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COVARIANT AND GAUGE-INVARIANT COSMOLOGICAL PERTURBATIONS

Thesis submitted for the degree of

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Supervisor:

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Preface

In this thesis I present a summary of the work I have been doing at *S.I.S.S.A.* After two chapters that review some necessary preliminaries, the rest of the thesis contains the material that has been published (or is going to be published) in a series of papers: Ellis and Bruni [35] (1989, EB from now on); Ellis, Hwang and Bruni [37] (1989, EHB); Ellis Bruni and Hwang [36] (1990, EBH); Bruni and Ellis [9] (1991, BE); Bruni, Ellis and Dunsby [10] (1991, BED); Bruni, Dunsby and Ellis [8] (1991, Paper I); Dunsby, Bruni and Ellis [22] (1991, Paper II). This series of papers represents a new approach to the study of cosmological perturbations. The work presented in them, carried out in collaboration with various colleagues, is original. The synthesis presented in this thesis is new.

We assume that the reader has a basic familiarity with classical general relativity theory and in particular with the homogeneous isotropic models of Friedman - Lemaître - Robertson - Walker and the standard treatment of perturbations in these models (e.g. see the books of Weinberg [127] and Kolb and Turner [70]).

The notation we use (unless otherwise stated) for vector and tensor is the abstract index notation: tensor equations look like components equations, except that the basis is not specified and tensor equations are valid in any basis (see section 2.4 of Wald's book [126]).

The bibliography is at the end, in alphabetic order: although references are called with a number in square brackets, sometime the author's name appear explicitely.

Acknowledgements

I want to acknowledge in the first place my supervisor George Ellis, for guiding me toward the subject treated in this thesis, and for his continuous friendly encouragement. I am obviously indebted to my colleagues, Peter Dunsby and Jai-chan Hwang: working with them has been a pleasure, and the many stimulating discussions we had often brought me to a deeper understanding of many of the topics addressed in this thesis. I am also grateful to Roberto Innocente for helping me with TEX problems, to Paul Haines for english improvements, and to Silvia Mollerach and Valerio Faraoni for enlightening discussions.

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Abstract

The aim of this thesis is to present a possible approach to the relativistic theory of cosmological perturbations and their linear evolution, and also to discuss the relation this approach has with the standard treatment of the problem.

The first chapter exposes the gauge problem existing in the classical theory of cosmological perturbations: this somehow motivate the alternative approach to gauge invariance presented in this thesis, i.e. the *covariant, geometrical* theory of cosmological perturbations.

Chapter 2 reviews the covariant approach to general relativity with matter described as a fluid: Bianchi identities are regarded as field equations, hydrodynamic equations follow from the Ricci identities, and Einstein equations algebrically relate curvature with the matter content at any spacetime point.

Chapter 3 is the core of the thesis, and is devoted to develop the covariant and gauge-invariant theory of cosmological perturbations. I introduce *covariantly defined, gauge-invariant variables*, and present some new exact equations for the most important of them, firstly derived in EB. Then I discuss a linearization procedure for the equations previously considered, and present the linear equations for the gauge-invariant variables as they have been derived in EHB, EBH and Paper I. Finally, I briefly consider the simplest solutions for the density perturbations derived in EBH.

The fourth chapter of this thesis follows from Paper I, and is devoted to the comparison of the covariant approach to gauge-invariant perturbations with the *coordinate based* gauge-invariant theory of Bardeen [1] (1980). It is shown how Bardeen's variables follow from a first -order expansion of the covariant variable. This give a new physical or geometrical significance to Bardeen's variables; at the same time the equations of Bardeen also follow directly from this expansion.

Chapters 5 and 6 contain two applications: the first is to a universe dominated by a scalar field (BED), thus is related to perturbations in inflationary models; the second generalize the theory previously presented to a multi-component fluid, and consider various applications (Paper II).

Contents

Conventions and abbreviations

Signature: $[-,+,+,+]$

Riemann tensor: $V^a{}_{;d;c} - V^a{}_{;c;d} \equiv R^a{}_{bcd}V^b$

where ; denotes covariant differentiation with respect to the metric;

Ricci tensor: $R_{ab} \equiv R^c{}_{acb}$

units: $c = 1$

gravitational constant: $\kappa = 8\pi G$

scale factor: a

energy density: μ

pressure: p

for a tensor $T^{a...b}{}_{c...d...e...f}$ we have

symmetrization: $T^{a...b}{}_{c...(d...e)...f}$

skew - symmetrization: $T^{a...b}{}_{c...[d...e]...f}$

respect to the indices $d...e$, where latin letters denote 4 - dimensional indices, and greek letters will denotes 3 - dimensional indeces for spatial components;

the completely skew symmetric pseudotensor is defined by $\eta^{0123} = -(-g)^{-\frac{1}{2}}$

GI gauge - invariant

FLRW Friedmann - Lemaître - Robertson - Walker

LRF local rest frame

INTRODUCTION

Since the discovery of the Planckian cosmic microwave background [17, 107] the relativistic hot Big Bang cosmology has developed into a mature and believable physical model - actually, the standard model of cosmology - thanks to its many successful predictions and interpretations of observations.[1] The fact that this simple model, based on the homogeneous and isotropic Friedmann - Lemaître - Robertson - Walker spacetimes of general relativity (FLRW hereafter), is so successful appears even more remarkable if we consider that the matter is distributed in structures at each length - scale on which the universe is observed [48]. However we have good reasons - plus our philosophical prejudice, i.e. the Copernican principle - to believe that the overall structure of the observable universe is very well described by the FLRW models. In particular, the extreme degree of isotropy observed in the cosmic microwave background - which has been confirmed and improved over the years [89] - puts severe limits on the inhomogeneity of the matter distribution at early epochs. Therefore all the scenarios advocated to explain the observed large - scale structure[2] assume that galaxies, clusters, superclusters and voids evolved somehow (either through gravitational instability or some astrophysical process such as cosmic explosions) from initially small density perturbations in a FLRW background. These initial fluctuations can be either *primordial* (fed in the initial conditions of the classical universe, i.e. at Planck time) or *spontaneous* (i.e., they arose inevitably at a very early epoch through some physical process, such as the quantum fluctuations advocated in the inflationary scenario); in any case, they evolved *linearly* for a long while. A basic step towards the understanding of structure formation is therefore the formulation of a relativistic theory of linear perturbations of the expanding, isotropic and homogeneous FLRW universe models: this thesis concerns a possible approach to this problem.

[1] For a recent review of the successes of the hot Big Bang model see Peebles *et al.* (1991) [106].

[2] For a classification of the top - five scenarios see Peebles and Silk (1990) [105].

Press and Vishniac (1980) remind us that the classical relativistic theory of cosmological perturbations "springs into existence virtually full-grown with the work of Lifshitz"[108];[3] as is well known, this theory has the disadvantage that truly physical results can be worked out only once a particular correspondence between the real perturbed universe and the background FLRW spacetime is completely specified, that is, a definite *gauge choice* has been made.[4] If part of this correspondence is left arbitrary, we say that we are left with a remaining *gauge freedom*: correspondingly, we are left with some unphysical *gauge modes* in the evolution of the perturbations. This arbitrariness is inherent to the *gauge-invariance* of the linearized Einstein equations with respect to a *gauge transformation* (Sachs (1964)[111]), i.e. a change in the correspondence between the perturbed and the background spacetimes: simply because the equations are linear, for any given physical solution a linear combination of this latter and a gauge mode is also a mathematically acceptable solution. This problem was pointed out by Lifshitz himself (see Lifshitz and Khalatnikov 1963) [76]. An attempt to circumvent it by use of covariant methods is due to Hawking (1966)[55], and his work was extended by Olson (1976)[98]. These authors based their analysis on curvature variables rather than on the metric but, although they used to this end gauge-invariant (often GI in the following) variables, their analysis of density perturbations is based on the gauge-dependent density contrast $\delta\mu/\mu$. A fully GI theory of cosmological linear perturbations has been developed by Bardeen (1980) in his seminal paper [1]; unfortunately, most of Bardeen's GI variables do not have a simple geometrical-physical interpretation, as they are defined with respect to a particular coordinate chart (Stewart 1990 [117]). This may be seen as a consequence of the fact that in the standard approach to perturbations (also followed by Bardeen) any tensorial quantity T is usually split into a background part T_0 and a perturbation δT: $T = T_0 + \delta T$, where the perturbation δT is treated as a propagating field in the background metric. However such a splitting is meaningful only with respect to a given coordinate system (see e.g. Faraoni (1991) [44]), because δT is not a tensor field with respect to coordinate transformations in the background. Bardeen's GI variables are thus constructed as GI linear combinations of gauge-dependent perturbations, knowing the transfor-

[3]The theory of cosmological perturbations in the Newtonian approximation was developed by Bonnor (1956) [5].

[4]Indeed the paper of Press and Vishniac was devoted to the somehow ambiguous synchronous gauge.

mation rule of these latter under a gauge transformation: the physical and geometrical meaning of the resulting quantities is often obscure, unless a particular hypersurface condition (i.e. a time gauge) is specified (Bardeen (1988) [2]).

In this thesis we shall follow a different approach to gauge-invariant cosmological perturbations, which in our view provides a clearer picture of the almost FLRW model we use to describe the real universe. The thesis is organized in six chapters: the first two review known material, although chapter 1 is partially based on EB [35] and the whole synthesis given there is original; in chapters 3 - 6 we present an original synthesis of the work carried out with various colleagues and that has been published (or is going to be published) in a series of papers (EB [35], EHB [37], EBH [36], BE [9], BED [10], Paper I [8] and Paper II [22]).

In chapter 1, we shall give a description of the gauge problem existing in cosmological perturbations, and we shall provide a brief discussion of the possible gauge choices. As we shall see in sections 1.1 and 1.3, gauge transformations can be regarded both from a coordinate and from a geometrical point of view. Here we recall only that the effect of a gauge transformation induced by an infinitesimal vector field ξ on a tensorial quantity T in the perturbed universe equals the Lie derivative of the background value T_0 of T along ξ:

$$T' = T + \mathcal{L}_\xi T_0 \quad \Rightarrow \quad \delta T' = \delta T + \mathcal{L}_\xi T_0 \,, \quad \mathcal{L}_\xi T_0 = 0 \quad \Rightarrow \quad T = \delta T.$$

From this follows the Stewart and Walker [118] (1973) Lemma: perturbations to a geometrical background quantity T_0 will be GI if and only if T_0 is either (1) a constant scalar, or (2) vanishes, or (3) is a linear combination of products of Kroneker deltas with constant coefficients. The gauge-invariant formalism we shall introduce is based on this Lemma.

In chapter 2 we shall review what we can call the covariant fluid approach to general relativity as is presented for example in the papers of Hawking [55] and Ellis [28, 29].[5] In this approach Bianchi identities are regarded as field equations, hydrodynamic equations follow from the Ricci identities, and the Einstein equations algebraically relate

[5]These papers are in turn based on the work of Ehlers [25, 26]

curvature with the matter content at any spacetime point. The presentation given here is an attempt to satisfy a requirement of self-consistency of this thesis, avoiding details irrelevant to the content of the following chapters.

Chapter 3 is the core of the thesis: it provides a derivation of the covariant and gauge-invariant formalism introduced in EB [35], EHB [37] and EBH [36], and extended to treat viscous fluids in Paper I [8].[6]

From the point of view of the Stewart and Walker Lemma quoted above, what seems unsatisfactory regarding GI variables introduced by Bardeen and other authors [7] is that although they are first order GI perturbations by construction, it is often not clear *to which quantity T they correspond.* The quoted Lemma suggests however a different approach (partially considered by Hawking (1966) [55] and extended to density perturbations in EB [35]) to treat perturbations gauge-invariantly. The basic idea is to introduce covariantly defined exact variables T (i.e. these variables are meaningful in any spacetime) such that their values T_0 in a FLRW universe vanish. In this way the quantity itself is a GI perturbation in an *almost* FLRW universe,[8] and its physical significance is apparent through the covariant definition.

We find that the most important of these covariant GI variables is the *comoving fractional density gradient* $\mathcal{D}_a$, together with the *expansion gradient* $\mathcal{Z}_a$: we shall introduce these variables in section 3.1, deriving exact equations for them in section 3.2.

More or less explicitly, Bardeen's approach is based on a 3+1 (ADM) *arbitrary* slicing of spacetime (see Bardeen (1988) [2]), and the resulting description is that of the Eulerian observers sitting in the slicing (York 1979). Instead we carry out a 1+3 splitting based on the natural threading of the problem,[9] i.e. the congruence of world

[6] Following the work presented in EB [35], Hwang and Vishniac [63] (1990) and Dunsby [18] (1991) provided exact equations for density inhomogeneities in an imperfect fluid universe. A similar approach to that of EB, EBH has been followed by Woszczyna and Kulak (1989) with the aim of extending Olson's (1976) work to non flat universes. Lyth and Stewart (1990) also used the covariant approach to derive equations in the comoving gauge.

[7] An alternative derivation of Bardeen's formalism based on a variational principle was provided by Brandemberger *et al.* [7] (1983), and Kodama and Sasaki [69] (1984) extended Bardeen's work to a multi-component fluid. This latter and the recent paper by Mukhanov *et al.* (1991)[95] are both detailed reviews on the construction of the standard gauge-invariant formalism.

[8] We shall define an almost FLRW universe as a spacetime in which the covariant GI variables do not vanish, but terms in the equations quadratic in these variables can be neglected (see section 3.5).

[9] See Jantzen and Carini [66] (1991) for a systematic comparison of threading and slicing.

lines of matter with tangent u^a. Thus the point of view adopted here is that of observers comoving with matter: since spatial vectors and tensors are defined projecting orthogonal to u^a, we may call this a Lagrangian point of view. As we shall see, for a viscous fluid the definition adopted for the matter four velocity u^a is crucial: different possible choices leading to GI variables are discussed in section 3.4, and in section 3.5 we introduce a complete set of GI variables, with special emphasis on curvature variables (which are naturally GI, as opposed to metric perturbations) such as the *3-curvature gradient* C_a. In section 3.6 we derive linear evolution equations for $\mathcal{D}_a$ and $\mathcal{Z}_a$, which lead to the analysis of density perturbations. Scalar perturbations are introduced in section 3.7 through a new local splitting (EBH [36]): we find that the GI variable that covariantly characterizes matter clumping (i.e. scalar density perturbations) is the divergence Δ of $\mathcal{D}_a$. This variable is, within our formalism, the equivalent of the variable ε_m of Bardeen [1];[10] in the same way, we introduce scalar GI variables for curvature (C) and expansion ($\mathcal{Z}$) perturbations. In section 3.8.1 we also derive a Jeans instability criterion. In section 3.8 we derive the large-scale evolution for $\mathcal{D}_a$, in the simple case of a perfect fluid but showing the effect of vorticity on this density perturbation variable (EBH [36]). Finally, in sections 3.9 we derive the whole set of covariant linearized hydrodynamical and gravitational equations (Paper I [8]) corresponding to the exact equations of chapter 2 (thus including viscous terms), giving the evolution and constraints of a full set of GI variables describing the curvature of the perturbed spacetime and the kinematical behaviour of matter, as well as matter inhomogeneities. Using the local splitting of section 3.7 we derive in section 3.10 the equations for our GI scalar variables: in particular we derive a second order equation for Δ (Paper I [8]), equivalent to the main equation of Bardeen [1] for his variable ε_m.

Chapter 4 is devoted to a systematic comparison of the two different approaches to GI perturbations, i.e. what we can call the Bardeen approach and the geometrical approach presented in chapter 3.

In section 4.1 we briefly review the Bardeen formalism, introducing explicitly a perturbed metric and the set of Bardeen's GI variables. While Bardeen used directly a harmonic decomposition for every quantity, we systematically decompose each variable both in the coordinate space, using the *non local* ADM splitting for 3-vectors and 3-

[10] This will be explicitely shown in section 4.2.

tensors [117], and in the Fourier space, using standard harmonics. In our view, although working in Fourier space has the advantage of reducing equations to algebraic relations, the presentations of the same equations in the coordinate space simplify somehow the physical interpretation.

Then, in section 4.2, we systematically expand to first - order the main covariant variables introduced in chapter 3. Since these variables are GI by themselves because they vanish in the background, we may expect to recover Bardeen's variables to first - order. Indeed, we show that *Bardeen's variables are the first - order - components of the covariant variables.* This gives to all of them a physical or geometrical meaning, without the need to specify a gauge; moreover all the equations of Bardeen are immediately recovered through this first - order expansion of the covariant variables (section 4.3).

The final two chapters consider applications and further extensions of the formalism of chapter 3. Although there are many cross - references with the material presented in the other chapters of this thesis, the presentation given in these two chapter is rather self - contained.

The aim of chapter 5 is to apply the covariant GI formalism of chapter 3 to a FLRW universe dominated by a classical minimally coupled scalar field ϕ (this chapter is based on Ref. [10] (BED) and [9] (BE)).

Scalar field dominated universes have attained prominence in the last decade through the *Inflationary Universe* idea [53, 77, 97], and perturbations of such universes are potentially important as seeds of galaxy growth.[11]

As in chapter 3, emphasis is given here on curvature perturbations (cf. [55]), which are naturally GI, rather than metric perturbations (as in [1, 69, 95]) which play no explicit role.[12] The background curvature K is maintained throughout for generality; moreover the general formalism presented here (cf. [2, 95, 60]) could be extended to consider situations different from inflation in which a scalar field dominates (see e.g. [94]).

[11] Various authors have applied Bardeen's formalism to the inflationary universe situation (e.g. [3], cf. also [95] and references therein), actually working either in the comoving gauge [3, 82, 83] or in the uniform Hubble constant gauge [3].

[12] The link between our GI curvature variables and the GI metric potentials of Bardeen is shown in chapter 4.

In sections 5.1 and 5.2.1 we set up the formalism, based on the natural slicing of the problem $\{\phi = const.\}$ and on its geometric characterization through the unit vector u^a orthogonal to these surfaces, and we present a set of exact covariant results valid in any curved spacetime with a minimally coupled scalar field (cf. [86]).

In section 5.2.3 we define the GI dimensionless gradient Ψ_a of the momentum $\psi = \dot{\phi}$ of the field ϕ, and its divergence Ψ: in our approach these variables incorporate the whole matter perturbation, because the spatial gradient of ϕ vanishes through our geometrical choice of the frame u^a. Indeed we show that the density perturbation Δ is simply proportional to the momentum perturbation Ψ (cf. [3, 81]), and that it characterizes matter clumping.

In section 5.2.5 we present various possible pairs of equations coupling the evolution of any of the matter perturbation variables with that of the curvature perturbation C, or with that of a related quantity, $\tilde{C}$. We also discuss if and when C or $\tilde{C}$ are conserved quantities on scales larger than the Hubble horizon (cf. [83, 63] and BE, EBH[9, 36]). The second order evolution equations equivalent to the above mentioned systems of first-order equations are also derived at the end of this section.

In section 5.3 we present solutions in simple cases, comparing them with standard results in the literature [3, 83]. We also examine perturbations in a coasting, scalar field dominated, FLRW universe with general curvature K. Open models seem particularly interesting, since any previously existing perturbation is erased during evolution, while the density parameter Ω stays constant: this could naturally provide the "clean slate" necessary for a successive "minimal" inflation (not driving $\Omega_0 = 1$) in order to satisfy constraints from the observed large-scale isotropy of the cosmic microwave background (cf. [110]).

The purpose of chapter 6 is to extend the formalism of chapter 3 to treat the more general case of a mixture of interacting viscous fluids (DBE [22]).[13] An analysis of such systems is important in order to have a more realistic picture of matter perturbations. In such situations, perturbations in the densities and velocities of individual compo-

[13]Perturbations in multi-component fluid systems have been studied before, indeed a rather complete analysis was presented by Kodama and Sasaki (1984) [69], based on the GI approach of Bardeen [1]; Dunsby [18] (1991) has considered the case of a mixture of non interacting perfect fluids. However, it seems beneficial to reformulate the theory using the covariant GI variables introduced in chapter 3.

nents behave differently due to a difference in their dynamical properties, especially the sound velocities.

In section 6.1, we define covariant and GI variables that characterize the time evolution of density and velocity perturbations in a multi-component fluid medium (including viscous terms). In section 6.2, we derive equations for both the total fluid and its constituent components. We also present the equations for the relative perturbation variables, which are very useful in differentiating between adiabatic and isothermal perturbations.

In section 6.3, we use harmonic analysis to relate the geometrical variables we use to those of Bardeen [1] and Kodama and Sasaki [69], concentrating only on the most important multi-component variables.

In section 6.4, we discuss if and when the curvature variables C and $\tilde{C}$ are conserved on scales larger than the Hubble horizon. In particular, we demonstrate that when the background FLRW model is flat, both these variables are conserved even in the presence of entropy perturbations and imperfect fluid source terms. We use this result to write down a general solution for the total energy density perturbation. We also briefly consider spatially open models.

In section 6.5, we consider the first of three applications. We examine the case where the background is described by a spatially flat FLRW universe model filled with a mixture of non-interacting dust and radiation and obtain solutions for density and velocity perturbations in the small scale limit (cf. Groth and Peebles [52] (1975)). In section 6.6 we study perturbations in a photon-baryon system, taking explicitly into account the interaction between components (Thompson scattering), and examining in detail the coupling between isothermal and adiabatic perturbations (cf. Kodama and Sasaki [69]). In section 6.7, we briefly consider an application to a system of two non-interacting scalar fields and obtain the standard results (see Mollerach (1990) [93], and BED [10]).

Finally, in appendix A we present the tilt angle formalism, following King and Ellis [67] (1973), and in appendix B we present the definitions and properties of covariant harmonic functions, and relate them to the standard harmonics used in the literature [1, 69].

Chapter 1

PERTURBATIONS: THE GAUGE DEPENDENCE

The relativistic hot Big Bang cosmology, based on FLRW models, has developed into a mature and believable physical model, thanks to its successful predictions and interpretations of observations [106]. In particular, the high degree of isotropy observed in the cosmic microwave background puts severe limits on the inhomogeneity of the matter distribution at early epochs, therefore all the most important scenarios advocated to explain the large-scale structure assume that this originated through the action of gravity from small inhomogeneities in an almost-FLRW universe. A basic step towards the understanding of such processes is then the formulation of a relativistic theory of linear perturbations of the expanding, isotropic and homogeneous Friedmann-Lemaître-Robertson-Walker models (FLRW from now on).

Such a theory was developed by Lifshits [75], who was mainly interested in the dynamical stability of FLRW models with respect to perturbations. His theory has become standard and is presented in many text-books on cosmology (e.g. see Börner [6], Peebles [104], and Weinberg [127]). However, as we shall see, such a theory suffers from gauge problems, as Lifshits himself [75, 76] and other authors [112] have pointed out. The usual approach is then to fix a particular gauge, and work within that; however this procedure is not free of problems, and the literature on the subject is full of examples of erroneous results or, at least, ambiguous interpretations (for comments on this point see e.g. [108] and [50]).

Another, different approach to the problem was pioneered by Hawking in 1966 [55]. His approach is fully covariant, but nevertheless gauge-affected [28, 98].

Finally, one can formulate a gauge-invariant theory of cosmological perturbations, avoiding the gauge problems. This has been accomplished by Bardeen in 1980 [1], who introduced a set of gauge-invariant quantities describing perturbations in the matter and in the geometry. However these variables are essentially based on coordinates (see chapter 4), and therefore their physical interpretation depends on the choice of one particular hypersurface condition, i.e. most of Bardeen's variables acquire a clear physical meaning only in some particular gauge [2].

In this chapter, we shall outline the gauge problem existing in perturbing general relativistic universe model (section 1.1.1), giving first the usual coordinate-based description of gauge transformations (section 1.1.2). Then in section 1.2 we give a more geometrical description of the gauge specification, while in section 1.3 the idea of gauge invariance is formalized in the Stewart and Walker Lemma [118]. In the final section we briefly discuss various possible choices of gauge.

1.1 The gauge problem

1.1.1 Perturbations in general relativity

Perturbations in general relativity are usually regarded as a subject that has not reached the level of sophistication of perturbation theory in, e.g. hydrodynamics. The first reason for this is technical, and is obviously due to the fact that one has to deal with ten gravitational potentials, besides matter variables. The second is more fundamental, and can be referred to as *the gauge problem.*

General relativity is a theory which is required to be *covariant* under general changes of coordinates; using a more geometrical language, this can be stated saying that general relativity is a theory about differential manifolds with no preferred coordinate charts [117]. However, in considering a particular class of models we usually use some preferred coordinates: e.g. FLRW spacetimes are usually introduced in textbooks has those models for which the metric can be written as[1]

$$ds^2 = -dt^2 + a^2 d\Sigma^2 \ , \tag{1.1}$$

[1]We refer to standard text-books on cosmology [6, 127, 70] (see also [109, 79]) for a detailed presentation of FLRW models. A characterization of FLRW models in the covariant fluid approach adopted in this thesis is given in section 3.5.1 (see also [28, 29]).

where $d\Sigma^2$ is the line element of the surfaces $t = constant$, i.e. the 3-spaces of constant curvature $K = 0, \pm 1$; in spherical polar coordinates

$$d\Sigma^2 = dr^2 + f^2(r)(d\theta^2 + sin^2\theta d\phi^2) \;, \tag{1.2}$$

and $f(r)$ takes one of the forms

$$f(r) = \begin{cases} sinr & K = +1 \quad (closed) \\ r & K = 0 \quad (flat) \\ sinhr & K = -1 \quad (open) \;. \end{cases} \tag{1.3}$$

To see what the gauge problem is, let us consider an idealised universe model $\bar{S}$.[2] Each quantity in this model will be indicated with an overbar, e.g. the energy density will be denoted by $\bar{\mu}$, the pressure by $\bar{p}$ and the metric by $\bar{g}_{ab}$; then the spacetime $\bar{S}$ will be given by the metric $\bar{g}_{ab}$ and the manifold $\bar{M}$: $\bar{S} \equiv \{\bar{g}_{ab}, \bar{M}\}$. We perturb this model to obtain a "realistic" or "lumpy" universe S, where any quantity will be denoted by the same symbols as in $\bar{S}$ but without overbars (e.g. the energy density is μ, the pressure is p and the metric is g_{ab}): then $S \equiv \{g_{ab}, M\}$. The perturbation in each quantity is then the difference between the value which it has at a given point in the physical space-time S and the value at *the corresponding point* in the background $\bar{S}$. Considering all points, the perturbation field is determined. For example, the metric perturbation is

$$\delta g_{ab} = g_{ab} - \bar{g}_{ab} \;, \tag{1.4}$$

while for the perturbation in the energy momentum tensor we have

$$\delta T_{ab} = T_{ab} - \bar{T}_{ab} \;, \tag{1.5}$$

where the perturbation fields are given respect to the coordinate chart (1.1). Therefore, adopting Bardeen's notation [1], we may write the perturbed metric in the form

$$ds^2 = a^2(\eta)\{-(1+2A)d\eta^2 - 2B_\alpha dx^\alpha d\eta + [(1+2H_L)\gamma_{\alpha\beta} + 2H_{T\alpha\beta}]dx^\alpha dx^\beta\} \;, \tag{1.6}$$

where η is the conformal time, and the spatial coordinates are left arbitrary.[3] Two assumptions are implicit in writing the above relations: one is obvious, while the other

[2]Here this will be a FLRW model, but the following discussion of the gauge problem is valid in general, for perturbations of any spacetime.

[3]In section 4.1 we shall outline the significance of the various metric component perturbations in (1.6); in section 1.4 we shall use these components to characterize the main possible gauge choices.

is rather obscure. The first is that the unperturbed metric is a solution of the Einstein equations with the unperturbed energy momentum tensor as the source term (we shall call this the "zero-order" solution). The second is that the perturbations are "small". Following these two assumptions one substitutes g_{ab} and T_{ab} in the Einstein equations, subtracts the zero-order solution, neglects higher-order terms, and obtains *linear* equations for the metric perturbation (1.4) with the energy momentum perturbation as the source term (1.5). Also, one carries out the same procedure with the energy momentum conservation equations to obtain equations of motion for the matter field perturbations.

However, the two assumptions above are typical of any perturbation theory: the fact that is peculiar to general relativity is that we must perturb spacetime itself, so that the barred (e.g. $\bar{\mu}$) and unbarred field (e.g. μ) are actually defined on different manifolds $\bar{M}$ and M. Because of this, the procedure outlined above makes sense only if the correspondence between points in $\bar{M}$ and M is fixed, i.e. if we have a point identification map, so that points in $\bar{M}$ and M are "the same", and operations such subtraction of vectors or tensors (such (1.4)) are well defined. Otherwise, even if we embed M and $\bar{M}$ in a higher dimensional manifold N, we would be trying to subtract vector or tensor defined at different points, an ill-defined operation.

The choice of a particular map between the background spacetime $\bar{S}$ and the perturbed universe S is usually referred to as a *choice of gauge*. Such a map is in general *completely arbitrary*, although particular ones may be suitable for some purposes; this arbitrariness is the *gauge freedom* of perturbation theory.

1.1.2 Gauge transformations

The gauge freedom outlined above is usually described in terms of coordinates. Within this description, the *gauge transformations*, are represented by infinitesimal coordinate transformations such that

$$\bar{x}^a \rightarrow \bar{x}'^a = \bar{x}^a - \varepsilon^a(x) , \tag{1.7}$$

where $\varepsilon^a(x)$ is an arbitrary infinitesimal vector field. The transformation (1.7) can be regarded in two ways, usually referred to as: 1) *passive* and 2) *active*. The first can be regarded as a mere relabeling of the point x, and we can compute the change induced by this in any vector or tensor with the usual transformation rules (neglecting second order terms in ε^a); in the second case, to compute the effect of (1.7) on any quantity $\bar{\mathcal{T}}$ (scalars, vectors, tensors) we expand it about x. Then we compare the results *at the same coordinate point* x; it follows that the gauge transformation (1.7) induces a change in any tensor $\bar{\mathcal{T}}$ such that,

$$\bar{\mathcal{T}}'(x) = \bar{\mathcal{T}}(x) + \mathcal{L}_\varepsilon \bar{\mathcal{T}}(x) \ , \tag{1.8}$$

where $\mathcal{L}_\varepsilon \bar{\mathcal{T}}$ is the Lie derivative of $\bar{\mathcal{T}}$ along ε [115]. For scalars, vectors and tensors of second rank we have[4]

$$\mathcal{L}_\varepsilon f \equiv f_{;a}\varepsilon^a \ , \tag{1.9}$$

$$\mathcal{L}_\varepsilon V_a \equiv V^b \varepsilon_{b;a} + V_{a;b}\varepsilon^b \ , \tag{1.10}$$

$$\mathcal{L}_\varepsilon T_{ab} \equiv T_{ab;c}\varepsilon^c + T_{ac}\varepsilon^c{}_{;b} + T_{cb}\varepsilon^c{}_{;a} \ , \tag{1.11}$$

and analogous expressions hold for tensors of higher rank (cf. Weinberg [127]). Thus the *gauge problem* is that, since the transformation (1.7) is a diffeomorfism[5] the solutions $\bar{g}_{ab}$ with source $\bar{T}_{ab}$ and $\bar{g}'_{ab}$ with source $\bar{T}'_{ab}$ are physically equivalent, but at the given coordinate point, they have different values. Consequently, the value of the perturbation in each quantity is also changed by the transformation (1.7). It is straightforward to verify that the fictitious perturbation $\mathcal{L}_\varepsilon \bar{g}_{ab}$ is a solution of the linearized Einstein equations with source term $\mathcal{L}_\varepsilon \bar{T}_{ab}$, therefore, *because of the linearity* of the equations, we can always find other solutions of the form $\delta g_{ab} + \mathcal{L}_\varepsilon \bar{g}_{ab}$ for any given solution δg_{ab}. Thus the linearized Einstein equations are said to be *gauge - invariant* with respect to the transformation (1.7), and the *gauge freedom* here is the freedom we have in choosing coordinates.

Quoting Bardeen [1], it clearly follows from (1.8) and (1.9) that

> " even if a quantity is a scalar under coordinate transformations, the value of the *perturbation* in the quantity will *not* be invariant under gauge transformations if the quantity is non - zero and position dependent in the background."

[4] We omit the overbar here, since these *definitions* of Lie derivatives have nothing to do with the background. However, in the present context, it is clear from (1.8) that we are considering Lie derivatives of the background quantities.

[5] Actually, is an infinitesimal diffeomorfism. See Wald [126], appendix C.

We already see from (1.8) and (1.10), (1.11) that the perturbation in a tensorial quantity is gauge-invariant only if the unperturbed quantity vanishes in the background, the same being true for a tensor of any rank: this result will be formally stated through a Lemma (due to Stewart and Walker [118]) in section 1.3. We have outlined here the gauge problem within a coordinate approach. However, as we have stated at the end of the previous section, a gauge choice is a specification of a map of the background spacetime $\bar{S}$ into the physical model S: therefore we pass now to consider this map in greater detail, through a geometrical description.

1.2 Gauge specification

It is very easy to be misled by the "obvious" way of investigating perturbations: following the procedure sketched above the energy density perturbation is

$$\delta\mu \equiv \mu - \bar{\mu}. \tag{1.12}$$

However this approach obscures the real situation. It suggests that there is something very special about the way the original model $\bar{S}$ is related to the lumpy model, whereas in reality this is not so. Suppose we consider the lumpy universe model S, not knowing how the model $\bar{S}$ was used to make the construction; can we uniquely recover $\bar{S}$ from S ? Without further restriction, the answer is No; for without a specific prescription for approximating the lumpy model by the smooth one, the quantities in the background model $\bar{S}$ are not uniquely determined from the lumpy model S (in equation (1.4), the only restriction relating the two models is that δg_{ab} is "small" in some suitable sense; it is far from obvious how one can extract $\bar{g}_{ab}$ from g_{ab} in a unique way). In fact the definition of the background model in S is equivalent to defining a map Φ from $\bar{S}$ to S, mapping the density in $\bar{S}$ into a background density $\bar{\mu}$ in S (for notational convenience, we use the same symbol for quantities in $\bar{S}$ and their images in S, e.g. the image $\Phi(\bar{\mu})$ in S of $\bar{\mu}$ in $\bar{S}$ is simply denoted by $\bar{\mu}$). The perturbations defined are completely dependent on how that map is chosen (Figure 1.1). This is the gauge freedom in defining the perturbation.

As delineated in section 1.1.2, the situation is usually expressed in terms of the coordinate choice in S, it being understood that the coordinates in S correspond to coordinates chosen in $\bar{S}$, so that a choice of coordinates determines a map from $\bar{S}$ into

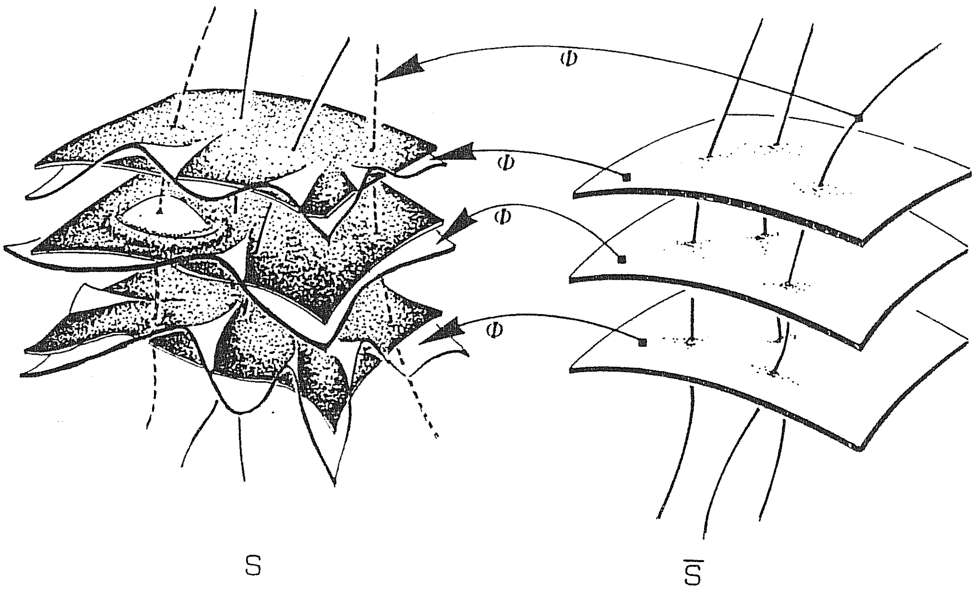

Figure 1.1: The perturbed density $\delta\mu$ is defined by a mapping Φ of an idealised world model $\bar{S}$ into a more accurate world model S; for Φ maps surfaces $\{\bar{\mu} = const\}$ from $\bar{S}$ into S, where they can be compared with the actual surfaces $\{\mu = const\}$.

S; thus the gauge freedom is represented as a freedom of coordinate choice in S (see equation 1.7). However, we want here to specifically consider the map Φ from $\bar{S}$ into S, noting that we have coordinate freedom both in $\bar{S}$ and in S which we can usefully adapt to the chosen map Φ.

Thus the actual situation is that what we are given to study is the real (lumpy) universe S (this is all we can measure), and we define the perturbed quantities and their evolution by the way we specify a mapping Φ of the (fictitious) idealised space-time $\bar{S}$ into S. The determination of the best way to make this correspondence can be called the "Fitting problem" for cosmology [43, 31]; there are various ways to do this, so the answer is not unique. Once we completely specify the map Φ, there is no arbitrariness in $\delta\mu$; insofar as Φ is unspecified, that quantity is arbitrary. It is convenient to think of this map as having four aspects (Figure 1.2):

(A) We define a family of world lines $\bar{\gamma}$ in $\bar{S}$ and a corresponding family of world lines $\bar{\gamma}$ in S. This determines the world lines in each space-time along which we will compare the evolution of density fluctuations. There is an obvious choice in $\bar{S}$, namely

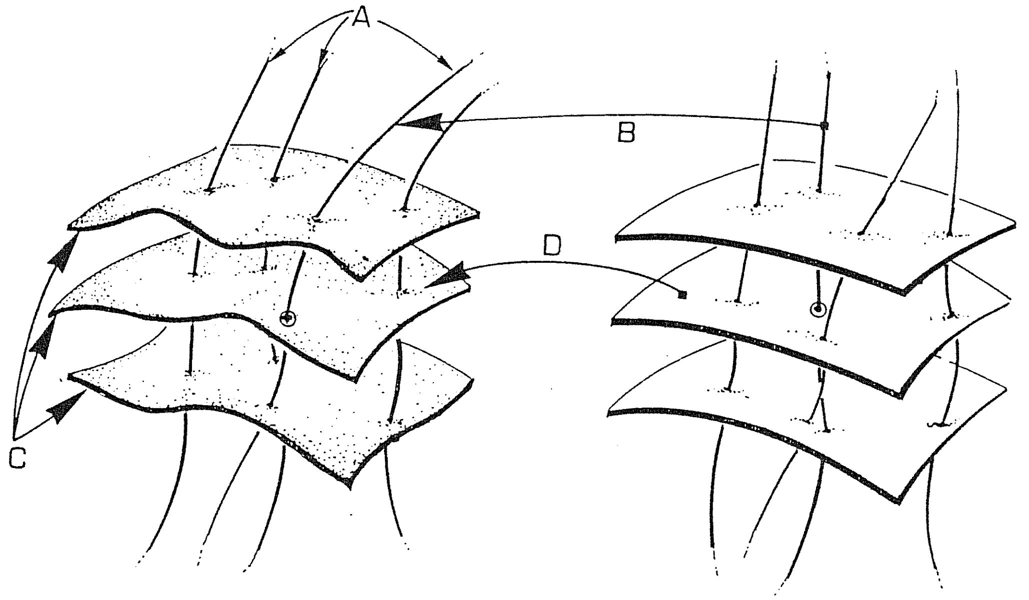

Figure 1.2: The map Φ has four aspects: (A) choice of a family of time lines in each space-time; (B) choice of a particular perturbations in general relativity correspondence of time lines in the family in $\bar{S}$ to particular time lines in the family in S; (C) choice of a family of spacelike surfaces in each space-time; (D) choice of a particular correspondence of surfaces from the family in $\bar{S}$ to surfaces in the family in S.

the fundamental flow lines; this will often be the best choice in S also, but others (e.g. normals to a chosen set of surfaces) may be convenient.

(B) We define a specific correspondence between individual world lines $\bar{\gamma}_i$ in $\bar{S}$ and individual world lines $\bar{\gamma}_i$ in S. This specifies which specific observer's observations we shall compare with which. In the case where $\bar{S}$ is an FLRW universe, this choice does not matter because of the spatial homogeneity of those models.

(C) We define a family of spacelike surfaces $\bar{\Sigma}$ in $\bar{S}$ and a corresponding family $\bar{\Sigma}$ in S; these are the "time surfaces" in each space-time. There is an obvious choice in $\bar{S}$, namely the surfaces of homogeneity $\{\bar{t} = const\}$; this means the image of these surfaces in S (that is, the surfaces $\{\bar{t} = const\}$ in S) *are* the idealised surfaces of constant density $\{\bar{\mu} = const\}$ we use to define the density perturbations. There is a variety of choice for the surfaces $\bar{\Sigma}$ in S, as discussed in depth by Bardeen [1].

(D) We define a correspondence between particular surfaces $\bar{\Sigma}_i$ in the family $\bar{\Sigma}$ in $\bar{S}$ and particular surfaces $\bar{\Sigma}_i$ in the family $\bar{\Sigma}$ in S, and so assign particular time values $\bar{t}$ to each event q in S. This is crucial: this specifies which specific point q in S corresponds to a point $\bar{q}$ in $\bar{S}$, and completes the specification of the map Φ. In particular, the time evolution of a density perturbation $\delta\mu$ is now defined, because this choice, by assigning particular values $\bar{\mu}$ to each surface $\bar{\Sigma}_i$ in S (the "unperturbed value" of the density) defines $\delta\mu$ via equation (1.12).

If we follow the normal convention, we understand (C) to *define* the coordinate surfaces $\{t = const\}$ in S (taking them as the same as the surfaces $\{\bar{t} = const\}$); and (D) to assign particular values to t at each event q in S by this map: $t_q = \bar{t}_q$. However this choice is not forced on us. Note that in general neither t nor $\bar{t}$ will measure proper time along the world lines in S.

1.3 Geometrical description of perturbations of spacetimes

In section 1.1.1 we have outlined the concept of the choice of gauge as a mapping between the background model $\bar{S}$ and the perturbed model S, and in section 1.2 we have seen various aspect of this mapping. The effect of changing the gauge choice, i.e. of gauge transformations, on scalar, vectors and tensors has been described in section 1.1.2 in terms of coordinates; here we shall give a brief coordinate - independent discussion of perturbing spacetimes, using an index - free notation, following Sachs (1964) [111], Stewart and Walker (1973) [118] and Stewart (1990) [117]; for a more formal and complete description see these papers, expecially [118].

Let us consider a one - parameter family of 4-manifolds M_ε embedded in a 5-dimensional manifold N. Each M_ε represent the corresponding spacetime: therefore here we shall take M_0 as being the manifold corresponding to the background spacetime $\bar{S}$ (previously we denoted this manifold $\bar{M}$), and for small value of ε the manifold M_ε corresponds to the perturbed spacetime S. Consistently, we shall denote with Φ_ε the *point identification map*

$$\Phi_\varepsilon : M_0 \rightarrow M_\varepsilon , \tag{1.13}$$

which says which point in the perturbed manifold M_ε is the "same" as a given point in the background M_0.

Consider now any smooth, nowhere - vanishing vector field X on the 5-dimensional manifold N, everywhere transverse (nowhere tangent) to the M_ε. We will then say that a point of M_ε is the same, with respect to X, as a point of M_0, if they lie on the same integral curve γ of X. The map (1.13) is then the identification map associated with X (see Fig. 1.3).

We have already said that a *choice of gauge* is given by the specification of a particular map: thus now is X that represent an *arbitrary* gauge choice. We can also introduce coordinates x^A on N ($A = 0...4$), and parametrize the curves γ by ε: then we can set

$$\frac{dx^A}{d\varepsilon} = X^A \; . \tag{1.14}$$

Now consider a geometrical field in the perturbed spacetime: we can denote it as $\mathcal{T}_\varepsilon$, a field defined on each M_ε. For small ε we can expand $\mathcal{T}_\varepsilon$ along γ,

$$h_\varepsilon^*(\mathcal{T}_\varepsilon) = \mathcal{T}_0 + \varepsilon(\mathcal{L}_X\mathcal{T})_0 + \mathcal{O}(\varepsilon^2) \; , \tag{1.15}$$

where $\mathcal{L}_X$ is the Lie derivative along X and h_ε^* is the pullback of M_ε to M_0.[6] It is clear from (1.15) that the perturbation $\delta\mathcal{T} = \bar{\mathcal{T}} - \mathcal{T}$ of section 1.1.2 is given here by

$$\delta\mathcal{T} = \varepsilon(\mathcal{L}_X\mathcal{T})_0 \; , \tag{1.16}$$

where now the dependence of this perturbation on the choice of gauge, given here by X, is made explicit. Since X is completely arbitrary, we could repeat the above arguments for another vector Y on N, obtaining this time $\delta\mathcal{T} = \varepsilon(\mathcal{L}_Y\mathcal{T})_0$. Given the properties of the Lie derivative, is then immediate to get the difference between the two choices, X and Y, as $\Delta\delta\mathcal{T} = \varepsilon(\mathcal{L}_{X-Y}\mathcal{T})_0 = (\mathcal{L}_\xi\mathcal{T})_0$, where $\xi = \varepsilon(X - Y)$ is a vector field in each M_ε.[7] Finally, we have to take the limit of the difference $\Delta\delta\mathcal{T}$ for $\varepsilon \to 0$, so that on M_0 we get

$$\Delta\delta\mathcal{T} = \mathcal{L}_{\xi_0}\mathcal{T} \; , \tag{1.17}$$

which is equivalent to (1.8). Thus (1.17) is *the effect of change of gauge* from X to Y (see Fig. 1.3), i.e. the geometrical equivalent of the gauge transformations described in coordinates by (1.7)).

[6] See e.g. Wald (1984) [126], appendix C.

[7] This can be seen adopting local coordinates $x^A = (x^a, \varepsilon)$, with $a = 0...3$: then from $X^4 = Y^4 = 1$ we get $\xi^4 = 0$.

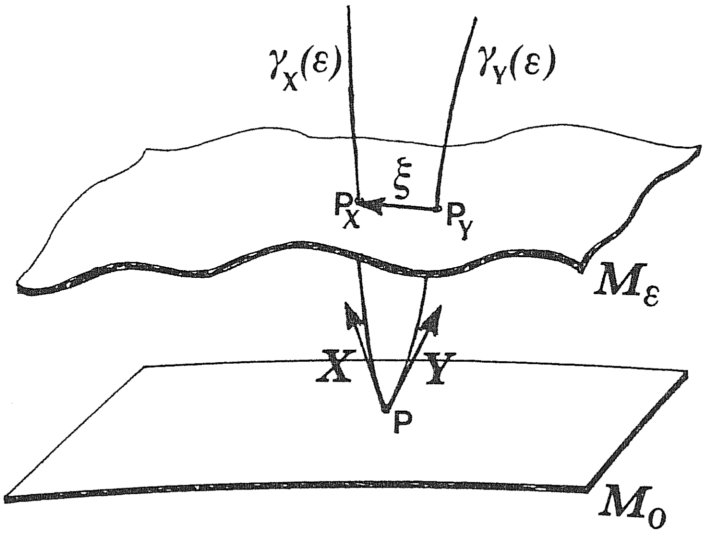

Figure 1.3: The significance of a gauge choice and the effect of gauge transformations can be visualized embedding the 4-dimensional manifolds M_0 (the background) and M_ε (the perturbed universe) in a 5-dimensional manifold N (see text). Here the point P in the background is identified respectively with P_X and P_Y in the perturbed universe by the two different gauge choices X and Y. ξ represent the effect of the change of gauge.

Taking into account (1.17) (or its equivalent form (1.8)), and the explicit expressions of the Lie derivative for scalars, vectors and tensors (1.9), (1.10) and (1.11), we may now state the following Lemma, due to Stewart and Walker (Lemma 2.2)(1973) [118]:

> **Lemma.** The linear perturbation $\delta\mathcal{T}$ of a quantity $\mathcal{T}_0$ on the background spacetime $\bar{S} \equiv \{M_0, g_0\}$ is gauge invariant if and only if one of the following holds:
> *(i)* $\mathcal{T}_0$ vanishes;
> *(ii)* $\mathcal{T}_0$ is a constant scalar;
> *(iii)* $\mathcal{T}_0$ is a constant linear combination of products of Kronecker deltas.

In this thesis, we shall follow the covariant approach outlined here, and we shall use covariantly defined quantities that vanish in the FLRW background, so that they are gauge invariant in accordance with the Lemma above.

1.4 Gauge choices

1.4.1 The arbitrariness of $\delta\mu$

Using again the description given in section 1.2, the problem is that the definition of $\delta\mu$ depends both on the choice of the surfaces $\bar{\Sigma}$ in S and on the allocation of density values to these surfaces. We can for example choose $t = \bar{t}$ and then set the dependence of $\delta\mu$ on the spatial coordinates to zero through the gauge freedom (C), by choosing the surfaces $\bar{\Sigma}$ as surfaces of constant density μ in S; because these surfaces are regarded as surfaces of constant reference density, we will then have $\delta\mu$ constant on these surfaces (they will be spacelike if the universe S is sufficiently like a FLRW universe), and as they are also surfaces of constant $\bar{t}$, we will find $\delta\mu = \delta\mu(t)$. In many ways this is an obvious choice for the time surfaces (the constant density surfaces are covariantly defined in S, and correspond precisely to the surfaces of homogeneity in the idealised model $\bar{S}$, which are also surfaces of constant density).

Furthermore, given a choice of the family of surfaces $\bar{\Sigma}$ in S, we can still assign any value we like to $\delta\mu$ at a particular event through the gauge freedom (D), by changing the assignation of values $\bar{\mu}$ to the surfaces $\bar{\Sigma}$. Thus in particular, given any choice whatever of the time surfaces, we can set $\delta\mu$ to zero at an event q at $t = t_0$ on any world line γ, by choosing $\bar{\mu}_q = \mu_q$; this is a possible assignation of a values of the "ideal" density $\bar{\mu}$ to the event q where $t = t_0$ intersects γ (Figure 1.4).

How this propagates along the chosen time lines then depends on the gauge choice and the fluid equation of state. We can choose a gauge where $\delta\mu$ vanishes at every point of γ by assigning the mapping of densities to satisfy the condition $\mu(t) = \bar{\mu}(t)$ on γ. This choice is obtained in Bardeen's formalism [1] by choosing the arbitrary function $T(\tau)$ (his notation, see his equation (3.1)) to be given (in terms of his variables) by

$$T = -\frac{\delta}{3(1+w)(\dot{S}/S)}$$

on γ, where the right hand side will only depend on the conformal time τ along any chosen world-line γ. Then his equation (3.7) shows $\tilde{\delta} = 0$, i.e. the energy density perturbation vanishes along γ in the new gauge.

If we combine these two choices, we will have chosen a gauge where $\delta\mu = 0$ identically; we map the FLRW model into the lumpy universes by mapping surfaces of

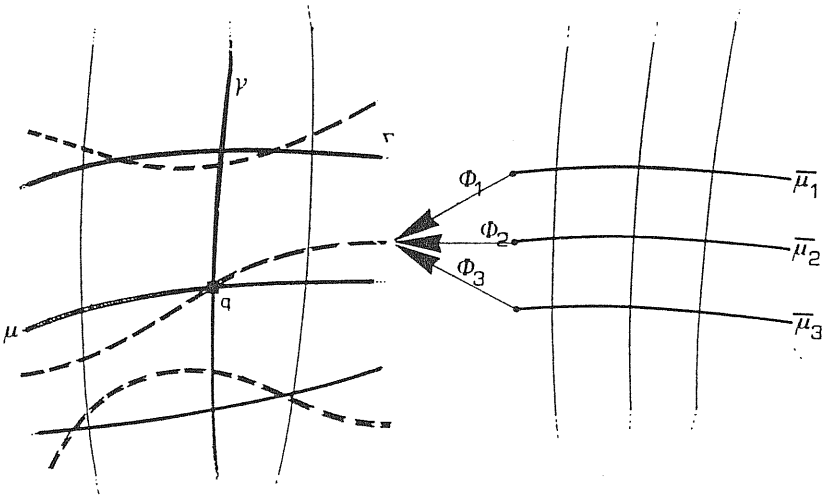

Figure 1.4: By varying the assignation (D) of particular surfaces in $\bar{S}$ to surfaces in S, we can give the density perturbation $\delta\mu = \mu - \bar{\mu}$ at the event q in S (where the world line γ intersects the surface $\{\bar{\mu} = const\}$) any value we like.

constant density $\bar{\mu}$ into surfaces of constant density μ with the same numerical values (Figure 1.5).

We might call this the *zero density-perturbation gauge.* This possibility will not of course mean that there are no spatial variations of density; in this gauge, inhomogeneities will be represented by the fact that the proper time separating a surface of coordinate time $\bar{t}_1$ from a surface of coordinate time $\bar{t}_2$, measured along the normals to these surfaces, varies spatially (corresponding to the normals to these surfaces being non-geodesic).

The basic problem, then, is this arbitrariness in definition of $\delta\mu$, because $\delta\mu$ (a) is not gauge invariant: it can be assigned any value we like at any event by appropriate gauge choice; and (b) is not observable even in principle, unless the gauge is fully specified by an *observationally based* procedure (as otherwise $\bar{\mu}$ is not an observable quantity).

As a result, if we are to use $\delta\mu$ in a satisfactory way to describe density perturbations, we must either leave some gauge freedom, and keep full track of the consequences

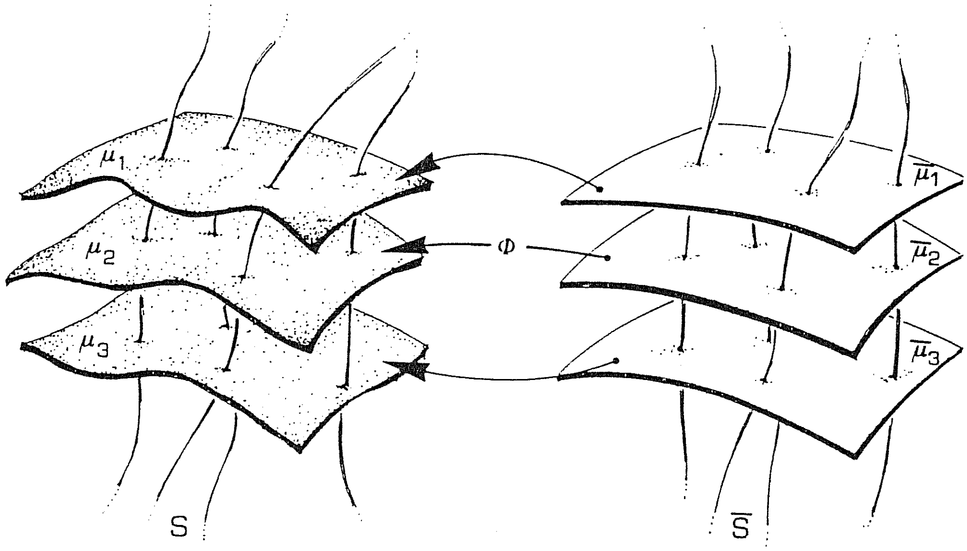

Figure 1.5: By choosing Φ so that the surfaces $\{\bar{\mu} = const\}$ in S are the same as the surfaces $\{\mu = const\}$, and then choosing the correspondence (D) to assign the same numerical values to $\bar{\mu}$ on each surface as μ has on it, we obtain a *zero density-perturbation* gauge. Note that the proper time τ between any two of these surfaces in S will vary spatially, in general; the physical density variation is coded in this spatial variation of $dt/d\tau$.

of *all* this freedom; or find a satisfactory, unique way of making the gauge choices (A)-(D) discussed above. The alternative is to look for gauge-invariant quantities that code the information we want.

1.4.2 Fixing a gauge

One way of approaching the problem is to choose a satisfactory specific gauge (specifying completely (A)-(D) above). We mention five possibilities.[8]

In each case we choose the corresponding world lines in S and $\bar{S}$ to be the fundamental flow lines. The issue then is the choice of time surfaces, i.e. the choice of slicing, and then a specific correspondence between these surfaces. It is standard to charac-

[8]We omit one of Bardeen's option [1], i.e. we do not consider surfaces of simultaneity determined by radar, because such surfaces in $\bar{S}$ do not coincide with the surfaces $\{\bar{t} = const\}$ there [41].

terize these slicing with conditions on the metric components in (1.6) and on other perturbation variables such the velocity perturbation v,[9] and to specify a particular gauge within the slicing with further conditions; here we give a brief outline of these gauge specifications, more details can be found in Kodama and Sasaki (1984) [69].

1) Proper time slicing: $A = 0$

One possibility is to define clearly equivalent proper times in the two models. For example a possible choice (cf. Olson [98]) is to choose proper time along the fluid flow lines from the big bang in both models. This is conceptually a clean solution to the problem, provided we can start at the big bang and follow the evolution of each model from then on.

Within this slicing, two further gauge specification are commonly used in the literature; both leave the gauge non completely specified.
1a) Synchronous gauge: $A = B = 0$; this is perhaps the most commonly used gauge [76, 104, 127]; here proper time is measured along the normals of the $t = constant$ hypersurfaces. The problem here, as pointed out by Bardeen [1], is that the definition is non-local. If we observe the universe *today*, this proposal means we cannot define $\delta\mu$ directly from these observations but have to do so by integrating the field equations all the way back to the big bang and then deducing from this integration what $\delta\mu$ is today. Apart from issues of practicality, this is clearly an unsatisfactory procedure;
1b) Comoving proper time gauge: $A = v = 0$; here proper time is measured along flow lines, which are tilted with respect to the surfaces of $t = constant$.

2) Flow - orthogonal slicing: $v = B$

A second possibility is to choose the surfaces of constant time as surfaces orthogonal to the fluid flow. However this choice (called *comoving hypersurfaces* by Bardeen) is only possible if the fluid vorticity is zero, so it is not a generic strategy (it is satisfactory in analyzing purely scalar perturbations, see chapter 4). Typical further gauge specifications are:

[9] Here we are necessarily anticipating some of the material presented in chapter 4; in particular, we denote the gauge specifications using the harmonic component of the metric and of the velocity variables; a prime indicates derivative with respect to conformal time and k is a wavenumber (for details, see in particular section 4.1).

2a) Comoving time orthogonal gauge: $v = B = 0$ here coordinate time is measured along the flow lines; there is some gauge freedom left [69].
2b) Velocity orthogonal isotropic gauge: $v = B$, $H_T = 0$ there is no gauge freedom left; the formalism in this gauge is the closest to the gauge invariant formalism of Bardeen.

3) Equivalent scalars

A third possibility is to identify equivalent scalars in S and $\bar{S}$, that define spacelike surfaces in $\bar{S}$. The obvious choices are the energy density μ (leading to the "zero density-perturbations" discussed above, with $\bar{\mu}_q = \mu_q$)) or the fluid expansion Θ (giving Bardeen' s *uniform-Hubble-constant hypersurfaces*, with $\bar{\Theta}_q = \Theta_q$). In this
3 Uniform Hubble slicing: $\delta\Theta = 0$ (see Eq. (4.24), (4.25)) there is no residual gauge freedom in the time coordinate; the gauge is further specified by the two following conditions.
3a) Time - orthogonal uniform Hubble gauge: $B = \delta\Theta = 0$ with a remaining gauge freedom, and
3b) Comoving uniform Hubble gauge: $v = \delta\Theta = 0$, again with some gauge freedom left.

4) Newtonian or zero - shear slicing: $\sigma_g = (1/k)H_T' - B = 0$

In this slicing [10] the perturbation in the expansion is isotropic, and there is no residual gauge freedom in the time coordinate. Two common further gauge specification are:
4a) Longitudinal gauge: $B = H_T' = 0$, a very common gauge in the literature [95]; there is a residual gauge freedom.
4b) Comoving Newtonian gauge: $B = (1/k)H_T'$, $v = 0$ again, there is a residual gauge freedom.

5) Spatial averaging

Another approach is to define the ideal density $\bar{\mu}$ in the lumpy model S as a suitable *average density* in S: $\bar{\mu} = <\mu>$, where $< . >$ denotes some suitable spatial average (cf.

[10] This is in a sense quite a controversial slicing, because zero - surfaces are not invariantly defined, and cannot exist in most space - times; furthermore in general such surfaces in the background $\bar{S}$ do not correspond to the surfaces $\{\bar{t} = const\}$ [90].

Lyth and Mukherjee [83]). This is equivalent to specifying a fitting procedure of the fictitious model to the real universe based on this averaging. This is indeed a reasonable thing to do [43, 31], and may be expected to lead to integral conditions such as the Traschen integral constraints [121, 122, 120], as discussed by Ellis and Jaklitsch [38].

This procedure may well give us the physical information we want. However one will then have to take seriously the problems associated with averaging in general relativity, for example the degree to which averaging commutes with the Einstein field equations [30, 57]. It also demands investigation of how this average depends on the choice of space-sections over which the average is taken.

The results obtained for the evolution of $\delta\mu/\mu$ from the various gauge choices are different (see Bardeen's paper [1] for an extensive discussion; and see also Goode [51]). In each of the last three cases considered, we have to concern ourselves with the relation between coordinate time and proper time along the fluid flow lines. In the first three cases, clearly the definitions are such that they have the correct correspondence limit: if S *is* a FLRW model, they define as surfaces $\{\bar{\mu} = const\}$ the surfaces $\{\mu = const\}$ in those universes. However the fourth approach is the most fundamental: it tackles the major issue, on what scale is the real universe approximated by the FLRW model [30]. From the viewpoint adopted here, the averaging implied is a sophisticated way of comparing evolution along neighbouring world lines in the real fluid.

1.4.3 Gauge invariant variables

We have seen in section 1.3 that the fundamental requirement for a gauge invariant quantity is that it be invariant under the choice of the mapping Φ. The simplest case is a scalar $\bar{f}$ that is constant in the unperturbed space-time $\bar{S}$ ($\bar{f} = const$) (see equation 1.9), or any tensor $\bar{f}^{ab}{}_{cd}$ that vanishes in $\bar{S}$: $\bar{f}^{ab}{}_{cd} = 0$ (see equations (1.8), (1.10). The reason is that in each case the mapped quantity $\bar{f}$ in S will also be constant, so *the choice of correspondence Φ does not matter; they will all define the same perturbation* $\delta f = f - \bar{f}$. The only other possibility for gauge invariant quantities is a tensor that is a constant linear combination of products of Kronecker deltas.

What are the simple covariantly defined gauge invariant quantities in a FLRW universe? We can easily determine them by writing down a list of all the simple covariantly defined quantities in a general fluid flow, and then seeing which ones vanish in a FLRW universe model (the other two options in the Stewart and Walker lemma are not useful in our context, as the only invariantly defined constant in the FLRW universes is the cosmological constant, and no tensors that are constant products of Kronecker deltas occur naturally).

To carry this out, it is convenient to use the general formalism developed by Schücking, Ehlers, and Trümper. We turn to this in the next chapter.

Chapter 2

THE COVARIANT APPROACH TO COSMOLOGY

The aim of this chapter is to briefly review the covariant fluid approach to cosmology. It is assumed that the description of the matter content of the universe as a continuous fluid is a good approximation; this fluid can be thought of as being divided into small volume elements. At each point of the spacetime we can assign a 4-velocity vector u^a representing the velocity of the volume element of fluid surrounding that point. This description of matter is complementary to the particle distribution function representation (see e.g. Ehlers (1971)[27]), and we can regard the 4-velocity of the fluid element as the average velocity of the particles in that volume. However for the viscous fluid we shall consider the definition of this 4-velocity is somehow not unique: here we shall simply assume that some reasonable choice has been made.[1]

To characterize this fluid we introduce the covariant approach to general relativity as is presented for example in the papers of Hawking [55] and Ellis [28, 29] (see also the paper by Ehlers [26]). The presentation given here is an attempt to satisfy a requirement of self-consistency of this thesis, avoiding details irrelevant to the content of the following chapters. The unsatisfied reader can refer to the papers quoted above, and references therein.

[1] In section 3.4 we will discuss various possible choices and their relations to GI variables.

2.1 Kinematics

2.1.1 Observers

Let n^a and u^a be two unit future directed timelike smooth vectors fields ($n^a n_a = u^a u_a = -1$): these can be regarded as the four velocities of two sets of observers $\mathcal{O}_n$ and $\mathcal{O}_u$. At each point p of the spacetime we have a subspace H_p of the tangent space T_p at p which is orthogonal to u^a (or to n^a); then we can define the projection tensors into these subspaces

$$h_{ab} \equiv g_{ab} + u_a u_b \ , \quad \tilde{h}_{ab} \equiv g_{ab} + n_a n_b \ , \tag{2.1}$$

which define the spatial part of the local rest frames (LRF) of these observers. These tensors are the metric in the subspaces H_p of T_p which are orthogonal to the corresponding vector: if this is hypersurface orthogonal, the relative projector is the metric in the surface. We follow King and Ellis (1973)[67] on characterizing the relation between n^a and u^a by the hyperbolic angle of tilt β

$$u^a n_a = -cosh\beta \ , \quad \beta \geq 0 \ , \tag{2.2}$$

and the direction of tilt: this can be specified either by the direction $\tilde{c}^a$ of the motion of $\mathcal{O}_u$ (the projection of u^a) in the LRF of $\mathcal{O}_n$, or by the the direction $-c^a$ of the motion of $\mathcal{O}_n$ (the projection of n^a) in the LRF of $\mathcal{O}_u$.[2] More details and useful relations are given in the appendix: what is important to the following discussion is that the tilt angle β is related to the relativistic contraction factor γ by

$$\gamma \equiv cosh\beta = (1 - v^2)^{-\frac{1}{2}} \ , \quad v = tanh\beta \ , \tag{2.3}$$

so that $\beta \simeq v \ll 1$ correspond to a non relativistic relative velocity v between $\mathcal{O}_u$ and $\mathcal{O}_n$. In this case

$$d^a = u^a - n^a \simeq \beta\tilde{c}^a \simeq \beta c^a \simeq \tilde{V}^a \simeq -V^a \ , \quad \tilde{h}_{ab} \simeq h_{ab} + 2u_{(a}V_{b)} \ . \tag{2.4}$$

From now on, we shall refer to the change between two arbitrary frames u^a and n^a with a small relative velocity as a change of first order in β; the tilt angle will turn out to be particularly useful in chapter 5, in considering single components four - velocities and their relations with the total fluid velocity.

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[2]For the signs of these directions we conform here to the choice of King and Ellis (1973) [67].

2.1.2 Kinematical quantities

In the context of cosmology, there will always be a preferred family of world-lines (the *fundamental world lines*) representing the motion of observers in the universe ("fundamental observers") which are at rest with respect to our volume element of fluid. We will often refer to the flow lines as "fluid flow lines", since we will use the fluid approximation. We may note however that for an imperfect (viscous) fluid the definition of the fluid 4-velocity is somewhat arbitrary [64]. In this chapter we shall assume that *some* reasonable definition of the fluid 4-velocity has been made, delaying a discussion of this issue to section 3.4. We shall now define kinematical quantities for the fluid 4-velocity, but clearly the same definitions could be made for any arbitrary timelike unit vector n^a: when needed, we shall denote such quantities with a tilde.

Let the normalized 4-velocity vector tangent to these world lines be

$$u^a = \frac{dx^a}{d\tau} \quad \Rightarrow \quad u^a u_a = -1 \ , \tag{2.5}$$

where τ is proper time along the fluid flow lines: at any point of the spacetime u^a is the 4-velocity of the volume-element of fluid surrounding that point. The projection tensor into H_p (the LRF of a comoving observer $\mathcal{O}_u$) is

$$h_{ab} \equiv g_{ab} + u_a u_b \quad \Rightarrow \quad h^a{}_b h^b{}_c = h^a{}_c \ , \quad h_a{}^b u_b = 0 \ . \tag{2.6}$$

It must be noted that the 3-subspaces H_p defined at each point by h_{ab} do not in general mesh together to form 3-surfaces in the spacetime (see section 2.2.5).

The time derivative of any tensor $T^{a...b}{}_{c...d}$ along the fluid flow lines is simply the covariant derivative along u^a

$$\dot{T}^{a...b}{}_{c...d} \equiv T^{a...b}{}_{c...d;e} u^e \ . \tag{2.7}$$

It is important to note that, because of (2.5), this is the derivative with respect to *proper time* defined along these lines: in other words $\dot{T}^{a...b}{}_{c...d}$ is the rate of change of $T^{a...b}{}_{c...d}$ as measured by a fundamental observer. Using $h_a{}^b$ we can also define the spatial derivative in the LRF of $\mathcal{O}_u$

$$\dot{T}^{a..b}{}_{c..d} \equiv T^{a..b}{}_{c..d;e} u^e \tag{2.8}$$

$${}^{(3)}\nabla_a T^{b..c}{}_{d..e} \equiv h_a{}^f h^b{}_g .. h^c{}_i h_d{}^l .. h_e{}^m T^{g..i}{}_{l..m;f} \ . \tag{2.9}$$

Thus the ${}^{(3)}\nabla_a$ operator is a useful tool to avoid a plethora of indices; a certain care is needed if the tensor T itself is not totally orthogonal to u^a, but this will not be a problem here.

The 4 - acceleration is defined as

$$a^a \equiv \dot{u}^a = u^a{}_{;b}u^b \ , \tag{2.10}$$

and from the second of (2.5) it follows that $a^a u_a = 0$.

A relevant quantity in the fluid - flow picture is the *connecting vector* η^a, joining any two given flow lines at all time. It can be show that η^a is *Lie dragged* [115] along u^a, i.e., its Lie derivative along the fluid flow lines vanishes. This implies

$$\dot{\eta}^a = \eta^a{}_{;b}u^b = u^a{}_{;b}\eta^b \ , \tag{2.11}$$

so that a significant quantity in our approach is the covariant derivative of the 4 - velocity. Hence it is convenient to split $u_{a;b}$, for which we will need to define new variables. This we take up next.

The *expansion scalar (volume expansion)* Θ is the trace of $u_{a;b}$

$$\Theta \equiv u^a{}_{;a} \ , \tag{2.12}$$

which represents the isotropic part of the expansion of the fluid. For instance, the action of Θ alone during a small time interval on a sphere of fluid changes the latter in a larger (smaller) sphere with the same orientation.

The *shear tensor* is the spatial trace - free symmetric part of $u_{a;b}$

$$\sigma_{ab} \equiv h_a{}^c h_b{}^d u_{(c;d)} - \tfrac{1}{3}\Theta h_{ab} \quad \Rightarrow \quad \sigma_{ab}u^b = 0 \ , \quad \sigma^a{}_a = 0 \ , \tag{2.13}$$

Its action distorts the sphere leaving unchanged its volume and the directions of the shear principal axis.The shear magnitude is

$$\sigma^2 \equiv \tfrac{1}{2}\sigma_{ab}\sigma^{ab} \geq 0 \ , \quad \sigma = 0 \quad \Leftrightarrow \quad \sigma_{ab} = 0 \ . \tag{2.14}$$

The *vorticity tensor* ω_{ab} is the skew - symmetric spatial part of $u_{a;b}$

$$\omega_{ab} \equiv h_a{}^c h_b{}^d u_{[c;d]} \quad \Rightarrow \quad \omega_{ab}u^b = 0 \ , \tag{2.15}$$

with magnitude

$$\omega^2 \equiv \tfrac{1}{2}\omega_{ab}\omega^{ab} \geq 0 \ . \tag{2.16}$$

Since ω_{ab} is skew-symmetric, all the information contained in it can be put in a vector, the *vorticity vector*

$$\omega^a \equiv \tfrac{1}{2}\eta^{abcd}u_b\omega_{cd} \quad \Leftrightarrow \quad \omega_{ab} = \eta_{abcd}\omega^c u^d \ , \tag{2.17}$$

$$\omega_a u^a = 0 \ , \quad \omega = 0 \ \Leftrightarrow \ \omega^a = 0 \ \Leftrightarrow \ \omega_{ab} = 0 \ ,$$

where η^{abcd} is the totally skew-symmetric tensor:

$$\eta^{abcd} = \eta^{[abcd]} \ , \quad \eta^{1234} = (-g)^{-\frac{1}{2}} \ , \quad g \equiv det(g_{ab}) \ . \tag{2.18}$$

The action of ω^a alone rotates the sphere, leaving its shape and volume unchanged.

With the definitions given above, the first covariant derivative of the 4-velocity vector is completely determined

$$u_{a;b} = {}^{(3)}\nabla_b u_a - a_a u_b \ , \quad {}^{(3)}\nabla_b u_a = \omega_{ab} + \Theta_{ab} \ , \quad \Theta_{ab} = \tfrac{1}{3}h_{ab}\Theta + \sigma_{ab} \ ; \tag{2.19}$$

if the vorticity $\omega_{ab} = \omega_{[ab]}$ vanishes, u^a is hypersurface orthogonal (see section 2.2.5) and $-\Theta_{ab} = -h_a{}^c h_b{}^d u_{(c;d)}$ is the second fundamental form of this surface (i.e., its extrinsic curvature).

It is convenient to define a representative *length scale* $S(\tau)$ by the relation

$$\dot{S}/S = H = \tfrac{1}{3}\Theta \quad \Leftrightarrow \quad \Theta = \frac{1}{S^3}\frac{d(S^3)}{d\tau} \ , \tag{2.20}$$

which determines S up to a constant factor along each world line, and where H the familiar Hubble parameter $H(\tau)$ when we consider a FLRW model. Hence, the volume of any fluid element varies as S^3 along the flow lines (this quantity is the generalization to arbitrary anisotropic flows of the Robertson-Walker scale parameter), so that S represents the *average* distance behaviour of the fluid. This can be understood if we refer to the definition of Θ and σ_{ab}: in general a sphere of fluid will expand in an anisotropic way during each small time interval, but if we *average* the expansion along different directions over this time interval the shear effect will cancel out and the resulting effect is described by Θ (S) alone.

2.2 Geometry and matter

2.2.1 Ricci identities

The Riemann tensor R_{abcd} (*Riemann* from now on) describes the curvature of space-time. It is defined by the commutation relation satisfied by the covariant derivatives of any arbitrary 4 - vector, i.e. by the Ricci identities; for the 4 - velocity vector u^a these are

$$u_{a;d;c} - u_{a;c;d} = R_{abcd}u^b \ . \tag{2.21}$$

Riemann satisfies the symmetry properties

$$R_{[ab][cd]} = R_{abcd} = R_{cdab}, \quad R_{a[bcd]} = 0 \ , \tag{2.22}$$

giving 20 independent components. *Riemann* can be decomposed into its "trace", i.e., the Ricci tensor (*Ricci*, 10 independent components)

$$R_{ab} \equiv R^c{}_{acb} \ , \tag{2.23}$$

and its "trace - free" part, the Weyl tensor C_{abcd} (*Weyl*, the remaining 10 components)

$$C^{ab}{}_{cd} \equiv R^{ab}{}_{cd} - 2g^{[a}{}_{[c}R^{b]}{}_{d]} + \tfrac{1}{3}Rg^{[a}{}_{[c}g^{b]}{}_{d]} \quad \Rightarrow \quad C^{ab}{}_{ad} = 0 \ , \tag{2.24}$$

where $R = R^a{}_a$ is the Ricci scalar. We can further spit *Weyl* into its "electric" and "magnetic" parts,[3] respectively defined by

$$E_{ac} \equiv C_{abcd}u^bu^d \ , \quad H_{ac} \equiv \tfrac{1}{2}\eta_{ab}{}^{gh}C_{ghcd}u^bu^d \ ; \tag{2.25}$$

$$E_{ab} = E_{(ab)} \ , \quad H_{ab} = H_{(ab)} \ , \quad E^a{}_a = H^a{}_a = 0 \ , \quad E_{ab}u^b = H_{ab}u^b = 0 \ .$$

Then *Weyl* can be written as

$$C_{abcd} = (\eta_{abpq}\eta_{cdrs} + g_{abpq}g_{cdrs})u^pu^rE^{qs} - (\eta_{abpq}g_{cdrs} + g_{abpq}\eta_{cdrs})u^pu^rH^{qs} \ , \tag{2.26}$$

$$g_{abcd} \equiv g_{ac}g_{bd} - g_{ad}g_{bc} \ .$$

The physical interpretation of the gravitational field E_{ab} is clarified by its Newtonian counterpart [4]

$$E_{\alpha\beta} = \phi_{,\alpha\beta} - \tfrac{1}{3}h_{\alpha\beta}\phi^{,\gamma}{}_{,\gamma} \ , \tag{2.27}$$

[3] The reason for this terminology is that E_{ab} and H_{ab} satisfy a "Maxwellian form" of the Bianchi's identity (see section 2.3.3).

[4] The Newtonian analogue of the general - relativistic fluid approximation is developed in detail in Ellis (1971) [28]. The extension of the covariant fluid analysis of cosmological density inhomogeneities to its Newtonian analogue is given in Ellis (1989) [33].

where ϕ is the Newtonian potential. thus E_{ab} represents the tidal force, inducing shear in the fluid flow lines (see equation (2.48)); H_{ab} has no Newtonian counterpart.

2.2.2 Bianchi identities

Riemann satisfies the Bianchi identities

$$R_{ab[cd;e]} = 0 \ ; \tag{2.28}$$

in 4 dimensions these are equivalent to (see Hawking and Ellis (1973) [56], page 43)

$$C^{abcd}{}_{;d} = R^{c[a;b]} - \tfrac{1}{6} g^{c[a} R^{;b]} \ . \tag{2.29}$$

Written in this form, they are differential equations relating the components of *Ricci* and *Weyl*. Contraction of (2.29) implies

$$G^{ab}{}_{;b} = 0 \ , \quad G_{ab} \equiv R_{ab} - \tfrac{1}{2} R \, g_{ab} \ , \tag{2.30}$$

where G_{ab} is the Einstein tensor.

2.2.3 Einstein equations

Within the covariant approach we are following, Einstein field equations establish at each spacetime point an algebric relation between curvature and the matter content represented by the energy momentum tensor T_{ab}

$$G_{ab} + \Lambda g_{ab} = R_{ab} - \tfrac{1}{2} g_{ab} R + \Lambda g_{ab} = \kappa T_{ab} \ , \tag{2.31}$$

where we include the cosmological constant term Λ for generality. The cosmological constant problem is a controversial one, but not at all old - fashioned, see e.g. the recent review by Weinberg (1989) [128]. If we substitute (2.31) in the Bianchi identities (2.29), we see that *Weyl* represents the "free gravitational field", determined non - locally by matter and suitable boundary conditions.

2.2.4 Matter: conservation equations

If T^{ab} is the energy momentum stress tensor, the covariant form of the energy momentum conservation equations is

$$T^{ab}{}_{;b} = 0 \ ; \tag{2.32}$$

within the covariant approach followed here these follows from (2.30) and (2.31). From now on we will assume a one component imperfect fluid unless otherwise specified; in this case the decomposition of T^{ab} with respect to u^a gives the well-known form

$$T_{ab} = \mu u_a u_b + p h_{ab} + 2q_{(a}u_{b)} + \pi_{ab} \ , \tag{2.33}$$

where

$$\mu \equiv T_{ab}u^a u^b \ , \qquad p \equiv \tfrac{1}{3}h^{ab}T_{ab} \ , \tag{2.34}$$

$$q_a \equiv -h_a{}^c T_{cd}u^d \ , \qquad \pi_{ab} \equiv h_a{}^c h_b{}^d T_{cd} - \tfrac{1}{3}(h^{cd}T_{cd})h_{ab} \ , \tag{2.35}$$

are respectively the energy density, the pressure, the energy flux and the anisotropic pressure in the LRF of $\mathcal{O}_u$; in general μ and p will be related through an equation of state. For a perfect fluid the fluid four velocity u^a can be uniquely specified: indeed in this case (2.33) admits the form

$$T^{ab} = \mu u^a u^b + p h^{ab} = (\mu + p)u^a u^b + p g^{ab} \ , \tag{2.36}$$

when it is decomposed with respect to its eigenvector $u^a = u^a_E$; more details will be given in section 3.4. In section 2.3.1 we will separate equation (2.32) in its time and space components.

2.2.5 Intrinsic 3-curvature when $\omega = 0$

When the fluid vorticity vanishes (and only then) there exists a family of 3-surfaces $\Sigma_\perp$ everywhere orthogonal to the fluid flow vector u^a. Indeed it is possible to show that

$$\omega = 0 \quad \Leftrightarrow \quad u_{[a}u_{b;c]} = 0$$

$$\Leftrightarrow \quad \exists \ \textit{locally} \ f, g : \ u_a = f g_{;a} \ . \tag{2.37}$$

In other words, for the surfaces $\Sigma_\perp$ to exist, it must be possible to write u_a as a 4-gradient. Then the surfaces $\Sigma_\perp \equiv \{g = \textit{constant}\}$ are instantaneous surfaces of simultaneity for all the fundamental observers, i.e., the surfaces $\Sigma_\perp$ define a cosmic time. However this can be locally normalized to measure proper time along each flow line only if $a_a = 0$.

Since for $\omega = 0$ these surfaces exist, one can define an intrinsic curvature tensor for them from the Ricci identity in $\Sigma_\perp$. For any vector V^a in $\Sigma_\perp : V^a u_a = 0$, we have

$$ {}^{(3)}\nabla_c {}^{(3)}\nabla_b V_a - {}^{(3)}\nabla_b {}^{(3)}\nabla_c V_a = V_d {}^{(3)}R^d{}_{abc} \; . \tag{2.38} $$

Finally, *Ricci* for these 3-spaces can be written as

$$ \begin{aligned} {}^{(3)}R_{ab} = h_a{}^f h_b{}^g \left(-S^{-3}(S^3\sigma_{fg})^{\cdot} + a_{(f;g)} \right) + a_a a_b + \\ + \kappa\pi_{ab} + \tfrac{1}{3} h_{ab}(-\tfrac{2}{3}\Theta^2 + 2\sigma^2 + 2\kappa\mu + 2\Lambda - A) \; . \end{aligned} \tag{2.39} $$

Now, in a general fluid flow, we can define the quantity

$$ \mathcal{K} \equiv 2(-\tfrac{1}{3}\Theta^2 + \sigma^2 + \kappa\mu + \Lambda) \; . \tag{2.40} $$

Then, when $\omega = 0$, this quantity acquires a special significance: it is the Ricci scalar ${}^{(3)}R$ of the 3-dimensional spaces $\Sigma_\perp$; that is, $\omega = 0 \Rightarrow {}^{(3)}R = \mathcal{K}$.[5] We shall introduce in section 3.3.1 two new tensor that generalize ${}^{(3)}R_{abcd}$ and ${}^{(3)}R_{ab}$ to the case of non vanishing fluid vorticity $\omega \neq 0$.

2.2.6 Spatial gradients

We may define spatial gradients (i.e. orthogonal to u^a) in the LRF of the observers $\mathcal{O}_u$ for any scalar function f

$$ {}^{(3)}\nabla_a f \; ; \tag{2.41} $$

[5] What is the meaning of $\mathcal{K}$ when $\omega \neq 0$? When $\omega \neq 0$, there are no surfaces orthogonal to the family of fluid flow lines, but we can find normalised comoving coordinates $\{t, y^\nu\}$ (see Ehlers [25] Treciokas and Ellis [123]). Using such coordinates, the surfaces $\{t = const\}$ can be set orthogonal to a particular chosen world line γ and almost orthogonal to neighbouring world lines, by the remaining gauge freedom (e.g. if we choose an initial surface $\{t = t_0\}$ to be generated by orthogonal geodesics emanating from γ). Then $\mathcal{K}$, given by (2.40), will be nearly the Ricci-scalar of these 3-spaces on and near γ. Note however these surfaces do not directly correspond to the FLRW surfaces $\{t = const\}$ when there are spatial density gradients, because if $X_a \neq 0$ the surfaces $\{\mu = const\}$ do not lie orthogonal to the world-lines; similarly if $Z_a \neq 0$ the surfaces $\{\Theta = const\}$ do not lie orthogonal to the world lines.

More generally, if u^a is not too different from the normals n^a to a family of surfaces, then $\mathcal{K}$ will be not too different from the Ricci scalar of those 3-spaces. The meaning of "not too different" can be made precise by either using (a) a formalism equivalent to the ADM lapse and shift formalism (cf. Bardeen [1] section VI and section 4.1), (b) the tilted flow vector formalism of King and Ellis (1973) [67], or (c) adapted comoving coordinates mentioned above.

in particular we find useful to define the gradients of energy density, pressure and expansion

$$X_a \equiv {}^{(3)}\nabla_a \mu \ , \quad Y_a \equiv {}^{(3)}\nabla_a p \ , \quad Z_a \equiv {}^{(3)}\nabla_a \Theta \ ; \tag{2.42}$$

also

$$A \equiv a^a{}_{;a} \ , \qquad A_a \equiv {}^{(3)}\nabla_a A \ , \tag{2.43}$$

are the acceleration divergence and its gradient, and

$$\mathcal{K}_a \equiv {}^{(3)}\nabla_a \mathcal{K} \tag{2.44}$$

is the spatial gradient of the quantity $\mathcal{K}$ defined in (2.40), i.e. the 3 - curvature gradient when $\omega = 0$. As we shall see in chapter 3, these quantities play a crucial role in the covariant theory of GI cosmological perturbations.

2.3 Dynamic non - linear equations

2.3.1 Conservation equations

The conservation equations (2.32) can be separated into time (energy conservation) and space (momentum conservation) components. For an imperfect fluid the energy - momentum tensor has the form (2.33). Inserting this in equation (2.32) and projecting along u_a gives the
energy conservation equation

$$\dot{\mu} + (\mu + p)\Theta + \pi_{ab}\sigma^{ab} + q^a{}_{;a} + q^a a_a = 0 \, . \tag{2.45}$$

In the same way, using the projection tensor h_{ab}, one has the
momentum conservation equation

$$(\mu + p)a_a + Y_a + h_a{}^c(\pi_c{}^b{}_{;b} + \dot{q}_c) + (\omega_a{}^b + \sigma_a{}^b + \tfrac{4}{3}\Theta h_a{}^b)q_b = 0 \, . \tag{2.46}$$

The time-evolution of p is determined by (2.45) when we specify an equation of state determining p from μ.

2.3.2 Hydrodynamic equations

The evolution equations for the kinematical quantities follow from the Ricci identities (2.21).

Raychaudhuri equation

The evolution of Θ along the fluid flow lines is given by the *Raychaudhuri equation*:

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 + 2(\sigma^2 - \omega^2) - A + \tfrac{1}{2}\kappa(\mu + 3p) - \Lambda = 0 \ , \tag{2.47}$$

where A is defined by (2.43). This equation was derived by Raychaudhuri (1955) [109] in the case of dust and generalized by Ehlers (1961) [26] to the case of non-vanishing pressure. It is also the *trace* of the Ricci identity (2.21) (actually it is obtained by contracting the spatial part of (2.21) projected along u^d). The Raychaudhuri equation is the fundamental equation of gravitational attraction that establishes that in general relativity $(\mu + 3p)$ is the active gravitational mass of the fluid.[6] For $\mu + 3p > 0$ we have from (2.47) a volume contraction; we also see from (2.47) that Λ contributes as a constant repulsive force, and a similar repulsive role is played by the acceleration divergence A and by the vorticity. On the other end the shear term tends to shrink the volume.

Shear and vorticity equations

Propagation equations for σ_{ab} and ω_{ab} are also obtained via the Ricci identity (2.21).

If we project (2.21) along u^d and then we take the spatial part of the resulting expression, we obtain a tensor (let us call it P_{ab}) the *symmetric trace-free part* of which is the *shear evolution equation*:

$$\begin{aligned} & h_a{}^f h_b{}^g(\sigma_{fg})^{\cdot} - h_a{}^f h_b{}^g a_{(f;g)} - a_a a_b + \omega_a\omega_b + \sigma_{af}\sigma^f{}_b + \\ & + \tfrac{2}{3}\Theta\sigma_{ab} + \tfrac{1}{3}h_{ab}(A - \omega^2 - 2\sigma^2) + E_{ab} = +\tfrac{1}{2}\pi_{ab} \ . \end{aligned} \tag{2.48}$$

The *vorticity evolution equation*

$$h^a{}_b(\omega^b)^{\cdot} + \tfrac{2}{3}\Theta\omega^a - \sigma^a{}_b\omega^b - \tfrac{1}{2}\eta^{abcd}u_b a_{c;d} = 0 \tag{2.49}$$

is then the *skew symmetric part* of P_{ab}.

[6] This is the role played by the mass density *only* in the Newtonian theory: the p term in the active gravitational mass is responsible for the *regeneration of pressure* enhancing gravitational collapse in general relativity.

Contraint equations

Three further equations follow from the Ricci identity (2.21).

They can be regarded as constraint equations, since they do not involve time derivatives:

$$h^e{}_b(\omega^{bc}{}_{;c} - \sigma^{bc}{}_{;c} + \tfrac{2}{3}Z^b) + (\omega^e{}_b + \sigma^e{}_b)a^b = q^e \,, \tag{2.50}$$

$$\omega^a{}_{;a} = 2\omega^a a_a \,, \tag{2.51}$$

$$H_{ad} = 2a_{(a}\omega_{d)} - h_a{}^e h_d{}^g \left(\omega_{(e}{}^{b;c} + \sigma_{(e}{}^{b;c}\right)\eta_{g)fbc}u^f \,. \tag{2.52}$$

2.3.3 Maxwell - like gravitational field equations

We have seen in section 2.2.2 that the Bianchi identities (2.28) obeyed by the Riemann tensor can be cast in the form (2.29), relating the Weyl tensor components to the Ricci tensor components. As the Weyl tensor can be decomposed into components E_{ab} and H_{ab} (see eq. (2.26)) and the Ricci tensor is related to the energy - momentum tensor via the Einstein equations (2.31), one can substitute (2.26) and (2.31) in (2.29), obtaining four tensorial equations for E_{ab} and H_{ab} that are rather similar to the Maxwell equations. In the perfect fluid case these *Maxwell - like Bianchi identities* are:

div E :

$$h^t{}_a E^{as}{}_{;d} h_s{}^d - \eta^{tbpq}u_b\sigma_p{}^d H_{qd} + 3H^t{}_s\omega^s = \tfrac{1}{3}\kappa X^t \,, \tag{2.53}$$

div H :

$$h^t{}_a H^{as}{}_{;d} h_s{}^d + \eta^{tbpq}u_b\sigma_p{}^d E_{qd} - 3E^t{}_s\omega^s = \kappa(\mu + p)\omega^t \,, \tag{2.54}$$

$\dot{E}$:

$$\begin{aligned} h_a{}^m h_c{}^t (E^{ac})^{\cdot} + h_a{}^{(m}\eta^{t)rsd}u_r H^a{}_{s;d} - 2H_q{}^{(t}\eta^{m)bpq}u_b a_p + \Theta E^{mt} + \\ + h^{mt}(\sigma^{ab}E_{ab}) - 3E_s{}^{(m}\sigma^{t)s} - E_s{}^{(m}\omega^{t)s} = -\tfrac{1}{2}\kappa(\mu + p)\sigma^{tm} \,, \end{aligned} \tag{2.55}$$

$\dot{H}$:

$$\begin{aligned} h^{ma}h^{tc}(H_{ac})^{\cdot} - h_a{}^{(m}\eta^{t)rsd}u_r E^a{}_{s;d} + 2E_q{}^{(t}\eta^{m)bpq}u_b a_p + \\ + h^{mt}(\sigma^{ab}H_{ab}) + \Theta H^{mt} - 3H_s{}^{(m}\sigma^{t)s} - H_s{}^{(m}\omega^{t)s} = 0 \,. \end{aligned} \tag{2.56}$$

The form of these equations is the reason for calling E_{ab} and H_{ab} the electric and magnetic component of the Weyl tensor.

If there are imperfect fluid terms in the energy momentum tensor, the right hand side of the above equations change to

$$DivE \ : \tag{2.57}$$

$$= \kappa \left[\tfrac{1}{3} X^t - \tfrac{1}{2} h^t{}_c \pi^{cb}{}_{;b} - \tfrac{3}{2} \omega^t{}_b q^b + \tfrac{1}{2} \sigma^t{}_b q^b + \tfrac{1}{2} \pi^t{}_b a^b - \tfrac{1}{3} \Theta q^t \right] \ , \tag{2.58}$$

$$DivH \ : \tag{2.59}$$

$$= \kappa \left[(\mu + p) \omega^t + \tfrac{1}{2} \eta^{tbef} u_b q_{[e;f]} + \tfrac{1}{2} \eta^{tbef} u_b \pi_{ec} (\omega^c{}_f + \sigma^c{}_f) \right] \ , \tag{2.60}$$

$$\dot{E} \ : \tag{2.61}$$

$$= \kappa \left[-\tfrac{1}{2} (\mu + p) \sigma^{tm} - a^{(t} q^{m)} - \tfrac{1}{2} h^t{}_a h^m{}_c q^{(a;c)} - \tfrac{1}{2} h^t{}_a h^m{}_c \dot{\pi}^{ac} \right. \tag{2.62}$$

$$\left. -\tfrac{1}{2} \pi^{b(m} \omega_b{}^{t)} - \tfrac{1}{2} \pi^{b(m} \sigma_b{}^{t)} - \tfrac{1}{6} \pi^{tm} \Theta + \tfrac{1}{6} (q^b{}_{;b} + a_b q^b + \pi^{cd} \sigma_{cd}) h^{mt} \right] \ , \tag{2.63}$$

$$\dot{H} \ : \tag{2.64}$$

$$= \kappa \left[\tfrac{1}{2} \sigma^{(t}{}_e \eta^{m)bef} u_b q_f - \tfrac{1}{2} h_c{}^{(t} \eta^{m)bef} u_b \pi^c{}_{e;f} + \tfrac{1}{2} (h^{mt} \omega_c q^c - 3 \omega^{(m} q^{t)}) \right] \ . \tag{2.65}$$

In the next chapter we shall consider a perturbed FLRW model, and we shall linearize the hydrodynamic and gravitational equations presented so far.

Chapter 3

COVARIANT THEORY OF GI COSMOLOGICAL PERTURBATION

This chapter is the core of the thesis: we develop here the covariant theory of gauge-invariant cosmological perturbations. The first part contains various preliminary (although original) material: I introduce two new inhomogeneity variables (which later will turn out to be our main GI variables in analyzing density perturbations) and I present new non-linear equations for them (as they where originally derived in EB). Then I introduce new tensors that describe the spatial curvature properties in the case of non-vanishing vorticity, and derive useful commutation rules for the 3-derivative ${}^{(3)}\nabla_a$ (EBH). Section 3.4 is devoted to discuss the various possible choices of frame (i.e. of the fluid 4-velocity) we have when the fluid is described by an energy-momentum tensor which contain viscous terms, and in section 3.5 the whole set of covariant GI variables is considered (Paper I). In section 3.6, 3.7 and 3.8 the attention is restricted to a perfect fluid, for which we derive the linear equations governing the evolution of our vectorial density perturbation variable $\mathcal{D}_a$. We derive a Jeans instability criterion (EHB), correcting previous result in the literature [65], and we show how vorticity can affect the evolution of $\mathcal{D}_a$ (EBH).

In the last part of this chapter we turn to the more general case of an imperfect fluid: in section 3.9 we derive the whole set of hydrodynamic and gravitational equations that follow on linearizing the equations presented in chapter 2, and in section 3.10 we consider the equations for the locally defined GI scalar variables introduced in section

3.7. In particular, we derive a second order equations for the evolution of our GI scalar variable Δ: this equation is the equivalent within our formalism of the main equation of Bardeen [1] for his for GI density perturbation variable.

The approach to GI perturbations we shall follow is based on the Stewart and Walker Lemma of chapter 1 (see section (1.3)): from now on, we shall refer to the gauge invariance of any variable as following from this Lemma. Throughout this chapter, we shall consider a set of exactly defined covariant quantities which have significance in any spacetime; unless otherwise specified these vanish in a FLRW model, and therefore constitute a set of covariant GI quantities in an almost FLRW universe.

3.1 Key GI variables

In the previous chapter we have reviewed the classical covariant hydrodynamical and gravitational equations [25, 27, 28], and we have explicitely introduced in section 2.2.6 a new notation for spatial (i.e. orthogonal to the fluid flow vector u^a) gradients. Since our goal is to derive a GI theory of cosmological perturbations, with special emphasis on density inhomogeneities, we wish to complete that set of equations with those governing the evolution of the density and expansion gradients X_a and Z_a.

Let us consider the meaning of these variables.[1]

X_a is measurable in the sense that (a) it can be determined from virial theorem estimates (indeed, dynamical mass estimates determine precisely spatial density gradients), and (b) the contribution to it from luminous matter can be found by observing gradients in the numbers of observed sources and estimating the mass to light ratio (Kristian and Sachs [72], equation (39)). It describes the density inhomogeneities which we wish to investigate, for if there is an overdensity which is a viable proto - galaxy, this will be evidenced by a non-zero value of X_a (the magnitude of X_a directly indicating how rapid the spatial variation of density is). Thus X_a seems to encapsulate much of the information we want.

However, the usual variable that characterize density perturbation is the dimensionless $\delta\mu/\mu$: within the covariant theory, we may expect that a similar quantity should be employed. Indeed we normally would like to compare the density gradient with

[1]The density gradient X_a naturally arises in the covariant fluid approach (see (1966) [55], and Olson (1976) [98]) as a term in the equations (see section 2.3). However it was not recognized as the GI variable in terms of which the whole perturbation problem can be formulated.

the existing density, to characterize it significance; moreover X_a is not dimensionless. This is essentially related to the fact that when we consider the time evolution of the fluid, X_a represents the change in density to a *fixed distance*, whereas in the context of considering the growth of proto-fluctuations we want to consider density variations at a *fixed comoving scale.* Noting these points, we define the *comoving fractional density gradient*

$$\mathcal{D}_a \equiv \frac{a}{\mu} X_a \ , \tag{3.1}$$

which is GI and dimensionless. The time variation of this quantity precisely reflects the relative growth of density in neighbouring fluid comoving volumes, and this is what we wish to investigate.

While both are observable in principle, it is a moot point whether X_a or $\mathcal{D}_a$ is more easily observable in practice.[2]

The vector $\mathcal{D}_a$ can be separated into a direction e_a and magnitude $\mathcal{D}$, where

$$\mathcal{D}_a = \mathcal{D} e_a, \ e_a e^a = 1, \ e_a u^a = 0 \quad \Rightarrow \quad \mathcal{D} = (\mathcal{D}^a \mathcal{D}_a)^{1/2}. \tag{3.2}$$

Given $\mathcal{D}_a$, its magnitude is $\mathcal{D}$ the GI variable that most closely corresponds to the intention of the usual $(\delta\mu/\mu)$ in representing the fractional density increase in a comoving density fluctuation. The crucial difference from the usual definition is that $\mathcal{D}$ represents a (real) spatial fluctuation, rather than a (fictitious) time fluctuation.

The vectors X_a, Y_a and Z_a defined in section 2.2.6 are dynamically dependent on each other, as will be shown in the following section. But we have also defined the comoving density gradient $\mathcal{D}_a$, therefore we find useful to define the *comoving expansion gradient*

$$\mathcal{Z}_a \equiv a Z_a \tag{3.3}$$

as the natural companion variable of $\mathcal{D}_a$. All these variables are GI, and directly determinable (at any desired scale) from a description of the real (lumpy) universe

[2]Both these vectors can be used to determine the spatial variation of the energy density μ. One important point should be noticed. In an arbitrary spacetime, in the case where $\omega = 0$, they will characterize the distribution of the density μ in the 3-spaces $\Sigma_\perp$ orthogonal to the fluid flow (which might naturally be chosen as the surfaces $\{t = const\}$). However when $\omega \neq 0$, no such orthogonal 3-surfaces exist. These vectors still characterize the gradient of μ orthogonal to u^a, but cannot be immediately integrated to give the distribution of density in the surfaces $\{t = const\}$ for a suitable set of coordinates [123, 24] because these surfaces cannot be everywhere orthogonal to the fluid flow lines. Even if $\omega = 0$, the time t such that the surfaces $\{t = const\}$ are orthogonal to the fluid flow will not measure proper time τ along the fluid flow lines unless the acceleration is zero also, that is, unless there are no pressure gradients.

at that scale. Thus our further analysis will concentrate for the moment on these quantities.

3.2 New non-linear evolution equations

3.2.1 Equations for a_a, X_a and Z_a

We pass now to sketch the derivations of the exact non-linear evolution equations for the acceleration a_a, the density gradient X_a and the expansion gradient Z_a as they were originally derived in EB [35]. In this section we assume a perfect fluid for simplicity, since the derivation of these equation for an imperfect fluid follows exactly the same line, with only few additional viscous terms.

Acceleration propagation equation

In order to derive an evolution equation for the acceleration we express a^a using the momentum conservation equation (2.46) simplified for a perfect fluid:

$$a^a = -\frac{Y^a}{(\mu + p)} \; . \tag{3.4}$$

Differentiating with respect to the proper time and projecting orthogonally to u^a we have

$$h_a{}^c(a_c)^{\cdot} = -\frac{h_a{}^c(Y_c)^{\cdot}}{(\mu + p)} + a_a\left(1 + \frac{dp}{d\mu}\right)\Theta \; , \tag{3.5}$$

where we substituted for $\dot{\mu}$ from the energy conservation equation (2.45) and we used $\dot{p} = \frac{dp}{d\mu}\dot{\mu}$, with $dp/d\mu$ taken along the fluid flow lines. Now with the same methods and after some algebra we can write

$$\begin{aligned} h_a{}^c(Y_c)^{\cdot} = &-(\mu + p)\left(\frac{dp}{d\mu}\Theta\right)_{,b} h^b{}_a - \\ &- \Theta\left(1 + \frac{dp}{d\mu}\right)Y_a - h_a{}^b p_{,c} u^c{}_{;b} - a_a\frac{dp}{d\mu}\Theta(\mu + p) \; . \end{aligned}$$

Using (2.19) to express $u^c{}_{;b}$ and substituting in (3.5) we finally obtain

$$h_a{}^c(a_c)^{\cdot} = a_a\Theta\left(\frac{dp}{d\mu} - \frac{1}{3}\right) + h_a{}^b\left(\frac{dp}{d\mu}\Theta\right)_{,b} - a_c(\omega^c{}_a + \sigma^c{}_a) \; . \tag{3.6}$$

Density gradient equation

To obtain a propagation equation for the spatial gradients of the energy density we could proceed from its definition, $X_a = h_a{}^b \mu_{,b}$, differentiating with respect to the proper time and projecting orthogonally to u^a.[3]

However it is more interesting to proceed directly from (2.45), because the propagation equation we derive for X_a is the *spatial variation of the energy conservation equation.* Taking the spatial gradient of (2.45) we obtain

$$h_a{}^b(\dot{\mu})_{;b} + (\mu + p)Z_a + \Theta X_a + \Theta Y_a = 0 \ . \tag{3.7}$$

Now we can write

$$\begin{aligned} h_a{}^b(\dot{\mu})_{;b} &= h_a{}^b(\mu_{;c}u^c)_{;b} = \\ &= h_a{}^b\mu_{;b;c}u^c + h_a{}^b\mu_{;c}(\sigma^c{}_b + \omega^c{}_b + \tfrac{1}{3}\Theta h^c{}_b) \ , \end{aligned}$$

where we used $\mu_{;c;b} = \mu_{;b;c}$ (μ is a scalar) and we substituted for $u^a{}_{;b}$ from (2.19). It is useful to express the first term in the last part of the previous equality as

$$\begin{aligned} h_a{}^b\mu_{;b;c}u^c &= h_a{}^d(h_d{}^b\mu_{;b})_{;c}u^c - h_a{}^d(h_d{}^b)_{;c}u^c\mu_{;b} = \\ &= \left[h_a{}^b(X_b)^{\cdot} - Y_a\Theta\right] \ , \end{aligned}$$

where in the last step we again used (2.45) and (3.4).

Substituting in (3.7) and using the definitions (2.42), we finally obtain

$$h_a{}^b(X_b)^{\cdot} + X_b(\sigma^b{}_a + \omega^b{}_a) + \tfrac{4}{3}\Theta X_a + (\mu + p)Z_a = 0 \ . \tag{3.8}$$

We can cast this equation in the following form:

$$a^{-4}h_c{}^a(a^4X_a)^{\cdot} = -(\mu + p)Z_c - (\omega^a{}_c + \sigma^a{}_c)X_a \ , \tag{3.9}$$

showing that the time variation of X_a is determined by the source term Z_a and by the non-linear term coupling X_a with the shear and vorticity.

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[3]in this case a key step would be $(\mu_{,c})^{\cdot} = (\dot{\mu})_{,c} - \mu_{,b}u^b{}_{;c}$, followed by the substitution for $u^a{}_{;b}$ from (2.19), a^a from (3.4) and $\dot{\mu}$ from the energy conservation equation (2.45).

Expansion gradient equation

We now want to derive an evolution equation for the spatial gradient of the expansion. As for X_a, we could start from the definition $Z_a = h_a{}^b\Theta_{;b}$ and spatially-project its time derivative. However we proceed from (2.47), because the equation we obtain for Z_a is the *spatial variation of the Raychaudhuri equation.* Taking the spatial gradient of (2.47) we have

$$h_a{}^b(\dot{\Theta})_{,b} + \tfrac{2}{3}\Theta Z_a + \tfrac{1}{2}\kappa X_a + \tfrac{3}{2}Y_a + h_a{}^b\left[2(\sigma^2)_{,b} - 2(\omega^2)_{,b} - A_a\right] = 0 \ , \qquad (3.10)$$

and the first term of this equation can be re-expressed as

$$\begin{aligned} &h_a{}^b(\Theta_{,c}u^c)_{;b} = \\ &\quad h_a{}^d(h_d{}^b\Theta_{,b})_{;c}u^c - h_a{}^d\Theta_{,b}(h_d{}^b)_{;c}u^c + Z_b(\sigma^b{}_a + \omega^b{}_a) + \tfrac{1}{3}\Theta Z_a = \\ &\quad h_a{}^b(Z_b)^{\cdot} - \dot{\Theta}a_a + Z_b(\sigma^b{}_a + \omega^b{}_a) + \tfrac{1}{3}\Theta Z_a \ . \end{aligned}$$

As before we used (2.19) to express $u^a{}_{;b}$, and the definitions (2.42). If now we substitute the last expression in (3.10), using (2.47) to express $\dot{\Theta}$, we obtain

$$h_a{}^b(Z_b)^{\cdot} + \Theta Z_a - a_a\mathcal{R} + h_a{}^b\left(\tfrac{1}{2}\kappa X_b + 2(\sigma^2)_{,b} - 2(\omega^2)_{,b} - A_b\right) + Z_b(\sigma^b{}_a + \omega^b{}_a) = 0 \ , \quad (3.11)$$

where we defined

$$\begin{aligned} \mathcal{R} &\equiv -\tfrac{1}{3}\Theta^2 - 2\sigma^2 + 2\omega^2 + A + \mu + \Lambda = \\ &= \tfrac{1}{2}\mathcal{K} + A - 3\sigma^2 + 2\omega^2 \ , \end{aligned} \qquad (3.12)$$

and $\mathcal{K}$ (defined by (2.40)) is the Ricci curvature ${}^{(3)}R$ of the surfaces orthogonal to the fluid flow when $\omega = 0$ (see section 2.2.5).

Equation (3.11) can be put in the form

$$a^{-3}h_a{}^b(a^3 Z_b)^{\cdot} = a_a\mathcal{R} + h_a{}^b\left[-\tfrac{1}{2}\kappa X_b - 2(\sigma^2)_{,b} + 2(\omega^2)_{,b} + A_b\right] - Z_b(\sigma^b{}_a + \omega^b{}_a) \ , \quad (3.13)$$

with $\mathcal{R}a_a$, X_a, A_a, $h_a{}^b(\sigma^2)_{,b}$, and $h_a{}^b(\omega^2)_{,b}$ acting as source terms, while the last non-linear term couples Z_a with the shear and vorticity.

Pressure and curvature gradients

We could derive the equation for the evolution of the pressure gradient Y_a proceeding from its definition, but this would not be an independent equation. Indeed, when the equation of state of the fluid is known, the evolution of Y_a will follow from that for X_a.

We could also derive a propagation equation for $\mathcal{K}_a$ (2.44); however, the resulting equation would be rather cumbersome, involving also the time derivative of σ. Therefore, we postpone the derivation of an equation for $\mathcal{K}_a$ (actually for a related quantity) to section 3.6.1, where we consider the linear approximation.

3.2.2 Equations $\mathcal{D}_a$ and $\mathcal{Z}_a$

The evolution equations for the comoving fractional density gradient $\mathcal{D}_a$ could be derived starting from its definition (3.1), spatially projecting its derivatives with respect to the proper time.

However, $\mathcal{D}_a$ is simply related to X_a; therefore, it is more convenient to use this interdependence to express the relation between the time derivative of $\mathcal{D}_a$ and the time derivative of X_a, since we have already derived the evolution equation for this latter.

We have

$$\mathcal{D}_a = \frac{aX_a}{\mu} \quad \Rightarrow \quad (\mathcal{D}_a)^{\cdot} = \frac{a}{\mu}(X_a)^{\cdot} + \left(\frac{4}{3} + \frac{p}{\mu}\right)\Theta\mathcal{D}_a \,. \tag{3.14}$$

Accordingly, the propagation equation for $\mathcal{D}_a$ follows from (3.8)

$$h_c{}^a(\mathcal{D}_a)^{\cdot} = \frac{p}{\mu}\Theta\mathcal{D}_c - \left(1 + \frac{p}{\mu}\right)\mathcal{Z}_c - \mathcal{D}_a(\omega^a{}_c + \sigma^a{}_c) \,, \tag{3.15}$$

where we have used $\mathcal{Z}_a = SZ_a$. The equation for this latter variables can be derived from that for Z_a (3.11) in the same way, and we write it here for completeness:

$$\begin{aligned} a^{-2}h_a{}^b(a^2\mathcal{Z}_b)^{\cdot} = &-\frac{1}{2}\kappa\mu\mathcal{D}_a - \mathcal{Z}_b(\sigma^b{}_a + \omega^b{}_a) + \\ &+ a\left\{a_a\mathcal{R} + A_a + h_a{}^b\left[2(\omega^2)_{,b} - 2(\sigma^2)_{,b}\right]\right\} \,. \end{aligned} \tag{3.16}$$

3.2.3 Some remarks on exact equations

It should be emphasized that, given the perfect fluid assumption, the equations presented in this section are *exact propagation equations*, valid in any fluid flow whatever: they maybe seen as completing the set of equations in section 2.3.

We see from (3.9), (3.15) and from (3.13), (3.16) that the density gradients X_a and $\mathcal{D}_a$ are coupled with the expansion gradients Z_a and $\mathcal{Z}_a$. For example, a significant feature follows immediately from (3.9): provided $(\mu + p) \neq 0$, $Z_a \neq 0 \Rightarrow X_a \neq 0$. The converse result ($X_a \neq 0 \Rightarrow Z_a \neq 0$) will hold in general as well (if $X_a \neq 0$, $Z_a = 0$ then the right hand side of (3.13) must be zero; this is unlikely to remain true even if it is true at some initial time). Indeed, the equations for $X_a, \mathcal{D}_a$, Z_a and $\mathcal{Z}_a$ contain non-linear terms coupling these quantities with shear, vorticity, acceleration, as well as the acceleration divergence A and its gradient A_a. Therefore, to consider a closed non-linear system of equations, one should take into account the evolution of all these quantities. Also, Maxwell-like equations should be included, because E_{ab} appear as a source term in the shear equation (2.48), and H_{ab} is coupled with E_{ab} in the $\dot{E}$ equation (2.55). Finally, the constraint equations (2.50; 2.51; 2.52) must be satisfied.

It is not surprising that to consider the full non-linear equation for the density gradient involve taking into account so many other quantities. After all, the full non-linear system is equivalent to the complete content of Einstein equations. We only chosen new variables, more suitable for the study of the growth in time of spatial density inhomogeneities. In order to solve the equations and determine this growth, one has therefore to adopt some restrictive and physically motivated hypothesis. The first step in this direction is the linear approximation. Indeed, the two equations for X_a (or $\mathcal{D}_a$) and Z_a ($\mathcal{Z}_a$) decouple from the others for an almost FLRW universe. We shall consider this case in section 3.6.2.

3.3 Intrinsic 3-curvature when $\omega \neq 0$

In section 2.2.5 we have reviewed some basics about the intrinsic curvature of 3-hypersurfaces in spacetime, as they arise naturally when the fluid flow vorticity vanishes, $\omega = 0$, and the flow is therefore hypersurface orthogonal. We wish however consider the general case of non-vanishing flow vorticity $\omega \neq 0$, continuing to maintain our description based on the fluid flow threading (the 1+3 description), rather than introduce an arbitrary slicing of spacetime as in the 3+1 ADM [4] formalism. To this end, we shall introduce here two new tensors which, at each point of spacetime, generalize the 3-Riemann and 3-Ricci tensors of section 2.2.5 to the case when $\omega \neq 0$.

[4] Acronym for: Arnowitt, Deser, and Misner. See for example: York (1979) [131], Wald (1984) [126].

3.3.1 Properties of the spatial derivative

Defect tensor

Given the smooth 4-velocity field u^a ($u^a u_a = -1$) at each point p of the spacetime we have a subspace H_p of the tangent space at p, T_p, which is orthogonal to u^a, and $h_{ab} \equiv g_{ab} + u_a u_b$ is a metric in H_p. The collection of these subspace H_p can be called a *distribution* D (see Crampin and Pirani [14], page 141) or a *smooth specification* (see Wald [126], Appendix B.3). When the vorticity is non zero we have, for two vectors $X^a, Y^a \in D$

$$[X,Y]^a - h_b{}^a [X,Y]^b = u^a \omega_{bc} X^b Y^c \ , \tag{3.17}$$

where the *defect tensor* [78]

$$D^a{}_{bc} = u^a \omega_{bc} \ , \tag{3.18}$$

expresses the fact that the vector $[X,Y]^a$ does not live in D. In this case Frobenius's theorem [126] tells us that D does not possess integrable submanifolds, i.e. surfaces orthogonal to u^a.

The spatial derivative

By definition, acting on scalars, vectors orthogonal to u^a, and tensors orthogonal to u^a, the *orthogonal covariant derivative* ${}^{(3)}\nabla_a$ is given by:[5]

$${}^{(3)}\nabla_a f = h_a{}^b \nabla_b f = h_a{}^b f_{,b} \ , \tag{3.19}$$

$${}^{(3)}\nabla_a X_b = h_a{}^c h_b{}^d \nabla_c X_d = h_a{}^c h_b{}^d X_{d;c} \ , \tag{3.20}$$

$${}^{(3)}\nabla_a T_{bc} = h_a{}^d h_b{}^e h_c{}^f \nabla_d T_{ef} = h_a{}^d h_b{}^e h_c{}^f T_{ef;d}. \tag{3.21}$$

This compact notation is a convenient way of avoiding a plethora of indices:

$${}^{(3)}\nabla_a S^{fg..lm}{}_{bc..de} = h_a{}^t h^f{}_n h^g{}_o .. h^l{}_u h^m{}_v h_b{}^s h_c{}^r .. h_d{}^p h_e{}^q \nabla_t S^{no..uv}{}_{sr..pq} \ . \tag{3.22}$$

It follows from the above definition that ${}^{(3)}\nabla_a$ preserves the orthogonal metric h_{bc}: that is, ${}^{(3)}\nabla_a h_{bc} = 0$. Consequently, we can raise and lower indices through equations acted on by ${}^{(3)}\nabla_a$ by use of h_{ab}, h^{ab}. However we cannot simply treat this operator as the

[5] We can extend the definition to vectors and tensors not orthogonal to u^a by projecting them orthogonal to u^a before allowing the derivative to act, see [78].

standard covariant derivative of a 3-space, because the defect tensor will be non-zero when $\omega \neq 0$. Thus we cannot assume the usual commutation relations; rather we must use the expressions given in the following sections.

3.3.2 Commutators and 3 - curvature

From these definitions, we can calculate the commutator of the 3-derivatives when acting on scalars, vectors, and tensors. The key point in the first case is to note that, for any function $f(x^i)$:

$$^{(3)}\nabla_a(^{(3)}\nabla_b f) = h_a{}^c h_b{}^d \nabla_c(^{(3)}\nabla_d f) = h_a{}^c h_b{}^d \nabla_c(h_d{}^e \nabla_e f) \;;$$

and then use Leibniz rule on the last bracket, together with the definition $h_{ab} = g_{ab} + u_a u_b$ and (2.19) in the form

$$^{(3)}\nabla_b u_a = \omega_{ab} + \Theta_{ab} = k_{ab} \tag{3.23}$$

where $\Theta_{ab} = \sigma_{ab} + \frac{1}{3}\Theta h_{ab} = \Theta_{ba}$ is the expansion tensor ($\Theta_{ab}u^b = 0$). We obtain,

$$^{(3)}\nabla_{[a}{}^{(3)}\nabla_{b]} f = -D^c{}_{ab}{}^{(3)}\nabla_c f = -\omega_{ab}\dot{f} \;. \tag{3.24}$$

Similarly, totally projecting the derivatives in the vector commutator, we find that for all vector fields X_a orthogonal to u^a ($X_a u^a = 0$),

$$^{(3)}\nabla_{[a}{}^{(3)}\nabla_{b]} X_c + \omega_{ab}(h_c{}^t \dot{X}_t) = \tfrac{1}{2}\; ^{(3)}R_{dcba} X^d \;, \tag{3.25}$$

where, using the above defined k_{ab}

$$^{(3)}R_{abcd} = (R_{abcd})_\perp + k_{ad}k_{bc} - k_{ac}k_{bd} \Rightarrow {}^{(3)}R^{ab}{}_{cd} = (R^{ab}{}_{cd})_\perp - 2k^{[a}{}_{[c}k^{b]}{}_{d]} \;. \tag{3.26}$$

When $\omega = 0$, $k_{ab} = \Theta_{ab}$ and this is the 3-curvature of the spaces orthogonal to u^a and will have the usual curvature tensor symmetries. In the case of non vanishing vorticity instead we have

$$^{(3)}R_{abcd} = {}^{(3)}R_{[ab][cd]} \;, \quad {}^{(3)}R^a{}_{[bcd]} = 2k^a{}_{[b}\omega_{cd]} \tag{3.27}$$

and

$$^{(3)}R^{ab}{}_{cd} - {}^{(3)}R_{cd}{}^{ab} = -8\omega^{[a}{}_{[c}\Theta^{b]}{}_{d]} \;. \tag{3.28}$$

Note that other definitions are possible for this '3-curvature'; for example (see [78], eq. 2.26, [11, 46, 47] and [66])

$$ {}^{(3)}\tilde{R}^{a}{}_{bcd} = {}^{(3)}R^{a}{}_{bcd} - 2k^{a}{}_{b}\omega_{cd} \Rightarrow \tilde{R}^{a}{}_{[bcd]} = 0 \ , \tag{3.29} $$

and the "usual symmetries 3-curvature"

$$ {}^{(3-sym)}R^{ab}{}_{cd} = {}^{(3)}R^{ab}{}_{cd} + 4\omega^{[a}{}_{[c}\Theta^{b]}{}_{d]} - 2\omega^{ab}\omega_{cd} \ , \tag{3.30} $$

$$ {}^{(3-sym)}R^{ab}{}_{cd} = (R^{ab}{}_{cd})_{\perp} - \{2\Theta^{[a}{}_{[c}\Theta^{b]}{}_{d]}\} - \{2\omega^{[a}{}_{[c}\omega^{b]}{}_{d]} + 2\omega^{ab}\omega_{cd}\} \ , \tag{3.31} $$

where each of the terms in curly brackets individually has all of the usual symmetries of the Riemann tensor.

Here we use the definition (3.25) because of the simple form it gives (3.25) and (3.32). Further, for each tensor field T_{ab} orthogonal to u^a ($T_{ab}u^a = 0 = T_{ab}u^b = 0$),

$$ {}^{(3)}\nabla_{[a}{}^{(3)}\nabla_{b]}T_{cd} + \omega_{ab}(h_c{}^{t}h_d{}^{s}\dot{T}_{ts}) = \tfrac{1}{2}({}^{(3)}R_{scba}T^{s}{}_{d} + {}^{(3)}R_{sdba}T_{c}{}^{s}). \tag{3.32} $$

It follows from the above relations that the corresponding "3 - Ricci tensor" is

$$ {}^{(3)}R_{ac} \equiv {}^{(3)}R_{a}{}^{b}{}_{cb} = {}^{(3)}R^{b}{}_{abc} = h^{bd}(R_{abcd})_{\perp} - \Theta k_{ac} + k_{ab}k^{b}{}_{c} \tag{3.33} $$

with skew part

$$ {}^{(3)}R_{[cb]} = \tfrac{1}{3}\omega_{bc}\Theta + (\omega_{db}\sigma_{c}{}^{d} - \omega_{dc}\sigma_{b}{}^{d}) \tag{3.34} $$

and the "Ricci scalar" is

$$ {}^{(3)}R \equiv {}^{(3)}R^{a}{}_{a} = R + 2R_{bd}u^{b}u^{d} - \tfrac{2}{3}\Theta^{2} + 2\sigma^{2} - 2\omega^{2}. \tag{3.35} $$

3.3.3 Zero - order relations

When (3.25), (3.32) are applied to a first-order quantity, the time-derivative term can be neglected and we only need the zero- order curvature tensor term in these expressions to get the correct first-order result. From the field equations, the zero-order expression for the curvature tensor (see (28) in [43]) is

$$ R_{abcd} = \tfrac{1}{2}\kappa(\mu + p)(u_a u_c g_{bd} + u_b u_d g_{ac} - u_a u_d g_{bc} - u_b u_c g_{ad}) $$
$$ + \tfrac{1}{3}(\kappa\mu + \Lambda)(g_{ac}g_{bd} - g_{ad}g_{bc}). \tag{3.36} $$

Thus the zero-order version of the 3-curvature quantities (remembering that to zero order, $u_{a;b} = \frac{1}{3}\Theta h_{ab}$) are

$$^{(3)}R_{abcd} = \frac{K}{a^2}(h_{ac}h_{bd} - h_{ad}h_{bc}), \tag{3.37}$$

$$^{(3)}R_{ac} = 2\frac{K}{a^2}h_{ac} = R_{ca}\ , \quad ^{(3)}R = 6\frac{K}{a^2}\ , \tag{3.38}$$

where

$$\frac{K}{a^2} = \frac{1}{3}(-\frac{1}{3}\Theta^2 + \kappa\mu + \Lambda) \tag{3.39}$$

These expressions can substituted for the 3-curvatures above when performing systematic approximations of the equations.

3.3.4 First - order relations

We may note that, for any first - order X_a and Y_a the right hand side of (3.17) vanish, which means that at first - order the commutator $[X,Y]^a$ live in the subspace D orthogonal to u^a.

3 - divergences

It follows from (3.32) that ,

$$^{(3)}\nabla_a{}^{(3)}\nabla_b T^{[ab]} = -\omega_{ab}\dot{T}^{ab} + {}^{(3)}R_{[ab]}T^{ab}, \tag{3.40}$$

which shows that in particular

$$^{(3)}\nabla_a{}^{(3)}\nabla_b \omega^{ab} = -\omega_{ab}\dot{\omega}^{ab} + {}^{(3)}R_{[ab]}\omega^{ab}\ , \tag{3.41}$$

is non-zero in general, but vanishes to first - order in an almost-FLRW universe model.

Time derivatives

Calculating

$$^{(3)}\nabla_a(\dot{f}) - ({}^{(3)}\nabla_a f)^{\cdot}_{\perp} = (f_{;c}u^c)_{;b}h^b{}_a - (f_{;b}h^b{}_d)_{;c}u^c h^d{}_a \tag{3.42}$$

we find

$$^{(3)}\nabla_a(\dot{f}) - ({}^{(3)}\nabla_a f)^{\cdot}_{\perp} = -\dot{f}a_a + \frac{1}{3}\Theta{}^{(3)}\nabla_a f + {}^{(3)}\nabla_d f(\sigma^d{}_a + \omega^d{}_a) \tag{3.43}$$

where the last two terms are second order if ${}^{(3)}\nabla_a f$ is first - order, and so can be ignored to first - order. Similarly, for a first - order vector field orthogonal to u^a, we find that to first - order

$$ {}^{(3)}\nabla_a(\dot{X}_b) - ({}^{(3)}\nabla_a X_b)^{\cdot}_{\perp} = \tfrac{1}{3}\Theta\,{}^{(3)}\nabla_a X_b \;\Rightarrow\; a\,{}^{(3)}\nabla_a \dot{X}_b = (a\,{}^{(3)}\nabla_a X_b)^{\cdot}_{\perp} \;, \tag{3.44} $$

where we have used (3.36). Similar results will then hold for a first - order tensor, e.g. if T_{bc} is orthogonal to u^a then to first - order

$$ {}^{(3)}\nabla_a(\dot{T}_{bc}) - ({}^{(3)}\nabla_a T_{bc})^{\cdot}_{\perp} = \tfrac{1}{3}\Theta\,{}^{(3)}\nabla_a T_{bc} \;\Rightarrow\; a\,{}^{(3)}\nabla_a \dot{T}_{bc} = (a\,{}^{(3)}\nabla_a T_{bc})^{\cdot}_{\perp} \;. \tag{3.45} $$

We can contract this equation to obtain the result for a divergence:

$$ {}^{(3)}\nabla^c(\dot{T}_{bc}) - ({}^{(3)}\nabla^c T_{bc})^{\cdot}_{\perp} = \tfrac{1}{3}\Theta\,{}^{(3)}\nabla^c T_{bc} \;\Rightarrow\; a\,{}^{(3)}\nabla^c \dot{T}_{bc} = (a\,{}^{(3)}\nabla^c T_{bc})^{\cdot}_{\perp}. \tag{3.46} $$

3.3.5 Curvature gradient

In section 2.2.5 (see EB) we defined the quantity $\mathcal{K} = {}^{(3)}R + 2\omega^2$, which reduces to the 3-curvature scalar of the hypersurfaces orthogonal to u^a when $\omega = 0$. However we have now defined the new 3 - scalar ${}^{(3)}R$ (3.35), therefore we may now introduce the quantity

$$ C_a = S^3\,{}^{(3)}\nabla_a {}^{(3)}R = a^3\mathcal{K}_a - 2a^3\,{}^{(3)}\nabla_a(\omega^2) \;, \tag{3.47} $$

where $\mathcal{K}_a = {}^{(3)}\nabla_a\mathcal{K}$ was defined in section 2.2.6. When $\omega = 0$, C_a is the (exactly defined) curvature gradient of the surfaces orthogonal to u^a. However the last term in the equation above is second order, so C_a is the comoving curvature gradient in H_p at linear order.

3.4 Matter description

Up to now we have assumed that some reasonable choice for the 4 - velocity u^a of our viscous fluid has been made, although we already pointed out at the beginning of the previous chapter that this choice is not unique for such a fluid. We shall now discuss this issue in more details, because - as we shall see in section 3.5.3 - this choice is crucial in order to find out whether a spatial (i.e. orthogonal to *some* timelike vector) is GI or not.

In relativistic thermodynamics (Israel 1976 [64], de Groot *et al.* 1980 [16]) the arbitrary (non equilibrium) state of the fluid is described by the the energy momentum T^{ab}, the particle flux N^a and the entropy flux S^a; T_{ab} and S^a respectively satisfy the energy - momentum conservation laws and the second law of thermodynamics:

$$T^{ab}{}_{;b} = 0 \ , \quad S^a{}_{;a} \geq 0 \ ; \tag{3.48}$$

also N^a is a conserved current in appropriate circumstances: $N^a{}_{;a} = 0$. With the assumption that the energy density is non-negative, i.e. $T_{ab}V^aV^b \geq 0$ for all timelike V^a, T^{ab} has a unique timelike and unit eigenvector [119] (Synge 1964 [119])

$$u^a_E \ , \Leftrightarrow h^E_{ab} T^{bc} u^E_c = 0 \ ; \tag{3.49}$$

where, with obvious notation

$$h^E_{ab} = u^E_a u^E_b + g_{ab} \tag{3.50}$$

is the projector tensor orthogonal to u^a_E. This latter is not the only unit timelike vector one can define from the thermodynamic variables: indeed one may define u^a_N as the unit timelike vector parallel to N^a,

$$u^a_N = \frac{N^a}{\sqrt{-N_b N^b}} \ , \tag{3.51}$$

and

$$h^N_{ab} = u^N_a u^N_b + g_{ab} \ , \tag{3.52}$$

will be the relative projector tensor.

3.4.1 The perfect fluid case

When the fluid is in equilibrium or is perfect S^a, u^a_E and u^a_N are all parallel, defining a *unique* hydrodynamical 4-velocity u^a for the fluid flow, and an associated local Lorentz rest frame (LRF), together with the projector tensor $h_{ab} = g_{ab} + u_a u_b$ ($h_a{}^b u_b = 0$). In this case the decomposition of the three fundamental physical quantities in terms of u^a has a special status, u^a being the only timelike vector for which T_{ab} takes the usual perfect fluid form:

$$T_{ab} = \mu u_a u_b + p h_{ab} \ , \quad N^a = n u^a \ , \quad S^a = S u^a \ , \tag{3.53}$$

where $\mu = T_{ab}u^a u^b$ and $p = \frac{1}{3}h_{ab}T^{ab}$ are the energy density and pressure in the LRF of $\mathcal{O}_u$, and $n = -N_a u^a$ and $S = -S_a u^a$ are the particle density and entropy density; in any other frame n^a an energy flux $\tilde{q}_a = -\tilde{h}_a{}^b T_{bc} n^c$ and a particle drift $\tilde{j}^a = \tilde{h}^a{}_b N^b$ will appear in the above expressions [67] (King and Ellis 1973). It is usually assumed that (3.53) also hold in standard FLRW spacetimes (but see footnote 7), even if the fluid is not barotropic,[6] so that the matter four velocity u^a is *uniquely* defined in these universe models.

3.4.2 The imperfect fluid case

If the fluid is not perfect, the choice of the hydrodynamical 4-velocity u^a is *arbitrary*. The decomposition of T_{ab} with respect to this *arbitrary* u^a gives

$$T_{ab} = \mu u_a u_b + p h_{ab} + 2q_{(a}u_{b)} + \pi_{ab} \ , \tag{3.54}$$

where $q_a = -h_a{}^b T_{bc} u^c$ is the energy flux in the frame u^a ($q^a u_a = 0$), $\pi_{ab} = h_a{}^c h_b{}^d T_{cd} - \frac{1}{3}h_{ab}(h_{cd}T^{cd})$ is the anisotropic pressure in this frame and p includes here a possible contribution from bulk viscosity: $p = p_t + B$, $B = -\xi\Theta$ (p_t is the thermodynamic pressure). Decomposing N^a with respect to u^a one has

$$N^a = n u^a + j^a \ , \tag{3.55}$$

where $j^a = h^a{}_b N^b$ ($j^a u_a = 0$) is the particle drift in this frame.[7]

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[6] There can be an entropy production due to a non vanishing bulk viscosity; for a study of this class of models see e.g. Belinskiĭ *et al.* (1979)[4]. FLRW inflationary universes with bulk viscosity have also been considered (Padmanabhan and Chitre 1987 [99], Pacher *et al.* 1987 [100]).

[7] In a strict interpretation of the cosmological principle the fundamental observers don't only measure an isotropic universe using rods (this is the isotropy of the metric), but actually any physical phenomena must appears isotropic to them (this is the point of view of Weinberg 1973 [127]). However, even in an exact FLRW universe one can have a particle flux N^a tilted with respect to the frame u^a_E of the fundamental observers, although T_{ab} still has the perfect fluid form (3.53) (Ellis *et al.* 1983 [42]). This effect clearly violates the "strict" cosmological principle, as the fundamental observers in the EF can pick up a preferred direction through the particle drift j^a they measure. It is possible if the macroscopic quantities (such N^a) are derived from kinetic theory: imposing on the distribution function f only invariance with respect to the quasi-translations of the FLRW metric, one ends up with an homogeneous but anisotropic f; Einstein's equations then put constraints on some, but not all, the moments of f, so that one can have $j^a \neq 0$. This is an example, in the context of cosmology, of the known fact that Einstein's equations must by supplied by prescriptions on the matter content in order for the solution to be uniquely specified (e.g. see Feinstein *et al.* 1989 [45], for an example in a different context). The correct phenomenological description of this effect (which anyhow assumes a small tilt: $\beta^N_E \ll 1$) is not given by the standard phenomenological equations (using which one would

The two frames u^a_E and u^a_N have a special status: from its definition (3.49) in the frame u^a_E there is no energy flux ($q^a_E = -h^E_{ab}T^b{}_c u^c_E = 0$), while in the frame u^a_N there is not particle drift ($j^a_N = h^a_{Nb}N^b = 0$). For this reason the Landau - Lifshitz [74] (1963) choice of dynamical four velocity u^a_E is dubbed as the *energy frame* (EF), while the Eckart [23] (1940) kinematical choice is referred to as the *particle frame* (PF); hereafter indices E and N on a quantity shall mean that that quantity is defined either in the EF or in the PF.

In general, for an arbitrary non equilibrium state, there is not a relationship which interrelates the primary variables S^a, N^a and T^{ab}.[8] Any relativistic thermodynamics - hydrodynamics theory which attempts to characterize states of the viscous fluid by N^a and T^{ab} only is therefore limited to consider small deviations from local equilibrium: $(j^a_E/n, \pi^E_{ab}) \sim O_1$, which is equivalent to assuming $u^a_N - u^a_E \sim O_1$ (Israel 1976 [64]); this latter condition is clearly expressed by $\beta^N_E \ll 1$, where β^N_E is the tilt angle between u^a_E and u^a_N. As is shown in Israel (1976) [64], it is possible to formulate the thermodynamic theory in a *frame invariant* way, i.e. using an arbitrary four velocity u^a which has a small tilt $\beta_E \ll 1$ with the EF u^a_E such that the variables of the theory are invariant at first order in β under the change of frame $u^a \to n^a$

$$\mu \simeq \tilde{\mu}\ , \quad p \simeq \tilde{p}\ , \quad n \simeq \tilde{n}\ , \quad \pi_{ab} \simeq \tilde{\pi}_{ab}\ . \tag{3.56}$$

The energy flux q^a and the particle drift j^a instead change at first order, but one can use the frame invariant (at first order in β) combinations

$$q^a_N = q^a + (\mu + p)(u^a - u^a_N) = \tilde{q}^a + (\mu + p)(n^a - u^a_N)\ , \tag{3.57}$$

$$j^a_E = j^a + n(u^a - u^a_E) = \tilde{j}^a + n(n^a - u^a_E)\ , \tag{3.58}$$

i.e. the PF energy flux and the EF particle drift. We cannot use this frame freedom in formulating a GI perturbation theory based on covariant GI variables, since we need

immediately derive $j^a = 0$ from homogeneity), but by causal transport equations such those provided by Israel (1976) [64] [see equations (8) in his paper]. It follows from these that in general j^a decay with time: perhaps this effect can thus be interpreted as a decaying solenoidal ($j^a_{;a} = 0$) "perturbation" in the particle motion in a geometrically isotropic FLRW universe, although this "perturbation" will be very special having a non vanishing spatial average value. We shall in general assume that in our background $u^a_N = u^a_E$.

[8] This is clearly seen in the context of kinetic theory: N^a and T^{ab} are the first and second moment (in the momentum space) of the particle distribution function, and in general two moments only do not provide enough information to determine S^a.

not an arbitrary matter four velocity, but a four velocity which coincides with u^a_E in the background (the reason for this will be clarified in next section). Thus natural choices are either $u^a = u^a_E$ itself, or $u^a = u^a_N$ (if we exclude the models of Ellis *et al.*, see footnote 7). We note however that these are not the unique four velocities we can use to define covariant GI variables; for example we could take $u^a = \alpha u^a_E + (1-\alpha)u^a_N$ $(0 \leq \alpha \leq 1)$ as the hydrodynamical four velocity of the fluid, and this will again coincide with the EF in the background. Finally, we want to point out that in certain cases the preferred four velocity to be used is suggested by the problem itself, while it could be difficult to find the eigenvector u^a_E, or to define a particle flux N^a. An example of this is given by a scalar field ϕ: when ϕ is minimally coupled we can describe it as a perfect fluid with four velocity $u^a_{(\phi)} = \phi^a/\sqrt{-\phi^b\phi_b}$ which corresponds to the eigenvector of T^{ab} (Bruni, Ellis and Dunsby 1991 [10], see chapter 5); this choice is also the simplest in the case of non - minimal coupling, in which however it no longer coincides with u^a_E [although in both cases $u^a_{(\phi)} = u^a_E$ in a FLRW universe, (see Madsen 1988 [86]) so that this is a satisfactory choice in order to define covariant GI variables].

The EF choice was adopted by Bardeen (1980) and (1988) [1, 2], and by Kodama and Sasaki (1984) [69], while the PF has been used by Hwang and Vishniac (1990) [63] and by Dunsby (1991a) [18]; Hwang (1990, 1991) [59, 61] also uses the energy frame.

We shall not explicitly explore this possibility here, and we shall in general assume that the fluid four velocity is $u^a = u^a_N$, or any other choice such that in the background $u^a = u^a_E$, unless otherwise specified.

In the following, viscous fluid terms are treated as known functions: they can be eventually related to kinematical quantities through a phenomenological theory: within the cosmological perturbation theory the standard first - order theory (see e.g. Ellis 1971 and 1973 [27, 28]) will be usually a sufficient approximation.

3.5 Covariantly defined GI variables

In section 3.1 we have introduced two new GI variables, namely the density and expansion gradients $\mathcal{D}_a$ and $\mathcal{Z}_a$; in section 3.2 we have derived the exact non - linear equations governing their evolution in a generic spacetime. Our aim is however to restrict our attention to *almost* FLRW universes, and to derive linear equations for a larger set of covariant GI variables we shall consider shortly. We shall now characterize covariantly a FLRW model, and shall clarify the linearization procedure we shall apply to the exact equations presented so far.

3.5.1 Characterization of FLRW models

In order to derive linear equations for GI variables, we have to characterize the background spacetime we are going to use, i.e. FLRW universe models within the covariant approach.

Let $u^a = u^a_E$ be the fluid flow vector: an exact FLRW is covariantly characterized by the vanishing of the shear and the vorticity of u^a and by the vanishing of the spatial gradients (i.e. orthogonal to u^a) of any scalar f:

$$\sigma = \omega = 0 \ , \quad {}^{(3)}\nabla_a f = 0 \ ; \tag{3.59}$$

in particular the gradients of energy density, pressure and expansion vanish

$$X_a = 0 \ , \quad Y_a = 0 \ , \quad Z_a = 0 \ , \tag{3.60}$$

where $Y_a = 0 \Rightarrow a_a = 0$ and these models are spatially homogeneous and isotropic since there are no directions defined in the 3-space orthogonal to u_a. Then $\mu = \mu(t)$, $p = p(t)$ and $\Theta = \Theta(t) = 3H(t)$ depend only on the cosmic time t defined (up to a constant) by the FLRW fluid flow vector $u_a = -t_{,a}$. FLRW models are also conformally flat (their metric can be written as $g = \Omega\eta$, where η is the metric of flat space), i.e. the Weyl tensor vanishes, so that

$$E_{ab} = 0 \ , \quad H_{ab} = 0 \ . \tag{3.61}$$

The energy momentum tensor necessarily has the perfect fluid form (3.53), so that the anisotropic pressure π_{ab} vanishes, the energy flux q^a identically vanish given our choice $u^a = u^a_E$; if we exclude the tilted FLRW models of Ellis *et al.* (1983) [42] (see section 3.4) (3.53) hold, and $u^a_N = u^a_E$, so that also $q^a_N = 0$ (3.57) and $j^a_E = 0$ (3.58), and (3.59) and (3.60) also hold for quantities defined in the PF u^a_N.

It follows that these models are completely determined by an equation of state $p = p(\mu)$, the energy conservation

$$\dot{\mu} + 3\mu H(\mu + p) = 0 \ , \tag{3.62}$$

and the Friedmann equations

$$3\dot{H} + 3H^2 + \tfrac{1}{2}\kappa(\mu + 3p) - \Lambda = 0 \; , \tag{3.63}$$

$$H^2 + \tfrac{K}{a^2} = \tfrac{1}{3}\kappa\mu + \tfrac{1}{3}\Lambda \; , \tag{3.64}$$

where the latter is a first integral of the former (when $H \neq 0$); we note that within the approach to cosmology followed here, (3.63) and (3.64) are respectively (2.47) and (2.40) (with $\mathcal{K} = 6K/a^2$) for a homogeneous and isotropic spacetime.

Following a now standard notation [1] we define

$$w \equiv \tfrac{p}{\mu} \; , \qquad c_s^2 \equiv \tfrac{dp}{d\mu} = \tfrac{\dot{p}}{\dot{\mu}} \; , \tag{3.65}$$

$$\Rightarrow \dot{w} = -(1+w)(c_s^2 - w)\Theta \; , \tag{3.66}$$

where c_s^2 is the speed of sound (see also Eq. (3.76)).

If we do assume standard comoving coordinates, the metric of FLRW takes the form

$$ds^2 = -(u_0)^2(dx^0)^2 + h_{\alpha\beta}dx^\alpha dx^\beta \tag{3.67}$$

$$(u_0)^2(dx^0)^2 = dt^2 = a^2 d\eta^2 \; , \quad h_{\alpha\beta} = a^2\gamma_{\alpha\beta} \; , \tag{3.68}$$

where x^0 can be either t (proper time, $u_0 = -1$) or η (conformal time, $u_0 = -a$), and $\gamma_{\alpha\beta}$ is the metric of the 3-surfaces of constant curvature $K = 0, \pm 1$.

3.5.2 Linearization procedure

The variables we have been considering so far are exactly defined covariant quantities, thus they have a physical or geometrical meaning in an arbitrary spacetime. the set of exact covariant equations governing their evolution have been briefly reviewed in section 2.3, and in sections 3.1 and 3.2 we have introduced new variables and derived exact equations for them. However we want now restrict our attention to a real physical spacetime which is close to a FLRW universe: we shall therefore refer to such a spacetime as an *almost FLRW universe*. In other words, instead of starting from an exact FLRW model and perturb it in the standard way, we want to approach these universe models from a general spacetime, considering them as defining a subset of the whole space of solutions of Einstein equations surrounding (or containing) the smaller set of exact FLRW models.

How can we make this concept more precise? Following from the Stewart and Walker Lemma [112] (see section 1.3, page 29) and the covariant characterization of FLRW models given in the previous section, those of such variables that vanish in a FLRW model are GI. We can therefore speak of two subset of variables:

zero - order variables such as

$$\mu\ ,\quad p\ ,\quad \Theta = 3H\ ; \tag{3.69}$$

these are the variables that do not vanish in the FLRW background, i.e. in the spacetime that is the the zero - order approximation to the physical almost FLRW universe;

first order GI variables these are the variables that *do vanish* in the FLRW background: in considering them as *first - order* quantities, we automatically define the *almost* FLRW universes as the spacetimes in which these variables are non - vanishing, but terms quadratic in these variables are negligible; for example terms such as

$$\mathcal{D}_b\sigma^b{}_a\ ,\quad \mathcal{Z}_b\omega^b{}_a\ ,\quad (\sigma^2)_{;b}\ , \tag{3.70}$$

occurring in (3.15) and (3.16), and similar terms occurring in the exact equations of section 2.3.

Our aim is to derive now linear equations for these GI variables, therefore we have to establish a linearization procedure of the above mentioned exact non - linear equations. However, given the above characterization of our variables, such a procedure is trivial: *a)* variables such μ, p and Θ that always appear in the exact equations of sections 2.3 and 3.2 as coefficients of GI variables are needed only to zero - order as it is determined in the background FLRW model: they are treated as known functions in the equations; *b)* terms such (3.70) which are quadratic in the GI variables are dropped from the equations.

3.5.3 Covariant GI variables

We shall now consider in detail the whole set of covariant GI variables, also in the light of what we said in section 3.4 about the choice of frame, i.e. on the choice of a 4 - velocity for the fluid when this is viscous.

Spatial gradients and the choice of frame

It follows from (3.59) that the shear and vorticity of $u^a = u^a_E$ are GI variables, together with the spatial gradients orthogonal to it; in general, shear, vorticity and spatial gradients defined with respect to any timelike unit vector u^a that coincides with u^a_E in the FLRW spacetime vanish in this background, so they are GI as well. This clarifies the importance of the choice of the fluid 4-velocity.

In particular are GI the two key variables

$$\mathcal{D}_a = \tfrac{a}{\mu}{}^{(3)}\nabla_a\mu \ , \qquad \mathcal{Z}_a = a^{(3)}\nabla_a\Theta \ , \tag{3.71}$$

i.e. the comoving fractional density gradient and the comoving gradient of expansion; as we said in section 3.1 these quantities have a direct link with observations (Kristian and Sachs 1966 [72]). As we shall see, the analysis of density perturbations within the covariant approach is based on these two variables and their evolution equations.

To fix the ideas, suppose that $\mathcal{D}_a$ and $\mathcal{Z}_a$ in (3.71) are defined in the EF: with obvious notation the gradient in the EF and in the PF at linear order in β^N_E are related by

$${}^{(3)}\nabla^E_a f = {}^{(3)}\nabla^N_a f + \tfrac{\dot{f}}{\mu+p} q^N_a \tag{3.72}$$

on using (2.4); but from (3.57) $q^N_a/(\mu+p) = -{}^E V^N_a$, where ${}^E V^N_a = u^N_a - u^E_a$ is the GI relative velocity of the two frames, thus

$$\mathcal{D}^E_a = \mathcal{D}^N_a + 3aH(1+w)^E V^N_a \ , \tag{3.73}$$

and an analogous relation holds for $\mathcal{Z}_a$.[9]

The equations below for $\mathcal{D}_a$ (3.167) and $\mathcal{Z}_a$ (3.168) are derived on taking the gradient of the energy conservation (3.152) and Raychaudhuri equation (3.155). In doing this (more details on the derivation of these equations can be found in EB and EBH) the momentum equation (3.153) is involved, which implies the appareance of pressure gradient terms in the equations. However if the equation of state is $p = p(\mu, s)$, where

[9] Note that for a general frame n^a

$$\mathcal{D}^E_a = \widetilde{\mathcal{D}}_a + 3aH(1+w)(u^E_a - n_a) \ ,$$

where now $\widetilde{\mathcal{D}}_a$ and $\widetilde{V}_a = u^E_a - n_a$ *are not* GI; this relation is an hint to the significance of Bardeen's GI variable ε_m [see Eq. (4.15) in section 4.1].

s is the entropy density, we can substitute for the pressure gradient Y_a in terms of energy density and entropy gradients.

Defining

$$P_a = \frac{a}{p}{}^{(3)}\nabla_a p \ , \qquad \mathcal{E}_a = \frac{a}{p}\left(\frac{\partial p}{\partial s}\right)_{(\mu)}{}^{(3)}\nabla_a s \ , \tag{3.74}$$

we have

$$p\mathcal{E}_a = pP_a - c_s^2\mu\mathcal{D}_a \ , \tag{3.75}$$

where $\mathcal{E}_a$ is the entropy perturbation. Note that if the background is a standard FLRW universe with vanishing bulk viscosity B

$$c_s^2 = \frac{\dot{p}}{\dot{\mu}} = \left(\frac{\partial p}{\partial \mu}\right)_{(s)} \ , \tag{3.76}$$

and $\dot{s} = 0$ along flow lines [see (3.162)]; however if $B \neq 0$ in the FLRW background (see footnote 6) the second equality in (3.74), (3.76) do not holds, but we can still take (3.75) as a definition of the entropy perturbation including a bulk viscosity contribution. In the following $\mathcal{E}_a$ will be taken in general as defined by (3.75), and $c_s^2 = \dot{p}/\dot{\mu}$.

Curvature variables and the choice of frame

In the standard approach to perturbations one looks at the gravitational potentials, i.e. at metric perturbations (see section 4.1). Following Hawking (1966) [55] we focus instead on the curvature, i.e. on the Riemann tensor $R^a{}_{bcd}$, its trace $R_{ab} = R^c{}_{acb}$ (*Ricci*), and its trace-free part, i.e. the Weyl tensor C_{abcd} (2.24) representing the free gravitational field determined non locally by matter (see (2.29) and section 2.2.3). FLRW spacetimes are conformally flat, $C_{abcd} = 0$, so that the Weyl tensor is GI; we may note in this respect an interesting difference with spatial gradients, shear and vorticity: while the latter are GI only when orthogonal to a unit timelike vector that coincide with u^a_E in the background, the Weyl tensor is GI in any frame. This means that any possible decomposition of C_{abcd} gives GI variables:[10] in particular the electric and magnetic parts of the Weyl tensor are covariant GI variables

$$E_{ab} = C_{acbd}u^c u^d \ , \quad H_{ab} = \frac{1}{2}C_{acst}u^c\eta^{st}{}_{bd}u^d \ , \tag{3.77}$$

where u^a here is *any completely arbitrary* timelike unit vector. We now fix $u^a = u^a_E$ in (3.77), so that the above defined E_{ab} and H_{ab} are the electric and magnetic part of the

[10] For example, we may consider the Weyl scalars defined in a proper tetrad of vectors (as in the Newman - Penrose formalism, see Chandrasekhar 1983 [13], Kramer *et al.* 1980 [71]) as GI variables.

Weyl tensor in the EF: then if $\tilde{E}_{ab}$ and $\widetilde{H}_{ab}$ are defined in any other arbitrary frame n^a (any other timelike unit vector) we get, for a small tilt angle β

$$\tilde{E}_{ab} \simeq E_{ab}\ , \quad \widetilde{H}_{ab} \simeq H_{ab}\ , \tag{3.78}$$

i.e. these quantities are also *frame invariant* to first order in β: while E_{ab} represents a purely tidal force, H_{ab} has not Newtonian counterpart [see Ellis (1971)[27] and section 4.2].

We have already seen in section (3.4) that the anisotropic pressure is frame invariant, $\pi_{ab} \simeq \tilde{\pi}_{ab}$: vanishing in a FLRW model is also GI. Using the Einstein equations we have

$$\kappa\pi_{ab} = h_a{}^c h_b{}^d R_{cd} - \tfrac{1}{3} h_{ab} h_{cd} R^{cd}\ , \tag{3.79}$$

therefore π_{ab} may be regarded as an anisotropic part of the gravitational field.

Given a unit timelike vector u^a, we can define at each point 3-curvature tensor ${}^{(3)}R^a{}_{bcd}$ and its trace ${}^{(3)}R_{ac} \equiv {}^{(3)}R^b{}_{abc} = {}^{(3)}R_a{}^b{}_{cb}$ as we did in section 3.3. Let u^a be the fluid four velocity: for the purpose of defining GI variables we can split ${}^{(3)}R_{ab}$ into its trace ${}^{(3)}R$ (3.35) and its GI trace-free part

$${}^{(3)}\mathcal{R}_{ab} = -H(\sigma_{ab} + \omega_{ab}) + E_{ab} + \tfrac{1}{2}\kappa\pi_{ab} - \tfrac{2}{3} h_{ab}(\sigma^2 - \omega^2) + Q_{ab} \tag{3.80}$$

where

$$Q_{ab} = \sigma_{ac}\sigma^c{}_b + \omega_{ac}\omega^c{}_b + \omega_{ac}\sigma^c{}_b - \omega_{bc}\sigma^c{}_a\ . \tag{3.81}$$

Note that the last three relations are exact, i.e. these are definitions valid in any spacetime; in an almost FLRW universe we shall neglect in these expressions terms quadratic in the GI variables, i.e. in the shear and vorticity, while we shall retain terms such as H to zero order (the background value): thus this is an explicit example of the linearization procedure outlined above.

${}^{(3)}R$ is a GI variable only if the background FLRW is flat: we can define a GI variable from it on taking its gradient (3.47); after substitution of R_{ab} with T_{ab} from the Einstein equations in (3.35), we get

$${}^{(3)}R = 2(-\tfrac{1}{3}\Theta^2 + \sigma^2 - \omega^2 + \kappa\mu + \Lambda)\ . \tag{3.82}$$

Thus to linear order we obtain [11]

$$C_a = a^3\, {}^{(3)}\nabla_a\, {}^{(3)}R = -4a^2 H \mathcal{Z}_a + 2\kappa\mu a^2 \mathcal{D}_a\ . \tag{3.83}$$

[11] This is also equivalent to taking the gradient of (3.156).

Thus C_a may be regarded as a supplementary quantity for $\mathcal{D}_a$ and $\mathcal{Z}_a$, and plays in our formalism the same role that δk does in Lyth and Mukherjee [83] (compare their equation (23) and (3.96) below).

We find also convenient to define

$$\tilde{C}_a = C_a - \tfrac{4K}{1+w}\mathcal{D}_a \; ; \tag{3.84}$$

this turns out to be conserved at large scales in a more general set of circumstances than C_a, and reduces to this latter for $K = 0$.

We shall see in the next chapter how the variables presented in this section are related to Bardeen's metric potentials Φ_A and Φ_H.

3.6 Linear evolution equations for spatial gradients

3.6.1 Linear equations: a first step

Using the linearization procedure outlined in section 3.5.2 we find from (3.15), (3.16) and (3.83)

$$\dot{\mathcal{D}}_a = w\Theta\mathcal{D}_a - (1+w)\mathcal{Z}_a \; , \tag{3.85}$$

$$\dot{\mathcal{Z}}_a = -\tfrac{2}{3}\Theta\mathcal{Z}_a - \tfrac{1}{2}\kappa\mu\mathcal{D}_a + a\left(\tfrac{3K}{a^2}a_a + A_a\right) \; , \tag{3.86}$$

$$\dot{C}_a = \tfrac{6K}{a^2}\,\Theta^{-1}\left(\tfrac{1}{2}C_a - \kappa\mu a^2\mathcal{D}_a\right) - \tfrac{3}{4}\Theta a^3\left(\tfrac{3K}{a^2}a_a + A_a\right) \; , \tag{3.87}$$

where the covariant derivatives (implied by the superscript dot) may all be taken in the background (zero-order) model, and $A_a = {}^{(3)}\nabla_a(a^c{}_{;c})$ is the gradient of the acceleration divergence. From the definition (2.43) of A_a and the momentum conservation equation (2.46) we see that to first order

$$A_a = -\,\frac{{}^{(3)}\nabla_a\,{}^{(3)}\nabla^2 p}{(\mu+p)} \;=\; -c_s^2\frac{{}^{(3)}\nabla_a{}^{(3)}\nabla^b\mathcal{D}_b}{a(1+w)} \; , \tag{3.88}$$

where we use the notation ${}^{(3)}\nabla^2 \equiv {}^{(3)}\nabla_b{}^{(3)}\nabla^b$, the second equality following from (3.66) and the assumption of adiabatic evolution used throughout this section (which implies the perturbation is adiabatic).

The dynamics of our basic variable, $\mathcal{D}_a$, is given by (3.85) in combination with one of the two equations (3.86, 3.87) (on using (3.87), one should trivially substitute for

$\mathcal{Z}_a$ in (3.85) from (3.83)), or by the linear second order equation which follows directly from (3.85) (3.86) using (3.88). This equation is

$$\ddot{\mathcal{D}}_a + \mathcal{A}(t)\dot{\mathcal{D}}_a - \mathcal{B}(t)\mathcal{D}_a - c_s^2\,{}^{(3)}\nabla_a({}^{(3)}\nabla^b\mathcal{D}_b) = 0 \;, \tag{3.89}$$

where the coefficients

$$\mathcal{A}(t) = (\tfrac{2}{3} - 2w + c_s^2)\Theta \;, \tag{3.90}$$

$$\mathcal{B}(t) = (\tfrac{1}{2} + 4w - \tfrac{3}{2}w^2 - 3c_s^2)\kappa\mu + (c_s^2 - w)\tfrac{12K}{a^2} + (5w - 3c_s^2)\Lambda \tag{3.91}$$

are determined from the background model. This form of the equations allows for a variation of $w = p/\mu$ with time. However if $w = const$, then from (3.66), $c_s^2 = w$, and the coefficients simplify to

$$\mathcal{A}(t) = (\tfrac{2}{3} - w)\Theta, \quad \mathcal{B}(t) = \tfrac{1}{2}(1 - w)(1 + 3w)\kappa\mu + 2w\Lambda \;. \tag{3.92}$$

3.6.2 The inhomogeneous equations: the vorticity source terms

The problem with (3.89) is that although it is a homogeneous equation, the last term is in an awkward form. We wish to commute the derivatives to bring the equation to a more standard form, with the spatial Laplacian acting on $\mathcal{D}_a$. When the vorticity is non-zero, there are no surfaces in space-time orthogonal to the fluid flow, and consequently these partial derivatives do not commute; rather, from (3.24) we get

$${}^{(3)}\nabla_b{}^{(3)}\nabla_a p - {}^{(3)}\nabla_a{}^{(3)}\nabla_b p = 2\omega_{ab}\dot{p} = -2c_s^2(1 + w)\mu\Theta\omega_{ab}. \tag{3.93}$$

From this it follows that

$${}^{(3)}\nabla_a({}^{(3)}\nabla^b\mathcal{D}_b) = \left(-\tfrac{2K}{a^2} + {}^{(3)}\nabla^2\right)\mathcal{D}_a + 2\Theta(1 + w)a\,{}^{(3)}\nabla^b\omega_{ab} \;. \tag{3.94}$$

First - order equations

The final linear first-order equation we obtain from (3.86) (3.87) are

$$\dot{\mathcal{Z}}_a = -\tfrac{2}{3}\Theta\mathcal{Z}_a - \tfrac{1}{2}\kappa\mu\mathcal{D}_a - \tfrac{c_s^2}{(1+w)}\left(\tfrac{K}{a^2} + {}^{(3)}\nabla^2\right)\mathcal{D}_a - 2c_s^2 a\Theta\,{}^{(3)}\nabla^b\omega_{ab} \;, \tag{3.95}$$

$$\dot{C}_a = \tfrac{6K}{a^2}\Theta^{-1}\left(\tfrac{1}{2}C_a - \kappa\mu a^2\mathcal{D}_a\right) + \tfrac{4}{3}\Theta a^2\tfrac{c_s^2}{1+w}\left(\tfrac{K}{a^2} + {}^{(3)}\nabla^2\right)\mathcal{D}_a + \tfrac{8}{3}c_s^2 a^3\Theta^2\,{}^{(3)}\nabla_b\omega_a{}^b \;. \tag{3.96}$$

The last term on the RHS of (3.95), (3.96) is the (first-order) term (omitted in EHB) giving the effect of vorticity on the expansion and curvature gradients, and so on the density gradient, measured by a fundamental observer.

The second - order equation

The final version of the linearised second order equation (3.89) follows directly from (3.85), (3.95) or from (3.89), (3.94). We find

$$\ddot{\mathcal{D}}_a + \mathcal{A}(t)\dot{\mathcal{D}}_a - \mathcal{B}(t)\mathcal{D}_a + \mathcal{L}(t)\mathcal{D}_a - \mathcal{C}(t)\,{}^{(3)}\nabla^b\omega_{ab} = 0 \;, \tag{3.97}$$

where the coefficients $\mathcal{A}(t)$, $\mathcal{B}(t)$ (given by (3.90)-(3.92)),

$$\mathcal{C}(t) = c_s^2 2a(1+w)\Theta \;, \tag{3.98}$$

and the operator

$$\mathcal{L}(t) = c_s^2\{\tfrac{2K}{a^2} - {}^{(3)}\nabla^2\} \tag{3.99}$$

are determined from the background model. The last term in (3.97) is the extra term due to (3.93), (3.94), omitted in EHB. The key point here is that it is the operator $\mathcal{L}(t)$ that determines the effective wavelength of density inhomogeneities through its eigenvalues and eigenfunctions (unlike the last term on the right hand side of (3.89)).

3.6.3 The interaction

The linking of vorticity to the time-evolution of the density gradient is through the 3-divergence of the vorticity *tensor* (i.e. the 3-curl of the vorticity vector). The effect is non-zero provided ${}^{(3)}\nabla^c\omega_{ac} \neq 0$; if vorticity is non-zero but this divergence vanishes, there is necessarily a density gradient associated with the vorticity, but the *growth* of this gradient is unaffected by the extra term. The same divergence is related to the 3-divergence of the shear and the 3-gradient of the expansion through the $(0,\nu)$ constraint equation (2.50) in its linearised form

$${}^{(3)}\nabla^b\omega_{ab} - {}^{(3)}\nabla^b\sigma_{ab} + \tfrac{2}{3}Z_a = 0 \;. \tag{3.100}$$

This already shows that the vorticity and density gradients are linked in the linear approximation, because expansion and density gradients are intimately related (see (3.85) (3.86) above). Equation (3.100) restricts how initial data can be chosen, while the extra term in (3.97) shows how (consistent with this) there is a direct effect of vorticity, in the linear approximation, as a source of growth of density gradients. However the coefficient $\mathcal{C}(t)$ of this extra term vanishes if the speed of sound is zero, or the universe is static.[12] The term does not occur in the case of Newtonian theory, because in that theory the hyperplanes orthogonal to the fluid flow are always tangent to hypersurfaces of absolute time, and are therefore integrable; so the equations in [32] correctly include the case of combined vorticity and density perturbations.

Equation (3.41) show that, to linear order, the total divergence of the extra term in (3.97) vanishes:

$$ {}^{(3)}\nabla^a({}^{(3)}\nabla^b \omega_{ab}) = 0 \ . \tag{3.101}$$

This means that the contribution the extra term induces in the density gradient will have vanishing spatial divergence: ${}^{(3)}\nabla^a \mathcal{D}_a = 0$ (if this divergence and its first derivative vanish initially, the effect of the vorticity term is to leave it zero, see equation (3.120) below), so the induced growth of inhomogeneity in one direction will be compensated by a lessening in other directions. The geometric meaning of this result will be discussed below.

The growth of the vorticity source term

At first glance this extra term seems to imply that the equations do not close at the second order anymore, because the covariant derivative of the vorticity along the fluid flow lines involves the shear [111, 49]. However this is not the case because vorticity propagation decouples in the linear approximation [67], so we can (to this level of accuracy) determine the evolutionary behaviour of the extra term.[13] In more detail, a perfect fluid with $p = w\mu$, $w = w(\mu)$ (see (3.66)) has an acceleration potential r [25, 28], where

$$ r = \exp \int_{p_i}^{p} \frac{dp}{\mu(1+w)} \ , \tag{3.102}$$

and the vorticity evolution equation (in the linear approximation) is

$$ \dot{\omega}_{ac} + \tfrac{2}{3}\Theta\omega_{ac} = {}^{(3)}\nabla_{[c}a_{a]} = -\tfrac{1}{\mu+p}{}^{(3)}\nabla_{[c}{}^{(3)}\nabla_{a]}p = -\tfrac{\dot{r}}{r}\omega_{ac} \ . \tag{3.103}$$

Thus

$$ (a^2 r \omega_{ac})^{\cdot} = 0 \ . \tag{3.104}$$

[12] Or the universe has the exceptional (inflationary) equation of state $w = -1$, when a perfect fluid description is not really valid: see chapter 5.

[13] This can be extended to the non-linear case if the propagation equation is written in terms of the Lie-derivative rather than covariant derivative along the fluid flow lines.

When $w = constant$, $\mu = M_1/a^{3(1+w)}$, $M_1 \equiv \mu_o a_o^{3(1+w)}$, $\dot{M}_1 = 0$, where a_o is the present value of the scale factor, and the acceleration potential is

$$r = \left(\frac{p}{p_i}\right)^{\frac{w}{1+w}} \quad \Rightarrow \quad r = \left(\frac{M_1 w}{p_i}\right)^{\frac{w}{1+w}} a^{-3w} \ . \tag{3.105}$$

Hence

$$\omega_{ac} = \frac{\Omega_{ac}}{a^{2-3w}} \ , \quad \dot{\Omega}_{ac} = 0 \ , \quad \Omega_{ac} = \Omega_{[ac]} \tag{3.106}$$

where all multiplying constants are now included in $\Omega_{ac} = r a^2 \omega_{ac} (\frac{M_1 w}{p_i})^{-\frac{w}{1+w}}$. Finally the equations (3.43) - (3.46) show that $(a\,^{(3)}\nabla^a)$ is the orthogonal derivative operator which, acting on a purely spatial first-order tensor that is covariantly constant along the fluid flow lines, preserves time independence. In particular, to first order, the divergence term obeys

$$(a\,^{(3)}\nabla^b \omega_{ab})^{\cdot} = a\,^{(3)}\nabla^b(\dot{\omega}_{ab}) \ , \tag{3.107}$$

thus

$$^{(3)}\nabla^a \omega_{ac} = \frac{(a^{(3)}\nabla^a \Omega_{ac})}{a^{3(1-w)}} \ , \quad (a\,^{(3)}\nabla^a \Omega_{ac})^{\cdot} = 0 \ , \quad ^{(3)}\nabla^c (a\,^{(3)}\nabla^a \Omega_{ac}) = 0 \tag{3.108}$$

(the last condition expressing the vanishing-divergence property (3.101)). The nature of the interaction depends (a) on the equation of state, and (b) on the initial value $^{(3)}\nabla^a \Omega_{ac}$ of the spatial vorticity divergence; the interaction term always decays as the universe expands, if $w < 1$.

As a simple example, in the case of a flat background with vanishing cosmological constant ($k = 0$, $\Lambda = 0$), the scale factor of the background model obeys

$$a = (\beta t)^{\frac{2}{3(1+w)}} \ , \qquad \beta \equiv \tfrac{3}{2}(1+w)\sqrt{\frac{\kappa}{3} M_1} \tag{3.109}$$

where t is proper time along the fluid flow lines, and so the vorticity goes as

$$\omega_{ab} = \Omega_{ab}\, (\beta t)^{-\frac{2(2-3w)}{3(1+w)}} \ . \tag{3.110}$$

The divergence goes as

$$^{(3)}\nabla^c \omega_{ac} = \Omega_a\, (\beta t)^{-2\frac{1-w}{1+w}} \ , \tag{3.111}$$

where we defined

$$\Omega_a \equiv a\,^{(3)}\nabla^c \Omega_{ac} \ , \quad \dot{\Omega}_a = 0 \ , \quad ^{(3)}\nabla^a \Omega_a = 0 \ . \tag{3.112}$$

3.7 Invariant decomposition

The standard harmonic representation [1, 72], which combines the ADM tensor decomposition with a (logically independent) harmonic analysis, should be regarded with some reservations because it is non-local, whereas the physics we are concerned with is essentially local. We present an alternative local decomposition and discuss its geometrical meaning. While it is not necessary to introduce the usual harmonic decomposition to derive our equations, it is instructive to consider how they relate to harmonic representations and the ADM decomposition.

3.7.1 The ADM decomposition

This non-local splitting can be applied to vectors and second-rank tensors in a standard manner [130, 15, 117]. In the case of a vector field V_a, independent of any Fourier analysis, it represents V_a in terms of "scalar" and "vector" parts $\tilde{\nabla}_a\phi$, B_a relative to a chosen family of 3-surfaces:

$$V_a = \tilde{\nabla}_a\phi + B_a \ , \quad \tilde{\nabla}^a B_a = 0 \ , \tag{3.113}$$

where $\tilde{\nabla}_a$ is the covariant derivative in these 3-spaces. If appropriate boundary conditions are satisfied [117] (which could be problematic if the background model has $k = 0$), and $k \neq -1$, then B_a is unique and ϕ unique up to a constant [15]. As the first term has vanishing curl but non-vanishing 3-divergence, whereas the second has vanishing 3-divergence (it is 'solenoidal'), if we take the 3-divergence of V_a we obtain an equation involving only the first term, while if we take its curl we obtain an equation involving only the second.

We shall return on this standard way of splitting tensors and its relation with harmonic analysis in section 4.1. We now turn to an alternative (local) decomposition whose meaning is more immediate.

3.7.2 A local decomposition

The spatial variation of the density (orthogonal to the fluid flow) is characterised by $\mathcal{D}_a$. A unique local splitting can be attained by considering the spatial derivative of this vector (multiplied by the scale factor a for convenience), and splitting this derivative into parts in analogy with (2.19):

$$a\,{}^{(3)}\nabla_b\mathcal{D}_a \equiv \Delta_{ab} = W_{ab} + \Sigma_{ab} + \tfrac{1}{3}\Delta h_{ab} \ , \tag{3.114}$$

where

$$W_{ab} \equiv \Delta_{[ab]} \ , \quad \Sigma_{ab} \equiv \Delta_{(ab)} - \tfrac{1}{3}\Delta h_{ab} \ , \quad \Sigma_{ab} = \Sigma_{(ab)} \ , \quad \Sigma^a{}_a = 0 \ . \tag{3.115}$$

The skew-symmetric part is

$$W_{ab} = \tfrac{a^2}{\mu}\,{}^{(3)}\nabla_{[b}{}^{(3)}\nabla_{a]}\mu = \tfrac{a^2}{\mu}\omega_{ab}\dot{\mu} = -a^2(1+w)\Theta\omega_{ab} \ , \tag{3.116}$$

where we neglect a second order term from ${}^{(3)}\nabla_a a$. This skew part by itself represents spatial variation of $\mathcal{D}_a$ in which its magnitude is preserved (i.e. rotations of this vector), e.g. that associated with the "tilt" of the fluid flow vector relative to the surfaces of constant density in homogeneous universes (i.e. the velocity of the matter relative to these surfaces). Thus although the associated density gradients exist and are observable [49, 67, 34] they are essentially *dipole-like* in character and are not directly associated with formation of local inhomogeneities.

By contrast, the spatial divergence

$$\Delta \equiv \Delta^a{}_a = a\,{}^{(3)}\nabla^a\mathcal{D}_a = \tfrac{a^2}{\mu}\,{}^{(3)}\nabla^2\mu \tag{3.117}$$

by itself is related to spherically symmetric spatial variation of μ where density is accumulated, i.e. to *spatial aggregation of matter* that we might expect to reflect existence of high-density structures in the universe. Finally, the trace-free symmetric part

$$\Sigma_{ab} = a\,{}^{(3)}\nabla_{(b}\mathcal{D}_{a)} - \tfrac{1}{3}\Delta h_{ab} \tag{3.118}$$

by itself is associated with spatial variations of $\mathcal{D}_a$ which do not represent spatial clumping of matter (as the associated divergence of $\mathcal{D}_a$ is zero) but rather represent change in the spatial anisotropy pattern of this gradient field. This seems to be what one might associate with existence of *pancake-like* or *cigar-like* structures.

A general pattern of inhomogeneity will have all the components (3.116) - (3.118) non-zero, for example implying aggregation ($\Delta > 0$) in a pancake-like structure ($\Sigma_{ab} \neq 0$) and with turbulence present ($W_{ab} \neq 0$).

Evolution equations

Now the evolution equations for these quantities follow from (3.97) and their definitions.

(i) Antisymmetric part :

Taking $a\,{}^{(3)}\nabla_b$ of (3.97) and antisymmetrizing over indices $[b, a]$, gives

$$\dot{W}_{ab} - \left(w\Theta + \frac{\dot{\Theta}}{\Theta} \right) W_{ab} = 0 \; . \tag{3.119}$$

Since $W_{ab} \propto \omega_{ab}$, this is equivalent to the vorticity conservation equation (3.103) (it is clear this must be so from (3.116)). Thus, we check the consistency of our equations, and so confirm the form (3.97) of the effect of the vorticity on the density anisotropy. This equation is the law controlling the dipole part of the observed density gradient, e.g. through governing the way the tilt angle of the surfaces of constant density in a Bianchi universe changes with time (see equations (1.17), (1.31) and (1.32) in [67]).

(ii) Symmetric part :

(ii-a) Trace:

Take the divergence of (3.97), keeping only the linear terms that arise. While the divergence of the vorticity term is non-zero, it is second order (see section 3.3 and (3.101)), so to linear order we obtain

$$\ddot{\Delta} + \mathcal{A}(t)\dot{\Delta} - \mathcal{B}(t)\Delta - c_s^2\,{}^{(3)}\nabla^2\Delta = 0 \; . \tag{3.120}$$

for the scalar $\Delta \equiv a\,{}^{(3)}\nabla^b\mathcal{D}_b$. This is like (3.97) except that (a) it is a scalar equation (for Δ), (b) the linear operator $\mathcal{L}(t)$ is replaced by a simpler Laplacian term, and (c) the vorticity term does not appear. Thus we attain a "scalar mode" equation (See Woszczyna and Kulak [129] for a similar result). independent of the vorticity source term. That part of the density evolution relating to spherical aggregation of matter (and so to growth of local density inhomogeneities) is expressed in this equation (equivalent to the Bardeen [1] scalar harmonic equation, see [63]): therefore, there is no contradiction between the presence of the vorticity source term in equation (3.97) and the standard results of cosmological perturbation theory in which only scalar modes contribute to describing density clumping.

(ii-b) Trace - free symmetric part :

We now take the symmetric, trace-free part of the spatial gradient of equation (3.97), finding

$$\ddot{\Sigma}_{ab} + \mathcal{A}(t)\dot{\Sigma}_{ab} \; - \mathcal{B}(t)\Sigma_{ab} \; + \; \tilde{\mathcal{L}}(t)\Sigma_{ab} \; + \; \mathcal{C}(t)\,{}^{(3)}\nabla_{(b}{}^{(3)}\nabla^c\omega_{a)c} = 0 \; , \tag{3.121}$$

where

$$\tilde{\mathcal{L}}(t) = c_s^2 \left\{ \tfrac{6K}{a^2} - \nabla^2 \right\} . \tag{3.122}$$

This equation (similar to (26)) governs the growth of pancake-like or cigar-like density inhomogeneities, because it will alter Σ_{ab} in time. The effect of vorticity in this equation will be non-zero provided the initial conditions satisfy ${}^{(3)}\nabla_{(b}{}^{(3)}\nabla^c\Omega_{a)c} \neq 0$; and there seems to be no reason why this term should vanish, in general.

Alternatively, operating by $a\,{}^{(3)}\nabla_b$ on equation (3.89), symmetrizing on the indices (b, a), and taking the trace-free part, we have

$$\ddot{\Sigma}_{ab} + \mathcal{A}(t)\dot{\Sigma}_{ab} - \mathcal{B}(t)\Sigma_{ab} - c_s^2 \left[{}^{(3)}\nabla_{(b}{}^{(3)}\nabla_{a)} - \tfrac{1}{3}h_{ab}{}^{(3)}\nabla^2 \right] \Delta = 0 \tag{3.123}$$

giving a form of the equations for Σ_{ab} that does not explicitly contain the vorticity. Instead there is an inhomogeneous source term involving gradients of Δ. This means that if we know the evolution of Δ (and its spatial variation) from (3.120), we do not need to explicitly introduce the vorticity term; we have enough information to find the evolution of Σ_{ab}. However it is simpler to use the form (3.121) that explicitly refers to the vorticity (because of the vorticity conservation equations discussed above). The kinematic and physical effects described by these two forms of the equation are of course the same.

We do not necessarily need to specifically consider equations (3.119) - (3.121), for all the information we need is in the original equation (3.97). However if we do wish to further reduce our equations, by contrast with applying a non-local decomposition (3.113) to them, the above procedure is (a) locally well-defined and (b) independent of large-scale boundary conditions which may or may not be satisfied in the real universe.

3.7.3 Harmonic analysis

If we apply a harmonic analysis, we can do so either to the full equation (3.97) or to the set of derived equations (3.119) - (3.121). The basic point is to expand every quantity in terms of eigenfunctions of the Helmholtz equations

$${}^{(3)}\nabla^2 Q^{(0)} + \frac{k^2}{a^2}Q^{(0)} = 0\,, \quad {}^{(3)}\nabla^2 Q_b^{(1)} + \frac{k^2}{a^2}Q_b^{(1)} = 0\,, \quad {}^{(3)}\nabla^2 Q_{bc}^{(2)} + \frac{k^2}{a^2}Q_{bc}^{(2)} = 0\,, \tag{3.124}$$

obtaining effective wavelengths from the eigenvalues. Following BI, superscripts $^{(0)}$, $^{(1)}$ and $^{(2)}$ denote respectively scalar, vector and tensor harmonics; a sum of some kind is understood in the Fourier expansion of any quantity. From $Q^{(0)}$ and $Q^{(1)}$ we can define in a standard way a vector $Q^{(0)}_a$ and tensors $Q^{(0)}_{ab}$ and $Q^{(1)}_{ab}$, to be used in decomposing general vectors and tensors. All these definitions and useful relations for the harmonics defined by (3.124) are provided in appendix B. The harmonics Q we introduce here are covariantly defined, using the ${}^{(3)}\nabla_a$ derivatives, in order to be covariantly constant along flow lines

$$\dot{Q}^{(0)} = 0\ , \quad \dot{Q}^{(1)}_a = 0\ , \quad \dot{Q}^{(2)}_{ab} = 0\ . \tag{3.125}$$

Instead, the harmonics usually employed (we shall denote them as Y) are independent of coordinate time (see section 4.1); in appendix B we provide the relationship between the standard harmonics Y and the covariant harmonics Q used here.

If we apply the harmonic analysis to (3.120), we obviously get

$$\ddot{\Delta} + \mathcal{A}(t)\dot{\Delta} - \mathcal{B}(t)\Delta + c_s^2\,\frac{k^2}{a^2}\Delta = 0\ , \tag{3.126}$$

where $\Delta_{(k)}$ denotes the k component of Δ. A final remark: whatever splitting or harmonic analysis is applied to the propagation equation (3.97) should also be applied to the constraint equation (3.100), where the same vorticity term occurs; (indeed the ADM splitting was precisely developed to analyse the constraint equations [130], see [15] for the cosmological case). We will again find a linking of vorticity to density gradients, but this time in terms of initial data.

3.8 Large scale evolution

No difference arises from the vorticity term in the case of pressure-free matter (EB [35]), rotation-free matter (EHB [37]), or the Newtonian limit [32]. The new term takes effect when $(\mu + p)c_s^2 \neq 0$, $\Theta \neq 0$, and the fluid is rotating with ${}^{(3)}\nabla^a\omega_{ab} \neq 0$. This is the generic case for a fluid with non-vanishing pressure; that is, the new term will almost always have a physical effect. However, the homogeneous (source-free) solutions are unaltered, so the speed of sound is unaltered. Before to find out the effects of the vorticity term on the evolution of $\mathcal{D}_a$ (this term can conceivably dominate the equations: presumably this will only occur under conditions of extreme turbulence) we turn now on the the Jeans length criterion of stability we can derive from the homogeneous part of our second order equation (3.97) for $\mathcal{D}_a$.

3.8.1 Jeans instability

To determine explicitly the solutions of the second-order equations we have obtained, we have to substitute for μ, Θ and S from the zero-order equations.

Speed of sound

We can examine solutions in the case where the divergence term is the dominant term, by examining the case where Θ, $\kappa\mu$, k/S^2 and Λ can be neglected. We see then directly from (3.97) that c_s introduced above *is* the speed of sound (and that imaginary values of c_s, that is, negative values of $dp/d\mu$, lead to exponential growth or decay rather than oscillations).

Instability criterion

The Jeans' criterion is that gravitational collapse will tend to occur if the combination of the matter term and the divergence term in (3.97) or (3.120) is positive. Restricting our attention to the case $w = const$ ($c_s^2 = w$) and $\Lambda = 0$, we have from (3.97), (refeq:coeff3) for $\mathcal{D}_a$

$$\tfrac{1}{2}(1-w)(1+3w)\kappa\mu\mathcal{D}_a > \ w\left(\tfrac{2K}{a^2}\mathcal{D}_a - {}^{(3)}\nabla^2\mathcal{D}_a\right) \ , \tag{3.127}$$

where we include the curvature term also, because it comes from the divergence term A_a; for Δ from (3.120) we have

$$\tfrac{1}{2}(1-w)(1+3w)\kappa\mu\Delta > \ -w^{(3)}\nabla^2\Delta \ . \tag{3.128}$$

Using the harmonic decomposition, both these relations can be expressed in terms of an equivalent scale: from (3.126), gravitational collapse tends to occur for a mode $\Delta^{(k)}$ if

$$\tfrac{1}{2}(1-w)(1+3w)\kappa\mu > \ w\tfrac{k^2}{a^2} \ , \tag{3.129}$$

that is

$$\left\{(1-w)\left(\tfrac{1}{w}+3\right)\tfrac{\kappa\mu}{2}\right\}^{1/2} \ > \tfrac{k}{a} \ . \tag{3.130}$$

In terms of wavelengths, the Jeans' length is defined by

$$\lambda_J \equiv \frac{2\pi a}{k_J} = c_s\sqrt{\pi}[G\mu(1-w)(1+3w)]^{-\frac{1}{2}} \tag{3.131}$$

where we have expressed the result in terms of the usual gravitational constant G. Thus gravitational collapse will occur for small k (wavelengths longer than λ_J), but not for sufficiently large k (wavelengths less than λ_J), for the pressure gradients are then large enough to resist the collapse and lead to oscillations instead (cf. Jackson [65], but his answer appears to be in error; we here present a corrected version of his result).

3.8.2 Long-wavelength solutions

Suppose we can ignore the "Laplacian part" of the second-order equation, that is $\mathcal{L}(t)\mathcal{D}_a$ can be ignored relative to the other terms in (3.97) (and consequently the Laplacian term in (3.120) can also be ignored). We shall call this the *long-wavelength limit*. This does not necessarily mean we can ignore the term A_a in our equations, for (3.94) (3.99) show that now we can have

$$^{(3)}\nabla_a\Delta \;\simeq\; 2\Theta a^2(1+w)\,^{(3)}\nabla^b\omega_{ab}\;, \tag{3.132}$$

a spatial gradient in Δ occurring in conjunction with the vorticity source term. Thus in general we cannot assume we can ignore the latter in the long wave-length limit, but only $^{(3)}\nabla^2\Delta$. In this limit, (3.120) becomes an ordinary homogeneous differential equation; with the solution of the latter, we can then consistently integrate (3.89), (3.123), or use the vorticity law (36) to integrate (3.97), (3.121), neglecting (3.122) (and so effectively using (3.132)). However in this section we prefer to solve for $\mathcal{D}_a$ through the system of first order equations introduced in section 3.6.

A conserved quantity on large scale

While the curvature variable C_a introduced previously (see (3.83)) is a geometrically natural quantity which is useful in discussing the long-wavelength limit, it turns out [63] that $\widetilde{C}_a$ (3.84) is physically significant because it is conserved in a more general set of circumstances; in particular it is suited to examining the long-wavelength limit for general k and Λ. The dynamics of our basic variable $\mathcal{D}_a$ can be determined through the system of two first order linear equations for $\mathcal{D}_a$ and $\widetilde{C}_a$ that follows from (3.85) (3.95) and (3.84):

$$\dot{\mathcal{D}}_a = \left\{w\Theta - \left[\tfrac{3}{2}\kappa\mu(1+w) - \tfrac{3K}{a^2}\right]\Theta^{-1}\right\}\mathcal{D}_a + \tfrac{3}{4}\tfrac{(1+w)}{a^2\Theta}\widetilde{C}_a\;, \tag{3.133}$$

$$\dot{\widetilde{C}}_a = \frac{4}{3}\frac{c_s^2 a^2 \Theta}{(1+w)}\left({}^{(3)}\nabla^2 - \frac{2K}{a^2}\right)\mathcal{D}_a + \frac{8}{3}c_s^2 a^3 \Theta^2\, {}^{(3)}\nabla^b \omega_{ab}\ . \tag{3.134}$$

In the assumed large scale limit, the first term in (3.134) vanishes; thus if there is no vorticity term, $\widetilde{C}_a$ is a conserved quantity on large length scales, for any value of K or Λ (and so C_a is conserved if $K = 0$).

A scalar type variable $a\,{}^{(3)}\nabla^a \widetilde{C}_a$ is a conserved quantity on the large scale even considering the vorticity term (in this case, an integral solution in the large scale case can be found in [63]), so the aggregation of matter to form spherically symmetric high-density concentrations (proto-structures) is unaffected. Notice that this is valid for general k and Λ, thus it generalizes the conserved quantity in [83, 3]. However the vector variable $\widetilde{C}_a$ can have a contribution from the vorticity even in the large scale case, as for example when there are homogeneous (Bianchi) perturbations. The effect of the vorticity is analysed below for the case $K = 0 = \Lambda$.

The evolution of the density gradient

In the case of flat background ($K = 0$), the above defined variable coincides with the previously introduced curvature gradient: i.e. $\widetilde{C}_a = C_a$. In this case equations (3.133, 3.134) coincide with (3.85, 3.96) and we can proceed to integrate them, neglecting the Laplacian term. Remembering that $\frac{1}{3}\Theta^2 = \kappa\mu$ is the zero-order equation when $K = 0 = \Lambda$, the equations for $\mathcal{D}_a$ and C_a become

$$\dot{\mathcal{D}}_a + \tfrac{1}{2}(1-w)\Theta\mathcal{D}_a = \tfrac{3}{4}\Theta^{-1}(1+w)\tfrac{C_a}{a^2}\ , \tag{3.135}$$

$$\dot{C}_{\perp a} = \tfrac{8}{3}\, c_s^2 S^3\, \Theta^2\, {}^{(3)}\nabla^b \omega_{ab}\ . \tag{3.136}$$

It is clear from the RHS of equation (3.136) that C_a is no longer a constant of motion, but can be determined from the source term ${}^{(3)}\nabla^b \omega_{ab}$. Then it acts as a source for $\mathcal{D}_a$. Now we can use

$$\Theta = \frac{2}{(1+w)t} \tag{3.137}$$

(following from (3.109)) to rewrite (3.135) as

$$\dot{\mathcal{D}}_{\perp a} + \frac{1-w}{1+w}\frac{\mathcal{D}_a}{t} = \frac{3}{8}\frac{(1+w)^2}{\beta^{\frac{4}{3(1+w)}}} t^{\frac{3w-1}{3(1+w)}} C_a\ , \tag{3.138}$$

while from the vorticity equation (34), assuming $c_s^2 = w$, equation (3.136) becomes

$$\dot{C}_{\perp a} = \frac{32}{3}\frac{w}{(1+w)^2}\beta^{\frac{2w}{1+w}}\Omega_a t^{-\frac{2}{1+w}}\ , \tag{3.139}$$

where Ω_a was defined in (3.112). From the above equation we have

$$w \neq 1 \quad \Rightarrow \quad C_a = C_a^{(\infty)} \left[1 - \left(\frac{t}{t_i}\right)^{-\frac{1-w}{1+w}} \right] + C_a^{(i)} \,, \tag{3.140}$$

where we have included an explicit initial time t_i, and $C_a^{(i)} = C_a(t_i)$ is the initial value of C_a (the constant of motion when $\omega = 0$) and

$$C_a^{(\infty)} \equiv \frac{32}{3} \frac{w}{1-w^2} \beta^{\frac{2w}{1+w}} \Omega_a t_i^{-\frac{1-w}{1+w}} \tag{3.141}$$

is the asymptotic value of C_a. Thus equation (3.140) shows how the decaying vorticity term in the RHS of (3.139) induces an asymptotically growing mode in C_a (for $w \neq 1$). Note that in the dust case ($w = 0$), $C_a^{(\infty)} = 0$ by the above definition. For $w = 1$ we obtain

$$w = 1 \quad \Rightarrow \quad C_a = 8\Omega_a \sqrt{\frac{\kappa}{3} M_1} \ln\left(\frac{t}{t_i}\right) + C_a^{(i)} \,. \tag{3.142}$$

Using (3.140) and (3.142) we can look at the time behaviour of "curvature perturbations": then we see that for $w < 1$ the extra mode induced by the vorticity term in the RHS of (3.139) grows up to an asymptotic value, while for $w > 1$ (not allowed physically!) there is a growing mode. Finally, C_a grows logarithmically if $w = 1$.

With (3.140) we can now integrate (3.138) when $w \neq 1$. The general solution for $\mathcal{D}_a$ is

$$\begin{aligned} \mathcal{D}_a = {} & \mathcal{D}_a^{(i)} \left(\frac{t}{t_i}\right)^{-\frac{1-w}{1+w}} + \frac{9(1+w)^3}{8(5+3w)} \frac{C_a^{(i)} t_i^2}{(\beta t_i)^{\frac{4}{3(1+w)}}} \left(\frac{t}{t_i}\right)^{\frac{2(1+3w)}{3(1+w)}} \left[1 - \left(\frac{t}{t_i}\right)^{-\frac{5+3w}{3(1+w)}} \right] + \\ & + (1+w)^3 \frac{9}{8} \frac{C_a^{(\infty)}}{(\beta t_i)^{\frac{4}{3(1+w)}}} \left\{ \frac{t_i^2}{5+3w} \left(\frac{t}{t_i}\right)^{\frac{2(1+3w)}{3(1+w)}} \left[1 - \left(\frac{t}{t_i}\right)^{-\frac{5+3w}{3(1+w)}} \right] - \right. \\ & \left. - \frac{t_i^2}{2(1+3w)} \left(\frac{t}{t_i}\right)^{\frac{9w-1}{3(1+w)}} \left[1 - \left(\frac{t}{t_i}\right)^{-\frac{2(1+3w)}{3(1+w)}} \right] \right\} . \end{aligned} \tag{3.143}$$

A similar expression can be found for the variable

$$\Phi_a \equiv \kappa \mu a^2 \mathcal{D}_a \tag{3.144}$$

introduced in EHB in analogy with Bardeen's variable Φ_H [1]; it is

$$\begin{aligned} \Phi_a = {} & \Phi_a^{(i)} \left(\frac{t}{t_i}\right)^{-\frac{5+3w}{3(1+w)}} + C_a^{(i)} \frac{3(1+w)}{2(5+3w)} \left[1 - \left(\frac{t}{t_i}\right)^{-\frac{5+3w}{3(1+w)}} \right] + \\ & + \frac{C_a^{(\infty)}}{2} \left\{ \frac{3(1+w)}{(5+3w)} \left[1 - \left(\frac{t}{t_i}\right)^{-\frac{5+3w}{3(1+w)}} \right] - \frac{3(1+w)}{2(1+3w)} \left(\frac{t}{t_i}\right)^{-\frac{1-w}{1+w}} \left[1 - \left(\frac{t}{t_i}\right)^{-\frac{2(1+3w)}{3(1+w)}} \right] \right\} . \end{aligned} \tag{3.145}$$

In both of the above expressions, the first term comes from the homogeneous equation, the second arises from the constant mode in C_a (3.140), and the last is due to the vorticity source term in (3.139). The solution for Δ is immediately obtained by applying $a^{(3)}\nabla^a$ to (3.143); then the last term disappears, for ${}^{(3)}\nabla^a C_a^{(\infty)} = 0$ by definition (3.141) and (40), and the second term is the term that comes from the scalar conserved quantity $C = a^{(3)}\nabla^a C_a^{(i)}$ which exists in the long-wavelength limit even if $\omega \neq 0$ (see previous section and [83, 63, 3]). Also we point out that in the very particular case in which $C_a^{(i)}$ and $C_a^{(\infty)}$ (thus Ω_a defined in (3.112)) have values such that $C_a^{(i)} = -C_a^{(\infty)}$, two of the growing modes in the above equations cancel. For the special case $w = 1$, $\beta = \sqrt{3\kappa M_1}$ we obtain from (3.138, 3.142)

$$\mathcal{D}_a = \mathcal{D}_a^{(i)} + \frac{9}{8} C_a^{(i)} t_i^2 \left(\frac{1}{\beta t_i}\right)^{\frac{2}{3}} \left(\frac{t}{t_i}\right)^{\frac{4}{3}} \left[1 - \left(\frac{t}{t_i}\right)^{-\frac{4}{3}}\right] + \\ + 3\Omega_a \beta^{\frac{1}{3}} t_i^{\frac{4}{3}} \left(\frac{t}{t_i}\right)^{\frac{4}{3}} \left\{\ln\left(\frac{t}{t_i}\right) - \frac{3}{4}\left[1 - \left(\frac{t}{t_i}\right)^{-\frac{4}{3}}\right]\right\}, \qquad (3.146)$$

showing that in this case the vorticity induced mode dominates. For the variable Φ_a in this case we have

$$\Phi_a = \Phi_a^{(i)} \left(\frac{t}{t_i}\right)^{-\frac{4}{3}} + \frac{3}{8} C_a^{(i)} \left[1 - \left(\frac{t}{t_i}\right)^{-\frac{4}{3}}\right] + \Omega_a \beta \left\{\ln\left(\frac{t}{t_i}\right) - \frac{3}{4}\left[1 - \left(\frac{t}{t_i}\right)^{-\frac{4}{3}}\right]\right\}. \qquad (3.147)$$

Radiation

The case of pure radiation is of particular relevance to the early universe. In this case, $\gamma = 4/3$, $w = 1/3 = c_s^2$, $\beta = 2\sqrt{\frac{\kappa}{3} M_1}$; then we find from (3.143)

$$\mathcal{D}_a = \mathcal{D}_a^{(i)} \left(\frac{t}{t_i}\right)^{-\frac{1}{2}} + \frac{4}{9}\frac{t_i}{\beta}(C_a^{(i)} + C_a^{(\infty)}) \left(\frac{t}{t_i}\right) \left[1 - \left(\frac{t}{t_i}\right)^{-\frac{3}{2}}\right] - \frac{2}{3}\frac{t_i}{\beta} C_a^{(\infty)} \left(\frac{t}{t_i}\right)^{\frac{1}{2}} \left[1 - \left(\frac{t}{t_i}\right)^{-1}\right], \qquad (3.148)$$

where we explicitly see that the growing mode induced by $C_a^{(i)}$ and one of those induced by $C_a^{(\infty)}$ (the faster growing mode) can eventually reciprocally cancel if $C_a^{(i)} = -C_a^{(\infty)}$. An analogous expression for Φ_a can be found from (3.146).

3.9 Dynamic equations for the imperfect fluid

In this section we generalize what we have done so far in this chapter in two directions: *a)* we consider the complete set of linear equations following from the covariant exact equations reviewed in chapter 2 and *b)* we include in our equations imperfect fluid terms, thus we explicitely take into account the discussion on the choice of frame given in section 3.4.

3.9.1 Hydrodynamic and gravitational equations

Let us briefly recall the covariant theory of chapter 2. Our field equations for the free gravitational field, the Weyl tensor C_{abcd}, are the Bianchi identities (see page 43)

$$R_{ab[cd;e]} = 0 \quad \Leftrightarrow \quad C^{abcd}{}_{;d} = R^{c[a;b]} - \tfrac{1}{6} g^{c[a} R^{;b]} \; . \tag{3.149}$$

The Einstein equations algebrically determine the Ricci tensor at each spacetime point

$$R_{ab} = \kappa(T_{ab} - \tfrac{1}{2} g_{ab} T^{a}{}_{a}) + \Lambda g_{ab} \; , \tag{3.150}$$

thus these latter are used in this approach to substitute for *Ricci* in the various equations with the matter tensor T_{ab}: then matter acts as source term in the field equations (3.149), as well as in the hydrodynamic equations (i.e. the evolution equations for the kinematical fluid quantities) which comes from the Ricci identities for the 4-velocity of the fluid

$$u_{a;d;c} - u_{a;c;d} = R_{abcd} u^{b} \; , \tag{3.151}$$

where from now on the fluid 4-velocity will be taken as any possible u^a constrained to coincide with u^a_E in the background, so that variables and equations are GI in what follows. In particular, for $u^a = u^a_E$ we must set $q^a = 0$ in the equations, and for $u^a = u^a_N$ we have $q^a = q^a_N$. Finally, the conservation equations $T^{ab}{}_{;b} = 0$ are regarded in this context as following from the contraction of (3.149).

The exact equations equivalent to (3.149), (3.150), (3.151) and the contracted Bianchi identities have been reviewed in chapter 2:[14] we shall now consider the covariantly expressed general relativistic dynamic equations which come from linearizing these exact equations about an exact FLRW model: we regard these equations as describing the evolution of the various covariant GI variables (as well as the constraints between them) in an almost FLRW universe.

[14]More details can be found in Ellis (1971, 1973) [27, 28]; see also the appendix of Hwang and Vishniac 1990 [63], for a compact presentation and a comparison with the ADM equations.

Firstly, we find convenient to rewrite the conservation equations (2.45) (2.46): retaining only terms up to the first order we have

$$\dot{\mu} + 3hH + h^{(3)}\nabla_a \Psi^a = 0 \ , \tag{3.152}$$

$$ha_a + Y_a + h\left[F_a + \Pi_a\right] = 0 \ , \tag{3.153}$$

where for later convenience we define

$$F_a = \dot{\Psi}_a - \left(3c_s^2 - 1\right) H\Psi_a \ , \quad \Pi_a = \frac{1}{h}{}^{(3)}\nabla^b \pi_{ab} \ , \quad \Psi_a = \frac{q_a}{h} \ , \quad h = (\mu + p) \ . \tag{3.154}$$

The first hydrodynamic equation we consider from those following from (3.151) is the linearized Raychaudhuri equation

$$3\dot{H} + 3H^2 - A + \tfrac{1}{2}\kappa\left(\mu + 3p\right) - \Lambda = 0 \tag{3.155}$$

which determines the evolution of the Hubble parameter H along flow lines, where $A = a^a{}_{;a}$ is the divergence of the acceleration.

If we use Einstein's equations to substitute R_{ab} with T_{ab} in (3.35) and linearize, we get the energy constraint

$$\tfrac{1}{6}{}^{(3)}R = -H^2 + \tfrac{1}{3}\kappa\mu + \tfrac{1}{3}\Lambda \ ; \tag{3.156}$$

this, together with (3.152) and (3.155) above, becomes the standard set of equations for homogeneous isotropic FLRW models on dropping the perturbative terms in them and putting ${}^{(3)}R = \frac{6K}{a^2}$.

Three other constraints follow from (3.151): these are the $(0, \nu)$ components of the field equations, which is conveniently rewritten as

$$\tfrac{2}{3}\mathcal{Z}^e + a^{(3)}\nabla_f \omega^{ef} - a^{(3)}\nabla_f \sigma^{ef} = \kappa a q^e \ , \tag{3.157}$$

the vorticity constraint

$${}^{(3)}\nabla^a \omega_a = 0 \quad \Leftrightarrow \quad {}^{(3)}\nabla^a(\eta_{abcd}u^b\omega^{cd}) = 0 \ , \tag{3.158}$$

the H_{ab} constraint

$$H_{fd} = h_f{}^t h_d{}^s \left[\omega_{(t}{}^{b;c} + \sigma_{(t}{}^{b;c}\right] \eta_{s)abc} u^a \ . \tag{3.159}$$

Two other hydrodynamic equations govern the evolution of the shear and the vorticity; they are:

$$(\sigma_{ab})^{\cdot} + E_{ab} + 2H\sigma_{ab} + \frac{c_s^2}{a(1+w)}\left[{}^{(3)}\nabla_{(a}\mathcal{D}_{b)} - \tfrac{1}{3}h_{ab}{}^{(3)}\nabla^c\mathcal{D}_c\right] =$$
$$-\frac{w}{a(1+w)}\left[{}^{(3)}\nabla_{(a}\mathcal{E}_{b)} - \tfrac{1}{3}h_{ab}{}^{(3)}\nabla^c\mathcal{E}_c\right]$$
$$-\left[{}^{(3)}\nabla_{(a}(F_{b)} + \Pi_{b)}) - \tfrac{1}{3}h_{ab}{}^{(3)}\nabla^c(F_c + \Pi_c)\right] \tag{3.160}$$

$$\dot{\omega}_{ab} + 2H\omega_{ab} = {}^{(3)}\nabla_{[b}a_{a]}$$
$$= -\frac{1}{\mu+p}{}^{(3)}\nabla_{[b}{}^{(3)}\nabla_{a]}p - {}^{(3)}\nabla_{[b}\left(F_{a]} + \Pi_{a]}\right)$$
$$= \left[3c_s^2 H - \left(\frac{\partial p}{\partial s}\right)_\mu \frac{\dot{s}}{\mu+p}\right]\omega_{ab} - {}^{(3)}\nabla_{[b}\left(F_{a]} + \Pi_{a]}\right) \ , \tag{3.161}$$

where we used (3.93); the last line in this equation requires a special comment: the $\dot{s}$ term will be usually of second order, $\dot{s}$ being in general a first order quantity due to bulk viscosity B

$$T\varrho\dot{s} = -(B\Theta + q^a{}_{;a}) = \xi\Theta^2 - q^a{}_{;a} \tag{3.162}$$

it will not be negligible however if the background is a FLRW universe with bulk viscosity.

Finally, the "Maxwell-like" gravitational field equations for E_{ab} and H_{ab} following from (3.149) take the linear form

Div$E$ equation

$$a^{(3)}\nabla^b E_{ab} = \tfrac{1}{3}\kappa\mu\mathcal{D}_a - \tfrac{1}{2}\kappa a\Pi_a - \kappa a H q_a \ , \tag{3.163}$$

$\dot{E}$ equation

$$\dot{E}_{ab} + 3HE_{ab} + h_{(a}{}^f\eta_{b)cde}u^c H_f{}^{d;e} + \tfrac{1}{2}\kappa h\sigma_{ab} =$$
$$-\tfrac{1}{2}\kappa\left[{}^{(3)}\nabla_{(a}q_{b)} - \tfrac{1}{3}h_{ab}{}^{(3)}\nabla^a q_a\right] - \tfrac{1}{2}\kappa\left(H\pi_{ab} + \dot{\pi}_{ab}\right) \ , \tag{3.164}$$

Div$H$ equation

$${}^{(3)}\nabla^b H_{ab} = \tfrac{1}{2}\kappa\eta_{abcd}\left[hu^b\omega^{cd} + q^{[c;d]}\right] \ , \tag{3.165}$$

$\dot{H}$ equation

$$\dot{H}_{ab} + 3HH_{ab} - h_{(a}{}^f\eta_{b)cde}u^c E_f{}^{d;e} = \tfrac{1}{2}\kappa h_{c(a}\eta_{b)def}u^d\pi^{ce;f} \ , \tag{3.166}$$

To complete the set of equations, we need to specify how to construct the metric tensor from the kinematic quantities. We will not pursue the issue further here, simply noting that a suitable way to do this is implied by section 3 (see particularly Theorem 2) of Treciokas and Ellis [123].

We may note that, although Hawking considered these equations, he didn't completely linearized them, since he didn't attempt to eliminate acceleration terms from his equations: in doing this we have explicitly coupled the shear equation to $\mathcal{D}_a$.

3.9.2 The structure of the equations

The full set of equations given so far consists of two different kinds: propagation equations (involving time-derivatives of the kinematic or dynamic quantities) and constraint equations (involving only their spatial derivatives). An important issue arising is the consistency of these equations. Consider for example the constraint equation (3.157), say C1. We can take the time derivative of this equation, and then (using commutation relations for time and space derivatives where necessary) substitute for all the time derivatives occurring, from the propagation equations. The result will be a new constraint equation, say C2 (because we have eliminated all the time derivatives). Now it may be that C2 is identically satisfied; then the constraint C1 is conserved in time. On the other hand C2 could be a genuinely new equation; in this case we can take its time derivative to obtain a further constraint equation C3. This in turn may be identically satisfied, or may be a further constraint equation that has to be satisfied in a non-trivial way. If too many non-trivial constraints arise in this way, we will have proved that the set of equations is inconsistent.

In fact, *the set of linear equations above is consistent*: that is, the time derivative of each constraint equation is identically satisfied as a consequence of the other equations that hold. Thus this is a consistent set of equations, in the sense that once a set of initial data has been found that satisfies the required initial conditions (the set of constraint equations), these equations will hold at all later times. This is of course known to be a property of the full Einstein equations (because of the contracted Bianchi identities and the conservation of energy and momentum); the linearisation introduced here is consistent in that it preserves this property.

Because the equations have been obtained by linearisation, we can regard their solutions as consisting additively of different parts that each themselves solve these linear equations.

As it is well known, (3.161) shows that rotational perturbations evolve independently of variables other than ω_{ab}, which however in general affect the evolution of other quantities: since the equations are linear, this just means that in the general case all the vectorial and tensorial variables have a rotational ("vector") contribution. Thus the vorticity terms in the above equations can be regarded as known source terms. Gravitational waves are represented by those parts of the Weyl tensor components E_{ab} and H_{ab} which do not arise from rotational ("vector") and density ("scalar") perturbations, i.e. by their TT (transverse traceless) parts ${}^{(3)}\nabla^b E_{ab} = 0$ and ${}^{(3)}\nabla^b H_{ab} = 0$. Both rotational and gravitational waves perturbations were discussed within the covariant approach by Hawking [55] (1966); we turn now on density perturbations.

3.9.3 Two alternative pairs of equations

Coupling between density and expansion gradients

In order to investigate density perturbations we need to complete the above set of equations with an evolution equation for the GI density variable. In Bardeen (1980) ,[1] and Kodama and Sasaki (1984) [69] this is coupled with an equation for a variable associated with the shear of matter. In EB [35] the GI density variable is $\mathcal{D}_a$, and its evolution equation is coupled with an equation for the gradient of expansion $\mathcal{Z}_a$. The equations for $\mathcal{D}_a$ and $\mathcal{Z}_a$ for the perfect fluid case have been derived and examined in section 3.6.2; for an imperfect fluid we obtain

$$\dot{\mathcal{D}}_a - 3Hw\mathcal{D}_a + (1+w)\,\mathcal{Z}_a = 3a\,(1+w)\,H\,[F_a + \Pi_a] - a\,(1+w)\,{}^{(3)}\nabla_a{}^{(3)}\nabla^b\Psi_b\,, \quad (3.167)$$

$$\dot{\mathcal{Z}}_a + 2H\mathcal{Z}_a + \tfrac{1}{2}\kappa\mu\mathcal{D}_a + \tfrac{c_s^2}{(1+w)}\left(\tfrac{K}{a^2} + {}^{(3)}\nabla^2\right)\mathcal{D}_a + \tfrac{w}{(1+w)}\left(\tfrac{K}{a^2} + {}^{(3)}\nabla^2\right)\mathcal{E}_a$$
$$= -a^{(3)}\nabla_a{}^{(3)}\nabla^b\,[F_b + \Pi_b] + a\left(\tfrac{3}{2}h - \tfrac{3K}{a^2}\right)[F_a + \Pi_a] - 6aHc_s^{2\,(3)}\nabla^b\omega_{ab}\,,\quad (3.168)$$

on taking the gradient of the energy conservation (3.152) and Raychaudhuri equation (3.155).

These equations have already been derived in section 3.5.2 for the case of a perfect fluid. If we look at them in the context of the full set of hydrodynamical and gravitational equations, it should be emphasized that these equations do not contain new dynamical information (they are implied by those already given). Rather, they extract the specific information we want (the propagation of the GI variables along the fluid flow lines) from the full set of equations. We can determine the behaviour we are interested in from these new equations alone; they are of course consistent with the full set of equations.

As was shown previously (see section 3.6.2 and EBH), it follows from these evolution equations for the density gradient $\mathcal{D}_a$ that although this is obviously not affected by gravitational wave (TT tensors) contributions, a rotational mode in the density gradient can be generated by ω_{ab}.

Coupling between density gradient and shear

As in BI and KS we can also couple the $\mathcal{D}_a$ evolution equation with an equation for the shear variable: on taking the divergence of the shear equation and using the following first - order identities for any vector V_a and tensor T_{ab}

$$a^{(3)}\nabla_a \dot{V}_b = (a^{(3)}\nabla_a V_b)^{\cdot}\,, \qquad a^{(3)}\nabla_a \dot{T}_{bc} = (a^{(3)}\nabla_a T_{bc})^{\cdot}\,, \tag{3.169}$$

we obtain

$$\begin{aligned} &(a^{(3)}\nabla^b\sigma_{ab})^{\cdot} + \tfrac{1}{3}\kappa\mu\mathcal{D}_a + 2H(a^{(3)}\nabla^b\sigma_{ab}) + \tfrac{2c_s^2}{3(1+w)}\left(\tfrac{K}{a^2} + {}^{(3)}\nabla^2\right)\mathcal{D}_a \\ &= -\tfrac{2w}{3(1+w)}\left(\tfrac{K}{a^2} + {}^{(3)}\nabla^2\right)\mathcal{E}_a + a\Pi_a + aHq_a - aH^{(3)}\nabla^b\omega_{ab} \\ &- \left[\tfrac{a}{2}\left({}^{(3)}\nabla^2 + \tfrac{2K}{a^2}\right)(F_a + \Pi_a) + \tfrac{1}{6}{}^{(3)}\nabla_a(F+\Pi)\right]\;; \end{aligned} \tag{3.170}$$

the equation for $\mathcal{D}_a$ is then coupled to the above equation for $a^{(3)}\nabla^b\sigma_{ab}$

$$\begin{aligned} &\dot{\mathcal{D}}_a - 3Hw\mathcal{D}_a + \tfrac{3}{2}(1+w)(a^{(3)}\nabla^b\sigma_{ab}) \\ &= a(1+w)\left[3H(F_a + \Pi_a) + \tfrac{3}{2}q_a - {}^{(3)}\nabla_a{}^{(3)}\nabla^b\Psi_b\right] \\ &+ \tfrac{3}{2}(1+w)a^{(3)}\nabla^b\omega_{ab}\;, \end{aligned} \tag{3.171}$$

where we used the constraint equation (3.157) to substitute for $\mathcal{Z}_a$ in (3.167).

These equations emphasize that when there is a density gradient there will in general be distortion taking place, and *vice versa*. However we do not need to know the distortion in order to determine the evolution of the density inhomogeneity (because of the equations derived in the previous subsection).

3.10 Equations for the scalar variables

3.10.1 Scalar variables

Up to now we have been dealing with true tensors in the real physical almost FLRW universe. In treating cosmological perturbations it is however standard to split them into "scalar", "vector" and "tensor" parts through a non local decomposition (see Stewart 1990 [117], and section 4.1), where scalar density perturbations are solely responsible for the formation of structure in the universe.

As we have seen in section 3.7.2 for $\mathcal{D}_a$, we may *locally* decompose the covariant derivative of any vector V_a in analogy with (2.19). For the density perturbations, $\Delta \equiv \Delta^a{}_a$ is the scalar GI variable representing the clumping of matter: we shall explicitly show in section 4.2 that Δ in the present approach is equivalent to Bardeen's ε_m. More generally, a *locally* defined scalar variable is obtained from any tensor by taking its total divergence.

In particular we shall derive equations for the following set of GI scalar variables

$$\Delta = a^{(3)}\nabla_a \mathcal{D}^a \ , \quad \mathcal{Z} = a^{(3)}\nabla_a \mathcal{Z}^a \ , \quad C = {}^{(3)}\nabla_a C^a \ , \tag{3.172}$$

$$\tilde{C} = a^{(3)}\nabla_a \tilde{C}^a \ , \quad a^{2(3)}\nabla_a{}^{(3)}\nabla_b \sigma^{ab} \ , \quad \mathcal{E} = a^{(3)}\nabla_a \mathcal{E}^a \ , \tag{3.173}$$

that follow from the analogous gradients previously introduced; clearly

$$C = -4a^2 H\mathcal{Z} + 2\kappa\mu a^2 \Delta \ , \qquad \tilde{C} = C - \tfrac{4K}{1+w}\Delta \ . \tag{3.174}$$

3.10.2 The evolution equations

We pass now to consider the equations for the scalar variables that concern directly the growth of structures in the universe. Taking the divergence of (3.157) one gets

$$\mathcal{Z} = \tfrac{3}{2}a^2({}^{(3)}\nabla_a{}^{(3)}\nabla_b \sigma^{ab} + {}^{(3)}\nabla_a q^a) \ ; \tag{3.175}$$

since the divergence of the vorticity term appearing in (3.157) disappears to linear order (see EBH)[36], the latter is a constraint between the GI scalar perturbations in expansion and shear: actually, this is the GI "scalar perturbation" equivalent of the ADM momentum constraint. Another scalar constraint is provided by (3.174), i.e. the "scalar perturbation" equivalent of the ADM energy constraint (see York 1979) [131].

Note that the appearance of the energy flux q^a on the RHS of (3.157),(3.175) is due to our choice of the PF, and it would disappear in the EF.

True density perturbations related with the growth of matter clumping are represented by Δ. We could actually consider the scalar equivalent of the whole set of hydrodynamic and gravitational equations in the previous section on taking their divergences and using (3.169), but this is not directly needed. Indeed it is possible to show that all the scalar variables we can obtain (on taking divergences of the vector and tensor variables previously used) are determined to first order by Δ through the relative equations:[15] thus we concentrate on the evolution equations for Δ. Defining

$$\Pi = a^{(3)}\nabla^a \Pi_a \ , \qquad F = a^{(3)}\nabla^a F_a \ , \tag{3.176}$$

we take the divergence of (3.167), (3.168), using (3.169), to obtain

$$\dot{\Delta} - 3Hw\Delta + (1+w)\,\mathcal{Z} = 3a\,(1+w)\,H\,[F+\Pi] - a\,(1+w)\,{}^{(3)}\nabla^2\Psi \ , \tag{3.177}$$

$$\begin{aligned}\dot{\mathcal{Z}} + 2H\mathcal{Z} + \tfrac{1}{2}\kappa\mu\Delta + \tfrac{c_s^2}{(1+w)}\left({}^{(3)}\nabla^2 + \tfrac{3K}{a^2}\right)\Delta \\ + \tfrac{w}{(1+w)}\left({}^{(3)}\nabla^2 + \tfrac{3K}{a^2}\right)\mathcal{E} = -a\left({}^{(3)}\nabla^2 + \tfrac{3K}{a^2}\right)[F+\Pi] + \tfrac{3}{2}h\,[F+\Pi] \ .\end{aligned} \tag{3.178}$$

From the practical point of view it is however convenient to couple the evolution of Δ with that of the variable $\tilde{C}$, which turns out to be conserved in various cases of interest;[16] $\tilde{C}$ satisfies

$$\begin{aligned}\dot{\tilde{C}} &= \tfrac{4a^2Hc_s^2}{(1+w)}{}^{(3)}\nabla^2\Delta + \tfrac{4a^2Hw}{(1+w)}\left[{}^{(3)}\nabla^2 + \tfrac{3K}{a^2}\right]\mathcal{E} \\ &\quad +4a^3H{}^{(3)}\nabla^2\,[F+\Pi] + \left[4Ka - 2a^3h\right]{}^{(3)}\nabla^2\Psi \ ,\end{aligned} \tag{3.179}$$

and the coupled equation for Δ is (3.177) on substituting for $\mathcal{Z}$ using (3.174):

$$\begin{aligned}&\dot{\Delta} - \left\{3Hw - \left[\tfrac{\kappa h}{2} - \tfrac{K}{a^2}\right]H^{-1}\right\}\Delta - \tfrac{(1+w)}{4a^2H}\tilde{C} \\ &= 3a\,(1+w)\,H\,[F+\Pi] - a\,(1+w)\,\nabla^2\Psi.\end{aligned} \tag{3.180}$$

[15] The equivalent of this statement within the Bardeen formalism has been proved by Goode: there, all the scalar perturbation variables are determined once ϵ_m is known.

[16] We refer to section 6.4 for a discussion of conserved quantities in a general case. See also EBH [36], Hwang and Vishniac (1990) [63], Hwang (1990, 1991) [59, 62] and references therein, and section refsec-variables, BE [9] and BED [10] for an application to scalar fields. Dunsby and Bruni (1991) [20] discuss the existence and use of generalized conserved quantities.

An alternative pair of equations

The scalar counterpart of (3.170), (3.171) are immediately obtained on taking their divergences, or on using (3.177), (3.178) and (3.175):

$$\dot{\Delta} - 3Hw\Delta + \tfrac{3}{2}(1+w)\left(a^{2\,(3)}\nabla^{a(3)}\nabla^b\sigma_{ab}\right)$$
$$= 3a(1+w)\left[H(F+\Pi) + \tfrac{1}{2}a^{(3)}\nabla^a q_a\right] - a(1+w)\,^{(3)}\nabla^2\Psi\,, \qquad (3.181)$$

$$\left(a^{2(3)}\nabla^{a(3)}\nabla^b\sigma_{ab}\right)^{\cdot} = -\tfrac{1}{3}\kappa\mu\Delta - 2H\left(s^{2(3)}\nabla^{a(3)}\nabla^b\sigma_{ab}\right) - \tfrac{2c_s^2}{3(1+w)}\left(\tfrac{3K}{a^2} + {}^{(3)}\nabla^2\right)\Delta$$
$$-\tfrac{2w}{3(1+w)}\left(\tfrac{3K}{a^2} + {}^{(3)}\nabla^2\right)\mathcal{E} - \tfrac{2a}{3}\left(\tfrac{3K}{a^2} + {}^{(3)}\nabla^2\right)(F+\Pi) + a\Pi + a^2H^{(3)}\nabla^a q_a \qquad (3.182)$$

Second order equation

In those cases where a conserved quantity does not exist, the evolution of Δ can be computed directly from a second order equation

$$\ddot{\Delta} + \left(2 + 3c_s^2 - 6w\right)H\dot{\Delta}$$
$$-\left[\left(\tfrac{1}{2} + 4w - \tfrac{3}{2}w^2 - 3c_s^2\right)\kappa\mu + \left(5w - 3c_s^2\right)\Lambda + \left(c_s^2 - w\right)\tfrac{12K}{a^2}\right]\Delta - c_s^{2(3)}\nabla^2\Delta$$
$$-w\left({}^{(3)}\nabla^2 + \tfrac{3K}{a^2}\right)\mathcal{E} = a(1+w)\left[-3w\mu + 3\Lambda + \left({}^{(3)}\nabla^2 - \tfrac{3K}{a^2}\right)\right][F+\Pi]$$
$$+a(1+w)\left[3H\left(\dot{F} + \dot{\Pi}\right) - \left(2H^{(3)}\nabla^2\Psi + {}^{(3)}\nabla^2\dot{\Psi}\right)\right]\;; \qquad (3.183)$$

this follows from any of the above pairs of equations coupling Δ with either $\mathcal{Z}$, $\tilde{C}$ or $a^{2(3)}\nabla^{a(3)}\nabla^b\sigma_{ab}$, and is equivalent to the equation for ε_m in BI (see Eq. 4.9 there) and generalize (3.120) to the imperfect fluid case.

3.10.3 Solutions and harmonic components

We already said that rotational perturbations evolve independently, so that vorticity terms can be considered as known source terms in the vector and tensor equations of sections 3.9.1, 3.9.3 producing a rotational mode in the corresponding variable. Passing to consider scalar equations, we may note that when we take the full divergence of our equations the gravitational (TT tensor) and rotational modes disappear, as ${}^{(3)}\nabla^{a(3)}\nabla^b\omega_{ab} = 0$ to first order. From the point of view of the initial value problem, this means that even if the fluid vorticity doesn't vanish, we can solve the scalar equations as if $\omega_{ab} = 0$: since ω_{ab} does not contribute to the scalar initial value constraints (3.156), (3.174) and (3.175) we can effectively set up initial values even on a comoving hypersurface orthogonal to u^a as if $\omega_{ab} = 0$.

To actually solve the equations (unless $p = 0$) it is standard to assume that time and spatial dependence in each variable are separable, expanding each quantity in spatial harmonics Q, as we already have seen in section 3.7.3.

The solution of the homogeneous part of (3.183) for a perfect fluid in a flat $K = 0$ background and for any wavelength can be found in terms of Bessel functions, as shown by Bardeen [1], who also considered the effect of entropy and anisotropic pressure perturbations; other solutions can be found in particular cases. Here we remind only that in a flat universe $K = 0$ we have:

dust: $p = 0 \Rightarrow w = c_s^2 = 0$

$$\Delta_+ \propto t^{\frac{2}{3}} , \quad \Delta_- \propto t^{-1} ; \tag{3.184}$$

radiation: $p = \frac{1}{3}\mu \Rightarrow w = c_s^2 = \frac{1}{3}$; in the long wavelength limit

$$\Delta_+ \propto t , \quad \Delta_- \propto t^{-\frac{1}{2}} ; \tag{3.185}$$

here Δ_+ and Δ_- represent growing and decaying modes respectively.

We turn now to a systematic comparison between the covariant GI formalism discussed up to now with that of Bardeen.

Chapter 4

COVARIANT FORMALISM VERSUS BARDEEN'S FORMALISM

Up to now we have considered the covariant approach to gauge - invariant cosmological perturbations. Aim of this chapter is to provide a comparison with the standard approach to gauge invariant perturbations elaborated by Bardeen [1] (1980) (often BI in the following), and followed more or less strictly by other authors such as Kodama and Sasaki [69] (1984), Brandenberger, Kahn and Press [7] (1983), Mukhanov, Feldman and Brandenberger [95]. Although Brandenberger, Kahn and Press introduced gauge invariant variables in their own original way (through a variational principle), and Kodama and Sasaki defined new variables, we assimilate their approach to that of Bardeen. Indeed the main variables they use are the same, i.e. linear combinations of gauge - dependent first - order quantities constructed *ad hoc* to be gauge - invariant first - order variables. What is important here is to realize that the approach followed by these authors is based on coordinates, and therefore most of their variables aquire a physical or geometrical significance only once a specific gauge choice has been made (see chapter 1).

We first review the Bardeen formalism briefly, introducing explicitely a perturbed metric and the set of Bardeen's GI variables. While Bardeen used directly a harmonic decomposition for every quantity, we systematically decompose each variable both in the coordinate space, using the *non local* ADM splitting for 3-vector and 3-tensors [117], and in the Fourier space, using standard harmonics. In our view, although working in Fourier space has the advantage of reducing equations to algebric relations, the presentations of the same equations in the coordinate space simplify somehow the physical interpretation.

Then we systematically expand at first-order the main covariant variables used in the previous chapter. Since these variables are GI by themselves because they vanish in the background, we may expect to recover Bardeen's variable at first-order: indeed, this is the case. Thus Bardeen's variables are first-order components of the covariant variables. This gives to all of them a physical or geometrical meaning, without the need to specify a gauge. Moreover all the equations of Bardeen are immediately recovered through this first-order expansion of the covariant variables.

4.1 The standard approach to GI perturbations

The role of the gravitational potential is played in general relativity by the metric tensor g_{ab}. The standard approach to perturbations of FLRW spacetime starts exactly from g_{ab}: if $\bar{g}_{ab}$ is the background metric in the standard coordinate system (3.67), $g_{ab} = \bar{g}_{ab} + \delta g_{ab}$ defines the metric perturbations δg_{ab} in these coordinates.

Following Bardeen's notation, we may write the perturbed metric in the form [to be compared with (1.1), (3.67)]

$$ds^2 = a^2(\eta)\{-(1+2A)d\eta^2 - 2B_\alpha dx^\alpha d\eta + [(1+2H_L)\gamma_{\alpha\beta} + 2H_{T\alpha\beta}]dx^\alpha dx^\beta\}\ , \qquad (4.1)$$

where η is the conformal time, and the spatial coordinates are left arbitrary.

From the point of view of the 3+1 (ADM) formalism (York 1979 [131]) the almost FLRW spacetime (4.1) is described by the foliation $\{\Sigma_\eta\}$ that arises locally as the level surfaces Σ_η of constant time η, i.e. the normal to Σ_η is $n_a = -N\eta_{,a}$ ($\Leftrightarrow \tilde{\omega}_{ab} = 0$), where $N = a(1+A)$ is the lapse function measuring the ratio between the proper time measured along the normal wordlines (with tangent n^a) and the coordinate time η. Given an initial-data surface Σ_{η_0}, data are propagated along an *arbitrary* congruence of curves with tangent η^a threading the slices Σ_η: these worldlines then are the curves $x^\alpha = const.$, and the condition $\eta^a\eta_{,a} = 1$ ensures that η is the parameter along the curves. Then $N^a = \tilde{h}^a{}_b\eta^b$ is the shift vector measuring the coordinate velocity of the eulerian observers $\mathcal{O}_n$ traveling with four velocity n^a and having a LRF coinciding with Σ_η: in the notation used for the metric (4.1) $N_\alpha = a^2 B_\alpha$ (see Fig. 4.1).

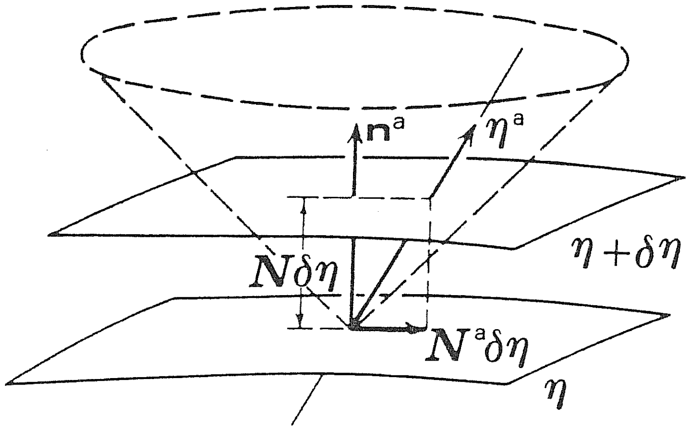

Figure 4.1: Illustrated are parts of two nearby slices of the foliation $\{\Sigma_\eta\}$. The time vector is $\eta^a = Nn^a + N^a$, where N is the lapse function, N^a is the shift vector, and n^a the unit normal (see text). The dashed figure represent a local light cone. Not shown are: *i)* the acceleration corresponding to the four velocity n^a of the Eulerian observers, which is tangent to the first slice together with N^a, and *ii)* the four velocity of matter u^a, which should appear as a third distinct vector within the light cone.

Thus A and B_α are respectively the perturbation in the lapse function and in the shift vector, while $2H_L\gamma_{\alpha\beta}+2H_{T\alpha\beta}$ is the perturbation in the metric of the Σ_η surfaces.

These and the other perturbation quantities (see below) are treated as 3-fields propagating in the background 3-geometry, as it is specified by the unperturbed metric. One thus has 3-scalars, 3-vectors and 3-tensors: under appropriate hypotheses on the boundary conditions they have to satisfy (see Stewart 1990 [117], and the comments in section 3.7.1), 3-vectors and 3-tensors such B_α and $H_{T\alpha\beta}$ can be uniquely decomposed as

$$B_\alpha = B_{|\alpha} + B^S_\alpha = B^{(0)}(\eta)Y^{(0)}_\alpha + B^{(1)}(\eta)Y^{(1)}_\alpha \ , \tag{4.2}$$

$$H_{T\alpha\beta} = \nabla_{\alpha\beta}H_T + H^S_{T(\alpha|\beta)} + H^{TT}_{T\alpha\beta} = H^{(0)}_T(\eta)Y^{(0)}_{\alpha\beta} + H^{(1)}_T(\eta)Y^{(1)}_{\alpha\beta} + H^{(2)}_T(\eta)Y^{(2)}_{\alpha\beta} \ , \tag{4.3}$$

where the slash indicates the covariant derivative determined by the metric $\gamma_{\alpha\beta}$, and

$$\nabla^2 f = f^{|\gamma}{}_{|\gamma}\ , \quad \nabla_{\alpha\beta} f = f_{|\beta\alpha} - \tfrac{1}{3}\nabla^2 f\ , \tag{4.4}$$

are respectively the Laplacian and the trace-free second derivative operator with respect to $\gamma_{\alpha\beta}$. The superscript S on a vector means it is solenoidal ($B^{S|\alpha}_{\alpha} = 0$), and TT tensors are transverse ($H^{TT\beta}_{T\alpha}{}_{|\beta} = 0$) and trace-free. Accordingly, it is standard to call *scalar* perturbations those quantities which are 3-scalars, or are derived from a scalar through linear operations involving only the metric $\gamma_{\alpha\beta}$ and its $|$ derivative; quantities derived from similar operations on solenoidal vectors and on TT tensors are dubbed *vector* and *tensor* perturbations. Scalar perturbations are relevant to matter clumping, i.e. correspond to density perturbations, while vector and tensor perturbations correspond to rotational perturbations and gravitational waves. Furthermore, given the homogeneity and isotropy of the background, we have separated in each variable the time and spatial dependence, where this latter is given by the spatial harmonics Y.[1] It is important to point out that these decompositions are *non local*, and that 3-scalar fields such B and H_T are defined up to a constant [117]. The key property of linear perturbation theory arising from the unicity of the splitting (4.2), (4.3) is that in any vector and tensor equation the scalar, vector and tensor parts on each side are separately equal.

The minimal set of perturbation variables is completed by the energy density and velocity perturbations

$$\mu = \bar{\mu} + \delta\mu\ , \quad \delta \equiv \delta\mu/\bar{\mu}\ , \quad \delta = \delta(\eta)Y\ , \tag{4.5}$$

$$u^a = \bar{u}^a + \delta u^a\ , \quad \delta u^\alpha = \bar{u}^0 v^\alpha\ , \quad \delta u^0 = -\bar{u}^0 A\ , \tag{4.6}$$

where

$$v_\alpha = v_{|\alpha} + v^S_\alpha = v^{(0)}(\eta)Y^{(0)}_\alpha + v^{(1)}(\eta)Y^{(1)}_\alpha\ , \tag{4.7}$$

[1]As in BI [1] and Goode (1989) [51] (see also Goode (1983) [50]) these harmonics are defined in the conformal 3-spaces of constant curvature. They do not depend on coordinate time: $u^a\partial_a Y = 0$. These authors denote them with Q: here we use Y as in Kodama and Sasaki (1984) [69] and Hwang and Vishniac (1990) [63], to distinguish them from the covariantly defined harmonics Q; the relations between the two conformal sets are given in the appendix. Note also the difference between the Laplacian ∇^2 used in this section, determined through the metric $\gamma_{\alpha\beta}$, and the covariant Laplacian ${}^{(3)}\nabla^2$ defined in the real physical space; in practice, in operating on a first order quantity T, ${}^{(3)}\nabla^2 T = a^{-2}\nabla^2 T$.

together with the energy flux q_a and the anisotropic pressure π_{ab} which are GI by themselves: since a direct check shows they have vanishing time components we may write

$$q_\alpha = q_{|\alpha} + q^S_\alpha = pf^{(0)}Y^{(0)}_\alpha + pf^{(1)}Y^{(1)}_\alpha \tag{4.8}$$

$$\pi_{\alpha\beta} = a^2\left[\nabla_{\alpha\beta}\pi + \pi^s_{(\alpha|\beta)} + \pi^{TT}_{\alpha\beta}\right] = a^2\left[p\pi^{(0)}_T Y^{(0)}_{\alpha\beta} + \pi^{(1)}_T Y^{(1)}_{\alpha\beta} + p\pi^{(2)}_T Y^{(2)}_{\alpha\beta}\right] \ . \tag{4.9}$$

From the above given gauge dependent 3-fields Bardeen constructed a set of GI variables. The *ad hoc* procedure used is to use the gauge transformations rules for these quantities to construct gauge - independent linear combinations. In other words, once we know how a given quantity (e.g. $\delta\mu/\bar{\mu}$) change under gauge transformations, we can add to it other quantities (such as v^α) with an appropriate time - dependent coefficient (such as $\frac{a'}{a}$), so that the resulting linear combination of first - order quantities is gauge independent. We give below this set of GI variables.

4.1.1 Scalar - perturbations

These are the GI perturbations fields that can be derived using only the above introduced scalar potentials.

GI metric perturbations As pointed out by Bardeen himself, there are only two independent GI metric perturbation variables: Bardeen introduced the two metric potentials

$$\Phi_A = A - \left(B' + \tfrac{a'}{a}B\right) - \left(H_T'' + \tfrac{a'}{a}H_T'\right) \tag{4.10}$$

$$= \left\{A + \tfrac{1}{k}\left(B^{(0)\prime} + \tfrac{a'}{a}B^{(0)}\right) - \tfrac{1}{k^2}\left(H_T^{(0)\prime\prime} + \tfrac{a'}{a}H_T^{(0)\prime}\right)\right\}Y = \Phi_A(\eta)Y \ , \tag{4.11}$$

$$\Phi_H = H_L - \tfrac{1}{3}\nabla^2 H_T - \tfrac{a'}{a}\left(B + H_T'\right) \tag{4.12}$$

$$= \left\{H_L + \tfrac{1}{3}H_T^{(0)} + \tfrac{a'}{ka}\left(B^{(0)} - \tfrac{1}{k}H_T^{(0)\prime}\right)\right\}Y = \Phi_H(\eta)Y \ , \tag{4.13}$$

where from now on the prime denotes derivative with respect to the conformal time η. Kodama and Sasaki [69] introduced other potentials, which are linear combinations of Φ_A and Φ_H.

GI matter perturbations There is one GI velocity perturbation variable, but the energy density perturbation is not uniquely defined

$$V_S = v + H_T' \quad \Rightarrow \quad V_{S|\alpha} = V_S^{(0)}(\eta)Y_\alpha^{(0)} \qquad V_S^{(0)}(\eta) = v^{(0)} - \tfrac{1}{k}H_T^{(0)\prime} \,, \tag{4.14}$$

$$\varepsilon_m = \delta - 3(1+w)\tfrac{a'}{a}(v-B) = \varepsilon_m(\eta)Y \quad \varepsilon_m(\eta) = \delta + 3(1+w)\tfrac{a'}{ka}(v^{(0)} - B^{(0)}) \,, \tag{4.15}$$

$$\varepsilon_g = \varepsilon_m + 3(1+w)\tfrac{a'}{a}V_S = \varepsilon(\eta)Y \,, \qquad \varepsilon_g(\eta) = \varepsilon_m(\eta) - 3(1+w)\tfrac{a'}{ka}V_S^{(0)} \,. \tag{4.16}$$

As pointed out by Kodama and Sasaki (1984) [69], there is not a preferred choice of GI density perturbation in this context, as many other GI combinations are possible: each of them is constructed in order to reduce to δ in a particular time slicing. For example the above given ε_m reduce to δ in the slicing $v = B$ (the velocity orthogonal slicing [1, 69]), and ε_g reduce to δ in the slicing in which $B = H_T'$ (the zero-shear slicing [1, 69]).

4.1.2 GI vector perturbations

GI metric perturbation There is only one GI metric perturbation, the "frame dragging potential"

$$\Psi_\alpha = B_\alpha^S + H_{T\alpha}^{S\prime} = \Psi(\eta)Y_\alpha^{(1)} \,, \quad \Psi(\eta) = B^{(1)} - \tfrac{1}{k}H_T^{(1)\prime} \,; \tag{4.17}$$

the reason for this denomination is that it appears as the potential for the vorticity variable V_C in the vector part of the GI version of the ADM momentum constraint (see section 4.3).

GI matter perturbations There are two GI matter velocity variables, related to shear and vorticity respectively

$$V_{S\alpha} = v_\alpha^S + H_{T\alpha}^{S\prime} = V_S^{(1)}(\eta)Y_\alpha^{(1)} \,, \quad V_S^{(1)}(\eta) = v^{(1)} - \tfrac{1}{k}H_T^{(1)\prime} \,, \tag{4.18}$$

$$V_{C\alpha} = v_\alpha^S - B_\alpha^S = V_C(\eta)Y_\alpha^{(1)} \,, \qquad V_C(\eta) = v^{(1)} - B^{(1)} \,; \tag{4.19}$$

given the above defined potential Ψ_α, these two variables are related by

$$\Psi_\alpha = V_{S\alpha} - V_{C\alpha} \,, \quad \Psi(\eta) = V_S^{(1)} - V_C^{(1)} \,. \tag{4.20}$$

4.1.3 GI tensor perturbation

The TT part of the metric $H^{TT}_{T\alpha\beta}$ is GI by itself; the TT part of the anisotropic pressure $\pi^{TT}_{\alpha\beta}$ is the GI matter tensor perturbation.

4.2 Expansion of GI perturbation variables

The variables covariantly defined in section 3.5.3 are, by themselves, exact quantities (defined in any spacetime) and can of course be expanded in terms of gauge dependent perturbations; however these variables are GI by themselves, therefore to first order we expect them to appear as linear combinations of the GI variables above, introduced by Bardeen. For spatial (i.e. orthogonal to u^a) vectors and tensors, it turns out that (0) and $(0,\alpha)$ components vanish; moreover, for the sake of comparison with Bardeen's variables, we want to use here the slash derivative rather than ${}^{(3)}\nabla$, and the Y harmonics rather than the covariantly constant (in proper time) harmonics Q. The following relations therefore should be taken as expressing the spatial components of 4-vectors and 4-tensors in terms of 3-vectors and 3-tensors in the conformal background 3-space with metric $\gamma_{\alpha\beta}$; raising and lowering of indices should be carried out with the metric $h_{\alpha\beta}$, giving extra a factors on the right hand sides (see appendix B).

4.2.1 Expansion procedure

To expand at first-order the covariant variables, we have to use their definitions in terms of basic matter, velocity and metric variables. We don't give here all the details of these simple but length and cumbersome calculations: rather we want just to give an example. The perturbations of the basic curvature variables, i.e. the Christoffel symbols, Riemann and Ricci tensor components can be found in the appendix of Kodama and Sasaki [69] (1984): our calculations are based on these expressions.

As an example, consider the expansion scalar Θ; we have

$$
\begin{aligned}
\Theta \equiv u^a{}_{;a} &= u^a{}_{,a} + \Gamma^a_{ab} u^b \\
&= \bar{u}^a; a + \delta u^a{}_{,a} + \bar{\Gamma}^a_{ab}\delta u^b + \bar{u}^b \delta\Gamma^a_{ab} \\
&= \bar{\Theta} + \bar{u}^0 \delta\Gamma^a_{a0} + \partial_0 \delta u^0 + \partial_\alpha \delta u^\alpha + \delta u^0 \bar{\Gamma}^a_{a0} + \delta u^\alpha \bar{\Gamma}^a_{a\alpha} \ ,
\end{aligned}
$$

i.e.

$$\Theta = \bar{\Theta} + \bar{u}^0 \delta\Gamma^a_{a0} + \partial_0 \delta u^0 + \delta u^0 \bar{\Gamma}^a_{a0} + \delta u^\alpha{}_{|\alpha} \, . \tag{4.21}$$

At this point one has to substitute for the Christoffel symbols and their perturbations in terms of the metric components, and use $\delta u^\alpha = a^{-1} v^\alpha$ and $\delta u^0 = -a^{-1} A$ from (4.6) and the fact that we use the conformal time (η) coordinates.

Another important quantity to consider is the projection tensor $h_a{}^b$: at fist order we have

$$h_a{}^b = \bar{h}_a{}^b + \bar{u}_a \delta u^b + \bar{u}^b \delta u_a \; ; \tag{4.22}$$

again, one has to substitute from (4.6). We turn now on the results we get on applying the first - order expansion procedure sketched here to our set of covariant variables.

4.2.2 Kinematical quantities

The expansion scalar is obviously gauge dependent, as it can be explicitely seen in (4.21), where there is a non - vanishing background contribution. Acceleration, shear and vorticity of the fluid flow are GI; expanding them at first - order and expressing the resulting expressions in terms of Bardeen's variables we get

$$\Theta = \bar{\Theta} + \delta\Theta \, , \tag{4.23}$$

$$\delta\Theta = \tfrac{3}{a}\left[H_L' - \tfrac{a'}{a} A + \tfrac{1}{3}\nabla^2 v\right] \tag{4.24}$$

$$= \tfrac{3}{a}\left[H_L' - \tfrac{a'}{a} A + \tfrac{k}{3} v^{(0)}\right] Y \, , \tag{4.25}$$

$$a_\alpha = \left[\Phi_{A|\alpha} + V_{S|\alpha}' + \tfrac{a'}{a} V_{S|\alpha}\right] + \left[V_{C\alpha}' + \tfrac{a'}{a} V_{C\alpha}\right] \tag{4.26}$$

$$= \left[V_S^{(0)\prime} + \tfrac{a'}{a} V_S^{(0)} - k\Phi_A\right] Y_\alpha^{(0)} + \left[V_C' + \tfrac{a'}{a} V_C\right] Y_\alpha^{(1)} \, , \tag{4.27}$$

$$\sigma_{\alpha\beta} = a\left[\nabla_{\alpha\beta} V_S + V_{S(\alpha|\beta)} + H_{T\alpha\beta}^{TT\,\prime}\right] \tag{4.28}$$

$$= a\left[-k V_S^{(0)} Y_{\alpha\beta}^{(0)} - k V_S^{(1)} Y_{\alpha\beta}^{(1)} + H_T^{(2)\prime} Y_{\alpha\beta}^{(2)}\right] \, , \tag{4.29}$$

$$\omega_{\alpha\beta} = a V_{C[\alpha|\beta]} = a V_C Y_{[\alpha|\beta]}^{(1)} \, . \tag{4.30}$$

The harmonic parts of these equations have also been obtained by Goode [50, 51].

4.2.3 Curvature variables

The 3-curvature scalar is GI only for a flat background model; the trace-free part of the 3-Ricci tensor is GI, together with the electric and magnetic part of the Weyl tensor. For these quantities we obtain

$$^{(3)}R = {}^{(3)}\bar{R} + \delta^{(3)}R = \tfrac{6K}{a^2} + \delta^{(3)}R\,, \tag{4.31}$$

$$\delta^{(3)}R = a^{-2}\left[-4\tfrac{a'}{a}\nabla^2(v-B) - 4(\nabla^2+3K)H_L + 2(\nabla_{\alpha\beta}H_T)^{|\alpha\beta}\right] \tag{4.32}$$

$$= a^{-2}\left\{-4\tfrac{a'}{a}k(v^{(0)}-B^{(0)}) + 4(k^2-3K)\left[H_L+\tfrac{1}{3}H_T^{(0)}\right]\right\}Y \tag{4.33}$$

$$= a^{-2}\left\{-4\left[(\nabla^2+3K)\left(\Phi_H+\tfrac{a'}{a}V_S\right)\right] + 12K\tfrac{a'}{a}(v-B)\right\} \tag{4.34}$$

$$= a^{-2}\left\{4(k^2-3K)\left[\Phi_H - \tfrac{a'}{ka}V_S^{(0)}\right] - 12K\tfrac{a'}{ka}(v^{(0)}-B^{(0)})\right\}Y\,, \tag{4.35}$$

$$^{(3)}\mathcal{R}_{\alpha\beta} = -\tfrac{a'}{a}\nabla_{\alpha\beta}V_S + \tfrac{1}{2}\nabla_{\alpha\beta}(\Phi_A-\Phi_H) + \tfrac{a^2}{2}\nabla_{\alpha\beta}\pi + (2K-\nabla^2)H^{TT}_{T\alpha\beta} \tag{4.36}$$

$$-\tfrac{a'}{a}V_{S(\alpha|\beta)} - \tfrac{1}{2}\Psi'_{(\alpha|\beta)} + \tfrac{1}{2}a^2\pi^S_{(\alpha|\beta)} - \tfrac{a'}{a}V_{C[\alpha|\beta]} \tag{4.37}$$

$$= \left[k\tfrac{a'}{a}V_S^{(0)} + \tfrac{k^2}{2}(\Phi_A-\Phi_H) + \tfrac{a^2}{2}p\pi_T^{(0)}\right]Y^{(0)}_{\alpha\beta} + (k^2+2K)H_T^{(2)}Y^{(2)}_{\alpha\beta} \tag{4.38}$$

$$+\left[k\tfrac{a'}{a}V_S^{(1)} + \tfrac{1}{2}k\Psi' + \tfrac{a^2}{2}p\pi_T^{(1)}\right]Y^{(1)}_{\alpha\beta} - \tfrac{a'}{a}V_C Y^{(1)}_{[\alpha|\beta]}\,, \tag{4.39}$$

$$E_{\alpha\beta} = \tfrac{1}{2}\left\{\nabla_{\alpha\beta}(\Phi_A-\Phi_H) - \Psi'_{(\alpha|\beta)} - [H^{TT}_{T\alpha\beta}{}'' + (\nabla^2-2K)H_{T\alpha\beta}]\right\} \tag{4.40}$$

$$= \tfrac{1}{2}\left\{k^2(\Phi_A-\Phi_H)Y^{(0)}_{\alpha\beta} + k\Psi' Y^{(1)}_{\alpha\beta} - [H_T^{(2)\prime\prime} - (k^2+2K)H_T^{(2)}]Y^{(2)}_{\alpha\beta}\right\}\,, \tag{4.41}$$

$$H_{\alpha\beta} = -a^{-2}\left[\tfrac{1}{2}\Psi^{\gamma}{}_{|(\alpha}{}^{|\delta}\eta_{\beta)0\gamma\delta} + H^{TT\prime\gamma|\delta}_{T(\alpha}\eta_{\beta)0\gamma\delta}\right] \tag{4.42}$$

$$= a^{-2}\left[\tfrac{1}{2}\Psi Y^{(1)\gamma}{}_{|(\alpha}{}^{|\delta}\eta_{\beta)0\gamma\delta} + H_T^{(2)\prime}Y^{(2)\gamma|\delta}_{(\alpha}\eta_{\beta)0\gamma\delta}\right]\,. \tag{4.43}$$

4.2.4 GI gradients

As we have seen in the previous chapter, spatial gradients (i.e. orthogonal to u^a) have a particular relevance within the covariant approach to GI perturbations. The spatial gradient of any scalar function $f = \bar{f} + \delta f$ with non constant background value $\bar{f}$ can be expanded to first order to give

$$^{(3)}\nabla_\alpha f = (\delta f)_{,\alpha} + \bar{u}^0\partial_0\bar{f}\,\delta u_\alpha = [\delta f + \bar{f}'(v-B)]_{,\alpha} + \bar{f}'\,V_{C\alpha} \tag{4.44}$$

$$= -k[\delta f(\eta) - \tfrac{\bar{f}'}{k}(v^{(0)}-B^{(0)})]Y^{(0)}_\alpha + \bar{f}'V_C Y^{(1)}_\alpha\,. \tag{4.45}$$

In particular we can express in this way the gradients of energy density, expansion and 3-curvature scalar ($h = \mu + p$)

$$\mathcal{D}_\alpha = a\,\varepsilon_{m|\alpha} - 3a'(1+w)V_{C\alpha} \tag{4.46}$$

$$= -ka\,\varepsilon_m(\eta)Y^{(0)}_\alpha - 3a'(1+w)V_C(\eta)Y^{(1)}_\alpha\,, \tag{4.47}$$

$$\mathcal{Z}_\alpha = 3\left[\Phi'_{H|\alpha} - \tfrac{a'}{a}\Phi_{A|\alpha}\right] + (\nabla^2 V_S)_{|\alpha} + \left[3K - \tfrac{3}{2}\kappa h a^2\right]V_{S|\alpha} + \left[3K - \tfrac{3}{2}\kappa h a^2\right]V_{C\alpha} \tag{4.48}$$

$$= \left\{-3k\left(\Phi'_H - \tfrac{a'}{a}\Phi_A\right) + \left[(3K - k^2) - \tfrac{3}{2}\kappa h a^2\right]V^{(0)}_S\right\}Y^{(0)}_\alpha \tag{4.49}$$

$$+ \left[3K - \tfrac{3}{2}\kappa h a^2\right]V_C(\eta)Y^{(1)}_\alpha\,, \tag{4.50}$$

$$C_\alpha = a^3\,{}^{(3)}\nabla_\alpha{}^{(3)}R = a^3 R^*_{|\alpha} - 12K\tfrac{a'}{a}V_{C\alpha} \tag{4.51}$$

$$= a\left\{-4[(\nabla^2 + 3K)(\Phi_H + \tfrac{a'}{a}V_S)]_{|\alpha} - 12K\tfrac{a'}{a}V_{C\alpha}\right\} \tag{4.52}$$

$$= a\left\{-k\left[4(k^2 - 3K)\left(\Phi_H - \tfrac{a'}{ka}V^{(0)}_S\right)\right]Y^{(0)}_\alpha - 12K\tfrac{a'}{a}V_C Y^{(1)}_\alpha\right\}\,. \tag{4.53}$$

4.2.5 Viscous fluid terms

We can express the energy flux q_a and the anisotropic pressure π_{ab} in terms of the GI metric and velocity potential introduced in section 4.1; using Einstein's equations to substitute the energy momentum tensor T^{ab} with the Ricci tensor in the q^a and π_{ab} definitions we get

$$\kappa q_a = -h_a{}^c R_{cd} u^d\,, \tag{4.54}$$

$$\kappa q_\alpha = -a\left\{\kappa h V_{S|\alpha} - \tfrac{2}{a^2}\left[\Phi'_{H|\alpha} - \tfrac{a'}{a}\Phi_{A|\alpha}\right] + \kappa h V_{C\alpha} + \tfrac{1}{2a^2}(\nabla^2 + 2K)\Psi_\alpha\right\} \tag{4.55}$$

$$= -a\left\{\left[\kappa h V^{(0)}_S + \tfrac{2k}{a^2}\left(\Phi'_H - \tfrac{a'}{a}\Phi_A\right)\right]Y^{(0)}_\alpha - \left[\kappa h V_C - \tfrac{1}{2a^2}(k^2 - 2K)\Psi\right]\right\}Y^{(1)}_\alpha\,, \tag{4.56}$$

$$\kappa\pi_{ab} = h_a{}^c h_b{}^d R_{cd} - \tfrac{1}{3}h_{ab}h_{cd}R^{cd}\,, \tag{4.57}$$

$$\kappa\pi_{\alpha\beta} = -\nabla_{\alpha\beta}(\Phi_H + \Phi_A) + [\Psi'_{(\alpha|\beta)} + 2\tfrac{a'}{a}\Psi_{(\alpha|\beta)}] \tag{4.58}$$

$$+[H^{TT\prime\prime}_{T\alpha\beta} + 2\tfrac{a'}{a}H^{TT\prime}_{T\alpha\beta} + (2K - \nabla^2)H^{TT}_{T\alpha\beta}] \tag{4.59}$$

$$= \left\{-\tfrac{k^2}{a^2}(\Phi_H + \Phi_A)\right\}Y^{(0)}_{\alpha\beta} + \left\{\Psi' + 2\tfrac{a'}{a}\Psi\right\}Y^{(1)}_{(\alpha|\beta)} \tag{4.60}$$

$$+\left\{H^{(2)\prime\prime}_T + 2\tfrac{a'}{a}H^{(2)\prime}_T + (k^2 + 2K)H_T\right\}Y^{(2)}_{\alpha\beta}\,. \tag{4.61}$$

4.2.6 Locally defined GI scalar variables

In section 3.5.3 we have introduced locally defined 4-scalar GI variables as nothing but the divergence of the corresponding spatial GI gradients or the total divergence of the corresponding GI tensors. Since the solenoidal and TT parts of the above expressions have, by definition, vanishing divergences, the expressions of the 4-scalars in terms of Bardeen's variables are nothing but the scalar part of the corresponding vectors and tensors; in Fourier space, their harmonics are just proportional through k factors. For example

$$\Delta = a^{(3)}\nabla^a\mathcal{D}_a = \nabla^2\varepsilon_m = -k^2\varepsilon_m(\eta)Y\ , \quad \Delta(\eta) = -k^2\varepsilon_m(\eta)\ , \tag{4.62}$$

$$C = a^{(3)}\nabla^a C_a = a^2\,\nabla^2 R^* = -k^2a^2\,R^*(\eta)Y\ , \quad C(\eta) = -k^2a^2R^*(\eta)\ . \tag{4.63}$$

4.3 Interpretation of Bardeen's variables

The relations above can be used to give an intrinsic physical and geometrical meaning to Bardeen's variables, and also to recover his equations: indeed, although Bardeen correctly pointed out that "*only gauge - invariant quantities have any inherent physical meaning*", he was able to interpret most of his variables only within specific gauge choices. In effect, equations (4.29), (4.30) were presented by Bardeen, thus giving an intrinsic significance to the variables $V_S^{(0)}$, $V_S^{(1)}$ and V_C as the harmonic components of the shear and vorticity: in equations (4.28) and (4.30) we have derived these quantities from the scalar velocity potential V_S and from the vector velocity $V_{S\alpha}$ and $V_{C\alpha}$. In our view, although working in Fourier space has the advantage of reducing equations to algebric relations, the presentations of the same equations in the coordinate space simplify somehow the physical interpretation.

The variable Ψ_α (4.17) was interpreted by Bardeen as a frame dragging potential because it appears as the potential for the vorticity variable V_C in the vector part of the GI version of the ADM momentum constraint, his Eq. (4.12) [the $(0,\alpha)$ Einstein equation (A2b) in BI]. This appears here directly, as the vector part of the GI energy flux, Eq. (4.56): in effect, we get exactly his equation if we set to zero the left hand side (i.e. if we pass from the PF to the EF), and we get the PF version of his equation on using Eq. (4.8) and equating the vector parts of the resulting equation.

The variable Ψ_α (4.17) acquires here new geometrical meaning through Eq. (4.42), (4.43): it is the vector part of the magnetic component of the Weyl tensor (this relation was also derived by Goode 1989 [51], and Hwang and Vishniac 1990 [63]). Note that an evolution equation for Ψ_α appears here directly as the vector part of (4.58), (4.60), on using (4.9) and equating the vector parts of the resulting equation.

Perhaps the more interesting results here are those regarding the GI scalar density and potential perturbations. The variables ε_m (4.15), interpreted by Bardeen as the usual density perturbation $\delta\mu/\mu$ within the comoving gauges $v^{(0)} - B^{(0)} = 0$, acquire a covariant significance as the scalar "potential" for the fractional density gradient $\mathcal{D}_a$ (4.46), i.e. its scalar harmonic component, without the need of specifying any time slicing condition; obviously, it is also the potential for the divergence Δ (4.62) of $\mathcal{D}_a$ (or its harmonic component). We could also expand P_a and $\mathcal{E}_a$ in (3.74), (3.75) using (4.44), (4.45): clearly, it turns out that $\mathcal{E}_a$ is analogous to Bardeen's entropy perturbation η.

As we have seen, Bardeen defined two independent GI metric potentials Φ_A (4.11) and Φ_H (4.13), for which we give here also the expressions (4.10) and (4.12) in coordinate space (these expressions have been given also by Stewart 1990 [117]); many other GI combinations are possible (see e.g. Kodama and Sasaki 1984 [69]). For a perfect fluid ($\pi_{ab} = 0$) $\Phi_A = -\Phi_H$, but in general, from (4.9), (4.58), (4.60)

$$a^2\nabla_{\alpha\beta}\pi = -\nabla_{\alpha\beta}(\Phi_H + \Phi_A) \quad \Leftrightarrow \quad a^2 p\pi_T^{(0)} = -k^2(\Phi_H + \Phi_A) \ , \tag{4.64}$$

i.e. Eq. (4.4) in BI. It seems therefore that the two potentials which have a direct physical interpretation are

$$\Phi_\pi = \tfrac{1}{2}(\Phi_H + \Phi_A) \ , \quad \Phi_N = \tfrac{1}{2}(\Phi_A - \Phi_H) \ ; \tag{4.65}$$

indeed while the former Φ_π is a stress potential through (4.64), the latter Φ_N plays exactly the role of a Newtonian gravitational potential. This interpretation follows directly through (4.40), where the scalar part of $E_{\alpha\beta}$ has exactly the same form it has in Newtonian theory $E_{\alpha\beta} = \nabla_{\alpha\beta}\Phi_N$ (see Ellis 1971 [27]), independently of any gauge choice. We may say that (the scalar part of) $E_{\alpha\beta}$ is the part of the curvature which depends purely on Φ_N and represents a purely tidal force, while in general other parts such as $\mathcal{R}_{\alpha\beta}$, ${}^{(3)}R$ and its gradient C_a are also affected by Φ_π and the shear V_S, as can be seen directly from (4.36), and from (4.34), (4.52) on using (4.64) and (4.65).

Finally, we note that (4.51), (4.53) give geometrical significance to the variable R^* defined by Goode (1989) [51] as the scalar part of the 3-curvature gradient C_a, while its variable $S_{\alpha\beta}$ appears here as the part of the trace-free 3-Ricci $\mathcal{R}_{\alpha\beta}$ which does not depend on vector perturbations, without assuming a vanishing fluid vorticity.

4.3.1 Bardeen's equations

All the equations in BI can be directly derived using the relations in section 4.2 and remembering that it follows from the unicity of the splitting (4.2), (4.3) that in any equation the scalar, vector and tensor parts are separately equal. In particular:

scalar equations: we already obtained (BI 4.4), i.e. (4.64); (BI 4.3) follows here from (the scalar parts of) (4.41), (4.64) and (3.163); (BI 4.5) follows from (4.26) plus (3.153); (BI 4.7) follows from (4.56) and (4.64); (BI 4.9) can be obtained substituting (4.62) in (3.183);

vector equations: as we already said (BI 4.12) follows from (the vector part of) (4.56); (BI 4.13) is obtained substituting (4.30) in (3.161);

tensor equation: there is one tensor equation for the tensor metric potential $H_T^{(2)}$: this follows here using the tensor part of (4.9) on the left hand side of (4.61).

Chapter 5

PERTURBATIONS IN A SCALAR FIELD DOMINATED UNIVERSE

Aim of this chapter is to apply the covariant Gi formalism of chapter 3 to a FLRW universe dominated by a classical minimally coupled scalar field ϕ.[1] Scalar field dominated universes have attained prominence in the last decade through the *Inflationary Universe* idea [53, 77, 97], and perturbations of such universes are potentially important as seeds of galaxy growth. Bardeen's formalism has been applied to the inflationary universe situation by various authors (e.g. [3], cf. also [95] and references therein), actually working either in the comoving gauge [3, 82, 83] or in the uniform Hubble constant gauge [3].

As in chapter 3, emphasis is given here on curvature perturbations (cf. [55]), which are naturally GI, rather than metric perturbations (as in [1, 69, 95]) which play no explicit role.[2] The background curvature K is maintained throughout: there are indeed both observational [113, 58, 124] and theoretical arguments [101, 102, 107] in favour of a density parameter $\Omega_0 < 1$, despite the prediction $\Omega_0 = 1$ of standard inflationary models [97]. It is therefore in principle interesting to look at "minimal" inflationary models [80, 125, 31] (see also [116, 39]) in which the inflationary phase do not last enough to drive to $\Omega_0 = 1$, and curvature effects on the perturbations evolution cannot

[1] This chapter is based on Ref. [10] (BED) and [9] (BE). Although there are many cross references with the material presented in the other chapters of this thesis, the presentation given here is rather self-contained.

[2] The link between our GI curvature variables and the GI metric potentials of Bardeen has been shown in chapter 4.

be neglected;[3] moreover the general formalism presented here (cf. [2, 95, 60]) could be extended to consider situations different from inflation in which a scalar field dominates (see e.g. [94]).

In sections 5.1 and 5.2.1 we set up the formalism, based on the natural slicing of the problem $\{\phi = const.\}$ and on its geometric characterization through the unit vector u^a orthogonal to these surfaces, and we present a set of exact covariant results valid in a any curved spacetime with a minimally coupled scalar field (cf. [86]).

In section 5.2.3 we define the GI dimensionless gradient Ψ_a of the momentum $\psi = \dot{\phi}$ of the field ϕ, and its divergence Ψ: in our approach these variables incorporate the whole matter perturbation, because the spatial gradient of ϕ vanishes through our geometrical choice of the frame u^a. We show that the density perturbation Δ is simply proportional to the momentum perturbation Ψ ($\Delta = \gamma\Psi$, cf. [3, 81]), and that it characterize matter clumping. We also introduce a variable Φ for ease of comparison with other works [3, 81, 83].

In section 5.2.5 we present various possible pairs of equations coupling the evolution of any of the matter perturbation variables with that of the curvature perturbation C, or with that of a related quantity, $\tilde{C}$. We also discuss if and when C or $\tilde{C}$ are conserved quantities on scales larger than the Hubble horizon: while in this limit C is a constant of motion in a flat background [83, 36], none of them is conserved if $K \neq 0$ in general, contrary to what happens for a barotropic fluid (when $K \neq 0$, $\tilde{C}$ is conserved for such a fluid [63, 36]). The second order evolution equations equivalent to the above mentioned systems of first-order equations are also derived at the end of this section.

In section 5.3 we present solutions in simple cases, comparing them with standard results in the literature [3, 83]. We also examine perturbations in a coasting, scalar field dominated, FLRW universe with general curvature K. Open models seem particularly interesting, since any previously existing perturbation is erased during evolution, while Ω stay constant: this could naturally provide the "clean slate" necessary for a successive "minimal" inflation (not driving $\Omega_0 = 1$) in order to satisfy constraints from the observed large-scale isotropy of the cosmic microwave background [110].

[3]We do not discuss *the probability* of an inflationary scenario with $\Omega_0 \neq 1$, but we assume that this is a possibility.

5.1 Preliminaries

5.1.1 The scalar field

In a completely general spacetime with metric g_{ab} and signature $(-\ +\ ++)$, let us consider a minimally coupled scalar field with Lagrangian density (conventions as in [126])

$$\mathcal{L}_\phi = -\sqrt{-g}\left[\tfrac{1}{2}\nabla_a\phi\nabla^a\phi + V(\phi)\right] , \tag{5.1}$$

where $V(\phi)$ is a general (effective) potential expressing the self interaction of the scalar field; ∇_a is the covariant derivative with respect to the metric g_{ab}.[4]

Then the equation of motion for the field ϕ following from $\mathcal{L}_\phi$ is the Klein - Gordon equation

$$\nabla_a\nabla^a\phi - V'(\phi) = 0 , \tag{5.2}$$

(from now on the prime indicate a derivative with respect to ϕ), and ϕ has an energy - momentum tensor of the form

$$T_{ab} = \nabla_a\phi\nabla_b\phi - g_{ab}\left[\tfrac{1}{2}\nabla_c\phi\nabla^c\phi + V(\phi)\right] ; \tag{5.3}$$

provided $\phi_{,a} \neq 0$, equation (5.2) follows from the conservation equation

$$\nabla_b T^{ab} = 0 . \tag{5.4}$$

We shall now assume that in the open region U of spacetime we consider, the *momentum density* $\nabla^a\phi$ is *timelike*:

$$\nabla_a\phi\nabla^a\phi < 0 . \tag{5.5}$$

This requirement implies two features: first, ϕ is not constant in U, and so $\{\phi = const.\}$ specifies well-defined surfaces in spacetime. When this is not true (i.e., ϕ is constant in U), then by (5.3) in U,

$$\nabla_a\phi = 0 \quad \Leftrightarrow \quad T_{ab} = -g_{ab}V(\phi) \quad \Rightarrow \quad V = const , \tag{5.6}$$

(the last being necessarily true due to the conservation law (5.4)), and we have an effective cosmological constant in U rather than a dynamical scalar field; the equations

[4] We shall assume in the following that ∇_a acts on the first argument on its right only [e.g., $\nabla^e T^{a\cdots b}{}_{c\cdots d}\nabla_f Q^{g\cdots h}{}_{l\cdots m} = (\nabla^e T^{a\cdots b}{}_{c\cdots d})(\nabla_f Q^{g\cdots h}{}_{l\cdots m})$]. When ∇_a acts on a group of arguments, this will enclosed in parenthesis.

may be handled accordingly.[5] In the case considered here, (5.5) implies unique normals are defined by the surfaces $\{\phi(x^j) = const.\}$. Secondly, (5.5) implies these surfaces are *spacelike*. This will be true in scalar-fields dominated exact FLRW models, and so will remain true in Universes in which the GI variables we use (see section 5.2.3) are small; we define such a Universe to be *almost* FLRW, thus (5.5) is a necessary condition for this latter model to be "close" to a FLRW model.

5.1.2 Kinematical quantities

It is our aim to give a formal description of the scalar field in terms of fluid quantities, therefore we have to assign a 4-velocity vector u^a to the scalar field itself. This will then define the dot derivative, i.e. the *proper time* derivative along the flow lines: $\dot{T}^{a\cdots b}{}_{c\cdots d} \equiv u^e\nabla_e T^{a\cdots b}{}_{c\cdots d}$. Now given the assumption (5.5), we can choose the 4-velocity field u^a as the unique timelike vector with unit magnitude parallel to the normals of the surfaces $\{\phi = const.\}$ [86] ,

$$u^a \equiv -\psi^{-1}\nabla^a\phi\ , \quad u^a u_a = -1\ , \quad \psi \equiv \dot{\phi} = u^a\nabla_a\phi = (-\nabla_a\phi\nabla^a\phi)^{1/2}\ , \tag{5.7}$$

where we have defined the field $\psi = \dot{\phi}$ to denote the momentum density magnitude (simply momentum from now on). The choice (5.7) defines u^a as the unique timelike eigenvector of the energy-momentum tensor (5.3).[6]

The kinematical quantities associated with the "flow vector" u^a can then be obtained as in chapter 2. Then the expansion, shear and acceleration are respectively given by

$$\Theta = -\nabla_a(\psi^{-1}\nabla^a\phi) = -\psi^{-1}\left[V'(\phi) + \dot{\psi}\right] \tag{5.8}$$

$$\sigma_{ab} = -\psi^{-1}\, h_a{}^c h_b{}^d \nabla_{(c}\left[\nabla_{d)}\phi\right] + \tfrac{1}{3}h_{ab}\nabla_c(\psi^{-1}\nabla^c\phi) \tag{5.9}$$

$$a_a = -\psi^{-1}\, {}^{(3)}\nabla_a\psi = -\psi^{-1}(\nabla_a\psi + u_a\dot{\psi})\ , \tag{5.10}$$

[5] Note particularly that in this case there is no preferred timelike vector field in U defined by the matter stress tensor; so the choice of u^a is arbitrary.

[6] The quantity ψ will be positive or negative depending on the initial conditions and the potential V; in general ϕ could oscillate and change sign even in an expanding phase, and the determination of u^a by (5.7) will be ill-defined on those surfaces where $\nabla_a\phi = 0 \Rightarrow \psi = 0$ (including the surfaces of maximum expansion in an oscillating Universe). This will not cause us a problem however, as we assume the solution is differentiable and (5.5) holds almost everywhere, so determination of u^a almost everywhere by this equation will extend (by continuity) to determination of u^a everywhere in U.

where the last equality in (5.8) follows on using the Klein-Gordon equation (5.2). We can see from (5.10) that ψ is an *acceleration potential* for the fluid flow [28]. Note also that the vorticity vanishes:

$$\omega_{ab} = -h_a{}^c h_b{}^d \nabla_{[d}\left(\psi^{-1}\nabla_{c]}\phi\right) = 0 \ , \tag{5.11}$$

an obvious result with the choice (5.7), so that ${}^{(3)}\nabla_a$ is the covariant derivative operator in the 3-spaces orthogonal to u^a, i.e. in the surfaces $\{\phi = const.\}$. As usual it is useful to introduce a scale length factor a along each flow-line by

$$\frac{\dot{a}}{a} \equiv \tfrac{1}{3}\Theta = H \ , \tag{5.12}$$

where H is the usual Hubble parameter if the Universe is homogeneous and isotropic. Finally, it is important to stress that

$${}^{(3)}\nabla_a\phi = 0 \tag{5.13}$$

follows from our choice of u^a via equation (5.7), a result that will be important for the choice of GI variables and for the perturbations equations.

5.1.3 Scalar field as a perfect fluid

It follows from our choice of the four velocity (5.7) that we can represent a minimally coupled scalar field as a perfect fluid; the energy-momentum tensor (5.3) takes the usual form for perfect fluids

$$T_{ab} = \mu u_a u_b + p h_{ab} \ , \tag{5.14}$$

the energy density μ and pressure p of the scalar field "fluid" being respectively:

$$\mu = \tfrac{1}{2}\psi^2 + V(\phi) = T + V \ , \tag{5.15}$$

$$p = \tfrac{1}{2}\psi^2 - V(\phi) = T - V \ , \tag{5.16}$$

where T denotes the kinetic term, $T = \frac{1}{2}\dot{\phi}^2$. If the scalar field is not minimally coupled this simple representation is no longer valid, but it is still possible to have an imperfect fluid form for the energy-momentum tensor [86].

Using the perfect fluid energy-momentum tensor (5.14) in (5.4) one obtains the energy and momentum conservation equations

$$\dot{\mu} + \Theta(\mu + p) = 0 \ , \tag{5.17}$$

$$a_a(\mu + p) + {}^{(3)}\nabla_a p = 0 \ . \tag{5.18}$$

If we now substitute μ and p from (5.15) and (5.16) in (5.17) we obtain the 1+3 form of the Klein - Gordon equation (5.2):

$$\ddot{\phi} + \Theta\dot{\phi} + V'(\phi) = 0 \ , \tag{5.19}$$

an exact ordinary differential equation for ϕ in any space - time with the choice (5.7) for the four - velocity. With the same substitution, (5.18) becomes an identity for the acceleration potential ψ. It is convenient to relate p and μ by the *index* γ defined by

$$p = (\gamma - 1)\mu \quad \Leftrightarrow \quad \gamma = \tfrac{p+\mu}{\mu} \ = \ \tfrac{\psi^2}{\mu} \ . \tag{5.20}$$

This index would be constant in the case of a simple one-component fluid, but in general will vary with time in the case of a scalar field:

$$\tfrac{\dot{\gamma}}{\gamma} = \Theta(\gamma - 2) - 2\tfrac{V'}{\psi} \ . \tag{5.21}$$

Finally, it is standard to *define* a speed of sound as [1]

$$c_s^2 \equiv \tfrac{\dot{p}}{\dot{\mu}} = \gamma - 1 - \tfrac{\dot{\gamma}}{\Theta\gamma} \ . \tag{5.22}$$

5.2 Gauge - invariant perturbations and their dynamics

5.2.1 Exact gravitational equations

As we have seen in chapter 2, Einstein's field equations are equivalent to a system of exact evolution and constraint equations for a set of covariantly defined quantities [27]: these are well defined physical fields in any space - time, and for an almost FLRW Universe most of them are natural GI variables (see chapter 3, EB, PaperI and [55, 63]).

We shall need here only two of these equations as well as the conservation equations (5.17) and (5.18). The first is the Raychaudhuri equation (2.47), governing the evolution of the expansion Θ (5.8). This exact equation for a scalar field takes the form

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 + 2\sigma^2 - A + \kappa[\psi^2 - V(\phi)] = 0 \ , \tag{5.23}$$

where $\kappa = 8\pi G = 8\pi M_{Pl}^{-2}$ is the gravitational constant, and

$$A \equiv \ \nabla_a a^a \ = \ {}^{(3)}\nabla_a a^a + a_a a^a \ = \ {}^{(3)}\nabla^a(\psi^{-1(3)}\nabla_a\psi) + \psi^{-2(3)}\nabla^a\psi^{(3)}\nabla_a\psi \ , \tag{5.24}$$

is the acceleration divergence. Thus the kinetic term ψ^2 contributes, with the shear term, to pulling the flow lines together, while the potential term V (assumed here to be always $V \geq 0$) tends to push them apart. The acceleration term A does not have a definite sign [but see the comment after (5.40)].

The second is the Gauss - Codacci equation (2.40), showing that the Ricci scalar ${}^{(3)}R$ of the 3-spaces orthogonal to u^a is

$$ {}^{(3)}R = 2\{-\tfrac{1}{3}\Theta^2 + \sigma^2 + \kappa[\tfrac{1}{2}\psi^2 + V(\phi)]\} \,. \tag{5.25} $$

It follows from the equations already given that the time derivative of the cosmological density parameter

$$ \Omega \equiv \kappa\mu/3H^2 = 3\kappa(\tfrac{1}{2}\psi^2 + V(\phi))/\Theta^2 \tag{5.26} $$

is

$$ \dot{\Omega} = \Omega(\Omega - 1)(\gamma - \tfrac{2}{3})\Theta + 2(2\sigma^2 - A)\Omega/\Theta \,. \tag{5.27} $$

We point out that all the equations this far are the exact equations for a classical scalar field in a curved spacetime, with no restriction on the field or spacetime other than (5.5).

5.2.2 Background dynamics

The field equations and conservation equations for an isotropic and spatially homogeneous cosmological model, whose matter content consists of a classical scalar field which is also spatially homogeneous, are the specialization of the equations in previous section to the case $\sigma = \omega = {}^{(3)}\nabla_a f = 0$, where f is any scalar quantity (see section 3.5.1).

Expressing the equations (5.23), (5.25), (5.19)≡(5.17) in terms of the Hubble parameter $H(t)$, the background (zero - order) equations are [40]

$$ 3\dot{H} + 3H^2 = \kappa[V(\phi) - \psi^2] \,, \tag{5.28} $$

$$ 3H^2 + 3\tfrac{K}{a^2} = \kappa[\tfrac{1}{2}\psi^2 + V(\phi)] \,, \tag{5.29} $$

$$ \dot{\psi} + 3H\psi + V'(\phi) = 0 \quad \Leftrightarrow \quad \dot{\mu} + 3H\psi^2 = 0 \,, \tag{5.30} $$

where all variables are a function of cosmic time t only.

We remind that the curvature parameter K is related to the density parameter Ω by the Friedmann equation (5.29) in the form:

$$\frac{K}{a^2} = H^2(\Omega - 1)\,, \;\Leftrightarrow\; \Omega = 1 + \frac{K}{a^2H^2} \tag{5.31}$$

and the density parameter Ω obeys the differential equation [87]

$$\dot{\Omega} = (2 - 3\gamma)H\Omega(1 - \Omega)\,. \tag{5.32}$$

5.2.3 Gauge - invariant variables

Spatial gradients

We have seen in chapter 3 that we can define exact variables that characterize inhomogeneities in any space - time, and also derive exact non - linear equations for them (EB). The most important of these variables are the *comoving fractional spatial gradient of the energy density* $\mathcal{D}_a$, the *comoving gradient of expansion* $\mathcal{Z}_a$, and the *comoving 3-curvature gradient* C_a. These quantities exactly characterize the inhomogeneity of any fluid; however we may want specifically characterize the inhomogeneity of the scalar field: we have already seen that, with our choice (5.7), the spatial gradient ${}^{(3)}\nabla_a\phi$ identically vanishes in any space - time [eq. (5.13)] (thus is also GI). It follows that in our approach the inhomogeneities in the matter field are completely incorporated in the spatial variation of the momentum $\dot{\phi}$,[7] i.e. in the gradient ${}^{(3)}\nabla_a\psi$, which appear as the natural GI matter perturbation variable. Thus we can define the dimensionless gradient

$$\Psi_a \equiv \frac{a}{\psi}{}^{(3)}\nabla_a\psi\,, \tag{5.33}$$

which is related to $\mathcal{D}_a$ by

$$\mathcal{D}_a = \frac{\psi^2}{\mu}\Psi_a = \gamma\Psi_a\,, \tag{5.34}$$

where we have used (5.15) and γ is given by (5.20); comparing (5.33) and (5.10) we see that Ψ_a is proportional to the acceleration: it is a GI measure of the spatial variation of proper time along the flow lines of u^a between two surfaces $\phi = consts.$ (cf. [27, 3]).

Up to now we have considered a completely general spacetime the matter content of which is given by a scalar field: from now on we shall restrict our considerations to

[7] The same result appears adopting the comoving gauge [3, 81]; our derivation is independent from any gauge choice, and follows from the geometrical choice of frame (5.7).

those Universes in which the magnitude of our GI variables is small; this automatically defines these models to be *almost* FLRW (see section 3.5.2).

In treating the perturbations evolution problem it is useful to have at hand a conserved quantity for those wavelengths that are larger than the Hubble radius: in a flat FLRW background the curvature gradient C_a is conserved in this latter limit. If $K \neq 0$ this is no longer true, but as we have seen in section 3.8.2 in certain cases of interest the related quantity $\tilde{C}_a$

$$\tilde{C}_a \equiv -\tfrac{4}{3}a^2\Theta\mathcal{Z}_a + 2\kappa\mu a^2\mathcal{D}_a\left(1 - \tfrac{2K}{a^2\kappa\mu\gamma}\right) , \tag{5.35}$$

is conserved if the fluid is barotropic and twist - free [36]; when $K = 0$, this reduces to C_a. We can express the GI variables $\mathcal{Z}_a$, C_a and $\tilde{C}_a$ in terms of Ψ_a: at linear order

$$\begin{aligned} \mathcal{Z}_a &= -(\Psi_a)^{\cdot} + (\Theta + 2\psi^{-1}V')\Psi_a , \\ C_a &= \tfrac{4}{3}\Theta a^2(\Psi_a)^{\cdot} + 2a^2\left[\kappa\psi^2 - \tfrac{2}{3}\Theta(\Theta + 2\psi^{-1}V')\right]\Psi_a , \\ \tilde{C}_a &= \tfrac{4}{3}\Theta a^2(\Psi_a)^{\cdot} + 4\left[2K - a^2\left(\tfrac{2}{3}\Theta\psi^{-1}V' + \kappa V\right)\right]\Psi_a . \end{aligned} \tag{5.36}$$

These can be regarded as ordinary linear equations for Ψ_a with $\mathcal{Z}_a$, C_a and $\tilde{C}_a$ as source terms, where the first can be also directly derived on taking the spatial gradient of the equation (5.30).

Scalar gauge - invariant variables

In treating cosmological inhomogeneities, the analysis is commonly restricted to scalar perturbations, as these are the only relevant to galaxies formation. We have seen in section 4.1 that this is usually achieved through a non local splitting [117]; instead in EBH (see section 3.7, Eq. (3.114), (3.116)) we defined a local decomposition for the gradient of $\mathcal{D}_a$. We note here that W_{ab} (3.117) is proportional to the vorticity, and therefore identically vanishes in a scalar field dominated universe.

We remind here that applying the operator $a^{(3)}\nabla_a$ on any of our GI gradients one obtains in the same way the corresponding decomposition [i.e., the analogue of (3.114)], the trace of which is a GI scalar variable, namely the comoving divergence of the gradient itself. Thus we have a set of scalar GI variables

$$\Delta \equiv a^{(3)}\nabla^a\mathcal{D}_a , \quad \mathcal{Z} \equiv a^{(3)}\nabla^a\mathcal{Z}_a , \quad C \equiv a^{(3)}\nabla^a C_a , \tag{5.37}$$

respectively giving the energy density, expansion and 3-curvature scalar perturbations; these are related by

$$C = -\tfrac{4}{3}\Theta a^2 \mathcal{Z} + 2\kappa\mu a^2 \Delta \; . \tag{5.38}$$

In the same way the scalar field

$$\Psi \equiv a^{(3)}\nabla^a \Psi_a \tag{5.39}$$

is a natural GI scalar variable characterizing the spatial distribution of the momentum $\dot\psi$, and related to Δ by

$$\Delta = \gamma\Psi \; . \tag{5.40}$$

From this and (5.33), (5.39) we see that $A = \Psi/S^2 = \Delta/(\gamma S^2)$ at first order; this clarify the gravitational role of Δ in (5.23): for $\gamma \geq 0$, when $\Delta < 0$ we have a *local* energy density enhancement that tends to slow down the expansion, while a local void ($\Delta > 0$) has the opposite effect. From this it is clear that the *comoving divergence* Δ of $\mathcal{D}_a$ is the scalar variable that characterizes matter clumping, therefore we shall focus here on this quantity and its companion variables

$$\tilde{C} \equiv a^{(3)}\nabla^a \tilde{C}_a = C - 4K\gamma^{-1}\Delta = C - 4K\Psi \; , \tag{5.41}$$

and on Ψ. Note that $\tilde{C}$ is a conserved quantity for scales larger than Hubble radius when the fluid is barotropic, even if $K \neq 0$ [63, 36] and $\omega \neq 0$ [36]; however, as we shall see, this is no longer true in general for a scalar field.

Expressions for $\mathcal{Z}, C$ and $\tilde{C}$ in terms of Ψ, analogous to (5.40), can be found, on taking the comoving divergence of (5.36) and using the relation (see section 3.3 for details):

$$a\,^{(3)}\nabla_a \dot{X}_b = (a\,^{(3)}\nabla_a X_b)^{\cdot} , \tag{5.42}$$

where X_a is any first-order vector orthogonal to u^a, $X_a u^a = 0$.

Finally, we find it useful to define the variable Φ for ease of comparison with the evolution of the Bardeen variable Φ_H[8]

$$\Phi \equiv \kappa\mu a^2 \Delta = \kappa\mu a^2 \gamma \Psi \; ; \tag{5.43}$$

this variable turns out to be particularly useful during epochs in which $\gamma \simeq$ *const.* and $\Omega \simeq 1$ (see section 5.3.1).

[8] We defined the vectorial variables $\Phi_a \equiv \kappa\mu S^2 \mathcal{D}_a$ corresponding to Φ in EHB (see section 3.8.2). Note that the variable Φ_H is defined by Bardeen [1] directly in the Fourier space, and has the same time behaviour as the variable Z used in [3] and [83].

5.2.4 Entropy perturbations

The minimally coupled scalar field we are examining in this chapter is formally equivalent to a perfect fluid. However the perfect fluid perturbation equations of chapter 3 do not describe the evolution of perturbations in a scalar field dominated universe, since they assume the perfect fluid to be *barotropic* also in the physical perturbed universe, i.e. they assume *adiabatic* perturbations. For the scalar field instead one has entropy perturbations. Although these have been assumed in the equations in section 3.9, it is worth to explicitely consider the situation for the scalar field case. The entropy perturbation follows from the perfect fluid equation of state $p = p(\mu, s)$ from which

$$a^{(3)}\nabla_a p = \left(\frac{\partial p}{\partial \mu}\right)_{(s)} a^{(3)}\nabla_a \mu + \left(\frac{\partial p}{\partial s}\right)_{(\mu)} a^{(3)}\nabla_a s \ , \tag{5.44}$$

where we have the usual thermodynamic partial derivatives at constant density and entropy. Note that we take advantage of having a perfect fluid, for which *entropy is constant along flow lines*[9] so that

$$c_s^2 \equiv \left(\frac{\partial p}{\partial \mu}\right)_{(s)} = \frac{\dot{p}}{\dot{\mu}} \ ; \tag{5.45}$$

therefore the usual definition of the speed of sound (5.22) [1] coincide with the standard thermodynamic definition.

Now we can recall that in section 3.5.3 we defined the comoving fractional pressure gradient and an entropy perturbation dimensionless variable respectively as

$$P_a \equiv \frac{a}{p}{}^{(3)}\nabla_a p \ , \quad \mathcal{E}_a \equiv \left(\frac{\partial p}{\partial s}\right)_{(\mu)} \frac{a}{p}{}^{(3)}\nabla_a s = P_a - \frac{c_s^2}{w}\mathcal{D}_a \ , \tag{5.46}$$

so that we have

$$pP_a = c_s^2 \mu \mathcal{D}_a + p\mathcal{E}_a \ . \tag{5.47}$$

We defined $\mathcal{E}_a$ in this way to have it as close as possible to the definition of Bardeen variable η [1, 3].[10] However for the scalar field we have

$$\mathcal{E}_a = (1 - c_s^2)\mu\mathcal{D}_a \quad \Rightarrow \quad pP_a = \mu\mathcal{D}_a \quad \Leftrightarrow \quad {}^{(3)}\nabla_a p = {}^{(3)}\nabla_a \mu \ , \tag{5.48}$$

an obvious result that can be obtained from the expression of μ and p directly, taking into account that ${}^{(3)}\nabla_a \phi = 0$.

[9] This is strictly true only if we can neglect the effect of bulk viscosity, which will be assumed here.

[10] In his paper Bardeen *define* η as the difference between pressure and density perturbation when the perturbation is not adiabatic, i.e. the second equality in (5.46) is used as a definition.

Identical relations hold for the scalar variables corresponding to $\mathcal{D}_a$, P_a and $\mathcal{E}_a$ and defined as described in section 5.2.3, i.e. for Δ, $P = a^{(3)}\nabla^a P_a$ and $\mathcal{E} = a^{(3)}\nabla^a \mathcal{E}_a$.

Thus for the scalar field

$$\mathcal{E} = (1 - c_s^2)\mu\Delta \ , \quad pP = \mu\Delta \ . \tag{5.49}$$

5.2.5 Perturbation equations

First - order linear equations

We shall examine here the linear evolution equations for the scalar our GI variables, specialized to the scalar field case.

We can determine the evolution of the matter perturbation variables Δ, Ψ and Φ through a system of two first - order (in proper time derivative) linear equations coupling their evolution to that of C or $\tilde{C}$.[11] The scalar field fluid is vorticity - free, $\omega = 0$: since in the linear approximation the equality (5.42) holds, spatial gradients and their comoving divergences satisfy the same evolution equations, the only change being that the operator ${}^{(3)}\nabla^2$ should be substituted by ${}^{(3)}\nabla^2 - 2K/a^2$ (see section 3.7.2) in the equation for the spatial gradient. This means that the harmonic components of the vectorial (e.g., $\mathcal{D}_a$) and scalar (e.g., Δ) variables satisfy the same ordinary differential equation, because ${}^{(3)}\nabla^2$ and ${}^{(3)}\nabla^2 - 2K/a^2$ have the same eigenvalue $-k^2/a^2$ acting on, respectively, a scalar harmonic $Q^{(k)}$ and its spatial gradient $Q_a^{(k)} \equiv {}^{(3)}\nabla_a Q^{(k)}$.

The equations we obtain for Δ and $\tilde{C}$ are:

$$\dot{\Delta} = \frac{3}{4}\frac{\gamma\tilde{C}}{a^2\Theta} + \left\{(\gamma - 1)\Theta - \left[\frac{3}{2}\kappa\mu\gamma - 3\frac{K}{a^2}\right]\Theta^{-1}\right\}\Delta \ , \tag{5.50}$$

$$\dot{\tilde{C}} = \frac{4}{3}\frac{\Theta a^2}{\gamma}{}^{(3)}\nabla^2\Delta + 8\frac{\Theta K}{\gamma}\Delta\left(\frac{\dot{\psi}}{\psi\Theta} + 1\right) \ . \tag{5.51}$$

The system for Δ and C is

$$\dot{\Delta} = \frac{3}{4}\frac{\gamma C}{a^2\Theta} + \left[(\gamma - 1)\Theta - \frac{3}{2}\kappa\mu\gamma\Theta^{-1}\right]\Delta \ , \tag{5.52}$$

$$\dot{C} = \frac{3KC}{a^2\Theta} + \frac{4}{3}\frac{\Theta a^2}{\gamma}{}^{(3)}\nabla^2\Delta + 4K\left(\frac{\Theta}{\gamma} - \frac{3}{2}\frac{\kappa\mu}{\Theta}\right)\Delta \ . \tag{5.53}$$

[11] As we have seen in chapter 3, through the constraint (5.38) and the definition (5.41) one can alternatively couple the first order evolution equation for any of the matter variables to the equation for $\mathcal{Z}$. We choose C and $\tilde{C}$ because then we can compare the scalar field case with the barotropic fluid case for $K \neq 0$, discussing conservation of this variable.

Because of the last term in (5.51), $\tilde{C}$ is not conserved in the long wavelength limit in the general case. To see this, it is convenient to rewrite the above equations in terms of the harmonic components of the variables we use. Since the eigenvalue k (introduced to have a unified notation for different values of K) does not directly correspond to physical wavelengths (unless $K = 0$), and the correspondence changes with K, we shall write here harmonic components equations only for the case $K = -1, 0$. In this case k is related to physical wavelengths by the time independent wavenumber ν:

$$k^2 = \nu^2 - K \ , \tag{5.54}$$

where $\nu \geq 0$ is a real number, and physical wavelengths are defined by [12]

$$\lambda \equiv \frac{a}{\nu} \, . \tag{5.55}$$

Then we have

$$\dot{\tilde{C}}_{(\nu)} = -4\frac{H^3 a^2}{\gamma}\left[\frac{\nu^2}{H^2 a^2} - \frac{6K}{H^2 a^2}\left(\frac{\dot{\psi}}{3H\dot{\psi}} + \frac{7}{6}\right)\right]\Delta_{(\nu)} \ , \tag{5.56}$$

where $K = -1, 0$; in the long wavelength limit the first term ν^2/H^2a^2 in the parenthesis is negligible, but the second will not be small in general. The latter vanishes however if $K = 0$, in which case $\tilde{C}$ reduces to the 3-curvature GI variable C [see (5.37) and (5.38)], which is then conserved in the long wavelength limit.

In the case $K = -1, 0$, the equation for the harmonic component $C_{(\nu)}$ of C is

$$\dot{C}_{(\nu)} = \frac{KC_{(\nu)}}{a^2 H} - \frac{4H^3 a^2}{\gamma}\left[\frac{\nu^2}{a^2 H^2} - \frac{K}{a^2 H^2}\left(4 - \frac{\gamma}{2}\frac{\kappa\mu}{H^2}\right)\right]\Delta_{(\nu)} \, . \tag{5.57}$$

We can write an equation for Ψ deriving it directly on applying the comoving derivative operator $a^{(3)}\nabla^a$ on any of the equations (5.36) [using also (5.42)]; an equation for $\Psi_{(\nu)}$ immediately follows on passing to the Fourier space. For example the equation coupling $\Psi_{(\nu)}$ to the variable $\tilde{C}_{(\nu)}$ is

$$\dot{\Psi}_{(\nu)} = \frac{\tilde{C}_{(\nu)}}{4Ha^2} - \left[H^{-1}\left(\frac{2K}{a^2} - \kappa V\right) - 2\dot{\psi}^{-1}V'\right]\Psi_{(\nu)} \, . \tag{5.58}$$

[12] Minimum values for k^2 are: $k^2 = 0$ for $K = 0$, $k^2 = 1$ for $K = -1$, $k^2 = 3$ for $K = 1$, where for this latter $k^2 = \nu(\nu + 2)$, $\nu = 1, 2...$ [54], see also [69, 76]. One can however assume the Universe is homogeneous over a scale L, and divide the space in volumes of this characteristic size; then k is an integer (e.g., see [83]). Often a factor 2π appears in the definition of λ, but this is not really needed here.

The companion equation then is (5.56), on substituting $\Delta_{(\nu)} = \gamma\Psi_{(\nu)}$; we note that the resulting pair of equations is particularly suited for treating an almost De Sitter inflationary phase, in which $\gamma \simeq 0$ and (5.53), (5.56) and (5.57) tend to blow up. In this sense, we may regard Ψ and C [or $\tilde{C}$, see (5.41)] as the fundamental perturbations variables, while $\Delta = \gamma\Psi$ will be correspondingly smaller in a De Sitter like phase.

Finally, the first order equation for $\Phi_{(\nu)}$ is

$$\dot{\Phi}_{(\nu)} + \left[H + H^{-1}\left(\frac{1}{2}\kappa\mu\gamma - \frac{K}{a^2}\right)\right]\Phi_{(\nu)} = \frac{\kappa\mu\gamma}{4H}\tilde{C}_{(\nu)} \ , \tag{5.59}$$

which follows on substituting (5.43) in (5.58). The equation corresponding to (5.59) (but for the vectorial variable Ψ_a) was derived in EHB, where it was coupled to C_a; (5.59) generalize to non flat universes equation (11) in [83].

Conserved quantities at $\lambda \gg H^{-1}$

As we already said, the curvature gradient C_a and its divergence C are conserved when the background universe is flat, $K = 0$. For $K \neq 0$ we can look at the evolution of $\tilde{C}$, which is conserved at large scales when the fluid is barotropic [36, 63]. Here we briefly discuss this point (see BE [9] see also [20, 62]), looking in particular at the two special cases of slow rolling, and $\gamma \approx$ *const.*

The *slow rolling approximation* is $\dot{\psi} \ll 3H\psi$. This can result from the evolution of the scalar field in certain effective potentials with a *plateau*; we see from (5.56) that $\tilde{C}$ is then conserved if

$$\frac{\nu^2}{H^2a^2} \ll 1 \ \ and \ \ \frac{7}{H^2a^2} \ll 1 \ , \tag{5.60}$$

i.e., if the long wavelength limit is valid up to $\nu^2 \leq 7$. It is immediate from (5.31) that the second inequality in (5.60) holds only if Ω is close enough to 1, but then also C is approximately conserved.

The second case is when γ *is slowly varying* (but not necessarily small): $H^{-1}|\dot{\gamma}/\gamma| \ll 1$. For simplicity, take $\gamma =$ *const.* along the flow lines, i.e. $\dot{\gamma} = 0$.[13]

[13] However γ will be not spatially constant:

$$\Gamma_a \equiv a^{(3)}\nabla_a\gamma = (2-\gamma)\mathcal{D}_a \quad \Gamma \equiv a^{(3)}\nabla^a\Gamma_a = (2-\gamma)\Delta \ , \tag{5.61}$$

are two GI variables describing the spatial variation of γ and simply related to the energy density GI variables.

Then it follows that the last term in (5.56) in this case is also a constant:

$$\frac{V'}{3H\dot{\psi}} = \frac{\gamma - 2}{2} \, . \tag{5.62}$$

Then for those scales $\nu < 1$ that never crossed the horizon (i.e. they have been always larger than H^{-1}) one should have $|7 - 3\gamma| \ll 1$ to have $\tilde{C}$ conserved in a phase when $1 - \Omega \simeq 1$, which is clearly impossible, because the hardest equation of state one can have is that of stiff matter, obtained when the potential is negligible with respect to the kinetic term, so $p \approx \mu$. Therefore also in this case $\tilde{C}$ is conserved out of the Hubble radius only if Ω is close enough to unity, and again also C will be conserved.

Note however that in general $\tilde{C}$ (or C for $K = 0$) will in fact be conserved in the long wavelength limit only if $(\tilde{C}H)^{-1}\dot{\tilde{C}} \approx -\nu^2/H^2a^2$, a condition of consistency that must be verified *a posteriori* once one has derived the mode in Ψ or Δ related to $\tilde{C}$ (C for $K = 0$, see [83]) General conserved quantities related to $\tilde{C}$, valid for any value of K and k, can be calculated for particular inflationary models, and will be presented elsewhere [20].

Second - order linear equations

When there is not a conserved quantity, its useful to derive a second order equation in order to investigate the evolution of the quantity we are interested in. For Δ such an equation would be the equivalent of the system (5.50) and (5.51). In general this equation has the form

$$\ddot{\Delta} + \mathcal{A}\dot{\Delta} - \mathcal{B}\Delta - {}^{(3)}\nabla^2\Delta = 0 \, , \tag{5.63}$$

where

$$\mathcal{A} \equiv \left(\tfrac{5}{3} - \gamma\right)\Theta - \tfrac{\dot{\gamma}}{\gamma} \, , \tag{5.64}$$

$$\mathcal{B} \equiv \left(1 - \tfrac{\gamma}{2}\right)\left[\Theta^2\left(\gamma - \tfrac{2}{3}\right) + 9\gamma\tfrac{K}{a^2}\right] + \Theta\tfrac{\dot{\gamma}}{\gamma} \, , \tag{5.65}$$

and $\dot{\gamma}/\gamma$ is given by (5.21). Then it is immediate to derive the equation for Ψ in a similar form:

$$\ddot{\Psi} + \tilde{\mathcal{A}}\dot{\Psi} - \tilde{\mathcal{B}}\Psi - {}^{(3)}\nabla^2\Psi = 0 \, , \tag{5.66}$$

where

$$\tilde{\mathcal{A}} \equiv \left(\tfrac{5}{3} - \gamma\right)\Theta + \tfrac{\dot{\gamma}}{\gamma} \, , \tag{5.67}$$

$$\tilde{\mathcal{B}} \equiv \left(1 - \tfrac{\gamma}{2}\right)\left[\Theta^2\left(\gamma - \tfrac{2}{3}\right) + 9\gamma\tfrac{K}{a^2}\right] + \tfrac{\dot{\gamma}}{\gamma}\left[\left(\gamma - \tfrac{2}{3}\right)\Theta - \left(\tfrac{\dot{\gamma}}{\gamma}\right)\right] \, . \tag{5.68}$$

In the next section, we shall look at perturbations evolution in simple cases using Φ, which again satisfies an equation similar to (5.63):

$$\ddot{\Phi} + \hat{\mathcal{A}}\dot{\Phi} - \hat{\mathcal{B}}\Phi - {}^{(3)}\nabla^2\Phi = 0 \; ; \tag{5.69}$$

where

$$\hat{\mathcal{A}} \equiv \Theta\left(\tfrac{1}{3}+\gamma\right) - \tfrac{\dot{\gamma}}{\gamma} \, , \tag{5.70}$$

$$\hat{\mathcal{B}} \equiv \tfrac{2K}{a^2}\left(1+\tfrac{3}{2}\gamma\right) + \tfrac{\Theta}{3}\tfrac{\dot{\gamma}}{\gamma} \, . \tag{5.71}$$

Passing in the Fourier space, the harmonic components of Δ, Ψ and Φ satisfy the equations above with the trivial substitution ${}^{(3)}\nabla^2 \to -K^2/a^2$. The resulting equation for $\Phi_{(k)}$ generalizes to $K \neq 0$ the equations for Z (see footnote 8) (2.20) in [3] and (8-30) in [83].

Finally we point out that also the evolution of the curvature perturbation variable C can be decoupled from that of other variables, i.e. a second order equation for C can be derived. Thus in our approach both matter variables (Δ, Ψ, and Φ) and the geometry-related variable (C) satisfy a second order linear equation (cf. [95]).

5.2.6 Independent variables and equations

In this section we have introduced various GI variables, relations between them, and equations governing their evolution. It is perhaps worth pointing out again that only two of these variables are independent: if we take $\mathcal{C}_a$ and $\mathcal{D}_a$ as independent variables, then $\mathcal{Z}_a$ is determined through the dynamical constraint (3.83), which has been derived on taking the spatial gradient of the Gauss-Codacci equation (5.25). All other variables have been introduced for convenience through "kinematical relationships" such as (5.33), (5.34), (5.35), (5.39) (5.40) and (5.43). Equation (5.38) is the dynamical constraint for scalar-type variables. The equations considered are enough to determine scalar-type perturbations; the whole set of covariant equations equivalent to Einstein linearized equations has been presented in chapter 3 (and has been derived in BDE[8]).

5.3 Solutions

We sketch here the solutions for the perturbation equations in some very simple cases, referring to the corresponding solutions for the background given in [40, 39].

5.3.1 $\gamma = constant$

In the case $\gamma = const.$ one obviously expects Ψ and Δ to have the same modes: indeed in this case $\tilde{\mathcal{A}} = \mathcal{A}$ from (5.67) (5.64) and $\tilde{\mathcal{B}} = \mathcal{B}$ from (5.68) and (5.65). The equation (5.69) takes a particularly simple form if expressed using the conformal time $d\eta = dt/a$:

$$\Phi''_{(k)} + 3\gamma\dot{a}(\eta)\Phi'_{(k)} - \left[2K\left(1 + \tfrac{3}{2}\gamma\right) - k^2\right]\Phi_{(k)} = 0 \; , \tag{5.72}$$

where a prime denotes derivative with respect to η and we have written the equation for the harmonic component $\Phi_{(k)}$ of Φ, in order to have an expression valid for any value of K (for each K, the appropriate substitution for the wavenumber ν is needed).

If in addition we have $K = 0$, it is immediate from (5.72) that $\Phi_{(k)}$ has a constant mode in the long wavelength limit ($k = \nu \to 0$), while the other mode is decaying. In this case it follows from (5.57) that the curvature perturbation $C_{(k)}$ is conserved, and (5.59) is a first integral for (5.72). From this one obtain the constant mode for these scales larger than the horizon [83, 37]

$$\Phi_{(k)} = \frac{C_{(k)}}{2(1 + \frac{2}{3\gamma})} \; , \tag{5.73}$$

a relation that holds also for barotropic fluids, and thus can be used to connect different epochs (with different $\gamma \approx const.$) in the history of a perturbation evolution outside the horizon.

Standard inflation: $\gamma \simeq 0$

The standard inflationary case corresponds to $\gamma \simeq 0$, which provides a De Sitter phase which lasts long enough to produce $\Omega_0 = 1$ today. In this scenario quantum generation of fluctuations takes place for wavelengths 'within the horizon' during inflation [77, 97]; then all the perturbation scales of interest exit the horizon in the last part of inflation, when already $\Omega = 1$ is a very good approximation. Thus we may reproduce the usual results [3, 83], which in the main follow from the conserved quantities discussed above (5.73) specialized to $\gamma \simeq 0$. Using (5.43) we can eliminate $\gamma_1 \simeq 0$ from (5.73) (thus avoiding the consequent blow up) and connect the inflationary phase with a subsequent epoch characterized by $\gamma_2 \simeq const.$ (e.g. a following radiation dominated era)

$$\Phi^{(1)}_{(k)}\left(1 + \tfrac{2}{3\gamma_1}\right) = \kappa\left[\mu a^2 \Psi_{(n)}\right]^{(1)}\left(\gamma_1 + \tfrac{2}{3}\right) \simeq \tfrac{2}{3}\kappa\left[\mu a^2 \Psi_{(n)}\right]^{(1)} = \Phi^{(2)}_{(k)}\left(1 + \tfrac{2}{3\gamma_2}\right) \; , \tag{5.74}$$

where $[\mu a^2 \Psi_{(n)}]^{(1)}$ is constant when γ_1 and $\Phi^{(1)}_{(k)}$ are constants.

$\gamma = 2/3$

For the case $\gamma = 2/3$ we have coasting solution for any value of K, i.e. the scale factor a grows linearly: $a = At$, $A = \dot{a}_C$. Since in this case $\mu a^2 = const.$, Φ has the same modes as Ψ and Δ, because of (5.43). Also, it follows from (5.32) that $\Omega = \Omega_C$ is constant during such a phase, and

$$A = [K(\Omega_C - 1)]^{-\frac{1}{2}} ; \tag{5.75}$$

for $K = 0 \Leftrightarrow \Omega = 1$ A is arbitrary. A peculiarity of the coasting phase is that the ratio of physical wavelengths to the Hubble scale $\lambda_H = H^{-1}$ is a constant, given by

$$\frac{\lambda}{\lambda_H} = \frac{aH}{k} = \frac{A}{k} . \tag{5.76}$$

Equation (5.72) further simplifies to

$$\Phi''_{(k)} + 2A\Phi'_{(k)} - (4K - k^2)\Phi_{(k)} = 0 , \tag{5.77}$$

which is then immediately solved. We distinguish three cases, where again we think of k as a label for the corresponding wavenumber ν valid for any value of K ($k^2 = \nu^2 - K$ for $K = 0, 1$, and $k^2 = \nu(\nu + 2)$ for $K = 1$ [54]):

a) $k^2 < A^2 + 4K$; then, defining $Q = \sqrt{\frac{4K-k^2}{A^2} + 1}$, we have

$$\Phi_{(k)} = \Phi^A_{(k)} \left(\frac{t}{t_i}\right)^{-1+Q} + \Phi^B_{(k)} \left(\frac{t}{t_i}\right)^{-1-Q} , \tag{5.78}$$

where we expressed the solution using proper time (t_i is some initial time), and $\Phi^A_{(k)}$ $\Phi^B_{(k)}$ are two constants.

b) $k^2 = A^2 + 4K$; in this case one has

$$\Phi_{(k)} = \left(\frac{t}{t_i}\right)^{-1} \left[\Phi^A_{(k)} + \Phi^B_{(k)} A^{-1} \ln\left(\frac{t}{t_i}\right)\right] , \tag{5.79}$$

i.e., only decaying modes.

c) $k^2 > A^2 + 4K$; these wavelength correspond to damped oscillatory modes:

$$\Phi_{(k)} = \left(\frac{t}{t_i}\right)^{-1} \left\{\Phi^A_{(k)} \cos\left[\tilde{Q} \ln\left(\frac{t}{t_i}\right)\right] + \Phi^B_{(k)} \sin\left[\tilde{Q} \ln\left(\frac{t}{t_i}\right)\right]\right\} , \tag{5.80}$$

where we defined $\tilde{Q} = \sqrt{\frac{k^2-4K}{A^2} - 1}$.

When $K = 0 \Leftrightarrow \Omega_C = 1$, we have the case *b)* for the scale $\lambda = H^{-1}$ on the horizon, thus *a)* corresponds to purely decaying modes for scales larger than the horizon, while scales inside it correspond to the damped oscillations of the case *c)*.

When $K = -1 \Leftrightarrow \Omega_C < 1$, we can distinguish two subcases. When $\Omega_C \leq 4/5$ we have damped oscillations given by (5.80) (with $\tilde{Q} = \sqrt{(\nu^2+5)/A^2 - 1}$) at *all scales*, irrespective of what is their ratio to the horizon. For $\Omega_C > 4/5$ there are purely decaying modes given by (5.78) in the long wavelength limit $\nu^2 \ll A^2$; more precisely, with $Q = \sqrt{1-(\nu^2+5)/A^2}$, we have the modes (5.78) for scales $\nu^2 < A^2$ if $\Omega_C > (\nu^2+4)/(\nu^2+5)$, while there are decaying modes (5.79) for the scale $\nu^2 = (5\Omega_C - 4)/(1-\Omega_C)$.

In the case of a closed universe $K = 1 \Leftrightarrow \Omega_C > 1$ we have $k^2 = \nu(\nu+2)$, where $\nu \geq 1$ is a positive integer now; as it is intuitive the longest perturbation wavelength correspond to $\lambda = a$ (for $\nu = 1$). To this scale correspond a growing and a decaying mode, given by (5.78) for $\pm Q$, $Q = \sqrt{\Omega_C}$.[14] However the next scale, given by $\nu = 2$, will have the same modes (5.78) only for $\Omega \leq 5/4$. In general, there will be growing and decaying modes corresponding to (5.78) with $Q = \sqrt{1+(4-k^2)/A^2}$ only if Ω_C is close enough to unity: $\Omega_C \leq (k^2-3)/(k^2-4)$. Thus, given Ω_C, most of the modes will have the damped oscillations given by (5.80) also for a closed universe.

The interesting application of these results is to a *coasting universe phase* with $K = -1$, resulting from an exponential potential (see [39]). As we see from the above, *all scales greater than the horizon scale are smoothed out by such an expansion.* Thus such an epoch, while can solve the horizon problem, is also very effective in erasing previous memory of the universe and creating a 'clean slate' for the start of a subsequent truly inflationary phase. In particular this suggest that in such a scenario, even if $\Omega \neq 1$ today, there will be no problem about large-scale microwave background anisotropy (as suggested by Rees [110] for a standard but "minimal" inflationary scenario, lasting not enough to produce $\Omega_0 = 1$), because all perturbation scales during this coasting epoch will be made very smooth by this expansion [33]

[14] The physical reality of the mode $\nu = 1$ for closed models is doubtful: see the comments after equations (4.9) and (6.27) in [1].

5.3.2 $\gamma = \gamma(t)$

In this case the second order equation for $\Phi_{(k)}$ is rather more complicated and cannot in general be solved analytically:

$$\ddot{\Phi}_{(k)} + \left[H\left(1+3\gamma\right) - \tfrac{\dot{\gamma}}{\gamma}\right]\dot{\Phi}_{(k)} - \left[\tfrac{2K}{S^2}\left(1+\tfrac{3}{2}\gamma\right) - \tfrac{k^2}{a^2} + H\tfrac{\dot{\gamma}}{\gamma}\right]\Phi_{(k)} = 0. \tag{5.81}$$

As a simple example to illustrate this case, we look at power-law inflation [81] where the scale factor grows like $a(t) = At^p$, $\dot{A} = 0$, $p \geq 1$ (an extensive analysis of these models when $K \neq 0$ can be found in [22]). In this case γ is given by [40]:

$$\gamma = \frac{2}{3}\left[\frac{t^{2p-2}+\beta}{pt^{2p-2}+\beta}\right] \tag{5.82}$$

where $\beta = K/(pA^2)$. When $p = 1$, $\gamma = 2/3$ and we have the coasting solution discussed in section 4.1.

5.4 Summary

In the first part of this chapter we have presented a set of exact results (partly given in [86]) valid in any spacetime dominated by a minimally coupled classical scalar field ϕ, formally described as a fluid (on assuming ϕ has a timelike momentum $\nabla_a\phi$). From this, introducing a set of covariantly defined variables, we have worked out a scheme to describe an almost FLRW universe in which this variables are automatically gauge-invariant (GI) with respect to an exact FLRW background; for sake of generality this latter is not assumed to be flat.

In our formalism (cf. [2, 95], see also [59, 60]) a crucial role is played by the surfaces $\{\phi = constant\}$ and by their unit normal vector u^a, which geometrically characterize these surfaces and plays the role of the fluid 4-velocity (see section 3.4). As long as we use a covariant description, this *does not* necessarily correspond to use the comoving gauge: indeed the perturbation variables we use are GI and physically meaningful on their own; however, if we attach spatial coordinates to the surfaces $\phi = const.$ and we choose coordinate time to flow along their normals, then our GI variables assume their values in the comoving time orthogonal gauge [69] (used e.g. in [3, 81]). More generally, we can as well introduce an arbitrary slicing and attach coordinates to these surfaces in order to relate our variables to standard GI variables (see chapter 4, PaperI [8] and [63, 62]).

Given our choice of u^a, the inhomogeneities of the matter field are incorporated in the covariant gradient ${}^{(3)}\nabla_a\psi$ of the momentum of the scalar field ($\psi = \dot{\phi}$), while ${}^{(3)}\nabla_a\phi$ vanishes exactly and gauge-invariantly (cf. [3, 81]). Then the dimensionless gradient $\Psi_a = \psi^{-1}S^{(3)}\nabla_a\psi$ and its comoving divergence Ψ are the natural GI variables of the problem and the GI density perturbation Δ that represent matter clumping is simply proportional to Ψ: $\Delta = \gamma\Psi$. Rather than metric perturbations, we deal with curvature perturbations (cf. [55], the relations between GI curvature variables and the GI metric potentials of Bardeen has been clarified in chapter 4). Thus the spatial gradient C_a of the 3-curvature scalar of the $\phi = const.$ surfaces and its divergence C are the natural associated GI variables, together with the related quantity $\tilde{C}$. We have derived a set of first order (in time derivative) evolution equations for our variables, showing how the equation for any of the matter perturbation variables (Δ, Ψ or the related $\Phi = \kappa\mu S^2\Delta$) is coupled to an equation for C or $\tilde{C}$; these turn out to be conserved in certain cases of interest for large scales ($\tilde{C} = C$ if $K = 0$), and in this case the evolution of the matter variable is given directly by its first order equation. In any case, the problem of the evolution of either the momentum perturbation Ψ or the density perturbation Δ (or Φ) is closed at second order (in the time derivatives). In the same way, one can decouple the evolution of the curvature perturbation C (or $\tilde{C}$) deriving a second order equation for it. All the equations we derived are valid also for non flat ($K \neq 0$) universes, thus while they generalize equations presented elsewhere [3, 83] (see also [95] and references therein), they can be used to analyze the behaviour of perturbations in unconventional scalar - field - dominated universe models (not necessarily inflationary, e.g. see [94]). When applied to standard inflation, we re-establish known results but in a covariant and gauge-invariant manner. When applied to $K = -1$ coasting models, we establish the decay of perturbations at all scales; this suggests that no problem should arises concerning large-scale microwave background radiation anisotropy in a scenario in which a "minimal" inflationary phase (*not* driving to $\Omega_0 = 1$) is preceded by a coasting era [39].

Chapter 6

PERTURBATIONS IN A MULTI-FLUID COSMOLOGICAL MEDIUM

The purpose of the present chapter is to extend the formalism of chapter 3 to treat multi-component systems of interacting fluids. An analysis of such systems is important, since during the evolution of the universe there are epochs when the matter is more accurately described by a mixture of several fluid components, e.g. radiation, baryonic matter and neutrinos. In such situations, perturbations in the densities and velocities of individual components behave differently due to a difference in their dynamical properties, especially the sound velocities.

Perturbations in multi-component fluid systems have been studied before, indeed a rather complete analysis was presented by Kodama and Sasaki (1984) [69] (KS from now on) based on the GI approach of Bardeen (BI). However, because the physical and geometrical meaning of the Bardeen variables is rather obscure (see chapter 4), it seems beneficial to reformulate the theory using the covariant GI variables introduced in chapter 3. Dunsby (1991a) [18] has considered the case of a mixture of non interacting perfect fluids: this chapter presents the more general case of a mixture of interacting viscous fluids considered in DBE [21].

This chapter is organized as follows. In section 6.1.2, we define covariant and GI variables that characterize the time evolution of density and velocity perturbations in a multi-component fluid medium (including viscous terms). In section 6.2, we derive equations for both the total fluid and it's constituent components. We also present the equations for the relative perturbation variables, which are very useful in differentiating between adiabatic and isothermal perturbations.

In section 6.3, we use harmonic analysis to relate the geometrical variables we use to those of BI and KS. Comparison of the two methods is discussed is great detail in Paper I [8], so here we concentrate only on the most important multi-component variables.

In section 6.4, we discuss if and when the curvature variables C and $\tilde{C}$ are conserved on scales larger than the Hubble horizon. In particular, we demonstrate that when the background FLRW model is flat, both these variables are conserved even in the presence of entropy perturbations and imperfect fluid source terms. We use this result to write down a general solution for the total energy density perturbation. We also briefly consider spatially open models.

In section 6.5, we consider the first of three applications. We examine the case where the background is described by a spatially flat FLRW universe model filled with a mixture of non-interacting dust and radiation and obtain solutions for density and velocity perturbations in the small scale limit. In section 6.6 we study perturbations in a photon-baryon system, taking explicitly into account the interaction between components, which arises through Thompson scattering. We examine in detail the coupling between isothermal and adiabatic perturbations and correct a number of errors in the previous literature (KS [69]). In section 6.7, we briefly consider an application to a system of two non-interacting scalar fields and obtain the standard results (Mollerach, 1990; BED) [93, 10].

6.1 Multi-component fluids

6.1.1 Choice of the frames

In section 3.4 we have seen that a relativistic fluid is described in general by three main variables: the energy momentum tensor (EMT) T_{ab}, the particle flux N^a and the entropy flux S^a [64, 16]. If the fluid is imperfect (e.g. when we include non-equilibrium terms arising from perturbing the fluid), then the fluid hydrodynamic 4-velocity is no longer unique. In this case, decomposing the EMT and the particle flux with respect to a general u^a we obtain:

$$T_{ab} = \mu u_a u_b + p h_{ab} + q_a u_b + q_b u_a + \pi_{ab} \;, \tag{6.1}$$

and

$$N^a = nu^a + j^a \ , \tag{6.2}$$

where q_a and π_{ab} is the energy flux and anisotropic pressure in this frame and j^a is the particle drift. In this case two frames u^a_E and u^a_N have a special status: in the energy frame defined by $u^a = u^a_E$ (Landau and Lifshitz (1963) [74]) there is no energy flux $q_a = q^E_a = 0$, while in the particle frame (Eckart (1940) [23]) $u^a = u^a_N$ there is no particle drift $j^a = j^a_N = 0$, and the EMT takes the form of equation (6.1) with $u^a = u^a_N$.

When we study multi-component fluids, new degrees of freedom arise in the matter variables. We assume that the total energy momentum tensor is given by

$$T_{ab} = \sum_{(i)} T^{(i)}_{ab} \ , \tag{6.3}$$

where, as in equation (6.1), the energy momentum tensor of each component takes the form

$$T^{(i)}_{ab} = \mu_{(i)} u^{(i)}_a u^{(i)}_b + p_{(i)} h^{(i)}_{ab} + q^{(i)}_a u^{(i)}_b + q^{(i)}_b u^{(i)}_a + \pi^{(i)}_{ab} \ . \tag{6.4}$$

Here $u^{(i)}_a$ is a normalized fluid 4-velocity vector for the i^{th} component $(u^{(i)}_a u^a_{(i)} = -1)$ and $h^{(i)}_{ab} = g_{ab} + u^{(i)}_a u^{(i)}_b$ is the projection tensor into the LRF of an observer $\mathcal{O}_{u_{(i)}}$ moving with that velocity $(u^a_{(i)} h^{(i)}_{ab} = 0)$. We can also define a particle flux $N^a_{(i)}$ for the i^{th} fluid component:

$$N^a_{(i)} = n_{(i)} u^a_{(i)} + j^a_{(i)} \ . \tag{6.5}$$

As before, we can fix each component velocity $u^a_{(i)}$ by either choosing the energy frame $u^a_{(i)} = u^a_{E(i)} \Rightarrow q^{(i)}_a = q^{E(i)}_a = 0$ or the particle frame $u^a_{(i)} = u^a_{N(i)} \Rightarrow j^a_{(i)} = j^a_{N(i)} = 0$ for that component.

Following King and Ellis (1973) [67] and Dunsby (1991a) [18], we look at the situation in a frame defined by the hydrodynamic 4-velocity u^a, which we will call the total fluid 4-velocity, again stressing that at this point we have not fixed u^a, which could for example be u^a_N or u^a_E.

In this frame the relation between the 4-velocities u^a and $u^a_{(i)}$ is determined by the hyperbolic angle of tilt $\beta_{(i)}$ and the direction of tilt is specified, either by the direction $\tilde{c}^a_{(i)}$ of the projection of $u^a_{(i)}$, or by the direction $-c^a_{(i)}$ of u^a perpendicular to $u^a_{(i)}$. Exact and linear relations, making this precise are given in the appendix A (see also King and Ellis (1973) [67], Dunsby (1991a) [18] and Paper I [8]). Here we will restrict ourselves to the linear theory by assuming that $\beta_{(i)} \ll 1$ which corresponds to looking only at small deviations from local equilibrium. In this case $\cosh\beta_{(i)} \approx 1$ and $\sinh\beta_{(i)} \approx \beta_{(i)}$ and u^a and $u^a_{(i)}$ satisfy the following relations:

$$u_a u^a_{(i)} \approx -1 , \tag{6.6}$$

$$h^a{}_b u^b_{(i)} \approx -\beta_{(i)} c^a_{(i)} \Rightarrow c^a_{(i)} u_a = 0 , \quad c^{(i)}_a c^a_{(i)} = 1 , \tag{6.7}$$

and

$$h^a_{(i)b} u^b \approx \beta_{(i)} \tilde{c}^a_{(i)} \Rightarrow \tilde{c}^a_{(i)} u^{(i)}_a = 0 , \quad \tilde{c}^a_{(i)} \tilde{c}^{(i)}_a = 1 . \tag{6.8}$$

These relations yield the following

$$u^a_{(i)} - u^a \approx \beta_{(i)} \tilde{c}^a_{(i)} = V^a_{(i)} , \tag{6.9}$$

so $V^{(i)}_a$ is the velocity of the i^{th} fluid component relative to an observer $\mathcal{O}_u$. We assume that all the fluid components share the same hydrodynamical 4-velocity $u^a_{(i)} = u^a_{E(i)}$ in the background FLRW model. It follows from the Stewart and Walker Lemma (1974) [118] (see chapter 1) that $\beta_{(i)}$ is GI (see Dunsby, 1991a) [18], and so $V^a_{(i)}$ is also GI. We will see later that this is the variable that describes velocity perturbations.

Decomposing the total EMT, equation (6.3), with respect to u^a, we find on using (6.1) and (6.4) the total energy flux is given by

$$q_a = \sum_{(i)} \tilde{q}^{(i)}_a , \tag{6.10}$$

$$\tilde{q}^{(i)}_a = q^{(i)}_a + (\mu_{(i)} + p_{(i)})\beta_{(i)} \tilde{c}^{(i)}_a = q^{(i)}_a + (\mu_{(i)} + p_{(i)}) V^{(i)}_a , \tag{6.11}$$

and the total energy density, pressure and anisotropic pressure are

$$\mu = \sum_{(i)} \mu_{(i)} , \quad p = \sum_{(i)} p_{(i)} , \quad \pi_{ab} = \sum_{(i)} \pi^{(i)}_{ab} . \tag{6.12}$$

Decomposing the the total particle drift with respect to this arbitrary u^a one has

$$j^a = \sum_{(i)} \left(j^{(i)}_a + n_{(i)} V^{(i)}_a \right) . \tag{6.13}$$

Now, if we make the Landau and Lifshitz choice, to fix $u^a_{(i)} = u^a_{E(i)}$ and $u^a = u^a_E$, i.e. choosing the energy frame for the component and total velocities, then $q^{(i)}_a = q_a = 0$ and the relative velocities $V^{(i)}_a$ satisfy

$$\sum_{(i)} \left(\mu_{(i)} + p_{(i)} \right) V^{(i)}_a = 0 . \tag{6.14}$$

Alternatively, choosing the particle frame which is the Eckart choice, fixes $u^a_{(i)} = u^a_{N(i)}$ and $u^a = u^a_N$, so $j^a_{(i)} = j^a = 0$ and in this case the relative velocities $V^{(i)}_a$ satisfy

$$\sum_{(i)} n_{(i)} V^{(i)}_a = 0 \ . \tag{6.15}$$

Since we wish to consider interactions between the components we define

$$T^{ab}_{(i);b} = J^a_{(i)} \ , \quad \sum_{(i)} J^a_{(i)} = 0 \ , \tag{6.16}$$

where the second relation is just a consequence of the conservation of the total energy momentum tensor: $T^{ab}{}_{;b} = 0$. It is convenient to decompose the interaction term $J^{(i)}_a$ into components parallel and perpendicular to the fluid 4-velocity u_a:

$$J^{(i)}_a = \epsilon_{(i)} u_a + f^{(i)}_a \ , \quad u_a f^a_{(i)} = 0 \ , \quad h^a{}_b u_a \epsilon_{(i)} = 0 \ . \tag{6.17}$$

These equations will be used later for determining the behaviour perturbations of the individual fluid components.

Finally we note that the above choices of u^a, either the energy frame or the particle frame, are not the only choices we could make. What is crucial is that, since we wish to develop a GI formulation based on gradients orthogonal to the fluid flow, we require that u^a be covariantly defined and coincides with u^a_E in the background; clearly both u^a_E and u^a_N are natural choices that satisfy this requirement. If a choice of u^a was made such that it didn't coincide with u^a_E in the background, then gradient variables defined with respect to u^a would NOT be gauge invariant. (See Paper I [8] for further discussion of this point). In the next section we will define a set of GI variables that characterize the fluid inhomogeneity.

6.1.2 The inhomogeneity variables

The variables we use that describe the inhomogeneity of the total fluid have been introduced in chapter 3 (see in particular sections 3.5.3 and 3.10), following the Stewart and Walker Lemma of chap[ter 1. In the case of a multi-component fluid, we need to define additional quantities that characterize the spatial variation of the density $\mu_{(i)}$ and the volume expansion $\Theta_{(i)}$ of the individual components. We can do this in two ways, either by defining spatial gradients of each component with respect to the total matter rest frame, taken to be $u^a = u^a_N$, so the fractional density gradient and the comoving gradient of the expansion of the i^{th} component relative to this frame are given by

$$\mathcal{D}_a^{(i)} = a\frac{X_a^{(i)}}{\mu_{(i)}} , \tag{6.18}$$

and

$$\mathcal{Z}_a^{(i)} = aZ_a^{(i)} , \tag{6.19}$$

where $X_a^{(i)} = h^b{}_a\mu_{(i);b}$ and $Z_a^{(i)} = h^b{}_a\Theta_{(i);b}$, or we could define gradients for the individual components with respect to the matter rest frame of the components themselves, $u^a_{(i)} = u^a_{N(i)}$, so in this case we have

$${}^N\mathcal{D}_a^{(i)} = a\frac{{}^NX_a^{(i)}}{\mu_{(i)}} , \tag{6.20}$$

and

$${}^N\mathcal{Z}_a^{(i)} = a^NZ_a^{(i)} , \tag{6.21}$$

where ${}^NX_a^{(i)} = h^b_{(i)a}\mu_{;b}$ and ${}^NZ_a^{(i)} = h^b_{(i)a}\Theta_{(i);b}$.

Relations between the various sets of gradients can be obtained using the results of section 6.1.1 (see also appendix A), thus for example the linear transformation law linking $\mathcal{D}_a^{(i)}$ to ${}^N\mathcal{D}_a^{(i)}$ is

$${}^N\mathcal{D}_a^{(i)} = \mathcal{D}_a^{(i)} - 3Ha\left(1 + w_{(i)}\right)V_a^{(i)} + \tfrac{a}{\mu_{(i)}}\epsilon_{(i)}V_a^{(i)} . \tag{6.22}$$

It turns out that for most purposes, it is best to use gradients defined with respect to the total matter rest frame (after all, in this case we have one frame to use for all our calculations instead of a whole set of frames, one for each fluid component) and in the next section we will derive a complete set of equations for these variables.

6.2 Equations

6.2.1 Total fluid equations

The equations for the total fluid density perturbation variables $\mathcal{D}_a$ and the companion variables $\mathcal{Z}_a$, C_a and $\tilde{C}_a$ have been derived in chapter 3.[1] Here we recall the main results: the evolution of $\mathcal{D}_a$ can be obtained either by a pair of first-order equations

[1] For the imperfect fluid case, these equations were presented by Dunsby [18].

for $\mathcal{D}_a$ and any of the variables $\mathcal{Z}_a$, C_a and $\tilde{C}$, or by a second-order equation for $\mathcal{D}_a$. To derive the large scale evolution, the pair of equations for $\mathcal{D}_a$ and $\tilde{C}_a$ is the most useful

$$\begin{aligned} \dot{\tilde{C}}_a &= \tfrac{4a^2Hc_s^2}{(1+w)}\left(\nabla^2 - \tfrac{2K}{a^2}\right)\mathcal{D}_a + 24c_s^2a^3H^2\nabla^b\omega_{ab} + \tfrac{4a^2Hw}{(1+w)}\left(\nabla^2 - \tfrac{2K}{a^2}\right)\mathcal{E}_a \\ &+ \tfrac{12KHw}{(1+w)}\mathcal{E}_a + 4a^3H\nabla_a\nabla^b\left[F_a + \Pi_b\right] + \left[4Ka - 2a^3h\right]\nabla_a\nabla^b\Psi_b, \end{aligned} \tag{6.23}$$

$$\begin{aligned} &\dot{\mathcal{D}}_a - \left\{3Hw - \left[\tfrac{\kappa h}{2} - \tfrac{K}{a^2}\right]H^{-1}\right\}\mathcal{D}_a - \tfrac{(1+w)}{4a^2H}\tilde{C}_a \\ &\quad = 3a(1+w)H\left[F_a + \Pi_a\right] - a(1+w)\nabla_a\nabla^b\Psi_b\,, \end{aligned} \tag{6.24}$$

because at large scales $\tilde{C}_a$ can be a conserved quantity, so that the evolution of $\mathcal{D}_a$ can be obtained directly through (6.24) (see section 6.4). In general however the dynamics of $\mathcal{D}_a$ is determined by a second order equation, which follows directly from the equations above. This equation is

$$\begin{aligned} &\ddot{\mathcal{D}}_a + \left(2 + 3c_s^2 - 6w\right)H\dot{\mathcal{D}}_a \\ &- \left[\left(\tfrac{1}{2} + 4w - \tfrac{3}{2}w^2 - 3c_s^2\right)\kappa\mu + \left(5w - 3c_s^2\right)\Lambda + \left(c_s^2 - w\right)\tfrac{12K}{a^2}\right]\mathcal{D}_a \\ &- c_s^2\left(-\tfrac{2K}{a^2} + \nabla^2\right)\mathcal{D}_a - w\left(\tfrac{K}{a^2} + \nabla^2\right)\mathcal{E}_a - 6a(1+w)Hc_s^2\nabla^b\omega_{ab} \\ &= a(1+w)\left[\left(-3w\mu + 3\Lambda - \tfrac{3K}{a^2}\right)\left[F_a + \Pi_a\right] + 3H\left[\dot{F}_a + \dot{\Pi}_a\right]\right. \\ &\left.+ \nabla_a\nabla^b\left[F_b + \Pi_b\right] - 2H\nabla_a\nabla^b\Psi_b - \nabla_a\nabla^b\dot{\Psi}_b\right]\,. \end{aligned} \tag{6.25}$$

It has the form of a wave equation with extra terms due to the expansion of the universe, gravity, the spatial curvature, the cosmological constant, the divergence of the vorticity and imperfect fluid source terms. When $F_a = \Psi_a = \Pi_a = 0$ and we have a barotropic equation of state, $p = p(\mu) \Rightarrow \mathcal{E}_a = 0$, and this equation reduces to equation (3.97) in chapter 3 (see also EBH [36]).

6.2.2 Component equations

To study the behaviour of the individual components it suffice to have the linear conservation equations for energy and momentum for each component in the total matter rest frame ($u^a = u^a_N$). From equation (6.16) and (6.17) we can derive these as follows:

$$\dot{\mu}_{(i)} + 3h_{(i)}H + h_{(i)}\nabla_a\Psi^a_{(i)} = \epsilon_{(i)} \tag{6.26}$$

and

$$h_{(i)}a_a + Y_a^{(i)} + h_{(i)}\left[F_a^{(i)} + \Pi_a^{(i)}\right] + \left(1 + c_{s(i)}^2\right)\epsilon_{(i)}\Psi_a^{(i)} = f_a^{(i)} \,, \tag{6.27}$$

where (analogously to (3.154))

$$F_a^{(i)} = \dot{\Psi}_a^{(i)} - \left(3c_{s(i)}^2 - 1\right)H\Psi_a^{(i)}, \quad \Pi_a^{(i)} = \frac{1}{h_{(i)}}\nabla^b\pi_{ab}^{(i)} \,, \tag{6.28}$$

$$\Psi_a^{(i)} = \frac{\tilde{q}_a^{(i)}}{h_{(i)}} = \frac{q_a^{(i)}}{h_{(i)}} + V_a^{(i)} \tag{6.29}$$

and

$$h_{(i)} = \left(\mu_{(i)} + p_{(i)}\right) \,. \tag{6.30}$$

When we have a perfect fluid $q_a^{(i)} = 0$ and so $\Psi_a^{(i)} = V_a^{(i)}$ is just the relative velocity of the i^{th} fluid component relative to an observer moving with the total fluid (cf. equation (16)).

We can now use the above equations to calculate an equation for the gradient $\mathcal{D}_a^{(i)}$ for each component, defined in section 6.1.2. This is done in the same way as we did for the total fluid, except that now we will have extra terms due to the presence of interactions between components. We obtain

$$\begin{aligned}\dot{\mathcal{D}}_a^{(i)} - 3Hw_{(i)}\mathcal{D}_a^{(i)} + \left(1 + w_{(i)}\right)\mathcal{Z}_a &= 3a\left(1 + w_{(i)}\right)H\left[F_a^{(i)} + \Pi_a^{(i)}\right] \\ &+ \tfrac{3aH}{\mu_{(i)}}\left(1 + c_{s(i)}^2\right)\epsilon_{(i)}\Psi_a^{(i)} - a\left(1 + w_{(i)}\right)\nabla_a\nabla_b\Psi_{(i)}^b \\ &- \tfrac{3Ha}{\mu_{(i)}}f_a^{(i)} + \tfrac{a}{\mu_{(i)}}\nabla_a\epsilon_{(i)} + \tfrac{a}{\mu_{(i)}}\epsilon_{(i)}a_a - \tfrac{1}{\mu_{(i)}}\epsilon_{(i)}\mathcal{D}_a^{(i)} \,,\end{aligned} \tag{6.31}$$

where the main difference with the analogous equation for $\mathcal{D}_a$ is the explicit appearance of an acceleration term: this term vanishes if there is no interaction (i.e. if $\epsilon_{(i)} = 0$). Here $f_a^{(i)}$ and $\epsilon_{(i)}$ represent respectively the perturbation in the mean momentum and energy transfer rates between components due to interactions (see equations (6.16), (6.17). If the interactions are specified, $f_a^{(i)}$ can always be written in terms of $V_a^{(i)}$. This will be illustrated in section 6.6, when we study perturbations in a photon - baryon system. The above equation is coupled to those for the total fluid through the comoving spatial gradient of the expansion $\mathcal{Z}_a$.

An equation for $\Psi_a^{(i)}$ in terms of the total fluid quantities follows from equation (6.27) by substituting for the acceleration a_a and using the total momentum conservation equation (3.153). This momentum equation (relative to the frame u_N^a) is

$$\dot{\Psi}_a^{(i)} - \left(3c_{s(i)}^2 - 1\right)H\Psi_a^{(i)} + \tfrac{1}{h_{(i)}}\left(1 + c_{s(i)}^2\right)\epsilon_{(i)}\Psi_a^{(i)} = F_a + \Pi_a - \Pi_a^{(i)}$$

$$+\tfrac{1}{ahh_{(i)}}\left[h_{(i)}c_s^2\mu\mathcal{D}_a - hc_{s(i)}^2\mu_{(i)}\mathcal{D}_a^{(i)} + h_{(i)}p\mathcal{E}_a - hp_{(i)}\mathcal{E}_a^{(i)}\right] + \tfrac{1}{h_{(i)}}f_a^{(i)} \;, \qquad (6.32)$$

where we have used equation (3.74) and its component equivalent given by

$$aY_a^{(i)} = c_{s(i)}^2\mu_{(i)}\mathcal{D}_a^{(i)} + p_{(i)}\mathcal{E}_a^{(i)} \;, \qquad (6.33)$$

$\mathcal{E}_a^{(i)}$ being entropy perturbation of the i^{th} component. Using the momentum equations (3.153) and (6.27) we can write the first order equation (6.31) for the component density gradient in the following useful form:

$$\begin{aligned}\dot{\mathcal{D}}_a^{(i)} - 3\left(w_{(i)} - c_{s(i)}^2\right)H\mathcal{D}_a^{(i)} + \left(1 + w_{(i)}\right)\mathcal{Z}_a &= 3aH\left(1 + w_{(i)}\right)[F_a + \Pi_a] \\ &+\tfrac{1}{\mu_{(i)}h}\left(3Hh_{(i)} - \epsilon_{(i)}\right)[c_s^2\mu\mathcal{D}_a + p\mathcal{E}_a] - a\left(1 + w_{(i)}\right)\nabla_a\nabla^b\Psi_b^{(i)} \\ &-3Hw_{(i)}\mathcal{E}_a^{(i)} + \tfrac{a}{\mu_{(i)}}\nabla_a\epsilon_{(i)} - \tfrac{a}{\mu_{(i)}}\epsilon_{(i)}[F_a + \Pi_a] - \tfrac{1}{\mu_{(i)}}\epsilon_{(i)}\mathcal{D}_a^{(i)} \;, \qquad (6.34)\end{aligned}$$

where

$$p\mathcal{E}_a = \sum_{(i)}\left(c_{s(i)}^2\mu_{(i)}\mathcal{D}_a^{(i)} + p_{(i)}\mathcal{E}_a^{(i)}\right) - c_s^2\mu\mathcal{D}_a \;, \qquad (6.35)$$

and

$$c_s^2 = \frac{1}{h}\sum_{(i)} h_{(i)}c_{s(i)}^2 \;. \qquad (6.36)$$

In order to see the physical meaning of the variable $\mathcal{E}_a$, we can rewrite equation (6.35) using the expression for c_s^2:

$$p\mathcal{E}_a = \sum_{(i)} p_{(i)}\mathcal{E}_a^{(i)} + \frac{1}{2}\sum_{(i,j)} \tfrac{h_{(i)}h_{(j)}}{h}\left(c_{s(i)}^2 - c_{s(j)}^2\right)S_a^{(ij)} \;, \qquad (6.37)$$

where

$$S_a^{(ij)} = \tfrac{\mu_{(i)}}{h_{(i)}}\mathcal{D}_a^{(i)} - \tfrac{\mu_{(j)}}{h_{(j)}}\mathcal{D}_a^{(j)} \;. \qquad (6.38)$$

This shows that the entropy perturbation consists of two parts; a part coming from the entropy perturbation of each component $\mathcal{E}_a^{(i)}$ and a part coming from the difference of the dynamical behaviour of the components $S^{(ij)}$. This latter part turns out to be the variable characterizing the time evolution of isothermal perturbations (KS [69]).

When there are no interactions equations (6.32) and (6.34) reduce to

$$\begin{aligned}\dot{\Psi}_a^{(i)} - \left(3c_{s(i)}^2 - 1\right)H\Psi_a^{(i)} &= F_a + \Pi_a - \Pi_a^{(i)} \\ &+\tfrac{1}{ahh_{(i)}}\left[h_{(i)}c_s^2\mu\mathcal{D}_a - hc_{s(i)}^2\mu_{(i)}\mathcal{D}_a^{(i)} + h_{(i)}p\mathcal{E}_a - hp_{(i)}\mathcal{E}_a^{(i)}\right] \;, \qquad (6.39)\end{aligned}$$

and

$$\dot{\mathcal{D}}_a^{(i)} - 3\left(w_{(i)} - c_{s(i)}^2\right) H\mathcal{D}_a^{(i)} + \left(1 + w_{(i)}\right)\mathcal{Z}_a = 3aH\left(1 + w_{(i)}\right)[F_a + \Pi_a]$$
$$+\tfrac{3H}{h}\left(1 + w_{(i)}\right)[c_s^2\mu\mathcal{D}_a + p\mathcal{E}_a] - a\left(1 + w_{(i)}\right)\nabla_a\nabla^b\Psi_b^{(i)} - 3Hw_{(i)}\mathcal{E}_a^{(i)} \,. \quad (6.40)$$

6.2.3 Relative equations

In the last section we defined the variable $S_a^{(ij)}$. This is one of a set of relative variables, that is, variables relating features of pairs of different fluid components, dependent on the choice of the individual velocities but independent of the choice of the overall frame (for this choice does not enter their definition), which allows us to clearly distinguish between isothermal and adiabatic perturbations (KS [69]). It is perhaps worth to point out again that we assume that the single component fluid 4-velocities $u_{(i)}^a$ coincide in the background, so that these relative variables are automatically GI (as for $V_{(i)}^a$ (6.9)); they can be defined in terms of the component variables used in the previous section:

$$S_a^{(ij)} = \tfrac{\mu_{(i)}}{h_{(i)}}\mathcal{D}_a^{(i)} - \tfrac{\mu_{(j)}}{h_{(j)}}\mathcal{D}_a^{(j)}\,, \quad \Psi_a^{(ij)} = \Psi_a^{(i)} - \Psi_a^{(j)}\,, \quad (6.41)$$

$$\mathcal{E}_a^{(ij)} = \tfrac{p_{(i)}}{h_{(i)}}\mathcal{E}_a^{(i)} - \tfrac{p_{(j)}}{h_{(j)}}\mathcal{E}_a^{(j)}, \quad f_a^{(ij)} = \tfrac{f_a^{(i)}}{h_{(i)}} - \tfrac{f_a^{(j)}}{h_{(j)}}\,, \quad (6.42)$$

$$\epsilon_{(ij)} = \tfrac{\epsilon_{(i)}}{h_{(i)}} - \tfrac{\epsilon_{(j)}}{h_{(j)}}, \quad \Pi_a^{(ij)} = \Pi_a^{(i)} - \Pi_a^{(j)}\,, \quad (6.43)$$

$$V_a^{(ij)} = V_a^{(i)} - V_a^{(j)}\,. \quad (6.44)$$

Using the equations for $\mathcal{D}_a^{(i)}$ and $\mathcal{D}_a^{(j)}$ derived in the last section, it is straightforward to find an evolution equation for $S_a^{(ij)}$. After some algebra we find

$$\dot{S}_a^{(ij)} + \tfrac{1}{h_{(j)}}(1 + c_{s(j)})\epsilon_{(j)}S_a^{(ij)} + a\nabla_a\nabla^b\Psi_b^{(ij)} + 3H\mathcal{E}_a^{(ij)}$$
$$= -\tfrac{1}{h}\epsilon_{(ij)}[c_s^2\mu\mathcal{D}_a + p\mathcal{E}_a] - a\epsilon_{(ij)}[F_a + \Pi_a] + a\nabla_a\epsilon_{(ij)}$$
$$- \tfrac{h_{(j)}}{h_{(i)}^2}\mu_{(i)}[h_{(j)}(1 + c_{s(i)}^2)\epsilon_{(i)} - h_{(i)}(1 + c_{s(j)}^2)\epsilon_{(j)}]\mathcal{D}_a^{(i)}\,. \quad (6.45)$$

Similarly, the evolution equations for $\Psi_a^{(i)}$ and $\Psi_a^{(j)}$ give

$$\dot{\Psi}_a^{(ij)} - \left[\left(3c_{s(j)}^2 - 1\right)H - \tfrac{1}{h_{(j)}}\left(1 + c_{s(j)}^2\right)\epsilon^{(j)}\right]\Psi_a^{(ij)}$$
$$- \left[3\left(c_{s(i)}^2 - c_{s(j)}^2\right)H + \tfrac{1}{h_{(i)}}\left(c_{s(i)}^2 + 1\right)\epsilon_{(i)} - \tfrac{1}{h_{(j)}}\left(1 + c_{s(j)}^2\right)\epsilon_{(j)}\right]\Psi_a^{(i)}$$
$$= -\tfrac{1}{ah_{(i)}}\left(c_{s(i)}^2 - c_{s(j)}^2\right)\mu_{(i)}\mathcal{D}_a^{(i)} - \tfrac{1}{a}c_{s(j)}^2 S_a^{(ij)} - \tfrac{1}{a}\mathcal{E}_a^{(ij)} - \Pi_a^{(ij)} + f_a^{(ij)}\,. \quad (6.46)$$

Using the following relations for the relative variables:

$$\sum_{(l)} h_{(l)}\Psi_a^{(il)} = h\left(\Psi_a^{(i)} - \Psi_a\right) , \tag{6.47}$$

and

$$\sum_{(l)} h_{(l)}S_a^{(il)} = h\tfrac{\mu_{(i)}}{h_{(i)}}\mathcal{D}_a^{(i)} - \mu\mathcal{D}_a , \tag{6.48}$$

we can write the above equations in terms of just total matter and relative variables:

$$\begin{aligned}
&\dot{\Psi}_a^{(ij)} - \left[3\left(c_{s(i)}^2 + c_{s(j)}^2\right) - 1\right] H\Psi_a^{(ij)} + \frac{3H}{h}\sum_{(l)} h_{(l)}\left[c_{s(j)}^2\Psi_a^{(il)} - c_{s(i)}^2\Psi_a^{(jl)}\right] \\
&+ \frac{1}{h}\sum_{(l)} h_{(l)}\left[\tfrac{1}{h_{(i)}}\left(c_{s(i)}^2 + 1\right)\epsilon_{(i)}\Psi_a^{(il)} - \tfrac{1}{h_{(j)}}\left(1 + c_{s(j)}^2\right)\epsilon_{(j)}\Psi_a^{(jl)}\right] \\
&= 3H\left(c_{s(i)}^2 - c_{s(j)}^2\right)\Psi_a + \left[\tfrac{1}{h_{(j)}}\left(1 + c_{s(j)}^2\right)\epsilon_{(j)} - \tfrac{1}{h_{(i)}}\left(1 + c_{s(i)}^2\right)\epsilon_{(i)}\right]\Psi_a \\
&- \tfrac{1}{ah}\left(c_{s(i)}^2 - c_{s(j)}^2\right)\mu\mathcal{D}_a - \tfrac{1}{a}\left(c_{s(i)}^2 - c_{s(j)}^2\right)S_a^{(ij)} - \tfrac{1}{a}\mathcal{E}_a^{(ij)} \\
&+ \frac{1}{ah}\sum_{(l)} h_{(l)}\left(c_{s(j)}^2 S_a^{(il)} - c_{s(i)}^2 S_a^{(jl)}\right) - \Pi_a^{(ij)} + f_a^{(ij)} ,
\end{aligned} \tag{6.49}$$

i.e. the relative velocity equation, and

$$\begin{aligned}
&\dot{S}_a^{(ij)} + \frac{1}{h}\sum_{(l)} h_{(l)}\left[\tfrac{1}{h_{(i)}}\left(1 + c_{s(i)}^2\right)\epsilon_{(i)}S_a^{(il)} - \tfrac{1}{h_{(j)}}\left(1 + c_{s(j)}^2\right)\epsilon_{(j)}S_a^{(jl)}\right] \\
&= -\tfrac{1}{h}\left[\tfrac{1}{h_{(i)}}\left(1 + c_{s(i)}^2\right)\epsilon_{(i)} - \tfrac{1}{h_{(j)}}\left(1 + c_{s(j)}^2\right)\epsilon_{(j)}\right]\mu\mathcal{D}_a - \tfrac{1}{h}\epsilon_{(ij)}\left[c_s^2\mu\mathcal{D}_a + p\mathcal{E}_a\right] \\
&- a\epsilon_{(ij)}\left[F_a + \Pi_a\right] + a\nabla_a\epsilon_{(ij)} - a\nabla_a\nabla^b\Psi_b^{(ij)} - 3H\mathcal{E}_a^{(ij)} ,
\end{aligned} \tag{6.50}$$

i.e. the relative density perturbation equation. These two equations are the most important in the study of perturbations in multi - component fluids. They show that in general, *adiabatic or total density perturbations are coupled to isothermal perturbations and that either one can be generated from each other.* A nice illustration of this will be discussed in section 6.6.

When there are no interactions these equations reduce to

$$\dot{S}_a^{(ij)} + a\nabla_a\nabla^b\Psi_b^{(ij)} + 3H\mathcal{E}_a^{(ij)} = 0 , \tag{6.51}$$

and

$$\dot{\Psi}_a^{(ij)} - \left[3\left(c_{s(i)}^2 + c_{s(j)}^2\right) - 1\right] H\Psi_a^{(ij)} + \frac{3H}{h}\sum_{(l)} h_{(l)}\left[c_{s(j)}^2\Psi_a^{(il)} - c_{s(i)}^2\Psi_a^{(jl)}\right]$$
$$= 3H\left(c_{s(i)}^2 - c_{s(j)}^2\right)\Psi_a - \tfrac{1}{ah}\left(c_{s(i)}^2 - c_{s(j)}^2\right)\mu\mathcal{D}_a - \tfrac{1}{a}\left(c_{s(i)}^2 + c_{s(j)}^2\right)S_a^{(ij)}$$
$$- \frac{1}{a}\mathcal{E}_a^{(ij)} + \frac{1}{ah}\sum_{(l)} h_{(l)}\left(c_{s(j)}^2 S_a^{(il)} - c_{s(i)}^2 S_a^{(jl)}\right) - \Pi_a^{(ij)} . \qquad (6.52)$$

Two-component fluids

Since it may be useful for many applications, let us write out these equations explicitly for two components. In this case the relations (6.47) and (6.48) give

$$\Psi_a^{(1)} = \Psi_a + \tfrac{h_{(2)}}{h}\Psi_a^{(12)} , \qquad (6.53)$$

and

$$\mu_{(1)}\mathcal{D}_a^{(1)} = \tfrac{h_{(1)}}{h}\left[\mu\mathcal{D}_a + h_{(2)}S_a^{(12)}\right] . \qquad (6.54)$$

So equations (6.51) and (6.52) become

$$\dot{S}_a^{(12)} + a\nabla_a\nabla^b\Psi_b^{(12)} + 3H\mathcal{E}_a^{(12)} = 0 , \qquad (6.55)$$

and

$$\dot{\Psi}_a^{(12)} - \left(3c_z^2 - 1\right)H\Psi_a^{(12)} - 3\left(c_{s(1)}^2 - c_{s(2)}^2\right)H\Psi_a$$
$$= -\tfrac{1}{ah}\left(c_{s(1)}^2 - c_{s(2)}^2\right)\mu\mathcal{D}_a - \tfrac{1}{a}c_z^2 S_a^{(12)} - \tfrac{1}{a}\mathcal{E}_a^{(12)} - \Pi_a^{(12)} , \qquad (6.56)$$

where

$$c_z^2 = \frac{1}{h}\left(h_{(2)}c_{s(1)}^2 + h_{(1)}c_{s(2)}^2\right) . \qquad (6.57)$$

For a mixture of two perfect fluids, $\Pi_a^{(12)} = \Psi_a = 0$ and $\Psi_a^{(12)} = V_a^{(12)}$, so these equations further simplify to:

$$\dot{S}_a^{(12)} + a\nabla_a\nabla^b V_a^{(12)} + 3H\mathcal{E}_a^{(12)} = 0 , \qquad (6.58)$$

and

$$\dot{V}_a^{(12)} - \left(3c_z^2 - 1\right)HV_a^{(12)} = -\tfrac{1}{ah}\left(c_{s(1)}^2 - c_{s(2)}^2\right)\mu\mathcal{D}_a - \tfrac{1}{a}c_z^2 S_a^{(12)} - \tfrac{1}{a}\mathcal{E}_a^{(12)} . \qquad (6.59)$$

A very important consequence of these two equations is that *the concept of adiabatic perturbations makes sense only if the two components share the same speed of sound*

(Kodama, 1983a [68], KS [69]). For the perturbation to be adiabatic, both $\mathcal{E}_a^{(12)}$ and $S_a^{(12)}$ must vanish in general. It follows from equation (6.58) that $V_a^{(12)}$ must also vanish and from equation (6.59), this implies that $\left(c_{s(1)}^2 - c_{s(2)}^2\right)\mu\mathcal{D}_a = 0$. Since $\mathcal{D}_a \neq 0$, we find that $c_{s(1)}^2 = c_{s(2)}^2$.

6.2.4 Scalar equations

So far we have only considered the evolution of gradient gauge-invariant variables, for example $\mathcal{D}_a$ which describes the spatial variation of the density orthogonal to the fluid flow. $\Delta = a\nabla^a\mathcal{D}_a$ is the part of the density evolution relating to the aggregation of matter. Since we are interested in matter clumping in this chapter we will concentrate on this variable and define companion scalar variables by taking divergences as follows: For the total fluid the scalar variables we need are

$$\mathcal{Z} = a\nabla^a\mathcal{Z}_a\ , \quad \tilde{C} = a\nabla^a\tilde{C}_a\ , \quad \Psi = a\nabla^a\Psi_a\ , \tag{6.60}$$

$$F = a\nabla^a F_a\ , \quad \Pi = a\nabla^a\Pi_a\ . \tag{6.61}$$

Similarly for the fluid components we have

$$\Delta_{(i)} = a\nabla^a\mathcal{D}_a^{(i)}\ , \quad \Psi_{(i)} = a\nabla^a\Psi_a^{(i)}\ , \quad \Pi_{(i)} = a\nabla^a\Pi_a^{(i)}\ , \tag{6.62}$$

$$\mathcal{E}_{(i)} = a\nabla^a\mathcal{E}_a^{(i)}\ , \quad f_{(i)} = a\nabla^a f_a^{(i)}\ , \tag{6.63}$$

and from these we can construct relative scalar variables:

$$S_{(ij)} = a\nabla^a S_a^{(ij)}\ , \quad \Psi_{(ij)} = a\nabla^a\Psi_a^{(ij)}\ , \quad \Pi_{(ij)} = a\nabla^a\Pi_a^{(ij)}\ , \tag{6.64}$$

$$\mathcal{E}_{(ij)} = a\nabla^a\mathcal{E}_a^{(ij)}\ , \quad f_{(ij)} = a\nabla^a f_a^{(ij)}\ . \tag{6.65}$$

As we already seen in chapter 3, evolution equations for the total perturbation variables follow by taking the divergence of the corresponding gradient equations (see the previous sections) and keeping only the linear terms that arise. We repeat here only the basic equations. Taking the divergence of equation (6.24) we obtain a scalar equation for Δ in terms of the curvature variable $\tilde{C}$:

$$\begin{aligned} &\dot{\Delta} - \left\{3Hw - \left[\tfrac{\kappa h}{2} - \tfrac{K}{a^2}\right]H^{-1}\right\}\Delta - \tfrac{(1+w)}{4a^2H}\tilde{C} \\ &= \ 3a(1+w)H\left[F + \Pi\right] - a(1+w)\nabla^2\Psi\ , \end{aligned} \tag{6.66}$$

while the evolution of $\tilde{C}$ itself is governed by

$$\begin{aligned}\dot{\tilde{C}} &= \tfrac{4a^2Hc_s^2}{(1+w)}{}^{(3)}\nabla^2\Delta + \tfrac{4a^2Hw}{(1+w)}\left[{}^{(3)}\nabla^2 + \tfrac{3K}{a^2}\right]\mathcal{E} \\ &\quad +4a^3H^{(3)}\nabla^2\left[F+\Pi\right] + \left[4Ka - 2a^3h\right]{}^{(3)}\nabla^2\Psi \,. \end{aligned} \tag{6.67}$$

The evolution of Δ can be also directly determined by the second order equation (3.183). In addition to the above equations, we need those for the single components scalar variables. For the fluid components we take the divergence of equations (6.32) and (6.34), which gives

$$\begin{aligned}&\dot{\Psi}_{(i)} - \left(3c_{s(i)}^2 - 1\right)H\Psi_{(i)} + \tfrac{1}{h_{(i)}}\left(1 + c_{s(i)}^2\right)\epsilon_{(i)}\Psi_{(i)} = F + \Pi - \Pi_{(i)} \\ &+ \tfrac{1}{ahh_{(i)}}\left[h_{(i)}c_s^2\mu\Delta - hc_{s(i)}^2\mu_{(i)}\Delta_{(i)} + h_{(i)}p\mathcal{E} - hp_{(i)}\mathcal{E}_{(i)}\right] + \tfrac{1}{h_{(i)}}f_{(i)}\,,\end{aligned} \tag{6.68}$$

and

$$\begin{aligned}&\dot{\Delta}_{(i)} - 3\left(w_{(i)} - c_{s(i)}^2\right)H\Delta_{(i)} + \left(1 + w_{(i)}\right)\mathcal{Z} = 3aH\left(1 + w_{(i)}\right)\left[F+\Pi\right] \\ &+ \tfrac{1}{\mu_{(i)}h}\left(3Hh_{(i)} - \epsilon_{(i)}\right)\left[c_s^2\mu\Delta + p\mathcal{E}\right] - a\left(1 + w_{(i)}\right)\nabla^2\Psi_{(i)} \\ &+ \tfrac{a^2}{\mu_{(i)}}\nabla^2\epsilon_{(i)} - 3Hw_{(i)}\mathcal{E}_{(i)} - \tfrac{a}{\mu_{(i)}}\epsilon_{(i)}\left[F+\Pi\right] - \tfrac{1}{\mu_{(i)}}\epsilon_{(i)}\Delta_{(i)}\,.\end{aligned} \tag{6.69}$$

In the same way, we take the divergence of equations (6.50) and (6.49) for the relative variables to obtain

$$\begin{aligned}&\dot{S}^{(ij)} + \frac{1}{h}\sum_{(l)} h_{(l)}\left[\tfrac{1}{h_{(i)}}\left(1 + c_{s(i)}^2\right)\epsilon_{(i)}S^{(il)} - \tfrac{1}{h_{(j)}}\left(1 + c_{s(j)}^2\right)\epsilon_{(j)}S^{(jl)}\right] \\ &= -\tfrac{1}{h}\left[\tfrac{1}{h_{(i)}}\left(1 + c_{s(i)}^2\right)\epsilon_{(i)} - \tfrac{1}{h_{(j)}}\left(1 + c_{s(j)}^2\right)\epsilon_{(j)}\right]\mu\Delta - \tfrac{1}{h}\epsilon(ij)\left[c_s^2\mu\Delta + p\mathcal{E}\right] \\ &- a\epsilon_{(ij)}\left[F+\Pi\right] + a\nabla^2\epsilon_{(ij)} - a\nabla^2\Psi_{(ij)} - 3H\mathcal{E}_{(ij)}\,,\end{aligned} \tag{6.70}$$

and

$$\begin{aligned}&\dot{\Psi}^{(ij)} - \left[3\left(c_{s(i)}^2 + c_{s(j)}^2\right) - 1\right]\Psi^{(ij)} + \frac{3H}{h}\sum_{(l)} h_{(l)}\left[c_{s(j)}^2\Psi^{(il)} - c_{s(i)}^2\Psi^{(jl)}\right] \\ &+ \frac{1}{h}\sum_{(l)} h_{(l)}\left[\tfrac{1}{h_{(i)}}\left(c_{s(i)}^2 + 1\right)\epsilon_{(i)}\Psi^{(il)} - \tfrac{1}{h_{(j)}}\left(1 + c_{s(j)}^2\right)\epsilon_{(j)}\Psi^{(jl)}\right] \\ &= 3H\left(c_{s(i)}^2 - c_{s(j)}^2\right)\Psi + \left[\tfrac{1}{h_{(j)}}\left(1 + c_{s(j)}^2\right)\epsilon_{(j)} - \tfrac{1}{h_{(i)}}\left(1 + c_{s(i)}^2\right)\epsilon_{(i)}\right]\Psi\end{aligned}$$

$$
\begin{aligned}
&- \tfrac{1}{ah}\left(c^2_{s(i)} - c^2_{s(j)}\right)\mu\Delta - \tfrac{1}{a}\left(c^2_{s(i)} - c^2_{s(j)}\right)S^{(ij)} - \tfrac{1}{a}\mathcal{E}^{(ij)} \\
&+ \frac{1}{ah}\sum_{(l)} h_{(l)}\left(c^2_{s(j)}S^{(il)} - c^2_{s(i)}S^{(jl)}\right) - \Pi^{(ij)} + f^{(ij)} . \qquad (6.71)
\end{aligned}
$$

In the case when we can neglect interactions and specializing to two components (see equations (6.55) and (6.56) in the previous section), these equations reduce to

$$
\dot{S}^{(12)} + a\nabla^2\Psi_{(12)} + 3H\mathcal{E}_{(12)} = 0 , \qquad (6.72)
$$

and

$$
\begin{aligned}
&\dot{\Psi}_{(12)} - \left(3c_z^2 - 1\right)H\Psi_{(12)} - 3\left(c^2_{s(1)} - c^2_{s(2)}\right)H\Psi \\
&= -\tfrac{1}{ah}\left(c^2_{s(1)} - c^2_{s(2)}\right)\mu\Delta - \tfrac{1}{a}c_z^2 S_{(12)} - \tfrac{1}{a}\mathcal{E}_{(12)} - \Pi_{(12)} . \qquad (6.73)
\end{aligned}
$$

Finally for two perfect fluids the above equations reduce to

$$
\dot{S}^{(12)} + a\nabla^2 V_{(12)} + 3H\mathcal{E}_{(12)} = 0 , \qquad (6.74)
$$

and

$$
\dot{V}_{(12)} - \left(3c_z^2 - 1\right)HV_{(12)} = -\tfrac{1}{ah}\left(c^2_{s(1)} - c^2_{s(2)}\right)\mu\Delta - \tfrac{1}{a}c_z^2 S_{(12)} - \tfrac{1}{a}\mathcal{E}_{(12)} . \qquad (6.75)
$$

6.3 Relation to other approaches

In our approach we use a covariant formalism, and geometrically defined variables which are exactly defined and GI at any order, since they vanish in a FLRW universe model. In chapter 4 (Paper I, see also Goode 1983, 1989 [50, 51]) we have related these quantities (in the case of single fluid) to those constructed by Bardeen [1]. Here we will only consider the most important variables used in the study of multi-components, for example the component and relative velocity perturbation variables.

Let us first consider the velocity perturbation variable $V_a^{(i)}$ for the i^{th} fluid component, which we defined in section 6.1.1):

$$
V_a^{(i)} = u_a^{(i)} - u_a . \qquad (6.76)
$$

From this we define the scalar velocity perturbation $V_{(i)}$, simply by taking the divergence

$$
\begin{aligned}
V_{(i)} \equiv a\nabla^a V_a^{(i)} &= a\nabla^a\left(u_a^{(i)} - u_a\right) \\
&= a\left(\Theta_{(i)} - \Theta\right) , \qquad (6.77)
\end{aligned}
$$

to first order. We therefore have

$$\Theta_{(i)} = \Theta + a^{-1}V_{(i)} \ . \tag{6.78}$$

Taking the spatial gradient orthogonal to $u_a^{(i)}$, namely ∇_a^N, we obtain a relation between $Z_a^{(i)} = \nabla_a^N\Theta_{(i)}$ and Z_a:

$$Z_a^{(i)} = Z_a + \left[\tfrac{3K}{a^2} - \tfrac{3}{2}\kappa h\right] V_a^{(i)} + a^{-1}\nabla_a V_{(i)} \ , \tag{6.79}$$

where we have used the zero-order relation

$$\dot{\Theta} = \tfrac{3K}{a^2} - \tfrac{3}{2}\kappa h \ . \tag{6.80}$$

Taking the divergence of (6.79) we obtain

$$\mathcal{Z}^{(i)} - \mathcal{Z} = a\left(\nabla^2 + \tfrac{3K}{a^2}\right) V_{(i)} - \tfrac{3}{2}\kappa h a V_{(i)} \ . \tag{6.81}$$

Now, using the Ricci identity for $u_a^{(i)}$, we can find the $(0,\nu)$ constraint equation for the i^{th} component:

$$\tfrac{2}{3}Z_a^{(i)} + \nabla^b\omega_{ab}^{(i)} - \nabla^b\sigma_{ab}^{(i)} = \kappa h\Psi_a - \kappa h V_a^{(i)} \ , \tag{6.82}$$

while the $(0,\nu)$ constraint for the total fluid, equation (3.157), can be written as

$$\tfrac{2}{3}Z_a + \nabla^b\omega_{ab} - \nabla^b\sigma_{ab} = \kappa h\Psi_a \ . \tag{6.83}$$

Taking the divergence of these two constraint equations and eliminating the energy flux term Ψ, we obtain

$$\tfrac{2}{3}\left(\mathcal{Z} - \mathcal{Z}_{(i)}\right) - a^2\left[\nabla^a\nabla^b\sigma_{ab} - \nabla^a\nabla^b\sigma_{ab}^{(i)}\right] = a\kappa h V_{(i)} \ . \tag{6.84}$$

Finally, substituting for $\mathcal{Z} - \mathcal{Z}_{(i)}$ from equation (6.79), we obtain the following relation:

$$a\left[\nabla^a\nabla^b\sigma_{ab} - \nabla^a\nabla^b\sigma_{ab}^{(i)}\right] = -\tfrac{2}{3}\left(\nabla^2 + \tfrac{3K}{a^2}\right) V_{(i)} \ . \tag{6.85}$$

This clearly shows that there is a relation between the scalar relative velocity variable $V_{(i)}$ and the total divergence of the difference between the shear of the total matter and the shear of the i^{th} component.

Harmonic analysis

Let us now harmonically analyse this relation using the covariant harmonics defined in appendix B. The harmonic components of σ_{ab}, $\sigma^{(i)}_{ab}$ and $V_{(i)}$ are:

$$\sigma_{ab} = \tfrac{k}{a}\left(\sigma^{(0)}_{m} Q^{(0)}_{ab} + \sigma^{(1)}_{m} Q^{(1)}_{ab}\right) , \tag{6.86}$$

$$\sigma^{(i)}_{ab} = \tfrac{k}{a}\left(\sigma^{(0)}_{m(i)} Q^{(0)}_{ab} + \sigma^{(1)}_{m(i)} Q^{(1)}_{ab}\right) , \tag{6.87}$$

and

$$V_{(i)} = V^{(k)}_{(i)} Q , \tag{6.88}$$

where, following KS [69], we denoted by $\sigma_m = -V_{KS}$ the shear harmonic component (see equation 3.45 in their paper), and we have put the subscript KS to distinguish this "velocity perturbation" from our V.[2] Clearly, the vector parts of the above expressions do not contribute to scalar perturbations, so substituting the scalar parts into equation (6.85) and using the identity (6.85) we obtain:

$$V^{(k)}_{(i)} = k\left(\sigma^{(o)}_{m} - \sigma^{(o)}_{m(i)}\right) . \tag{6.89}$$

The relationship between the two variables is therefore clearly given by:

$$V^{(k)}_{(i)} \equiv -k\left(V_{KS} - V^{(i)}_{KS}\right) , \tag{6.90}$$

and the relative velocity variables are related as follows:

$$V^{(k)}_{(ij)} \equiv k V^{KS}_{(ij)} . \tag{6.91}$$

We now turn to the density perturbation variables. In chapter 4, (Paper I [8]) we found that the scalar variable Δ we use is related to the GI Bardeen variable ε_m (denoted also by Δ in KS [69]) as follows:

$$\Delta \equiv -k^2 \varepsilon_m \equiv -k^2 \Delta_{KS} , \tag{6.92}$$

where again we use the subscript KS to distinguish between Kodama and Sasaki and us. It follows that the relationship between the relative variable $S_{(ij)}$ we use, defined in section 3.4 equation (6.2.4) and its equivalent Kodama and Sasaki variable is:

$$S_{(ij)} = -k^2 S^{KS}_{(ij)} . \tag{6.93}$$

[2]For the total fluid, the variable V_{KS} is nothing but the Bardeen's variable $V^{(0)}_{S}$ introduced in chapter 4 (Paper I [8]).

Using the above relations and the ones derived in Paper I [8] (see sections 4.2, 4.3) it is now straightforward to derive the equations in BI [1] and KS [69] from our equations. For example, equations 5.53 and 5.57 in KS [69] follow from equations (6.70) and (6.71), with $q_a^{(i)} = 0$ since Kodama and Sasaki work in the energy frame.

6.4 Conserved quantities

In chapter 3 we introduced two variables, C_a and $\tilde{C}_a$, and their divergences, C and $\tilde{C}$, which are useful in the discussion of the large scale evolution of energy density perturbations. In this section we will briefly discuss some of their properties and the equations which they satisfy.[3] Such conserved quantities are useful in connecting different epochs (e.g. radiation dominated and matter dominated eras) in the evolution of the total density perturbation.

Consider the scalar variables C and $\tilde{C}$: as we have already seen in section 3.10) they are related by

$$\tilde{C} \equiv a\nabla^a \tilde{C}_a = C - \tfrac{4K}{1+w}\Delta \ , \tag{6.94}$$

so they coincide when $K = 0$. Writing equation (3.179) in terms of harmonic components the evolution equation for $\tilde{C}^{(k)}$ is

$$\begin{aligned} \dot{\tilde{C}}^{(k)} &= -\tfrac{4a^2H^3c_s^2}{(1+w)}\left(\tfrac{k^2}{a^2H^2}\right)\Delta^{(k)} + \tfrac{4a^2H^3w}{(1+w)}\left[\tfrac{3K}{a^2} - \tfrac{k^2}{a^2H^2}\right]\mathcal{E}^{(k)} \\ &- 4a^3H^3\left(\tfrac{k^2}{a^2H^2}\right)\left[F^{(k)} + \Pi^{(k)}\right] - H^2\left[4Ka - 2a^3h\right]\left(\tfrac{k^2}{a^2H^2}\right)\Psi^{(k)} \ . \end{aligned} \tag{6.95}$$

Now, the eigenvalue k does not directly correspond to physical wavelengths unless $K = 0$ and the correspondence changes with different values of K. Considering only the $K = -1,\ 0$ cases, k is related to physical wavelengths by the time independent wavenumber ν:

$$k^2 = \nu^2 - K \ , \tag{6.96}$$

where $\nu \geq 0$ is a real number, and physical wavelengths are defined by[4]

$$\lambda \equiv \frac{a}{\nu} \ , \tag{6.97}$$

[3]These quantities are conserved only at scales larger than the horizon; it is possible however to derive quantities which are conserved at any scale; details and an application to inflationary universes will be presented in Dunsby and Bruni (1991) [20]

[4]We recall that in the $K = 1$ case k is related to ν by $k^2 = \nu(\nu+2)$, $\nu = 1, 2\ldots$ (see Harrison, 1967 [54]). Minimum values for k^2 are: $k^2 = 0$ for $K = 0$, $k^2 = 1$ for $K = -1$ and $k^2 = 3$ for $K = 1$.

so for a wavenumber ν equation (6.95) becomes

$$\begin{aligned}\dot{\tilde{C}}^{(\nu)} &= -\tfrac{4a^2H^3c_s^2}{(1+w)}\left(\tfrac{\nu^2}{a^2H^2}-\tfrac{K}{a^2H^2}\right)\Delta^{(\nu)}+\tfrac{4a^2H^3w}{(1+w)}\left[\tfrac{4K}{a^2H^2}-\tfrac{\nu^2}{a^2H^2}\right]\mathcal{E}^{(\nu)}\\ &- 4a^3H^3\left(\tfrac{\nu^2}{a^2H^2}-\tfrac{K}{a^2H^2}\right)\left[F^{(\nu)}+\Pi^{(\nu)}\right]\\ &- H^2\left[4Ka-2a^3h\right]\left(\tfrac{\nu^2}{a^2H^2}-\tfrac{K}{a^2H^2}\right)\Psi^{(\nu)}\,. \end{aligned} \tag{6.98}$$

For wavelengths $\lambda \gg H^{-1} \Rightarrow \frac{\nu^2}{a^2H^2} \ll 1$ this equation reduces to

$$\begin{aligned}\dot{\tilde{C}}^{(\nu)} &= +\tfrac{4a^2H^3c_s^2}{(1+w)}\left(\tfrac{K}{a^2H^2}\right)\Delta^{(\nu)}+\tfrac{4a^2H^3w}{(1+w)}\left[\tfrac{4K}{a^2H^2}\right]\mathcal{E}^{(\nu)}\\ &+ 4a^3H^3\left(\tfrac{K}{a^2H^2}\right)\left[F^{(\nu)}+\Pi^{(\nu)}\right]+H^2\left[4Ka-2a^3h\right]\left(\tfrac{K}{a^2H^2}\right)\Psi^{(\nu)}\,. \end{aligned} \tag{6.99}$$

Let us now examine the consequences of this equation for a number of different cases.

If the background is flat, $K = 0$, we immediately see that $\tilde{C} = C$ *is conserved even when entropy perturbations and imperfect fluid source terms are present.* This is a very important result, since we can now use this conserved quantity to write down a general solution of the first order equation for $\Delta^{(\nu)}$, (3.177), in the long wavelength limit:

$$\Delta^{(\nu)} = \int_{t_0}^{t}\left[\int_{t_1}^{t} e^{A(t_2)}dt_2\right]B^{(\nu)}(t_1)dt_1\ , \tag{6.100}$$

where

$$A(t) = 3Hw - \tfrac{\kappa h}{2H}\ , \tag{6.101}$$

$$B^{(\nu)}(t) = \tfrac{(1+w)}{4a^2H}\tilde{C}^{(\nu)} + 3a\,(1+w)\,H\left[F^{(\nu)}+\Pi^{(\nu)}\right]\ , \tag{6.102}$$

and t_0 is the epoch at which an initial condition for $\Delta^{(\nu)}$ is specified.

The conservation of $\tilde{C}$ is also very important in the study of perturbations in both standard and non-standard inflationary models (based on generalised gravity theories.[5]) since it can be used to directly connect the amplitude of present day large scale structure, which came inside the horizon during the matter dominated era, to the initial conditions just after horizon-crossing during the inflationary era.

[5]The fluid flow approach to perturbations can also be used to study perturbations in generalised gravity theories. The basic idea is to treat all additional contributions to the field equations except, the Einstein tensor part, as contributions to an effective energy momentum tensor (see [88]) Effective fluid quantities are then easy to compute and GI quantities based on spatial gradients can be defined as usual (see Hwang, 1990 [59]).

We now turn to the case when the background is an open ($K = -1$) model. Suppose we can neglect entropy perturbations and the imperfect fluid source terms $\Psi^{(\nu)}$, $F^{(\nu)}$ and $\Pi^{(\nu)}$, then $\tilde{C}$ is conserved if the following inequalities hold:

$$\frac{\nu^2}{a^2H^2} \ll 1 \quad \mathit{and} \quad \frac{1}{a^2H^2} \ll 1 \, , \tag{6.103}$$

i.e. if the long wavelength limit is valid up to $\nu = 1$. However, in an open universe the cosmological density parameter Ω is given by

$$\Omega = 1 - \frac{1}{a^2H^2} \, , \tag{6.104}$$

so the second inequality only holds if Ω is close enough to 1, but then C is also approximately conserved. The properties of C and $\tilde{C}$ are discussed in the context of scalar field dominated universes in BED [8] (see section 5.2.5).

6.5 Perturbations in a radiation - dust universe

6.5.1 Background model

In this section we will discuss a simple application of the equations presented in the first part of this chapter. We will assume that the background model is described by a flat ($K = 0$) FLRW, containing a mixture of non-interacting dust (pressureless matter) and radiation with vanishing cosmological constant ($\Lambda = 0$). By non-interacting we mean that each fluid component satisfies the background or zero-order conservation equation with vanishing interaction source term. For dust and radiation these equations are:

$$\dot{\mu}_{(d)} + 3H\mu_{(d)} = 0 \, , \tag{6.105}$$

and

$$\dot{\mu}_{(r)} + 3Hh_{(r)} = 0 \, , \tag{6.106}$$

where $h_{(r)} = \mu_{(r)} + p_{(r)} = \frac{4}{3}\mu_{(r)}$ and the subscripts d and r stand for dust and radiation respectively. These two equations integrate to give $\mu_{(d)} = Ma^{-3}$ and $\mu_{(r)} = Ra^{-4}$, so the total energy density and presure is given by

$$\mu = \tfrac{3}{\beta^2}a^{-4}(1 + \alpha a) \, , \tag{6.107}$$

and

$$p = \frac{1}{\beta^2} a^{-4} \ , \tag{6.108}$$

so h is simply

$$h = \frac{1}{\beta^2} a^{-4} (4 + 3\alpha a) \ , \tag{6.109}$$

where $M = \mu_{(d)_0} a_0^3$, $\dot{M} = 0$, $R = \mu_{(r)_0} a_0^4$, $\dot{R} = 0$, $\alpha = \frac{M}{R}$ represents the fraction of dust to radiation, $\beta = \sqrt{\frac{3}{R}}$ and a_0 is the present value of the scale factor. The Friedmann equation is

$$H^2 = \frac{1}{3}\kappa\mu = \frac{1}{\beta^2} a^{-4} (1 + \alpha a) \ , \tag{6.110}$$

so H is given by

$$H = \frac{1}{\beta a^2} (1 + \alpha a)^{1/2} \ , \tag{6.111}$$

where we have taken $\kappa = 1$. Finally $c_s^2 = \frac{\dot{p}}{\dot{\mu}}$ is given by

$$c_s^2 = \frac{4}{3(4 + 3\alpha a)} \ . \tag{6.112}$$

Note, this is not the speed of sound in the fluid mixture since the components are un - coupled, i.e. there are no interactions. Having derived the background quantities we now turn to the equations that describe the evolution of fluid inhomogeneities.

6.5.2 Small scale solutions

The study of density perturbations in a radiation - dust universe is very instructive since, apart from exotic stages, the matter in the universe is well described by such a mixture. Long wavelength solutions for a dust and radiation background have been discussed in KS [69] and Dunsby (1991a) [18]. Let us now look at the case where the characteristic size of the fluid inhomogeneities is much less than the Jeans length for the radiation but is still larger than the mean free path of the photon enabling us to neglect interactions between the fluid components. In this limit we can take the radiation energy density to be approximately homogeneous so we can assume that $\Delta_{(r)}$ can be neglected (Groth and Peebles, 1975 [52], Peebles, 1980 [103]). This reduces the problem to one of studying the evolution of matter fluctuations on a homogeneous radiation background. Although the radiation causes negligible gravitational perturbation in the matter density, it does effect the growth of the matter fluctuations by speeding up the expansion of the universe.

Neglecting the interaction and imperfect fluid source terms, and writing in terms of harmonic components, the basic equations for the matter and velocity perturbations are (6.69), (3.178) and (6.68):

$$\dot{\Delta}^{(k)}_{(d)} + \mathcal{Z}^{(k)} = \frac{3H}{h}\left[c_s^2\mu\Delta^{(k)} + p\mathcal{E}^{(k)}\right] + a\left(\frac{k^2}{a^2}\right)V^{(k)}_{(d)} , \tag{6.113}$$

$$\dot{\mathcal{Z}}^{(k)} + 2H\mathcal{Z}^{(k)} + \frac{1}{2}\mu\Delta^{(k)} - \frac{1}{h}\left(\frac{k^2}{a^2}\right)\left[c_s^2\mu\Delta^{(k)} + p\mathcal{E}^{(k)}\right] = 0 , \tag{6.114}$$

and

$$\dot{V}^{(k)}_{(d)} + HV^{(k)}_{(d)} = \frac{1}{ah}\left[c_s^2\mu\Delta^{(k)} + p\mathcal{E}^{(k)}\right] , \tag{6.115}$$

where we have used the fact that $w_{(d)} = 0$ and $\Psi^{(k)}_{(d)} = V^{(k)}_{(d)}$. We can also write down an equation for the relative velocity $V^{(k)}_{(dr)}$ of the dust and radiation. It is:

$$\dot{V}^{(k)}_{(dr)} - \left(3c_z^2 - 1\right)HV^{(k)}_{(dr)} = \frac{1}{3ah}\mu\Delta - \frac{1}{a}c_z^2 S^{(k)}_{(dr)} , \tag{6.116}$$

where c_z^2, for dust and radiation is given by $c_z^2 = \frac{\mu_{(d)}}{3h}$. Now, since $\Delta_{(r)}$ can be neglected, we have the following relationships:

$$c_s^2\mu\Delta^{(k)} + p\mathcal{E}^{(k)} = \frac{1}{3}\mu_{(r)}\Delta^{(k)}_{(r)} \approx 0 , \tag{6.117}$$

and

$$S^{(k)}_{(dr)} \approx \Delta^{(k)}_{(d)} , \tag{6.118}$$

so the above equations reduce to

$$\dot{\Delta}^{(k)}_{(d)} + \mathcal{Z}^{(k)} - a\left(\frac{k^2}{a^2}\right)V^{(k)}_{(d)} = 0 , \tag{6.119}$$

$$\dot{\mathcal{Z}} + 2H\mathcal{Z}^{(k)} + \frac{1}{2}\mu_{(d)}\Delta^{(k)}_{(d)} = 0 , \tag{6.120}$$

$$\dot{V}^{(k)}_{(d)} + HV^{(k)}_{(d)} = 0 , \tag{6.121}$$

and the relative velocity equation becomes

$$\dot{V}^{(k)}_{(dr)} + \frac{4}{3}\frac{\mu_{(r)}}{h}HV^{(k)}_{(dr)} = 0 . \tag{6.122}$$

Equations (6.119), (6.120) and (6.121) can be combined to give a second order equation for $\Delta^{(k)}_{(d)}$:

$$\ddot{\Delta}^{(k)}_{(d)} + 2H\dot{\Delta}^{(k)}_{(d)} - \frac{1}{2}\mu_{(d)}\Delta^{(k)}_{(d)} = 0 . \tag{6.123}$$

This is simply the equation for the evolution of matter perturbations in a radiation-dust background.

Substituting for the background quantities H and $\mu_{(d)}$ and changing the time variable to a, this equation becomes

$$\frac{d^2\Delta^{(k)}_{(d)}}{da^2} + \frac{2+3a}{2a(1+a)}\frac{d\Delta^{(k)}_{(d)}}{da} - \frac{3}{2a(1+a)}\Delta^{(k)}_{(d)} = 0 \ . \tag{6.124}$$

where we have taken $\alpha = 1$, so that $a = 1$ at dust and radiation equi - density. This equation was obtained by Mészáros (1974) [91] and has the particular integral

$$\Delta^{(k)}_{(d)_1} = 1 + \tfrac{3}{2}a \ , \tag{6.125}$$

so the *general solution* is

$$\Delta^{(k)}_{(d)} = A^{(k)}\left(1+\tfrac{3}{2}a\right) - \tfrac{1}{4}B^{(k)}\left[\left(1+\tfrac{3}{2}a\right) ln\left(\tfrac{\sqrt{a+1}+1}{\sqrt{a+1}-1}\right) - 3\,(a+1)^{1/2}\right] \ , \tag{6.126}$$

where $A^{(k)}$ and $B^{(k)}$ are positive constants. This solution was first obtained by Groth and Peebles (1975) [52] using the Newtonian approximation. For large values of a, which corresponds to the dust dominated stage, the growing mode of this solution is proportional to the scale factor a, which agrees with the Einstein - de Sitter model.

Let us now consider the velocity perturbation equations (6.121) and (6.122). Substituting for the background quantities and as before changing the time variable to a, we obtain

$$\frac{dV^{(k)}_{(d)}}{da} + \frac{1}{a}V^{(k)}_{(d)} = 0 \ , \tag{6.127}$$

and

$$\frac{dV^{(k)}_{(dr)}}{da} + \frac{4}{a(4+3a)}V^{(k)}_{(dr)} = 0 \ . \tag{6.128}$$

The solutions are

$$V^{(k)}_{(d)} = \frac{V^{(k)}_0}{a} \ , \tag{6.129}$$

and

$$V^{(k)}_{(dr)} = E^{(k)}\left[\frac{4+3a}{a}\right] \ , \tag{6.130}$$

where $V_0^{(k)}$ and $E^{(k)}$ are constants. These two solutions show that in general for large values of the scale factor a (normalised here to equi-density) the 4-velocity of matter will coincide with the total 4-velocity, while the relative velocity $V_{(dr)}$ will tend to a constant value.

6.6 Perturbations in a photon - baryon system

6.6.1 Assumptions and background evolution

According to the standard model of the universe, between the epochs of cosmological nuclear synthesis, when the light elements such as deuterium and helium are produced, and the hydrogen recombination, the cosmic matter can be well described by a mixture of photons and baryons in the form of nuclei, electrons, and neutrinos with constant relative abundances. Clearly photons are important since radiation dominates the cosmic energy density and pressure during this period and baryons are of course responsible for the large-scale structures we see today (or at least for the luminous part of it). Although the contribution of electrons to the total energy density can be neglected, they do provide the necessary coupling between photons and baryons through Thompson scattering. Furthermore, because there is a small neutron-fraction of baryons in the standard model of the universe, we can regard baryons as being totally composed of protons. Given these basic assumptions, the total energy density is given by

$$\mu = \mu_{(r)} + \mu_{(m)} \ , \tag{6.131}$$

where

$$\mu_r = 4\sigma_b T^4, \quad \mu_{(m)} = n_B m_p \ , \tag{6.132}$$

σ_b is the Stephan-Boltzmann constant and m_p is the proton mass; and the total pressure is

$$p = p_{(r)} + p_{(m)} \ , \tag{6.133}$$

where

$$p_{(r)} = \tfrac{1}{3}\mu_{(r)}, \quad p_{(m)} = (1 + x_e) n_B T \ , \tag{6.134}$$

and $x_e \equiv \frac{n_e}{n_B}$ is the fractional hydrogen ionization where n_e is the number density of free electrons. Note that in the above equations, unlike the radiation and dust case discussed in the previous section, we include the pressure of the matter. Although its contribution to the total pressure is negligible, so in practice we will assume that $p_{(m)} \approx 0$, it does play an important role in the form of the sound speed $c_{(m)}$, defined by $c^2_{(m)} = \frac{\dot{p}_{(m)}}{\dot{\mu}_{(m)}}$, when one considers matter density perturbations on scales comparable or less than the matter sound horizon.

With these arguments in mind, we can now write down the background evolution equations. These are respectively the energy conservation equations for matter and radiation:

$$\dot{\mu}_{(r)} + 3H\left(\mu_{(r)} + p_{(r)}\right) = 0 , \tag{6.135}$$

$$\dot{\mu}_{(m)} + 3H\mu_{(m)} = 0 , \tag{6.136}$$

and assuming the background is flat ($K = 0$), the Friedmann equation:

$$H^2 = \tfrac{1}{3}\mu , \tag{6.137}$$

where $\mu = \mu_{(r)} + \mu_{(m)}$ is the total cosmic energy density and we have taken $\kappa = 1$.

6.6.2 Interaction terms

Before we go on to consider energy density and velocity perturbations in photon-baryon systems, we need to discuss how to treat the interactions between components which come about through Thompson scattering. Collisions between protons, electrons and photons result in a transfer of momentum and this is described by including a source term $f_{(i)}$ in the momentum conservation equation for each component. For matter and radiation these equations are:

$$\mu_{(m)}a_a + Y_a^{(m)} + \mu_{(m)}F_a^{(m)} = f_a^{(m)} , \tag{6.138}$$

and

$$h_{(r)}a_a + Y_a^{(r)} + h_{(r)}F_a^{(r)} = f_a^{(r)} , \tag{6.139}$$

where $h_{(r)} \equiv \mu_{(r)} + p_{(r)} = \frac{4}{3}\mu_{(r)}$ and $f_a^{(m)} + f_a^{(r)} = 0$ expressing the conservation of total momentum.

Explicit expressions for $f_a^{(m)}$ and $f^{(r)}$ can be found using relativistic kinetic theory (see for example the book by De Groot, Van Leeuwen and Van Weert, 1980 [16]). We will only present the results of this calculation here and refer the interested reader to appendix E of KS [69] for a detailed derivation.

It is found that the momentum transfer rate for matter and radiation is given by

$$\begin{aligned} f_a^{(m)} &= \tfrac{4}{3}\mu_{(r)}HR_c\left(V_a^{(r)} - V_a^{(m)}\right) \\ &= -f_a^{(r)} , \end{aligned} \tag{6.140}$$

and so the relative interaction variable $f_a^{(mr)}$ defined by equation (6.42) is found to be

$$\begin{aligned} f_a^{(mr)} &= \frac{f_a^{(m)}}{h_{(m)}} - \frac{f_a^{(r)}}{h_{(r)}} \\ &= -\frac{Hh}{\mu_{(m)}} R_c V_a^{(mr)} , \end{aligned} \tag{6.141}$$

where R_c, defined by

$$R_c = \frac{1}{H\tau_c} , \tag{6.142}$$

is the ratio of the horizon size to the mean free path for photons colliding with electrons (τ_c is the mean collision time of photons with electrons). Since we are primarily interested in scalar perturbations we can simply compute the scalar relative interaction variable by taking the divergence of equation (6.141):

$$\begin{aligned} f_{(mr)} &= a\nabla^a f_a^{(mr)} \\ &= -\frac{Hh}{\mu_{(m)}} R_c V_{(mr)} . \end{aligned} \tag{6.143}$$

This is the variable we will need for the rest of this section.

6.6.3 Isothermal versus adiabatic perturbations

As we discussed earlier, when one considers the coupling between isothermal and adiabatic perturbations, it is much more convenient to use the equations for the relative variables $V_{(ij)}$ and $S_{(ij)}$. Substituting for the background quantities in to equations (6.71) and (6.70) and writing in terms of harmonic components, we obtain the following:

$$\begin{aligned} &\dot{V}_{(mr)}^{(k)} + H\left[\frac{4}{3}\frac{\mu_{(r)}}{h}\left(1 - 3c_{s(m)}^2\right) + \frac{h}{\mu_{(m)}}R_c\right] V_{(mr)}^{(k)} \\ &= -\frac{1}{ah}\left(c_{s(m)}^2 - \frac{1}{3}\right)\mu\Delta^{(k)} - \frac{1}{ah}\left[\frac{1}{3}\mu_{(m)} + \frac{4}{3}c_{s(m)}^2\mu_{(r)}\right] S_{(mr)}^{(k)} - \Pi_{(mr)}^{(k)} , \end{aligned} \tag{6.144}$$

and

$$\dot{S}_{(mr)}^{(k)} = a\left(\frac{k^2}{a^2}\right) V_{(mr)}^{(k)} , \tag{6.145}$$

where $h = \mu_{(r)} + p_{(r)} + \mu_{(m)}$. From these two equations we can derive a second order equation for $S_{(mr)}^{(k)}$:

$$\ddot{S}_{(mr)}^{(k)} + H\left[\frac{4}{3}\frac{\mu_{(r)}}{h}\left(1 - 3c_{s(m)}^2\right) + \frac{h}{\mu_{(m)}}R_c + 1\right] \dot{S}_{(mr)}^{(k)}$$

$$+\frac{1}{3h}\left(\frac{k^2}{a^2}\right)\left[\mu_{(m)}+4c_{s(m)}^2\mu_{(r)}\right]S_{(mr)}^{(k)}$$
$$=\frac{\mu}{3h}\left(1-3c_{s(m)}^2\right)\left(\frac{k^2}{a^2}\right)\Delta^{(k)}-a\left(\frac{k^2}{a^2}\right)\Pi_{(mr)}^{(k)}\,. \qquad (6.146)$$

This equation shows how isothermal perturbations are coupled to adiabatic perturbations. To close the system of equations we need an equation for the total density fluctuation:

$$\Delta=\frac{1}{\mu}\left(\mu_{(r)}\Delta_{(r)}+\mu_{(m)}\Delta_{(m)}\right)\,. \qquad (6.147)$$

Using equation (3.183) we can write it in form useful for the present case:

$$\left(\mu a^3\Delta^{(k)}\right)^{\cdot\cdot}+H\left(2+3c_s^2\right)\left(\mu a^3\Delta^{(k)}\right)^{\cdot}$$
$$+\left[\left(\frac{k^2}{a^2}\right)c_s^2-\frac{1}{2}h\right]\left(\mu a^3\Delta^{(k)}\right)+\frac{4}{9}\left(\frac{k^2}{a^2}\right)\frac{\mu_{(r)}\mu_{(m)}a^3}{h}\left(3c_{s(m)}^2-1\right)S_{(mr)}^{(k)}$$
$$=ha^4\left[3H\dot{\Pi}-\left\{\mu_{(r)}+\left(\frac{k^2}{a^2}\right)\right\}\Pi\right]\,, \qquad (6.148)$$

where c_s^2 is the sound velocity in the total fluid (since the fluid is coupled) given by

$$c_s^2=\frac{4\mu_{(r)}+9c_{s(m)}^2\mu_{(m)}}{3\left(4\mu_{(r)}+3\mu_{(m)}\right)}\,, \qquad (6.149)$$

and we have used equation (6.37) to express the total entropy perturbation $\mathcal{E}^{(k)}$ in terms of $S_{(mr)}^{(k)}$:

$$p\mathcal{E}^{(k)}=\frac{4}{9}\frac{\mu_{(r)}\mu_{(m)}}{h}\left(3c_{s(m)}^2-1\right)S_{(mr)}^{(k)}\,. \qquad (6.150)$$

Many authors regard perturbations of the total energy density as purely "adiabatic", however this is not quite correct since we have to separate the contribution of $S_{(mr)}^{(k)}$ from $\Delta^{(k)}$. This can be done using the relation (6.48). For matter and radiation this is:

$$\mu\Delta^{(k)}=\frac{3}{4}h\Delta_{(r)}^{(k)}+\mu_{(m)}S_{(mr)}^{(k)}\,. \qquad (6.151)$$

However, since initially radiation is the dominant fluid component, we have $\Delta^{(k)}\approx\Delta_{(r)}^{(k)}$, so during this stage we can take $\Delta^{(k)}$ as being the amplitude of the adiabatic perturbation.

This suggests that it is much more convenient to regard $S_{(mr)}^{(k)}$ and $\Delta_{(r)}^{(k)}$ as the dynamical variables of the problem, so replacing $\Delta^{(k)}$ in equation (6.146) using the above relation gives

$$\ddot{S}_{(mr)}^{(k)}+H\left[\frac{4}{3}\frac{\mu_{(r)}}{h}\left(1-3c_{s(m)}^2\right)+\frac{h}{\mu_{(m)}}R_c+1\right]\dot{S}_{(mr)}^{(k)}+\left(\frac{k^2}{a^2}\right)c_{s(m)}^2S_{(mr)}^{(k)}$$
$$=\frac{1}{4}\left(\frac{k^2}{a^2}\right)\left(1-3c_{s(m)}^2\right)\Delta_{(r)}-a\left(\frac{k^2}{a^2}\right)\Pi_{(mr)}^{(k)}\,. \qquad (6.152)$$

This equation clearly shows that *isothermal perturbations can only have oscillatory behaviour on scales smaller than the matter sound horizon.* We can also substitute for $\Delta^{(k)}$ in equation (6.148) and this gives

$$\left(ha^3\Delta^{(k)}_{(r)}\right)^{\cdot\cdot} + H\left(2+3c_s^2\right)\left(ha^3\Delta^{(k)}_{(r)}\right)^{\cdot} + \left[\tfrac{1}{3}\left(\tfrac{k^2}{a^2}\right) - \tfrac{1}{2}h\right]\left(ha^3\Delta^{(k)}_{(r)}\right)$$
$$= \tfrac{4}{3}a^3\mu_{(m)}\left[\tfrac{1}{2}hS^{(k)}_{(mr)} - H\left(1+3c^2_{s(m)} - \tfrac{h}{\mu_{(m)}}R_c\right)\dot{S}^{(k)}_{(mr)}\right]$$
$$+\tfrac{4}{3}ha^4\left[3H\dot{\Pi}^{(k)} - \mu_{(r)}\Pi^{(k)} - \left(\tfrac{k^2}{a^2}\right)\Pi^{(k)}_{(r)}\right] , \tag{6.153}$$

where we have substituted for $\ddot{S}^{(k)}_{(mr)}$ from equation (6.152) and used the following relations:

$$\Pi^{(k)}_{(mr)} = \Pi^{(k)}_{(m)} - \Pi^{(k)}_{(r)}, \qquad \Pi^{(k)} = \tfrac{1}{h}\left(h_{(m)}\Pi^{(k)}_{(m)} + h_{(r)}\Pi^{(k)}_{(r)}\right) . \tag{6.154}$$

Finally, substituting for $\Delta^{(k)}$ in the relative velocity equation (6.144) we obtain

$$\dot{V}^{(k)}_{(mr)} + H\left[\tfrac{4}{3}\tfrac{\mu_r}{h}\left(1-3c^2_{s(m)}\right) + \tfrac{h}{\mu_{(m)}}R_c\right]V^{(k)}_{(mr)}$$
$$= \tfrac{1}{4}\tfrac{1}{a}\left(1-3c^2_{s(m)}\right)\Delta^{(k)}_{(r)} - \tfrac{1}{a}c^2_{s(m)}S^{(k)}_{(mr)} - \Pi^{(k)}_{(mr)} . \tag{6.155}$$

6.6.4 Long wavelength solutions

In this section we will investigate the behaviour of perturbations on scales much larger than the horizon, i.e. we will assume that $\frac{k^2}{a^2H^2} \ll 1$. In this limit we can assume that the matter sound velocity can be neglected so we can set $c^2_{s(m)} = 0$ and the system becomes essentially equivalent to a radiation-dust universe, the only difference is that we explicitly take into account the interactions between photons and baryons.

As in KS [69], it is convenient to introduce the following set of variables:

$$X^{(k)} = ha^3\Delta^{(k)}_{(r)} = \left(\tfrac{4}{3}\mu_{(r)} + \mu_{(m)}\right)\Delta^{(k)}_{(r)} , \tag{6.156}$$

and

$$Y^{(k)} = \mu_{(m)}a^3S^{(k)}_{(mr)} . \tag{6.157}$$

Using these variables, setting $c^2_{s(m)} = 0$ and neglecting the viscosity source terms for simplicity, equations (6.153), (6.152) and (6.155) become

$$\ddot{X}^{(k)} + H\left(2+3c_s^2\right)\dot{X} + \left[\tfrac{1}{3}\left(\tfrac{k^2}{a^2}\right) - \tfrac{1}{2}h\right]X^{(k)}$$
$$= \tfrac{4}{3}\left[\tfrac{1}{2}hY^{(k)} - H\left(1 - \tfrac{h}{\mu_{(m)}}R_c\right)\dot{Y}^{(k)}\right] ;, \tag{6.158}$$

$$\ddot{Y}^{(k)} + H\left[\frac{4}{3}\frac{\mu_{(r)}}{h} + \frac{h}{\mu_{(m)}}R_c + 1\right]\dot{Y}^{(k)} = \frac{1}{4}\left(\frac{k^2}{a^2}\right)\frac{\mu_{(m)}}{h}X^{(k)} , \tag{6.159}$$

and

$$\dot{V}^{(k)}_{(mr)} + H\left[\frac{4}{3}\frac{\mu_{(r)}}{h} + \frac{h}{\mu_{(m)}}R_c\right]V^{(k)}_{(mr)} = \frac{1}{4}\frac{1}{ha^4}X^{(k)} . \tag{6.160}$$

Let us first look at the homogeneous part of equation (6.158) by setting it's right - hand side to zero. Substituting for the background quantities and changing the time variable to $z = 1 + a$ we obtain

$$\frac{d^2X^{(k)}}{dz^2} + \left[\frac{1}{2z} + \frac{2}{z-1} - \frac{3}{3z+1}\right]\frac{dX^{(k)}}{dz} - \left[\frac{1}{2z} - \frac{1}{2(z-1)} + \frac{2}{(z-1)^2}\right]X^{(k)} = 0 . \tag{6.161}$$

To solve this equation, change the unknown function $X^{(k)}$ to $u^{(k)}$ defined by

$$X^{(k)} = uz^{1/2}(z-1)^{-2} , \tag{6.162}$$

so equation (6.161) can be reduced to a pair of first order equations for

$$v = \frac{du}{dz} \tag{6.163}$$

$$\frac{dv}{dz} + \left[\frac{3}{2z} - \frac{2}{z-1} - \frac{3}{3z+1}\right]v = 0 , \tag{6.164}$$

which can easily be integrated to give

$$u(z) = c_1 Q(z) + c_2 , \tag{6.165}$$

where

$$Q(z) = z^{-1/2}\left(z^3 - \frac{25}{9}z^2 + \frac{5}{3}z - \frac{5}{3}\right) , \tag{6.166}$$

and c_1 and c_2 are arbitrary constants. Hence *the general solution* of equation (6.161) is

$$X^{(k)} = A_1^{(k)}X_1(z) + A_2^{(k)}X_2(z) , \tag{6.167}$$

where

$$X_1(z) = z^{1/2}(z-1)^{-2} , \tag{6.168}$$

$$X_2(z) = [Q(z) + x]\,X_1(z) , \tag{6.169}$$

$A_1^{(k)}$ and $A_2^{(k)}$ are arbitrary constants and x is a constant to be determined (See KS [69]; Chernin, 1966 [12]; Nariai, Tomita and Kato, 1967 [96]).

Let us now examine the limiting values of these solutions. For $a \ll 1 \Leftrightarrow z \sim 1$, corresponding to the radiation dominated stage we find

$$X_1(a) = \tfrac{1}{a^2} + \tfrac{1}{2}\tfrac{1}{a} - \tfrac{1}{8} + \dots , \tag{6.170}$$

and

$$X_2(a) = \left(\tfrac{1}{a^2} + \tfrac{1}{2}\tfrac{1}{a} - \tfrac{1}{8} + \dots\right)\left(\tfrac{16}{9} + \tfrac{10}{9}a^3\right) + x\left(\tfrac{1}{a^2} + \tfrac{1}{2}\tfrac{1}{a} - \tfrac{1}{8} + \dots\right) , \tag{6.171}$$

so that if we choose the constant x to be $x = -\frac{16}{9}$, we obtain

$$X_2(a) = \tfrac{10}{9}a + \dots , \tag{6.172}$$

which is purely growing. So in the radiation dominated era we can write the solution for $X^{(k)}$ as

$$X^{(k)} = \tfrac{10}{9}A_2^{(k)}a + A_1^{(k)}a^{-2} . \tag{6.173}$$

For $a \gg 1 \Leftrightarrow z \gg 1$, corresponding to the matter dominated stage we find

$$X_1(a) \sim a^{-3/2} , \tag{6.174}$$

and

$$X_2(a) \sim a . \tag{6.175}$$

So in this case the solution for $X^{(k)}$ is

$$X^{(k)} = A_2^{(k)}a + A_1^{(k)}a^{-3/2} . \tag{6.176}$$

Turning to the homogeneous parts of equations (6.159) and (6.160), we see that, because of their simple structure, they can be integrated to give

$$Y^{(k)} = Y_0^{(k)} + B^{(k)}\int \tfrac{1}{Ha^2}e^{-P(a)}da , \tag{6.177}$$

and

$$V_{(mr)}^{(k)} = E^{(k)}e^{-P(a)} , \tag{6.178}$$

where the function $P(a)$ is given by

$$P(a) = \int \tfrac{1}{a}\left(\tfrac{4}{3}\tfrac{\mu_{(r)}}{h} + \tfrac{h}{\mu_{(m)}}R_c\right)da , \tag{6.179}$$

and $E^{(k)}$ is an arbitrary constant. Substituting for the background quantities $P(a)$ becomes

$$P(a) = \int \tfrac{1}{a} \left(\tfrac{4}{4+3a} + \tfrac{4+3a}{3a} R_c \right) da \ . \tag{6.180}$$

Now R_c, the ratio of the horizon size to the photon mean free path, is much greater than unity at the cosmological stage of our interest, so we have $P(a) \gg 1$. This means that the solution for $Y^{(k)}$ settles down to a constant value immediately after it is provoked and so *the isothermal perturbation has essentially one mode which is constant with time.* The reason for this is that the matter is so tightly coupled with the radiation that it cannot move relative to the radiation. Indeed, this fact can easily be seen by looking at the solution for the relative velocity, equation (6.178), which tends very rapidly to zero. It is also interesting to note that the solution for $Y^{(k)}$ is independent of the scale under consideration so this constant isothermal mode remains true on almost all scales of interest, provided of course $R_c \gg 1$.

Consider now the case when $Y^{(k)}$ is not present from the beginning i.e. there are initially no isothermal perturbations but $X^{(k)}$ is non zero. We then have the interesting possibility that *isothermal perturbations may be generated from adiabatic perturbations.*

First consider the equation for $Y^{(k)}$. This can be integrated with the help of the homogeneous solution to give

$$Y^{(k)} = \tfrac{1}{2\mathcal{H}_{eq}^2} \int_{a_0}^{a} \int_{a_0}^{a_1} \exp\left[- \int_{a_2}^{a_1} \tfrac{1}{a_3} \left(\tfrac{4}{4+3a_3} + \tfrac{4+3a_3}{3a_3} R_c \right) da_3 \right] S^{(k)}(a_2) \tfrac{da_2}{\sqrt{1+a_2}} \tfrac{da_1}{\sqrt{1+a_1}} \ , \tag{6.181}$$

where

$$S^{(k)}(a) = \tfrac{\mu_{(m)}}{h} X^{(k)}(a) = \tfrac{3}{3a+4} X^{(k)}(a) \ , \tag{6.182}$$

and $\mathcal{H}_{eq}^2 = \left(\frac{Ha}{k} \right)_{a=1}$ is the ratio of the reduced proper wavelength to the horizon size at matter and radiation equi - density. Now since $X^{(k)}(a)$ is a slowly varying function of the scale factor (see equations (6.173) and (6.176)) and $R_c \gg 1$ the above solution reduces to

$$Y^{(k)}(a) = \tfrac{1}{2\mathcal{H}_{eq}^2} \int_{a_0}^{a} \tfrac{9a_1^3}{(a_1+1)(3a_1+4)^2} \tfrac{X^{(k)}(a_1)}{R_c a_1} da_1 \ . \tag{6.183}$$

Now let us consider the case where $X^{(k)}(a)$ has initially only a growing mode. Then, in this case, from equations (6.173) and (6.176) we have

$$X^{(k)} = \tfrac{10}{9} A_2^{(k)} a \ , \qquad a \ll 1 \ , \tag{6.184}$$

and

$$X^{(k)} = A_2^{(k)} a \ , \qquad a \gg 1 \ . \tag{6.185}$$

From equation (6.142) we can express R_c in terms of the scale factor a:[6]

$$R_c = \frac{\sqrt{2}a^2}{(1+a)^{1/2}} R_{eq} \ , \tag{6.186}$$

where R_{eq} is the value of R_c at matter and radiation equi-density. Substituting for R_c and $X^{(k)}$ in equation (6.183) we get

$$Y^{(k)}(a) = \left(\frac{5}{48}\right) \frac{A_2^{(k)}}{\sqrt{2}\mathcal{H}_{eq}^2 R_{eq}} a^3 \ , \quad a \ll 1 \ , \tag{6.187}$$

and

$$Y^{(k)}(a) = \frac{1}{\sqrt{2}} \frac{A_2^{(k)}}{\mathcal{H}_{eq}^2 R_{eq}} a^{1/2} \ , \quad a \gg 1 \ , \tag{6.188}$$

and this leads to an expression for $S_{(mr)}^{(k)}$ in terms of $\Delta_{(r)}$, R_c and $\mathcal{H} = \frac{Ha}{n}$:

$$S_{(mr)}^{(k)}(a) = \frac{1}{16} \frac{\Delta_{(r)}^{(k)}}{\mathcal{H}^2 R_c} a \ , \quad a \ll 1 \ , \tag{6.189}$$

and

$$S_{(mr)}^{(k)}(a) = \frac{\Delta_{(r)}^{(k)}}{2R_c\mathcal{H}^2} \ , \quad a \gg 1 \ , \tag{6.190}$$

so when the wavelength comes into the horizon ($\mathcal{H} = 1$) we obtain

$$S_{(mr)}^{(k)}(a_H) = \frac{1}{16} \frac{\Delta_{(r)}^{(k)}(a_H)}{R_c(a_H)} a_H \ , \quad a_H \ll 1 \ , \tag{6.191}$$

and

$$S_{(mr)}^{(k)}(a_H) = \frac{\Delta_{(r)}^{(k)}(a_H)}{2R_c(a_H)} \ , \quad a_H \gg 1 \ , \tag{6.192}$$

where a_H is the value of the scale factor when $\mathcal{H} = 1$. These results, although different to the ones obtained by KS [69], give basically the same physical result, that is that the generated amplitude of the isothermal perturbation is depressed by a factor of order $1/R_c(a_H)$ compared with the amplitude of the adiabatic perturbation. Since $R_c(a_H) \gg 1$ this is probably totally negligible, however it is of some conceptual interest. The generation of isothermal perturbations from adiabatic perturbations has been discussed in detail in the context of universe models dominated by two scalar fields by Mollerach (1990) [93].

[6]Note, there is an error in the expression for R_c in Kodama and Sasaki (1984) [69] and this leads to different expressions for $Y^{(k)}(a)$, $S_{(mr)}^{(k)}(a)$ and $S_{(mr)}^{(k)}(a_H)$.

6.7 Perturbations in two scalar field systems

In this section we will study the evolution of the fluctuations in a two-component scalar field system $(\phi_{(1)}, \phi_{(2)})$ in a spatially flat ($k=0$) universe model (an application is considered e.g. in Mollerach (1990) [93]). The fluctuations can be characterized in two different ways; either by using the scalar gauge-invariant variables $\Delta_{(i)}$, $(i=1,2)$, Δ, which correspond to energy density fluctuations of each component and the total fluid and $V_{(i)}$, which corresponds to the velocity perturbation of each component, or by using the relative scalar variables $S_{(ij)}$ and $V_{(ij)}$ defined by (6.41), (6.42), (6.43) and (6.44). As in the previous example we will concentrate on the relative variables.

For two components the equations we need are (6.74) and (6.75):

$$\dot{S}_{(12)} + S\nabla^2 V_{(12)} + 3H\mathcal{E}_{(12)} = 0 \ , \tag{6.193}$$

and

$$\dot{V}_{(12)} - \left(3c_z^2 - 1\right) HV_{(12)} = -\tfrac{1}{Sh}\left(c_{s(1)}^2 - c_{s(2)}^2\right)\mu\Delta - \tfrac{1}{S}c_z^2 S_{(12)} - \tfrac{1}{S}\mathcal{E}_{(12)} \ . \tag{6.194}$$

We will assume that the energy density of each field is dominated by the potential term, so $|1+w_{(1)}| \ll 1$ and $|1+w_{(2)}| \ll 1$. It is easy to obtain an expression for $c_{s(1)}^2$ for the field $\phi_{(1)}$:

$$c_{s(1)}^2 = \frac{3H\dot{\phi}_{(1)} + 2V'_{(1)}}{3H\dot{\phi}_{(1)}} \ . \tag{6.195}$$

A similar result can be found for $\phi_{(2)}$.

In the slow rolling approximation we have $3H\dot{\phi}_{(1)} \simeq -V'(\phi_{(1)})$, so we can take $c_{s(1)}^2 = c_{s(2)}^2 \simeq -1$. Another very important point is that since we are dealing with scalar fields, we cannot neglect the individual entropy perturbations $\mathcal{E}_{(i)}$. For each component $i = 1,2$ we have

$$\mathcal{E}_{(i)} = \left(1 - c_{s(i)}^2\right)\mu_{(i)}{}^N\!\Delta_{(i)} = 2\mu_{(i)}{}^N\!\Delta_{(i)} \ , \tag{6.196}$$

where ${}^N\!\Delta_{(i)}$ is the energy density fluctuation of the i^{th} fluid component defined in it's own matter rest frame (cf. KS [69] , Mollerach (1990) [93] and BDE [8]). However what we want to look at are the relative perturbations defined in the total matter rest frame, so using the linear transformation law

$${}^N\!\Delta_{(i)} = \Delta_{(i)} - 3SHV_{(i)} \ , \tag{6.197}$$

we get an expression for the relative entropy perturbation $\mathcal{E}_{(12)}$ in terms of $V_{(12)}$ and $S_{(12)}$:

$$\mathcal{E}_{(12)} = 2S_{(12)} - 6HSV_{(12)} \ . \tag{6.198}$$

Changing the time variable to the scale factor a, equations (6.193) and (6.194), in terms of harmonic components become:

$$S\frac{dS^{(k)}_{(12)}}{da} + 6S^{(k)}_{(12)} = SH\left[18 + \left(\frac{n}{aH}\right)^2\right]V^{(k)}_{(12)} \ , \tag{6.199}$$

and

$$S\frac{dV^{(k)}_{(12)}}{da} - 2V^{(k)}_{(12)} = -\left(\frac{1}{aH}\right)S^{(k)}_{(12)} \ . \tag{6.200}$$

These two equations can be combined to give a second order differential equation for $V^{(k)}_{(12)}$:

$$S\frac{d^2V^{(k)}_{(12)}}{da^2} + \left[6 - \frac{3}{2}(1+w)\right]\frac{dV^{(k)}_{(12)}}{da} + \left[4 + 3(1+w) + \left(\frac{n}{aH}\right)^2\right]\frac{1}{a}V^{(k)}_{(12)} = 0 \ . \tag{6.201}$$

For scales much larger than the Hubble radius, $\frac{k^2}{a^2H^2} \ll 1$, this equation admits power law solutions. So assuming that $|1+w| \ll 1$, we obtain

$$V^{(k)}_{(12)} = V^{(k)}_A a^{-1-\frac{3}{2}(1+w)} + V^{(k)}_B a^{-4+3(1+w)} \ , \tag{6.202}$$

and

$$S^{(k)}_{(12)} = -\left[3 + \frac{3}{2}(1+w)\right]V^{(k)}_A a^{-3(1+w)} - \left[6 - 3(1+w)\right]V^{(k)}_B a^{-3+\frac{3}{2}(1+w)} \ , \tag{6.203}$$

where $V^{(k)}_A$ and $V^{(k)}_B$ are constants.

We can also look at the fluctuations in the total fluid. For constant w in the large scale limit, equation (3.183) becomes

$$S\frac{d^2\Delta}{da^2} + \left[6 - \frac{9}{2}(1+w)\right]\frac{d\Delta}{da} - \left[12(1+w) - 6\right]\frac{\Delta}{a} = 0 \ . \tag{6.204}$$

The solution is

$$\Delta_{(k)} = \Delta^{(k)}_A a^{-2+3(1+w)} + \Delta^{(k)}_B a^{-3+\frac{3}{2}(1+w)} \ , \tag{6.205}$$

with $\Delta^{(k)}_A$ and $\Delta^{(k)}_B$ constants. It is also interesting to note that since $k = 0$, the scalar curvature variable C is a conserved quantity for scales larger than the Hubble radius. For details see section 5.

These equations thus lead, using our GI variables, to established results for inflationary universes with two scalar fields (Mollerach, 1990) [93].

6.8 Summary

In this chapter (mainly based on Paper II [21]) we have developed a theory of cosmological density perturbations in a multi-component fluid medium, based on the GI approach presented in chapter 3 (and derived in EB [35]). We have introduced here GI variables characterizing density and velocity perturbations in the single fluid components, and relative perturbations as well. Then we derived a complete set of linear equations, both for the single fluid components and for relative perturbations variables.

As we did in chapter 4 for the total fluid perturbation variables, we have show (using harmonic analysis) that the equations of Kodama and Sasaki (1984) [69] can be recovered using the fact that their variables are a first order approximation to our exactly defined covariant variables. In our view, this both clarifies and gives a physical and geometrical meaning to their variables.

We then applied the theory to three physically interesting examples. First we considered perturbations in a radiation-dust universe and derived small scale solutions for both the density and velocity perturbations [52]. We then studied perturbations in a Baryon-Photon system, taking explicitly into account the interactions between components which arise through Thompson scattering. We examined in detail the coupling between adiabatic and isothermal perturbations in the large scale limit and corrected a number of errors in the previous literature [69]. Finally we discussed perturbations in systems dominated by two scalar fields, obtaining standard results [93, 10].

These examples demonstrate the utility of this approach in understanding situations of importance in cosmology. The equations obtained here can be applied in many situations of cosmological interest because they are completely general in terms of fluid properties and interactions; and (by construction) they give a covariant and gauge invariant description of properties of perturbed FLRW universes.

CONCLUSIONS

In this thesis we have presented a new approach to the relativistic theory of cosmological perturbations and their linear evolution, and we have discussed the relation this approach has with the standard treatment of the problem; finally, we considered some applications.

The thesis is organized in six chapters: the first two review known material, although chapter 1 is partially based on EB [35] and the whole synthesis given there is original; in chapters 3 - 6 we have presented an original synthesis of the work carried out with various colleagues and that as been published (or is going to be published) in a series of papers (EB [35], EHB [37], EBH [36], BE [9], BED [10], Paper I [8] and Paper II [21]).

Chapter 1 was aimed to provide an introduction to the gauge problems affecting the standard approach to density perturbations in cosmology. Therefore we discussed gauge - transformations both from a coordinate point of view (section 1.1), and adopting a more geometrical approach: in particular in section 1.3 we presented the Stewart and Walker [118] Lemma, on which we base our definitions of gauge - invariant variables. In section 1.4 we also provided a brief discussion of the usual gauge choices made in the literature (cf. [69]).

In chapter 2 we summarized the covariant fluid flow approach to cosmology [27, 28]. In this approach Bianchi identities are regarded as field equations, hydrodynamic equations follow from the Ricci identities, and Einstein equations algebrically relate curvature with the matter content at any spacetime point. The exact equations of this chapter provide the framework for the following treatment of cosmological perturbations.

Indeed in chapter 3 (which is the core of this thesis) we introduced new *covariant and gauge - invariant* variables characterizing density inhomogeneities in cosmology (section 3.1), and derived exact non - linear evolution equations for them (section 3.2), coupled with the equations of the previous chapter. These variables have a straightforward physical gauge-independent interpretation: they represent the *spatial* variation of energy density in the real universe [72]. In particular, we have identified the quantity $\mathcal{D}_a$, the *comoving fractional density gradient* as the covariant GI quantities which embody most closely the intention of the usual (gauge-dependent) variable $\delta\mu/\mu$.

In section 3.4 we considered the problem of the definition of the 4-velocity of the fluid when this is imperfect: only an appropriate covariant choice leads to define GI spatial variables as those introduced in section 3.5. Then we outlined a linearization procedure in almost Robertson - Walker universes, and applied it to our equations, obtaining linear first - order propagation equations for our variables (sections 3.6). In section 3.7 we introduced a new local decomposition that allows us to define GI scalar variables, the most important being the divergence Δ of $\mathcal{D}_a$: Δ represents gauge-invariantly matter clumping, i.e. scalar density perturbations. In section 3.8 we derive: *a)* a Jeans criterion for gravitational instability, correcting a previous result by Jackson [65]; *b)* the long-wavelength limit of equations, and corresponding first integrals [3, 83] that exist for $\Lambda = K = \omega = 0$; *c)* the extra mode induced in the density gradient $\mathcal{D}_a$ by vorticity (EBH [36]). In the final part of this chapter we derived the whole set of linear gravitational and hydrodynamic equations corresponding to the exact equations of chapter 2; in section 3.10 we gave the equations for the scalar variables previously introduced: in particular we derived the second order equation for Δ, which corresponds to the main equation given by Bardeen [1] for his GI density perturbation variable.

Chapter 4 was devote to compare the approach to GI perturbations followed here with that of Bardeen: after a brief summary of his formalism, we systematically expanded (to first - order) our covariant variables, showing that *Bardeen's variables are the first - order - components of the covariant variables.* From this expansion, we also get the whole set of Bardeen's equation.

The last two chapters (5 and 6) have been devoted to further extensions of the formalism, considering applications. In chapter 5 we considered the case of a scalar field dominated universe, which is relevant to an inflationary stage [97]. Contrary to the usual assumption, we maintained a non vanishing curvature throughout, introducing a new variable and deriving perturbations equations in full generality: in this way these equations can be also applied to situations different than the inflationary epoch [94].

We also derived in a transparent way various results in the literature (cf. [3, 81, 82]). Finally, in chapter 6, we considered a mixture of interacting, imperfect fluids. We have introduced a set of variables appropriate for this situation, and derived equations for them. We applied our formalism to three cases of interest: a mixture of dust and radiation (cf. [52]), a mixture of baryons and radiation (cf. [69]), and a a pair of scalar fields (cf. [93]).

There are various topics related to those treated in this thesis that have not been tackled here, such for example perturbation spectra and the relations between a GI treatment of perturbations and specific gauge choices. These issues are however treated extensively in the literature in connection with Bardeen's formalism (e.g. see [70] and [1, 3, 61]), and the relations provided in chapter 4 between covariant GI variables and Bardeen's variables can be used to obtain e.g., the evolution of one of the covariant GI variables in a specific gauge. Here the focus was on presenting original material as it has been obtained in the papers mentioned above, with a resulting inhomogeneity (!) in the treatment of the various topics related to cosmological perturbations.

There are in our view three main advantages in following the *covariant - geometrical approach* presented here: *a)* it provides a unified treatment for the exact and the linearized theory, thus giving a clearer picture of the almost FLRW model we use to describe the real universe; *b)* the Stewart and Walker Lemma is valid for any background spacetime, so one can often use the same GI variables in perturbing different universe models: in particular GI variables can be easily identified in perturbing homogeneous anisotropic spacetimes, and in this case the geometrical approach can be much simpler than the standard perturbative technique (Dunsby 1991b)[19]; *c)* as pointed out by Stewart and Walker themselves and remarked by Mukhanov *et al.* [95] (1991) the use of exactly defined covariant variables has the advantage that if they vanish in the background FLRW universe they are also GI under large gauge transformations.

In conclusion, we point out that the formalism presented here could be easily applied to different situations, such for example in perturbations of different background models, or extended to treat perturbations in generalized gravity theories, as these latter can be formally related to general relativity with viscous fluid source terms [88, 59, 60].

Appendix A

Tilt angle formalism

In section 2.1.1 we have sketched the relation existing between two timelike unit vectors, as they are expressed using the tilt angle between them. Here we consider these relations in more details, considering also the case when the tilt is small, $\beta \ll 1$.

A.1 Exact relations

Let n^a and u^a be two unit future directed timelike vectors fields

$$n^a n_a = u^a u_a = -1 \ ; \tag{A.1}$$

these can be regarded as the four velocities of two sets of observers $\mathcal{O}_n$ and $\mathcal{O}_u$, and the projection tensors

$$h_{ab} = g_{ab} + u_a u_b \ , \quad \tilde{h}_{ab} = g_{ab} + n_a n_b \tag{A.2}$$

define the spatial part of the local rest frames (LRF) of these observers; these tensors are the metric in the subspace of the tangent space which is orthogonal to the corresponding vector: if this is hypersurface orthogonal, the relative projector is the metric in the surface. We follow King and Ellis (1973)[67] on characterizing the relation between n^a and u^a by the hyperbolic angle of tilt β

$$u^a n_a = -cosh\beta \ , \quad \beta \geq 0 \ , \tag{A.3}$$

and the direction of tilt: this can be specified either by the direction $\tilde{c}^a$ of the motion of $\mathcal{O}_u$ (the projection of u^a) in the LRF of $\mathcal{O}_n$, or by the the direction $-c^a$ of the motion of $\mathcal{O}_n$ (the projection of n^a) in the LRF of $\mathcal{O}_u$ (the signs of these directions are chosen as in King and Ellis [67]). Thus we have

$$\tilde{h}^a{}_b u^b = sinh\beta\tilde{c}^a = \tilde{V}^a \quad \Rightarrow \quad \tilde{V}^a n_a = \tilde{c}_a n^a = 0 \ , \quad \tilde{c}_a\tilde{c}^a = 1 \ , \tag{A.4}$$

$$h^a{}_b n^b = -sinh\beta c^a = V^a \quad \Rightarrow \quad V^a u_a = c_a u^a = 0 \ , \quad c_a c^a = 1 \ , \tag{A.5}$$

plus the following useful algebric relations

$$u^a = cosh\beta n^a + sinh\beta\tilde{c}^a \ , \qquad n^a = cosh\beta u^a - sinh\beta c^a \ , \tag{A.6}$$

$$c^a = sinh\beta n^a + cosh\beta\tilde{c}^a \ , \qquad \tilde{c}^a = -sinh\beta u^a + cosh\beta c^a \tag{A.7}$$

$$\tilde{c}_a u^a = sinh\beta = -c_a n^a \qquad c_a\tilde{c}^a = cosh\beta \ , \tag{A.8}$$

and

$$V^a n_a = \tilde{V}^a u_a = sinh^2\beta \ . \tag{A.9}$$

Note that in (A.4,A.5) we have introduced $\tilde{V}^a$ (V^a) as the covariantly defined spatial component of the four velocity u^a (n^a) of $\mathcal{O}_u$ ($\mathcal{O}_n$) in the LRF of $\mathcal{O}_n$ ($\mathcal{O}_u$). The physical meaning of the above relations is further clarified introducing the usual special relativistic contraction factor γ (see e.g. [73]), related to β by:

$$cosh\beta \equiv \gamma = (1-v^2)^{-\frac{1}{2}} \ , \quad v = tanh\beta = -\frac{\tilde{V}^a\tilde{c}_a}{u^a n_a} = \frac{V^a c_a}{u^a n_a} \ ; \tag{A.10}$$

where v is the magnitude of the three dimensional relative velocities $v\tilde{c}^a$ and $-vc^a$ measured by $\mathcal{O}_n$ and $\mathcal{O}_u$ respectively. Then (A.6) are nothing but a compact notation for the standard splitting of four velocity vectors in a time component plus a spatial part, e.g.

$$u^a = \frac{n^a}{\sqrt{1-v^2}} + \frac{v\tilde{c}^a}{\sqrt{1-v^2}} \ , \tag{A.11}$$

except that this is a covariant splitting.

We can now define the spacelike difference vector d^a satisfying the following relations:

$$d^a = u^a - n^a \ , \quad d^a d_a = 2(cosh\beta - 1) \geq 0 \ , \tag{A.12}$$

$$d^a = (cosh\beta - 1)n^a + sinh\beta\tilde{c}^a = (1 - cosh\beta)u^a + sinh\beta c^a \ , \tag{A.13}$$

$$d^a c_a = d^a\tilde{c}_a = sinh\beta \ , \quad d^a n_a = -d^a u_a = 1 - cosh\beta \ , \tag{A.14}$$

$$h^a{}_b d^d = -h^a{}_b n^b = -V^a \ , \quad \tilde{h}^a{}_b d^b = \tilde{h}^a{}_b u^b = \tilde{V}^a \ . \tag{A.15}$$

Finally, the following relations hold between $h^a{}_b$ and $\tilde{h}^a{}_b$:

$$\tilde{h}_{ab} = h_{ab} - 2d_{(a}u_{b)} + d_a d_b \qquad h_{ab} = \tilde{h}_{ab} + 2d_{(a}n_{b)} + d_a d_b \tag{A.16}$$

$$h_a{}^b \tilde{h}_{bc} = h_{ac} - V_a n_c = \tilde{h}_{ac} + \tilde{V}_c u_a \tag{A.17}$$

$$h_a{}^c h_b{}^d \tilde{h}_{cd} = h_{ab} - V_a V_b \qquad \tilde{h}_a{}^c \tilde{h}_b{}^d h_{cd} = \tilde{h}_{ab} + \tilde{V}_a \tilde{V}_b \tag{A.18}$$

Clearly, all the above definitions and relations are valid for any pair of unit timelike vectors. In particular they hold when we introduce the four velocities $u^a_{(i)}$ of the fluid components (as in chapter 6), and it is understood that quantities as those in this section are defined for the relation between u^a (intended as the total fluid four velocity) and $u^a_{(i)}$, except that the new quantities carry an index (i).

A.2 Linearization with respect to β

The definitions and relations in the previous section are parametrized by the tilt angle β, so they can be linearized in a very precise sense when $\beta \ll 1$, i.e. the relative velocities of $\mathcal{O}_n$ and $\mathcal{O}_u$ are not relativistic. If we systematically apply this procedure we obtain the following set of useful relations:

$$u^a n_a \simeq -1 - \tfrac{1}{2}\beta^2 \qquad v \simeq \beta \tag{A.19}$$

$$\tilde{h}^a{}_b u^b = \tilde{V}^a \simeq \beta \tilde{c}^a \qquad h^a{}_b n^b = V^a = -\beta c^a \tag{A.20}$$

$$u^a \simeq n^a + \beta \tilde{c}^a = n^a + \tilde{V}^a \qquad n^a \simeq u^a - \beta c^a = u^a + V^a \tag{A.21}$$

$$c^a \simeq \beta n^a + \tilde{c}^a \qquad \tilde{c}^a \simeq -\beta u^a + c^a \tag{A.22}$$

$$\tilde{c}_a u^a = -c_a n^a \simeq \beta \qquad V_a n^a = \tilde{V}_a u^a \simeq \beta^2 \tag{A.23}$$

$$c_a \tilde{c}^a \simeq 1 + \tfrac{1}{2}\beta^2 \tag{A.24}$$

$$d^a \simeq \beta \tilde{c}^a \simeq \beta c^a \simeq -V^a \simeq \tilde{V}^a \tag{A.25}$$

$$d^a d_a \simeq \beta^2 \tag{A.26}$$

$$d^a c_a = d^a \tilde{c} \simeq \beta \qquad -d^a n_a = d^a u_a \simeq \tfrac{1}{2}\beta^2 \tag{A.27}$$

$$h^a{}_b d^b \simeq \tilde{h}^a{}_b d^b \simeq \tilde{V}^a \simeq -V^a \simeq \beta c^a \simeq d^a \tag{A.28}$$

The relations between $h^a{}_b$ and $\tilde{h}^a{}_b$ reduce to:

$$\tilde{h}_{ab} \simeq h_{ab} - 2d_{(a}u_{b)} \simeq h_{ab} + 2V_{(a}u_{b)} \tag{A.29}$$

$$h_{ab} \simeq \tilde{h}_{ab} + 2d_{(a}n_{b)} \simeq \tilde{h}_{ab} + 2\widetilde{V}_{(a}n_{b)} \tag{A.30}$$

$$h_a{}^b\tilde{h}_{bc} = h_{ac} - V_a n_c = \tilde{h}_{ac} + \widetilde{V}_c u_a \tag{A.31}$$

$$h_a{}^c h_b{}^d \tilde{h}_{cd} \simeq h_{ab} \qquad \tilde{h}_a{}^c \tilde{h}_b{}^d h_{cd} \simeq \tilde{h}_{ab} \tag{A.32}$$

Appendix B

Harmonics

Here we examine the relation between the harmonics Y used in the Bardeen formalism and the harmonics Q used in the covariant formalism, thus providing the proper foundation for examining the relation between these formalisms.

B.1 Generalities

The standard harmonic decomposition of first-order perturbations is usually carried out explicitly using harmonic functions defined as eigenfunctions of certain differential operators on well established spaces. Usually, in cosmology we deal with harmonics which are eigenfuctions of a Laplace-Beltrami operator on 3-hypersurfaces of constant curvature, i.e. on the homogeneous spatial sections of FLRW universes with metric $\gamma_{\alpha\beta}$ (3.68); e.g. scalar harmonics are defined by (Kodama and Sasaki 1984)[69]

$$\nabla^2 Y^{(k)} = -k^2 Y^{(k)} \ , \tag{B.1}$$

where $Y^{(k)}$ is the harmonic of order k,[1] and as in chapter 4 $\nabla^2 Y = Y^{|\gamma}{}_{|\gamma}$ is the Laplacian in the 3-spaces with metric $\gamma_{\alpha\beta}$. This is because in the standard approach to linear perturbations one explicitly separates in each physical quantity the zero order part (the FLRW background value of this quantity) from the first-order perturbative part, and uses derivatives with respect to the background metric [1, 69, 63]. In the covariant approach instead we consider, as long as we can, only quantities defined in the real

[1]We use this rather standard notation that unifies the three possible different cases; k is however the wavenumber only when the spatial sections are flat. Minimum values for k^2 are: $k^2 = 0$ for $K = 0$, $k^2 = 1$ for $K = -1$, $k^2 = 3$ for $K = 1$ (see Harrison 1967[54])

almost FLRW universe; in doing this we emphasize the fluid four velocity u^a rather than an arbitrarily chosen spatial slicing, and we define spatial quantities on projecting orthogonal to u^a with h^{ab}. In a general spacetime, we have a spatial derivative ${}^{(3)}\nabla_a$ with ${}^{(3)}\nabla_a h_{bc} = 0$ (see section 3.3, Wald 1984 [126] and the EBH appendix for details), which however is not a derivative in a hypersurface, unless $\omega = 0$; in addition to this we have a proper time derivative along flow lines. Accordingly, we want to use harmonics defined through operators constructed with the ${}^{(3)}\nabla_a$ derivatives, and constant along flow lines (i.e. independent of proper time). Here and elsewhere we shall often denote with Q the whole set of covariant harmonics defined in the next sections: using a standard but misleading terminology (see chapter 4) we have scalars Q, $Q_a^{(0)}$, $Q_{ab}^{(0)}$, vectors $Q_a^{(1)}$, $Q_{ab}^{(1)}$, and tensors $Q_{ab}^{(2)}$.[2]

In dealing with an almost FLRW model, we can use these harmonics in expanding first-order quantities; for example, for a scalar T[3]

$$T = \sum T_{(k)} Q^{(k)} \;, \tag{B.2}$$

where the $T_{(k)}$ are the (first order) components of T. Then we can relate ${}^{(3)}\nabla_a$ and the stroke derivative when ${}^{(3)}\nabla_a$ acts on a first-order quantity; for the scalar in (3.34) we have[4]

$${}^{(3)}\nabla_a T = T_{|a} = \sum T_{(k)} Q^{(k)}_{|a} \;. \tag{B.3}$$

We stress that (3.34) and (B.3) are meaningful and unambiguous only for GI first-order quantities: in this case one clearly needs the harmonics at zero order only, as the $T_{(k)}$

[2]In what follows, we shall omit the type index: for example $Q_a^{(0)}$ should denote the vector constructed with the scalar harmonic Q, to distinguish it from the solenoidal vector $Q_a^{(1)}$ (see below): we use this index only when we have to use the two vector (or tensor) harmonics together, as in chapter 4.

[3]We use a $\sum$ as a symbol for a sum that could actually be a summation over a discrete set or an integral over a continuously varying index. This will depend both on the curvature of the spatial slicing and on our choice: for example one can be interested in scales smaller than the curvature radius and thus consider finite volumes with corresponding discrete sums, or Fourier integrals on spatially flat sections, or appropriate integrals on a negatively curved slicing as well as a sum on the discrete set of eigenfunctions in a closed model. As usual, in chapter 4 the summation symbol is understood.

[4]We are neglecting a term in the expansion (B.3)

$${}^{(3)}\nabla_a T_{(0)} = 0 \;,$$

because $T_{(k)} = T_{(k)}(t)$. There is no contradiction with the existence of a non-vanishing vorticity in the physical universe, as ${}^{(3)}\nabla_a T_{(k)} = 0$ only implies $\bar{\omega} = 0$, ${}^{(3)}\nabla_a$ being the derivative with respect to the background 3-metric in acting on a first order quantity.

are unambiguously zero in the background (Hawking 1966) [55] (except T_0 if T is a constant scalar in the background).

We now look successively at scalar, vector, and tensor harmonics.

B.2 Scalar harmonics

B.2.1 Definitions

As in Hawking (1966) [55] we can therefore take the scalar harmonics as being eigenfunctions of the covariantly defined Laplace-Beltrami operator

$$ {}^{(3)}\nabla^2 Q \equiv {}^{(3)}\nabla_a {}^{(3)}\nabla^a Q = -\frac{k^2}{a^2} Q \ , \tag{B.4}$$

where from now on we drop the index (k) from the Q's. This scalar harmonic can be used to expand scalars; with this we can however define a vector[5]

$$ Q_a = -\frac{a}{k} {}^{(3)}\nabla_a Q \tag{B.5}$$

and a trace-free symmetric tensor

$$ Q_{ab} = \frac{a^2}{a^2} {}^{(3)}\nabla_b {}^{(3)}\nabla_a Q + \frac{1}{3} h_{ab} Q \ . \tag{B.6}$$

These harmonics are defined in order to have

$$ \dot{Q} = \dot{Q}_a = \dot{Q}_{ab} = 0 \ ; \tag{B.7}$$

using this property and applying the commutation relations in the appendix of EBH [36] we get

$$ {}^{(3)}\nabla_{[a} {}^{(3)}\nabla_{b]} Q = -\omega_{ab} \dot{Q} = 0 \ , \tag{B.8}$$

$$ {}^{(3)}\nabla_{[a} {}^{(3)}\nabla_{b]} Q_c = \tfrac{K}{2a^2} (h_{ac} Q_b - h_{bc} Q_a) \ , \tag{B.9}$$

$$ {}^{(3)}\nabla_{[a} {}^{(3)}\nabla_{b]} Q_{cd} = \tfrac{K}{2a^2} [(h_{ac} Q_{bd} - h_{bc} Q_{ad}) + (h_{ad} Q_{bc} - h_{bd} Q_{ac})] \ . \tag{B.10}$$

[5] This definition differs from that in EHB; we adopt here definitions that are easily matched with those in BI[1] and Kodama and Sasaki (1984)[69].

B.2.2 Properties

Using the derivative ${}^{(3)}\nabla$ we derive various properties satisfied by the above defined scalar harmonics.

$$a^{(3)}\nabla_a Q^a = kQ \tag{B.11}$$

$$a^{2(3)}\nabla^2 Q_a = -(k^2 - 2K)Q_a \tag{B.12}$$

$$a^{(3)}\nabla_b Q_a = -k(Q_{ab} - \tfrac{1}{3}h_{ab}Q) \tag{B.13}$$

$$Q^a{}_a = 0 \tag{B.14}$$

$$a^{(3)}\nabla^b Q_{ab} = -\tfrac{2}{3}k^{-1}(3K - k^2)Q_a \tag{B.15}$$

$$a^{2(3)}\nabla^{a(3)}\nabla^b Q_{ab} = -\tfrac{2}{3}(3K - k^2)Q \tag{B.16}$$

$$a^{2(3)}\nabla_b{}^{(3)}\nabla^c Q_{ac} = \tfrac{2}{3}(3K - k^2)(Q_{ab} - \tfrac{1}{3}h_{ab}Q) \tag{B.17}$$

$$a^{2(3)}\nabla^2 Q_{ab} = -(k^2 - 6K)Q_{ab} \tag{B.18}$$

B.3 Vector harmonics

B.3.1 Definitions

These are again defined as eigenfunctions of the Helmholtz equation

$$a^{2(3)}\nabla^2 Q_a = -k^2 Q_a\ , \tag{B.19}$$

Q_a being a solenoidal vector[6]

$${}^{(3)}\nabla_a Q^a = 0\ . \tag{B.20}$$

With this we can construct a symmetric trace-free tensor

$$Q_{ab} = -\frac{a}{2k}({}^{(3)}\nabla_b Q_a + {}^{(3)}\nabla_a Q_b)\ ; \tag{B.21}$$

Both these harmonics are covariantly constant along flow lines

$$\dot{Q}_a = \dot{Q}_{ab} = 0\ . \tag{B.22}$$

[6]So we cannot construct a scalar from it, see also (B.24).

B.3.2 Properties

The following relations hold for the tensor Q_{ab} (B.21) defined from the vector Q_a.

$$Q^a{}_a = 0 \tag{B.23}$$

$$a^{(3)}\nabla^b Q_{ab} = \tfrac{1}{2}k^{-1}(k^2 - 2K)Q_a \tag{B.24}$$

$$a^{2(3)}\nabla_b{}^{(3)}\nabla^c Q_{ac} + a^{2(3)}\nabla_a{}^{(3)}\nabla^c Q_{bc} = -(k^2 - 2K)Q_{ab} \tag{B.25}$$

$$a^{2(3)}\nabla^2 Q_{ab} = -(k^2 - 4K)Q_{ab} \tag{B.26}$$

Q_a and Q_{ab} also respectively satisfy the relations (B.9),(B.10).

B.4 Tensor harmonics

Again we have

$$a^{2(3)}\nabla^2 Q_{ab} = -k^2 Q_{ab} \ , \tag{B.27}$$

and

$$\dot{Q}_{ab} = 0 \ , \tag{B.28}$$

where this tensor is trace-free and divergenceless

$$Q^a{}_a = 0 \ , \tag{B.29}$$

$$^{(3)}\nabla^b Q_{ab} = 0 \ , \tag{B.30}$$

and satisfies (B.10).

B.5 Relations between the Q's and the Y's

To obtain the relation between the two sets of harmonics, the above derived relations for the Q's can be compared with those in appendix C of Kodama and Sasaki (1984)[69] for the Y's if we explicitly use comoving coordinates (either t or η time coordinates in which $u^a = \delta^a_0 u^0$). Then the spatial metric h_{ab} is conformally related to the 3-metric $\gamma_{\alpha\beta}$ of the constant curvature hypersurfaces by

$$h_{ab} = a^2 \delta^\alpha_a \delta^\beta_b \gamma_{\alpha\beta} \ , \quad h_a{}^b = \delta^\alpha_a \delta^\beta_b \gamma_\alpha{}^\beta \ , \quad h^{ab} = a^{-2}\delta^a_\alpha \delta^b_\beta \gamma^{\alpha\beta} \ ; \tag{B.31}$$

it follows that the spatial derivative operator ${}^{(3)}\nabla_a$ (defined with respect to h_{ab}) is related to the covariant derivative in the 3-surfaces by

$$ {}^{(3)}\nabla_a T = \delta_a^\alpha T_{|\alpha} \ , \quad {}^{(3)}\nabla^a T = a^{-2}\delta_\alpha^a T^{|\alpha} \ . \tag{B.32}$$

The Q's are defined to be constant with respect to proper time along flow lines: using the relations above one can immediately show that the requirement for the Y's to be independent of coordinate time is satisfied if the following relations hold between the two sets of harmonics

$$ Q = Y \tag{B.33}$$

$$ Q_a = s\delta_a^\alpha Y_\alpha \qquad\qquad Q^a = a^{-1}\delta_\alpha^a Y^\alpha \tag{B.34}$$

$$ Q_{ab} = a^2\delta_a^\alpha\delta_b^\beta Y_{\alpha\beta} \quad Q^a{}_b = \delta_\alpha^a\delta_b^\beta Y^\alpha{}_\beta \quad Q^{ab} = a^{-2}\delta_\alpha^a\delta_\beta^b Y^{\alpha\beta} \ . \tag{B.35}$$

Using these relations we thus have[7]

$$ \dot{Q} = 0 \quad \Leftrightarrow \quad u^0\partial_0 Y = 0 \ , \tag{B.36}$$

$$ \dot{Q}_a = 0 \quad \Leftrightarrow \quad u^0\partial_0 Y_\alpha = 0 \ , \tag{B.37}$$

$$ \dot{Q}_{ab} = 0 \quad \Leftrightarrow \quad u^0\partial_0 Y_{\alpha\beta} = 0 \ . \tag{B.38}$$

The Y harmonics therefore satisfy the same relations we gave for the Q's, on dropping the scale factors a and on substituting ${}^{(3)}\nabla^a$ derivatives with the slash derivatives.

[7]We also use

$$ \dot{V}_a = u^0\partial_0 V_a - \frac{\dot{a}}{a}V_a \ , \quad \dot{T}_{ab} = u^0\partial_0 T_{ab} - 2\frac{\dot{a}}{a}T_{ab}$$

for a vector V_a and a tensor T_{ab}.

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