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Aspects of special Kähler geometry and moduli space theory in string compactifications and (2,2) superconformal models

Thesis submitted for the degree of
"Doctor Philosophiae"

Elementary Particle Sector

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Supervisor:
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Contents

Introduction

The appearance of peculiar geometrical structures [1, 2, 3, 4, 5, 6, 7, 8] in superstring compactifications and $(2,2)$ superconformal models stimulated, in the last few years, many interesting progresses in the search for a candidate superstring vacuum [9].

The string vacuum is degenerate, since the effective four dimensional theory one gets from a suitable compactification depends on scalar neutral fields which do not contribute to the scalar self–interaction. These "flat directions" of the scalar potential are usually called moduli [2, 5, 10, 11, 12, 13].

From the point of view of the underlying superconformal field theory, which defines the compactified string, this means that such a theory is not isolated. Moduli parametrize marginal perturbations of the superconformal theory [14, 15]. Geometrically this reflects into the fact that also the compact manifold is not isolated. Moduli correspond to allowed deformation parameters of such a manifold.

Special Kähler geometry arises naturally from heterotic superstring compactification on Kähler manifolds with vanishing first Chern class $c_1 = 0$ (Calabi–Yau manifolds) [16, 17]. Moduli spaces, described by Kähler and complex structure deformations, preserving the Ricci flatness, are special Kähler manifolds [4, 7]. Historically, however, the concept of special Kähler geometry was introduced in the context of $N = 2$ supergravity, while solving the problem of coupling an arbitrary number of vector multiplets [3, 18, 19, 20].

The connection between these rather different contexts, which give rise to special geometry, is easily understood from the observation that on the same Calabi–Yau space one can compactify either the heterotic superstring or the type II superstring, displaying N=1 and N=2 supersymmetry, respectively [5, 6]. The moduli space geometry is therefore compatible with N=2 supersymmetry, and this is why it is special.

Let n be the complex dimension of the moduli space. The structure of special geometry is encoded in a homogeneous, degree two holomorphic function $F(X^\Lambda)$, where X^Λ $(\Lambda = (0,i) = 0,1,\cdots,n)$ are holomorphic sections of a line bundle whose first Chern class coincides with the Kähler class. A natural $2(n+1)$–dimensional symplectic structure defined by the section $(X^\Lambda, i\partial_\Lambda F)$ is the characterizing property of special

Kähler manifolds [4, 8, 21, 22]. "Special" coordinates can be obtained by defining the moduli space coordinates as $z^i = \frac{X^i}{X^0}$ [20, 23]. From a physical point of view these properties allow to extract informations on Yukawa couplings for the chiral families in the effective lagrangian, and for the matter fields metric involved in the kinetics terms.

If one considers a d–dimensional Calabi–Yau manifold as a target space of an N=2 σ–model, one expects, on general grounds, a correspondence between the critical point of this model and a $(2,2)$ superconformal model with $c = 3d$ [14]. It then follows that moduli of a Calabi–Yau can also be interpreted as marginal perturbations of the corresponding superconformal theory.

It is well known [15] that the geometry of the moduli space for a generic conformal theory has a Riemannian structure. For c=3d, $(2,2)$ superconformal theories, marginal perturbations that belong to the so called chiral ring correspond to complex structure deformations of the Calabi–Yau manifold and, as a consequence, exhibit a special Kähler structure [11]. A similar statement can be said for Kähler structure deformations, that correspond to the antichiral ring.

Starting from a $(2,2)$ theory one can always get, with a suitable "twist" of the superconformal algebra, a topological conformal theory, which is characterized by a nilpotent BRST operator [24, 25, 26, 27]. The cohomology of such operator defines the physical space, which coincides with the chiral (or the antichiral) ring of the original superconformal theory. The main advantage of a topological field theory is that it can be solved exactly, and for this particular case it can give informations on the moduli space special geometry.

A very powerful technique to study moduli spaces of $(2,2)$ superconformal models and of their topological counterparts is represented by 2–dimensional Landau–Ginzburg models with two supersymmetries [28, 29, 30, 31, 32, 33, 34, 35, 36, 37] . All relevant properties of these models are encoded in a polynomial superpotential W, which defines the structure of the chiral ring. On the other hand, the superpotentials, seen as analytic functions, are exactly the same functions appearing in the construction of Calabi–Yau manifolds [38] . This establishes a well defined relation between modal deformations of the superpotentials, moduli of superconformal (topological) theories and moduli of Calabi–Yau spaces.

The global structure of moduli spaces is in general modified by discrete symmetries, called duality transformations [39, 40, 41]. These symmetries generalize the $R \rightarrow \frac{1}{2R}$ transformation, which states the equivalence between a string compactified on circles of radius R and $\frac{1}{2R}$. Since moduli enter the effective action as massless neutral fields, the action should be invariant under duality transformations, which are also denoted as target space modular transformations. Automorphic functions for these symmetries give exact non–perturbative results for self–couplings in the effective four dimensional action (automorphic superpotentials) [42].

In this thesis we are concerned with some aspects of special Kähler geometry arising in string theory and superconformal models.

In the first chapter we present a general discussion on moduli and moduli spaces, with the purpose of clarifying some topics sketched in this introduction. In the second one, after a suitable definition of special Kähler geometry, we consider orbifolds of homogeneous special Kähler manifolds, namely varieties of the type $\mathcal{M} = \mathcal{M}'/\Gamma$ where $\mathcal{M}'$ is a special Kähler coset manifold G/H and $\Gamma \subset G$ is a discrete subgroup of its isometry group. Varieties of this type appear as moduli spaces in orbifold compactification of superstrings, where Γ plays the role of target space modular group.

We show that the construction of the homogeneous function $F(X)$, encoding the special geometry of $\mathcal{M}'$, can be systematically derived from the symplectic embedding of the isometry group G into $Sp(2n+2,R)$, n being the complex dimension of $\mathcal{M}'$ [43]. This is actually related to the Gaillard-Zumino [39] construction of Lagrangians with duality symmetries. Different embeddings yield different $F(X)$. For the case of $\mathcal{M}' = \frac{SU(1,1)}{U(1)} \times \frac{SO(2,n)}{SO(2)\times SO(n)}$, we show that it is possible to obtain a new symplectic section $\Omega = (X, i\partial F(X))$, generating a new set of special coordinates. They transform linearly under $SO(n)$, differently from the old special coordinates that transform linearly only under $SO(n-1)$. This solves an apparent paradox in superstring compactifications.

From the embedding of G into $Sp(2n+2,R)$ one retrieves the embedding of Γ into $Sp(2n+2,Z)$. This embedding yields the explicit rule to give a formal definition of a $PSL(2,Z) \times SO(2,n,Z)$ automorphic superpotential for any n.

As a second application we consider the duality group $\Gamma = SU(3,3,Z)$ [44] for the Narain lattice [45] of the T^6/Z_3 orbifold and its action on the corresponding moduli space $\mathcal{M}_{3.3}/\Gamma$, where $\mathcal{M}_{3.3} = \frac{SU(3,3)}{SU(3)\times SU(3)\times U(1)}$. A symplectic embedding of the momenta and winding numbers allows to connect the orbifold lattice to the special geometry of $\mathcal{M}_{3.3}$. A formal expression for an automorphic function, which is a candidate for a non–perturbative superpotential, is given.

In the last chapter we consider the realization of $(2,2)$ superconformal models in terms of free first-order (b,c,β,γ)-systems [46, 47], and show that an arbitrary Landau-Ginzburg interaction with quasi-homogeneous potential can be introduced without spoiling the $(2,2)$ superconformal invariance [48]. We discuss the topological twisting and the renormalization group properties of these theories, and compare them to the conventional topological Landau-Ginzburg models [49].

By deforming the theory with relevant and marginal operators it is possible to define perturbed correlation functions, and a suitable metric in the coupling constant space. After a proper bosonization of the first-order systems, explicit calculations of perturbed topological correlation functions are performed by using standard Coulomb gas techniques [50, 51, 52]. In the coupling constant space of topological field theories there is a preferred coordinate frame in which the metric is constant. These "flat coordinates" are strongly connected with special coordinate systems in special geometry. We show that in our formulation the parameters multiplying deformation terms in the potential are flat coordinates. We retrieve known results for minimal models and for the $c = 3$ cubic torus. The extension of the techniques presented here to a general $c = 3d$ theory should give more information on the special geometry structure of the moduli spaces.

At the end of the chapter we elaborate, for a particular example, on the relation between Picard–Fuchs equations and topological Landau–Ginzburg models

Finally in the appendices we deserve some technical remarks on the topics treated in the thesis.

Chapter 1

Overview on the subject

1.1 Discussion on moduli and moduli spaces

The action for a string moving in a non trivial metric and antisymmetric tensor background is [9]:

$$S = -\frac{1}{2\pi}\int d^2\sigma \left[\eta^{\alpha\beta}\partial_\alpha X^{\hat{\mu}}\partial_\beta X^{\hat{\nu}} G_{\hat{\mu}\hat{\nu}}(X) + \epsilon^{\alpha\beta}\partial_\alpha X^{\hat{\mu}}\partial_\beta X^{\hat{\nu}} B_{\hat{\mu}\hat{\nu}}(X)\right] , \qquad (1.1)$$

where we have chosen the conformal gauge and kept only bosonic degrees of freedom. The index $\hat{\mu}$ runs on the range $\hat{\mu} = (\mu, \hat{\imath})$, where μ is the space–time index and $\hat{\imath}$ is the internal (compactified) one.

Moduli space is the space parametrized by the allowed deformations of a given background on the internal manifold.

In the more general context of conformal field theories moduli are truly marginal operators, i.e. operators with conformal dimension $(h, \bar{h}) = (1, 1)$, with the property that perturbations generated by these operators do not act to change their own dimensions. A very interesting aspect of two dimensional superconformal field theories is the possibility of describing the abstract space of all such theories in standard geometrical terms. In particular the space of conformal field theories is equipped with a natural Riemannian structure [15]. For the application, one is particularly interested in the connected components of this space, which are the moduli spaces of the various superconformal field theories.

A very powerful tool [10] to construct the moduli space and to compute their geometrical properties is to study the low energy field theory for a superstring compactified on the given superconformal theory. Roughly speaking, the moduli space is just the manifold of classical vacua for the low–energy theory. Moduli appear as massless neutral scalar fields M_i whose vacuum expectation values $< M_i >$ are left undetermined by the equations of motion, and they represent the free parameters for the internal metric and anti- symmetric tensor field. The method is particularly useful when the resulting low–energy effective theory is supersymmetric, since the spacetime SUSY gives rather severe restrictions on the space of possible vacua.

For physical applications, the most important case is that of the (2,2) superconformal theories with $c = 9$. They can be used to compactify the heterotic string down to four dimensions, while preserving space–time supersymmetry [12]. Indeed this class of superconformal systems can be viewed [14] as sigma models on some kind of Calabi–Yau manifold [16].

To study the moduli space of (2,2) superconformal systems it is more convenient [10, 2] to use them as internal spaces for type II rather than heterotic superstring. The reason being that in this case we get $N = 2$ space–time supersymmetry, and thus stronger SUSY constraints.

The relation between the scalar moduli and the underlying two–dimensional conformal field theory is best seen from their interpretation as flat directions of the scalar potential. If we denote by P any other scalar field which may be charged or neutral under the gauge group but which comes from the gauge degrees of freedom (as we will see below), and by $V(M, P)$ the scalar potential of the theory, then the moduli fields satisfy identically the following property:

$$\frac{\partial V}{\partial M} = 0 \ , \tag{1.2}$$

while for the field P the equation $\frac{\partial V}{\partial P} = 0$ fixes its value at some point P_0. The moduli thus parametrize flat directions of the potential. In the background field approach [53] moduli appear as "coupling constant" in a underlying two–dimensional σ model. The requirement of conformal invariance beyond the tree level, that is the statement that the β–functions associated to these couplings vanish, is nothing but their effective space–time equation of motion. i.e.

$$\beta_{g_{\mu\nu}} = 0 \quad \rightarrow \quad R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu} \tag{1.3}$$

and for a generic scalar field ϕ

$$\beta_\phi = 0 \quad \rightarrow \quad \frac{\partial V}{\partial \phi} = 0 \ \text{for} \ \phi = <\phi> \ . \tag{1.4}$$

From eq. (1.4) we see that flat directions correspond to a coupling constant $<\phi>$ of the underlying conformal field theory for which the theory is *exactly conformal invariant*. If we call V_M the marginal operator which corresponds to moduli massless excitation then V_M is a conformally invariant perturbation for all M. The motion in the space of conformal field theories is given by the geometry of the M manifold, i.e. the "coupling constant space".

Let us assume that this space has some differentiable structure. Zamolodchikov has shown [15] that it can be regarded as a Riemannian space with metric given by[1]:

$$\langle V_{M_i}(1)V_{M_j}(0)\rangle = g_{ij}(M) \ . \tag{1.5}$$

It may be shown that in the effective lagrangian the M_i kinetic term is given by:

$$g_{ij}\partial_\mu M_i \partial_\mu M_j \tag{1.6}$$

and moreover

$$V(M) = 0 \ . \tag{1.7}$$

when we set all other non–moduli fields to their vacuum expectation values. The characterization of the metric g_{ij} is the main problem to be solved, since it gives most of the information on the geometry of the moduli space. Let us analyze in some detail the case of moduli space, obtained by superstring compactification on Calabi– Yau spaces.

1.2 Moduli space of Calabi–Yau manifolds

In any string compactification to four dimension we require $N = 1$ space– time supersymmetry to be unbroken. This requirement allows to solve the hierarchy problem of the weak interaction scale [54], provided that some still mysterious mechanism will generate both weak and supersymmetry scale breaking at energies below 1 Tev.

An interesting class of compactifications leading to $N = 1$ space– time supersymmetry are those on Calabi-Yau manifolds [16]. The latter are defined as compact Kähler manifolds of complex dimension d (three in the case of physical compactification down to four dimensions) with vanishing first Chern class.

More generally, Calabi-Yau (CY) compactifications are referred as (2,2) vacua, because of the famous Gepner conjecture which states the correspondence of a (2,2) superconformal theory with $c = 9$ with a critical point of an n=2 σ–model on a target Calabi-Yau space [14]. In the context of type II string theories this means that the internal conformal field theory with central charge $(c, \bar{c}) = (9,9)$ has left and right moving $n = 2$ superconformal symmetries [2]. Such a theory can be mapped to a heterotic theory with the same internal superconformal system [55, 56]. The missing 13

[1] In this chapter we denote the moduli space metric by lower–case letter, to be clearly distinguished by the "capital" metric G, which refers to the space time or to the internal CY space

[2] From now on we will refer to capital N for spacetime SUSY and to small n for the world sheet one

units to the left–moving conformal anomaly are provided by 13 free bosons moving on the maximal torus of $E_8 \times SO(10)$. The $U(1)$ current of the left moving n=2 algebra combines with the SO(10) to E_6. The gauge group of heterotic Calabi–Yau compactifications is thus in general $E_8 \times E_6$. For type II Calabi–Yau compactifications the spacetime supersymmetry in an $N = 2$ [12, 57, 58, 59]. In heterotic compactifications the right moving n=2 algebra is necessary for N=1 space–time supersymmetry. The left moving n=2 algebra now establishes a one-to-one correpondence between those massless multiplets whose scalar components are moduli, and the matter multiplets charged under the gauge group [10, 11, 59, 60]. This relation between N=1 heterotic and N=2 type II theories compactified on the same CY space will be exploited in the following.

A CY manifold [17] is characterized by some topological numbers $h_{p,q}$ $(p, q = 0,1,2,3)$, which describes the number of harmonic (p,q) forms on the manifold. For CY threefolds the $h_{1,1}$ and $h_{2,1}$ harmonic forms have the geometrical meaning of deformation parameters of the Kähler structure and of the complex structure respectively. Indeed, if one varies the metric on the CY space such as to preserve Ricci flatness, one can show that $i\delta G_{ij^*}dy^i \wedge dy^{j^*}$ have to be harmonic (1,1) forms, whereas $\Omega^{l^*}_{ij}\delta G_{l^*k^*}dy^i \wedge dy^j \wedge dy^{k^*}$ have to be harmonic (2,1) forms. Here $y^i = y^i, y^{i^*}$ are the complex coordinates on the internal CY threefold and $\Omega_{ijk} = g_{kl^*}\Omega^{l^*}_{ij}$ is the unique covariantly constant holomorphic (3,0) form, which can always be shown to exist. If we denote by $\omega^a_{(1,1)}(y)$ $(a = 1, \cdots h_{(1,1)})$ and $\omega^\alpha_{(2,1)}(y)$, $\alpha = 1, \cdots h_{(2,1)}$ the bases for harmonic (1,1) and (2,1) forms respectively, we can expand any variation of the Kähler and complex structure in terms of them. The expansion parameters are just the moduli. We let they depend on the uncompactified space–time dimensions x^μ and we get:

$$\begin{aligned} i\delta G_{ij^*}(x,y)dy^i \wedge dy^{j^*} &= \sum_a M^a_1(x)\omega^a_{ij^*}dy^i \wedge dy^{j^*} \, , \\ \Omega^{l^*}_{ij}\delta G_{l^*k^*}(x,y)dy^i \wedge dy^j \wedge dy^{k^*} &= \sum_\alpha N^\alpha(x)\omega^\alpha_{ijk^*}dy^i \wedge dy^j \wedge dy^{k^*} \, . \end{aligned} \tag{1.8}$$

The fields M^a_1 (real) and N^α (complex) appear as massless scalar fields with vanishing potential in Minkowski space. This follows from the equation of motion for the internal components of the metric.

The total number of real moduli fields we extract from (1.8) is $h_{1,1} + 2h_{2,1}$. However, in a string theory compactified on a CY manifold we have additional scalar degrees of freedom from the non gauge sector, namely those coming from the internal components of the antisymmetric tensor B_{ij^*} (which are exactly $h_{1,1}$) and two more coming from the dilaton field and the space time components $B_{\mu\nu}$ of the antisymmetric tensor (the axion field). Therefore in any superstring compactified on a CY threefold the non gauge sector gives $2(h_{1,1}+h_{2,1}+1)$ degrees of freedom. The real fields M^a_2, which are associated with changes of B_{ij^*} (whose zero modes are also harmonic (1,1) forms), combine with the fields M_1^a to complex massless scalar fields M. The $1 + h_{1,1} + h_{2,1}$ complex moduli become coordinates of a Kähler manifold. It can be proved [11], using superconformal Ward identities, that the moduli manifold has the product structure:

$$\mathcal{M} = \frac{SU(1,1)}{U(1)} \times \mathcal{M}_{h_{1,1}} \times \mathcal{M}_{h_{2,1}} \,, \tag{1.9}$$

where $\mathcal{M}_{h_{1,1}}$, $\mathcal{M}_{h_{2,1}}$ are two Kähler manifolds of complex dimension $h_{1,1}$ and $h_{2,1}$ respectively, and $SU(1,1)/U(1)$ is the moduli manifold associated to the dilaton- -axion field. This is also shown in [10, 5, 6], where $N = 2$ space–time SUSY is explicitly utilized, via the connection between heterotic and type II theories above mentioned. This also gives additional insights in the structure of the moduli space. Since the moduli metric does not know which specific superstring theory one is compactifying, the kinetic term in (1.6) in the effective Lagrangian is common to heterotic and type II theories, but in the latter case it has to satisfy the additional constraint coming from the second space–time supersymmetry. The same constraint can be recovered by Ward–identities of the underlying (2,2) superconformal algebra [11]

As already mentioned above, the left moving $n = 2$ superconformal algebra relates moduli to charged matter fields. Each (1,1) modulus is accompanied by a 27 and each (2,1) modulus by a $\overline{27}$ left–handed family of the E_6 gauge group (singlet with respect to the residual E_8). The Euler number is now simply given by $2(h_{1,1} - h_{2,1})$ and a model is therefore chiral if such number is nonzero. For CY compactifications to four dimensions we can also see the degrees of freedom in a pure space–time picture assuming the compactification scale R much larger that the string size $\sqrt{\alpha'}$. In this regime we may use the point field limit of 10 –dimensional superstring. For heterotic superstring we have 10D supergravity coupled to a Yang–Mills $E_8 \times E_8$ multiplet. For type II strings we have type IIA (non–chiral) and type IIB (chiral) supergravity [9] The bosonic fields which give rise to scalars in four dimensions are :

$$G_{\hat\mu\hat\nu},\, B_{\hat\mu\hat\nu},\, \phi,\, A^I_{\hat\mu} \tag{1.10}$$

for heterotic superstring,

$$G_{\hat\mu\hat\nu},\, B_{\hat\mu\hat\nu},\, \phi,\, A_{\hat\mu},\, A_{\hat\mu\hat\nu\hat\rho} \tag{1.11}$$

for type IIA superstring, and

$$G_{\hat\mu\hat\nu},\, B^c_{\hat\mu\hat\nu},\, \phi^c,\, A_{\hat\mu\hat\nu\hat\rho\hat\sigma} \tag{1.12}$$

for type IIB superstring. Here $B^c_{\hat\mu\hat\nu}, \phi^c$ denote complex antisymmetric tensor and scalar field in ten dimension, $A^I_{\hat\mu}$ are gauge fields and $A_{\hat\mu\hat\nu\hat\rho}$ and $A_{\hat\mu\hat\nu\hat\rho\hat\sigma}$ are antisymmetric tensors, the latter being self–dual. In CY compactifications the harmonic (1,1), (2,1) forms come as follows (with the splitting of the internal index $\hat\imath = (i, i^*), i = 1, 2, 3$):

$$G_{ij^*}, B_{ij^*}\,;\; G_{ij} \tag{1.13}$$

for the heterotic case,

$$G_{ij^*}, B_{ij^*}, A_{\mu ij^*} \; ; \; G_{ij}, A_{ij^*k^*} \tag{1.14}$$

for type IIA strings, (where suitable contraction with the metric and the (0,3)–(3,0) forms of the CY are understood, to get the correct index content for (2,1) forms) and

$$G_{ij^*}, B^c_{ij^*}, A_{\mu\nu ij^*} \; ; \; G_{ij}, A_{\mu ij^*k^*} \tag{1.15}$$

for type IIB string. The reason we have as many 27, $\overline{27}$ families as (1,1), (2,1) forms is because we identify the $SU(3)$ [16] holonomy connection of the Calabi–Yau manifold with the $SU(3)$ gauge connection in the decomposition of $E_8 \to E_6 \times SU(3)$. The full spectrum of the scalar fields in the three theories compactified on the same CY space is as follows:
Heterotic case:

$$M_a, N_\alpha, \phi^I_a, \phi^{I^*}_\alpha, S \; , \tag{1.16}$$

where M_a corresponds to G_{ij^*}, B_{ij^*}; N_α to G_{ij}; S to ϕ and $B_{\mu\nu}$ and $a = 1, \cdots h_{1,1}$, $\alpha = 1, \cdots, h_{2,1}$, $I \in 27, I^* \in \overline{27}$.
Type IIA case

$$M_a, N_\alpha, C_\alpha, S, C \; , \tag{1.17}$$

where C_α correspond to the $A_{ij^*k^*}$ modes and S to A_{ijk} mode.
Type IIB case:

$$M_a, C_a, S_1, S_2, N_\alpha \; , \tag{1.18}$$

where M_a, C_a correspond to $G_{ij^*}, B^c_{ij^*}, A_{\mu\nu ij^*}$ and S_1, S_2 correspond to $\phi^c, B^c_{\mu\nu}$.

Since in type IIA theories there are 4 degrees of freedom for each (2,1) form and in type IIB for each (1,1) form, we can show [10, 5] that (2,1) and (1,1) moduli belong to N=2 hypermultiplets respectively in type IIA and type IIB theories. In the chirality reversed theory the same moduli belong to vector multiplets. Indeed in type IIA theories there are $h_{2,1} + 1$ gauge vectors coming from $A_{\mu ijk^*}$ and $A_{\mu ijk}$. The additional vector is the graviphoton. The only interaction of vector multiplets and hypermultiplet consistent with N= 2 supergravity is a non linear σ–model of the form:

$$\mathcal{M}_{SK} \times Q \; , \tag{1.19}$$

where $\mathcal{M}_{SK}$ is a very "special" (Kähler) manifold (see in the following) for the vector multiplets and Q is a quaternionic manifold for the hypermultiplet [18, 19, 61]. By writing explicitly the dimension of these manifolds we get [62]:

$$\mathcal{M}^A = \mathcal{M}^A_{h_{1,1}} \times Q^A_{h_{2,1}+1} \; , \tag{1.20}$$

$$\mathcal{M}^B = \mathcal{M}^B_{h_{2,1}} \times Q^B_{h_{1,1}+1} \; , \tag{1.21}$$

the additional multiplet for the Q manifold coming from the space– time dilaton and antisymmetric tensor sector.

Notice that while the $\mathcal{M}$ Kähler manifolds contain the same moduli fields which appear in heterotic strings the Q manifolds are obtained by gluing together moduli scalars with non–moduli scalars.

The first observation at this point is that the manifolds $\mathcal{M}^A$ and $\mathcal{M}^B$ must coincide with the submanifolds of the heterotic string when we freeze one of the two set of topologically distinct moduli. The fact that the full manifold is a product space given by eq. (1.9) comes by setting to zero the non–moduli fields in type II theories. In the type IIA example, setting $C_a = C = 0$ we get

$$Q^A_{h_{2,1}+1} \rightarrow \mathcal{M}_{h_{2,1}} \times \frac{SU(1,1)}{U(1)} \ . \tag{1.22}$$

We conclude by pure space–time arguments that eq. (1.9) is true. The moduli spaces $\mathcal{M}_{h_{1,1}}$, $\mathcal{M}_{h_{2,1}}$ have a very rich geometrical structure: they are "special Kähler" manifolds. We give precise definition of special geometry in the following chapter, here we simply mention that the curvature of a special manifolds satisfies the constraint [3, 4, 21]:

$$\mathcal{R}_{\bar{a}b\bar{c}d} = -g_{\bar{c}(b}g_{d)\bar{a}} + \frac{1}{2}e^{2G}w_{\bar{a}\bar{c}\bar{p}}w_{qdb}g^{\bar{p}q} \ , \tag{1.23}$$

where w_{abc} is a holomorphic 3-index symmetric tensor [3] and G is the Kähelr potential. The tensor w_{abc} has the meaning of Yukawa coupling for (27) (or $\overline{27}$ families) [11].

$$w_{abc}(27)^3 \quad , \quad w_{\alpha\beta\gamma}(\overline{27})^3 \ . \tag{1.24}$$

A metric which satisfies (1.23) can be found in a special coordinate system which is the one used in N=2 supergravity tensor calculus [3]. We will describe this special system in the following chapter.

There are profound implications for superstring dynamics which come from this specific structure of the moduli space and its relation to the Yukawa couplings. The first one is that $(27)^3$ and $(\overline{27})^3$ couplings can only depend on their separate moduli (the M moduli for the (27) and the N moduli for the $\overline{27}$). Moreover it has been shown that Yukawa couplings for 27 families are just constant and cannot depend on the moduli parameters [63]

Another consequence of (1.23), which is worth to mention here, is the relation between the moduli and the matter metric. In heterotic string the full scalar (moduli + matter) self–couplings in N=1 supergravity action are determined by the function [64]

$$\hat{G} = G + \log|W|^2 \ , \tag{1.25}$$

[3] Here we are abusing of the definition of "tensor", since w_{abc} is actually a section of a bundle which will be defined in the next chapter

where W is the superpotential. In this case we can write:

$$W(M^a, N^\alpha, \phi^a, \phi^\alpha) = w_{abc}(M)\phi^a\phi^b\phi^c + w_{\alpha\beta\gamma}(N)\phi^\alpha\phi^\beta\phi^\gamma \qquad (1.26)$$

(E_6 gauge indexes are suppressed for convenience). Here ϕ^a and ϕ^α are the matter fields related to the (1,1) and (2,1) moduli respectively. From eq: (1.23) we learn that under Kähler transformation of the moduli space we must have:

$$\begin{array}{rcl} G_{h_{1,1}} & \rightarrow & G_{h_{1,1}} - \Lambda_{h_{1,1}} - \bar{\Lambda}_{h_{1,1}} \quad , \quad w_{abc} \rightarrow w_{abc} e^{2\Lambda_{h_{1,1}}} \ , \\ G_{h_{2,1}} & \rightarrow & G_{h_{2,1}} - \Lambda_{h_{2,1}} - \bar{\Lambda}_{h_{2,1}} \quad , \quad w_{\alpha\beta\gamma} \rightarrow w_{\alpha\beta\gamma} e^{2\Lambda_{h_{2,1}}} \ , \end{array} \qquad (1.27)$$

where $\Lambda_{h_{1,1}} = \Lambda_{h_{1,1}}(M)$ and $\Lambda_{h_{2,1}} = \Lambda_{h_{2,1}}(N)$ are holomorphic parameters of the moduli. The full Kähler potential of the moduli + matter field has the form:

$$G = G_{h_{1,1}} + G_{h_{2,1}} + O(\phi^2) + \text{high. order terms} \ . \qquad (1.28)$$

The crucial fact is that the matter dependent part must be Kähler inert under the Kähler transformation of the moduli subspace. In order for the function $\hat{G}$ to be invariant both terms in W must scale as $We^{\Lambda_{h_{1,1}}+\Lambda_{h_{2,1}}}$. This is achieved by using the following Kähler transformations for the ϕ fields:

$$\phi_a \rightarrow \phi_a e^{(\Lambda_{h_{2,1}} - \Lambda_{h_{1,1}})/3} \quad , \quad \phi_\alpha \rightarrow e^{(\Lambda_{h_{1,1}} - \Lambda_{h_{2,1}})/3} \ . \qquad (1.29)$$

It is now easy to see that the only possible metrics for the matter fields, which are Kähler inert are given by:

$$\begin{array}{rcl} g_{\phi_a\bar{\phi}_{\bar{b}}} & = & g_{a\bar{b}} e^{(G_{h_{1,1}} - G_{h_{2,1}})/3} \\ g_{\phi_\alpha\bar{\phi}_{\bar{\beta}}} & = & g_{\alpha\bar{\beta}} e^{(G_{h_{2,1}} - G_{h_{1,1}})/3} \end{array} \qquad (1.30)$$

a result derived from conformal field theory argument in ref. [11]

1.3 Duality symmetry and target modular invariance

So far we have only considered the local structure of the moduli space of string compactifications. As we will see, the global structure is modified by discrete symmetries, called duality transformations [41, 40] We will demonstrate this in the simplest case, namely the compactification of the closed bosonic string on a circle of radius R , i.e. we choose $\mathcal{M}_{int} = S^1$. If we denote by X the string coordinate, compactification on a circle means that we have to identify $X = X + 2\pi R$. This implies that the center mass momentum is quantized, $p = \frac{m}{R}$, $m \in Z$. Moreover the string may wrap around the circle, so that:

$$X(\sigma + \pi) = X(\sigma) + 2\pi n R \ , \qquad (1.31)$$

where $n \in Z$ is the winding number (the number of times the string wrap around the circle). The internal string coordinate can be expanded as:

$$\begin{aligned} X(\sigma,\tau) &= x_0 + 2\sigma n R + p\tau + \text{oscillators} \\ &= x_0 + p_L(\tau+\sigma) + p_R(\tau-\sigma) + \text{oscillators} \ , \end{aligned} \tag{1.32}$$

where

$$p_{L,R} = \frac{m}{2R} \pm nR \ . \tag{1.33}$$

The mass string state is:

$$\begin{aligned} \frac{1}{4}m^2 &= \frac{1}{8}m_L^2 + \frac{1}{8}m_R^2 \\ &= \tfrac{1}{2}(p_L^2 + p_R^2) + N_L + N_R - 2 \\ &= \frac{m^2}{4R^2} + n^2R^2 + N_L + N_R - 2 \ , \end{aligned} \tag{1.34}$$

where $N_{L,R}$ counts the number of oscillator excitations and we have also included the normal ordering constant appropriate for the closed bosonic string. We see that under a simultaneous interchange of momentum and winding quantum numbers $m \leftrightarrow n$, $R \leftrightarrow \frac{1}{2R}$ the mass of a given state is invariant. This symmetry is called *duality symmetry.* It leaves the spectrum invariant and is consequently a symmetry of the partition function. To show that it is really a symmetry of the theory we also have to make sure that it is a symmetry of the interactions. In particular $e^{ip\cdot X} = e^{i(p_L\cdot X_L + p_R\cdot X_R)}$ must be invariant. Indeed, one finds that if one accompanies the interchange of momentum and winding quantum numbers and the replacement of the compactifying radius R by its dual radius $\frac{1}{2R}$ by $X_L \to X_L$, $X_R \to -X_R$, then all the S–matrix elements are invariant [65]. This symmetry of the theory means that we cannot distinguish a string theory compactified on a large circle radius R from one compactified on a small circle with radius $\frac{1}{2R}$. The value $R = \frac{1}{\sqrt{2}}$ is the fixed point of the duality symmetry.

For this example any radius R is possible. Naively one would conclude that the moduli space for circle compactification is $R > 0$. However, due to duality symmetry, the moduli space is actually either $R \leq \frac{1}{\sqrt{2}}$ or $R \geq \frac{1}{\sqrt{2}}$. Each encompass all possible distinct compactifications on a circle. The duality symmetry for this case is simply a Z_2 symmetry.

Let us now generalize the discussion to the more interesting case of compactifications on a d–dimensional flat torus with constant metric and antisymmetric tensor field background [40, 45]. The metric is defined in terms of the basis vectors $e_i, i = 1, \cdots, d$ of the lattice Γ_d that generates the torus $T_d = R^d/\Gamma_d$:

$$G_{ij} = e_i \cdot e_j \ . \tag{1.35}$$

By compactification assumption we have to identify [4] $X^i \equiv X^i + 2\pi n^i, n^i \in Z$, and winding states satisfy $X^i(\sigma+\pi) = X^i(\sigma) + 2\pi n^i$. The compact string coordinates are then:

$$X^i(\sigma,\tau) = x^i + 2n^i\sigma + p^i\tau + \text{oscill.} \; . \tag{1.36}$$

The canonical mass momentum:

$$\pi_i = \int d\sigma \partial S/\partial \dot{X}^i = G_{ij}p^j + 2B_{ij}n^j \tag{1.37}$$

must be quantized. Note that even though the B_{ij} term is a total derivative and does not modify the equation of motion, it does enter the definition of the canonical momentum. Single valuedness of $e^{i\pi_i x^i}$ requires that $\pi_i = m_i \in Z$. We again define the left–right momenta $p^i_{L,R} = \frac{1}{2}p^i \pm n^i$ for which we get:

$$p^i_{L,R} = \frac{1}{2}G^{ij}m_j - G^{ij}B_{jk}n^k \pm n^i \quad ; \quad n^i, m_j \in Z \; . \tag{1.38}$$

Using this in the mass formula with $p^2_{L,R} = p^i_{L,R}G_{ij}p^j_{L,R}$ we find:

$$\begin{aligned} \frac{1}{8}m^2_{L,R} &= \frac{1}{2}[\frac{1}{4}\mathbf{m}^T\mathbf{G}^{-1}\mathbf{m} + \mathbf{n}^T(\mathbf{G}-\mathbf{B}\mathbf{G}^{-1}\mathbf{B})\mathbf{n} + \tfrac{1}{2}\mathbf{n}^T\mathbf{B}\mathbf{G}^{-1}\mathbf{m} \\ &- \tfrac{1}{2}\mathbf{m}^T\mathbf{G}^{-1}\mathbf{B}\mathbf{n} \pm \mathbf{n}^T\mathbf{m}] + N_{L,R} - 1 \; , \end{aligned} \tag{1.39}$$

where we have employed a matrix notation. The expression $m^2_{L,R}$ are manifestly invariant under simultaneous interchanges of

$$\mathbf{m} \leftrightarrow \mathbf{n} \quad \text{and} \quad \begin{aligned} &\tfrac{1}{4}\;\mathbf{G}^{-1} \leftrightarrow \mathbf{G} - \mathbf{B}\mathbf{G}^{-1}\mathbf{B} \\ &\mathbf{B}\;\mathbf{G}^{-1} \leftrightarrow -\mathbf{G}^{-1}\mathbf{B} \; , \end{aligned} \tag{1.40}$$

which is equivalent to

$$\mathbf{m} \leftrightarrow \mathbf{n} \quad \text{and} \quad (\mathbf{G}+\mathbf{B}) \leftrightarrow \frac{1}{4}(\mathbf{G}+\mathbf{B})^{-1} \; . \tag{1.41}$$

This generalize the duality transformations $R \to \frac{1}{2R}$ of the one–dimensional circle case, which corresponds to $B_{ij} = 0$, $G_{ij} = R^2$.

The discussion on the moduli space is most easily done by using the correspondence between metric and antisymmetric tensor background with Lorentzian lattices . One finds that locally the moduli space is $\frac{O(d,d)}{O(d)\times O(d)}$ with dimension d^2 equal to the number of components of G_{ij}, B_{ij}. They can be used to parametrize the moduli space. The group of duality transformations generated by the symmetries mentioned above can be shown to be $O(d,d,Z)$ [40]. We do not proof this result, but we explain in some

[4] X^i are the components with respect to the lattice Γ_d. The components with respect to R^d are $X^\mu = e^\mu_i X^i$

detail the example of the compactification on a two–dimensional torus. The generating lattice of the torus is spanned by the two basis vectors e_1, e_2. They define the metric $G_{ij} = e_i \cdot e_j$, with three independent components. The antisymmetric tensor has only one component $B_{ij} = b\epsilon_{ij}$. Together we have four real moduli which can be rearranged into two complex moduli as follows:

$$\sigma = \frac{|e_1|}{|e_2|}e^{i\phi} \quad , \quad \tau = 2(b + iA) \ , \tag{1.42}$$

where ϕ is the angle between the two basis vectors and $A = \sqrt{|\det\, G|} > 0$ is the area of the unit cell lattice. Both the moduli parametrize the upper half complex plane, which is isomorphic to $\frac{SU(1,1)}{U(1)}$. σ is a complex structure modulus and is usually called the Teichmuller parameter, while τ parametrize the different Kähler structures. To compare with the general result we recall the isomorphism:

$$\frac{O(2,2)}{O(2)\times O(2)} \sim \frac{SU(1,1)}{U(1)} \times \frac{SU(1,1)}{U(1)} \ . \tag{1.43}$$

Naively one would conclude that the moduli space for the two– dimensional torus compactification is just two copies of the upper half plane. Let us see how this is modified by discrete duality symmetries. We express the metric component and antisymmetric tensor in terms of $\sigma = \sigma_1 + i\sigma_2$ and $\tau = \tau_1 + i\tau_2$:

$$b = \tfrac{1}{2}\tau_1 \quad , \quad G = \frac{\tau_2}{2\sigma_2}\begin{pmatrix} |\sigma|^2 & \sigma_1 \\ \sigma_1 & 1 \end{pmatrix} \tag{1.44}$$

and obtain

$$\begin{aligned} p_L^2 &= \frac{1}{2\sigma_2\tau_2}|(m_1 - \sigma m_2) - \tau(n_2 + \sigma n_1)|^2 \ , \\ p_R^2 &= \frac{1}{2\sigma_2\tau_2}|(m_1 - \sigma m_2) - \bar{\tau}(n_2 + \sigma n_1)|^2 \ . \end{aligned} \tag{1.45}$$

It is now easy to see that with a suitable redefinition of the momentum and winding quantum numbers we have the following symmetries of p_L^2 and p_R^2. A $Z_2 \times Z_2$ given by $\sigma \longrightarrow \tau$ and $(\sigma,\tau) \leftrightarrow (-\bar{\sigma},-\bar{\tau})$ and a $SL(2,Z) \sim SU(1,1,Z)$ given by: $\tau \longrightarrow \frac{a\tau+b}{c\tau+d}$, $ad - bc = 1$, $a,b,c,d \in Z$. The combination of the previous discrete symmetries implies a $SL(2,Z)$ symmetry for σ. Due to these discrete symmetries the moduli space is no longer the product of two copies of the upper half plane. If we identify all points on the upper half plane which are related by a SL(2,Z) transformation, we arrive at a fundamental region $\mathcal{F} = \frac{H_+}{SL(2,Z)}$, $H_+ \sim \frac{SU(1,1)}{U(1)}$. The moduli space for the two–dimensional torus compactification is thus $\frac{\mathcal{F}\times\mathcal{F}}{Z_2\times Z_2}$.

What we have seen in some specific examples is how discrete symmetries affect the global structure of the moduli space of string compactification. We know that moduli enter the effective action as massless neutral scalar fields with vanishing potential. The action should then be invariant under duality transformations which are discrete coordinate transformations in the moduli space which is the target space of the four dimensional σ–model for the moduli. For this reason duality transformations are also referred as target space modular transformations. Instead of torus compactifications we can have orbifold compactifications, were the torus is modded out by a discrete symmetry . The moduli spaces can be obtained by performing suitable truncations of the original moduli space for the torus compactification. In particular Z_n orbifolds correspond to particular choices of the following special symmetric moduli spaces [5, 66].

$$\begin{aligned}
\mathcal{M}_{h_{1,1}{}^0} = (\frac{SU(1,1)}{U(1)})^3 \quad &, \quad \frac{SU(1,1)}{U(1)} \times \frac{SU(2,2)}{SU(2)\times SU(2)\times U(1)} \\
&, \quad \frac{SU(3,3)}{SU(3)\times SU(3)\times U(1)}\,; \\
\mathcal{M}_{h_{2,1}{}^0} \;=\; 1 \text{ or } &\frac{SU(1,1)}{SU(1)}\,,
\end{aligned} \tag{1.46}$$

where $h_{1,1}{}^0, h_{2,1}{}^0$ refers to the (1,1), (2,1) moduli in the untwisted sector, i. e. "proper" moduli of the orbifold, whereas the twisted moduli are moduli of the smooth Calabi–Yau space of which the orbifold is a singular limit (the so called "blowing up" of the orbifold singularities). As we will see in next chapter the duality groups corresponding to the moduli spaces in (1.46) can be analyzed in detail.

Finally it is worth to mention the orbifold compactifications that are compatible with the N=1 supergravity effective actions for the untwisted (2,2) sector. We have the following generic cases [66],

$$\begin{aligned}
&1)\, 3h^0_{1,1} + l h^0_{2,1} \quad \text{moduli:} \quad \mathcal{M} = \left[\frac{SO(2,2)}{SO(2)\times SO(2)}\right]^l \times \left[\frac{SU(1,1)}{U(1)}\right]^{3-l}\,, \; l = 0,1,2,3 \\
&2)\, 5h^0_{1,1} + h^0_{2,1} \quad \text{moduli:} \quad \mathcal{M} = \frac{SO(2,2)}{SO(2)\times SO(2)} \times \frac{SU(2,2)}{SU(2)\times SU(2)\times U(1)}\,, \\
&3)\, 5h^0_{1,1} \quad \text{moduli:} \quad \mathcal{M} = \frac{SU(2,2)}{SU(2)\times SU(2)\times U(1)} \times \frac{SU(1,1)}{SU(1)}\,, \\
&4)\, 9h^0_{1,1} \quad \text{moduli:} \quad \mathcal{M} = \frac{SU(3,3)}{SU(3)\times SU(3)\times U(1)}\,,
\end{aligned} \tag{1.47}$$

which are in correspondence with particular abelian orbifolds [67], obtained by modding suitable products of Z_N discrete subgroups.

Chapter 2

Symplectic embeddings and special Kähler geometry

2.1 Introduction

Originally the concept of special Kähler geometry was introduced in the context of N=2 supergravity, while solving the problem of coupling an arbitrary number n of vector multiplets [18, 3, 19]. It was found that the n complex scalars z^i, corresponding to the lowest spin components of these multiplets, had to parametrize a Kähler manifold of a restricted type, where the Kähler potential $G(z,\bar{z})$ is obtained from a holomorphic prepotential $f(z)$ through the formula[1]

$$G(z,\bar{z}) = -log\left[2\left(f + \bar{f}\right) - \left(\partial_i f - \partial_{i^*}\bar{f}\right)\left(z^i - \bar{z}^{i^*}\right)\right] . \tag{2.1}$$

Furthermore the same holomorphic function $f(z)$, that through eq. (2.1) determines the scalar kinetic terms, appears also in the vector kinetic terms. Clearly eq.(2.1) is a coordinate dependent statement and, for a long time, the intrinsic geometric characterization of special Kähler manifolds remained unknown. Nonetheless it was early realized [3, 19, 20] that, in all these manifolds, the Riemann tensor satisfies the following identity :

$$\mathcal{R}_{i^*j\ell^*k} = -g_{\ell^*(j}g_{k)i^*} + \frac{1}{2}e^{2G}w_{i^*\ell^*s^*}w_{tkj}g^{s^*t} . \tag{2.2}$$

In [20] all the homogeneous symmetric Kähler manifolds with such a property were classified and a suitable $f(z)$ function was found, for each of them, in appropriate *special* coordinates. A very strong result was also proved [23], namely that the only manifold fulfilling eq.(2.2), which is also a direct product of manifolds, is necessarily

[1]To avoid any particular reference $h_{1,1}$ and $h_{2,1}$ labels, here we use the indices i, i^*, which are not ,in the following, internal space indices, but span a *generic* special Kähler manifold

of the following form:

$$SK(n+1) = \frac{SU(1,1)}{U(1)} \times \frac{SO(2,n)}{SO(2) \times SO(n)} . \tag{2.3}$$

As we are going to illustrate, the manifold $SK(n+1)$ plays an important role in various contexts. In the following we will be particularly concerned with its structure and we solve an apparent paradox related with its geometry. The other case that we will study in detail is the special Kähler manifold given by:

$$\mathcal{M}_{3,3} = \frac{SU(3,3)}{SU(3) \times SU(3) \times U(1)} , \tag{2.4}$$

which is also in the list of ref. [20]

To explain this matter we have to resume our discussion, concerning the intrinsic geometric characterization of special manifolds.

As sketched in Chapter 1 special Kähler manifolds $\mathcal{M}'$ are regarded as moduli spaces of Calabi-Yau complex 3-folds, describing either the $n = h_{1,1}$-dimensional space of Kähler class deformations, or the $n = h_{2,1}$-dimensional space of complex structure deformations [4]. In both cases, the relation with Dolbeault cohomology of the underlying 3-fold puts into evidence a crucial symplectic $Sp(2+2n, R)$ structure, corresponding to change of bases for the homology cycles, which eventually can be taken as an intrinsic definition of special Kähler geometry. We shall review this definition in section 2.2.

In [21, 22] special manifolds $\mathcal{M}'$ were instead regarded as the σ-model manifolds $\mathcal{VM}$, associated with the vector multiplets of an arbitrary N=2, D=4 Supergravity: utilizing superspace Bianchi identities and the rheonomy approach, rather than the conformal tensor calculus, the authors of [21, 22] were able to obtain the just mentioned intrinsic definition of special geometry as a pure consequence of N=2 target supersymmetry, without using special coordinate systems. From the supergravity viewpoint, the symplectic $Sp(2+2n, R)$ structure is related with duality transformations that fit in the general scheme developed in [39, 1]. Furthermore it is shown that the symplectic structure and the existence of a w_{ijk} satisfying eq.(2.2) are equivalent properties implying each other.

The relation between the supergravity and the Calabi-Yau view point, was provided [5, 6] through the observation that, on the same 3-fold one can compactify either the heterotic superstring or the type II superstring, displaying N=1 and N=2 supersymmetry, respectively. The moduli-space geometry must therefore be compatible with N=2 SUSY and that is the reason why it is special. Actually in [11] eq. (2.2) has been directly derived from (2,2) superconformal ward identities. This has been the direct proof that special Kähler geometry is the geometry of the moduli space for

(2,2) superconformal field-theories. New insight in this direction has recently been obtained [31, 68, 33, 49] by the topological reinterpretation (through twisting) of n=2 2D theories. Indeed it appears that the holomorphic prepotential of special geometry is directly related to the topological free-energy and w_{ijk} is related to the topological 3-point function. On the other hand w_{ijk} expresses the Yukawa couplings, when the (2,2) theory is utilized to compactify the heterotic superstring. We will resume this discussion in chapter 3. Here we begin the main body of the present chapter by giving characterization of the special geometry.

2.2 Special Kähler manifolds: definition and applications

A special Kähler manifold is a Kähler manifold with additional restrictions on its Kähler structure.

Definition 1 *A manifold M_n, of complex dimension n, is Kählerian if it has a complex structure and a hermitean metric*

$$ds^2 = g_{ij^*}(z,\bar{z})dz^i \otimes d\bar{z}^{j^*} \;, \tag{2.5}$$

such that the (1,1)-form

$$K = ig_{ij^*}(z,\bar{z})dz^i \wedge d\bar{z}^{j^*} \tag{2.6}$$

is closed ($dK = 0$).

As it is well known, K cannot be globally exact, yet it is certainly locally exact. Indeed in every coordinate patch we can find a real function $G(z,\bar{z})$ (named the Kähler potential) such that

$$\begin{aligned} g_{ij^*} &= \partial_i\partial_{j^*}G(z,\bar{z}) \;, \\ K &= dQ \;, \\ Q &= -\frac{i}{2}(\partial_i G dz^i - \partial_{i^*}G d\bar{z}^{i^*}) \;. \end{aligned} \tag{2.7}$$

Under a Kähler transformation

$$G \to G + \alpha(z) + \bar{\alpha}(\bar{z}) \tag{2.8}$$

the 1-form Q transforms as

$$Q \to Q + d(Im\alpha). \tag{2.9}$$

Therefore Q is a $U(1)$ connection.

The $U(1)$ covariant differential of a field $\Phi(z,\bar{z})$ of weight p is

$$\nabla\Phi = (d + ipQ)\Phi \ , \tag{2.10}$$

or in components

$$\begin{aligned} \nabla_i\Phi &= (\partial_i + \frac{1}{2}p\partial_i G)\Phi \ , \\ \nabla_{i^*}\Phi &= (\partial_{i^*} - \frac{1}{2}p\partial_{i^*} G)\Phi \ . \end{aligned} \tag{2.11}$$

A covariantly holomorphic field of weight p is defined by

$$\nabla_{i^*}\Phi = 0 \ . \tag{2.12}$$

By a change of trivialization the real $U(1)$-bundle can be reduced to a holomorphic $U(1)$ line bundle $\mathcal{L}$. Indeed setting

$$\tilde{\Phi} = e^{-pG/2}\Phi \tag{2.13}$$

we have

$$\begin{aligned} \nabla_i\tilde{\Phi} &= (\partial_i + p\partial_i G)\tilde{\Phi} \ , \\ \nabla_{i^*}\tilde{\Phi} &= \partial_{i^*}\Phi \quad . \end{aligned} \tag{2.14}$$

In particular, if Φ is a covariantly holomorphic section with respect to the Q-connection, $\tilde{\Phi}$ is a holomorphic section with respect to the holomorphic connection $\partial_i G$.

Definition 2 *If the $U(1)$ line bundle is such that the first Chern class $c_1(\mathcal{L})$ coincides with the Kähler class $[K]$ then the Kähler manifold is of restricted type or a Hodge manifold.*

A special Kähler manifold is a Hodge manifold obeying additional restrictions that we shall presently illustrate. To this effect we point out that in addition to the $U(1)$-holomorphic connection $\partial_i G$ one has the holomorphic Levi Civita connection

$$\begin{aligned} \Gamma^i_j &= \Gamma^i_{kj} dz^k \ , \\ \Gamma^i_{kj} &= -g^{i\ell^*}(\partial_j g_{k\ell^*}) \ , \\ \Gamma^{i^*}_{j^*} &= \Gamma^{i^*}_{k^*j^*} d\bar{z}^{k^*} \ , \\ \Gamma^{i^*}_{k^*j^*} &= -g^{i^*\ell}(\partial_{j^*} g_{k^*\ell}) \end{aligned} \tag{2.15}$$

and its curvature

$$\begin{aligned} \mathcal{R}^i_j &= \mathcal{R}^i_{jk^*\ell} d\bar{z}^{k^*} \wedge dz^\ell \ , \\ \mathcal{R}^i_{jk^*\ell} &= \partial_{k^*}\Gamma^i_{j\ell} \ , \\ \mathcal{R}^{i^*}_{j^*} &= \mathcal{R}^{i^*}_{j^*k\ell^*} dz^k \wedge d\bar{z}^{\ell^*} \ , \\ \mathcal{R}^{i^*}_{j^*k\ell+} &= \partial_k\Gamma^{i^*}_{j^*\ell^*} \ . \end{aligned} \tag{2.16}$$

Definition 3 *By definition a restricted Kähler manifold is special if and only if there exist a completely symmetric holomorphic 3-index section w_{ijk} of $(T^*)^3 \otimes \mathcal{L}^2$ (and its antiholomorphic conjugate $w_{i^*j^*k^*}$) such that*

$$\partial_{m^*} w_{ijk} = 0 \quad , \quad \partial_m w_{i^*j^*k^*} = 0 \ ; \tag{2.17}$$

$$\nabla_{[m} w_{i]jk} = 0 \quad , \quad \nabla_{[m} w_{i^*]j^*k^*} = 0 \ ; \tag{2.18}$$

$$\mathcal{R}_{i^*j\ell^*k} = -g_{\ell^*(j}g_{k)i^*} + \frac{1}{2}e^{2G} w_{i^*\ell^*s^*} w_{tkj} g^{s^*t} \ . \tag{2.19}$$

In the equations above ∇ denotes the derivative covariant with respect to *both* the Levi Civita and the $U(1)$ holomorphic connection.

In the case of the w_{ijk}, $p = 2$. In [4, 21, 22] it was shown that on a n–dimensional special Kähler manifold one can always introduce a *a* $(n+1)$- dimensional holomorphic vector bundle whose holomorphic sections we denote by X^Λ $(\Lambda = 1, ..., n+1)$

$$\partial_{i^*} X^\Lambda = 0 \tag{2.20}$$

and a function $F(L)$ which is holomorphic and homogeneous of degree two in the transformed section

$$L^\Lambda(z, \bar{z}) = e^{1/2G(z,\bar{z})} X^\Lambda(z) \ . \tag{2.21}$$

This means that $F(L) = e^{G(z,\bar{z})} F(X(z))$ so that $F(X)$ is a holomorphic section of $\mathcal{L}^2$. For $L^\Lambda(z, \bar{z})$ we have:

$$\nabla L^\Lambda = dL^\Lambda + iQL^\Lambda = \nabla_i L^\Lambda dz + \nabla_{i^*} L^\Lambda dz^{i^*} \ , \tag{2.22}$$

$$\nabla_{i^*} L^\Lambda = \partial_{i^*} L^\Lambda - \frac{1}{2}\partial_{i^*} G L^\Lambda = 0 \ . \tag{2.23}$$

Eq. (2.23) follows from eq. (2.20) and eq.(2.21).

The geometry of the special manifold is completely determined by the sections $\{X^\Lambda\}$ and by the analytic function $F(L)$ or equivalently by $F(X)$. Define

$$F_{\Lambda_1 \ldots \Lambda_n} = \frac{\partial}{\partial L^{\Lambda_1}} \frac{\partial}{\partial L^{\Lambda_2}} \cdots \frac{\partial}{\partial L^{\Lambda_n}} F(L) \tag{2.24}$$

and set

$$N_{\Lambda\Sigma} = F_{\Lambda\Sigma} + \bar{F}_{\Lambda\Sigma} \ , \tag{2.25}$$

$$f_i^\Lambda = \nabla_i L^\Lambda \equiv \partial_i L^\Lambda + \frac{1}{2} G_i L^\Lambda \equiv e^{G/2}(\delta^\Lambda_\Sigma - \frac{X^\Lambda (N\bar{X})_\Sigma}{XN\bar{X}}) \partial_i X^\Sigma \ , \tag{2.26}$$

$$f_{i^*}^\Lambda = \nabla_{i^*} \bar{L}^\Lambda = \partial_{i^*} \bar{L}^\Lambda + \frac{1}{2} G_{i^*} \bar{L}^\Lambda \equiv e^{G/2}(\delta^\Lambda_\Sigma - \frac{\bar{X}^\Lambda (NX)_\Sigma}{XN\bar{X}}) \partial_{i^*} \bar{X}^\Sigma \ , \tag{2.27}$$

$$S = -\frac{1}{4} N_{\Lambda\Sigma} L^\Lambda L^\Sigma \ , \tag{2.28}$$

then one finds

$$\begin{aligned}
g_{ij^*} = -f_i^\Lambda f_{j^*}^\Sigma N_{\Lambda\Sigma} &= \partial_i X^\Lambda \partial_j \bar{X}^\Sigma \partial_\Lambda \partial_{\bar\Sigma}^* G(X,\bar{X}) , & (2.29)\\
e^G w_{ijk} = \nabla_i \nabla_j \nabla_k S &= f_i^\Lambda f_j^\Sigma f_k^\Gamma F_{\Lambda\Sigma\Gamma} = \\
&= e^G \partial_i X^\Lambda \partial_j X^\Sigma \partial_k X^\Gamma F_{\Lambda\Sigma\Gamma}(X) , & (2.30)\\
N_{\Lambda\Sigma} L^\Lambda \bar{L}^\Sigma &= e^G N_{\Lambda\Sigma} X^\Lambda \bar{X}^\Sigma = 1 , & (2.31)\\
f_i^\Lambda \bar{L}^\Sigma N_{\Lambda\Sigma} &= 0 , & (2.32)\\
f_{i^*}^\Lambda L^\Sigma N_{\Lambda\Sigma} &= 0 , & (2.33)\\
\mathcal{U}^{\Lambda\Sigma} \equiv g^{ij^*} f_i^\Lambda f_{j^*}^\Sigma &= -(N^{-1})^{\Lambda\Sigma} + L^\Lambda \bar{L}^\Sigma , & (2.34)\\
N_{\Lambda\Sigma} U^{\Sigma\Pi} N_{\Pi\Gamma} &= e^G \partial_\Lambda \partial_{\bar\Sigma} G , & (2.35)
\end{aligned}$$

where $\partial_\Lambda \equiv \frac{\partial}{\partial X^\Lambda}, \partial_{\bar\Lambda} \equiv \frac{\partial}{\partial \bar{X}^\Lambda}$.

In ref. [4] it was shown that $\Omega = \{X^\Lambda, i\frac{\partial F}{\partial X^\Sigma}\}$ can be viewed as the cross-section of a flat holomorphic, $Sp(2n+2,R)$–bundle $\mathcal{H}$, whose existence is in fact equivalent to eq. (2.2). In the case where we are considering CY compactifications there is a deep relation between this symplectic structure and the algebraic geometry of the underlying CY manifold. We give some detail on this point in the appendix A.

The Kähler 2-form is given by:

$$K = -\partial\bar\partial log \left(-i < \bar\Omega|\Omega >\right) , \qquad (2.36)$$

where $-i < \bar\Omega|\Omega >$ denotes the compatible hermitean metric on $\mathcal{H}$:

$$< \bar\Omega|\Omega > = \Omega^\dagger \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Omega = i\left(\bar{X}^\Lambda \partial_\Lambda F + \bar\partial_\Lambda \bar{F} X^\Lambda\right) . \qquad (2.37)$$

Recalling the definitions (2.6), (2.7), (2.25) and the fact that $F(X)$ is homogeneous of degree two, equations (2.36) and (2.37) are equivalent to the following formula relating the Kähler potential G to the norm of the holomorphic section Ω in the $Sp(2n+2)$ flat bundle :

$$G = -log \parallel \Omega \parallel^2 \equiv -log\left(-i < \bar\Omega|\Omega >\right) = -\log(N_{\Lambda\Sigma} X^\Lambda \bar{X}^\Sigma) . \qquad (2.38)$$

Special coordinates for the underlying Kähler manifold correspond to the choice $z^i = X^i/X^0$ $(i = 1,..,n)$ and the prepotential $f(z)$ mentioned at the beginning is related to F through

$$f(z) = \left(X^0\right)^{-2} F(X) . \qquad (2.39)$$

With this identification the formula (2.1) for the Kähler potential match perfectly with (2.37)

In this context duality transformations associated with a discrete subgroup $\Gamma \subset Sp(2+2n,Z) \subset Sp(2+2n,R)$ of the above mentioned symplectic group have become a focus of interest [42, 41, 40]. Γ is the target space modular group and corresponds to the action on the Calabi-Yau homology basis of global diffeomorphisms. For the same reason as explained in chapter 1, Γ must be modded out and it turns out to be an exact non perturbative symmetry of the effective low energy lagrangian. This provides a powerful tool to obtain exact non perturbative results for the superpotential $W(z)$. Indeed in order to maintain Γ symmetry $W(z)$ has to be a Γ-automorphic function.

In [42] a general formula has been proposed to express the automorphic superpotential $W(z)$ as a sum over a Γ-homogeneous lattice Λ_Γ:

$$
\begin{aligned}
log \quad ||W(z)||^2 &= log \left[|W(z)|^2 e^{G(z,\bar{z})} \right] = \\
&= \left[- \sum_{(M_\Sigma, N^\Sigma) \in \Lambda_\Gamma} log \frac{|M_\Sigma X^\Sigma + i N^\Sigma \partial_\Sigma F|^2}{X^\Sigma \bar{\partial}_\Sigma \bar{F} + \bar{X}^\Sigma \partial_\Sigma F} \right]_{reg} ,
\end{aligned}
\tag{2.40}
$$

where a suitable regularization of the infinite sum is understood. From (2.40) we see that in order to obtain an explicit evaluation of the automorphic superpotential we need two informations:

i) An explicit form for the homogeneous function $F(X)$

ii) An explicit embedding of the homogeneous Γ lattice Λ_Γ into the homogeneous $Sp(2+2n,Z)$ lattice $\Lambda_{Sp(2+2n,Z)}$ which corresponds to an unrestricted sum over all the integers M_Σ, N^Σ

The embedding $\Lambda_\Gamma \subset \Lambda_{Sp(2+2n,Z)}$ is clearly determined once we know the embedding $\Gamma \subset Sp(2+2n,Z)$. A crucial observation is the following. when the special variety $\mathcal{M} = \frac{\mathcal{M}'}{\Gamma}$ is an orbifold of a special Kähler coset manifold $\mathcal{M}' = G/H$ with respect to the action of a discrete subgroup $\Gamma \subset G$ of the isometry group, both problems (i) and (ii) can be solved in one stroke.

Indeed the function F(X) is completely determined by the embedding of G into $Sp(2+2n,R)$ and of H into the maximal compact subgroup $U(n+1) \subset Sp(2+2n,R)$. This is just the general construction of continuous duality transformations according to [39, 1]. Different choices of F(X) correspond to different embeddings. Furthermore the embedding of the modular group Γ into $Sp(2+2n,Z)$ is obviously realized by restricting from R to Z.

2.3 The case of $SK(n+1) = \frac{SU(1,1)}{U(1)} \times \frac{SO(2,n)}{SO(2)\times SO(n)}$

As already mentioned the list of homogeneous symmetric special Kähler manifolds was found in [20]. It includes the series $SK(n+1)$ defined in equation (2.3), and the series:

$$CP_{n-1,1} = \frac{SU(1,n)}{SU(n)\times U(1)} , \tag{2.41}$$

corresponding to the so called minimal coupling. In addition we have the four sporadic cases

$$\frac{Sp(6,R)}{U(3)}; \quad \frac{SU(3,3)}{SU(3)\times U(3)}; \quad \frac{SO^*(12)}{U(6)}; \quad \frac{E_{7(-26)}}{E_6\times U(1)} . \tag{2.42}$$

The F-functions for these manifolds obtained in [3, 20] are all of the following form

$$F(X) = i d_{\Lambda\Sigma\Gamma} \frac{X^\Lambda X^\Sigma X^\Gamma}{X^0} , \tag{2.43}$$

where $d_{\Lambda\Sigma\Gamma}$ are constant coefficients. In the $SK(n+1)$ case, we have:

$$d_{\Lambda\Sigma\Gamma} = d_{1ij} = -\tfrac{1}{2}\eta_{ij} \quad ; \quad 0 \ \text{ otherwise} , \tag{2.44}$$

where the index Λ range is as follows

$$\Lambda = 0, I; \quad I = 1, i; \quad i = 2, \cdots n+1 \tag{2.45}$$

and

$$\eta_{ij} = diag\{+,-,-,\cdots,-\} . \tag{2.46}$$

From equation (2.46) one observes that F is invariant under an $SO(n-1,1)$ group and not under an $SO(n)$ group. This means that the holomorphic symplectic section $(X, i\partial F)$ does not transform linearly under $SO(n)$. Comparing with the results of [22] one concludes that if $SK(n+1)$ is utilized as the vector multiplet manifold $\mathcal{VM}$ for an $N=2$ supergravity model, one can not gauge an n–dimensional semisimple group. Apparently one always needs at least two spectator multiplets: one is sitting in $SU(1,1)/U(1)$ and the other in $SO(2,n)/SO(n)\times SO(2)$. Indeed the condition to gauge an $n+1$–dimensional group G of isometries of $\mathcal{VM}$ is :

$$l_\Lambda X^\Gamma = -f^\Gamma_{\Lambda\Sigma} X^\Sigma , \tag{2.47}$$

where l_Λ is the Lie derivative generating an infinitesimal G–isometry, and $f^\Gamma_{\Lambda\Sigma}$ are the G structure constants.

On the other hand, from string theory, one knows that there are $N=2$ models based on $\mathcal{VM} = SK(n+1)$ where an n–dimensional semisimple group is gauged, the only spectator being the vector multiplet that contains the dilaton and the axion ($SU(1,1)/U(1)$ factor). For instance in the free fermion construction of superstring vacua, one obtains $N = 2$ models by means of a Z_2 projection on $N = 4$ models such that the massless vector multiplets are all untwisted states. This implies that the corresponding $N = 2$ manifold $\mathcal{VM}$ is obtained by disintegration of the known $N = 4$ scalar manifold $M^{N=4} = SO(6, dimG^{N=4})/SO(6) \times SO(dimG^{N=4})$ and one finds $\mathcal{VM} = SK(dimG^{N=2} + 1)$ [69, 70].

If $G^{N=2}$ is semisimple we are in trouble. This trouble occurs in hundreds of examples: one example is described in detail in the appendix of [82], where $n = 418 = 3 + 3 + 3 + 28 + 133 + 248$ and $G^{N=2} = SU(2)^3 \times SO(8) \times E_7 \times E_8'$.

What we have described is the apparent paradox mentioned at the beginning of this chapter. Its obvious solution is that there must be a different description of the special structure of $SK(n+1)$ in terms of a different section $(X, i\partial F)$ such that the $SO(n)$ transformations are linearly realized.

In what follows we derive this new section, constructing the corresponding embedding of $SU(1,1) \times SO(2,n)$ into $Sp(2n+4, R)$. we show that the transformation mapping the old into the new section is symplectic. Our construction allows a derivation of an explicit formula for automorphic superpotentials under the modular group $\Gamma = SO(2,n,Z) \times SL(2,Z)$ which in the case $n = 2$ reproduces the result of [42] for the 3–torus.

2.3.1 The symplectic embedding of $SK(n+1)$

The derivation we present here is related to the Gaillard–Zumino construction of lagrangians possessing duality rotations on the vector fields. Let us recall some crucial points of this construction [39].

Consider lagrangian densities of the following form:

$$\mathcal{L} = -4Re(\mathcal{N}_{\Lambda\Sigma}(\phi))F^{\Lambda}_{ab}F^{\Sigma ab} - 2iIm(\mathcal{N}_{\Lambda\Sigma}(\phi))F^{\Lambda}_{ab}F^{\Sigma}_{cd}\epsilon^{abcd} + g_{\alpha\beta}\partial_\mu\phi^\alpha\partial_\mu\phi^\beta \ . \tag{2.48}$$

They describe a system of $n+1$ vector fields A^{Λ}_a ($\Lambda = 0, \cdots, n$) and 2m scalar fields ϕ_α ($\alpha = 1, ..2m$). They can be obtained from the general formula of the $N = 2$ Lagrangian [22] by setting to zero the gauge coupling(s), by defining $z^i = \phi^i + i\phi^{m+i}$ ($i = 1....m$), and by deleting the gravitational and hypermultiplet sectors. $\mathcal{N}_{\Lambda\Sigma}$ is a $(n+1)\times(n+1)$ symmetric matrix functionally depending on the coordinate of the manifold, namely on the scalar fields. In the case of an $N = 2$ supergravity $\mathcal{N}_{\Lambda\Sigma}$ is given by

$$\mathcal{N}_{\Lambda\Sigma} = -\bar{F}_{\Lambda\Sigma} + \frac{1}{N_{\Gamma\Delta}L^\Gamma L^\Delta} N_{\Lambda\Pi}L^\Pi N_{\Sigma\Delta}L^\Delta \ . \tag{2.49}$$

If we assume that the scalar manifold is a coset $\frac{G}{H}$ and that $g_{\alpha\beta}$ is its own invariant metric, then the scalar sector of the lagrangian is invariant under G–isometries. Howewer in the case of (2.48) there is something more than the G–isometries. According to the results of [39, 1] the isometry group G of the scalar σ–model acts on the $n+1$–vector fields as a group of duality transformations. This means the following:

i)There is an inclusion mapping

$$G \rightarrow Usp(n+1,n+1) = U(n+1,n+1) \cap Sp(2n+2,C) \ , \tag{2.50}$$

$$H \rightarrow U(n+1) \ , \tag{2.51}$$

which to each element $g \in G$ associates a $(2n+2)\times(2n+2)$ complex matrix $S(g)$ with the following block structure :

$$S(g) = \begin{pmatrix} \psi_0(g) & \psi_1^*(g) \\ \psi_1(g) & \psi_0^*(g) \end{pmatrix} \ , \tag{2.52}$$

where the $(n+1)\times(n+1)$ matrices ψ_o, ψ_1 fulfill the conditions:

$$\psi_0^\dagger\psi_0 - \psi_1^\dagger\psi_1 \ = \mathbf{1} \ , \tag{2.53}$$

$$\psi_0^\dagger\psi_1^* - \psi_1^\dagger\psi_0^* \ = 0 \ . \tag{2.54}$$

Equations (2.53) and (2.54) guarantee that $S(g)$ is simultaneously pseudo–unitary and symplectic, as required by (2.50).

ii) The equation of motion and Bianchi identities of the n+1–vector field A_a^Λ can be written as a system of $2n+2$–equations of the following type:

$$\begin{aligned} \partial^a F_{ab}^+ &= 0 \ , \\ \frac{1}{2}\partial_a \frac{\partial \mathcal{L}}{\partial F_{ab}^+} &= 0 \ , \end{aligned} \tag{2.55}$$

where F_{ab}^+ denotes the self dual part of F_{ab}. Analogous equations hold for the antiself dual part. Equations (2.55) are left invariant by the transformations of G defined through the action of the matrix $S(g)$.

Given a parametrization $L(\phi)$ of the scalar coset manifold $\frac{G}{H}$, we can construct immediately $S(L(\phi))$. The lagrangian, which yields equations of motion with the invariance properties (2.55) has, in terms of $\psi_0(L(\phi)) = \psi_0$ and $\psi_1(L(\phi)) = \psi_1$, a universal structure. Indeed one must have:

$$\mathcal{L} = -4(F_{\Lambda ab}^+ F_{\Sigma ab}^+ \mathcal{N}^{\Lambda\Sigma} + c.c) \ , \tag{2.56}$$

where $\mathcal{N}^{\Lambda\Sigma} = Re(\mathcal{N}^{\Lambda\Sigma}) + iIm(\mathcal{N}^{\lambda\Sigma})$ and

$$-4\mathcal{N} = [\psi_o^\dagger + \psi_1^\dagger]^{-1}[\psi_0^\dagger - \psi_1^\dagger] \ . \tag{2.57}$$

In all this construction a crucial role is played by the embedding of $\frac{G}{H}$ into $Usp(n+1,n+1)$, which defines the right structure of duality rotations in connection with G–isometries.

In the same way we are going to show that the right duality transformations on the section $(X, i\partial F)$ for the coset $SK(n+1)$ are induced by the G–isometries in the appropriate $Sp(2(n+2),R)$ embedding.

We start by recalling some basic definitions and introducing our notation.

A general matrix of $Sp(2(n+2),R)$ can be written in the following block structure (each block is a $(n+2)\times(n+2)$ matrix):

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \tag{2.58}$$

and it is defined by the condition

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \tag{2.59}$$

which implies $\mathcal{A}^T\mathcal{C} - \mathcal{C}^T\mathcal{A} = 0$; $\mathcal{B}^T\mathcal{D} - \mathcal{D}^T\mathcal{B} = 0$; $\mathcal{A}^T\mathcal{D} - \mathcal{C}^T\mathcal{B} = \mathbf{1}$

The manifold $\frac{SO(2,n)}{SO(n)\times SO(2)}$ can be described by the following equation in $CP(1,n)$:

$$\eta_{\Lambda\Sigma}Y^\Lambda Y^\Sigma = 0 \quad , \quad \eta_{\Lambda\Sigma}Y^\Lambda \bar{Y}^\Sigma = \frac{1}{2} , \tag{2.60}$$

where we set $\Lambda = 0,1,\alpha$; $\alpha = 2,\cdots n+1$; $\eta_{\Lambda\Sigma} = (+,+,-,\cdots,-)$. The constraints (2.60) are easily solved by choosing the so called Calabi–Visentini parametrization, namely:

$$\begin{aligned} Y^0 &= \frac{1}{2}(1+y^\alpha y^\alpha)/J_1^{\frac{1}{2}} , \\ Y^1 &= \frac{i}{2}(1-y^\alpha y^\alpha)/J_1^{\frac{1}{2}} , \\ Y^\alpha &= y^\alpha/J_1^{\frac{1}{2}} , \\ J_1 &= (1 - 2y^\alpha\bar{y}^\alpha + y^\alpha y^\alpha \bar{y}^\beta\bar{y}^\beta) . \end{aligned} \tag{2.61}$$

Finally $SU(1,1)/U(1)$ is parametrized by choosing two complex numbers ϕ_0, ϕ_1 such that

$$|\phi_0|^2 - |\phi_1|^2 = \frac{1}{2} . \tag{2.62}$$

In particular equation (2.62) is automatically satisfied if we choose

$$\phi_0 = -\frac{D+i}{2J_2^{\frac{1}{2}}} ,$$

$$\begin{aligned} \phi_1 &= -\frac{D-i}{2J_2^{\frac{1}{2}}} , \\ J_2 &= -i(D-\bar{D}) , \end{aligned} \tag{2.63}$$

where $D = iS$ is the complex field containing the dilaton and the axion fields. As already stated, in the old parametrization, the special structure of the manifold $SK(n+1)$ was realized by an F–function of type (2.43) with $d_{\Lambda\Sigma\Gamma}$ as in (2.44), where $X^0 = 1$, $X^1 = D$, $X^i = z^i$ (i runs from $2 \cdots n+1$ with lorentzian metric $(+,-,\cdots,-)$). The special coordinates were defined as follows

$$\begin{aligned} z^1 &= \frac{X^1}{X^0} = D , \\ z^i &= \frac{X^i}{X^0} \end{aligned} \tag{2.64}$$

and the Kähler potential was given by:

$$G(D,\bar{D},z,\bar{z}) = -log[\frac{i}{2}(D-\bar{D}))(z^i-\bar{z}^i)\eta_{ij}(z^j-\bar{z}^j)] . \tag{2.65}$$

The z–coordinates are an alternative parametrization of the manifold $\frac{SO(2,n)}{SO(n)\times SO(2)}$ and they are related to the Calabi–Visentini frame through an appropriate holomorphic coordinate transformation [71], which, as we will see, is actually induced by a symplectic transformation . In terms of the Calabi–Visentini variables, G is given by:

$$G(D,\bar{D},y,\bar{y}) = -log[\frac{i}{2}(\bar{D}-D)(1-2y^\alpha\bar{y}^\alpha + y^\alpha y^\alpha\bar{y}^\alpha\bar{y}^\alpha)] . \tag{2.66}$$

Obviously the special coordinates (2.64) are not unique; they depend on the choice of the section we consider. This means that there is a different choice of special coordinates which as well as the z^i's reveal the special structure of $SK(n+1)$. Let us construct the new coordinates (or equivalently the new $Sp(2(n+2),R)$ section). We consider a general $SO(2,n)$ matrix :

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} , \tag{2.67}$$

where A is a 2×2 matrix; B a $2\times n$; C a $n\times 2$ and D a $n\times n$ matrix. A,B,C,D satisfy the following conditions:

$$\begin{aligned} A^TA - C^TC &= \mathbf{1}_{2\times 2} , \\ A^TB - C^TD &= 0 , \\ B^TB - D^TD &= -\mathbf{1}_{n\times n} . \end{aligned} \tag{2.68}$$

As can be easily checked the $2(n+2) \times 2(n+2)$ matrix

$$A_1 = \begin{pmatrix} A & 0_{2\times n} & 0_{2\times 2} & -B \\ 0_{n\times 2} & D & -C & 0_{n\times n} \\ 0_{2\times 2} & -B & A & 0_{2\times n} \\ -C & 0_{n\times n} & 0_{n\times 2} & D \end{pmatrix} \tag{2.69}$$

is in $Sp(2(n+2),R)$, and it is a good candidate for the embedding we search. Moreover let us take the $SL(2,R) \sim SU(1,1)$ matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} , \tag{2.70}$$

where $ad - bc = 1$. SL(2,R) can be easily embedded into $Sp(2(n+2),R)$ by writing (in the same block structure notation):

$$A_2 = \begin{pmatrix} a\mathbf{1}_2 & 0 & b\mathbf{1}_2 & 0 \\ 0 & d\mathbf{1}_n & 0 & c\mathbf{1}_n \\ c\mathbf{1}_2 & 0 & d\mathbf{1}_2 & 0 \\ 0 & b\mathbf{1}_n & 0 & a\mathbf{1}_n \end{pmatrix} \tag{2.71}$$

with $\mathbf{1}_2 = \mathbf{1}_{2\times 2}, \mathbf{1}_n = \mathbf{1}_{n\times n}$. One can check that the two $Sp(2(n+2),R)$ matrices A_1, A_2 commute, and that they close the $SO(2,n) \times SL(2,R)$ algebra. Let us introduce the shorthand notation $\tilde{Y} \equiv (Y^0 J_1^{\frac{1}{2}}, Y^1 J_1^{\frac{1}{2}})$, $\hat{Y} = \{Y^\alpha J_1^{\frac{1}{2}}\}$ for the $\frac{SO(2,n)}{SO(n)\times SO(2)}$ variables (suitably rescaled). Under the action of an $SO(2,n)$ isometry $(\tilde{Y},\hat{Y})$ transforms as a vector. Analogously $(\phi_0 J_2^{\frac{1}{2}}, \phi_1 J_2^{\frac{1}{2}})$ transform as a vector under the action of an $SU(1,1)$ matrix. Equivalently $(\eta_0 = (\phi_0 + \phi_1) J_2^{\frac{1}{2}}, \eta_1 = i(\phi_0 - \phi_1) J_2^{\frac{1}{2}})$ transform under the action of the SL(2,R) matrix (2.70). If we impose that the section $(X, i\partial F)$ transforms as a vector under the particular $Sp(2(n+2),R)$ transformation given by the product $A_1 A_2$, we find the following relations:

$$\begin{aligned} \tilde{X} &= \tilde{Y}\eta_0 , \\ \hat{X} &= -\hat{Y}\eta_1 , \\ i\tilde{\partial F} &= \tilde{Y}\eta_1 , \\ i\hat{\partial F} &= -\hat{Y}\eta_0 . \end{aligned} \tag{2.72}$$

If we use the parametrizations (2.61) and (2.63) (with rescaled variables) we get:

$$\begin{aligned} X^0 &= -\frac{D}{2}(1+y^2) , \\ X^1 &= -i\frac{D}{2}(1-y^2) , \\ X^\alpha &= -y^\alpha , \end{aligned}$$

$$\begin{aligned} i\partial_0 F &= \frac{1}{2}(1+y^2) , \\ i\partial_1 F &= \frac{i}{2}(1-y^2) , \\ i\partial_\alpha F &= D y^\alpha , \end{aligned} \tag{2.73}$$

where $y^2 = y^\alpha y^\alpha$. The relations (2.72), (2.73) are actually a set of differential equations for F. To see this it suffices to utilize the constraints (2.60) in eq. (2.72). The solution of this system of equations is:

$$F(X) = i\sqrt{(X^0X^0 + X^1X^1)X^\alpha X^\alpha} . \tag{2.74}$$

It is an easy algebraic calculation to verify that the function F(X) in (2.74) gives the right Kähler potential. A different (but equivalent) solution can be obtained by interchanging the role of X and $i\partial F$, that is:

$$\begin{aligned} X^0 &= \frac{1}{2}(1+y^2) , \\ X^1 &= \frac{i}{2}(1-y^2) , \\ X^\alpha &= D y^\alpha , \\ i\partial_0 F &= -\frac{D}{2}(1+y^2) , \\ i\partial_1 F &= -i\frac{D}{2}(1-y^2) , \\ i\partial_\alpha F &= -y^\alpha . \end{aligned} \tag{2.75}$$

It is now a standard matter to introduce the new special coordinates using for instance (2.75). We obtain:

$$\begin{aligned} \pi^1 &= \frac{X^1}{X^0} = i\frac{(1-y^2)}{1+y^2} , \\ \pi^\alpha &= \frac{X^\alpha}{X^0} = 2D\frac{y^\alpha}{1+y^2} \end{aligned} \tag{2.76}$$

and the new holomorphic prepotential:

$$f(\pi) = i\sqrt{(1+\pi^1\pi^1)(\pi^\alpha\pi^\alpha)} . \tag{2.77}$$

In terms of the variables π the Kähler potential is:

$$\begin{aligned} G(\pi,\bar\pi) &= -log\{i[\sqrt{(1+\pi^1\pi^1)(\pi^\alpha\pi^\alpha)} - c.c.] - i[\pi^1\sqrt{\frac{\pi^\alpha\pi^\alpha}{1+\pi^1\pi^1}} + c.c.][\pi^1 - \bar\pi^1] \\ &\quad + i\pi^\alpha\bar\pi^\alpha[\sqrt{\frac{1+\pi^1\pi^1}{\pi^\alpha\pi^\alpha}} - c.c.]\} . \end{aligned} \tag{2.78}$$

A lengthy but straightforward calculation shows that (2.78), once expressed in terms of the old special coordinates has the form (2.65) (modulo the logarithm of a real function). In the same way, by defining [4, 21, 22]:

$$c_{IJK} = e^{G(\pi,\bar{\pi})}\partial_I\partial_J\partial_K f(\pi) \tag{2.79}$$

one can obtain the Yukawa couplings $w_{IJK} = \partial_I\partial_J\partial_K f(\pi)$, which transform into the old one (modulo a phase), once we use the right tensor property transformations of c_{IJK}. Explicitly one has:

$$\begin{aligned}
c_{111} &= -3ie^{G(\pi,\bar{\pi})}\frac{\pi^1\sqrt{\pi^\alpha\pi^\alpha}}{(1+\pi^1\pi^1)^{\frac{5}{2}}} , &(2.80)\\
c_{\alpha\beta\gamma} &= 3ie^{G(\pi,\bar{\pi})}\frac{\sqrt{1+\pi^1\pi^1}}{(\pi^\delta\pi^\delta)^{\frac{5}{2}}}[\pi^{(\alpha}\pi^\beta\pi^{\gamma)} - \delta^{(\alpha\beta}\pi^{\gamma)}\pi^\delta\pi^\delta] , &(2.81)\\
c_{1\alpha\beta} &= e^{G(\pi,\bar{\pi})}\frac{i\pi^1}{(\pi^\delta\pi^\delta)^{\frac{3}{2}}(1+\pi^1\pi^1)^{\frac{1}{2}}}[\delta^{\alpha\beta}\pi^\delta\pi^\delta - \pi^\alpha\pi^\beta] , &(2.82)\\
c_{11\alpha} &= e^{G(\pi,\bar{\pi})}\frac{i\pi^\alpha}{(\pi^\delta\pi^\delta)^{\frac{1}{2}}(1+\pi^1\pi^1)^{\frac{3}{2}}} . &(2.83)
\end{aligned}$$

The choice of the Calabi–Visentini parametrization for $\tilde{Y}, \hat{Y}$ is of course arbitrary. One can use any $\frac{SO(2,n)}{SO(n)\times SO(2)}$ parametrization. Let us take:

$$\begin{aligned}
Y^0 &= -\frac{1}{2}(1-\eta_{ij}z^iz^j) ,\\
Y^1 &= z^2 ,\\
Y^{\tilde{\alpha}} &= z^{\tilde{\alpha}+1} ,\\
Y^{n+1} &= \frac{1}{2}(1+\eta_{ij}z^iz^j)
\end{aligned} \tag{2.84}$$

where we set: $\tilde{\alpha} = 2, 3, \cdots, n$; $i = 2, \cdots, n+1$. we concentrate, to fix ideas on the case $n = 2$, the general case being an obvious generalization. In these coordinates the section $(X, i\partial F)$ is expressed by the formula:

$$\begin{aligned}
X^0 &= \frac{D}{2}(1-\eta_{ij}z^iz^j) \quad , \quad X^1 = -Dz^2 ,\\
X^2 &= -z^3 \quad , \quad X^3 = -\frac{1}{2}(1+\eta_{ij}z^iz^j) ,\\
i\partial_0 F &= -\frac{1}{2}(1-\eta_{ij}z^iz^j) \quad , \quad i\partial_1 F = z^2 ,\\
i\partial_2 F &= Dz^3 \quad , \quad i\partial_3 F = \frac{D}{2}(1+\eta_{ij}z^iz^j) ,
\end{aligned} \tag{2.85}$$

which is again solved by the square–root function (2.74). The Kähler potential coming from (2.85) is precisely (2.65). we know that the same Kähler potential is given by the old special section $(X', \partial F')$:

$$\begin{aligned} X^{0'} &= 1 \quad , & X^{1'} = D\ , \quad X^{2'} = z^2\ , \quad X^{3'} = z^3\ , \\ i\partial_0 F' &= -\frac{D}{2}\eta_{ij}z^i z^j \quad , & i\partial_1 F' = \frac{\eta_{ij}z^i z^j}{2}\ , \\ i\partial_2 F' &= Dz^2 \quad , & i\partial_3 F' = -Dz^3\ . \end{aligned} \tag{2.86}$$

In the spirit of special geometry we expect that the two sections should be related by a $Sp(2(n+2), R)$ transformation. This is precisely what happens, since the matrix:

$$\mathcal{U} = \begin{pmatrix} 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & -1/2 & 1/2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{2.87}$$

is symplectic and transforms the new section into the old one: $(X', i\partial F') = \mathcal{U}(X, i\partial F)$

2.3.2 Automorphic function for $SK(n+1)$

The construction of the previous section can be utilized also to define the automorphic function (2.40) for the manifold $SK(n+1)$. In this case the modular group is well known [41, 40], and it is given by $SO(2,n,Z) \times PSL(2,Z)$. The only so far undefined point is how to restrict the sum over the integers M_Λ, N^Λ appearing in (2.40), which define $Sp(2(n+2), Z)$ orbits, to unrestricted integers, defining orbits of the true modular group $SO(2,n,Z) \times PSL(2,Z) \subset Sp(2(n+2), Z)$.

We consider, to perform this restriction, the conjugate transformation of (M_Λ, N^Λ) under $Sp(2(n+2), Z)$:

$$\begin{pmatrix} M' \\ N' \end{pmatrix} = \begin{pmatrix} \mathcal{D} & -\mathcal{C} \\ -\mathcal{B} & \mathcal{A} \end{pmatrix} \begin{pmatrix} M \\ N \end{pmatrix} \tag{2.88}$$

in such a way the expression (2.40) is invariant under symplectic transformations. If we denote by m^0, m^1, m^α the integers transforming as a vector under the (conjugate) $SO(2,n,Z)$ matrix; s_0, s_1 those transforming under $SL(2,Z)$ and if we use the explicit form of the matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$, we get (with $\tilde{m} = (m_0, m_1)$, $\hat{m} = \{m_\alpha\}$):

$$\begin{aligned} \tilde{M} &= \tilde{m} s_0\ , \\ \hat{M} &= -\hat{m} s_1\ , \\ \tilde{N} &= \tilde{m} s_1\ , \\ \hat{N} &= -\hat{m} s_0\ . \end{aligned} \tag{2.89}$$

Equation (2.89) gives the explicit dependence of the capital integers M, N in terms of the lower case m, s, and the rule to restrict the sum in (2.40) to the appropriate SK(n+1) modular group orbits. In this way one can make explicit the general formula (2.40) for the automorphic superpotential W. In terms of the symplectic section $\Omega = (X, i\partial F)$ eq. (2.40) can be reinterpreted as follows:

$$log \parallel W \parallel^2 = log\,|W|^2 e^G \;=\; -\sum_{L\in\Lambda_\Gamma} log\, i\,\frac{|< L|\Omega >|^2}{<\bar{\Omega}|\Omega>}\,, \tag{2.90}$$

where $L = (N, -M)$ is a vector in Λ_Γ. In our case we have $L = (N, -M)$ given by eq.(2.89), and the summation on Λ_Γ corresponds to a summation over unrestricted integers (s_0, s_1) (spanning a lattice of $PSL(2, Z)$ orbits) and over integers (m_0, m_1, m_α) such that:

$$m_0^2 + m_1^2 - m_\alpha m_\alpha = 0\ . \tag{2.91}$$

Eq (2.91) clearly defines a lattice of $SO(2, n, Z)$ orbits and it is immediately recognizable as the level matching condition, equating the the left and the right masses in the $\Gamma_{2,n}$ Narain Lattice.

What is still missing to make (2.90) well defined is the specification of a regularization procedure. In general this can be done via a ζ–function regularization scheme:

$$\begin{aligned} log\,|W|^2 e^G &= -\lim_{s\to 0}\frac{d}{ds}\zeta(s) \\ \zeta(s) &= \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\sum_{L\in\Lambda_\Gamma} exp - i\, t\frac{|< L|\Omega >|^2}{<\Omega|\Omega>}\,. \end{aligned} \tag{2.92}$$

This procedure always provides a correct definition for the infinite sum appearing in the right–hand side of equation (2.90). What is not obvious, in the general case, is how to extract from (2.92) the squared modulus of a holomorphic function WW^* modulo the holomorphic anomaly $-i <\bar{\Omega}|\Omega>$. However, in some specific cases one can verify this holomorphic factorization explicitly. This is for instance the case of $n = 2$ case, where $SK(3) \sim (SU(1,1)/U(1))^3$. There the automorphic function is given by [42]:

$$log \parallel W(D_i, \bar{D}_i) \parallel^2 = -\sum_{i=1}^{3} log[\,|\eta(-iD_i)|^4 i(\bar{D}_i - D_i)]\,, \tag{2.93}$$

where η is the Dedekind function, and D_1, D_2, D_3 are the complex fields parametrizing the three $SU(1,1)/U(1)$ manifolds.

The connection between our result and the one found in [42] is retrieved once we know the embedding (actually the isomorphism) of $(SU(1,1)/U(1))^2$ into $SO(2,2)/SO(2) \times SO(2)$ which is explicitly realized by the choice:

$$Y^0 \;=\; -D_2 - D_3\,,$$

$$\begin{aligned} Y^1 &= (1 - D_2 D_3) , \\ Y^2 &= (-1 - D_2 D_3) , \\ Y^3 &= (D_3 - D_2) , \end{aligned} \tag{2.94}$$

where the Y are the (rescaled) SO(2,2) coordinates which satisfies:

$$\begin{aligned} \eta_{IJ} Y^I Y^J &= 0 , \\ \eta_{IJ} Y^I \bar{Y}^J &= -2(D_2 - \bar{D}_2)(D_3 - \bar{D}_3) . \end{aligned} \tag{2.95}$$

All the other steps are just algebraic. We construct the section $(X, i\partial F)$ following the above procedure. We get:

$$\begin{aligned} X^0 &= D_1(D_2 + D_3) , \\ X^1 &= -D_1(1 - D_2 D_3) . \\ X^2 &= (1 + D_2 D_3) , \\ X^3 &= (D_2 - D_3) , \\ i\partial_0 F &= -D_2 - D_3 , \\ i\partial_1 F &= (1 - D_2 D_3) , \\ i\partial_2 F &= -D_1(1 + D_2 D_3) , \\ i\partial_3 F &= (D_3 - D_2) D_1 \end{aligned} \tag{2.96}$$

Then we write down the explicit dependence of M, N on $r_0, r_1; t_0, t_1; s_0, s_1$ for the three SU(1,1)/U(1) factors, that is formula (2.89) with $\hat{m}, \tilde{m}$ given by (see eq. (2.94)):

$$\begin{aligned} \tilde{m} &= (r_1 t_0 + r_0 t_1, r_1 t_1 - r_0 t_0) , \\ \hat{m} &= (-r_0 t_0 - r_1 t_1, r_0 t_1 - r_1 t_0) . \end{aligned} \tag{2.97}$$

Note that eq (2.97) parametrize the four integers m_0, m_1, m_2, m_3, obeying the constraint (2.91), in terms of the four integers r_0, r_1, t_0, t_1 in the orbit of $[SL(2,Z)]^2$. By writing down the sum:

$$\sum_{(s_0,s_1,r_0,r_1,t_0,t_1)} -log \left(\frac{|M_\Sigma(r,s,t) X^\Sigma + i N^\Sigma(r,s,t) \partial_\Sigma F|^2}{X \bar{\partial} \bar{F} + \bar{X} \partial F} \right) \tag{2.98}$$

and comparing with:

$$\sum_{(s_0,s_1,r_0,r_1,t_0,t_1)} -log \left(\frac{|(s_1 - s_0 D_1)(r_1 - r_0 D_2)(t_1 - t_0 D_3)|^2}{i(D_1 - \bar{D}_1)(D_2 - \bar{D}_2)(D_3 - \bar{D}_3)} \right) \tag{2.99}$$

we get the same result (modulo logarithm of a constant). This shows that the final form of the automorphic function is exactly the same as in [42] (with the only change $-iD_i = T_i$), in terms of the same regularization procedure.

2.4 The case of $\mathcal{M}_{3,3} = \frac{SU(3,3)}{SU(3)\times SU(3)\times U(1)}$

As we know from the previous discussions the list of homogeneous symmetric special Kähler manifolds $\mathcal{M}$ contains in particular the manifold:

$$\mathcal{M}_{3,3} = \frac{SU(3,3)}{SU(3)\times SU(3)\times U(1)}. \tag{2.100}$$

In this section we extend the symplectic embedding technique to the case of the manifold $\mathcal{M}_{3,3}$, that is the Teichmuller covering of the moduli space for the T^6/Z_3 orbifold. In this way we exhaust the analysis of automorphic superpotentials for the Z orbifold compactification of superstrings. Indeed, comparing with the list appearing in eq: (1.47) we see that, with the exception of $\mathcal{M}_{3,3}$, all the other (untwisted) orbifold moduli spaces correspond to special values of n in the $SK(n+1)$ series (we have just to remember the isomorphism between the manifolds $\frac{SU(2,2)}{SU(2)\times U(2)}$ and $\frac{SO(2,4)}{SO(2)\times SO(4)}$).

Our goal is to exhibit the special geometry of $\mathcal{M}_{3.3}$ and to construct the appropriate infinite sum defining the $SU(3,3,Z)$ automorphic superpotential.

As for any other special manifold $\mathcal{M}$, the special geometry of $\mathcal{M}_{3,3}$ is encoded in a homogeneous of degree two holomorphic function $F(X)$. The F function proposed in ref. [3], via supergravity considerations, for the case of $\mathcal{M}_{3.3}$ is of the form:

$$F(X) \sim i\frac{\det X}{X^0}, \tag{2.101}$$

where X is a three by three matrix. Equation (2.101) is just a particular case of the general formula $F(X) = id_{\Lambda\Sigma\Delta}\frac{X^\Lambda X^\Sigma X^\Delta}{X^0}$, where the $d_{\Lambda\Sigma\Delta}$ are constant coefficients, valid for any $\mathcal{M}$ in the list of homogeneous symmetric special Kähler manifolds, except $CP_{n-1,1}$.

As pointed out in the previous sections one should be able to derive systematically the F function from the embedding of $\mathcal{M}$ into $Sp(2\,dim\mathcal{M}+2,R)$. Here we show that we can get, in a rigorous way, a symplectic section Ω corresponding to the F function (2.101).

Let Γ denote the (target) space modular group of $\mathcal{M}$. Using the general formula (2.40), we are able to construct the Γ automorphic function (the superpotential) for our case, writing it as a sum over integers describing a modular lattice.

2.4.1 Construction of the section Ω for $\mathcal{M}_{3,3}$

We start our programme by writing the coset representative of the manifold $\mathcal{M}_{3,3} = \frac{G}{H}$ in projective coordinates [1]:

$$M = \begin{pmatrix} (1-ZZ^\dagger)^{-\frac{1}{2}} & (1-ZZ^\dagger)^{-\frac{1}{2}}Z \\ Z^\dagger(1-ZZ^\dagger)^{-\frac{1}{2}} & 1+Z^\dagger(1-ZZ^\dagger)^{-1}Z \end{pmatrix} \tag{2.102}$$

where Z is a complex 3×3 matrix. Let us denote by A the 6×3 matrix given by

$$A = ((1-ZZ^\dagger)^{-\frac{1}{2}}, (1-ZZ^\dagger)^{-\frac{1}{2}}Z)\,, \tag{2.103}$$

where the indices of A^I_i run as follows: $I = (i, i^*)$; $i, i^* = 1,2,3$ (i corresponds to the plus signs of the metric and i^* to the minus signs). Following the general procedure discussed in [43, 1] we have to embed G into the symplectic group of dimension $(9+1)\times 2 = 20$. If we consider the isometry group $G = SU(3,3)$, it is easily recognized that the three–index antisymmetric representation of G has the required dimension. Hence let us define:

$$t^{IJK} = \epsilon^{ijk} A^I_i A^J_j A^K_k\,. \tag{2.104}$$

The three index antisymmetric tensor t^{IJK} is acted on by the matrix $\mathcal{B} = \mathcal{U}^{[I'}_I \mathcal{U}^{J'}_J \mathcal{U}^{K']}_K$, where $\mathcal{U} \in SU(3,3)$. One can verify that:

$$\mathcal{U}^T C \mathcal{U} = C\,, \tag{2.105}$$

where the matrix C satisfies $C^T = -C$, $C^2 = -\mathbf{1}$, and can be viewed as acting on the triplet IJK as the Levi Civita symbol: $[Ct]^{IJL} = \epsilon^{IJKLMN} t_{LMN}$. Moreover one has

$$\mathcal{U}^\dagger E \mathcal{U} = E, \tag{2.106}$$

where $E^2 = 1$, $E^\dagger = E$ and where E acts on the three–index tensors as $E = \eta_{IL'}\eta_{JM'}\eta_{KN'}$ ($\eta = (+,+,+,-,-,-)$), antisymmetrized with respect to L', M', N'.

Equations (2.105) and (2.106) show that t^{IJK} realizes a symplectic embedding of G into a symplectic group of dimension 20. However, due to the signature of the "metric" E this group is $Usp(10,10)$ rather than $Sp(20,R)$. The use of a generalized Cayley matrix allows us to transform the $Usp(10,10)$ representation into the real symplectic one. Explicitly we set:

$$T^{LMN} = \epsilon^{ijk} A^I_i \mathcal{C}^L_I A^J_j \mathcal{C}^M_J A^K_k \mathcal{C}^N_K \equiv \hat{\mathcal{C}}t\,, \tag{2.107}$$

where

$$\mathcal{C} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1_3 & -i1_3 \\ 1_3 & i1_3 \end{pmatrix} \tag{2.108}$$

is the "Cayley matrix" in the 3 + 3 space (i.e. in the fundamental representation of $SU(3,3)$) and $\hat{\mathcal{C}}$ as defined by equation (2.107), is the generalized Cayley matrix in the 20 space (i.e. in the three index antisymmetric representation of $SU(3,3)$). Note that both $\mathcal{C}$ and $\hat{\mathcal{C}}$ are defined up to a phase. The three–index representation is now written as (if we drop an overall $\frac{1}{\sqrt{2}}$ factor)

$$\begin{aligned}
T^{ijk} &= \det\frac{1+Z}{(1-ZZ^\dagger)^{\frac{1}{2}}}\epsilon^{ijk} , \\
T^{i^*ik} &= \epsilon^{ijk}\det\frac{1+Z}{(1-ZZ^\dagger)^{\frac{1}{2}}}\left(\frac{i(Z-1)}{Z+1}\right)_i^{i^*} , \\
T^{ij^*k^*} &= \det\frac{1+Z}{(1-ZZ^\dagger)^{\frac{1}{2}}}\epsilon^{irs}\left(\frac{i(Z-1)}{Z+1}\right)_r^{j^*}\left(\frac{i(Z-1)}{Z+1}\right)_s^{k^*} , \\
T^{i^*j^*k^*} &= \det\frac{1+Z}{(1-ZZ^\dagger)^{\frac{1}{2}}}\det\frac{i(Z-1)}{Z+1} ,
\end{aligned} \tag{2.109}$$

where we adopt the convention: $\det M_{3\times3} = \frac{1}{3!}\epsilon^{ijk}\epsilon_{i^*j^*k^*}M_i^{i^*}M_j^{j^*}M_k^{k^*}$. Equation (2.109) gives the explicit form of the $Sp(20,R)$ section Ω, and each line of such an equation has to be interpreted as X^Λ or as $i\partial_\Lambda F(X)$. If we divide by the overall factor $\det(1+Z)$, which gives, together with its antiholomorphic counterpart, a real function contribution to the Kähler potential, we can read the first 10 components of Ω as the elements of the matrix $X_i^{i^*} = [\frac{i(Z-1)}{Z+1}]_i^{i^*}$ together with $X^0 = 1$.

By recalling the definition $L^\Lambda = e^{\frac{1}{2}G}X^\Lambda$, and remembering that for $\mathcal{M}_{3,3}$ the Kähler potential G is expressed by [1]

$$G(Z,Z^\dagger) = -log\, det(1-ZZ^\dagger) , \tag{2.110}$$

we obtain from (2.109) the following identifications:

$$\begin{aligned}
T^{ijk} &= \epsilon^{ijk}L^0 , \\
T^{i^*jk} &= \epsilon^{ijk}L_i^{i^*} , \\
T^{ij^*k^*} &= \epsilon^{i^*j^*k^*}\frac{\partial}{\partial L_i^{i^*}}\left(-\frac{\det L}{L^0}\right) = \epsilon^{i^*j^*k^*}\frac{\partial}{\partial L_i^{i^*}}(iF(L)) , \\
T^{i^*j^*k^*} &= \epsilon^{i^*j^*k^*}\frac{\partial}{\partial L^0}\left(-\frac{\det L}{L^0}\right) = \epsilon^{i^*j^*k^*}\frac{\partial}{\partial L^0}(iF(L)) .
\end{aligned} \tag{2.111}$$

The last two eq.s (2.111) have to be interpreted as differential equations satisfied by the F function. These are solved by:

$$F(L) = i\frac{\det L}{L^0} \tag{2.112}$$

(the same expression, of course, holds for $F(X)$). If we introduce the "special coordinates" $S_i^{i^*} = L_i^{i^*}/L^0 = X_i^{i^*}/X^0$, we immediately recover the standard expression of the

Kähler potential $G(S,\bar{S}) = -\log\det(S-\bar{S})$, which is explicitly obtained from (2.38) and which coincides with the $-\log\det(1-ZZ^\dagger)$, while posing $S = \frac{i(Z-1)}{Z+1}$ (modulo the real part of a holomorphic function).

Our next step consists in searching an explicit formula for an automorphic superpotential in the case of T^6/Z_3. For sake of clarity we recall the general formula for the superpotential.

$$\log||W||^2 = \log\left[|W|^2 e^G\right] = \left[-\sum_{(M_\Sigma,N^\Sigma)\in\Lambda_\Gamma} \log\frac{|M_\Sigma X^\Sigma + iN^\Sigma\partial_\Sigma F|^2}{X^\Sigma\bar{\partial}_\Sigma\bar{F} + \bar{X}^\Sigma\partial_\Sigma F}\right]_{reg}, \quad (2.113)$$

where the integers (M,N) belong to a homogeneous lattice Λ_Γ associated with the target space modular group $\Gamma \in Sp(20,Z)$, where Γ corresponds to a suitable definition of $SU(3,3,Z)$ (see next section). The only difficult point, to make formula (2.113) explicit, is to find the explicit parametrization of the capital integers (M,N) in terms of the small integers n spanning the Narain lattice for the T^6/Z_3 orbifold. In particular, for our case, we need the formula relating 20 "integers" M^{IJK} in the three-times antisymmetric representation of $Sp(20,R)$ (or equivalently $Usp(10,10)$) to the "integers" l^I in the fundamental 6-dimensional representation of $SU(3,3)$. We write "integers" in quotes because both M^{IJK} and l^I are not integers, rather they are complex numbers parametrized by a double number of integers (real and imaginary parts). The solution of this problem is given by splitting the (complexified) momentum lattice of T^6/Z_3 into three conjugacy classes, and by constructing the symplectic integers in terms of the integers belonging to these classes. Let us give in some detail the analysis of the momentum lattice and its behaviour under the modular group $SU(3,3,Z)$.

2.4.2 The momentum lattice of T^6/Z_3 orbifold

Following a well–established literature we define a T^{2n}/Z_N orbifold via a two–step process [45, 72, 66]. First we introduce a 2n dimensional torus T^{2n} by identifying points in R^{2n} with respect to the action of a lattice group Λ_R:

$$X^\mu \sim X^\mu + v^\mu \quad ; \quad v^\mu \in \Lambda_R \quad (2.114)$$

and we define T^{2n}/Z_N by identifying points in T^{2n} with respect to the action of a point group $\mathcal{P} \sim \mathcal{Z}_N$ that acts cristallographically on the lattice Λ_R and that is a subgroup of $SO(2n)$:

$$(\Theta X)^\mu \sim X^\mu \quad ; \quad \Theta \in SO(2n) \quad (2.115)$$

$$(\Theta v)^\mu \in \Lambda_R \; if \; v^\mu \in \Lambda_R . \quad (2.116)$$

In the case of T^6/Z_3, the standard choice of Λ_R corresponds to $\Lambda_R = RA_2 \otimes RA_2 \otimes RA_2$, where RA_2 is the root lattice of the simply laced Lie algebra A_2 [72]. In this way one easily obtains an $SO(6)$ rotation matrix Θ, which maps Λ_R into itself and such that $\Theta^3 = 1$.

This construction has been discussed in the literature [45, 72] but we need to recall it here. Indeed we have to illustrate the properties of the momentum lattice we shall utilize in the derivation of the $SU(3,3,Z)$ modular group and of the coefficients M^{IJK}. We begin by introducing a complex structure in R^6. This is done by substituting three complex coordinates Z^i to the six real coordinates X^μ via the relation:

$$X^\mu = Z^i e_i^\mu + \bar{Z}^{i^*} e_{i^*}^\mu \tag{2.117}$$

$(i, i^* = 1,2,3)$, where $\{e_i^\mu, e_{i^*}^\mu\}$ is a basis of six complex, linear, independent vectors fulfilling the conditions [73]:

$$\begin{aligned} (e_i^\mu)^* &= e_{i^*}^\mu \\ (e_i, e_j) &= (e_{i^*}, e_{j^*}) = 0 \quad (e_i, e_{j^*}) = g_{ij^*} \, . \end{aligned} \tag{2.118}$$

In (2.118) the scalar product (,) is defined with respect to some constant metric $g_{\mu\nu}$ with (+,+,+,+,+,+) signature:

$$(v, w) = v^\mu w^\nu g_{\mu\nu} \, . \tag{2.119}$$

The Hermitian form:

$$g_{ij^*} = (e_i, e_{j^*}) = g_{j^* i} \tag{2.120}$$

defines a Hermitian metric in R^6 equipped with the complex structure (2.117):

$$g_{\mu\nu} X^\mu X^\nu = 2 Z^i \bar{Z}^{j^*} g_{ij^*} \, . \tag{2.121}$$

The torus T^6 is obtained by setting the following identification of points in R^6 [72]:

$$Z^i = Z^i + (n^i + \Theta m^i)\sqrt{2} \tag{2.122}$$

where n^i, $m^i \in Z$

$$\Theta = e^{2\pi i/3} . \tag{2.123}$$

Equation (2.122) corresponds to the modding by a lattice $\Lambda_R = RA_2 \otimes RA_2 \otimes RA_2$, as claimed at the beginning. Indeed for the algebra A_2 a system of simple roots is given by the two–dimensional vectors:

$$\alpha_1 = (\sqrt{2}, 0) = \sqrt{2} \quad \alpha_2 = \left(-\frac{\sqrt{2}}{2}, \sqrt{\frac{3}{2}} \right) = \sqrt{2} e^{2\pi i/3} , \tag{2.124}$$

so that an element of the root lattice RA_2 can be represented by the following complex number:

$$n\alpha_1 + m\alpha_2 = \sqrt{2}(n + m\Theta) \quad (n, m \in Z)\,. \tag{2.125}$$

The dual–weight lattice WA_2 is spanned by the simple weights:

$$\lambda_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}\right) = \sqrt{\frac{2}{3}} e^{\pi i/6}\,, \tag{2.126}$$

$$\lambda_2 = \left(0, \sqrt{\frac{2}{3}}\right) = \sqrt{\frac{2}{3}} e^{\pi i/2} \tag{2.127}$$

and a generic element of this lattice is represented by the following complex number:

$$p\lambda_1 + q\lambda_2 = \sqrt{\frac{2}{3}}(p\omega_1 + q\omega_2)\,, \tag{2.128}$$

where $\omega_1 = e^{\pi i/6}$ and $\omega_2 = e^{\pi i/2}$.

The metric g_{ij^*} defined by eq. (2.120) enters, together with an antisymmetric two–form B_{ij^*}, the two-dimensional σ model action on the T^6 torus [40, 66]:

$$S = \int d^2\xi \partial_\alpha Z^i \partial^\alpha \bar{Z}^{j^*} (g_{ij^*} + B_{ij^*})\,. \tag{2.129}$$

The nine complex parameters encoded in the complex 3×3 matrix

$$M_{ij^*} = g_{ij^*} + B_{ij^*} \tag{2.130}$$

parametrize the orbifold T^6/Z_3 moduli space whose special Kähler geometry we have described in the previous section. For a generic T^6 torus we would have 36 moduli corresponding to an arbitrary $g_{\mu\nu}$ metric and an arbitrary $B_{\mu\nu}$ two–form. On the contrary, for the orbifold, we just have the freedom of choosing g_{ij^*} and B_{ij^*}, since the complex structure (2.117) cannot be deformed. Indeed in addition to the identification (2.122) under the lattice group Λ_R we also have the identification under the point group Z_3. The generator Θ of Z_3 acts on the complex coordinates Z^i as a multiplication by Θ:

$$Z^i \sim \Theta Z^i = e^{2\pi i/3} Z^i\,. \tag{2.131}$$

Equations (2.122) and (2.131) are compatible just because Θ acts cristallographically on the lattice Λ_R. Indeed:

$$\Theta[\sqrt{2}(n^i + m^i)] = \sqrt{2}(n^{i'} + \Theta m^{i'})\,, \tag{2.132}$$

where $n^{i'} = -m^i$, $m^{i'} = n^i - m^i$, which follows from

$$\Theta^2 = e^{4\pi i/3} = -1 - \Theta\,. \tag{2.133}$$

The momentum lattice is introduced in the usual way by considering the plane waves $\exp(iP_\mu X^\mu)$ and demanding that they are single–valued on the torus T^6 with the complex structure (2.117). This implies:

$$P_\mu = g_{\mu\nu}(P_i e^{i\nu} + \bar{P}_{i^*} e^{i^*\nu}) , \tag{2.134}$$
$$P_i = \sqrt{\frac{2}{3}}(p_i\omega_1 + q_i\omega_2) \quad , \quad (p_i, q_i \in Z) , \tag{2.135}$$

where $\{e^{i\nu}, e^{i^*\nu}\}$ $i, i^* = 1,2,3$ form the dual basis to the basis (2.118). Following a standard procedure the winding modes can be included into the momentum lattice, which becomes the Lorentzian 12 dimensional Narain lattice [45] Λ_W with signature $g_{\tilde{\mu}\tilde{\nu}} = diag(+,+,+,+,+,+,-,-,-,-,-,-)$. In complete analogy to Eq.s (2.135), one writes:

$$P_{\tilde{\mu}} = g_{\tilde{\mu}\tilde{\nu}}(P_I e^{I\tilde{\nu}} + \bar{P}_{I^*} e^{I^*\tilde{\nu}}) \quad , \quad (I = 1, \cdots 6) , \tag{2.136}$$
$$P_I = \sqrt{\frac{2}{3}}(p_I\omega_1 + q_I\omega_2) \quad , \quad (p_I, q_I \in Z) , \tag{2.137}$$

where $e^{I\tilde{\nu}}, e^{I^*\tilde{\nu}}$ are the basis vectors of the Narain lattice Λ_W, and $P_{\tilde{\mu}}$ its elements. We have:

$$(e^I, e^J) = 0 \quad , \quad (e^{I^*}, e^{J^*}) = 0 \tag{2.138}$$
$$(e^I, e^{J^*}) = g^{IJ^*} . \tag{2.139}$$

The metric g^{IJ^*} is a Hermitian metric with signature $diag(+,+,+,-,-,-)$; hence the sesquilinear form $v^\dagger g w$ is invariant against the transformations of a group isomorphic to $SU(3,3)$. This is the origin of the $SU(3,3)$ symmetry discussed in the previous section. Its role is clarified by considering the level– matching condition in the Narain lattice [45] (i.e. the equality of the left and right masses):

$$0 = P^{\tilde{\mu}} P^{\tilde{\nu}} g_{\tilde{\mu}\tilde{\nu}} = \frac{2}{3} g^{IJ^*}(p_I p_J + q_I q_J + p_I q_J) . \tag{2.140}$$

Equation (2.140) follows upon straightforward substitution of eq. (2.135) and (2.128) into $P^{\tilde{\mu}} P^{\tilde{\nu}} g_{\tilde{\mu}\tilde{\nu}}$. By means of a similarity transformation, the metric g^{IJ^*} could be reduced to the standard $SU(3,3)$ metric $\eta^{IJ^*} = diag(+,+,+,-,-,-)$. Indeed there exists a non–singular 6×6 matrix Ω such that:

$$g^{IJ^*} = (\Omega^\dagger \eta \Omega)^{IJ^*} . \tag{2.141}$$

Consider now the matrix S given by:

$$S = \begin{pmatrix} \frac{1}{\sqrt{2}}\mathbf{1} & 0 \\ \frac{1}{\sqrt{6}}\mathbf{1} & \frac{2}{\sqrt{6}}\mathbf{1} \end{pmatrix} , \tag{2.142}$$

where $\mathbf{1}$ is the unit matrix in the six dimensions. Equation (2.140) can be rewritten as follows:

$$0 = u^T g u + v^T g v \tag{2.143}$$

where

$$\begin{pmatrix} u \\ v \end{pmatrix} = S \begin{pmatrix} p \\ q \end{pmatrix} . \tag{2.144}$$

The quadratic form (2.140) is the standard $SO(6,6)$ invariant form. The elements of $SO(6,6)$ have the generic form:

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.145}$$

where the 6×6 blocks fulfil the following conditions:

$$\begin{aligned} A^T g A + C^T g C &= g \, , \\ A^T g B + C^T g D &= 0 \, , \\ B^T g B + D^T g D &= g \, . \end{aligned} \tag{2.146}$$

In the (u,v) basis the Z_3 generator Θ is given by the matrix:

$$\Theta_{(u,v)} = \begin{pmatrix} \cos(\frac{2\pi}{3})\mathbf{1} & -\sin(\frac{2\pi}{3})\mathbf{1} \\ \sin(\frac{2\pi}{3})\mathbf{1} & \cos(\frac{2\pi}{3})\mathbf{1} \end{pmatrix} . \tag{2.147}$$

The normalizer of Z_3 in $SO(6,6)$ is the group $SU(3,3)$. It is composed of those matrices that have the special form:

$$\mathcal{A} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} . \tag{2.148}$$

with:

$$A^T g A + B^T g B = 0 \quad ; \quad A^T g B = B^T g A \, , \tag{2.149}$$

In the (p,q) basis the Z_3 generator is integer–valued:

$$\Theta_{(p,q)} = S^{-1} \Theta_{(u,v)} S = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} , \tag{2.150}$$

a proof that Z_3 acts cristallographically also on the Narain lattice. In the torus compactification the modular group is $\Gamma(T^6) = SO(6,6,Z)$, namely the subgroup of $SO(6,6)$ that maps the Narain lattice into itself. In the orbifold case the modular group $\Gamma(T^6/Z_3)$ is the subgroup of of $SU(3,3)$ that maps the Narain lattice into itself. We name this group $SU(3,3,Z)$ and we easily identify its elements. In the (p,q) basis an $SU(3,3)$ element is obtained from eq (2.148) via conjugation with the matrix S. We get:

$$\mathcal{A}_{(p,q)} = S^{-1} \mathcal{A} S = \begin{pmatrix} A - \frac{B}{\sqrt{3}} & -2\frac{B}{\sqrt{3}} \\ 2\frac{B}{\sqrt{3}} & A + \frac{B}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} H & H - K \\ -H + K & K \end{pmatrix} , \tag{2.151}$$

where we have set:

$$H = A - \frac{B}{\sqrt{3}} \quad ; \quad K = A + \frac{B}{\sqrt{3}} \,. \tag{2.152}$$

In terms of the blocks H, K the conditions (2.149) become:

$$K^T g K \;=\; H^T g H \,, \tag{2.153}$$

$$H^T g H \;+\; K^T g K - \frac{1}{2} K^T g H - \frac{1}{2} H^T g K = g \,. \tag{2.154}$$

The group $SU(3,3,Z)$ is obtained by demanding that H, K should be integer–valued: $H_{IJ}, K_{IJ} \in Z$. Since $\det \mathcal{A}_{(p,q)} = \det \mathcal{A} = 1$, this condition is compatible with the group structure and $SU(3,3,Z)$ is well defined. Equivalently we can say that the group $SU(3,3,Z)$ is composed by all the pseudo–unitary 6×6 matrices $\mathcal{U}$:

$$\mathcal{U}^\dagger g \mathcal{U} = g \quad ; \quad \det \mathcal{U} = 1 \tag{2.155}$$

that have the special form:

$$\mathcal{U} = \tfrac{1}{2}(K + H) + i \frac{\sqrt{3}}{2}(K - H) \,, \tag{2.156}$$

K, H being integer–valued matrices. In this case eq.s (2.153) and(2.154) follow from insertion of (2.156) into (2.155). The matrices $\mathcal{U}$ have the property that acting on complex vectors of the form:

$$l^I = \frac{1}{\sqrt{2}} p^I + \frac{i}{\sqrt{6}} (p^I + 2q^I) \quad , \quad p^I, q^I \in Z \tag{2.157}$$

map them into complex vectors of the same form. Equations (2.156), (2.157) are the final parametrization of the $SU(3,3,Z)$ modular group and of the Narain lattice for the T^6/Z_3 orbifold. They are the starting point for the construction of the M^{IJK} coefficients appearing in the automorphic superpotential formula.

2.4.3 Construction of the M^{IJK} coefficients

Na¨ively l^I are in the six of $SU(3,3)$ while M^{IJK} are in the three–times antisymmetric representation, where we are considering the $Usp(10,10)$ symplectic group. The only possibility of constructing M^{IJK} out of the l^I momenta would be:

$$M^{IJK} = l^{[I} l^J l^{K]} \tag{2.158}$$

which unfortunately is zero! The way out of this riddle results from the properties of the Narain weight lattice Λ_W which, while modded with respect to its root sublattice $\Lambda_R \subset \Lambda_W$, splits into three conjugacy classes that are separately invariant under the action of $SU(3,3,Z)$. Working in the (p,q) basis (related to the actual momenta l^I via eq.(2.157) we define Λ_R as the sublattice, where p,q have the form:

$$p^I = 2n^I - m^I , \tag{2.159}$$
$$q^I = 2m^I - n^I , \tag{2.160}$$

where $n^I, m^I \in Z$. This definition is inspired by the relation between simple roots and simple weights in the A_2 case:

$$\alpha_1 = 2\lambda_1 + \lambda_2 \quad ; \quad \alpha_2 = -\lambda_1 + 2\lambda_2 , \tag{2.161}$$

so that

$$n\alpha_1 + m\alpha_2 = (2n-m)\lambda_1 + (2m-n)\lambda_2 . \tag{2.162}$$

Equation (2.160) is equivalent to the condition:

$$\frac{1}{3}(p^I - q^I) \in Z \tag{2.163}$$

or

$$n^I = \frac{1}{3}(2p^I + q^I) \in Z , \tag{2.164}$$
$$m^I = \frac{1}{3}(p^I + 2q^I) \in Z . \tag{2.165}$$

An important result is the following:

Lemma: *The modular group $SU(3,3,Z)$ maps the root sublattice Λ_R into itself.*

This follows straightforwardly from eq.(2.151)

$$\frac{1}{3}(p' - q') = \frac{1}{3}[Hp + (H-K)q + (K-H)p - Kq] = H\frac{1}{3}(2p+q) - K\frac{1}{3}(2q+p) \tag{2.166}$$

If the condition (2.163) (implying (2.165)) is fulfilled by (p,q), the same condition is fulfilled by the transformed (p',q'). We can now write the complete Narain lattice Λ_W as the sum of three sublattices

$$\Lambda_W = \Lambda_0 + \Lambda_1 + \Lambda_2 , \tag{2.167}$$

where $\Lambda_0 = \Lambda_R$ is the already defined root lattice while Λ_1 and Λ_2 are defined below:

Definition 4 *Let $\alpha = 1,2$. A vector $(p,q) \in \Lambda_W$ belongs to $\Lambda_\alpha \subset \Lambda_W$ if and only if there exists an integer–valued non–zero six–vector $x^I \in Z$ such that:*

$$\left(\frac{1}{3}(p^I - q^I) + \frac{\alpha}{3}x^I\right) \in Z . \tag{2.168}$$

The reason why the above is a good definition and why (2.167) is a good decomposition is the following. For each value of the index I the difference $p^I - q^I$ can be $0, 1, 2$ mod 3. The root lattice is that sublattice such that $p^I - q^I = 0$ mod 3 for all the values of I. Λ_1 is composed by those vectors such that $p^I - q^I = 1$ mod 3 for some values I (at least one value) and $p^I - q^I = 0$ mod 3 in all the other cases. An analogous definition is given for Λ_2. Since we have exhausted all the possibilities, any vector $(p, q) \in \Lambda_W$ can be written as the sum of a vector in Λ_0 plus a vector in Λ_1, plus a vector in Λ_2. We have now the following:

Theorem 1 *The lattices Λ_α are invariant under the action of the modular group - $SU(3,3,Z)$.*

proof: Using the definitions (2.165) for each $(p, q) \in \Lambda_\alpha$ we can write:

$$n^I = \frac{1}{3}(2p^I + q^I) = \bar{n}^I + \frac{\alpha}{3}x^I \tag{2.169}$$

$$m^I = \frac{1}{3}(2q^I + p^I) = \bar{m}^I - \frac{\alpha}{3}x^I , \tag{2.170}$$

where $\bar{n}^I \bar{m}^I \in Z$. Under the action of $SU(3,3,Z)$ we get:

$$\frac{1}{3}(p' - q') = Hn - Km = H\bar{n} - \bar{m} + \frac{\alpha}{3}(H + K)x ; \tag{2.171}$$

since $H\bar{n} - K\bar{m} \in Z$ it follows that:

$$\frac{1}{3}(p' - q') + \frac{\alpha}{3}x' \tag{2.172}$$

where

$$x' = -(H + K)x . \tag{2.173}$$

Therefore, provided $x' \neq 0$, the image of a vector in Λ_α is still in Λ_α. On the other hand x' cannot be zero. Indeed if x' were zero then the image of $(p, q) \in \Lambda_\alpha$, under the $SU(3,3,Z)$ group element γ we consider, would be in Λ_0. Consider now the inverse group element γ^{-1}: we obtain $\gamma^{-1}\gamma(p, q) = (p, q) \in \Lambda_\alpha$, with $\alpha = 1, 2$. This would imply that the image of the Λ_0 element $\gamma(p, q)$ under the $SU(3,3,Z)$ transformation γ^{-1} is not in Λ_0, contrary to the lemma we have shown. Hence $x' \neq 0$ and the theorem is proved.

Relying on this theorem we can now conclude the construction of the M^{IJK} coefficients. Extending the index α to the value $\alpha = 0$ corresponding to the root sublattice we can set:

$$M^{IJK} = \epsilon^{\alpha\beta\gamma} l^I_\alpha l^J_\beta l^K_\gamma , \tag{2.174}$$

where the $l^I_\alpha \in \Lambda_\alpha$ is given in terms of p^I, q^I by eq. (2.157). The final formula for the automorphic superpotential (where we are considering the $Usp(10,10)$ representation) is encoded in the following ζ–function regularization:

$$\begin{aligned} \log|W|^2 e^G &= -\lim_{s\to 0}\frac{d}{ds}\zeta(s) \\ \zeta(s) &= \frac{1}{\Gamma(s)}\int_0^\infty dt t^{s-1} \sum_{l^I_\alpha\in\Lambda_\alpha} e^{-it|M^{IJK}t_{IJK}|^2} \end{aligned} \tag{2.175}$$

As can be seen, by summing independently on the three sublattices Λ_α we are actually summing on Λ_W. The coefficients (2.174) transform as a three–index antisymmetric representation of $SU(3,3,Z)$ because of our theorem. Utilizing the embedding of $SU(3,3)$ into $Usp(10,10)$ (or via Cayley in $Sp(20,R)$) we discussed in Section 2.4.1 we also see that $SU(3,3,Z)$ is a suitable discrete subgroup of $Usp(10,10)$ and that M^{IJK} span the corresponding 20 dimensional symplectic representation of this modular group.

Chapter 3

n=2 first order systems

3.1 Introduction

In the previous chapters we recalled that the $N = 1$ low energy effective field theory is largely determined by two holomorphic functions $f_{1,2}$ of the Kähler and complex structure moduli, respectively, in term of which Yukawa couplings of the matter fields and Kähler potentials are given by:

$$w^{1,2}_{ijk} = \partial_i\partial_j\partial_k f_{1,2}\,, \tag{3.1}$$

$$K_{1,2} = -\log\left[\sum_{1}^{N_1,N_2} -(\partial_i f_{1,2} - \partial_{i^*}\bar{f}_{1,2})(z^i_{1,2} - \bar{z}^{i^*}_{1,2}) + 2(f_{1,2} + \bar{f}_{1,2})\right]\,. \tag{3.2}$$

where $N_1 = h_{1,1}, N_2 = h_{2,1}$ and $z^i, \bar{z}^{i^*}$ are special coordinates of the moduli spaces for complex and Kähler deformations. As we know the geometry of such moduli spaces is of the special Kähler type. The moduli space of (2,2) superconformal field theories with central charge $c = 9$ (or in general $c = 3d$) exhibit a very similar structure. Calabi–Yau threefolds are examples of $c = 9$ SCFT and thus their moduli spaces are special Kähler.

A crucial property that we will extensively use in this section is that substantial informations about $(2,2)$ SCFT can be obtained from a purely topological sector of the theory. Topological conformal field theories obtained by Twisting n=2 superconformal theories have this topological sector as physical sector [24, 31, 32, 25, 28, 29]. The main advantage of topological theories is that they can be solved completely, by computing all correlation functions at any order. Then one conclude that the study of topological conformal field theories can give essential information on N=1 low energy effective actions, giving an additional technique to characterize it.

For sake of completeness we give a brief review on the basic formalism of the topological field theories. We first recall the definition of (two–dimensional) topological theories [25].

Given a collection of physical operators in a quantum field theory, their correlation functions depend, in general, on the positions of the operators and on the metric defined in the manifold $\mathcal{M}$, on which we consider the correlators. The first characteristic feature of a topological field theory is that the correlation functions of the observables are independent of the metric on $\mathcal{M}$, and therefore independent of the position of the operators. The fact that the physical correlation functions are metric independent is the consequence of a symmetry of topological QFT which reduces the Hilbert space $\mathcal{H}$ to the space $\mathcal{H}_{Phys}$ of physical states, and causes the stress–tensor $T_{\alpha\beta}$ to decouple from physical correlation functions. The symmetry that is responsible for all this is generated by a nilpotent BRST-like operator Q satisfying:

$$Q^2 = 0 \ . \tag{3.3}$$

The physical states are characterised by the cohomology of the operator Q. The space H of the physical states is equal to:

$$\mathcal{H} = \frac{\mathrm{Ker}Q}{\mathrm{Im\ Q}} \ , \tag{3.4}$$

so that physical observables are defined up to Q–commutators:

$$\phi_i \equiv \phi_i + \{Q, \phi_i\} \ . \tag{3.5}$$

The Q invariance of the theory implies that physical correlators are independent on the representative of each ϕ_i. Moreover the stress –energy tensor of topological theories is a Q–commutator.

$$T_{\alpha\beta} = \{Q, G_{\alpha\beta}\}, \tag{3.6}$$

and thus vanishes inside the correlation functions. This ensures that the physical correlators are indeed independent on the two dimensional metric $g_{\alpha\beta}$. The second important property of topological field theory is that correlation functions can be factorized by inserting a complete set of states in the intermediate channels. This amounts to the equation:

$$\mathbf{1}_{phys} = \sum_{i,j} |\phi_i\rangle \eta^{ij} \langle \phi_j| \ , \tag{3.7}$$

where η^{ij} is the metric in $\mathcal{H}_{phys}$ defined by the inverse of the two point function on the sphere

$$\langle \phi_i \phi_j \rangle = \eta_{ij} \ . \tag{3.8}$$

The class of Topological Conformal Field Theories (TCFT) is singled out by requiring the additional property that the energy momentum is traceless. The combined presence of conformal invariance and topological symmetry implies that the generator Q can be decomposed into holomorphic and antiholomorphic components.

It is well known that given an n=2 superconformal algebra generated by $T(z)$, $G^{\pm}(z)$ and $J(z)$, one obtains a topological conformal algebra by "twisting" the currents according to [25, 26, 27]

$$\begin{aligned} \hat{T}_{\pm}(z) &= T(z) \pm \frac{1}{2}\partial J(z) \ , \\ \hat{J}_{\pm}(z) &= \pm J(z) \ , \\ Q_{\pm}(z) &= G^{\pm}(z) \ , \\ G(z) &= G^{\mp}(z) \ . \end{aligned} \tag{3.9}$$

As a result of the modification of the energy-momentum tensor the supercurrent of the $n = 2$ algebra acquire conformal spins $\frac{3}{2} \pm \frac{1}{2}$. In this case $Q_{\pm}(z)$ is interpreted as a BRST current and the cohomology classes of the BRST charge

$$Q_{\pm}^{\mathrm{BRST}} = \oint dz \ Q_{\pm}(z) \tag{3.10}$$

are identified with the physical fields of the topological theory. To fix ideas let us specialize to the + case in (3.9) (we drop for convenience the "+" superscript) The fundamental operator product expansion are:

$$\begin{aligned} \hat{T}(z)\,\hat{T}(w) &= \frac{2\,\hat{T}(w)}{(z-w)^2} + \frac{\partial\hat{T}(w)}{(z-w)} + reg \ , \\ \hat{T}(z)\,G(w) &= \frac{2}{(z-w)^2}G(w) + \frac{\partial G(w)}{(z-w)} + reg \ , \\ \hat{T}(z)\,Q(w) &= \frac{1}{(z-w)^2}Q(w) + \frac{\partial Q(w)}{(z-w)} + reg \ , \\ \hat{T}(z)\,J(w) &= -\frac{c}{3}\frac{1}{(z-w)^3} + \frac{1}{(z-w)^2}J(w) + \frac{1}{(z-w)}\partial J(w) \\ Q(z)\,G(w) &= \frac{2}{3}c\frac{1}{(z-w)^3} + 2\frac{J(w)}{(z-w)^2} + 2\frac{\hat{T}(z)}{(z-w)} + reg \ , \\ J(z)\,G(w) &= -\frac{G(w)}{(z-w)} + reg \ , \\ J(z)\,Q(w) &= \frac{Q(w)}{(z-w)} + reg \ , \\ J(z)\,J(w) &= \frac{c}{3}\frac{1}{(z-w)^2} + reg \ . \end{aligned} \tag{3.11}$$

Notice that the central extension of the Virasoro subalgebra vanishes, but the U(1) current is anomalous, coupled to a central extension $\frac{c}{3}$. The cohomology of the BRST operator defines a ring which is in one to one correspondence with the states in the chiral ring of the original n=2 conformal field theory. In the case where we choose the "−" twist the correspondence is with the antichiral ring of the $n = 2$. To choose a unique representative in each cohomology class we can indeed impose the "Hodge" conditions:

$$G_0|\phi\rangle = L_0|\phi\rangle = 0, \tag{3.12}$$

which, in the correspondence with $n = 2$ models are precisely the conditions defining the so–called chiral primary fields. In general there are only a finite number of such fields, and their $U(1)$ charges are positive and bounded by $\frac{c}{3} \equiv d$:

$$J_0|\phi_i\rangle = q_i|\phi_i\rangle \quad 0 \leq q_i \leq d\ . \tag{3.13}$$

To each chiral primary, we can associate a (chiral $n = 2$) superfield

$$A(z, \bar{z}, \theta, \bar{\theta}) = \phi^{(0)} + \theta\phi^{(1)} + \bar{\theta}\bar{\phi}^{(1)} + \theta\bar{\theta}\phi^{(2)}\ , \tag{3.14}$$

which contains the "zero form" (first component) $\phi^{(0)} \equiv \phi$, as well as the "one" and "two form" components $(\phi^{(1)}, \bar{\phi}^{(1)})$ and $\phi^{(2)}$ of the physical field. We refer to superfield components as to "form" for a very simple reason (which will be completely clarified in the following sections): by Q symmetry θ and $\bar{\theta}$ transform as differentials dz, $d\bar{z}$, since $\delta z = \theta$ and $\delta\bar{z} = \bar{\theta}$. These components are obtained by acting with the superconformal generators G and $\bar{G}$. In particular:

$$\phi^{(2)} = \oint G \oint \bar{G}\phi^{(0)}, \tag{3.15}$$

which shows that the two–form has $U(1)$ charge $(q-1, \bar{q}-1)$.

As can be easily verified using $\hat{T}$ in (3.11) the physical operators $\phi_i = \phi_i^{(0)}$ have conformal dimension $(h, \bar{h}) = (0, 0)$, and their operator product expansions are nonsingular:

$$\phi_i\phi_j = c_{ij}^k\phi_k\ , \tag{3.16}$$

where

$$c_{ijk} = c_{ij}^l\eta_{lk} =< \phi_i\phi_j\phi_k > \tag{3.17}$$

are the three point correlation functions between the primary fields.

Let us suppose now that our (generical) topological field theory can be described by an action S. Consider the following family of actions:

$$S(t) = S(0) - \sum t_n \int \phi_n^{(2)}\ . \tag{3.18}$$

They are obtained by deforming the action $S = S(0)$, describing the original topological theory, with the operators $\int \phi_i^{(2)}$, corresponding to the coupling constants t_i. These perturbations respect the nilpotency symmetry Q and therefore preserve the topological properties of the theory, such as the metric independence and the factorization. In correspondence to eq. (3.18) we can write the perturbed correlation functions:

$$c_{ijk}(t) = \langle\phi_i\phi_j\phi_k exp(\sum_n t_n \int \phi_n^{(2)}\rangle\ . \tag{3.19}$$

In the case of TCFT, by exploiting the conformal invariance of the $t_n = 0$ point, we can show that the coefficients $c_{ijk}(t)$ satisfy an important integrability condition. Namely

$$\frac{\partial c_{ijk}(t)}{\partial t_l} = \frac{\partial c_{ijl}(t)}{\partial t_k} . \tag{3.20}$$

This equation shows that it is possible to integrate the $c_{ijk}(t)$, and obtain a single function $F(t)$ that satisfies:

$$c_{ijk}(t) = \frac{\partial^3 F(t)}{\partial t_i \partial t_j \partial t_k} . \tag{3.21}$$

Another important consequence of the integrability equation is that the metric $\eta_{ij} = c_{0ij}$ is in fact independent of the couplings t_i.

$$\frac{\partial \eta_{ij}}{\partial t_k} = 0 . \tag{3.22}$$

The coordinates t_i form a distinguished basis in the space of couplings: the "flat" basis. They correspond to the directions in the space of all TCFT that are perturbed by the scaling operators. However we can take arbitrary directions $s_1, \cdots s_n$, where $s_i = s_i(t_j)$. In this case we must replace in the above equations the ordinary derivatives by covariant ones defined by the metric η_{ij} (which is no more constant). This is very reminescent of the fact that in special geometry, only in "special" coordinates one can write (3.1), where in a general system we must use covariant derivatives. It is obvious that there should be a relation between the flat geometry of topological theories and the special geometry. Since in special geometry we are in particular dealing with moduli–spaces of Calabi–Yau manifolds such a relation can be fully established using superconformal theories with $c = 9$ (or in general $c = 3d$) , which precisely corresponds to space–time compactification of CY manifolds. In this case the "moduli" (chiral primary marginal perturbation) of the n=2 superconformal theory corresponds to the moduli of the CY manifold, and Yukawa couplings of (3.1) can be computed using the marginal perturbations in (3.18) [74].

The best way to analyze the general properties of TCFT, whether in connection with CY compactification or not, was mainly based on the Landau–Ginzburg formulation of topological models, with deep connections on singularity theory and algebraic geometry [28, 29, 30, 75, 31, 32, 33, 34]. Indeed in the LG formulation of (2,2)-supersymmetric models, the superconformal theory is viewed as the infrared fixed point of a two-dimensional n=2 Wess-Zumino model with a polynomial superpotential W. When W is an analytic quasi-homogeneous function of the chiral superfields, we can assign a well defined $U(1)$ charge to these fields, which are in one to one correspondence with the chiral primary fields of n=2 superconformal theory. Furthermore, the polynomials W's can be identified, in the particular cases where $c = 3d$, with those used in the construction of Calabi-Yau d-folds. It can be shown [38, 30] that a superconformal model with $c = 3d$, corresponding to a LG potential W, is the same as that associated to a σ-model on the Calabi-Yau d-fold defined by the polynomial constraint $W(X_i) = 0$ in a suitable projective or weighted projective space [1].

In such a formulation the parameters entering the Landau-Ginzburg superpotential corresponds either to "versal" deformations of the holomorphic (antiholomorphic) superpotential, or to "modal" deformations. In terms of quantum field theory this means that we are perturbing the theory, which is known to correspond to a well defined n=2 model, with some relevant (versal) or marginal (modal) deformations. The coupling constants parametrizing the deformations are interpreted as coordinates of some space (a moduli space in the case of marginal ones).

As was shown before, once given the chiral ring of the n= 2 superconformal theory (i.e. the topological sector of the model), we can consider the perturbed three point functions $c_{ijk}(t)$ as well as the metric $\eta_{ij}(t) = \eta_{ij}(0)$, which defines in the coupling space a *flat constant geometry.* This is not so in the Landau–Ginsburg formulation of the n=2 superconformal theories, just because the versal and modal deformations of the superpotential do not corresponds directly to deformations around the conformal point; rather they are related to the latter by the solution of a uniformization problem, which in general involves higher transcendental functions. This is easily undestood with the following consideration. The n=2 Landau–Ginsburg action can be put into correspondence with a n=2 supeconformal theory only at its infrared fixed point. In particular the chiral primary ring of the conformal theory can be identified with the quotient ring

$$\mathcal{R} = \frac{C[X_I]}{dW} \tag{3.23}$$

of polynomials in X_I modulo the vanishing relations $dW = 0$ (where the index I runs over the number of Landau–Ginzburg fields X_I). When we turn on a coupling δs_i to one of the operators $\phi_i(X_J)$ we change the potential as:

$$\delta W = \delta s_j \phi_j \ . \tag{3.24}$$

This modifies also the chiral ring, which as a consequence will effect the potential W to the next order. This reflects to the fact that the metric defined by residue pairing relation of the perturbed potential is no more constant in the couplings s, as shown in [31, 32, 33, 34, 76, 77, 78]. In particular in ref. [49] is studied the problem of twisting LG models into topological LG models and of computing arbitrary correlation functions in the topological theory. At genus zero one finds the following residue pairing metric

$$g_{ij} = \langle \phi_i(X)\phi_j(X) \rangle = \sum_{dW=0} \frac{\phi_i(X)\phi_j(X)}{\partial^2 W} = \text{res}\ \frac{\phi_i(X)\phi_j(X)}{\partial W}\ , \tag{3.25}$$

[1] In general $W(X_i)$ is actually the sum of several terms $W(X_i) = \sum_\alpha W_\alpha(X_i)$ and the Calabi-Yau d-fold is given by the complete intersection $W_1 = W_2 = \cdots = W_n = 0$ [38].

(where the residue involves taking a contour at large radius). As stated above g_{ij} is no more constant. Moreover there exists a coordinate transformation $s_i = s_i(t_j)$ such that the new metric is constant. In the more complicated cases (i.e. when moduli are present), the search for such a transformation is the afore mentioned uniformization problem.

In the following sections we present an alternative approach to topological models where the relation to singularity theory is directly obtained in a natural system of flat coordinates [48]. At the same time, these are the parameters of a Landau-Ginzburg superpotential as well as the deformations around the conformal point. The first step of our construction is the use of free first-order (b, c, β, γ)-systems to describe n=2 superconformal theories as proposed in [47]. We then show that an arbitrary interaction of the Landau-Ginzburg type – *i.e.* characterized by a polynomial potential V – can be added to the free Lagrangian without spoiling the superconformal invariance if V is a quasi-homogeneous function. The deformation parameters of the potential are the flat coordinates simply because they corresponds to deformations by primary fields around the conformal point. In this way we loose something: in the presence of a deformed potential we cannot use the residue pairing metric to define the perturbed correlation functions. We have to do any computation in the context of conformal field theory, using for example bosonization techniques.

3.2 Lagrangian formulation of n=2 theories via first order systems

In this section we consider the realization of (2,2)-supersymmetric models in terms of free (b, c, β, γ)-systems recently introduced in [47], and generalize it to include interactions of the Landau-Ginzburg (LG) type. We show that it is possible to add a polynomial interaction V of the LG type to a collection of free first-order (b, c, β, γ)-systems in such a way that, if V is a quasi-homogeneous function, the theory possesses an n=2 superconformal symmetry already at the classical level. We also show that the interaction potential unambiguously fixes the weights of the pseudo-ghost fields. As in the standard LG case, also here we can recover the ADE classification of the n=2 minimal models from ADE classification of the interaction potential [30, 75]; however in our case the theory is always manifestly superconformal invariant. Our formulation allows us to add all relevant perturbations (versal deformations of the potential) and to study the renormalization group flows in a very simple way. Whenever we use a quasi-homogeneous potential with modality different from zero, we can study marginal deformations and eventually Zamolodchikov's metric on the associated moduli space. Alternatively, we can consider topological models by "twisting" the generators of the superconformal algebra and compute topological correlation functions. Our formulation provides valuable methods to evaluate these latter.

We start this program by defining our model. We consider a collection of pseudo-ghost fields $\{b_\ell, c_\ell, \beta_\ell, \gamma_\ell; \tilde{b}_r, \tilde{c}_r, \tilde{\beta}_r, \tilde{\gamma}_r\}$ where $\ell = 1, \ldots, N_{\rm L}$ and $r = 1, \ldots, N_{\rm R}$. β_ℓ and γ_ℓ form a bosonic first-order system with weights λ_ℓ and $1-\lambda_\ell$ respectively whereas b_ℓ and c_ℓ form a fermionic first-order system with weights $\lambda_\ell + \frac{1}{2}$ and $\frac{1}{2} - \lambda_\ell$ respectively. The same can be said for the tilded fields with λ_ℓ replaced by $\tilde{\lambda}_r$. The action is

$$S = \int d^2z\ \mathcal{L} = \int d^2z\ (\mathcal{L}_0 + \Delta\mathcal{L}) \tag{3.26}$$

where

$$\begin{aligned} \mathcal{L}_0 &= \sum_\ell \left[-\lambda_\ell \beta_\ell \bar{\partial}\gamma_\ell + (1-\lambda_\ell)\gamma_\ell\bar{\partial}\beta_\ell - (\lambda_\ell + \tfrac{1}{2})\ b_\ell\bar{\partial}c_\ell + (\lambda_\ell - \tfrac{1}{2})\ c_\ell\bar{\partial}b_\ell\right] \\ &+ \sum_r \left[-\tilde{\lambda}_r\,\tilde{\beta}_r\partial\tilde{\gamma}_r + (1-\tilde{\lambda}_r)\,\tilde{\gamma}_r\partial\tilde{\beta}_r - (\tilde{\lambda}_r + \tfrac{1}{2})\,\tilde{b}_r\partial\tilde{c}_r + (\tilde{\lambda}_r - \tfrac{1}{2})\,\tilde{c}_r\partial\tilde{b}_r\right] \end{aligned} \tag{3.27}$$

and

$$\Delta\mathcal{L} = \sum_{\ell,r} b_\ell\,\tilde{b}_r\,\partial_\ell V(\beta)\,\tilde{\partial}_r\tilde{V}(\tilde{\beta}) \quad . \tag{3.28}$$

Here and in the following we use the short-hand notations $\partial_\ell \equiv \partial/\partial\beta_\ell$ and $\tilde{\partial}_r \equiv \partial/\partial\tilde{\beta}_r$. $\mathcal{L}_0$ in (3.27) represents the standard free Lagrangian for first-order systems of the given weights and $\Delta\mathcal{L}$ in (3.28) defines an interaction of the LG type when V and $\tilde{V}$ are polynomial functions of β_ℓ and $\tilde{\beta}_r$ respectively. From (3.26) one can derive the following equations of motion

$$\begin{aligned} \bar{\partial}\beta_\ell &= 0 \ , \ \ \bar{\partial}b_\ell = 0 \ , \\ \partial\tilde{\beta}_\ell &= 0 \ , \ \ \partial\tilde{b}_\ell = 0 \ , \\ \bar{\partial}c_\ell &= \sum_r \tilde{b}_r\partial_\ell V(\beta)\tilde{\partial}_r\tilde{V}(\tilde{\beta}) \ , \\ \bar{\partial}\gamma_\ell &= \sum_{m,r} b_m\tilde{b}_r\partial_\ell\partial_m V(\beta)\tilde{\partial}_r\tilde{V}(\tilde{\beta}) \ , \\ \partial\tilde{c}_r &= -\sum_\ell b_\ell\partial_\ell V(\beta)\tilde{\partial}_r\tilde{V}(\tilde{\beta}) \ , \\ \partial\tilde{\gamma}_r &= \sum_{\ell,s} b_\ell\partial_\ell V(\beta)\tilde{b}_s\tilde{\partial}_r\tilde{\partial}_s\tilde{V}(\tilde{\beta}) \ . \end{aligned} \tag{3.29}$$

first two lines of (3.29) show that β_ℓ, b_ℓ, $\tilde{\beta}_r$ and b_r satisfy the same equations as in the free case, whereas c_ℓ, γ_ℓ, $\tilde{c}_r$ and $\tilde{\gamma}_r$ have no longer a definite holomorphic or anti-holomorphic character in the presence of the interaction. We can write formal solutions to the equations for c_ℓ and γ_ℓ as follows [88]:

$$c_\ell(z,\bar{z}) = c_\ell^0(z) + \int_\Delta \frac{d^2w}{2\pi\mathrm{i}}\frac{1}{w-z}\sum_r \tilde{b}_r(\bar{w})\partial_\ell V(\beta(w))\tilde{\partial}_r\tilde{V}(\tilde{\beta}(\bar{w})) \ , \tag{3.30}$$

$$\gamma_\ell(z,\bar{z}) = \gamma_\ell^0(z) + \int_\Delta \frac{d^2w}{2\pi\mathrm{i}}\frac{1}{w-z}\sum_{m,r} b_m(w)\tilde{b}_r(\bar{w})\partial_\ell\partial_m V(\beta(w))\tilde{\partial}_r\tilde{V}(\tilde{\beta}(\bar{w})) \tag{3.31}$$

where Δ is a disk containing w, and $c_\ell^0$and γ_ℓ^0 are arbitrary holomorphic fields. Similar formal expressions (with the obvious changes) hold also for $\tilde{c}_r$ and $\tilde{\gamma}_r$. It is fairly easy to realize that under canonical quantization of (3.26) the fundamental operator product expansions are the same as in the free case. Indeed, even in the presence of the interaction, we have

$$\begin{aligned} \beta_\ell(z)\,\gamma_m(w,\bar{w}) &= -\frac{\delta_{\ell m}}{z-w} + \cdots \quad , \\ b_\ell(z)\,c_m(w,\bar{w}) &= \frac{\delta_{\ell m}}{z-w} + \cdots \quad , \end{aligned} \tag{3.32}$$

and similarly for the tilded fields. Of course, the interaction is not immaterial and it has to be carefully analyzed in a complete quantum treatment as we will do in Section 3.2.1.

It is well-known that $\mathcal{L}_0$ in (3.27) describes a (2,2)-superconformal field theory with central charges

$$c_{\rm L} = \sum_\ell (3 - 12\lambda_\ell) \quad , \quad c_{\rm R} = \sum_r (3 - 12\tilde{\lambda}_r) \quad , \tag{3.33}$$

for the left and the right sectors respectively. We will now show that the addition of the interaction $\Delta\mathcal{L}$ does not destroy this (2,2)-superconformal invariance if V and $\tilde{V}$ are quasi-homogeneous functions, *i.e.* if for any $a \in \mathbf{R}^+$,

$$V(a^{\omega_\ell}\beta_\ell) = a\,V(\beta_\ell) \quad , \quad \tilde{V}(a^{\tilde{\omega}_r}\tilde{\beta}_r) = a\,\tilde{V}(\tilde{\beta}_r) \quad . \tag{3.34}$$

The parameters ω_ℓ and $\tilde{\omega}_r$ are called the homogeneous weights of β_ℓ and $\tilde{\beta}_r$ respectively. By enforcing the requirement that the interaction Lagrangian $\Delta\mathcal{L}$ have the correct dimensions, one can see that

$$\omega_\ell = 2\lambda_\ell \quad , \quad \tilde{\omega}_r = 2\tilde{\lambda}_r \quad ; \tag{3.35}$$

the parameters λ_ℓ and $\tilde{\lambda}_r$ of the free Lagrangian (3.27) are therefore *fixed* by the interaction terms. When (3.34) and (3.35) are satisfied, the action S in (3.26) is invariant under the following n=2 holomorphic supersymmetry transformations

$$\begin{aligned} \delta\beta_\ell &= 2\sqrt{2}\epsilon^- b_\ell \quad , \\ \delta b_\ell &= \frac{1}{\sqrt{2}}\epsilon^+\partial\beta_\ell + \sqrt{2}\lambda_\ell\,\partial\epsilon^+\beta_\ell \quad , \\ \delta c_\ell &= 2\sqrt{2}\epsilon^-\gamma_\ell \quad , \\ \delta\gamma_\ell &= \frac{1}{\sqrt{2}}\epsilon^+\partial c_\ell - \sqrt{2}\,(\lambda_\ell - \tfrac{1}{2})\,\partial\epsilon^+ c_\ell \quad , \\ \delta\tilde{\beta}_r &= 0 \quad , \\ \delta\tilde{b}_r &= 0 \quad , \end{aligned} \tag{3.36}$$

$$\delta\tilde{c}_r = -\frac{1}{\sqrt{2}}\epsilon^+ V(\beta)\tilde{\partial}_r\tilde{V}(\tilde{\beta}) \ ,$$
$$\delta\tilde{\gamma}_r = \frac{1}{\sqrt{2}}\epsilon^+ V(\beta)\sum_s \tilde{\partial}_r\tilde{\partial}_s\tilde{V}(\tilde{\beta})\tilde{b}_s \ ,$$

where $\epsilon^\pm$ are arbitrary holomorphic functions ($\bar{\partial}\epsilon^\pm = 0$). The action S is also invariant under n=2 anti-holomorphic symmetries which are similar to the ones defined in (3.36), with the exchange of the tilded and untilded quantities, and the replacement of $\epsilon^\pm$ with arbitrary anti-holomorphic functions $\bar{\epsilon}^\pm$ ($\partial\bar{\epsilon}^\pm = 0$). Moreover, if we relax the hypothesis that V and $\tilde{V}$ are quasi-homogeneous, the transformations (3.36) and their $\bar{\epsilon}$-analogues remain symmetries of (3.26) provided $\epsilon^\pm$ and $\bar{\epsilon}^\pm$ are constant parameters. This means that our model has a global n=2 supersymmetry for any choice of V and $\tilde{V}$, and an n=2 superconformal invariance for quasi-homogeneous potentials. Using Noether's theorem, we can calculate the conserved currents associated to (3.36).

Let $\mathcal{L}(\phi_i, \partial\phi_i, \bar{\partial}\phi_i)$ be a 2d– Lagrangian for a collection of fields ϕ_i and let us assume that under a variation

$$\delta\phi_i = \epsilon_\Lambda T^\Lambda(\phi) \tag{3.37}$$

we have:

$$\delta\mathcal{L} = \epsilon_\Lambda(\bar{\partial}f^\Lambda_z + \partial f^\Lambda_{\bar{z}}) \ . \tag{3.38}$$

The corresponding currents are given by the formula:

$$\begin{aligned} j^\Lambda_z &= T^\Lambda_i(\phi)\frac{\partial\mathcal{L}}{\partial(\bar{\partial}\phi)} - f^\Lambda_z \ , \\ j^\Lambda_{\bar{z}} &= T^\Lambda_i(\phi)\frac{\partial\mathcal{L}}{\partial(\partial\phi)} - f^\Lambda_{\bar{z}} \end{aligned} \tag{3.39}$$

and are conserved

$$\bar{\partial}j^\Lambda_z + \partial j^\Lambda_{\bar{z}} = 0 \ . \tag{3.40}$$

If one of the two components of j vanishes, the other is holomorphic (respectively antiholomorphic).

For our lagrangian the above procedure leads to the following conserved currents [2]

$$\begin{aligned} G^+_z &= \sqrt{2}\sum_\ell \left[\left(\tfrac{1}{2} - \lambda_\ell\right) c_\ell\partial\beta_\ell - \lambda_\ell\beta_\ell\partial c_\ell\right] \ , \\ G^+_{\bar{z}} &= \sqrt{2}\sum_\ell \left[\lambda_\ell\beta_\ell\bar{\partial}c_\ell + \left(\lambda_\ell - \tfrac{1}{2}\right)\bar{\partial}\beta_\ell c_\ell\right] - \frac{1}{\sqrt{2}}\sum_r V(\beta)\tilde{b}_r\tilde{\partial}_r\tilde{V}(\tilde{\beta}) \ , \\ G^-_z &= 2\sqrt{2}\sum_\ell \gamma_\ell b_\ell \ , \\ G^-_{\bar{z}} &= 0 \ . \end{aligned} \tag{3.41}$$

[2] From now on, to avoid repetitions we will discuss only the left sector and understand that similar considerations can be made in the right sector, with some obvious change of signs.

If we use the equations of motion (3.29) for quasi-homogeneous potentials, we see that $G^+_{\bar z}$ vanishes on-shell; thus from the conservation laws we deduce that G^+_z and G^-_z are holomorphic currents even if they contain the non-holomorphic fields c_ℓ and γ_ℓ. We denote these currents by $G^\pm(z)$.

The general superconformal transformations (3.36) are retrieved from the structure of the supercurrents utilizing the general formula

$$\begin{aligned}\delta\phi(w,\bar w) &= \oint_w \frac{dz}{2\pi i}[\epsilon^+(z)G^+_z(z,\bar z)+\epsilon^-(z)G^-_z(z,\bar z)]\phi(w,\bar w)\\ &+ \oint_{\bar w}\frac{d\bar z}{2\pi i}[\epsilon^+(z)G^+_{\bar z}(z,\bar z)+\epsilon^-(z)G^-_{\bar z}(z,\bar z)]\phi(w,\bar w)\ ,\end{aligned}\tag{3.42}$$

which holds for any field $\phi(w,\bar w)$.

The action (3.26) is also invariant under holomorphic conformal reparametrizations and $U(1)$-rescalings of the fields; the conserved Noether's currents associated to such symmetries are the stress-energy tensor $T_{\mu\nu}$ and the $U(1)$-current J_μ. For homogeneous potentials it is not difficult to see that the trace of $T_{\mu\nu}$ and the $\bar z$-component of J_μ are zero on-shell (see also Section 3.2.1). Therefore, from the conservation laws, we deduce that

$$T_{zz}=\sum_\ell\left[-\lambda_\ell\beta_\ell\partial\gamma_\ell+(1-\lambda_\ell)\gamma_\ell\partial\beta_\ell-\left(\lambda_\ell+\tfrac{1}{2}\right)b_\ell\partial c_\ell-\left(\tfrac{1}{2}-\lambda_\ell\right)c_\ell\partial b_\ell\right]\ ,\tag{3.43}$$

and

$$J_z=\sum_\ell\left[(2\lambda_\ell-1)b_\ell c_\ell+2\lambda_\ell\beta_\ell\gamma_\ell\right]\tag{3.44}$$

are holomorphic currents. We denote them by $T(z)$ and $J(z)$ respectively.

Using the OPE's in (3.32), it is straightforward to check that $T(z)$, $G^\pm(z)$ and $J(z)$ close an n=2 superconformal algebra

$$\begin{aligned}T(z)\,T(w) &= \frac{c}{2}\frac{1}{(z\ -\ w)^4}+\frac{2\,T(w)}{(z\ -\ w)^2}+\frac{\partial T(w)}{(z\ -\ w)}+reg\ ,\\ T(z)\,G^\pm(w) &= \frac{3}{2}\frac{1}{(z\ -\ w)^2}G^\pm(w)+\frac{\partial G^\pm(w)}{(z\ -\ w)}+reg\ ,\\ T(z)\,J(w) &= \frac{1}{(z\ -\ w)^2}J(w)+\frac{1}{(z\ -\ w)}\partial J(w)+reg\ ,\\ G^+(z)\,G^-(w) &= \frac{2}{3}c\frac{1}{(z\ -\ w)^3}+2\frac{J(w)}{(z\ -\ w)^2}+2\frac{T(w)+\frac{1}{2}\partial J(w)}{(z\ -\ w)}+reg\ ,\\ J(z)\,G^\pm(w) &= \pm\frac{G^\pm(w)}{(z\ -\ w)}+reg\ ,\\ J(z)\,J(w) &= \frac{c}{3}\frac{1}{(z\ -\ w)^2}+reg\ ,\end{aligned}\tag{3.45}$$

with central charge

$$c = c_{\rm L} = \sum_{\ell}(3 - 12\lambda_\ell) \ . \tag{3.46}$$

Thus, we have shown that the interaction $\Delta\mathcal{L}$ with homogeneous polynomials V and $\tilde{V}$ does not spoil the superconformal properties of $\mathcal{L}_0$.

In our formulation the ADE classification of n=2 superconformal models is an immediate consequence of ADE classification of homogeneous polynomials of zero modality [38, 30, 75]. The latter are

$$\begin{aligned}
A_n &: \quad V = \frac{1}{n+1}\beta^{n+1} \quad \Rightarrow \lambda = \frac{1}{2n+2} \quad n \geq 1 \ , \\
D_n &: \quad V = \frac{1}{n-1}\beta_1^{n-1} + \frac{1}{2}\beta_1\beta_2^2 \quad \Rightarrow \lambda_1 = \frac{1}{2n-2} \ , \lambda_2 = \frac{n-2}{4n-4} \quad n \geq 2 \ , \\
E_6 &: \quad V = \frac{1}{3}\beta_1^3 + \frac{1}{4}\beta_2^4 \quad \Rightarrow \lambda_1 = \frac{1}{6} \ , \lambda_2 = \frac{1}{8} \ , \\
E_7 &: \quad V = \frac{1}{3}\beta_1^3 + \frac{1}{3}\beta_1\beta_2^3 \quad \Rightarrow \lambda_1 = \frac{1}{6} \ , \ \lambda_2 = \frac{1}{9} \ , \\
E_8 &: \quad V = \frac{1}{3}\beta_1^3 + \frac{1}{5}\beta_2^5 \quad \Rightarrow \lambda_1 = \frac{1}{6} \ , \ \lambda_2 = \frac{1}{10} \ .
\end{aligned} \tag{3.47}$$

We remark that the values of λ_ℓ's listed in (3.47) are fixed by the homogeneous weights of β_ℓ's according to (3.35). If we now insert such values into (3.46) we obtain the correct central charges for the n=2 minimal models in the ADE classification, namely

$$c(A_n) = \frac{3n-3}{n+1} \ , c(D_n) = \frac{3n-6}{n-1} \ , \ c(E_6) = \frac{5}{2} \ , \ c(E_7) = \frac{8}{3} \ , \ c(E_8) = \frac{14}{5} \ . \tag{3.48}$$

It is also interesting to observe that the ring determined by the potential V, which contains all polynomials in β_ℓ's modulo the vanishing relations $\partial_\ell V = 0$, coincides with the ring of chiral primary operators of the n=2 minimal model associated to V. Indeed, using 3.43 and (3.44), one can easily check that the $U(1)$-charge of $(\beta_\ell)^n$ is twice its conformal dimension.

In order to compare our formulation of n=2 supersymmetric models with the standard LG approach and to establish a clear correspondence with the topological conformal field theories, it is convenient to specialize our system to the case of a complete symmetry between the left and the right sectors ($N_{\rm L} = N_{\rm R} = N$). We shall then consider interactions of the form

$$\Delta\mathcal{L} = \sum_{i,j=1}^{N} b_i\tilde{b}_j\,\partial_i\tilde{\partial}_j W \tag{3.49}$$

where W is a quasi-homogeneous function of the variables

$$X_i \ = \ \beta_i\tilde{\beta}_i \quad , \quad i = 1, \ldots, N \ . \tag{3.50}$$

This is clearly a very special case of (3.28). Under these conditions, the Lagrangian (3.26) describes the infrared fixed point of an ordinary n=2 LG model with superpotential W. This equivalence will be fully illustrated in the next sections. Here instead, we discuss the topological formulation of our models. It is interesting to study the consequences of the twist (3.9) on our (b, c, β, γ) systems. We first analyze the + case. From (3.43) and (3.44) we simply get

$$\hat{T}_+ = T + \frac{1}{2}\partial J = \sum_{i=1}^{N} (\partial c_i \, b_i + \gamma_i \, \partial\beta_i) \quad . \tag{3.51}$$

This is the canonical stress-energy tensor for a collection of N commuting (β,γ)-systems of weight $\lambda = 0$, and N anticommuting (b,c)-systems of weight $\lambda = 1$. $\hat{T}$ in (3.51) closes a Virasoro algebra with vanishing central charge. Indeed, the central charge of a first-order system of weight λ is

$$c_\lambda = \varepsilon \left(1 - 3Q^2\right) \tag{3.52}$$

where

$$Q = \varepsilon\,(1 - 2\lambda) \tag{3.53}$$

is a "background charge" and $\varepsilon = 1$ or -1, depending on whether the system is anticommuting or commuting. In our case both the (β,γ)-systems and the (b,c)-systems have $Q = -1$, but since their statistics is different, their central charges exactly cancel.

To fully appreciate the effects of this topological twist on our models, we now write the topological Lagrangian and its BRST symmetries. The Lagrangian is

$$\mathcal{L}_{\rm top} = \sum_{i=1}^{N} \left[\gamma_i\bar\partial\beta_i + \tilde\gamma_i\partial\tilde\beta_i - b_i\bar\partial c_i - \tilde b_i\partial\tilde c_i\right] + \sum_{i,j=1}^{N} \left[b_i\tilde b_j\partial_i\bar\partial_j W\,(X)\right] \ . \tag{3.54}$$

The BRST transformations which leave (3.54) invariant, can be obtained from (3.36) and their analogues by identifying the BRST parameter θ with $\frac{\epsilon^+}{\sqrt{2}} = \frac{\bar\epsilon^+}{\sqrt{2}}$ (the factor of $1/\sqrt{2}$ is introduced for convenience). These transformations are most conveniently exhibited as the action of the nilpotent Slavnov operator s on all fields, namely

$$\begin{aligned}
s\,\beta_i &= 0 \ , \\
s\,\tilde\beta_i &= 0 \ , \\
s\,b_i &= \partial\beta_i \ , \\
s\,\tilde b_i &= \bar\partial\tilde\beta_i \ , \\
s\,\gamma_i &= \partial c_i - \sum_{j=1}^{N} b_j\,\partial_i\partial_j W \ ,
\end{aligned}$$

$$\begin{aligned} s\,\tilde{\gamma}_i &= \bar{\partial}\tilde{c}_i + \sum_{j=1}^{N} \tilde{b}_j\,\tilde{\partial}_i\tilde{\partial}_j W \ , \\ s\,c_i &= +\partial_i W \ , \\ s\,\tilde{c}_i &= -\tilde{\partial}_i W \ . \end{aligned} \tag{3.55}$$

Using (3.55) it is quite easy to construct the representatives of the BRST-cohomology classes and the corresponding integrated invariants. According to the general theory, we have to consider multiplets composed by a 0-form Φ_P, a 1-form $\Phi_P^{(1)}$ and a 2-form $\Phi_P^{(2)}$ which satisfy the following descent equations

$$\begin{aligned} s\,\Phi_P &= 0 \ , \\ s\,\Phi_P^{(1)} &= -\,d\,\Phi_P \ , \\ s\,\Phi_P^{(2)} &= -\,d\,\Phi_P^{(1)} \ , \\ d\,\Phi_P^{(2)} &= 0 \ . \end{aligned} \tag{3.56}$$

Moreover the 0-form Φ_P must belong to a non-trivial BRST-cohomology class, *i.e.* it should not be BRST-exact. The solutions of the descent equations (3.55) provide the local physical observables Φ_P appearing in correlation functions as well as the integrated invariants $\Phi_P^{(2)}$ which can be used to deform the theory. Thus, the general form of a perturbed topological correlation function is

$$c_{P_1,\ldots,P_m}(t_1,\ldots,t_M) = \left\langle \Phi_{P_1}(z_1)\cdots\Phi_{P_m}(z_m)\ \exp\left[\sum_{k=1}^{M} t_k \int \Phi_{P_k}^{(2)}\right]\right\rangle_{\mathrm{top}} \tag{3.57}$$

where $\langle\cdots\rangle_{\mathrm{top}}$ means functional integration with the measure provided by the umperturbed Lagrangian $\mathcal{L}_{\mathrm{top}}$ and t_k are coupling constants parametrizing its deformations $\int \Phi_{P_k}^{(2)}$.

In our (b, c, β, γ) formulation the general solution of the descent equations (3.56) is

$$\begin{aligned} \Phi_P &= P(X) \ , \\ \Phi_P^{(1)} &= -\sum_{i=1}^{N}\left[b_i\,\partial_i P\,dz + \tilde{b}_i\,\tilde{\partial}_i P\,d\bar{z}\right] \ , \\ \Phi_P^{(2)} &= \sum_{i,j=1}^{N}\left[b_i\tilde{b}_j\,\partial_i\tilde{\partial}_j P\right]\,dz\wedge d\bar{z} \ , \end{aligned} \tag{3.58}$$

where $P(X)$ is any polynomial in the variables $X_i = \beta_i\tilde{\beta}_i$ corresponding to a non trivial element of the local ring determined by the superpotential W of the Lagrangian (3.54). Indeed if the polynomial $P(X)$ is proportional to the vanishing relations (*i.e.* if $P(X) = \sum_i p^i(X)\frac{\partial W}{\partial X_i}$), then using the BRST transformations (3.55), we easily see that $P(X) = s\,K$ and so Φ_P would be exact. (For the proof it suffices to set $K = p^i(X)\,\frac{\partial\beta_j}{\partial X_i}\,c_j$.) Thus, the physical observables in the topological theory are simply *local* polynomials of β_i and $\tilde\beta_i$, which correspond to chiral primary fields of the original n=2 superconformal theory.

On the other hand, comparing the expression of the 2-form $\Phi_P^{(2)}$ in (3.58) with the topological Lagrangian (3.54), it is easy to see that a deformation of the potential with some element $P(X)$ of the local ring, *i.e.*

$$W(X) \longrightarrow W(X) - t_P\,P(X) \quad , \tag{3.59}$$

corresponds to a perturbation of the action with $\int \Phi_P^{(2)}$, *i.e.*

$$\int d^2z\ \mathcal{L}_{\rm top} \longrightarrow \int d^2z\ \mathcal{L}_{\rm top} - t_P \int \Phi_P^{(2)} \quad . \tag{3.60}$$

Thus, the possible perturbations of the theory are in one-to-one correspondence with the possible deformations of the potential. As we are going to see , something similar happens also in the ordinary LG models, but only up to BRST-exact terms.

For the sake of comparison we now write the general form of the Lagrangian, of the supersymmetry transformations and, after twisting, of the topological BRST-transformations of an ordinary n=2 LG model [28, 29, 49]. A short rheonomic derivation of the results hereinafter reported is given in Appendix B1. Let $X^i(z,\bar z)$ be N complex scalar fields, $X^{i^*}(z,\bar z)$ their complex conjugates, ψ^i and ψ^{i^*} their left-moving anticommuting superpartners, and $\tilde\psi^i$ and $\tilde\psi^{i^*}$ their right-moving anticommuting superpartners. The Lagrangian for a LG model with superpotential W is

$$\begin{aligned}
\mathcal{L} &= -\left[\partial X^i\,\bar\partial X^{j^*} + \bar\partial X^i\,\partial X^{j^*}\right]\eta_{ij^*} + 8\,\partial_i W\,\partial_{j^*}\bar W\,\eta^{ij^*} \\
&+ 4\,\mathrm{i}\left[\psi^i\,\bar\partial\psi^{j^*} + \tilde\psi^i\,\partial\tilde\psi^{j^*}\right]\eta_{ij^*} \\
&+ 8\left[\partial_i\partial_j W\,\psi^i\tilde\psi^j - \partial_{i^*}\partial_{j^*}\bar W\,\psi^{i^*}\tilde\psi^{j^*}\right]
\end{aligned} \tag{3.61}$$

where η_{ij^*} is the flat Kählerian metric of $\mathbf{C}^n$. Here we have understood summations over repeated indices, and used the short-hand notations $\partial_i \equiv \partial/\partial X^i$ and $\partial_{i^*} \equiv \partial/\partial X^{i^*}$. The Lagrangian above is invariant against the following global n=2 supersymmetry transformations

$$\begin{aligned}
\delta X^i &= -\varepsilon^-\psi^i - \tilde\varepsilon^-\tilde\psi^i \quad , \\
\delta X^{i^*} &= +\varepsilon^+\psi^{i^*} + \tilde\varepsilon^+\tilde\psi^{i^*} \quad , \\
\delta\psi^i &= -\frac{\mathrm{i}}{2}\partial X^i\,\varepsilon^+ + \eta^{ij^*}\partial_{j^*}\bar W\,\tilde\varepsilon^- \quad , \\
\delta\tilde\psi^i &= -\frac{\mathrm{i}}{2}\bar\partial X^i\,\tilde\varepsilon^+ - \eta^{ij^*}\partial_{j^*}\bar W\,\varepsilon^- \quad , \\
\delta\psi^{i^*} &= \frac{\mathrm{i}}{2}\partial X^{i^*}\,\varepsilon^- + \eta^{ji^*}\partial_j W\,\tilde\varepsilon^+ \quad , \\
\delta\tilde\psi^{i^*} &= \frac{\mathrm{i}}{2}\bar\partial X^{i^*}\,\tilde\varepsilon^- - \eta^{ji^*}\partial_j W\,\varepsilon^+ \quad .
\end{aligned} \tag{3.62}$$

Contrary to our (b,c,β,γ) formulation, the global supersymmetries (3.62) do not extend to classical superconformal symmetries of the action (3.61), even when the superpotential $W(X)$ is a quasi-homogeneous function. Indeed, it is only after quantization that one can argue the equivalence of (3.61) at its infrared fixed point with a (2,2) superconformal model. Our theory in (3.26) instead, is superconformal already at the classical level whenever the potentials V and $\tilde{V}$ are quasi homogeneous. Of course, this applies in particular to the left-right symmetric case we are discussing where we have a single potential $W(\beta\tilde{\beta})$ that can be identified with the superpotential $W(X)$ of the LG theory.

Performing the topological twist does not modify the Lagrangian (3.61) but merely changes the spin of the fields [49]. If we choose as BRST-parameter $\theta = \varepsilon^+ = \tilde{\varepsilon}^+$ (as is appropriate for the + twist), the action of the topological Slavnov operator on the LG fields turns out to be

$$\begin{aligned} s\,X^i &= 0 \ , \\ s\,X^{i^*} &= \psi^{i^*} + \tilde{\psi}^{i^*} \ , \\ s\,\psi^i &= -\frac{\mathrm{i}}{2}\,\partial X^i \ , \\ s\,\tilde{\psi}^i &= -\frac{\mathrm{i}}{2}\,\bar{\partial} X^i \ , \\ s\,\psi^{i^*} &= \eta^{i^*j}\,\partial_j W \ , \\ s\,\tilde{\psi}^{i^*} &= -\eta^{i^*j}\,\partial_j W \ . \end{aligned} \tag{3.63}$$

Using (3.63), we can easily solve the descent equations (3.55) and find

$$\begin{aligned} \Phi_P &= P(X) \ , \\ \Phi_P^{(1)} &= -2\,\mathrm{i}\partial_i\,P\left(\psi^i\,dz + \tilde{\psi}^i\,d\bar{z}\right) \ , \\ \Phi_P^{(2)} &= -4\left[\partial_i\partial_j P\,\psi^i\tilde{\psi}^j \,+\, \partial_k P\,\partial_{l^*}\bar{W}\,\eta^{kl^*}\right] dz\wedge d\bar{z} \ , \end{aligned} \tag{3.64}$$

where $P(X)$ is a polynomial corresponding to some non trivial element of the local ring determined by the superpotential $W(X)$. Indeed, if $P(X)$ is proportional to the vanishing relations (*i.e.* if $P(X) = \sum_i p^i(X)\frac{\partial W}{\partial X_i}$), then using the BRST transformations (3.63), one can see that $P(X) = s\,K$ and so Φ_P would be exact. (For the proof it suffices to set $K = p^i(X)\psi^{j^*}\eta_{ij^*}$.)

It is interesting to observe that under the deformation

$$W \longrightarrow W - \frac{1}{2}t_P\,P(X) \ , \tag{3.65}$$

where $P(X)$ is some element of the local ring and t_P is the corresponding coupling constant, the (topological) LG action changes as follows

$$\int d^2z\;\mathcal{L} \longrightarrow \int d^2z\;\mathcal{L} - t_P\int \Phi_P^{(2)} - \bar{t}_P\int \bar{\Phi}_P^{(2)} \tag{3.66}$$

where $\mathcal{L}$ is given in (3.61), $\Phi_P^{(2)}$ in (3.64) and $\bar{\Phi}_P^{(2)}$ is the complex conjugate 2-form. These equations have to be compared with the analogous ones (3.58) and (3.59) of the (b,c,β,γ) theory. At first sight, in the LG models there seem to be a problem in identifying the topological perturbations of the Lagrangian with the deformations of the superpotential because of the last term in (3.66). However, this problem does not exist because the 2-form $\bar{\Phi}_P^{(2)}$ is BRST-exact, and so adding or not its integral to the action is completely irrelevant. In fact, using the BRST-transformations (3.63), one can check that

$$\bar{\Phi}_P^{(2)} = s\left(-4\,\partial_{j^*}\bar{P}\,\psi^{j^*}\right) \ . \tag{3.67}$$

We want to emphasize that in the (b,c,β,γ) formulation instead, there is no counterpart of this BRST-trivial part and deformations of the superpotential identically coincide with topological deformations of the Lagrangian.

We conclude this section by briefly commenting on the other choice of sign in the topological twist for our (b,c,β,γ)-system. If one chooses in (3.9) the $-$ sign, from (3.43) and (3.44) one obtains

$$\begin{aligned}\hat{T}_- &= T - \frac{1}{2}\partial J \\ &= \sum_{i=1}^{N}(1-2\lambda_i)\,\partial b_i\,c_i - 2\lambda_i\,b_i\partial c_i + (1-2\lambda_i)\,\gamma_i\,\partial\beta_i - 2\lambda_i\,\beta_i\,\partial\gamma_i \ .\end{aligned} \tag{3.68}$$

This is the canonical stress-energy tensor for N commuting (β,γ)-systems with weight $2\lambda_i$ and N anticommuting (b,c)-systems also with weight $2\lambda_i$. It is straightforward to check that T_- closes a Virasoro algebra with zero central charge; indeed the bosonic and fermionic contributions to the central charge exactly cancel each other. However, the cohomology classes of the BRST charge $Q_-^{\rm BRST}$ correspond to anti-chiral primary fields of the original n=2 algebra and these do not have a simple and local representation in terms of the elementary fields appearing in the Lagrangian: indeed, to describe the anti-chiral operators one has to resort to the bosonization of the (b,c,β,γ)-systems (see Section 3.2.2). On the other hand, as we explicitly show in Appendix B.2, after performing the topological twist, the Lagrangian is BRST-exact, *i.e.* it is of the form $\mathcal{L}_{\rm top} = \left[\ Q_-^{\rm BRST}\ ,\ \mathcal{L}'\ \right]$ for some local functional $\mathcal{L}'$. Using the terminology of [24], this means that the $-$ twist defines a topological field theory of the Witten-type. On the contrary, the $+$ twist leads to the Lagrangian (3.54) which is not BRST exact with respect to $Q_+^{\rm BRST}$; thus the $+$ twist defines a topological field theory of the Schwarz-type. As pointed out in [49], also the ordinary topological LG models are theories of the Schwarz-type.

In conclusion, we have shown that n=2 LG models admit a (b,c,β,γ)-formulation which is already superconformal at the classical level. After topological twisting, there is a natural correspondence between the deformations of the LG potential and the abstract topological deformations. In the next sections, after discussing the renormalization group properties of our theory, we shall illustrate how one can use this explicit formulation to calculate (perturbed) topological correlation functions in LG models.

3.2.1 On the quantum properties of the system

In the previous section we discussed the classical properties of the action (3.26) and showed that with a suitable choice of the interaction potential, the theory exhibits a non trivial (2,2)-superconformal invariance. However, the presence of interactions can in principle spoil this invariance at the quantum level and one has eventually to restore it after a suitable renormalization [89].

For the sake of clarity, we begin by considering a single left-right symmetric (b,c,β,γ)-system of weight $\lambda=\tilde{\lambda}$ with potential

$$W=\frac{1}{(n+1)^2}(\beta\tilde{\beta})^{n+1} \quad . \tag{3.69}$$

This corresponds to the A_n minimal model of the n=2 discrete series if $\lambda=1/(2n+2)$ (see (3.47)). We are going to give now two different "proof" that the interaction (3.69) preserve all the classical properties of the $(2,2)$ action, and in particular the conformal invariance. The first one is supported by a so called "criterion for integrability" of marginal operator that is strongly used in the literature (see for example [79]). It gives the recipe to state when a perturbing operator is "truly marginal" (i.e. it does not modify its conformal dimension at quantum level). Let us explain it.

From a purely conformal field theory point of view, once requiring that the interaction term has the right (1,1) dimension, we are perturbing the theory with a "candidate" marginal operator. The mere existence of a (1,1) operator is not sufficient to guarantee the existence of a fixed line. We have to require additional "integrability"conditions (see [79]), so that the perturbation generated by the marginal operator does not act to change its own conformal weight from (1,1). In our case, where we have collected the potential term as a single marginal operator O with coupling g, this reduce to one loop level to the requirement that in the operator product expansion of O with itself there are no term of the form:

$$O(z,\bar{z})O(w,\bar{w})=c_{OOO}(z-w)^{-1}(\bar{z}-\bar{w})^{-1}O \ . \tag{3.70}$$

If this condition is not satisfied the conformal weight of O is shifted by a quantity proportional to the three point function c_{OOO}: O would not remain marginal away from the point of departure, and could not be used to generate a family of conformal field theories. To higher orders we need to require as well the vanishing of integrals of the $(n+2)$–point functions of the O's. If this is the case O is called a "truly marginal operator". It is easily verified that our interaction satisfies this requirement (even in the general case), due to the operator product expansion of the b, β fields. We will do some more comment on this point in Appendix B.3, where we will discuss some different possible choices of marginal perturbations.

The second "proof" is more direct, we explicitly compute the stress–energy tensor and show that the interaction considered does not give any loop correction to it. For the time being we leave the weight λ unfixed. The Lagrangian for this system is $\mathcal{L} = \mathcal{L}_0 + \Delta\mathcal{L}$ where $\mathcal{L}_0$ is as in (3.27) and the interaction term is

$$\Delta\mathcal{L} = g\ b\tilde{b}\,\beta^n\tilde{\beta}^n \tag{3.71}$$

where g is a coupling constant. Since the weight of β and $\tilde{\beta}$ is arbitrary, g is a quantity with dimension

$$[g] = (1 - 2\lambda(n+1)) \ . \tag{3.72}$$

To study the scaling properties of this system, we compute the trace of the stress-energy tensor which turns out to be [3]

$$T_{z\bar{z}} = \left[-\lambda\beta\bar{\partial}\gamma + (1-\lambda)\gamma\bar{\partial}\beta - (\lambda+\tfrac{1}{2})b\bar{\partial}c - (\tfrac{1}{2}-\lambda)c\bar{\partial}b + g\ b\tilde{b}\beta^n\tilde{\beta}^n\right] + \text{c.c.} \ . \tag{3.73}$$

After using the equations of motion (3.29), we have

$$\Theta \equiv -T_{z\bar{z}} = g(2(n+1)\lambda - 1)\, b\tilde{b}\,\beta^n\tilde{\beta}^n \ , \tag{3.74}$$

so that our system is classically invariant under scale transformations (*i.e.* $\Theta = 0$) either if

$$g = 0 \qquad \text{for any} \quad \lambda \ , \tag{3.75}$$

or if

$$\lambda = \frac{1}{2(n+1)} \qquad \text{for} \ \ g \neq 0 \ . \tag{3.76}$$

Discarding the case (3.75) which corresponds to a free theory, we see from (3.76) that λ must be fixed by the homogeneous weight of the potential (cf. (3.35)); when (3.76) is satisfied of course g becomes dimensionless and the operator $b\tilde{b}\,\beta\tilde{\beta}$ becomes marginal, so that no dimensionful parameters are left in the model.

Let us now quantize this system by using perturbation theory in g. From the explicit expression of the Lagrangian $\mathcal{L}$, we see that the propagators are

$$\begin{aligned} \langle\gamma(z,\bar{z})\beta(w,\bar{w})\rangle &= \langle b(z,\bar{z})c(w,\bar{w})\rangle = \frac{1}{z-w} \ , \\ \langle\tilde{\gamma}(z,\bar{z})\tilde{\beta}(w,\bar{w})\rangle &= \langle\tilde{b}(z,\bar{z})\tilde{c}(w,\bar{w})\rangle = \frac{1}{\bar{z}-\bar{w}} \ , \end{aligned} \tag{3.77}$$

[3]Here and in the following "c.c." means exchanging the untilded fields with the tilded ones and ∂ with $\bar{\partial}$.

so that it is obvious that even when the interaction (3.71) is present, it is impossible to form loops. Therefore we conclude that there are no (perturbative) quantum corrections to the classical results simply because there are no loops! These considerations imply in particular that Θ in (3.74) is also the *quantum* trace of the stress-energy tensor and hence the coefficient of the spinless operator $b\tilde{b}\beta^n\tilde{\beta}^n$ appearing in (3.74) can be interpreted as a renormalization group β–function [15], namely

$$\beta(g) = g\,(2(n+1)\lambda - 1) \quad . \tag{3.78}$$

The zeroes of $\beta(g)$ identify the conformal fixed points and these are given precisely by (3.75) and (3.76).

It is now interesting to see what happens when a second interaction

$$\Delta\mathcal{L}' = g' b\tilde{b}\beta^m\tilde{\beta}^m \qquad (m < n) \tag{3.79}$$

is added to the original system. We now assume that $\lambda = 1/(2n+2)$ so that (3.79) can be considered as a perturbation around a conformal theory. Following the same procedure as above, we compute the β–function $\beta(g')$ and find

$$\beta(g') = g'\left(\frac{m+1}{n+1} - 1\right) = \frac{m-n}{n+1}g' \quad . \tag{3.80}$$

It is clear from (3.80) that the new model does not have any non-trivial fixed point; indeed the only solution to $\beta(g') = 0$ is $g' = 0$ which is achieved in the ultraviolet regime for $m < n$. Hence we cannot have a renormalization group flow to another n=2 superconformal field theory, in agreement with the conclusions of [90].

The extension of these results to the generic case of quasi-homogeneous potentials is an easy task. To this end let us first recall that if f is a quasi–homogeneous polynomial in N variables with weights $(\omega_1, \cdots \omega_N)$, $(\omega_i \in Q,\, \omega_i > 0)$ and

$$f = \sum_\rho a_\rho x^\rho \tag{3.81}$$

where $\rho \equiv (\rho_1, \cdots \rho_N)$, $X^\rho \equiv X_1^{\rho_1} \cdots X_N^{\rho_N}$, $\rho_i \in Z^+$ and $a_\rho \neq 0$, then

$$\rho_1\omega_1 + \cdots \rho_N\omega_N = 1 \quad . \tag{3.82}$$

Let us now consider the following interaction term

$$\Delta\mathcal{L} = g\sum_{i,j} b_i\tilde{b}_j\partial_i V\tilde{\partial}_j\tilde{V} \tag{3.83}$$

where $V(\beta)$ and $\tilde{V}(\tilde{\beta})$ are quasi–homogeneous potentials satisfying (3.81) and (3.82). For simplicity, we take $V(\beta) = \tilde{V}(\tilde{\beta})$ and assume that the weights $\lambda_i = \tilde{\lambda}_i$ are unconstrained. Then, the trace of the stress–energy tensor, upon using the equations of motion (3.29), turns out to be

$$\Theta = -T_{z\bar{z}} = - g\sum_{i,j}\left(b_i\tilde{b}_j\partial_i V\tilde{\partial}_j\tilde{V} - 2\lambda_i\beta_i\tilde{b}_j\partial_i V\tilde{\partial}_j\tilde{V}\right) \tag{3.84}$$

$$- \sum_{i,j,l}\left(\lambda_i\beta_i b_l\tilde{b}_j\partial_i\partial_l V\tilde{\partial}_j\tilde{V} - \lambda_i\tilde{\beta}_i b_j\tilde{b}_l\tilde{\partial}_i\tilde{\partial}_l\tilde{V}\partial_j V\right) \quad . \tag{3.85}$$

Using (3.81), after some algebra, the trace (3.85) can be rewritten as

$$\Theta = g\left(2\sum_k \lambda_k\rho_k - 1\right)\left(\sum_{i,\rho} a_\rho b_i\rho_i\beta_1^{\rho_1}\cdots\beta_N^{\rho_N}\right)\left(\sum_{j,\rho}\tilde{a}_\rho\tilde{b}_j\rho_j\tilde{\beta}_1^{\rho_1}\cdots\tilde{\beta}_N^{\rho_N}\right) \quad . \tag{3.86}$$

If $g \neq 0$, the system is invariant under scale transformations only if

$$\sum_i \lambda_i\rho_i = \frac{1}{2} \tag{3.87}$$

Comparing (3.87) with (3.81) and (3.82) we see that the weights λ_i must be one half of the homogeneous weights of the potential ω_i. Since there are no loop corrections, this result extends automatically to the quantum theory. Furthermore, we point out that the same conclusion is obtained in a similar way when $V(\beta)$ and $\tilde{V}(\tilde{\beta})$ are different, or when the interaction depends on a single quasi–homogeneous function W in the variables $X_i = \beta_i\tilde{\beta}_i$ with weight ω_i.

Finally, in our formulation it is easy to realize that the potential

$$\hat{V}(\beta^{(i,A)}) = V(\beta^i) + \sum_{A=n+1}^{m+n}(\beta^A)^2 \tag{3.88}$$

defines the same conformal theory as the potential V (as one should expect from the notion of stable singularity [30, 75]). Indeed, a (b, c, β, γ)-system with $\lambda_A = \frac{1}{4}$ gives a $c = 0$ conformal field theory.

3.2.2 Bosonization of the (b, c, β, γ)-system

The result of our previous analysis is that for any value $\lambda = (2n+2)^{-1}$ with $n = 2, 3, \ldots$, we have a realization of the n=2 superconformal algebra with central charge

$$c = 3 - 12\lambda = \frac{3(n-1)}{n+1} \quad . \tag{3.89}$$

The conformal weights h and the $U(1)$-charges q of the pseudo-ghost fields that define such a realization are given by

$$h(\beta) = \frac{1}{2(n+1)} \quad , \quad q(\beta) = \frac{1}{n+1} \quad ; \tag{3.90}$$

$$h(\gamma) = \frac{2n+1}{2(n+1)} \quad , \quad q(\gamma) = -\frac{1}{n+1} \quad ; \tag{3.91}$$

$$h(b) = \frac{n+2}{2(n+1)} \quad , \quad q(b) = -\frac{n}{n+1} \quad ; \tag{3.92}$$

$$h(c) = \frac{n}{2(n+1)} \quad , \quad q(c) = \frac{n}{n+1} \quad . \tag{3.93}$$

The Fock space generated by the modes of the almost-free fields b, c, β, γ and their spin fields, contains the irreducible representations of the n=2 minimal models. Such representations can be obtained from the Fock space through a suitable projection like in the case of the standard free-field realization of the minimal models as given for instance in [50]. In this section our aim is to make contact with this Coulomb gas formalism, which, as we will see in the sequel, enables us to calculate explicitly (perturbed) topological correlation functions in the presence of a LG interaction.

Adopting the conventions of [50], an n=2 minimal model with central charge as in (3.89), can be described in terms of three scalar fields ϕ_0, ϕ_1 and ϕ_2 with mode expansions

$$\phi_i(z) = \hat{q}_i - \hat{p}_i \ln z + \sum_{k\neq 0} \frac{\hat{a}^i_k}{k} z^{-k} \quad , \quad i = 0,1,2 \quad , \tag{3.94}$$

where

$$[\hat{q}_i, \hat{p}_j] = \delta_{ij} \quad , \quad [\hat{a}^i_k, \hat{a}^j_\ell] = k\, \delta_{k+\ell,0}\, \delta^{ij} \quad . \tag{3.95}$$

While ϕ_0 and ϕ_2 are really free fields, ϕ_1 is coupled to a background charge

$$Q_1^{(n)} = \sqrt{\frac{2}{n+1}} \quad . \tag{3.96}$$

In this realization the holomorphic currents of the n=2 superconformal algebra are [4]

$$T = \frac{1}{2}\left[(\partial\phi_0)^2 + (\partial\phi_1)^2 + (\partial\phi_2)^2\right] - \frac{1}{2} Q_1^{(n)}\, \partial^2\phi_1 \quad , \tag{3.97}$$

$$J = \sqrt{\frac{n-1}{n+1}}\, \partial\phi_0 \quad , \tag{3.98}$$

$$G^\pm = \sqrt{\frac{2n-2}{n+1}}\, \Psi^\pm \exp\left[\pm\sqrt{\frac{n+1}{n-1}}\, \phi_0\right] \quad , \tag{3.99}$$

$$\Psi^\pm = \frac{1}{\sqrt{2}} \exp\left[\pm \mathrm{i}\sqrt{\frac{2}{n-1}}\, \phi_2\right] \left(\sqrt{\frac{n+1}{n-1}}\partial\phi_1 \pm \mathrm{i}\, \partial\phi_2\right) \quad . \tag{3.100}$$

The field ϕ_0 bosonizes the U(1)-current and its exponentials realize the well known n=2 spectral flow [38]. The operators $\Psi^\pm$ in (3.100) are, instead, parafermionic currents

[4] Here and in the following, any exponential of free fields is understood as normal ordered.

and generate the non trivial part of the n=2 algebra. The complete Fock space which embeds the n=2 irreducible modules is generated by the vertex operators

$$V_{q,\ell,m} = \exp\left[\frac{q}{\sqrt{n^2-1}}\phi_0 + \frac{\ell}{\sqrt{2(n+1)}}\phi_1 + \mathrm{i}\frac{m}{\sqrt{2(n-1)}}\phi_2\right] \tag{3.101}$$

and their derivatives.

In particular the n=2 primary fields are given by

$$\Lambda^{(n)}_{\ell,m;s} = \frac{1}{\sqrt{2}}\exp\left[\frac{m+sn-s}{\sqrt{n^2-1}}\phi_0 + \frac{\ell}{\sqrt{2(n+1)}}\phi_1 + \mathrm{i}\frac{m}{\sqrt{2(n-1)}}\phi_2\right] \tag{3.102}$$

where ℓ takes the integer values $0 \leq \ell \leq n-1$ and m takes the integer values $m = -l, -l+2, ..., l$. The quantum number s represents the sector and is 0 in the Neveu-Schwarz sector and $\pm 1/2$ in the Ramond sector. The conformal weight h and the $U(1)$-charge q of $\Lambda^{(n)}_{\ell,m;s}$ are given by the standard formulas

$$\begin{aligned} h(\ell,m;s) &= \frac{\ell(\ell+2)}{4(n+1)} - \frac{m^2}{4n-4} + \frac{(m+sn-s)^2}{2(n^2-1)} \quad , \\ q(m;s) &= \frac{m+sn-s}{n+1} \quad . \end{aligned} \tag{3.103}$$

As we will see later, it is convenient to factor out the ϕ_0 contribution and rewrite the primary fields (3.102) as

$$\Lambda^{(n)}_{\ell,m;s} = \exp\left[\sqrt{\frac{3}{c}}q(m,s)\ \phi_0\right]\ \varphi^{\ell}_{m} \tag{3.104}$$

where

$$\varphi^{\ell}_{m} = \exp\left[\frac{\ell}{\sqrt{2(n+1)}}\phi_1 + \mathrm{i}\frac{m}{\sqrt{2(n-1)}}\phi_2\right] \quad .$$

The operators φ^{ℓ}_{m} are the principal primary fields of the $\mathbf{Z}_{n-1}$ parafermion algebra and must be identified according to

$$\varphi^{\ell}_{m} \sim \varphi^{n-1-\ell}_{m\pm(n-1)} \quad . \tag{3.105}$$

In fact the Hilbert space created by φ^{ℓ}_{m} is isomorphic to the one created by $\varphi^{n-1-\ell}_{m\pm(n-1)}$ due to the existence of a map between the two that commutes with all generators of the algebra [50].

In order to relate this realization of the n=2 minimal models to the one provided by our (b,c,β,γ)-system, we bosonize the latter according to the standard rules and write[5]

$$b = e^{-\pi_1} \quad , \quad \beta = e^{i\pi_2 - \pi_3} \quad , \quad c = e^{\pi_1} \quad , \quad \gamma = e^{-i\pi_2 + \pi_3}\,\partial\pi_3 \tag{3.106}$$

where the π_i's are scalar fields coupled to the following background charges

$$\tilde{Q}_1^{(n)} = -\frac{1}{n+1} \quad , \quad \tilde{Q}_2^{(n)} = i\frac{n}{n+1} \quad , \quad \tilde{Q}_3^{(n)} = -1 \ . \tag{3.107}$$

These numbers are explained as follows: π_1 bosonizes the anticommuting (b,c)-system whose weight is $\lambda + \frac{1}{2} = \frac{n+2}{2(n+1)}$. Insertion of this value in the general formula (3.53) yields $\tilde{Q}_1^{(n)}$ as listed in (3.107). The field $i\pi_2$ bosonizes the commuting (β, γ)-system according to the rule

$$\gamma = e^{-i\pi_2}\,\partial\xi \quad , \quad \beta = e^{i\pi_2}\,\eta \tag{3.108}$$

where ξ and η form an anticommuting first-order system of weight $\lambda_{\xi\eta} = 1$. The background charge $\tilde{Q}_2^{(n)}$ of the field π_2 follows from (3.53) with $\lambda_{\beta\gamma} = \lambda = \frac{1}{2n+2}$. Finally π_3 is the scalar field that bosonizes the (ξ, η)-system and its background charge $\tilde{Q}_3^{(n)}$ also follows from (3.53) upon use of the value $\lambda_{\xi\eta} = 1$.

Consequently, in terms of the fields π_i's , the stress-energy tensor of the n=2 model is

$$T = \frac{1}{2}\left[(\partial\pi_1)^2 + (\partial\pi_2)^2 + (\partial\pi_3)^2\right] - \frac{1}{2}\left(-\frac{1}{n+1}\,\partial^2\pi_1 + i\frac{n}{n+1}\,\partial^2\pi_2 - \partial^2\pi_3\right) \ . \tag{3.109}$$

Similarly, using (3.106) in (3.41) and (3.44) we obtain

$$J = \frac{1}{n+1}\left(n\partial\pi_1 - i\partial\pi_2\right) \ , \tag{3.110}$$

$$G^- = 2\sqrt{2}\exp\left[-\pi_1 - i\,\pi_2 + \pi_3\right]\partial\pi_3 \ . \tag{3.111}$$

Comparing (4.16) and the last two equations with (3.97)–(3.100), we obtain the relation between the π's and the ϕ's, namely

$$\pi_1 = \frac{n}{\sqrt{n^2-1}}\,\phi_0 - \frac{1}{\sqrt{2(n+1)}}\,\phi_1 + i\frac{1}{\sqrt{2(n-1)}}\,\phi_2 \ , \tag{3.112}$$

$$i\,\pi_2 = \frac{1}{\sqrt{n^2-1}}\,\phi_0 - \frac{n}{\sqrt{2(n+1)}}\,\phi_1 + i\frac{n}{\sqrt{2(n-1)}}\,\phi_2 \ , \tag{3.113}$$

$$\pi_3 = -\sqrt{\frac{n+1}{2}}\,\phi_1 + i\sqrt{\frac{n-1}{2}}\,\phi_2 \ . \tag{3.114}$$

[5]Notice that the bosonization rules we are giving are actually true for the "free" holomorphic part of the c, γ fields. However, as we are going to see, we only need of the bosonized b, β fields (which are correctly expressed by (3.106), (3.115) and (3.116) in application to topological correlation functions

One can proceed even further and use (4.13) and (3.112)– (3.114) to identify the pseudo-ghost fields with the operators of the abstract n=2 superconformal model. Explicitly one finds

$$\beta = \exp\left[\frac{1}{\sqrt{n^2-1}}\phi_0 + \frac{1}{\sqrt{2(n+1)}}\phi_1 + \mathrm{i}\frac{1}{\sqrt{2(n-1)}}\phi_2\right] , \tag{3.115}$$

$$b = \exp\left[\frac{-n}{\sqrt{n^2-1}}\phi_0 + \frac{1}{\sqrt{2(n+1)}}\phi_1 + \mathrm{i}\frac{-1}{\sqrt{2(n-1)}}\phi_2\right] , \tag{3.116}$$

$$c = \exp\left[\frac{n}{\sqrt{n^2-1}}\phi_0 + \frac{-1}{\sqrt{2(n+1)}}\phi_1 + \mathrm{i}\frac{1}{\sqrt{2(n-1)}}\phi_2\right] , \tag{3.117}$$

$$\gamma = \exp\left[\frac{-1}{\sqrt{n^2-1}}\phi_0 + \frac{-1}{\sqrt{2(n+1)}}\phi_1 + \mathrm{i}\frac{-1}{\sqrt{2(n-1)}}\phi_2\right] \times$$

$$\times \sqrt{\frac{n-1}{2}}\left(-\sqrt{\frac{n+1}{n-1}}\partial\phi_1 + \mathrm{i}\partial\phi_2\right) . \tag{3.118}$$

From (3.115) one realizes that β is a chiral primary field and is given by

$$\beta = \Lambda^{(n)}_{1,1;0} . \tag{3.119}$$

More generally one can write

$$\beta^\ell = \Lambda^{(n)}_{\ell,\ell;0} \quad \text{for } \ell = 0, \dots, n-1 , \tag{3.120}$$

which shows that at the quantum level the general chiral primary field is simply the ℓ-th power of β and the vanishing relation is recovered by enforcing the bound $\ell \leq n-1$. Moreover b is the first component of a chiral primary superfield and can be explicitly obtained by $\frac{1}{2\sqrt{2}} \oint_z G^-(w)\beta(w) = b(z)$; the same is true for fields of the form $b\beta^{l-1}$. On the contrary γ and c are in the Fock space of the three scalar fields, but not in the n=2 irreducible module.

It could be interesting to mention that in the bosonized formalism the operator $b\beta^n$, which appear in the interaction term is simply expressed by:

$$b\beta^n = e^{\sqrt{\frac{n+1}{2}}\phi_1 - i\sqrt{\frac{n-1}{2}}\phi_2} = \eta \tag{3.121}$$

where η is the fields appearing in (3.108). Equation (3.121) shows that $b\beta^n$ coincides precisely with the screening operator $S_+(z)$ in ref [50], which can be used to construct the Felder complex [91], and the Fock space. The Felder BRST Q_+ charge is given by $Q_+ = \oint dz S_+(z) = \eta_0$, where η_0 is the zero mode for η.

We now consider the case when the theory is topologically twisted with

$$Q_+(z) = G^+(z) . \tag{3.122}$$

As mentioned in Section 3.2, we have a new (b, c, β, γ)-system with $\lambda_\beta = 0$, $\lambda_b = 1$ whose Lagrangian is given in (3.54). These new pseudo-ghost fields are still bosonized as in (3.106), but now the background charges of the π_i's become

$$\tilde{Q}_1^{(n)} = -1 \ , \ \tilde{Q}_2^{(n)} = \mathrm{i} \ , \ \tilde{Q}_3^{(n)} = -1 \ . \tag{3.123}$$

The new stress-energy tensor is then given by

$$\hat{T} = \frac{1}{2}\left[(\partial\pi_1)^2 + (\partial\pi_2)^2 + (\partial\pi_3)^2\right] - \frac{1}{2}\left(-\partial^2\pi_1 + \mathrm{i}\,\partial^2\pi_2 - \partial^2\pi_3\right) \ , \tag{3.124}$$

whereas the $U(1)$-current J is the same as in (3.110). On the other hand, the twist of the stress-energy tensor is given by $\hat{T} = T + \frac{1}{2}J$, and hence, using (3.97) , (3.98) we get

$$\hat{T} = \frac{1}{2}\left[(\partial\phi_0)^2 + (\partial\phi_1)^2 + (\partial\phi_2)^2\right] - \frac{1}{2}\sqrt{\frac{2}{n+1}}\partial^2\phi_1 + \frac{1}{2}\sqrt{\frac{n-1}{n+1}}\partial^2\phi_0 \ . \tag{3.125}$$

If we now compare (3.124) and (3.125) and observe that J is the same before and after the twist, so that

$$J = \sqrt{\frac{n-1}{n+1}}\partial\phi_0 = \frac{n}{n+1}\partial\pi_1 - \frac{i}{n+1}\partial\pi_2 \ , \tag{3.126}$$

we can realize that the relations (3.112)–(3.114) and the identifications (3.115)–(3.118) and (3.120) hold true also in the topological field theory, giving us a complete characterization of the fields b, c, β and γ at the quantum level. Moreover, using (3.115)–(3.118) and taking into account both left and right movers, we can easily check the descent equations (3.56) in the complete bosonized formalism. As is clear from (3.125) in comparison with (3.100), the net effect of the topological twist is simply to switch on a background charge for ϕ_0 given by

$$Q_0^{(n)} = \frac{1-n}{\sqrt{n^2-1}} \ . \tag{3.127}$$

Therefore, even if the bosonized expressions for the topological b, c, β and γ are still given by (3.115)–(3.118), their conformal dimensions change with the twist. In particular the chiral primary fields β^ℓ loose their conformal weight and become dimensionless, as is appropriate for the physical operators of a topological field theory. Furthermore, the $U(1)$ current J acquires an anomaly proportional to $Q_0^{(n)}$.

Let us briefly mention that if we perform the topological twist with $Q_- = G^-$ instead of $Q_+ = G^+$, not only the the stress energy tensor but also the lagrangian becomes a BRST commutator (see Appendix B.2 for details). In this case, however, there is no identification of the chiral fields in terms of *local* expressions of b, c, β or γ: they can only be written in a bosonized form. As is well known, chiral fields with respect to the BRST charge $Q_- = G^-$ are antichiral fields with respect to the BRST charge $Q_+ = G^+$. This means that the bosonized expression of these fields could be obtained via spectral flow from (3.120).

The complete bosonization of the (b, c, β, γ)-system we have just presented is the technical tool which enables us to make explicit calculations of (perturbed) correlation functions for single minimal models as well as for tensor products thereof. It is also very useful in establishing the precise relationship between the correlation functions of topological minimal models and the chiral Green functions of LG theories as computed in [29].

To this end, let us first introduce the following notation

$$|q, \ell, m\rangle \equiv \lim_{z \to 0} V_{q,\ell,m}(z)\ |0,0,0\rangle \tag{3.128}$$

where $V_{q,\ell,m}$ is defined in (3.101) and $|0,0,0\rangle$ is the $Sl(2,\mathbf{C})$ invariant vacuum of ϕ_0, ϕ_1 and ϕ_2. Before the topological twist only ϕ_1 has a background charge and the dual conjugate of $|q, \ell, m\rangle$ is $\langle -q, -2-\ell, -m|$. After the twist also ϕ_0 acquires a background charge and so the dual conjugate of $|q, \ell, m\rangle$ becomes $\langle n-1-q, -2-\ell, -m|$.

In the LG theory with superpotential $W \sim X^{n+1}$, the supersymmetric vacua $|m\rangle$ $(m = 0, \ldots, n-1)$ are identified at the conformal point with the Ramond vacua of the minimal model A_n. According to (3.102), the Ramond chiral primary fields of such a model are

$$R_m(z) = \mathcal{N}_m\ \Lambda^{(n)}_{m,m;-\frac{1}{2}}(z) \tag{3.129}$$

where $m = 0, \ldots, n-1$ and the normalization factor $\mathcal{N}_m$ is introduced to enforce the standard structure constants of the n=2 operator algebra [6]. This normalization can be computed using different techniques [29, 51] and is given by

$$\mathcal{N}_m{}^2 = \frac{1}{(2n+2)^{\frac{m}{n+1}}} \sqrt{\frac{\sin\left(\frac{\pi}{n+1}\right)}{\sin\left(\frac{\pi(m+1)}{n+1}\right)} \frac{\Gamma\left(\frac{1}{n+1}\right)}{\Gamma\left(\frac{m+1}{n+1}\right)}}\ . \tag{3.130}$$

Therefore at the conformal point, the supersymmetric vacua of the LG theory are

$$\begin{aligned} |m\rangle &= \lim_{z\to 0} R_m(z)|0,0,0\rangle_{\rm L} \times \lim_{\bar z \to 0} R_m(\bar z)|0,0,0\rangle_{\rm R} \\ &= \mathcal{N}_m{}^2 |m-(n-1)/2, m, m\rangle_{\rm L} \times |m-(n-1)/2, m, m\rangle_{\rm R} \end{aligned} \tag{3.131}$$

where the subscripts L and R refer to the holomorphic and anti-holomorphic components respectively. The vacua $\langle m|$ are obtained by taking the dual conjugate of (3.131)

[6] Notice that the Ramond fields R_m do not have a local expression in terms of the fields b, c, β, γ of our model, contrary to the Neveu-Schwarz chiral primaries which are simply powers of the bosonic field β.

and remembering that only ϕ_1 has a background charge (indeed the topological twist has not been performed yet).

It is now straightforward to compute the correlation functions of a string of chiral primary fields between two supersymmetric vacua. From Eqs. (3.120) and (3.131) we have

$$\langle m_1|\prod_{i=1}^{N}\left(\beta(z_i)\tilde{\beta}(\bar{z}_i)\right)^{\ell_i}|m_2\rangle = \frac{\mathcal{N}_{m_2}{}^2}{\mathcal{N}_{m_1}{}^2}\prod_{i=1}^{N}(z_i\bar{z}_i)^{\frac{-\ell_i}{2n+2}}\ \delta\left(\sum_i \ell_i + m_2 - m_1\right) \qquad (3.132)$$

where the δ-function arises from charge conservation. Apart from the z-dependent factor, this result coincides with the LG chiral Green functions computed in [29] using quantum field theory techniques. To make the precise comparison, however, one has to remember that in [29] the LG theory was defined on a cylinder, whereas our formula (3.132) applies to the plane. This difference is easily eliminated by mapping the plane to the cylinder, under which the chiral primary fields transform as

$$\beta^{\ell_i}(z_i) \longrightarrow \beta^{\ell_i}(w_i)\ z_i^{\frac{\ell_i}{2n+2}} \quad . \qquad (3.133)$$

Here w_i are the cylinder coordinates. An analogous expression holds also for the $\tilde{\beta}$ fields. The z-dependent factors of (3.132) are therefore cancelled in going from the plane to the cylinder and we can conclude that

$$\langle m_1|\prod_{i=1}^{N}\left(\beta\tilde{\beta}\right)^{\ell_i}|m_2\rangle\Big|_{\rm cyl} = \frac{\mathcal{N}_{m_2}{}^2}{\mathcal{N}_{m_1}{}^2}\ \delta\left(\sum_i \ell_i + m_2 - m_1\right) \qquad (3.134)$$

exactly coincides (normalization factors included) with the chiral Green function of the LG theory of [29]. Once more we see that the L.G. field X has to be identified with the product $(\beta\tilde{\beta})$ (see also Eq. (3.50)).

We now proceed to establish the relationship with the topological conformal field theories. To this end let us consider a particular case of (3.132), namely the correlation functions between the lowest vacuum $|0\rangle$ and the highest one $\langle n-1|$,

$$\langle n-1|\ \prod_{i=1}^{N}\left(\beta(z_i)\tilde{\beta}(\bar{z}_i)\right)^{\ell_i}|0\rangle\ =$$

$$\frac{\mathcal{N}_0}{\mathcal{N}_{n-1}}\quad {}_{\rm L}\langle(1-n)/2,-1-n,1-n|\prod_{i=1}^{N}\beta^{\ell_i}(z_i)|(1-n)/2,0,0\rangle_{\rm L}\times(\text{c.c.}) \qquad (3.135)$$

Since

$$|(1-n)/2,0,0\rangle = \exp\left[\frac{(1-n)/2}{\sqrt{n^2-1}}\hat{q}_0\right]\ |0,0,0\rangle \qquad (3.136)$$

and

$$\beta^{\ell}(z)\ \exp\left[\frac{(1-n)/2}{\sqrt{n^2-1}}\hat{q}_0\right] = \exp\left[\frac{(1-n)/2}{\sqrt{n^2-1}}\hat{q}_0\right]\ \beta^{\ell}(z)\ z^{\frac{-\ell}{2n+2}}\quad , \qquad (3.137)$$

from (4.34) and (4.31), we get

$$_{\mathrm{L}}\langle 1-n,-1-n,1-n|\prod_{i=1}^{N}\beta^{\ell_i}(z_i)|0,0,0\rangle_{\mathrm{L}}\times(\mathrm{c.c.})=\delta\left(\sum_i \ell_i-n+1\right) \quad . \tag{3.138}$$

This is the natural candidate for a topological correlation function. Notice that (3.138) is independent of z_i as any topological correlator should be. Indeed all the z-dependent factors are canceled in flowing from the lowest vacuum $|0\rangle$ of the Ramond sector to the $Sl(2,\mathbf{C})$ invariant vacuum $|0,0,0\rangle$ of the Neveu-Schwarz sector. However, the fields ϕ_0, ϕ_1 and ϕ_2 which implicitly appear in (3.138) are those which bosonize the original (b,c,β,γ)-system *before* the topological twist. To obtain a more adequate characterization of topological correlation functions in the bosonized formalism, it is more appropriate to use the fields which bosonize a twisted (b,c,β,γ)-system. As we have seen, only very few things change; most notably the field ϕ_0 acquires the background charge (3.127). Thus, in (3.138) instead of the state $\langle 1-n,-1-n,1-n|$, which is the dual conjugate of $|n-1,n-1,n-1\rangle$ before the twist, we should have the state $\langle 0,-1-n,1-n|$ which is the conjugate of $|n-1,n-1,n-1\rangle$ after the twist. If we define the topological vacuum $|0\rangle_{\mathrm{top}}$ as

$$|0\rangle_{\mathrm{top}}\equiv|0,0,0\rangle \tag{3.139}$$

and its dual conjugate $_{\mathrm{top}}\langle\infty|$ as

$$_{\mathrm{top}}\langle\infty|\equiv\langle n-1,-2,0| \quad , \tag{3.140}$$

we can rewrite the holomorphic part of (3.138) and define the topological correlator as follows [7]

$$\begin{aligned}\langle\prod_{i=1}^{N}\beta^{\ell_i}(z_i)\rangle_{\mathrm{top}} &\equiv \langle 0,-1-n,1-n|\prod_{i=1}^{N}\beta^{\ell_i}(z_i)|0,0,0\rangle \\ &= {}_{\mathrm{top}}\langle\infty|\Omega^\dagger\prod_{i=1}^{N}\beta_i^{\ell}(z_i)|0\rangle_{\mathrm{top}} \\ &= \delta\left(\sum_i \ell_i-n+1\right)\end{aligned} \tag{3.141}$$

where

$$\Omega=\exp\left[\frac{n-1}{\sqrt{n^2-1}}\hat{q}_0+\frac{n-1}{\sqrt{2(n+1)}}\hat{q}_1+\mathrm{i}\frac{n-1}{\sqrt{2(n-1)}}\hat{q}_2\right] \tag{3.142}$$

and the † operation is defined as $\left(\mathrm{e}^{\alpha\hat{q}_i}\right)^\dagger=\mathrm{e}^{-\alpha\hat{q}_i}$ (see e.g. [50]). Notice that Ω is simply the zero-mode part (the only one which survives on the left vacuum) of the top chiral primary field β^{n-1}.

[7] Notice that in the topological case, a string of β between the "bra" and "ket" does not give zero, as in the non–topological case. This is due to the definition of the dual conjugate of the topological vacuum

Therefore our conclusion is that in the bosonized formalism a topological correlation function of a string of fields is obtained by taking the expectation value between the topological vacua $|0,0,0\rangle$ and $\langle 0,-1-n,1-n|$.

3.2.3 Explicit calculations of topological correlation functions

In this section we show in a few examples how to use the (b,c,β,γ) representation to compute *explicitly* some (perturbed) topological correlation functions. We also verify that the parameters of the deformed LG potential for the (b,c,β,γ)-system are the flat coordinates of the topological field theories. We stress that our techniques can be applied to single minimal models as well as to their tensor products, since there is practically no difference between the two cases. Even though the final goal is to use our methods in the interesting case of the Calabi-Yau 3-fold, for the sake of clarity here we will limit ourselves to the simpler cases of the minimal models and the torus.

We start by considering the simplest possible situation: the A_2 minimal model, which corresponds to the potential

$$W = \frac{1}{9}(\beta\tilde{\beta})^3 \quad . \tag{3.143}$$

In this model, besides the identity $\Phi_0 = 1$, there is only one other chiral primary field: $\Phi_1 = (\beta\tilde{\beta})$ with $U(1)$-charge $q = 1/3$. As one can see from (3.58) the 2-form operator associated to $(\beta\tilde{\beta})$ is simply $(b\tilde{b})$. Therefore, $\int d^2w\ b(w)\tilde{b}(\bar{w})$ is the only relevant deformation which can be used to perturb the minimal model A_2. The resulting Lagrangian is then

$$\mathcal{L} = \mathcal{L}_{\rm top} - t\int d^2w\ b(w)\tilde{b}(\bar{w}) \tag{3.144}$$

where $\mathcal{L}_{\rm top}$ is the Lagrangian for a topological (b,c,β,γ)-system as given in (3.54), and t is a dimensionful coupling constant parametrizing its perturbation. Using the rules explained in the previous section and in particular enforcing the anomalous $U(1)$ charge conservation, it is not difficult to realize that at $t=0$ the only non-vanishing topological 3-point function for this model is

$$c_{001} = \langle \Phi_0\ \Phi_0\ \Phi_1(z,\bar{z})\rangle_{\rm top} = 1 \quad . \tag{3.145}$$

However, things change when $t \neq 0$. The perturbed topological 3-point functions (see (3.57)) are in fact given by

$$c_{\ell_1\ell_2\ell_3}(t) \equiv \left\langle \Phi_{\ell_1}(z_1,\bar{z}_1)\ \Phi_{\ell_2}(z_2,\bar{z}_2)\ \Phi_{\ell_3}(z_3,\bar{z}_3)\ e^{t\int d^2w\ b(w)\tilde{b}(\bar{w})}\right\rangle_{\rm top} \tag{3.146}$$

where ℓ_1, ℓ_2, ℓ_3 can be either 0 or 1. A simple analysis reveals that the only interesting case is the correlation $c_{111}(t)$; all other correlators are indeed zero because of charge conservation. To compute $c_{111}(t)$ we expand the exponential and evaluate each term using the bosonization rules of Section 3.2.2. In particular, once again because of charge conservation, all terms in this expansion vanish except for the first-order one. Thus we obtain

$$\begin{aligned} c_{111}(t) &= t \left\langle \Phi_1(z_1,\bar{z}_1)\ \Phi_1(z_2,\bar{z}_2)\ \Phi_1(z_3,\bar{z}_3) \int d^2w\ b(w)\tilde{b}(\bar{w}) \right\rangle_{\rm top} \\ &= t \int d^2w\ \langle \beta(z_1)\beta(z_2)\beta(z_3)\,b(w)\rangle_{\rm top} \left\langle \tilde{\beta}(\bar{z}_1)\tilde{\beta}(\bar{z}_2)\tilde{\beta}(\bar{z}_3)\,\tilde{b}(\bar{w})\right\rangle_{\rm top} \end{aligned} \quad (3.147)$$

Let us now turn to the calculation of the conformal blocks appearing in the integrand of (3.147). We will focus just on the holomorphic piece, since the anti-holomorphic one is simply obtained by complex conjugation. First of all let us use (3.115)-(3.118) and (3.2.2) for $n = 2$ and write

$$\begin{aligned} \beta &= \exp\left[\frac{1}{\sqrt{3}}\phi_0\right]\ \varphi_1^1 \ , \\ b &= \exp\left[-\frac{2}{\sqrt{3}}\phi_0\right]\ \varphi_{-1}^1 \ . \end{aligned} \quad (3.148)$$

Then, using (3.147) together with the definition of topological correlation functions given at the end of Section 3.2.2, we have

$$\begin{aligned} \langle\beta(z_1)\ \ \beta(z_2)\ \ &\beta(z_3)\,b(w)\rangle_{\rm top} \\ \equiv\ \ & \langle 0,-3,-1|\beta(z_1)\beta(z_2)\beta(z_3)\,b(w)|0,0,0\rangle \\ =\ \ & \langle 0|e^{\left[\frac{1}{\sqrt{3}}\phi_0(z_1)\right]}\ e^{\left[\frac{1}{\sqrt{3}}\phi_0(z_2)\right]}\ e^{\left[\frac{1}{\sqrt{3}}\phi_0(z_3)\right]}\ e^{\left[-\frac{2}{\sqrt{3}}\phi_0(w)\right]}|0\rangle\ \times \\ \times\ \ & \langle -3,-1|\varphi_1^1(z_1)\ \varphi_1^1(z_2)\ \varphi_1^1(z_3)\ \varphi_{-1}^1(w)|0,0\rangle\ . \end{aligned} \quad (3.149)$$

The ϕ_0-contribution in (3.149) is immediate: the charges exactly soak up the background anomaly $Q_0^{(2)} = -1/\sqrt{3}$ and so we get

$$\begin{aligned} \langle\ 0\ \ &|e^{\left[\frac{1}{\sqrt{3}}\phi_0(z_1)\right]}\ e^{\left[\frac{1}{\sqrt{3}}\phi_0(z_2)\right]}\ e^{\left[\frac{1}{\sqrt{3}}\phi_0(z_3)\right]}\ e^{\left[-\frac{2}{\sqrt{3}}\phi_0(w)\right]}|0\rangle \\ =\ \ & [(z_1-z_2)(z_1-z_3)(z_2-z_3)]^{\frac{1}{3}}\ [(z_1-w)(z_2-w)(z_3-w)]^{-\frac{2}{3}}\ . \end{aligned} \quad (3.150)$$

The parafermion contribution

$$\langle -3,-1|\varphi_1^1(z_1)\ \varphi_1^1(z_2)\ \varphi_1^1(z_3)\ \varphi_{-1}^1(w)|0,0\rangle \quad (3.151)$$

is also easily computed. The most efficient way is perhaps the following: if we take into account the identification (3.105), we see that the fields φ_1^1 and φ_{-1}^1 are both proportional to $\varphi_0^0 = 1$. Moreover, the vacuum $\langle -3,-1|$ is proportional to $\langle -2,0|$ since their dual conjugates $|1,1\rangle$ and $|0,0\rangle$ are equivalent because of (3.105). Thus, (3.151) is simply the vacuum expectation value of the identity and so it is a constant. One can also verify this result by explicitly computing (3.151) using for example the method of the screening charges [50, 91]. In particular, using two of the above identifications for $\varphi_1^1(z_2), \varphi_1^1(z_3)$ we have to compute:

$$B = \langle -2, 0 | \varphi_1^1(z_1) \varphi_{-1}^1(w) | 0, 0 \rangle \ . \tag{3.152}$$

Now we see that in order to compute B we have to insert one screening operator

$$S(u) = -\frac{1}{\sqrt{2}} (\sqrt{3}\partial\phi_1(u) + i\partial\phi_2(u)) e^{-\frac{2}{\sqrt{6}}\phi_1(u)} \ , \tag{3.153}$$

thus we have

$$\begin{aligned} B &= \frac{-1}{\sqrt{2}} \oint du \langle -2, 0 | e^{\frac{1}{\sqrt{6}}\phi_1(z_1) + i\frac{i}{\sqrt{2}}\phi_2(z_1)} (\sqrt{3}\partial\phi_1(u) \\ &+ i\partial\phi_2(u)) \, e^{\frac{-2}{\sqrt{6}}\phi_1(u)} \, e^{\frac{1}{\sqrt{6}}\phi_1(w) - \frac{i}{\sqrt{2}}\phi_2(w)} | 0, 0 \rangle \ . \end{aligned} \tag{3.154}$$

Putting everything together, and remembering that

$$\langle e^{\alpha\phi(x)} \partial\phi(u) e^{\beta\phi(w)} \rangle = (x - w)^{\alpha\beta} \left[\frac{-\alpha}{x - u} + \frac{\beta}{u - w} \right] \delta(\alpha + \beta) \tag{3.155}$$

we obtain:

$$\begin{aligned} &\langle \beta(z_1) \quad \beta(z_2) \quad \beta(z_3) \, b(w) \rangle_{\text{top}} \\ &\sim \quad [(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)]^{\frac{1}{3}} \ [(z_1 - w)(z_2 - w)(z_3 - w)]^{-\frac{2}{3}} \end{aligned} \tag{3.156}$$

Notice that this topological correlation function *does* depend on the coordinates z_i where the chiral fields β are inserted. This is not at all a surprise because (3.156) is not a correlator of only physical fields.

The full topological correlation function $c_{111}(t)$, which of course should be independent of z_i, can now be easily computed. If we substitute (3.156) and its complex conjugate into (3.147), we get

$$c_{111}(t) \sim t \ (|z_1 - z_2| \, |z_1 - z_3| \, |z_2 - z_3|)^{\frac{2}{3}} \ \int d^2w \ (|z_1 - w| \, |z_2 - w| \, |z_3 - w|)^{-\frac{4}{3}} \ . \tag{3.157}$$

The integral I in (3.157) is evaluated using elementary techniques and the result is

$$I \sim (|z_1 - z_2| \, |z_1 - z_3| \, |z_2 - z_3|)^{-\frac{2}{3}} \ . \tag{3.158}$$

Absorbing all numerical factors of (3.156) and (3.157) into a rescaling of t, we can conclude that

$$c_{111}(t) = t \ . \tag{3.159}$$

Notice that only after doing the integral over d^2w, the correlation function (3.157) becomes independent of the coordinates z_i of the physical fields, as it should. In the minimal model A_2, $c_{111}(t) = t$ and $c_{001}(t) = 1$ are the only non vanishing topological 3-point functions.

As it is well known, the 2-point function $\eta_{\ell_1\ell_2}(t) \equiv c_{0\ell_1\ell_2}(t)$ serves as a metric in the space of coupling constants. This metric is flat and hence there exist special flat coordinates in which it is constant. For the minimal model A_2 we simply have

$$\eta_{00}(t) = \eta_{11}(t) = 0 \quad , \quad \eta_{01}(t) = 1 \quad . \tag{3.160}$$

Since $\eta(t)$ is not only flat but also constant, the parameter t in (3.144) is a flat coordinate.

This example is however too trivial to let us reach any conclusion, and one should test our methods in more complicated cases. This can be easily done and in several non-trivial examples for single minimal models we have checked that the parameters entering the deformed LG potential in our formulation are indeed flat coordinates. This is to be contrasted with the usual formulation where the parameters of the deformed LG potential are *not* flat coordinates but are related to these by more or less complicated transformations. The origin of this difference is that we compute the perturbed correlation functions without using the residue pairing defined by the perturbed potential [31] and stay in the context of perturbed conformal field theory Thus we can always maintain in a natural way a frame of flat coordinates.

We have verified these properties also in the case of the torus, which can be described by the tensor product of three minimal models A_2 deformed by its marginal operator. Therefore, the potential we should consider is

$$W = \frac{1}{9}(\beta_x\tilde{\beta}_x)^3 + \frac{1}{9}(\beta_y\tilde{\beta}_y)^3 + \frac{1}{9}(\beta_z\tilde{\beta}_z)^3 - t\,(\beta_x\tilde{\beta}_x)(\beta_y\tilde{\beta}_y)(\beta_z\tilde{\beta}_z) \tag{3.161}$$

The dimensionless parameter t in (3.161) is (related to) the modulus of the torus. The chiral ring of the tensor product of three A_2 minimal models is generated by $\Phi_0 = \mathbf{1}$, $(\Phi_1, \Phi_2, \Phi_3) = (\beta_x\tilde{\beta}_x, \beta_y\tilde{\beta}_y, \beta_z\tilde{\beta}_z)$, $\Phi_4 = \beta_x\tilde{\beta}_x\beta_y\tilde{\beta}_y$, $\Phi_5 = \beta_y\tilde{\beta}_y\beta_z\tilde{\beta}_z$, $\Phi_6 = \beta_x\tilde{\beta}_x\beta_z\tilde{\beta}_z$, and $\Phi_7 = \beta_x\tilde{\beta}_x\beta_y\tilde{\beta}_y\beta_z\tilde{\beta}_z$. Φ_7 is a marginal operator and is the one appearing in (3.161) as a deformation. The Lagrangian corresponding to (3.161) is

$$\mathcal{L} = \mathcal{L}_{\rm top} - t\int \Phi_7^{(2)} \tag{3.162}$$

where $\mathcal{L}_{\rm top}$ is the Lagrangian for three topological (b, c, β, γ)-systems as in (3.54), and $\Phi_7^{(2)}$ is the 2-form associated to Φ_7. According to (3.61), we have

$$\int \Phi_7^{(2)} = \int d^2w\,[(b_x\beta_y\beta_z + b_y\beta_x\beta_z + b_z\beta_x\beta_y)(w) \times ({\rm c.c})] \quad . \tag{3.163}$$

The simplest way to check whether the parameter t in (3.162) is a flat coordinate or not, is to compute the topological metric

$$\eta_{ij}(t) = \left\langle \Phi_i\, \Phi_j\, e^{t\int \Phi_7^{(2)}} \right\rangle_{\text{top}} \tag{3.164}$$

and see whether it is constant or not. To this end it is enough to consider one component, for example

$$\eta_{07}(t) = \left\langle \Phi_0\, \Phi_7\, e^{t\int \Phi_7^{(2)}} \right\rangle_{\text{top}} \quad . \tag{3.165}$$

By looking at the $U(1)$-charges of the operators in (3.165), it is easy to realize that when expanding the exponential, only terms with $3n$ insertions of $\int \Phi_7^{(2)}$ will satisfy charge conservation. Thus, (3.165) can be rewritten as

$$\eta_{07}(t) = \sum_{n=0}^{\infty} \frac{1}{(3n)!} a_{3n}\, t^{3n} \tag{3.166}$$

where

$$a_{3n} = \left\langle \Phi_0\, \Phi_7 \left(\int \Phi_7^{(2)}\right)^{3n} \right\rangle_{\text{top}} \quad . \tag{3.167}$$

It is immediate to see that $a_0 = 1$. The first non-trivial contribution is a_3 which, spelled out in detail, is

$$a_3 = \int d^2w_1 \int d^2w_2 \int d^2w_3\, f(u, w_1, w_2, w_3) \bar{f}(\bar{u}, \bar{w}_1, \bar{w}_2, \bar{w}_3) \tag{3.168}$$

where

$$f(u, w_1, w_2, w_3) = \left\langle (\beta_x \beta_y \beta_z)(u) \prod_{i=1}^{3} \left[(b_x \beta_y \beta_z + b_y \beta_x \beta_z + b_z \beta_x \beta_y)(w_i) \right] \right\rangle_{\text{top}} \tag{3.169}$$

and $\bar{f}$ is its complex conjugate. To compute f, we split the r.h.s. of (3.169) into a sum of factorized correlation functions for each of the three minimal models and enforce on them (anomalous) conservation of the $U(1)$ charges. During this process the b fields in (3.169) must be suitably rearranged and proper minus signs arise from their anticommutation relations. After some straightforward algebra, it turns out that $f = 0$, which implies

$$a_3 = 0 \tag{3.170}$$

Actually it is very easy to generalize this result to all higher-order coefficients, and eventually conclude that

$$\eta_{07}(t) = 1 \tag{3.171}$$

The other entries of the metric $\eta_{ij}(t)$ can be computed in a similar way and all of them turn out to be constants. Thus the parameter t in (3.162) is a flat coordinate.

We want to emphasize that instead, in the standard LG formulation of the torus described by the potential

$$W = \frac{1}{3}X^3 + \frac{1}{3}Y^3 + \frac{1}{3}Z^3 - sXYZ \tag{3.172}$$

the topological correlation $\langle \Phi_0 \, \Phi_7 \rangle(s)$ *is* a non-trivial function of the LG parameter s, which therefore cannot be a flat coordinate [8].

It is interesting now to investigate the relation between s and t. On general grounds [33], it is possible to show that

$$s(t) = \frac{c_{111}(t)}{c_{123}(t)} \tag{3.173}$$

where

$$c_{ijk}(t) = \left\langle \Phi_i \, \Phi_j \, \Phi_k \, e^{t \int \Phi_7^{(2)}} \right\rangle_{\rm top} \quad . \tag{3.174}$$

In our formulation it is easy to compute these perturbed 3-point functions as a power series in the flat coordinate t. Let us briefly see how $c_{123}(t)$ is evaluated. Expanding the exponential and looking for terms which satisfy charge conservation, we get

$$c_{123}(t) = \sum_{n=0}^{\infty} \frac{1}{(3n)!} \tilde{a}_{3n} \, t^{3n} \tag{3.175}$$

where

$$\tilde{a}_{3n} = \left\langle \Phi_1 \, \Phi_2 \, \Phi_3 \left(\int \Phi_7^{(2)} \right)^{3n} \right\rangle_{\rm top} \quad . \tag{3.176}$$

It is immediate to see that $\tilde{a}_0 = 1$. The next contribution is

$$\tilde{a}_3 = \int d^2w_1 \int d^2w_2 \int d^2w_3 \, g(x,y,z,w_1,w_2,w_3) \bar{g}(\bar{x},\bar{y},\bar{z},\bar{w}_1,\bar{w}_2,\bar{w}_3) \tag{3.177}$$

where

$$g(x,y,z,w_1,w_2,w_3) = \left\langle \beta_x(x)\beta_y(y)\beta_z(z) \prod_{i=1}^{3} \left[(b_x\beta_y\beta_z + b_y\beta_x\beta_z + b_z\beta_x\beta_y)(w_i) \right] \right\rangle_{\rm top} \quad . \tag{3.178}$$

By splitting the r.h.s. of (3.178) into a sum of factorized terms and using the explicit results derived earlier for each of the three minimal models, one can prove that the integrand of (3.177) is

$$|x-y|^2 |x-z|^2 |y-z|^2 \prod_{i=1}^{3} \left(|x-w_i| \, |y-w_i| \, |z-w_i| \right)^{-\frac{4}{3}} \tag{3.179}$$

[8]It turns out that $\langle \Phi_0 \, \Phi_7 \rangle(s) = \left[F(\frac{1}{3}, \frac{1}{3}; \frac{2}{3}; s^3) \right]^{-2}$, where F is the hypergeometric function.

Using (3.158) and rescaling t to absorb all numerical constants, we conclude that $\tilde{a}_3 = 1$, and hence

$$c_{123}(t) = 1 + \frac{1}{6}t^3 + O(t^6) \tag{3.180}$$

Similarly one can check that

$$c_{111}(t) = t + O(t^7) \tag{3.181}$$

so that from (3.173) it follows

$$s(t) = t - \frac{1}{6}t^4 + O(t^7) \tag{3.182}$$

These are precisely the first terms in the power series expansion of the solution of the schwarzian differential equation

$$\{s; t\} = -\frac{1}{2}\frac{(8 + s^3)}{(1 - s^3)^2}(s')^2 s \tag{3.183}$$

where

$$\{s; t\} = \frac{s'''}{s'} - \frac{3}{2}\left(\frac{s''}{s'}\right)^2 \tag{3.184}$$

and the primes denote t-derivatives. We will show in next section that the condition (3.183) is equivalent to the requirement that t be a flat coordinate [33, 34, 78], whenever we use to compute the metric in the coupling constant space the standard Landau–Ginzburg approach. The methods we have just presented provide automatically the solution to (3.183) as a power series.

3.3 Digression: The Landau–Ginzburg approach and Picard Fuchs equations

In this digression we recall how equation (3.183) is retrieved in the Landau-Ginsburg approach, utilizing algebraic geometry techniques. This establishes the relationship between our superconformal approach and the theory of Picard–Fuchs differential equations satisfied by the periods of the holomorphic n–forms on $c_1 = 0$ n–folds. [81, 77, 35].

We skip many technical details for the general theory, restricting our attention to the specific example of the cubic torus and trying to clarify some obscure points found in the literature.

Consider a CP^{n+1} projective space and denote X^A $A = 1, 2 \cdots, n+2$ its projective coordinates. Next consider the projective variety V defined as the zero locus of the homogeneous potential W

$$W = 0 \longleftrightarrow X \in V \; . \tag{3.185}$$

The degree of the potential W will be denoted by ν. Consider the $(n+1)$–differential form in $CP^{(n+1)}$

$$\omega = X^{A_1} dX^{A_2} \wedge \cdots \wedge dX^{A_{n+2}} \epsilon_{A_1 \cdots A_{n+2}} \tag{3.186}$$

and set

$$\Omega_B = \epsilon_{BA_1 \cdots A_{n+1}} dX^{A_1} \wedge dX^{A_2} \wedge \cdots \wedge dX^{A_{n+1}} \; , \tag{3.187}$$

then

$$dX^{A_1} \wedge dX^{A_2} \wedge \cdots \wedge dX^{A_{n+1}} = \frac{(-)^{(n+1)}}{(n+1)!} \epsilon^{A_1 \cdots A_{n+1} B} \Omega_B \; . \tag{3.188}$$

Consider now an n–differential form ϕ defined by:

$$\phi = \frac{1}{W^l} \epsilon_{MNA_1 \cdots A_n} X^M Y^N dX^{A_1} \wedge dX^{A_2} \wedge \cdots \wedge dX^{A_n} \; , \tag{3.189}$$

where Y^M are $(n+2)$ homogeneous polynomial of degree q. If we compute the exterior derivative of ϕ we get after some algebra:

$$d\phi = \frac{1}{W^{l+1}} \left[W Y^Q \left(-1 - \frac{q}{n+1} + \frac{\nu l}{n+1}\right) + \frac{W}{n+1} \partial_M Y^M X^Q - \frac{l}{n+1} X^Q Y^M \partial_M W \right] \Omega_Q \tag{3.190}$$

hence if

$$q = l\nu - (n+1) \tag{3.191}$$

we obtain

$$d\phi = -\frac{1}{W^{l+1}} \left[\frac{l}{n+1} Y^M \partial_M W - \frac{1}{n+1} W \partial_M Y^M \right] \omega \; . \tag{3.192}$$

The result in eq. (3.192) can be expressed in a more convenient way for our purpose. Let us consider an n–form $\Omega(Y)$ defined by:

$$\Omega(Y) = \int_\gamma \frac{Y^A \partial_A W}{W^{(l+1)}} \omega \tag{3.193}$$

where Y^A is a $n+2$ vector of homogeneous polynomials with degrees:

$$\begin{aligned} \deg Y^A &= l\nu - (n+1) \ , \\ \deg W &= \nu \end{aligned} \tag{3.194}$$

and γ is a small circle winding around the hypersurface $W(X) = 0$. Then $\Omega(Y)$ is cohomologous to

$$\Omega(\partial \cdot Y) = \frac{1}{l} \int_\gamma \frac{\partial_A Y^A}{W^l} \omega \ . \tag{3.195}$$

This is because

$$\Omega(Y) - \Omega(\partial \cdot Y) = -\frac{n+1}{l} d\tilde{\phi} \ , \tag{3.196}$$

with $\tilde{\phi} = \int_\gamma \phi$.

Consider the chiral ring $\mathcal{R} = C(X)/\partial W$ associated to the superpotential W and consider the subring consisting of polynomials of degree $0, \nu, 2\nu \cdots n\nu$.

$$\{1, p^\nu, p^{2\nu}, \cdots, p^{n\nu}\} \ . \tag{3.197}$$

Clearly this set defines a subring of $\mathcal{R}$. Indeed $p^{k_1\nu} p^{k_2\nu} = p^{(k_1+k_2)\nu}$ until we get the polynomial of maximum degree $n\nu$. From a superconformal point of view the maximum degree polynomial is the top chiral primary field with maximal U(1) charge content $\frac{c}{3}$. In the singularity theory approach it can be singled out by computing:

$$p^{n\nu} \sim det\ (\partial_i \partial_j W(X)) \ . \tag{3.198}$$

We can put the elements of this subring into correspondence with the order n cohomology group:

$$F^n = \sum_{k=0}^{n} H^{(n-k,k)} \ , \tag{3.199}$$

where $H^{(n-k,k)}$ are the Dolbeault cohomology groups. Indeed we can set [96]

$$\Omega_\alpha^{(n-k,k)} = \int_\gamma \frac{p_\alpha^{(k\nu)}(X)}{W^{(k+1)}} \omega \ , \tag{3.200}$$

where $p_\alpha^{(k\nu)}(X)$ is a homogeneous polynomial of degree $k\nu$, $k = 0, \cdots, n$, which is not a trivial element of the chiral ring. It can indeed be shown that (3.200) defines a closed (n-k,k) form which is exact if and only if the polynomial belong to the ideal $[\partial W]$ of the chiral ring. The explicit proof of this statement is given in [96].

Here we give an application of eq. (3.200), which allows us to derive a differential equation for the perturbation parameter s in the case of the cubic torus:

$$W = \frac{1}{3}(X^3 + Y^3 + Z^3) - sXYZ \ . \tag{3.201}$$

The equation we find is known in algebraic geometry with the name of Picard–Fuchs equation.

Picard–Fuchs equations are differential equation satisfied by the periods of a differential form Ω_α. In general periods are defined to be the integrals of Ω_α over elements of a basis of integral homology in V.

$$\Pi_\alpha^\beta = \int_{\Gamma_\beta} \Omega_\alpha \ , \tag{3.202}$$

where Γ_β is a representative of a homology basis in $H_{n+1}(CP^{(n+1)} - V, Z)$. The curve Γ_β may be thought of as a tube over the corresponding cycle in $H_n(V,Z)$. In our case we fix Γ_β to γ and we consider the vector $\Omega_\alpha \equiv \Pi_\alpha^{(\gamma)}$.

Our purpose is to show how to generate these differential equations directly from W in (3.201).

Following our identification we define the $H^{(1,0)}$ and $H^{(0,1)}$ representatives as follows:

$$\Omega_1 = \int \frac{\omega}{W} \quad , \quad \Omega_2 = \int \frac{XYZ}{W^2}\omega \tag{3.203}$$

with

$$\omega = 2(XdY \wedge dZ + YdZ \wedge dX + ZdX \wedge dY) \ . \tag{3.204}$$

The "vanishing relations" which define the trivial element in the chiral ring are:

$$\begin{aligned} X^2 &= sYZ + \partial_X W \ , \\ Y^2 &= sXZ + \partial_Y W \ , \\ Z^2 &= sXY + \partial_Z W \ . \end{aligned} \tag{3.205}$$

Using eq.s. (3.203) and (3.205) we get:

$$\frac{\partial}{\partial s}\Omega_1 = \Omega_2 \ , \tag{3.206}$$

$$\frac{\partial}{\partial s}\Omega_2 = 2\int \frac{X^2Y^2Z^2}{W^3}\omega \tag{3.207}$$

and

$$X^2Y^2Z^2 = \frac{1}{(1-s^3)} Y^A \partial_A W \ , \tag{3.208}$$

where $\vec{Y} = (sXZ^3, X^2Z^2, s^2XYZ^2)$ is a vector with degree four polynomial components. According to (3.195) we have:

$$\frac{\partial}{\partial s}\Omega_2 = \frac{1}{1-s^3}\int \frac{\partial_A Y^A}{W^2}. \quad (3.209)$$

Using again the vanishing relations we finally find:

$$(1-s^3)\partial_s\Omega_2 = s\Omega_1 + 3s^2\Omega_2 \ , \quad (3.210)$$

which combined with (3.206) yields:

$$[(1-s^3)\partial_s^2 - 3s^2\partial_s - s]\Omega_1 = 0 \ . \quad (3.211)$$

For the moment we will not discuss in detail the properties of the solution of (3.211). We are interested in the connection of the Picard–Fuchs equation (3.211) with the Schwarzian equation we wrote in chapter 3. In particular we are going to illustrate how this is related to the constant flatness of the metric in the coupling constant space as defined in section 3.1 of this chapter. We will use this discussion to introduce the concepts of duality and monodromy groups associated to the potential W and to the differential Picard Fuchs equation (3.211). Our starting point is the full perturbed potential:

$$\begin{aligned} W &= \frac{1}{3}(X^3+Y^3+Z^3) - s_0 - s_1X - s_2Y - s_3Y \\ &- s_4XY - s_5XZ - s_6YZ - sXYZ \ . \end{aligned} \quad (3.212)$$

In general using the residue pairing relation (3.25) we can write:

$$\langle g(X_I)\rangle = \gamma(s_I)h \ , \quad (3.213)$$

where $g(X_I)$ is an arbitrary monomial in X_I and h is the vacuum expectation value of the Hessian $h =< det\ \partial_I\partial_j W >=< \rho >$. For a generic potential, γ is a polynomial function of the relevant couplings (as can be understood by U(1) charge conservation), but as well as h, it can depend in a nontrivial way from the marginal perturbations (if any). The above result is due to the fact that with the help of the vanishing relations, any monomial $g(X_I)$ can be decomposed as follows:

$$g(X_I) = \gamma(s_I)\rho + G^I\partial_I W + (q < c/3 \text{ fields}) \ , \quad (3.214)$$

of which in the topological correlation functions only the first term survives.

Starting with (3.212) we can calculate the metric in the coupling constant space g_{ij} as:

$$g_{ij} =< f_i(X)f_j(X) > \ . \quad (3.215)$$

We find that restricting the vacuum expectation value of the hessian to be constant will result in a nonvanishing curvature tensor. For g_{ij} to be flat we have to allow h to be a function of the marginal coupling s, and to impose the vanishing of the Riemann tensor. Because we are interested in the s dependence, we keep only linear and quadratic terms in relevant couplings, as we will send it to zero in computing the Riemann tensor. Actually, in many cases, to perform such calculation, one can limit himself to a subring of relevant and marginal perturbations. We show how to do it for a particular case in the Appendix B.4. All the non–vanishing components of the Riemann tensor are proportional to the left hand side of the equation:

$$(1-s^3)y''(s) - 3s^2y'(s) - sy(s) = 0 \ , \tag{3.216}$$

where we set $h = y^{-2}$. The differential equation (3.216) is the same as the Picard– Fuchs equation (3.211). To find flat coordinates (the ones in which the metric is constant), we should write the most general coordinate transformation $s_I = s_I(t)$ compatible with the U(1) symmetry and discrete symmetries of the potential and insert it into:

$$\eta^{ij}\frac{\partial s_i}{\partial t_k}\frac{\partial s_j}{\partial t_l} = g^{kl}(s(t)) \ . \tag{3.217}$$

Alternatively we can observe that the choice of the top chiral field $\phi_{top} = XYZ$ is ambiguous in the following sense: any linear combination of the chiral primary field with highest charge ($\beta\phi_{top}$ with $\beta \neq 0$) with fields of lower charges can be used, resulting in a conformally related metric $g_{ij} = \frac{1}{\beta}g_{ij}$. However, conformal perturbation theory, as explained, gives a natural set of flat coordinates t_i, in which $g_{0\,top} = 1$. This means that to get the flat metric for (3.212) as a function of the flat coordinate t, we should take the top chiral field proportional to a "flattening" factor $\frac{ds}{dt}$, i.e.

$$\phi_{top} = \frac{ds}{dt}XYZ \ . \tag{3.218}$$

Our purpose to insert the t–dependence, due to the flattening factor, in the computation of the metric and Riemann tensor. We find that the flatness requirement for the metric $g_{ij} = \eta_{i,j} = \delta_{i+j,top}$ implies:

$$\{t,s\} = \frac{1}{2}\frac{8+s^3}{(1-s^3)^2}s \ , \tag{3.219}$$

which is precisely the one we wrote in (3.183) (by using $\{s,t\} = -(s')^2\{t,s\}$).

For the case of the torus, as well as for more complicated cases we can establish some interesting relations with the theory of differential equations [35, 99, 98]. Given a differential equation of the form:

$$D_x\xi(x) = (d_x^n + \sum_{j=2}^{n} a_j(x)d_x^{n-j})(x)\xi(x) \ , \tag{3.220}$$

it can be shown that, under a coordinate transformation $x = x(t)$, the coefficient a_{n-2} transforms with the law

$$a_{n-2}(t) = a_{n-2}(x)(\frac{dx}{dt})^2 + n(n^2-1)\{x,t\} \ . \tag{3.221}$$

A possible term like $a_1(x)d_x^{n-1}$ in (3.220) can always be reabsorbed, without any loss of generality, with a suitable redefinition of ξ. In our case defining $\xi_1 = \Omega_1 exp(\frac{-1}{2}\int \frac{3s^2}{1-s^3})$, from (3.211) we immediately get

$$\partial^2\xi + \frac{1}{4}\frac{8+s^3}{(1-s^3)^2}s\xi = 0 \ . \tag{3.222}$$

Using equation (3.221), where we set $n = 2$, and requiring that $a_0(t) = a(t) = 0$ we get precisely the Schwarzian equation (3.183). This requirement for the case of the torus is perfectly equivalent to the flatness requirement for the metric g_{ij}. Moreover it is quite easy to verify that keeping two distinct solutions of the equation (3.211), their ratio t satisfies precisely the nonlinear Schwarzian equation (3.183)

$$\{t,s\} = 2I \quad , \quad I = \frac{1}{4}\frac{8+s^3}{(1-s^3)^2}s \ . \tag{3.223}$$

This is a very general result for linear, second order differential equations, strongly related to the construction of nonlinear differential equation starting from the ratio of two solutions of the linear one [98]. For example, the linear second order equation

$$\frac{d^2z}{dx^2} + p(x)\frac{dz}{dx} + q(x)z \tag{3.224}$$

gives rise to the nonlinear, Schwarzian equation:

$$\{\xi,x\} = 2I \quad , \quad I = q - \frac{1}{4}p^2 - \frac{1}{2}\frac{dp}{dx} \ , \tag{3.225}$$

where $\xi = z_1/z_2$ is the ratio of two independent solution of the linear equation above. The quantity I is referred as the invariant of (3.211) since it is unchanged if one replaces (3.211) by the linear equation for $f(s)\Omega(s)$ (where $f(s)$) is an arbitrary function).

What we have shown in this example is that flat coordinates are associated with appropriate invariant equations, which can be derived from the linear Picard–Fuchs ones. This property reflects only partially to more complicated cases. In general we can always find a system of "schwarzian" coordinates in which the coefficient a_{n-2} is set to zero, but not necessarily it corresponds to flat coordinates. When the degree of the differential equation increases (as for example for Calabi–Yau threefolds), there are other "invariants" that can be defined. In this case one can arrange the coefficients of the linear equation to get W–algebra generators, which have well known tensor properties transformations [99, 77] (notice that the schwarzian derivative is associated to the transformation rule of a stress energy–tensor, i.e. a W_2).

The understanding of these properties of Picard–Fuchs equations should play a fundamental role (and indeed it does in the torus case) in the study the so called duality group of the potential W, denoted by Γ_W [74].

Γ_W consists of those transformations of the moduli that are induced through quasihomogeneous changes of the variables X^A, that leave the form of the superpotential unchanged up to an overall factor, i.e.

$$W(X'(X), a_i) = W'(X, a') = C(a)W(X, a'_i) , \tag{3.226}$$

where a_i are moduli parameters, and $C(a)$ is a moduli dependent factor. This means that the deformations in W' are still described by marginal operators from the chiral ring. The factor $C(a)$ can be eliminated by rescaling the X–fields.

If one makes such a change of variables in the period integrals of Ω then the result is changed only by an overall factor. It follows that the linear system of differential equations must be covariant with respect to the duality group Γ_W of the superpotential. The corresponding system of non–linear equations must be invariant with respect to this duality group. This duality covariance and invariance can be very instructive in understanding the properties of the linear and nonlinear equations. On the other side one can get informations on Γ_W from the differential equations. In particular given a second order linear equation we can often read the solution of associated Schwarzian differential equations in terms of triangle functions [98, 41, 35, 34] $s(\alpha,\beta,\gamma)$. The parameters α,β,γ then determine the duality group of the superpotential. For triangle functions, this group can be generated by reflections in the sides of hyperpolic triangles (with angles $\pi\alpha, \pi\beta, \pi\gamma$) that cover the upper half plane.

It is thus interesting to investigate the relationships between the foregoing duality group Γ_W, the monodromy group Γ_M associated to the linear equations and the modular group Γ of the surface defined by $W = 0$. The integral homology basis undergoes an integral symplectic transformation when it is transported around singular points in the moduli space of the manifold. Consequently the periods of the differential forms undergo such symplectic transformations about these singular points. This reflects into the monodromy around Fuchsian singular points of the solutions of the differential equation. The set of all such monodromies will generate a subgroup Γ_M of the full modular group Γ. The set of duality transformations Γ_W maps the surface back to itself and will extend the group Γ_M to an even larger subgroup of Γ. In the case of the torus (3.212) this extension is all of Γ, and then the duality group of W is $\Gamma_W = \Gamma/\Gamma_M$. The Schwarzian equation (3.223) is solved by the triangular function $s(t) = s(\frac{1}{2}, \frac{1}{3}, \frac{1}{3}; J(t))$ [35, 98] , where $J(t)$ is the absolute modular invariant. $s(t)$ is a modular form of $\Gamma(3)$[9] , and the full modular group of the torus is $\Gamma = PSL(2,Z)$. The duality group of the torus can be easily found by imposing the conditions (3.226), with the following transformations on the X, Y, Z fields (up to an irrelevant minus sign)

$$A \ : \ \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{3.227}$$

and

$$B \ : \ \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{3.228}$$

where $\alpha = e^{\frac{2\pi i}{3}}$. These transformations produce for the parameter s the following ones

$$\begin{aligned} A \ : \ s \ &\longrightarrow \ e^{\frac{2\pi i}{3}} s \\ B \ : \ s \ &\longrightarrow \ \frac{s+2}{s-1} \end{aligned} \tag{3.229}$$

The transformations defined in (3.229) generates the tetrahedral group $\Gamma_W = T$, while the ones acting on the LG fields correspond to a central extension of such a group. From an abstract point of view the tetrahedral group is characterized by two generators A, B satisfying the relations:

$$A^2 = 1; \quad B^3 = 1; \quad (AB)^3 = 1 \tag{3.230}$$

In the case under consideration the full modular group of the torus, which is $PSL(2,Z)$ is obtained as a semidirect product of Γ_W with $\Gamma(3)$. $\Gamma(3)$ precisely corresponds to the monodromy group of the Picard–Fuchs equation, as can be intuitively understood from the observation that $s(t)$ is a modular form of $\Gamma(3)$

Actually this last observation requires some subtleties that is worth to mention here.

The first point is to understand what are the monodromy transformations we are interested in. If we consider the manifold $\mathcal{M}$ defined by the polynomial (3.212), it is crucial to note that there are special values of s for which $\mathcal{M}$ is singular. This occurs when the conditions:

$$\frac{\partial W}{\partial X_i} = 0 \quad , \quad (X_1, X_2, X_3) = (X, Y, Z) \tag{3.231}$$

are simultaneously satisfied. This equation implies that

$$X^3 = Y^3 = Z^3 = s^3 XYZ \tag{3.232}$$

$$(XYZ)^3 = s^3 (XYZ)^3 \tag{3.233}$$

[9] We remember that $\Gamma(N)$ is defined as the subgroup of $SL(2,Z)$ with the property $\gamma \in \Gamma(N)$ if $\gamma = \pm\mathbf{1} \bmod N$, where mod N means that any entry is defined modulo N.

If s is finite then none of the X_i may be zero for if one were zero then by the above equations they would be all zero. It follows that equation (3.231) can only be satisfied if $s^3 = 1$. The value $s = \infty$ corresponds to the singular cubic $XYZ = 0$.

The periods are in general multivalued about the singular points of s . Transport around these points generates transformations on the periods that are called monodromy transformations. This reflects to the monodromy of the Picard–Fuchs equation around its regular singular points, which are given as well by $s = 1, \alpha, \alpha^2, \infty$. Notice that the tetrahedral group (3.229) permute among themselves these four singular points.

If we perform a suitable change of variable in eq (3.211) we can trasform such equation into an ordinary hypergeometric differential equation [35]. Thus one is very tempted to analyze the monodromy transformations properties on such a solution in the new variable. This would represent a great advantage, since it allows to study the monodromy transformations around standard fuchsian singular points $(0, 1, \infty)$ of the hypergeometric equation, utilizing well known results in mathematics [98]. This however does not give the correct result. The monodromy group one finds has two generators that do not define $\Gamma(3)$.

The correct way to proceed in finding the monodromy group of the differential equation (3.211) is to work explicitly with the variable s, and to take carefully into account the behaviour of the solution around singular points, following the procedure suggested by Candelas [81] for the mirror quintic case.

In the vicinity of infinity the regular solution of eq. (3.211) is given by:

$$\omega_0(s) = {}_2F_1(\frac{1}{3}, \frac{2}{3}, 1; \frac{1}{s^3})\frac{1}{s} \; . \tag{3.234}$$

The use of the Barne's integral representation [98] allows to represent ω_0 as :

$$\omega_0(s) = \frac{1}{2\pi i} \oint_C dl \frac{\Gamma(-l)\Gamma(3l+1)}{\Gamma^2(l+1)(-)^l} (\frac{1}{3s})^{3l+1} \; , \tag{3.235}$$

where $0 < \arg s < \frac{2\pi}{3}$ and $|s| > 1$. To recover (3.234) as a sum over residues one has to close the contour C to include the poles of $\Gamma(-l)$ on the positive real axis. On the other hand one can define the analytic continuation of $\omega_0(s)$ in the region $|s| < 1$, by closing C to the left to include poles of $\Gamma(3l + 1)$. Thus we find:

$$\omega_0(s) = -\frac{1}{3} \sum_{m=1}^{\infty} \alpha^m \frac{\Gamma(\frac{m}{3})(3s)^{m-1}}{\Gamma^2(1-\frac{m}{3})\Gamma(m)} \equiv \frac{1}{s}\xi_0(s) \; , \tag{3.236}$$

where we have introduced the quantity $\xi_0(s)$, for pure convenience. Looking at the sum in eq. (3.236) we easily realize that an independent set of solutions can be singled out by setting:

$$\omega_1(s) = \omega_0(\alpha s) \tag{3.237}$$

as well as

$$\omega_2(s) = \omega_0(\alpha^2 s) \ . \qquad (3.238)$$

Obviously, since we are dealing with a second order equation the three solutions $\omega_i(s)$, $s = 0, 1, 2$ are not all independent, and indeed they satisfy the constraint

$$\omega_0 + \alpha\omega_1 + \alpha^2\omega_2 = 0 \ . \qquad (3.239)$$

Notice that defining $\xi_i = \alpha^i \xi_0$ we get $\xi_0 + \xi_1 + \xi_2 = 0$. Now we want to compute the action on the basis $\xi_i(s)$ of the transvection around $s = 1$. The indicial equation for (3.211), which displays a double root $r = 0$, shows that in the neighborhood of $s = 1$ we have two solutions, one of which is regular and the other has a logarithmic behaviour. This means that, in the vicinity of $s = 1$, any solution has the form:

$$\xi_i(s) = c_i ln(s-1) g_1(s) + f_i(s) \ , \qquad (3.240)$$

where $g_1(s)$ is an analytic solution of eq (3.211) and $f_i(s)$ are analytic functions ($i = 0, 1$). Notice that the factor $\frac{1}{s}$ in the ω_i basis with respect to the ξ_i basis is not essential in writing the logarithimic behaviour in the vicinity of s=1, but it contribute in computing the monodromy transformations, as we will point out later on. For $|s| > 1$ real we have that:

$$\xi_i(s - i\epsilon) - \xi_i(s + i\epsilon) = 2\pi i c_i g(s) \ . \qquad (3.241)$$

On the other hand considering the solution (3.234) for $|s| > 1$ and the identification $\xi_i = \alpha^i \xi_0$ it follows that

$$\xi_1(s - i\epsilon) = \xi_0(\alpha(s - i\epsilon)) = \xi_0(s + i\epsilon) \qquad (3.242)$$

hence

$$2\pi c_1 g(s) = \xi_0 - \xi_1 \ . \qquad (3.243)$$

If we consider the action of a transvection T_1 about $s = 1$ on the ξ_i we find:

$$T_1 \xi_i = \xi_i + c_i g(s) = \xi_i + \frac{c_i}{c_1}(\xi_0 - \xi_1) \ . \qquad (3.244)$$

For the torus case the matrix representing T is immediately found if we observe that $g(s)$ is a power series in s, and as a consequence its (trivial) monodromy transformation implies from (3.243) that $\frac{c_0}{c_1} = 1$. Hence we get

$$T_1 = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \ . \qquad (3.245)$$

In the ξ_i basis is easy to find also the action of the transformation $s \to \alpha s$, which is not a monodromy transformation. This is represented by the matrix A, with $A^3 = 1$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \ . \qquad (3.246)$$

It can be easily seen that all others monodromy transformations T_2, T_3, T_4 around the singular points α, α^2, ∞ are obtained as

$$T_k = A^{-k} T_1 A^k \quad , \quad k = 2, 3, 4 \tag{3.247}$$

which gives:

$$T_2 = \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix} , T_3 = \begin{pmatrix} -1 & -1 \\ 4 & 3 \end{pmatrix} , T_4 = \begin{pmatrix} 1 & 0 \\ 9 & 1 \end{pmatrix} . \tag{3.248}$$

Of course in passing from the variables ω_i to ξ_i the matrices T_k and A have their entries changed by some α factor. However the two sets of matrices we get are perfectly equivalent since they are each other conjugated in $SL(2, C)$.

Now let us come to the final point of this rather technical procedure. If we look at the monodromy group generated by T_1, T_2, T_3, T_4 we should find the group $\Gamma(3)$. However, looking at eq. (3.248), this does not apparently seem to be the case. The answer is that the group generated by T_k is conjugated in $SL(2, C)$ to $\Gamma(3)$ through the conjugation matrix:

$$\mathcal{M} = \begin{pmatrix} i\sqrt{3} & 0 \\ \frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} \end{pmatrix} , \tag{3.249}$$

which takes the generators T_k to the standard Klein– Fricke $\Gamma(3)$ generators [101]. Indeed we can easily show that under conjugation T_1, T_2, T_3, T_4 go respectively into:

$$\begin{aligned} V_1 &= \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \quad , \quad V_x = \begin{pmatrix} -5 & 12 \\ -3 & 7 \end{pmatrix} , \\ V_3 &= \begin{pmatrix} -2 & 3 \\ -3 & 4 \end{pmatrix} \quad , \quad V_2 = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \end{aligned} \tag{3.250}$$

where V_1, V_2, V_3 are three generator for the standard basis, while V_x is connected to the fourth standard generator V_4 by the relation $V_4 = V_1 V_x V_1^{-1}$.

The matrix A does not belong to $\Gamma(3)$, as it should be, because it belongs to the duality group of the potential. In this specific example the duality group $\Gamma/\Gamma(3)$ contains one more generator B, with $B^2 = 1$, which permutes the singular points of the Picard–Fuchs equation.

In more complicated cases, say for example for the mirror Calabi–Yau quintic [81], the structure of the duality group is quite involved, and in general it is difficult to completely characterize it. However it is always possible to follow the procedure above explained and find the monodromy group, and at least the generator of the duality group which corresponds to A [81]. It is not clear, in the general case, how to characterize other possible duality group generators.

However it should be interesting to find the analogue of the cubic torus for other "Platonic" cases, namely to find to which Landau–Ginzburg polynomials correspond the Dihedral, Octahedral and Icosahedral groups as duality groups (for the Dihedral group the answer is actually known [34]). This problem is presently under investigation.

Appendix A

Aspects of algebraic-geometry in special manifolds

The definition of special geometry we have given in chapter 2 does not depend on the fact that we are considering a Calaby–Yau compactification or not . However in the case of CY compactifications we can find an interesting relation with algebraic geometry [18, 4]. This relation arises with the interpretation of the Kähler metric $g_{\alpha\beta^*}$ for (2,1) moduli as the corresponding metric for the moduli space for the complex structure deformations, i.e. the Weil– Peterson metric:

$$\hat{g}^{\mathrm{wp}}_{\alpha\bar{\beta}} = i\int \rho_\alpha \wedge \bar{\rho}_{\bar{\beta}} \ , \tag{A.1}$$

where ρ_α is a moduli dependent basis for the $H^{(2,1)}$ Dolbeault cohomology:

$$\rho_\alpha = \rho_{\alpha,ijk^*} dx^i \wedge dx^j \wedge d\bar{x}^{k^*} \ , \tag{A.2}$$

(x are complex coordinates on Calabi-Yau manifold).

Indeed, if we consider the deformed holomorphic threeform [1] $\Omega(\psi^\alpha)$ and take the derivative with respect ψ^α in some point ψ^α_0 we get:

$$\frac{\partial\Omega}{\partial\psi^\alpha} \in H^{(3,0)} + H^{(2,1)} \ . \tag{A.3}$$

Eq. (A.3) is easily understood if we think of variables x^i as functions of the deformations ψ^α, and we project the derivative of dx^i into a holomorphic and antiholomorphic part. $H^{(3,0)}$ has dimension one for a CY threefold. We can explicitly show that [18]:

$$\frac{\partial\Omega}{\partial\psi^\alpha} = k_\alpha\Omega + \rho_\alpha \ , \tag{A.4}$$

[1] We limit ourself to the case of "physical" CY threefolds, but what we say is completely general and holds true for all CY n-folds

where k_α may depend on ψ_α but not on the CY coordinates. From (A.4), using

$$\int \Omega \wedge \rho_\alpha = \int \Omega \wedge \bar{\rho}_{\bar{\alpha}} = 0 \tag{A.5}$$

we immediately find:

$$g_{\alpha\bar{\beta}} = \frac{1}{i\int \Omega \wedge \bar{\Omega}} \hat{g}^{\mathrm{wp}}_{\alpha\bar{\beta}} = -\partial_\alpha \partial_{\bar{\beta}} \log \left(i \int \Omega \wedge \bar{\Omega}\right) , \tag{A.6}$$

The holomorphic sections $X^\Lambda(\psi), F_\Lambda(\psi)$ of the special geometry are identified with the periods of Ω along the $b_3 = 2(h_{2,1}+1)$ homology cycles A_Λ and B^Λ:

$$\Omega(\psi) = X^\Lambda \alpha_\Lambda + i F_\Lambda \beta^\Lambda \tag{A.7}$$

where $\alpha_\Lambda, \beta^\Lambda$ is a fixed cohomology basis in H^3 dual to homology cycles A_Λ, B^Σ:

$$\begin{aligned} \int \alpha_\Lambda \wedge \beta^\Sigma &= -\int \beta^\Sigma \wedge \alpha_\Lambda = \delta^\Sigma_\Lambda , \\ \int \alpha_\Lambda \wedge \alpha_\Sigma &= \int \beta^\Lambda \wedge \beta^\Sigma = 0 . \end{aligned} \tag{A.8}$$

This means that, from an abstract point of view, we define X^Λ, F_Λ as:

$$X^\Lambda = \int_{A^\Lambda} \Omega \quad , \quad F_\Lambda = \int_{B_\Lambda} \Omega . \tag{A.9}$$

It can be shown that, locally in the moduli space, the complex structure of the CY manifold is completely determined by the ψ^α, so that $F_\Lambda = F_\Lambda(\psi)$. It is clear that a rescaling $X \to \lambda X$ by a nonzero λ corresponds to a rescaling of Ω. In other words we may regard the X^Λ as projective coordinates for the complex structure and Ω as being homogeneous of degree one in these coordinates.

$$\Omega(\lambda X) = \lambda \Omega . \tag{A.10}$$

Now, in virtue of the eq. (A.4) we have:

$$\int \Omega \wedge \frac{\partial \Omega}{\partial \psi^\alpha} = 0 . \tag{A.11}$$

This gives the relation :

$$2F_\Lambda = \frac{\partial}{\partial X^\Lambda}(X^\Sigma F_\Sigma) , \tag{A.12}$$

thus F_Λ is the gradient of an homogeneous function of degree two.

$$F_\Lambda = \frac{\partial F}{\partial X^\Lambda} \quad , \quad F(\lambda X) = \lambda^2 F(X) . \tag{A.13}$$

It is easy to show that the Kähler potential is written as in chapter 2:

$$G = -\log \left(i \int \Omega \wedge \bar{\Omega}\right) = -\log \left[X^\Lambda \bar{F}_\Lambda + \bar{X}^\Lambda F_\Lambda\right] . \tag{A.14}$$

Moreover the holomorphic three index section $w_{\alpha\beta\gamma}$ is expressed by:

$$w_{\alpha\beta\gamma} = i \int \Omega \wedge \frac{\partial\Omega}{\partial\psi_\alpha \partial\psi_\beta \partial\psi_\gamma} \,. \tag{A.15}$$

The symplectic transformations are retrieved as transformations associated to the change of canonical homology cycles.

Guided by this discussion we can give the following definition of special geometry (completely equivalent to the one presented in chapter 2)

Definition 5 *A special manifold is an n–dimensional Kähler manifold of restricted type such that on each path U_i of a good cover there exist complex projective coordinates X_i^Λ and a homogeneous, degree two holomorphic function $F(X)_i$, related to the Kähler potential by*

$$G_i = -log[X_i^\Lambda \bar{\partial}_\Lambda F_i + \bar{X}_i^\Lambda \partial_\Lambda F_i] \,. \tag{A.16}$$

On the intersection of adiacent patches U_i and U_j, $\partial_\Lambda F$ and X^Λ are related by special coordinate transformations :

$$\begin{pmatrix} X \\ i\partial F \end{pmatrix}_i = e^{f_{ij}} M_{ij} \begin{pmatrix} X \\ i\partial F \end{pmatrix}_j \tag{A.17}$$

where the f_{ij} are holomorphic and M_{ij} is a constant element of $Sp(2n+2, R)$. The transition functions are subject to the usual consistency conditions on triple overlap:

$$\begin{aligned} e^{f_{ij}+f_{jk}+f_{ki}} &= 1 \,, \\ M_{ij} M_{jk} M_{ki} &= 1 \,. \end{aligned} \tag{A.18}$$

This definition refers to a particular coordinate system. A coordinate independent definition is given by:

Definition 6 *Let L denote the complex line bundle whose first Chern class equals the Kähler form K, of an n–dimensional Kähler manifold $\mathcal{M}$ of restricted type. Let $\mathcal{H}$ denote a holomorphic $Sp(2n+2, R)$ vector bundle over $\mathcal{M}$ and $-i < \ | \ >$ the compatible hermitian metric on $\mathcal{H}$. $\mathcal{M}$ is a special manifold if, for some choice of $\mathcal{H}$, there exists a holomorphic section Ω of $\mathcal{H} \otimes L$ with the property:*

$$K = -\partial\bar{\partial} log\,(-i\langle\Omega\,|\,\bar{\Omega}\rangle) \,. \tag{A.19}$$

Note that the transition functions of a holomorphic $Sp(2n+2, R)$ vector bundle are necessarily constant on each overlap. The compatible Hermitian metric can be defined as in chapter 2.

The equivalence of all the definitions of special geometry we give are easily understood by thinking of the section Ω as expressed by $\Omega = (X, i\partial F)$ in each coordinate patch and by utilizing eq.s (2.20)–(2.36) from chapter 2

Appendix B

Technical remarks concerning Chapter 3

B.1 Landau-Ginzburg action and transformation rules in component formalism

In this appendix we present the explicit form of the Landau-Ginzburg lagrangian in component formalism and use the rheonomic approach to find the N=2 supersymmetry transformations. In the notations of [1], we write the following curvatures

$$\begin{aligned} T^a &= \mathcal{D}V^a - \frac{i}{2}\bar{\xi} \wedge \gamma^a \xi \ , \\ \rho &= \mathcal{D}\xi \ , \\ F &= dA - i\bar{\xi} \wedge \xi \ , \\ R^{ab} &= d\omega^{ab} \ , \end{aligned} \tag{B.1}$$

where V^a is the zweibein, ξ is the gravitino one-form, A is the $U(1)$ connection, ω^{ab} is the spin connection and $\mathcal{D}$ is the Lorentz covariant derivative. The gravitino ξ is a Dirac spinor. In general we can write

$$\xi = e^{-i\pi/4} \begin{pmatrix} \zeta \\ \tilde{\zeta} \end{pmatrix} \tag{B.2}$$

with $\zeta \neq \zeta^*$, $\tilde{\zeta}^* \neq \zeta$. More precisely, if we set

$$\begin{aligned} e^{\pm} &= \frac{1}{2}(V^0 \pm V^1) \ , \\ \omega^{ab} &= \epsilon^{ab}\omega \ , \end{aligned} \tag{B.3}$$

we obtain

$$T^a = dV^a - \omega^{ab} \wedge V^b - \frac{i}{2}\bar{\xi} \wedge \gamma^a \xi \ , \tag{B.4}$$

or

$$T^{\pm} = de^{\pm} \pm \omega \wedge e^{\pm} - \frac{\mathrm{i}}{2}\bar{\xi} \wedge \gamma^{\pm}\xi \quad , \tag{B.5}$$

where $\gamma^{\pm} \equiv \frac{1}{2}(1 \pm \gamma_3)$. Using (B.2), we have

$$\begin{aligned} T^{+} &= de^{+} + \omega \wedge e^{+} - \frac{\mathrm{i}}{2}\zeta^{*} \wedge \zeta \quad , \\ T^{-} &= de^{-} + \omega \wedge e^{-} - \frac{\mathrm{i}}{2}\tilde{\zeta}^{*} \wedge \tilde{\zeta} \quad . \end{aligned} \tag{B.6}$$

Similarly, we get

$$F = dA - \zeta^{*} \wedge \tilde{\zeta} + \tilde{\zeta}^{*} \wedge \zeta \quad . \tag{B.7}$$

Following the general recipes of the rheonomic procedure [1], we write the background Maurer-Cartan equations

$$\begin{aligned} de^{+} &+ \omega \wedge e^{+} - \frac{\mathrm{i}}{2}\zeta^{+} \wedge \zeta^{-} = 0 \quad , \\ de^{-} &- \omega \wedge e^{-} - \frac{\mathrm{i}}{2}\tilde{\zeta}^{+} \wedge \tilde{\zeta}^{-} = 0 \quad , \\ d\zeta^{+} &+ \frac{1}{2}\omega \wedge \zeta^{+} = 0 \quad , \\ d\tilde{\zeta}^{+} &- \frac{1}{2}\omega \wedge \tilde{\zeta}^{+} = 0 \quad , \\ d\omega &= 0 \quad , \\ dA &- \zeta^{-} \wedge \tilde{\zeta}^{+} + \tilde{\zeta}^{+} \wedge \zeta^{-} = 0 \quad , \end{aligned} \tag{B.8}$$

where we have set $\zeta^{-} = \zeta$ and $\zeta^{+} = \zeta^{*}$. Using these notations, we can write the general form of the LG lagrangian in components. From the Bianchi identities $d^2X^i = d^2\psi^i = d^2\psi^{i^*} = 0$ and from (B.8) one derives the following rheonomic parametrizations

$$\begin{aligned} dX^i &= \partial_{+}X^i e^{+} + \partial_{-}X^i e^{-} + \psi^i\zeta^{-} + \tilde{\psi}^i\tilde{\zeta}^{-} \quad , \\ dX^{i^*} &= \partial_{+}X^{i^*} e^{+} + \partial_{-}X^{i^*} e^{-} - \psi^{i^*}\zeta^{+} - \tilde{\psi}^{i^*}\tilde{\zeta}^{+} \quad , \\ d\psi^i &= \partial_{+}\psi^i e^{+} + \partial_{-}\psi^i e^{-} - \frac{\mathrm{i}}{2}\partial_{+}X^i\zeta^{+} + \eta^{ij^*}\partial_{j^*}\bar{W}\zeta^{-} \quad , \\ d\tilde{\psi}^i &= \partial_{+}\tilde{\psi}^i e^{+} + \partial_{-}\tilde{\psi}^i e^{-} - \frac{\mathrm{i}}{2}\partial_{-}X^i\tilde{\zeta}^{+} - \eta^{ij^*}\partial_{j^*}\bar{W}\zeta^{-} \quad , \end{aligned} \tag{B.9}$$

where $X^i, X^{i^*} = (X^i)^*$ are complex coordinates in a flat Kähler manifold, η^{ij^*} is the flat metric and ψ^i, $\tilde{\psi}^i$ are the complex spin-$\frac{1}{2}$ fermionic partners of X^i's. The parametrizations of $d\psi^{i^*}$ and $d\tilde{\psi}^{i^*}$ are obtained by complex conjugation. Using standard techniques, one finds that the action, from which (B.9) follow as field equations in the vertical directions, is

$$S = \int \mathcal{L} \tag{B.10}$$

where

$$
\begin{aligned}
\mathcal{L} &= \eta_{ij^*}(dX^i - \psi^i\zeta^- - \tilde{\psi}^i\tilde{\zeta}^-) \wedge (\Pi_+^{j^*} e^+ - \Pi_-^{j^*} e^-) \\
&+ \eta_{j^*i}(dX^{j^*} + \psi^{j^*}\zeta^+ + \tilde{\psi}^{j^*}\tilde{\zeta}^+) \wedge (\Pi_+^i e^+ - \Pi_-^i e^-) \\
&+ \eta_{ij^*}(\Pi_+^i \Pi_-^{j^*} + \Pi_-^i \Pi_+^{j^*}) e^+ \wedge e^- - (4\mathrm{i}\eta_{ij^*}\psi^i d\psi^{j*} + \frac{\mathrm{i}}{2}\psi^k \partial_k W \tilde{\zeta}^+) \wedge e^+ \\
&+ (4\mathrm{i}\eta_{ij^*}\tilde{\psi}^i d\tilde{\psi}^{j*} + \frac{\mathrm{i}}{2}\tilde{\psi}^k \partial_k W \zeta^+) \wedge e^- \\
&+ \frac{\mathrm{i}}{2}\psi^{k^*}\partial_{k^*}\bar{W}\tilde{\zeta}^- \wedge e^+ - \frac{\mathrm{i}}{2}\tilde{\psi}^{k^*}\partial_{k^*}\bar{W}\zeta^- \wedge e^- \\
&+ (8\psi^i\tilde{\psi}^j\partial_i\partial_j W + 8\tilde{\psi}^{i^*}\tilde{\psi}^{j^*}\partial_{i^*}\partial_{j^*}\bar{W} + 8\eta^{ij^*}\partial_i W \partial_{j^*}\bar{W}) e^+ \wedge e^- \\
&+ dX^{j^*} \wedge \psi^i\zeta^-\eta_{ij^*} - dX^{j^*} \wedge \tilde{\psi}^i\tilde{\zeta}^-\eta_{ij^*} \\
&+ dZ^i \wedge \psi^{j^*}\zeta^+\eta_{ij^*} - dZ^i \wedge \tilde{\psi}^{j^*}\tilde{\zeta}^+\eta_{ij^*} \quad . \qquad \text{(B.11)}
\end{aligned}
$$

The lagrangian (B.11) is written in first-order formalism and the auxiliary fields $\pi_\pm^i, \pi_\pm^{i^*}$ can be eliminated through their equations of motion: $\Pi_\pm^i = \partial_\pm X^i$ and $\Pi_\pm^{i^*} = \partial_\pm X^{i^*}$. The lagrangian (3.61) of Section 3.2 is obtained from (B.11) by restricting it to the bosonic surface (namely discarding all terms containing the gravitino forms $\zeta^\pm, \tilde{\zeta}^\pm$) and substituting back the above mentioned equations for $\Pi_\pm^{i,i^*}$. Form the curvature parametrization (B.9), we easily recover the N=2 supersymmetry transformations

$$
\begin{aligned}
\delta X^i &= -\varepsilon^-\psi^i - \tilde{\varepsilon}^-\tilde{\psi}^i \quad , &\text{(B.12)}\\
\delta X^{i^*} &= \varepsilon^+\psi^{i^*} + \tilde{\varepsilon}^+\tilde{\psi}^{i^*} \quad , &\text{(B.13)}\\
\delta\psi^i &= -\frac{\mathrm{i}}{2}\partial X^i\,\varepsilon^+ + \eta^{ij^*}\partial_{j^*}\bar{W}\,\tilde{\varepsilon}^- \quad , &\text{(B.14)}\\
\delta\tilde{\psi}^i &= -\frac{\mathrm{i}}{2}\bar{\partial} X^i\,\tilde{\varepsilon}^+ - \eta^{ij^*}\partial_{j^*}\bar{W}\,\varepsilon^- \quad , &\text{(B.15)}\\
\delta\psi^{i^*} &= \frac{\mathrm{i}}{2}\partial X^{i^*}\,\varepsilon^- + \eta^{ji^*}\partial_j W\,\tilde{\varepsilon}^+ \quad , &\text{(B.16)}\\
\delta\tilde{\psi}^{i^*} &= \frac{\mathrm{i}}{2}\bar{\partial} X^{i^*}\,\tilde{\varepsilon}^- - \eta^{ji^*}\partial_j W\,\varepsilon^+ \quad , &\text{(B.17)}
\end{aligned}
$$

which coincide precisely with the ones in (3.63) of Section 3.2.

B.2 The lagrangian for the topological (b, c, β, γ)-system

In Section 3.2 we gave the explicit form of the descent equations (3.58), using (3.55) as BRST tranformations. We know that this case corresponds to the twisted stress-energy tensor

$$\hat{T}_+ = T + \frac{1}{2}\partial J = \partial c b + \gamma\partial\beta \tag{B.18}$$

which is associated to a (b, c, β, γ)-system with $\lambda_\beta = 0$, $\lambda_b = 0$. For simplicity we consider the case of only one collection of pseudo-ghost fields and as usual, we understand the expressions for the tilded operators. The main advantage of this approach is that we can write the chiral primary fields of the N=2 theory in terms of the *local* fields appearing in the lagrangian, and hence we can construct the representatives of the BRST cohomology in a rather simple way. The lagrangian, however, is not a BRST commutator: we are dealing with a topological theory of the Schwartz-type [24].

As pointed out in Section 3.2, there is another possibilty and one can take as twisted stress-energy tensor the following expression

$$\begin{aligned} \hat{T}_- &= T - \frac{1}{2}\partial J \\ &= (1-2\lambda)\,\partial b\, c \,-2\lambda\, b\partial c \,+\, (1-2\lambda)\,\gamma\,\partial\beta \,-\, 2\lambda\beta\,\partial\gamma \ . \end{aligned} \tag{B.19}$$

This corresponds to a commuting (β, γ)-system with weights $\lambda_\beta = 2\lambda$, $\lambda_\gamma = 1 - 2\lambda$, and to an anticommuting (b, c)-system with weights $\lambda_b = 2\lambda$ and $\lambda_c = 1 - 2\lambda$. In this case the BRST tranformations are

$$\begin{aligned} s\beta &= 2b \ , \quad s\tilde{\beta} = 2\tilde{b} \ , \\ sc &= 2\gamma \ , \quad s\tilde{c} = 2\tilde{\gamma} \ , \\ sb &= 0 \ , \quad s\tilde{b} = 0 \ , \\ s\gamma &= 0 \ , \quad s\tilde{\gamma} = 0 \ . \end{aligned} \tag{B.20}$$

Eq.s (B.20) are obtained from the supersymmetry tranformations (3.36) by setting $\epsilon^+ = \bar{\epsilon}^+ = 0$ and choosing as BRST parameter $\theta = \sqrt{2}\epsilon^- = \sqrt{2}\bar{\epsilon}^-$ This corresponds to taking

$$Q = \frac{1}{\sqrt{2}}\left(\oint G^-(z)dz + \oint \bar{G}^-(\bar{z})d\bar{z}\right) \tag{B.21}$$

as BRST charge. The stress-energy tensor (B.19) is a BRST commutator, namely

$$\hat{T} = s\left[-(\lambda - \frac{1}{2})c\partial\beta - \lambda\beta\partial c\right] \tag{B.22}$$

where, as usual, we have defined $s\phi = [Q, \phi]$ for a generic field ϕ. Now we want to show that the twist (B.19) can be seen directly at the lagrangian level, or equivalently that we can write the lagrangian (including the interaction term) as a BRST commutator. If this is the case, then we have a topological theory of Witten-type [24]. To understand this point, we first recall that the supercurrent G^+, as shown in (3.41), is actually composed of a G^+_z term and a $G^+_{\bar{z}}$ term. The latter is zero on shell, but it has to be taken into account in defining the superconformal transformations [47]. Keeping this in mind, we recognize that

$$\begin{aligned} \mathcal{L} &= s(-G^+_{\bar{z}} - \bar{G}^+_z) \\ &= s[-(\lambda - \tfrac{1}{2})c\bar{\partial}\beta - \lambda\beta\bar{\partial}c + \tfrac{1}{2}V\tilde{b}\bar{\partial}\tilde{V} - (\lambda - \tfrac{1}{2})\tilde{c}\partial\tilde{\beta} - \lambda\tilde{\beta}\partial\tilde{c} - \tfrac{1}{2}\tilde{V}b\partial V] \end{aligned} \quad \text{(B.23)}$$

Using (B.22) , we get

$$\mathcal{L} = \left[-2\lambda\beta\bar{\partial}\gamma + (1-2\lambda)\gamma\bar{\partial}\beta + (1-2\lambda)\bar{\partial}bc - 2\lambda b\bar{\partial}c + b\tilde{b}\partial V\bar{\partial}\tilde{V}\right] + \text{c.c} \qquad \text{(B.24)}$$

which is the expected lagrangian for the twisted λ's. It is suggestive to point out that, if we define

$$\begin{aligned} G^+ &= -(\lambda - \frac{1}{2})c\; d\beta - \lambda\beta\; dc + \frac{1}{2}V\tilde{b}\bar{\partial}\tilde{V}d\bar{z} \;, \\ \bar{G}^+ &= -(\lambda - \frac{1}{2})\tilde{c}\; d\tilde{\beta} - \lambda\tilde{\beta}\; d\tilde{c} - \frac{1}{2}\tilde{V}b\partial V dz \;, \end{aligned} \qquad \text{(B.25)}$$

where for a generic pseudo-ghost field ϕ

$$d\phi = \partial\phi dz + \bar{\partial}\phi d\bar{z} \qquad \text{(B.26)}$$

denotes the space-time part of the rheonomic parametrizations (that is we disregard the gravitino contributions), we get

$$S = \int \mathcal{L}_{\text{top}}\; dz \wedge d\bar{z} = \int dz \wedge s(G^+) + \int s(\bar{G}^+) \wedge d\bar{z} \;. \qquad \text{(B.27)}$$

Finally, if we regard (B.24) as a topological lagrangian without special requirements on the interaction term, we can ask when this model defines a conformal field theory. The answer to this question is almost obvious, even if the calculation is a little different: only for quasi-homogeneous potential $V(\beta)$ with $2\lambda = \omega$ we get a conformal field theory.

B.3 Adding marginal perturbations to the free first order lagrangian

As pointed out in the first chapters the moduli space of a (2,2) superconformal theory, describing a Calabi–Yau compactification theory, is the direct product of two special Kähler manifolds $\mathcal{S}_{(1,1)} \times \mathcal{S}_{(2,1)}$, with dimensions h_{11}, h_{21}. From the abstract (2,2) viewpoint $\mathcal{S}_{(2,1)}$ is the parameter space of marginal deformations induced by chiral–chiral primary fields characterized by conformal weights $h = \bar{h} = \frac{1}{2}$ and $U(1)$ charges $q = \bar{q} = 1$, while $\mathcal{S}_{(1,1)}$ is the parameter space of the deformations induced by chiral–antichiral primary fields having $h = \bar{h} = \frac{1}{2}$ and $q = -\bar{q} = 1$. In our first order $b--c--\beta--\gamma$ approach we are not able to produce local deformations corresponding to the antichiral primary fields, but we can analize the chiral sector. In this context, given a Landau–Ginzburg "characterization" of a first order system with $c \geq 3$, the only perturbations that correspond to the chiral primary operators are the modal deformations of the defining polynomial. Let us take for example the case of the quintic polynomial

$$W = \sum_{i=1}^{5} X_i^5 \ , \ X_i = \beta_i \tilde{\beta}_i \ . \tag{B.28}$$

As shown in [47] the perturbations which corresponds to chiral primary operators are the 101 algebraic moduli of the polynomial (B.28). This exhaust all the $h_{21} = 101$ complex structure deformations. Of course there could exist marginal perturbations that do not correspond to (second component of) chiral–primary operators of the n=2 theory. If we consider, say, the A_n models, the interaction term in the first order lagrangian is as well a marginal perturbation. In this case however there are no moduli that corresponds to chiral primary fields, simply because the A_n potential has modality zero. We are just speaking of marginal perturbations of (2,2) theories which define fixed lines of such theories, without corresponding directly to the geometrical interpretation of complex–Kähler deformations.

In this case it could happen that there there exist other marginal operators, which move the *free* theory along different fixed lines, and preserve as well the N=2 superconformal invariance. It is known [93] that perturbing an N=2 superconformal theory with of the operator

$$\mathcal{O}(z, \bar{z}) = J(z)\tilde{J}(\bar{z}) \ , \tag{B.29}$$

where $J(z)$ is the U(1) current, we move it along a one parameter fixed line, to which belong also the SU(2) Wess-Zumino-Witten models. Howewer the perturbation (B.29) explicitly break the N=2 invariance. This can be easily understood form the observation that it cannot be written as the second component of a N=2 superfield, or more simply by looking at the OPE expansion of the $J(z)$ with $G^{\pm}(z)$.

At the classical level it is not difficult to select a marginal operator which preserves both conformal and susy invariance, playing a similar role of our interaction term (except the fundamental one of fixing the nature of the model through fixing its conformal dimension). Let us consider the simple framework of just one collection of b, c, β, γ fields. If we take the following combination [95, 100]:

$$\begin{aligned} O_2(z,\bar{z}) &= -J^3(z)\bar{J}^3(\bar{z}) = -\oint G^-(w) \oint \bar{G}^-(\bar{w})\beta(z)c(z)\tilde{\beta}(\bar{z})\tilde{c}(\bar{z}) \\ &= (bc+\beta\gamma)(\tilde{\beta}\tilde{\gamma}+\tilde{b}\tilde{c}) \end{aligned} \tag{B.30}$$

(the minus sign in (B.30) is conventional) and we add it to the free first order action with a small coupling δg such as:

$$\mathcal{L} = \mathcal{L}_0 + \delta g O_2 \ , \tag{B.31}$$

we get a theory which is classically invariant under the following $N = 2$ holomorphic transformations:

$$\begin{aligned} \delta\beta &= 2\sqrt{2}\epsilon^- b \ , \\ \delta b &= \frac{1}{\sqrt{2}}\epsilon^+\partial\beta + \sqrt{2}\lambda\,\partial\epsilon^+\beta \ , \\ \delta c &= 2\sqrt{2}\epsilon^-\gamma \\ \delta\gamma &= \frac{1}{\sqrt{2}}\epsilon^+\partial c - \sqrt{2}\,(\lambda - \tfrac{1}{2})\,\partial\epsilon^+ c \ , \\ \delta\tilde{\beta} &= \frac{1}{\sqrt{2}}\epsilon^+\delta g\beta c\tilde{\beta} \ , \\ \delta\tilde{b} &= \frac{1}{\sqrt{2}}\epsilon^+\delta g\beta c\tilde{b} \ , \\ \delta\tilde{c} &= -\frac{1}{\sqrt{2}}\epsilon^+\delta g\beta c\tilde{c} \ , \\ \delta\tilde{\gamma}_r &= -\frac{1}{\sqrt{2}}\epsilon^+\delta g\beta c\tilde{\gamma} \ , \end{aligned} \tag{B.32}$$

where $\epsilon^\pm$ are arbitrary holomorphic functions ($\bar{\partial}\epsilon^\pm = 0$). The action is also invariant under N=2 anti-holomorphic symmetries which are similar to the ones defined above. In this case the equation of motion are completely symmetric for all the fields; they can be written in a compact way as:

$$\begin{aligned} (\bar{\partial} - \delta g\bar{J}^3)\phi &= 0 \ , \\ (\partial - \delta g J^3)\tilde{\phi} &= 0 \ , \end{aligned} \tag{B.33}$$

for *any* tilded and untilded fields in the lagrangian. As a consequence of (B.33) there is no field in the perturbed lagrangian which preserve its holomorfic (antihlomorphic) character. Using the same procedure as in Section 3.2 we find that the Noether procedure gives the same holomorphic conserved currents G^+_z, G^-_z, $G^-_{\bar z}$ as in (3.41), and changes $G^+_{\bar z}$ to:

$$G^+_{\bar z} = \sqrt{2}[\lambda\beta\bar\partial c + (\lambda - \tfrac{1}{2})\bar\partial\beta c] + \frac{1}{\sqrt{2}}\delta g\beta c[\tilde b\tilde c + \tilde\beta\tilde\gamma] \ , \tag{B.34}$$

which is again zero on shell. To ask whether or not the operator (B.30) is truly marginal, we can refer to the integrability criterion exposed in chapter 3. It is immediately verified that the OPE of $O_2(z.\bar z)$ with itself is regular, as a consequence of the cancellation of singular terms between the bc and $\beta\gamma$ contributions, so that there is no one loop correction to the conformal dimension of O_2 itself. The operator $O_2(z,\bar z)$ is truly marginal. So we expect that also at the quantum level either the conformal symmetry (B.32), either the N=2 susy is preserved. However, due to the presence of loop corrections in this non trivially interacting theory, the quantum case is far more complicated than our model [100], and it has to be treated very carefully. Here we limit ourself to simple classical considerations.

Before going into further discussions let us make a comment on the use of the "current" notation $J^3(z)$. This is a consequence of the fact that for a $b-c-\beta-\gamma$ system we can contruct [95] a $Sl(2)$ current algebra as follow:

$$\begin{aligned} J^+ &= -\gamma \ , \\ J^3 &= -\beta\gamma - bc \ , \\ J^- &= \beta^2\gamma + 2\beta bc \ , \end{aligned} \tag{B.35}$$

with (left) conformal dimensions: $1-\lambda$, 1 and $1+\lambda$. J^3 is precisely the combination used in (B.30).

The natural question that now arise from the discusion is what happens when also the Landau–Ginzburg potential term is present. For sake of simplicity we consider the A_n model. The interaction part of the lagrangian is now given by:

$$\begin{aligned} \Delta\mathcal{L} &= \delta g_1 b\beta^n\tilde b\tilde\beta^n - \delta g_2(bc+\beta\gamma)(\tilde\beta\tilde\gamma + \tilde b\tilde c) \\ &= g_1 O_1(z\bar z) + g_2 O_2(z,\bar z) \ . \end{aligned} \tag{B.36}$$

It is tedious, but straighforward to show that even at the classical level the trasformations obtained by a suitable combination of (B.32) and (3.36) are no more symmetries of the full lagrangian. Hovewer, since we are adding "classical" marginal perturbations we may ask if the conformal invariance is preserved at the quantum level. The answer is negative. Indeed with a little generalization of the citerion exposed in chapter 3 [94] we should require that in the operator product of $O_i, i = 1,2$ no pole terms are present. Hovewer, if we consider:

$$O_2(z,\bar z)O_1(w,\bar w) = (n+1)^2\frac{O_1(w,\bar w)}{(z-w)(\bar z - \bar w)} + \text{reg.} \tag{B.37}$$

we find such a term, which destroys, at one loop level, the conformal properties of the model [79]. Notice that if we consider instead of O_2 the combination O in (B.29) the "no pole" condition is satisfied, since:

$$O(z,\bar{z})O_1(w,\bar{w}) = \left[(-\frac{n}{n+1}bc + \frac{1}{n+1}\beta\gamma)\, b\beta^n\right] \times [\text{c.c.}] = \text{reg.} \tag{B.38}$$

B.4 The flat metric method: an example

The basic idea of this appendix is to give an explicit sample of calculation for the metric and Riemann tensor in the coupling constant space for a general parametrization of the Landau–Ginsburg potential, and impose flatness.

We consider the example of the $c = 3$ model defined by:

$$W = \frac{X^4}{4} + \frac{Y^4}{4} - sX^2Y^2 \; . \tag{B.39}$$

We are interested in finding the differential equation on the variable s, that represents the flatness condition, in complete analogy to eq (3.216) The usual computation requires to write down the fully perturbed potential

$$W = \frac{X^4}{4} + \frac{Y^4}{4} - \mu_0 - \mu_1 X - \mu_2 X^2 - \mu_3 XY^2 - \mu_4 X^2Y^2 - \mu_5 X^2 Y - \mu_6 Y^2 - sX^2Y^2 \tag{B.40}$$

and to compute all metric elements in this eight–dimensional space . However we are going to show that in this case it suffices, in performing calculations, to consider only a subring of the chiral ring, resulting in a considerable semplification of the technical machinery.

Let us consider the action on (B.39) of the Z_4 symmetry generated by

$$h = (\beta, \beta^3) \quad , \quad \beta = e^{\frac{\pi i}{2}} \; , \tag{B.41}$$

with the first entry of h multiplying the X coordinate and the second the Y coordinate. The only elements in the chiral ring that are left invariant by this symmetry are $\phi_0 = 1$, $\phi_4 = XY$ and $\phi_8 = \phi_{top} = X^2Y^2$. Our purpose is to compute the metric and the Riemann tensor for the reduced perturbed potential:

$$W(\mu, s) = \frac{X^4}{4} + \frac{Y^4}{4} - \mu\phi_4 - s\phi_8 \; . \tag{B.42}$$

Since we are interested only in the s dependence for the potential (B.39), we will keep only linear and quadratic terms in the relevant coupling μ, as we will send it to zero in the Riemann tensor computation. The vanishing relations associated to (B.42) are:

$$\begin{aligned} X^3 &= \mu Y + 2sXY^2 \; , \\ Y^3 &= \mu X + 2sX^2Y \; . \end{aligned} \tag{B.43}$$

Using (B.39) it is immediate to find

$$\rho = \det\, \partial_i \partial_j W = (1 - 4s^2) X^2 Y^2 \; , \tag{B.44}$$

so that
$$\langle X^2Y^2\rangle(s) = \frac{1}{1-4s^2}\langle\rho\rangle = \frac{1}{1-4s^2}h(s) \ . \tag{B.45}$$
Formula (B.45) gives the g_{08} component of the metric, since:
$$g_{08} =< \phi_0\phi_8 >=< X^2Y^2 > (s) \ . \tag{B.46}$$
For the other components we have to use repeatedly the vanishing relations (B.43). As an example we write explicitly g_{48}. The operator product gives
$$\phi_4\phi_8 = X^3Y^3 = (\mu Y + 2sXY^2)(\mu X + 2sX^2Y) \ , \tag{B.47}$$
so that
$$(1-4s^2)X^3Y^3 = \mu^2 XY + 4sX^2Y^2 \ . \tag{B.48}$$
In the vacuum expectation value only the second term survives and
$$g_{48} = \frac{4\mu s}{1-4s^2}f(s) \ , \tag{B.49}$$
where we have posed $f(s) = \frac{h(s)}{1-4s^2}$. Our final result is expressed by the following following three by three matrix:
$$g = f(s)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & \frac{4\mu s}{1-4s^2} \\ 1 & \frac{4\mu s}{1-4s^2} & \frac{\mu^2}{1-4s^2} + \frac{16s^2\mu^2}{(1-4s^2)^2} \end{pmatrix} \ . \tag{B.50}$$
If we compute the R^4_{884} component of the Riemann tensor and send the relevant coupling μ to zero, we find:
$$R^4_{884} = 2(1-4s^2)f\frac{d^2f}{ds^2} - s(1-4s^2)(\frac{df}{ds})^2 - 64s^2f^2 - 12f^2 \ . \tag{B.51}$$
Imposing the flatness requirement and expressing the differential equation for $h(s) = (1-4s^2)f(s)$ we get
$$2(1-4s^2)h\frac{d^2h}{ds^2} - 3(1-4s^2)(\frac{dh}{ds})^2 - 16sh\frac{dh}{ds} + 4h^2 = 0 \ , \tag{B.52}$$
which precisely coincides with the one appearing in ref.s [34, 78]. Moreover, introducing the flattening factor for ϕ_8, i.e.
$$\phi_8 = \frac{ds}{dt}X^2Y^2 \ , \tag{B.53}$$
we can verify that (B.52) is equivalent to the following Schwarzian equation [78]:
$$\{t,s\} = \frac{8s^2+6}{(1-4s^2)^2} \tag{B.54}$$
Also for the case in (B.39) one can analize the duality group starting from the properties of the solutions of (B.52) (B.54). In this case the duality group is given by $\Gamma_W = \frac{\Gamma}{\Gamma(2)}$. Γ_W coincides with the Dihedral group [34]. This result can be easily obtained by performing the same calculations as in the cubic torus case, finding the Picard–Fuchs equation associated to (B.39) and studying its monodromy group.

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Bibliography

[1] For a review see L. Castellani, R. D' Auria and P. Frè, "Supergravity and superstrings: a geometric prospective", World Scientific, Singapore (1991).

[2] S. Ferrara and S. Theisen, "Moduli Spaces, Effective Actions and Duality Symmetry in String Compactifications" Proceedings of the Third Hellenic Summer School, 1989.

[3] E. Cremmer, C. Kounnas, A. Van Proeyen, J.P. Derendinger, S. Ferrara, B. de Wit and L. Girardello, Nucl. Phys. **B250** (1985) 385.

[4] A. Strominger, Comm. Math. Phys. **133** (1990) 163.

[5] S. Cecotti, S. Ferrara and L. Girardello, Int. Jour. Mod . Phys **A10** (1989) 2475 and Phys. Lett. **213B** (1988) 443.

[6] S. Ferrara and A. Strominger, Strings 89, Eds. R. Arnowitt, R. Bryan, M. Duff, D. Nanopulos and C. Pope, World Scientific, Singapore, 1989.

[7] P. Candelas, X. de la Ossa, Nucl. Phys. **B355** (1991) 455.

[8] S. Cecotti Comm. Math. Phys. **131** (1990) 517

[9] For a review see M.B. Green, J. Schwarz and E. Witten, "Superstring theory", Cambridge University Press, 1987.

[10] N. Seiberg, Nucl. Phys. **B303** (1988) 286.

[11] L. J. Dixon, V. Kaplunowski and J. Louis, Nucl. Phys. **B329** (1990) 27.

[12] A.Sen, Nucl. Phys. **B278** (1986) 289 and Nucl. Phys. **B284** (1987) 423; T. Banks, L.J. Dixon, D. Friedan and E. Martinec, Nucl. Phys. **B299** (1988) 613 and Nucl. Phys. **B307** (88) 93.

[13] P. Candelas, T. Hübsh and R. Schimmrig, Nucl. Phys. **B239** (1990) 582.

[14] D. Gepner, Nucl. Phys. **B296** (1988) 757 and Phys. Lett. **199B** (1987) 380.

[15] A. B. Zamolodchikov, JEPT Lett. **43** (1986) 730 and Sov.A J. Nucl. Phys. **46** (1987) 1090.

[16] P. Candelas, C.T. Horowitz, A. Strominger and E. Witten, Nucl. Phys. **B258** (1985) 46.

[17] P. Candelas "Lectures on Complex manifolds" Proceedings of Trieste Spring School on Superstrings, 1987

[18] B. de Wit and A. Van Proeyen, Nucl. Phys. **B245** (1984) 89.

[19] B. de Wit, P.G. Lauwers and A. Van Proeyen, Nucl. Phys. **B255** (1985) 560.

[20] E. Cremmer and A. Van Proeyen, Class. and Quantum Grav. **2** (1985) 485.

[21] L. Castellani, R. D'Auria and S. Ferrara, Phys Lett **B241** (1990) 57 and Class. and Quantum Grav. **1** (1990) 1767.

[22] R. D'Auria, S. Ferrara and P. Frè, Nucl. Phys. **B359** (1991) 705.

[23] S. Ferrara, A. Van Proeyen, Class. and Quantum Gravity **6** (1989) *L*243.

[24] For a review see D. Birmingham, M. Blau and M. Rakowski, Phys. Rep. **209** (1991) 129.

[25] E. Witten, Comm. Math. Phys. **118** (1988) 411 and Nucl. Phys. **B340** (1990) 281,
T. Eguchi and S.K. Yang Mod Phys Lett **A5** (1990) 1693.

[26] L. Baulieu and E. M. Singer, Comm. Math. Phys. **125** (1989) 125.

[27] E. Witten, "Mirror manifolds and topological field theory", Essays on Mirror manifolds, ed. S-T Yau, Intern. Press (1992).

[28] For a review see A. Pasquinucci, Ph.D. Thesis SISSA/EP 1990.

[29] S. Cecotti, L. Girardello and A. Pasquinucci, Nucl. Phys. **B328** (1989) 701 and Int. J. Mod. Phys. **A6** (1991) 2427.

[30] N.P. Warner, Lectures at Trieste Spring School 1988, World Scientific, Singapore;
E. Martinec, Phys. Lett. **217B** (1989) 431;
For a review see also "Criticality, Catastrophe and Compactification", V.G. Knizhnik memorial volume, 1989 .

[31] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. **B352** (1991) 59.

[32] B. Block and A. Varchenko, Int. Jour. Mod. Phys. **A7** (1992) 1647.

[33] E. Verlinde and N.P. Warner, Phys. Lett. **269B** (1991) 96.

[34] A. Klemm, S. Theisen and M. Schmidt, "Correlation functions for topological Landau–Ginzburg models with $c < 3$" Prepr. TUM-TP-129/91, KA-THEP-91-00, HD-THEP-91-32.

[35] W. Lerche, D.J. Smit and N.P. Warner Nucl. Phys. **B372** (1992) 87.

[36] S. Cecotti Nucl. Phys. **B355** (1991) 1986 and Int. Jour. Mod. Phys. **A6** (1991) 1749.

[37] S. Cecotti and C. Vafa Nucl. Phys. **B367** (1991) 359

[38] W. Lerche, C. Vafa and N.P. Warner, Nucl. Phys. **B324** (1989) 427 ,
B. Greene, C. Vafa and N.P. Warner, Nucl. Phys. **B324** (1989) 371.

[39] M.K, Gaillard, B. Zumino, Nucl. Phys. **B193** (1981) 221.

[40] A. Giveon, E. Rabinovici and G. Veneziano, Nucl. Phys. **B322** (1989) 167.

[41] K.Kikkawa and M. Yamasaki, Phys. Lett. **149B** (1984) 357.
N. Sakai and L. Senda, Progr. Theor. Phys. **75** (1986) 692.
V.P. Nair, A. Shapere, A. Strominger and F. Wilczek, Nucl. Phys. **B287** (1987) 402.
M. Dine, P. Huet and N. Seiberg, Nucl. Phys. **B322** (1989) 301.
A. Shapere and F. Wilczek, Nucl. Phys. **B320** (1989) 167.
A. Giveon and M. Porrati, Phys. Lett. **246B** (1990) 54 and Nucl. Phys. **355** (1991) 422.
J. Lauer, J. Mas and H.P. Nilles, Phys. Lett. **B226** (1989) 251 and Nucl. Phys. **B351** (1991) 353.
W. Lerche, D. Lüst and N.P. Warner Phys. Lett. **B231** (1989) 417.
M. Duff Nucl. Phys. **B335** (1990) 610.
A. Giveon, N. Malkin and E. Rabinovici, Phys. Lett. **B238** (1990) 57.
J. Erler, D. Jungnickel and H.P. Nilles, Preprint MPI-Ph/91-90 (1991).
S. Ferrara, D. Lüst, A. Shapere and S. Theisen, Phys. Lett. **B233** (1989) 147.
S. Ferrara, D. Lüst and S. Theisen, Phys. Lett. **B233** (1989) 147 and Phys. Lett. **B242** (1990) 39.
J. Schwartz, "Superstring compactifications and target space duality", Proceedings of the Stony–Brook conference "Strings and Symmetries" and Phys. Lett. **B272** (1991) 239.
J. Erler, D. Jungnickel and H.P. Nilles, Preprint MPI-Ph/91-81 (1991).

[42] S. Ferrara, C. Kounnas, D. Lüst and F. Zwirner, Nucl. Phys. **B365** (1991) 431.

[43] P. Frè and P. Soriani, Nucl. Phys. **B371** (1992) 659.
P. Frè and P. Soriani "Moduli spaces and the geometries of N=2 SUSY" Proceedings of Trieste Spring School and Workshop on String Theory and Quantum Gravity, 1991.

[44] S. Ferrara, P. Frè and P. Soriani Class. and Quantum Grav. **9** (1992) 1649.

[45] K.S. Narain, Phys. Lett. **B169** (1986) 369.
K.S. Narain, M.H. Sarmadi and E. Witten, Nucl. Phys. **B361** (1987) 414.
L.J. Dixon, lectures given at 1987 ICTP Summer Workshop in High Energy Physics and Cosmology.

[46] D.Friedan, E. Martinec and S. Shenker, Nucl. Phys. **B271** (1986) 93.

[47] P. Frè, F. Gliozzi, R. Monteiro and A. Piras, Class. and Quantum Grav **8** (1991) 1455.

[48] P. Frè, L. Girardello, A. Lerda and P. Soriani, " Topological first-order systems with Landau-Ginzburg interactions" SISSA/ISAS 28/92/EP, ITP-SB-92-7, IFUM 416/FT, March 92, to appear in Nucl. Phys B.
P. Frè and P. Soriani, "N=2 First order systems, Landau- Ginsburg potentials and topological twists" Prepr. SISSA/ISAS 165/92/EP , Talk given at Trieste Spring School on String Theory and Quantuom Gravity.

[49] C. Vafa, Mod. Phys. Lett. **A6** (1991) 337.

[50] M. Frau, J.G. McCarthy, A. Lerda, S. Sciuto and J. Sidenius, Phys. Lett. **254B** (1991) 381 and Phys. Lett. **245B** (1990) 453.

[51] G. Mussardo, G. Sotkov and M. Stanishkov, Int. J. Mod. Phys. **A4** (1986) 1135.

[52] V.S. Dotsenko and V.A. Fateev, Nucl. Phys. **B240** (1984) 312.

[53] E. S. Fradkin and A.A. Tseytlin, Nucl. Phys. **B261** (1985) 1,
C.G. Callan, D. Friedan, E. Martinec and M. Perry, Nucl. Phys. **B262** (1985) 593.

[54] For a review see H. P. Nilles, Phys. Rep. **C110** (1984) 1

[55] F. Englert, H. Nicolai and A.N. Schellekens, Nucl. Phys. **B274** (1986) 315.

[56] W. Lerche D. Lüst and A.N. Schellekens, Nucl. Phys. **B287** (1987) 477.

[57] A.Sen, Phys. Rev. Lett. **D32** (1985) 2102 and Phys Rev. Lett. **55** (1985) 1846.

[58] J. Lauer, D. Lüst and S. Theisen, Nucl. Phys. **B304** (1988) 236 and Nucl. Phys. **B309** (1988) 771.

[59] D. Lüst and S. Theisen, Int. Jour. Mod. Phys. **A4** (1989) 4513.

[60] M. Dine and N. Seiberg, Nucl. Phys. **B301** (1988) 357.

[61] J. Bagger and E. Witten, Nucl. Phys. **B222** (1983) 1,
J. Bagger, A. Galperin, E. Ivanov and V. Ogievetski, Nucl. Phys. **B303** (1988) 522

[62] T. Eguchi, H. Ouguri, A. Taormina and S. K. Yang, Nucl. Phys. **B315** (1989) 192.

[63] J. Distler and B. Greene, Nucl. Phys. **B309** (1989) 295.

[64] E. Cremmer, B. Julia, J. Scherk, S. Ferrara, L. Girardello and P. van Nieuwenhuizen, Nucl. Phys. **B147** (1979) 105,
E. Cremmer, S. Ferrara, L. Girardello and A. Van Proeyen, Nucl. Phys. **B212** (1983) 413.

[65] R. Brandenberger and C. Vafa, Nucl. Phys. **B316** (1989) 391.

[66] E. Witten, Phys. Lett. **B155** (1985) 151.
S. Ferrara, C. Kounnas and M. Porrati, Phys. Lett. **B181** (1986) 263.
M. Cvetic, J. Louis and B. Ovrut, Phys. Lett. **B206** (1988) 227.
S. Ferrara and M. Porrati, Phys. Lett. **B216** (1989) 289.
M. Cvetic, J. Molera and B. Ovrut, Phys. Lett. **D40** (1989) 1140.

[67] L. Dixon, J. Harvey, C. Vafa and E. Witten, Nucl. Phys. **B355** (1991) 422 and Nucl. Phys. **B274** (1986) 54.

[68] R. Dijkgraaf, E.Verlinde, H.Verlinde, Lectures at Trieste Spring School 90.

[69] S. Ferrara, C. Kounnas, L. Girardello and M. Porrati, Phys. Lett. **B194** (1987) 368.

[70] S. Ferrara, C. Kounnas, L.Girardello and M. Porrati, Phys.Lett. **B194** (1987) 358.

[71] S. Ferrara, C. Kounnas, M. Porrati and F. Zwirner, Nucl. Phys. **B318** (1989) 75.

[72] D. Shevitz, Nucl. Phys. **B338** (1990) 283.

[73] W. Lerche, A. Schellekens and N. Warner, Phys. Lett. **177** (1989) 1.

[74] A. Giveon and D.J. Smit, Nucl. Phys. **B349** (1991) 168.

[75] V.I. Arnold, S.M. Gusein-Zade, A.N. Varchenko, "Singularities of differentiable maps", Vol I, II Birkäuser, Boston.

[76] S. Ferrara, J. Louis, Phys. Lett. **B279** (1992) 240 Prepr. CERN-TH-6334/91.

[77] A. Ceresole, R. D'Auria, S. Ferrara, W. Lerche and J. Louis, CERN-TH-6441/92.

[78] Z. Maassarani, Phys. Lett. **273B** (1991) 457.

[79] P. Ginsparg, "Applied conformal field theory" Les Houches 1988.

[80] P. Candelas and X. de la Ossa, Nucl. Phys. **B342** (1990) 246.

[81] P. Candelas, X. de la Ossa, P. Green, and L. Parkes, Nucl. Phys. **B359** (1991) 21.

[82] L. Castellani, P. Fré, F. Gliozzi and M. Rego Monteiro, Int. Jour. Mod. Phys. **A30** (1991) 55.

[83] M. Ademollo, L. Brink, A. D'Adda, R. D'Auria, E. Napolitano, S. Sciuto, E. del Giudice, P. di Vecchia, S. Ferrara, F. Gliozzi, R. Musto and R. Pettorino, Phys. Lett. **62B** (1976) 105.

[84] P. Candelas, A.M. Dale, C.A. Lutken and R. Schimmrick, Nucl. Phys. **B298** (1988) 493.

[85] M. Linker and R. Schimmrick, Phys. Lett. **208B** (1988) 216 and Phys. Lett. **215B** (1988) 681.

[86] C.A. Lutken and G.C. Ross, Phys. Lett. **213B** (1988) 152.

[87] P. Zoglin, Phys. Lett. **218B** (1989) 444.

[88] P. Griffiths and J. Harris, "Principles of algebraic geometry" ed.s John Wiley and sons.

[89] M.T. Grisaru, A. Lerda, S. Penati and D. Zanon, Nucl. Phys. **B342** (1990) 564 and Phys. Lett. **234B** (1990) 88.

[90] E. Gava and M. Stanishkov, Mod. Phys. Lett. **27** (1990) 2261.

[91] G. Felder, Nucl. Phys. **B317** (1989) 215 and Nucl. Phys. **B324** (1989) 548E.

[92] M. Cvetic, J. Molera and B. Ovrut, Phys. Lett. **B248** (1990) 83.

[93] S. Yang, Phys. Lett. **209B** (1988) 242.

[94] S. Chaudhuri and J. A. Schwartz Phys. Lett. **B219** (1989) 291

[95] H. Yoshii, Phys. Lett. **B275** (1992) 70.

[96] P. Griffiths, Ann. Math. **90** (1969) 460.

[97] P. Candelas, Nucl. Phys. **B298** (1988) 458.

[98] H. Bateman, "Higher Trascendental Functions" Vol. 1, 2, 3 ed. Mac Graw–Hill.

[99] P. Di Francesco, C. Itzykson and J.B. Zuber, Comm. Math. Phys. **140** (1991) 543.

[100] D. Z. Freedman, E. Gath and K. Pilch Phys Rev. D **39** (1989) 3703,
D. Z. Freedman, and K. Pilch Int. Jour. Mod. Phys. **A20** (1989) 5553.

[101] F. Klein "Vorlesungen uber die theorie der elliptiscen modulfunzionen" Teubner, Leipzig 1890.

[102] G. Mazzeo, Magister thesis, SISSA/CM/1990