fact from Theorem 4.2 we deduce the following result.

Corollary 4.1. *Suppose a, b, p satisfy (4.2) and $a \neq b$. If $b < h_1(a, 0)$, then $S(a, b)$ in (4.4) is attained by a non-radially symmetric function.*

With respect to Theorem 4.1 by Catrina and Wang, the above corollary provides a curve starting from the origin below which symmetry breaking occurs; such a curve is explicitly given by $h_1(a, 0)$, i.e. the first curve at which degeneray occurs, and stays above the curve found by Catrina and Wang. A comparison between the symmetry breaking result by Catrina and Wang and ours is done in pictures 4.2 and 4.3 below.
Concerning step two, the one-dimensional reduction, we follow closely the abstract

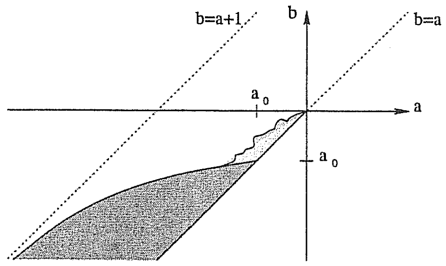

Fig. 4.2. region of non-radial minimizers in [34]

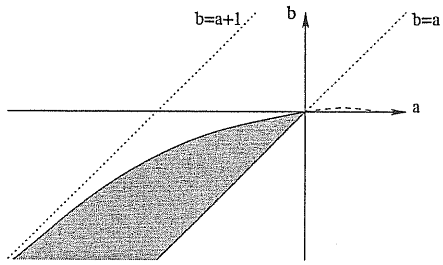

Fig. 4.3. region of non-radial minimizers given by $h_1(\cdot, 0)$

scheme in [7] presented in Chapter 3 and construct a perturbed manifold

$$Z^{\varepsilon}_{a,b,\lambda} = \{z^{a,b,\lambda}_{\mu} + w(\varepsilon,\mu) \mid \mu > 0\},$$

such that any critical point of f_ε restricted to $Z^{\varepsilon}_{a,b,\lambda}$ is a solution to $(\mathcal{P}_{a,b,\lambda})$. We emphasize that in contrast to the local approach in [7] we construct a manifold which is globally diffeomorphic to the unperturbed one such that we may estimate the difference $\|w(\varepsilon,\mu)\|$ when $\mu \to \infty$ or $\mu \to 0$ (see also [13, 19]). More precisely we show under assumption (4.11) below that $\|w(\varepsilon,\mu)\|$ vanishes as $\mu \to \infty$ or $\mu \to 0$. We will prove the following existence results.

Theorem 4.3. *Suppose (4.2), (4.7), (4.8), and (4.10) hold. Then problem* $(\mathcal{P}_{a,b,\lambda})$ *has a solution for all* $|\varepsilon|$ *sufficiently small if*

$$k(\infty) := \lim_{|x|\to\infty} k(x) \text{ exists and } k(\infty) = k(0) = 0. \tag{4.11}$$

Theorem 4.4. *Assume (4.2), (4.7), (4.8), (4.10) and*

$$k \in C^2(\mathbb{R}^N),\ |\nabla k| \in L^\infty(\mathbb{R}^N) \text{ and } |D^2 k| \in L^\infty(\mathbb{R}^N). \tag{4.12}$$

Then $(\mathcal{P}_{a,b,\lambda})$ *is solvable for all small* $|\varepsilon|$ *under each of the following conditions*

$$\limsup_{|x|\to\infty} k(x) \le k(0) \text{ and } \Delta k(0) > 0, \tag{4.13}$$

$$\liminf_{|x|\to\infty} k(x) \ge k(0) \text{ and } \Delta k(0) < 0. \tag{4.14}$$

Problem (4.6), the non-perturbative version of $(\mathcal{P}_{a,b,\lambda})$, was studied by [77] in the case $a = b = 0$ and $0 < \lambda < (N-2)^2/4$. A variational minimax method combined with a careful analysis and construction of Palais-Smale sequences shows that in dimension $N = 4$ equation (4.6) has a positive solution $u \in D_a^{1,2}(\mathbb{R}^N)$ if $K \in C^2$ is positive and satisfies an analogous condition to (4.11), namely $K(0) = \lim_{|x|\to\infty} K(x)$. In our perturbative approach we need not to impose any condition on the space dimension N. The nonperturbative case will be discussed in Chapter 5. The remainder of this chapter is devoted to the proof of Theorems 4.2, 4.3, 4.4, and Corollary 4.1.

4.3 Preliminaries

In [34] it is proved that, for $b = a+1$, $S(a,a+1) = \left(\frac{N-2-2a}{2}\right)^2$ (see Section 4.1). Then we obtain for $-\infty < \lambda < \left(\frac{N-2-2a}{2}\right)^2$ a norm, equivalent to norm (4.3), given by

$$||u|| = \left[\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2\,dx - \lambda\int_{\mathbb{R}^N} \frac{u^2}{|x|^{2(1+a)}}\,dx\right]^{1/2}. \tag{4.15}$$

We denote $D_{a,\lambda}^{1,2}(\mathbb{R}^N)$ the Hilbert space equipped with the scalar product induced by $||\cdot||$

$$(u,v) = \int_{\mathbb{R}^N} |x|^{-2a}\nabla u\cdot\nabla v\,dx - \lambda\int_{\mathbb{R}^N}\frac{u\,v}{|x|^{2(1+a)}}\,dx.$$

Moreover, we denote by $\mathcal{C}$ the cylinder $\mathbb{R}\times S^{N-1}$. It is is shown in [34, Prop. 2.2] that the transformation

$$u(x) = |x|^{-\frac{N-2-2a}{2}}v\Big(-\ln|x|, \frac{x}{|x|}\Big) \tag{4.16}$$

induces a Hilbert space isomorphism from $D_{a,\lambda}^{1,2}(\mathbb{R}^N)$ to $H_\lambda^{1,2}(\mathcal{C})$, where the scalar product in $H_\lambda^{1,2}(\mathcal{C})$ is defined by

$$(v_1,v_2)_{H_\lambda^{1,2}(\mathcal{C})} := \int_{\mathcal{C}} \nabla v_1\cdot\nabla v_2 + \left(\left(\frac{N-2-2a}{2}\right)^2 - \lambda\right)v_1 v_2.$$

Using the canonical identification of the Hilbert space $D_{a,\lambda}^{1,2}(\mathbb{R}^N)$ with its dual induced by the scalar-product and denoted by $\mathcal{K}$, i.e.

$$\begin{aligned}\mathcal{K} :&(D_{a,\lambda}^{1,2}(\mathbb{R}^N))' \to D_{a,\lambda}^{1,2}(\mathbb{R}^N),\\ &(\mathcal{K}(\varphi),u) = \varphi(u)\quad \forall\,(\varphi,u)\in (D_{a,\lambda}^{1,2}(\mathbb{R}^N))'\times D_{a,\lambda}^{1,2}(\mathbb{R}^N),\end{aligned}$$

we shall consider $f_\varepsilon'(u)$ as an element of $D_{a,\lambda}^{1,2}(\mathbb{R}^N)$ and $f_\varepsilon''(u)$ as one of $\mathcal{L}(D_{a,\lambda}^{1,2}(\mathbb{R}^N))$. If we test $f_\varepsilon'(u)$ with $u_- = \max\{-u,0\}$ we get

$$\begin{aligned}(f_\varepsilon'(u),u_-) &= \int_{\mathbb{R}^N}|x|^{-2a}\nabla u\cdot\nabla u_- - \lambda\int_{\mathbb{R}^N}\frac{uu_-}{|x|^{2(1+a)}} - \int_{\mathbb{R}^N}\big(1+\varepsilon k(x)\big)\frac{u_+^{p-1}u_-}{|x|^{bp}}\\ &= -||u_-||^2\end{aligned}$$

and see that any critical point of f_ε is nonnegative. The maximum principle applied in $\mathbb{R}^N\setminus\{0\}$ shows that any nontrivial critical point is positive in that region. We cannot expect more since the radial solutions to the unperturbed problem ($\varepsilon = 0$) vanish at the origin if $\lambda < 0$. Moreover from standard elliptic regularity theory, solutions to $(\mathcal{P}_{a,b,\lambda})$ are $C^{1,\alpha}(\mathbb{R}^N \setminus \{0\})$, $\alpha > 0$. The unperturbed functional f_0 is given by

$$f_0(u) := \frac{1}{2}\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2\,dx - \frac{\lambda}{2}\int_{\mathbb{R}^N} \frac{u^2}{|x|^{2(1+a)}}\,dx - \frac{1}{p}\int_{\mathbb{R}^N} \frac{u_+^p}{|x|^{bp}}\,dx$$

for any $u \in D^{1,2}_{a,\lambda}(\mathbb{R}^N)$, and we may write $f_\varepsilon(u) = f_0(u) - \varepsilon G(u)$, where

$$G(u) := \frac{1}{p}\int_{\mathbb{R}^N} k(x)\frac{u_+^p}{|x|^{bp}}. \tag{4.17}$$

4.4 The unperturbed problem

Critical points of the unperturbed functional f_0 solve the equation

$$\begin{cases} -\operatorname{div}\left(|x|^{-2a}\nabla u\right) - \dfrac{\lambda}{|x|^{2(1+a)}}\,u = \dfrac{1}{|x|^{bp}}\,u^{p-1} \\ u \in D^{1,2}_{a,\lambda}(\mathbb{R}^N), \quad u > 0 \text{ in } \mathbb{R}^N\setminus\{0\}. \end{cases} \tag{4.18}$$

To find all radially symmetric solutions u of (4.18), i.e. $u(x) = u(r)$, where $r = |x|$, we follow [34] and note that if u is radial, then equation (4.18) can be written as

$$-\frac{u''}{r^{2a}} - \frac{N-2a-1}{r^{2a+1}}\,u' - \frac{\lambda}{r^{2(a+1)}}\,u = \frac{1}{r^{bp}}\,u^{p-1}. \tag{4.19}$$

Making now the change of variable

$$u(r) = r^{-\frac{N-2-2a}{2}}\varphi(\ln r), \tag{4.20}$$

we come to the equation

$$-\varphi'' + \left[\left(\frac{N-2-2a}{2}\right)^2 - \lambda\right]\varphi - \varphi^{p-1} = 0. \tag{4.21}$$

All positive solutions of (4.21) in $H^{1,2}(\mathbb{R})$ are the translates of

$$\varphi_1(t) = \left[\frac{N(N-2-2a)\sqrt{(N-2-2a)^2-4\lambda}}{4(N-2(1+a-b))}\right]^{\frac{N-2(1+a-b)}{4(1+a-b)}} \cdot \left(\cosh\frac{(1+a-b)\sqrt{(N-2-2a)^2-4\lambda}}{N-2(1+a-b)}\,t\right)^{-\frac{N-2(1+a-b)}{2(1+a-b)}},$$

namely $\varphi_\mu(t) = \varphi_1(t - \ln\mu)$ for some $\mu > 0$ (see [34]). Consequently all radial solutions of (4.18) are dilations of

$$z_1^{a,b,\lambda}(x) = \left[\frac{N(N-2-2a)\sqrt{(N-2-2a)^2-4\lambda}}{N-2(1+a-b)}\right]^{\frac{N-2(1+a-b)}{4(1+a-b)}} \cdot$$
$$\cdot \left[|x|^{\left(1-\frac{\sqrt{(N-2-2a)^2-4\lambda}}{N-2-2a}\right)\frac{(N-2-2a)(1+a-b)}{N-2(1+a-b)}}\left[1+|x|^{\frac{2(1+a-b)\sqrt{(N-2-2a)^2-4\lambda}}{N-2(1+a-b)}}\right]\right]^{-\frac{N-2(1+a-b)}{2(1+a-b)}} \tag{4.22}$$

and given by

$$z_\mu^{a,b,\lambda}(x) = \mu^{-\frac{N-2-2a}{2}} z_1^{a,b,\lambda}\left(\frac{x}{\mu}\right), \quad \mu > 0.$$

Using the change of coordinates in (4.20), respectively (4.16), and the exponential decay of $z_\mu^{a,b,\lambda}$ in these coordinates it is easy to see that the map $\mu \mapsto z_\mu^{a,b,\lambda}$ is at least twice continuously differentiable from $(0,\infty)$ to $D_{a,\lambda}^{1,2}(\mathbb{R}^N)$ and we obtain

Lemma 4.2. *Suppose a, b, λ, p satisfy (4.2) and (4.7). Then the unperturbed functional f_0 has a one dimensional C^2-manifold of critical points $Z_{a,b,\lambda}$ given by $\{z_\mu^{a,b,\lambda} \mid \mu > 0\}$. Moreover, $Z_{a,b,\lambda}$ is exactly the set of all radially symmetric, positive solutions of (4.18) in $D_{a,\lambda}^{1,2}(\mathbb{R}^N)$.*

In order to apply the abstract perturbation method we need to show that the manifold $Z_{a,b,\lambda}$ satisfy a non-degeneracy condition. This is the content of Theorem 4.2.

Proof of Theorem 4.2. The inclusion $T_{z_\mu^{a,b,\lambda}} Z_{a,b,\lambda} \subseteq \ker D^2 f_0(z_\mu^{a,b,\lambda})$ always holds and is a consequence of the fact that $Z_{a,b,\lambda}$ is a manifold of critical points of f_0. Consequently, we have only to show that $\ker D^2 f_0(z_\mu^{a,b,\lambda})$ is one dimensional. Fix $u \in \ker D^2 f_0(z_\mu^{a,b,\lambda})$. The function u is a solution of the linearized problem

$$-\text{div}\,(|x|^{-2a}\nabla u) - \frac{\lambda}{|x|^{2(a+1)}} u = \frac{p-1}{|x|^{bp}} (z_\mu^{a,b,\lambda})^{p-2} u. \tag{4.23}$$

We expand u in spherical harmonics

$$u(r\vartheta) = \sum_{i=0}^{\infty} \mathbf{v}_i(r)\mathbf{Y}_i(\vartheta), \quad r \in \mathbb{R}^+, \quad \vartheta \in \mathbb{S}^{N-1},$$

where $\mathbf{v}_i(r) = \int_{\mathbb{S}^{N-1}} u(r\vartheta)\mathbf{Y}_i(\vartheta)\, d\vartheta$ and $\mathbf{Y}_i$ denotes the orthogonal i-th spherical harmonic jet satisfying for all $i \in \mathbb{N}_0$

$$-\Delta_{\mathbb{S}^{N-1}} \mathbf{Y}_i = i(N+i-2)\mathbf{Y}_i. \tag{4.24}$$

Since u solves (4.23) the functions $\mathbf{v}_i$ satisfy for all $i \geq 0$

$$-\frac{\mathrm{v}_i''}{r^{2a}}\mathrm{Y}_i - \frac{N-1-2a}{r^{2a+1}}\mathrm{v}_i{}'\mathrm{Y}_i - \frac{\mathrm{v}_i}{r^{2(a+1)}}\Delta_\vartheta \mathrm{Y}_i - \frac{\lambda}{r^{2(a+1)}}\mathrm{v}_i\,\mathrm{Y}_i = \frac{p-1}{r^{bp}}(z_\mu^{a,b,\lambda})^{p-2}\mathrm{v}_i\,\mathrm{Y}_i$$

and hence, in view of (4.24),

$$-\frac{\mathrm{v}_i''}{r^{2a}} - \frac{N-1-2a}{r^{2a+1}}\mathrm{v}_i{}' + \frac{i(N+i-2)}{r^{2(a+1)}}\mathrm{v}_i - \frac{\lambda}{r^{2(a+1)}}\mathrm{v}_i = \frac{p-1}{r^{bp}}(z_\mu^{a,b,\lambda})^{p-2}\mathrm{v}_i. \quad (4.25)$$

Making in (4.25) the transformation (4.20) we obtain the equations

$$-\varphi_i'' - \beta\cosh^{-2}\left(\gamma(t-\ln\mu)\right)\varphi_i = \left(\lambda - \left(\frac{N-2-2a}{2}\right)^2 - i(N+i-2)\right)\varphi_i, \quad i \in \mathbb{N}_0,$$

where

$$\beta = \frac{N(N+2(1+a-b))((N-2-2a)^2-4\lambda)}{4(N-2(1+a-b))^2} \quad \text{and}$$
$$\gamma = \frac{(1+a-b)\sqrt{(N-2-2a)^2-4\lambda}}{N-2(1+a-b)},$$

which is equivalent, through the change of variable $\zeta(s) = \varphi(s+\ln\mu)$, to

$$-\zeta_i'' - \beta\cosh^{-2}(\gamma s)\zeta_i = \left(\lambda - \left(\frac{N-2-2a}{2}\right)^2 - i(N+i-2)\right)\zeta_i, \quad i \in \mathbb{N}_0. \quad (4.26)$$

It is known (see [56],[68, p. 74]) that the negative part of the spectrum of the problem

$$-\zeta'' - \beta\cosh^{-2}(\gamma s)\zeta = \nu\zeta$$

is discrete, consists of simple eigenvalues and is given by

$$\nu_j = -\frac{\gamma^2}{4}\left(-(1+2j) + \sqrt{1+4\beta\gamma^{-2}}\right)^2, \ j \in \mathbb{N}_0, \ 0 \le j < \frac{1}{2}\left(-1+\sqrt{1+4\beta\gamma^{-2}}\right).$$

Thus we have for all $i \ge 0$ that zero is the only solution to (4.26) if and only if

$$A_i(a,\lambda) \ne B_j(a,b,\lambda) \text{ for all } 0 \le j < \frac{N}{2(1+a-b)}, \quad (4.27)$$

where

$$A_i(a,\lambda) = \lambda - \left(\frac{N-2-2a}{2}\right)^2 - i(N+i-2)$$

and

$$B_j(a,b,\lambda) = -\frac{((N-2-2a)^2-4\lambda)(1+a-b)^2}{4(N-2(1+a-b))^2}\left(-2j+\frac{N}{1+a-b}\right)^2.$$

Note that $A_0(a,\lambda) = B_1(a,b,\lambda)$, $A_i(a,\lambda) \ge A_{i+1}(a,\lambda)$ and $B_j(a,b,\lambda) \le B_{j+1}(a,b,\lambda)$, as it is shown in the figure below.

Hence (4.27) is satisfied for $i \ge 1$ if and only if $B_0(a,b,\lambda) \ne A_i(a,b,\lambda)$, which is equivalent to $b \ne h_i(a,\lambda)$. On the other hand for $i = 0$ equation (4.26) has a one dimensional space of nonzero solutions. Hence, $\ker D^2 f_0(z_\mu^{a,b,\lambda})$ is one dimensional if and only if $b \ne h_i(a,\lambda)$ for any $i \ge 1$, which proves the claim. □

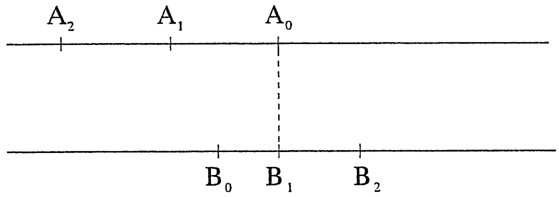

Fig. 4.4. $A_0 = B_1$, $A_i \geq A_{i+1}$ and $B_j \leq B_{j+1}$

Proof of Corollary 4.1. We define I on $D_a^{1,2}(\mathbb{R}^N)\backslash\{0\}$ by

$$I(u) := \frac{||\nabla u||_a^2}{||u||_{p,b}^2}$$

where

$$||\nabla u||_a = \left(\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2\right)^{\frac{1}{2}} \quad \text{and} \quad ||u||_{p,b} = \left(\int_{\mathbb{R}^N} |x|^{-bp}|u|^p\right)^{\frac{1}{p}}.$$

I is twice continuously differentiable and

$$(I'(u),\varphi) = \frac{2}{||u||_{p,b}^2}\left(\int_{\mathbb{R}^N} |x|^{-2a}\nabla u \nabla\varphi - \frac{||\nabla u||_a^2}{||u||_{p,b}^p}\int_{\mathbb{R}^N} |x|^{-bp}|u|^{p-2}u\varphi\right).$$

Moreover, for positive critical points u of I a short computation leads to

$$\begin{aligned}(I''(u)\varphi_1,\varphi_2) = &\frac{2}{||u||_{p,b}^2}\left(\int_{\mathbb{R}^N} |x|^{-2a}\nabla\varphi_1\nabla\varphi_2 - \frac{||\nabla u||_a^2}{||u||_{p,b}^p}(p-1)\int_{\mathbb{R}^N} |x|^{-bp}u^{p-2}\varphi_1\varphi_2\right)\\ &+ (p-2)\frac{2||\nabla u||_a^2}{||u||_{p,b}^{2p+2}}\left(\int_{\mathbb{R}^N} |x|^{-bp}u^{p-1}\varphi_1\right)\left(\int_{\mathbb{R}^N} |x|^{-bp}u^{p-1}\varphi_2\right).\end{aligned}$$

Obviously I is constant on $Z_{a,b,0}$ and we get for $z_1 := z_1^{a,b,0}$ and all $\varphi_1, \varphi_2 \in D_{a,\lambda}^{1,2}(\mathbb{R}^N)$

$$\begin{aligned}(I'(z_1),\varphi_1) &= \frac{2}{||z_1||_{p,b}^2}(f_0'(z_1),\varphi_1) = 0,\\ (I''(z_1)\varphi_1,\varphi_2) &= \frac{2}{||u||_{p,b}^2}(f_0''(z_1)\varphi_1,\varphi_2)\\ &+ (p-2)\frac{2}{||z_1||_{p,b}^{p+2}}\left(\int_{\mathbb{R}^N} |x|^{-bp}z_1^{p-1}\varphi_1\right)\left(\int_{\mathbb{R}^N} |x|^{-bp}z_1^{p-1}\varphi_2\right). \qquad (4.28)\end{aligned}$$

From the proof of Theorem 4.2 we know that for $b < h_1(a,0)$ there exist functions $\hat{\phi} \in D_a^{1,2}(\mathbb{R}^N)$ of the form $\hat{\phi}(x) = \bar{\phi}(|x|)Y_1(x/|x|)$, where Y_1 denotes one of the first spherical harmonics, such that $(f_0''(z_1)\hat{\phi},\hat{\phi}) < 0$. By (4.28) we get $(I''(z_1)\hat{\phi},\hat{\phi}) < 0$ because the integral $\int |x|^{-bp}z_1^{p-1}\hat{\phi} = 0$. Consequently $\mathcal{C}_{a,b}{}^{-1}$ is strictly smaller than

$I(z_1) = I(z_\mu^{a,b,0})$. Since all positive radial solutions of (4.18) are given by $z_\mu^{a,b,0}$ (see Lemma 4.2) and the infimum in (4.4) is attained (see [34, Thm 1.2]) the minimizer must be non-radial. □

As a particular case of Theorem 4.2 we can state

Corollary 4.2.

(i) If $0 < a < \frac{N-2}{2}$ *and* $0 \le \lambda < \left(\frac{N-2-2a}{2}\right)^2$ *then* $Z_{a,b,\lambda}$ *is non-degenerate for any* b *between* a *and* $a+1$.
(ii) If $a = 0$ *and* $0 \le \lambda < \left(\frac{N-2-2a}{2}\right)^2$, *then* $Z_{0,b,\lambda}$ *is degenerate if and only if* $b = \lambda = 0$.

Remark 4.1. If $a = b = \lambda = 0$, equation (4.18) is invariant not only by dilations but also by translations. The manifold of critical points is in this case $N+1$-dimensional and given by the translations and dilations of $z_1^{0,0,0}$. Hence the one dimensional manifold $Z_{0,0,0}$ is degenerate. However, the full $N+1$-dimensional critical manifold is non-degenerate in the case $a = b = \lambda = 0$ (see [11]).

4.5 The finite dimensional reduction

We follow the perturbative method developed in [7] (see Chapter 3) and show that a finite dimensional reduction of our problem is possible whenever the critical manifold is non-degenerated. For simplicity of notation we write z_μ instead of $z_\mu^{a,b,\lambda}$ and Z instead of $Z_{a,b,\lambda}$ if there is no possibility of confusion.

Lemma 4.3. *Suppose* a, b, λ, p *satisfy (4.2) and (4.7) and* v *is a measurable function such that* $\int_{\mathbb{R}^N} |x|^{-bp}|v|^{\frac{p}{p-2}}$ *is finite. Then the operator* $J_v : D^{1,2}_{a,\lambda}(\mathbb{R}^N) \to D^{1,2}_{a,\lambda}(\mathbb{R}^N)$, *defined by*

$$J_v(u) := \mathcal{K}\Big(\int_{\mathbb{R}^N} |x|^{-pb} vu\, \cdot\Big), \tag{4.29}$$

is compact.

Proof. Fix a sequence $(u_n)_{n\in\mathbb{N}}$ converging weakly to zero in $D^{1,2}_{a,\lambda}(\mathbb{R}^N)$. To prove the assertion it is sufficient to show that up to a subsequence $J_v(u_n) \to 0$ as $n \to \infty$. Using the Hilbert space isomorphism given in (4.16) we see that the corresponding sequence $(v_n)_{n\in\mathbb{N}}$ converges weakly to zero in $H^{1,2}_\lambda(\mathcal{C})$. Since $(v_n)_{n\in\mathbb{N}}$ converges strongly in $L^2(\Omega)$ for all bounded domains Ω in $\mathcal{C}$, we may extract a subsequence that converges to zero pointwise almost everywhere. Going back to $D^{1,2}_{a,\lambda}(\mathbb{R}^N)$ we may assume that this also holds for $(u_n)_{n\in\mathbb{N}}$. By Hölder's inequality and (4.1)

$$
\begin{aligned}
||J_v(u_n)|| &\le \sup_{||h||_{D^{1,2}_{a,\lambda}(\mathbb{R}^N)}\le 1} \int_{\mathbb{R}^N} |x|^{-pb}|v||u_n||h| \\
&\le \sup_{||h||_{D^{1,2}_{a,\lambda}}\le 1} \Big(\int_{\mathbb{R}^N} |x|^{-pb}|h|^p\Big)^{1/p}\Big(\int_{\mathbb{R}^N} |x|^{-pb}|v|^{\frac{p}{p-1}}|u_n|^{\frac{p}{p-1}}\Big)^{(p-1)/p} \\
&\le C\Big(\int_{\mathbb{R}^N} |x|^{-pb}|v|^{\frac{p}{p-1}}|u_n|^{\frac{p}{p-1}}\Big)^{(p-1)/p}.
\end{aligned}
$$

To show that the latter integral converges to zero we use Vitali's convergence theorem given for instance in [61, 13.38]. Obviously the functions $|\cdot|^{-pb}|v|^{\frac{p}{p-1}}|u_n|^{\frac{p}{p-1}}$ converge pointwise almost everywhere to zero. For any measurable $\Omega \subset \mathbb{R}^N$ we may estimate using Hölder's inequality

$$
\begin{aligned}
\int_\Omega |x|^{-pb}|v|^{\frac{p}{p-1}}|u_n|^{\frac{p}{p-1}} &\le \Big(\int_\Omega |x|^{-pb}|v|^{\frac{p}{p-2}}\Big)^{(p-2)/(p-1)}\Big(\int_\Omega |x|^{-pb}|u_n|^p\Big)^{1/(p-1)} \\
&\le C\Big(\int_\Omega |x|^{-pb}|v|^{\frac{p}{p-2}}\Big)^{(p-2)/(p-1)}
\end{aligned}
$$

for some positive constant C. Taking Ω a set of small measure or the complement of a large ball and the use of Vitali's convergence theorem prove the assertion. □

Lemma 4.3 immediately leads to

Corollary 4.3. *For all $z \in Z$ the operator $f_0''(z) : D^{1,2}_{a,\lambda}(\mathbb{R}^N) \to D^{1,2}_{a,\lambda}(\mathbb{R}^N)$ may be written as $f_0''(z) = \text{Identity} - J_{|z|^{p-2}}$ and is consequently a self-adjoint Fredholm operator of index zero.*

Define for $\mu > 0$ the map $U_\mu : D^{1,2}_{a,\lambda}(\mathbb{R}^N) \to D^{1,2}_{a,\lambda}(\mathbb{R}^N)$ by

$$U_\mu(u) := \mu^{-\frac{N-2-2a}{2}} u\Big(\frac{x}{\mu}\Big).$$

It is easy to check that U_μ conserves the norms $||\cdot||$ and $||\cdot||_{p,b}$, thus for every $\mu > 0$

$$(U_\mu)^{-1} = (U_\mu)^t = U_{\mu^{-1}} \text{ and } f_0 = f_0 \circ U_\mu \tag{4.30}$$

where $(U_\mu)^t$ denotes the adjoint of U_μ. Twice differentiating the identity $f_0 = f_0 \circ U_\mu$ yields for all $h_1, h_2, v \in D^{1,2}_{a,\lambda}(\mathbb{R}^N)$

$$(f_0''(v)h_1, h_2) = (f_0''(U_\mu(v))U_\mu(h_1), U_\mu(h_2)),$$

that is

$$f_0''(v) = (U_\mu)^{-1} \circ f_0''(U_\mu(v)) \circ U_\mu \quad \forall v \in D^{1,2}_{a,\lambda}(\mathbb{R}^N). \tag{4.31}$$

Differentiating (4.30) we see that $U(\mu, z) := U_\mu(z)$ maps $(0,\infty) \times Z$ into Z, hence

$$\frac{\partial U}{\partial z}(\mu, z) = U_\mu : T_z Z \to T_{U_\mu(z)} Z \text{ and } U_\mu : (T_z Z)^\perp \to (T_{U_\mu(z)} Z)^\perp. \tag{4.32}$$

If the manifold Z is non-degenerated the self-adjoint Fredholm operator $f_0''(z_1)$ maps the space $D^{1,2}_{a,\lambda}(\mathbb{R}^N)$ into $T_{z_1} Z^\perp$ and $f_0''(z_1) \in \mathcal{L}(T_{z_1} Z^\perp)$ is invertible. Consequently, using (4.31) and (4.32), we obtain in this case

$$\|(f_0''(z_1))^{-1}\|_{\mathcal{L}(T_{z_1} Z^\perp)} = \|(f_0''(z))^{-1}\|_{\mathcal{L}(T_z Z^\perp)} \quad \forall z \in Z. \tag{4.33}$$

Lemma 4.4. *Suppose a, b, p, λ satisfy (4.2), (4.7), and (4.8) holds. Then there exists a constant $C_1 = C_1(\|k\|_\infty, a, b, \lambda) > 0$ such that for any $\mu > 0$ and $w \in D^{1,2}_{a,\lambda}(\mathbb{R}^N)$*

$$|G(z_\mu + w)| \le C_1 \left(\||k|^{1/p} z_\mu\|^p_{p,b} + \|w\|^p\right) \tag{4.34}$$

$$\|G'(z_\mu + w)\| \le C_1 \left(\||k|^{1/p} z_\mu\|^{p-1}_{p,b} + \|w\|^{p-1}\right) \tag{4.35}$$

$$\|G''(z_\mu + w)\| \le C_1 \left(\||k|^{1/p} z_\mu\|^{p-2}_{p,b} + \|w\|^{p-2}\right). \tag{4.36}$$

Moreover, if $\lim_{|x|\to\infty} k(x) =: k(\infty) = 0 = k(0)$ then

$$\||k|^{1/p} z_\mu\|_{p,b} \to 0 \text{ as } \mu \to \infty \text{ or } \mu \to 0. \tag{4.37}$$

Proof. (4.34)-(4.36) are consequences of (4.1) and Hölder's inequality. We will only show (4.36) as (4.34)-(4.35) follow analogously. By Hölder's inequality and (4.1)

$$\begin{aligned}
\|G''(z_\mu + w)\| &\le (p-1) \sup_{\|h_1\|,\|h_2\|\le 1} \int_{\mathbb{R}^N} \frac{|k(x)|}{|x|^{bp}} |z_\mu + w|^{p-2} |h_1||h_2| \\
&\le (p-1)\||k|^{1/p}\|^2_\infty \sup_{\|h_1\|,\|h_2\|\le 1} \||k|^{1/p}(z_\mu + w)\|^{p-2}_{p,b} \|h_1\|_{p,b} \|h_2\|_{p,b} \\
&\le c(\|k\|_\infty, a, b, \lambda)\, \||k|^{1/p}(z_\mu + w)\|^{p-2}_{p,b}.
\end{aligned}$$

Using the triangle inequality and again (4.1) we obtain (4.36).
Under the additional assumption $k(0) = k(\infty) = 0$ estimate (4.37) follows by the dominated convergence theorem and

$$\int_{\mathbb{R}^N} \frac{|k(x)|}{|x|^{bp}} z_\mu^p = \int_{\mathbb{R}^N} \frac{|k(\mu x)|}{|x|^{bp}} z_1^p.$$

□

Lemma 4.5. *Suppose a, b, p, λ satisfy (4.2), (4.7), (4.8), and (4.9) hold. Then there exist constants $\varepsilon_0, C > 0$ and a smooth function*

$$w = w(\mu, \varepsilon) : \quad (0, +\infty) \times (-\varepsilon_0, \varepsilon_0) \longrightarrow D^{1,2}_{a,\lambda}(\mathbb{R}^N)$$

such that for any $\mu > 0$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$

$$w(\mu, \varepsilon) \text{ is orthogonal to } T_{z_\mu} Z \tag{4.38}$$

$$f_\varepsilon'(z_\mu + w(\mu, \varepsilon)) \in T_{z_\mu} Z \tag{4.39}$$

$$\|w(\mu, \varepsilon)\| \le C\,|\varepsilon|. \tag{4.40}$$

Moreover, if (4.11) holds then

$$\|w(\mu,\varepsilon)\| \to 0 \text{ as } \mu \to 0 \text{ or } \mu \to \infty. \tag{4.41}$$

Proof. Define $H : (0,\infty) \times D^{1,2}_{a,\lambda}(\mathbb{R}^N) \times \mathbb{R} \times \mathbb{R} \to D^{1,2}_{a,\lambda}(\mathbb{R}^N) \times \mathbb{R}$

$$H(\mu, w, \alpha, \varepsilon) := (f'_\varepsilon(z_\mu + w) - \alpha\dot{\xi}_\mu, (w, \dot{\xi}_\mu)),$$

where $\dot{\xi}_\mu$ denotes the normalized tangent vector $\frac{d}{d\mu}z_\mu$. If $H(\mu,w,\alpha,\varepsilon) = (0,0)$ then w satisfies (4.38)-(4.39) and $H(\mu,w,\alpha,\varepsilon) = (0,0)$ if and only if $(w,\alpha) = F_{\mu,\varepsilon}(w,\alpha)$, where

$$F_{\mu,\varepsilon}(w,\alpha) := -\Big(\frac{\partial H}{\partial(w,\alpha)}(\mu,0,0,0)\Big)^{-1} H(\mu,w,\alpha,\varepsilon) + (w,\alpha).$$

We prove that $F_{\mu,\varepsilon}(w,\alpha)$ is a contraction in some ball $B_\rho(0)$, where we may choose the radius $\rho = \rho(\varepsilon) > 0$ independent of $z \in Z$. To this end we observe that

$$\begin{aligned}\Big(\Big(\frac{\partial H}{\partial(w,\alpha)}(\mu,0,0,0)\Big)(w,\beta), (f''_0(z_\mu)w - \beta\dot{\xi}_\mu, (w,\dot{\xi}_\mu))\Big)& \\ = \|f''_0(z_\mu)w\|^2 + \beta^2 + |(w,\dot{\xi}_\mu)|^2&,\end{aligned} \tag{4.42}$$

where

$$\Big(\frac{\partial H}{\partial(w,\alpha)}(\mu,0,0,0)\Big)(w,\beta) = (f''_0(z_\mu)w - \beta\dot{\xi}_\mu, (w,\dot{\xi}_\mu)).$$

From Corollary 4.3 and (4.42) we infer that $\big(\frac{\partial H}{\partial(w,\alpha)}(\mu,0,0,0)\big)$ is an injective Fredholm operator of index zero, hence invertible and by (4.33) and (4.42) we obtain

$$\Big\|\Big(\frac{\partial H}{\partial(w,\alpha)}(\mu,0,0,0)\Big)^{-1}\Big\| \le \max\big(1, \|(f''_0(z_\mu))^{-1}\|\big) = \max\big(1, \|(f''_0(z_1))^{-1}\|\big) =: C_*. \tag{4.43}$$

Suppose $(w,\alpha) \in B_\rho(0)$. We use (4.31) and (4.43) to see

$$\begin{aligned}\|F_{\mu,\varepsilon}(w,\alpha)\| &\le C_* \Big\|\Big(H(\mu,w,\alpha,\varepsilon) - \Big(\frac{\partial H}{\partial(w,\alpha)}(\mu,0,0,0)\Big)(w,\alpha)\Big)\Big\| \\ &\le C_*\|f'_\varepsilon(z_\mu + w) - f''_0(z_\mu)w\| \\ &\le C_* \int_0^1 \|f''_0(z_\mu + tw) - f''_0(z_\mu)\| \,\mathrm{dt}\, \|w\| + C_*|\varepsilon|\|G'(z_\mu + w)\| \\ &\le C_* \int_0^1 \|f''_0(z_1 + tU_{\mu^{-1}}(w)) - f''_0(z_1)\| \,\mathrm{dt}\, \|w\| + C_*|\varepsilon|\|G'(z_\mu + w)\| \\ &\le C_*\rho \sup_{\|w\|\le\rho} \|f''_0(z_1 + w) - f''_0(z_1)\| + C_*|\varepsilon| \sup_{\|w\|\le\rho} \|G'(z_\mu + w)\|.\end{aligned} \tag{4.44}$$

Analogously we get for $(w_1, \alpha_1), (w_2, \alpha_2) \in B_\rho(0)$

$$\frac{\|F_{\mu,\varepsilon}(w_1,\alpha_1) - F_{\mu,\varepsilon}(w_2,\alpha_2)\|}{C_*\|w_1 - w_2\|} \le \frac{\|f'_\varepsilon(z_\mu + w_1) - f'_\varepsilon(z_\mu + w_2) - f''_0(z_\mu)(w_1 - w_2)\|}{\|w_1 - w_2\|}$$
$$\le \int_0^1 \|f''_\varepsilon(z_\mu + w_2 + t(w_1 - w_2)) - f''_0(z_\mu)\|\, dt$$
$$\le \int_0^1 \|f''_0(z_\mu + w_2 + t(w_1 - w_2)) - f''_0(z_\mu)\|\, dt$$
$$+ |\varepsilon| \int_0^1 \|G''(z_\mu + w_2 + t(w_1 - w_2))\|\, dt$$
$$\le \sup_{\|w\|\le 3\rho} \|f''_0(z_1 + w) - f''_0(z_1)\| + |\varepsilon| \sup_{\|w\|\le 3\rho} \|G''(z_\mu + w)\|.$$

We may choose $\rho_0 > 0$ such that

$$C_* \sup_{\|w\|\le 3\rho_0} \|f''_0(z_1 + w) - f''_0(z_1)\| < \frac{1}{2}$$

and $\varepsilon_0 > 0$ such that

$$\begin{cases} 4\varepsilon_0 C_* C_1 \||k|^{1/p} z_\mu\|_{p,b}^{p-1} < \min\left\{\rho_0, \left(\frac{1}{8\varepsilon_0 C_1 C_*}\right)^{\frac{1}{p-2}}\right\} \\ 3\varepsilon_0 < \left(\sup_{z\in Z, \|w\|\le 3\rho_0} \|G''(z + w)\|\right)^{-1} C_*^{-1} \\ 3\varepsilon_0 < \left(\sup_{z\in Z, \|w\|\le \rho_0} \|G'(z + w)\|\right)^{-1} C_*^{-1}\rho_0, \end{cases} \tag{4.45}$$

where C_1 is given in Lemma 4.4. With these choices and the above estimates it is easy to see that for every $z_\mu \in Z$ and $|\varepsilon| < \varepsilon_0$ the map $F_{\mu,\varepsilon}$ maps $B_{\rho_0}(0)$ in itself and is a contraction there. Thus $F_{\mu,\varepsilon}$ has a unique fixed-point $(w(\mu,\varepsilon), \alpha(\mu,\varepsilon))$ in $B_{\rho_0}(0)$ and it is a consequence of the implicit function theorem that w and α are continuously differentiable.

From (4.44) we also infer that $F_{z,\varepsilon}$ maps $B_\rho(0)$ into $B_\rho(0)$, whenever $\rho \le \rho_0$ and

$$\rho > 2|\varepsilon|\left(\sup_{\|w\|\le\rho} \|G'(z + w)\|\right)C_*.$$

Consequently due to the uniqueness of the fixed-point we have

$$\|(w(z,\varepsilon), \alpha(z,\varepsilon))\| \le 3|\varepsilon|\left(\sup_{\|w\|\le\rho_0} \|G'(z + w)\|\right)C_*,$$

which gives (4.40). Let us now prove (4.41). Set

$$\rho_\mu := 4\varepsilon_0 C_* C_1 \||k|^{1/p} z_\mu\|_{p,b}^{p-1}.$$

From (4.45) it follows that

$$\rho_\mu < \min\left\{\rho_0, \left(\frac{1}{8\varepsilon_0 C_1 C_*}\right)^{\frac{1}{p-2}}\right\}.$$

In view of (4.35) we have that for any $|\varepsilon| < \varepsilon_0$ and $\mu > 0$

$$2|\varepsilon|C_* \sup_{\|w\|\le\rho_\mu} \|G'(z_\mu + w)\| \le 2|\varepsilon|C_*C_1\||k|^{1/p}z_\mu\|_{p,b}^{p-1} + 2|\varepsilon|C_*C_1\rho_\mu^{p-2}\rho_\mu.$$

Since $\rho_\mu^{p-2} < \frac{1}{8\varepsilon_0 C_1 C_*}$ we have,

$$2|\varepsilon|C_* \sup_{\|w\|\le\rho_\mu} \|G'(z_\mu + w) < 2|\varepsilon|C_*C_1\||k|^{1/p}z_\mu\|_{p,b}^{p-1} + \frac{1}{4}\rho_\mu < \frac{3}{4}\rho_\mu < \rho_\mu,$$

so that, by the above argument, we can conclude that $F_{\mu,\varepsilon}$ maps $B_{\rho_\mu}(0)$ into $B_{\rho_\mu}(0)$. Consequently due to the uniqueness of the fixed-point we have

$$\|w(\mu,\varepsilon)\| \le \rho_\mu.$$

Since by (4.37) we have that $\rho_\mu \to 0$ for $\mu \to 0$ and for $\mu \to +\infty$, we get (4.41). □

Under the assumptions of Lemma 4.5 we may define for $|\varepsilon| < \varepsilon_0$

$$Z^\varepsilon_{a,b,\lambda} := \left\{u \in D^{1,2}_{a,\lambda}(\mathbb{R}^N) \mid u = z_\mu^{a,b,\lambda} + w(\mu,\varepsilon),\ \mu \in (0,\infty)\right\}. \tag{4.46}$$

Note that Z^ε is a one dimensional manifold.

Lemma 4.6. *Under the assumptions of Lemma 4.5 we may choose $\varepsilon_0 > 0$ such that for every $|\varepsilon| < \varepsilon_0$ the manifold Z^ε is a natural constraint for f_ε, i.e. every critical point of $f_\varepsilon|_{Z^\varepsilon}$ is a critical point of f_ε.*

Proof. Fix $u \in Z^\varepsilon$ such that $f_\varepsilon|'_{Z^\varepsilon}(u) = 0$. In the following we use a dot for the derivation with respect to μ. Since $(\dot z_\mu, w(\mu,\varepsilon)) = 0$ for all $\mu > 0$ we obtain

$$(\ddot z_\mu, w(\mu,\varepsilon)) + (\dot z_\mu, \dot w(\mu,\varepsilon)) = 0. \tag{4.47}$$

Moreover differentiating the identity $z_\mu = U_\sigma z_{\mu/\sigma}$ with respect to μ we obtain

$$\dot z_\sigma = \frac{1}{\sigma}U_\sigma \dot z_1 \text{ and } \ddot z_\sigma = \frac{1}{\sigma^2}U_\sigma \ddot z_1. \tag{4.48}$$

From (4.39) we get that $f'_\varepsilon(u) = c_1\dot z_\mu$ for some $\mu > 0$. By (4.47) and (4.48)

$$\begin{aligned} 0 &= (f'_\varepsilon(u), \dot z_\mu + \dot w(\mu,\varepsilon)) = c_1(\dot z_\mu, \dot z_\mu + \dot w(\mu,\varepsilon)) \\ &= c_1\mu^{-2}\left(\|\dot z_1\|^2 - (\ddot z_1, U_{\mu^{-1}}w(\mu,\varepsilon))\right) = c_1\mu^{-2}\left(\|\dot z_1\|^2 - \|\ddot z_1\|O(1)\varepsilon\right). \end{aligned}$$

Finally we see that for small $\varepsilon > 0$ the number c_1 must be zero and the assertion follows. □

In view of the above result we end up facing a finite dimensional problem as it is enough to find critical points of the functional $\Phi_\varepsilon : (0,\infty) \to \mathbb{R}$ given by $f_\varepsilon|_{Z^\varepsilon}$.

4.6 Study of Φ_ε

In this section we will assume that the critical manifold is non-degenerate, i.e. (4.9), such that the functional Φ_ε is defined. To find critical points of $\Phi_\varepsilon = f_\varepsilon|_{Z^\varepsilon}$ it is convenient to introduce the functional Γ given below.

Lemma 4.7. *Suppose a, b, p, λ satisfy (4.2), (4.7), and (4.8) holds. Then*

$$\Phi_\varepsilon(\mu) = f_0(z_1) - \varepsilon\Gamma(\mu) + o(\varepsilon), \tag{4.49}$$

where $\Gamma(\mu) = G(z_\mu)$. In particular, there is $C > 0$, independent of μ and ε, such that

$$\begin{aligned}|\Phi_\varepsilon(\mu) - &(f_0(z_1) - \varepsilon\Gamma(\mu))| \\ &\le C\big(\|w(\varepsilon,\mu)\|^2 + (1+|\varepsilon|)\|w(\varepsilon,\mu)\|^p + |\varepsilon|\|w(\varepsilon,\mu)\|\big).\end{aligned} \tag{4.50}$$

Consequently, if there exist $0 < \mu_1 < \mu_2 < \mu_3 < \infty$ such that

$$\Gamma(\mu_2) > \max(\Gamma(\mu_1), \Gamma(\mu_3)) \text{ or } \Gamma(\mu_2) < \min(\Gamma(\mu_1), \Gamma(\mu_3)) \tag{4.51}$$

then Φ_ε will have a critical point, if $\varepsilon > 0$ is sufficiently small.

Proof. Note that for all $\mu > 0$ we have $f_0(z_\mu) = f_0(z_1)$,

$$\|z_\mu\|^2 = \int_{\mathbb{R}^N} \frac{z_\mu^p}{|x|^{bp}} \text{ and } \big(z_\mu, w(\varepsilon,\mu)\big) = \int_{\mathbb{R}^N} \frac{z_\mu^{p-1} w(\varepsilon,\mu)}{|x|^{bp}}. \tag{4.52}$$

From (4.52) we infer

$$\Phi_\varepsilon(\mu) = \frac{1}{2}\int_{\mathbb{R}^N} \frac{z_\mu^p}{|x|^{bp}} + \frac{1}{2}\|w(\varepsilon,\mu)\|^2 + \int_{\mathbb{R}^N} \frac{z_\mu^{p-1} w(\varepsilon,\mu)}{|x|^{bp}} - \frac{1}{p}\int_{\mathbb{R}^N} \frac{(1+\varepsilon k)\big(z_\mu + w(\varepsilon,\mu)\big)^p}{|x|^{bp}}$$

and

$$f_0(z_1) = f_0(z_\mu) = \frac{1}{2}\|z_\mu\|^2 - \frac{1}{p}\int_{\mathbb{R}^N} \frac{z_\mu^p}{|x|^{bp}} = \left(\frac{1}{2} - \frac{1}{p}\right)\int_{\mathbb{R}^N} \frac{z_\mu^p}{|x|^{bp}}.$$

Hence

$$\Phi_\varepsilon(\mu) = f_0(z_1) - \varepsilon\Gamma(\mu) + \frac{1}{2}\|w(\varepsilon,\mu)\|^2 - \frac{1}{p}H_\varepsilon(\mu), \tag{4.53}$$

where

$$H_\varepsilon(\mu) = \int_{\mathbb{R}^N} \frac{\big(z_\mu + w(\varepsilon,\mu)\big)^p - z_\mu^p - p\, z_\mu^{p-1} w(\varepsilon,\mu) + \varepsilon k\big((z_\mu + w(\varepsilon,\mu))^p - z_\mu^p\big)}{|x|^{bp}}.$$

Using the inequality

$$(z+w)^{s-1} - z^{s-1} - (p-1)z^{s-2}w \le \begin{cases} C(z^{s-3}w^2 + w^{s-1}) & \text{if } s \ge 3 \\ C\, w^{s-1} & \text{if } 2 < s < 3, \end{cases}$$

where $C = C(s) > 0$, with $s = p + 1$ and Hölder's inequality we have for some $c_2, c_3 > 0$

$$\begin{aligned}
|H_\varepsilon(\mu)| \le& \int_{\mathbb{R}^N} \frac{\left|(z_\mu + w(\varepsilon,\mu))^p - z_\mu^p - p\, z_\mu^{p-1} w(\varepsilon,\mu)\right|}{|x|^{bp}} \\
&+ |\varepsilon| \int_{\mathbb{R}^N} \frac{|k| \left((z_\mu + w(\varepsilon,\mu))^p - z_\mu^p\right)}{|x|^{bp}} \\
\le& c_2 \left[\int_{\mathbb{R}^N} \frac{z_\mu^{p-2} w^2(\varepsilon,\mu)}{|x|^{bp}} + \int_{\mathbb{R}^N} \frac{|w(\varepsilon,\mu)|^p}{|x|^{bp}} \right. \\
&\left. + |\varepsilon| \int_{\mathbb{R}^N} \frac{z_\mu^{p-1} |w(\varepsilon,\mu)|}{|x|^{bp}} + |\varepsilon| \int_{\mathbb{R}^N} \frac{|w(\varepsilon,\mu)|^p}{|x|^{bp}} \right] \\
\le& c_3 \left[\|w(\varepsilon,\mu)\|^2 + (1 + |\varepsilon|) \|w(\varepsilon,\mu)\|^p + |\varepsilon| \|w(\varepsilon,\mu)\| \right]
\end{aligned}$$

and the claim follows. □

Although it is convenient to study only the reduced functional Γ instead of Φ_ε, it may lead in some cases to a loss of information, i.e. Γ may be constant even if k is a non-constant function. This is due to the fact that the critical manifold consists of radially symmetric functions. Thus Γ is constant for every k that has constant mean-value over spheres, i.e.

$$\frac{1}{r^{N-1}} \int_{\partial B_r(0)} k(x)\, dS(x) \equiv \text{const} \qquad \forall r > 0.$$

In this case we have to study the functional $\Phi_\varepsilon(\mu)$ directly.

Proof of Theorem 4.3. By (4.11), (4.37), (4.41) and (4.50)

$$\lim_{\mu \to 0^+} \Phi_\varepsilon(\mu) = \lim_{\mu \to +\infty} \Phi_\varepsilon(\mu) = f_0(z_1).$$

Hence, either the functional $\Phi_\varepsilon \equiv f_0(z_1)$, and we obtain infinitely many critical points, or $\Phi_\varepsilon \not\equiv f_0(z_1)$ and Φ_ε has at least a global maximum or minimum. In any case Φ_ε has a critical point that provides a solution of $(\mathcal{P}_{a,b,\lambda})$. □

The next lemma shows that it is possible (and convenient) to extend the C^2- functional Γ by continuity to $\mu = 0$. The proof of this fact is analogous to the one in [11, Lem. 3.4] and we omit it here.

Lemma 4.8. *Under the assumptions of Lemma 4.7*

$$\Gamma(0) := \lim_{\mu \to 0} \Gamma(\mu) = k(0) \frac{1}{p} \|z_1\|_{p,b}^p \quad \textit{and} \tag{4.54}$$

$$\frac{1}{p} \liminf_{|x| \to \infty} k(x) \|z_1\|_{p,b}^p \le \liminf_{\mu \to \infty} \Gamma(\mu) \le \limsup_{\mu \to \infty} \Gamma(\mu) \le \frac{1}{p} \limsup_{|x| \to \infty} k(x) \|z_1\|_{p,b}^p. \tag{4.55}$$

If, moreover, (4.12) holds we obtain

$$\Gamma'(0) = 0 \text{ and } \Gamma''(0) = \frac{\Delta k(0)}{Np} \int |x|^2 \frac{z_1(x)^p}{|x|^{bp}}. \tag{4.56}$$

Proof of Theorem 4.4. To see that assumptions (4.13) and (4.14) give rise to a critical point we use the functional Γ. Condition (4.13) and Lemma 4.8 imply that Γ has a global maximum strictly bigger than $\Gamma(0)$ and $\limsup_{\mu\to\infty} \Gamma(\mu)$. Consequently Φ_ε has a critical point in view of Lemma 4.7. The same reasoning yields a critical point under condition (4.14). □

5 Blow-up analysis for Caffarelli-Kohn-Nirenberg equations

In this chapter we continue the study of degenerate critical elliptic equations of Caffarelli-Kohn-Nirenberg type by presenting the compactness and existence results obtained by M. Schneider and the author in [49]. By means of blow-up analysis techniques, we first prove an a-priori estimate in a weighted space of continuous functions. Then from this compactness result, the existence of a solution to our problem is proved by exploiting the homotopy invariance of the Leray-Schauder degree.

As in the previous chapter, let us consider the following equation in $\mathbb{R}^N$ in dimension $N \geq 3$,

$$-\operatorname{div}(|x|^{-2\alpha}\nabla v) - \frac{\lambda}{|x|^{2(1+\alpha)}} v = K(x)\frac{v^{p-1}}{|x|^{\beta p}}, \quad v \geq 0, \quad v \in \mathcal{D}^{1,2}_{\alpha}(\mathbb{R}^N) \setminus \{0\}, \tag{5.1}$$

where $K \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is positive and

$$\alpha < \frac{N-2}{2}, \quad \alpha \leq \beta < \alpha + 1, \tag{5.2}$$

$$\lambda < \left(\frac{N-2-2\alpha}{2}\right)^2, \quad p = p(\alpha, \beta) = \frac{2N}{N-2(1+\alpha-\beta)}. \tag{5.3}$$

We look for weak solutions in the space $\mathcal{D}^{1,2}_{\alpha}(\mathbb{R}^N)$ (see page 30 for the definition of $\mathcal{D}^{1,2}_{\alpha}(\mathbb{R}^N)$). Let us define

$$a(\alpha, \lambda) := \frac{N-2}{2} - \sqrt{\left(\frac{N-2-2\alpha}{2}\right)^2 - \lambda} \text{ and } b(\alpha, \beta, \lambda) := \beta + a(\alpha, \lambda) - \alpha. \tag{5.4}$$

The change of variable $u(x) = |x|^{a-\alpha} v(x)$ shows that equation (5.1) is equivalent to

$$-\operatorname{div}(|x|^{-2a}\nabla u) = K(x)\frac{u^{p-1}}{|x|^{bp}}, \quad u \geq 0, \quad u \in D^{1,2}_{a}(\mathbb{R}^N) \setminus \{0\}, \tag{5.5}$$

where $a = a(\alpha, \lambda)$ and $b = b(\alpha, \beta, \lambda)$. More precisely the following lemma holds.

Lemma 5.1. *v is a solution to (5.1) if and only if $u(x) = |x|^{a-\alpha} v(x)$ solves (5.5), where $a = a(\alpha, \lambda)$ and $b = b(\alpha, \beta, \lambda)$ are given in* (5.4).

Proof. By standard elliptic regularity u and v are $C^2(\mathbb{R}^N \setminus \{0\})$. Consequently we may compute for $x \in \mathbb{R}^N \setminus \{0\}$

$$\operatorname{div}(|x|^{-2a}\nabla u(x)) = (a-\alpha)(N-a-\alpha-2)|x|^{-a-\alpha-2}v(x) + |x|^{\alpha-a}\operatorname{div}(|x|^{-2\alpha}\nabla v)$$

and hence, in view of (5.1)

$$-\operatorname{div}(|x|^{-2a}\nabla u(x)) = \big[\lambda + (\alpha-a)(N-2-\alpha-a)\big]\frac{u(x)}{|x|^{2a+2}} + K(x)\frac{u^{p-1}}{|x|^{p(a-\alpha+\beta)}}.$$

From (5.4) we have that $\lambda + (\alpha-a)(N-2-\alpha-a) = 0$ and $a-\alpha+\beta = b$. Since $C^\infty(\mathbb{R}^N \setminus \{0\})$ is dense in $\mathcal{D}_\alpha^{1,2}(\mathbb{R}^N)$ and $D_a^{1,2}(\mathbb{R}^N)$ (see [34]), the lemma is thereby proved. □

Clearly, if we replace α by a and β by b then (5.2)-(5.3) still hold and $p(\alpha,\beta) = p(a,b)$. We will write in the sequel for short that a, b and p satisfy (5.2)-(5.3). We will mainly deal with equation (5.5) and look for weak solutions in $D_a^{1,2}(\mathbb{R}^N)$. The advantage of working with (5.5) instead of (5.1) is that we know from the regularity results discussed in Section 5.1 that weak solutions of (5.5) are Hölder-continuous in $\mathbb{R}^N$ whereas solutions to (5.1), as our analysis shows, behave (possibly singular) like $|x|^{\alpha-a}$ at the origin. The main difficulty in facing problem (5.5) is the lack of compactness as p is the critical exponent in the related Caffarelli-Kohn-Nirenberg inequality. More precisely, if K is a positive constant equation (5.5) is invariant under the action of the non-compact group of dilations, in the sense that if u is a solution of (5.5) then for any positive μ the dilated function

$$\mu^{-\frac{N-2-2a}{2}}u(x/\mu)$$

is also a solution with the same norm in $D_a^{1,2}(\mathbb{R}^N)$. As already pointed out in the previous chapter, the dilation invariance gives rise to a non-compact, one dimensional manifold of solutions for $K \equiv K(0)$ (see in (5.15) below).

The first theorem in this chapter, provides sufficient conditions on K ensuring compactness of the set of solutions by means of an a-priori bound in a weighted space E defined by

$$E := D_a^{1,2}(\mathbb{R}^N) \cap C^0(\mathbb{R}^N, (1+|x|^{N-2-2a})),$$

where

$$C^0(\mathbb{R}^N, (1+|x|^{N-2-2a})) := \{u \in C^0(\mathbb{R}^N) : u(x)(1+|x|^{N-2-2a}) \in L^\infty(\mathbb{R}^N)\}$$

is equipped with the norm

$$||u||_{C^0(\mathbb{R}^N,(1+|x|^{N-2-2a}))} := \sup_{x\in\mathbb{R}^N} |u(x)|(1+|x|^{N-2-2a}).$$

We endow E with the norm

$$||u||_E = ||u||_{D_a^{1,2}(\mathbb{R}^N)} + ||u||_{C^0(\mathbb{R}^N,(1+|x|^{N-2-2a}))}.$$

The uniform bound in E of the set of solutions to (5.5) will provide the necessary compactness needed in the sequel. We formulate the compactness result in terms of α, β and v the parameters of equation (5.1), where we started from. Let us set

$$\tilde{K}(x) := K(x/|x|^2). \tag{5.6}$$

Theorem 5.1. (Compactness) *Let α, β, λ satisfy (5.2)-(5.3) and*

$$\lambda \geq -\alpha(N-2-\alpha), \tag{5.7}$$

$$\left(\frac{N-2-2\alpha}{2}\right)^2 - 1 < \lambda, \tag{5.8}$$

$$\beta > \alpha, \quad p > \frac{2}{\sqrt{\left(\frac{N-2-2\alpha}{2}\right)^2 - \lambda}}. \tag{5.9}$$

Suppose $K \in C^2(\mathbb{R}^N)$ satisfies

$$\tilde{K} \in C^2(\mathbb{R}^N), \text{ where } \tilde{K}(x) \text{ is defined in (5.6)}, \tag{5.10}$$

$$\nabla K(0) = 0, \quad \Delta K(0) \neq 0, \quad \text{and} \quad \nabla \tilde{K}(0) = 0, \quad \Delta \tilde{K}(0) \neq 0, \tag{5.11}$$

and for some positive constant A_1

$$1/A_1 \leq K(x), \quad \forall x \in \mathbb{R}^N. \tag{5.12}$$

Then there is $C_K > 0$ such that for any $t \in (0,1]$ and any solution v_t of

$$\begin{aligned} -\operatorname{div}(|x|^{-2\alpha}\nabla v) - \frac{\lambda}{|x|^{2(1+\alpha)}} v = (1 + t\,(K(x) - 1))\frac{v^{p-1}}{|x|^{\beta p}}, \\ v \geq 0, \ v \in \mathcal{D}^{1,2}_{\alpha}(\mathbb{R}^N) \setminus \{0\}, \end{aligned} \tag{5.13$_t$}$$

we have $\||x|^{a-\alpha} v_t\|_E < C_K$ and

$$C_K^{-1} < |x|^{a-\alpha}(1 + |x|^{N-2-2a}) v_t(x) < C_K \text{ in } \mathbb{R}^N \setminus \{0\}. \tag{5.14}$$

To prove the above compactness result we adapt the arguments of [69] to carry out a fine blow-up analysis for (5.5). Assumptions (5.7)-(5.9) imply

$$(5.7) \implies a \geq 0, \qquad (5.8) \implies \tfrac{N-4}{2} < a < \tfrac{N-2}{2}$$

$$(5.9) \implies \tfrac{4}{N-2-2a} < p < 2^* = \tfrac{2N}{N-2}.$$

A key ingredient is the exact knowledge of the solutions to the limit problem with $K \equiv$ const, which is only available for $a \geq 0$. In [37] (see also [82]) it is shown through the method of moving planes that if $a \geq 0$ then any locally bounded positive solution in $C^2(\mathbb{R}^N \setminus \{0\})$ of (5.5) with $K \equiv K(0)$ is of the form

$$z^{a,b}_{K(0),\mu} := \mu^{-\frac{N-2-2a}{2}} z^{a,b}_{K(0)}\left(\frac{x}{\mu}\right), \quad \mu > 0, \tag{5.15}$$

where $z^{a,b}_{K(0)} = z^{a,b}_1\big(x\,K(0)^{\frac{2}{(p-2)(N-2-2a)}}\big)$ and $z^{a,b}_1$ is explicitly given by

$$z^{a,b}_1(x) = \left[1 + \frac{N-2(1+a-b)}{N(N-2-2a)^2}|x|^{\frac{2(1+a-b)(N-2-2a)}{N-2(1+a-b)}}\right]^{-\frac{N-2(1+a-b)}{2(1+a-b)}}.$$

As we pointed out in the previous chapter, for $a < 0$ the set of positive solutions becomes more and more complicated as $a \to -\infty$ due to the existence of non-radially symmetric solutions (see Theorem 4.1 and Corollary 4.1). Up to now, our blow-up analysis is only available for $p < 2^*$; the case $p = 2^*$ presents additional difficulties because besides the blow-up profile $z^{a,b}_1$ a second blow-up profile described by the usual Aubin-Talenti instanton of Yamabe-type equations may occur. The further restrictions on a, p and K should be compared to the so-called flatness-assumptions in problems of prescribing scalar curvature.

Non-existence results for equation (5.5) can be obtained using a Pohozaev-type identity, i.e. any solution u to (5.5) satisfies the following identity

$$\int_{\mathbb{R}^N} (\nabla K(x)\cdot x)\frac{u^p}{|x|^{bp}}\,dx = 0,$$

provided the integral is convergent and K is bounded and smooth enough (see Corollary 5.3). This implies that there are no such solutions if $\nabla K(x)\cdot x$ does not change sign in $\mathbb{R}^N$ and K is not constant.

The above compactness result allows us to exploit the homotopy invariance of the Leray-Schauder degree to pass from t small to $t = 1$ in $(5.13)_t$. We compute the degree of positive solutions to $(5.13)_t$ for small t using the Melnikov-type function introduced in [7, 8] (see Chapter 3) and show that it equals (see Theorem 5.10)

$$-\frac{\operatorname{sgn}\Delta K(0) + \operatorname{sgn}\Delta \tilde{K}(0)}{2}.$$

In particular, we prove the following existence result.

Theorem 5.2. (Existence) *Under the assumptions of Theorem 5.1, if, moreover,* $p > 3$ *and*

$$\operatorname{sgn}\Delta K(0) + \operatorname{sgn}\Delta \tilde{K}(0) \neq 0$$

then equation (5.1) has a positive solution v *such that* $|x|^{a-\alpha}v \in B_{C_K}(0) \subset E$ *and* v *satisfies (5.14).*

The assumption $p > 3$ is essentially technical and yields C^3 regularity of the functional associated to the problem which is needed in the computation of the degree.

Remark 5.1. If we drop the assumption $\alpha < \frac{N-2}{2}$ we may still change the variables $u(x) = |x|^{a-\alpha}v(x)$, where a is given in (5.4), and we still obtain weak solutions u of (5.5) in $\mathcal{D}^{1,2}_a(\mathbb{R}^N)$. But in this case the transformation $v(x) = |x|^{\alpha-a}u(x)$ gives rise only to classical solutions of (5.1) in $\mathbb{R}^N \setminus \{0\}$ but not to distributional solutions in the whole $\mathbb{R}^N$.

The chapter is organized as follows. In Section 5.1 we establish Hölder continuity of weak solutions to (5.5). In Section 5.2 we prove a Pohozaev type identity for equation (5.5). In Section 5.3 we introduce the notion of isolated and isolated simple blow-up point which was first introduced by Schoen [75] and provide the main local blow-up analysis. In Section 5.4 we prove Theorem 5.1 by combining the Pohozaev type identity with the results of our local blow-up analysis. The last part is devoted to the computation of the Leray-Schauder degree and to the proof of the existence theorem.

5.1 Regularity for equations of Caffarelli-Kohn-Nirenberg type

In this section we present the regularity results for degenerate elliptic equations of Caffarelli-Kohn-Nirenberg type obtained by M. Schneider and the author in [50]. Our purpose is to establish Hölder continuity of weak solutions to

$$-\mathrm{div}\,(|x|^{-2a}\nabla u) = \frac{f}{|x|^{bp}}, \text{ in } \Omega \subset \mathbb{R}^N, \tag{5.16}$$

where Ω is an open set, $N \geq 3$ and a, b, and p satisfy (5.2) and (5.3). For a given weight ω we denote by $L^p(\Omega, \omega)$ the space of functions u such that

$$||u||^p_{L^p(\Omega,\omega)} := \int_\Omega |u|^p \omega(x) < \infty.$$

The space $H^1_a(\Omega)$ is defined to be the closure of $C^\infty(\bar{\Omega})$ with respect to

$$||u||^2_{H^1_a(\Omega)} := \int_\Omega |x|^{-2a} \left(|\nabla u|^2 + |u|^2\right).$$

In order to study problem (5.5) for non-constant functions K using for instance a degree argument (as we will do in Section 5.5), Hölder estimates for weak solutions of (5.16) are an important tool. Regularity properties of weak solution to degenerate elliptic problems with more general weighted operators of the form $\mathrm{div}\,(\omega(x)\nabla(\cdot))$ are studied in [42, 43, 60] (see also the references mentioned there). The classes of weights ω treated there include the class (QC) of weights

$$\omega(x) = |\mathrm{det} T'|^{1-2/N},$$

where $T : \mathbb{R}^N \to \mathbb{R}^N$ is quasi-conformal (see [43, 60] for a definition). In fact our weights $|\cdot|^{-2a}$ are associated with quasi-conformal transformations $T_a(x) := x|x|^{-2a/(N-2)}$. The right-hand sides studied in [42, 43, 60] are either zero or in divergence form, e.g. Hölder continuity of weak solutions to

$$-\mathrm{div}\,(|x|^{-2a}\nabla u) = \mathrm{div}\,(F) \text{ in } \Omega$$

is established in [42] assuming $|F||x|^{2a} \in L^p(\Omega, |x|^{-2a})$ for some $p > \max\{N - 2a, N, 2\}$. We derive Hölder estimates for weak solutions to (5.16) in terms of f, because a sharp relation of the integrability of f and its representation in divergence form F in the various weighted spaces is not obvious. We compare weak solutions of (5.16) with μ_a-harmonic functions, which are by definition weak solutions of

$$-\mathrm{div}\,(|x|^{-2a}\nabla u) = 0 \text{ in } \Omega,$$

and for which Hölder regularity is known (see for instance [60]) and prove

Theorem 5.3. *Suppose $\Omega \subset \mathbb{R}^N$ is a bounded domain and $u \in H^1_a(\Omega)$ weakly solves (5.16), that is*

$$\int_\Omega |x|^{-2a}\nabla u \cdot \nabla\varphi\, dx = \int_\Omega |x|^{-bp} f\varphi\, dx \quad \forall \varphi \in H^1_{0,a}(\Omega).$$

Assume a, b and p satisfy (5.2) and (5.3), $b < a+1$, and $f \in L^s(\Omega, |x|^{-bp})$ for some $s > p/(p-2)$. Then $u \in C^{0,\alpha}$ for any $\alpha \in (0,1)$ satisfying

$$\alpha < \min(\alpha_h, 1) \text{ and } \alpha < \begin{cases} \left(\frac{N-2}{2} - a\right)\left(p - 2 - \frac{p}{s}\right) & \text{if } b \geq 0 \\ \frac{N}{p}\left(p - 2 - \frac{p}{s}\right) & \text{if } b < 0 \end{cases},$$

where α_h is the regularity of μ_a-harmonic functions given in Theorem 5.5 below. Moreover, for any $\Omega' \subset\subset \Omega$ there is a constant $C = C(N, a, \alpha, \Omega, dist(\Omega', \Omega), s)$ such that

$$\sup_{\Omega'} |u| + \sup_{\substack{x,y\in\Omega' \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \leq C\left\{\|u\|_{L^2(\Omega,\, d\mu_a)} + \|f\|_{L^s(\Omega,|x|^{-bp})}\right\}.$$

For the nonlinear problem (5.5) we use a De Giorgi-Moser type iteration procedure as in [26] and obtain

Theorem 5.4. *Let a, b and p satisfy (5.2) and (5.3) and $u \in D^{1,2}_a(\mathbb{R}^N)$ be a weak solution to*

$$-\mathrm{div}\,(|x|^{-2a}\nabla u) = K(x)\frac{|u|^{p-2}u}{|x|^{bp}}, \quad x \in \Omega \tag{5.17}$$

where $K \in L^\infty(\Omega)$. Then $u \in L^s_{\mathrm{loc}}(\Omega, |x|^{-bp})$ for any $s \in [p, +\infty[$. Moreover, u is Hölder continuous with Hölder exponent given in Theorem 5.3.

Let us mention that in [39] weighted q-Laplacian equations of the form

$$-\mathrm{div}\,(|x|^{-qa}|\nabla u|^{q-2}\nabla u) = g$$

are studied. Under assumption (5.2) and (5.3) Hölder regularity of weak solutions to equation

$$-\mathrm{div}\,(|x|^{-qa}\nabla u) = \frac{f}{|x|^{bp}}, \text{ in } \Omega \subset \mathbb{R}^N,$$

is shown if $a = b$, $a > -1$, and $f \in L^s(\Omega, |x|^{-bp})$ for some $s > p/(p-2)$. Theorem 5.3 extends this result to the full range for a and b in the case $q = 2$.

5.1.1 Properties of weighted measures

We collect some properties of the weighted measure $\mu_a := |x|^{-2a}\,dx$ and μ_a-harmonic functions. We refer to [43, 60] for the proofs.

- The measure μ_a satisfies the doubling property, i.e. for every $\tau \in (0,1)$ there exists a constant $C_{(5.18)}(\tau)$ such that

$$\mu_a(B(x,r)) \le C_{(5.18)}(\tau)\mu_a(B(x,\tau r)) \tag{5.18}$$

- A Poincaré-type inequality holds, i.e. there is a positive constant $C_{(5.19)}$ such that any $u \in D_a^{1,2}(\mathbb{R}^N)$ satisfies

$$⨍_{B_r(x)} |u - u_{x,r}|^2\, d\mu_a \le C_{(5.19)}\, r^2 ⨍_{B_r(x)} |\nabla u|^2\, d\mu_a, \tag{5.19}$$

  where $u_{x,r}$ denotes the weighted mean-value

$$u_{x,r} := ⨍_{B_r(x)} u\, d\mu_a = \frac{1}{\mu_a(B_r(x))} \int_{B_r(x)} u(x)\, d\mu_a.$$

Concerning μ_a- harmonic functions we have the following results.

Theorem 5.5 (Thm. 3.34 in [60](p. 65), Thm. 6.6 in [60](p. 111)). *There are constants $C_{(5.20)}(N,a)$ and $\alpha_h = \alpha_h(N,a) \in (0,1)$ such that if u is μ_a-harmonic in $B_r(x_0) \subset \mathbb{R}^N$ and $0 < \rho < r$ then*

$$\text{ess-sup}_{B(x_0,\frac{r}{2})}|u| \le C_{(5.20)} ⨍_{B(x_0,r)} |u|^2\, d\mu_a, \tag{5.20}$$

$$osc(u, B_\rho(x_0)) \le 2^{\alpha_h} \left(\frac{\rho}{r}\right)^{\alpha} osc(u, B_r(x_0)), \tag{5.21}$$

where $osc(u, B_\rho(x_0)) := \sup_{B_\rho(x_0)} u - \inf_{B_\rho(x_0)} u$ denotes the oscilllation of u. Consequently, μ_a-harmonic functions are Hölder continuous.

We will call a function $u \in D^{1,2}_{a,loc}(\mathbb{R}^N)$ weakly super μ_a-harmonic in Ω, if for all nonnegative $\varphi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega |x|^{-2a}\nabla u \nabla \varphi \ge 0. \tag{5.22}$$

Theorem 5.6 (Thm 3.51 in [60](p. 70)). *There exist positive constants $s = s(N,a)$ and $C_{(5.23)} = C_{(5.23)}(N,a)$ such that if u is nonnegative and weakly super μ_a-harmonic in Ω and $B_{2r}(x_0) \subset \Omega$ we have*

$$\text{ess inf}_{B_{\frac{r}{2}}(x_0)} u \ge C_{(5.23)} \left(⨍_{B_r(x_0)} u^s\, d\mu_a \right)^{\frac{1}{s}}. \tag{5.23}$$

We use the two theorems above to derive

Lemma 5.2. *For any ball $B_r(x_0)$ there is a constant $C_{(5.24)}(B_r(x_0))$ such that any μ_a-harmonic function u in $B_r(x_0)$ satisfies*

$$\fint_{B_\rho(x_0)} |\nabla u|^2 \, d\mu_a \leq C_{(5.24)} \left(\frac{\rho}{r}\right)^{2\alpha_h - 2} \fint_{B_r(x_0)} |\nabla u|^2 \, d\mu_a, \tag{5.24}$$

where $\alpha_h \in (0,1)$ is given in Theorem 5.5.

Proof. To prove the claim we may assume that $0 < \rho < (1/4)r$ and that u has mean-value zero in $B_r(x_0)$. We take a cut-off function $\xi \in C_c^\infty(B_{2\rho}(x_0))$ such that $\xi \equiv 1$ in $B_\rho(x_0)$, $0 \leq \xi \leq 1$, $\|\nabla \xi\|_\infty \leq 2\rho^{-1}$ and define $\phi := \xi^2(u - u(x_0))$. Testing with ϕ and using Hölder's inequality we get

$$\int_{B_r(x_0)} |\nabla u|^2 \xi^2 \, d\mu_a \leq \int_{B_r(x_0)} |\nabla \xi|^2 (u - u(x_0))^2 \, d\mu_a \tag{5.25}$$

$$\leq \|u - u(x_0)\|^2_{\infty, B_{2\rho}(x_0)} \mu_a\big(B_{2\rho}(x_0)\big)\rho^{-2}. \tag{5.26}$$

From (5.26) and (5.21) we infer

$$\fint_{B_\rho(x_0)} |\nabla u|^2 \, d\mu_a \leq C\left(\frac{\rho}{r}\right)^{2\alpha} osc(u, B_{\frac{r}{2}}(x_0))^2 \rho^{-2} \leq C\left(\frac{\rho}{r}\right)^{2\alpha} \rho^{-2} \fint_{B_r(x_0)} |u|^2 \, d\mu_a.$$

Finally, since u has mean-value zero in $B_r(x_0)$ the Poincaré inequality (5.19) yields the claim. □

5.1.2 Growth of local integrals

We give a weighted version of the Campanato-Morrey characterization of Hölder continuous functions.

Theorem 5.7. *Suppose $\Omega \subset \mathbb{R}^N$ is a bounded domain and $u \in L^2(\Omega, d\mu_a)$ satisfies*

$$\fint_{B_r(x)} |u(y) - u_{x,r}|^2 \, d\mu_a \leq M^2 r^{2\alpha} \quad \text{for any } B_r(x) \subset \Omega \tag{5.27}$$

and some $\alpha \in (0,1)$. Then $u \in C^{0,\alpha}(\Omega)$ and for any $\Omega' \subset\subset \Omega$ there holds

$$\sup_{\Omega'} |u| + \sup_{\substack{x,y \in \Omega' \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq C\Big\{M + \|u\|_{L^2(\Omega, |x|^{-2a})}\Big\}$$

where $C = C(N, a, \alpha, \Omega, dist(\Omega', \Omega))$ is a positive constant independent of u.

Proof. Denote $R_0 = \operatorname{dist}(\Omega', \partial\Omega)$. Using the triangle inequality and integrating in $B_{r_1}(x_0)$ we have for any $x_0 \in \Omega'$ and $0 < r_1 < r_2 \le R_0$

$$\begin{aligned}&\left|u_{x_0,r_1} - u_{x_0,r_2}\right|^2 \\ &\quad \le \frac{2}{\mu_a(B_{r_1}(x_0))}\left\{\int_{B_{r_1}(x_0)} \left|u(x) - u_{x_0,r_1}\right|^2 d\mu_a + \int_{B_{r_2}(x_0)} \left|u(x) - u_{x_0,r_2}\right|^2 d\mu_a\right\}.\end{aligned}$$

Using assumption (5.27) we obtain

$$\left|u_{x_0,r_1} - u_{x_0,r_2}\right|^2 \le \frac{2M^2}{\mu_a(B_{r_1}(x_0))}\{\mu_a(B_{r_1}(x_0))r_1^{2\alpha} + \mu_a(B_{r_2}(x_0))r_2^{2\alpha}\}. \tag{5.28}$$

For any $R \le R_0$ we take $r_1 = 2^{-(i+1)}R$ and $r_2 = 2^{-i}R$ in (5.28). The doubling property (5.18) then gives

$$\left|u_{x_0,2^{-(i+1)}R} - u_{x_0,2^{-i}R}\right| \le 2M^2\Big(1 + C_{(5.18)}(N,a)2^{2a}\Big)2^{-2(i+1)\alpha}R^{2\alpha}.$$

We sum up and get for $h < k$

$$\left|u_{x_0,2^{-h}R} - u_{x_0,2^{-k}R}\right| \le \frac{C(N,a,\alpha)M}{2^{h\alpha}}R^{\alpha}. \tag{5.29}$$

The above estimates prove that $\{u_{x_0,2^{-i}R}\}_{i\in\mathbb{N}} \subset \mathbb{R}$ is a Cauchy sequence in $\mathbb{R}$, hence it converges to some limit, denoted as $\hat{u}(x_0)$. The value of $\hat{u}(x_0)$ is independent of R, which may be seen by analogous estimates. Consequently, from (5.29) we have that

$$\left|u_{x_0,r} - \hat{u}(x_0)\right| \le C(N,a,\alpha)Mr^{\alpha} \quad \forall\, x_0 \in \Omega'. \tag{5.30}$$

By the Lebesgue theorem we infer

$$u_{x,r} = \frac{|B_r(x)|}{\int_{B_r(x)} |y|^{-2a}\,dy} \cdot \frac{\int_{B_r(x)} |y|^{-2a}u(y)\,dy}{\frac{1}{|B_r(x)|}} \underset{r\to 0+}{\longrightarrow} u(x), \quad \text{a. e. in } \Omega'.$$

Hence $\hat{u} = u$ a. e. in Ω' and (5.30) gives

$$\left|u_{x_0,r} - u(x_0)\right| \le C(N,a,\alpha)Mr^{\alpha} \quad \forall\, x_0 \in \Omega', \tag{5.31}$$

which implies that $u_{x,r}$ converges to u uniformly in Ω'. Since $x \mapsto u_{x,r}$ is a continuous function, we conclude that u is continuous in Ω'. From (5.31) we have

$$|u(x)| \le C(N,a,\alpha)MR^{\alpha} + |u_{x,R}| \quad \forall\, x \in \Omega',\ \forall\, R \le R_0.$$

Thus u is bounded in Ω' with the estimate

$$\sup_{\Omega'} |u| \le c(N,a,\alpha,\Omega,\operatorname{dist}(\Omega',\Omega))\Big\{M + \|u\|_{L^2(\Omega,|x|^{-2a})}\Big\}. \tag{5.32}$$

Let us now prove that u is Hölder continuous. Let $x, y \in \Omega'$ with $|x - y| = R < \frac{R_0}{2}$. Assume that $|x| < |y|$. Then we have

$$|u(x) - u(y)| \le |u(x) - u_{x,2R}| + |u(y) - u_{y,2R}| + |u_{x,2R} - u_{y,2R}|.$$

The first two terms are estimated by (5.31), whereas for the last term we have

$$|u_{x,2R} - u_{y,2R}|^2 \le 2\{|u_{x,2R} - u(\xi)|^2 + |u(\xi) - u_{y,2R}|^2\}$$

and integrating with respect to ξ over $B_{2R}(x) \cap B_{2R}(y) \supseteq B_R(x)$ we obtain

$$|u_{x,2R} - u_{y,2R}|^2 \le \frac{2}{\mu_a(B_R(x))}\Big(M^2\mu_a(B_{2R}(x))2^{2\alpha}R^{2\alpha} + M^2\mu_a(B_{2R}(y))2^{2\alpha}R^{2\alpha}\Big).$$

Since x is closer to 0 than y, we have that $\mu_a(B_{2R}(y)) \le \mu_a(B_{2R}(x))$ and hence

$$|u(x) - u(y)| \le C(N,a,\alpha)M|x-y|^\alpha.$$

If $|x-y| > \frac{R_0}{2}$ we can use estimate (5.32) thus finding

$$|u(x) - u(y)| \le 2\sup_{\Omega'}|u| \le c2^\alpha\left[M + \frac{1}{R_0^\alpha}\|u\|_{L^2(\Omega,|x|^{-2a})}\right]|x-y|^\alpha.$$

The proof is thereby complete. □

Corollary 5.1. *Suppose $\Omega \subset \mathbb{R}^N$ is a bounded domain and $u \in H_a^1(\Omega)$ satisfies*

$$\fint_{B_r(x)} |\nabla u|^2\, d\mu_a \le M^2 r^{2\alpha-2} \quad \text{for any } B_r(x) \subset \Omega$$

and some $\alpha \in (0,1)$. Then $u \in C^{0,\alpha}(\Omega)$ and for any $\Omega' \subset\subset \Omega$ there holds

$$\sup_{\Omega'}|u| + \sup_{\substack{x,y\in\Omega'\\ x\ne y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \le c\Big\{M + \|u\|_{L^2(\Omega,|x|^{-2a})}\Big\}$$

where $c = c(N,a,\alpha,\Omega,dist(\Omega',\Omega)) > 0$.

Proof. The proof follows from Theorem 5.7 and the Poincaré type inequality in (5.19). □

Lemma 5.3. *Suppose Φ be a nonnegative and nondecreasing functions on $[0,R]$ such that*

$$\Phi(\rho) \le A_1\ \mu_a(B_\rho(x))\mu_a(B_r(x))^{-1}\left(\frac{\rho}{r}\right)^{-\alpha}\Phi(r) + A_2\ \mu_a(B_r(x))r^{-\beta}, \tag{5.33}$$

for any $0 < \rho \le r \le R$, where A_1, A_2, α and β are positive constants satisfying $\alpha < \beta$. Then for any $\gamma \in (\alpha,\beta)$ there exists a constant $C_{(5.34)} = C_{(5.34)}(A_1,\alpha,\beta,\gamma)$ independent of x and r such that for $0 < \rho \le r \le R$

$$\Phi(\rho) \le C_{(5.34)}\left(\mu_a(B_\rho(x))\mu_a(B_r(x))^{-1}\left(\frac{\rho}{r}\right)^{-\gamma}\Phi(r) + A_2\ \mu_a(B_\rho(x))\rho^{-\beta}\right). \tag{5.34}$$

Proof. Fix $\gamma \in (\alpha, \beta)$ and set $\tau := \min(A_1^{-1/(\gamma-\alpha)}, 1/2)$. Then we have for $0 < r \leq R$

$$\Phi(\tau r) \leq \mu_a(B_{\tau r}(x))\mu_a(B_r(x))^{-1}\tau^{-\gamma}\Phi(r) + A_2 r^{-\beta}\mu_a(B_r(x)).$$

Hence we may estimate for $k \in \mathbb{N}$

$$\begin{aligned}\Phi(\tau^{k+1}r) &\leq \mu_a(B_{\tau^{k-1}r}(x))\mu_a(B_{\tau^k r}(x))^{-1}\tau^{\beta-\gamma})\\ &\leq \mu_a(B_{\tau^{k+1}r}(x))\mu_a(B_r(x))^{-1}\tau^{-(k+1)\gamma}\Phi(r) + A_2(\tau^k r)^{-\beta}\mu_a(B_{\tau^k r}(x))\\ &\quad \cdot \sum_{j=0}^{k} \underbrace{\mu_a(B_{\tau^{k+1}r}(x))\mu_a(B_{\tau^k r}(x))^{-1}}_{\leq 1}\ \underbrace{\mu_a(B_{\tau^{k-j}r}(x))\mu_a(B_{\tau^{k-j+1}r}(x))^{-1}}_{\leq C_{(5.18)}(\tau)\text{ by (5.18)}}\tau^{(\beta-\gamma)j}\\ &\leq C_{(5.18)}(\tau)\mu_a(B_{\tau^{k+2}r}(x))\mu_a(B_r(x))^{-1}\tau^{-(k+1)\gamma}\Phi(r)\\ &\quad + \frac{A_2 C_{(5.18)}(\tau)}{1-\tau^{\beta-\gamma}}(\tau^k r)^{-\beta}\mu_a(B_{\tau^k r}(x)).\end{aligned}$$

For $0 < \rho \leq r$ we may choose $k \in \mathbb{N}$ such that $\tau^{k+2}r < \rho < \tau^{k+1}r$ and obtain

$$\begin{aligned}\Phi(\rho) &\leq \Phi(\tau^{k+1}r)\\ &\leq C_{(5.18)}(\tau)\mu_a(B_\rho(x))\mu_a(B_r(x))^{-1}\left(\frac{\rho}{r}\right)^{-\gamma}\Phi(r) + \frac{A_2 C^3_{(5.18)}(\tau)}{\tau(1-\tau^{\beta-\gamma})}\mu_a(B_\rho(x))\rho^{-\beta}.\end{aligned}$$

The proof is now complete. □

In order to prove Theorem 5.3 we need the following lemma.

Lemma 5.4. *Let a, b and p satisfy (5.2) and (5.3) and $\varepsilon > 0$. Then we have*

$$\left(\int_{B_\rho(x_0)} |x|^{-bp}\right)^{2/p+\varepsilon} \leq C_{(5.35)}(N)\,\rho^{-2+\varepsilon N}(\max(\rho, |x_0|))^{-\varepsilon bp}\int_{B_\rho(x_0)} |x|^{-2a}. \quad (5.35)$$

Proof. Let us distinguish two cases.
Case 1: $\rho \geq |x_0|/2$. Since $(N-bp)(2/p+\varepsilon) = N-2-2a+\varepsilon(N-bp)$ we obtain

$$\left(\int_{B_\rho(x_0)} |x|^{-bp}\right)^{2/p+\varepsilon} \leq \left(\int_{B_{3\rho}(0)} |x|^{-bp}\right)^{2/p+\varepsilon} = C_1(N)\rho^{N-2-2a+\varepsilon(N-bp)}.$$

From the doubling property (5.18) and the fact that $B_\rho(0) \subset B_{4\rho}(x_0)$ we infer,

$$\begin{aligned}\rho^{\varepsilon(N-bp)-2}\int_{B_\rho(x_0)} |x|^{-2a} &\geq c\rho^{\varepsilon(N-bp)-2}\int_{B_{4\rho}(x_0)} |x|^{-2a}\\ &\geq c\rho^{\varepsilon(N-bp)-2}\int_{B_\rho(0)} |x|^{-2a} = C_2(N)\rho^{N-2-2a+\varepsilon(N-bp)}\end{aligned}$$

and the claim follows in Case 1.
Case 2: $\rho < |x_0|/2$. We have for all $x \in B_r(x_0)$ that $1/2 \leq |x|/|x_0| \leq 3$. Consequently,

$$\left(\int_{B_\rho(x_0)} |x|^{-bp}\right)^{2/p+\varepsilon} \le C_3(N)\rho^{N(2/p+\varepsilon)}|x_0|^{-2b-\varepsilon bp}$$
$$\le C_3(N)\rho^{N-2}|x_0|^{-2a}\rho^{2N/p-N+2}|x_0|^{-2(b-a)}\rho^{N\varepsilon}|x_0|^{-\varepsilon bp}$$

From $r < |x_0|/2$ we get

$$\left(\int_{B_\rho(x_0)} |x|^{-bp}\right)^{2/p+\varepsilon} \le C_3(N)\rho^{N-2}|x_0|^{-2a}\rho^{N\varepsilon}|x_0|^{-\varepsilon bp}$$
$$\le C_4(N)\left(\int_{B_\rho(x_0)} |x|^{-2a}\right)\rho^{N\varepsilon}|x_0|^{-\varepsilon bp},$$

which ends the proof. □

Proof of Theorem 5.3. Let $w \in u + H^1_{0,a}(B_r(x_0))$ be the unique solution to the Dirichlet problem

$$\begin{cases} -\mathrm{div}\,(|x|^{-2a}\nabla w) = 0 & \text{in } B_r(x_0) \\ w\big|_{\partial B_r(x_0)} = u. \end{cases} \tag{5.36}$$

Clearly the function $v = u - w \in H^1_{0,a}(B_r(x_0))$ weakly solves

$$-\mathrm{div}\,(|x|^{-2a}\nabla v) = \frac{f}{|x|^{bp}} \quad \text{in } B_r(x_0).$$

Testing the above equation with v and using Hölder's inequality and (4.1), we get

$$\int_{B_r(x_0)} |\nabla v|^2\, d\mu_a x \le C_{a,b,N}\left(\int_{B_r(x_0)} |x|^{-bp}|f|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\left(\int_{B_r(x_0)} |\nabla v|^2\, d\mu_a\right)^{\frac{1}{2}}.$$

Since $f \in L^s(\Omega, |x|^{-bp})$ for some $s > p/(p-2)$ we may use Hölder's inequality with conjugate exponents $s(p-1)/p$ and

$$\frac{s(p-1)}{s(p-1)-p} = \frac{p-1}{1+(p-2-(p/s))}$$

and Lemma 5.4 with $\varepsilon = 2(p-2-p/s)/p$ to obtain

$$\int_{B_r(x_0)} |\nabla v|^2\, d\mu_a \le C^2_{a,b,N}\left(\int_{B_r(x_0)} |x|^{-bp}|f|^s\right)^{\frac{2}{s}}\left(\int_{B_r(x_0)} |x|^{-bp}\right)^{\frac{2}{p}+\varepsilon}$$
$$\le Cr^{-2+N\varepsilon}\max(r, |x_0|)^{-bp\varepsilon}\mu_a(B_r(x_0))\left(\int_{B_r(x_0)} |x|^{-bp}|f|^s\right)^{2/s}. \tag{5.37}$$

From (5.24) we deduce for any $0 < \rho \le r$

$$\begin{aligned}
&\fint_{B_\rho(x_0)} |\nabla u|^2 \, d\mu_a \le 4 \fint_{B_\rho(x_0)} |\nabla w|^2 \, d\mu_a + 4 \fint_{B_\rho(x_0)} |\nabla v|^2 \, d\mu_a \\
&\le 4C_{(5.24)} \left(\frac{\rho}{r}\right)^{2\alpha_h - 2} \fint_{B_r(x_0)} |\nabla w|^2 \, d\mu_a + 4\mu_a(B_\rho(x_0))^{-1} \int_{B_r(x_0)} |\nabla v|^2 \, d\mu_a.
\end{aligned} \tag{5.38}$$

Since w minimizes the Dirichlet integral we may replace w in (5.38) by u. If we further estimate the integral containing v in (5.38) using (5.37) we get

$$\begin{aligned}
\int_{B_\rho(x_0)} |\nabla u|^2 \, d\mu_a \le C\Big(& \mu_a(B_\rho(x_0))\mu_a(B_r(x_0))^{-1} \left(\frac{\rho}{r}\right)^{-2+2\alpha_h} \int_{B_r(x_0)} |\nabla u|^2 \, d\mu_a \\
&+ r^{-2+N\varepsilon} \max(r, |x_0|)^{-bp\varepsilon} \mu_a(B_r(x_0)) \|f\|^2_{L^s(B_r(x_0), |x|^{-bp})} \Big).
\end{aligned}$$

We estimate the term $\max(r, |x_0|)^{-bp\varepsilon}$ by $r^{-bp\varepsilon}$ if $b \ge 0$ and in the case $b < 0$ by a constant $C(\Omega)$. For the rest of the proof we will consider the more interesting situation $b \ge 0$. The case $b < 0$ may be treated analogously.

Lemma 5.3 with $\Phi(\rho) := \int_{B_\rho(x_0)} |\nabla u|^2 \, d\mu_a$ gives for $0 < \rho < r \le r_0 := \operatorname{dist}(x_0, \partial\Omega)$

$$\begin{aligned}
\int_{B_\rho(x_0)} |\nabla u|^2 \, d\mu_a \le C(\alpha)\Big(& \frac{\mu_a(B_\rho(x_0))}{\mu_a(B_r(x_0))} \left(\frac{\rho}{r}\right)^{-2+2\alpha} \int_{B_r(x_0)} |\nabla u|^2 \, d\mu_a \\
&+ \rho^{-2+(N-bp)\varepsilon} \mu_a(B_\rho(x_0) \|f\|^2_{L^s(\Omega, |x|^{-bp})} \Big).
\end{aligned}$$

We take a cut-off function $\xi \in C_c^\infty(B_r(x_0))$ such that $\xi \equiv 1$ in $B_{r/2}(x_0)$, $0 \le \xi \le 1$, $\|\nabla \xi\|_\infty \le 2r^{-1}$ and define $\phi := \xi^2 u$. Testing with ϕ and using (4.1) and Hölder's inequality we get

$$\begin{aligned}
\int_{B_r(x_0)} |\nabla u|^2 \xi^2 \, d\mu_a \le\ & C_{a,b,N} \|f\|_{L^{\frac{p}{p-1}}(\Omega, |x|^{-bp})} \|\xi^2 u\|_{\mathcal{D}_a^{1,2}(\Omega)} \\
&+ \|u \nabla \xi\|_{L^2(B_r(x_0),\, d\mu_a)} \|\nabla u \xi\|_{L^2(B_r(x_0),\, d\mu_a)}.
\end{aligned}$$

We divide by $\|\nabla u \xi\|_{L^2(B_r(x_0),\, d\mu_a)}$ and obtain

$$\begin{aligned}
\int_{B_r(x_0)} |\nabla u|^2 \xi^2 \, d\mu_a &\le \frac{2C^2_{a,b,N} \|f\|^2_{L^{\frac{p}{p-1}}(\Omega, |x|^{-bp})} \|\xi^2 u\|^2_{\mathcal{D}_a^{1,2}(\Omega)}}{\|\nabla u \xi\|^2_{L^2(B_r(x_0),\, d\mu_a)}} + \|u \nabla \xi\|^2_{L^2(B_r(x_0),\, d\mu_a)} \\
&\le \frac{2C^2_{a,b,N} \|f\|^2_{L^{\frac{p}{p-1}}(\Omega, |x|^{-bp})} \|\xi^2 u\|^2_{\mathcal{D}_a^{1,2}(\Omega)}}{\|\nabla u \xi\|^2_{L^2(B_r(x_0),\, d\mu_a)}} + 4r^{-2} \|u\|^2_{L^2(B_r(x_0),\, d\mu_a)} \\
&\le 2C^2_{a,b,N} \|f\|^2_{L^{\frac{p}{p-1}}(\Omega, |x|^{-bp})} \left(\frac{8r^{-2} \|u\|^2_{L^2(B_r(x_0),\, d\mu_a)}}{\|\nabla u \xi\|^2_{L^2(B_r(x_0),\, d\mu_a)}} + 2 \right) \\
&\quad + 4r^{-2} \|u\|^2_{L^2(B_r(x_0),\, d\mu_a)}.
\end{aligned}$$

Thus

$$\int_{B_r(x_0)} |\nabla u|^2 \xi^2 \, d\mu_a \le 8\mathcal{C}_{a,b,N}^2 \|f\|^2_{L^{\frac{p}{p-1}}(\Omega,|x|^{-bp})} + 4r^{-2}\|u\|^2_{L^2(B_r(x_0),\, d\mu_a)}.$$

Taking $r = r_0$ we have for $0 < \rho \le r_0/2$

$$⨍_{B_\rho(x_0)} |\nabla u|^2 \, d\mu_a \le C(N, a, \Omega, r_0)\left(\int_\Omega |u|^2 \, d\mu_a + \|f\|^2_{L^{\frac{2p}{p-2}}(\Omega,|x|^{-bp}\,dx)}\right)\rho^{-2+2\alpha}.$$

From the above estimate, Corollary 5.1 and, the fact that $(N - bp)2/p = N - 2 - 2a$ we derive the desired conclusion. □

5.1.3 A Brezis-Kato type Lemma

As in [26] we prove the following lemma to start an iteration procedure.

Lemma 5.5. *Let $\Omega \subset \mathbb{R}^N$ be open, a, b and p satisfy (5.2) and (5.3), and $q > 2$. Suppose $u \in D_a^{1,2}(\mathbb{R}^N) \cap L^q(\Omega, |x|^{-bp})$ is a weak solution of*

$$-\mathrm{div}\left(|x|^{-2a}\nabla u\right) - \frac{V(x)}{|x|^{bp}}\, u = \frac{f(x)}{|x|^{bp}} \quad \text{in } \Omega, \tag{5.39}$$

where $f \in L^q(\Omega, |x|^{-bp})$ and V satisfies for some $\ell > 0$

$$\int_{|V(x)|\ge\ell} |x|^{-bp}|V|^{\frac{p}{p-2}} + \int_{\Omega\setminus B_\ell(0)} |x|^{-bp}|V|^{\frac{p}{p-2}} \le \min\left\{\frac{1}{8}\mathcal{C}_{a,b}^{-1}, \frac{2}{q+4}\mathcal{C}_{a,b,N}^{-1}\right\}^{\frac{p}{p-2}}. \tag{5.40}$$

Then for any $\Omega' \subset\subset \Omega$

$$\|u\|_{L^{\frac{pq}{2}}(\Omega',|x|^{-bp})} \le C(\ell, q, \Omega')\|u\|_{L^q(\Omega,|x|^{-bp})} + \|f\|_{L^q(\Omega,|x|^{-bp})}. \tag{5.41}$$

If, moreover, $u \in D_a^{1,2}(\mathbb{R}^N)$ then (5.41) remains true for $\Omega' = \Omega$.

Proof. Hölder's inequality, (4.1) and (5.40) give for any $w \in D_a^{1,2}(\mathbb{R}^N)$

$$\begin{aligned}\int_\Omega |x|^{-bp}|V(x)|w^2 &\le \ell \int_{\substack{|V(x)|\le\ell \text{ and}\\ x\in\Omega\cap B_\ell(0)}} |x|^{-bp}w^2 + \int_{\substack{|V(x)|\ge\ell \text{ or}\\ x\in\Omega\setminus B_\ell(0)}} |x|^{-b(p-2)}|V||x|^{-2b}w^2\\
&\le \ell\int_{\Omega\cap B_\ell(0)} |x|^{-bp}w^2 + \left(\int_\Omega \frac{w^p}{|x|^{bp}}\right)^{\frac{2}{p}}\left(\int_{\substack{|V(x)|\ge\ell \text{ or}\\ x\in\Omega\setminus B_\ell(0)}} |x|^{-bp}|V|^{\frac{p}{p-2}}\right)^{\frac{p-2}{p}}\\
&\le \ell\int_{\Omega\cap B_\ell(0)} |x|^{-bp}w^2 + \min\left(\frac{1}{8}, \frac{2}{q+4}\right)\int_\Omega |x|^{-2a}|\nabla w|^2.\end{aligned} \tag{5.42}$$

Suppose now that $u \in L^q(\Omega, |x|^{-bp})$. Fix $\Omega' \subset\subset \Omega$ and a nonnegative cut-off function η, such that $\operatorname{supp}(\eta) \subset\subset \Omega$ and $\eta \equiv 1$ on Ω'. Set $u^n := \min(n, |u|) \in D_a^{1,2}(\mathbb{R}^N)$ and test (5.39) with $u(u^n)^{q-2}\eta^2 \in D_a^{1,2}(\mathbb{R}^N)$. This leads to

$$\begin{aligned}(q-2)\int_\Omega |x|^{-2a}\eta^2|\nabla u^n|^2(u^n)^{q-2} + \int_\Omega |x|^{-2a}\eta^2(u^n)^{q-2}|\nabla u|^2 \\ = \int_\Omega |x|^{-bp}V(x)\eta^2u^2(u^n)^{q-2} + \int_\Omega |x|^{-bp}f\eta^2(u^n)^{q-2}u \\ - 2\int_\Omega |x|^{-2a}\nabla u\eta(u_n)^{q-2}\nabla\eta u.\end{aligned}$$

We use the elementary inequality $2ab \le 1/2a^2 + 4b^2$ and obtain

$$(q-2)\int_\Omega |x|^{-2a}\eta^2|\nabla u^n|^2(u^n)^{q-2} + \frac{1}{2}\int_\Omega |x|^{-2a}\eta^2(u^n)^{q-2}|\nabla u|^2 \tag{5.43}$$

$$\begin{aligned}\le \int_\Omega |x|^{-bp}V(x)\eta^2u^2(u^n)^{q-2} + \int_\Omega |x|^{-bp}f\eta^2(u^n)^{q-2}u \\ + 4\int_\Omega |x|^{-2a}|\nabla\eta|^2u^2(u_n)^{q-2}.\end{aligned} \tag{5.44}$$

Furthermore, an explicit calculation gives

$$\begin{aligned}\left|\nabla\left((u^n)^{\frac{q}{2}-1}u\eta\right)\right|^2 \le \frac{(q+4)(q-2)}{4}(u^n)^{q-2}\eta^2|\nabla u^n|^2 + 2(u^n)^{q-2}|\nabla u|^2\eta^2 \\ + 2(u^n)^{q-2}u^2|\nabla\eta|^2 + \frac{q-2}{2}(u^n)^q|\nabla\eta|^2.\end{aligned} \tag{5.45}$$

Let $C(q) := \min\left\{\frac{1}{4}, \frac{4}{q+4}\right\}$. From (5.43) and (5.45) we get

$$\begin{aligned}C(q)\int_\Omega \frac{\left|\nabla\left((u^n)^{\frac{q}{2}-1}u\eta\right)\right|^2}{|x|^{2a}} \le 2(2+C(q))\int_\Omega \frac{(u^n)^{q-2}u^2|\nabla\eta|^2}{|x|^{2a}} \\ + C(q)\frac{q-2}{2}\int_\Omega \frac{(u^n)^q|\nabla\eta|^2}{|x|^{2a}} \\ + \int_\Omega \frac{f(x)}{|x|^{bp}}\eta^2(u^n)^{q-2}u + \int_\Omega \frac{V(x)}{|x|^{bp}}\eta^2(u^n)^{q-2}u^2.\end{aligned} \tag{5.46}$$

Estimate (5.42) applied to $\eta(u^n)^{\frac{q}{2}-1}u$ gives

$$\begin{aligned}\int_\Omega \frac{V^+\left(\eta(u^n)^{\frac{q}{2}-1}u\right)^2}{|x|^{bp}} \le \frac{C(q)}{2}\int_\Omega \frac{\left|\nabla(\eta(u^n)^{\frac{q}{2}-1}u)\right|^2}{|x|^{2a}} \\ + \ell\int_{\Omega\cap B_\ell(0)} \frac{(u^n)^{q-2}u^2\eta^2}{|x|^{bp}}.\end{aligned} \tag{5.47}$$

By Hölder's inequality and convexity we arrive at

$$\int_\Omega \frac{|f|\eta}{|x|^{\frac{bp}{q}}} \frac{(u^n)^{q-2}u\eta}{|x|^{\frac{bp(q-1)}{q}}} \leq \frac{q-1}{q} \int_\Omega |x|^{-bp} \eta^{\frac{q}{q-1}} |u^n|^{q\frac{q-2}{q-1}} |u|^{\frac{q}{q-1}} + \frac{1}{q} \int_\Omega |x|^{-bp} |f|^q \eta^q. \tag{5.48}$$

We use (5.47) and (5.48) to estimate the terms with f and V in (5.46), then (4.1) yields

$$\begin{aligned}
\left(\int_\Omega |x|^{-bp} |u^n|^{(\frac{q}{2}-1)p} |u|^p \eta^p \right)^{2/p} &\leq \frac{2\mathcal{C}_{a,b,N}(q-1)}{C(q)q} \int_\Omega |x|^{-bp} \eta^{\frac{q}{q-1}} |u^n|^{q\frac{q-2}{q-1}} |u|^{\frac{q}{q-1}} + \frac{2\mathcal{C}_{a,b}}{C(q)q} \int_\Omega |x|^{-bp} |f(x)|^q \eta^q \\
&+ \frac{2\ell \mathcal{C}_{a,b,N}}{C(q)} \int_{\Omega \cap B_\ell(0)} |x|^{-bp} \eta^2 |u^n|^{q-2} u^2 \\
&+ \frac{4\mathcal{C}_{a,b}(2 + C(q))}{C(q)} \int_\Omega |x|^{-2a} |u^n|^{q-2} u^2 |\nabla \eta|^2 \\
&+ \mathcal{C}_{a,b,N}(q-2) \int_\Omega |x|^{-2a} |u^n|^q |\nabla \eta|^2.
\end{aligned}$$

Letting $n \to \infty$ in the above inequality (5.41) follows. Observe that if $u \in D_a^{1,2}(\mathbb{R}^N)$ then we need not to use the cut-off η and the same analysis as above gives the estimate (5.41) for $\Omega' = \Omega$. The lemma is thereby proved. □

Remark 5.2. By Vitali's theorem V belongs to $L^{p/(p-2)}(\Omega, |x|^{-bp})$ if and only if there exists ℓ such that (5.40) is satisfied. But the constant in (5.41) depends uniformly on ℓ and not on the norm of V in $L^{p/(p-2)}(\Omega, |x|^{-bp})$.

Proof of Theorem 5.4. We apply Lemma 5.5 with $f = 0$ and $V(x) = K(x)|u|^{p-2}$. Starting with $q = p$, the lemma gives $u \in L_{\text{loc}}^{\frac{p^2}{2}}(\Omega, |x|^{-bp})$. Taking $q = \frac{p^2}{2}$, we find $u \in L_{\text{loc}}^{\frac{p^3}{4}}(\Omega, |x|^{-bp})$. Iterating the process, we obtain that $u \in L_{\text{loc}}^{p^{k+1}/2^k}(\Omega, |x|^{-bp})$ for any k. Let $k_0 \in \mathbb{N}$ be such that $(p/2)^{k_0} \geq 2(p-1)/(p-2)$, then after k_0 steps we find that $u \in L_{\text{loc}}^{\frac{2p(p-1)}{p-2}}(\Omega)$. Having this high integrability we may use Theorem 5.3 with $f(x) = K(x)|u|^{p-2}u$ to get the desired regularity of u. □

5.2 A Pohozaev-type identity

Theorem 5.8. *Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \geq 3$) with smooth boundary, a, b, and p satisfy (5.2)-(5.3), $K \in C^1(\overline{\Omega})$ and $u \in D_a^{1,2}(\mathbb{R}^N)$ be a weak positive solution of*

$$-\text{div}\,(|x|^{-2a} \nabla u) = K(x) \frac{u^{p-1}}{|x|^{bp}}, \quad x \in \Omega. \tag{5.49}$$

There holds

$$\frac{1}{p}\int_{\Omega}(\nabla K\cdot x)\,\frac{u^p}{|x|^{bp}}\,dx-\frac{1}{p}\int_{\partial\Omega}K(x)\frac{u^p}{|x|^{bp}}x\cdot\nu=\frac{N-2-2a}{2}\int_{\partial\Omega}|x|^{-2a}u\nabla u\cdot\nu$$
$$-\frac{1}{2}\int_{\partial\Omega}|x|^{-2a}|\nabla u|^2x\cdot\nu+\int_{\partial\Omega}|x|^{-2a}(x\cdot\nabla u)(\nabla u\cdot\nu)$$

where ν denotes the unit normal of the boundary.

Proof. Note that

$$\int_0^1 ds\int_{\partial B_s(0)}\left[\frac{|K(x)|u^p}{|x|^{bp}}+\frac{|\nabla u|^2}{|x|^{2a}}\right]=\int_{B_1(0)}\left[\frac{|K(x)|u^p}{|x|^{bp}}+\frac{|\nabla u|^2}{|x|^{2a}}\right]<\infty$$

which implies that there exists a sequence $\varepsilon_n\to 0^+$ such that

$$\varepsilon_n\int_{\partial B_{\varepsilon_n}(0)}\left[\frac{|K(x)|u^p}{|x|^{bp}}+\frac{|\nabla u|^2}{|x|^{2a}}\right]\to 0 \tag{5.50}$$

as $n\to\infty$. Let $\Omega_{\varepsilon_n}:=\Omega\backslash B_{\varepsilon_n}(0)$. Multiplying equation (5.49) by $x\cdot\nabla u$ and integrating over Ω_{ε_n} we obtain

$$-\sum_{j,k=1}^{N}\int_{\Omega_{\varepsilon_n}}\frac{\partial}{\partial x_j}\left(|x|^{-2a}\frac{\partial u}{\partial x_j}\right)x_k\,\frac{\partial u}{\partial x_k}\,dx=\sum_{k=1}^{N}\int_{\Omega_{\varepsilon_n}}x_k\,\frac{\partial u}{\partial x_k}\,K(x)\frac{u^{p-1}}{|x|^{bp}}\,dx. \tag{5.51}$$

Let us first consider the right-hand side of (5.51). Integrating by parts we have

$$\sum_{k=1}^{N}\int_{\Omega_{\varepsilon_n}}x_k\,\frac{\partial u}{\partial x_k}\,K(x)\frac{u^{p-1}}{|x|^{bp}}\,dx=\left(b-\frac{N}{p}\right)\int_{\Omega_{\varepsilon_n}}K(x)\frac{u^p}{|x|^{bp}}\,dx$$
$$-\frac{1}{p}\sum_{k=1}^{N}\int_{\Omega_{\varepsilon_n}}u^px_k\frac{\partial K}{\partial x_k}|x|^{-bp}+\frac{1}{p}\sum_{k=1}^{N}\int_{\partial\Omega_{\varepsilon_n}}x_k\nu_k\frac{K(x)u^p}{|x|^{bp}}. \tag{5.52}$$

Integrating by parts in the left-hand side of (5.51), we obtain

$$-\sum_{j,k=1}^{N}\int_{\Omega_{\varepsilon_n}}\frac{\partial}{\partial x_j}\left(|x|^{-2a}\frac{\partial u}{\partial x_j}\right)x_k\,\frac{\partial u}{\partial x_k}\,dx=-\frac{N-2-2a}{2}\int_{\Omega_{\varepsilon_n}}|x|^{-2a}|\nabla u|^2\,dx$$
$$+\frac{1}{2}\int_{\partial\Omega_{\varepsilon_n}}|x|^{-2a}|\nabla u|^2x\cdot\nu-\int_{\partial\Omega_{\varepsilon_n}}|x|^{-2a}(x\cdot\nabla u)(\nabla u\cdot\nu). \tag{5.53}$$

From (5.51), (5.52), and (5.53), we have

$$\begin{aligned}&\left(b-\frac{N}{p}\right)\int_{\Omega_{\varepsilon_n}} K(x)\frac{u^p}{|x|^{bp}}\,dx-\frac{1}{p}\sum_{k=1}^{N}\int_{\Omega_{\varepsilon_n}} u^p x_k\frac{\partial K}{\partial x_k}|x|^{-bp}\\&+\frac{1}{p}\sum_{k=1}^{N}\int_{\partial\Omega_{\varepsilon_n}} x_k\nu_k\frac{K(x)u^p}{|x|^{bp}}\\&=-\frac{N-2-2a}{2}\int_{\Omega_{\varepsilon_n}}|x|^{-2a}|\nabla u|^2\,dx+\frac{1}{2}\int_{\partial\Omega_{\varepsilon_n}}|x|^{-2a}|\nabla u|^2 x\cdot\nu\\&-\int_{\partial\Omega_{\varepsilon_n}}|x|^{-2a}(x\cdot\nabla u)(\nabla u\cdot\nu).\end{aligned}$$

Because of the integrability of $|x|^{-bp}u^p$ and of $|x|^{-2a}|\nabla u|^2$, it is clear that

$$\begin{aligned}&\left(b-\frac{N}{p}\right)\int_{\Omega_{\varepsilon_n}} K(x)\frac{u^p}{|x|^{bp}}\,dx-\frac{1}{p}\sum_{k=1}^{N}\int_{\Omega_{\varepsilon_n}} u^p x_k\frac{\partial K}{\partial x_k}|x|^{-bp}\\&\underset{\varepsilon\to0^+}{\longrightarrow}\left(b-\frac{N}{p}\right)\int_{\Omega} K(x)\frac{u^p}{|x|^{bp}}\,dx-\frac{1}{p}\sum_{k=1}^{N}\int_{\Omega} u^p x_k\frac{\partial K}{\partial x_k}|x|^{-bp}\end{aligned}$$

and

$$\int_{\Omega_{\varepsilon_n}}|x|^{-2a}|\nabla u|^2\,dx\underset{\varepsilon\to0^+}{\longrightarrow}\int_{\Omega}|x|^{-2a}|\nabla u|^2\,dx.$$

Hence, in view of (5.50), we have

$$\begin{aligned}&\left(b-\frac{N}{p}\right)\int_{\Omega} K(x)\frac{u^p}{|x|^{bp}}\,dx-\frac{1}{p}\sum_{k=1}^{N}\int_{\Omega} u^p x_k\frac{\partial K}{\partial x_k}|x|^{-bp}\\&+\frac{1}{p}\sum_{k=1}^{N}\int_{\partial\Omega} x_k\nu_k\frac{K(x)u^p}{|x|^{bp}}\\&=-\frac{N-2-2a}{2}\int_{\Omega}|x|^{-2a}|\nabla u|^2\,dx+\frac{1}{2}\int_{\partial\Omega}|x|^{-2a}|\nabla u|^2 x\cdot\nu\\&-\int_{\partial\Omega}|x|^{-2a}(x\cdot\nabla u)(\nabla u\cdot\nu).\end{aligned}\tag{5.54}$$

Multiplying equation (5.49) by u and integrating by parts, we have

$$\int_{\Omega}|x|^{-2a}|\nabla u|^2\,dx=\int_{\partial\Omega}|x|^{-2a}u\frac{\partial u}{\partial\nu}+\int_{\Omega}K(x)\frac{u^p}{|x|^{bp}}\,dx.\tag{5.55}$$

The conclusion follows from (5.54), (5.55), and from the identity $\frac{N-bp}{p}-\frac{N-2-2a}{2}=0$. □

Corollary 5.2. *If a, b, and p satisfy (5.2)-(5.3), $K\in C^1(\overline{B}_\sigma)$ and u be a weak positive solution in $D_a^{1,2}(\mathbb{R}^N)$ of*

$$-\operatorname{div}(|x|^{-2a}\nabla u) = K(x)\frac{u^{p-1}}{|x|^{bp}}, \quad x \in B_\sigma := \{x \in \mathbb{R}^N : |x| < \sigma\} \tag{5.56}$$

then

$$\frac{1}{p}\int_{B_\sigma} (\nabla K \cdot x)\frac{u^p}{|x|^{bp}}\,dx - \frac{\sigma}{p}\int_{\partial B_\sigma} K(x)\frac{u^p}{|x|^{bp}} = \int_{\partial B_\sigma} B(\sigma, x, u, \nabla u) \tag{5.57}$$

where

$$B(\sigma, x, u, \nabla u) = \frac{N-2-2a}{2}|x|^{-2a}u\frac{\partial u}{\partial \nu} - \frac{\sigma}{2}|x|^{-2a}|\nabla u|^2 + \sigma|x|^{-2a}\left(\frac{\partial u}{\partial \nu}\right)^2.$$

Corollary 5.3. *Let u be a weak positive solution in $D_a^{1,2}(\mathbb{R}^N)$ of*

$$-\operatorname{div}(|x|^{-2a}\nabla u) = K(x)\frac{u^{p-1}}{|x|^{bp}}, \quad x \in \mathbb{R}^N$$

where a, b, and p satisfy (5.2)-(5.3) and $K \in L^\infty \cap C^1(\mathbb{R}^N)$, $|\nabla K(x)\cdot x| \le$ const. Then

$$\int_{\mathbb{R}^N} (\nabla K(x)\cdot x)\frac{u^p}{|x|^{bp}}\,dx = 0. \tag{5.58}$$

Proof. Since

$$\int_0^{+\infty} ds \int_{\partial B_s}\left[\frac{|K(x)|u^p}{|x|^{bp}} + \frac{|\nabla u|^2}{|x|^{2a}}\right] = \int_{\mathbb{R}^N}\left[\frac{|K(x)|u^p}{|x|^{bp}} + \frac{|\nabla u|^2}{|x|^{-2a}}\right] < \infty$$

there exists a sequence $R_n \to +\infty$ such that

$$R_n\int_{\partial B_{R_n}}\left[\frac{|K(x)|u^p}{|x|^{bp}} + \frac{|\nabla u|^2}{|x|^{2a}}\right] \underset{n\to\infty}{\longrightarrow} 0. \tag{5.59}$$

From Corollary 5.2 we have that

$$\begin{aligned}\frac{1}{p}\int_{B_{R_n}} (\nabla K \cdot x)\frac{u^p}{|x|^{bp}}\,dx =& \frac{R_n}{p}\int_{\partial B_{R_n}} K(x)\frac{u^p}{|x|^{bp}} + \frac{N-2-2a}{2}\int_{\partial B_{R_n}} |x|^{-2a}u\frac{\partial u}{\partial \nu}\\ &- \frac{R_n}{2}\int_{\partial B_{R_n}} |x|^{-2a}|\nabla u|^2 + R_n\int_{\partial B_{R_n}} |x|^{-2a}\left(\frac{\partial u}{\partial \nu}\right)^2.\end{aligned} \tag{5.60}$$

In view of (5.59) and noting that from Hölder inequality

$$\begin{aligned}\int_{\partial B_{R_n}} |x|^{-2a}u\frac{\partial u}{\partial \nu} &= R_n^{b-a}\int_{\partial B_{R_n}} \frac{u}{|x|^b}\cdot\frac{\nabla u\cdot\nu}{|x|^a}\\ &\le |\mathbb{S}^N|^{\frac{p-2}{2p}} R_n^{b-a+\frac{(N-1)(p-2)}{2p}-\frac{1}{p}-\frac{1}{2}}\left(R_n\int_{\partial B_{R_n}}\frac{u^p}{|x|^{bp}}\right)^{\frac{1}{p}}\left(R_n\int_{\partial B_{R_n}}\frac{|\nabla u|^2}{|x|^{2a}}\right)^{\frac{1}{2}}\\ &= |\mathbb{S}^N|^{\frac{p-2}{2p}}\left(R_n\int_{\partial B_{R_n}}\frac{u^p}{|x|^{bp}}\right)^{\frac{1}{p}}\left(R_n\int_{\partial B_{R_n}}\frac{|\nabla u|^2}{|x|^{2a}}\right)^{\frac{1}{2}}\end{aligned}$$

we can pass to the limit in (5.60) thus obtaining the claim. □

It is easy to check that the boundary term $B(\sigma, x, u, \nabla u)$ has the following properties.

Proposition 5.1.

(i) *For* $u(x) = |x|^{2+2a-N}$, $\sigma > 0$, $B(\sigma, x, u, \nabla u) = 0$ *for all* $x \in \partial B_\sigma$.

(ii) *For* $u(x) = |x|^{2+2a-N} + A + \zeta(x)$, *with* $A > 0$ *and* $\zeta(x)$ *some function differentiable near* 0 *satisfying* $\zeta(0) = 0$, *there exists* $\bar{\sigma}$ *such that*

$$B(\sigma, x, u, \nabla u) < 0 \quad \text{for all } x \in \partial B_\sigma \text{ and } 0 < \sigma < \bar{\sigma}$$

and

$$\lim_{\sigma \to 0} \int_{\partial B_\sigma} B(\sigma, x, u, \nabla u) = -\frac{(N-2-2a)^2}{2} A |\mathbb{S}^{N-1}|.$$

5.3 Local blow-up analysis

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, a, b, and p satisfy (5.2)-(5.3), and $\{K_i\}_i \subset C(\Omega)$ satisfy, for some constant $A_1 > 0$,

$$1/A_1 \le K_i(x) \le A_1, \quad \forall x \in \Omega \quad \text{and} \quad K_i \to K \text{ uniformly in } \Omega. \tag{5.61}$$

Moreover, we will assume throughout this section that $a \ge 0$. We are interested in the family of problems

$$-\mathrm{div}\,(|x|^{-2a}\nabla u) = K_i(x)\frac{u^{p-1}}{|x|^{bp}} \quad \text{weakly in } \Omega, \quad u > 0 \text{ in } \Omega, \; u \in D_a^{1,2}(\mathbb{R}^N). \tag{P_i}$$

Definition 5.1. *Let* $\{u_i\}_i$ *be a sequence of solutions of* (P_i). *We say that* $0 \in \Omega$ *is a blow-up point of* $\{u_i\}_i$ *if there exists a sequence* $\{x_i\}_i$ *converging to* 0 *such that*

$$u_i(x_i) \to +\infty \quad \text{and} \quad u_i(x_i)^{\frac{2}{N-2-2a}}|x_i| \to 0 \quad \text{as } i \to +\infty. \tag{5.62}$$

Definition 5.2. *Let* $\{u_i\}_i$ *be a sequence of solutions of* (P_i). *The point* 0 *is said to be an isolated blow-up point of* $\{u_i\}_i$ *if there exist* $0 < \bar{r} < dist(0, \partial\Omega)$, $\bar{C} > 0$, *and a sequence* $\{x_i\}_i$ *converging to* 0 *such that* $u_i(x_i) \to +\infty$, $u_i(x_i)^{\frac{2}{N-2-2a}}|x_i| \to 0$ *as* $i \to +\infty$, *and for any* $x \in B_{\bar{r}}(x_i)$

$$u_i(x) \le \bar{C}\,|x - x_i|^{-\frac{N-2-2a}{2}}$$

where $B_{\bar{r}}(x_i) := \{x \in \Omega : \; |x - x_i| < \bar{r}\}$.

If 0 is an isolated blow-up point of $\{u_i\}_i$ we define

$$\bar{u}_i(r) = \fint_{\partial B_r(x_i)} u_i = \frac{1}{|\partial B_r(x_i)|}\int_{\partial B_r(x_i)} u_i, \quad r > 0$$

and

$$\bar{w}_i(r) = r^{\frac{N-2-2a}{2}}\bar{u}_i(r), \quad r > 0. \tag{5.63}$$

Definition 5.3. *The point* 0 *is said to be an isolated simple blow-up point of* $\{u_i\}_i$ *if it is an isolated blow-up point and there exist some positive* $\rho \in (0, \bar{r})$ *independent of* i *and* $\tilde{C} > 1$ *such that*

$$\bar{w}_i'(r) < 0 \quad \text{for } r \text{ satisfying } \tilde{C}\, u_i(x_i)^{-\frac{2}{N-2-2a}} \leq r \leq \rho. \tag{5.64}$$

Let us now introduce the notion of blow-up at infinity. To this aim, we consider the Kelvin transform,

$$\tilde{u}_i(x) = |x|^{-(N-2-2a)} u_i\left(\frac{x}{|x|^2}\right), \tag{5.65}$$

which is an isomorphism of $D_a^{1,2}(\mathbb{R}^N)$. If u_i solves (P_i) in a neighborhood of ∞, i.e. $\Omega = \mathbb{R}^N \setminus D$ for some compact set D, then $\tilde{u}_i$ is a solution of (P_i) where K_i is replaced by $\tilde{K}_i(x) = K_i(x/|x|^2)$ and Ω by $\tilde{\Omega} = \mathbb{R}^N \setminus \{x/|x|^2 \,|\, x \in D\}$, a neighborhood of 0.

Definition 5.4. *Let* $\{u_i\}_i$ *be a sequence of solutions of* (P_i) *in a neighborhood of* ∞. *We say that* ∞ *is a blow-up point (respectively an isolated blow-up point, an isolated simple blow-up point) if* 0 *is a blow-up point (respectively an isolated blow-up point, an isolated simple blow-up point) of the sequence* $\{\tilde{u}_i\}_i$ *defined by the Kelvin transform (5.65).*

Remark 5.3. It is easy to see that ∞ is a blow-up point of $\{u_i\}_i$ if and only if there exists a sequence $\{x_i\}_i$ such that $|x_i| \to \infty$ as $i \to +\infty$ and

$$|x_i|^{N-2-2a} u_i(x_i) \underset{i\to+\infty}{\longrightarrow} \infty \quad \text{and} \quad |x_i| u_i(x_i)^{\frac{2}{N-2-2a}} \underset{i\to+\infty}{\longrightarrow} 0.$$

In the sequel we will use the notation c to denote a positive constant which may vary from line to line.

In order to prove a Harnack type inequality (see Lemma 5.7), we need the following lemma.

Lemma 5.6. *Suppose* a, b, p *satisfy (5.7) and (5.4). Let* $(z_i)_{i\in\mathbb{N}} \subset \mathbb{R}^N$ *and consider the measures* $\mu_i := |x - z_i|^{-2a}\, dx$, *then we have for* $0 < r < 2$ *as* $r \to 0$

$$\sup_{x\in B_2(0),\, i\in\mathbb{N}} \int_{B_r(x)} |y - z_i|^{-bp} \int_{|x-y|}^{8} \frac{s\, ds}{\mu_i(B_s(x))}\, dy \to 0.$$

Proof. We use as c a generic constant that may change its value from line to line. Fix $x \in B_2(0)$. From the doubling property of the measure μ_i (see [60]) we find

$$\begin{aligned} M_i(x, |x-y|) &:= \int_{|x-y|}^{8} \frac{s\, ds}{\mu_i(B_s(x))} \\ &\leq c \begin{cases} |x-y|^{-N+2a+2}, & \text{if } |x-y| > \frac{1}{2}|x - z_i| \\ |x-y|^{-N+2}|x-z_i|^{2a} + |x-z_i|^{-N+2a+2}, & \text{if } |x-y| \leq \frac{1}{2}|x-z_i|. \end{cases} \end{aligned}$$

An easy calculation shows that $2a - bp > -2$ and that if $a \geq 0$ then $2a - bp \leq 0$. Hence, we may estimate for $0 < r \leq \frac{1}{2}|x - z_i|$ and $y \in B_r(x)$

$$|y - z_i| \geq |x - z_i| - |x - y| \geq \frac{1}{2}|x - z_i|$$

and

$$\int_{B_r(x)} |y - z_i|^{-bp} M_i(x, |x - y|)\, dy \leq c r^{2+2a-bp}.$$

Since $-bp > -2 - 2a > -N$ we may use the above estimate to derive

$$\int\limits_{B_{2|x-z_i|}(x)} |y - z_i|^{-bp} M_i(x, |x - y|)\, dy \leq c|x - z_i|^{2+2a-bp}.$$

Consequently we obtain for $\frac{1}{2}|x - z_i| \leq r \leq 2|x - z_i|$

$$\int_{B_r(x)} |y - z_i|^{-bp} M_i(x, |x - y|)\, dy \leq c|x - z_i|^{2+2a-bp} \leq c r^{2+2a-bp}.$$

Finally we obtain for $2|x - z_i| < r \leq 2$ and $|x - y| > 2|x - z_i|$

$$|y - z_i| \geq |y - x| - |x - z_i| \geq \frac{1}{2}|y - x|$$

and

$$\int_{B_r(x)} |y - z_i|^{-bp} M_i(x, |x - y|)\, dy \leq c r^{2+2a-bp},$$

which ends the proof. □

Lemma 5.7. *Let $(K_i)_{i\in\mathbb{N}}$ satisfy (5.61), $\{u_i\}_i$ satisfy (P_i) and $x_i \to 0$ be an isolated blow up point. Then there is a positive constant $C = C(N, \bar{C}, A_1)$ such that for any $0 < r < \min(\bar{r}/3, 1)$ there holds*

$$\max_{x\in B_{2r}(x_i)\setminus B_{r/2}(x_i)} u_i(x) \leq C \min_{x\in B_{2r}(x_i)\setminus B_{r/2}(x_i)} u_i(x). \tag{5.66}$$

Proof. We define $v_i(x) := r^{\frac{N-2-2a}{2}} u_i(rx + x_i)$. Then v_i satisfies in $B_3(0)$

$$0 < v_i(x) < \bar{C}|x|^{-\frac{N-2-2a}{2}}, \tag{5.67}$$

and

$$\begin{aligned} -\mathrm{div}\left(|x + r^{-1}x_i|^{-2a}\nabla v_i(x)\right) &= -r^{\frac{N-2-2a}{2}+2+2a}\mathrm{div}\left(|\cdot|^{-2a}\nabla u_i(\cdot)\right)(rx + x_i) \\ &= K_i(rx + x_i)\ |x + r^{-1}x_i|^{-bp}\ v_i^{p-1}(x), \end{aligned}$$

since

$$\frac{N-2-2a}{2} + 2 + 2a - bp - (p-1)\frac{N-2-2a}{2} = N - p\Big(\frac{N - 2(1 + a - b)}{2}\Big) = 0.$$

To prove the claim we use a weighted version of Harnack's inequality applied to v_i and

$$-\operatorname{div}\left(|x+r^{-1}x_i|^{-2a}\nabla v_i(x)\right) - W_i(x)v_i(x) = 0 \quad \text{in } B_{9/4}(0)\setminus B_{1/4}(0),$$

where $W_i(x) := K_i(rx+x_i)\ |x+r^{-1}x_i|^{-bp}\ v_i^{p-2}(x)$. From (5.67) the function v_i is uniformly bounded in $B_{9/4}(0)\setminus B_{1/4}(0)$ and the claim follows from Harnack's inequality in [58]. We mention that $|\cdot + r^{-1}x_i|^{-bp}$ belongs to the class of potentials required in [58] (see Lemma 5.6). □

Proposition 5.2. *Let $\{K_i\}_i$ satisfy (5.61), $\{u_i\}_i$ satisfy (P_i) and $x_i \to 0$ be an isolated blow up point. Then for any $R_i \to \infty$, $\varepsilon_i \to 0^+$, we have, after passing to a subsequence that:*

$$R_i u_i(x_i)^{-\frac{2}{N-2-2a}} \to 0 \ \text{as } i\to\infty, \tag{5.68}$$

$$\|u_i(x_i)^{-1}u_i\big(u_i(x_i)^{-\frac{2}{N-2-2a}}\cdot + x_i\big) - z^{a,b}_{K(0)}(\cdot)\|_{C^{0,\gamma}(B_{2R_i}(0))} \le \varepsilon_i, \tag{5.69}$$

$$\|u_i(x_i)^{-1}u_i\big(u_i(x_i)^{-\frac{2}{N-2-2a}}\cdot + x_i\big) - z^{a,b}_{K(0)}(\cdot)\|_{H^1_a(B_{2R_i}(0))} \le \varepsilon_i, \tag{5.70}$$

where $H^1_a(B_{2R_i}(0))$ is the weighted Sobolev space defined at page 53.

Proof. Consider

$$\phi_i(x) = u_i(x_i)^{-1}u_i\big(u_i(x_i)^{-\frac{2}{N-2-2a}}x + x_i\big), \quad |x| < \bar r u_i(x_i)^{\frac{2}{N-2-2a}}.$$

We have

$$\begin{aligned}-\operatorname{div}&\left(\left|x+u_i(x_i)^{\frac{2}{N-2-2a}}x_i\right|^{-2a}\nabla\phi_i(x)\right)\\ &= K_i\big(u_i(x_i)^{-\frac{2}{N-2-2a}}x + x_i\big)\ \left|x+u_i(x_i)^{\frac{2}{N-2-2a}}x_i\right|^{-bp}\ \phi_i^{p-1}(x).\end{aligned}$$

Moreover, from the definition of isolated blow-up

$$\phi_i(0)=1, \quad 0<\phi_i(x)\le \bar C|x|^{-\frac{N-2-2a}{2}} \quad \text{for } |x| < \bar r\, u_i(x_i)^{\frac{2}{N-2-2a}}. \tag{5.71}$$

Lemma 5.7 shows that for large i and for any $0<r<1$ we have

$$\max_{\partial B_r}\phi_i \le C\min_{\partial B_r}\phi_i, \tag{5.72}$$

where $C = C(N,\bar C, A_1)$. Since

$$-\operatorname{div}\left(\left|x+u_i(x_i)^{\frac{2}{N-2-2a}}x_i\right|^{-2a}\nabla\phi_i(x)\right) \ge 0 \text{ and } \phi_i(0)=1$$

we may use (5.72) and the minimum principle for $|x|^{-2a}$-superharmonic functions in [60, Thm 7.12] to deduce that

$$\phi_i(x) \le C \quad \text{in } B_1(0). \tag{5.73}$$

From (5.71), (5.73) and regularity results in [50] the functions ϕ_i are uniformly bounded in $C^{0,\gamma}_{loc}(\mathbb{R}^N)$ and $H^1_{a,loc}(\mathbb{R}^N)$ for some $\gamma \in (0,1)$. Since point-concentration is ruled out by the L^∞-bound, there is some positive function $\phi \in C^{0,\gamma'}_{loc}(\mathbb{R}^N) \cap H^1_{a,loc}(\mathbb{R}^N)$ and some $\gamma' \in (0,1)$ such that

$$\begin{aligned}
&\phi_i \to \phi \text{ in } C^{0,\gamma'}_{loc}(\mathbb{R}^N) \cap H^1_{a,loc}(\mathbb{R}^N),\\
&-\operatorname{div}(|x|^{-2a}\nabla\phi) = \lim_{i\to\infty} K_i(x_i)\frac{\phi^{p-1}}{|x|^{bp}}\\
&\phi(0) = 1.
\end{aligned}$$

By uniqueness of the solutions proved in [37] we deduce that $\phi = z^{a,b}_{K(0)}$. □

Remark 5.4. From the proof of Proposition 5.2 one can easily check that if $x_i \to 0$ is an isolated blow-up point then there exists a positive constant C, depending on $\lim_{i\to\infty} K_i(x_i)$ and a, b, and N, such that the function $\bar{w}_i$ defined in (5.63) is strictly decreasing for $Cu_i(x_i)^{-2/(N-2-2a)} \le r \le r_i = R_i u_i(x_i)^{-\frac{2}{N-2-2a}}$, as the following Lemma shows.

Lemma 5.8. *Let $\{K_i\}_i$ satisfy (5.61), $(u_i)_{i\in\mathbb{N}}$ satisfy (P_i) and $x_i \to 0$ be an isolated blow up point. Then for any $R_i \to \infty$, there exists a positive constant C depending on $\lim_{i\to\infty} K_i(x_i)$ and a, b, and N such that after passing to a subsequence the function $\bar{w}_i$ defined in (5.63) is strictly decreasing for $Cu_i(x_i)^{-2/(N-2-2a)} \le r \le r_i$ where $r_i = R_i u_i(x_i)^{-\frac{2}{N-2-2a}}$.*

Proof. Making the change of variable $y = u_i(x_i)^{-\frac{2}{N-2-2a}}x + x_i$, there results

$$\begin{aligned}
\bar{w}_i(r) &= \frac{r^{\frac{N-2-2a}{2}}}{|\partial B_r(x_i)|}\int_{\partial B_r(x_i)} u_i(y)\\
&= r^{\frac{N-2-2a}{2}} \fint_{\partial B_{ru_i(x_i)^{2/(N-2-2a)}}(0)} u_i\big(u_i(x_i)^{-\frac{2}{N-2-2a}}x + x_i\big).
\end{aligned}$$

From the proof of Proposition 5.2 we have for some function $g_i \in C^{0,\gamma}(B_{2R_i}(0))$

$$u_i\big(u_i(x_i)^{-\frac{2}{N-2-2a}}x + x_i\big) = u_i(x_i)\big(z^{a,b}_{K(0)}(x) + g_i(x)\big)$$

where $\|g_i\|_{C^2(B_{2R_i}(0)\setminus B_C(0))} \le \varepsilon_i$. Being $z^{a,b}_{K(0)}$ a radial function, from above we find

$$\begin{aligned}
\bar{w}_i(r) &= r^{\frac{N-2-2a}{2}} u_i(x_i) \fint_{\partial B_{ru_i(x_i)^{2/(N-2-2a)}}(0)} \big(z^{a,b}_{K(0)}(x) + g_i(x)\big)\\
&= r^{\frac{N-2-2a}{2}} u_i(x_i)\big[z^{a,b}_{K(0)}(ru_i(x_i)^{2/(N-2-2a)}) + \textstyle\fint_{\partial B_{ru_i(x_i)^{2/(N-2-2a)}}(0)} g_i\big].
\end{aligned}$$

A direct computation shows that

$$\frac{d}{dr}\bar{w}_i(r) = u_i(x_i)r^{\frac{N-4-2a}{2}}(z^{a,b}_{K(0)}(ru_i(x_i)^{2/(N-2-2a)}))^{\frac{p}{2}} \times$$
$$\times \Big[\tfrac{N-2-2a}{2}\Big(1-K(0)u_i(x_i)^{p-2}r^{\frac{(p-2)(N-2-2a)}{2}}\Big) + r(\textstyle\fint g_i)' z^{a,b}_{K(0)}(ru_i(x_i)^{2/(N-2-2a)})^{-\frac{p}{2}}$$
$$+ \tfrac{N-2-2a}{2}(\textstyle\fint g_i) z^{a,b}_{K(0)}(ru_i(x_i)^{2/(N-2-2a)})^{-\frac{p}{2}}\Big].$$

Since for $Cu_i(x_i)^{-2/(N-2-2a)} \le r \le r_i$, there results $C \le ru_i(x_i)^{2/(N-2-2a)} \le R_i$, we have that

$$\fint_{\partial B_{ru_i(x_i)^{2/(N-2-2a)}}(0)} g_i \le \varepsilon_i, \quad \frac{d}{dr}\fint_{\partial B_{ru_i(x_i)^{2/(N-2-2a)}}(0)} g_i \le \varepsilon_i.$$

Moreover for $C = \big(\frac{1+\delta}{K(0)}\big)^{\frac{2}{(p-2)(N-2-2a)}}$ we have $1-K(0)u_i(x_i)^{p-2}r^{\frac{(p-2)(N-2-2a)}{2}} \le -\delta$. Choosing $\varepsilon_i = o\Big(R_i^{-\frac{p(N-2-2a)}{2}}\Big)$ the claim follows. □

Lemma 5.9. *Let $x_i \to 0$ be a blow-up point. Then for any x such that $|x-x_i| \ge r_i = R_i u_i(x_i)^{-\frac{2}{N-2-2a}}$ we have*

$$|x-x_i| = |x|(1+o(1)).$$

In particular, $x_i \in B_{r_i}(0)$.

Proof. The assumption $\Big|x_i u_i(x_i)^{\frac{2}{N-2-2a}}\Big| = o(1)$ implies that $|x_i| = r_i\, o(1)$. Hence

$$|x| \ge |x-x_i| - |x_i| \ge r_i - r_i\, o(1) = r_i(1+o(1)).$$

Therefore

$$\frac{|x_i|}{|x|} \le \frac{r_i\, o(1)}{r_i(1+o(1))} = o(1)$$

and hence

$$\left|\frac{x-x_i}{|x|}\right| = \left|\frac{x}{|x|} - \frac{x_i}{|x|}\right| \underset{i\to+\infty}{\longrightarrow} 1$$

thus proving the lemma. □

Proposition 5.3. *Suppose $\{K_i\}_i \subset C^1_{\text{loc}}(B_2)$ satisfy (5.61) with $\Omega = B_2$ and*

$$|\nabla K_i(x)| \le A_2 \text{ for all } x \in B_2. \tag{5.74}$$

Let u_i satisfy (P_i) with $\Omega = B_2$ and suppose that $x_i \to 0$ is an isolated simple blow-up point such that

$$|x-x_i|^{\frac{N-2-2a}{2}} u_i(x) \le A_3 \text{ for all } x \in B_2. \tag{5.75}$$

Then there exists $C = C(N,a,b,A_1,A_2,A_3,\bar{C},\rho) > 0$ such that

$$u_i(x) \le C\, u_i(x_i)^{-1}|x-x_i|^{2+2a-N} \quad \textit{for all } |x-x_i| \le 1. \tag{5.76}$$

Furthermore there exists a Hölder continuous function $B(x)$ (smooth outside 0) satisfying div $(|x|^{-2a}\nabla B) = 0$ *in B_1, such that, after passing to a subsequence,*

$$u_i(x_i)u_i(x) \to h(x) = A|x|^{2+2a-N} + B(x) \quad \textit{in } C^2_{\mathrm{loc}}(B_1 \setminus \{0\})$$

where

$$A = \frac{K(0)}{(N-2-2a)|\mathbb{S}^N|}\int_{\mathbb{R}^N} \frac{\left(z^{a,b}_{K(0)}\right)^{p-1}}{|x|^{bp}}\,dx.$$

Lemma 5.10. *Under the assumption of Proposition 5.3 without (5.74) there exist a positive $\delta_i = O\Big(R_i^{\frac{-2(1+a-b)(N-2-2a)}{N-2(1+a-b)}}\Big)$ and $C = C(N,a,b,A_1,A_2,\bar{C},\rho) > 0$ such that*

$$u_i(x) \le C\, u_i(x_i)^{-\lambda_i}|x-x_i|^{2+2a-N+\delta_i} \quad \textit{for all } r_i \le |x-x_i| \le 1, \tag{5.77}$$

where $\lambda_i := 1 - 2\delta_i/(N-2-2a)$.

Proof. It follows from Proposition 5.2 that

$$u_i(x) \le c u_i(x_i) R_i^{2a+2-N} \text{ for } |x-x_i| = r_i. \tag{5.78}$$

From the definition of isolated simple blow-up in (5.64) there exists $\rho > 0$ such that

$$r^{\frac{N-2-2a}{2}}\bar{u}_i \text{ is strictly decreasing in } r_i < r < \rho. \tag{5.79}$$

From (5.78), (5.79) and Lemma 5.7 it follows that for all $r_i \le |x-x_i| < \rho$

$$|x-x_i|^{\frac{N-2-2a}{2}}u_i(x) \le c|x-x_i|^{\frac{N-2-2a}{2}}\bar{u}_i(|x-x_i|) \le c r_i^{\frac{N-2-2a}{2}}\bar{u}_i(r_i) \le c R_i^{\frac{2+2a-N}{2}}.$$

Therefore for $r_i < |x-x_i| < \rho$

$$u_i(x)^{\frac{4}{N-2-2a}} \le c R_i^{-2}|x-x_i|^{-2}. \tag{5.80}$$

Consider the following degenerated elliptic operator

$$\mathcal{L}_i\varphi = \operatorname{div}(|x|^{-2a}\nabla\varphi) + K_i(x)|x|^{-bp}u_i(x)^{p-2}\varphi.$$

Clearly $u_i > 0$ solves $\mathcal{L}_i u_i = 0$. Hence $-\mathcal{L}_i$ is nonnegative and the maximum principle holds for $\mathcal{L}_i$. Direct computations show for any $0 \le \mu \le N-2-2a$

$$\operatorname{div}(|x|^{-2a}\nabla(|x|^{-\mu})) = -\mu(N-2-2a-\mu)|x|^{-2-2a-\mu} \text{ for } x \ne 0. \tag{5.81}$$

From (5.80), (5.81) and Lemma 5.9 we infer

$$\mathcal{L}_i(|x|^{-\mu}) \le \Big(-\mu(N-2-2a-\mu) + c R_i^{\frac{-2(1+a-b)(N-2-2a)}{N-2(1+a-b)}}\Big)|x|^{-2-2a-\mu}.$$

We can choose $\delta_i = O(R_i^{\frac{-2(1+a-b)(N-2-2a)}{N-2(1+a-b)}})$ such that

$$\max(\mathcal{L}_i(|x|^{-\delta_i}), \mathcal{L}_i(|x|^{2a+2-N+\delta_i})) \le 0. \tag{5.82}$$

Set $M_i := 2\max_{\partial B_\rho(x_i)} u_i$, $\lambda_i = 1 - 2\delta_i/(N-2-2a)$, and

$$\phi_i(x) := M_i \rho^{\delta_i}|x|^{-\delta_i} + A u_i(x_i)^{-\lambda_i}|x|^{2+2a-N+\delta_i} \text{ for } r_i \le |x-x_i| \le \rho, \tag{5.83}$$

where A will be chosen later. We will apply the maximum principle to compare ϕ_i and u_i. By the choice of M_i and Lemma 5.9 we infer for i sufficiently large

$$\phi_i(x) \ge \frac{M_i}{2} \ge u_i(x) \text{ for } |x-x_i| = \rho.$$

On the inner boundary $|x-x_i| = r_i$ we have by (5.78) and for A large enough

$$\begin{aligned}\phi_i(x) &\ge A(1+o(1))u_i(x_i)^{-\lambda_i} r_i^{2+2a-N+\delta_i} \\ &= A(1+o(1))R_i^{2+2a-N+\delta_i} u_i(x_i)^{2-\frac{2\delta_i}{N-2-2a}-\lambda_i} \\ &\ge A(1+o(1))R_i^{2+2a-N} u_i(x_i) \ge u_i(x).\end{aligned}$$

Now we obtain from the maximum principle in the annulus $r_i \le |x-x_i| \le \rho$ that

$$u_i(x) \le \phi_i(x) \text{ for all } r_i \le |x-x_i| \le \rho. \tag{5.84}$$

It follows from (5.79), (5.84) and Lemma 5.7 that for any $r_i \le \theta \le \rho$ we have

$$\begin{aligned}\rho^{\frac{N-2-2a}{2}} M_i &\le c\rho^{\frac{N-2-2a}{2}} \bar{u}_i(\rho) \le c\theta^{\frac{N-2-2a}{2}} \bar{u}_i(\theta) \\ &\le c\theta^{\frac{N-2-2a}{2}} \left(M_i \rho^{\delta_i}\theta^{-\delta_i} + A u_i(x_i)^{-\lambda_i}\theta^{2+2a-N+\delta_i}\right).\end{aligned}$$

Choose $\theta = \theta(\rho, c)$ such that

$$c\theta^{\frac{N-2-2a}{2}} \rho^{\delta_i}\theta^{-\delta_i} < \frac{1}{2}\rho^{\frac{N-2-2a}{2}}.$$

Then we have

$$M_i \le c u_i(x_i)^{-\lambda_i},$$

which, in view of (5.84) and the definition of ϕ_i in (5.83), proves (5.77) for $r_i \le |x-x_i| \le \rho$. The Harnack inequality in Lemma 5.7 allows to extend (5.77) for $r_i \le |x-x_i| \le 1$. □

In order to prove Proposition 5.3 we also need to prove a Bôcher-type theorem for μ_a-harmonic functions. A function u will be called μ_a-harmonic in $\Omega \subset \mathbb{R}^N$, if $u \in D^{1,2}_{a,loc}(\mathbb{R}^N)$ and for all $\varphi \in C_c^\infty(\Omega)$ there holds

$$\int_\Omega |x|^{-2a} \nabla u \nabla \varphi = 0.$$

Theorem 5.9. *Let u be a nonnegative μ_a-harmonic function in $\mathbb{R}^N \setminus \{0\}$. Then there exist a constant $A \geq 0$ and a Hölder continuous function B, μ_a-harmonic in $\mathbb{R}^N$, such that*

$$u(x) = A\,|x|^{2+2a-N} + B(x).$$

Proof. We distinguish two cases.
Case 1: there exists a sequence $x_n \to 0$ and a positive constant M such that $|u(x_n)| \leq M$. In this case the Harnack Inequality (Theorem 6.2 of [60]) implies that u is bounded. Moreover from [60, Lemma 6.15] u can be continuously extended to 0 and is a weak solution of

$$-\mathrm{div}\,(|x|^{-2a}\nabla u) = 0 \quad \text{in } \mathbb{R}^N,$$

see [34, Lemma 2.1]. Therefore from the Liouville Theorem [60, Theorem 6.10] we get that u is constant and the theorem holds with $A = 0$ and $B \equiv$ const.
Case 2: $u(x_n) \to +\infty$ for any sequence $x_n \to 0$. We can extend u in 0 to be $u(0) := +\infty$, thus obtaining a lower semi-continuous function in $\mathbb{R}^N$. Moreover [60, Theorem 7.35] implies that u is super-harmonic in the sense of the definition of [60, Chapter 7], i.e.

- u is lower semi-continuous,
- $u \not\equiv \infty$ in each component of $\mathbb{R}^N$,
- for each open $D \subset\subset \mathbb{R}^N$ and each $h \in C^0(\mathbb{R}^N)$ μ_a-harmonic in D the inequality $u \geq h$ on ∂D implies $u \geq h$ in D.

Let us remark that in order to apply Theorem 7.35 in [60] we need to prove that 0 has capacity 0 with respect to our weight; indeed

$$\mathrm{cap}_{|x|^{-2a}}(\{0\}, \mathbb{R}^N) := \inf_{\substack{u \in C_0^\infty(\mathbb{R}^N),\ u \equiv 1 \\ \text{in a neighborhood of } 0}} \int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \leq \mathrm{cap}_{|x|^{-2a}}(B_r, \mathbb{R}^N)$$
$$\leq \mathrm{cap}_{|x|^{-2a}}(B_r, B_{2r}) \leq c r^{N-2-2a}$$

for any $r > 0$, where we have used [60, Lemma 2.14]. Then $\mathrm{cap}_{|x|^{-2a}}(\{0\}, \mathbb{R}^N) = 0$. From [60, Corollary 7.21] there holds

$$-\mathrm{div}\,(|x|^{-2a}\nabla u) \geq 0 \quad \text{in the sense of distributions on } \mathbb{R}^N$$

hence from the Riesz Theorem there exists a Radon measure $\mu \geq 0$ in $\mathbb{R}^N$ such that

$$\langle -\mathrm{div}\,(|x|^{-2a}\nabla u), \varphi \rangle = \int_{\mathbb{R}^N} \varphi \, d\mu \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N).$$

Since $\langle -\mathrm{div}\,(|x|^{-2a}\nabla u), \varphi\rangle = 0$ for any $\varphi \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$, μ must be supported in $\{0\}$ and so $\mu = A\delta_0$ for a nonnegative constant A. Since the Green's function $G_a(x) := |x|^{2+2a-N}$ satisfies

$$-\mathrm{div}\,(|x|^{-2a}\nabla G_a) = \delta_0 \quad \text{in the sense of distributions on } \mathbb{R}^N,$$

we have that

$$-\mathrm{div}\,(|x|^{-2a}\nabla(u - AG_a)) = 0$$

in the sense of distributions on $\mathbb{R}^N$. Theorem 3.70 and Lemma 6.47 in [60] imply that $B := u - AG_a$ is Hölder continuous. □

Proof of Proposition 5.3. The inequality of Proposition 5.3 for $|x - x_i| \le r_i$ follows immediately for Proposition 5.2. Let $e \in \mathbb{R}^N$, $|e| = 1$ and consider the function

$$v_i(x) = u_i(x_i + e)^{-1} u_i(x).$$

Clearly v_i satisfies the equation

$$-\mathrm{div}\,(|x|^{-2a}\nabla v_i) = u_i(x_i + e)^{p-2} K_i(x) \frac{{v_i}^{p-1}}{|x|^{bp}} \quad \text{in } B_{4/3}. \tag{5.85}$$

Applying the Harnack inequality of Lemma 5.7 on v_i, we obtain that v_i is bounded on any compact set not containing 0. By standard elliptic theories, it follows that, up to a subsequence, $\{v_i\}_i$ converges in $C^2_{\mathrm{loc}}(B_2 \setminus \{0\})$ to some positive function $v \in C^2(B_2 \setminus \{0\})$. Since $u_i(x_i + e) \to 0$ due to Lemma 5.10, we can pass to the limit in (5.85) thus obtaining

$$-\mathrm{div}\,(|x|^{-2a}\nabla v) = 0 \quad \text{in } B_2 \setminus \{0\}.$$

We claim that v has a singularity at 0. Indeed, from Lemma 5.7 and standard elliptic theories, for any $0 < r < 2$ we have that

$$\lim_{i\to\infty} u_i(x_i + e)^{-1} r^{\frac{N-2-2a}{2}} \bar{u}_i(r) = r^{\frac{N-2-2a}{2}} \bar{v}(r)$$

where $\bar{v}(r) = \frac{1}{|\partial B_r|}\int_{\partial B_r} v$. Since the blow-up is simple isolated, $r^{\frac{N-2-2a}{2}}\bar{v}(r)$ is non-increasing for $0 < r < \rho$ and this would be impossible in the case in which v is regular at 0. It follows that v is singular at 0 and hence from Theorem 5.9

$$v(x) = a_1 |x|^{2+2a-N} + b_1(x)$$

where $a_1 > 0$ is some positive constant and $b_1(x)$ is some Hölder continuous function in B_2 such that $-\mathrm{div}\,(|x|^{-2a}\nabla b_1) = 0$.

Let us first establish the inequality in Proposition 5.3 for $|x - x_i| = 1$. Namely we prove that

$$u_i(x_i + e) \le c\, u_i(x_i)^{-1}. \tag{5.86}$$

By contradiction, suppose that (5.86) fails. Then along a subsequence, we have

$$\lim_{i\to\infty} u_i(x_i + e) u_i(x_i) = \infty. \tag{5.87}$$

Multiplying (P_i) by $u_i(x_i + e)^{-1}$ and integrating on B_1, we get

$$-\int_{\partial B_1} |x|^{-2a} \frac{\partial v_i}{\partial \nu} = \int_{B_1} K_i(x) \frac{u_i^{p-1}}{|x|^{bp}} u_i(x_i + e)^{-1}\, dx. \tag{5.88}$$

From the properties of b_1 and the convergence of v_i to v, we know that

$$\lim_{i\to\infty}\int_{\partial B_1}|x|^{-2a}\frac{\partial v_i}{\partial\nu}=\int_{\partial B_1}|x|^{-2a}\frac{\partial}{\partial\nu}\left(a_1|x|^{2+2a-N}+b_1(x)\right)$$
$$=-a_1(N-2-2a)|\mathbb{S}^N|<0. \qquad (5.89)$$

From Proposition 5.2 there holds

$$\int_{|x-x_i|\le r_i}\frac{K_i(x)u_i^{p-1}}{|x|^{bp}}\,dx\le C\,u_i(x_i)^{-1} \qquad (5.90)$$

while from Lemma 5.10 and Lemma 5.9 we have that

$$\int_{r_i\le|x-x_i|\le 1}\frac{K_i(x)u_i^{p-1}}{|x|^{bp}}\,dx\le c\int_{r_i\le|x-x_i|\le 1}u_i(x_i)^{-\lambda_i(p-1)}\frac{|x-x_i|^{(2+2a-N+\delta_i)(p-1)}}{|x|^{bp}}$$
$$\le c\,u_i(x_i)^{-\lambda_i(p-1)}r_i^{(2+2a-N+\delta_i)(p-1)-bp+N}$$
$$=c\,u_i(x_i)^{-1}R_i^{(2+2a-N+\delta_i)(p-1)-bp+N}=o(1)u_i(x_i)^{-1}. \qquad (5.91)$$

Finally, (5.87), (5.89), (5.90), and (5.91) lead to a contradiction. Since we have established (5.86), the inequality in Proposition 5.3 has been established for $\rho\le|x-x_i|\le 1$ (due to Lemma 5.7). It remains to treat the case $r_i\le|x-x_i|\le\rho$. To this aim we scale the problem to reduce it to the case $|x-x_i|=1$. By contradiction, suppose that there exists a subsequence $\tilde{x}_i$ satisfying $r_i\le|\tilde{x}_i-x_i|\le\rho$ and

$$\lim_{i\to+\infty}u_i(\tilde{x}_i)u_i(x_i)|\tilde{x}_i-x_i|^{N-2-2a}=+\infty. \qquad (5.92)$$

Set $\tilde{r}_i=|\tilde{x}_i-x_i|$ and $\tilde{u_i}(x)=\tilde{r}_i^{\frac{N-2-2a}{2}}u_i(\tilde{r}_ix)$. We have that $\tilde{u_i}$ satisfies the equation

$$-\mathrm{div}\,(|x|^{-2a}\nabla\tilde{u_i}(x))=K_i(\tilde{r}_ix)\frac{\tilde{u_i}(x)^{p-1}}{|x|^{bp}}.$$

Since $|x_i|=r_i\,o(1)$ and $\tilde{r}_i\ge r_i$ we have that $x_i/\tilde{r}_i\to 0$. We have that $x_i/\tilde{r}_i$ is an isolated simple blow-up point for $\{\tilde{u_i}\}_i$. From (5.86), we have that

$$\tilde{u_i}\left(\frac{x_i}{\tilde{r}_i}+\frac{\tilde{x}_i-x_i}{\tilde{r}_i}\right)\le c\tilde{u_i}\left(\frac{x_i}{\tilde{r}_i}\right)^{-1}$$

which gives

$$\tilde{r}_i^{N-2-2a}u_i(\tilde{x}_i)u_i(x_i)\le c.$$

The above estimate and (5.92) give rise to a contradiction. The inequality in Proposition 5.3 is thereby established.

We compute A by multiplying (P_i) by $u_i(x_i)$ and integrating over B_1. From the divergence theorem,

$$-\int_{\partial B_1} |x|^{-2a}\frac{\partial}{\partial\nu}(u_i(x_i)u_i) = u_i(x_i)\int_{B_1} K_i(x)\frac{u_i^{p-1}}{|x|^{bp}}\,dx. \qquad (5.93)$$

Let $w_i(x) = u_i(x_i)u(x)$. We have that w_i satisfies

$$-\operatorname{div}(|x|^{-2a}\nabla w_i) = u_i(x_i)^{2-p}K_i(x)\frac{w_i^{p-1}}{|x|^{bp}}.$$

Moreover the inequality (5.76) implies that w_i is bounded on any compact set not containing 0. Hence $w_i \to w$ in $C^2_{\mathrm{loc}}(\mathbb{R}^N\setminus\{0\})$ where w satisfies

$$-\operatorname{div}(|x|^{-2a}\nabla w) = 0 \quad \text{in } \mathbb{R}^N\setminus\{0\}.$$

From the Bôcher-type theorem 5.9, we find that $w(x) = A|x|^{2+2a-N} + B(x)$ where $B(x)$ is Hölder continuous in $\mathbb{R}^N$ and satisfies $-\operatorname{div}(|x|^{-2a}\nabla B) = 0$ in $\mathbb{R}^N$. Hence

$$\begin{aligned}\lim_{i\to\infty}\int_{\partial B_1}|x|^{-2a}\frac{\partial}{\partial\nu}(u_i(x_i)u_i) &= \int_{\partial B_1}|x|^{-2a}\frac{\partial}{\partial\nu}(A|x|^{2+2a-N}+B(x))\\ &= A(2+2a-N)|\mathbb{S}^N|. \qquad (5.94)\end{aligned}$$

On the other hand from (5.91) and Proposition 5.2

$$\begin{aligned}u_i(x_i)\int_{B_1}K_i(x)\frac{u_i^{p-1}}{|x|^{bp}}\,dx &= u_i(x_i)\int_{|x-x_i|\le r_i}K_i(x)\frac{u_i^{p-1}}{|x|^{bp}}\,dx + o(1)\\ &= K_i(0)\int_{|y|\le R_i}\frac{\left(z^{a,b}_{K(0)}\right)^{p-1}}{\left|y+u_i(x_i)^{\frac{2}{N-2-2a}}x_i\right|^{bp}}\,dy + o(1)\\ &= K(0)\int_{\mathbb{R}^N}\frac{\left(z^{a,b}_{K(0)}\right)^{p-1}}{|y|^{bp}}\,dy + o(1). \qquad (5.95)\end{aligned}$$

By (5.93), (5.94), and (5.95) the value of A is computed and Proposition 5.3 is thereby established. □

Using Proposition 5.2 and the upper bound in Proposition 5.3 it is easy to see that the following estimates hold.

Lemma 5.11. *Under the assumptions of Proposition 5.3 we have for* $s = s_1 + s_2$

$$\int_{|x-x_i|\le r_i}|x-x_i|^{s_1}|x|^{s_2}|x|^{-bp}u_i(x)^p = \begin{cases} u_i(x_i)^{\frac{-2s}{N-2-2a}}\left(o(1)+\int_{\mathbb{R}^N}|x|^{s-bp}z^p_{1,K_i(x_i)}\right) & \text{if } -N+bp<s<N-bp,\\ O(u_i(x_i)^{-p}\log u_i(x_i)) & \text{if } s=N-bp,\\ o\left(u_i(x_i)^{-p}\right) & \text{if } s>N-bp.\end{cases}$$

and

$$\int_{r_i\le|x-x_i|\le 1}|x-x_i|^{s_1}|x|^{s_2}|x|^{-bp}u_i(x)^p$$
$$\le\begin{cases} o\big(u_i(x_i)^{\frac{-2s}{N-2-2a}}\big) & \text{if } -N+bp<s<N-bp,\\ O(u_i(x_i)^{-p}\log u_i(x_i)) & \text{if } s=N-bp,\\ O\big(u_i(x_i)^{-p}\big) & \text{if } s>N-bp.\end{cases}$$

Proposition 5.4. *Let $a\in\left[\frac{N-4}{2},\frac{N-2}{2}\right[$. Suppose that $\{K_i\}_i$ satisfy (5.61) with $\Omega=B_2\subset\mathbb{R}^N$ for some positive constant A_1, $\nabla K_i(0)=0$, $\{K_i\}_i$ converge to K in $C^2(B_2)$, $\{u_i\}_i$ satisfy (P_i) with $\Omega=B_2(0)$ and $x_i\to 0$ is an isolated blow-up point with (5.75) for some positive constant A_3. Then it has to be an isolated simple blow-up point.*

Proof. From Remark 5.4 there exists a constant c such that $r^{\frac{N-2-2a}{2}}\bar u_i(r)$ is decreasing in $cu_i(x_i)^{-2/(N-2-2a)}\le r\le r_i$. Arguing by contradiction, let us suppose that the blow-up is not simple. Hence for any i there exists $\mu_i\ge r_i$, $\mu_i\to 0$, such that μ_i is the first point after r_i in which the function $r^{\frac{N-2-2a}{2}}\bar u_i(r)$ becomes increasing. In particular μ_i is a critical point of such a function. Set

$$\xi_i(x)=\mu_i^{\frac{N-2-2a}{2}}u_i(\mu_i x),\quad \text{for } |\mu_i x-x_i|\le 1.$$

Clearly ξ_i satisfies

$$-\operatorname{div}(|x|^{-2a}\nabla\xi_i)=K_i(\mu_i x)\frac{\xi_i^{p-1}}{|x|^{bp}},\quad \text{for } |\mu_i x-x_i|\le 1.$$

Note that $\mu_i^{-1}\le R_i^{-1}u_i(x_i)^{\frac{2}{N-2-2a}}\le u_i(x_i)^{\frac{2}{N-2-2a}}$ and hence

$$\mu_i^{-1}|x_i|\le u_i(x_i)^{\frac{2}{N-2-2a}}|x_i|\to 0$$

in view of (5.62). Moreover (5.75) implies that

$$|x-\mu_i^{-1}x_i|^{\frac{N-2-2a}{2}}\xi_i(x)\le \text{const}\quad \text{for } |x-\mu_i^{-1}x_i|\le 1/\mu_i.$$

It is also easy to verify that

$$\lim_{i\to\infty}\xi_i(\mu_i^{-1}x_i)=\lim_{i\to\infty}\mu_i^{\frac{N-2-2a}{2}}u_i(x_i)=\infty.$$

On the other hand

$$\fint_{\partial B_r(\mu_i^{-1}x_i)}\xi_i=\mu_i^{\frac{N-2-2a}{2}}\fint_{\partial B_{r\mu_i}(x_i)}u_i=\mu_i^{\frac{N-2-2a}{2}}\bar u_i(\mu_i r).$$

Hence

$$r^{\frac{N-2-2a}{2}}\bar\xi_i(r)=\bar w_i(\mu_i r)$$

and the function $r^{\frac{N-2-2a}{2}}\bar\xi_i(r)$ is decreasing in $c\xi_i(\mu_i^{-1}x_i)^{-\frac{2}{N-2-2a}}<r<1$ so that 0 is an isolated simple blow-up point for $\{\xi_i\}$. From Proposition 5.3 we have that

$$\xi_i(\mu_i^{-1}x_i)\xi_i(x) \to h(x) = A|x|^{2+2a-N} + B(x) \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$$

where $B(x)$ is Hölder continuous in $\mathbb{R}^N$ and satisfies $-\text{div}\,(|x|^{-2a}\nabla B) = 0$ in $\mathbb{R}^N$. Since $h \geq 0$, the Harnack inequality implies that B is bounded and from the Liouville Theorem (see [60]) we find that B must be constant. Since

$$\frac{d}{dr}\{h(r)r^{\frac{N-2-2a}{2}}\}|_{r=1} = 0$$

we have that $A = B > 0$. From the Taylor expansion, (5.62) and the assumption $\nabla K_i(0) = 0$ we find

$$|\nabla K_i(\mu_i^{-1}x_i)| \leq \text{const}|\mu_i^{-1}x_i| = o\Big(\xi_i(\mu_i^{-1}x_i)^{-\frac{2}{N-2-2a}}\Big). \tag{5.96}$$

Using Lemma 5.11, (5.96), and the assumption on a, we have

$$\begin{aligned}
&\int_{B_\sigma(\mu_i^{-1}x_i)} \mu_i \nabla K_i(\mu_i x)\cdot x\frac{\xi_i^p}{|x|^{bp}} \\
&= \int_{B_\sigma(\mu_i^{-1}x_i)} \mu_i\Big[\nabla K_i(\mu_i^{-1}x_i) + O(\mu_i x - \mu_i^{-1}x_i)\Big]\cdot x\frac{\xi_i^p}{|x|^{bp}} \\
&= \int_{B_\sigma(\mu_i^{-1}x_i)} \mu_i\Big[\nabla K_i(\mu_i^{-1}x_i) + O(|x| + |x-\mu_i^{-1}x_i|)\Big]\cdot x\frac{\xi_i^p}{|x|^{bp}} \\
&= \left\{\begin{array}{ll} \mu_i O\Big(\xi_i(\mu_i^{-1}x_i)^{-\frac{4}{N-2-2a}}\Big) & \text{if } p > \frac{4}{N-2-2a} \\ \mu_i O\Big(\xi_i(\mu_i^{-1}x_i)^{-p}\log u_i(x_i)\Big) & \text{if } p = \frac{4}{N-2-2a} \\ \mu_i O\Big(\xi_i(\mu_i^{-1}x_i)^{-p}\Big) & \text{if } p < \frac{4}{N-2-2a} \end{array}\right\} = o(\xi_i(\mu_i^{-1}x_i)^{-2}).
\end{aligned}$$

Hence, from Corollary 5.2 and (5.76), we have that for any $0 < \sigma < 1$

$$\begin{aligned}
&\int_{\partial B_\sigma(0)} B(\sigma, x, \xi_i, \nabla\xi_i) \\
&= \frac{1}{p}\int_{B_\sigma(0)} \mu_i\nabla K_i(\mu_i x)\cdot x\frac{\xi_i^p}{|x|^{bp}} - \frac{\sigma}{p}\int_{\partial B_\sigma(0)} K_i(\mu_i x)\frac{\xi_i^p}{|x|^{bp}} \\
&= \frac{1}{p}\int_{B_\sigma(\mu_i^{-1}x_i)} \mu_i\nabla K_i(\mu_i x)\cdot x\frac{\xi_i^p}{|x|^{bp}} + O(\xi_i(\mu_i^{-1}x_i)^{-p}) \\
&= o(\xi_i(\mu_i^{-1}x_i)^{-2}).
\end{aligned}$$

Multiplying by $\xi_i(\mu_i^{-1}x_i)^2$ and letting $i \to \infty$ we find that

$$\int_{\partial B_\sigma} B(\sigma, x, h, \nabla h) = 0.$$

On the other hand Proposition 5.1 implies that for small σ the above integral is strictly negative, thus giving rise to a contradiction. The proof is now complete. □

5.4 A-priori estimates

To prove the a-priori estimates we first locate the possible blow-up points as in [76]. To this end we use the Kelvin transform defined in (5.65). We recall that if u solves (5.5) then $\tilde{u} = |x|^{-(N-2-2a)}u(x/|x|^2)$ solves (5.5) with K replaced by $\tilde{K}(x) = K(x/|x|^2)$. Since weak solutions to (5.5) are Hölder continuous (see [50]) we infer that

$$\lim_{|x|\to\infty} |x|^{N-2-2a}u(x) \text{ exists.} \tag{5.97}$$

Let us define $\omega_a(x) := (1 + |x|^{N-2-2a})^{-1}$.

Lemma 5.12. *Suppose $a \geq 0$, $2 < p < 2^*$, and $K \in C^2(\mathbb{R}^N)$ satisfies (5.10) and for some positive constants A_1, A_2 condition (5.12) and*

$$\|\nabla K\|_{L^\infty(B_2(0))} + \|\nabla \tilde{K}\|_{L^\infty(B_2(0))} \leq A_2. \tag{5.98}$$

Then for any $\varepsilon \in (0,1)$, $R > 1$, there exists $C_0 = C_0(\varepsilon, R, N, a, b, A_1, A_2) > 0$, such that if u is a solution of (5.5) and $\mathcal{K} = \{q_1, \dots, q_k\} \subset \mathbb{R}^N \cup \{\infty\}$ with

$$\begin{cases} \max\limits_{x\in\mathbb{R}^N\cup\{\infty\}} \frac{u(x)}{\omega_a(x)} dist(x,\mathcal{K})^{\frac{N-2-2a}{2}} > C_0, \\ u(q_i)|q_i|^{\frac{2}{N-2-2a}} < \varepsilon, \text{ and for all } 1 \leq i \leq k \\ \max\limits_{x\in\mathbb{R}^N\cup\{\infty\}} \frac{u(x)}{\omega_a(x)} dist(x,\{q_1,\dots,q_{i-1}\})^{\frac{N-2-2a}{2}} \leq \frac{u(q_i)}{\omega_a(q_i)} dist(q_i,\{q_1,\dots,q_{i-1}\})^{\frac{N-2-2a}{2}}, \end{cases} \tag{5.99}$$

then there exists $q^* \notin \mathcal{K}$ such that q^* is a maximum point of $(u/\omega_a) dist(\cdot,\mathcal{K})^{\frac{N-2-2a}{2}}$ and

(A) if $|q^*| \leq 1$

$$\left\| \frac{u(u(q^*)^{-\frac{2}{N-2-2a}}x + q^*)}{u(q^*)} - z^{a,b}_{K(q^*)} \right\|_{\substack{C^{0,\gamma}(B_{2R}(0))\cap \\ H^1_a(B_{2R}(0))}} + |q^*|u(q^*)^{\frac{2}{N-2-2a}} < \varepsilon \tag{5.100}$$

(B) if $|q^*| > 1$

$$\left\| \frac{\tilde{u}(\tilde{u}(\tilde{q}^*)^{-\frac{2}{N-2-2a}}x + \tilde{q}^*)}{\tilde{u}(\tilde{q}^*)} - z^{a,b}_{K(q^*)} \right\|_{\substack{C^{0,\gamma}(B_{2R}(0))\cap \\ H^1_a(B_{2R}(0))}} + |\tilde{q}^*|\tilde{u}(\tilde{q}^*)^{\frac{2}{N-2-2a}} < \varepsilon \tag{5.101}$$

where $\tilde{q}^* = Inv(q^*) := q^*/|q^*|^2$, $\tilde{u}$ is the Kelvin transform of u, $dist(\cdot,\cdot)$ is the distance on $\mathbb{R}^N \cup \{\infty\}$ induced by the standard metric on the sphere through the stereo-graphic projection, and $dist(\cdot,\emptyset) \equiv 1$.

Proof. Fix $\varepsilon > 0$ and $R > 1$. Let C_0 and C_1 be positive constants depending on $\varepsilon, R, a, b, N, A_1, A_2$ which shall be appropriately chosen in the sequel.
Let $q^* \in \mathbb{R}^N \cup \{\infty\}$ be the maximum point of $u/\omega_a \mathrm{dist}(x,\mathcal{K})^{\frac{N-2-2a}{2}}$. By (5.97) this maximum is achieved. From the first in (5.99) we have that

$$u(q^*)/\omega_a(q^*)\mathrm{dist}(q^*,\mathcal{K})^{\frac{N-2-2a}{2}} > C_0.$$

First we treat the case $|q^*| \le 1$. We claim that there exists a constant C_1, depending only on $\varepsilon, R, a, b, N, A_1, A_2$, such that $|q^*|^{\frac{N-2-2a}{2}}u(q^*) < C_1$. If not, there exist solutions u_i of (5.5) and finite sets $\mathcal{K}_i = \{q_1^i, \dots, q_{k_i}^i\}$ satisfying (5.99) above, such that for the maximum points q_i^* of $u_i/\omega_a \mathrm{dist}(\cdot,\mathcal{K}_i)^{\frac{N-2-2a}{2}}$ there holds

$$|q_i^*| \le 1 \text{ and } |q_i^*|^{\frac{N-2-2a}{2}}u_i(q_i^*) \to \infty.$$

Consider the functions v_i, defined by

$$v_i(x) := u_i(q_i^*)^{-1}u_i\big(|q_i^*|^{1+\frac{(N-2-2a)(2-p)}{4}}u(q_i^*)^{\frac{2-p}{2}}x + q_i^*\big),$$

which satisfy

$$\begin{aligned}
&-\operatorname{div}\Big(\big||q_i^*|^{\frac{(N-2-2a)(2-p)}{4}}u_i(q_i^*)^{\frac{2-p}{2}}x + |q_i^*|^{-1}q_i^*\big|^{-2a}\nabla v_i\Big)\\
&= K\big(|q_i^*|^{1+\frac{(N-2-2a)(2-p)}{4}}u_i(q_i^*)^{\frac{2-p}{2}}x + q_i^*\big)\frac{v_i^{p-1}}{\big||q_i^*|^{\frac{(N-2-2a)(2-p)}{4}}u_i(q_i^*)^{\frac{2-p}{2}}x + |q_i^*|^{-1}q_i^*\big|^{bp}}.
\end{aligned}$$

Let $p_i = q_{j_i}^i \in \mathcal{K}_i$ be such that $\mathrm{dist}(q_i^*,\mathcal{K}_i) = \mathrm{dist}(q_i^*,p_i)$ and set $\hat{\mathcal{K}}_i = \{q_1^i,\dots,q_{j_i-1}^i\}$. From (5.99) we infer

$$\begin{aligned}
&\mathrm{dist}(p_i,\hat{\mathcal{K}}_i) \le \mathrm{dist}(p_i,q_i^*) + \mathrm{dist}(q_i^*,\hat{\mathcal{K}}_i) \le 2\mathrm{dist}(q_i^*,\hat{\mathcal{K}}_i),\\
&u_i(p_i)|p_i|^{\frac{2}{N-2-2a}} < \varepsilon, \quad u_i(q_i^*) \le u_i(p_i)\left(\frac{\mathrm{dist}(p_i,\hat{\mathcal{K}}_i)}{\mathrm{dist}(q_i^*,\hat{\mathcal{K}}_i)}\right)^{\frac{N-2-2a}{2}}\frac{\omega_a(q_i^*)}{\omega_a(p_i)},
\end{aligned}$$

and finally that if $|p_i| \le 2$

$$\begin{aligned}
\varepsilon\left(\frac{|q_i^*|}{|p_i|}\right)^{\frac{2}{N-2-2a}} &> u_i(p_i)|q_i^*|^{\frac{2}{N-2-2a}}\varepsilon u_i(q_i^*)|q_i^*|^{\frac{2}{N-2-2a}}\left(\frac{\mathrm{dist}(q_i^*,\hat{\mathcal{K}}_i)}{\mathrm{dist}(p_i,\hat{\mathcal{K}}_i)}\right)^{\frac{N-2-2a}{2}}\frac{\omega_a(p_i)}{\omega_a(q_i^*)}\\
&\ge \mathrm{const}\, u_i(q_i^*)|q_i^*|^{\frac{2}{N-2-2a}} \to \infty.
\end{aligned}$$

Consequently there exists a positive constant c such that $|q_i^*|^{-1}\mathrm{dist}(q_i^*,\mathcal{K}_i) > c$, which is trivial in the case $|p_i| > 2$ and follows from the above estimate if $|p_i| \le 2$. Thus

$$\begin{aligned}
|q_i^*|^{-1-\frac{(N-2-2a)(2-p)}{4}}u_i(q_i^*)^{-\frac{2-p}{2}}\mathrm{dist}(q_i^*,\mathcal{K}_i)\varepsilon\big(u_i(q_i^*)|q_i^*|^{\frac{N-2-2a}{2}}\big)^{\frac{p-2}{2}}|q_i^*|^{-1}\mathrm{dist}(q_i^*,\mathcal{K}_i)&\\
\ge c\big(u_i(q_i^*)|q_i^*|^{\frac{N-2-2a}{2}}\big)^{\frac{p-2}{2}} \to \infty.&
\end{aligned}$$

For $|x| \le \frac{c}{4}|q_i^*|^{-\frac{(N-2-2a)(2-p)}{4}}u_i(q_i^*)^{-\frac{2-p}{2}}$ we have that

$$\begin{aligned}
v_i(x) &= u_i(q_i^*)^{-1}u_i\big(|q_i^*|^{1+\frac{(N-2-2a)(2-p)}{4}}u(q_i^*)^{\frac{2-p}{2}}x + q_i^*\big)\\
&\le u_i(q_i^*)^{-1}\omega_a\big(|q_i^*|^{1+\frac{(N-2-2a)(2-p)}{4}}u(q_i^*)^{\frac{2-p}{2}}x + q_i^*\big)\frac{u_i(q_i^*)}{\omega_a(q_i^*)}\\
&\le c\sup_{|x|\le\frac{c}{4}}\omega_a(x+q_i^*)\omega_a(q_i^*)^{-1} \le \mathrm{const}.
\end{aligned}$$

Up to a subsequence, we have that $q_i^* \to \bar{q}_1$ and v_i converges in $C^2_{\mathrm{loc}}(\mathbb{R}^N)$ to a solution of

$$-\Delta w = K(\bar{q}_1) w^{p-1} \text{ in } \mathbb{R}^N, \quad w(0) = 1.$$

This is impossible since the above equation has no solution for $p < 2^*$. The claim is thereby proved. The function v_1, defined by

$$v_1(x) := u(q^*)^{-1} u\big(u(q^*)^{-\frac{2}{N-2-2a}} x + q^*\big),$$

satisfies

$$\begin{aligned} &-\operatorname{div}\big(|x + u(q^*)^{\frac{2}{N-2-2a}} q^*|^{-2a} \nabla v_1\big) = \frac{K\big(u(q^*)^{-\frac{2}{N-2-2a}} x + q^*\big) v_1^{p-1}}{|x + u(q^*)^{\frac{2}{N-2-2a}} q^*|^{bp}}, \\ &v_1(0) = 1. \end{aligned}$$

For $|x| \le C_0^{-\frac{1}{N-2-2a}} u(q^*)^{\frac{2}{N-2-2a}} \operatorname{dist}(q^*, \mathcal{K})$ we obtain

$$\begin{aligned} \operatorname{dist}(u(q^*)^{-\frac{2}{N-2-2a}} x + q^*, \mathcal{K}) &\ge \operatorname{dist}(q^*, \mathcal{K}) - c C_0^{-\frac{1}{N-2-2a}} \operatorname{dist}(q^*, \mathcal{K}) \\ &\ge \operatorname{dist}(q^*, \mathcal{K})\big(1 - c C_0^{-\frac{1}{N-2-2a}}\big) \end{aligned}$$

and

$$\begin{aligned} v_1(x) &= u(q^*)^{-1} u\big(u(q^*)^{-\frac{2}{N-2-2a}} x + q^*\big) \\ &\le u(q^*)^{-1} \omega_a\big(u(q^*)^{-\frac{2}{N-2-2a}} x + q^*\big) \frac{u(q^*)}{\omega_a(q^*)} \left(\frac{\operatorname{dist}(q^*, \mathcal{K})}{\operatorname{dist}(u(q^*)^{-\frac{2}{N-2-2a}} x + q^*, \mathcal{K})}\right)^{\frac{N-2-2a}{2}} \\ &\le \omega_a(q^*)^{-1} \big(1 - c C_0^{-\frac{1}{N-2-2a}}\big)^{-\frac{N-2-2a}{2}}. \end{aligned}$$

Notice that $|q^*| < \text{const}\, C_1^{\frac{2}{N-2-2a}} C_0^{-\frac{2}{N-2-2a}}$ and

$$C_0^{-\frac{1}{N-2-2a}} u(q^*)^{\frac{2}{N-2-2a}} \operatorname{dist}(q^*, \mathcal{K}) > \left(\frac{1}{4} C_0\right)^{\frac{1}{N-2-2a}}.$$

Hence for any $\delta > 0$ we may choose C_0, depending on $a, b, N, \varepsilon, R, A_1, A_2, C_1$, such that

$$\omega_a(q^*)^{-1} \big(1 - C_0^{-\frac{1}{N-2-2a}}\big)^{-\frac{N-2-2a}{2}} \le 1 + \delta$$

and v_1 is $\varepsilon/4$-close in $C^{0,\gamma}(B_{2R}(0))$ to a solution of

$$\begin{aligned} &-\operatorname{div}\big(|x + u(q^*)^{\frac{2}{N-2-2a}} q^*|^{-2a} \nabla w\big) = K(q^*) \frac{w^{p-1}}{|x + u(q^*)^{\frac{2}{N-2-2a}} q^*|^{bp}} \text{ in } \mathbb{R}^N, \\ &w(0) = 1, \quad 0 \le w(x) \le 1 + \delta. \end{aligned}$$

If we choose δ small enough, depending on ε and R, then it is easy to see that any solution of the above equation is $\varepsilon/4$-close in $C^{0,\gamma}(B_{2R}(0)) \cap H^1_a(B_{2R}(0))$ to $z^{a,b}_{K(q^*)}$ and $u(q^*)^{\frac{2}{N-2-2a}} |q^*| \le \varepsilon/2$. This gives estimate (5.100). Case (B) can be reduced to case (A) using the Kelvin transform. □

Proposition 5.5. *Under the assumptions and notations of Lemma 5.12 there exists for any $0 < \varepsilon < 1$ and $R > 1$ a constant $C_0 = C_0(\varepsilon, R, N, a, b, A_1, A_2) > 0$ such that if u is a solution of (5.5) with*

$$\max_{x \in \mathbb{R}^N \cup \{\infty\}} \frac{u(x)}{\omega_a(x)} > C_0$$

then there exist $1 \leq k = k(u) < \infty$ and a set $\mathcal{S}(u) = \{q_1, q_2, \dots, q_k\} \subset \mathbb{R}^N \cup \{\infty\}$ such that for each $1 \leq j \leq k$ we have

(A) if $|q_j| \leq 1$

$$\left\| \frac{u(u(q_j)^{-\frac{2}{N-2-2a}} x + q_j)}{u(q_j)} - z^{a,b}_{K(q_j)} \right\|_{\substack{C^{0,\gamma}(B_{2R}(0)) \cap \\ H^1_a(B_{2R}(0))}} + |q_j| u(q_j)^{\frac{2}{N-2-2a}} < \varepsilon \tag{5.102}$$

(B) if $|q_j| > 1$

$$\left\| \frac{\tilde{u}(\tilde{u}(\tilde{q}_j)^{-\frac{2}{N-2-2a}} x + \tilde{q}_j)}{\tilde{u}(\tilde{q}_j)} - z^{a,b}_{K(q_j)} \right\|_{\substack{C^{0,\gamma}(B_{2R}(0)) \cap \\ H^1_a(B_{2R}(0))}} + |\tilde{q}_j| \tilde{u}(\tilde{q}_j)^{\frac{2}{N-2-2a}} < \varepsilon. \tag{5.103}$$

Moreover the sets

$$U_j := \begin{cases} B_{Ru(q_j)^{-\frac{2}{N-2-2a}}}(q_j) & \text{in case (A)} \\ Inv(B_{R\tilde{u}(\tilde{q}_j)^{-\frac{2}{N-2-2a}}}(\tilde{q}_j)) & \text{in case (B)} \end{cases} \quad \text{are disjoint.}$$

Furthermore, u satisfies

$$u(x) \leq C_0 \omega_a(x) \max_{1 \leq j \leq k} dist(x, q_j)^{-\frac{N-2-2a}{2}}.$$

Proof. Fix $\varepsilon > 0$ and $R > 1$. Let C_0 be as in Lemma 5.12. First we apply Lemma 5.12 with $\mathcal{K} = \emptyset$ and find $q_1 \in \mathbb{R}^N \cup \{\infty\}$ the maximum point of u/ω_a. If $u(x) \leq C_0 \omega_a(x) \mathrm{dist}(x, q_1)^{-\frac{N-2-2a}{2}}$ holds we stop here. Otherwise we apply again Lemma 5.12 to obtain q_2. From estimates (5.102) and (5.103) it follows that U_1 and U_2 are disjoint. We continue the process. Since $u \in L^p(\mathbb{R}^N, |x|^{-bp})$ and

$$\int_{U_j} \frac{K(x)}{|x|^{bp}} u(x)^p \, dx \geq \frac{1}{2A_1} \int_{B_R(0)} \frac{(z^{a,b}_{K(q_j)})^p}{|y + \varepsilon q_j/|q_j||^{bp}} \, dy \geq c(a, b, N),$$

where $c(a, b, N)$ is independent of q_j, u, $R > 1$ and $\varepsilon < 1$, we will stop after a finite number of steps. □

Proposition 5.6. *Under the assumptions and notations of Lemma 5.12 there exist for any $0 < \varepsilon < 1$ and $R > 1$ some positive constants $C_0 = C_0(\varepsilon, R, N, a, b, A_1, A_2)$ and $\delta = \delta(\varepsilon, R, N, a, b, A_1, A_2)$ such that if u is a solution of (5.5) with*

$$\max_{x\in\mathbb{R}^N\cup\{\infty\}} \frac{u(x)}{\omega_a(x)} > C_0$$

then

$$dist(q_j, q_\ell) \geq \delta \text{ for all } 1 \leq j \neq \ell \leq k,$$

where $q_j = q_j(u)$, $q_\ell = q_\ell(u)$ *and* $k = k(u)$ *are given in Proposition 5.5.*

Proof. To obtain a contradiction we assume that for some constants ε, R, A_1 and A_2 there exist sequences K_i and u_i satisfying the assumptions of Proposition 5.6 such that

$$\lim_{i\to\infty} \min_{j\neq\ell} \operatorname{dist}(q_j(u_i), q_\ell(u_i)) = 0.$$

We may assume that

$$\sigma_i := \operatorname{dist}(q_1(u_i), q_2(u_i)) = \min_{j\neq\ell} \operatorname{dist}(q_j(u_i), q_\ell(u_i)) \to 0 \text{ as } i \to \infty. \tag{5.104}$$

Let us denote $q_j(u_i)$ by q_j^i. Since $U_1(u_i)$ and $U_2(u_i)$ are disjoint and (5.104) holds we have that $u_i(q_1^i) \to \infty$ and $u_i(q_2^i) \to \infty$. Therefore we can pass to a subsequence still denoted by $\{u_i\}$ and find $R_i \to \infty$, $\varepsilon_i \to 0$ such that either $q_1^i = q_1(u_i) \to 0$ or $|q_1^i| \to \infty$, and for $j = 1, 2$

$$\left\| \frac{u_i\big(u_i(q_j^i)^{\frac{2}{N-2-2a}} x + q_j^i\big)}{u_i(q_j^i)} - z^{a,b}_{K(q_j^i)} \right\|_{C^{0,\gamma}(B_{2R_i}(0))} + |q_j^i| u_i(q_j^i)^{\frac{2}{N-2-2a}} < \varepsilon_i \text{ if } q_1^i \to 0$$

$$\left\| \frac{\tilde u_i\big(\tilde u_i(\tilde q_j^i)^{\frac{2}{N-2-2a}} x + \tilde q_j^i\big)}{\tilde u_i(\tilde q_j^i)} - z^{a,b}_{K(\tilde q_j^i)} \right\|_{C^{0,\gamma}(B_{2R_i}(0))} + |\tilde q_j^i| \tilde u_i(\tilde q_j^i)^{\frac{2}{N-2-2a}} < \varepsilon_i \text{ if } |q_1^i| \to \infty. \tag{5.105}$$

We first consider the case $q_1^i \to 0$. Since $U_1(u_i)$ and $U_2(u_i)$ are disjoint we have that

$$\sigma_i > c(N) \max_{j=1,2} \{R_i u_i(q_j^i)^{-\frac{2}{N-2-2a}}\}. \tag{5.106}$$

From (5.105) and (5.106) we get that $\sigma_i^{-1}|q_j^i| < \frac{\varepsilon_i}{c(N)R_i} \to 0$ for $j = 1, 2$ and obtain the contradiction

$$\frac{1}{2} < |\sigma_i^{-1}(q_2^i - q_1^i)| \to 0.$$

Performing the same analysis as above for the Kelvin transform $\tilde u$ of u leads to a contradiction if $\tilde q_1^i \to 0$. □

Remark 5.5. Propositions 5.5 and 5.6 imply that there are only finitely many blow-up points and all of them are isolated.

Proposition 5.7. *Suppose* $\{K_i\}$ *and* $a \in](N-4)/2, (N-2)/2[$ *satisfy the assumptions of Lemma 5.12 and Proposition 5.4. Let* $\{u_i\}$ *be solutions of* (P_i) *with* $\Omega = \mathbb{R}^N$. *Then after passing to a subsequence either* $\{u_i/\omega_a\}$ *stays bounded in* $L^\infty(\mathbb{R}^N)$ *or* $\{u_i\}$ *has precisely one blow-up point, which can be at 0 or at* ∞.

Proof. Suppose that $\{u_i/\omega_a\}$ is not uniformly bounded in $L^\infty(\mathbb{R}^N)$, otherwise there is nothing to prove. Consequently we may apply Proposition 5.5 and Proposition 5.6 to obtain isolated points $\{q_1^i, \dots, q_{k(i)}^i\}$ satisfying (5.102) and (5.103) with $R_i \to \infty$ and $\varepsilon_i \to 0$. To obtain a contradiction, we assume that up to a subsequence $k(i) \geq 2$. Since $u(q_j^i)/\omega_a(q_j^i) \to \infty$ for $j = 1, 2$ and $\mathrm{dist}(q_1^i, q_2^i) \geq \delta > 0$ we may assume $q_1^i \to 0$ and $q_2^i \to \infty$ and $k(i) = 2$ as $i \to \infty$. From Proposition 5.4 and Remark 5.5 they are isolated simple blow-up points. From Proposition 5.3 we infer that

$$\lim_{i\to\infty} u_i(q_1^i)u_i(x) = h(x) \text{ in } C^0_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\}),$$
$$\mathrm{div}\,(|x|^{-2a}\nabla h) = 0 \text{ in } \mathbb{R}^N \setminus \{0\}.$$

Using Theorem 5.9 for h and its Kelvin transform and the maximum principle we obtain for some $a_1, a_2 > 0$

$$h(x) = a_1|x|^{2+2a-N} + a_2.$$

We may now proceed as in the proof of Proposition 5.4 to see that

$$\int_{\partial B_\sigma(q_1^i)} B(\sigma, x, u_i, \nabla u_i) = o(u_i(q_1^i)^{-2}).$$

Multiplying by $u_i(q_1^i)^2$ and letting $i \to \infty$ we find that

$$\int_{\partial B_\sigma} B(\sigma, x, h, \nabla h) = 0.$$

This contradicts for small σ Proposition 5.1 and completes the proof. □

Proposition 5.8. *Suppose $K \in C^2(\mathbb{R}^N)$ satisfies (5.10)-(5.12),*

$$a \geq 0, \quad \frac{N-4}{2} < a < \frac{N-2}{2}, \quad \text{and} \quad \frac{4}{N-2-2a} < p < 2^*.$$

Then there exists $C_K > 0$ such that for any $t \in (0,1]$ and any solution u_t of

$$-\mathrm{div}\,(|x|^{-2a}\nabla u) = (1 + t\,(K(x)-1))\frac{u^{p-1}}{|x|^{bp}}, \quad u > 0 \text{ in } D_a^{1,2}(\mathbb{R}^N) \tag{P_t}$$

there holds

$$\|u_t\|_E < C_K \tag{5.107}$$

and

$$C_K^{-1} < u_t\omega_a^{-1} < C_K. \tag{5.108}$$

Proof. The bound in (5.108) follows from (5.107) and Harnack's inequality in [58]. The estimate in Lemma 5.6 shows that $(1+t(K(x)-1))u^{p-2}|x|^{-bp}$ belongs to the required class of potentials in [58]. To show that u_t/ω_a is bounded in $L^\infty(\mathbb{R}^N)$ we argue by contradiction and may assume in view of Proposition 5.7 that there exists a sequence $\{t_i\} \subset (0,1]$ converging to $t_0 \in [0,1]$ as $i \to \infty$ such that u_{t_i} has precisely one blow-up point (x_i), which can be supposed to be zero using the Kelvin transform. Corollary 5.3 yields

$$0 = \int_{\mathbb{R}^N} x \cdot \nabla K(x) \frac{u_{t_i}(x)^p}{|x|^{bp}}\, dx.$$

Since 0 is assumed to be the only blow-up point, the Harnack inequality and (5.76) yield, for any $\sigma \in (0,1)$,

$$\begin{aligned}\left|\int_{B_\sigma(x_i)} x \cdot \nabla K(x) \frac{u_{t_i}(x)^p}{|x|^{bp}}\, dx\right| &= \left|\int_{\mathbb{R}^N \setminus B_\sigma(x_i)} x \cdot \nabla K(x) \frac{u_{t_i}(x)^p}{|x|^{bp}}\, dx\right| \\ &\le C(\sigma)\Big(u_{t_i}(x_i)^{-p}\Big).\end{aligned}$$

We have that from Taylor expansion, (5.62), and (5.11)

$$|\nabla K(x_i)| \le \mathrm{const}|x_i| = o\Big(u_{t_i}(x_i)^{-\frac{2}{N-2-2a}}\Big) \tag{5.109}$$

and

$$\begin{aligned}&\left|\int_{B_\sigma(x_i)} \nabla K(x) \cdot x \frac{u_{t_i}^p}{|x|^{bp}}\, dx\right| \\ &= \left|\int_{B_\sigma(x_i)} \nabla K(x_i) \cdot x \frac{u_{t_i}^p}{|x|^{bp}}\, dx + \int_{B_\sigma(x_i)} D^2K(x_i)(x-x_i) \cdot x \frac{u_{t_i}^p}{|x|^{bp}}\, dx\right. \\ &\quad \left. + \int_{B_\sigma(x_i)} o(|x-x_i|) \cdot x \frac{u_{t_i}^p}{|x|^{bp}}\, dx\right|.\end{aligned}$$

From Lemma 5.11 and (5.109) we infer

$$\left|\int_{B_\sigma(x_i)} \nabla K(x_i) \cdot x \frac{u_{t_i}^p}{|x|^{bp}}\, dx + \int_{B_\sigma(x_i)} o(|x-x_i|) \cdot x \frac{u_{t_i}^p}{|x|^{bp}}\, dx\right| = o\Big(u_{t_i}(x_i)^{-\frac{4}{N-2-2a}}\Big).$$

Hence

$$\int_{B_\sigma(x_i)} D^2K(x_i)(x-x_i) \cdot x \frac{u_{t_i}^p}{|x|^{bp}}\, dx = o\Big(u_{t_i}(x_i)^{-\frac{4}{N-2-2a}}\Big).$$

Since by Lemma 5.11

$$\int_{r_i \le |x-x_i| \le \sigma} D^2K(x_i)(x-x_i) \cdot x \frac{u_{t_i}^p}{|x|^{bp}}\, dx = o\Big(u_{t_i}(x_i)^{-\frac{4}{N-2-2a}}\Big)$$

we have

$$\int_{B_{r_i}(x_i)} D^2K(x_i)(x-x_i)\cdot x\frac{u_{t_i}^p}{|x|^{bp}}\,dx = o\Big(u_{t_i}(x_i)^{-\frac{4}{N-2-2a}}\Big). \tag{5.110}$$

Making in (5.110) the change of variables $x = u_{t_i}(x_i)^{-2/(N-2-2a)}y + x_i$ and using Proposition 5.2

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} D^2K(0)y\cdot y|y|^{-bp} z^{a,b}_{1+t_0(K(0)-1)}(y)^p\,dy \\ &= \Delta K(0)\int_{\mathbb{R}^N} |y|^{2-bp} z^{a,b}_{1+t_0(K(0)-1)}(y)^p\,dy \end{aligned}$$

which is not possible in view (5.11). □

Proof of Theorem 5.1. It follows from Proposition 5.8 and Lemma 5.1. □

We define $f_{K,\varepsilon} : D_a^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ by

$$\begin{aligned} f_{K,\varepsilon}(u) &= f_0(u) - \varepsilon G_K(u) \\ f_0(u) &= \frac{1}{2}\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2 - \frac{1}{p}\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^{bp}} \\ G_K(u) &= \frac{1}{p}\int_{\mathbb{R}^N}\frac{K(x)|u|^p}{|x|^{bp}}. \end{aligned}$$

We will use the notation f_ε (respectively G) instead of $f_{K,\varepsilon}$ (respectively G_K) whenever there is no possibility of confusion. Let us denote by Z the manifold

$$Z = \{z_\mu = z^{a,b}_{1,\mu} : \mu > 0\}$$

of the solutions to (5.5) with $K \equiv 1$.

Lemma 5.13. *Suppose $p > 3$. There exist constants $\rho_0, \varepsilon_0, C > 0$, and smooth functions*

$$\begin{aligned} w = w(\mu,\varepsilon) : &\quad (0,+\infty)\times(-\varepsilon_0,\varepsilon_0) \longrightarrow D_a^{1,2}(\mathbb{R}^N) \\ \eta = \eta(\mu,\varepsilon) : &\quad (0,+\infty)\times(-\varepsilon_0,\varepsilon_0) \longrightarrow \mathbb{R} \end{aligned}$$

such that for any $\mu > 0$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$

$$w(\mu,\varepsilon) \text{ is orthogonal to } T_{z_\mu}Z \tag{5.111}$$

$$f'_\varepsilon(z_\mu + w(\mu,\varepsilon)) = \eta(\mu,\varepsilon)\dot{\xi}_\mu \tag{5.112}$$

$$|\eta(\mu,\varepsilon)| + \|w(\mu,\varepsilon)\|_{D_a^{1,2}(\mathbb{R}^N)} \le C\,|\varepsilon| \tag{5.113}$$

$$\|\dot{w}(\mu,\varepsilon)\|_{D_a^{1,2}(\mathbb{R}^N)} \le C(1+\mu^{-1})\,|\varepsilon|, \tag{5.114}$$

where $\dot{\xi}_\mu$ denotes the normalized tangent vector $\frac{d}{d\mu}z_\mu$ and $\dot{w}$ stands for the derivative of w with respect to μ. Moreover, (w,η) is unique in the sense that there exists $\rho_0 > 0$ such that if $(v,\tilde{\eta})$ satisfies $\|v\|_{D_a^{1,2}(\mathbb{R}^N)} + |\tilde{\eta}| < \rho_0$ and (5.111)-(5.112) for some $\mu > 0$ and $|\varepsilon| \le \varepsilon_0$, then $v = w(\mu,\varepsilon)$ and $\tilde{\eta} = \eta(\mu,\varepsilon)$.

Proof. Existence, uniqueness, and estimate (5.113) are proved in Chapter 4. In fact w and η are implicitly defined by $H(\mu, w, \eta, \varepsilon) = (0,0)$ where

$$H : (0,\infty) \times D^{1,2}_a(\mathbb{R}^N) \times \mathbb{R} \times \mathbb{R} \to D^{1,2}_a(\mathbb{R}^N) \times \mathbb{R}$$
$$H(\mu, w, \eta, \varepsilon) := (f'_\varepsilon(z_\mu + w) - \eta\dot{\xi}_\mu, (w, \dot{\xi}_\mu)).$$

Let us now show estimate (5.114). There exists a positive constant C_* such that for any $\mu > 0$ (see [48])

$$\left\| \left(\frac{\partial H}{\partial(w,\eta)}(\mu, 0, 0, 0) \right)^{-1} \right\| \le C_*.$$

Since $\dot{w}$ satisfies

$$\begin{pmatrix} \dot{w} \\ \dot{\eta} \end{pmatrix} = -\left(\frac{\partial H}{\partial(w,\eta)} \right)^{-1} \Bigg|_{(\mu,w,\eta,\varepsilon)} \cdot \frac{\partial H}{\partial \mu}\Bigg|_{(\mu,w,\eta,\varepsilon)}$$

we have for ε small using (5.113) and the fact that $f_0 \in C^3$

$$\begin{aligned}
\|\dot{w}(\mu,\varepsilon)\| &\le C_* \frac{\partial H}{\partial \mu}\Bigg|_{(\mu,w,\eta,\varepsilon)} \\
&\le C_* \left(\left\| f''_\varepsilon(z_\mu + w(\mu,\varepsilon))\dot{z}_\mu - \eta(\mu,\varepsilon)\frac{d}{d\mu}\dot{\xi}_\mu \right\| + \left| \left(w(\mu,\varepsilon), \frac{d}{d\mu}\dot{\xi}_\mu \right) \right| \right) \\
&\le C(1+\mu^{-1})|\varepsilon| + \|f''_0(z_\mu + w(\mu,\varepsilon))\dot{z}_\mu\| \\
&\le C(1+\mu^{-1})|\varepsilon| + O(\|w(\mu,\varepsilon)\|)\|\dot{z}_\mu\| \\
&\le C(1+\mu^{-1})|\varepsilon|.
\end{aligned}$$

This ends the proof. □

Corollary 5.4. *Suppose $p > 3$ and K satisfies the assumptions of Proposition 5.8. Then there exist $t_0 > 0$ and $R_0 > 0$ such that any solution u_t of (P_t) for $t \le t_0$ is of the form $z_\mu + w(\mu, t)$, where $1/R_0 < \mu < R_0$.*

Proof. First we show that there exists $R_1 > 0$ and $t_1 > 0$ such that any solution u_t of (P_t) for $t < t_1$ satisfies

$$\operatorname{dist}(u_t, Z_{R_1}) < \rho_0,$$

where by dist we mean the distance in the $D^{1,2}_a(\mathbb{R}^N)$-norm, ρ_0 is given in Lemma 5.13, and $Z_{R_1} := \{z_\mu \,|\, 1/R_1 < \mu < R_1\}$. By contradiction, assume there exist $R_i \to \infty$, $t_i \to 0$, and solutions u_{t_i} of (P_t) such that $\operatorname{dist}(u_{t_i}, Z_{R_i})\varepsilon\rho_0$. From (5.107) we can pass to a subsequence converging weakly in $D^{1,2}_a(\mathbb{R}^N)$ to some $\bar{u}$; since in view of the regularity results of [50] $\{u_t\}$ is bounded in $C^{0,\gamma}$ and such a bound excludes any possibility of concentration, the convergence is actually strong and $\operatorname{dist}(\bar{u}, Z) \ge \rho_0$. Furthermore, $\bar{u}$ solves (P_t) with $t = 0$ and hence $\bar{u} \in Z$, which is impossible.
Fix a solution u_t of (P_t) for some $t < t_1$. A short computation shows

$$\lim_{\mu\to 0} \mathrm{dist}(z_\mu, u_t)^2 = \lim_{\mu\to\infty} \mathrm{dist}(z_\mu, u_t)^2 = \|z_1\|^2 + \|u_t\|^2 > \rho_0^2.$$

Consequently there exists $R_0 > 0$ independent of t and $z_\mu \in Z_{R_0}$ such that

$$\mathrm{dist}(u_t, Z) = \|u_t - z_\mu\| \text{ and } u_t - z_\mu \in T_{z_\mu} Z^\perp.$$

Since u_t solves (P_t) we have $f_t'(z_\mu + u_t - z_\mu) = 0$ and the uniqueness in Lemma 5.13 yields the claim. □

5.5 Leray-Schauder degree

We introduce the Melnikov function

$$\Gamma_K(\tau) = \frac{1}{p} \int_{\mathbb{R}^N} K(x) \frac{z_\tau^p}{|x|^{bp}}.$$

It is known (for details see [48]) that it is possible to extend the C^2- function Γ_K by continuity to $\tau = 0$ and

$$\Gamma_K'(0) = 0 \text{ and } \Gamma_K''(0) = \frac{\Delta K(0)}{Np} \int_{\mathbb{R}^N} |x|^2 \frac{z_1(x)^p}{|x|^{bp}}. \tag{5.115}$$

Furthermore, using the Kelvin transform, we find

$$\Gamma_K(\tau) = \Gamma_{\tilde{K}}(\tau^{-1}) \quad \text{where} \quad \tilde{K}(x) = K(x/|x|^2). \tag{5.116}$$

We define for small t the function $\Phi_{K,t}(\mu) := f_{K,t}(z_\mu + w(\mu, t))$ and will denote it by Φ_t whenever there is no possibility of confusion.

Lemma 5.14. *Let $p > 3$ and assume Γ_K has only non-degenerate critical points. Then there exists $t_1 > 0$ such that for any $0 < t < t_1$ any solution u_t of (P_t) is of the form $u_t = z_{\mu_t} + w(\mu_t, t)$, where $\Phi_{K,t}'(\mu_t) = 0$ and $\mu_t \in (R_0^{-1}, R_0)$ for some positive R_0. Moreover, up to a subsequence as $t \to 0$*

$$|\mu_t - \bar{\mu}| = O(t), \tag{5.117}$$

where $\bar{\mu}$ is a critical point of Γ_K. Viceversa, for any critical point $\bar{\mu}$ of Γ_K and for any $0 < t < t_1$ there exists one and only one critical point μ_t of $\Phi_{K,t}$ such that (5.117) holds.

Proof. By Corollary 5.4 any solution u_t of (P_t) is of the form $u_t = z_{\mu_t} + w(\mu_t, t)$, where $\Phi_t'(\mu_t) = 0$ and $R_0^{-1} < \mu_t < R_0$. Using the Taylor expansion and (5.113) - (5.114), we have that for $R_0^{-1} < \mu < R_0$

$$\begin{aligned}
\Phi_t'(\mu) &= f_t'(z_\mu + w(\mu,t))(\dot{z}_\mu + \dot{w}(\mu,t)) \\
&= f_t'(z_\mu)(\dot{z}_\mu + \dot{w}(\mu,t)) + (f_t''(z_\mu)w(\mu,t), \dot{z}_\mu + \dot{w}(\mu,t)) + O(\|w(\mu,t)\|^2)) \\
&= -tG'(z_\mu)(\dot{z}_\mu + \dot{w}(\mu,t)) + (f_0''(z_\mu)w(\mu,t), \dot{w}(\mu,t))) \\
&\quad - t(G''(z_\mu)w(\mu,t), \dot{z}_\mu + \dot{w}(\mu,t)) + O(\|w(\mu,t)\|^2)) \\
&= -t\Gamma'(\mu) + O(t^2).
\end{aligned} \tag{5.118}$$

Fix a sequence (t_n) converging to 0. Since μ_t is bounded, we may assume that (μ_{t_n}) converges to $\bar{\mu}$. From expansion (5.118) we have that

$$0 = \Phi'_{t_n}(\mu_{t_n}) = -t_n(\Gamma'(\mu_{t_n}) + O(t_n))$$

hence $\bar{\mu}$ is a critical point of Γ. A further expansion yields

$$0 = \Phi'_{t_n}(\mu_{t_n}) - t_n(\Gamma''(\bar{\mu})(\mu_{t_n} - \bar{\mu}) + o(\mu_{t_n} - \bar{\mu})) + O(t_n^2)$$

which gives for $n \to \infty$

$$(\mu_{t_n} - \bar{\mu})(\Gamma''(\bar{\mu}) + o(1)) = O(t_n)$$

proving (5.117) for $\Gamma''(\bar{\mu}) \neq 0$. Viceversa let $\bar{\mu}$ be a critical point of Γ. Arguing as above we find as $\mu \to \bar{\mu}$ and for any $0 < t < t_1$

$$\Phi'_t(\mu) = t(\mu - \bar{\mu})(\Gamma''(\bar{\mu}) + o(1)) + O(t^2)$$

hence there exists μ_t such that

$$\mu_t = \bar{\mu} - (\Gamma''(\bar{\mu}) + o(1))^{-1} O(t) \quad \text{and} \quad \Phi'_t(\mu_t) = 0.$$

To prove uniqueness of such a μ_t, we follow [20] and expand Φ_t in a critical point μ_t

$$\begin{aligned}
\Phi''_t(\mu_t) &= \big(f''_t(z_{\mu_t} + w(\mu_t, t))(\dot{z}_{\mu_t} + \dot{w}(\mu_t, t)), (\dot{z}_{\mu_t} + \dot{w}(\mu_t, t))\big) \\
&= \big(f''_0(z_{\mu_t} + w(\mu_t, t))(\dot{z}_{\mu_t} + \dot{w}(\mu_t, t)), (\dot{z}_{\mu_t} + \dot{w}(\mu_t, t))\big) \\
&\quad - t\big(G''(z_{\mu_t} + w(\mu_t, t))(\dot{z}_{\mu_t} + \dot{w}(\mu_t, t)), (\dot{z}_{\mu_t} + \dot{w}(\mu_t, t))\big) \\
&= (f''_0(z_{\mu_t})\dot{w}(\mu_t, t), \dot{w}(\mu_t, t)) + (f'''_0(z_{\mu_t})w(\mu_t, t)(\dot{z}_{\mu_t} + \dot{w}(\mu_t, t)), \dot{z}_{\mu_t} + \dot{w}(\mu_t, t)) \\
&\quad - t\big(G''(z_{\mu_t} + w(\mu_t, t))(\dot{z}_{\mu_t} + \dot{w}(\mu_t, t)), (\dot{z}_{\mu_t} + \dot{w}(\mu_t, t))\big) \\
&= (f'''_0(z_{\mu_t})w(\mu_t, t)\dot{z}_{\mu_t}, \dot{z}_{\mu_t}) - t\big(G''(z_{\mu_t})\dot{z}_{\mu_t}, \dot{z}_{\mu_t}\big) + O(t^2).
\end{aligned} \tag{5.119}$$

Since any critical point μ_t of Φ_t gives rise to a critical point $z_{\mu_t} + w(\mu_t, t)$ of f_t, we have that

$$\begin{aligned}
0 &= (f'_t(z_{\mu_t} + w(\mu_t, t)), \ddot{z}_{\mu_t}) \\
&= (f'_t(z_{\mu_t}) + f''_t(z_{\mu_t})w(\mu_t, t) + O(\|w(\mu_t, t)\|^2), \ddot{z}_{\mu_t}) \\
&= -t(G'(z_{\mu_t}), \ddot{z}_{\mu_t}) + (f''_0(z_{\mu_t})w(\mu_t, t), \ddot{z}_{\mu_t}) + O(t^2).
\end{aligned} \tag{5.120}$$

Differentiating $f''_0(z_{\mu_t})\dot{z}_{\mu_t} = 0$ and testing with $w(\mu_t, t)$ we obtain

$$0 = (f'''_0(z_{\mu_t})\dot{z}_{\mu_t}\dot{z}_{\mu_t}, w(\mu_t, t)) + (f''_0(z_{\mu_t})\ddot{z}_{\mu_t}, w(\mu_t, t)). \tag{5.121}$$

Putting together (5.120) and (5.121) we get

$$(f'''_0(z_{\mu_t})\dot{z}_{\mu_t}\dot{z}_{\mu_t}, w(\mu_t, t)) = -t(G'(z_{\mu_t}), \ddot{z}_{\mu_t}) + O(t^2)$$

hence in view of (5.119)

$$\begin{aligned}\Phi_t''(\mu_t) &= -t(G'(z_{\mu_t}),\ddot{z}_{\mu_t}) - t\big(G''(z_{\mu_t})\dot{z}_{\mu_t},\dot{z}_{\mu_t}\big) + O(t^2)\\ &= -t\Gamma''(\mu_t) + O(t^2).\end{aligned} \tag{5.122}$$

To prove uniqueness, we choose $\delta > 0$ such that $\mathrm{sgn}\Gamma''(\mu) = \mathrm{sgn}\Gamma''(\bar\mu) \neq 0$ for any $|\mu - \bar\mu| < \delta$. From (5.122), there exists $t(\delta) > 0$ such that if $t < t(\delta)$ and μ_t is a critical point of Φ_t such that $|\mu_t - \bar\mu| < \delta$, then

$$\mathrm{sgn}\Phi_t''(\mu_t) = -\mathrm{sgn}\Gamma''(\bar\mu).$$

From (5.118) we have that for $t < t(\delta)$

$$\begin{aligned}\mathrm{sgn}\Gamma''(\bar\mu) &= \deg(\Gamma', B_\delta(\bar\mu), 0) = \deg(-\Phi_t', B_\delta(\bar\mu), 0)\\ &= -\sum_{\substack{y\in B_\delta(\bar\mu)\\ \Phi_t'(y)=0}} \mathrm{sgn}\Phi_t''(y) = \#\{y\in B_\delta(\bar\mu):\ \Phi_t'(y)=0\}\,\mathrm{sgn}\Gamma''(\bar\mu).\end{aligned}$$

Hence $\#\{y \in B_\delta(\bar\mu):\ \Phi_t'(y) = 0\} = 1$, proving uniqueness. □

Lemma 5.15. *For any $K \in L^\infty(\mathbb{R}^N)$ the operator*

$$L_K:\ u \mapsto \big(-\operatorname{div}(|x|^{-2a}\nabla)\big)^{-1}\frac{K(x)}{|x|^{bp}}|u|^{p-2}u$$

is compact from E to E.

Proof. Let $\{u_n\}$ be a bounded sequence in E and set $v_n = L_K(u_n)$, i.e.

$$-\operatorname{div}(|x|^{-2a}\nabla v_n) = \frac{K(x)}{|x|^{bp}}|u_n|^{p-2}u_n.$$

By Caffarelli-Kohn-Nirenberg inequality, $\{v_n\}$ is bounded in $D_a^{1,2}(\mathbb{R}^N)$ and passing to a subsequence we may assume that it converges weakly in $D_a^{1,2}(\mathbb{R}^N)$ and pointwise almost everywhere to some limit $v \in D_a^{1,2}(\mathbb{R}^N)$. Since $\{u_n\}$ is uniformly bounded in $L^\infty(B_3(0))$, from [50] the sequence $\{v_n\}$ is uniformly bounded in $C^{0,\gamma}(B_2(0))$. Using the Kelvin transform we arrive at

$$\begin{aligned}-\operatorname{div}(|x|^{-2a}\nabla\tilde v_n) &= |x|^{-(N+2+2a)+bp}K(x/|x|^2)\,|u_n(x/|x|^2)|^{p-2}u_n(x/|x|^2)\\ &= K(x/|x|^2)\frac{|\tilde u_n|^{p-2}\tilde u_n}{|x|^{bp}}.\end{aligned}$$

Since $\{u_n\}$ is uniformly bounded in E, $\{\tilde u_n\}$ is uniformly bounded in $L^\infty(B_3(0))$ and hence from [50] the sequence $\{\tilde v_n\}$ is uniformly bounded in $C^{0,\gamma}(B_2(0))$. Since a uniform bound in $C^{0,\gamma}(B_2(0))$ implies equicontinuity and

$$||(v_n - v_m)\omega_a^{-1}||_{C^0(\mathbb{R}^N\setminus B_1(0))} \le \mathrm{const}\,||\tilde v_n - \tilde v_m||_{C^0(B_1(0))}$$

from the Ascoli-Arzelà Theorem there exists a subsequence $\{v_n\}$ strongly converging in $C^0(\mathbb{R}^N,\omega_a)$ to v. Moreover, the $C^0(\mathbb{R}^N,\omega_a)$-convergence excludes any possibility of concentration at 0 or at ∞ and $\{v_n\}$ converges strongly in $D_a^{1,2}(\mathbb{R}^N)$. □

From Proposition 5.8, there exists a positive constant C_K such that $\|u\|_E < C_K$ and $C_K^{-1} < u\omega_a^{-1}$ for any solution u of (P_t) uniformly with respect to $t \in (0,1]$. By the above lemma, the Leray-Schauder degree $\deg(Id - L_K, \mathcal{B}_K, 0)$ is well-defined, where $\mathcal{B}_K := \{u \in E : \|u\|_E < C_K, C_K^{-1} < u\omega_a^{-1}\}$.

Theorem 5.10. *Under the assumptions of Proposition 5.8 and for $p > 3$ we have*

$$\deg(Id - L_K, \mathcal{B}_K, 0) = -\frac{\mathrm{sgn}\Delta K(0) + \mathrm{sgn}\Delta \tilde{K}(0)}{2}.$$

Proof. By transversality, we can assume that Γ_K has only non-degenerate critical points. If not, we proceed with a small perturbation of K. By Proposition 5.8 and the homotopy invariance of the Leray-Schauder degree, for $0 < t < t_1$

$$\deg(Id - L_K, \mathcal{B}_K, 0) = \deg(Id - L_{tK}, \mathcal{B}_K, 0).$$

By Lemma 5.14 we have

$$\deg(Id - L_{tK}, \mathcal{B}_K, 0) = \sum_{\mu \in (\Phi'_{t,K})^{-1}(0)} (-1)^{\mathfrak{m}(z_\mu + w(\mu,t), f_{t,K})}$$

where $\mathfrak{m}(z_\mu + w(\mu,t), f_{t,K})$ denotes the Morse index of $f_{t,K}$ in $z_\mu + w(\mu,t)$. We will only sketch the computation of $\mathfrak{m}(z_\mu + w(\mu,t), f_{t,K})$ and refer to [7, 20, 67] for details. The spectrum of $f_0''(z_\mu)$ is completely known (see [48]) and $D_a^{1,2}(\mathbb{R}^N)$ is decomposed in $\langle z_\mu \rangle \oplus T_{z_\mu}Z \oplus \langle z_\mu, T_{z_\mu}Z\rangle^\perp$, where z_μ is an eigenfunction of $f_0''(z_\mu)$ with corresponding eigenvalue $-(p-2)$, $T_{z_\mu}Z = \ker(f_0''(z_\mu))$, and $f_0''(z_\mu)$ restricted to the orthogonal complement of $\langle z_\mu, T_{z_\mu}Z\rangle$ is bounded below by a positive constant. Consequently, to compute the Morse index $\mathfrak{m}(z_\mu + w(\mu,t), f_{t,K})$ for small t it is enough to know the behavior of $f''_{t,K}(z_\mu + w(\mu,t))$ along $T_{z_\mu}Z$. From the expansion

$$f_{t,K}(z_\mu + w(\mu,t)) = f_0(z_\mu) - t\Gamma_K(\mu) + o(t^2) = \text{const} - t\Gamma_K(\mu) + o(t^2)$$

we have that for t small

$$\mathfrak{m}(z_\mu + w(\mu,t), f_{t,K}) = 1 + \begin{cases} 1 & \text{if } \Gamma_K''(\mu) > 0 \\ 0 & \text{if } \Gamma_K''(\mu) < 0. \end{cases} \tag{5.123}$$

From (5.123) and Lemma 5.14, we know that for t small

$$\begin{aligned}\sum_{\mu \in (\Phi'_{t,K})^{-1}(0)} (-1)^{\mathfrak{m}(z_\mu + w(\mu,t), f_{t,K})} &= -\sum_{\mu \in (\Gamma_K')^{-1}(0)} (-1)^{\mathfrak{m}(\mu, -\Gamma_K)} \\ &= \deg(\Gamma_K', ((R_0+1)^{-1}, R_0+1), 0),\end{aligned}$$

where R_0 is given in Lemma 5.14. From (5.115) we obtain for $\mu \to 0$

$$\Gamma_K'(\mu) = \Gamma_K''(0)\mu + o(\mu) = \text{const}\Delta K(0)\mu + o(\mu).$$

Hence $\mathrm{sgn}\Gamma'_K((R_0+1)^{-1}) = \mathrm{sgn}\Delta K(0)$. Using (5.116) for obtain for $\mu \to \infty$

$$\Gamma'_K(\mu) = -\mu^{-2}\Gamma'_{\tilde{K}}(\mu^{-1}) = -\mathrm{const}\Delta\tilde{K}(0)\mu^{-3} + o(\mu^{-3}).$$

Therefore $\mathrm{sgn}\Gamma'_K((R_0+1)) = -\mathrm{sgn}\Delta\tilde{K}(0)$ and

$$\deg(\Gamma'_K, ((R_0+1)^{-1}, R_0+1), 0) = -\frac{\mathrm{sgn}\Delta K(0) + \mathrm{sgn}\Delta\tilde{K}(0)}{2},$$

which proves the claim. □

Proof of Theorem 5.2. It follows directly from Theorem 5.10 and Lemma 5.1. □

6 About an equation involving Hardy inequality and critical Sobolev exponent

In this chapter we present the results proved in [2] about the following class of problems

$$\begin{cases} -\Delta u = \Big(\dfrac{A+h(x)}{|x|^2}\Big)u + k(x)u^{2^*-1}, \ x \in \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N, \ \text{ and } u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \end{cases} \tag{6.1}$$

where $N \geq 3$, $2^* = \frac{2N}{N-2}$ and h, k are continuous bounded functions, for which we will state appropriate complementary hypotheses. Here $\mathcal{D}^{1,2}(\mathbb{R}^N)$ denotes the closure space of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$||u||_{\mathcal{D}^{1,2}(\mathbb{R}^N)} := \Big(\int_{\mathrm{Re}^N} |\nabla u|^2 dx\Big)^{1/2}.$$

By the Sobolev inequality we can see that $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is the class of functions in $L^{2^*}(\mathbb{R}^N)$ the distributional gradient of which satisfies $\big(\int_{\mathbb{R}^N} |\nabla u|^2 dx\big)^{1/2} < \infty$.

For $h \equiv 0, k \equiv 1$ the problem is studied by S. Terracini in [82]. In chapter 4, following [48], we proved the existence of a positive solution in the case $h = 0$ by using the perturbative method by Ambrosetti-Badiale in [8], even for a more general class of differential operators related to the Caffarelli-Kohn-Nirenberg inequalities that contains the operator in equation (6.1). By the perturbative nature of the method, the solutions found in [48] are close to some radial solutions to the unperturbed problem. On the other hand, in [77] Smets obtains the existence of a positive solution for problem (6.1) with $h = 0$, k bounded, $k(0) = \lim_{|x|\to\infty} k(x)$ and dimension $N = 4$.

In this chapter we study the existence of positive solutions in the case in which either $h \equiv 0$ and $k \not\equiv 1$ or $k \equiv 1$ and $h \not\equiv 0$ satisfying suitable assumptions. Our results hold in any dimension and are proved using the *concentration-compactness* arguments by P.L. Lions, see Chapter 2.

It is known that the general problem has an obstruction provided by a Pohozaev type identity that shows us the particularity of this problem, that is: *the existence of a positive solution depends not only on the size of the functions h and k but also on their shape.* More precisely, assume that u is a variational solution to our equation with $h, k \in C^1$. Multiplying the equation by $\langle x, \nabla u\rangle$ and with a convenient argument of approximation we get that necessarily

$$\frac{\lambda}{2}\int \langle \nabla h(x), x\rangle \frac{u^2}{|x|^2}dx + \frac{1}{2^*}\int \langle \nabla k(x), x\rangle |u|^{2^*} dx = 0.$$

This behaviour makes the problem more interesting to be analyzed. The existence part of the present chapter is mainly based on the *concentration-compactness arguments* by P.L. Lions (see [70] and [71]) and involves some qualitative properties of the coefficients that avoids the Pohozaev type obstruction. We also obtain multiplicity of positive solutions by using variational and topological arguments.

The present chapter is organized as follows. First we study the problem of nonexistence and existence for $k \equiv 1$ and h satisfying suitable conditions. As pointed out above, we mainly use the *concentration-compactness* principle by P.L. Lions. The main result in this part is Theorem 6.2. Then we deal with the existence and multiplicity results for the case in which $h \equiv 0$ and k satisfies some convenient conditions. In this part we will use techniques that previously had been introduced to study related problems by Tarantello in [81] and refined by Cao- Chabrowsky in [32] (see also the references therein). We use this approach in the case that the function k achieves its maximum at a finite number of points. The main result in this direction is Theorem 6.4. Finally we study a more general class of functions k, i.e. we treat the case in which k can reach its maximum at infinitely many points, but having only accumulation points at finite distance to the origin. To analyze this case we use the *Lusternik-Schnirelman category.* This point of view is inspired by the study of multiplicity of positive solutions to subcritical problems done by R. Musina in [72].

6.1 Perturbation in the linear term

We will study perturbations of a class of elliptic equations in $\mathbb{R}^N$ related to a Hardy inequality interacting with a nonlinear term involving the critical Sobolev exponent . Precisely we will consider the following problem

$$\begin{cases} -\Delta u = \dfrac{A + h(x)}{|x|^2} u + u^{2^*-1}, \ x \in \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N, \text{ and } u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \end{cases} \tag{6.2}$$

where $N \geq 3$ and $2^* = \frac{2N}{N-2}$. Hypotheses on h will be given below. Let us denote by $\Lambda_N := C_N^{-1} = \frac{(N-2)^2}{4}$, where C_N is the best constant in the Hardy inequality (see Lemma 4.1). The case $h = 0$ of (6.2) has been studied by S. Terracini in [82]; she shows, in particular, that

1. if $A \geq \Lambda_N$, then problem (6.2) has no positive solution in $\mathcal{D}'(\mathbb{R}^N)$;
2. if $A \in (0, \Lambda_N)$ then problem (6.2) has the one-dimensional $\mathcal{C}^2$ manifold of positive solutions
$$Z_A = \left\{ w_\mu \mid w_\mu(x) = \mu^{-\frac{N-2}{2}} w^{(A)}\left(\frac{x}{\mu}\right), \ \mu > 0 \right\}, \tag{6.3}$$
where
$$w^{(A)}(x) = \frac{\left(N(N-2)\nu_A^2\right)^{\frac{N-2}{4}}}{\left(|x|^{1-\nu_A}(1 + |x|^{2\nu_A})\right)^{\frac{N-2}{2}}}, \text{ and } \nu_A = \left(1 - \frac{A}{\Lambda_N}\right)^{\frac{1}{2}}. \tag{6.4}$$

Moreover, if we set $Q_A(u) = \int_{\mathbb{R}^N} |\nabla u|^2 dx - A \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx$, then we obtain that

$$\overline{S} \equiv \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{Q_A(u)}{\|u\|_{2^*}^2} = \frac{Q_A(w_\mu)}{\|w_\mu\|_{2^*}^2} = S\left(1 - \frac{A}{\Lambda_N}\right)^{\frac{N-1}{N}}, \tag{6.5}$$

where S is the best constant in the Sobolev inequality. Notice that $\overline{S}$ is attained exactly in the family w_μ defined in (6.3).

6.1.1 Nonexistence results

We begin by proving some nonexistence results that show the fact that in this kind of problems both the size and the shape of the perturbation are important. Define

$$Q(u) = \int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} \left(\frac{A + h(x)}{|x|^2}\right) u^2 dx, \tag{6.6}$$

$\mathcal{K} = \{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} |u|^{2^*} dx = 1\}$, and consider $I_1 = \inf_{u \in \mathcal{K}} Q(u)$.

Lemma 6.1. *Problem (6.2) has no positive solution in the following cases:*

(a) If $A + h(x) \geq 0$ in some ball $B_\delta(0)$ and $I_1 < 0$.
(b) If h is a differentiable function such that $\langle h'(x), x\rangle$ has a fixed sign.

Proof. We begin by proving nonexistence under hypothesis (a). Suppose that $I_1 < 0$, and let u be a positive solution to (6.2). By classical regularity results for elliptic equations we obtain that $u \in \mathcal{C}^\infty(\mathbb{R}^N \setminus\{0\})$. On the other hand, since $A + h(x) \geq 0$ in $B_\delta(0)$, we obtain that $-\Delta u \geq 0$ in $\mathcal{D}'(B_\delta(0))$. Therefore, since $u \geq 0$ and $u \not\equiv 0$, by the strong maximum principle we obtain that $u(x)\varepsilon c > 0$ in some ball $B_\eta(0) \subset\subset B_\delta(0)$. Let $\phi_n \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, $\phi_n \geq 0$, $\|\phi_n\|_{2*} = 1$, be a minimizing sequence of I_1. By using $\frac{\phi_n^2}{u}$ as a test function in equation (6.2) we obtain

$$\int_{\mathbb{R}^N} \nabla\left(\frac{\phi_n^2}{u}\right) \nabla u = \int_{\mathbb{R}^N} \frac{A + h(x)}{|x|^2} \phi_n^2 + \int_{\mathbb{R}^N} \phi_n^2 u^{2^*-2}.$$

A direct computation gives

$$2\int_{\mathbb{R}^N} \frac{\phi_n}{u} \nabla\phi_n \nabla u dx - \int_{\mathbb{R}^N} \frac{\phi_n^2}{u^2} |\nabla u|^2 dx = \int_{\mathbb{R}^N} \frac{A + h(x)}{|x|^2} \phi_n^2 + \int_{\mathbb{R}^N} \phi_n^2 u^{2^*-2}, \tag{6.7}$$

and since

$$2\frac{\phi_n}{u} \nabla\phi_n \nabla u - \frac{\phi_n^2}{u^2} |\nabla u|^2 \leq |\nabla\phi_n|^2,$$

we conclude that

$$\int_{\mathbb{R}^N} |\nabla\phi_n|^2 dx \geq \int_{\mathbb{R}^N} \frac{A + h(x)}{|x|^2} \phi_n^2 + \int_{\mathbb{R}^N} \phi_n^2 u^{2^*-2}.$$

On the other hand, $I_1 < 0$ implies that we can find an integer n_0 such that if $n \geq n_0$,

$$\int_{\mathbb{R}^N} |\nabla \phi_n|^2 - \int_{\mathbb{R}^N} \frac{A + h(x)}{|x|^2} \phi_n^2 < 0.$$

As a consequence $\int_{\mathbb{R}^N} \phi_n^2 u^{2^*-2} < 0$, for $n \geq n_0$, which contradicts the hypothesis $u > 0$.
Let us now prove (b). By using the Pohozaev multiplier $\langle x, \nabla u \rangle$, we obtain that if u is a positive solution to (6.2), then

$$\int_{\mathbb{R}^N} \frac{\langle h'(x), x \rangle}{|x|^2} u^2 dx = 0,$$

which is not possible if $\langle h'(x), x \rangle$ has a fixed sign and $u \not\equiv 0$. □

Corollary 6.1. *Assume either*

i) $A > \Lambda_N$ *and* $h\varepsilon 0$, *or*
ii)$A > \Lambda_N$ *and* $1 \leq \frac{4A}{(N-2)^2 ||h||_\infty}$,

then problem (6.2), has no positive solution.

6.1.2 The local Palais-Smale condition: existence results

To prove the existence results we will use a variational approach for the associated functional

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} \frac{A + h(x)}{|x|^2} u^2 dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} dx. \tag{6.8}$$

We suppose that h verifies the following hypotheses

(h0) $A + h(0) > 0$.
(h1) $h \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.
(h2) For some $c_0 > 0$, $A + ||h||_\infty \leq \Lambda_N - c_0$.

Critical points of J in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ are solutions to equation (6.2). We begin by proving a local Palais-Smale condition for J. Precisely we prove the following Theorem.

Theorem 6.1. *Suppose that* $(h0)$, $(h1)$, $(h2)$ *hold and denote* $h(\infty) \equiv \limsup_{|x|\to\infty} h(x)$.

Let $\{u_n\} \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ *be a Palais-Smale sequence for* J, *namely*

$$J(u_n) \to c < \infty, \; J'(u_n) \to 0.$$

If

$$c < c^* = \frac{1}{N} S^{\frac{N}{2}} \min \left\{ \left(1 - \frac{A + h(0)}{\Lambda_N}\right)^{\frac{N-1}{2}}, \left(1 - \frac{A + h(\infty)}{\Lambda_N}\right)^{\frac{N-1}{2}} \right\},$$

then $\{u_n\}$ *has a converging subsequence.*

Proof. Let $\{u_n\}$ be a Palais-Smale sequence for J, then according to $(h2)$, $\{u_n\}$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Then, up to a subsequence, *i)* $u_n \rightharpoonup u_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, *ii)* $u_n \to u_0$ a.e., and *iii)* $u_n \to u_0$ in L^{α}_{loc}, $\alpha \in [1, 2^*)$. Therefore, by using the *concentration compactness principle* by P. L. Lions, (see [70] and [71]), there exists a subsequence (still denoted by $\{u_n\}$) which satisfies

1. $|\nabla u_n|^2 \rightharpoonup d\mu \geq |\nabla u_0|^2 + \sum_{j\in\mathcal{J}} \mu_j \delta_{x_j} + \mu_0\delta_0$,
2. $|u_n|^{2^*} \rightharpoonup d\nu = |u_0|^{2^*} + \sum_{j\in\mathcal{J}} \nu_j \delta_{x_j} + \nu_0\delta_0$,
3. $S\nu_j^{\frac{2}{2^*}} \leq \mu_j$ for all $j \in \mathcal{J} \cup \{0\}$, where $\mathcal{J}$ is at most countable,
4. $\dfrac{u_n^2}{|x|^2} \rightharpoonup d\gamma = \dfrac{u_0^2}{|x|^2} + \gamma_0\delta_0$,
5. $\Lambda_N \gamma_0 \leq \mu_0$.

To study the concentration at infinity of the sequence we will also need to consider the following quantities

$$\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |u_n|^{2^*} dx, \quad \mu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |\nabla u_n|^{2^*} dx$$

and

$$\gamma_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} \frac{u_n^2}{|x|^2} dx.$$

We claim that $\mathcal{J}$ is finite and for $j \in \mathcal{J}$, either $\nu_j = 0$ or $\nu_j \geq S^{N/2}$. We follow closely the arguments in [12]. Let $\varepsilon > 0$ and let ϕ be a smooth cut-off function centered at x_j, $0 \leq \phi(x) \leq 1$ such that

$$\phi(x) = \begin{cases} 1, & \text{if } |x - x_j| \leq \dfrac{\varepsilon}{2}, \\ 0, & \text{if } |x - x_j| \geq \varepsilon, \end{cases}$$

and $|\nabla\phi| \leq \dfrac{4}{\varepsilon}$. So we get

$$\begin{aligned} 0 &= \lim_{n\to\infty} \langle J'(u_n), u_n\phi\rangle \\ &= \lim_{n\to\infty} \left(\int_{\mathbb{R}^N} |\nabla u_n|^2\phi + \int_{\mathbb{R}^N} u_n \nabla u_n \nabla\phi - \int_{\mathbb{R}^N} \frac{A + h(x)}{|x|^2} u_n^2\phi - \int_{\mathbb{R}^N} \phi|u_n|^{2^*} \right). \end{aligned}$$

From 1), 2) and 4) and since $0 \notin \operatorname{supp}(\phi)$ we find that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2\phi = \int_{\mathbb{R}^N} \phi d\mu, \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2^*}\phi = \int_{\mathbb{R}^N} \phi d\nu$$

and

$$\lim_{n\to\infty} \int_{B_\varepsilon(x_j)} \frac{A + h(x)}{|x|^2} u_n^2\phi = \int_{B_\varepsilon(x_j)} \frac{A + h(x)}{|x|^2} u_0^2\phi.$$

Taking limits as $\varepsilon \to 0$ we obtain

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left|\int_{\mathbb{R}^N} u_n\nabla u_n\nabla\phi\right|\to 0.$$

Hence

$$0=\lim_{\varepsilon\to 0}\lim_{n\to\infty}\langle J'(u_n),u_n\phi\rangle=\mu_j-\nu_j.$$

By 3) we have that $S\nu_j^{\frac{2}{2^*}}\le\mu_j$, then we obtain that either $\nu_j=0$ or $\nu_j\ge S^{N/2}$, which implies that $\mathcal{J}$ is finite. The claim is proved.

Let us now study the possibility of concentration at $x=0$ and at ∞. Let ψ be a regular function such that $0\le\psi(x)\le 1$,

$$\psi(x)=\begin{cases}1, & \text{if } |x|>R+1\\ 0, & \text{if } |x|<R,\end{cases}$$

and $|\nabla\psi|\le\frac{4}{R}$. From (6.5) we obtain that

$$\frac{\int_{\mathbb{R}^N}|\nabla(u_n\psi)|^2dx-(A+h(\infty))\int_{\mathbb{R}^N}\frac{\psi^2u_n^2}{|x|^2}dx}{\left(\int_{\mathbb{R}^N}|\psi u_n|^{2^*}\right)^{2/2^*}}\ge S\Big(1-\frac{A+h(\infty)}{\Lambda_N}\Big)^{\frac{N-1}{N}}. \tag{6.9}$$

Hence

$$\int_{\mathbb{R}^N}|\nabla(u_n\psi)|^2dx-(A+h(\infty))\int_{\mathbb{R}^N}\frac{\psi^2u_n^2}{|x|^2}dx \ge S\Big(1-\frac{A+h(\infty)}{\Lambda_N}\Big)^{\frac{N-1}{N}}\Big(\int_{\mathbb{R}^N}|\psi u_n|^{2^*}\Big)^{2/2^*}.$$

Therefore we conclude that

$$\int_{\mathbb{R}^N}\psi^2|\nabla u_n|^2dx+\int_{\mathbb{R}^N}u_n^2|\nabla\psi|^2dx+2\int_{\mathbb{R}^N}u_n\psi\nabla u_n\nabla\psi dx \ge(A+h(\infty))\int_{\mathbb{R}^N}\frac{\psi^2u_n^2}{|x|^2}dx+S\Big(1-\frac{A+h(\infty)}{\Lambda_N}\Big)^{\frac{N-1}{N}}\Big(\int_{\mathbb{R}^N}|\psi u_n|^{2^*}\Big)^{2/2^*}.$$

We claim that

$$\lim_{R\to\infty}\limsup_{n\to\infty}\left\{\int_{\mathbb{R}^N}u_n^2|\nabla\psi|^2dx+2\int_{\mathbb{R}^N}|u_n|\psi|\nabla u_n||\nabla\psi|dx\right\}=0.$$

Using Hölder inequality we obtain

$$\int_{\mathbb{R}^N}|u_n|\psi|\nabla u_n||\nabla\psi|dx \le\Big(\int_{R<|x|<R+1}|u_n|^2|\nabla\psi|^2dx\Big)^{1/2}\Big(\int_{R<|x|<R+1}|\nabla u_n|^2dx\Big)^{1/2}.$$

Hence

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n||\psi||\nabla u_n||\nabla\psi|dx \le C\Big(\int_{R<|x|<R+1}|u_0|^2|\nabla\psi|^2dx\Big)^{1/2}$$
$$\le C\Big(\int_{R<|x|<R+1}|u_0|^{2^*}dx\Big)^{2/2^*}\Big(\int_{R<|x|<R+1}|\nabla\psi|^N dx\Big)^{2/N}$$
$$\le \overline{C}\Big(\int_{R<|x|<R+1}|u_0|^{2^*}dx\Big)^{2/2^*}.$$

Therefore we conclude that

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n||\psi||\nabla u_n||\nabla\psi|dx \le \overline{C}\lim_{R\to\infty}\Big(\int_{R<|x|<R+1}|u_0|^{2^*}dx\Big)^{2/2^*} = 0.$$

Using the same argument we can prove that

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}u_n^2|\nabla\psi|^2 = 0.$$

Then we get

$$\mu_\infty - (A+h(\infty))\gamma_\infty \ge S\Big(1-\frac{A+h(\infty)}{A_N}\Big)^{\frac{N-1}{N}}\nu_\infty^{2/2^*}. \tag{6.10}$$

Since $\lim_{R\to\infty}\lim_{n\to\infty}\langle J'(u_n),u_n\psi\rangle = 0$, we obtain that $\mu_\infty - (A+h(\infty))\gamma_\infty \le \nu_\infty$. Therefore we conclude that either $\nu_\infty = 0$ or $\nu_\infty \ge S^{\frac{N}{2}}(1-\frac{A+h(\infty)}{A_N})^{\frac{N-1}{2}}$. The same holds for the concentration in $x_0 = 0$, namely that either

$$\nu_0 = 0 \text{ or } \nu_0 \ge S^{\frac{N}{2}}\Big(1-\frac{A+h(0)}{A_N}\Big)^{\frac{N-1}{2}}.$$

As a conclusion we obtain

$$c = J(u_n) - \frac{1}{2}\langle J'(u_n),u_n\rangle + o(1)$$
$$= \frac{1}{N}\int_{\mathbb{R}^N}|u_n|^{2*}dx + o(1) = \frac{1}{N}\Big\{\int_{\mathbb{R}^N}|u_0|^{2*}dx + \nu_0 + \nu_\infty + \sum_{j\in\mathcal{J}}\nu_j\Big\}.$$

If we assume the existence of $j \in \mathcal{J}\cup\{0,\infty\}$ such that $\nu_j \ne 0$, then we obtain that $c \ge c^*$ a contradiction with the hypothesis, then up to a subsequence $u_n \to u_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. □

In order to find solutions we look for some path in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ along which the maximum of $J(\gamma(t))$ is less than c^*. To do that, for $H = \max\{h(0),h(\infty)\}$, we consider $\{w_\mu\}$ the one parameter family of minimizer to problem (6.5) where A is replaced by $A+H$. Then we have the following result.

Theorem 6.2. *Suppose that (h0), (h1) and (h2) hold. Assume the existence of $\mu_0 > 0$ such that*

$$\int_{\mathbb{R}^N} h(x)\frac{w_{\mu_0}^2(x)}{|x|^2}dx > H\int_{\mathbb{R}^N}\frac{w_{\mu_0}^2(x)}{|x|^2}dx, \tag{6.11}$$

then (6.2) has at least a positive solution.

Proof. Let μ_0 be as in the hypothesis, then if we set

$$\begin{aligned} f(t) &= J(tw_{\mu_0}) \\ &= \frac{t^2}{2}\left(\int_{\mathbb{R}^N}|\nabla w_{\mu_0}|^2dx - \int_{\mathbb{R}^N}\frac{A+h(x)}{|x|^2}w_{\mu_0}^2dx\right) - \frac{t^{2^*}}{2^*}\int_{\mathbb{R}^N}|w_{\mu_0}|^{2^*}dx,\ t\geq 0 \end{aligned}$$

we can see easily that f achieves its maximum at some $t_0 > 0$ and we can prove the existence of $\rho > 0$ such that $J(tw_{\mu_0}) < 0$ if $||tw_{\mu_0}|| \geq \rho$. By a simple calculation we obtain that

$$t_0^{2^*-2} = \frac{\int_{\mathbb{R}^N}|\nabla w_{\mu_0}|^2dx - \int_{\mathbb{R}^N}\frac{A+h(x)}{|x|^2}w_{\mu_0}^2dx}{\int_{\mathbb{R}^N}|w_{\mu_0}|^{2^*}dx},$$

and

$$J(t_0w_{\mu_0}) = \max_{t\geq 0} J(tw_{\mu_0}) = \frac{1}{N}\left(\frac{\int_{\mathbb{R}^N}|\nabla w_{\mu_0}|^2dx - \int_{\mathbb{R}^N}\frac{A+h(x)}{|x|^2}w_{\mu_0}^2dx}{\left(\int_{\mathbb{R}^N}|w_{\mu_0}|^{2^*}dx\right)^{2/2^*}}\right)^{N/2}. \tag{6.12}$$

Using (6.11) we obtain that

$$\begin{aligned} J(t_0w_{\mu_0}) &< \frac{1}{N}\left(\frac{\int_{\mathbb{R}^N}|\nabla w_{\mu_0}|^2dx - (A+H)\int_{\mathbb{R}^N}\frac{w_{\mu_0}^2}{|x|^2}dx}{\left(\int_{\mathbb{R}^N}|w_{\mu_0}|^{2^*}dx\right)^{2/2^*}}\right)^{N/2} \\ &= \frac{1}{N}S^{\frac{N}{2}}\left(1-\frac{A+H}{\Lambda_N}\right)^{\frac{N-1}{2}} \leq c^*. \end{aligned}$$

We set

$$\Gamma = \{\gamma \in C([0,1], \mathcal{D}^{1,2}(\mathbb{R}^N)) :\ \gamma(0) = 0 \text{ and } J(\gamma(1)) < 0\}.$$

Let

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)).$$

Since $J(t_0w_{\mu_0}) < c^*$, then we get a mountain pass critical point u_0. Then we have just to prove that we can choose $u_0 \geq 0$. We give two different proofs.
First proof. Consider the Nehari manifold,

$$\begin{aligned} M &\equiv \{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) :\ u \neq 0 \text{ and } \langle J'(u), u\rangle = 0\} \\ &= \left\{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) :\ u \neq 0 \text{ and } \int_{\mathbb{R}^N}|\nabla u|^2dx = \int_{\mathbb{R}^N}\frac{A+h(x)}{|x|^2}u^2dx + \int_{\mathbb{R}^N}|u|^{2^*}dx\right\}. \end{aligned}$$

Notice that u_0, $|u_0| \in M$. Since u_0 is a mountain pass solution to problem (6.2) then one can prove easily that $c \equiv J(u_0) = \min_{u\in M} J(u)$ (see [83]). Moreover as $J(|u_0|) = \min_{u\in M} J(u)$, then $|u_0|$ is also a critical point of J.
Second proof. Here we use a variation of the deformation lemma. Since u_0 is a mountain pass critical point of J, which is even, we have

$$c = J(u_0) = J(|u_0|) = \max_{t>0} J(t|u_0|).$$

Let $t_1 > 0$ be such that $J(t_1|u_0|) < 0$. We set $\gamma_0(t) = t(t_1|u_0|)$ for $t \in [0,1]$. Notice that $\gamma_0 \in \Gamma$ and

$$c = J(|u_0|) = \max_{t\in[0,1]} J(\gamma_0(t)).$$

If $|u_0|$ is a critical point to J, then we have done. If not then using Lemma 3.7 of [55] we obtain that γ_0 can be deformed to a path $\gamma_1 \in \Gamma$ with $\max_{t\in[0,1]} J(\gamma_1(t)) < c$, a contradiction with the definition of c as a *min-max value.*

Hence we have nonnegative solution to problem (6.2). The positivity of the solution u_0 is an application of the strong maximum principle by using hypotheses $(h0)$ and $(h1)$. □

We give now some sufficient condition on h to have hypothesis (6.11).

Lemma 6.2. *Suppose one of the following hypotheses holds*
(1) $h(x) \geq h(0) + c_1|x|^{\nu_{A+H}(N-2)}$ *for* $|x|$ *small and* $c_1 > 0$ *if* $h(0) \geq h(\infty)$, *or*
(2) $h(x) \geq h(\infty) + c_2|x|^{-\nu_{A+H}(N-2)}$ *for* $|x|$ *large and* $c_2 > 0$ *if* $h(\infty) \geq h(0)$,
then there exists $\mu_0 > 0$ *such that (6.11) holds.*

Proof. Let $\delta > 0$ be small such that if $|x| < \delta$ then $h(x) \geq h(0) + c_1|x|^{\nu_{A+H}(N-2)}$. For simplicity of notation we set $\nu_{A+H} = \nu$. Let

$$I_{\delta,\mu} = \int_{|x|<\delta} (h(x) - H)\frac{dx}{|x|^{(1-\nu)N+2\nu}(\mu^{2\nu} + |x|^{2\nu})^{N-2}},$$

then

$$I_{\delta,\mu} \geq c_1 \int_{|x|<\delta} \frac{|x|^{\nu(N-2)}dx}{|x|^{(1-\nu)N+2\nu}(\mu^{2\nu} + |x|^{2\nu})^{N-2}}.$$

Since $\nu(N-2) - [(1-\nu)N + 2\nu + 2\nu(N-2)] = -N$, we conclude that $I_{\delta,\mu} \to \infty$ as $\mu \to 0$. On the other hand

$$\int_{|x|\geq\delta} |h(x) - H|\frac{dx}{|x|^{(1-\nu)N+2\nu}(\mu^{2\nu} + |x|^{2\nu})^{N-2}} \leq C\int_{|x|\geq\delta} \frac{dx}{|x|^{(1+\nu)N-2\nu}} \leq C(\delta).$$

Therefore we get the existence of $\mu_0 > 0$ such that

$$\int_{\mathbb{R}^N} (h(x)-H)\frac{w_{\mu_0}^2(x)}{|x|^2}dx \geq \int_{|x|<\delta} (h(x) - H)\frac{w_{\mu_0}^2(x)}{|x|^2}dx - \int_{|x|\geq\delta} |h(x) - H|\frac{w_{\mu_0}^2(x)}{|x|^2}dx > 0.$$

Then the result follows. The second case follows by using the same argument near infinity. □

6.2 Perturbation of the nonlinear term: multiplicity of positive solutions

In this subection we deal with the following problem

$$\begin{cases} -\Delta u = \dfrac{\lambda}{|x|^2}u + k(x)u^{2^*-1}, \ x \in \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N, \ \text{and } u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \end{cases} \tag{6.13}$$

where $N \geq 3$, $0 < \lambda < \Lambda_N$ and k is a positive function.

6.2.1 Existence

Assume that k verifies the hypothesis

$$(K0) \quad k \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N) \ \text{ and } ||k||_\infty > \max\{k(0), k(\infty)\}, \\ \text{where } k(\infty) \equiv \limsup_{|x|\to\infty} k(x).$$

We associate to problem (6.13) the following functional

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\lambda}{2}\int_{\mathbb{R}^N} \frac{u^2}{|x|^2}dx - \frac{1}{2^*}\int_{\mathbb{R}^N} k(x)|u|^{2^*} dx. \tag{6.14}$$

As in the previous section we have the following Lemma.

Lemma 6.3. *Let $\{u_n\} \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ be a Palais-Smale sequence for J_λ, namely*

$$J_\lambda(u_n) \to c < \infty, \ J'_\lambda(u_n) \to 0.$$

If

$$c < \tilde{c}(\lambda)$$

for

$$\tilde{c}(\lambda) = \frac{1}{N}S^{\frac{N}{2}} \min\left\{ ||k||_\infty^{-\frac{N-2}{2}}, \ (k(0))^{-\frac{N-2}{2}}\left(1 - \frac{\lambda}{\Lambda_N}\right)^{\frac{N-1}{2}}, (k(\infty))^{-\frac{N-2}{2}}\left(1 - \frac{\lambda}{\Lambda_N}\right)^{\frac{N-1}{2}} \right\}$$

then $\{u_n\}$ has a converging subsequence.

The proof is similar to the proof of Theorem 6.1.
In the case in which k is a radial positive function, we can prove the following improved Palais-Smale condition.

Lemma 6.4. *Define*

$$\tilde{c}_1(\lambda) = \frac{1}{N}S^{\frac{N}{2}}\left(1 - \frac{\lambda}{\Lambda_N}\right)^{\frac{N-1}{2}} \min\left\{(k(0))^{-\frac{N-2}{2}}, \ (k(\infty))^{-\frac{N-2}{2}}\right\}.$$

If $\{u_n\} \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ is a Palais-Smale sequence for J_λ, namely

$$J_\lambda(u_n) \to c, \quad J'_\lambda(u_n) \to 0,$$

and $c < \tilde{c}_1$, then $\{u_n\}$ has a converging subsequence.

Remark 6.1. This follows from the fact that the inclusion of

$$H_r^1(\Omega) \equiv \{u \in L^2(\Omega) : |\nabla u| \in L^2(\Omega),\ u \text{ radial}\}$$

where $\Omega = \{x \in \mathbb{R}^N : R_1 < |x| < R_2\}$, in $L^q(\Omega)$ is compact for all $1 \le q < \infty$ and in particular for $q = 2^*$, see [66].

As a consequence we obtain the following existence result.

Theorem 6.3. *Let k be a positive radial function such that $(K0)$ is satisfied. Assume that there exists $\mu_0 > 0$ such that*

$$\int_{\mathbb{R}^N} k(x) w_{\mu_0}^{2^*}(x)dx > \max\{k(0), k(\infty)\} \int_{\mathbb{R}^N} w_{\mu_0}^{2^*}(x)dx, \tag{6.15}$$

where w_{μ_0} is a solution to problem

$$\begin{cases} -\Delta w = \dfrac{\lambda}{|x|^2} w + w^{2^*-1}, \ x \in \mathbb{R}^N, \\ w > 0 \text{ in } \mathbb{R}^N, \text{ and } w \in \mathcal{D}^{1,2}(\mathbb{R}^N). \end{cases}$$

Then (6.13) has at least a positive solution.

Proof. Since the proof is similar to the proof of Theorem 6.2, we omit it. □

Remark 6.2. Assume that one of the following hypotheses holds

(1) $k(x) \ge k(0) + c_1|x|^{2\nu_\lambda}$ for $|x|$ small and $c_1 > 0$ if $k(0) \ge k(\infty)$, or
(2) $k(x) \ge k(\infty) + c_2|x|^{-2\nu_\lambda}$ for $|x|$ large and $c_2 > 0$ if $k(\infty) \ge k(0)$,

then there exists $\mu_0 > 0$ such that (6.15) holds.

Let us set

$$b(\lambda) \equiv \begin{cases} +\infty & \text{if } k(0) = k(\infty) = 0 \\ \min\left\{ (k(0))^{-\frac{N-2}{2}} \left(1 - \dfrac{\lambda}{\Lambda_N}\right)^{\frac{N-1}{2}}, (k(\infty))^{-\frac{N-2}{2}} \left(1 - \dfrac{\lambda}{\Lambda_N}\right)^{\frac{N-1}{2}} \right\} & \text{otherwise.} \end{cases}$$

Lemma 6.5. *If $(K0)$ holds, there exists $\varepsilon_0 > 0$ such that $||k||_\infty^{-\frac{N-2}{2}} \le b(\varepsilon_0)$ and*

$$\tilde{c}(\lambda) = \tilde{c} \equiv \frac{1}{N} S^{N/2} ||k||_\infty^{-\frac{N-2}{2}} \tag{6.16}$$

for any $0 < \lambda \le \varepsilon_0$.

Proof. From $(K0)$ it follows that if ε_0 is sufficiently small then $||k||_\infty^{-\frac{N-2}{2}} \le b(\varepsilon_0)$ and hence from the definition of $\tilde{c}(\lambda)$ we obtain the result. □

6.2.2 Multiplicity

To find multiplicity results for problem (6.13) we need the following extra hypotheses on k:

($K1$) the set $\mathcal{C}(k) = \left\{a \in \mathbb{R}^N \,\middle|\, k(a) = \max_{x\in\mathbb{R}^N} k(x)\right\}$ is finite,
say $\mathcal{C}(k) = \{a_j \mid 1 \le j \le \operatorname{Card}(\mathcal{C}(k))\}$;

($K2$) there exists $2 < \theta < N$ such that if $a_j \in \mathcal{C}(k)$ then $k(a_j) - k(x) = o(|x - a_j|)^\theta$ as $x \to a_j$.

Consider $0 < r_0 \ll 1$ such that $B_{r_0}(a_j) \cap B_{r_0}(a_i) = \emptyset$ for $i \ne j$, $1 \le i, j \le \operatorname{Card}(\mathcal{C}(k))$. Let $\delta = \frac{r_0}{3}$ and for any $1 \le j \le \operatorname{Card}(\mathcal{C}(k))$ define the following function

$$T_j(u) = \frac{\int_{\mathbb{R}^N} \psi_j(x)|\nabla u|^2 dx}{\int_{\mathbb{R}^N} |\nabla u|^2 dx} \quad \text{where} \quad \psi_j(x) = \min\{1, |x - a_j|\}. \tag{6.17}$$

Notice that if $u \not\equiv 0$ and $T_j(u) \le \delta$, then

$$\begin{aligned} r_0 \int_{\mathbb{R}^N \setminus B_{r_0}(a_j)} |\nabla u|^2 dx &\le \int_{\mathbb{R}^N \setminus B_{r_0}(a_j)} \psi_j(x)|\nabla u|^2 dx \\ &\le \int_{\mathbb{R}^N} \psi_j(x)|\nabla u|^2 dx \le \delta \int_{\mathbb{R}^N} |\nabla u|^2 dx = \frac{r_0}{3} \int_{\mathbb{R}^N} |\nabla u|^2 dx. \end{aligned}$$

Hence we have the following property.

Lemma 6.6. *Let $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ be such that $T_j(u) \le \delta$, then*

$$\int_{\mathbb{R}^N} |\nabla u|^2 \ge 3 \int_{\mathbb{R}^N \setminus B_{r_0}(a_j)} |\nabla u|^2 dx.$$

As a consequence we obtain the following separation result.

Corollary 6.2. *Consider $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, $u \not\equiv 0$, such that $T_i(u) \le \delta$ and $T_j(u) \le \delta$, then $i = j$.*

Proof. By Lemma 6.6 we obtain that

$$2 \int_{\mathbb{R}^N} |\nabla u|^2 dx \ge 3 \left(\int_{\mathbb{R}^N \setminus B_{r_0}(a_i)} |\nabla u|^2 dx + \int_{\mathbb{R}^N \setminus B_{r_0}(a_j)} |\nabla u|^2 dx \right).$$

If $i \ne j$ we find that

$$2 \int_{\mathbb{R}^N} |\nabla u|^2 dx \ge 3 \int_{\mathbb{R}^N} |\nabla u|^2 dx,$$

a contradiction if $u \not\equiv 0$. □

Consider the Nehari manifold,

$$M(\lambda) = \{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) : \ u \not\equiv 0 \text{ and } \langle J'_\lambda(u), u\rangle = 0\}. \tag{6.18}$$

Therefore if $u \in M(\lambda)$

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx - \lambda \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx = \int_{\mathbb{R}^N} k(x)|u|^{2^*} dx.$$

Notice that for all $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, $u \not\equiv 0$, there exists $t > 0$ such that $tu \in M(\lambda)$ and for all $u \in M(\lambda)$ we have

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx - \lambda \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx < (2^* - 1) \int_{\mathbb{R}^N} k(x)|u|^{2^*} dx,$$

hence, there exists $c_1 > 0$ such that

$$\forall u \in M(\lambda), \quad ||u||_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \geq c_1.$$

Definition 6.1. *For any* $0 < \lambda < \Lambda_N$ *and* $1 \leq j \leq \operatorname{Card}(\mathcal{C}(k))$, *let us consider*

$$M_j(\lambda) = \{u \in M(\lambda) : \ T_j(u) < \delta\}$$

and its boundary

$$\Gamma_j(\lambda) = \{u \in M(\lambda) : \ T_j(u) = \delta\}.$$

We define

$$m_j(\lambda) = \inf\{J_\lambda(u) : \ u \in M_j(\lambda)\} \text{ and } \eta_j(\lambda) = \inf\{J_\lambda(u) : \ u \in \Gamma_j(\lambda)\}.$$

The following two Lemmas give the behaviour of the functional with respect to the critical level $\tilde{c}$.

Lemma 6.7. *Suppose that* $(K0)$, $(K1)$, *and* $(K2)$ *hold, then* $M_j(\lambda) \neq \emptyset$ *and there exists* $\varepsilon_1 > 0$ *such that*

$$m_j(\lambda) < \tilde{c} \quad \text{for all } 0 < \lambda \leq \varepsilon_1 \text{ and } 1 \leq j \leq \operatorname{Card}(\mathcal{C}(k)). \tag{6.19}$$

Proof. We set

$$v_{\mu,j}(x) = \frac{1}{(\mu^2 + |x - a_j|^2)^{\frac{N-2}{2}}} \quad \text{and} \quad u_{\mu,j} = \frac{v_{\mu,j}}{||v_{\mu,j}||_{2^*}}, \tag{6.20}$$

then $||u_{\mu,j}||_{2^*} = 1$ and $\int_{\mathbb{R}^N} |\nabla u_{\mu,j}|^2 dx = S$. If

$$t_{\mu,j}(\lambda) = \left(\frac{\int_{\mathbb{R}^N} |\nabla u_{\mu,j}|^2 dx - \lambda \int_{\mathbb{R}^N} |x|^{-2} u_{\mu,j}^2 dx}{\int_{\mathbb{R}^N} k(x)|u_{\mu,j}|^{2^*} dx} \right)^{\frac{N-2}{4}},$$

then $t_{\mu,j}(\lambda) u_{\mu,j} \in M(\lambda)$. Making the change of variable $x - a_j = \mu y$, we obtain

$$T_j(t_{\mu,j}(\lambda)u_{\mu,j}) = \frac{\int_{\mathbb{R}^N}\psi_j(x)|\nabla u_{\mu,j}|^2dx}{\int_{\mathbb{R}^N}|\nabla u_{\mu,j}|^2dx} = \frac{\int_{\mathbb{R}^N}\psi_j(a_j+\mu y)|\nabla u_0(y)|^2dy}{\int_{\mathbb{R}^N}|\nabla u_0(y)|^2dy},$$

where $u_0(x)$ is $u_{\mu,j}$ to scale $\mu = 1$ and concentrated in the origin. Then

$$\lim_{\mu\to 0} T_j(t_{\mu,j}(\lambda)u_{\mu,j}) = \frac{\int_{\mathbb{R}^N}\psi_j(a_j)|\nabla u_0(y)|^2dy}{\int_{\mathbb{R}^N}|\nabla u_0(y)|^2dy} = \psi_j(a_j) = 0,$$

uniformly in λ. Hence we get the existence of μ_0 independent of λ such that if $\mu < \mu_0$, then $t_{\mu,j}(\lambda)u_{\mu,j} \in M_j(\lambda)$. Notice that

$$t_{\mu,j}(\lambda) \geq t_1(\lambda) \equiv \left(||k||_\infty^{-1}\left(1-\frac{\lambda}{\Lambda_N}\right)S\right)^{\frac{N-2}{4}}.$$

In order to prove (6.19), it is sufficient to show the existence of $\mu < \mu_0$ such that if $0 < \lambda < \varepsilon_1$ then

$$\max_{t\geq t_1(\lambda)} J_\lambda(tu_{\mu,j}) = J_\lambda(t_{\mu,j}(\lambda)u_{\mu,j}) < \tilde{c}.$$

We have

$$\max_{t\geq t_1(\lambda)} J_\lambda(tu_{\mu,j})$$
$$\leq \max_{t>0}\left\{\frac{t^2}{2}\int_{\mathbb{R}^N}|\nabla u_{\mu,j}|^2dx - \frac{t^{2^*}}{2^*}\int_{\mathbb{R}^N}k(x)|u_{\mu,j}|^{2^*}dx\right\} - \frac{1}{2}\lambda t_1^2(\lambda)\int_{\mathbb{R}^N}\frac{u_{\mu,j}^2}{|x|^2}dx$$

and

$$\max_{t>0}\left\{\frac{t^2}{2}\int_{\mathbb{R}^N}|\nabla u_{\mu,j}|^2dx - \frac{t^{2^*}}{2^*}\int_{\mathbb{R}^N}k(x)|u_{\mu,j}|^{2^*}dx\right\}$$
$$= \frac{1}{N}\left(\frac{S}{\left(\int_{\mathbb{R}^N}k(x)|u_{\mu,j}|^{2^*}dx\right)^{2/2^*}}\right)^{N/2}.$$

In view of assumption $(K2)$ we have that for some positive constants $\bar{c}_1, \bar{c}_2$

$$\int_{\mathbb{R}^N}k(x)|u_{\mu,j}|^{2^*}dx = ||k||_\infty - \int_{\mathbb{R}^N}(k(a_j)-k(x))|u_{\mu,j}|^{2^*}dx$$
$$= ||k||_\infty - \bar{c}_1\mu^N\int_{\mathbb{R}^N}\frac{k(a_j)-k(x)}{(\mu^2+|x-a_j|^2)^N}dx$$
$$\geq ||k||_\infty - \bar{c}_1\mu^N\left\{\int_{B_\delta(a_j)}\frac{\bar{c}_2|x-a_j|^\theta dx}{(\mu^2+|x-a_j|^2)^N} + 2||k||_\infty\int_{\mathbb{R}^N\setminus B_\delta(a_j)}\frac{dx}{(\mu^2+|x-a_j|^2)^N}\right\}$$
$$\geq ||k||_\infty - \bar{c}_1\mu^N\left\{\mu^{\theta-N}\bar{c}_2\int_{\mathbb{R}^N}\frac{|y|^\theta dy}{(1+|y|^2)^N} + 2||k||_\infty\int_{|y|\geq\delta}\frac{dy}{|y|^{2N}}\right\}$$
$$= ||k||_\infty + O(\mu^\theta).$$

Then we obtain that

$$\max_{t\geq t_1(\lambda)} J(tu_{\mu,j}) \leq \frac{1}{N}\frac{S^{N/2}}{||k||_\infty^{\frac{N-2}{2}} + O(\mu^\theta)} - \frac{1}{2}\lambda t_1^2(\lambda)\int_{\mathbb{R}^N}\frac{u_{\mu,j}^2}{|x|^2}dx$$
$$\leq \frac{1}{N}\frac{S^{N/2}}{||k||_\infty^{\frac{N-2}{2}}} + O(\mu^\theta) - \frac{1}{2}\lambda t_1^2(\lambda)\int_{\mathbb{R}^N}\frac{u_{\mu,j}^2}{|x|^2}dx.$$

Using estimate A.6 from [77] we obtain that for some positive constant c

$$\int_{\mathbb{R}^N}\frac{u_{\mu,j}^2}{|x|^2}dx \geq c\mu^2 \text{ as } \mu \to 0.$$

Therefore we get

$$\max_{t\geq t_1(\lambda)} J(tu_{\mu,j}) \leq \frac{1}{N}\frac{S^{N/2}}{||k||_\infty^{\frac{N-2}{2}}} + O(\mu^\theta) - \frac{1}{2}c\lambda t_1^2(\lambda)\mu^2$$
$$\leq \tilde{c} + \bar{c}_3\mu^\theta - \frac{1}{2}c\lambda t_1^2(\lambda)\mu^2,$$

where $\bar{c}_3$ is a positive constant. Since from $(K2)$ we have $2 < \theta < N$, we get the existence of ε_1 and μ_0 such that if $\mu < \mu_0$ and $0 < \lambda < \varepsilon_1$, then $\max_{t\geq t_1(\lambda)} J(tu_{\mu,j}) < \tilde{c}$ and the result follows. □

We prove now the next result.

Lemma 6.8. *Suppose that $(K0)$, $(K1)$, and $(K2)$ are satisfied, then there exists ε_2 such that for all $0 < \lambda < \varepsilon_2$ we have*

$$\tilde{c} < \eta_j(\lambda).$$

Proof. We argue by contradiction. We assume the existence of $\lambda_n \to 0$ and $\{u_n\}$ such that $u_n \in \Gamma_j(\lambda_n)$ and $J_{\lambda_n}(u_n) \to c \leq \tilde{c} = \frac{1}{N}S^{N/2}||k||_\infty^{-\frac{N-2}{2}}$. We can easily prove that $\{u_n\}$ is bounded. Then up to a subsequence we get the existence of $l > 0$ such that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^2 dx = \lim_{n\to\infty}\int_{\mathbb{R}^N} k(x)|u_n|^{2^*}dx = l.$$

Notice that $l\varepsilon S^{N/2}||k||_\infty^{-\frac{N-2}{2}}$. On the other hand, by the definition of $\{u_n\}$ we have,

$$\frac{1}{N}l + o(1) = J_{\lambda_n}(u_n)$$
$$= \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u_n|^2 dx - \frac{\lambda_n}{2}\int_{\mathbb{R}^N}\frac{u_n^2}{|x|^2} - \frac{1}{2^*}\int_{\mathbb{R}^N}k(x)|u_n|^{2^*}dx$$
$$\leq \frac{1}{N}S^{N/2}||k||_\infty^{-\frac{N-2}{2}} + o(1).$$

Then we conclude that $l = S^{N/2}||k||_\infty^{-\frac{N-2}{2}}$ and therefore,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(||k||_\infty - k(x))|u_n|^{2^*}dx = 0. \tag{6.21}$$

We set $w_n = \dfrac{u_n}{||u_n||_{2^*}}$, then $||w_n||_{2^*} = 1$ and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla w_n|^2 dx = S.$$

Hence by using the concentration compactness arguments by P.L. Lions (see also Proposition 5.1 and 5.2 in [82]), we get the existence of $w_0 \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that w_n converges to w_0 weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ (up to a subsequence) and one of the following alternatives holds

1. $w_0 \not\equiv 0$ and $w_n \to w_0$ strongly in the $\mathcal{D}^{1,2}(\mathbb{R}^N)$.
2. $w_0 \equiv 0$ and either
   i) $|\nabla w_n|^2 \rightharpoonup d\mu = S\delta_{x_0}$ and $|w_n|^{2^*} \to d\nu = \delta_{x_0}$ or
   ii) $|\nabla w_n|^2 \rightharpoonup d\mu_\infty = S\delta_\infty$ and $|w_n|^{2^*} \rightharpoonup d\nu_\infty = \delta_\infty$.

The last case means that

$$\nu_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>R}|w_n|^{2^*}dx = 1$$

and

$$\mu_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>R}|\nabla w_n|^{2^*}dx = S.$$

If the first alternative holds, from (6.21) we obtain that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(||k||_\infty - k(x))w_n^{2^*}dx = \int_{\mathbb{R}^N}(||k||_\infty - k(x))w_0^{2^*}dx = 0,$$

a contradiction with the fact that k is not a constant. Assume that we have the alternative 2 i), then since $T_j(w_n) = T_j(u_n) = \delta$, we conclude that

$$\delta = T_j(w_n) = \lim_{n\to\infty}\frac{\int_{\mathbb{R}^N}\psi_j(x)|\nabla w_n|^2 dx}{\int_{\mathbb{R}^N}|\nabla w_n|^2 dx} = \psi_j(x_0).$$

Hence the concentration is impossible in any point $a_j \in \mathcal{C}(k)$. On the other hand from (6.21) we obtain that

$$0 = \lim_{n\to\infty}\int_{\mathbb{R}^N}(||k||_\infty - k(x))w_n^{2^*}dx = ||k||_\infty - k(x_0),$$

a contradiction. To analyze concentration at ∞, consider a regular function ξ satisfying

$$\xi(x) = \begin{cases} 1, & \text{if } |x| > R+1 \\ 0, & \text{if } |x| < R, \end{cases}$$

where R is chosen in a such way that $|a_j| < R - 1$ for all j. Then we have

$$\delta = T_j(w_n) = \lim_{n\to\infty} \frac{\int_{\mathbb{R}^N} \psi_j(x)|\nabla w_n|^2 dx}{\int_{\mathbb{R}^N} |\nabla w_n|^2 dx}$$
$$= \lim_{R\to\infty} \lim_{n\to\infty} \frac{\int_{\mathbb{R}^N} \xi(x)|\nabla w_n|^2 dx + \int_{\mathbb{R}^N}(1-\xi(x))\psi_j(x)|\nabla w_n|^2 dx}{\int_{\mathbb{R}^N} |\nabla w_n|^2 dx}.$$

Since $\lim_{n\to\infty} \int_{\mathbb{R}^N}(1-\xi(x))\psi_j(x)|\nabla w_n|^2 dx = 0$, we conclude that

$$\delta = \lim_{R\to\infty} \lim_{n\to\infty} \frac{\int_{\mathbb{R}^N} \xi(x)|\nabla w_n|^2 dx}{\int_{\mathbb{R}^N} |\nabla w_n|^2 dx} = 1,$$

a contradiction if we choose $\delta < 1$. So we conclude. □

We need now the following lemma that is suggested by the work of Tarantello [81]. See also [32].

Lemma 6.9. *Assume that* $0 < \lambda < \min\{\varepsilon_1, \varepsilon_2\}$ *where* ε_1, ε_2 *are given by Lemmas 6.7 and 6.8. Then for all* $u \in M_j(\lambda)$ *there exists* $\rho_u > 0$ *and a differentiable function*

$$f : B(0,\rho_u) \subset \mathcal{D}^{1,2}(\mathbb{R}^N) \to \mathrm{Re}$$

such that $f(0) = 1$ *and for all* $w \in B(0,\rho_u)$ *we have* $f(w)(u-w) \in M_j(\lambda)$. *Moreover for all* $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ *we have*

$$\langle f'(0), v\rangle = -\frac{2\int_{\mathbb{R}^N} \nabla u \nabla v dx - 2\lambda \int_{\mathbb{R}^N} \frac{uv}{|x|^2} dx - 2^* \int_{\mathbb{R}^N} k(x)|u|^{2^*-2} uv dx}{\int_{\mathbb{R}^N} |\nabla u|^2 dx - \lambda \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx - (2^*-1) \int_{\mathbb{R}^N} k(x)|u|^{2^*} dx}. \tag{6.22}$$

Proof. Let $u \in M_j(\lambda)$ and let $G : \mathrm{Re} \times \mathcal{D}^{1,2}(\mathbb{R}^N) \to \mathrm{Re}$ be the function defined by

$$G(t,w) = t\Big(\int_{\mathbb{R}^N} |\nabla(u-w)|^2 dx - \lambda \int_{\mathbb{R}^N} \frac{(u-w)^2}{|x|^2} dx\Big) - t^{2^*-1} \int_{\mathbb{R}^N} k(x)|u-w|^{2^*} dx.$$

Then $G(1,0) = 0$ and

$$G_t(1,0) = \int_{\mathbb{R}^N} |\nabla u|^2 dx - \lambda \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx - (2^*-1) \int_{\mathbb{R}^N} k(x)|u|^{2^*} dx \neq 0$$

(since $u \in M_j(\lambda)$). Then by using the Implicit Function Theorem we get the existence of $\rho_u > 0$ small enough and of a differentiable function f satisfying the required property. Moreover, notice that

$$\langle f'(0), v\rangle = -\frac{\langle G_w(1,0), v\rangle}{G_t(1,0)}$$
$$= -\frac{2\int_{\mathbb{R}^N} \nabla u \nabla v dx - 2\lambda \int_{\mathbb{R}^N} \frac{uv}{|x|^2} dx - 2^* \int_{\mathbb{R}^N} k(x)|u|^{2^*-2} uv dx}{\int_{\mathbb{R}^N} |\nabla u|^2 dx - \lambda \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx - (2^*-1) \int_{\mathbb{R}^N} k(x)|u|^{2^*} dx}.$$

The lemma is thereby proved. □

We are now in position to prove the main result.

Theorem 6.4. *Assume that* $(K0)$, $(K1)$, *and* $(K2)$ *hold, then there exists* ε_3 *small such that for all* $0 < \lambda < \varepsilon_3$ *equation (6.13) has* $\text{Card}\,(C(k))$ *positive solutions* $u_{j,\lambda}$ *such that*

$$|\nabla u_{j,\lambda}|^2 \to S^{N/2}||k||_\infty^{-(N-2)/2}\delta_{a_j} \text{ and } |u_{j,\lambda}|^{2^*} \to S^{N/2}||k||_\infty^{-N/2}\delta_{a_j} \text{ as } \lambda \to 0. \quad (6.23)$$

Proof. Assume that $0 < \lambda < \varepsilon_3 = \min\{\varepsilon_0, \varepsilon_1, \varepsilon_2\}$, where ε_0, ε_1 and ε_2 are given by Lemmas 6.5, 6.7 and 6.8. Let $\{u_n\}$ be a minimizing sequence to J_λ in $M_j(\lambda)$, that is, $u_n \in M_j(\lambda)$ and $J_\lambda(u_n) \to m_j(\lambda)$ as $n \to \infty$. Since $J_\lambda(u_n) = J_\lambda(|u_n|)$, we can choose $u_n \geq 0$. Notice that we can prove the existence of c_1, c_2 such that $c_1 \leq ||u_n||_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \leq c_2$. By the Ekeland variational principle we get the existence of a subsequence denoted also by $\{u_n\}$ such that

$$J_\lambda(u_n) \leq m_j(\lambda) + \frac{1}{n} \text{ and } J_\lambda(w) \geq J_\lambda(u_n) - \frac{1}{n}||w - u_n|| \text{ for all } w \in M_j(\lambda).$$

Let $0 < \rho < \rho_n \equiv \rho_{u_n}$ and $f_n \equiv f_{u_n}$, where ρ_{u_n} and f_{u_n} are given by Lemma 6.9. We set $v_\rho = \rho v$ where $||v||_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = 1$, then $v_\rho \in B(0, \rho_n)$ and we can apply Lemma 6.9 to obtain that $w_\rho = f_n(v_\rho)(u_n - v_\rho) \in M_j(\lambda)$. Therefore we get

$$\begin{aligned}\frac{1}{n}||w_\rho - u_n|| &\geq J_\lambda(u_n) - J_\lambda(w_\rho) = \langle J_\lambda'(u_n), u_n - w_\rho\rangle + o(||u_n - w_\rho||)\\ &\geq \rho f_n(\rho v)\langle J_\lambda'(u_n), v\rangle + o(||u_n - w_\rho||).\end{aligned}$$

Hence we conclude that

$$\langle J_\lambda'(u_n), v\rangle \leq \frac{1}{n}\frac{||w_\rho - u_n||}{\rho f_n(\rho v)}(1 + o(1)) \text{ as } \rho \to 0.$$

Since $|f_n(\rho v)| \to |f_n(0)| \geq c$ as $\rho \to 0$ and

$$\begin{aligned}\frac{||w_\rho - u_n||}{\rho} &= \frac{||f_n(0)u_n - f_n(\rho v)(u_n - \rho v)||}{\rho}\\ &\leq \frac{||u_n|||f_n(0) - f_n(\rho v)| + |\rho||f_n(\rho v)|}{\rho} \leq C|f_n'(0)|||v|| + c_3 \leq c.\end{aligned}$$

Then we conclude that $J_\lambda'(u_n) \to 0$ as $n \to \infty$. Hence $\{u_n\}$ is a Palais-Smale sequence for J_λ. Since $m_j(\lambda) < \tilde{c}$ and $\tilde{c} = \tilde{c}(\lambda)$ for $\lambda \leq \varepsilon_0$, then from Lemma 6.3 we get the existence result.

To prove (6.23) we follow the proof of Lemma 6.8. Assume $\lambda_n \to 0$ as $n \to \infty$ and let $u_n \equiv u_{j_0,\lambda_n} \in M_{j_0}(\lambda_n)$ be a solution to problem (6.13) with $\lambda = \lambda_n$. Then up to a subsequence we get the existence of $l_1 > 0$ such that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^2 dx = \lim_{n\to\infty}\int_{\mathbb{R}^N}k(x)|u_n|^{2^*}dx = l_1.$$

Therefore as in the proof of Lemma 6.8 we obtain that $l_1 = S^{N/2}||k||_\infty^{-\frac{N-2}{2}}$ and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(||k||_\infty - k(x))u_n^{2^*}\,dx = 0.$$

We set $w_n = \dfrac{u_n}{||u_n||_{2^*}}$, then $||w_n||_{2^*} = 1$ and $\lim_{n\to\infty}||w_n||^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = S$. Hence we get the existence of $w_0 \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that one of the following alternatives holds

1. $w_0 \not\equiv 0$ and $w_n \to w_0$ strongly in the $\mathcal{D}^{1,2}(\mathbb{R}^N)$.
2. $w_0 \equiv 0$ and either
   i) $|\nabla w_n|^2 \rightharpoonup d\mu = S\delta_{x_0}$ and $|w_n|^{2^*} \to d\nu = \delta_{x_0}$ or
   ii) $|\nabla w_n|^2 \rightharpoonup d\mu_\infty = S\delta_\infty$ and $|w_n|^{2^*} \rightharpoonup d\nu_\infty = \delta_\infty$.

As in Lemma 6.8, the alternative 1 and the alternative 2 ii) do not hold. Then we conclude that the unique possible behaviour is the alternative 2. i), namely, we get the existence of $x_0 \in \mathbb{R}^N$ such that

$$|\nabla w_n|^2 \rightharpoonup d\mu = S\delta_{x_0} \text{ and } |w_n|^{2^*} \rightharpoonup d\nu = \delta_{x_0}.$$

Since

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla w_n|^2dx &= S + o(1) = S\int_{\mathbb{R}^N}|w_n|^{2^*}dx + o(1)\\ &= \frac{S}{||k||_\infty}\int_{\mathbb{R}^N}k(x)|w_n|^{2^*}dx + o(1) = \frac{S}{||k||_\infty}k(x_0) + o(1),\end{aligned}$$

then we obtain that $x_0 \in \mathcal{C}(k)$. Using Corollary 6.2, we conclude that $x_0 = a_{j_0}$ and the result follows. □

Remark 6.3. As in [23], we can prove the same kind of results under more general condition on k. For instance, we can assume that k changes sign and the following conditions hold,

$(K'1)$ $\max_{x\in\mathbb{R}^N} k(x) > 0$ and $\mathcal{C}'(k) = \{a \in \mathbb{R}^N \mid k(a) = \max_{x\in\mathbb{R}^N} k(x)\}$ is a finite set.

$(K'2)$ $(K2)$ holds.

In this case the level at which the Palais-Smale conditions fails becomes

$$\hat{c}(\lambda) = \frac{1}{N}S^{\frac{N}{2}}\min\left\{||k_+||_\infty^{-\frac{N-2}{2}}, (k_+(0))^{-\frac{N-2}{2}}\left(1-\frac{\lambda}{\Lambda_N}\right)^{\frac{N-1}{2}}, (k_+(\infty))^{-\frac{N-2}{2}}\left(1-\frac{\lambda}{\Lambda_N}\right)^{\frac{N-1}{2}}\right\}.$$

6.3 Category setting

In this section we use the Lusternik-Schnirelman category theory to get multiplicity results for problem (6.13), we refer to [6] for a complete discussion. We follow the argument by Musina in [72]. We assume that k is a nonnegative function and that $0 < \lambda < \overline{\varepsilon}_0$ where $\overline{\varepsilon}_0$ is chosen in a such way that $\left(1 - \frac{\overline{\varepsilon}_0}{\Lambda_N}\right)^{\frac{N-1}{2}} > \frac{1}{2}$ and $\overline{\varepsilon}_0 \le \varepsilon_0$, being ε_0 given in Lemma 6.5. We set for $\delta > 0$

$$\mathcal{C}(k) = \{a \in \mathbb{R}^N \mid k(a) = ||k(x)||_\infty\} \text{ and } \mathcal{C}_\delta(k) = \{x \in \mathbb{R}^N : dist(x, \mathcal{C}(k)) \leq \delta\}.$$

We suppose that $(K2)$ holds and

$$(K3) \quad \text{there exist } R_0,\, d_0 > 0 \text{ such that } \sup_{|x|>R_0} |k(x)| \leq ||k||_\infty - d_0.$$

Let $M(\lambda)$ be defined by (6.18). Consider

$$\tilde{M}(\lambda) \equiv \{u \in M(\lambda) : J_\lambda(u) < \tilde{c}\}.$$

Then we have the following local Palais-Smale condition.

Lemma 6.10. *Let $\{v_n\} \subset M(\lambda)$ be such that*

$$J_\lambda(v_n) \to c < \tilde{c} \text{ and } J'_{\lambda|_{M(\lambda)}}(v_n) \to 0, \tag{6.24}$$

then $\{v_n\}$ contains a converging subsequence.

Proof. Assume that $\{v_n\}$ satisfies (6.24), then there exists $\{\alpha_n\} \subset \text{Re}$ such that

$$J'_\lambda(v_n) - \alpha_n G'_\lambda(v_n) \to 0 \text{ as } n \to \infty \text{ in } \mathcal{D}^{-1,2}(\mathbb{R}^N) \tag{6.25}$$

where $G_\lambda(u) = \langle J'_\lambda(u), u\rangle$. Since $\{v_n\} \subset M(\lambda)$ and $J_\lambda(v_n) \leq \tilde{c}$, we have $r_1 \leq ||v_n||_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \leq r_2$ for some constants $r_1, r_2 > 0$. Using v_n as a test function in (6.25) we conclude that $\alpha_n \to 0$ as $n \to \infty$. Hence $\{v_n\}$ is a Palais-Smale sequence for J_λ at the level $c < \tilde{c}$ and then the result follows by using Lemma 6.3. □

To prove that $\tilde{M}(\lambda) \neq \emptyset$ we give the next result.

Lemma 6.11. *There exists $\bar{\varepsilon}_1 > 0$ such that if $0 < \lambda < \lambda_0 := \min\{\bar{\varepsilon}_0, \bar{\varepsilon}_1\}$, then $\tilde{M}(\lambda) \neq \emptyset$. Moreover for any $\{\lambda_n\} \subset \text{Re}_+$ such that $\lambda_n \to 0$ as $n \to \infty$ and $\{v_n\} \subset \tilde{M}(\lambda_n)$, there exist $\{x_n\} \subset \mathbb{R}^N$ and $\{r_n\} \subset \text{Re}_+$ such that $x_n \to x_0 \in \mathcal{C}(k), r_n \to 0$ as $n \to \infty$ and*

$$v_n - \left(\frac{S}{||k||_\infty}\right)^{\frac{N-2}{4}} u_{r_n}(\cdot - x_n) \to 0 \text{ in } \mathcal{D}^{1,2}(\mathbb{R}^N), \tag{6.26}$$

where

$$u_r(x) = \frac{C_r}{(r^2 + |x|^2)^{\frac{N-2}{2}}} \tag{6.27}$$

and C_r is the normalizing constant to be $||u_r||_{2^*} = 1$.

Proof. The first assertion follows by using the same argument as in Lemma 6.7 since we have

$$\max_{t>0} J_\lambda(tw_{\mu,x}) \leq \frac{1}{N}\frac{S^{N/2}}{||k||_\infty^{\frac{N-2}{2}}} + O(\mu^\theta) - c\lambda\mu^2 < \tilde{c} \text{ for } \mu \text{ small and } 2 < \theta < N,$$

where $w_{\mu,x}(y) = \dfrac{C}{(\mu^2 + |y - x|^2)^{\frac{N-2}{2}}}$, $x \in \mathcal{C}(k)$ and C is the normalizing constant such that $||w_{\mu,x}||_{2^*} = 1$ (see also [27]). As a consequence, there exists λ_0 such that for all $0 < \lambda < \lambda_0$ the set $\tilde{M}(\lambda)$ is not empty. To prove the second part of the Lemma, eventually passing to a subsequence we set

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} k(x)|v_n|^{2^*} dx = l.$$

Then as in Lemma 6.8 we can prove that $l = S^{N/2}||k||_\infty^{-\frac{N-2}{2}}$ and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (||k||_\infty - k(x))v_n^{2^*} dx = 0. \tag{6.28}$$

Consider the normalized function $w_n = \dfrac{v_n}{||v_n||_{2^*}}$ and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla w_n|^2 dx = S.$$

Using the concentration-compactness arguments by P.L. Lions, we obtain the existence of $\{x_n\} \subset \mathbb{R}^N$ and $\{r_n\} \subset \mathrm{Re}_+$ such that

$$w_n - u_{r_n}(\cdot - x_n) \to 0 \text{ in } \mathcal{D}^{1,2}(\mathbb{R}^N), \tag{6.29}$$

and $w_n \rightharpoonup w_0 \in \mathcal{D}^{1,2}(\mathbb{R}^N)$. Moreover by the same argument as in the proof of Lemma 6.8 the weak limit is $w_0 = 0$. We will show now that the concentration at infinity is not possible. Indeed if concentration at ∞ occurs, by using (6.28) and $(K3)$ we obtain

$$\begin{aligned} ||k||_\infty &= \int_{\mathbb{R}^N} k(x)w_n^{2^*} dx + o(1) = \int_{\mathbb{R}^N \setminus B_{R_0}(0)} k(x)w_n^{2^*} dx + o(1) \\ &\le \sup_{|x|>R_0} |k(x)| \int_{\mathbb{R}^N \setminus B_{R_0}(0)} w_n^{2^*} dx + o(1) \le (||k||_\infty - d_0) + o(1), \end{aligned}$$

which is a contradiction. Then the unique possible concentration is at some point $x_0 \in \mathbb{R}^N$. Hence we conclude that, up to a subsequence, $r_n \to 0$ and

$$|\nabla u_{r_n}(x - x_n)|^2 \rightharpoonup S\delta_{x_0}.$$

Using (6.28) it is easy to obtain that $x_0 \in \mathcal{C}(k)$. □

Remark 6.4. Notice that as a consequence of the above Lemmas we obtain the existence of at least $\mathrm{cat}(\tilde{M}(\lambda))$ solutions that eventually can change sign.

Hereafter we concentrate our study on the analysis of $\mathrm{cat}(\tilde{M}(\lambda))$, the behaviour of the energy, and the positivity of solutions.

If R_0 is like in hypothesis $(K3)$, we define

$$\xi(x) = \begin{cases} x & \text{if } |x| \le R_0, \\ R_0 \dfrac{x}{|x|} & \text{if } |x| \ge R_0, \end{cases}$$

and for $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that $u \neq 0$ we set

$$\Xi(u) = \frac{\int_{\mathbb{R}^N} \xi(x)|\nabla u|^2 dx}{\int_{\mathbb{R}^N} |\nabla u|^2 dx}. \tag{6.30}$$

We recall that for $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that $u \neq 0$ we have $t_\lambda(u)u \in M(\lambda)$ where $t_\lambda(u)$ is given by

$$t_\lambda(u) = \left(\frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx - \lambda \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx}{\int_{\mathbb{R}^N} k(x)|u|^{2^*} dx} \right)^{\frac{N-2}{4}}.$$

Let $\Psi_\lambda : \mathbb{R}^N \to \mathcal{D}^{1,2}(\mathbb{R}^N)$ be given by

$$\Psi_\lambda(x) = t_\lambda(u_{\mu_\lambda}(\cdot - x))u_{\mu_\lambda}(\cdot - x),$$

where u_{μ_λ} is given by (6.20), $\mu_\lambda \equiv g(\lambda)$ such that $g(\lambda) \to 0$ as $\lambda \to 0$. Notice that if $x \in \mathcal{C}(k)$ and λ is sufficiently small, then

$$J_\lambda(\Psi_\lambda(x)) = \max_{t>0} J_\lambda(tu_{\mu_\lambda}(\cdot - x)) \le \frac{1}{N} \frac{S^{N/2}}{||k||_\infty^{\frac{N-2}{2}}} + O(\mu_\lambda^\theta) - c\lambda\mu_\lambda^2 < \tilde{c}. \tag{6.31}$$

Then we can prove the existence of $\lambda_0, c_1, c_2 > 0$ such that for all $0 < \lambda < \lambda_0$ we have $\Psi_\lambda(x) \in \tilde{M}(\lambda)$, $J_\lambda(\Psi_\lambda(x)) = \tilde{c} + o(1)$ as $\lambda \to 0$, and $c_1 < t_\lambda(u_{\mu_\lambda}(\cdot - x)) < c_2$ for all $x \in \mathcal{C}(k)$. As a consequence, taking limits for $\lambda \to 0$ we obtain by Lemma 6.11 that for any $x \in \mathcal{C}(k)$

$$|\nabla\Psi_\lambda(x)|^2 \rightharpoonup d\mu = S^{N/2}||k||_\infty^{-\frac{N-2}{2}}\delta_x \text{ and } |\Psi(x)|^{2^*} \rightharpoonup d\nu = (S||k||_\infty^{-1})^{N/2}\delta_x. \tag{6.32}$$

We prove now the next result.

Lemma 6.12. *For $\lambda \to 0$ we have*

1. $\Xi(\Psi_\lambda(x)) = x + o(1)$ *uniformly for* $x \in B_{R_0}(0)$.
2. $\sup\{dist(\Xi(u), \mathcal{C}(k)) : u \in \tilde{M}(\lambda)\} \to 0$.

Proof. Let $x \in B_{R_0}(0)$, then by (6.32) we obtain that

$$\Xi(\Psi_\lambda(x)) = \frac{\int_{\mathbb{R}^N} \xi(y)|\nabla\Psi_\lambda(x)|^2 dy}{\int_{\mathbb{R}^N} |\nabla\Psi_\lambda(x)|^2 dy} = \frac{\int_{\mathbb{R}^N} \xi(y) d\mu}{\int_{\mathbb{R}^N} d\mu} + o(1) = x + o(1) \quad \text{as } \lambda \to 0.$$

To prove the second assertion we take $\lambda_n \to 0$ and let $v_n \in \tilde{M}(\lambda_n)$, then by Lemma 6.11 we get the existence of $\{x_n\} \subset \mathbb{R}^N$ and $\{r_n\} \subset \mathrm{Re}_+$ such that $x_n \to x_0$, for some $x_0 \in \mathcal{C}(k), r_n \to 0$ as $n \to \infty$ and

$$v_n - \left(\frac{S}{||k||_\infty}\right)^{\frac{N-2}{4}} u_{r_n}(\cdot - x_n) \to 0 \text{ in } \mathcal{D}^{1,2}(\mathbb{R}^N).$$

Since Ξ is a continuous function we obtain that

$$\Xi(v_n) = \frac{\int_{\mathbb{R}^N} \xi(x)|\nabla v_n|^2 dx}{\int_{\mathbb{R}^N} |\nabla v_n|^2 dx} = \frac{\int_{\mathbb{R}^N} \xi(x)|\nabla u_{r_n}(\cdot - x_n)|^2 dx}{\int_{\mathbb{R}^N} |\nabla u_{r_n}(\cdot - x_n)|^2 dx} + o(1) = \xi(x_0) + o(1).$$

Since $x_0 \in \mathcal{C}(k) \subset B_{R_0}(0)$ we conclude that $\xi(x_0) = x_0$ and the result follows. □

We are now able to prove the main result.

Theorem 6.5. *Assume that hypotheses* $(K0)$, $(K2)$ *and* $(K3)$ *hold and let* $\delta > 0$. *Then there exists* $\lambda_0 > 0$ *such that for all* $0 < \lambda < \lambda_0$, *equation* (6.13) *has at least* $\text{cat}_{\mathcal{C}_\delta(k)} \mathcal{C}(k)$ *solutions.*

Proof. Given $\delta > 0$ there exists $\lambda_0(\delta) > 0$ such that by Lemma 6.12 and (6.31), for $0 < \lambda < \lambda_0(\delta)$ we have that $\Psi_\lambda(x) \in \tilde{M}(\lambda)$ for any $x \in \mathcal{C}(k)$, and

$$|\Xi(\Psi_\lambda(x)) - x| < \delta \text{ for all } x \in B_{R_0}(0) \text{ and } \Xi(u) \in \mathcal{C}_\delta(k) \text{ for all } u \in \tilde{M}(\lambda).$$

Let $\mathcal{H}(t,x) = x + t(\Xi(\Psi_\lambda(x)) - x)$ where $(t,x) \in [0,1] \times \mathcal{C}(k)$, then $\mathcal{H}$ is a continuous function and $dist(\mathcal{H}(t,x), \mathcal{C}(k)) \leq \delta$ for all $(t,x) \in [0,1] \times \mathcal{C}(k)$. Hence

$$\mathcal{H}([0,1] \times \mathcal{C}(k)) \subset \mathcal{C}_\delta(k).$$

Since $\mathcal{H}(0,x) = x$ and $\mathcal{H}(1,x) = \Xi(\Psi_\lambda(x))$, then we conclude that $\Xi \circ \Psi_\lambda$ is homotopic to the inclusion $\mathcal{C}(k) \hookrightarrow \mathcal{C}_\delta(k)$. Since J_λ satisfies the Palais-Smale condition below the level $\tilde{c}$, to prove the Theorem we need just to prove that $\text{cat}(\tilde{M}(\lambda)) \geq \text{cat}_{\mathcal{C}_\delta(k)} \mathcal{C}(k)$. Suppose that $\{M_i\}$, $i = 1, ..., n_0$, is a closed covering of $\tilde{M}(\lambda)$, then for any $i = 1, ..., n_0$ there exists a homotopy

$$\mathcal{H}_i : [0,1] \times M_i \to \tilde{M}(\lambda)$$

such that

$$\mathcal{H}_i(0,u) = u \text{ for all } u \in M_i \text{ and } \mathcal{H}_i(1,\cdot) = \text{constant for } i = 1, ..., n_0.$$

Notice that from (6.31), we obtain that $\Psi_\lambda(\mathcal{C}(k)) \subset \tilde{M}(\lambda)$. We set $\mathcal{C}_i = \Psi_\lambda^{-1}(M_i)$, then $\mathcal{C}_i$ is closed in $\mathcal{C}_\delta(k)$ and $\mathcal{C}(k) \subset \cup_i \mathcal{C}_i \subset \mathcal{C}_\delta(k)$. Then we have just only to show that $\mathcal{C}_i$ are contractible in $\mathcal{C}_\delta(k)$. We set

$$\mathcal{G}_i : [0,1] \times \mathcal{C}_i \to \mathcal{C}_\delta(k), \quad \text{where} \quad \mathcal{G}_i(t,x) = \Xi(\mathcal{H}_i(t, \Psi_\lambda(x))).$$

Then

$$\mathcal{G}_i(0,x) = \Xi \circ \Psi_\lambda(x) \text{ for all } x \in \mathcal{C}_i \text{ and } \mathcal{G}_i(1,\cdot) = \text{ constant for } i = 1, ..., n_0.$$

Since $\Xi \circ \Psi_\lambda$ is homotopic to the inclusion $\mathcal{C}(k) \hookrightarrow \mathcal{C}_\delta(k)$ we have that $\mathcal{C}_i$ are contractible in $\mathcal{C}_\delta(k)$. To complete the proof it remains to prove that any solution has a fixed sign. We follow the argument used in [38]. Assume that $u = u^+ - u^-$ with $u^+ \geq 0, u^- \geq 0$ and $u^+ \not\equiv 0, u^- \not\equiv 0$. Then we have

$$\int_{\mathbb{R}^N} |\nabla u^\pm|^2 dx - \lambda \int_{\mathbb{R}^N} \frac{|u^\pm|^2}{|x|^2} dx \geq S(1 - \frac{\lambda}{\Lambda_N})^{\frac{N-1}{N}} \Big(\int_{\mathbb{R}^N} |u^\pm|^{2^*} dx \Big)^{2/2^*}$$
$$\geq S(1 - \frac{\lambda}{\Lambda_N})^{\frac{N-1}{N}} ||k||_\infty^{-\frac{2}{2^*}} \Big(\int_{\mathbb{R}^N} k(x)|u^\pm|^{2^*} dx \Big)^{2/2^*}.$$

Since u is a solution to problem (6.13) we obtain that

$$\int_{\mathbb{R}^N} |\nabla u^\pm|^2 dx - \lambda \int_{\mathbb{R}^N} \frac{|u^\pm|^2}{|x|^2} dx = \int_{\mathbb{R}^N} k(x)|u^\pm|^{2^*} dx. \tag{6.33}$$

Therefore we conclude that

$$\tilde{c} > J_\lambda(u) = \frac{1}{N} \int_{\mathbb{R}^N} k(x)|u|^{2^*} dx = \frac{1}{N} \Big\{ \int_{\mathbb{R}^N} k(x)|u^+|^{2^*} dx + \int_{\mathbb{R}^N} k(x)|u^-|^{2^*} dx \Big\}$$
$$\geq \frac{2S^{\frac{N}{2}}}{N} \Big(1 - \frac{\lambda}{\Lambda_N} \Big)^{\frac{N-1}{2}} ||k||_\infty^{-\frac{N-2}{2}}.$$

Hence we obtain $2(1 - \frac{\lambda}{\Lambda_N})^{\frac{N-1}{2}} \leq 1$ which contradicts the choice of λ. □

Remark 6.5.

i) If $\mathcal{C}(k)$ is finite, then for λ small, equation (6.13) has at least Card($\mathcal{C}(k)$) solutions.

ii) We give now a typical example where equation (6.13) has infinity many solutions. Let $\eta : \mathrm{Re} \to \mathrm{Re}_+$ such that η is regular, $\eta(0) = 0$ and $\eta(r) = 1$ for $r \geq \frac{1}{2}$. We define k_1 on $[0,1] \subset \mathrm{Re}$ by

$$k_1(r) = \begin{cases} 0 & \text{if } r = 0, \\ \eta(r)(1 - |\sin(\frac{1}{r - \frac{1}{2}})|^\theta) & \text{if } 0 < r \leq 1, \end{cases}$$

where $2 < \theta < N$. Notice that k_1 has infinitely many global maximums archived on the set

$$\mathcal{C}(k_1) = \Big\{ r_n = \frac{1}{2} + \frac{1}{n\pi} \text{ for } n \geq 1 \Big\}.$$

Now we define k to be any continuous bounded function such that $k(x) = k_1(|x|)$ if $|x| \leq 1$, $||k||_\infty \leq 1$ and $\lim_{|x| \to \infty} k(x) = 0$. Since for all $m \in \mathbb{N}$, there exists $\delta(m)$ such that $\mathrm{cat}_{\mathcal{C}_\delta(k)}(\mathcal{C}(k)) = m$, then we conclude that equation (6.13) has at least m solutions for $0 < \lambda < \lambda(\delta)$.

iii) Let us note that if δ becomes larger, then $\mathrm{cat}_{\mathcal{C}_\delta(k)}(\mathcal{C}(k))$ decreases, so that Theorem 6.5 is interesting for δ small.

Part III

H-bubbles

7 $\mathbb{S}^2$-type surfaces with prescribed mean curvature

In this chapter we discuss the problem of existence of surfaces in $\mathbb{R}^3$ parametrized on the sphere $\mathbb{S}^2$ with prescribed mean curvature. We start by giving a short description of the models of capillarity phenomena motivating the study of this problem. For a more detailed discussion of properties of minimal surfaces we refer to [41].

7.1 Motivation: soap films and soap bubbles

The study of the configuration of soap films dates back to Plateau (1801-1883). Some of his physical experiments were actually very simple and familiar to everyone; any child has enjoyed himself in blowing soap bubbles through a tube or spanning a wire contour with soap films. In particular let us consider two kind of soap films:

1. *soap films* (with boundary) obtained by taking a wire contour out of soapy water avoiding sudden movements, see Fig. 7.1 a);
2. *soap bubbles* (without boundary) obtained by blowing through a tube that has previously been deeped in soapy water; such bubbles (stabilized by the addition of glycerine to the water) are formed and stay in equilibrium by the pressure of the air inside the bubble, see Fig. 7.1 b).

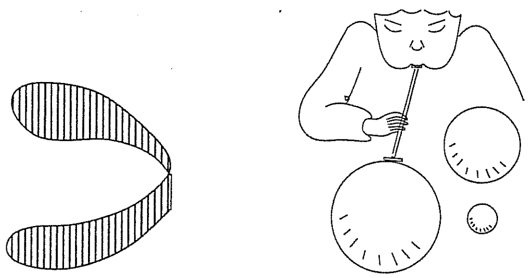

Fig. 7.1. a) soap film b) soap bubbles

The first type of soap films can be mathematically modeled by *minimal surfaces.* Indeed, from a physical point of view, the equilibrium of soap film is reached in the configuration with minimal energy. The energy of a soap film can be described by terms of the surface tension of a liquid, determined by the balance of the attraction forces between molecules on the boundary of the surface. Neglecting the force of gravity (we can do it for small sizes of the film; the larger the contour of the film is, the easier it breaks up under the force of gravity), the liquid film turns into an elastic surface that tends to minimize its own area and hence the energy of the surface tension. As a consequence, a mathematical model of soap films is a minimal surface, i.e. a surface which has the least possible area (locally) among all surfaces with a prescribed boundary (the wire contour). The problem of finding a surface of least area with a given boundary has been studied since the 18th century by Euler and Lagrange while exact solutions for some special boundary contours were found by Riemann, Schwarz and Weirstrass in the 19th century.

In the understanding of the configuration of the second type of film, i.e. soap bubbles, the first important step is due to Boyle, who studied the dependence between the size of a drop and its form. An important contribution to the study of the interface between media was given in 1828 by Poisson who understood the role of the *mean curvature* of the surface in the description of soap bubbles in equilibrium (see Section 7.2 for the definition of mean curvature), obtaining the following result (see [41] for the details).

Theorem 7.1 (Poisson). *The mean curvature H of a smooth two-dimensional surface in $\mathbb{R}^3$ which is the interface between two media in equilibrium is constant and*

$$H = h(p_1 - p_2)$$

where $1/h$ is the coefficient of surface tension and p_1, p_2 are the pressures of the two media.

From the above theorem, the physical reason why soap bubbles have spherical form becomes clear: the pressure of the gas (air) inside exceeds the external pressure ($p_1 > p_2$) and the equilibrium is a result of the action of the forces of the surface tension. Thus $H = h(p_1 - p_2) = \text{const} > 0$.

We will focus on the problem of existence of soap bubbles with prescribed non-constant mean curvature, which is physically related to the phenomenon of formation of an electrified drop. In fact it has been experimentally observed (see [53]) that an external electric field may affect the shape of the drop making surface mean curvature non constant. In the next section we present the mathematical formulation of the problem.

7.2 Analytical formulation

Let us start by recalling some basic definitions about surfaces. Let $S \subset \mathbb{R}^3$ be a surface in $\mathbb{R}^3$. Let us consider the Gauss map which maps each point $p \in S$ into a unit normal vector $n = n(p) \perp S$. For $p \in S$ let us take a vector $v \in T_pS$. The pair $(n(p), v)$ determines the normal plane P_v. Let us consider the curve $\alpha = P_v \cap S$ (*normal section*) which is locally smooth and let denote by $K(v,p)$ its curvature.

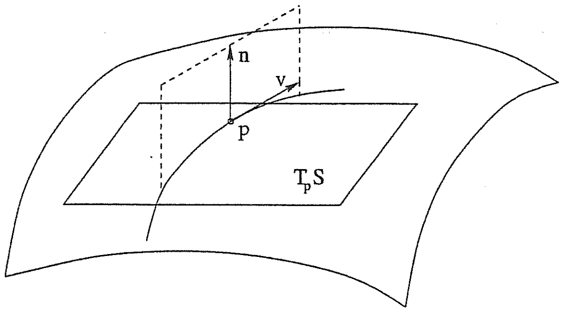

Fig. 7.2. the normal section

If $\alpha: \ s \mapsto \alpha(s)$ is the arc-length parameterization, then the curvature is given by $K(s) = |\alpha''(s)|$.

Definition 7.1. *The principal curvatures of S at p are defined as*

$$K_1 = \max_{v \in T_pS} K(v,p), \quad K_2 = \min_{v \in T_pS} K(v,p).$$

The Gaussian curvature is defined as the product $R = K_1 \cdot K_2$. The mean curvature is defined as $H = (K_1 + K_2)/2$.

The *Theorema Egregium* by Gauss states that the Gaussian curvature is invariant under local isometries. This means that the Gaussian curvature is an intrinsic quantity which depends on the surface and not on the parameterization. The conclusion of Theorema Egregium fails for the mean curvature as one can easily observe in the case of a cylinder of radius ρ: the principal curvatures are $K_1 = 1/\rho$ and $K_2 = 0$, so that the Gaussian curvature is 0 and the mean curvature is $\frac{1}{2\rho} \neq 0$. The cylinder is locally isometric to a plane, for which both the Gaussian curvature and the mean curvature are 0.

The principal curvatures can be defined also in another way. If $p \mapsto n_p$ is the Gauss map, we notice that its differential dn_p is symmetric. The principal curvatures are the eigenvalues of dn_p (in particular the Gaussian curvature is the determinant and the mean curvature is the trace divided by 2). This second definition allows us to extend the notion of Gaussian and mean curvature to higher dimension. In dimension N, the previous map has N eigenvalues $k_1, k_2, \ldots, k_N$ and we define the mean curvature H and the Gaussian curvature R as $H = (k_1 + k_2 + \cdots + k_N)/N$, $R = \prod k_i$.

Let us now focus our attention on surfaces of $\mathbb{S}^2$-type. Let us identify $\mathbb{R}^2 \cup \{\infty\}$ with the unit standard 2-dimensional sphere

$$\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 : \ x^2 + y^2 + z^2 = 1\}$$

through the stereographic projection Φ from the north pole $N = (0,0,1)$

$$\Phi : \mathbb{S}^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2, \quad (x,y,z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$

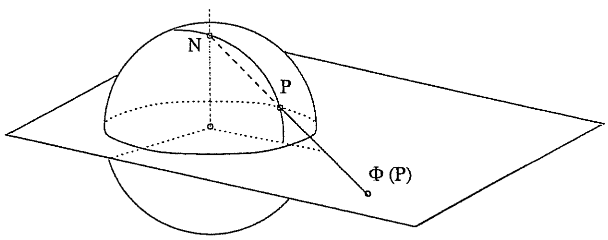

Fig. 7.3. the stereographic projection

Let $\omega : \mathbb{R}^2 \to \mathbb{S}^2$ be the inverse of the stereographic projection, i.e. $\omega : \mathbb{R}^2 \to \mathbb{S}^2$

$$\omega(x,y) = \big(\mu(x,y)x, \mu(x,y)y, 1 - \mu(x,y)\big)$$

where

$$\mu(x,y) = \frac{2}{1+x^2+y^2}, \ (x,y) \in \mathbb{R}^2.$$

We will identify maps defined on the sphere with maps defined on $\mathbb{R}^2$ through the stereographic projection

$$u : \mathbf{R}^2 \to \mathbf{R}^3 \quad \sim \quad v = u \circ \omega : \mathbb{S}^2 \to \mathbf{R}^3.$$

A parametric surface in $\mathbf{R}^3$ of type $\mathbb{S}^2$ is a map $u : \mathbf{R}^2 \to \mathbf{R}^3$ smooth as a map on $\mathbb{S}^2$. A point $(x,y) \in \mathbb{R}^2$ at which $u_x \wedge u_y = 0$ is said to be a *branch point* of the surface, where

$$u_x = \left(\frac{\partial u_1}{\partial x}, \frac{\partial u_2}{\partial x}, \frac{\partial u_3}{\partial x}\right), \quad u_y = \left(\frac{\partial u_1}{\partial y}, \frac{\partial u_2}{\partial y}, \frac{\partial u_3}{\partial y}\right),$$

and $\wedge$ denotes the exterior product in $\mathbb{R}^3$. For any point z of the surface which is not a branch point, consider the normal vector at point $u(z)$, i.e. the Gauss map

$$n(u(z)) = \frac{u_x \wedge u_y}{|u_x \wedge u_y|}.$$

The mean curvature of the surface parametrized by u at point $u(z)$, where z is not a branch point, is

$$H(u(z)) = \frac{1}{2}\frac{|u_y|^2 u_{xx} \cdot n + |u_x|^2 u_{yy} \cdot n - 2(u_x \cdot u_y)u_{xy} \cdot n}{|u_x|^2|u_y|^2 - (u_x \cdot u_y)^2}.$$

Definition 7.2. *We say that u is conformal if*

$$|u_x| = |u_y| \quad \textit{and} \quad u_x \cdot u_y = 0.$$

If u conformal then

$$H(u) = \frac{\Delta u \cdot n}{2|u_x \wedge u_y|}$$

where $\Delta\omega = \omega_{xx} + \omega_{yy}$ and since Δu is orthogonal to the tangent plane spanned by $\langle u_x, u_y \rangle$, we get

$$\Delta u = 2H(u)\, u_x \wedge u_y.$$

As a consequence, the geometric problem of finding a $\mathbb{S}^2$-type parametric surface in $\mathbb{R}^3$ having prescribed mean curvature H *(H-bubble)* admits the following analytical formulation: given $H \in C^1(\mathbb{R}^3)$, find a nonconstant function $u : \mathbb{R}^2 \to \mathbb{R}^3$ smooth as a map on $\mathbb{S}^2$ solving

$$\begin{cases} \Delta u = 2H(u)\, u_x \wedge u_y \\ \int_{\mathbf{R}^2} |\nabla u|^2 < +\infty \end{cases} \tag{P_H}$$

where $\nabla u = (u_x, u_y)$. Indeed, if u solves (P_H), then by regularity for H-systems [57] u is regular. Moreover a short computation shows that u is conformal. Indeed, let us define the map $\Psi : \mathbb{C} \to \mathbb{C}$, $\Psi(z) = \Psi(x+iy) = \alpha(x,y) + i\beta(x,y)$, where $\alpha(x,y) = |u_x|^2 - |u_y|^2$ and $\beta(x,y) = 2u_x \cdot u_y$. The functions α and β are regular and satisfy the Cauchy-Riemann condition

$$\begin{cases} \alpha_x = -\beta_y \\ \alpha_y = \beta_x. \end{cases}$$

Hence Ψ is holomorphic. Since Ψ is integrable, it must be $\Psi \equiv 0$, namely u must be conformal. Then, if u is a solution of (P_H), at any regular point $p = u(z)$, $H(p)$ is the mean curvature of the surface parametrized by u at the point p.

Remark 7.1. Problem (P_H) is invariant under the action of the conformal group of $\mathbb{S}^2 \sim \mathbf{R}^2 \cup \{\infty\}$, i.e. if g is a conformal diffeomorphism

$$|g_x| - |g_y| = 0 \quad \text{and} \quad g_x \cdot g_y = 0$$

and u solves (P_H), then also $u \circ g$ solves (P_H). This means that actually the unknown of the problem (P_H) is the *surface* rather than the *parametrization*, i.e. the image of u rather then the function u.

7.3 Existence of H-bubbles via perturbation methods

In this section we are going to present the results of [45] where the author studied the problem of existence of H-bubbles, which, as explained in the previous section, admits the following analytical formulation: given a function $H \in C^1(\mathbb{R}^3)$, find a smooth nonconstant function $\omega : \mathbb{R}^2 \to \mathbb{R}^3$ which is conformal as a map on $\mathbb{S}^2$ and solves the problem

$$\begin{cases} \Delta\omega = 2H(\omega)\, \omega_x \wedge \omega_y, \quad \text{in } \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} |\nabla\omega|^2 < +\infty. \end{cases} \tag{P_H}$$

Brezis and Coron [25] proved that for constant nonzero mean curvature $H(u) \equiv H_0$ the only nonconstant solutions are spheres of radius $|H_0|^{-1}$.

While the Plateau and the Dirichlet problems has been largely studied both for H constant and for H nonconstant (see [24, 25, 62, 64, 65, 78, 79, 80]), problem (P_H) in the case of nonconstant H has been investigated only recently, see [29, 30, 31, 73]. In [29] Caldiroli and Musina proved the existence of H-bubbles with minimal energy under the assumptions that $H \in C^1(\mathbb{R}^3)$ satisfies

$$(i) \quad \sup_{u \in \mathbb{R}^3} |\nabla H(u+\xi) \cdot u\, u| < 1 \quad \text{for some } \xi \in \mathbb{R}^3,$$

$$(ii) \quad H(u) \to H_\infty \quad \text{as } |u| \to \infty \quad \text{for some } H_\infty \in \mathbb{R},$$

$$(iii) \quad c_H = \inf_{\substack{u \in C_c^1(\mathbb{R}^2,\mathbb{R}^3)\\ u \neq 0}} \sup_{s>0} \mathcal{E}_H(su) < \frac{4\pi}{3H_\infty^2}$$

where $\mathcal{E}_H(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 + 2\int_{\mathbb{R}^2} Q(u) \cdot u_x \wedge u_y$ and $Q : \mathbb{R}^3 \to \mathbb{R}^3$ is any vector field such that $\operatorname{div} Q = H$.

The perturbative method introduced by Ambrosetti and Badiale and discussed in Chapter 3 was used in [31] to treat the case in which H is a small perturbation of a constant, namely

$$H(u) = H_\varepsilon(u) = H_0 + \varepsilon H_1(u),$$

where $H_0 \in \mathbb{R} \setminus \{0\}$, $H_1 \in C^2(\mathbb{R}^3)$, and ε is a small real parameter. This method allows to find critical points of a functional f_ε of the type $f_\varepsilon(u) = f_0(u) - \varepsilon G(u)$ in a Banach space by studying a finite dimensional problem. More precisely, if the unperturbed functional f_0 has a finite dimensional manifold of critical points Z which satisfies a nondegeneracy condition, it is possible to prove, for $|\varepsilon|$ sufficiently small, the existence of a smooth function $\eta_\varepsilon(z) : Z \to (T_z Z)^\perp$ such that any critical point $\bar{z} \in Z$ of the function

$$\Phi_\varepsilon : Z \to \mathbb{R}, \quad \Phi_\varepsilon(z) = f_\varepsilon\big(z + \eta_\varepsilon(z)\big)$$

gives rise to a critical point $u_\varepsilon = \bar{z} + \eta_\varepsilon(\bar{z})$ of f_ε, i.e. the perturbed manifold $Z_\varepsilon := \{z + \eta_\varepsilon(z) : \ z \in Z\}$ is a natural constraint for f_ε. Furthermore Φ_ε can be expanded as

$$\Phi_\varepsilon(z) = b - \varepsilon \Gamma(z) + o(\varepsilon) \quad \text{as } \varepsilon \to 0 \tag{7.1}$$

where $b = f_0(z)$ and Γ is the Melnikov function defined as the restriction of the perturbation G on Z, namely $\Gamma = G\big|_Z$. For problem (P_{H_ε}), Γ is given by

$$\Gamma : \mathbb{R}^3 \to \mathbb{R}, \quad \Gamma(p) = \int_{|p-q|<\frac{1}{|H_0|}} H_1(q)\, dq.$$

In [31] Caldiroli and Musina studied the functional Γ giving some existence results in the perturbative setting for problem (P_{H_ε}). They prove that for $|\varepsilon|$ small there exists a smooth H_ε-bubble if one of the following conditions holds

1) H_1 has a nondegenerate stationary point and $|H_0|$ is large;
2) $\max_{p\in\partial K} H_1(p) < \max_{p\in K} H_1(p)$ or $\min_{p\in\partial K} H_1(p) > \min_{p\in K} H_1(p)$
   for some nonempty compact set $K \subset \mathbb{R}^3$ and $|H_0|$ is large;
3) $H_1 \in L^r(\mathbb{R}^3)$ for some $r \in [1,2]$.

They prove that critical points of Γ give rise to solutions to (P_{H_ε}) for ε sufficiently small. Precisely the assumption that H_0 is large required in cases 1) and 2) ensures that if H_1 is not constant then Γ is not identically constant. If we let this assumption drop, it may happen that Γ is constant even if H_1 is not. This fact produces some loss of information because the first order expansion (7.1) is not sufficient to deduce the existence of critical points of Φ_ε from the existence of critical points of Γ. Instead of studying Γ we perform a direct study of Φ_ε which allows us to prove some new results. In the first one, we assume that H_1 vanishes at ∞ and has bounded gradient, and prove the existence of a solution without branch points. Let us recall that a branch point for a solution ω to (P_H) is a point where the surface parametrized by ω fails to be immersed.

Theorem 7.2. *Let $H_0 \in \mathbb{R}\setminus\{0\}$, $H_1 \in C^2(\mathbb{R}^3)$ such that*

$$(H1) \quad \lim_{|p|\to\infty} H_1(p) = 0;$$

$$(H2) \quad \nabla H_1 \in L^\infty(\mathbb{R}^3, \mathbb{R}^3).$$

Let $H_\varepsilon = H_0 + \varepsilon H_1$. Then for $|\varepsilon|$ sufficiently small there exists a smooth H_ε-bubble without branch points.

With respect to case 1) of [31] we require neither nondegeneracy of critical points of H_1 nor largeness of H_0. With respect to case 2) we do not assume that H_0 is large; on the other hand our assumption $(H1)$ implies 2). Moreover we do not assume any integrability condition of type 3). With respect to the result proved in [29], we have the same kind of behavior of H_1 at ∞ (see (ii) and assumption $(H1)$) but we do not need any assumption of type (iii); on the other hand in [29] it is not required that the prescribed curvature is a small perturbation of a constant.

The following results give some conditions on H_1 in order to have two or three solutions.

Theorem 7.3. *Let $H_0 \in \mathbb{R}\setminus\{0\}$, $H_1 \in C^2(\mathbb{R}^3)$ such that $(H1)$, $(H2)$,*

$(H3)$ $\operatorname{Hess} H_1(p)$ *is positive definite for any* $p \in B_{1/|H_0|}(0)$,

$(H4)$ $H_1(p) > 0$ *in* $B_{1/|H_0|}(0)$,

hold. Then for $|\varepsilon|$ sufficiently small there exist at least three smooth H_ε-bubbles without branch points.

Remark 7.2. If we assume $(H1)$, $(H2)$, and, instead of $(H3)-(H4)$, that $H_1(0) > 0$ and $\operatorname{Hess} H_1(0)$ is positive definite, then we can prove that for $|H_0|$ sufficiently large and $|\varepsilon|$ sufficiently small there exist at least three smooth H_ε-bubbles without branch points.

Theorem 7.4. *Let $H_0 \in \mathbb{R}\backslash\{0\}$, $H_1 \in C^2(\mathbb{R}^3)$ such that $(H1)$ and $(H2)$ hold. Assume that there exist $p_1, p_2 \in \mathbb{R}^3$ such that*

$$(H5) \quad \int_{B(p_1,1/|H_0|)} H_1(\xi)\, d\xi > 0 \quad \textit{and} \quad \int_{B(p_2,1/|H_0|)} H_1(\xi)\, d\xi < 0.$$

Then for $|\varepsilon|$ sufficiently small there exist at least two smooth H_ε-bubbles without branch points.

Remark 7.3. If we assume $(H1)$, $(H2)$, and, instead of $(H5)$, that there exist $p_1, p_2 \in \mathbb{R}^3$ such that $H_1(p_1) > 0$ and $H_1(p_2) < 0$, then we can prove that for $|H_0|$ sufficiently large and $|\varepsilon|$ sufficiently small there exist at least two smooth H_ε-bubbles without branch points.

7.3.1 Notation and known facts

In the sequel we will take $H_0 = 1$; this is not restrictive since we can do the change $H_1(u) = H_0\widetilde{H}_1(H_0 u)$. Hence we will always write

$$H_\varepsilon(u) = 1 + \varepsilon H(u),$$

where $H \in C^2(\mathbb{R}^3)$. Let us denote by ω the function $\omega : \mathbb{R}^2 \to \mathbb{S}^2$ defined as

$$\omega(x,y) = \big(\mu(x,y)x, \mu(x,y)y, 1 - \mu(x,y)\big) \quad \mu(x,y) = \frac{2}{1+x^2+y^2}, \ (x,y) \in \mathbb{R}^2.$$

Note that ω is a conformal parametrization of the unit sphere and solves

$$\begin{cases} \Delta\omega = 2\,\omega_x \wedge \omega_y & \text{on } \mathbb{R}^2 \\ \displaystyle\int_{\mathbb{R}^2} |\nabla\omega|^2 < +\infty. \end{cases} \tag{7.2}$$

Problem (7.2) has in fact a family of solutions of the form $\omega \circ \phi + p$ where $p \in \mathbb{R}^3$ and ϕ is any conformal diffeomorphism of $\mathbb{R}^2 \cup \{\infty\}$. For $s \in (1, +\infty)$, we will set $L^s := L^s(\mathbb{S}^2, \mathbb{R}^3)$, where any map $v \in L^s$ is identified with the map $v \circ \omega : \mathbb{R}^2 \to \mathbb{R}^3$ which satisfies

$$\int_{\mathbb{R}^2} |v \circ \omega|^s \mu^2 = \int_{\mathbb{S}^2} |v|^s.$$

We will use the same notation for v and $v \circ \omega$. By $W^{1,s}$ we denote the Sobolev space $W^{1,s}(\mathbb{S}^2, \mathbb{R}^3)$ endowed (according to the above identification) with the norm

$$||v||_{W^{1,s}} = \left[\int_{\mathbb{R}^2} |\nabla v|^s \mu^{2-s}\right]^{1/s} + \left[\int_{\mathbb{R}^2} |v|^s \mu^2\right]^{1/s}.$$

If s' is the conjugate exponent of s, i.e. $\frac{1}{s} + \frac{1}{s'} = 1$, the duality product between $W^{1,s}$ and $W^{1,s'}$ is given by

$$\langle v, \varphi \rangle = \int_{\mathbb{R}^2} \nabla v \cdot \nabla \varphi + \int_{\mathbb{R}^2} v \cdot \varphi \, \mu^2 \quad \text{for any } v \in W^{1,s} \text{ and } \varphi \in W^{1,s'}.$$

Let Q be any smooth vector field on $\mathbb{R}^3$ such that $\operatorname{div} Q = H$. The energy functional associated to problem

$$\begin{cases} \Delta u = 2\big(1 + \varepsilon H(u)\big)\, u_x \wedge u_y, & \text{in } \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} |\nabla u|^2 < +\infty, \end{cases} \tag{P_ε}$$

is given by

$$\mathcal{E}_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 + 2\mathcal{V}_1(u) + 2\varepsilon \mathcal{V}_H(u), \quad u \in W^{1,3},$$

where

$$\mathcal{V}_H(u) = \int_{\mathbb{R}^2} Q(u) \cdot u_x \wedge u_y$$

has the meaning of an algebraic volume enclosed by the surface parametrized by u with weight H (it is independent of the choice of Q); in particular

$$\mathcal{V}_1(u) = \frac{1}{3} \int_{\mathbb{R}^2} u \cdot u_x \wedge u_y.$$

In [31], Caldiroli and Musina studied some regularity properties of $\mathcal{V}_H$ on the space $W^{1,3}$. In particular they proved the following properties.

a) For $H \in C^1(\mathbb{R}^3)$, the functional $\mathcal{V}_H$ is of class C^1 on $W^{1,3}$ and the Fréchet differential of $\mathcal{V}_H$ at $u \in W^{1,3}$ is given by

$$d\mathcal{V}_H(u)\varphi = \int_{\mathbb{R}^2} H(u)\, \varphi \cdot u_x \wedge u_y \quad \text{for any } \varphi \in W^{1,3} \tag{7.3}$$

and admits a unique continuous and linear extension on $W^{1,3/2}$ defined by (7.3). Moreover for every $u \in W^{1,3}$ there exists $\mathcal{V}_H'(u) \in W^{1,3}$ such that

$$\langle \mathcal{V}_H'(u), \varphi \rangle = \int_{\mathbb{R}^2} H(u)\, \varphi \cdot u_x \wedge u_y \quad \text{for any } \varphi \in W^{1,3/2}. \tag{7.4}$$

b) For $H \in C^2(\mathbb{R}^3)$, the map $\mathcal{V}_H' : W^{1,3} \to W^{1,3}$ is of class C^1 and

$$\langle \mathcal{V}_H''(u) \cdot \eta, \varphi \rangle = \int_{\mathbb{R}^2} H(u)\, \varphi \cdot (\eta_x \wedge u_y + u_x \wedge \eta_y) + \int_{\mathbb{R}^2} (\nabla H(u) \cdot \eta) \varphi \cdot (u_x \wedge u_y)$$
$$\text{for any } u, \eta \in W^{1,3} \text{ and } \varphi \in W^{1,3/2}. \tag{7.5}$$

Hence for all $u \in W^{1,3}$, $\mathcal{E}_\varepsilon'(u) \in W^{1,3}$ and for any $\varphi \in W^{1,3/2}$

$$\langle \mathcal{E}_\varepsilon'(u), \varphi \rangle = \int_{\mathbb{R}^2} \nabla u \cdot \nabla \varphi + 2 \int_{\mathbb{R}^2} \varphi \cdot u_x \wedge u_y + 2\varepsilon \int_{\mathbb{R}^2} H(u)\, \varphi \cdot u_x \wedge u_y.$$

As remarked in [31], critical points of $\mathcal{E}_\varepsilon$ in $W^{1,3}$ give rise to bounded weak solutions to (P_ε) and hence by the regularity theory for H-systems (see [57]) to classical conformal solutions which are $C^{3,\alpha}$ as maps on $\mathbb{S}^2$.

The unperturbed problem, i.e. (P_ε) for $\varepsilon = 0$, has a 9-dimensional manifold of solutions given by

$$Z = \{R\omega \circ L_{\lambda,\xi} + p : \; R \in SO(3), \; \lambda > 0, \; \xi \in \mathbb{R}^2, \; p \in \mathbb{R}^3\}$$

where $L_{\lambda,\xi} z = \lambda(z-\xi)$ (see [63]). In [63] the nondegeneracy condition $T_u Z = \ker \mathcal{E}_0''(u)$ for any $u \in Z$ (where $T_u Z$ denotes the tangent space of Z at u) is proved (see also [36]).

As observed in [31], in performing the finite dimensional reduction, the dependence on the 6-dimensional conformal group can be neglected since any H-system is conformally invariant. Hence we look for critical points of $\mathcal{E}_\varepsilon$ constrained on a three-dimensional manifold Z_ε just depending on the translation variable $p \in \mathbb{R}^3$.

7.3.2 Proof of Theorem 7.2

We start by constructing a perturbed manifold which is a natural constraint for $\mathcal{E}_\varepsilon$.

Lemma 7.1. *Assume $H \in C^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and $\nabla H \in L^\infty(\mathbb{R}^3, \mathbb{R}^3)$. Then there exist $\varepsilon_0 > 0$, $C_1 > 0$, and a C^1 map $\eta : (-\varepsilon_0, \varepsilon_0) \times \mathbb{R}^3 \to W^{1,3}$ such that for any $p \in \mathbb{R}^3$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$*

$$\mathcal{E}_\varepsilon'(\omega + p + \eta(\varepsilon, p)) \in T_\omega Z, \tag{7.6}$$

$$\eta(\varepsilon, p) \in (T_\omega Z)^\perp, \tag{7.7}$$

$$\int_{\mathbb{S}^2} \eta(\varepsilon, p) = 0, \tag{7.8}$$

$$\|\eta(\varepsilon, p)\|_{W^{1,3}} \le C_1 |\varepsilon|. \tag{7.9}$$

Moreover if we assume that the limit of H at ∞ exists and

$$\lim_{|p| \to \infty} H(p) = 0 \tag{7.10}$$

we have that $\eta(\varepsilon, p)$ converges to 0 in $W^{1,3}$ as $|p| \to \infty$ uniformly with respect to $|\varepsilon| < \varepsilon_0$.

Proof. Let us define the map

$$F = (F_1, F_2) : \mathbb{R} \times \mathbb{R}^3 \times W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3 \to W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3$$

$$\langle F_1(\varepsilon, p, \eta, \lambda, \alpha), \varphi \rangle = \langle \mathcal{E}_\varepsilon'(\omega + p + \eta), \varphi \rangle - \sum_{i=1}^{6} \lambda_i \int_{\mathbb{R}^2} \nabla\varphi \cdot \nabla\tau_i + \alpha \cdot \int_{\mathbb{S}^2} \varphi,$$

$$F_2(\varepsilon, p, \eta, \lambda, \alpha) = \left(\int_{\mathbb{R}^2} \nabla\eta \cdot \nabla\tau_1, \dots, \int_{\mathbb{R}^2} \nabla\eta \cdot \nabla\tau_6, \int_{\mathbb{S}^2} \eta \right)$$

$\forall\varphi \in W^{1,3/2}$, where $\tau_1, \dots, \tau_6$ are chosen in $T_\omega Z$ such that

$$\int_{\mathbb{R}^2} \nabla\tau_i \cdot \nabla\tau_j = \delta_{ij} \quad \text{and} \quad \int_{\mathbb{S}^2} \tau_i = 0 \quad i,j = 1, \dots, 6$$

so that $T_\omega Z$ is spanned by $\tau_1, \dots, \tau_6, e_1, e_2, e_3$. It has been proved by Caldiroli and Musina [31] that F is of class C^1 and that the linear continuous operator

$$\mathcal{L} : W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3 \to W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3$$
$$\mathcal{L} = \frac{\partial F}{\partial(\eta, \lambda, \alpha)}(0, p, 0, 0, 0)$$

i.e.

$$\langle \mathcal{L}_1(v, \mu, \beta), \varphi\rangle = \langle \mathcal{E}_0''(\omega) \cdot v, \varphi\rangle - \sum_{i=1}^{6} \mu_i \int_{\mathbb{R}^2} \nabla\varphi \cdot \tau_i - \beta \int_{\mathbb{S}^2} \varphi \quad \forall\varphi \in W^{1,3/2}$$
$$\mathcal{L}_2(v, \mu, \beta) = \left(\int_{\mathbb{R}^2} \nabla v \cdot \nabla\tau_1, \dots, \int_{\mathbb{R}^2} \nabla v \cdot \nabla\tau_6, \int_{\mathbb{S}^2} v \right)$$

is invertible. Caldiroli and Musina applied the Implicit Function Theorem to solve the equation $F(\varepsilon, p, \eta, \lambda, \alpha) = 0$ locally with respect to the variables ε, p, thus finding a C^1-function η on a neighborhood $(-\varepsilon_0, \varepsilon_0) \times B_R \subset \mathbb{R} \times \mathbb{R}^3$ satisfying (7.6), (7.7), and (7.8). We will use instead the Contraction Mapping Theorem, which allows to prove the existence of such a function η globally on $\mathbb{R}^3$, thanks to the fact that the operator $\mathcal{L}$ does not depend on p and hence it is invertible uniformly with respect to $p \in \mathbb{R}^3$.

We have that $F(\varepsilon, p, \eta, \lambda, \alpha) = 0$ if and only if (η, λ, α) is a fixed point of the map $T_{\varepsilon,p}$ defined as

$$T_{\varepsilon,p}(\eta, \lambda, \alpha) = -\mathcal{L}^{-1} F(\varepsilon, p, \eta, \lambda, \alpha) + (\eta, \lambda, \alpha).$$

To prove the existence of η satisfying (7.6), (7.7), and (7.8), it is enough to prove that $T_{\varepsilon,p}$ is a contraction in some ball $B_\rho(0)$ with $\rho = \rho(\varepsilon) > 0$ independent of p, whereas the regularity of $\eta(\varepsilon, p)$ follows from the Implicit Function Theorem.

We have that if $\|\eta\|_{W^{1,3}} \le \rho$

$$\begin{aligned}
\|T_{\varepsilon,p}(\eta, \lambda, \alpha)\|_{W^{1,3}\times\mathbb{R}^6\times\mathbb{R}^3} &\le C_2 \|F(\varepsilon, p, \eta, \lambda, \alpha) - \mathcal{L}(\eta, \lambda, \alpha)\|_{W^{1,3}\times\mathbb{R}^6\times\mathbb{R}^3} \\
&\le C_2 \|\mathcal{E}_\varepsilon'(\omega + p + \eta) - \mathcal{E}_0''(\omega)\eta\|_{W^{1,3}} \\
&\le C_2 \left(\|\mathcal{E}_0'(\omega + \eta) - \mathcal{E}_0''(\omega)\eta\|_{W^{1,3}} + 2|\varepsilon| \|\mathcal{V}_H'(\omega + p + \eta)\|_{W^{1,3}} \right) \\
&\le C_2 \left(\int_0^1 \|\mathcal{E}_0''(\omega + t\eta) - \mathcal{E}_0''(\omega)\|_{W^{1,3/2}}\, dt \|\eta\|_{W^{1,3}} + 2|\varepsilon| \|\mathcal{V}_H'(\omega + p + \eta)\|_{W^{1,3}} \right) \\
&\le C_2 \rho \sup_{\|\eta\|_{W^{1,3}} \le \rho} \|\mathcal{E}_0''(\omega + \eta) - \mathcal{E}_0''(\omega)\|_{W^{1,3/2}} \\
&\quad + 2C_2|\varepsilon| \sup_{\|\eta\|_{W^{1,3}} \le \rho} \|\mathcal{V}_H'(\omega + p + \eta)\|_{W^{1,3}}
\end{aligned} \tag{7.11}$$

where $C_2 = \|\mathcal{L}^{-1}\|_{\mathcal{L}(W^{1,3}\times\mathbb{R}^6\times\mathbb{R}^3)}$. For $(\eta_1, \lambda_1, \alpha_1), (\eta_2, \lambda_2, \alpha_2) \in B_\rho(0) \subset W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3$ we have

$$\frac{\|T_{\varepsilon,p}(\eta_1,\lambda_1,\alpha_1)-T_{\varepsilon,p}(\eta_2,\lambda_2,\alpha_2)\|_{W^{1,3}\times\mathbb{R}^6\times\mathbb{R}^3}}{C_2\|\eta_1-\eta_2\|_{W^{1,3}}}$$
$$\le \frac{\|\mathcal{E}'_\varepsilon(\omega+p+\eta_1)-\mathcal{E}'_\varepsilon(\omega+p+\eta_2)-\mathcal{E}''_0(\omega)(\eta_1-\eta_2)\|_{W^{1,3}}}{C_2\|\eta_1-\eta_2\|_{W^{1,3}}}$$
$$\le \int_0^1 \|\mathcal{E}''_\varepsilon(\omega+p+\eta_2+t(\eta_1-\eta_2))-\mathcal{E}''_0(\omega)\|_{W^{1,3/2}}\,dt$$
$$\le \int_0^1 \|\mathcal{E}''_0(\omega+p+\eta_2+t(\eta_1-\eta_2))-\mathcal{E}''_0(\omega)\|_{W^{1,3/2}}\,dt$$
$$+2|\varepsilon|\int_0^1 \|\mathcal{V}''_H(\omega+p+\eta_2+t(\eta_1-\eta_2))\|_{W^{1,3/2}}\,dt$$
$$\le \sup_{\|\eta\|_{W^{1,3}}\le 3\rho}\|\mathcal{E}''_0(\omega+\eta)-\mathcal{E}''_0(\omega)\|_{W^{1,3/2}}+2|\varepsilon|\sup_{\|\eta\|_{W^{1,3}}\le 3\rho}\|\mathcal{V}''_H(\omega+p+\eta)\|_{W^{1,3/2}}.$$

From (7.4), (7.5), and the Hölder inequality it follows that there exists a positive constant C_3 such that for any $\eta\in W^{1,3}$, $p\in\mathbb{R}^3$

$$\|\mathcal{V}'_H(\omega+p+\eta)\|_{W^{1,3}}\le C_3\left[\left(\int_{\mathbb{R}^2}|H(\omega+p+\eta)|^{3/2}|\nabla\omega|^3\mu^{-1}\right)^{2/3}+\|\eta\|^2_{W^{1,3}}\right] \tag{7.12}$$

$$\|\mathcal{V}''_H(\omega+p+\eta)\|_{W^{1,3/2}}\le C_3\left[\left(\int_{\mathbb{R}^2}|H(\omega+p+\eta)|^2|\nabla(\omega+\eta)|^2\right)^{1/2}+\left(\int_{\mathbb{R}^2}|\nabla H(\omega+p+\eta)|^{3/2}|\nabla(\omega+\eta)|^3\mu^{-1}\right)^{2/3}\right]. \tag{7.13}$$

Choosing $\rho_0>0$ such that

$$C_2\sup_{\|\eta\|_{W^{1,3}}\le 3\rho_0}\left\|\mathcal{E}''_0(\omega+\eta)-\mathcal{E}''_0(\omega)\right\|_{W^{1,3/2}}<\frac{1}{2}$$

and $\varepsilon_0>0$ such that

$$8C_2C_3\varepsilon_0\|H\|_{L^\infty(\mathbb{R}^3)}\|\omega\|^2_{W^{1,3}}<\min\left\{1,\rho_0,\frac{1}{8C_2C_3\varepsilon_0}\right\}, \tag{7.14}$$

$$\sup_{\substack{\|\eta\|_{W^{1,3}}\le\rho_0\\ p\in\mathbb{R}^3}}\left\|\mathcal{V}'_H(\omega+p+\eta)\right\|_{W^{1,3}}<\frac{\rho_0}{6\varepsilon_0C_2}, \tag{7.15}$$

$$\sup_{\substack{\|\eta\|_{W^{1,3}}\le 3\rho_0\\ p\in\mathbb{R}^3}}\left\|\mathcal{V}''_H(\omega+p+\eta)\right\|_{W^{1,3/2}}<\frac{1}{8\varepsilon_0C_2}, \tag{7.16}$$

we obtain that $T_{\varepsilon,p}$ maps the ball $\overline{B_{\rho_0}(0)}$ into itself for any $|\varepsilon|<\varepsilon_0$, $p\in\mathbb{R}^3$, and is a contraction there. Hence it has a unique fixed point $\big(\eta(\varepsilon,p),\lambda(\varepsilon,p),\alpha(\varepsilon,p)\big)\in\overline{B_{\rho_0}(0)}$. From (7.11) we have that the following property holds

(∗) $T_{\varepsilon,p}$ maps a ball $\overline{B_\rho(0)} \subset W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3$ into itself whenever $\rho \le \rho_0$ and

$$\rho > 4|\varepsilon|C_2 \sup_{\|\eta\|_{W^{1,3}} \le \rho} \|\mathcal{V}'_H(\omega + p + \eta)\|_{W^{1,3}}.$$

In particular let us set

$$\rho_\varepsilon = 5|\varepsilon|C_2 \sup_{\substack{\|\eta\|_{W^{1,3}} \le \rho_0 \\ p \in \mathbb{R}^3}} \|\mathcal{V}'_H(\omega + p + \eta)\|_{W^{1,3}}. \tag{7.17}$$

In view of (7.15) and (7.17), we have that for any $|\varepsilon| < \varepsilon_0$ and for any $p \in \mathbb{R}^3$

$$\rho_\varepsilon \le \rho_0 \quad \text{and} \quad \rho_\varepsilon > 4|\varepsilon|C_2 \sup_{\|\eta\|_{W^{1,3}} \le \rho_\varepsilon} \|\mathcal{V}'_H(\omega + p + \eta)\|_{W^{1,3}}$$

so that, due to (∗), $T_{\varepsilon,p}$ maps $\overline{B_{\rho_\varepsilon}(0)}$ into itself. From the uniqueness of the fixed point we have that for any $|\varepsilon| < \varepsilon_0$ and $p \in \mathbb{R}^3$

$$\|(\eta(\varepsilon,p), \lambda(\varepsilon,p), \alpha(\varepsilon,p))\|_{W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3} \le \rho_\varepsilon \le C_1|\varepsilon| \tag{7.18}$$

for some positive constant C_1 independent of p and hence $\|\eta(\varepsilon,p)\|_{W^{1,3}} \le \rho_\varepsilon \le C_1|\varepsilon|$ thus proving (7.9). Assume now (7.10) and set for any $p \in \mathbb{R}^3$

$$\rho_p = 8C_2C_3\varepsilon_0 \left(\int_{\mathbb{R}^2} \sup_{|q-p| \le 1 + C_0} |H(q)|^{3/2} |\nabla\omega|^3 \mu^{-1} \right)^{2/3}$$

where C_0 is a positive constant such that $\|u\|_{L^\infty} \le C_0\|u\|_{W^{1,3}}$ for any $u \in W^{1,3}$. From (7.14) we have that

$$\rho_p < \min\left\{1, \rho_0, \frac{1}{8C_2C_3\varepsilon_0}\right\}.$$

Hence, due to (7.12), we have that for $|\varepsilon| < \varepsilon_0$ and $\|\eta\|_{W^{1,3}} \le \rho_p$

$$\begin{aligned} &4|\varepsilon|C_2\|\mathcal{V}'_H(\omega + p + \eta)\|_{W^{1,3}} \\ &\le 4\varepsilon_0C_2C_3 \left(\int_{\mathbb{R}^2} \sup_{|q-p| \le 1 + C_0} |H(q)|^{3/2} |\nabla\omega|^3 \mu^{-1} \right)^{2/3} + 4\varepsilon_0C_2C_3\rho_p^2 < \rho_p. \end{aligned}$$

From (∗) and the uniqueness of the fixed point, we deduce that $\|\eta(\varepsilon,p)\|_{W^{1,3}} \le \rho_p$ for any $|\varepsilon| < \varepsilon_0$ and $p \in \mathbb{R}^3$. On the other hand, since H vanishes at ∞, by the definition of ρ_p we have that $\rho_p \to 0$ as $|p| \to \infty$, hence

$$\lim_{|p| \to \infty} \eta(\varepsilon,p) = 0 \quad \text{in } W^{1,3} \text{ uniformly for } |\varepsilon| < \varepsilon_0.$$

The proof of Lemma 7.1 is now complete.

Remark 7.4. The map η given in Lemma 7.1 satisfies

$$\langle \mathcal{E}'_\varepsilon(\omega + p + \eta(\varepsilon,p)), \varphi\rangle - \sum_{i=1}^{6} \lambda_i(\varepsilon,p) \int_{\mathbb{R}^2} \nabla\varphi \cdot \nabla\tau_i + \alpha(\varepsilon,p) \cdot \int_{\mathbb{S}^2} \varphi, \quad \forall \varphi \in W^{1,3/2}$$

where $\big(\eta(\varepsilon,p), \lambda(\varepsilon,p), \alpha(\varepsilon,p)\big) \in \overline{B_{\rho_\varepsilon}(0)} \subset W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3$ being ρ_ε given in (7.17), hence

$$\begin{aligned}\int_{\mathbb{R}^2} &\nabla(\omega + \eta(\varepsilon,p)) \cdot \nabla\varphi + 2\int_{\mathbb{R}^2} \varphi \cdot (\omega + \eta(\varepsilon,p))_x \wedge (\omega + \eta(\varepsilon,p))_y \\ &+ 2\varepsilon \int_{\mathbb{R}^2} H(\omega + p + \eta(\varepsilon,p))\varphi \cdot (\omega + \eta(\varepsilon,p))_x \wedge (\omega + \eta(\varepsilon,p))_y \\ &= \sum_{i=1}^{6} \lambda_i(\varepsilon,p) \int_{\mathbb{R}^2} \nabla\varphi \cdot \nabla\tau_i - \alpha(\varepsilon,p) \cdot \int_{\mathbb{S}^2} \varphi, \quad \forall\varphi \in W^{1,3/2},\end{aligned}$$

i.e. $\eta(\varepsilon,p)$ satisfies the equation

$$\Delta\eta(\varepsilon,p) = F(\varepsilon,p)$$

where

$$\begin{aligned}F(\varepsilon,p) =& 2(\omega + \eta(\varepsilon,p))_x \wedge (\omega + \eta(\varepsilon,p))_y - 2\omega_x \wedge \omega_y + \lambda(\varepsilon,p) \cdot \Delta\tau - \alpha(\varepsilon,p)\mu^2 \\ &+ 2\varepsilon H(\omega + p + \eta(\varepsilon,p))(\omega + \eta(\varepsilon,p))_x \wedge (\omega + \eta(\varepsilon,p))_y \quad \text{in } \mathbb{R}^2.\end{aligned}$$

Since $F(\varepsilon,p) \in L^{3/2}$ and, in view of (7.9) and (7.18), $F(\varepsilon,p) \to 0$ in $L^{3/2}$ as $\varepsilon \to 0$ uniformly with respect to p, by regularity we have that

$$\eta(\varepsilon,p) \in W^{2,3/2} \quad \text{and} \quad \eta(\varepsilon,p) \to 0 \text{ in } W^{2,3/2}$$

hence, by Sobolev embeddings, $F(\varepsilon,p) \in L^3$ and $F(\varepsilon,p) \to 0$ in L^3 as $\varepsilon \to 0$ uniformly with respect to p. Again by regularity

$$\eta(\varepsilon,p) \in W^{2,3} \quad \text{and} \quad \eta(\varepsilon,p) \to 0 \text{ in } W^{2,3}$$

hence $\eta(\varepsilon,p) \in C^{1,1/3}$ and

$$\eta(\varepsilon,p) \to 0 \quad \text{in } C^{1,1/3} \text{ as } \varepsilon \to 0 \text{ uniformly with respect to } p. \tag{7.19}$$

For any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, let us define the perturbed manifold

$$Z_\varepsilon := \big\{\omega + p + \eta(\varepsilon,p) : \ p \in \mathbb{R}^3\big\}.$$

From [31], we have that Z_ε is a natural constraint for $\mathcal{E}_\varepsilon$, namely any critical point $p \in \mathbb{R}^3$ of the functional

$$\Phi_\varepsilon : \mathbb{R}^3 \to \mathbb{R}, \qquad \Phi_\varepsilon(p) = \mathcal{E}_\varepsilon\big(\omega + p + \eta(\varepsilon,p)\big)$$

gives rise to a critical point $u_\varepsilon = \omega + p + \eta(\varepsilon,p)$ of $\mathcal{E}_\varepsilon$.

Proposition 7.1. *Assume* $H \in C^2(\mathbb{R}^3)$, $\nabla H \in L^\infty(\mathbb{R}^3, \mathbb{R}^3)$, *and* $\lim_{|p|\to\infty} H(p) = 0$. *Then for any* $|\varepsilon| < \varepsilon_0$

$$\lim_{|p|\to\infty} \Phi_\varepsilon(p) = \text{const} = \mathcal{E}_0(\omega).$$

Proof. We have that

$$\begin{aligned}
\Phi_\varepsilon(p) &= \mathcal{E}_\varepsilon\big(\omega + p + \eta(\varepsilon,p)\big) = \mathcal{E}_0\big(\omega + p + \eta(\varepsilon,p)\big) + 2\varepsilon\mathcal{V}_H\big(\omega + p + \eta(\varepsilon,p)\big) \\
&= \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\omega|^2 + \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\eta(\varepsilon,p)|^2 + \int_{\mathbb{R}^2} \nabla\omega\cdot\nabla\eta(\varepsilon,p) \\
&\quad + \frac{2}{3}\int_{\mathbb{R}^2} \big(\omega + p + \eta(\varepsilon,p)\big)\cdot\big(\omega + \eta(\varepsilon,p)\big)_x \wedge \big(\omega + \eta(\varepsilon,p)\big)_y \\
&\quad + 2\varepsilon\big[\mathcal{V}_H(\omega+p) + \langle\mathcal{V}_H'(\omega+p), \eta(\varepsilon,p)\rangle + o\big(\|\eta(\varepsilon,p)\|_{W^{1,3}}\big)\big] \\
&= \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\omega|^2 + \frac{2}{3}\int_{\mathbb{R}^2} \omega\cdot\omega_x\wedge\omega_y + \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\eta(\varepsilon,p)|^2 + \int_{\mathbb{R}^2} \nabla\omega\cdot\nabla\eta(\varepsilon,p) \\
&\quad + \frac{2}{3}\int_{\mathbb{R}^2} \omega\cdot(\omega_x\wedge\eta(\varepsilon,p)_y + \eta(\varepsilon,p)_x\wedge\omega_y) \\
&\quad + \frac{2}{3}\int_{\mathbb{R}^2} \omega\cdot\eta(\varepsilon,p)_x\wedge\eta(\varepsilon,p)_y + \frac{2}{3}\int_{\mathbb{R}^2} \eta(\varepsilon,p)\cdot(\omega+\eta(\varepsilon,p))_x\wedge(\omega+\eta(\varepsilon,p))_y \\
&\quad + 2\varepsilon\mathcal{V}_H(\omega+p) + 2\varepsilon\langle\mathcal{V}_H'(\omega+p),\eta(\varepsilon,p)\rangle + 2\varepsilon\, o\big(\|\eta(\varepsilon,p)\|_{W^{1,3}}\big) \qquad (7.20)
\end{aligned}$$

where we have used the fact that

$$\int_{\mathbb{R}^2} p\cdot u_x\wedge u_y = 0 \quad \forall p\in\mathbb{R}^3,\ u\in W^{1,3},$$

(see [31], Lemma A.3). Notice that from Lemma 7.1 we have that

$$\int_{\mathbb{R}^2} |\nabla\eta(\varepsilon,p)|^2 \le \sqrt[3]{4\pi}\|\eta(\varepsilon,p)\|^2_{W^{1,3}} \underset{|p|\to\infty}{\longrightarrow} 0, \qquad (7.21)$$

$$\left|\int_{\mathbb{R}^2} \nabla\omega\cdot\nabla\eta(\varepsilon,p)\right| \le \sqrt[6]{4\pi}\left(\int_{\mathbb{R}^2}|\nabla\omega|^2\right)^{1/2}\|\eta(\varepsilon,p)\|_{W^{1,3}} \underset{|p|\to\infty}{\longrightarrow} 0, \qquad (7.22)$$

and, by the Hölder inequality and Lemma 7.1,

$$\left|\int_{\mathbb{R}^2} \omega\cdot(\omega_x\wedge\eta(\varepsilon,p)_y + \eta(\varepsilon,p)_x\wedge\omega_y)\right| \le 2\|\omega\|^2_{W^{1,3}}\|\eta(\varepsilon,p)\|_{W^{1,3}} \underset{|p|\to\infty}{\longrightarrow} 0, \qquad (7.23)$$

$$\left|\int_{\mathbb{R}^2} \omega\cdot(\eta(\varepsilon,p)_x\wedge\eta(\varepsilon,p)_y)\right| \le \|\omega\|_{W^{1,3}}\|\eta(\varepsilon,p)\|^2_{W^{1,3}} \underset{|p|\to\infty}{\longrightarrow} 0, \qquad (7.24)$$

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} \eta(\varepsilon,p)\cdot(\omega+\eta(\varepsilon,p))_x\wedge(\omega+\eta(\varepsilon,p))_y\right| \\
&\qquad\qquad \le \|\omega+\eta(\varepsilon,p)\|^2_{W^{1,3}}\|\eta(\varepsilon,p)\|_{W^{1,3}} \underset{|p|\to\infty}{\longrightarrow} 0. \qquad (7.25)
\end{aligned}$$

Moreover the Gauss-Green Theorem yields

$$\mathcal{V}_H(\omega + p) = -\int_{B_1} H(\xi + p)\, d\xi$$

so that by the Dominated Convergence Theorem we have that

$$\lim_{|p|\to\infty} \mathcal{V}_H(\omega + p) = 0. \tag{7.26}$$

From (7.4), Hölder inequality, and Lemma 7.1, we have that

$$\begin{aligned} |\langle \mathcal{V}_H'(\omega + p), \eta(\varepsilon, p)\rangle| &= \left| \int_{\mathbb{R}^2} H(\omega + p)\, \eta(\varepsilon, p) \cdot \omega_x \wedge \omega_y \right| \\ &\leq \|H\|_{L^\infty(\mathbb{R}^3)} \|\omega\|^2_{W^{1,3}} \|\eta(\varepsilon, p)\|_{W^{1,3}} \underset{|p|\to\infty}{\longrightarrow} 0. \end{aligned} \tag{7.27}$$

From (7.20) - (7.27), it follows that

$$\lim_{|p|\to\infty} \Phi_\varepsilon(p) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla \omega|^2 + \frac{2}{3}\int_{\mathbb{R}^2} \omega \cdot \omega_x \wedge \omega_y = \mathcal{E}_0(\omega).$$

The proposition is thereby proved.

Proof of Theorem 7.2. As already observed at the beginning of Section 2, it is not restrictive to take $H_0 = 1$. From Proposition 7.1 it follows that for $|\varepsilon| < \varepsilon_0$ either Φ_ε is constant (and hence we have infinitely many critical points) or it has a global maximum or minimum point. In any case Φ_ε has a critical point. Since Z_ε is a natural constraint for $\mathcal{E}_\varepsilon$, we deduce the existence of a critical point of $\mathcal{E}_\varepsilon$ for $|\varepsilon| < \varepsilon_0$ and hence of a solution to (P_ε). The H_ε-bubble ω_ε found in this way is of the form $\omega + p^\varepsilon + \eta(\varepsilon, p^\varepsilon)$ for some $p^\varepsilon \in \mathbb{R}^3$ where η is as in Lemma 7.1. Remark 7.4 yields that ω_ε is closed in $C^{1,1/3}(\mathbb{S}^2, \mathbb{R}^3)$-norm to the manifold $\{\omega + p :\ p \in \mathbb{R}^3\}$ for ε small. Since ω has no branch points, we deduce that ω_ε has no branch points. □

To prove Theorems 7.3 and 7.4, we need the following expansion for Φ_ε (see [31])

$$\Phi_\varepsilon(p) = \mathcal{E}_0(\omega) - 2\varepsilon\Gamma(p) + O(\varepsilon^2) \quad \text{as } \varepsilon \to 0 \text{ uniformly in } p \in \mathbb{R}^3. \tag{7.28}$$

Proof of Theorem 7.3. Let $\varepsilon > 0$ small. Assumption $(H4)$ implies that $\Gamma(0) > 0$ and hence from (7.28) we have that for ε small $\Phi_\varepsilon(0) < \mathcal{E}_0(\omega)$, whereas from assumption $(H3)$ we have that $\operatorname{Hess}\Gamma(0)$ is positive definite so that Γ has a strict local minimum in 0 and hence from (7.28) Φ_ε has a strict local maximum in $B_r(0)$ for some $r > 0$ such that $\Phi_\varepsilon(p) < \Phi_\varepsilon(0) - c_\varepsilon < \mathcal{E}_0(\omega)$ for $|p| = r$, where c_ε is some positive constant depending on ε. In particular Φ_ε has a mountain pass geometry. Moreover by Theorem 7.2 $\Phi_\varepsilon(p) \to \mathcal{E}_0(\omega)$ as $|p| \to \infty$, and so Φ_ε must have a global minimum point. If the minimum point and the mountain pass point coincide then Φ_ε has infinitely many critical points. Otherwise Φ_ε has at least three critical points: a local maximum point, a global minimum point, and a mountain pass. If $\varepsilon < 0$ we find the inverse inequalities and hence we find that Φ_ε has a local minimum point, a global maximum point, and a mountain pass. As a consequence (P_ε) has at least three solutions provided $|\varepsilon|$ is sufficiently small. □

As observed in Remark 7.2, if $H_1(0) > 0$ and $\text{Hess}\, H_1(0)$ is positive definite, by continuity we have that for H_0 sufficiently large $\Gamma(0) > 0$ and $\text{Hess}\, \Gamma(0)$ is positive definite, so that we can still prove the existence of three solutions arguing as above.

Proof of Theorem 7.4. Assumption $(H5)$ implies that $\Gamma(p_1) > 0$ and $\Gamma(p_2) < 0$. Since $\Phi_\varepsilon(p) = \mathcal{E}_0(\omega) + 2\varepsilon\big(-\Gamma(p) + o(1)\big)$ as $\varepsilon \to 0$, we have for ε sufficiently small

$$\Phi_\varepsilon(p_1) < \mathcal{E}_0(\omega) \quad \text{and} \quad \Phi_\varepsilon(p_2) > \mathcal{E}_0(\omega)$$

if $\varepsilon > 0$ and the inverse inequalities if $\varepsilon < 0$. Since by Theorem 7.2 $\Phi_\varepsilon(p) \to \mathcal{E}_0(\omega)$ as $|p| \to \infty$, we conclude that Φ_ε must have a global maximum point and a global minimum point in $\mathbb{R}^3$. Since Z_ε is a natural constraint for $\mathcal{E}_\varepsilon$, we deduce the existence of two critical points of $\mathcal{E}_\varepsilon$ for $|\varepsilon|$ sufficiently small and hence of two solutions to (P_ε). □

As observed in Remark 7.3, if $H_1(p_1) > 0$ and $H_1(p_2) < 0$, by continuity we have that for H_0 sufficiently large $\Gamma(p_1) > 0$ and $\Gamma(p_2) < 0$, so that we can still prove the existence of two solutions arguing as above.

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