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Issues in String Theory and Noncommutative Geometry

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Introduction

The discovery of D–branes (1995) [1] radically changed our way of looking at open strings. As well known, D–branes are defined as hyperplanes on which open strings are allowed to end. In fact, there is more than this: open strings have to end on a D–brane: there are no open strings without D–branes where to end, and, conversely, there always can be open strings attached to a single D–brane and D–branes interacting among each other via open strings stretched from one brane to another. Open strings that move across the entire space should be regarded as strings attached to a brane that fills the entire space: this is the D25–brane. The target space of bosonic string theory is not empty, it is filled completely by the D25–brane. Very early was realized in studies of the physics of D–branes that their low-energy effective field theory has configuration space which is described in terms of noncommuting, matrix-valued spacetime coordinate fields [2], the entries of these matrices actually representing open strings stretched among the D–branes. This entails in particular that at short brane distances space becomes noncommutative. Noncommutativity shows up from D–branes also in another way: when D–branes are in presence of a constant NSNS B-field. In this case, the low energy effective action of the open strings attached to the branes can be represented by a Euclidean field theory defined on a noncommutative spacetime (noncommutative field theory). Noncommutative field theories have many novel properties which are not exhibited by conventional quantum field theories. They should be properly understood as lying somewhere between ordinary field theory and string theory, and the hope is that from these models we may learn something about string theory, using the somewhat simpler techniques of quantum field theory.

There is another place in string theory where the idea of a noncommutative product is present: it is the Witten's Open String Field Theory [63]. This is a second quantized formulation of string theory where the operation of gluing together open strings is noncommutative. It should be stressed that this is a theory for open strings, and then D–branes, and we do not have to be surprised if in such a theory noncommutativity shows up since it is contained in the idea itself of D–brane. Moreover String Field Theory, being a second quantized theory, is well defined also off–shell, on the contrary of the usual first quantized string theory where string amplitudes are computed rigorously on–shell, being defined as 2d conformal field theory correlators of string vertex operators. This property allows the study of nonperturbative phenomena like tachyon condensation.

This thesis deals with two problems in string theory characterized by the presence of noncommutativity and B field. The first problem regards the renormalization properties of noncommutative field theories, the second one the possibility of defining Vacuum String Field Theory in the presence of a B field. Let us have a closer look at these two subjects.

The correspondence between D–branes in the presence of a constant background B field and noncommutative field theories, summarized by Seiberg and Witten in [4], was done at tree level: tree amplitudes computed at tree level in string theory and field theory completely agree. Noncommutative field theories show however a peculiar behaviour at one loop, known as UV/IR mixing. Also the definition of noncommutative gauge theories based on groups different than $U(N)$ is highly non trivial. It seemed to be important therefore to know exactly what are the properties of a noncommutative YM theory we can rely on. One of the basic properties is renormalizability. In Chapter 2, referring ourselves to the works [31, 49] done in collaboration with L. Bonora, we will explicitly show that noncommutative $U(N)$ gauge theory is one–loop renormalizable. Then we will move to the one–loop study of noncommutative gauge theories based on orthogonal and symplectic groups. These kind of theories are very difficult to define, both from the field and the string point of view, even at tree level. Starting from the proposal [29] we investigate the matching of the string and the field definition of such theories at one loop, concluding that a satisfactory solution to this problem is far from obvious. Chapter 1 contains an introduction to noncommutative field theory, explaining in details its string origin and its perturbative properties. Moreover a section is devoted to the study of classical (solitonic) solution of noncommutative scalar theories, and to its use for describing D–branes, [92, 94, 95, 96] .

The second part of the thesis deals with the possibility of defining Vacuum String Field Theory in the presence of a B field. Vacuum String Field Theory (VSFT) is the guessed form of String Field Theory at the closed string vacuum as proposed by Rastelli, Sen and Zwiebach [72]. To give an insight of what VSFT is we have to go back to the D–25 brane. The tachyon field present on its world volume renders the brane unstable, and the the condensation of the tachyon correspond to the decay of the D–25 brane. In particular, three conjectures have been made by Ashoke Sen [53, 54, 55], about how this decay process takes place:

1) the difference in energy between the maximum at the origin, corresponding to the negative mass^2 of the tachyon, and the perturbatively stable vacuum should be equal to the mass of the D25–brane.

This means that after tachyon condensation the D25–brane completely disappeared, and then

2) at the perturbatively stable vacuum there are no physical open string excitations: only *closed* strings are there.

3) Starting from the D25–brane, lower dimensional branes are realized as soliton configurations of the tachyon and other fields.

At the present moment there is no explicit knowledge of the tachyon potential. The main difficulty in studying the tachyon potential resides in the fact that the zero momentum tachyon is far off–shell, and hence unreachable by a description in terms of first quantized string theory which deals with only on–shell S matrix elements. Chapters 3 and 4 contain respectively an introduction to String Field Theory and its use to describe tachyon condensation. Recent reviews of the subject can be found in [59, 60]. Chapter 5 is a review of the works of Rastelli, Sen and Zwiebach that defined VSFT, [72, 73, 74]. The spirit of their approach was, starting from the guessed form of the theory at the true vacuum, to try to get back to the perturbative unstable tachyon vacuum, with the hope of discovering a path connecting the two vacua and then being able to follow it from the unstable to the stable vacuum and obtain a closed form for the tachyon potential. The first step was then to construct in VSFT the D25–brane, the momentum independent vacuum of the bosonic theory, [73]. Also lower dimensional branes have been constructed in VSFT [73] and afterward systems of multiple branes have been defined [74]. In Chapter 6 we will describe the works done with L. Bonora and D. Mamone [108, 109, 110], about the possibility of defining VSFT in the presence of a constant B field. It will be shown not only that a background B field is not an obstacle in defining lump solutions all with the correct tensions, but also that in some cases the B field is a useful device. Indeed in VSFT there arise several singularities. We will show that the known smoothing out effects of a B field may help in taming some of them [109].

Finally we will deal with question that could spontaneously arise after reading these few lines: since they are both noncommutative, does it exist any connection between the Witten star product and the Moyal product? It has been shown that a Moyal structure arises form Witten's star product in the large B field limit (low energy limit) [66, 67] and that the Witten product can be rewritten as a Moyal product on an auxiliary non physical space [101]. What we will show is that it is possible to define a sequence of VSFT D–branes whose low energy limit leads exactly to a corresponding sequence of GMS (Gopakumar, Minwalla, Strominger) solitons [110]. GMS solitons are classical solutions of noncommutative scalar field theories, that do not exist at $B = 0$, and that show a Laguerre polynomial structure. The same Laguerre polynomial structure is present in VSFT, even without B field, creating in this way a link between the two noncommutative products.

As usual let us perform a Wick rotation on the worldsheet in order to map it onto the upper half plane: $\tau \to -i\tau$, $z = e^{\tau+i\sigma}, \bar{z} = e^{\tau-i\sigma}$ $(0 \leq \sigma \leq \pi)$. The boundary conditions become

$$E_{\mu\nu}\partial_{\bar{z}}X^\nu = (E^T)_{\mu\nu}\partial_z X^\nu \tag{1.8}$$

that are imposed on the real axis $z = \bar{z}$. The propagator is

$$\begin{aligned}\langle X^\mu(z,\bar{z})X^\nu(z',\bar{z}')\rangle &= -\alpha'\Big[g^{\mu\nu}\ln|z-z'| - g^{\mu\nu}\ln|z-\bar{z}'| \\ &\quad +G^{\mu\nu}\ln|z-\bar{z}'|^2 + \frac{1}{2\pi\alpha'}\theta^{\mu\nu}\ln\frac{z-\bar{z}'}{\bar{z}-z'} + D^{\mu\nu}\Big]\end{aligned} \tag{1.9}$$

where

$$\begin{aligned}G^{\mu\nu} &= (E^{-1}\,g\,(E^T)^{-1})^{\mu\nu} \\ &= \left(\frac{1}{g+2\pi\alpha' B}\,g\,\frac{1}{g-2\pi\alpha' B}\right)^{\mu\nu} \\ \theta^{\mu\nu} &= (2\pi\alpha')^2(E^{-1}\,B\,(E^T)^{-1})^{\mu\nu} \\ &= -(2\pi\alpha')^2\left(\frac{1}{g+2\pi\alpha' B}\,B\,\frac{1}{g-2\pi\alpha' B}\right)^{\mu\nu}.\end{aligned} \tag{1.10}$$

On the upper half plane the mode expansion (1.7) becomes

$$\begin{aligned}X^{\mu\nu} &= x_0^\mu + \alpha'\Big[(E^{-1})^{\mu\nu}g_{\nu\rho}p^\rho\ln\bar{z} + (E^{-1T})^{\mu\nu}g_{\nu\rho}p^\rho\ln z\Big] \\ &\quad +i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\Big[(E^{-1})^{\mu\nu}g_{\nu\rho}\alpha_n^\rho\bar{z}^{-n} + (E^{-1T})^{\mu\nu}g_{\nu\rho}\alpha_n^\rho z^{-n}\Big].\end{aligned} \tag{1.11}$$

The indices of p^ρ and α_n^ρ were lowered by the metric $g_{\mu\nu}$ and not by the metric $G_{\mu\nu}$. From the definition of the propagator we can read the commutation rules for α_n^ρ, x_0, and p:

$$\langle X^\mu(z,\bar{z})X^\nu(z',\bar{z}')\rangle \equiv R(X^\mu(z,\bar{z})X^\nu(z',\bar{z}')) - N(X^\mu(z,\bar{z})X^\nu(z',\bar{z}')) \tag{1.12}$$

where R and N stand for the radial ordering and the normal ordering respectively. We assume that the normal ordering prescription for the product of x_0^μ with p_ν is $:x_0^\mu p_\nu:\equiv x_0^\mu p_\nu$. The vacuum is then defined as

$$p_\mu|0\rangle = \alpha_n^\mu|0\rangle = 0 \quad (n>0), \quad \langle 0|\alpha_n^\mu = 0 \quad (n<0) \tag{1.13}$$

From standard calculations of two dimensional conformal field theory, one can obtain the commutators

$$[\alpha_m^\mu, \alpha_n^\nu] = n\delta_{m+n} G^{\mu\nu} \qquad [x_0^\mu, p_\nu] = \delta_\nu^\mu \tag{1.14}$$

where, as usual, we set $\alpha_{0,\mu} \equiv \sqrt{2\alpha'} p_\mu$. The constant $D^{\mu\nu}$ is written as $\alpha' D^{\mu\nu} = -\langle 0|X_0^\mu X_0^\nu|0\rangle$, that is equivalent to set the normal ordering for $x_0^\mu x_0^\nu$ as

$$: x_0^\mu x_0^\nu := x_0^\mu x_0^\nu + \alpha' D^{\mu\nu} \tag{1.15}$$

If we fix, as in [4], $D^{\mu\nu}$ as $\alpha' D^{\mu\nu} = -\frac{1}{2}\theta^{\mu\nu}$, the coordinates x_0^μ become noncommutative :

$$[x_0^\mu, x_0^\nu] = i\theta^{\mu\nu} \tag{1.16}$$

It is important to notice however that the center of mass coordinates

$$\hat{x}_0^\mu \equiv x_0^\mu + \frac{1}{2}\theta^{\mu\nu} p_\nu \tag{1.17}$$

still commute among each other. In order to understand the physical meaning of the parameters $G^{\mu\nu}$ and $\theta^{\mu\nu}$ let us restrict the analysis to the boundary of the worldsheet. On the boundary the propagator is

$$\langle X^\mu(\tau) X^\nu(\tau')\rangle = -\alpha' G^{\mu\nu} \ln(\tau-\tau')^2 + \frac{i}{2}\theta^{\mu\nu}\epsilon(\tau-\tau') \tag{1.18}$$

where $\epsilon(\tau)$ is the function that is 1 or -1 for positive or negative $(\tau-\tau')$. $G^{\mu\nu}$ has then a simple and deep physical interpretation: it is the effective open metric seen by the open strings. We will then refer to $g_{\mu\nu}$ as to the *closed string metric* and to $G_{\mu\nu}$ as to the *open string metric*. To interpret the meaning of $\theta^{\mu\nu}$ is useful to re-express the string mode expansion in the following way:

$$\begin{aligned} X^\mu(\sigma,\tau) &= \hat{x}_0 + 2\alpha'\Big(G^{\mu\nu}\tau + \frac{1}{2\pi\alpha'}\theta^{\mu\nu}(\sigma-\pi/2)\Big)p_\nu \\ &\quad + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}e^{-in\tau}\Big[G^{\mu\nu}\cos(n\sigma) - i\frac{1}{2\pi\alpha'}\theta^{\mu\nu}\sin(n\sigma)\Big]\alpha_{n,\nu} \end{aligned} \tag{1.19}$$

Using the formula

$$\sum_{n=1}^{\infty}\frac{2}{n}\sin(n(\sigma+\sigma')) = \begin{cases} \pi-\sigma-\sigma' & (\sigma+\sigma'\neq 0, 2\pi) \\ 0 & (\sigma+\sigma' = 0, 2\pi) \end{cases} \tag{1.20}$$

Chapter 1

Open Strings in a Constant B Field and Noncommutative Field Theory

1.1 Open Strings in a Constant B Field

In this chapter we want to give a self contained introduction to the link between string theory in the presence of a B field and noncommutative field theories, in such a way it could be an introduction to both subjects of renormalization problems in $U(N)$ and $SO(N)$ noncommutative gauge theories and Vacuum String Field Theory in the presence of a B field. We will mainly follow the presentation given by [3, 4]

We consider an open string ending on a Dp-brane in the presence of a constant Neveu-Schwarz B field. Since the components of B outside the brane can always be gauged away we assume the rank r of B such that $r < p+1$. The worldsheet action is

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\left[g_{\mu\nu}\partial_a X^\mu \partial^a X^\nu + 2\pi\alpha'\epsilon^{ab}B_{\mu\nu}\partial_a X^\mu \partial_b X^\nu\right] \\ &= \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu + \frac{1}{2}\oint_{\partial\Sigma} B_{\mu\nu}X^\mu \partial_t X^\nu \ , \end{aligned} \tag{1.1}$$

where Σ is the string worldsheet that we take with Lorentz signature, and ∂_t is the derivative tangent to the worldsheet boundary $\partial\Sigma$. Since the term proportional to $B_{\mu\nu}$ can be written as total derivative term, it does not affect the equation of motion but does the boundary condition, which reads as

$$g_{\mu\nu}\partial_n X^\nu + 2\pi\alpha' B_{\mu\nu}\partial_t X^\mu\Big|_{\partial\Sigma} = 0 \ , \tag{1.2}$$

where ∂_n is the normal derivative to $\partial\Sigma$. A new dimensionless parameter is now present: $\alpha' B_{\mu\nu}$. The mutual behavior of $\alpha' B_{\mu\nu}$ with respect to $g_{\mu\nu}$ determines which type of boundary condition, Neumann or Dirichlet, is dominant in (1.2). If $\alpha' B_{\mu\nu} \gg g_{\mu\nu}$ along the spatial directions of the brane, the boundary conditions become Dirichlet. Indeed, in this limit, the second term in (1.2) dominates, and, with B being invertible, (1.2) reduces to $\partial_t X^j = 0$. Obviously for $B = 0$ the boundary conditions in (1.2) are Neumann conditions.

We have to stress that the presence of a B field has a physical effect only along the brane and not outside it. This is because outside the brane there is always the possibility of making a 'gauge transformation' $B \to B + d\Lambda$ that sets the condition $B = 0$. Along the brane lives also the $U(1)$ gauge field of the string endpoints. It is described by the action

$$S(A) = \frac{1}{2\pi\alpha'} \oint_{\partial\Sigma} d\tau A_\mu(X) \partial_\tau X^\mu = \frac{-1}{4\pi\alpha'} \int_\Sigma d^2\sigma \epsilon^{ab} F_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \ . \tag{1.3}$$

Therefore for open strings B and F always appear together in the combination $\mathcal{F} = B - F$, and the 'gauge transformation' above, when performed, has now physical effects on the $U(1)$ gauge field. The combination $\mathcal{F} = B - F$ is indeed invariant under both gauge transformations for the one-form gauge field A

$$A \to A + d\Lambda, \quad B \to B \tag{1.4}$$

and for the two-form gauge field B

$$B \to B + d\Lambda, \quad A \to A + \Lambda \ . \tag{1.5}$$

From now on we will restrict our analysis to the case of the presence of the B field only.

Boundary condition can be rewritten in the convenient form

$$E_{\mu\nu} \partial_- X^\nu = (E^T)_{\mu\nu} \partial_+ X^\nu \tag{1.6}$$

where $E_{\mu\nu} \equiv g_{\mu\nu} + 2\pi\alpha' B_{\mu\nu}$, and $\partial_\pm$ are derivatives with respect to the light cone variables $\sigma^\pm = \tau \pm \sigma$. The string coordinates $X^\mu(\sigma, \tau)$ satisfying the boundary condition (1.6) have the following expansion:

$$\begin{aligned} X^\mu &= x_0^\mu + \alpha' \Big[(E^{-1})^{\mu\nu} g_{\nu\rho} p^\rho \sigma^- + (E^{-1T})^{\mu\nu} g_{\nu\rho} p^\rho \sigma^+ \Big] \\ &\quad + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \Big[(E^{-1})^{\mu\nu} g_{\nu\rho} \alpha_n^\rho e^{-in\sigma^-} + (E^{-1T})^{\mu\nu} g_{\nu\rho} \alpha_n^\rho e^{-i\sigma^+} \Big] \end{aligned} \tag{1.7}$$

we see that the endpoints of the string become noncommutative :

$$[X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')] = \begin{cases} i\theta^{\mu\nu} & (\sigma=\sigma'=0) \\ -i\theta^{\mu\nu} & (\sigma=\sigma'=\pi) \\ 0 & (\text{otherwise}) \end{cases}$$

and, since the the brane is by definition the place where the string endpoints are forced to belong, the world volume of the brane itself becomes a noncommutative space. On the other hand the conjugate momenta have the mode expansion identical with that of the Neumann case:

$$\begin{aligned} P_\mu &= \frac{1}{2\pi\alpha'}(g_{\mu\nu}\partial_\tau - 2\pi\alpha' B_{\mu\nu}\partial_\sigma)X^\mu(\tau,\sigma) \\ &= \frac{1}{\pi}p_\mu + \frac{1}{\pi\sqrt{2\alpha'}}\sum_{n\neq 0} e^{-in\tau}\cos(n\sigma)\alpha_{n,\mu} \end{aligned} \qquad (1.21)$$

Now we want to see how the open string tree level amplitudes are changed by the presence of the B field. The connection between string S-matrix elements and field theory ones will be our main tool to investigate the links between the two theories. Let us consider an operator on the boundary of the worldsheet (we are interested in the emission of open string states) that is of the general form $P(\partial X(\tau), \partial^2 X(\tau), \ldots)e^{ip\cdot X(\tau)}$ where P is a polynomial in the derivatives of X and X are the coordinates along the Dp-brane. The second term in (1.18) is proportional to $\epsilon(\tau-\tau')$ and does not contribute to contractions of derivatives of X. Then, the correlation function of a product of k such operators, of momenta $p^1, \ldots, p^k$, satisfies

$$\left\langle \prod_{n=1}^{k} P_n(\partial X(\tau_n), \partial^2 X(\tau_n), \ldots)e^{ip^n\cdot X(\tau_n)} \right\rangle_{G,\theta} \qquad (1.22)$$

$$= \exp\Big[-\frac{i}{2}\sum_{n>m} p^m_\mu \theta^{\mu\nu} p^n_\nu \epsilon(\tau_n - \tau_m)\Big] \left\langle \prod_{n=1}^{k} P_n(\partial X(\tau_n), \partial^2 X(\tau_n), \ldots)e^{ip^n\cdot X(\tau_n)} \right\rangle_{G,\theta=0}$$

where $\langle\ldots\rangle_{G,\theta}$ is the expectation value with the propagator (1.18) parametrized by G and θ. The exponential prefactor in (1.22) depends only on the cyclic ordering of the points $\tau_1, \ldots, \tau_k$ along the boundary. We see that, when the description of the theory is given in terms of $G_{\mu\nu}$ and $\theta^{\mu\nu}$, rather then in terms of $g_{\mu\nu}$ and $B_{\mu\nu}$, the θ dependence of the correlators is very simple. $G_{\mu\nu}$ and $\theta^{\mu\nu}$ are then the *natural parameters* that should be used to describe *open strings* in the presence of a B field.

We want now to perform the low energy limit on the string amplitudes with B field. Without B field this limit is done by taking $\alpha' \to 0$. With B field we have to add to this condition also the request that the natural open strings parameters G and θ are kept fixed. By looking at the eqs.(1.10) we see that both the requests can be satisfied by imposing

$$\alpha' \to 0 \tag{1.23}$$

in such a way that

$$\begin{aligned} &\alpha' B_{\mu\nu} \sim \epsilon \to 0 \\ &g_{\mu\nu} \sim \epsilon^2 \to 0 \quad \text{for} \quad \mu, \nu = 1, \dots, r \end{aligned} \tag{1.24}$$

with B fixed. In this way eqs.(1.10) become

$$G^{\mu\nu} = -\frac{1}{(2\pi\alpha')^2}\left(\frac{1}{B}g\frac{1}{B}\right)^{\mu\nu} \tag{1.25}$$

$$G_{\mu\nu} = -(2\pi\alpha')^2 (B\, g^{-1}\, B)_{\mu\nu} \tag{1.26}$$

$$\theta^{\mu\nu} = \left(\frac{1}{B}\right)^{\mu\nu} \tag{1.27}$$

As one can immediately see, G and θ are finite in this limit. The conditions (1.24) guarantee that $\alpha' B_{\mu\nu} \gg g_{\mu\nu}$, and the boundary conditions (1.2) become more and more Dirichlet. There exists in the literature another form of this limit that is particularly used in the analysis of noncommutative solitons. It is characterized by $\alpha' B_{\mu\nu} \to \infty$ with $g_{\mu\nu}$ held fixed. α' can then be taken finite or can be sent to 0 in such a way still $\alpha' B_{\mu\nu} \to \infty$. In either form of the limit

$$\theta^{\mu\nu} = \left(\frac{1}{B}\right)^{\mu\nu}, \tag{1.28}$$

In order to avoid confusion we will always refer to the form (1.24) of the low energy limit.

In the low energy limit the boundary propagator becomes

$$\langle X^i(\tau) X^j(\tau') \rangle = \frac{i}{2}\theta^{ij}\epsilon(\tau - \tau') \tag{1.29}$$

and the action (1.1) reduces to

$$S \to -\frac{i}{2}\int_{\partial\Sigma} B_{\mu\nu} X^\mu \partial_t X^\nu \tag{1.30}$$

This action, regarded as a one-dimensional action, describes the motion of a charged particle in a large magnetic field. Indeed the action for such (nonrelativistic) point particle is

$$S = \int dt \left(\frac{1}{2} m \dot{x}^i \dot{x}^i + e B_{ij} x^i \dot{x}^j \right) \tag{1.31}$$

The conjugate momentum Π_i to x^i is

$$\Pi_i = m \dot{x}_i + e B_{ij} x^j \tag{1.32}$$

In the limit where the energy $\omega \ll e|B|/m$, the canonical commutation relations become simply

$$[x^i, x^j] = i (B^{-1})^{ij} \frac{m}{e} \tag{1.33}$$

Thus at energies much less than the cyclotron frequency $e|B|/m$, when one is in the lowest Landau level, one effectively has noncommuting coordinates.

With the propagator (1.18), normal ordered operators satisfy

$$: e^{ip_i x^i(\tau)} : \quad : e^{iq_i x^i(0)} := e^{-\frac{i}{2}\theta^{ij} p_i q_j \epsilon(\tau)} : e^{ipx(\tau) + iqx(0)} :, \tag{1.34}$$

or more generally

$$: f(x(\tau)) : \quad : g(x(0)) :=: e^{\frac{i}{2}\epsilon(\tau)\theta^{ij} \frac{\partial}{\partial x^i(\tau)} \frac{\partial}{\partial x^j(0)}} f(x(\tau)) g(x(0)) :, \tag{1.35}$$

and

$$\lim_{\tau \to 0^+} : f(x(\tau)) : \quad : g(x(0)) :=: f(x(0)) \star g(x(0)) :, \tag{1.36}$$

where

$$f(x) \star g(x) = e^{\frac{i}{2}\theta^{ij} \frac{\partial}{\partial \xi^i} \frac{\partial}{\partial \zeta^j}} f(x+\xi) g(x+\zeta)|_{\xi=\zeta=0} \tag{1.37}$$

is the product of functions on a noncommutative space that we will describe in details in the next section.

To be more specific in the connection with noncommutative field theories let us consider the three point function for gauge vectors, created by vertex operators

$$V(p, \xi) = \int \xi \cdot \partial X e^{ip \cdot X}, \qquad \xi \cdot p = p \cdot p = 0, \tag{1.38}$$

where the dot product is done with the open string metric G. The three point function is

$$\langle \xi^1 \cdot \partial X e^{ip^1 \cdot X(\tau_1)} \, \xi^2 \cdot \partial X e^{ip^2 \cdot X(\tau_2)} \, \xi^3 \cdot \partial X e^{ip^3 \cdot X(\tau_3)} \rangle$$

$$\sim \frac{1}{(\tau_1 - \tau_2)(\tau_2 - \tau_3)(\tau_3 - \tau_1)}$$

$$\cdot \left(\xi^1 \cdot \xi^2 p^2 \cdot \xi^3 + \xi^1 \cdot \xi^3 p^1 \cdot \xi^2 + \xi^2 \cdot \xi^3 p^3 \cdot \xi^1 + 2\alpha' p^3 \cdot \xi^1 p^1 \cdot \xi^2 p^2 \cdot \xi^3 \right)$$

$$\cdot \exp\left[-\frac{i}{2} \left(p^1_\mu \theta^{\mu\nu} p^2_\mu \epsilon(\tau_1 - \tau_2) + p^2_\mu \theta^{\mu\nu} p^3_\nu \epsilon(\tau_2 - \tau_3) + p^3_\mu \theta^{\mu\nu} p^1_\nu \epsilon(\tau_3 - \tau_1) \right) \right]$$

We still have to fix the SL(2;$\mathbb{R}$) conformal invariance that consists in removing the denominator $(\tau_1 - \tau_2)(\tau_2 - \tau_3)(\tau_3 - \tau_1)$. At the end we have the amplitude

$$\left(\xi^1 \cdot \xi^2 p^2 \cdot \xi^3 + \xi^1 \cdot \xi^3 p^1 \cdot \xi^2 + \xi^2 \cdot \xi^3 p^3 \cdot \xi^1 + 2\alpha' p^3 \cdot \xi^1 p^1 \cdot \xi^2 p^2 \cdot \xi^3 \right) e^{-\frac{i}{2} p^1_\mu \theta^{\mu\nu} p^2_\mu} \tag{1.39}$$

The first three terms are the color-ordered three point amplitude as derives from the noncommutative gauge theory described by the action

$$S = \frac{1}{4g^2_{YM}} \int \mathrm{Tr} F_{\mu\nu} \star F^{\mu\nu} \tag{1.40}$$

where g^2_{YM} is the effective Yang-Mills coupling and the space-time indices are contracted with the open string metric G. The fourth term in (1.39) is a stringy correction that indeed vanishes for $\alpha' \to 0$.

The last thing we need to determine is the expression of G_s in terms of the closed string variables g, B and g_s. The constant term in the effective lagrangian will do this job. For slowly varying fields, the effective Lagrangian is the Dirac-Born-Infeld lagrangian

$$\mathcal{L}_{DBI} = \frac{1}{g_s (2\pi)^2 (\alpha')^{\frac{p+1}{2}}} \sqrt{\det(g + 2\pi\alpha'(B - F))} \tag{1.41}$$

The constant part is

$$\mathcal{L}(F = 0) = \frac{1}{g_s (2\pi)^2 (\alpha')^{\frac{p+1}{2}}} \sqrt{\det(g + 2\pi\alpha' B)} \tag{1.42}$$

On the other side we know that when we describe the effective action in terms of open string quantities the whole θ dependence is contained in the $\star$ product. In this description the correspondent of (1.41) is

$$\mathcal{L}(\widehat{F}) = \frac{1}{G_s (2\pi)^2 (\alpha')^{\frac{p+1}{2}}} \sqrt{\det(G + 2\pi\alpha' \widehat{F})} \tag{1.43}$$

and the constant term is

$$\mathcal{L}(\widehat{F}=0) = \frac{1}{G_s(2\pi)^2(\alpha')^{\frac{p+1}{2}}}\sqrt{\det G} \tag{1.44}$$

Equating the two constant parts (1.42) and (1.44)

$$G_s = g_s\left(\frac{\det G}{\det(g+2\pi\alpha' B)}\right)^{\frac{1}{2}} \tag{1.45}$$

that in the $\alpha' \to 0$ limit becomes

$$G_s = g_s \det(2\pi\alpha' B g^{-1})^{\frac{1}{2}} \tag{1.46}$$

where the determinant is calculated only in the dimensions with nonzero B field.

1.2 Noncommutative field theory

In the usual quantum mechanics we have the well known commutation relations:

$$\begin{aligned} [\hat{x}_i, \hat{p}_j] &= i\hbar\delta_{ij} \text{ and} \\ [\hat{p}_i, \hat{p}_j] &= [\hat{p}_i, \hat{p}_j] = 0 \end{aligned} \tag{1.47}$$

However there is no evidence that at very short distances (or very high energies) these relations should still be true. Then a natural generalization of above is to take the coordinates which do not commute anymore,

$$[\hat{x}_i, \hat{x}_j] = i\theta_{ij}\,, \tag{1.48}$$

where θ_{ij} is a *constant* of dimension length^2. An immediate remark is that introducing this kind of commutation relation between coordinates the Lorentz invariance is spoiled explicitly. We should remember however that we assumed this feature to appear only at very short distances, i.e. for $\theta \to 0$ we should recover the Lorentz symmetry. This is one of the main constraints on our noncommutative field theories: at least at classical level, in the limit $\theta \to 0$ we should find a previously known commutative field theory. We will see that at loop level in some cases the limit $\theta \to 0$ may be singular. In general (1.48) can be extended to space-time coordinates:

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}\,. \tag{1.49}$$

Hereafter we call a space with the above commutation relations as a noncommutative space.

To construct the perturbative field theory formulation, it is more convenient to use fields which are some functions and not operator valued objects. To pass to such fields while keeping (1.49) property one should redefine the multiplication law of functional (field) space. This new multiplication is induced from (1.49) through the so called Weyl-Moyal correspondence. Roughly speaking the Weyl-Moyal correspondence means to declare that the product of operators and the product of the corresponding functions do share the same Fourier transform:

$$\hat{\Phi}(\hat{x}) \longleftrightarrow \Phi(x) \ ;$$

$$\begin{aligned} \hat{\Phi}(\hat{x}) &= \int_\alpha e^{i\alpha\hat{x}}\, \phi(\alpha)\, d\alpha \\ \phi(\alpha) &= \int e^{-i\alpha x}\, \Phi(x)\, dx\,, \end{aligned} \tag{1.50}$$

where α and x are real variables. Then,

$$\begin{aligned} \hat{\Phi}_1(\hat{x})\, \hat{\Phi}_2(\hat{x}) &= \int\!\!\int_{\alpha\,\beta} e^{i\alpha\hat{x}}\, \phi(\alpha)\, e^{i\beta\hat{x}}\, \phi(\beta)\, d\alpha\, d\beta \\ &= \int\!\!\int_{\alpha\,\beta} e^{i(\alpha+\beta)\hat{x}-\frac{1}{2}\alpha_\mu\beta_\nu[\hat{x}_\mu,\hat{x}_\nu]}\, \phi_1(\alpha)\, \phi_2(\beta)\, d\alpha\, d\beta\ , \end{aligned} \tag{1.51}$$

and hence,

$$\hat{\Phi}_1(\hat{x})\, \hat{\Phi}_2(\hat{x}) \longleftrightarrow \Big(\Phi_1 \star \Phi_2\Big)(x)\ , \tag{1.52}$$

with

$$\Big(\Phi_1 \star \Phi_2\Big)(x) \equiv \Big[e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\xi_\mu}\partial_{\eta_\nu}}\, \Phi_1(x+\xi)\, \Phi_2(x+\eta)\Big]_{\xi=\eta=0}\ . \tag{1.53}$$

This suggests that we can work on a usual commutative space for which the multiplication operation is modified to the so called star product. It is easy to check that the Moyal bracket (the commutator in which the product is modified with a star product) of two coordinates x_μ and x_ν gives exactly the desired commutation relations,

$$[x_\mu, x_\nu]_{MB} \;=\; i\theta_{\mu\nu} \tag{1.54}$$

We can summarize some useful identities of the star product algebra.

1. Star product between exponentials:

$$\begin{aligned} e^{ikx} \star e^{iqx} &= e^{i(k+q)x} e^{-\frac{i}{2}(k\theta q)}\text{, where} \\ k\theta p &\equiv k^\mu p^\nu \theta_{\mu\nu} \end{aligned} \tag{1.55}$$

2. Momentum space representation:
Let $\tilde{f}(k)$ and $\tilde{g}(k)$ be the Fourier components of f and g. Then using (1.55)

$$(f \star g)(x) = \int d^4k\, d^4q\, \tilde{f}(k)\, \tilde{g}(q)\, e^{-\frac{i}{2}(k\theta q)}\, e^{i(k+q)x}\,. \tag{1.56}$$

3. Associativity:

$$\Big[(f \star g) \star h\Big](x) = \Big[f \star (g \star h)\Big](x)\,, \tag{1.57}$$

which can be proved immediately going to momentum space.

$$\begin{aligned} \text{rhs} &= \int d^4k\, d^4q\, d^4p\, \tilde{f}(k)\, \tilde{g}(q)\, \tilde{h}(p)\, e^{-\frac{i}{2}(k\theta q)}\, e^{-\frac{i}{2}((k+q)\theta p)}\, e^{i(k+q+p)x}\,, \quad \text{and} \\ \text{lhs} &= \int d^4k\, d^4q\, d^4p\, \tilde{f}(k)\, \tilde{g}(q)\, \tilde{h}(p)\, e^{-\frac{i}{2}(q\theta p)} e^{-\frac{i}{2}(k\theta(q+p))}\, e^{i(k+q+p)x}\,. \end{aligned} \tag{1.58}$$

4. Star products under integral sign:

$$\int (f \star g)(x)\, d^4x = \int (g \star f)(x)\, d^4x = \int (f \cdot g)(x)\, d^4x\,. \tag{1.59}$$

Using (1.56) we can immediately perform the integration over x which will give a $\delta^4(k+q)$. Due to the antisymmetry of θ the exponent vanishes and so:

$$\begin{aligned} \int (f \star g)(x)\, d^4x &= \int d^4k\, \tilde{f}(k)\tilde{g}(-k) \\ &= \int (f \cdot g)(x)\, d^4x \end{aligned} \tag{1.60}$$

5. Cyclic property (from (1.59)):

$$\int (f_1 \star f_2 \star \ldots f_n)(x)\, d^4x = \int (f_n \star f_1 \star \ldots f_{n-1})(x)\, d^4x\,. \tag{1.61}$$

6. Complex conjugation:

$$(f \star g)^* = g^* \star f^*\,. \tag{1.62}$$

It is obvious that if f is a real function then $f \star f$ is also real.

1.3 Feynman Rules for Noncommutative field theories

As we have seen in the previous section the way to treat the noncommutative theories is to modify the usual product of fields with the star product. So, for example, the action for the noncommutative analog of the real ϕ^3 theory will be:

$$S[\phi] = \int d^6x \left[\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi - \frac{m^2}{2}\phi\star\phi - \frac{g}{3!}\phi\star\phi\star\phi \right] \tag{1.63}$$

Thanks to (1.59), the quadratic part of the action is the same as the commutative case:

$$\frac{1}{2}\int d^6x\, \partial_\mu\phi \star \partial^\mu\phi = \frac{1}{2}\int d^6x\, \partial_\mu\phi\partial^\mu\phi \tag{1.64}$$

$$\frac{m^2}{2}\int d^6x\, \phi\star\phi = \frac{m^2}{2}\int d^6x\, \phi\phi \tag{1.65}$$

Therefore the only thing which is modified is the interaction. This is a very important point to keep in mind: the free theory is *the same* as in the commutative case. Writing the interaction part of (1.63) in the momentum space we read the Feynman rules for the interaction vertex

$$S^g_{\rm int} = \frac{g}{3!}\int d^6k_1 d^6k_2 d^6k_3\, (2\pi)^6\delta(k_1+k_2+k_3)\,\tilde\phi(k_1)\tilde\phi(k_2)\tilde\phi(k_3)e^{-\frac{i}{2}k_1\theta k_2} \tag{1.66}$$

Summing over all the *inequivalent* permutations of the external legs we obtain

$$V(k_1,k_2) = \frac{g}{2}\left(e^{-\frac{i}{2}k_1\theta k_2} + e^{\frac{i}{2}k_1\theta k_2}\right) = g\cos\left(\frac{k_1\theta k_2}{2}\right) \tag{1.67}$$

Setting $\theta = 0$ we obtain the standard commutative vertex. It is also instructive to explicitly build the interaction vertex for the noncommutative ϕ^4 theory, in order to have a deeper feeling of the general rule that we will show in brief

$$\begin{aligned} S^\lambda_{\rm int} &= \frac{\lambda}{4!}\int d^4x\, \phi\star\phi\star\phi\star\phi \\ &= \frac{\lambda}{4!}\int d^4x\, (\phi\star\phi)\cdot(\phi\star\phi) \\ &= \frac{\lambda}{4!}\int d^4x \int d^4k_1 d^4k_2 d^4k_3 d^4k_4\, e^{-\frac{i}{2}k_1\theta k_2}e^{-\frac{i}{2}k_3\theta k_4} \end{aligned}$$

$$
\begin{aligned}
&\times \quad e^{i(k_1+k_2+k_3+k_4)x}\tilde{\phi}(k_1)\tilde{\phi}(k_2)\tilde{\phi}(k_3)\tilde{\phi}(k_4) \\
&= \quad \frac{\lambda}{4!}\int d^4k_1d^4k_2d^4k_3d^4k_4\, e^{-\frac{i}{2}k_1\theta k_2}e^{-\frac{i}{2}k_3\theta k_4} \\
&\times \quad (2\pi)^4\delta^4(k_1+k_2+k_3+k_4)\tilde{\phi}(k_1)\tilde{\phi}(k_2)\tilde{\phi}(k_3)\tilde{\phi}(k_4) \\
&= \quad \frac{\lambda}{3\cdot 4!}\int d^4k_1d^4k_2d^4k_3d^4k_4\,(2\pi)^4\delta^4(k_1+k_2+k_3+k_4) \\
&\times \quad \left[\cos\left(\frac{k_1\theta k_2}{2}\right)\cos\left(\frac{k_3\theta k_4}{2}\right)+\cos\left(\frac{k_1\theta k_3}{2}\right)\cos\left(\frac{k_2\theta k_4}{2}\right)\right. \\
&\qquad \left. +\cos\left(\frac{k_1\theta k_4}{2}\right)\cos\left(\frac{k_2\theta k_3}{2}\right)\right]
\end{aligned}
$$

It follows that the the vertex in the noncommutative ϕ^4 theory is

$$
\begin{aligned}
V(k_1,k_2,k_3) \quad = \quad & \frac{\lambda}{3}\left[\cos\left(\frac{k_1\theta k_2}{2}\right)\cos\left(\frac{k_3\theta k_4}{2}\right)+\cos\left(\frac{k_1\theta k_3}{2}\right)\cos\left(\frac{k_2\theta k_4}{2}\right)\right. \\
& \left. +\cos\left(\frac{k_1\theta k_4}{2}\right)\cos\left(\frac{k_2\theta k_3}{2}\right)\right]
\end{aligned}
$$

Again, for $\theta = 0$ we have the commutative vertex. Both the two vertices can be obtained by multiplying the standard, commutative vertex, by the factor

$$
V = \exp\left[\frac{i}{2}\sum_{i<j} k_i\theta k_j\right] \tag{1.68}
$$

This phase factor is nothing but the prefactor of (1.22). Due to it the interaction vertex has only cyclic symmetry, in contrast to the vertex in the ordinary real scalar field theory which is invariant under the whole permutation group. This is exactly what happens in the case of matrix field theories; it is then useful, following 't Hooft [58, 51], to introduce a 'double line' notation for the Feynman diagrams. In this way we can give a precise treatment of the noncommutative phase factor that accompanies the most general diagram. In the 't Hooft formulation of perturbative expansion of a matrix valued field theory the concept of planarity and non-planarity of the diagrams is fundamental: in the large N limit planar diagrams dominate. In noncommutative field theory the phase factor provides a similar matrix structure. We introduce the 'double line (ribbon) notation' writing the momentum flowing in a propagator as the sum of two momenta flowing in opposite directions belonging

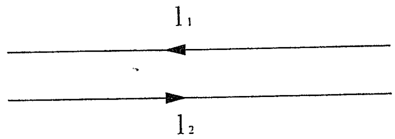


Figure 1.1: The propagator carrying momentum k_1 in the double line notation; $k_1 = l_1 - l_2$

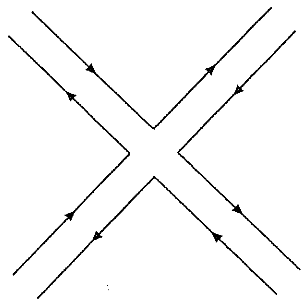


Figure 1.2: Double line representation for ϕ^4 vertex

to the edges of the now 'thick' propagator, see Fig. (1.1). This construction automatically enforces momentum conservation in at each of the vertices, and forces the vertices to retain only the cyclic symmetry in the external line.

Planar diagrams

For any vertex of the graph, let the momenta entering the vertex through the n propagators be $k_1, k_2, \ldots, k_n$, in cyclic order. Then we set $k_1 = l_{i_1} - l_{i_2}, k_2 = l_{i_2} - l_{i_3}, \ldots k_n = l_{i_n} - l_{i_1}$, in terms of which $\sum_{i<j} k_i \theta k_j = l_{i_1}\theta l_{i_2} + l_{i_2}\theta l_{i_3} + \ldots + l_{i_n}\theta l_{i_1}$. Thus the phase factor at any interaction point may be expressed as the product of n phase factors, one for each incoming propagator

$$V = \prod_{j=1}^{n} e^{-\frac{i}{2}(l_{i_j}\theta l_{i_{j+1}})} \tag{1.69}$$

With the phase factor written in this way is then easy to see that the phase associated with any internal propagator is equal and opposite at its two end vertices, and so cancels. Also tadpole diagrams give no phase factors [5]. We conclude that the

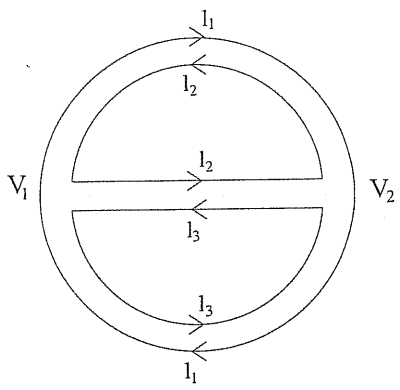


Figure 1.3: A planar graph in the double line notation.

phase factor associated with *any* planar diagram is indeed

$$V = \exp\Big[\frac{i}{2}\sum_{i<j} k_i\theta k_j\Big] \tag{1.70}$$

where the sum is taken over all external momenta in the correct cyclic order.

Non-planar diagrams

The previous construction is valid only for planar diagrams. Non-planar diagrams have, by definition, lines that do cross each other, and they give extra phase factors. Consider for instance two lines, with momenta k_i, k_j crossing each other. If, instead of crossing, the two lines had joined at a 4 point vertex, the graph would have had an additional phase factor of $\exp\big[-\frac{i}{2}(k_j\theta k_i - k_j\theta k_i - k_i\theta k_j + k_j\theta k_i)\big] = \exp\big[-ik_j\theta k_i\big]$. So, any non-planar diagram have an extra phase

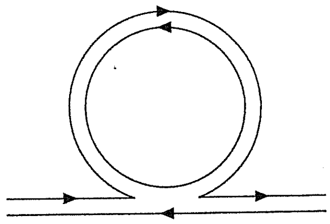
$$e^{+ik_j\theta k_i} \tag{1.71}$$

for each crossing of momenta k_i and k_j in addiction to the phase associated with the ordering of external momenta. We define C_{ij} as the intersection matrix of an oriented graph

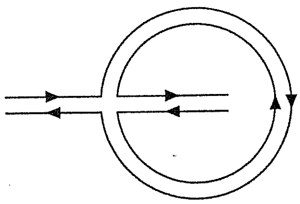
$$C_{ij} = \begin{cases} 1 & \text{if line } j \text{ crosses line } i \text{ from right} \\ -1 & \text{if line } j \text{ crosses line } i \text{ from left} \\ 0 & \text{if lines } j \text{ and } i \text{ do not cross} \end{cases}$$

An orientation is given by the sign conventions chosen for the momenta in the conservation conditions. Then the the most general noncommutative diagram is

$$V = e^{-\frac{i}{2}\sum_{i<j} p_i\theta p_j} e^{-\frac{i}{2}\sum_{i,j} C_{ij}\, k_i\theta k_j} \tag{1.72}$$



(a)



(b)

Figure 1.4: One loop correction to two–point functions in ϕ^4 theory

where the first exponential is the factor associated with the ordering of external momenta, and the second one takes into account the non-planar intersections.

UV/IR mixing

An intriguing UV/IR mixing is present at one–loop in noncommutative field theories [20, 22]. Consider ϕ^4 theory in four dimensions with the Euclidean action

$$S = \int d^4x \Big(\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}g^2\phi\star\phi\star\phi\star\phi \Big). \tag{1.73}$$

Consider the 1PI two point function, which at lowest order is simply the inverse propagator

$$\Gamma^{(2)}_0 = p^2 + m^2. \tag{1.74}$$

In the noncommutative theory, this receives corrections at one loop from the two diagrams in Figure (1.4), one planar and the other non-planar.
The two diagrams (which are identical in the $\theta = 0$ theory up to a symmetry factor) give

$$\Gamma^{(2)}_{1\ \mathrm{planar}} = \frac{g^2}{3(2\pi)^4}\int \frac{d^4k}{k^2+m^2}$$

$$\Gamma^{(2)}_{1\ \text{nonplanar}} = \frac{g^2}{6(2\pi)^4}\int\frac{d^4k}{k^2+m^2}e^{ik\theta p}$$

The planar diagram is proportional to the one loop mass correction of the commutative theory, and is quadratically divergent at high energies. In order to see the effect of the phase factor in the second integral we rewrite the expressions for the two integrals in terms of Schwinger parameters

$$\frac{1}{k^2+m^2} = \int_0^\infty d\alpha\ e^{-\alpha(k^2+m^2)}. \tag{1.75}$$

The k integrals are now Gaussian, and may be evaluated to yield

$$\begin{aligned}\Gamma^{(2)}_{1\ \text{planar}} &= \frac{g^2}{48\pi^2}\int\frac{d\alpha}{\alpha^2}e^{-\alpha m^2}\\ \Gamma^{(2)}_{1\ \text{nonplanar}} &= \frac{g^2}{96\pi^2}\int\frac{d\alpha}{\alpha^2}e^{-\alpha m^2-\frac{p\circ p}{\alpha}},\end{aligned} \tag{1.76}$$

where we have introduced new notation

$$p\circ q \equiv -p_\mu\theta^2_{\mu\nu}q_\nu = |p_\mu\theta^2_{\mu\nu}q_\nu| \tag{1.77}$$

(note that $p\circ p$ has dimension of length squared). In order to regulate the small α divergence we multiply the integrands in the expressions above by $\exp(-1/(\Lambda^2\alpha))$ to get

$$\begin{aligned}\Gamma^{(2)}_{1\ \text{planar}} &= \frac{g^2}{48\pi^2}\int\frac{d\alpha}{\alpha^2}e^{-\alpha m^2-\frac{1}{\Lambda^2\alpha}}\\ \Gamma^{(2)}_{1\ \text{nonplanar}} &= \frac{g^2}{96\pi^2}\int\frac{d\alpha}{\alpha^2}e^{-\alpha m^2-\frac{p\circ p+\frac{1}{\Lambda^2}}{\alpha}}.\end{aligned} \tag{1.78}$$

Therefore,

$$\begin{aligned}\Gamma^{(2)}_{1\ \text{planar}} &= \frac{g^2}{48\pi^2}\Big(\Lambda^2 - m^2\ln\Big(\frac{\Lambda^2}{m^2}\Big)+O(1)\Big)\\ \Gamma^{(2)}_{1\ \text{nonplanar}} &= \frac{g^2}{96\pi^2}\Big(\Lambda^2_{\text{eff}} - m^2\ln\Big(\frac{\Lambda^2_{eff}}{m^2}\Big)+O(1)\Big),\end{aligned} \tag{1.79}$$

where

$$\Lambda^2_{\text{eff}} = \frac{1}{1/\Lambda^2+p\circ p}. \tag{1.80}$$

In the limit $\Lambda \to \infty$, the nonplanar one loop graph remains finite, effectively regulated by the noncommutativity of spacetime. In this limit the effective cutoff $\Lambda^2_{\text{eff}} = \frac{1}{pop}$ goes to infinity when $\theta^{\mu\nu}p_\nu$ goes to zero (i.e. when $\theta \to 0$, or $p_{\text{nc}} \to 0$, where p_{nc} is the projection of p onto the noncommutative subspace).

The one loop 1PI quadratic effective action is

$$S^{(2)}_{1PI} = \int d^4p \, \frac{1}{2}\left(p^2 + M^2 + \frac{g^2}{96\pi^2(p \circ p + \frac{1}{\Lambda^2})} - \frac{g^2M^2}{96\pi^2}\ln\left(\frac{1}{M^2(p\circ p + \frac{1}{\Lambda^2})}\right) + \cdots + \mathcal{O}(g^4)\right)\phi(p)\phi(-p) \qquad (1.81)$$

where $M^2 = m^2 + \frac{g^2\Lambda^2}{48\pi^2} - \frac{g^2m^2}{48\pi^2}\ln\left(\frac{\Lambda^2}{m^2}\right)\ldots$ is the renormalized mass. Consider the two cases

- $p \circ p \ll \frac{1}{\Lambda^2}$, and in particular the zero momentum limit. Here $\Lambda_{\text{eff}} \approx \Lambda$, and we recover the effective action of the commutative theory,

$$S^{(2)}_{1PI} = \int d^4p \, \frac{1}{2}\left(p^2 + M'^2\right)\phi(p)\phi(-p), \qquad (1.82)$$

  where $M'^2 = M^2 + 3\frac{g^2\Lambda^2}{96\pi^2} - \frac{3g^2m^2}{96\pi^2}\ln\left(\frac{\Lambda^2}{m^2}\right)\ldots$. If M is fine tuned to be cutoff independent, then M' and also $S^{(2)}_{1PI}$ diverge as $\Lambda \to \infty$.

- $p \circ p \gg \frac{1}{\Lambda^2}$ and in particular the limit $\Lambda \to \infty$. Here $\Lambda^2_{\text{eff}} = \frac{1}{pop}$, and

$$S^{(2)}_{1PI} = \int d^4p \, \frac{1}{2}\left(p^2 + M^2 + \frac{g^2}{96\pi^2 p\circ p} - \frac{g^2M^2}{96\pi^2}\ln\left(\frac{1}{m^2 p\circ p}\right) + \ldots + \mathcal{O}(g^4)\right)\phi(p)\phi(-p). \qquad (1.83)$$

The fact that the limit $\Lambda \to \infty$ does not commute with the low momentum limit $p_{\text{nc}} \to 0$ demonstrates the interesting mixing of the UV ($\Lambda \to 0$) and IR ($p \to 0$) in this theory.

This UV/IR mixing is one of the most fascinating aspects of noncommutative quantum field theory. To recapitulate, we have seen that a divergent diagram in the $\theta = 0$ theory by the noncommutativity at $\theta \neq 0$ which renders it finite, but as $p \to 0$ the phases become ineffective and the diagram diverges at vanishing momentum. This property will be fundamental for the analysis of renormalization properties that will be presented in the next chapter, because it will allow us to discard all the non–planar diagram in the analysis of the UV divergences.

1.4 Noncommutative solitons

Another very intersting aspect of noncommutative field theories is that thay allow classical solutions otherwise forbidden in the commutative case [92, 96]. We consider a theory of a single scalar field in $2+1$ dimensions with noncommutativity in the two spatial directions. We parametrize the spatial $\mathbb{R}^2$ by complex coordinates $z, \bar{z}$. The energy functional is

$$E = \frac{1}{g^2}\int d^2z(\partial_z\phi\partial_{\bar{z}}\phi + V(\phi)_\star), \tag{1.84}$$

where $d^2z = dxdy$. Fields are multiplied using the Moyal star product, that in complex coordinates is

$$(f\star g)(z\bar{z}) = e^{\frac{\theta}{2}(\partial_z\partial_{\bar{z}'}-\partial_z\partial_{\bar{z}'})}f(z,\bar{z})g(z',\bar{z}')\Big|_{z=z'} \tag{1.85}$$

Before we look for classical solutions to this action, let us recall that the scalar theory without noncommutativity does not have any lump solutions. This is actually true for any bounded potential in spatial dimension greater than one, and follows from a simple scaling argument [50, 51]. Let $\phi_0(x)$ be an extremum of the energy functional (1.84) with $\theta = 0$. We consider the energy of the field configurations $\phi_\lambda(x) = \phi_0(\lambda x)$.

$$\begin{aligned} E(\lambda) &= \frac{1}{g^2}\int d^Dx\left(\frac{1}{2}(\partial\phi_0(\lambda x))^2 + V(\phi_0(\lambda x))\right) \\ &= \frac{1}{g^2}\int d^Dx\left(\frac{1}{2}\lambda^{2-D}(\partial\phi_0(x))^2 + \lambda^{-D}V(\phi_0(x))\right) \end{aligned} \tag{1.86}$$

Since $\phi_0(x)$ is an extremum, we require $\partial_\lambda E(\lambda)|_{\lambda=1} = 0$. This means

$$\int d^Dx\left(\frac{1}{2}(D-2)(\partial\phi_0(x))^2 + DV(\phi_0(x))\right) = 0 \tag{1.87}$$

For spatial dimension $D \geq 2$, for a potential bounded from below by zero, the only way this relation can hold is that the kinetic and the potential terms separately vanish. There are therefore no nontrivial configurations. This argument fails if one includes higher derivative terms. Instead, for $D = 2$, only the potential energy should be zero.

The limit of large noncommutativity is useful to simplify the search for finite energy (localized) solution of (1.84). We take $\theta \to \infty$. This is exactly the low energy limit we introduced before, seen from the point of view of the field theory. In this simple case we can indeed assume $\theta = |\theta|$. It is useful to rescale the coordinates $z \to z\sqrt{\theta}$, $\bar{z} \to \bar{z}\sqrt{\theta}$. In this way the commutation relations will not depend on θ. In the rescaled coordinates the energy functional becomes

$$E = \frac{1}{g^2}\int d^2z(\partial_z\phi\partial_{\bar{z}}\phi + \theta V(\phi)_\star) \tag{1.88}$$

In the limit $\theta \to \infty$ we can consider just the potential term in (1.88), and the energy

$$E = \frac{\theta}{g^2}\int d^2z V(\phi)_\star \tag{1.89}$$

is extremized by solving the equation

$$\frac{\partial V}{\partial \phi} = 0 \tag{1.90}$$

We will consider potentials in the polynomial form

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \sum_{j=3}^{r}\frac{b_j}{j}\phi^j \tag{1.91}$$

where the product among the fields is understood to be the Moyal one.

If $V(\phi)$ were the potential in a commutative scalar field theory, the only solutions would be the constant configurations

$$\phi = \lambda_i \tag{1.92}$$

where $\lambda_i \in \{\lambda_1, \lambda_2, \cdots, \lambda_k\}$ are the various real extrema of the function $V(x)$. For $V(\phi)$ as in (1.91), λ_i are the real roots of the equation $m^2x + \sum_{j=3}^{r} b_j x^{j-1} = 0$. As we shall see below, the derivatives in the definition of the star product allow for more interesting solutions of (1.90). The first non-trivial solution to (1.90) can be constructed starting from a function ϕ_0 that satisfies

$$(\phi_0 \star \phi_0)(x) = \phi_0(x) \tag{1.93}$$

For such a function the two following relation hold

$$\phi_0^n(x) = \phi(x), \qquad f(a\phi_0(x)) = f(a)\,\phi_0(x) \tag{1.94}$$

We can solve (1.90) with $\lambda_i\phi_0(x)$ where λ_i is an extremum of $V(x)$. The problem of solving (1.90) becomes then that one of finding a function f that squares to itself under $\star$ product. Such a function is

$$\psi(r) = 2e^{-r^2} \tag{1.95}$$

where $r^2 = x^2 + y^2$. Going to momentum space it is easy to compute

$$\tilde{\psi}(k) = \int d^2x\, \psi(x)\, e^{ik\cdot x} = 2\pi e^{-k^2/4} \tag{1.96}$$

and

$$\begin{aligned} \left(\tilde{\psi} \star \tilde{\psi}\right)(p) &= 4\pi^2 \int \frac{d^2k}{(2\pi)^2}\, \tilde{\psi}(k)\, \tilde{\psi}(p-k)\, e^{\frac{i}{2}\epsilon_{\mu\nu}k^\mu (p-k)^\nu} \\ &= 2\pi\, e^{-p^2/4} \end{aligned} \tag{1.97}$$

Going back to coordinate space

$$(\psi \star \psi)(r) = 2e^{-r^2} = \psi(r) \tag{1.98}$$

and $\lambda_i \psi(x)$ solves (1.90).

In order to find more general solutions of (1.90) we use again the Weyl-Moyal correspondence. We used it in section (1.2) to pass from the operator space where the coordinate operators do not commute, to the space of functions endowed with the $\star$ product. Now we want to use it in the opposite direction: starting from the coordinate $\mathbb{R}^2$ where the B field is turned on, we want to go to the corresponding single particle quantum mechanical Hilbert space, $\mathcal{H}$, and find there all the solutions of (1.90). We recall briefly here the definition of the Weyl-Moyal correspondence adapted to the present case. Given a C^∞ function $f(p,q)$ on $\mathbb{R}^2$ (thought of as the phase space of a one-dimensional particle), we assign to it an operator $O_f(\hat{p},\hat{q}) \in \mathcal{H}$:

$$O_f(\hat{q},\hat{p}) = \frac{1}{(2\pi^2)} \int d^2k\, \tilde{f}(k)\, e^{-i(k_q\hat{q}+k_p\hat{p})} \tag{1.99}$$

where

$$\tilde{f}(k) = \int d^2x\, e^{i(k_q q + k_p p)}\, f(q,p)\ , \qquad [\hat{q},\hat{p}] = i \tag{1.100}$$

The map given by (1.99) can also be inverted. Using

$$\mathrm{Tr}_{\mathcal{H}}\, e^{-i(k_q\hat{q}+k_p\hat{p})}\, e^{i(k'_q\hat{q}+k'_p\hat{p})} = 2\pi\, \delta(k_q - k'_q)\delta(k_p - k'_p) \tag{1.101}$$

we can project $\tilde{f}$ in (1.99) and then perform the Fourier transform to find

$$f(q,p) = \int dk'_p e^{-ipk'_p} \langle q + k'_p/2 | O_f(\hat{q},\hat{p}) | q - k'_p/2 \rangle \tag{1.102}$$

Remember that the Moyal product has been transformed into ordinary operator product:

$$O_{f\star g} = O_f \cdot O_g \tag{1.103}$$

A useful identity relates the integral of the phase space function to the trace of its Weyl transform

$$\begin{aligned}
\int dqdpf(p,q) &= \int dqdpdk'_p e^{-ipk'_p}\langle q + k'_p/2|O_f(\hat{q},\hat{p})|q - k'_p/2\rangle \\
&= \int dqdk'_p 2\pi\delta(k'_p)\langle q + k'_p/2|O_f(\hat{q},\hat{p})|q - k'_p/2\rangle \\
&= 2\pi \int dq\langle q|O_f|q\rangle \\
&= 2\pi \mathrm{Tr}_{\mathcal{H}} O_f
\end{aligned} \tag{1.104}$$

In order to solve any algebraic equation involving the star product, it is thus sufficient to determine all operator solutions to the equation in $\mathcal{H}$. The functions on phase space corresponding to each of these operators may then be read off from (1.102).

It is easy to see that $O = \lambda_i P$ is a solution to $V'(O) = 0$ if P is an arbitrary projection operator on some subspace of $\mathcal{H}$ and if λ_i is an extremum of $V(x)$. The energy of this solution is, using (1.104),

$$E = \frac{2\pi\theta}{g^2}\mathrm{Tr}V(O_\phi) = \frac{2\pi\theta}{g^2}V(\lambda_i)\mathrm{Tr}P \tag{1.105}$$

Thus the energy is finite if P is projector onto a finite dimensional subspace of $\mathcal{H}$. In fact, the most general solution to (1.90) has the form

$$O = \sum_j a_j P_j \tag{1.106}$$

where $\{P_j\}$ are mutually orthogonal projectors onto one dimensional subspaces

$$P_i P_j = \delta_{ij} P_j, \qquad \mathrm{Tr}_{\mathcal{H}} P_i = 1 \tag{1.107}$$

with a_j taking values in the set $\{\lambda_i\}$ of real extrema of $V(x)$. From now on we will restrict ourselves to a potential with one nontrivial minimum λ other than the one at the origin. we have a huge infinity of solutions of the form λP. To see what they mean, let us translate them into position space. It will be convenient for this purpose to choose a particular basis in $\mathcal{H}$. Let $|n\rangle$ represent the energy eigenstates of the one dimensional harmonic oscillator whose creation and annihilation operators are defined by

$$a = \frac{\widehat{q} + i\widehat{p}}{\sqrt{2}}, \quad a^\dagger = \frac{\widehat{q} - i\widehat{p}}{\sqrt{2}} \tag{1.108}$$

where obviously $a|n\rangle = \sqrt{n}|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. Any operator may be written as a linear combination of the basis operators $|m\rangle\langle n|$, that, in turn, may be expressed in terms of a and $a^\dagger$ as

$$|m\rangle\langle n| =: \frac{(a^\dagger)^m}{\sqrt{m!}} e^{-a^\dagger a} \frac{a^n}{\sqrt{n!}} : \tag{1.109}$$

where double dots denote normal ordering. We will first describe operators of the form (1.106) that correspond to radially symmetric functions in space. As $a^\dagger a \approx r^2/2$, operators corresponding to radially symmetric wave functions are functions of $a^\dagger a$. From (1.109), the only such operators are linear combinations of the diagonal projection operators $|n\rangle\langle n| = \frac{1}{n!} : a^{\dagger n} e^{-a^\dagger a} a^n :$. Hence all radially symmetric solutions of (1.90) correspond to operators of the form $O = \sum_n a_n |n\rangle\langle n|$, where the numbers a_n can take any values in the set $\{\lambda_i\}$.

We now translate these operator solutions back to field space. From the Baker-Campbell-Hausdorff formula

$$e^{-i(k_q\hat{q}+k_p\hat{p})} = e^{-i\left(k_{\bar{z}}a + k_z a^\dagger\right)} = e^{-\frac{k^2}{4}} : e^{-i\left(k_{\bar{z}}a + k_z a^\dagger\right)} :, \tag{1.110}$$

where

$$k_z = \frac{k_x + ik_y}{\sqrt{2}}, \quad k_{\bar{z}} = \frac{k_x - ik_y}{\sqrt{2}}, \quad k^2 = 2k_z k_{\bar{z}}.$$

Any operator O expressed as a normal ordered function of a and $a^\dagger$, $f_N(a, a^\dagger)$, can be rewritten in Weyl ordered form as follows. By definition,

$$O =: f_N(a, a^\dagger) := \frac{1}{(2\pi)^2} \int d^2k \tilde{f}_N(k) : e^{-i\left(k_{\bar{z}}a + k_z a^\dagger\right)} : . \tag{1.111}$$

Using (1.110), (1.111) may be rewritten as

$$O = \frac{1}{(2\pi)^2} \int d^2k \tilde{f}_N(k) e^{\frac{k^2}{4}} e^{-i\left(k_{\bar{z}}a + k_z a^\dagger\right)}. \tag{1.112}$$

Thus, the momentum space function $\tilde{f}$ associated with the operator O, is

$$\tilde{f}(k) = e^{\frac{k^2}{4}} \tilde{f}_N(k) . \tag{1.113}$$

For the operator $O_n = |n\rangle\langle n|$ we find, using (1.110) and (1.111), that the corresponding normal ordered function $\tilde{\phi}_N^{(n)}(k) = 2\pi e^{\frac{-k^2}{2}} L_n(\frac{k^2}{2})$. (1.113) then becomes

$$|n\rangle\langle n| = \frac{1}{(2\pi)} \int d^2k e^{\frac{-k^2}{4}} L_n\left(\frac{k^2}{2}\right) e^{-i(k_{\bar{z}} a + k_z a^\dagger)} \tag{1.114}$$

where $L_n(x)$ is the n^{th} Laguerre polynomial. The field $\phi_n(x,y)$ that corresponds to the operator $O_n = |n\rangle\langle n|$ is, therefore,

$$\phi_n(r^2 = x^2 + y^2) = \frac{1}{(2\pi)} \int d^2k e^{\frac{-k^2}{4}} L_n\left(\frac{k^2}{2}\right) e^{-ik.x} = 2(-1)^n e^{-r^2} L_n(2r^2). \tag{1.115}$$

Note that $\phi_0(r^2)$ is precisely the gaussian solution found in the previous section.

In summary, (1.90) has an infinite number of real radial solutions, given by

$$\sum_{n=0}^{\infty} a_n \phi_n(r^2) \tag{1.116}$$

where $\phi_n(r^2)$ is given by (1.115) and each a_n takes values in $\{\lambda_i\}$.

We also see from (1.105) that the action at $\theta \to \infty$ has a large symmetry $O_\phi \to UO_\phi U^\dagger$, where U is any unitary operator acting on $\mathcal{H}$. This $U(\infty)$ global symmetry generates new *nonradially* symmetric solutions out of the radially symmetric ones. The most general projection operator $O = \lambda P$, of rank k, is unitary related to a projection operator which is diagonal that is of the form $\lambda(\sum_{i=0}^{k-1} |i\rangle\langle i|)$. The corresponding solutions are all degenerate in energy. In fact, their energy is k times the energy of the minimal energy soliton $k = 1$.

1.5 D–branes as noncommutative solitons

In this section we present the description of D–branes as noncommutative solitons first proposed by Harvey, Kraus, Larsen and Martinec [94]. As we anticipated in the Introduction, in the bosonic string theory there are D-branes of all dimensions that are however unstable: they have a tachyon on their world volume. In particular, the space filling D25–brane is unstable, and reflects the instability of the bosonic open string in 26 dimensions. Ashoke Sen has made a series of definite conjectures about the fate of the tachyon. Firstly, the vacuum that the tachyon rolls down to, is expected to contain no open strings. Secondly the difference in energy per unit volume of this vacuum to the original unstable one is expected to be equal to the tension of the D25–brane. Thirdly, the lower dimensional D-branes are solitonic excitations of the tachyon potential. These conjectures will be described and studied at length in Chapter 4, for the moment we hope what we said is enough for the subject of noncommutative solitons.

The effective action for the tachyon field, obtained by integrating out the massive string fields, is expected to take the form

$$S = \frac{C}{g_s} \int d^{26}\sqrt{g}\left(\frac{1}{2}f(T)g^{\mu\nu}\partial_\mu T\partial_\nu T - U(T) + \cdots\right) \tag{1.117}$$

where the dots stand for omitted higher derivative terms and terms involving the massless modes. The potential $U(T)$ is a general potential having an unstable extremum at $T = T_*$ (the unstable vacuum), and a minimum that we choose at $T = 0$. The constant $C = g_s\tau_{25}$ is independent of g_s. With these conventions Sen's conjecture requires $U(T = T_*) = 1$: in this way $S(T = T_*) = \tau_{25}V_{26}$. Always according Sen's conjecture the whole action should vanish at the local minimum $U(T = 0) = 0$.

Let us turn on a B field in two spatial directions of the theory, say $x_{1,2}$. In the presence of a B field the action becomes

$$S = \frac{C}{G_s} \int d^{26}\sqrt{G}\left(\frac{1}{2}f(T)G^{\mu\nu}\partial_\mu T\partial_\nu T - U(T) + \cdots\right)_\star \tag{1.118}$$

The advantage of taking the limit of large B field, as we know, is that derivative terms can be neglected. The soliton solutions of (1.118) are then exactly the noncommutative solitons we described earlier. According to Sen's conjecture, these should be the D-branes of the bosonic string theory. The simplest noncommutative soliton solution to (1.118) is

$$T = T_*\phi_0(r^2) \equiv T_*\, 2e^{-r^2/\theta} \tag{1.119}$$

where $r^2 = x_1^2 + x_2^2$ and the dependence on θ has been reintroduced (in this section we do not make the rescaling $z \to z\sqrt{\theta}$). This is a codimension two object and a candidate for the D23–brane. Let us see the energy of such an object. In the large B field limit the action is

$$S = -\frac{C}{G_s} \int d^{26}\sqrt{G}\, U(T) \tag{1.120}$$

Inserting $T = T_*\phi_0(r^2)$ we have

$$S = -\frac{C\;U(T_*)}{G_s} \int d^{24}x \int d^2x\sqrt{G}\,\phi_0(r) = -\frac{(2\pi\theta)\,C\;U(T_*)}{G_s} \int d^{24}x\,\sqrt{G} \tag{1.121}$$

Using the relation (1.45) between G_s and g_s, that for large B field is

$$G_s = \frac{g_s\sqrt{G}}{2\pi\alpha' B\sqrt{g}}\ , \tag{1.122}$$

and keeping in mind that $\theta = 1/B$, and $U(T_*) = 1$, we have

$$S = -(2\pi)^2\alpha' \frac{C}{g_s}\int d^{24}x\sqrt{g} = -(2\pi)^2\alpha'\, \tau_{25}V_{24} \tag{1.123}$$

The tension of the soliton is then

$$\tau_{\mathrm{soliton}} \;=\; (2\pi)^2\alpha'\tau_{25} \;\equiv\; \tau_{23} \tag{1.124}$$

that is exactly the right tension of a D23–brane. The only information we needed to obtain the energy of the noncommutative soliton is the value of U at the extremum T_* that is a part of the potential that we have some information about from Sen's conjecture. Using noncommutativity in additional spatial directions, it is also possible to obtain branes of all even codimension as noncommutative solitons, all of them with the right tensions.

Chapter 2

Renormalization of noncommutative gauge theories

In Chapter 1 we saw that for D–branes in the presence of a constant NSNS B-field the low energy effective action of the open strings attached to the branes can be represented by a Yang-Mills theory defined on a noncommutative spacetime endowed with a Moyal bracket.

All this holds at a semiclassical level (i.e. tree amplitudes computed in the string theory and in the field theory setting compare well). However one can try to compare loop amplitudes calculated both in string theory and in the corresponding noncommutative field theory, in order to see how effective the noncommutative effective field theory is. Several calculations of this type have been carried out [35, 36, 14, 13, 15]. It seems to be important therefore to know exactly on what properties of a noncommutative YM theory we can rely. One of the basic properties is renormalizability. Works that cover this subject are [6, 7, 8, 9, 10, 11, 12, 19, 21, 33] . In this Chapter we will study renormalization of different noncommutative gauge field theories. We first consider the case of a noncommutative YM theory with $U(N)$ gauge group in 4D without matter and analyze its one loop renormalizability properties. Since non-planar singularities are rendered harmless by the noncommutative parameter θ, only planar one-loop contributions are relevant in the UV region. Actually we will explicitly show that noncommutative gauge theories are one-loop renormalizable, exactly as ordinary YM theories [31]. Next we consider noncommutative gauge theories whose Lie algebra are determined by orthogonal or symplectic groups. One can show that these theories can indeed be defined and correspond to the field theory limit of open string theories attached to D–branes at tree level in the presence of an orientifold [29]. We show however that the field theory limit of the one-loop string amplitudes disagree with the one-loop amplitudes calculated from the noncommutative field theory [49].

2.1 One–loop structure of noncommutative $U(N)$ gauge theories

Our noncommutative theory is specified by the action

$$S = \int d^4x \, \mathrm{Tr}\left(-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{\alpha}(\partial_\mu A^\mu)^2 + (i\bar{c} * \partial_\mu D^\mu c - i\partial_\mu D^\mu c * \bar{c})\right) \tag{2.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig(A_\mu * A_\nu - A_\nu * A_\mu) \tag{2.2}$$

and we will choose the Feynman gauge $\alpha = 1$. The action (2.1) is invariant under the gauge transformations

$$\begin{aligned} \delta A_\mu &= \partial_\mu \lambda + A_\mu \star A_\nu - A_\nu \star A_\mu \, , \\ \delta F_{\mu\nu} &= F_{\mu\nu} \star \lambda - \lambda \star F_{\mu\nu} \end{aligned}$$

It is possible to see immediately that the potential A_μ should be valued in the Lie algebra $u(N)$, i.e. should be an hermitian matrix. Let us suppose that $A_\mu = A_\mu^a T^a$ is an element of some Lie algebra with basis T^a. Then under a noncommutative gauge transformation

$$\begin{aligned} \delta A_\mu &= \partial_\mu \lambda + A_\mu^a * \lambda^b T^a T^b - \lambda^b * A_\mu^a T^b T^a \\ &= \partial_\mu \lambda + \frac{1}{2}(A_\mu^a * \lambda^b + \lambda^b * A_\mu^a)[T^a, T^b] + \frac{1}{2}(A_\mu^a * \lambda^b - \lambda^b * A_\mu^a)\{T^a, T^b\} \end{aligned} \tag{2.3}$$

The transformed field will belong to the same Lie algebra, only if the anticommutator $i\{T^a, T^b\}$ does, and this happens only for $u(N)$ algebra.

The possibility of defining noncommutative gauge theories corresponding to subgroup of $U(N)$ and a string/brane configurations that correspond to them, is then a highly non trivial task [29, 57, 56]. Let us consider for instance the simplest case: the $U(1)$ part of the $p+1$ dimensional supersymmetric $U(N)$ gauge theory that arises as the low energy effective theory of n coincident Dp-branes. At $B = 0$ this $U(1)$ part decouples from the open string dynamics and effectively we find an $SU(N)$ theory. Indeed the $U(1)$ dynamics is a free dynamics since it represents the center of mass of the stack of branes. String amplitudes at tree level and one loop take into account this decoupling (see for instance [14]). When a B field is turned on, separating the center of mass becomes impossible. Basically this is because the $U(1)$ part represents the interactions of the (D-brane) open strings with the bulk closed strings; when $B \neq 0$ the open string left and right modes contribute unequally, but in the closed string sector left and right modes always appear on an equal footing, see also [23] for a perturbative analysis of the $NC\,U(1)$ non decoupling.

In the next section we will investigate the one-loop structure of the noncommutative $SO(N)$ gauge theory proposed in [29], so we spend the rest of this section to show explicitly the one-loop renormalizability of noncommutative $U(N)$ gauge theory.

For the noncommutative $U(N)$ gauge theory the potential A_μ is valued in the Lie algebra $u(N)$, i.e. is an hermitian matrix. As is customary in dealing with 4D field theories, throughout the chapter we use a Minkowski formulation of the theory, although its brane origin is Euclidean.

Since the properties of the Lie algebra $u(N)$ tensors are crucial in our calculation, we devote the rest of this subsection to deriving them.

We use a basis t^a, $a = 1, \ldots, N^2 - 1$ of traceless hermitean matrices for the Lie algebra $su(N)$, with normalization

$$\mathrm{tr}(t^a t^b) = \frac{1}{2}\delta^{ab} \tag{2.4}$$

and structure constants f_{abc} defined by

$$[t^a, t^b] = i f_{abc} t^c \ . \tag{2.5}$$

We define also the third order ad-invariant completely symmetric tensor d_{abc} by means of

$$\{t^a, t^b\} = \frac{1}{N}\delta_{ab} + d_{abc} t^c \ . \tag{2.6}$$

Next we pass to the Lie algebra $u(N)$ by introducing the additional generator $t^0 = \frac{1}{\sqrt{2N}}\mathbf{1}_N$. Corresponding to any index a for $su(N)$ we introduce the index $A = (0, a)$, so that A runs from 0 to $N^2 - 1$. We have

$$[t^A, t^B] = i f_{ABC} t^C, \quad \{t^A, t^B\} = d_{ABC} t^C \tag{2.7}$$

where f_{ABC} is completely antisymmetric, f_{abc} is the same as for $su(N)$ and $f_{0BC} = 0$, while d_{ABC} is completely symmetric; d_{abc} is the same as for $su(N)$, $d_{0BC} = \sqrt{\frac{2}{N}}\delta_{BC}$,

$d_{00c} = 0$ and $d_{000} = \sqrt{\frac{2}{N}}$. We have also

$$\mathrm{Tr}(t^A t^B) = \frac{1}{2}\delta^{AB} . \tag{2.8}$$

The following identities hold and will be extensively used below

$$\begin{aligned} f_{ABX}\, f_{XCD} + f_{ACX}\, f_{XDB} + f_{ADX}\, f_{XBC} = 0 \\ f_{ABX}\, d_{XCD} + f_{ACX}\, d_{XDB} + f_{ADX}\, d_{XBC} = 0 \\ f_{ADX}\, f_{XBC} = d_{ABX}\, d_{XCD} - d_{ACX}\, d_{XDB} \end{aligned} \tag{2.9}$$

Next we define the matrices F_A, D_A as follows

$$(F_A)_{BC} = f_{BAC}, \qquad (D_A)_{BC} = d_{BAC} \tag{2.10}$$

In the evaluation of Feynman diagrams we need to know traces of two, three and four such matrices. We borrow from the literature, [26, 27, 28], the corresponding results for $su(N)$ and extend them to $u(N)$. Denoting by $\widehat{\mathrm{Tr}}$ the traces over the relevant $N^2 \times N^2$ space, we obtain

$$\begin{aligned} &\widehat{\mathrm{Tr}}(F_A F_B) = -N\, c_A\, \delta_{AB}, \qquad c_A = 1 - \delta_{A,0} \\ &\widehat{\mathrm{Tr}}(D_A D_B) = N d_A \delta_{AB}, \qquad d_A = 2 - c_A \\ &\widehat{\mathrm{Tr}}(F_A D_B) = 0 \end{aligned} \tag{2.11}$$

$$\begin{aligned} \widehat{\mathrm{Tr}}(F_A F_B F_C) &= -\frac{N}{2} f_{ABC} \\ \widehat{\mathrm{Tr}}(F_A F_B D_C) &= -\frac{N}{2} d_{ABC}\, c_A\, c_B\, d_C \\ \widehat{\mathrm{Tr}}(F_A D_B D_C) &= \frac{N}{2} f_{ABC} \\ \widehat{\mathrm{Tr}}(D_A D_B D_C) &= \frac{N}{2} \eta_{ABC}\, d_{ABC} \end{aligned} \tag{2.12}$$

where $\eta_{ABC} = d_A\, d_B\, d_C - 4\delta_{A+B+C,0}$. Finally

$$\begin{aligned} \widehat{\mathrm{Tr}}(F_A F_B F_C F_D) &= \Big[\frac{1}{2}\delta_{(AB}\delta_{CD)} + \frac{N}{8}\left(d_{ABX}\, d_{CDX} + d_{ADX}\, d_{BCX}\right) \\ &\quad + \frac{N}{8}\left(f_{ADX}\, f_{BCX} - f_{ABX}\, f_{CDX}\right)\Big] c_A\, c_B\, c_C\, c_D \\ \widehat{\mathrm{Tr}}(F_A F_B F_C D_D) &= -\frac{N}{4}\left(d_{ABX}\, f_{CDX} + f_{ABX}\, d_{CDX}\right) c_A\, c_B\, c_C\, d_D \end{aligned}$$

$$\begin{aligned}
\widehat{\mathrm{Tr}}(F_A F_B D_C D_D) &= c_A\, c_B \Big[c_C\, c_D \frac{1}{2} \left(\delta_{AC}\delta_{BD} - \delta_{AB}\delta_{CD} + \delta_{AD}\delta_{BC}\right) \\
&\quad + \frac{N}{8}\, d_C\, d_D \left(f_{ABX}\, f_{CDX} - f_{ADX}\, f_{BCX}\right. \\
&\quad\quad \left. - d_{ABX}\, d_{CDX} - d_{ADX}\, d_{BCX}\right) \Big] \\
\widehat{\mathrm{Tr}}(F_A D_B F_C D_D) &= \frac{1}{2} \left(\delta_{AB}\delta_{CD} + \delta_{AD}\delta_{BC} - \delta_{AC}\delta_{BD}\right) c_A\, c_B\, c_C\, c_D \\
&\quad + \frac{N}{8} \left(f_{ABX}\, f_{CDX} - f_{ADX}\, f_{BCX}\right. \\
&\quad\quad \left. - d_{ABX} d_{CDX} - d_{ADX}\, d_{BCX}\right) c_A\, d_B\, c_C\, d_D \\
\widehat{\mathrm{Tr}}(F_A D_B D_C D_D) &= \frac{N}{4} \left(f_{ABX}\, d_{CDX} + d_{ABX}\, f_{CDX}\right) c_A\, d_B\, d_C\, d_D \\
\widehat{\mathrm{Tr}}(D_A D_B D_C D_D) &= \frac{1}{2} \delta_{(AB}\delta_{CD)}\, c_A\, c_B\, c_C\, c_D \\
&\quad + \frac{N}{8} \left(f_{ADX}\, f_{BCX} - f_{ABX}\, f_{CDX}\right. \\
&\quad\quad \left. + d_{ABX}\, d_{CDX} + d_{ADX}\, d_{BCX}\right) \eta_{ABCD}
\end{aligned} \tag{2.13}$$

where $\eta_{ABCD} = d_A\, d_B\, d_C\, d_D - 8\delta_{A+B+C+D,0}$.

The Feynman rules we are going to use are collected in Appendix A. Evaluating the one–loop contributions is lengthy but straightforward. In this section we consider the planar part of the 2–, 3– and 4–point functions and, adopting the dimensional regularization ($\epsilon = 4 - D$, as usual), we extract first the planar part and, out of it, the divergent part. The relevant results are written down below. The 2– and 3–point functions are exactly parallel to the corresponding ones in ordinary gauge theories, and some of them are written down below only for the sake of comparison.

Gluons carry Lorentz indices $\mu, \nu, \ldots$, color indices $A, B, \ldots$, and momenta $p, q, \ldots$. Ghosts carry only the last two type of labels. All the momenta are entering, unless otherwise specified, and we use the notation $p \times q = \frac{1}{2} p_\mu \theta^{\mu\nu} q_\nu$.

2–point function. We have two nonvanishing contribution to the UV divergent part:

– gluons circulating inside the loop:

$$i \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \delta_{AB} N \left[\frac{19}{12} g_{\mu\rho} p^2 - \frac{11}{6} p_\mu p_\nu \right] \tag{2.14}$$

– ghosts circulating inside the loop:

$$i \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \delta_{AB} N \left[\frac{1}{12} g_{\mu\rho} p^2 + \frac{1}{6} p_\mu p_\nu \right] \tag{2.15}$$

Their sum is:

$$i\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\delta_{AB}N\frac{5}{3}\left[g_{\mu\rho}p^2 - p_\mu p_\nu\right] \tag{2.16}$$

which entails the usual renormalization constant

$$Z_3 = 1 + \frac{5}{3}g^2N\frac{1}{(4\pi)^2}\frac{2}{\epsilon} \quad . \tag{2.17}$$

3–point function. The external gluons carry labels (A,p,μ), (B,q,ν) and (C,k,λ) for the Lie algebra, momentum and Lorentz indices. They are ordered in anticlockwise sense. The triangle diagram gives

$$\begin{aligned}-\frac{13}{8}g^3N\frac{1}{(4\pi)^2}\frac{2}{\epsilon}&\left(\cos(p\times q)f_{ABC}+\sin(p\times q)d_{ABC}\right)\\ &\cdot\left((p-q)_\lambda g_{\mu\nu}+(q-k)_\mu g_{\nu\lambda}+(k-p)_\nu g_{\mu\lambda}\right)\end{aligned} \tag{2.18}$$

The diagram with one three gluon vertex and one four–gluon vertex gives:

$$\begin{aligned}\frac{9}{4}g^3N\frac{1}{(4\pi)^2}\frac{2}{\epsilon}&\left(\cos(p\times q)f_{ABC}+\sin(p\times q)d_{ABC}\right)\\ &\cdot\left((p-q)_\lambda g_{\mu\nu}+(q-k)_\mu g_{\nu\lambda}+(k-p)_\nu g_{\mu\lambda}\right)\end{aligned} \tag{2.19}$$

The contribution of the two ghost circulating diagrams is:

$$\begin{aligned}\frac{1}{24}g^3N\frac{1}{(4\pi)^2}\frac{2}{\epsilon}&\left(\cos(p\times q)f_{ABC}+\sin(p\times q)d_{ABC}\right)\\ &\cdot\left((p-q)_\lambda g_{\mu\nu}+(q-k)_\mu g_{\nu\lambda}+(k-p)_\nu g_{\mu\lambda}\right)\end{aligned} \tag{2.20}$$

The sum of the coefficients is

$$-\frac{13}{8}+\frac{9}{4}+\frac{1}{24}=\frac{2}{3} \tag{2.21}$$

Therefore, as in the ordinary YM theory, the renormalization constant Z_1 is

$$Z_1 = 1 + \frac{2}{3}g^2N\frac{1}{(4\pi)^2}\frac{2}{\epsilon} \quad . \tag{2.22}$$

4–point function. The external gluons carry labels (A,μ,p), (B,ν,q), (C,ρ,r) and (D,σ,s) for Lie algebra, Lorentz index and momentum, as shown in Figure 1.

There are four distinct graphs contributing to the 4–gluon vertex: the gluon box $\mathfrak{b}$, the ghost box $\mathfrak{g}$, the gluon triangle $\mathfrak{t}$ and the gluon candy $\mathfrak{c}$. There are two main type of contributions, distinguished by their Lie algebra tensor structure.

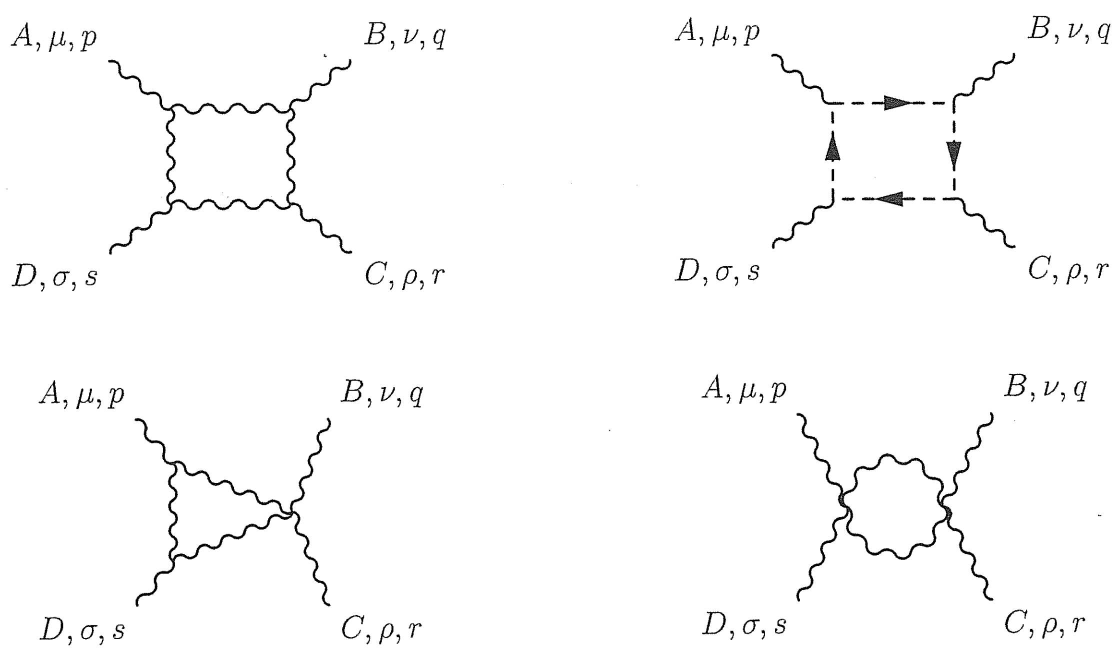


Figure 2.1: One loop contributions to the four-point function

The first is characterized by Kronecker delta functions in the Lie algebra indices, while the second consists of d and f tensors. The first type contributions, which are potentially dangerous for renormalizability, fortunately vanish.

The second type contributions have the general form

$$\begin{aligned} -ig^4\frac{2}{\epsilon}\frac{1}{(4\pi)^2} \Big[&\left(\frac{N}{8} \cos(p\times s - q\times r)\, L_{ABCD} + \frac{N}{8} \sin(p\times s - q\times r)\, M_{ABCD}\right) K^{\mathfrak{i}}_{\mu\nu\rho\sigma} \\ + &\left(\frac{N}{8} \cos(p\times r - q\times s)\, L_{BACD} - \frac{N}{8} \sin(p\times r - q\times s)\, M_{BACD}\right) K^{\mathfrak{i}}_{\nu\mu\rho\sigma} \\ + &\left(\frac{N}{8} \cos(p\times s + q\times r)\, L_{ACBD} + \frac{N}{8} \sin(p\times s + q\times r)\, M_{ACBD}\right) K^{\mathfrak{i}}_{\mu\rho\nu\sigma}\Big] \end{aligned} \tag{2.23}$$

where

$$\begin{aligned} L_{ABCD} &= d_{ABX}\, d_{CDX} + d_{ADX}\, d_{CBX} - f_{ABX}\, f_{CDX} + f_{ADX} f_{BCX} \\ M_{ABCD} &= d_{ABX}\, f_{CDX} + d_{ADX}\, f_{BCX} + f_{ABX}\, d_{CDX} - f_{ADX}\, d_{BCX} \end{aligned} \tag{2.24}$$

and

$$K^{\mathfrak{b}}_{\mu\nu\rho\sigma} = \frac{94}{3}\, g_{\mu\nu}\, g_{\rho\sigma} + \frac{94}{3}\, g_{\mu\sigma}\, g_{\nu\rho} + \frac{34}{3}\, g_{\mu\rho}\, g_{\nu\sigma}$$

$$\begin{aligned} K^{\mathfrak{g}}_{\mu\nu\rho\sigma} &= -\frac{1}{3}\,(g_{\mu\nu}\,g_{\rho\sigma} + g_{\mu\sigma}\,g_{\nu\rho} + g_{\mu\rho}\,g_{\nu\sigma}) \\ K^{\mathfrak{t}}_{\mu\nu\rho\sigma} &= -(46\,g_{\mu\nu}\,g_{\rho\sigma} + 46\,g_{\mu\sigma}\,g_{\nu\rho} - 32\,g_{\mu\rho}\,g_{\nu\sigma}) \\ K^{\mathfrak{c}}_{\mu\nu\rho\sigma} &= 16\,(7\,g_{\mu\nu}\,g_{\rho\sigma} + 7\,g_{\mu\sigma}\,g_{\nu\rho} - 8\,g_{\mu\rho}\,g_{\nu\sigma}) \end{aligned} \tag{2.25}$$

The 4–point vertex is now easily calculated by summing all the contributions with the appropriate symmetry factors. The contributions (2.23) give rise to

$$\begin{aligned} i\frac{N}{12}g^4\frac{2}{\epsilon}\frac{1}{(4\pi)^2}\Big[&\Big(\cos(p\times s - q\times r)\,L_{ABCD} + \sin(p\times s - q\times r)\,M_{ABCD}\Big)\,T_{\mu\nu\rho\sigma} \\ + &\Big(\cos(p\times r - q\times s)\,L_{BACD} - \sin(p\times r - q\times s)\,M_{BACD}\Big)\,T_{\nu\mu\rho\sigma} \\ + &\Big(\cos(p\times s + q\times r)\,L_{ACBD} + \sin(p\times s + q\times r)\,M_{ACBD}\Big)\,T_{\mu\rho\nu\sigma}\Big] \end{aligned} \tag{2.26}$$

where

$$T_{\mu\nu\rho\sigma} = g_{\mu\nu}\,g_{\rho\sigma} + g_{\mu\sigma}\,g_{\nu\rho} - 2\,g_{\mu\rho}\,g_{\nu\sigma} \tag{2.27}$$

Comparing (2.26) with eq.(A.5) in Appendix A, we see that the contribution (2.26) implies that the four–A term in the action is renormalized with a Z_4 given by

$$Z_4 = 1 - \frac{1}{3}\,g^2\,N\,\frac{1}{(4\pi)^2}\frac{2}{\epsilon} \quad . \tag{2.28}$$

This is the same renormalization that occurs in ordinary $U(N)$ Yang–Mills theories. Therefore, *the noncommutative $U(N)$ Yang–Mills theories are one–loop renormalizable.*

The $U(1)$ case must be treated separately. Using the corresponding Feynman rules (see Appendix A), one finds the 2– and 3–point contributions evaluated above with $f = 0$ and $d = 1$ and multiplied by $\frac{1}{2}$. As for the 4–point function, the term corresponding to (2.26) is obtained by setting $L = 2$ and $M = 0$ in the latter. Therefore all the renormalization constants satisfy the renormalization conditions, and, as a consequence, the noncommutative $U(1)$ gauge theory is one–loop renormalizable too, [9, 11].

Finally let us consider the restriction from the $U(N)$ to the $SU(N)$ case. It is still not known what a noncommutative $SU(N)$ gauge theory is, although attempts of defining it have been done, [30, 56]. In particular we do not know the explicit form of the action. Therefore we can only try to guess the relevant Feynman rules. The most obvious possibility one can envisage is that they are simply obtained from the Feynman rules of the noncommutative $U(N)$ theory by restricting everywhere the $U(N)$ indices $A, B, ...$ to the corresponding $SU(N)$ ones $a, b, ...$ As one can see in this case the renormalization constants do not coincide with the ones in the ordinary $SU(N)$ gauge theory. Strictly speaking this is not enough to conclude that the noncommutative $SU(N)$ theory is nonrenormalizable, unless one assumes that the $\theta \to 0$ limit of the quantum theory is smooth.

However, even allowing for such more general possibility, it is easy to show that the theory defined by such Feynman rules is not one–loop renormalizable, see also [23]. To this purpose it is sufficient to compare the ratio of the renormalization constants of gluon propagator, Z_3, and the three gluon vertex, Z_1, with the ratio of ghost propagator, $\tilde{Z}_3$, and ghost-ghost-gluon vertex, $\tilde{Z}_1$. If the $SU(N)$ theory were renormalizable, we should find $Z_3/Z_1 = \tilde{Z}_3/\tilde{Z}_1$. Instead we obtain

$$\begin{aligned}
Z_1 &= 1 + g^2 \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \frac{1}{4} \left(\frac{N^2 - 2}{N} \right) \\
Z_3 &= 1 + g^2 \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \frac{5}{3} \left(\frac{N^2 - 2}{N} \right) \\
\tilde{Z}_1 &= 1 - g^2 \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \frac{1}{2} \left(\frac{N^2 - 3}{N} \right) \\
\tilde{Z}_3 &= 1 - g^2 \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \frac{1}{2} \left(\frac{N^2 - 2}{N} \right) ,
\end{aligned}$$

where we used the traces over the $SU(N)$ indices that can be found in [27].

2.2 One–loop structure of noncommutative $SO(N)$ gauge theories

For the noncommutative $SO(N)$ gauge theory we recall that, even without resorting to an action, we can extract the gluon Feynman rules for this low energy field theory from the string tree amplitudes [29, 49]. A natural question that arises is whether by applying these Feynman rules to compute one–loop amplitudes one gets a renormalizable theory. The answer is that, if we apply Feynman rules in the ordinary way, we get a nonrenormalizable theory.

To illustrate the problem the simple $NC\,SO(2)$ case will do. A brief remark on $NC\,SO(2)$ theory is needed in order to show the peculiar behavior a noncommutative field theories. In the ordinary (commutative) case $NC\,SO(2)$ and $NC\,U(1)$ are the same (free) theory. This is not true anymore in the noncommutative case. This is due to the fact that, unlike the $u(N)$ Lie algebra, $so(N)$ does not possess a third order invariant symmetric tensor. In the noncommutative theory this difference is relevant, because the d^{abc} tensor enters the Feynman rules multiplied by a nonzero coefficient, i.e. $\sin(p\times q)$. In the ordinary limit $p\times q=0$ and we find the equivalence of the two theories.

The Feynman rules are very simple in the $NC\,SO(2)$ case since only the four–point vertex is nonvanishing, see Appendix A where we collected the complete Feynman rules for the noncommutative $SO(N)$ gauge theory. If p,q,r,s and μ,ν,ρ,σ are the momenta and the Lorentz indices of the four legs in clockwise order, the four–point vertex is:

$$\begin{aligned} -2ig^2 \quad & [\,\cos(p\times r - q\times s)\,(g_{ik}g_{jl}+g_{ij}g_{kl}-2g_{il}g_{jk}) \\ & +\cos(p\times s + q\times r)\,(g_{il}g_{jk}+g_{ik}g_{jl}-2g_{ij}g_{kl}) \\ & +\cos(p\times s - q\times r)\,(g_{ij}g_{kl}+g_{il}g_{jk}-2g_{ik}g_{jl})\,] \end{aligned} \tag{2.29}$$

The one–loop correction is infinite. So the theory needs a renormalization. What is worse is that the divergent part is not of the form (2.29), but

$$\begin{aligned} \sim \frac{g^4}{\epsilon} \quad & [\,\cos(p\times r - q\times s)\,(7g_{ik}g_{jl}+7g_{ij}g_{kl}-8g_{il}g_{jk}) \\ & +\cos(p\times s + q\times r)\,(7g_{il}g_{jk}+7g_{ik}g_{jl}-8g_{ij}g_{kl}) \\ & +\cos(p\times s - q\times r)\,(7g_{ij}g_{kl}+7g_{il}g_{jk}-8g_{ik}g_{jl})\,] \end{aligned} \tag{2.30}$$

In order to eliminate this divergence we need a counterterm of the form

$$\sim (7A_i * A^i * A_j * A^j - 4A_i * A_j * A^i * A^j) \tag{2.31}$$

Therefore the divergent part of the $NCSO(2)$ gauge field theory breaks the gauge symmetry.

Now let us look at the problem from the string theory point of view. It is well known that in the limit $\alpha' \to 0$ string theory reproduces gauge theory amplitudes (see for instance [45] and refs therein). Then, to try to have an answer to our problem we have to study the one-loop corrections of unoriented open string theory with orthogonal Chan–Paton factors in the presence of a background B field. Let us start without B field. In this case, besides the usual annulus contribution, we have to take into account the Möbius band amplitude. Both amplitudes have a common structure which can be written as

$$A^{(1)}(p_1,\ldots,p_M) = \frac{1}{2}\chi_M\, f_N^{a_1,a_2,\ldots,a_M}\, \frac{g_D^M}{(4\pi)^{\frac{D}{2}}}\,(2\alpha')^{-\frac{D}{2}} \int \prod_{r=2}^{M} d\nu_r d\tau e^{2\tau}\tau^{-\frac{D}{2}}$$

$$\times \prod_{n=1}^{\infty} \left(1-\eta_n e^{-2n\tau}\right)^{2-D} \exp\left[\sum_{r<s} p_r G_{\mathcal{A},\mathcal{M}}(\nu_{rs}) p_s\right]$$
$$\times \exp\left[\sum_{r\neq s}\left(p_s \partial_r G_{\mathcal{A},\mathcal{M}}(\nu_{sr})\epsilon_r + \frac{1}{2}\epsilon_r \partial_r \partial_s G_{\mathcal{A},\mathcal{M}}(\nu_{sr})\epsilon_s\right)\right]_{\text{m.l.}}$$

where $\chi_M = i\,(1)$ for M even (odd). $f_N^{a_1,a_2,\ldots,a_M}$ is the group theory factor. It equals $N\text{tr}(t^{a_1}\ldots t^{a_N})$ in the annulus case for planar amplitudes and $\text{tr}(t^{a_1}\ldots t^{a_N})$ in the Möbius strip case. The ν coordinates represent the insertion coordinates of the vertex operator on the boundary of the world-sheet, $p\,G\,q$ stands for $p_i G^{ij} q_j$, $\nu_{rs} = \nu_r - \nu_s$ and $\partial_r = \frac{\partial}{\partial \nu_r}$. The factor $\eta_n = 1$ in the orientable case, and $= (-1)^n$ in the non-orientable case. The suffix m.l. stands for multilinear, meaning that in the series expansion of the exponential we keep only the terms that are linear in each polarization. The propagator $G_{\mathcal{A},\mathcal{M}}$ is either the annulus ($\mathcal{A}$) or the Möbius strip ($\mathcal{M}$) propagator. Their explicit expressions are contained in [49].

The integrals over the ν variables are evaluated in the appropriate regions of integration (moduli space), and one can see that a large amount of information contained in a Möbius amplitude is captured by doubling the integration region. Collecting together the planar amplitudes and the Möbius ones one can single out the divergent parts that corresponds in field theory to one-particle irreducible diagrams. The result can be written

$$A^{(1)}(p_1,\ldots)\big|_{\text{div}} = -\frac{N-2}{2}\frac{g^2}{(4\pi)^2}\frac{11}{3}\frac{1}{\epsilon}A^{(0)}(p_1,\ldots) \quad , \tag{2.32}$$

where in the $(N-2)$ factor the N part comes from the annulus and the -2 comes from the Möbius strip. Switching on a constant B field amounts to replacing the propagators $G_{\mathcal{A},\mathcal{M}}$ with $G_{\mathcal{A},\mathcal{M}}(\rho-\rho') - \frac{i}{2}\theta^{ij}\epsilon(\rho-\rho')$, with $\rho = \exp(-2\nu)$. Inserting it into the general formula (2.32) has a simple effect. The addition of the second term $-\frac{i}{2}\theta^{ij}\epsilon(\rho-\rho')$ does not affect derivatives of propagators, while it modifies the term $\prod_{r<s} e^{p_r G(\rho_r-\rho_s)p_s}$. This modification turns out to be very simple since the insertion points along the boundary of $\mathcal{M}$ are ordered, so that the relevant ϵ function is always either $+1$ or -1. As a consequence the corresponding exponential factors can be extracted from the moduli integral. In other words, the gluon amplitudes are multiplied by a global (noncommutative) factor, i.e.

$$A^{(1)}(p_1,\ldots,p_m) \to \prod_{r<s} e^{ip_r\times p_s} A^{(1)}(p_1,\ldots,p_m) \quad , \tag{2.33}$$

where $A^{(1)}(p_1, \ldots, p_m)$ are the $B = 0$ amplitudes. We can therefore conclude that the structure of the divergent terms; as well as the renormalization constants, are the same as in the ordinary $SO(N)$ gauge theories. Therefore, if there exists a noncommutative gauge field theory that represents the low energy effective action of open strings with orthogonal CP factors in the presence of a constant B field, *this noncommutative gauge field theory is one–loop renormalizable*, see also [61] for related comments.

Let us come back to the $NC\,SO(2)$ case. From the string theory point of view it is rather easy to argue that this theory should not be UV divergent. The one–loop contributions to open string amplitudes with $SO(N)$ Chan–Paton factors are of three types: planar (P) and nonplanar (NP) with the world–sheet of the annulus, and nonorientable (NO) with the world–sheet of the Möbius strip. Due to the structure of the string propagators on the annulus and on the Möbius strip, we saw that the contributions in the presence and in the absence of the B field for P and NO differ only by overall noncommutative factors of the type $\cos(p \times q)$ or $\sin(p \times q)$. It follows that those contributions which become divergent in the field theory limit are the same whether B is there or not. Now in the ordinary (commutative) $SO(N)$ case the divergent part comes from the planar contribution with a factor of N in front, and from the non orientable contribution with a factor of -2. So altogether the divergent field theory part is proportional to $N-2$, and therefore vanishes in the case $N = 2$. This is obvious from the ordinary field theory side, because the theory is free. However, as we noticed above, this conclusion is also suggested by string theory in the noncommutative case. Therefore string theory tells us that $NCSO(2)$ theory should not give rise to UV divergences.

A proposal for solving this puzzle is based on the fact that the element where field theory and string theory diverge is not the Feynman rules themselves (or the action they come from) but their application in the one–loop calculation. We have applied them in the usual way, but that may be too naive. We would need a suitably modified set of rules, based on a 'deformation' of the Lie algebra.

Chapter 3

String Field Theory: a Primer

3.1 The String Field

The first building block of Witten's Cubic String Field Theory (SFT) is the String Field. It is defined as the *most general* state living in the Hilbert space $\mathcal{H}$ of the first-quantized open string theory:

$$\begin{aligned} |\Psi\rangle &= \Big(\phi(x) + A_\mu(x)\alpha^\mu_{-1} + B_{\mu\nu}(x)\alpha^\mu_{-1}\alpha^\mu_{-1} + \dots\Big)c_1|\Omega\rangle \\ &= \int d^{26}k\Big(\phi(k) + A_\mu(k)\alpha^\mu_{-1} + B_{\mu\nu}(k)\alpha^\mu_{-1}\alpha^\mu_{-1} + \dots\Big)c_1|k\rangle \end{aligned} \tag{3.1}$$

where $|k\rangle = e^{ik\cdot X(0)}|0\rangle$. $|\Omega\rangle = c_1|0\rangle$ is the ghost number 1 vacuum. It is defined by

$$\begin{aligned} \alpha^\mu_n\,|\Omega\rangle &= 0 \qquad (n>0) \\ b_n\,|\Omega\rangle &= 0 \qquad (n\geq 0) \\ c_n\,|\Omega\rangle &= 0 \qquad (n>0) \\ k^\mu\,|\Omega\rangle &= 0 \end{aligned} \tag{3.2}$$

$|0\rangle$ is the SL(2,$\mathbb{R}$) invariant vacuum, x and k are the center-of-mass coordinate and momentum. Thinking the coefficients functions in front of the basis states as spacetime particle fields, we can call $|\Psi\rangle$ a 'string field'. We define now some operation on the string field needed to build the string field action. They are a kinetic operator Q, a multiplication rule ($*$ product),and an inner product $\langle\,\cdot,\cdot\rangle$. They act on $\mathcal{H}$ in the following way

$$\begin{aligned} Q &: \mathcal{H} \to \mathcal{H} \\ * &: \mathcal{H}\otimes\mathcal{H} \to \mathcal{H} \\ \langle\,\cdot,\cdot\rangle &: \mathcal{H}\otimes\mathcal{H} \to \mathbb{C} \end{aligned} \tag{3.3}$$

The kinetic operator Q is defined to satisfy the following identities

$$\begin{aligned} Q^2 A &= 0 \\ Q(A * B) &= (QA) * B + (-1)^A A * (QB) \\ \langle QA, B \rangle &= -(-1)^A \langle A, QB \rangle \end{aligned} \tag{3.4}$$

The first one is the nilpotency condition. The second states that Q is an odd derivation of the star product. The third declares that Q is self-adjoint. Inner product and $*$ product should satisfy

$$\begin{aligned} \langle A, B \rangle &= (-1)^{AB} \langle B, A \rangle \\ \langle A, B * C \rangle &= \langle A * B, C \rangle \\ A * (B * C) &= (A * B) * C \end{aligned} \tag{3.5}$$

The first property is a symmetry condition, the second is a cyclicity property analogous to the similar property of trace operation, the third is the condition of associativity. Associativity condition is fundamental for the interpretation of string field theory as a theory of interacting *open* strings, [63]. We will see it when we will introduce Witten's original formulation of SFT. We equip the string fields with a Grassmanality through the ghost number. We declare that the string field has ghost number 1, completely taken into account by the action of c_1 on $|0\rangle$. Q is then an odd operator of degree 1

$$\mathrm{gh}(QA) = \mathrm{gh}(A) + 1 \tag{3.6}$$

and the $*$ product is an even operator of degree zero

$$\mathrm{gh}(A * B) = \mathrm{gh}(A) + \mathrm{gh}(B) \tag{3.7}$$

In the properties above $(-1)^A$ is $+1$ when A is Grassman even, and -1 when A is Grassman odd.

We are now ready to give the string action:

$$S(\Phi) = -\frac{1}{g_o^2}\left[\frac{1}{2}\langle \Phi, Q\Phi \rangle + \frac{1}{3}\langle \Phi, \Phi * \Phi \rangle\right] \tag{3.8}$$

where g_o is the open string coupling constant. Since $S(\Phi)$ should obviously be a real number, we need also to impose a reality condition on the string field. Unfortunately at the moment we do not have enough tools to define this condition, we will come back on it later. The quadratic part of (3.8) is the *free part* of the action and represents an evolving on shell string state from $\tau = -\infty$ to $\tau = +\infty$. The cubic term is the interaction vertex of three string states.

Using the properties (3.4), (3.5) and the ghost numbers assignments, is easy to see that the action (3.8) is invariant under the gauge transformation

$$\delta\Phi = Q\Lambda + \Phi * \Lambda - \Lambda * \Phi \tag{3.9}$$

where Λ is a Grassman even ghost number zero string field. Variation of the action (3.8) gives the field equation of motion

$$Q\Phi + \Phi * \Phi = 0 \tag{3.10}$$

From the second of the equations (3.5) we have

$$\langle A, B * C \rangle = (-1)^{A(B+C)} \langle B, C * A \rangle \tag{3.11}$$

and, since the string fields are all of ghost number 1, the above equation states that the cubic term $\langle \Phi, \Phi * \Phi \rangle$ in the action (3.8) is cyclic in the permutations of the fields. In a similar way is possible to show that

$$\langle \Phi_1, Q\Phi_2 \rangle = \langle \Phi_2, Q\Phi_1 \rangle \tag{3.12}$$

Twist Operator

Twist operation reverses the parametrization of a string. It is realized by an operator Ω satisfying the following properties:

$$\begin{aligned} \Omega(QA) &= Q(\Omega A) \\ \langle \Omega A, \Omega B \rangle &= \langle A, B \rangle \\ \Omega(A * B) &= (-1)^{AB+1} \Omega(A) * \Omega(B) \end{aligned} \tag{3.13}$$

The first property means that the BRST operator has zero twist, the second property states that the bilinear form is twist invariant, and the third one shows that, up to signs, twisting the $*$ product of string fields amounts to multiplying the twisted states in opposite order. For the string field Φ, that is Grassmann odd, the twist operator acts as

$$\Omega(\Phi * \Phi) = +(\Omega\Phi) * (\Omega\Phi) \tag{3.14}$$

with the plus sign. This result, together with the first two equation of (3.14) implies that the string field action is twist invariant

$$S(\Omega\Phi) = S(\Phi) \tag{3.15}$$

Identity String Field

The algebra of the $*$ product has an identity element, usually indicated as I

$$I * A = A * I = A \tag{3.16}$$

From the above definition follows immediately that I is a Grassmann even, ghost number zero string field. Under twist operation is a twist odd state:

$$\Omega(I * A) = (-1)^{A+1}(\Omega A) * (\Omega I) = -(\Omega A) * (\Omega I) \tag{3.17}$$

since the last term should simply be (ΩA), it follows

$$\Omega I = -I \tag{3.18}$$

Finally, any derivation D of the $*$ algebra should annihilate the identity:

$$D(I * A) = (DI) * A + I * DA = (DI) * A + DA \tag{3.19}$$

3.2 Conformal Field Theory point of view

Our main need is to find a way to do actual computations starting from the string field action (3.8). The kinetic operator is immediately chosen as the BRST operator Q_B: it satisfies all the properties (3.4) and the variation of the free (quadratic) part only of (3.8), with $Q = Q_B$ is nothing but the physical state condition on the first-quantized string theory states:

$$Q_B|\Psi\rangle = 0\ . \tag{3.20}$$

From now on the kinetic operator will be the BRST operator; without any risk of confusion we will continue to refer to it as Q instead of Q_B.

The action (3.8) can also be written by defining two states

$$\langle V_2| \in \mathcal{H}^* \otimes \mathcal{H}^* \tag{3.21}$$

and

$$\langle V_3| \in \mathcal{H}^* \otimes \mathcal{H}^* \otimes \mathcal{H}^* \tag{3.22}$$

such that

$$S(\Phi) = -\frac{1}{g_o^2}\left[\frac{1}{2}\,{}_{(12)}\langle V_2||\Phi\rangle_{(1)}|Q\Phi\rangle_{(2)} + \frac{1}{3}\,{}_{(123)}\langle V_3||\Phi\rangle_{(1)}|\Phi\rangle_{(2)}|\Phi\rangle_{(3)}\right] \tag{3.23}$$

where the pedices are introduced to distinguish explicitly among the different copies of the string Fock space referred to different strings. Our final task will be the explicit determination of $\langle V_2|$ and $\langle V_3|$. This will be done using the two dimensional conformal field theory (CFT) structure that underlines string theory.

First of all we need a recipe to define $\mathcal{H}^*$, the dual of the Hilbert space of first-quantized string states. This is done by means of the internal product defined through **bpz** conjugation [52]:

$$\begin{aligned}\text{bpz} &: \mathcal{H} \to \mathcal{H}^* \\ \text{bpz}\,|A\rangle &= \langle\, \text{bpz}\,(A)| \end{aligned} \tag{3.24}$$

To define bpz conjugation consider a primary field $\phi(z)$ of dimension d with mode expansion

$$\phi(z) = \sum_{n=-\infty}^{\infty} \frac{\phi_n}{z^{n+d}}\,, \quad \phi_n = \oint \frac{dz}{2\pi i} z^{n+d-1}\phi(z)\,. \tag{3.25}$$

By the state-operator correspondence, $\phi(z)$ creates in the far past ($\tau \to -\infty, z \to 0$) the state

$$|\phi\rangle = \lim_{z\to 0} \phi(z)|0\rangle \tag{3.26}$$

We define

$$\langle \text{bpz}\,(\phi)| = \langle 0| \lim_{z\to 0} \phi\Big(-\frac{1}{z}\Big) \tag{3.27}$$

The state $\langle 0|$ is the left (out) vacuum defined as the time evoluted $|0\rangle$ at $\tau \to \infty$ ($z \to \infty$). The transformation

$$\mathcal{I} : z \mapsto -\frac{1}{z} \tag{3.28}$$

is a SL(2,$\mathbb{C}$) transformation which sends the origin to infinity while taking the unit circle to itself. On the modes ϕ_n the inversion $\mathcal{I}$ acts as

$$\begin{aligned}\text{bpz}\,(\phi_n) &= \oint \frac{dz}{2\pi i} z^{n+d-1} \mathcal{I} \circ \phi(z) \\ &= \oint \frac{dz}{2\pi i} z^{n+d-1} \Big(\frac{1}{z^2}\Big)^h \phi\Big(-\frac{1}{z}\Big) \\ &= \oint \frac{dz}{2\pi i} z^{n-d-1} \sum_m \phi_m (-1)^{m+d} z^{m+d} \\ &= (-1)^{-n+d} \phi_{-n} \end{aligned} \tag{3.29}$$

Bpz conjugated of known operators are

$$\begin{aligned} \text{bpz}\,(L_n) &= (-1)^n L_{-n} \\ \text{bpz}\,(\alpha^\mu_{-n}) &= (-1)^{n+1}\alpha^\mu_n \end{aligned}$$

Equipped with bpz conjugation we can now discuss the reality condition on $|\Phi\rangle$. Hermitian conjugation (hc) just transforms a bra into a ket. Bpz and hc conjugation are used together to define complex conjugation:

$$|A^*\rangle = \text{bpz}^{-1} \circ \text{hc}\,|A\rangle = \text{hc}^{-1} \circ \text{bpz}\,|A\rangle \tag{3.30}$$

Reality condition is

$$|A^*\rangle = |A\rangle \quad \rightarrow \quad \text{hc}\,|A\rangle = \text{bpz}\,|A\rangle \tag{3.31}$$

This condition ensures the reality of the fields $\phi, A_\mu, B_{\mu\nu}, \ldots$ in the expansion (3.1).

The importance of using bpz conjugation instead of hc one, is in the conformal field theory character of the former. What we want to do now is indeed to give a conformal field theory prescription for calculating the vertices $\langle V_2|$ and $\langle V_3|$, and to be able to do actual computations.

Comparing the two forms (3.8) and (3.23) of the string field action it also follows that

$$\langle\, A, B\rangle = \langle \text{bpz}(A)|B\rangle = {}_{(12)}\langle V_2||A\rangle_{(1)}|B\rangle_{(2)} \tag{3.32}$$

and

$$\langle\, A, B * C\rangle = \langle \text{bpz}\,(A)|B * C\rangle = {}_{(123)}\langle V_3||A\rangle_{(1)}|B\rangle_{(2)}|C\rangle_{(3)}, \tag{3.33}$$

this means that the vertex $|V_3\rangle$ realizes the $*$ product:

$$(|A\rangle * |B\rangle)_3 = {}_1\langle \text{bpz}\,(A)|_2\langle \text{bpz}\,(B)||V_3\rangle_{123} \tag{3.34}$$

We now have a cft definition of $\langle\,\cdot,\cdot\rangle$ in terms of bpz conjugation. What we want is a correspondent cft definition of the $*$ product. It is useful to define the $*$ product through the interaction term of the action. Consider three generic string states A, B and C , and their corresponding vertex operators $\mathcal{O}_A(z), \mathcal{O}_B(z), \mathcal{O}_C(z)$. We define three conformal transformations $f_i(z), i = 1, 2, 3$ such that

$$\langle\, A, B * C\rangle \equiv \left\langle f_1^D \circ \mathcal{O}_A(0)\, f_2^D \circ \mathcal{O}_B(0)\, f_3^D \circ \mathcal{O}_C(0)\right\rangle_D \tag{3.35}$$

There is a crucial conceptual difference between the two sides of the above equation. The left-hand side is an inner product in the product space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ of the 3 Fock spaces $\mathcal{H}_i$, while the right-hand side is a correlation function of a single string conformal field theory in the z upper half plane. The index D indicates that the conformal transformations and the correlation function are defined on the disk (that is conformally equivalent to the upper half plane).

Remember the time evolution of a single free string when the worldsheet is parametrized by the upper half plane: in $z = 0$ the string starts to evolve, the real positive axis being the boundary $\sigma = \pi$ and the negative one the boundary $\sigma = 0$. The front of the evolving string is represented by half circumferences centered in the origin, all the points belonging to the same radius being points at the same time. The intersection of the front line of the string with the imaginary positive axis is the midpoint Q of the string ($\sigma = \pi/2$).

The basic idea for defining the conformal transformations f_i is then to map three upper-half disks into a single disk representing the interaction vertex of the three strings. The operation of $*$ product is then interpreted as a gluing of two string worldsheets. We start with three upper-half disks parametrized by their own local coordinates z_i. On each half disk we perform the following coordinate transformation

$$h : z_i \mapsto \zeta = \frac{1 + iz_i}{1 + iz_1} \tag{3.36}$$

This transformation maps the mid-string point Q of each string in the center $\zeta = 0$ of the unit disk, and the open string boundaries to the boundary of the unit disk. Then we shrink the half disks obtained by a factor 2/3, and rotate the first one counterclockwise by a $2\pi/3$ angle, and the third one clockwise by a $2\pi/3$ angle. The three transformations are

$$\begin{aligned} f_1(z_1) &= e^{-\frac{2\pi i}{3}} \left(\frac{1 + iz_1}{1 - iz_1}\right)^{\frac{2}{3}} \\ f_2(z_2) &= \left(\frac{1 + iz_2}{1 - iz_2}\right)^{\frac{2}{3}} \\ f_3(z_3) &= e^{\frac{2\pi i}{3}} \left(\frac{1 + iz_3}{1 - iz_3}\right)^{\frac{2}{3}} \end{aligned} \tag{3.37}$$

The *global* disk is now constructed gluing together the three world sheets: for instance the right part ($\pi/2 \leq \sigma \leq \pi$) of the front line of the first string is glued with the left part ($0 \leq \sigma \leq \pi/2$) of the of the front line of the second string and so

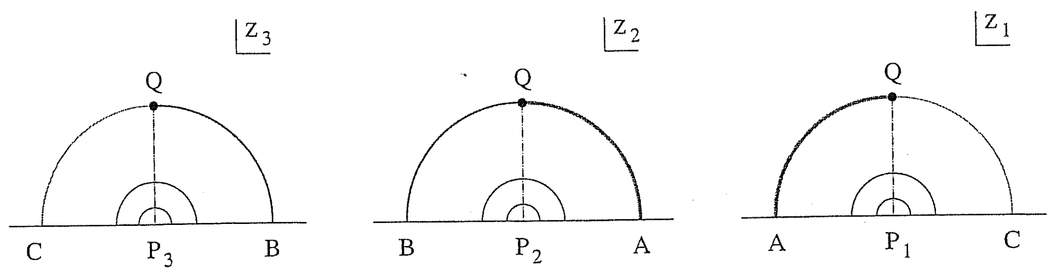


Figure 3.1: Representation of the cubic vertex as the gluing of 3 half–disks.

forth. Cyclicity of the cubic term in the action (3.8) is now manifest by construction. The open string worldsheet is also represented by the upper half plane; the SL(2,$\mathbb{C}$) transformation that sends the disk in the upper half plane is

$$h^{-1} : \zeta \mapsto z = -i\frac{\zeta - 1}{\zeta + 1} \tag{3.38}$$

Of course, since the correlator in (3.35) is SL(2,$\mathbb{C}$) invariant, computing it on the disk or on the plane gives the same result.

It is now straightforward to define an arbitrary n-point vertex through the transformations

$$f_k(z_k) = e^{\frac{2\pi i}{n}(k-1)}\left(\frac{1+iz_k}{1-iz_k}\right)^{\frac{2}{n}}, \quad 1 \le k \le n \tag{3.39}$$

Each f_k maps an upper half disk to a $2\pi/n$ wedge, and n such wedges gather to make a unit disk.

Also the CFT description of the quadratic term can be encoded in this formulation, with the $n = 2$ case of (3.39). Writing explicitly f_1 and f_2

$$h^{-1}f_1(z_1) = h^{-1}\left(\frac{1+iz_1}{1-iz_1}\right) = z_1 = I(z_1) \tag{3.40}$$

$$h^{-1}f_2(z_2) = h^{-1}\left(-\frac{1+iz_2}{1-iz_2}\right) = -\frac{1}{z_2} \equiv \mathcal{I}(z_2) \tag{3.41}$$

The quadratic term becomes

$$\begin{aligned} \langle \Phi, Q\Phi \rangle &= \langle f_2 \circ \Phi(0)\, f_1 \circ Q\Phi(0) \rangle \\ &= \langle h^{-1} \circ f_2 \circ \Phi(0)\, h^{-1} \circ f_1 \circ Q\Phi(0) \rangle \\ &= \langle \mathcal{I} \circ \Phi(0)\, Q\Phi(0) \rangle \end{aligned} \tag{3.42}$$

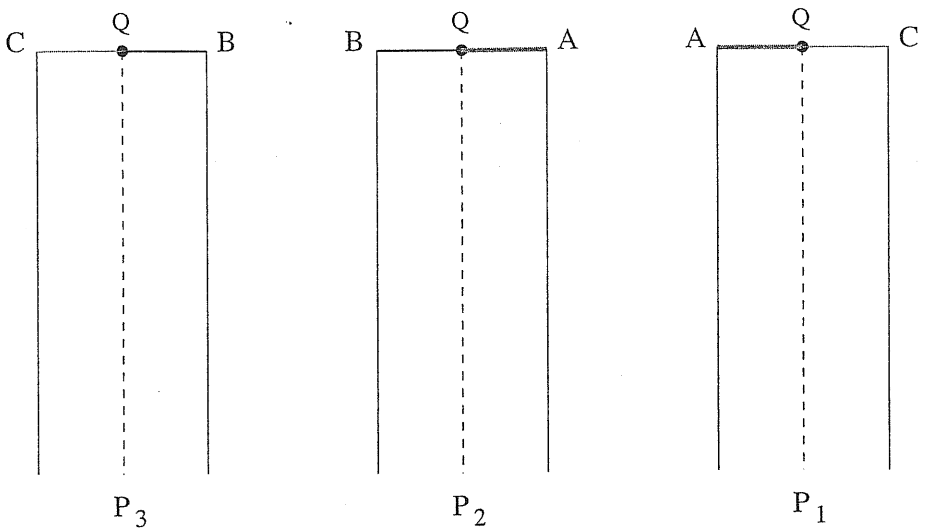


Figure 3.2: Representation of the cubic vertex as the gluing of 3 semi–infinite strips.

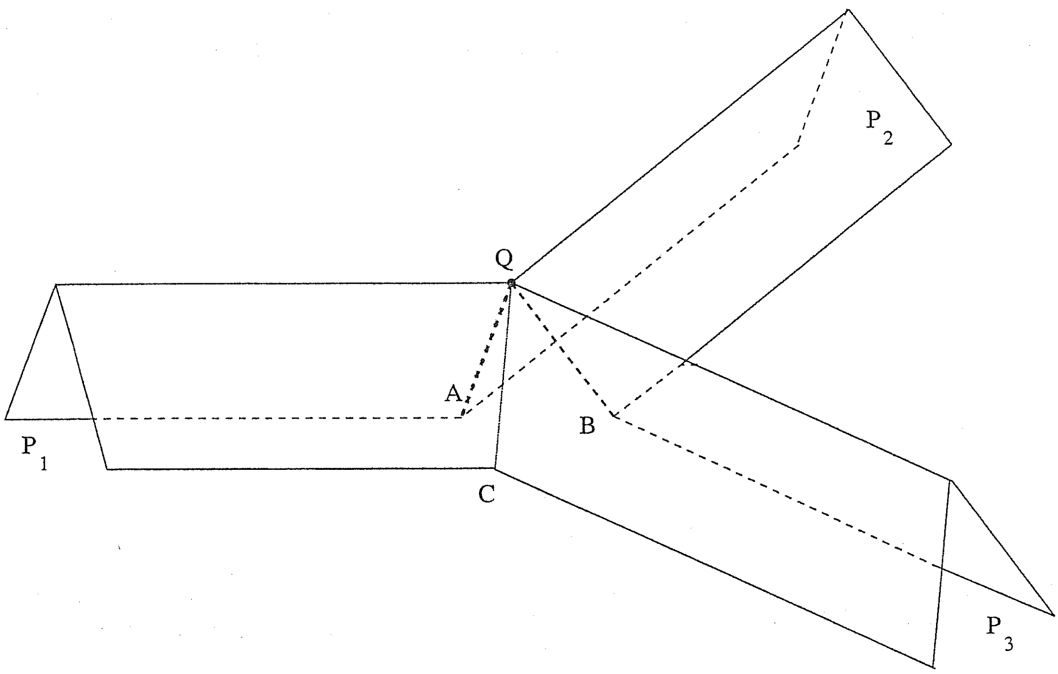


Figure 3.3: The result of gluing the 3 strips of Fig. 3.2.

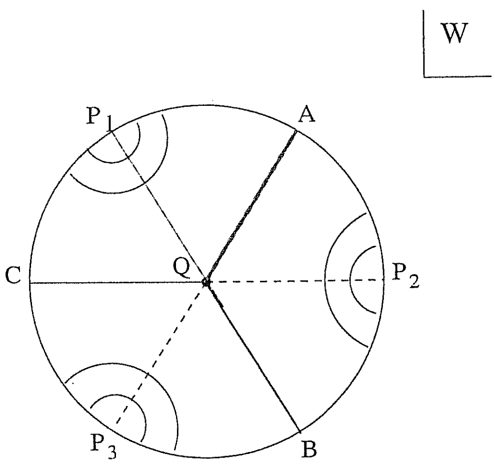


Figure 3.4: Representation of the cubic vertex as a 3–punctured unit disk.

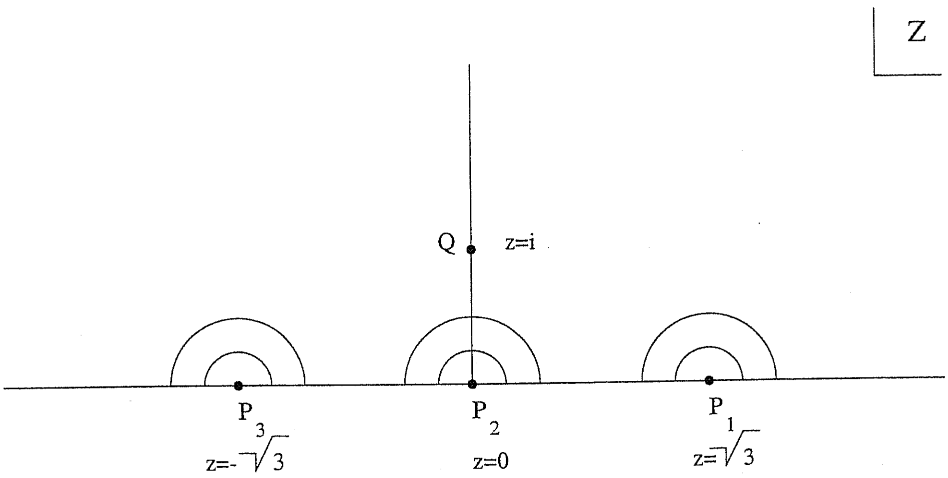


Figure 3.5: Representation of the cubic vertex as the upper–half plane with 3 punctures on the real axis.

The complete action, rewritten in terms of CFT correlators, is

$$S = -\frac{1}{g_o^2}\left[\frac{1}{2}\Big\langle \mathcal{I}\circ\Phi(0)\, Q\Phi(0)\Big\rangle + \frac{1}{3}\Big\langle f_1\circ\Phi(0)\, f_2\circ\Phi(0)\, f_3\circ\Phi(0)\Big\rangle\right] \tag{3.43}$$

We need the last step: the explicit expression of $\langle V_3|$ and $\langle V_2|$ in terms of the transformations f_i and $\mathcal{I}$. As usual we start with $\langle V_3|$. The ansatz for this vertex is

$$\begin{aligned}\langle V_3| &= \int d^{26}p_{(1)}d^{26}p_{(2)}d^{26}p_{(3)}\,\langle\tilde{0},p|_1\otimes\langle\tilde{0},p|_2\otimes\langle\tilde{0},p|_3\\ &\times \exp\Big(-\frac{1}{2}\sum_{r,s}\sum_{n,m\geq 0}\eta_{\mu\nu}\,\alpha_m^{(r)\mu}\,N^{rs}_{mn}\,\alpha_n^{(s)\nu}\Big)\\ &\times \exp\Big(\sum_{r,s}\sum_{\substack{m\geq 0\\ n\geq 1}} b_m^{(r)}\,X^{rs}_{mn}\,c_n^{(s)}\Big)\end{aligned} \tag{3.44}$$

where $\langle\tilde{0},p|_i = \langle p|_i\otimes\langle 0|_i c_{-1}c_0$. We want to find explicit expressions for the coefficients N^{rs}_{mn} and X^{rs}_{mn}, known as Neumann coefficients, in terms of the functions f_i defining the vertex. We begin the derivation considering the matter sector and setting the momenta to zero $(m,n>0)$; consider the expression

$$\mathcal{M} = \langle V_3|\, i\partial X^{(r)}(z)\, i\partial X^{(s)}(w)\,|\Omega\rangle_{(1)}|\Omega\rangle_{(2)}|\Omega\rangle_{(3)} \tag{3.45}$$

We compute it first using the contractions of the conformal fields $i\partial X(z)$, and then the oscillator form (3.44) of the vertex. $\mathcal{M}$ is rewritten as

$$\begin{aligned}\mathcal{M} &= \langle V_3|\, i\partial X^{(r)}(z)\, i\partial X^{(s)}(w)\, c^{(1)}(0)\, c^{(2)}(0)\, c^{(3)}(0)\,|0\rangle_{(1)}|0\rangle_{(2)}|0\rangle_{(3)}\\ &= \Big\langle f_r\circ\Big(i\partial X(z)c(0)\Big)\, f_s\circ\Big(i\partial X(w)c(0)\Big)\, f_t\circ c(0)\Big\rangle\end{aligned} \tag{3.46}$$

where $t\neq r,s$. The ghost part gives a constant that we will not have to calculate explicitly: $\langle f_r\circ c(0)\, f_s\circ c(0)\, f_t\circ c(0)\rangle \equiv \mathcal{N}$. Being $i\partial X$ a primary field it transforms as $f\circ i\partial X(z) = i\partial X(f(z))\frac{df}{dz}$, and we have

$$\begin{aligned}\mathcal{M} &= \mathcal{N}\, f_r'(z) f_s'(w)\Big\langle\, i\partial X(f_r(z))\, i\partial X(f_s(w))\,\Big\rangle\\ &= \mathcal{N}\,\frac{f_r'(z)f_s'(w)}{(f_r(z)-f_s(w))^2}\end{aligned} \tag{3.47}$$

Using the oscillator form of the vertex

$$\begin{aligned}\mathcal{M} &= \sum_{m,n} z^{m-1}w^{n-1}\langle V_3|\alpha^{(r)}_{-m}\alpha^{(s)}_{-m}c_1^{(1)}|0\rangle_{(1)}c_1^{(2)}|0\rangle_{(2)}c_1^{(3)}|0\rangle_{(3)}\\ &= -\mathcal{N}\sum_{m,n} z^{m-1}w^{n-1}\, mnN^{rs}_{mn}\end{aligned} \tag{3.48}$$

Comparing the equations (3.47) and (3.48) we have

$$\frac{f_r'(z)f_s'(w)}{(f_r(z)-f_s(w))^2} = -\sum_{m,n} z^{m-1}w^{n-1}\, mnN_{mn}^{rs} \tag{3.49}$$

that means

$$N_{mn}^{rs} = -\frac{1}{mn}\oint_0 \frac{dz}{2\pi i}\oint_0 \frac{dz}{2\pi i}\frac{1}{z^m w^n}\frac{f_r'(z)f_s'(w)}{(f_r(z)-f_s(w))^2} \tag{3.50}$$

Taking into account also the momenta we have to remember the transformation law for the exponentials $f\circ \exp(ip\cdot X(z)) = |f'(z)|^{p^2/2}\exp(ip\cdot X(f(z)))$ and the fact that the operators $i\partial X$ can also contract with factors $ip\cdot X$ in the exponentials. For the part of the vertex bilinear in the momenta we choose, as states in (3.35), A and B as tachyonic states of momenta p_r, p_s and C as a tachyonic state of zero momentum. Consider the expression $\mathcal{M}_{00}$

$$\mathcal{M}_{00} = \langle V_3|\, e^{ip_1\cdot X(0)}\, e^{ip_2\cdot X(0)}\, |\Omega\rangle_{(1)}|\Omega\rangle_{(2)}|\Omega\rangle_{(3)} \tag{3.51}$$

Evaluated through the oscillator form of the vertex is

$$\begin{aligned}\mathcal{M}_{00} &= \int d^{26}p_{(1)}d^{26}p_{(2)}d^{26}p_{(3)}\, \langle\tilde{0},p|_1\langle\tilde{0},p|_2\langle\tilde{0},p|_3 \\ &\quad \exp\Big(-\frac{1}{2}\sum_{u,v} p^u\, N_{00}^{uv}\, p^v\Big)c_1^{(1)}|0,p_r\rangle_{(1)}c_1^{(2)}|0,p_s\rangle_{(2)}c_1^{(3)}|0,0\rangle_{(3)}\end{aligned} \tag{3.52}$$

Calculated as a correlation function it has the form

$$\begin{aligned}\mathcal{M}_{00} &= \Big\langle f_r\circ\Big(e^{ip_r\cdot X(0)}c(0)\Big)\, f_s\circ\Big(e^{ip_s X(0)}c(0)\Big)\, f_t\circ c(0)\Big\rangle \\ &= \mathcal{N}\,|f_r'(0)|^{p_r^2/2}\,|f_s'(0)|^{p_s^2/2}\exp\left(p_r\cdot p_s\,\log|f_r(0)-f_s(0)|\right)\end{aligned} \tag{3.53}$$

It follows that N_{00} is

$$N_{00}^{rs} = \begin{cases} \log|f_r'(0)| & r=s \\ \log|f_r(0)-f_s(0)| + \frac{1}{2}\log|f_r'(0)f_s'(0)| & r\neq s\end{cases} \tag{3.54}$$

In a similar way it is possible to show that

$$N_{0m}^{rs} = -\oint_0 \frac{dw}{2\pi i}\frac{1}{w^m}\frac{\log|f_r'(0)|^{1/2}f_s'(w)}{(f_r(0)-f_s(w))} \tag{3.55}$$

For the ghost sector the coefficients X^{rs}_{mn} are computed equating two different ways of calculating the expression

$$\mathcal{G} = \langle V_3|\, b^{(s)}(z)\, c^{(r)}(w)\, c_1^{(1)}|0\rangle_{(1)} c_1^{(2)}|0\rangle_{(2)} c_1^{(3)}|0\rangle_{(3)} \tag{3.56}$$

Using the mode expansion for ghost and antighost and interpreting $\mathcal{G}$ as a correlator we find

$$\begin{aligned}\mathcal{G} &= \Big\langle f_s \circ b(z)\, f_r \circ c(w)\, f_1 \circ c(0)\, f_2 \circ c(0)\, f_3 \circ c(0) \Big\rangle \\ &= \frac{(f_s'(z))^2}{f_r'(w)} \frac{1}{f_1'(0) f_2'(0) f_3'(0)} \Big\langle\, b(f_s(z))\, c(f_r(w))\, c(f_1(0))\, c(f_2(0))\, c(f_3(0)) \Big\rangle\end{aligned} \tag{3.57}$$

The simpler way to calculate this correlator is to see its singular structure and derive its normalization from a special configuration. There must be zeroes when any pair of c fields approach to each other. This will give a factor $(f_1(0) - f_2(0))(f_1(0) - f_3(0))(f_2(0) - f_3(0))$ as for $\mathcal{N}$. There are also poles when the antighost approaches any ghost. These considerations imply that

$$\mathcal{G} = \mathcal{N} \frac{(f_s'(z))^2}{f_r'(w)} \frac{1}{f_s(z) - f_r(w)} \frac{\prod_{i=1}^3 (f_r(w) - f_i(0))}{\prod_{j=1}^3 (f_s(z) - f_j(0))} \tag{3.58}$$

Using instead the vertex (3.44) we find

$$\begin{aligned}\mathcal{G} &= \sum_{m,n} \frac{1}{z^{-n+2}} \frac{1}{w^{-m-1}} \langle V_3|\, b^{(s)}_{-n}\, c^{(r)}_{-m} c_1^{(1)}|0\rangle_{(1)} c_1^{(2)}|0\rangle_{(2)} c_1^{(3)}|0\rangle_{(3)} \\ &= \mathcal{N} \sum_{m,n} z^{-n+2} w^{-m-1}\, X^{rs}_{mn}\end{aligned} \tag{3.59}$$

Comparing the two expressions (3.58) and (3.59) we have

$$X^{rs}_{mn} = \oint \frac{dz}{2\pi i} \frac{1}{z^{n-1}} \oint \frac{dw}{2\pi i} \frac{1}{w^{m+2}} \frac{(f_s'(z))^2}{f_r'(w)} \frac{1}{f_s(z) - f_r(w)} \frac{\prod_{i=1}^3 (f_r(w) - f_i(0))}{\prod_{j=1}^3 (f_s(z) - f_j(0))} \tag{3.60}$$

The explicit expression for $\langle V_2|$ is found by calculating the product $\langle\, i\partial X(z), i\partial X(w)\rangle$:

$$\langle\, i\partial X(z), i\partial X(w)\rangle = -\langle \mathcal{I} \circ \partial X(z)\, \partial X(w)\rangle = -\Big\langle\, \partial X\Big(-\frac{1}{z}\Big)\, \partial X(w)\, \Big\rangle \tag{3.61}$$

Remembering the transformation law (3.30) of the oscillators α_n under the inversion $\mathcal{I}$, the form of the vertex $\langle V_2|$ is

$$\begin{aligned}\langle V_2| &= \int d^{26}p^{(1)} d^{26}p^{(2)}\, \langle \tilde{0}, p|_1 \otimes \langle \tilde{0}, p|_2\, \delta(p^{(1)} + p^{(2)}) \\ &\times\ \exp\left(-\frac{1}{n} \alpha_n^{(1)} C_{nm} \alpha_m^{(2)} - c_n^{(1)} C_{nm} b_m^{(2)} - c_n^{(2)} C_{nm} b_m^{(1)} \right)\end{aligned} \tag{3.62}$$

where

$$C_{nm} = \delta_{nm}(-1)^n \tag{3.63}$$

is the twist matrix.

Gluing and resmoothing theorem

We have calculated the Neumann vertices N^{rs}_{mn} and X^{rs}_{mn} for the most general conformal transformation f_i. They could be transformation different from (3.37) but always for the three (Witten's) vertex, or even for more general vertex for n strings. Although the presence of an n interaction vertex in the action (3.8) is forbidden by ghost number conservation, it is possible to give a very important theorem for the vertex $\langle V_n|$, the *gluing and resmoothing theorem*, that guarantees a rigorous proof of the gauge invariance of the String Field Theory action (3.8). It states that, given n states $\Phi_1, \Phi_2, \ldots, \Phi_n$, then

$$\langle V_n||\Phi_1\rangle|\Phi_2\rangle \ldots |\Phi_n\rangle = \langle V_{n-1}||\Phi_1\rangle \ldots (|\Phi_i\rangle * |\Phi_j\rangle) \ldots |\Phi_n\rangle \tag{3.64}$$

3.3 Witten's original formulation

The presentation we gave of String Field Theory respects the need of a pedagogical introduction that brings the reader to get acquainted with modern techniques of SFT needed for the study of tachyon condensation, but this was not the historical developing path of SFT. In particular the starting point [63] of SFT dealt with *string functionals* instead that with *string states.* We want to give here a brief introduction to Witten's original formulation of SFT. The starting point is the string functional $\Phi[X(\sigma), c(\sigma), b(\sigma)]$ defined as the Schrödinger representation of the first quantized string field $|\Phi\rangle$

$$\Phi[X(\sigma), c(\sigma), b(\sigma)] \equiv \langle X(\sigma), c(\sigma), b(\sigma)|\Phi\rangle \tag{3.65}$$

In this language the action (3.8) takes the form of a Chern-Simons type action

$$S = -\frac{1}{g_o^2}\int\left(\frac{1}{2}\Phi * Q\Phi + \frac{1}{3}\Phi * \Phi * \Phi\right) \tag{3.66}$$

where the operation $\int(A * B)$, 'integration of a $*$ product', substitutes the product $\langle A, B\rangle$. The Chern-Simons action is defined to be

$$S(A) = \frac{1}{2}\int_M A \wedge dA + \frac{1}{3}\int_M A \wedge A \wedge A \tag{3.67}$$

where M is a 3-manifold and A a 1-form. This action is invariant under the gauge transformation

$$\delta A = d\epsilon + A \wedge \epsilon - \epsilon \wedge A \tag{3.68}$$

where ϵ is a 0–form. We can write the correspondence between Witten's Cubic SFT and Chern-Simons theory with the help of the following 'dictionary':

Chern-Simons	Witten's open SFT
differential form	state in CFT
wedge product $\wedge$	$*$ product
degree of a differential form	ghost number of a state
gauge state A	string field Φ with ghost number 1
gauge parameter ϵ	state in the CFT with ghost number 0
exterior derivative d	BRST operator Q
integration $\int$	Witten's integration

In Witten's original approach is more manifest the interpretation of the $*$ product as 'gluing of strings'. The $*$ product is defined by

$$(\Phi_1 * \Phi_2)(X_0(\sigma)) = \int \mathcal{D}X_1(\sigma)\mathcal{D}X_2(\sigma)\, \Phi_1(X_1(\sigma))\, \Phi_2(X_2(\sigma)) \tag{3.69}$$

$$\prod_{0\leq\sigma\leq\pi/2} \delta(X_2(\sigma) - X_1(\pi-\sigma))\, \delta(X_1(\sigma) - X_0(\sigma))\, \delta(X_0(\pi-\sigma) - X_2(\pi-\sigma))\ ,$$

and the integration by

$$\int \Phi = \int \mathcal{D}X(\sigma)\Phi(X(\sigma)) \prod_{0\leq\sigma\leq\pi/2} \delta(X(\sigma) - X(\pi-\sigma)) \tag{3.70}$$

The definition (3.69) of the $*$ product should be interpreted in the following way: the functional $(\Phi_1 * \Phi_2)$ of the strings coordinates X_0 is given by gluing the left half of the first string with the right half of the second string (first δ function in (3.69)), and then imposing that the remaining halves of the strings X_1 and X_2 constitutes the whole X_0 string. The integration (3.70) means to take a string and then obtaining a number from it collapsing the two halves of the string on each other. With the two definitions (3.69, 3.70) the 3-string interaction vertex can be written as

$$\begin{aligned}\langle \Phi_1, \Phi_2 * \Phi_3\rangle &= \int \Phi_1 * \Phi_2 * \Phi_3 \\ &= \int \mathcal{D}X_1(\sigma)\mathcal{D}X_2(\sigma)\mathcal{D}X_3(\sigma)\Phi_1(X_1(\sigma))\,\Phi_2(X_2(\sigma))\,\Phi_3(X_3(\sigma)) \\ &\quad \prod_{0\leq\sigma\leq\pi/2} \delta(X_2(\sigma)-X_1(\pi-\sigma))\,\delta(X_3(\sigma)-X_2(\pi-\sigma))\,\delta(X_1(\sigma)-X_3(\pi-\sigma))\end{aligned} \tag{3.71}$$

The whole construction of SFT that we gave in the previous section can then be translated in the language of functional integration. Bpz conjugation, that reverses the σ orientation on the boundary of the unit disk, has the Schrödinger representation

$$\langle \mathrm{bpz}\,(\Phi)|X(\sigma)\rangle = \Phi[X(\pi-\sigma)] \tag{3.72}$$

and the reality condition on $\langle\Phi|$ translates into

$$\Phi[X(\sigma)] = \Phi^*[X(\pi-\sigma)] \tag{3.73}$$

Using the representation of the identity

$$1 = \int \mathcal{D}X(\sigma)|X(\sigma)\rangle\langle X(\sigma)| \tag{3.74}$$

we have for the quadratic term

$$\begin{aligned}\langle \Phi_1, Q\Phi_2\rangle &= \langle \mathrm{bpz}\,(\Phi_1)|Q\Phi_2\rangle \\ &= \int \mathcal{D}X(\sigma)\,\langle \mathrm{bpz}\,(\Phi_1)|X(\sigma)\rangle\,\langle X(\sigma)|Q\Phi_2\rangle \\ &= \int \mathcal{D}X(\sigma)\,\Phi_1(X(\pi-\sigma))\,Q\Phi_2(X(\sigma)) \\ &= \int \mathcal{D}X_1(\sigma)\mathcal{D}X_2(\sigma)\,\Phi_1(X_1(\sigma))\,Q\Phi_2(X_2(\sigma)) \prod_{0\leq\sigma\leq\pi/2} \delta(X_2(\sigma)-X_1(\pi-\sigma))\end{aligned}$$

The cubic term is

$$
\begin{aligned}
\langle\, \Phi_1, \Phi_2 * \Phi_3 \rangle &= \langle \mathrm{bpz}\,(\Phi_1)|\Phi_2 * \Phi_3\rangle \\
&= \int \mathcal{D}X\, \langle \mathrm{bpz}\,(\Phi_1)|X(\sigma)\rangle\langle X(\sigma)|\Phi_2 * \Phi_3\rangle \\
&= \int \mathcal{D}X_1(\sigma)\, \mathcal{D}X_2(\sigma)\mathcal{D}X_3(\sigma)\;\; \Phi_1(X_1(\sigma))\, \Phi_2(X_2(\sigma))\, \Phi_3(X_3(\sigma)) \\
&\quad \prod_{0\le\sigma\le\pi/2} \delta(X_2(\sigma) - X_1(\pi-\sigma))\, \delta(X_3(\sigma) - X_2(\pi-\sigma))\, \delta(X_1(\sigma) - X_3(\pi-\sigma))
\end{aligned}
$$

where we used the reality condition on $\Phi[X(\sigma)]$. The above equation is obviously equal to (3.71).

We end this chapter by writing the String Field Theory action in all the forms we met

$$
\begin{aligned}
S(\Phi) &= -\frac{1}{g_o^2}\left[\frac{1}{2}\langle\, \Phi, Q\Phi\rangle + \frac{1}{3}\langle\, \Phi, \Phi * \Phi\rangle\right] \\
&= -\frac{1}{g_o^2}\left[\frac{1}{2}\, {}_{(12)}\langle V_2||\Phi\rangle_{(1)}|Q\Phi\rangle_{(2)} + \frac{1}{3}\, {}_{(123)}\langle V_3||\Phi\rangle_{(1)}|\Phi\rangle_{(2)}|\Phi\rangle_{(3)}\right] \\
&= -\frac{1}{g_o^2}\left[\frac{1}{2}\Big\langle \mathcal{I}\circ\Phi(0)\, Q\Phi(0)\Big\rangle + \frac{1}{3}\Big\langle f_1\circ\Phi(0)\, f_2\circ\Phi(0)\, f_3\circ\Phi(0)\Big\rangle\right] \\
&= -\frac{1}{g_o^2}\int\left(\frac{1}{2}\Phi * Q\Phi + \frac{1}{3}\Phi * \Phi * \Phi\right) \qquad (3.75)
\end{aligned}
$$

Chapter 4

Applications to Tachyon Condensation

4.1 Sen's conjectures

Open String Field Theory presented in the previous chapter is particularly useful, even fundamental, for the study of tachyon condensation. Is a well known fact that bosonic D branes are unstable due to the presence of a tachyonic mode in the spectrum of the open strings ending on the brane itself. Ashoke Sen made three very definite conjectures about how the process of tachyon condensation takes place [53, 54, 55].

- The difference in the potential between the unstable vacuum and the perturbatively stable vacuum should be the mass of the D25-brane.
- Lower-dimensional D–branes should be realized as soliton configurations of the tachyon and other string fields.
- The perturbatively stable vacuum should correspond to the closed string vacuum. In particular, there should be no physical open strings excitations around this vacuum.

Let us see why String Field Theory is suitable for describing such conjectures. Sen showed that the tachyon potential has a universal form which is independent of the details of the theory describing the D–brane [55], and he also related, in the formalism of SFT, the open string coupling constant g_o to the D–brane tension [55]. This 'universality' of the tachyon potential means that we can choose the easiest background for the theory that we want describing the tachyon potential. In particular, using SFT, one can take the *conformal* background to be the Boundary Conformal Field Theory (BCFT) of any bosonic Dp-brane, with the flat 26 dimensional Minkowski space being just the space filling D25–brane. The study of Sen's conjectures becomes then the study of the fields at zero momentum, living at the bottom of the tachyonic potential, which are radically off-shell states. The simplest of these states is the zero momentum tachyon state. SFT can describe these states in a very natural way; moreover zero momentum tachyon is included in a subalgebra of the string $*$ algebra, thing that gives us a lot of computational advantages. The tachyon state at zero momentum is $tc_1|0\rangle$ where t is a constant. It belongs to the subspace $\mathcal{H}_1$ of the whole string Fock space $\mathcal{H}$ defined as the space of states of ghost number one obtained by acting on $|0\rangle$ with oscillators b_n, c_n and matter Virasoro generators L_n. The subspace $\mathcal{H}_1$ of $\mathcal{H}$ is a background independent subspace having the property that we can consistently set the component of the string field along $\mathcal{H} - \mathcal{H}_1$ to zero in looking for a solution of the equation of motion. $\mathcal{H}_1$ is background independent for the simple reason that there is no room in this theory for containing information on the boundary CFT which describes any brane. Since the fields in $\mathcal{H}_1$ have zero momenta, and hence are independent of the coordinates on the D–brane world-volume, the integration in the string field action over x gives the $(p+1)$-dimensional volume factor V_{p+1}. So the action can be written as

$$S(T) = V_{p+1}\mathcal{L}(T) = -V_{p+1}U(T) \tag{4.1}$$

where we defined the tachyon potential as the negative of the lagrangian.

The string field $|T\rangle = T(0)|0\rangle$ includes an infinite collection of variables corresponding to the coefficients of expansion of a state in $\mathcal{H}_1$ in some basis. The tension τ_{25} of the D25–brane in terms of the open string coupling constant g_0 is

$$\tau_{25} = \frac{1}{2\pi^2 g_o^2} \tag{4.2}$$

What Sen showed in [55] is that we can use the above (4.2) whatever the type of brane we are considering: the tachyon potential on it has the universal form

$$U(T) \;=\; 2\pi^2 M\left[\frac{1}{2}\langle\mathcal{I}\circ T(0)\,QT(0)\rangle + \frac{1}{3}\langle f_1\circ T(0)\,f_2\circ T(0)\,f_3\circ T(0)\rangle\right] \tag{4.3}$$

where we set $M = V_p\tau_p$ and the time interval over which we integrate is taken of unit length. Let us evaluate the the tachyon potential for the simplest field, the zero momentum tachyon $tc_1|0\rangle$.

The quadratic term gives the contribution

$$\begin{aligned}\langle T, QT\rangle &= t^2\langle 0|c_{-1}Qc_1|0\rangle \\ &= t^2\langle 0|c_{-1}c_0L_0c_1|0\rangle \\ &= -t^2\langle 0|c_{-1}c_0c_1|0\rangle \\ &= -t^2 \end{aligned}\tag{4.4}$$

The cubic term can be calculated using the cft definition.

$$\langle T, T*T\rangle = t^3\langle f_1^H\circ c(0) f_2^H\circ c(0) f_3^H\circ c(0)\rangle_H \tag{4.5}$$

where $f_i^H = h^{-1}\circ f_i^D$, and the index H means that we calculate the correlation function on the upper half plane. The field $c(z)$ is primary of dimension -1, so we have

$$f\circ c(0) = \frac{c(f(0))}{f'(0)} \tag{4.6}$$

Using equations (3.37) to read the values of $f_i^H(0)$ and $\frac{df_i^H}{d\xi}(0)$, we therefore get, for instance

$$f_1^H\circ c(0) = \frac{c(f_1^H(0))}{f_1^{H'}(0)} = \frac{c(\sqrt{3})}{8/3} \tag{4.7}$$

The other two insertions are dealt with similarly, and we find

$$\begin{aligned}\langle T, T*T\rangle &= t^3\Big\langle \frac{c(\sqrt{3})}{8/3}\,\frac{c(0)}{2/3}\,\frac{c(-\sqrt{3})}{8/3}\Big\rangle_H \\ &= \frac{3^3}{2^7}t^3\langle c(\sqrt{3})\,c(0)\,c(-\sqrt{3})\rangle_H \\ &= \Big(\frac{3\sqrt{3}}{4}\Big)^3 t^3 \\ &\equiv K^3t^3 \end{aligned}\tag{4.8}$$

Substituting into eq.(4.3) we get the first approximation to the tachyon potential

$$f^{(0)}(t) \equiv \frac{U(T = tc(z))}{M} = 2\pi^2\Big(-\frac{1}{2}t^2 + \frac{1}{3}\Big(\frac{3\sqrt{3}}{4}\Big)^3 t^3\Big) \tag{4.9}$$

This has a local minimum at

$$t = t_c = \Big(\frac{4}{3\sqrt{3}}\Big)^3 \simeq 0.456 \tag{4.10}$$

At this minimum

$$f(t_c) \simeq -0.684 \tag{4.11}$$

Remembering the first Sen's conjecture

$$U(T_0) + \tau_p = \tau_p(1 + f(T_0)) = 0 \tag{4.12}$$

we see that the tachyon state alone satisfies the condition (4.12) as much as 68% of the conjectured value. The tachyon field is said to be of level zero. The level l of a state is related to the L_0 eigenvalue as

$$l = L_0 + 1 \tag{4.13}$$

State levels are a very efficient criterion to organize an approximate but always more precise calculation of the tachyon potential. One indeed truncates the string field $|T\rangle$ at a finite number of terms. The more space-time components one keeps, the better the approximation. Doing a (m, n) approximation means to keep all fields up to level m and all interactions up to level n. The level of an interaction is defined to be the sum of the levels of all fields entering into it. What we just did is the $(0, 0)$ approximation. Next step should be the $(1, 3)$ approximation, but we saw that string field action is twist invariant. This means that the coefficients of states at odd levels above $c_1|0\rangle$ must always enter the action in pairs, and we can satisfy the equation of motion of these fields by setting them to zero. Thus we look for solutions where $|T\rangle$ contains only even level states. A further simplification is given by choosing the Feynman-Siegel gauge

$$b_0|T\rangle = 0 \tag{4.14}$$

The tachyon field up to level two is then given by

$$|T\rangle = tc_1|0\rangle + uc_{-1}|0\rangle + vL_2c_1|0\rangle \tag{4.15}$$

At level (2,4) we again find a stationary point at $t_c \simeq -0.541, u_c \simeq -0.173, v_0 \simeq 0.051$ which gives 0.948% of the exact answer.

4.2 Surface states and wedge states: Identity and Sliver

In this section we want to introduce the new concept of *surface states*, and by means of it, to describe from a new point of view the identity state $|I\rangle$, and to define the new *sliver state* $|\Xi\rangle$ that will play a fundamental role in Vacuum String Field Theory.

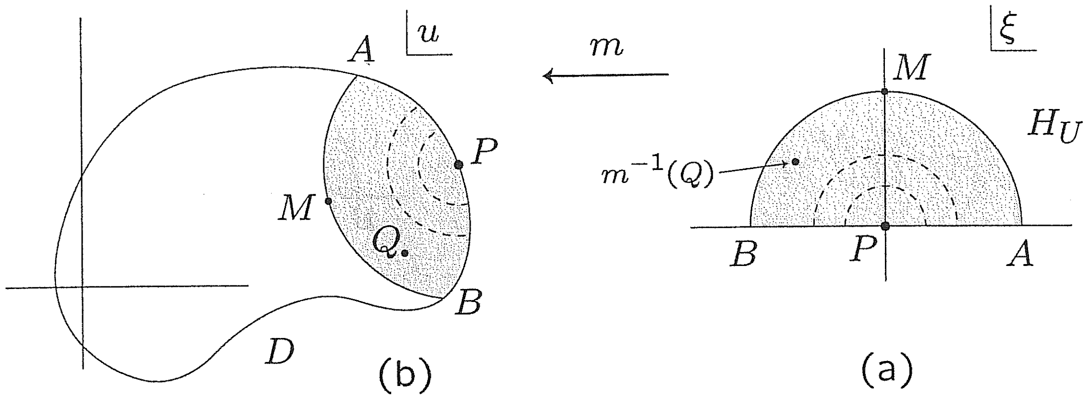


Figure 4.1: A punctured disk D with a local coordinate around the puncture P. The coordinate is defined through a map m from a canonical half disk H_U to the disk. The arcs AM and MB in D represent the left half and the right half of the open string respectively (the present figure and the following two are taken from [62]).

A *surface state* $\langle\Sigma|$ is a universal state, that means, independent of any BCFT, defined in association with a Riemann surface Σ with the topology of a disk D, with a marked point P, the puncture, lying on the boundary of D, and a local coordinate around it. A local coordinate at a puncture is obtained from an analytic map m taking a canonical half-disk H_U defined as

$$H_U : \{|\xi| \leq 1, \mathrm{Im}(\xi) \geq 0\} \tag{4.16}$$

into D, where $\xi = 0$ maps to the puncture P, and the image of the real segment $\{|\xi| \leq 1, \Im(\xi) = 0\}$ lies on the boundary of D. The coordinate ξ of the half disk is called the local coordinate. Using any global coordinate u on the disk D, the map m can be described by some analytic function s:

$$u = s(\xi), \quad u(P) = s(0) \tag{4.17}$$

Given this geometrical data, and a BCFT with state space $\mathcal{H}$, the state $\langle\Sigma| \in \mathcal{H}^*$ associated to the surface Σ is defined as follows. For any local operator $\phi(\xi)$, with associated state $|\phi\rangle = \lim_{\xi\to 0}\phi(\xi)|0\rangle$ we set

$$\langle\Sigma|\phi\rangle = \langle s\circ\phi(0)\rangle_D \tag{4.18}$$

where $\langle\ \rangle_D$ is the correlation function on D and $s\circ\phi(0)$ is the transform of the operator by the map $s(\xi)$. While computations of correlation functions involving

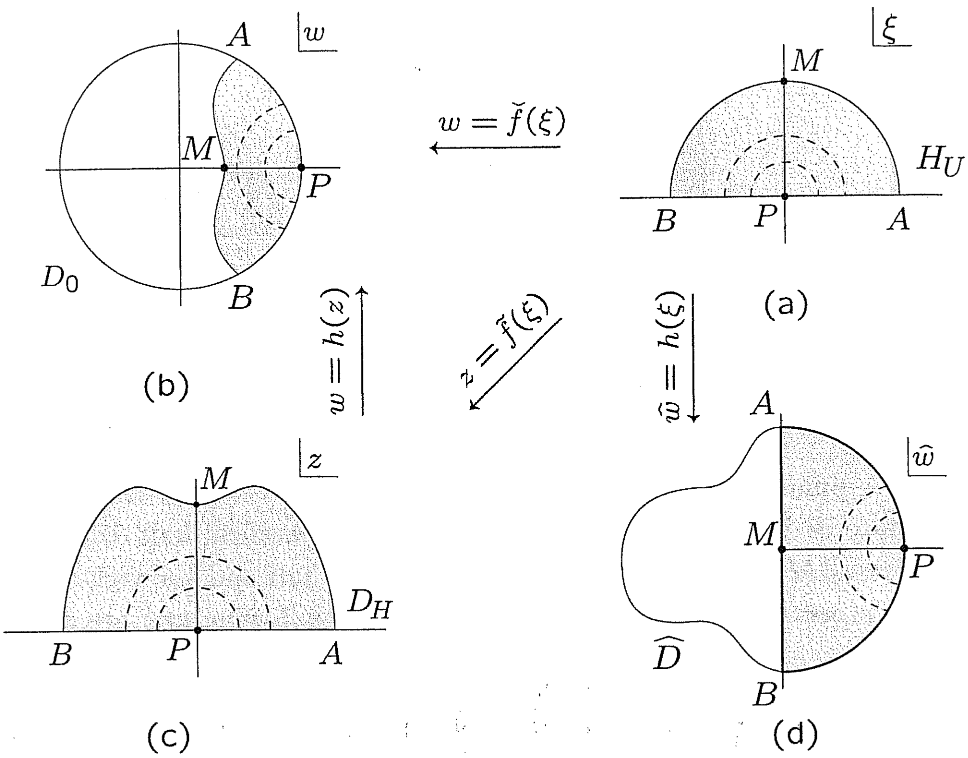


Figure 4.2: A punctured disk D with a local coordinate around the puncture P. The coordinate is defined through a map m from a canonical half disk H_U to the disk. The arcs AM and MB in D represent the left half and the right half of the open string respectively

states in $\mathcal{H}$ requires that the s be defined only locally around the puncture P, more general constructions, such as gluing of surfaces, requires that the full map of the half disk H_U into the disk D be well defined. Among all the possible surface states we define a sub-class of them called *wedge states*. The identity and the sliver are particular wedge states.

We start with the map

$$w_n = \check{f}_n(\xi) \equiv (h(\xi))^{2/n} = \left(\frac{1+i\xi}{1-i\xi}\right)^{2/n} \tag{4.19}$$

that sends the upper half-disk H_U into a wedge with the angle at $w_n = 0$ equal to $2\pi/n$. The transformation (4.19) can be rewritten as

$$w_n = \exp\left(i\frac{4}{n}\tan^{-1}(\xi)\right) \tag{4.20}$$

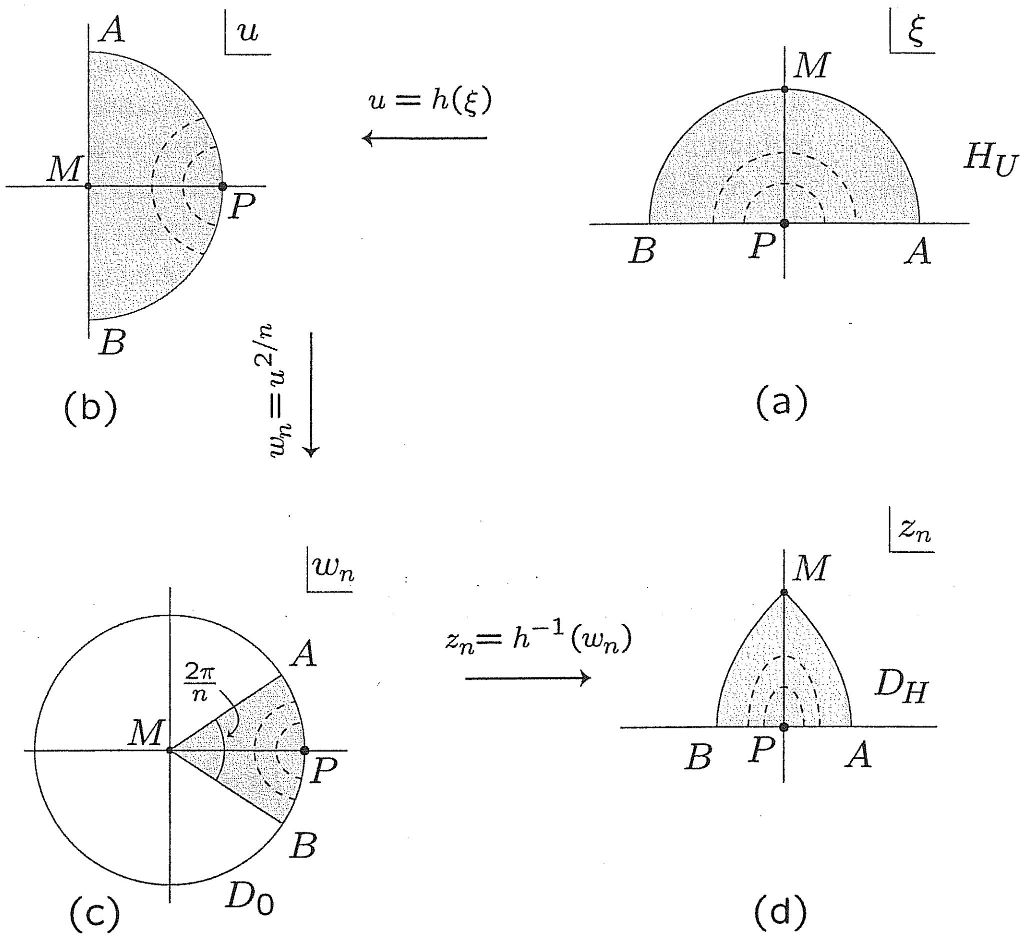


Figure 4.3: A punctured disk D with a local coordinate around the puncture P. The coordinate is defined through a map m from a canonical half disk H_U to the disk. The arcs AM and MB in D represent the left half and the right half of the open string respectively.

We define $\langle n|$ such that

$$\langle n|\phi\rangle \equiv \langle \check{f}_n \circ \phi(0)\rangle_{D_0} \tag{4.21}$$

The state that we obtain for $n=1$ is the identity state, that in the coordinates w_n is the full unit disk D_0 with a cut on the negative real axis. The left-half and the right-half of the string coincides along this cut. The $n=2$ is the vacuum state, and, in the w_n plane, the image of H_U covers the right half of the full unit disk D_0 in the w_n plane. The $n \to \infty$ limit is the sliver. It is an infinitely thin sliver of the disk D_0 around the positive real axis. In the next section we will see that $n=2$ is indeed the vacuum state and that the limit $n \to \infty$ gives rise to a well-defined state.

We describe now $|n\rangle$ taking back the wedge on the upper half plane. We define

$$z_n = h^{-1}(w_n) = i\frac{1-w_n}{1+w_n} = \tan\Big(-\frac{i}{2}\ln w_n\Big) \tag{4.22}$$

Putting together (4.20) and (4.22) we have

$$z_n = \tan\Big(\frac{2}{n}\tan^{-1}(\xi)\Big) \equiv \tilde{f}_n(\xi) \tag{4.23}$$

and

$$\langle n|\phi\rangle = \langle \tilde{f}_n \circ \phi(0)\rangle_{D_H} \tag{4.24}$$

The two description of the sliver (4.21) and (4.24)seems to be singular, in the sense that the maps $\check{f}_n(\xi)$ and $\tilde{f}_n(\xi)$ are singular in the $n \to \infty$ limit. This apparent singular behaviour is solved by noticing the SL(2,$\mathbb{R}$) invariance of the correlation functions on the upper half plane. Given any SL(2,$\mathbb{R}$) map $R(z)$ we have the relation

$$\langle \prod_i \mathcal{O}_i(x_i)\rangle_{D_H} = \langle \prod_i R \circ \mathcal{O}_i(x_i)\rangle_{D_H} \tag{4.25}$$

for any set of operators $\mathcal{O}_i$ and with D_H denoting the upper half plane. Since the sliver $|\Xi\rangle$ is defined through a correlation function, we can set

$$R_n(z) = \frac{n}{2}z \tag{4.26}$$

so that

$$\langle \Xi|\phi\rangle = \langle f \circ \phi(0)\rangle_{D_H} \tag{4.27}$$

where

$$\begin{aligned} f(\xi) &= \lim_{n\to\infty} R_n \circ \tilde{f}_n(\xi) \\ &= \lim_{n\to\infty} \frac{n}{2} \tan\left(\frac{2}{n}\tan^{-1}(\xi)\right) = \tan^{-1}\xi \end{aligned} \tag{4.28}$$

Since this map is non-singular at $\xi = 0$, we have a finite expression for $\langle\Xi|\phi\rangle$ for any state $|\phi\rangle$. We need now a prescription for $*$ multiplicating the surface states. $*$ multiplication is better understood in a third representation of the punctured disk D, where D itself is mapped into a disk $\widehat{D}$ having the special property that the local coordinate patch, *i.e.* the image of H_U in $\widehat{D}$, is nothing else that the vertical half-disk. This is done by taking, for $\xi \in H_U$

$$\hat{w} = h(\xi) = \frac{1+i\xi}{1-i\xi} \tag{4.29}$$

It is clear that in this representation the remaining part of $\widehat{D}$ may take a complicated form. Using eqs. (4.19) and (4.29) we see that

$$\widehat{w}_n = (w_n)^{n/2} \tag{4.30}$$

Under this map the unit disk D_0 in the w_n-coordinates is mapped to a cone in the $\widehat{w}_n$ coordinate, subtending an angle $n\pi$ at the origin $\widehat{w}_n = 0$. The disk D_0 mapped in this way represents then a wedge $|n\rangle$. We can give now the prescription for the $*$ product. Les us consider directly wedge states and remove the local coordinate patch from the disk D_0 in the w_n coordinate: the left over region becomes a sector of angle $\pi(n-1)$. If we denote by $|\mathcal{R}_\alpha\rangle$ a sector state arising from a sector of angle α, we have the identification of sector states with wedge states

$$|n\rangle = |\mathcal{R}_{\pi(n-1)}\rangle \tag{4.31}$$

We declare that the operation of $*$ multiplication of two wedge states $|m\rangle * |n\rangle$ is given by gluing together the two sector states $|\mathcal{R}_{\pi(m-1)}\rangle$ and $|\mathcal{R}_{\pi(n-1)}\rangle$ identifying the left-hand side of the string front of $|m\rangle$ with the right-hand side of the string front of $|n\rangle$. With this prescription we obtain the rule

$$|\mathcal{R}_a\rangle * |\mathcal{R}_b\rangle = |\mathcal{R}_{a+b}\rangle \tag{4.32}$$

that means

$$|m\rangle * |n\rangle = |m+n-1\rangle \tag{4.33}$$

The sliver state ($n \to \infty$) is a projector under the $*$ product:

$$|\mathcal{R}_\infty\rangle * |\mathcal{R}_\infty\rangle = |\mathcal{R}_\infty\rangle \tag{4.34}$$

We said that surface states, and then wedge states, are universal states. We want to give now an explicit background independent characterization to wedge states. This means that we want to find an operator $U = U(f_n)$ depending only on matter Virasoro generators L_n and ghost fields b and c such that $\langle n| = \langle 0|U$. Remember that a primary field of conformal weight d transforms under *finite* conformal transformation f as

$$f \circ \phi(z) = (f'(z))^d \phi(f(z)) \tag{4.35}$$

we would like to rewrite this transformation rule using the Virasoro generators L_n of the conformal group in the form

$$(f'(z))^d \phi(f(z)) = U_f \phi(z) U_f^{-1} \tag{4.36}$$

with

$$U_f = \exp[v_0 L_0] \exp\left[\sum_{n\geq 1} v_n L_n\right] \tag{4.37}$$

The coefficients v_n can be determined recursively from the Taylor expansion of f, by requiring

$$\begin{aligned} e^{v_0} &= f'(0) \\ \exp\left[\sum_{n\geq 1} v_n z^{n+1}\partial_z\right] &= (f'(0))^{-1} f(z) = z + a_2 z^2 + a_3 z^3 + \ldots \end{aligned} \tag{4.38}$$

For instance, for the first coefficients one finds

$$v_1 = a_2\,, \quad v_2 = -a_2^2 + a_3\,, \quad v_3 = \frac{3}{2}a_2^2 - \frac{5}{2}a_2 a_3 + a_4 \tag{4.39}$$

One can determine eqs.(4.38) in the following way. From the commutation relation

$$[L_m, \phi_n] = ((d-1)m - n)\phi_{m+n} \tag{4.40}$$

follows that

$$U_f \phi(z) U_f^{-1} = \exp[v(z)\partial_z + dv'(z)]\phi(z) \tag{4.41}$$

This equation is satisfied by a function $v(z)$ such that

$$e^{v(z)\partial_z} z = f(z) \tag{4.42}$$

from where follow eqs.(4.38). Choosing $\tilde{f}_n(z) = \tan\left(\frac{2}{n}\tan^{-1}(z)\right)$ to define the wedge states $|n\rangle$, we have

$$\begin{aligned} |n\rangle &= \exp\left[-\frac{n^2-4}{3n^2}L_{-2} + \frac{n^4-16}{30n^4} - \frac{(r^2-4)(176+128r^2+11r^4)}{1890r^6} \right.\\ &\left. + \frac{(r^2-4)(r^2+4)(16+32r^2+r^4)}{1260r^8}L_{-8} + \cdots\right]|0\rangle \end{aligned} \tag{4.43}$$

Particularly interesting states are

- The Identity State ($n=1$):

$$\begin{aligned} |I\rangle &\equiv |1\rangle \\ &= \exp\left[L_{-2} - \frac{1}{2}L_{-4} + \frac{1}{2}L_{-6} - \frac{7}{12}L_{-8} + \cdots\right]|0\rangle \end{aligned} \tag{4.44}$$

- The Vacuum State ($n=2$):

$$|0\rangle = |2\rangle \tag{4.45}$$

- The Sliver State ($n \to \infty$):

$$\begin{aligned} |\Xi\rangle &\equiv |\infty\rangle \\ &= \exp\left[-\frac{1}{3}L_{-2} + \frac{1}{30}L_{-4} - \frac{11}{1890}L_{-6} + \frac{1}{1260}L_{-8} + \cdots\right]|0\rangle \end{aligned} \tag{4.46}$$

Before going on and specialize our discussion in the next Chapter to the sliver state, it is useful to make a few comments about the algebra of the wedge states. We saw that the universal subspace $\mathcal{H}_{univ}$ of $\mathcal{H}$ containing zero momentum scalars

$$\mathcal{H}_{univ} \equiv \text{Span}\left\{L^m_{-j_1}\dots L^m_{-j_p}b_{-k_1}\dots b_{-k_q}c_{-l_1}\dots c_{-l_r}|0\rangle,\ j_i \geq 2,\ k_i \geq 2,\ l_i \geq -1\right\}$$

defines a subalgebra of the star-algebra of open string fields. Note also that $Q : \mathcal{H}_{univ} \to \mathcal{H}_{univ}$, since the BRST charge is built from matter Virasoro and ghost oscillators. $\mathcal{H}_{univ}$ can be splitted into a direct sum of spaces generated by states of a given ghost number:

$$\mathcal{H}_{univ} = \oplus_{n\in\mathbb{Z}} \mathcal{H}^{(n)}_{univ} \tag{4.47}$$

Since the ghost number is additive under star multiplication we have

$$\mathcal{H}^{(m)}_{univ} * \mathcal{H}^{(n)}_{univ} \subseteq \mathcal{H}^{(m+n)}_{univ} \tag{4.48}$$

It follows then that $\mathcal{H}^{(0)}_{univ}$ is a closed subalgebra of $\mathcal{H}_{univ}$. But there is an even smaller sub-algebra at ghost number zero. Consider indeed

$$\mathcal{H}^{(0)}(L) \equiv \mathrm{Span}\,\{L^{tot}_{-j_1} \dots L^{tot}_{-j_p}|0\rangle,\, j_i \geq 2\}$$

where L^{tot} denotes the combined matter and ghost ($c = 0$) Virasoro operators. Indeed since $|0\rangle * |0\rangle \in \mathcal{H}^0(L)$ it could be shown by conservation laws that any product of descendants of the vacuum is a descedant of $|0\rangle * |0\rangle \in \mathcal{H}^{(0)}(L)$ and thus a descendant of the vacuum. This confirms that $\mathcal{H}^{(0)}(L)$ is a subalgebra. Of course the space $\mathcal{H}_{wedge}$ of the wedge states is a subalgebra of $\mathcal{H}^{(0)}(L)$. The situation is summarized by

$$\mathcal{H}_{wedge} \subset \mathcal{H}^{(0)}(L) \subset \mathcal{H}^{(0)}_{univ} \subset \mathcal{H}_{univ} \tag{4.49}$$

Chapter 5

Vacuum String Field Theory

The results reported in the previous chapter, although very impressive, ultimately rely on numerical study of the solutions of the equations of motion using the level truncation scheme. We want now to propose an analytic approach developed by Rastelli, Sen and Zwiebach, known under the name of Vacuum String Field Theory (VSFT). This theory uses the open string tachyon vacuum to formulate the dynamics. Among all possible open string backgrounds the tachyon vacuum is particularly natural given its physically expected uniqueness as the endpoint of all processes of tachyon condensation.

5.1 An Ansatz for SFT after condensation

Let Φ_0 be the string field configuration describing the tachyon vacuum, a solution of the classical field equations following from the action in (3.8):

$$Q\Phi_0 + \Phi_0 * \Phi_0 = 0\,. \tag{5.1}$$

If $\widetilde{\Phi} = \Phi - \Phi_0$ denotes the shifted open string field, then the cubic string field theory action expanded around the tachyon vacuum has the form:

$$S(\Phi_0 + \widetilde{\Phi}) = S(\Phi_0) - \frac{1}{g_o^2}\left[\frac{1}{2}\langle\,\widetilde{\Phi}\,,\,\widehat{Q}\,\widetilde{\Phi}\rangle + \frac{1}{3}\langle\,\widetilde{\Phi}\,,\,\widetilde{\Phi} * \widetilde{\Phi}\rangle\right]. \tag{5.2}$$

Here $S(\Phi_0)$ is a constant, which according to the tachyon conjectures equals the mass M of the D-brane when the D-brane extends over a space-time of finite volume. As we saw in the previous chapters, the potential energy $V(\Phi_0) = -S(\Phi_0)$ associated to this string field configuration should equal minus the mass of the brane. The kinetic operator $\widehat{Q}$ is given in terms of Q and Φ_0 as:

$$\widehat{Q}\widetilde{\Phi} = Q\widetilde{\Phi} + \Phi_0 * \widetilde{\Phi} + \widetilde{\Phi} * \Phi_0 \,. \tag{5.3}$$

More generally, on arbitrary string fields one would define

$$\widehat{Q}A = QA + \Phi_0 * A - (-1)^A A * \Phi_0 \,. \tag{5.4}$$

The consistency of the action (5.2) is guaranteed from the consistency of the one in (3.8). Since neither the inner product nor the star multiplication have changed, the identities in (3.5) still hold. One can readily check that the identities in (3.4) hold when Q is replaced by $\widehat{Q}$. Just as (3.8) is invariant under the gauge trasformations (3.9), the action in (5.2) is invariant under $\delta\widetilde{\Phi} = \widehat{Q}\Lambda + \widetilde{\Phi} * \Lambda - \Lambda * \widetilde{\Phi}$ for any Grassmann-even ghost-number zero state Λ.

Since the energy density of the brane represents a positive cosmological constant, it is natural to add the constant $-M = -S(\Phi_0)$ to (3.8). This will cancel the $S(\Phi_0)$ term in (5.2), and will make manifest the expected zero energy density in the final vacuum without D–brane. For the analysis around this final vacuum it suffices therefore to study the action

$$S_0(\widetilde{\Phi}) \equiv -\frac{1}{g_o^2}\left[\frac{1}{2}\langle\,\widetilde{\Phi}\,,\,\widehat{Q}\,\widetilde{\Phi}\rangle + \frac{1}{3}\langle\,\widetilde{\Phi}\,,\,\widetilde{\Phi} * \widetilde{\Phi}\rangle\right]. \tag{5.5}$$

If we had a closed form solution Φ_0 available, the problem of formulating SFT around the tachyon vacuum would be significantly simplified, as we would only have to understand the properties of the new kinetic operator $\widehat{Q}$ in (5.4). In particular we would like to confirm that its cohomology vanishes in accordance with the expectation that all conventional open string excitations disappear in the tachyon vacuum. Even if we knew Φ_0 explicitly and constructed $S_0(\widetilde{\Phi})$ using eq.(5.5), this may not be the most convenient form of the action. Typically a nontrivial field redefinition is necessary to bring the shifted SFT action to the canonical form representing the new background. In fact, in some cases, such as in the formulation of open SFT for D-branes with various values of magnetic fields, it is simple to formulate the various SFT's directly, but the nontrivial classical solution relating theories with different magnetic fields are not known. This suggests that if a simple form exists for the SFT action around the tachyon vacuum it might be easier to guess it than to derive it.

In proposing a simple form of the tachyon action, we have in mind field redefinitions of the action in (5.5) that leave the cubic term invariant but simplify the operator $\widehat{Q}$ in (5.4) by transforming it into a simpler operator $\mathcal{Q}$. To this end we consider homogeneous field redefinitions of the type

$$\widetilde{\Phi} = e^{K}\,\Psi\,, \tag{5.6}$$

where K is a ghost number zero Grassmann even operator. In addition, we require

$$\begin{aligned} &K(A * B) = (KA) * B + A * (KB)\,,\\ &\langle\, KA, B\,\rangle = -\langle A, KB\rangle\,. \end{aligned} \tag{5.7}$$

These properties guarantee that the form of the cubic term is unchanged and that after the field redefinition the action takes the form

$$\mathcal{S}(\Psi) \equiv -\frac{1}{g_0^2}\left[\frac{1}{2}\langle\,\Psi\,,\,\mathcal{Q}\,\Psi\rangle + \frac{1}{3}\langle\,\Psi, \Psi * \Psi\rangle\right], \tag{5.8}$$

where

$$\mathcal{Q} = e^{-K}\widehat{Q}e^{K}\,. \tag{5.9}$$

Again, gauge invariance only requires:

$$\begin{aligned} &\mathcal{Q}^2 = 0,\\ &\mathcal{Q}(A * B) = (\mathcal{Q}A) * B + (-1)^{A}A * (\mathcal{Q}B)\,,\\ &\langle\,\mathcal{Q}A, B\,\rangle = -(-1)^{A}\langle A, \mathcal{Q}B\rangle\,. \end{aligned} \tag{5.10}$$

These identities hold by virtue of (5.7) and (5.9). We will proceed here postulating a $\mathcal{Q}$ that satisfies these identities as well as other conditions, since, lacking knowledge of Φ_0, the above field redefinitions cannot be attempted.

The choice of $\mathcal{Q}$ will be required to satisfy the following properties:

- The operator $\mathcal{Q}$ must be of ghost number one and must satisfy the conditions (5.10) that guarantee gauge invariance of the string action.
- The operator $\mathcal{Q}$ must have vanishing cohomology.
- The operator $\mathcal{Q}$ must be universal, namely, it must be possible to write without reference to the brane boundary conformal field theory.

We can satisfy the three requirements by letting $\mathcal{Q}$ be constructed purely from ghost operators. In particular we claim that the ghost number one operators

$$\mathcal{C}_n \equiv c_n + (-1)^n\,c_{-n}\,, \quad n = 0, 1, 2, \cdots \tag{5.11}$$

satisfy the properties

$$\begin{aligned} &\mathcal{C}_n\mathcal{C}_n = 0, \\ &\mathcal{C}_n(A*B) = (\mathcal{C}_nA)*B + (-1)^A A*(\mathcal{C}_nB), \\ &\langle\, \mathcal{C}_nA, B\,\rangle = -(-1)^A\langle A, \mathcal{C}_nB\rangle . \end{aligned} \tag{5.12}$$

The first property is manifest. The last property follows because under BPZ conjugation $c_n \to (-1)^{n+1}c_{-n}$. The second property follows from the conservation law

$$\langle V_3|(\mathcal{C}_n^{(1)} + \mathcal{C}_n^{(2)} + \mathcal{C}_n^{(3)}) = 0\,, \tag{5.13}$$

on the three string vertex [71].

Each of the operators $\mathcal{C}_n$ has vanishing cohomology since for each n the operator $\mathcal{B}_n = \frac{1}{2}(b_n + (-1)^n\, b_{-n})$ satisfies $\{\mathcal{C}_n, \mathcal{B}_n\} = 1$. It then follows that whenever $\mathcal{C}_n\psi = 0$, we have $\psi = \{\mathcal{C}_n, \mathcal{B}_n\}\psi = \mathcal{C}_n(\mathcal{B}_n\psi)$, showing that ψ is $\mathcal{C}_n$ trivial. Finally, since they are built from ghost oscillators, all $\mathcal{C}_n$'s are manifestly universal.

It is clear from the structure of the conditions (5.10) that they are satisfied for the general choice:

$$\mathcal{Q} = \sum_{n=0}^{\infty} a_n\, \mathcal{C}_n\,, \tag{5.14}$$

where the a_n's are constant coefficients.

There may be other choices of $\mathcal{Q}$ satisfying all the requirements stated above. Fortunately, the future analysis will not require the knowledge of the detailed form of $\mathcal{Q}$, as long as it does not involve any matter operators. To this end, it will be useful to note that since $\mathcal{Q}$ does not involve matter operators, we can fix the gauge by choosing a gauge fixing condition that also does not involve any matter operator. In such a gauge, the propagator will factor into a non-trivial operator in the ghost sector, and the identity operator in the matter sector.

5.1.1 Factorization property of the field equation

If (5.8) really describes the string field theory around the tachyon vacuum, then the equations of motion of this field theory:

$$\mathcal{Q}\Psi = -\Psi * \Psi\,, \tag{5.15}$$

must have a space-time independent solution describing the D25–brane, and also lump solutions of all codimensions describing lower dimensional D-branes. We shall look for solutions of the form:

$$\Psi = \Psi_m \otimes \Psi_g\,, \tag{5.16}$$

where Ψ_g denotes a state obtained by acting with the ghost oscillators on the SL(2,$\mathbb{R}$) invariant vacuum of the ghost CFT, and Ψ_m is a state obtained by acting with matter oscillators on the SL(2,$\mathbb{R}$) invariant vacuum of the matter CFT. Let us denote by $*^g$ and $*^m$ the star product in the ghost and matter sector respectively. Eq.(5.15) then factorizes as

$$\mathcal{Q}\Psi_g = -\Psi_g *^g \Psi_g\,, \tag{5.17}$$

and

$$\Psi_m = \Psi_m *^m \Psi_m\,. \tag{5.18}$$

Such a factorization is possible since $\mathcal{Q}$ is made purely of ghost operators. Note that we have used the freedom of rescaling Ψ_g and Ψ_m with λ and λ^{-1} to put eqs.(5.17), (5.18) in a convenient form.

In looking for the solutions describing D-branes of various dimensions we shall assume that Ψ_g remains the same for all solutions, whereas Ψ_m is different for different D-branes. Given two static solutions of this kind, described by Ψ_m and Ψ'_m, the ratio of the energy associated with these two solutions is obtained by taking the ratio of the actions associated with the two solutions. For a string field configuration satisfying the equation of motion (5.15), the action (5.8) is given by

$$\mathcal{S}|_\Psi = -\frac{1}{6g_0^2}\,\langle\,\Psi\,,\,\mathcal{Q}\,\Psi\,\rangle\,. \tag{5.19}$$

Thus with the ansatz (5.16) the action takes the form:

$$\mathcal{S}|_\Psi = -\frac{1}{6g_0^2}\,\langle\,\Psi_g\,|\,\mathcal{Q}\,\Psi_g\,\rangle_g\,\langle\Psi_m|\Psi_m\rangle_m \equiv K\langle\Psi_m|\Psi_m\rangle_m\,, \tag{5.20}$$

where $\langle\,|\,\rangle_g$ and $\langle\,|\,\rangle_m$ denote BPZ inner products in ghost and matter sectors respectively. $K = -(6g_0^2)^{-1}\langle\Psi_g|\mathcal{Q}\Psi_g\rangle_g$ is a constant factor calculated from the ghost sector which remains the same for different solutions. Thus we see that the ratio of the action associated with the two solutions is

$$\frac{\mathcal{S}|_{\Psi'}}{\mathcal{S}|_\Psi} = \frac{\langle\Psi'_m|\Psi'_m\rangle_m}{\langle\Psi_m|\Psi_m\rangle_m}\,. \tag{5.21}$$

It is worthwhile to notice that the ghost part drops out of this calculation.

What is more important, is that we already know two solutions to the equation of motion for the matter sector (5.18): they are the matter part of identity state

$|I\rangle$ and sliver state $|\Xi\rangle$. Even more important is the fact that the sliver state *is* the D25–brane, the unstable vacuum of open strings, as shown by Okawa [107]. Roughly speaking what we do is to 'build' on the not yet found *closed string vacuum* a state that is the old *perturbative vacuum.* Furthermore, from the sliver we can construct lump solutions of arbitrary co-dimension with the correct ratios of tensions of lower dimensional Dp–branes. The precise identification of $|\Xi_m\rangle$ with the D25–brane was done by Okawa using the surface state presentation (CFT) of wedge states that we gave in the previous chapter. We will not give in this thesis the complete Okawa's computation for lack of space. What we will treat exhaustively is the description of the sliver state through the so called 'operator formalism' first proposed by Kostelecky and Potting [70]. In the operator formalism, computations can be done explicitly and algebraically. This formalism involves, however, infinite dimensional matrices and their determinants, often making it necessary to rely on numerical test using level truncation. On the other hand, the CFT formalism gives a geometrical picture to various aspects of VSFT and often analytical computations are possible, but techniques are more abstract.

In the next section we will present the solution to $\Psi_m *^m \Psi_m = \Psi_m$ given by Kostelecky and Potting in the form of $|\Psi_m\rangle$, a *squezeed state*, that means, an exponential of bilinears of the string creation operators acting on the vacuum. Moreover Okuda [90] showed analitically that the squeezed state $|\Psi_m\rangle$ is indeed equal to the sliver $|\Xi\rangle$.

5.2 Squeezed state solutions to $\Psi_m *^m \Psi_m = \Psi_m$

5.2.1 A solution for the D25–brane

The three string vertex [63, 68, 69] of the Open String Field Theory is given by

$$|V_3\rangle = \int d^{26}p_{(1)}d^{26}p_{(2)}d^{26}p_{(3)}\,\delta^{26}(p_{(1)}+p_{(2)}+p_{(3)})\,\exp(-E)\,|0,p\rangle_{123} \qquad (5.22)$$

where

$$E = \sum_{r,s=1}^{3}\left(\frac{1}{2}\sum_{m,n\geq 1}\eta_{\mu\nu}a_m^{(r)\mu\dagger}\,V_{mn}^{rs}\,a_n^{(s)\nu\dagger} + \sum_{n\geq 1}\eta_{\mu\nu}p_{(r)}^{\mu}\,V_{0n}^{rs}\,a_n^{(s)\nu\dagger} + \frac{1}{2}\eta_{\mu\nu}p_{(r)}^{\mu}\,V_{00}^{rs}\,p_{(s)}^{\nu}\right) \qquad (5.23)$$

Summation over the Lorentz indices $\mu,\nu = 0,\ldots,25$ is understood and η denotes the flat Lorentz metric and the operators $a_m^{(r)\mu}, a_m^{(r)\mu\dagger}$ denote the non–zero modes matter oscillators of the r–th string, which satisfy

$$[a_m^{(r)\mu}, a_n^{(s)\nu\dagger}] = \eta^{\mu\nu}\delta_{mn}\delta^{rs}, \qquad m,n \geq 1 \tag{5.24}$$

$p_{(r)}$ is the momentum of the r–th string and $|0,p\rangle_{123} \equiv |p_{(1)}\rangle \otimes |p_{(2)}\rangle \otimes |p_{(3)}\rangle$ is the tensor product of the Fock vacuum states relative to the three strings. $|p_{(r)}\rangle$ is annihilated by the annihilation operators $a_m^{(r)\mu}$ and is eigenstate of the momentum operator $\hat{p}^\mu_{(r)}$ with eigenvalue $p^\mu_{(r)}$. The normalization is

$$\langle p_{(r)}|\, p'_{(s)}\rangle = \delta_{rs}\delta^{26}(p+p')$$

The coefficients V_{MN}^{rs} ($M(N)$ denotes from now on the couple $\{0,m\}$ ($\{0,n\}$)) have been computed in [68, 69]. We will use them in the notation of [73] reported in Appendix B.

Since we are interested in this section in *space-time translational invariant* solutions representing the D25–brane, we choose to ignore the momentum dependent factors in the vertex, and the relevant form of E is:

$$E = \frac{1}{2}\sum_{r,s} \eta_{\mu\nu} a^{(r)\mu\dagger} \cdot V^{rs} \cdot a^{(s)\nu\dagger}, \tag{5.25}$$

where the dots represent sums over mode numbers, and V^{rs}_{mn} for $m,n \geq 1$ is written as the V^{rs} matrix. For the analysis of lumps, however, we will need the full vertex.

Some appreciation of the properties reviewed in Appendix C is useful. Equation (C.15), in particular, gives

$$V^{rs} = \frac{1}{3}(C + \omega^{s-r}U + \omega^{r-s}\bar{U}), \tag{5.26}$$

where $\omega = e^{2\pi i/3}$, U and C are regarded as matrices with indices running over $m,n \geq 1$,

$$C_{mn} = (-1)^m\delta_{mn}, \quad m,n \geq 1, \tag{5.27}$$

and U satisfies (C.17)

$$\bar{U} \equiv U^* = CUC, \qquad U^2 = \bar{U}^2 = 1, \qquad U^\dagger = U, \quad \bar{U}^\dagger = \bar{U}. \tag{5.28}$$

The superscripts r,s are defined mod(3), and (5.26) manifestly implements the cyclicity property $V^{rs} = V^{(r+1)(s+1)}$. Also note the transposition property $(V^{rs})^T = V^{sr}$. Finally, eqs.(5.26), (5.28) allow one to show that

$$[CV^{rs}, CV^{r's'}] = 0 \quad \forall \quad r,s,r',s', \tag{5.29}$$

and

$$(CV^{12})(CV^{21}) = (CV^{21})(CV^{12}) = (CV^{11})^2 - CV^{11}\,,$$
$$(CV^{12})^3 + (CV^{21})^3 = 2(CV^{11})^3 - 3(CV^{11})^2 + 1\,. \tag{5.30}$$

Equations (5.26) up to (5.30) are all that we shall need to know about the matter part of the relevant star product to construct the translationally invariant solution. In fact, since (5.29), (5.30) follow from (5.26) and (5.28), these two equations are really all that is strictly needed. Such structure will reappear in the next section with matrices that also include $m = 0$ and $n = 0$ entries, and thus will guarantee the existence of a solution constructed in the same fashion as the solution to be obtained below.

We are looking for a space-time independent solution of eq.(5.18). The strategy of Kostelecky and Potting [70] is to take a trial solution of the form:

$$|\Psi_m\rangle = \mathcal{N}^{26} \exp\Big(-\frac{1}{2}\eta_{\mu\nu}\sum_{m,n\geq 1} S_{mn}\, a_m^{\mu\dagger} a_n^{\nu\dagger}\Big)|0\rangle\,, \tag{5.31}$$

where $|0\rangle$ is the SL(2,$\mathbb{R}$) invariant vacuum of the matter CFT, $\mathcal{N}$ is a normalization factor, and S_{mn} is an infinite dimensional matrix with indices m, n running from 1 to ∞. We shall take S_{mn} to be twist invariant. Due to this property the BPZ conjugate of the state $|\Psi_m\rangle$ is the same as its hermitian conjugate. Otherwise we need to keep track of extra $-$ signs coming from the fact that the BPZ conjugate of $a_m^\dagger$ is $(-1)^{m+1}a_m$. This is nothing but the requirement that the string state $|\Psi_m\rangle$ be hermitian.

$$CSC = S\,. \tag{5.32}$$

We shall check in the end that the solution constructed below is indeed twist invariant.

If we define

$$\Sigma = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}, \qquad \mathcal{V} = \begin{pmatrix} V^{11} & V^{12} \\ V^{21} & V^{22} \end{pmatrix}, \tag{5.33}$$

and

$$\chi^{\mu T} = \left(a^{(3)\mu\dagger} V^{31}\,,\ \ a^{(3)\mu\dagger} V^{32} \right), \qquad \chi^\mu = \begin{pmatrix} V^{13} a^{(3)\mu\dagger} \\ V^{23} a^{(3)\mu\dagger} \end{pmatrix}, \tag{5.34}$$

then using eqs.(5.22), (5.25) we get

$$|\Psi_m * \Psi_m\rangle_3 = \mathcal{N}^{52} \det\{(1-\Sigma\mathcal{V})^{-1/2}\}^{26} \times \exp\Big[-\frac{1}{2}\eta_{\mu\nu}\{\chi^{\mu T}[(1-\Sigma\mathcal{V})^{-1}\Sigma]\chi^{\nu} + a^{(3)\mu\dagger}\cdot V^{33}\cdot a^{(3)\nu\dagger}\}\Big]|0\rangle_3 \,. \tag{5.35}$$

In deriving eq.(5.35) we have used the general formula [70]

$$\langle 0|\exp\Big(\lambda_i a_i - \frac{1}{2}P_{ij}a_i a_j\Big)\exp\Big(\mu_i a_i^\dagger - \frac{1}{2}Q_{ij}a_i^\dagger a_j^\dagger\Big)|0\rangle = \det(K)^{-1/2}\exp\Big(\mu^T K^{-1}\lambda - \frac{1}{2}\lambda^T Q\, K^{-1}\lambda - \frac{1}{2}\mu^T K^{-1}P\mu\Big)\,, \quad K\equiv 1-PQ\,, \tag{5.36}$$

that is the multidimensional case of

$$\langle 0|\exp(\lambda a + \frac{1}{2}aSa)\exp\mu a^\dagger + \frac{1}{2}a^\dagger V a^\dagger|0\rangle = (1-SV)^{-1/2}\exp\Big[\lambda(1-VS)^{-1}\mu + \frac{1}{2}\lambda(1-VS)^{-1}V\lambda + \frac{1}{2}\mu(1-SV)^{-1}S\mu\Big] \tag{5.37}$$

In using the formula (5.35) we took the a_i to be the list of oscillators $(a_m^{(1)}, a_m^{(2)})$ with $m \geq 1$. (5.35) then follows from (5.36) by identifying P with Σ, Q with $\mathcal{V}$, μ with χ and setting λ to 0.

Demanding that the exponents in the expressions for $|\Psi_m\rangle$ and $|\Psi_m * \Psi_m\rangle$, given in eqs.(5.31) and (5.35) respectively, match, we get

$$S = V^{11} + (V^{12}, V^{21})(1-\Sigma\mathcal{V})^{-1}\Sigma\begin{pmatrix} V^{21} \\ V^{12}\end{pmatrix}\,, \tag{5.38}$$

where we have used the cyclicity property of the V matrices and the mod 3 periodicity of the indices r and s to write the equation in a convenient form. To proceed, we assume that

$$[CS, CV^{rs}] = 0 \quad \forall \quad r, s\,. \tag{5.39}$$

We shall check later that the solution obeys these conditions. We can now write eq.(5.38) in terms of

$$T \equiv CS = SC, \qquad M^{rs} \equiv CV^{rs}\,, \tag{5.40}$$

and because of (5.29), (5.39) we can manipulate the equation as if T and M^{rs} are numbers rather than infinite dimensional matrices. We first multiply (5.38) by C and write it as:

$$T = X + (M^{12}, M^{21})(1-\Sigma\mathcal{V})^{-1}\begin{pmatrix} TM^{21} \\ TM^{12}\end{pmatrix}\,, \tag{5.41}$$

where

$$X = M^{11} = CV^{11}\,. \tag{5.42}$$

We then note that since the submatrices commute:

$$\begin{aligned}(1-\Sigma\mathcal{V})^{-1} &= \begin{pmatrix} 1-TX & -TM^{12} \\ -TM^{21} & 1-TX \end{pmatrix}^{-1} \\ &= ((1-TX)^2 - T^2M^{12}M^{21})^{-1}\begin{pmatrix} 1-TX & TM^{12} \\ TM^{21} & 1-TX \end{pmatrix}\,. \end{aligned} \tag{5.43}$$

Finally, we record that

$$\det(1-\Sigma\mathcal{V}) = \det(1-2TX+T^2X)\,, \tag{5.44}$$

where use was made the first equation in (5.30) reading $M^{12}M^{21} = X^2 - X$.

It is now a simple matter to substitute (5.43) into (5.41) and expand out eliminating all reference of M^{12} and M^{21} in favor of X by use of eqs.(5.30). The result is the condition:

$$(T-1)(XT^2 - (1+X)T + X) = 0\,. \tag{5.45}$$

This gives the solution for S:[1]

$$S = CT, \qquad T = \frac{1}{2X}(1+X-\sqrt{(1+3X)(1-X)})\,. \tag{5.46}$$

We can now verify that S obtained this way satisfies equations (5.32) and (5.39). Indeed, since CS is a function of X, and since $X(\equiv CV^{11})$ commutes with CV^{rs}, CS also commutes with CV^{rs}. Furthermore, since V^{11} is twist invariant, so is X. It then follows that the inverse of X and any polynomial in X are twist invariant. Therefore T is twist invariant, and, as desired, S is twist invariant.

Demanding that the normalization factors in $|\Psi_m\rangle$ and $|\Psi_m * \Psi_m\rangle$ match gives

$$\mathcal{N} = \det(1-\Sigma\mathcal{V})^{1/2} = (\det(1-X)\det(1+T))^{1/2}\,, \tag{5.47}$$

where we have used eqn.(5.44) and simplified it further using (5.45). Thus the solution is given by

$$|\Psi_m\rangle = \{\det(1-X)^{1/2}\det(1+T)^{1/2}\}^{26}\exp\Big(-\frac{1}{2}\eta_{\mu\nu}\sum_{m,n\geq 1} S_{mn}\, a_m^{\mu\dagger}a_n^{\nu\dagger}\Big)|0\rangle\,. \tag{5.48}$$

[1] Of the two other solutions, $T=1$ gives the identity state $|I_m\rangle$, whereas the third solution has diverging eigenvalues and hence is badly behaved.

This is the matter part of the state found in [70] after suitable correction to the normalization factor. From eq.(5.20) we see that the value of the action associated with this solution has the form:

$$\mathcal{S}|_\Psi = K\mathcal{N}^{52}\langle 0|\exp\Big(-\frac{1}{2}\eta_{\mu'\nu'}\sum_{m',n'\geq 1} S_{m'n'}a^{\mu'}_{m'}a^{\nu'}_{n'}\Big)\exp\Big(-\frac{1}{2}\eta_{\mu\nu}\sum_{m,n\geq 1} S_{mn}a^{\mu\dagger}_{m}a^{\nu\dagger}_{n}\Big)|0\rangle\,.$$

By evaluating the matrix element using eq.(5.36), and using the normalization:

$$\langle 0|0\rangle = \delta^{(26)}(0) = \frac{V^{(26)}}{(2\pi)^{26}}\,, \tag{5.49}$$

where $V^{(26)}$ is the volume of the 26-dimensional space-time, we get the value of the action to be

$$\begin{aligned}\mathcal{S}|_\Psi &= K\frac{V^{(26)}}{(2\pi)^{26}}\mathcal{N}^{52}\{\det(1-S^2)^{-1/2}\}^{26}\\ &= K\frac{V^{(26)}}{(2\pi)^{26}}\{\det(1-X)^{3/4}\det(1+3X)^{1/4}\}^{26}\,.\end{aligned} \tag{5.50}$$

In arriving at the right hand side of eq.(5.50) we have made use of eqs.(5.46) and (5.47). Thus the tension of the D25–brane is given by

$$\tau_{25} = K\frac{1}{(2\pi)^{26}}\{\det(1-X)^{3/4}\det(1+3X)^{1/4}\}^{26}\,. \tag{5.51}$$

5.2.2 Lump solutions: lower dimensional branes

We begin by noting that the solution (5.48) representing the D25–brane has the form of a product over 26 factors, each involving the oscillators associated with a given direction. This suggests that in order to construct a solution of codimension k representing a D$(25-k)$-brane, we need to replace k of the factors associated with directions transverse to the D-brane by a different set of solutions, but the factors associated with directions tangential to the D-brane remains the same. Suppose we are interested in a D$(25-k)$-brane solution. Let us denote by $x^{\bar\mu}$ $(0\leq\bar\mu\leq(25-k))$ the directions tangential to the brane and by x^α $((26-k)\leq\alpha\leq 25)$ the directions transverse to the brane. We now use the representation of the vertex in the zero mode oscillator basis for the directions x^α, as given in Appendix C. For this we define, for each string,

$$a^\alpha_0 = \frac{1}{2}\sqrt{b}\,\hat p^\alpha - \frac{1}{\sqrt{b}}i\hat x^\alpha, \qquad a^{\alpha\dagger}_0 = \frac{1}{2}\sqrt{b}\,\hat p^\alpha + \frac{1}{\sqrt{b}}i\hat x^\alpha\,, \tag{5.52}$$

where b is an arbitrary constant dimensioned as a length, and $\hat{x}^\alpha$ and $\hat{p}^\alpha$ are the zero mode coordinate and momentum operators associated with the direction x^α. We also denote by $|\Omega_b\rangle$ the normalized state which is annihilated by all the annihilation operators a_0^α, and by $|\Omega_b\rangle_{123}$ the direct product of the vacuum $|\Omega_b\rangle$ for each of the three strings.

The relation between the momentum basis and the new oscillator basis is given by (for each string)

$$|\{p^\alpha\}\rangle = (2\pi/b)^{-k/4} \exp\Big[-\frac{b}{4}p^\alpha p^\alpha + \sqrt{b}a_0^{\alpha\dagger}p^\alpha - \frac{1}{2}a_0^{\alpha\dagger}a_0^{\alpha\dagger}\Big]|\Omega_b\rangle\,. \tag{5.53}$$

In the above equation $\{p^\alpha\}$ label momentum eigenvalues. Substituting eq.(5.53) into eq.(5.23), and integrating over $p^\alpha_{(i)}$, we can express the three string vertex as

$$\begin{aligned} |V_3\rangle \;=\; & \int d^{26-k}p_{(1)}d^{26-k}p_{(2)}d^{26-k}p_{(3)}\delta^{(26-k)}(p_{(1)}+p_{(2)}+p_{(3)}) \\ & \exp\Big(-\frac{1}{2}\sum_{\substack{r,s\\ m,n\geq 1}}\eta_{\bar{\mu}\bar{\nu}}a_m^{(r)\bar{\mu}\dagger}V^{rs}_{mn}a_n^{(s)\bar{\nu}\dagger} - \sum_{\substack{r,s\\ n\geq 1}}\eta_{\bar{\mu}\bar{\nu}}p^{\bar{\mu}}_{(r)}V^{rs}_{0n}a_n^{(s)\bar{\nu}\dagger} - \frac{1}{2}\sum_r \eta_{\bar{\mu}\bar{\nu}}p^{\bar{\mu}}_{(r)}V^{rr}_{00}p^{\bar{\nu}}_{(r)}\Big)|0,p\rangle_{123} \\ & \otimes\Big(\frac{\sqrt{3}}{(2\pi b^3)^{1/4}}(V^{rr}_{00}+\frac{b}{2})\Big)^{-k}\exp\Big(-\frac{1}{2}\sum_{\substack{r,s\\ M,N\geq 0}} a_M^{(r)\alpha\dagger}V'^{rs}_{MN}a_N^{(s)\alpha\dagger}\Big)|\Omega_b\rangle_{123}\,. \end{aligned} \tag{5.54}$$

In this expression the sums over $\bar{\mu},\bar{\nu}$ run from 0 to $(25-k)$, and sum over α runs from $(26-k)$ to 25. Note that in the last line the sums over M, N run from 0 to ∞. The coefficients V'^{rs}_{MN} have been given in terms of V^{rs}_{mn} in Appendix in C eq.(C.7).

In Appendix C it is shown that V'^{rs}, regarded as matrices with indices running from 0 to ∞, satisfy

$$V'^{rs} = \frac{1}{3}(C' + \omega^{s-r}U' + \omega^{r-s}\bar{U}')\,, \tag{5.55}$$

where we have dropped the explicit b dependence from the notation, $C'_{MN} = (-1)^M\delta_{MN}$ with indices M,N now running from 0 to ∞, and U', $\bar{U}'\equiv U'^*$ viewed as matrices with $M,N\geq 0$ satisfy the relations:

$$\bar{U}' = C'U'C', \qquad U'^2 = \bar{U}'^2 = 1, \qquad U'^\dagger = U'\,. \tag{5.56}$$

We note now the complete analogy with equations (5.26) and (5.28) [70]. It follows also that the V' matrices, together with C' will satisfy equations exactly analogous to (5.29), (5.30). Thus we can construct a solution of the equations of motion

(5.18) in an identical manner with the unprimed quantities replaced by the primed quantities. Taking into account the extra normalization factor appearing in the last line of eq.(5.54), we get the following form of the solution of eq.(5.18):

$$\begin{aligned} |\Psi'_m\rangle &= \{\det(1-X)^{1/2}\det(1+T)^{1/2}\}^{26-k}\exp(-\frac{1}{2}\eta_{\bar{\mu}\bar{\nu}}\sum_{m,n\geq 1} S_{mn}a_m^{\bar{\mu}\dagger}a_n^{\bar{\nu}\dagger})|0\rangle \\ &\otimes\Big(\frac{\sqrt{3}}{(2\pi b^3)^{1/4}}(V_{00}^{rr}+\frac{b}{2})\Big)^k\{\det(1-X')^{1/2}\det(1+T')^{1/2}\}^k \\ &\exp\Big(-\frac{1}{2}\sum_{M,N\geq 0} S'_{MN}a_M^{\alpha\dagger}a_N^{\alpha\dagger}\Big)|\Omega_b\rangle\,, \end{aligned} \tag{5.57}$$

where

$$S' = C'T'\,, \qquad T' = \frac{1}{2X'}(1+X'-\sqrt{(1+3X')(1-X')})\,, \tag{5.58}$$

$$X' = C'V'^{11}\,. \tag{5.59}$$

Using eq.(5.20) we can calculate the value of the action associated with this solution. It is given by an equation analogous to (5.50):

$$\begin{aligned} \mathcal{S}_{\Psi'} &= K\frac{V^{(26-k)}}{(2\pi)^{26-k}}\{\det(1-X)^{3/4}\det(1+3X)^{1/4}\}^{26-k} \\ &\times\Big(\frac{3}{(2\pi b^3)^{1/2}}(V_{00}^{rr}+\frac{b}{2})^2\Big)^k\{\det(1-X')^{3/4}\det(1+3X')^{1/4}\}^k\,, \end{aligned} \tag{5.60}$$

where $V^{(26-k)}$ is the D$(25-k)$–brane world-volume. This gives the tension of the D$(25-k)$–brane to be

$$\begin{aligned} \tau_{25-k} &= K\frac{1}{(2\pi)^{26-k}}\{\det(1-X)^{3/4}\det(1+3X)^{1/4}\}^{26-k} \\ &\times\Big(\frac{3}{(2\pi b^3)^{1/2}}(V_{00}^{rr}+\frac{b}{2})^2\Big)^k\{\det(1-X')^{3/4}\det(1+3X')^{1/4}\}^k\,. \end{aligned} \tag{5.61}$$

Clearly for $k=0$ this agrees with (5.51). From eq.(5.61) we get

$$\frac{\tau_{24-k}}{2\pi\tau_{25-k}} = \frac{3}{\sqrt{2\pi b^3}}\Big(V_{00}^{rr}+\frac{b}{2}\Big)^2\frac{\{\det(1-X')^{3/4}\det(1+3X')^{1/4}\}}{\{\det(1-X)^{3/4}\det(1+3X)^{1/4}\}} \equiv R\,. \tag{5.62}$$

Okuyama [88] has been able to prove that $R=1$; therefore VSFT describes the correct ratios of Dp–brane:

$$\frac{\tau_{24-k}}{2\pi\tau_{25-k}} = 1\,.$$

We will not present his exact calculation. What we will present is, instead, the modified calculation in the presence of a B field as given in [109]. The detailed calculations are given in Appendix E.1. It is then a quite straightforward exercise to obtain Okuyama's result by putting $B = 0$.

We end this section by noticing that there exist also a consistent boundary CFT formulation of lower dimensional branes [62]. For lower-dimensional Dp–branes, the Neumann sliver state $|\Xi\rangle$ is replaced by the sliver state with the Dirichlet boundary condition for each of the $25 - p$ transverse directions. It is similarly defined as in Section 4.2 by a limit of wedge states with the following boundary condition for the string coordinate $X^i(z)$ on the real axis t of the upper-half plane:

$$\partial X^i(t) = \bar{\partial} X^i(t) \qquad \text{for} \quad -\frac{n}{2}\tan\frac{\pi}{2n} \leq t \leq \frac{n}{2}\tan\frac{\pi}{2n},$$

$$X^i(t) = a^i \qquad \text{for} \quad t < -\frac{n}{2}\tan\frac{\pi}{2n}, \quad t > \frac{n}{2}\tan\frac{\pi}{2n}, \tag{5.63}$$

where a^i is the position of the D–brane in space-time. It is shown in [62] that ratios of tensions of various D–branes are correctly reproduced using this description.

5.2.3 Open String states and the D25–brane tension

In the previous section we saw that the ratios of tensions of D–branes can be reproduced, but we did not say anything about the tension of a *single* D25–brane. Another important problem is that we do not completely understand how to describe open string states around D–branes in the framework of VSFT. In [107] Okawa, following a path opened by Hata and Kawano [76], and precised by Rastelli, Sen and Zwiebach [81], provided a resolution to these two problems. The language he used is that one of Boundary CFT that we introduced in Section 4.2 to describe the surface states. A complete treatment of this technique goes beyond the purposes of this thesis. Nevertheless we will try to give to the reader an insight of this formulation of VSFT since it is the only presentation where, up to now, to two above mentioned problems has been solved. Actually, these two problems are closely related. Since the D25–brane tension τ_{25} is related to the on-shell three-tachyon coupling g_T through the relation [55]

$$\tau_{25} = \frac{1}{2\pi^2\alpha'^3 g_T^2}\,, \tag{5.64}$$

the energy density $\mathcal{E}_c$ of the classical solution corresponding to a single D25–brane must satisfy

$$\frac{\mathcal{E}_c}{\tau_{25}} = 2\pi^2\alpha'^3 g_T^2\, \mathcal{E}_c = 1\,. \tag{5.65}$$

However, the on-shell three-tachyon coupling g_T based on the earlier proposal for the tachyon state by Hata and Kawano [76], failed to reproduce the relation (5.64) [76, 89], and the ratio $\mathcal{E}_c/\tau_{25}$ turned out to be [81]

$$\frac{\mathcal{E}_c}{\tau_{25}} = \frac{\pi^2}{3}\left(\frac{16}{27\ln 2}\right)^3 \simeq 2.0558. \tag{5.66}$$

This was regarded as the most crucial problem with the earlier proposal for the tachyon state [76]. The proposal of [76] was to build a state $|\phi_t\rangle$ defined as

$$|\phi_t\rangle = \exp\Big(\sum_{n\geq 1} a_n^\dagger \cdot t_n \cdot p\Big)|\Xi\rangle \tag{5.67}$$

where p is the center-of-mass momentum of the string, and the product over space-time indices is understood. For this state the linearized field equations around the D25–background lead to the correct on–shell condition for the tachyon [76]. In [76] also a proposal for a state $|\phi_V\rangle$ representing the massless vector state on the perturbative vacuum was given:

$$|\phi_V\rangle = \Big(\sum_{n=1,3,5,\dots} d_n^\mu a_n^{\mu\dagger}\Big)|\phi_t\rangle \tag{5.68}$$

Unfortunately, the algebraic definition of such a states, since they involve infinite dimensional determinants, can lead only to numerical approximations of the values of the brane tensions. Rastelli, Sen and Zwiebach [81] maneged to re-express the state $|\phi_t\rangle$ in terms of boundary CFT, proving the relation (5.66). The CFT description of $|\phi_t\rangle$ was found to be the sliver state with the tachyon vertex operator e^{ikX} inserted at the midpoint of the boundary of the sliver state. The approach of Okawa for finding the correct boundary CFT formulation of the tachyon state is the following one. The massless scalar fields on a D-brane describing its fluctuation in the transverse directions are Goldstone modes associated with the broken translational symmetries, and should be identified with infinitesimal deformations of the collective coordinates. Since the collective coordinates are encoded as Dirichlet boundary conditions $X^i = a^i$ in (5.63), the massless scalar fields must be identified with infinitesimal deformations of the boundary condition. It is well-known in the open string sigma model that such a deformation is realized by an insertion of an integral of the vertex operator $\partial_\perp X^i e^{ikX}$ of the scalar field, where $\partial_\perp$ is the derivative normal to the boundary. Therefore, the massless scalar field should be described by the sliver state where the integral of the vertex operator is inserted along the boundary with the Dirichlet boundary condition. This identification of the scalar fields is generalized to other open string states using the relation between the deformation of the boundary condition and the insertion of an integrated vertex operator. Let us see how this approach works for the tachyon state. The tachyon field $T(k)$ on a D25–brane represented by the sliver state $|\Xi\rangle$ should be described at the linear order of $T(k)$ as follows:

$$|\Psi\rangle = |\Xi\rangle - \int d^{26}k\ T(k)|\chi_T(k)\rangle, \tag{5.69}$$

where $|\chi_T(k)\rangle$ is defined for any state in the matter Fock space $|\phi\rangle$ by

$$\langle\chi_T(k)|\phi\rangle = \lim_{n\to\infty} \mathcal{N} \left\langle \int dt\ e^{ikX(t)} h_n \circ \phi(0) \right\rangle_{\Sigma_n} . \tag{5.70}$$

Here a combination of a conformal transformation $h_n(\xi)$ and Riemann surface Σ_n should be conformally equivalent to that of $f_n(\xi)$ and the upper-half plane, and the integral of the tachyon vertex operator $e^{ikX(t)}$ is taken along the boundary of the wedge state from $h_n(1)$ to $h_n(-1)$ along the Dirichlet conditions.

Using this kind of states Okawa has been able to prove in [107] that the matter sliver $|\Xi\rangle$ has indeed the right tension of a D25–brane, and to check that mass of the state $|\chi_T(k)\rangle$ is that one of the tachyon.

5.3 Multiple D–branes

In this section we show, following [74], how it is possible to construct solutions to the projection equation (5.17) representing multiple D–branes. These kind of solutions will be particular important because will be the 'building blocks' of that particular solutions that will reduce to the GMS solitons in the low energy limit.

It is useful to collect a few properties of the matrices M and X:

$$\begin{aligned}
&X + M^{12} + M^{21} = 1\,, \\
&M^{12}M^{21} = X^2 - X\,, \\
&(M^{12})^2 + (M^{21})^2 = 1 - X^2\,, \\
&(M^{12})^3 + (M^{21})^3 = 2X^3 - 3X^2 + 1 = (1-X)^2(1+2X)\,,
\end{aligned} \tag{5.71}$$

that gives

$$(M^{12} - M^{21})^2 = (1-X)(1+3X)\,. \tag{5.72}$$

Besides we have the following additional relations between T and X following from the form of the solution and the equation that relates them (5.46):

$$\frac{1-TX}{1-X} = \frac{1}{1-T}, \quad \frac{1-T}{1+T} = \sqrt{\frac{1-X}{1+3X}}, \quad \frac{X}{1-X} = \frac{T}{(1-T)^2}. \tag{5.73}$$

We construct now a pair of projectors that will be used in the definition of the multiple D-branes states. We define the matrices

$$\begin{aligned} \rho_1 &= \frac{1}{(1+T)(1-X)}\Big[M^{12}(1-TX) + T(M^{21})^2\Big], \\ \rho_2 &= \frac{1}{(1+T)(1-X)}\Big[M^{21}(1-TX) + T(M^{12})^2\Big], \end{aligned} \tag{5.74}$$

That satisfy the following properties:

$$\rho_1^T = \rho_1, \quad \rho_2^T = \rho_2, \quad C\rho_1 C = \rho_2, \tag{5.75}$$

and more importantly

$$\begin{aligned} \rho_1 + \rho_2 &= 1, \\ \rho_1 - \rho_2 &= \frac{M^{12} - M^{21}}{\sqrt{(1-X)(1+3X)}}. \end{aligned} \tag{5.76}$$

From eq.(5.72) we see that the square of the second right hand side is the unit matrix. Thus $(\rho_1 - \rho_2)^2 = 1$, and this together with the squared version of the first equation gives

$$\rho_1 \rho_2 = 0. \tag{5.77}$$

This equation is also easily verified directly. Multiplying the first equation in (5.76) by ρ_1 and alternatively by ρ_2 we get

$$\rho_1 \rho_1 = \rho_1, \quad \rho_2 \rho_2 = \rho_2. \tag{5.78}$$

This shows that ρ_1 and ρ_2 are projection operators into orthogonal subspaces, and the C exchanges these two subspaces.

5.3.1 Computation of $*$ products of states

As seen earlier, the matter part of the sliver state is given by

$$|\Xi\rangle = \mathcal{N}^{26} \exp\Big(-\frac{1}{2} a^\dagger \cdot S \cdot a^\dagger\Big)|0\rangle. \tag{5.79}$$

Coherent states are defined by letting exponentials of the creation operator act on the vacuum. Treating the sliver as the vacuum we introduce coherent like states of the form

$$|\Xi_\beta\rangle = \exp\Big(\sum_{n=1}^{\infty}(-)^{n+1}\beta_{\mu n}a_n^{\mu\dagger}\Big)|\Xi\rangle = \exp(-a^\dagger\cdot C\beta)|\Xi\rangle\,. \tag{5.80}$$

As built, the states satisfy a simple BPZ conjugation property:

$$\langle\Xi_\beta| = \langle\Xi|\exp\Big(\sum_{n=1}^{\infty}\beta_{n\mu}a_n^{\mu}\Big) = \langle\Xi|\exp(\beta\cdot a)\,. \tag{5.81}$$

We compute the $*$ product of two such states using the procedure discussed in refs.[70, 73]. We begin by writing out the product using two by two matrices encoding the oscillators of strings one and two:

$$\begin{aligned}
\Big(|\Xi_{\beta_1}\rangle * |\Xi_{\beta_2}\rangle\Big)_{(3)} &= \Big(\exp(-a^\dagger\cdot C\beta_1)|\Xi\rangle * \exp(-a^\dagger\cdot C\beta_2)|\Xi\rangle\Big)_{(3)} \\
&= {}_{(1)}\langle\Xi|\exp(\beta_1\cdot a_{(1)})\;{}_{(2)}\langle\Xi|\exp(\beta_2\cdot a_{(2)})|V_{123}\rangle \\
&= \langle 0_{12}|\exp\Big(\beta\cdot a - \frac{1}{2}a\cdot\Sigma\cdot a\Big)\exp\Big(-\frac{1}{2}a^\dagger\cdot\mathcal{V}\cdot a^\dagger - \chi^T\cdot a^\dagger\Big)|0_{12}\rangle \\
&\quad\cdot\exp\Big(-\frac{1}{2}a^\dagger_{(3)}\cdot V^{11}\cdot a^\dagger_{(3)}\Big)|0_3\rangle\,,
\end{aligned} \tag{5.82}$$

where $a = (a_{(1)}, a_{(2)})$, and

$$\begin{aligned}
\Sigma &= \begin{pmatrix} S & 0 \\ 0 & S\end{pmatrix}, & \mathcal{V} &= \begin{pmatrix} V^{11} & V^{12} \\ V^{21} & V^{22}\end{pmatrix}, \\
\beta &= \begin{pmatrix}\beta_1 \\ \beta_2\end{pmatrix}, & \chi^T &= (a^\dagger_{(3)}V^{12}\,,\; a^\dagger_{(3)}V^{21})\,.
\end{aligned} \tag{5.83}$$

Explicit evaluation continues by using the equation

$$\begin{aligned}
&\langle 0|\exp\Big(\beta_i a_i - \frac{1}{2}P_{ij}a_i a_j\Big)\exp\Big(-\chi_i a_i^\dagger - \frac{1}{2}Q_{ij}a_i^\dagger a_j^\dagger\Big)|0\rangle \\
&= \det(K)^{-1/2}\exp\Big(-\chi^T K^{-1}\beta - \frac{1}{2}\beta^T Q\,K^{-1}\beta - \frac{1}{2}\chi^T K^{-1}P\chi\Big)\,,\quad K\equiv 1-PQ\,.
\end{aligned} \tag{5.84}$$

At this time we realize that since $|\Xi\rangle * |\Xi\rangle = |\Xi\rangle$ the result of the product is a sliver with exponentials acting on it; the exponentials that contain β. This gives

$$|\Xi_{\beta_1}\rangle * |\Xi_{\beta_2}\rangle = \exp\Big(-\chi^T\mathcal{K}^{-1}\beta - \frac{1}{2}\beta^T\mathcal{V}\mathcal{K}^{-1}\beta\Big)|\Xi\rangle\,,\quad \mathcal{K} = (1-\Sigma\mathcal{V})\,. \tag{5.85}$$

The expression for $\mathcal{K}^{-1}$, needed above is simple to obtain given that all the relevant submatrices commute. One finds that

$$\mathcal{K}^{-1} = (1-\Sigma\mathcal{V})^{-1} = \frac{1}{(1+T)(1-X)} \begin{pmatrix} 1-TX & TM^{12} \\ TM^{21} & 1-TX \end{pmatrix} . \tag{5.86}$$

We now recognize that the projectors ρ_1 and ρ_2 defined in (5.74) make an appearance in the oscillator term of (5.85)

$$\begin{aligned} -\chi^T\mathcal{K}^{-1}\beta &= -a^\dagger \cdot C\,(M^{12}, M^{21})\mathcal{K}^{-1}\beta = -a^\dagger \cdot C\,(\rho_1,\rho_2)\beta \\ &= -a^\dagger C\,\cdot(\rho_1\,\beta_1 + \rho_2\,\beta_2)\,. \end{aligned} \tag{5.87}$$

One can verify that

$$\begin{aligned} \mathcal{C}(\beta_1,\beta_2) &\equiv \frac{1}{2}\,(\beta_1,\beta_2)\,\mathcal{V}\mathcal{K}^{-1}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} \\ &= \frac{1}{2}\,(\beta_1,\beta_2)\frac{1}{(1+T)(1-X)}\begin{pmatrix} V^{11}(1-T) & V^{12} \\ V^{21} & V^{11}(1-T)\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} . \end{aligned} \tag{5.88}$$

Since the matrix in between is symmetric we have

$$\mathcal{C}(\beta_1,\beta_2) = \mathcal{C}(\beta_2,\beta_1)\,. \tag{5.89}$$

Using (5.87) and (5.88) we finally have:

$$|\Xi_{\beta_1}\rangle * |\Xi_{\beta_2}\rangle = \exp\Big(-\mathcal{C}(\beta_1,\beta_2)\Big)|\Xi_{\rho_1\beta_1+\rho_2\beta_2}\rangle\,. \tag{5.90}$$

This is a useful relation that allows one to compute $*$-products of slivers acted by oscillators by simple differentiation. In particular, using eq.(5.80) we get

$$\begin{aligned} (a^{\mu_1\dagger}_{m_1}\cdots a^{\mu_k\dagger}_{m_k}|\Xi\rangle) * (a^{\nu_1\dagger}_{n_1}\cdots a^{\nu_l\dagger}_{n_l}|\Xi\rangle) &= (-1)^{\sum_{i=1}^k(m_i+1)+\sum_{j=1}^l(n_j+1)} \\ & \Big(\frac{\partial}{\partial\beta_{1m_1\mu_1}}\cdots\frac{\partial}{\partial\beta_{1m_k\mu_k}}\frac{\partial}{\partial\beta_{2n_1\nu_1}}\cdots\frac{\partial}{\partial\beta_{2n_l\nu_l}}(|\Xi_{\beta_1}\rangle * |\Xi_{\beta_2}\rangle)\Big)_{\beta_1=\beta_2=0} . \end{aligned} \tag{5.91}$$

Since $\rho_1+\rho_2=1$, for $\beta_1=\beta_2$ eq.(5.90) reduces to

$$|\Xi_\beta\rangle * |\Xi_\beta\rangle = \exp\Big(-\mathcal{C}(\beta,\beta)\Big)|\Xi_\beta\rangle\,. \tag{5.92}$$

Using the definition of $\mathcal{C}$ in (5.88) one can show that $\mathcal{C}(\beta,\beta)$ simplifies down to

$$\mathcal{C}(\beta,\beta) = \frac{1}{2}\,\beta C(1-T)^{-1}\beta\,. \tag{5.93}$$

It follows from (5.92) that by adjusting the normalization of the Ξ_β state

$$P_\beta \equiv \exp\Big(\mathcal{C}(\beta,\beta)\Big)|\Xi_\beta\rangle\,, \tag{5.94}$$

we obtain projectors

$$P_\beta * P_\beta = P_\beta\,. \tag{5.95}$$

Using eq.(5.84) one can also check that

$$\langle P_\beta|P_\beta\rangle = \langle\Xi|\Xi\rangle\,. \tag{5.96}$$

5.3.2 Projection operators

We are now ready to define the multiple D-branes states. We start with the following ansatz:

$$|P\rangle = \Big(-\xi\cdot a^\dagger\zeta\cdot a^\dagger + k\Big)|\Xi\rangle \tag{5.97}$$

where $\zeta = C\xi$ in order to guarantee the Hermiticity of $|P\rangle$. ξ is constrained to satisfy

$$\rho_1\xi = 0\,,\quad \rho_2\xi = \xi\,, \tag{5.98}$$

Since $C\rho_1 C = \rho_2$ we have also

$$\rho_2\zeta = 0\,,\quad \rho_2\zeta = \zeta\,. \tag{5.99}$$

We want now to show that the state $|P\rangle$ satisfies the following properties

1. $|P\rangle * |\Xi\rangle = 0$
2. $|P\rangle * |P\rangle = |P\rangle$
3. $\langle P|P\rangle = \langle\Xi|\Xi\rangle$

The first property will be used to fix k, the second will normalize ξ, and the third will follow automatically from the first two.

1) The product $|P\rangle * |\Xi\rangle$ requires multiplying a sliver times a sliver with two oscillators acting on it. As we saw in the previous section one can obtain this result

by applying the differential operator $\frac{\partial^2}{\partial\beta_{1m\mu}\partial\beta_{1n\nu}}$ on both sides of eq.(5.90), and then setting β_1 and β_2 to zero. One finds

$$|P * \Xi\rangle = |\Xi * P\rangle = (k + \xi^T(\mathcal{V}\mathcal{K}^{-1})_{11}\zeta)|\Xi\rangle\,, \tag{5.100}$$

Since we require $|P\rangle * |\Xi\rangle = 0$, we need to set

$$k = -\xi^T(\mathcal{V}\mathcal{K}^{-1})_{11}\widetilde{\xi} = -\xi^T T(1-T^2)^{-1}\xi\,, \tag{5.101}$$

where the last equation in (5.73) was used to simplify the expression for $(\mathcal{V}\mathcal{K}^{-1})_{11}$.

2) We calculate $|P\rangle * |P\rangle$ by differentiating both sides of eq.(5.90) with respect to $\beta_{1m\mu}$ and $\beta_{2m\mu}$ appropriate number of times, and then setting β_1 and β_2 to zero. We get the result for the $*$ product to be:

$$\begin{aligned}|P\rangle * |P\rangle \;&=\; -(\xi^T(\mathcal{V}\mathcal{K}^{-1})_{12}\zeta)\,\xi\cdot a^\dagger\,\zeta\cdot a^\dagger|\Xi\rangle \\ &\quad+\Big((\xi^T(\mathcal{V}\mathcal{K}^{-1})_{11}\zeta)(\xi^T(\mathcal{V}\mathcal{K}^{-1})_{22}\zeta) + (\xi^T(\mathcal{V}\mathcal{K}^{-1})_{12}\zeta)(\zeta^T(\mathcal{V}\mathcal{K}^{-1})_{12}\xi) - \kappa^2\Big)|\Xi\rangle\,.\end{aligned} \tag{5.102}$$

Using (5.88), the last equation in (5.73), and (5.101) one finds that

$$\zeta^T(\mathcal{V}\mathcal{K}^{-1})_{12}\xi = -\xi^T T(1-T^2)^{-1}\xi = k\,. \tag{5.103}$$

Furthermore $(\mathcal{V}\mathcal{K}^{-1})_{11} = (\mathcal{V}\mathcal{K}^{-1})_{22}$. Using this and eqs.(5.101), (5.103), we see that eq.(5.102) can be written as

$$|P\rangle * |P\rangle = (\xi^T(\mathcal{V}\mathcal{K}^{-1})_{12}\zeta)\Big(-\xi\cdot a^\dagger\,\zeta\cdot a^\dagger + k\Big)|\Xi\rangle\,. \tag{5.104}$$

If we now normalize ξ such that

$$\xi^T\,(\mathcal{V}\mathcal{K}^{-1})_{12}\,\zeta = 1\,, \tag{5.105}$$

then eq.(5.104) reduces to the desired equation:

$$|P\rangle * |P\rangle = |P\rangle\,. \tag{5.106}$$

The normalization condition eq.(5.105) can be simplified using the first equation in (5.73) to obtain:

$$\xi^T(1-T^2)^{-1}\xi = 1\,. \tag{5.107}$$

3) We shall now show that the new solution $|P\rangle$ also represents a single D25–brane. For this we shall calculate the tension associated with this solution and try

to verify that it agrees with the tension of the brane described by the sliver. Since the tension of the brane associated to a given state is proportional to the BPZ norm of the state [73], all we need to show is that $\langle P|P\rangle$ is equal to $\langle\Xi|\Xi\rangle$. This is a straightforward calculation using the formula (5.36), that, rewritten for the present purposes is

$$\begin{aligned} &\langle 0|\exp\Big(-\frac{1}{2}a\cdot Sa+\lambda\cdot a\Big)\exp\Big(-\frac{1}{2}a\cdot Sa^{\dagger}+\beta\cdot a\Big)|0\rangle \qquad (5.108)\\ =\;& \det(1-S^2)^{-1}\exp\Big(\beta^T\cdot(1-S^2)^{-1}\cdot\lambda-\frac{1}{2}\beta^T\cdot S(1-S^2)^{-1}\cdot\beta \\ &-\frac{1}{2}\lambda^T\cdot S(1-S^2)^{-1}\cdot\lambda\Big), \end{aligned}$$

One then differentiates both sides of this equation with respect to components of λ and β to calculate the required correlator. The final result is

$$\langle P|P\rangle=\langle\Xi|\Xi\rangle\,. \qquad (5.109)$$

Thus the solution described by $|P\rangle$ has the same tension as the solution described by $|\Xi\rangle$. Similar calculation also yields:

$$\langle\Xi|P\rangle=0\,. \qquad (5.110)$$

The BPZ norm of $|\Xi\rangle+|P\rangle$ is $2\langle\Xi|\Xi\rangle$. This shows that $|\Xi\rangle+|P\rangle$ represents a configuration with twice the tension of a single D25–brane.

Consider now another projector $|P'\rangle$ built just as $|P\rangle$ but using a vector ξ':

$$|P'\rangle=\Big(-\xi'\cdot a^{\dagger}\,\zeta'\cdot a^{\dagger}+k'\Big)|\Xi\rangle\,, \qquad (5.111)$$

with

$$\rho_1\xi'=0, \qquad \rho_2\xi'=\xi'\,, \qquad (5.112)$$

k' given as

$$k'=-\xi'^{T}T(1-T^2)^{-1}\xi'\,, \qquad (5.113)$$

and normalization fixed by

$$\xi'^{T}(1-T^2)^{-1}\xi'=1\,. \qquad (5.114)$$

Thus $|P'\rangle$ is a projector orthogonal to $|P\rangle$. We now want to find the condition under which $|P'\rangle$ projects into a subspace orthogonal to $|P\rangle$ as well, *i.e.* the condition under which $|P\rangle * |P'\rangle$ vanishes. We can compute $|P\rangle * |P'\rangle$ in a manner identical to the one used in computing $|P\rangle * |P\rangle$ and find that it vanishes if:

$$\xi^T(1-T^2)^{-1}\xi' = 0\,. \tag{5.115}$$

Since this equation is symmetric in ξ and ξ', it is clear that $|P'\rangle * |P\rangle$ also vanishes when eq.(5.115) is satisfied. Given eqs.(5.114) and (5.115) we also have:

$$\langle P'|P'\rangle = 1, \quad \langle P|P'\rangle = \langle \Xi|P'\rangle = 0\,. \tag{5.116}$$

Thus $|\Xi\rangle + |P\rangle + |P'\rangle$ describes a solution with three D25–branes. This procedure can be continued indefinitely to generate solutions with arbitrary number of D25–branes.

The same procedure can be applied for the construction of multiple Dp–brane solutions for $p < 25$. Indeed we saw that the matrices M'_{rs}, V'_{rs}, X', C', and T' obey the same properties of the corresponding unprimed matrices. The procedure of the previous section can then be applied *mutatis mutandis.*

Chapter 6

Vacuum String Field Theory with B Field

Witten's star product and the Moyal product show both a noncommutative structure. It is then immediate to ask if there is some connection between the two products. This question has already been partially answered. Sugino [64] and Kawano and Takahashi [65], showed that Cubic String Field Theory is compatible with the introduction of a constant B field. They proved that when a B field is turned on, the kinetic term of the SFT action (3.8) is modified only by changing the closed string metric $g_{\mu\nu}$ with the open one $G_{\mu\nu}$, while the three string vertex changes, besides the substitution $g_{\mu\nu} \to G_{\mu\nu}$, being multiplied by the (cyclically invariant) noncommutative phase factor that we already met in Chapter 1. Then Witten [66] and Schnabl [67] proved that the two products are indeed compatible, since a Moyal structure emerges from Witten's star product in the low energy limit. In this chapter we want to address our investigations to the effects on the nonperturbative structure of SFT of turning on a B field. We will repeat the analysis of the previous chapter with a B field: exact solutions to VSFT equation of motion can be written down for tachyonic lumps much in the same way as one finds analogous solutions without B field. Also wedge-like states and orthogonal projectors will be defined in the presence of a B field. But what is more important is that a B field can be precious tool to regularize some of the several singularities that arise in VSFT. Moreover we saw that solitons solutions, the GMS solitons, otherwise forbidden, are present in scalar field theories when noncommutativity is turned on. We will show that the same structure, mainly given in terms of Laguerre polynomials, arises also in VSFT, giving a new perspective under which looking at the connection between Witten star and Moyal product.

This chapter is organized as follows. First we will write down the SFT vertex in the presence of a constant B field. This result was first found by Sugino [64] and Kawano and Takahashi [65], using the overlap conditions as in [68, 69]. We will give here an alternative derivation of their result, based on the LeClair, Peskin and Preitshopf construction of the vertex $\langle V_3|$. Then we will construct squeezed states solutions with B field, as much as done in the previous chapter. We will show analogies and differences with the $B = 0$ case by constructing wedge-like states and orthogonal projectors, and investigating the behaviour of the string midpoint with $B \neq 0$. Finally we construct a series of orthogonal projectors that in the low energy limit give exactly the GMS solitons.

6.1 String Field Theory with B field

It is useful to recall the form of the propagator and of the string field expansion when the B field is turned on. They are

$$\begin{aligned}\langle X^\mu(z,\bar z)X^\nu(z',\bar z')\rangle &= -\alpha'\Big[g^{\mu\nu}\ln|z-z'| - g^{\mu\nu}\ln|z-\bar z'| \\ &\quad + G^{\mu\nu}\ln|z-\bar z'|^2 + \frac{1}{2\pi\alpha'}\theta^{\mu\nu}\ln\frac{z-\bar z'}{\bar z - z'} + D^{\mu\nu}\Big] \qquad (6.1)\end{aligned}$$

and

$$\begin{aligned}X^\mu &= x_0^\mu + \alpha'\Big[(E^{-1})^{\mu\nu}g_{\nu\rho}p^\rho\ln\bar z + (E^{-1T})^{\mu\nu}g_{\nu\rho}p^\rho\ln z\Big] \\ &\quad + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\Big[(E^{-1})^{\mu\nu}g_{\nu\rho}\alpha_n^\rho\bar z^{-n} + (E^{-1T})^{\mu\nu}g_{\nu\rho}\alpha_n^\rho z^{-n}\Big] \qquad (6.2)\end{aligned}$$

where $E_{\mu\nu} = g_{\mu\nu} + 2\pi\alpha' B_{\mu\nu}$. The two and three string vertices were defined as correlation functions of string fields operators computed on the upper half plane. For instance the three string vertex was

$$\langle A, B * C\rangle = \langle f_1^D\circ\mathcal{O}_A(0)\, f_2^D\circ\mathcal{O}_B(0)\, f_3^D\circ\mathcal{O}_C(0)\rangle_D \qquad (6.3)$$

Inverting eq.(6.2) to obtain α_{-n}^μ as function of $X^\mu(z)$ we find

$$\alpha_{-n}^\mu = \oint\frac{dz}{2\pi i}z^{-n}(E)^{\mu\nu}g_{\nu\rho}\partial_z X^\rho(z) \qquad (6.4)$$

The operators $\mathcal{O}$ are products of polynomials in the creation operators α_{-n}^μ with exponentials $e^{ip\cdot X}$. Under conformal transformations the latter change as

$$f_i[\alpha_{-n}^\mu] = \oint\frac{dz}{2\pi i}z^{-n}(h_1'(z))(E^{-1})^{\mu\nu}g_{\nu\rho}\partial_z X^\rho(f_i(z))$$

$$f_i[e^{ip\cdot X(0)}] = |f_i'(0)|^{p^2/2}\, e^{ip\cdot X(f_i(0))}$$

The contraction of any two α^{μ}_{-n} is then

$$\begin{aligned}
&f_i[\ldots\alpha^{\mu}_{-m}\ldots]f_j[\ldots\alpha^{\nu}_{-n}\ldots]\\
&= \oint\frac{dz}{2\pi i}z^{-n}(f_i'(z))\oint\frac{dw}{2\pi i}w^{-m}(f_j'(w))\\
&\cdot(E^{-1})^{\mu\nu}g_{\nu\rho}(E^{-1})^{\nu\eta}g_{\eta\sigma}\langle\partial X^{\rho}(f_i(z))\partial X^{\sigma}(f_j(w))\rangle\\
&= \oint\frac{dz}{2\pi i}z^{-n}(f_i'(z))\oint\frac{dw}{2\pi i}w^{-m}(f_j'(w))(E^{-1})^{\mu\nu}g_{\nu\rho}(E^{-1})^{\nu\eta}g_{\eta\sigma}\frac{-g^{\rho\sigma}}{(f_i(z)-f_j(w))^2}\\
&= \oint\frac{dz}{2\pi i}z^{-n}(f_i'(z))\oint\frac{dw}{2\pi i}w^{-m}(f_j'(w))\frac{-G^{\mu\nu}}{(f_i(z)-f_j(w))^2} \qquad (6.5)
\end{aligned}$$

where the correlation function $\langle\,\partial X^{\mu}\partial X^{\nu}\,\rangle$ is obtained deriving eq.(6.1). We see that the modification induced by the B field on the part of the vertex with indices $m, n > 0$ is completely taken into account by substituting the *closed string metric* $g_{\mu\nu}$ with the *open string metric* $G_{\mu\nu}$. The same modification occurs in the kinetic term. Things are different when both the mode indices of the Neumann coefficients N_{MN} are zero: N_{00}. This corresponds to the contraction of two exponentials that are both forced to belong to the real axis. To do this contraction we need then the propagator for points belonging to the boundary of the worldsheet. It was given in (1.18) and is

$$\langle X^{\mu}(\tau)X^{\nu}(\tau')\rangle = -\alpha' G^{\mu\nu}\ln(\tau-\tau')^2 + \frac{i}{2}\theta^{\mu\nu}\epsilon(\tau-\tau') \qquad (6.6)$$

With this propagator the matrix element of exponentials becomes, up to normalization factors

$$\begin{aligned}
&\Big\langle \prod_i e^{ip_i\cdot X(f_i(0))}\Big\rangle\\
&= \exp\Big[\sum_{i<j}p_i^{\mu}G_{\mu\nu}p_j^{\nu}\log|f_i(0)-f_j(0)|\Big]\exp\Big[\frac{i}{2}\sum_{i<j}p_{\mu}\theta^{\mu\nu}p_{\nu}\,\epsilon(f_i(0)-f_j(0))\Big]
\end{aligned}$$

The B modified $|V_3\rangle$ is then

$$\begin{aligned}
|V_3\rangle &= \delta(p^{(1)}+p^{(2)}+p^{(3)})|\Omega_1\rangle\otimes|\Omega_3\rangle\otimes|\Omega_3\rangle\\
&\times\ \exp\left(\sum_{M,N=0}^{\infty}\frac{1}{2}G_{\mu\nu}\alpha^{(r)\mu}_{-n}N^{rs}_{NM}\alpha^{(s)\nu}_{-m} + \sum_{m=0,n=1}^{\infty}c^{(r)}_{-n}X^{rs}_{mn}b^{(s)}_{-m} - \frac{i}{2}\theta_{\mu\nu}p^{(1)\mu}p^{(2)\nu}\right)
\end{aligned}$$

6.2 The coefficients V^{rs}_{MN} with B field

Our next goal is to find the form of the coefficients V^{rs}_{MN} when a constant B field is switched on. We start from the simplest case, i.e. when B is nonvanishing in the two space directions, say the 24–th and 25–th ones. Let us denote these directions with the Lorentz indices α and β. Then, as we saw in the first chapter, in these two directions we have a new effective metric $G_{\alpha\beta}$, the open string metric, as well as an effective antisymmetric parameter $\theta_{\alpha\beta}$, given by

$$
\begin{aligned}
G^{\alpha\beta} &= \left(\frac{1}{\eta+2\pi\alpha' B}\,\eta\,\frac{1}{\eta-2\pi\alpha' B}\right)^{\alpha\beta},\\
\theta^{\alpha\beta} &= -(2\pi\alpha')^2\left(\frac{1}{\eta+2\pi\alpha' B}\,B\,\frac{1}{\eta-2\pi\alpha' B}\right)^{\alpha\beta}
\end{aligned}
$$

The three string vertex is modified only in the 24-th and 25-th direction, which, in view of the subsequent D-brane interpretation, we call the transverse directions. We split the three string vertex into the tensor product of the perpendicular part and the parallel part

$$
|V_3\rangle = |V_{3,\perp}\rangle \otimes |V_{3,\parallel}\rangle \tag{6.7}
$$

The parallel part is the same as in the ordinary case and will not be re-discussed here. On the contrary we will describe in detail the perpendicular part of the vertex. We rewrite the exponent E as $E = E_{\parallel} + E_{\perp}$, according to the above splitting. $E_{\perp}$ will be modified as follows

$$
\begin{aligned}
E_{\perp} \;\to\; E'_{\perp} = \sum_{r,s=1}^{3}\Bigg(&\frac{1}{2}\sum_{m,n\geq 1} G_{\alpha\beta}a_m^{(r)\alpha\dagger}V^{rs}_{mn}a_n^{(s)\beta\dagger} + \sum_{n\geq 1} G_{\alpha\beta}p^{\alpha}_{(r)}V^{rs}_{0n}a_n^{(s)\beta\dagger}\\
&+\frac{1}{2}G_{\alpha\beta}p^{\alpha}_{(r)}V^{rs}_{00}p^{\beta}_{(s)} + \frac{i}{2}\sum_{r<s}p^{(r)}_{\alpha}\theta^{\alpha\beta}p^{(s)}_{\beta}\Bigg)
\end{aligned}\tag{6.8}
$$

where we set $\alpha' = 1$. Next, as far as the zero modes are concerned, we pass from the momentum to the oscillator basis, [68, 69]. As before we define

$$
a_0^{(r)\alpha} = \frac{1}{2}\sqrt{b}\hat{p}^{(r)\alpha} - i\frac{1}{\sqrt{b}}\hat{x}^{(r)\alpha}, \qquad a_0^{(r)\alpha\dagger} = \frac{1}{2}\sqrt{b}\hat{p}^{(r)\alpha} + i\frac{1}{\sqrt{b}}\hat{x}^{(r)\alpha}, \tag{6.9}
$$

where $\hat{p}^{(r)\alpha}, \hat{x}^{(r)\alpha}$ are the zero momentum and position operator of the r–th string, and we have kept the 'gauge' parameter b of ref.[73] ($b \sim \alpha'$). It is understood that $p^{(r)\alpha} = G^{\alpha\beta}p^{(r)}_{\beta}$. We have

$$
[a_0^{(r)\alpha}, a_0^{(s)\beta\dagger}] = G^{\alpha\beta}\delta^{rs} \tag{6.10}
$$

Denoting by $|\Omega_{b,\theta}\rangle$ the oscillator vacuum ($a_0^\alpha|\Omega_{b,\theta}\rangle = 0$), the relation between the momentum basis and the oscillator basis is defined by

$$|p^{24}\rangle_{123} \otimes |p^{25}\rangle_{123} \equiv |\{p^\alpha\}\rangle_{123} =$$

$$\left(\frac{b}{2\pi\sqrt{\det G}}\right)^{\frac{3}{2}} \exp\left[\sum_{r=1}^{3}\left(-\frac{b}{4}p_\alpha^{(r)}G^{\alpha\beta}p_\beta^{(r)} + \sqrt{b}a_0^{(r)\alpha\dagger}p_\alpha^{(r)} - \frac{1}{2}a_0^{(r)\alpha\dagger}G_{\alpha\beta}a_0^{(r)\beta\dagger}\right)\right]|\Omega_{b,\theta}\rangle$$

Now we insert this equation inside $E'_\perp$ and try to eliminate the momenta along the perpendicular directions by integrating them out. To this end we rewrite $E'_\perp$ in the following way and, for simplicity, drop all the labels α, β and r, s:

$$E'_\perp = \frac{1}{2}\sum_{m,n\geq 1} a_m^\dagger G V_{mn} a_n^\dagger + \sum_{n\geq 1} p V_{0n} a_n^\dagger + \frac{1}{2}p\left[G^{-1}(V_{00} + \frac{b}{2}) + \frac{i}{2}\theta\epsilon\chi\right]p - \sqrt{b}pa_0^\dagger + \frac{1}{2}a_0^\dagger G a_0^\dagger$$

where we have set $\theta^{\alpha\beta} = \epsilon^{\alpha\beta}\theta$ and introduced the matrices ϵ with entries $\epsilon^{\alpha\beta}$ and χ with entries

$$\chi^{rs} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \tag{6.11}$$

At this point we impose momentum conservation. There are three distinct ways to do that and eventually one has to (multiplicatively) symmetrize with respect to them. Let us start by setting $p_3 = -p_1 - p_2$ in $E'_\perp$ and obtain an expression of the form

$$p\,X_{00}\,p + \sum_{N\geq 0} p\,Y_{0N}\,a_N^\dagger + \sum_{M,N\geq 0} a_M^\dagger\,Z_{MN}\,a_N^\dagger \tag{6.12}$$

where, in particular, X_{00} is given by

$$X_{00}^{\alpha\beta,rs} = G^{\alpha\beta}\,(V_{00} + \frac{b}{2})\,\eta^{rs} + i\frac{\theta}{4}\,\epsilon^{\alpha\beta}\,\epsilon^{rs} \tag{6.13}$$

Here the indices r, s take only the values 1,2, and

$$\eta = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}, \qquad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{6.14}$$

Now, as usual, we redefine p so as eliminate the linear term in (6.12). At this point we can easily perform the Gaussian integration over $p_{(1)}, p_{(2)}$, while the remnant of (6.12) will be expressed in terms of the inverse of X_{00}:

$$\left(X_{00}^{-1}\right)^{\alpha\beta,rs} = \frac{2A^{-1}}{4a^2+3}\left(\frac{3}{2}G^{\alpha\beta}\,(\eta^{-1})^{rs} - 2i\,a\,\hat{\epsilon}^{\alpha\beta}\,\epsilon^{rs}\right) \tag{6.15}$$

where

$$A = V_{00} + \frac{b}{2}, \qquad a = \frac{\theta}{4A}\sqrt{\mathrm{Det}G}, \qquad \epsilon^{\alpha\beta} = \sqrt{\mathrm{Det}G}\,\hat{\epsilon}^{\alpha\beta} \tag{6.16}$$

Let us use henceforth for the B field the explicit form

$$B_{\alpha\beta} = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix} \tag{6.17}$$

so that

$$\mathrm{DetG} = \left(1 + (2\pi B)^2\right)^2, \qquad \theta\sqrt{\mathrm{DetG}} = -(2\pi)^2 B, \qquad a = -\frac{\pi^2}{A}B \tag{6.18}$$

Now one has to symmetrize with respect to the three possibilities of imposing the momentum conservation. Remembering the factors due to integration over the momenta and collecting the results one gets for the three string vertex in the presence of a B field

$$|V_3\rangle' = |V_{3,\perp}\rangle' \otimes |V_{3,\|}\rangle \tag{6.19}$$

$|V_{3,\|}\rangle$ is the same as in the ordinary case (without B field), while

$$|V_{3,\perp}\rangle' = K_2\, e^{-E'}|\tilde{0}\rangle \tag{6.20}$$

with

$$K_2 = \frac{\sqrt{2\pi b^3}}{A^2(4a^2+3)}(\mathrm{Det}G)^{1/4}, \tag{6.21}$$

$$E' = \frac{1}{2}\sum_{r,s=1}^{3}\sum_{M,N\geq 0} a_M^{(r)\alpha\dagger}\mathcal{V}^{rs}_{\alpha\beta,MN}a_N^{(s)\beta\dagger} \tag{6.22}$$

and $|\tilde{0}\rangle = |0\rangle \otimes |\Omega_{b,\theta}\rangle$. The coefficients $\mathcal{V}^{\alpha\beta,rs}_{MN}$ are given by

$$\mathcal{V}^{\alpha\beta,rs}_{00} = G^{\alpha\beta}\delta^{rs} - \frac{2A^{-1}b}{4a^2+3}\left(G^{\alpha\beta}\phi^{rs} - ia\hat{\epsilon}^{\alpha\beta}\chi^{rs}\right) \tag{6.23}$$

$$\mathcal{V}^{\alpha\beta,rs}_{0n} = \frac{2A^{-1}\sqrt{b}}{4a^2+3}\sum_{t=1}^{3}\left(G^{\alpha\beta}\phi^{rt} - ia\hat{\epsilon}^{\alpha\beta}\chi^{rt}\right)V^{ts}_{0n} \tag{6.24}$$

$$\mathcal{V}^{\alpha\beta,rs}_{mn} = G^{\alpha\beta}V^{rs}_{mn} + \frac{2A^{-1}}{4a^2+3}\sum_{t,v=1}^{3}V^{rv}_{m0}\left(G^{\alpha\beta}\phi^{vt} - ia\hat{\epsilon}^{\alpha\beta}\chi^{vt}\right)V^{ts}_{0n} \tag{6.25}$$

where, by definition, $V^{rs}_{0n} = V^{sr}_{n0}$, and

$$\phi = \begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix} \tag{6.26}$$

while the matrix χ has been defined above (6.11). These two matrices satisfy the algebra

$$\chi^2 = -2\phi, \qquad \phi\chi = \chi\phi = \frac{3}{2}\chi, \qquad \phi^2 = \frac{3}{2}\phi \tag{6.27}$$

To end this section we would like to notice that the above results can be easily extended to the case in which the transverse directions are more than two (i.e. the 24–th and 25–th ones) and even. The canonical form of the transverse B field is

$$B_{\alpha\beta} = \begin{pmatrix} \begin{matrix} 0 & B_1 \\ -B_1 & 0 \end{matrix} & 0 & \dots \\ 0 & \begin{matrix} 0 & B_2 \\ -B_2 & 0 \end{matrix} & \dots \\ \dots & \dots & \dots \end{pmatrix} \tag{6.28}$$

It is not hard to see that each couple of conjugate transverse directions under this decomposition, can be treated in a completely independent way. The result is that each couple of directions $(26-i, 25-i)$, corresponding to the eigenvalue B_i, will be characterized by the same formulas (6.23, 6.24, 6.25) above with B replaced by B_i. The properties of the new coefficients $\mathcal{V}^{\alpha\beta,rs}_{MN}$ are reported in Appendix D

6.3 The squeezed state solution

A squeezed state in the present context is written as

$$|S\rangle = |S_\perp\rangle \otimes |S_{||}\rangle \tag{6.29}$$

where $|S_{||}\rangle$ has the ordinary form that we presented in the previous chapter, and is treated in the usual way, while

$$\langle S_\perp| = \mathcal{N}^2 \langle\tilde{0}| \exp\left(-\frac{1}{2}\sum_{M,N\geq 0} a^\alpha_M \, \tilde{\mathbb{S}}_{\alpha\beta,MN} \, a^\beta_N\right) \tag{6.30}$$

$$|S_\perp\rangle = \mathcal{N}^2 \exp\left(-\frac{1}{2}\sum_{M,N\geq 0} a^{\alpha\dagger}_M \, \mathbb{S}_{\alpha\beta,MN} \, a^{\beta\dagger}_N\right) |\tilde{0}\rangle \tag{6.31}$$

where $|\tilde{0}\rangle = |\Omega_{b,\theta}\rangle \otimes |0\rangle$. Here we have written down both bra and ket in order to stress the difference with the $B = 0$ case, which stems from the fact that, in view of (D.26), we assume $C'\mathcal{S}^{\alpha\beta}C' = (\mathcal{S}^{\alpha\beta})^* = \mathcal{S}^{\beta\alpha}$. The $*$ product of two such states, labeled $_1$ and $_2$, is carried out in the same way as in the ordinary case, see Chapter 5. Therefore we limit ourselves to writing down the result

$$|S'_\perp\rangle = |S_{1,\perp}\rangle * |S_{2,\perp}\rangle = \frac{K_2\,(\mathcal{N}_1\mathcal{N}_2)^2}{\mathrm{DET}(\mathbf{I}-\Sigma\mathcal{V})^{1/2}} \exp\left(-\frac{1}{2}\sum_{M,N\geq 0} a_M^{\alpha\dagger}\mathcal{S}'_{\alpha\beta,MN}a_N^{\beta\dagger}\right)|\tilde{0}\rangle \tag{6.32}$$

where, in matrix notation which includes both the indices N, M and α, β,

$$\mathcal{S}' = \mathcal{V}^{11} + (\mathcal{V}^{12}, \mathcal{V}^{21})(\mathbf{I}-\Sigma\mathcal{V})^{-1}\Sigma\begin{pmatrix}\mathcal{V}^{21}\\ \mathcal{V}^{12}\end{pmatrix} \tag{6.33}$$

In RHS of these equations

$$\Sigma = \begin{pmatrix}\tilde{\mathcal{S}}_1 & 0\\ 0 & \tilde{\mathcal{S}}_2\end{pmatrix}, \qquad \mathcal{V} = \begin{pmatrix}\mathcal{V}^{11} & \mathcal{V}^{12}\\ \mathcal{V}^{21} & \mathcal{V}^{22}\end{pmatrix}, \tag{6.34}$$

and $\mathbf{I}^{\alpha,rs}_{\beta,MN} = \delta^\alpha_\beta\,\delta_{MN}\,\delta^{rs}$, $r, s = 1, 2$. DET is the determinant with respect to all indices. In order to avoid confusion we remind the reader that we work with three kind of indices: $r, s = 1, 2, 3$ for the three strings, $\alpha, \beta = 24, 25$ for the space-time direction where the B field is switched on, and $m, n = 1, \ldots \infty$ for the string modes. We adopt the following notation for different identity operators:

$$\begin{aligned}\mathbf{I}^{\alpha,rs}_{\beta,MN} &= \delta^\alpha_\beta\,\delta_{MN}\,\delta^{rs}\\ \mathbb{I}^{\alpha}_{\beta,MN} &= \delta^\alpha_\beta\,\delta_{MN}\\ \mathbf{1}^\alpha_\beta &= \delta^\alpha_\beta\end{aligned} \tag{6.35}$$

To reach the form (6.33) one has to use cyclicity of $\mathcal{V}^{rs}$ (see Appendix D). The expression of $\mathcal{S}'$ is in fact a series, therefore some kind of condition on the coefficients $\mathcal{S}_i$ must be satisfied in order for it to make sense. The squeezed states $\mathcal{S}$ satisfying this condition form a subalgebra of the algebra defined by the $*$ product.

Let us now discuss the squeezed state solution of the equation $|\Psi\rangle * |\Psi\rangle = |\Psi\rangle$ in the matter sector. In order for this to be satisfied with the above states $|S\rangle$, we must first impose

$$\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S}' \equiv \mathcal{S}$$

and then suitably normalize the resulting state. Then (6.33) becomes an equation for $\mathcal{S}$, i.e.

$$\tilde{\mathcal{S}} = \mathcal{V}^{11} + (\mathcal{V}^{12}, \mathcal{V}^{21})(\mathbf{I}-\Sigma\mathcal{V})^{-1}\Sigma\begin{pmatrix}\mathcal{V}^{21}\\ \mathcal{V}^{12}\end{pmatrix} \tag{6.36}$$

where $\Sigma, \mathcal{V}$ are the same as above with $\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S}$. Eq.(6.36) has an obvious (formal) solution by iteration. However we saw that it is possible to obtain the solution in compact form by 'abelianizing' the problem. Notwithstanding the differences with that case, it is possible to reproduce the same trick on eq.(6.36), thanks to (D.23). We set

$$C'\mathcal{V}^{rs} = \mathcal{X}^{rs} \quad \text{and} \quad C'\mathcal{S} = \mathcal{T}\,,$$

and assume that

$$[\mathcal{X}^{rs}, \mathcal{T}] = 0$$

(of course this has to be checked *a posteriori*). Notice however that we cannot assume that C' commutes with $\mathcal{S}$, but we assume that

$$C'\mathcal{S}C' = \tilde{\mathcal{S}}\,.$$

By multiplying (6.36) from the left by C' we get:

$$\mathcal{T} = \mathcal{X}^{11} + (\mathcal{X}^{12}, \mathcal{X}^{21})(\mathbf{I} - \Sigma\mathcal{V})^{-1}\begin{pmatrix}\mathcal{T}\mathcal{X}^{21}\\ \mathcal{T}\mathcal{X}^{12}\end{pmatrix} \tag{6.37}$$

For instance $\tilde{\mathcal{S}}\mathcal{V}^{12} = \tilde{\mathcal{S}}C'C'\mathcal{V}^{12} = \mathcal{T}\mathcal{X}^{12}$, etc. In the same way,

$$(\mathbf{I} - \Sigma\mathcal{V})^{-1} = \begin{pmatrix}\mathbb{I} - \mathcal{T}\mathcal{X}^{11} & -\mathcal{T}\mathcal{X}^{12}\\ -\mathcal{T}\mathcal{X}^{21} & \mathbb{I} - \mathcal{T}\mathcal{X}^{11}\end{pmatrix}^{-1}$$

where $\mathbb{I}^{\alpha}_{\beta,MN} = \delta^{\alpha}_{\beta}\,\delta_{MN}$. Now all the entries are commuting matrices, so the inverse can be calculated straight away.

From now on everything is the same as in [70, 73], therefore we limit ourselves to a quick exposition. Using (D.24) and (D.25), one arrives at an equation only in terms of $\mathcal{T}$ and $\mathcal{X} \equiv \mathcal{X}^{11}$:

$$(\mathcal{T} - \mathbb{I})(\mathcal{X}\mathcal{T}^2 - (\mathbb{I} + \mathcal{X})\mathcal{T} + \mathcal{X}) = 0 \tag{6.38}$$

This gives two solutions:

$$\mathcal{T} = \mathbb{I} \tag{6.39}$$

$$\mathcal{T} = \frac{1}{2\mathcal{X}}\left(\mathbb{I} + \mathcal{X} - \sqrt{(\mathbb{I} + 3\mathcal{X})(\mathbb{I} - \mathcal{X})}\right) \tag{6.40}$$

The third solution, with a + sign in front of the square root, is not acceptable. In both cases we see that the solution commutes with $\mathcal{X}^{rs}$. Naturally we are talking about solutions of the abelianized eq.(6.37). The true solution we are looking for is, in both cases, $\mathcal{S} = C'\mathcal{T}$.

As for (6.39), it is easy to see that it leads to the identity state. Therefore, from now on we will consider (6.40) alone.

Now, let us deal with the normalization of $|S_\perp\rangle$. Imposing $|S_\perp\rangle * |S_\perp\rangle = |S_\perp\rangle$ we find

$$\mathcal{N}^2 = K_2^{-1}\,\mathrm{DET}\,(\mathbf{I} - \Sigma\mathcal{V})^{1/2}$$

Replacing in it the solution one finds

$$\mathrm{DET}(\mathbf{I} - \Sigma\mathcal{V}) = \mathrm{Det}\,((\mathbb{I} - \mathcal{X})(\mathbb{I} + \mathcal{T})) \tag{6.41}$$

Det denotes the determinant with respect to the indices α, β, M, N. Using this equation and (6.21), and borrowing from Chapter 3 the expression for $|S_{||}\rangle$, one finally gets for the 23-dimensional tachyonic lump:

$$\begin{aligned}
|S\rangle &= \left\{\det(1-X)^{1/2}\det(1+T)^{1/2}\right\}^{24} \exp\left(-\frac{1}{2}\eta_{\bar\mu\bar\nu}\sum_{m,n\geq 1} a_m^{\bar\mu\dagger} S_{mn} a_n^{\bar\nu\dagger}\right)|0\rangle \otimes \qquad (6.42)\\
&\frac{A^2(3+4a^2)}{\sqrt{2\pi b^3}(\mathrm{Det}G)^{1/4}}\left(\mathrm{Det}(\mathbb{I}-\mathcal{X})^{1/2}\mathrm{Det}(\mathbb{I}+\mathcal{T})^{1/2}\right)\exp\left(-\frac{1}{2}\sum_{M,N\geq 0} a_M^{\alpha\dagger}\mathcal{S}_{\alpha\beta,MN}a_N^{\beta\dagger}\right)|\tilde{0}\rangle,
\end{aligned}$$

where $\mathcal{S} = C'\mathcal{T}$ and $\mathcal{T}$ is given by (6.40). The quantities in the first line are defined in ref.[73] with $\bar\mu, \bar\nu = 0, \ldots 23$ denoting the parallel directions to the lump.

The value of the action corresponding to (6.42) is easily calculated

$$\begin{aligned}
\mathcal{S}_{\mathcal{S}} &= \mathcal{K}\frac{V^{(24)}}{(2\pi)^{24}}\left\{\det(1-X)^{3/4}\det(1+3X)^{1/4}\right\}^{24}\\
&\cdot\frac{A^4(3+4a^2)^2}{2\pi b^3(\mathrm{Det}G)^{1/2}}\,\mathrm{Det}(\mathbb{I}-\mathcal{X})^{3/4}\mathrm{Det}(\mathbb{I}+3\mathcal{X})^{1/4} \qquad (6.43)
\end{aligned}$$

where $V^{(24)}$ is the volume along the parallel directions and $\mathcal{K}$ is the constant of eq.(E.44).

Finally, let $\mathfrak{e}$ denote the energy per unit volume, which coincides with the brane tension when $B = 0$. Then one can compute the ratio of the D23–brane energy density $\mathfrak{e}_{23}$ to the D25–brane energy density $\mathfrak{e}_{25}$;

$$\frac{\mathfrak{e}_{23}}{\mathfrak{e}_{25}} = \frac{(2\pi)^2}{(\mathrm{Det}G)^{1/4}}\cdot\mathcal{R} \tag{6.44}$$

$$\mathcal{R} = \frac{A^4(3+4a^2)^2}{2\pi b^3(\mathrm{Det}G)^{1/4}}\frac{\mathrm{Det}(\mathbb{I}-\mathcal{X})^{3/4}\mathrm{Det}(\mathbb{I}+3\mathcal{X})^{1/4}}{\det(1-X)^{3/2}\det(1+3X)^{1/2}} \tag{6.45}$$

Since $\mathcal{R}$ equals 1 (see Appendix D), this equation is exactly what is expected for the ratio of a flat static D25-brane action and a D23–brane action per unit volume in the presence of the B field (6.17). In fact the DBI Lagrangian for a flat static Dp–brane is,

$$\mathcal{L}_{DBI} = \frac{1}{g_s(2\pi)^p}\sqrt{\mathrm{Det}(1+2\pi B)} \tag{6.46}$$

where g_s is the closed string coupling. Substituting (6.17) and taking the ratio the claim follows.

Let us briefly discuss the generalization of the above results to lower dimensional lumps. As remarked at the end of section 2, every couple of transverse directions corresponding to an eigenvalue B_i of the field B can be treated in the same way as the 24–th and 25–th directions. One has simply to replace in the above formulas B with B_i. The derivation of the above formulas for the case of $25-2i$ dimensional lumps is straightforward.

Switching on a constant B field on VSFT does not obstruct the possibility to find exact results. On the contrary, we have found that (matter) squeezed states representing tachyonic lumps are still solutions of the equations of motion, and that we can give compact explicit formulas for these solutions, much like in the $B=0$ case. Indeed these are still interpretable as (lower dimensional) D–branes.

6.4 Some results in VSFT with B field

In this section we present a couple of results which are natural extensions of analogous results with $B=0$, namely the possibility of defining wedge-like states and orthogonal projectors. But we investigate also a particular phenomenon, the confinement or not of the midpoint of the string, where the presence of the B field determines makes a strong difference with the $B=0$ case.

6.4.1 Wedge–like states

We saw that wedge states are geometrical states in that they can be defined simply by means of a conformal map of the unit disk to a portion of it. They are spanned by an integer n: the limit for $n \to \infty$ is the sliver $|\Xi\rangle$, which is interpreted as the D25–brane. Wedge states also admit a representation in terms of oscillators $a_n^\dagger$ with $n>0$,

$$|W_n\rangle = \mathcal{N}_n^{26} e^{-\frac{1}{2}a^\dagger CT_n a^\dagger}|0\rangle \tag{6.47}$$

which is specified by the matrix T_n, $n > 1$. It can be shown that, see [91], T_n satisfy a recursive relation which can be solved in terms of the matrix T characterizing the sliver state ($T = CS$, S being the sliver matrix). The normalization $\mathcal{N}$ can also be derived from a recursion relation. Since all these results are essentially based on equations which are generalized to the case when a B–field is present and are in fact reported in Appendix D, it is easy to deduce that analogous results hold also when a B field is turned on.

The generalized wedge states will be the tensor product of a factor like (6.47) for the the 24 directions in which the components of the B field are zero and

$$|\mathcal{W}_n\rangle = \mathcal{N}_n^2\, e^{-\frac{1}{2}a^\dagger C'\mathcal{T}_n a^\dagger}|\tilde{0}\rangle \tag{6.48}$$

for the other two directions. From now on we will be concerned with the determination of $\mathcal{T}_n$ and $\mathcal{N}_n$. We start from the hypothesis that

$$[\mathcal{X}^{rs}, \mathcal{T}_n] = 0, \qquad C'\mathcal{T}_n = \tilde{\mathcal{T}}_n C' \tag{6.49}$$

whose consistency we will verify a posteriori.

Now we define $\mathcal{T}_2 = 0$ and the sequence of states

$$|\mathcal{W}_{n+1}\rangle = |\mathcal{W}_n\rangle * |\mathcal{W}_2\rangle \tag{6.50}$$

Using eq.(6.32) and (6.36), with $\Sigma = \begin{pmatrix} C'\tilde{\mathcal{T}}_n & 0 \\ 0 & 0 \end{pmatrix}$, we find the recursion relation

$$\begin{aligned} \mathcal{T}_{n+1} &= \mathcal{X}^{11} + (\mathcal{X}^{12}, \mathcal{X}^{21}) \left(1 - \begin{pmatrix} \mathcal{T}_n\mathcal{X}^{11} & \mathcal{T}_n\mathcal{X}^{12} \\ 0 & 0 \end{pmatrix}\right)^{-1} \begin{pmatrix} \mathcal{T}_n\mathcal{X}^{21} \\ 0 \end{pmatrix} \\ &= \mathcal{X}\frac{1-\mathcal{T}_n}{1-\mathcal{T}_n\mathcal{X}} \end{aligned} \tag{6.51}$$

where use has been made of the second equation in (D.27). Solving this recursion relation, [91], we can write

$$\mathcal{T}_n = \frac{\mathcal{T} + (-\mathcal{T})^{n-1}}{1-(-\mathcal{T})^n} \tag{6.52}$$

Notice that this sequence of states can be extended to $|\mathcal{W}_1\rangle$ defined by $\mathcal{T}_1 = 1$. An analogous recursion relation applies also to the normalization factors. Once solved, it gives

$$\mathcal{N}_n = K_2^{-1}\det\left(\frac{1-\mathcal{T}^2}{1-(-\mathcal{T})^{n+1}}\right)^{1/2} \tag{6.53}$$

The constant K_2 is defined in eq.(2.19) of [108]. The relations (6.49) are now easy to verify.

The limit of $\mathcal{T}_n$ as $n \to \infty$ is $\mathcal{T}$ (i.e. the deformation of the lump), provided $\lim \mathcal{T}^n = 0$. In turn, the latter holds if the eigenvalues of $\mathcal{T}$ are in absolute value less then 1, as those of T are.

6.4.2 Orthogonal projectors

In the presence of a background B field it is also possible to construct other projectors than the one shown in (6.42). To show this we follow Chapter 5. The treatment is very close to what can be found there, and the main purpose of this subsection is to stress some differences with it. As usual we will be concerned only with the transverse part of the projectors, the parallel being exactly the same as in (5.74), and will denote the transverse part of the solution (6.42) by $|\mathcal{S}_\perp\rangle$.

We start by introducing the projection operators parallel to that ones of eq.(5.74)

$$\rho_1 = \frac{1}{(\mathbb{I}+\mathcal{T})(\mathbb{I}-\mathcal{X})}\left[\mathcal{X}^{12}(\mathbb{I}-\mathcal{T}\mathcal{X})+\mathcal{T}(\mathcal{X}^{21})^2\right] \tag{6.54}$$

$$\rho_2 = \frac{1}{(\mathbb{I}+\mathcal{T})(\mathbb{I}-\mathcal{X})}\left[\mathcal{X}^{21}(\mathbb{I}-\mathcal{T}\mathcal{X})+\mathcal{T}(\mathcal{X}^{12})^2\right] \tag{6.55}$$

They satisfy

$$\rho_1^2 = \rho_1, \qquad \rho_2^2 = \rho_2, \qquad \rho_1 + \rho_2 = \mathbb{I} \tag{6.56}$$

i.e. they project onto orthogonal subspaces. Moreover, if we use the superscript ^{T} to denote transposition with respect to the indices N, M and α, β, we have

$$\rho_1^T = \tilde{\rho}_1 = C'\rho_2 C', \qquad \rho_2^T = \tilde{\rho}_2 = C'\rho_1 C'. \tag{6.57}$$

Now, in order to find another solution of the equation $|\Psi\rangle * |\Psi\rangle = |\Psi\rangle$, distinct from $|\mathcal{S}_\perp\rangle$, we make the following ansatz:

$$|\mathcal{P}_\perp\rangle = \left(-\xi\tau a^\dagger\ \zeta\cdot a^\dagger + \kappa\right)|\mathcal{S}_\perp\rangle \tag{6.58}$$

where $\xi = \{\xi_N^\alpha\}$, $\zeta = C'\xi$ and τ is the matrix $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ acting on the indices α and β. κ is a constant to be determined and ξ is required to satisfy the constraints:

$$\rho_1\xi = 0, \qquad \rho_2\xi = \xi, \qquad \text{i.e.} \quad \tilde{\rho}_1\zeta = \zeta, \qquad \tilde{\rho}_2\zeta = 0 \tag{6.59}$$

Using (6.56,6.59) it is simple to prove that

$$\zeta^T f(\mathcal{X}^{rs},\mathcal{T})\,\xi = 0, \qquad \xi^T f(\tilde{\mathcal{X}}^{rs},\tilde{\mathcal{T}})\zeta = 0$$

for any function f. Now, imposing $|\mathcal{P}_\perp\rangle * |\mathcal{S}_\perp\rangle = 0$ we determine κ:

$$\kappa = -\frac{1}{2}\zeta^T\tau\left(\mathcal{V}\mathcal{K}^{-1}\right)_{11}\xi - \frac{1}{2}\xi^T\left(\mathcal{V}\mathcal{K}^{-1}\right)_{11}\tau\zeta \tag{6.60}$$

where

$$\mathcal{K} = \mathbb{I} - \mathcal{T}\mathcal{X}, \qquad \mathcal{V} = \begin{pmatrix} \mathcal{V}^{11} & \mathcal{V}^{12} \\ \mathcal{V}^{21} & \mathcal{V}^{22} \end{pmatrix} \tag{6.61}$$

Next we compute $|\mathcal{P}_\perp\rangle * |\mathcal{P}_\perp\rangle$. This gives

$$|\mathcal{P}_\perp\rangle * |\mathcal{P}_\perp\rangle = \frac{1}{2}\left(\xi^T\left(\mathcal{V}\mathcal{K}^{-1}\right)_{12}\tau\zeta + \zeta^T\tau\left(\mathcal{V}\mathcal{K}^{-1}\right)_{21}\xi\right)\left(-a^\dagger\tau\xi\; a^\dagger\cdot\zeta + \kappa\right)|\mathcal{S}_\perp\rangle \tag{6.62}$$

where use has been made of the identities

$$\begin{aligned}
\zeta^T\tau\left(\mathcal{V}\mathcal{K}^{-1}\right)_{11}\xi &= \zeta^T\tau\left(\mathcal{V}\mathcal{K}^{-1}\right)_{22}\xi = -\zeta^T\tau\left(\mathcal{V}\mathcal{K}^{-1}\right)_{12}\xi = \xi^T\tau\frac{\mathcal{T}}{\mathbb{I}-\mathcal{T}^2}\xi \\
\xi^T\left(\mathcal{V}\mathcal{K}^{-1}\right)_{11}\tau\zeta &= \xi^T\left(\mathcal{V}\mathcal{K}^{-1}\right)_{22}\tau\zeta = -\xi^T\left(\mathcal{V}\mathcal{K}^{-1}\right)_{21}\tau\zeta = \zeta^T\frac{\mathcal{T}}{\mathbb{I}-\mathcal{T}^2}\tau\zeta \\
\xi^T\left(\mathcal{V}\mathcal{K}^{-1}\right)_{12}\tau\zeta &= \zeta^T\frac{1}{\mathbb{I}-\mathcal{T}^2}\tau\zeta, \qquad \zeta^T\tau\left(\mathcal{V}\mathcal{K}^{-1}\right)_{21}\xi = \xi^T\tau\frac{1}{\mathbb{I}-\mathcal{T}^2}\xi
\end{aligned} \tag{6.63}$$

Similarly one can prove that

$$\zeta^T\frac{\mathcal{T}}{\mathbb{I}-\mathcal{T}^2}\tau\zeta = \xi^T\tau\frac{\mathcal{T}}{\mathbb{I}-\mathcal{T}^2}\xi, \qquad \zeta^T\frac{1}{\mathbb{I}-\mathcal{T}^2}\tau\zeta = \xi^T\tau\frac{1}{\mathbb{I}-\mathcal{T}^2}\xi \tag{6.64}$$

So, in order for $|\mathcal{P}_\perp\rangle$ to be a projector, we have to impose

$$\left(\xi^T\left(\mathcal{V}\mathcal{K}^{-1}\right)_{12}\tau\zeta + \zeta^T\tau\left(\mathcal{V}\mathcal{K}^{-1}\right)_{21}\xi\right) = 2\,\xi^T\tau\frac{1}{\mathbb{I}-\mathcal{T}^2}\xi = 2 \tag{6.65}$$

Using this and following [74], it is simple to prove that

$$\langle\mathcal{P}_\perp|\mathcal{P}_\perp\rangle = \left(\zeta^T\frac{1}{\mathbb{I}-\mathcal{T}^2}\tau\zeta\;\xi^T\tau\frac{1}{\mathbb{I}-\mathcal{T}^2}\xi\right)\langle\mathcal{S}_\perp|\mathcal{S}_\perp\rangle = \langle\mathcal{S}_\perp|\mathcal{S}_\perp\rangle \tag{6.66}$$

thanks to (6.64,6.65).

Therefore, under the condition

$$\xi^T\tau\frac{1}{\mathbb{I}-\mathcal{T}^2}\xi = 1 \tag{6.67}$$

the BPZ norm of $|\mathcal{P}_\perp\rangle + |\mathcal{S}_\perp\rangle$ is twice the norm of $|\mathcal{S}_\perp\rangle$. As a consequence the sum of these two states, once they are tensored by the corresponding 24–dimensional complements defined in Chapter 5, represent a couple of parallel D23–branes.

Similarly one can construct the more complicated brane configurations as we saw at the end of the Chapter 5.

6.4.3 The string midpoint

It was shown in [78] that, in the absence of a B field, the string midpoint in the lower dimensional lumps is confined to the hyperplane (D–brane) of vanishing transverse coordinates. Evaluating the exact string midpoint position in the full VSFT is in fact a nontrivial and interesting problem. We intend to address it in this subsection.

The oscillator expansion for the transverse string coordinates is, (1.20), setting $\alpha' = \frac{1}{2}$,

$$x^\alpha(\sigma) = x_0^\alpha + \frac{\theta^{\alpha\beta}}{\pi} p_{0,\beta}\left(\sigma - \frac{\pi}{2}\right) + \sqrt{2}\sum_{n=1}^{\infty}\left[x_n^\alpha \cos(n\sigma) + \frac{\theta^{\alpha\beta}}{\pi}\frac{1}{n} p_{n,\beta} \sin(n\sigma)\right] \tag{6.68}$$

Therefore the string midpoint is specified by

$$x^\alpha\left(\frac{\pi}{2}\right) = x_0^\alpha + \sqrt{2}\sum_{n=1}^{\infty}(-1)^n\left[x_{2n}^\alpha - \frac{\theta^{\alpha\beta}}{\pi}\frac{1}{2n-1} p_{2n-1,\beta}\right] \tag{6.69}$$

It is more convenient to pass to the operator basis $a_N^\alpha, a_N^{\alpha\dagger}$, which satisfies the algebra

$$[a_M^{(r)\alpha}, a_N^{(s)\beta\dagger}] = G^{\alpha\beta}\delta_{MN}\delta^{rs}$$

and are related to x_n, p_n by

$$x_n^\alpha = \frac{i}{\sqrt{2n}}\left(a_n^\alpha - a_n^{\alpha\dagger}\right), \qquad p_{n,\alpha} = \sqrt{\frac{n}{2}} G_{\alpha\beta}\left(a_n^\beta + a_n^{\beta\dagger}\right), \tag{6.70}$$

while the analogous relation for x_0, p_0 is given by eq.(6.9) with the specification that throughout this section, for simplicity, we fix $b = 2$.

Now, confinement of the string midpoint means

$$x^\alpha\left(\frac{\pi}{2}\right)|\mathbb{S}_\perp\rangle = 0 \tag{6.71}$$

Evaluating the LHS we get

$$\begin{aligned} x^\alpha\left(\frac{\pi}{2}\right)|\mathbb{S}_\perp\rangle &= -\frac{i}{\sqrt{2}}(a^\dagger + a^\dagger\mathbb{S})_0^\alpha|\mathbb{S}_\perp\rangle - i\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{2n}}(a^\dagger + a^\dagger\mathbb{S})_{2n}^\alpha|\mathbb{S}_\perp\rangle \\ &-\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{2n-1}}\frac{\theta^{\alpha\beta}}{\pi}G_{\beta\gamma}(a^\dagger - a^\dagger\mathbb{S})_{2n-1}^\gamma|\mathbb{S}_\perp\rangle \end{aligned} \tag{6.72}$$

Confinement requires that this vanish. In order to write this condition in compact form, we introduce the 2×2–matrix–valued vector

$$\Theta = |\nu\rangle\,\mathbf{1} + |\mu\rangle\,\mathbf{e} \tag{6.73}$$

where

$$
\begin{aligned}
|\nu\rangle &= \{\nu_0, \nu_{2n}\}, \qquad \nu_0 = \frac{1}{\sqrt{2}}, \qquad \nu_{2n} = \frac{(-1)^n}{\sqrt{2n}} \\
|\mu\rangle &= \{\mu_{2n-1}\}, \qquad \mu_{2n-1} = i\pi\, B\, \frac{(-1)^n}{\sqrt{2n-1}}
\end{aligned}
\tag{6.74}
$$

Now the confinement condition for the string midpoint can be written as

$$
\mathcal{S}C'\,\Theta = -\Theta, \qquad \text{or, equivalently,} \qquad \tilde{\mathcal{T}}\,\Theta = -\Theta, \quad \text{i.e.} \quad \mathcal{T}\,\tilde{\Theta} = -\tilde{\Theta}. \tag{6.75}
$$

Due to (6.40) an eigenvalue -1 of $\mathcal{T}$ corresponds to an eigenvalue $-\frac{1}{3}$ of $\mathcal{X}$ with the same eigenvector. Let us rewrite $\mathbb{I} + 3\mathcal{X}$ as

$$
\mathbb{I} + 3\mathcal{X} = \mathcal{Y}\,\mathbf{1} + \mathcal{Z}\,\mathbf{e} \tag{6.76}
$$

Then eq.(6.75) becomes $(\mathbb{I} + 3\mathcal{X})\,\tilde{\Theta} = 0$, which in turn corresponds to the two equations

$$
\mathcal{Y}\,|\nu\rangle + \mathcal{Z}\,|\mu\rangle = 0 \tag{6.77}
$$

$$
\mathcal{Z}\,|\nu\rangle - \mathcal{Y}\,|\mu\rangle = 0 \tag{6.78}
$$

It is useful to further split $\mathcal{Y}$ as $\mathcal{Y} = \mathcal{Y}_0 + \mathcal{Y}_1$, where $\mathcal{Y}_0 = \mathcal{Y}(B=0)$. Using (E.7) one obtains

$$
\mathcal{Y}_0 = \begin{pmatrix} 4(1 - A^{-1}) & -4A^{-1}\langle v_e| \\ -4A^{-1}|v_e\rangle & 1 + 3X - 4\,A^{-1}(|v_e\rangle\langle v_e| - |v_o\rangle\langle v_o|) \end{pmatrix} \tag{6.79}
$$

$$
\mathcal{Y}_1 = 12\,H \begin{pmatrix} 1 & \langle v_e| \\ |v_e\rangle & |v_e\rangle\langle v_e| - |v_o\rangle\langle v_o| \end{pmatrix} \tag{6.80}
$$

$$
\mathcal{Z} = 8\sqrt{3}\,i\,a\,K \begin{pmatrix} 0 & \langle v_o| \\ |v_o\rangle & |v_e\rangle\langle v_o| + |v_o\rangle\langle v_e| \end{pmatrix} \tag{6.81}
$$

where $H = \frac{4}{3}\frac{a^2 A^{-1}}{4a^2+3}$.

Now let us express the previous equations in a more explicit form.

$$
\begin{aligned}
|\nu\rangle &= \nu_0 \oplus |\nu_e\rangle, \\
|\mu\rangle &= -i\pi B|\lambda_o\rangle,
\end{aligned}
$$

where

$$|\nu_e\rangle_n = \frac{1+(-1)^n}{2}\nu_n, \qquad \nu_n = \frac{(-1)^{n/2}}{\sqrt{n}} \tag{6.82}$$

$$|\lambda_o\rangle_n = \frac{1-(-1)^n}{2}\lambda_n, \qquad \lambda_n = \frac{(-1)^{(n+1)/2}}{\sqrt{n}} \tag{6.83}$$

We remark that $|\nu\rangle$ is the eigenvector corresponding to the eigenvalue $-\frac{1}{3}$ of $\mathfrak{X}(B=0)$, introduced in [78]; and that $|\lambda_o\rangle$ is the eigenvector with eigenvalue $-\frac{1}{3}$ of X, introduced in [82]. As a consequence one has

$$\mathcal{Y}_0\,|\nu\rangle = 0, \qquad (1+3X)|\lambda_o\rangle = 0 \tag{6.84}$$

The first equation can be rewritten as

$$\langle v_e|\nu_e\rangle = V_{00}\nu_0 \tag{6.85}$$

$$(1+3X)|\nu_e\rangle = 4\nu_0|v_e\rangle \tag{6.86}$$

Remarkably enough, all the other equations from (6.77, 6.78), after using (6.85) and the second equation in (6.84), reduce to a single one

$$\langle v_o|\lambda_o\rangle = \sqrt{\frac{2}{3}}\pi \tag{6.87}$$

Therefore, since eqs.(6.84) have been proved independently, confinement of the string midpoint holds or not according to whether eq.(6.87) is true or not. Now, the LHS of this equation is

$$\langle v_o|\lambda_o\rangle = \sum_{n\ \mathrm{odd}} (-1)^{(n+1)/2}\frac{A_n}{n} \tag{6.88}$$

The latter series can be summed with standard methods and gives

$$\langle v_o|\lambda_o\rangle = \frac{9-2\sqrt{3}\pi}{6}$$

Therefore (6.87) is definitely not satisfied. So we can conclude that the string midpoint in the presence of a B field is *not confined* on the hyperplane that identifies the D23–brane.

In this section we have shown that the introduction of a B field in VSFT does not prevent us from obtaining parallel results to those obtained when $B=0$. Once the formalism is set up, the formal complications brought about by the B field are far from scaring.

On the other hand a nonvanishing background B field may have advantageous aspects. The smoothing out effects of B on the UV divergences of noncommutative field theories are well–known. We have verified that the singular geometry of the lump solutions, pointed out in [78], disappears in the presence of a B field, in particular the string midpoint is not confined any longer to stay on the D–brane.

We remark that this *deconfinement* might mean also that the left–right factorization characteristic of the sliver solution, [74, 85, 87], is not possible for lump solutions with B field. However it looks like there are other aspects of VSFT which may be fruitfully extended to VSFT with B field. For instance, the series of wedge–like states introduced before seem to suggest that the geometric nature of the wedge states, [71], persists also in the presence of a B field. This is confirmed by the results obtained in [97], where the presence of a B field has been dealt with entirely geometrically.

6.5 VSFT ancestors of the GMS solitons

In this section, starting from the squeezed state, we construct an infinite sequence of solutions of eq.(6.56), denoted $|\Lambda_n\rangle$ for any natural number n. $|\Lambda_n\rangle$ is generated by acting on a tachyonic lump solution $|\Lambda_0\rangle$ with $(-\kappa)^n L_n(\mathbf{x}/\kappa)$, where L_n is the n-th Laguerre polynomial, $\mathbf{x}$ is a quadratic expression in the string creation operators, see below eqs.(6.92, 6.93), and κ is an arbitrary constant. These states satisfy the remarkable properties

$$|\Lambda_n\rangle * |\Lambda_m\rangle = \delta_{n,m}|\Lambda_n\rangle \tag{6.89}$$

$$\langle\Lambda_n|\Lambda_m\rangle = \delta_{n,m}\langle\Lambda_0|\Lambda_0\rangle \ . \tag{6.90}$$

Each $|\Lambda_n\rangle$ represents a D23–brane, parallel to all the others. The field theory limit of $|\Lambda_n\rangle$ factors into the sliver state (D25–brane) and the n-th GMS soliton. The algebra (6.89) and the property (6.90) exactly reflect isomorphic properties of the GMS solitons (in terms of Moyal product). In other words, the GMS solitons are nothing but the relics of the $|\Lambda_n\rangle$ D23–branes in the low energy limit.

To define the states $|\Lambda_n\rangle$ we start from the lump solution (6.42). I.e. we take $|\Lambda_0\rangle = |\mathcal{S}\rangle$. However, in the following, we will limit ourselves only to the transverse part of it, the parallel one being universal and irrelevant for our construction. We will denote the transverse part by $|\mathcal{S}_\perp\rangle$.

First we introduce two 'vectors' $\xi = \{\xi_{N\alpha}\}$ and $\zeta = \{\zeta_{N\alpha}\}$, which are chosen to satisfy the conditions

$$\rho_1\xi = 0, \qquad \rho_2\xi = \xi, \qquad \text{and} \quad \rho_1\zeta = 0, \qquad \rho_2\zeta = \zeta, \tag{6.91}$$

Next we define

$$\mathrm{x} = (a^\dagger \tau \xi)\,(a^\dagger C' \zeta) = (a_N^{\alpha\dagger} \tau_\alpha{}^\beta \xi_{N\beta})(a_N^{\alpha\dagger} C'_{NM} \zeta_{M\alpha}) \tag{6.92}$$

where τ is the matrix $\tau = \{\tau_\alpha{}^\beta\} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and introduce the Laguerre polynomials $L_n(\mathrm{x}/\kappa)$. The definition of $|\Lambda_n\rangle$ is as follows

$$|\Lambda_n\rangle = (-\kappa)^n L_n\Big(\frac{\mathrm{x}}{\kappa}\Big)|\mathcal{S}_\perp\rangle \tag{6.93}$$

where κ is an arbitrary constant. Hermiticity requires that

$$(a\tau\xi^*)(aC'\zeta^*) = (a\tau C'\xi)(a\zeta) \tag{6.94}$$

Finally we impose that the two following conditions be satisfied

$$\xi^T \tau \frac{1}{\mathbb{I} - \mathcal{T}^2}\zeta = -1, \qquad \xi^T \tau \frac{\mathcal{T}}{\mathbb{I} - \mathcal{T}^2}\zeta = -\kappa \tag{6.95}$$

Let us spend a few words to motivate the definition of the states $|\Lambda_n\rangle$. The definition (6.93) is not, as one might suspect, dictated in the first place by the similarity with the form of the GMS solitons. Rather it has been selected due to its apparently unique role in the framework of Witten's star algebra.

In [109], on the wake of [74], starting from the (transverse) lump solution $|\mathcal{S}_\perp\rangle$ we introduced a new lump solution $|\mathcal{P}_\perp\rangle = (\mathrm{x}-\kappa)|\mathcal{S}_\perp\rangle$. Imposing that $|\mathcal{P}_\perp\rangle * |\mathcal{P}_\perp\rangle = |\mathcal{P}_\perp\rangle$ and $|\mathcal{P}_\perp\rangle * |\mathcal{S}_\perp\rangle = 0$ and, moreover, that $\langle\mathcal{P}_\perp|\mathcal{P}_\perp\rangle = \langle\mathcal{S}_\perp|\mathcal{S}_\perp\rangle$, we found the conditions (6.95).

The next most complicated state one is lead to try is of the form

$$|\mathcal{P}'\rangle = (\alpha + \beta\mathrm{x} + \gamma\mathrm{x}^2)|\mathcal{S}_\perp\rangle \tag{6.96}$$

The conditions this state has to satisfy turn out to be more restrictive than for $|\mathcal{P}\rangle$, but, nevertheless, are satisfied if, besides conditions (6.95), the following relations hold

$$-2(\alpha)^{1/2} = \beta, \qquad \gamma = \frac{1}{2} \tag{6.97}$$

and then, putting $\alpha = \kappa$

$$|\mathcal{P}'\rangle = \Big(\kappa^2 - 2\kappa\mathrm{x} + \frac{1}{2}\mathrm{x}^2\Big)|\mathcal{S}_\perp\rangle \tag{6.98}$$

The polynomial in the RHS is nothing but the second Laguerre polynomial of x/κ multiplied by κ^2. We deduce from this that the Laguerre polynomials must play a fundamental role in this problem and, as a consequence, put forward the general ansatz (6.93).

Proving the necessity of the conditions (6.95) for general n is very cumbersome, so we will limit ourselves to showing that these conditions are sufficient. However it is instructive and rather easy to see, at least, that the second condition (6.95) is necessary in general. In fact, by requiring that the state $|\Lambda_n\rangle$ be orthogonal to the 'ground state' $|\mathbb{S}_\perp\rangle$, we get:

$$\begin{aligned}
|\Lambda_n\rangle * |\mathbb{S}_\perp\rangle &= (-\kappa)^n \sum_{j=0}^{\infty} \binom{n}{j} \frac{(-\mathrm{x}/\kappa)^j}{j!} |\mathbb{S}_\perp\rangle * |\mathbb{S}_\perp\rangle \\
&= (-\kappa)^n \sum_{j=0}^{\infty} \binom{n}{j} (\kappa)^{-j} \\
&\cdot \; (\xi \tau C')^{\alpha_1}_{l_1} \dots (\xi\tau C')^{\alpha_j}_{l_j} \zeta^{\beta_1}_{k_1} \dots \zeta^{\beta_j}_{k_j} \frac{\partial}{\partial \mu^{\alpha_1}_{l_1}} \cdots \frac{\partial}{\partial \mu^{\alpha_j}_{l_j}} \frac{\partial}{\partial \mu^{\beta_1}_{k_1}} \cdots \frac{\partial}{\partial \mu^{\beta_j}_{k_j}} \\
&\cdot \; \exp\Big(- (\chi^T \mathcal{K}_1)^{-1} \mu - \frac{1}{2} \mu^T (\mathcal{V}\mathcal{K}^{-1})_{11} \mu \Big) |\mathbb{S}_\perp\rangle \Big|_{\mu=0} \\
&= (-\kappa)^n \sum_{j=0}^{\infty} \binom{n}{j} (\kappa)^{-j} \left(\xi^T \tau \frac{\mathcal{T}}{\mathbb{I} - \mathcal{T}^2} \zeta \right)^j |\mathbb{S} \perp\rangle \\
&= (-\kappa)^n \left(1 + \frac{1}{\kappa} \xi^T \tau \frac{\mathcal{T}}{\mathbb{I} - \mathcal{T}^2} \zeta \right)^n |\mathbb{S}_\perp\rangle = 0 \qquad (6.99)
\end{aligned}$$

which is true for the choice κ given by the second eq.(6.95).

The complete proof of eqs.(6.89) and (6.90) is presented in appendix D.

6.6 The field theory limit and the GMS solitons

We saw in Chapter 1 that soliton solutions of field theories defined on a noncommutative space describe Dp–branes. It is then interesting to see if we can recover such solutions using the Seiberg-Witten limit, that gives a noncommutative field theory from a string theory with a B field turned on.

To discuss this limit we first reintroduce the closed string metric $g_{\alpha\beta}$ as $g\,\delta_{\alpha\beta}$. Now we take $\alpha' B \gg g$, in such a way that G, θ and B are kept fixed. The limit is described by means of a parameter ϵ going to 0. ($\alpha' \sim \epsilon$). We could also choose to parametrize the $\alpha' B \gg g$ condition by sending B to infinity, keeping g and α' fixed and operating a rescaling of the string modes as in [67], of course at the end we get identical results. By looking at the exponential of the 3-string field theory vertex in the presence of a B field

$$\sum_{r,s=1}^{3}\left(\frac{1}{2}\sum_{m,n\geq 1} G_{\alpha\beta}a_m^{(r)\alpha\dagger}V_{mn}^{rs}a_n^{(s)\beta\dagger}+\sqrt{\alpha'}\sum_{n\geq 1}G_{\alpha\beta}p_{(r)}^{\alpha}V_{0n}^{rs}a_n^{(s)\beta\dagger}\right.$$
$$\left.+\alpha'\frac{1}{2}G_{\alpha\beta}p_{(r)}^{\alpha}V_{00}^{rs}p_{(s)}^{\beta}+\frac{i}{2}\sum_{r<s}p_{\alpha}^{(r)}\theta^{\alpha\beta}p_{\beta}^{(s)}\right) \tag{6.100}$$

we see that the limit is characterized by the rescalings

$$\begin{aligned} V_{mn} &\to V_{mn} \\ V_{m0} &\to \sqrt{\epsilon}V_{m0} \\ V_{00} &\to \epsilon V_{00} \end{aligned} \tag{6.101}$$

$G_{\alpha\beta}$ and $\theta^{\alpha\beta}$ are kept fixed. Their explicit dependence on g, α' and B will be reintroduced at the end of our calculations in the form

$$G_{\alpha\beta}=\frac{(2\pi\alpha' B)^2}{g}\delta_{\alpha\beta}, \qquad \theta=\frac{1}{B} \tag{6.102}$$

Substituting the leading behaviors of V_{MN} in eqs.(6.25), and keeping in mind that $A=V_{00}+\frac{b}{2}$, the coefficients $\mathcal{V}_{MN}^{\alpha\beta,rs}$ become

$$\mathcal{V}_{00}^{\alpha\beta,rs} \to G^{\alpha\beta}\delta^{rs}-\frac{4}{4a^2+3}\left(G^{\alpha\beta}\phi^{rs}-ia\hat{\epsilon}^{\alpha\beta}\chi^{rs}\right) \tag{6.103}$$
$$\mathcal{V}_{0n}^{\alpha\beta,rs} \to 0 \tag{6.104}$$
$$\mathcal{V}_{mn}^{\alpha\beta,rs} \to G^{\alpha\beta}V_{mn}^{rs} \tag{6.105}$$

We see that the squeezed state (6.42) factorizes in two parts: the coefficients $\mathcal{V}_{mn}^{\alpha\beta,11}$ reconstruct the full 25 dimensional sliver, while the coefficients $\mathcal{V}_{00}^{\alpha\beta,11}$ take a very simple form

$$\mathcal{S}_{00}^{\alpha\beta}=\frac{2|a|-1}{2|a|+1}G^{\alpha\beta}\equiv s\,G^{\alpha\beta} \tag{6.106}$$

The soliton lump with this choice of the coefficients $\mathcal{V}_{MN}^{\alpha\beta,rs}$ will be called $|\hat{\mathcal{S}}\rangle$

$$|\hat{\mathcal{S}}\rangle = \left\{\det(1-X)^{1/2}\det(1+T)^{1/2}\right\}^{24}\exp\left(-\frac{1}{2}\eta_{\bar\mu\bar\nu}\sum_{m,n\geq 1}a_m^{\bar\mu\dagger}S_{mn}a_n^{\bar\nu\dagger}\right)|0\rangle\otimes \tag{6.107}$$
$$\exp\left(-\frac{1}{2}G_{\alpha\beta}\sum_{m,n\geq 1}a_m^{\alpha\dagger}S_{mn}a_n^{\beta\dagger}\right)|0\rangle\otimes$$
$$\frac{A^2(3+4a^2)}{\sqrt{2\pi b^3}(\mathrm{Det}G)^{1/4}}\left(\mathrm{Det}(\mathbb{I}-\mathcal{X})^{1/2}\mathrm{Det}(\mathbb{I}+\mathcal{T})^{1/2}\right)\exp\left(-\frac{1}{2}sa_0^{\alpha\dagger}G_{\alpha\beta}a_0^{\beta\dagger}\right)|\Omega_{b,\theta}\rangle,$$

where $\bar{\mu}, \bar{\nu} = 0, \ldots 23$ and $\alpha, \beta = 24, 25$. In the low energy limit we have also

$$\mathrm{Det}(\mathbb{I} - \mathcal{X})^{1/2}\mathrm{Det}(\mathbb{I} + \mathcal{T})^{1/2} = \frac{4}{4a^2+3}\det(1-X)\,\frac{4a}{2a+1}\det(1+T) \qquad (6.108)$$

So the complete lump state becomes

$$\begin{aligned} |\hat{\mathbb{S}}\rangle &= \left\{\det(1-X)^{1/2}\det(1+T)^{1/2}\right\}^{26}\exp\left(-\frac{1}{2}G_{\mu\nu}\sum_{m,n\geq 1} a_m^{\mu\dagger} S_{mn} a_n^{\nu\dagger}\right)|0\rangle \otimes \\ &\frac{4a}{2a+1}\,\frac{b^2}{\sqrt{2\pi b^3}(\det G)^{1/4}}\exp\left(-\frac{1}{2}s a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}\right)|\Omega_{b,\theta}\rangle, \end{aligned} \qquad (6.109)$$

where $\mu, \nu = 0, \ldots 25$ and $G_{\mu\nu} = \eta_{\bar{\mu}\bar{\nu}} \oplus G_{\alpha\beta}$. The first line of (6.109) is the usual 25-dimensional sliver up to a simple rescaling of $a_n^{\alpha\dagger}$. The norm of the lump is now regularized by the presence of a which is directly proportional to B: $a = -\frac{\pi^2}{A}B$. Using

$$|x\rangle = \sqrt{\frac{2\sqrt{\det G}}{b\pi}}\exp\left[-\frac{1}{b}x^\alpha G_{\alpha\beta}x^\beta - \frac{2}{\sqrt{b}} i a_0^{\alpha\dagger} G_{\alpha\beta} x^\beta + \frac{1}{2} a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}\right]|\Omega_{b,\theta}\rangle$$

we can calculate the projection onto the basis of position eigenstates of the transverse part of the lump state

$$\langle x|e^{-\frac{s}{2}a_0^{\alpha\dagger}G_{\alpha\beta}a_0^{\beta\dagger}}|\Omega_{b,\theta}\rangle = \sqrt{\frac{2\sqrt{\det G}}{b\pi}}\,\frac{1}{1+s}\,e^{-\frac{1-s}{1+s}\frac{1}{b}x^\alpha x^\beta G_{\alpha\beta}} \qquad (6.110)$$

The transverse part of the lump state in the x representation is then

$$\langle x|\mathbb{S}_\perp\rangle = \frac{1}{\pi}e^{-\frac{1}{2|a|b}x^\alpha x^\beta G_{\alpha\beta}}\,. \qquad (6.111)$$

Finally, the lump state projected into the x representation is

$$\langle x|\hat{\mathbb{S}}\rangle = \frac{1}{\pi}\exp\left[-\frac{1}{2|a|b}x^\alpha x^\beta G_{\alpha\beta}\right]|\Xi\rangle = \frac{1}{\pi}\exp\left[-\frac{x^\alpha x^\beta \delta_{\alpha\beta}}{\theta}\right]|\Xi\rangle\,. \qquad (6.112)$$

$|\Xi\rangle$ is the sliver state (RHS of first line in eq.(6.109)) and $\theta = \frac{1}{B}$. We recall that B has been chosen nonnegative. The coefficient in front of the sliver $|\Xi\rangle$ is nothing but the simplest GMS soliton solution (1.95):

$$\psi(r) = 2e^{-r^2} \qquad (6.113)$$

which corresponds to the $|0\rangle\langle 0|$ projector in the harmonic oscillator Hilbert space. Strictly speaking there is a discrepancy between these coefficients and the corresponding GMS soliton, given by the normalizations which differ by a factor of 2π. This can be traced back to the traditional normalizations used for the eigenstates $|x\rangle$ and $|p\rangle$ in the SFT theory context and in the Moyal context, respectively. This discrepancy can be easily dealt with with a simple redefinition.

We notice that the profile and the normalization of $\langle x|\hat{\mathbb{S}}_\perp\rangle$ do not depend on b.

As compared to [78], the B field provides a natural realization of the regulator for the tachyonic soliton introduced ad hoc there. This beneficial effect of the B field is confirmed by the fact that the projector (6.109) is no longer annihilated by x_0

$$\begin{aligned} x_0 \exp\left(-\frac{1}{2} s a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}\right) |\Omega_{b,\theta}\rangle &= i\,\frac{\sqrt{b}}{2}\,(a_0 - a_0^\dagger) \exp\left(-\frac{1}{2} s a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}\right) |\Omega_{b,\theta}\rangle \\ &= -i\,\frac{\sqrt{b}}{2}\left[\frac{4a}{2a+1}\right] a_0^\dagger \exp\left(-\frac{1}{2} s a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}\right) |\Omega_{b,\theta}\rangle \end{aligned}$$

Therefore, also in the low energy limit, the singular structure found in [78] has disappeared.

In order to analyze the same limit for any $|\Lambda_n\rangle$, first of all we have to find the low energy limit of the projectors ρ_1, ρ_2. Also these two projectors factorize into the zero mode and non–zero mode part. The former is given by

$$(\rho_1)_{00}^{\alpha\beta} \to \frac{1}{2}\Big[G^{\alpha\beta} + i\epsilon^{\alpha\beta}\Big], \qquad (\rho_2)_{00}^{\alpha\beta} \to \frac{1}{2}\Big[G^{\alpha\beta} - i\epsilon^{\alpha\beta}\Big], \tag{6.114}$$

Now, in order to single out the appropriate limit of $|\Lambda_n\rangle$, we take, in the definition (6.92), $\xi = \hat{\xi} + \eta$ and $\zeta = \hat{\zeta} + \vartheta$, where η, ϑ vanish in the limit $\alpha' \to 0$. Then we make the choice $\hat{\xi}_n = \hat{\zeta}_n = 0\ \forall n > 0$. We will see that the two zero components $\hat{\xi}_0$ and $\hat{\zeta}_0$ are enough to define a consistent low energy limit. In the field theory limit the defining conditions (6.91) become

$$\hat{\xi}_{0,24} + i\hat{\xi}_{0,25} = 0, \qquad \hat{\zeta}_{0,24} + i\hat{\zeta}_{0,25} = 0, \tag{6.115}$$

From now on we set $\hat{\xi}_0 = \hat{\xi}_{0,25} = -i\hat{\xi}_{0,24}$ and, similarly, $\hat{\zeta}_0 = \hat{\zeta}_{0,25} = -i\hat{\zeta}_{0,24}$. The conditions (6.95) become

$$\xi^T \tau \frac{1}{\mathbb{I} - \mathcal{T}^2}\zeta \to -\frac{1}{1-s^2}\frac{2}{\sqrt{\det G}}\hat{\xi}_0\hat{\zeta}_0 = -1 \tag{6.116}$$

$$\xi^T \tau \frac{\mathcal{T}}{\mathbb{I} - \mathcal{T}^2}\zeta \to -\frac{s}{1-s^2}\frac{2}{\sqrt{\det G}}\hat{\xi}_0\hat{\zeta}_0 = -\kappa \tag{6.117}$$

Compatibility requires

$$\frac{2\hat{\xi}_0\hat{\zeta}_0}{\sqrt{\det G}} = 1 - s^2, \qquad \kappa = s \tag{6.118}$$

At the same time

$$(\xi \tau a^\dagger)(\zeta C' a^\dagger) \to -\hat{\xi}_0\hat{\zeta}_0((a_0^{24\dagger})^2 + (a_0^{25\dagger})^2) = -\frac{\hat{\xi}_0\hat{\zeta}_0}{\sqrt{\det G}} a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger} \tag{6.119}$$

Hermiticity (6.94) requires that the product $\hat{\xi}_0\hat{\zeta}_0$ be real. In order to be able to compute $\langle x|\Lambda_n\rangle$ in the field theory limit, we have to evaluate first

$$\begin{aligned}\langle x| \left(a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}\right)^k e^{-\frac{s}{2} a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}} |\Omega_{b,\theta}\rangle &= (-2)^k \frac{d^k}{ds^k} \left(\langle x| e^{-\frac{s}{2} a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}} |\Omega_{b,\theta}\rangle\right) \\ &= (-2)^k \frac{d^k}{ds^k} \left(\sqrt{\frac{2\sqrt{\det G}}{b\pi}} \frac{1}{1+s} e^{-\frac{1-s}{1+s}\frac{1}{b} x^\alpha G_{\alpha\beta} x^\beta}\right)\end{aligned} \tag{6.120}$$

An explicit calculation gives

$$\begin{aligned}&\frac{d^k}{ds^k} \left(\frac{1}{1+s} e^{-\frac{1-s}{1+s}\frac{1}{b} x^\alpha x^\beta G_{\alpha\beta}}\right) = \\ &= \sum_{l=0}^{k} \sum_{j=0}^{k-l} \frac{(-1)^{k+j}}{(1-s)^j (1+s)^{k+1}} \frac{k!}{j!} \binom{k-l-1}{j-1} \langle x, x\rangle^j e^{-\frac{1}{2}\langle x,x\rangle}\end{aligned} \tag{6.121}$$

where we have set

$$\langle x, x\rangle = \frac{1}{ab} x^\alpha G_{\alpha\beta} x^\beta = \frac{2r^2}{\theta} \tag{6.122}$$

with $r^2 = x^\alpha x^\beta \delta_{\alpha\beta}$. In this equation it must be understood that, by definition, the binomial coefficient $\binom{-1}{-1}$ equals 1.

Now, inserting (6.121) in the definition of $|\Lambda_n\rangle$, we obtain after suitably reshuffling the indices:

$$\begin{aligned}&\langle x|(-\kappa)^n L_n\left(\frac{\mathrm{X}}{\kappa}\right) e^{-\frac{1}{2} s a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}} |\Omega_{b,\theta}\rangle \\ &\to \langle x|(-s)^n L_n\left(-\frac{1-s^2}{2s} a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}\right) e^{-\frac{1}{2} s a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}} |\Omega_{b,\theta}\rangle \\ &= \frac{(-s)^n}{(1+s)} \sum_{j=0}^{n} \sum_{k=j}^{n} \sum_{l=j}^{k} \binom{n}{k} \binom{l-1}{j-1} \frac{1}{j!} \frac{(1-s)^k}{(1+s)^j s^k} \\ &\cdot (-1)^j \langle x, x\rangle^j \, e^{-\frac{1}{2}\langle x,x\rangle} \sqrt{\frac{2\sqrt{\det G}}{b\pi}}\end{aligned} \tag{6.123}$$

The expression can be evaluated as follows. First one uses the result

$$\sum_{l=j}^{k} \binom{l-1}{j-1} = \binom{k}{j} \tag{6.124}$$

Inserting this into (6.123) one is left with the following summation, which contains an evident binomial expansion,

$$\sum_{k=j}^{n} \binom{n}{k} \binom{k}{j} \left(\frac{1-s}{s}\right)^k = \binom{n}{j} \frac{(1-s)^j}{s^n} \tag{6.125}$$

Replacing this result into (6.123) we obtain

$$\begin{aligned} &\langle x|(-\kappa)^n L_n\left(\frac{\mathrm{X}}{\kappa}\right) e^{-\frac{1}{2}s a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}} |\Omega_{b,\theta}\rangle \\ &\to \frac{2|a|+1}{4|a|}\sqrt{\frac{2\sqrt{\det G}}{b\pi}}(-1)^n \sum_{j=0}^{n} \binom{n}{j} \frac{1}{j!}\left(-\frac{2r^2}{\theta}\right)^j e^{-\frac{r^2}{\theta}} \end{aligned}$$

Recalling now that the definition of $|\hat{\mathrm{S}}\rangle$ includes an additional numerical factor (see eq.(6.109)), we finally obtain

$$\begin{aligned} \langle x|\Lambda_n\rangle &\to \frac{1}{\pi}(-1)^n \sum_{j=0}^{n} \binom{n}{j} \frac{1}{j!}\left(-\frac{2r^2}{\theta}\right)^j e^{-\frac{r^2}{\theta}} |\Xi\rangle \\ &= \frac{1}{\pi}(-1)^n L_n\left(\frac{2r^2}{\theta}\right) e^{-\frac{r^2}{\theta}} |\Xi\rangle \end{aligned} \tag{6.126}$$

The coefficient in front of the sliver $|\Xi\rangle$ is the $n-th$ GMS solution.

In Chapter 1 it was shown that a generic noncommutative scalar field theory with polynomial interaction allows for solitonic solutions in any space dimension. The solutions are very elegantly constructed in terms of harmonic oscillators eigenstates $|n\rangle$. In particular, solitonic solutions correspond to projectors $P_n = |n\rangle\langle n|$. Via the Weyl transform these projectors can be mapped to classical functions $\psi_n(x,y)$ of two variables x, y, in such a way that the operator product in the Hilbert space correspond to the Moyal product in (x,y) space.

This construction is rather universal and does not depend in any essential way on the form of the potential. Now, as we have noticed in the introduction, the low energy effective tachyonic field theory derived from SFT in the presence of a background B field is a noncommutative scalar field theory of the type described above. Therefore it is endowed with the GMS noncommutative solitons. It is reasonable to expect that these solitons may emerge from soliton–type solutions of the SFT, which has the noncommutative scalar tachyonic field theory as its low energy effective action. Therefore the low energy GMS solitons we found in the previous sections are no surprise. What is surprising however is the isomorphism we find between the lump solutions $|\Lambda_n\rangle$ in VSFT and the corresponding GMS solitons. Setting $r^2 = x^2 + y^2$ and $\psi_n(x,y) = 2(-1)^n L_n(\frac{2r^2}{\theta})\, e^{-\frac{r^2}{\theta}}$, we have in fact the following correspondences

$$\begin{array}{ccccc} |\Lambda_n\rangle & \longleftrightarrow & P_n & \longleftrightarrow & \psi_n(x,y) \\ |\Lambda_n\rangle * |\Lambda_{n'}\rangle & \longleftrightarrow & P_n P_{n'} & \longleftrightarrow & \psi_n \star \psi_{n'} \end{array} \tag{6.127}$$

where $\star$ denotes the Moyal product. Moreover

$$\langle \Lambda_n|\Lambda_{n'}\rangle \longleftrightarrow \mathrm{Tr}(P_n P_{n'}) \longleftrightarrow \int dxdy\, \psi_n(x,y)\psi_{n'}(x,y) \tag{6.128}$$

up to normalization (see (6.90)). This correspondence seems to indicate that the Laguerre polynomials hide a universal structure of these noncommutative algebras.

It is evident from the above that the GMS solitons are the low energy remnants of corresponding D–branes in SFT. This explains many features of the former: why, for instance, the energy of the soliton given by $\sum_{k=0}^{n-1} |k\rangle\langle k|$ is n time the energy of the soliton $|0\rangle\langle 0|$; this is nothing but a low energy relic of the same property for the tensions of the corresponding D–branes.

Appendix A

Feynman rules for noncommutative gauge theories

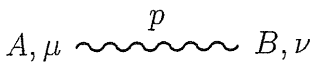
A.1 Feynman rules for noncommutative $U(N)$ gauge theory

In this Appendix we report the Feynman rules for noncommutative gauge theories defined with groups $U(N)$ and $SO(N)$ [31, 49].

Gluons carry Lorentz indices $\mu, \nu, ...$, color indices $A, B, ...$, and momenta $p, q, ...$. Ghosts carry only the last two type of labels. All the momenta are entering unless otherwise specified.

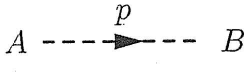
The propagators are untouched by the noncommutativity as seen in section 1.3.

gluon propagator.



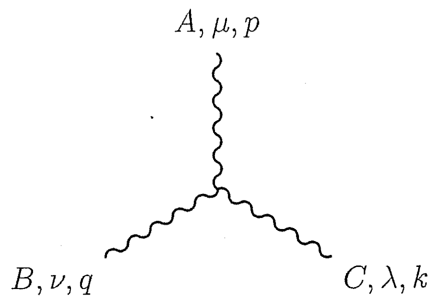
$$-\frac{i}{p^2}\delta_{AB}g_{\mu\nu} \tag{A.1}$$

ghost propagator.



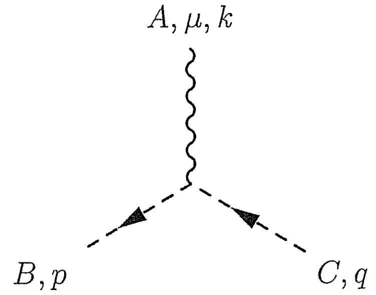
$$\frac{i}{p^2}\delta_{AB} \tag{A.2}$$

3–gluon vertex. The external gluons carry labels (A, μ, p), (B, ν, q) and (C, λ, k) for the Lie algebra, momentum and Lorentz indices and are ordered in anticlockwise sense:



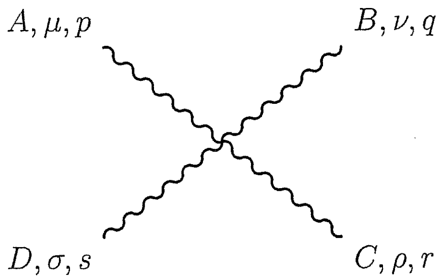
$$-g\,(f_{ABC}\,\cos(p\times q)+d_{ABC}\,\sin(p\times q))\,(g_{\mu\nu}\,(p-q)_\lambda+g_{\nu\lambda}\,(q-k)_\mu+\,g_{\lambda\mu}(k-p)_\nu) \quad (A.3)$$

ghost vertex. The gluon carries label (A,μ,k), the ghosts (B,p) and (C,q):



$$-g\,p_\mu\,(f_{ABC}\,\cos(p\times q)-d_{ABC}\,\sin(p\times q)) \quad (A.4)$$

4–gluon vertex. The gluons carry labels (A,μ,p), (B,ν,q), (C,ρ,r) and (D,σ,s) for Lie algebra, Lorentz index and momentum. They are clockwise ordered:



$$\begin{aligned}-ig^2\ \Big[\ &(f_{ABX}\,\cos(p\times q)+d_{ABX}\,\sin(p\times q))\\ &\quad\cdot(f_{XCD}\,\cos(r\times s)+d_{XCD}\,\sin(r\times s))\,(g_{\mu\rho}\,g_{\nu\sigma}-g_{\mu\sigma}\,g_{\nu\rho})\\ +\ &(f_{ACX}\,\cos(p\times r)+d_{ACX}\,\sin(p\times r))\\ &\quad\cdot(f_{XDB}\,\cos(s\times q)+d_{XDB}\,\sin(s\times q))\,(g_{\mu\sigma}\,g_{\nu\rho}-g_{\mu\nu}\,g_{\rho\sigma})\\ +\ &(f_{ADX}\,\cos(p\times s)+d_{ADX}\,\sin(p\times s))\\ &\quad\cdot(f_{XBC}\,\cos(q\times r)+d_{XBC}\,\sin(q\times r))\,(g_{\mu\nu}\,g_{\rho\sigma}-g_{\mu\rho}\,g_{\nu\sigma})\Big]\end{aligned}$$

With elementary manipulations we can rewrite this as follows:

$$-\frac{i}{4}g^2 \Big[\Big(\cos(p\times s - q\times r)\,L_{ABCD} + \sin(p\times s - q\times r)\,M_{ABCD}\Big)\,T_{\mu\nu\rho\sigma} + \Big(\cos(p\times r - q\times s)\,L_{BACD} - \sin(p\times r - q\times s)\,M_{BACD}\Big)\,T_{\nu\mu\rho\sigma} + \Big(\cos(p\times s + q\times r)\,L_{ACBD} + \sin(p\times s + q\times r)\,M_{ACBD}\Big)\,T_{\mu\rho\nu\sigma}\Big] \quad \text{(A.5)}$$

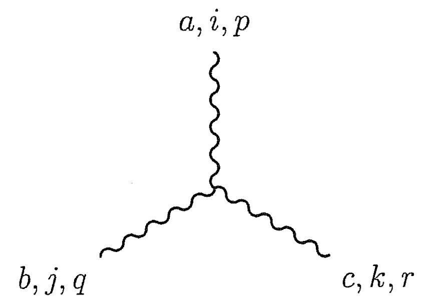
The tensors M, L, T are defined in Chapter 2.

The Feynman rules for $U(1)$ are formally obtained from the above ones by setting $\theta = 0$, the tensor $f = 0$ and $d = 1$ (therefore, in particular, $L = 2$, $M = 0$).

A.2 Feynman rules for noncommutative $SO(N)$ gauge theory

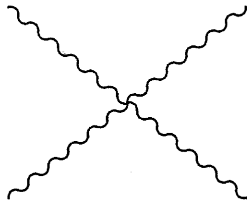
The free part of NC $SO(N)$ theory is common to that one of NC $U(N)$.

3–gluon vertex. The external gluons carry labels (a, i, p), (b, j, q) and (c, k, r) for the Lie algebra, Lorentz indices and momentum and are ordered in anticlockwise sense:



$$-gf^{abc}\cos(p\times q)\,(g_{ij}\,(p-q)_k + g_{jk}\,(q-r)_i + g_{ki}(r-p)_j) \quad \text{(A.6)}$$

4–gluon vertex. The gluons carry labels (a, i, p), (b, j, q), (c, k, r) and (d, l, s) for Lie algebra, Lorentz index and momentum. They are clockwise ordered:



$$
\begin{aligned}
-ig^2 \quad & \Big\{ \Big[f^{xab} f^{xcd} \cos(p\times q)\cos(r\times s) \\
& - \left(4d^{abcd} - \frac{1}{3}(f^{xac}f^{xbd} + f^{xbc}f^{xad})\right) \sin(p\times q)\sin(r\times s)\Big] (g_{ik}g_{jl} - g_{il}g_{jk}) \\
& + \Big[f^{xac}f^{xdb}\cos(p\times r)\cos(s\times q) \\
& - \left(4d^{abcd} - \frac{1}{3}(f^{xcd}f^{xab} + f^{xcb}f^{xad})\right) \sin(p\times r)\sin(s\times q)\Big] (g_{il}g_{jk} - g_{ij}g_{kl}) \\
& + \Big[f^{xad}f^{xbc}\cos(p\times s)\cos(q\times r) \\
& - \left(4d^{abcd} - \frac{1}{3}(f^{xdb}f^{xac} + f^{xba}f^{xdc})\right) \sin(p\times s)\sin(q\times r)\Big] (g_{ij}g_{kl} - g_{ik}g_{jl}) \Big\}
\end{aligned}
\tag{A.7}
$$

We recall that this last vertex can be obtained from the string four–gluon amplitude only after subtracting two suitable tree one–particle reducible diagrams.

One can verify that the above Feynman diagrams can be obtained from the action suggested in [29]. From that action, which was called $NCSO(N)$, one can in addition extract the Feynman rules for the ghost fields.

Appendix B

The coefficients V_{mn}^{rs}

In this Appendix we give the coefficients V_{mn}^{rs} introduced in Chapter 5. These results are taken from refs.[68, 69, 73]. First we define the coefficients A_n and B_n for $n \geq 0$ through the relations:

$$\begin{aligned}
\left(\frac{1+ix}{1-ix}\right)^{1/3} &= \sum_{n\,\text{even}} A_n x^n + i \sum_{n\,\text{odd}} A_n x^n \,, \\
\left(\frac{1+ix}{1-ix}\right)^{2/3} &= \sum_{n\,\text{even}} B_n x^n + i \sum_{n\,\text{odd}} B_n x^n \,.
\end{aligned} \tag{B.1}$$

In terms of A_n and B_n we define the coefficients $N_{mn}^{r,\pm s}$ as follows:

$$\begin{aligned}
N_{nm}^{r,\pm r} &= \frac{1}{3(n \pm m)}(-1)^n (A_n B_m \pm B_n A_m) \quad \text{for} \quad m+n \text{ even}, \; m \neq n \,, \\
&= 0 \quad \text{for} \quad m+n \text{ odd} \,, \\
N_{nm}^{r,\pm(r+1)} &= \frac{1}{6(n \pm m)}(-1)^{n+1} (A_n B_m \pm B_n A_m) \quad \text{for} \quad m+n \text{ even}, \; m \neq n \,, \\
&= \frac{1}{6(n \pm m)} \sqrt{3}\, (A_n B_m \mp B_n A_m) \quad \text{for} \quad m+n \text{ odd} \,, \\
N_{nm}^{r,\pm(r-1)} &= \frac{1}{6(n \mp m)}(-1)^{n+1} (A_n B_m \mp B_n A_m) \quad \text{for} \quad m+n \text{ even}, \; m \neq n \,, \\
&= -\frac{1}{6(n \mp m)} \sqrt{3}\, (A_n B_m \pm B_n A_m) \quad \text{for} \quad m+n \text{ odd} \,.
\end{aligned} \tag{B.2}$$

The coefficients V_{mn}^{rs} are then given by

$$\begin{aligned}
V_{nm}^{rs} &= -\sqrt{mn}\,(N_{nm}^{r,s} + N_{nm}^{r,-s}) \quad \text{for} \quad m \neq n,\, m, n \neq 0 \,, \\
V_{nn}^{rr} &= -\frac{1}{3}[2\sum_{k=0}^{n}(-1)^{n-k} A_k^2 - (-1)^n - A_n^2], \quad \text{for} \quad n \neq 0 \,,
\end{aligned}$$

$$
\begin{aligned}
V^{r(r+1)}_{nn} &= V^{r(r+2)}_{nn} = \frac{1}{2}[(-1)^n - V^{rr}_{nn}] \quad \text{for} \quad n \neq 0\,, \\
V^{rs}_{0n} &= -\sqrt{2n}\,(N^{r,s}_{0n} + N^{r,-s}_{0n}) \quad \text{for} \quad n \neq 0\,, \\
V^{rr}_{00} &= \ln(27/16)\,.
\end{aligned}
\tag{B.3}
$$

The value of V^{rr}_{nn} quoted above corrects the result for $N^{rr}_{nn}(\equiv -V^{rr}_{nn}/n)$ quoted in eqn.(1.18) of [69]. In writing down the expressions for V^{rs}_{0n} and V^{rr}_{00} has been taken into account the fact that we are using $\alpha' = 1$ convention, as opposed to the $\alpha' = 1/2$ convention used in refs.[68, 69].

Appendix C

Conversion from momentum to oscillator basis

We start with the three string vertex in the matter sector as given in Chapter 5:

$$|V_3\rangle = \int d^{26}p_{(1)}d^{26}p_{(2)}d^{26}p_{(3)}\delta^{(26)}(p_{(1)} + p_{(2)} + p_{(3)}) \exp(-E)|0,p\rangle_{123} \tag{C.1}$$

where

$$E = \frac{1}{2}\sum_{\substack{r,s\\ m,n\geq 1}} \eta_{\mu\nu} a_m^{(r)\mu\dagger} V_{mn}^{rs} a_n^{(s)\nu\dagger} + \sum_{\substack{r,s\\ n\geq 1}} \eta_{\mu\nu} p_{(r)}^{\mu} V_{0n}^{rs} a_n^{(s)\nu\dagger} + \frac{1}{2}\sum_{r} \eta_{\mu\nu} p_{(r)}^{\mu} V_{00}^{rr} p_{(r)}^{\nu} \,. \tag{C.2}$$

Note that using the freedom of redefining V_{00}^{rs} using momentum conservation, we have chosen V_{00}^{rs} to be zero for $r \neq s$. Due to the same reason, a redefinition $V_{0n}^{rs} \to V_{0n}^{rs} + A_n^s$ by some r independent constant A_n^s leaves the vertex unchanged. We shall use this freedom to choose:

$$\sum_r V_{0n}^{rs} = 0 \,. \tag{C.3}$$

It can be easily verified that V_{0n}^{rs} given in eq.(B.3) satisfy these conditions.

We now pass to the oscillator basis for a subset of the space-time coordinates x^α ($(26-k) \leq \alpha \leq 25$), by relating the zero mode operators $\hat{x}^\alpha$ and $\hat{p}^\alpha$ to oscillators a_0^α and $a_0^{\alpha\dagger}$. For this one writes:

$$a_0^\alpha = \frac{1}{2}\sqrt{b}\,\hat{p}^\alpha - \frac{1}{\sqrt{b}} i\hat{x}^\alpha, \qquad a_0^{\alpha\dagger} = \frac{1}{2}\sqrt{b}\,\hat{p}^\alpha + \frac{1}{\sqrt{b}} i\hat{x}^\alpha \,, \tag{C.4}$$

where b is an arbitrary constant. Then a_0^α, $a_0^{\alpha\dagger}$ satisfy the usual commutation rule $[a_0^\alpha, a_0^{\beta\dagger}] = \delta^{\alpha\beta}$ (we are assuming that the directions x^α are space-like; otherwise we shall need $\eta^{\alpha\beta}$), and we can define a new vacuum state $|\Omega_b\rangle$ such that $a_0^\alpha|\Omega_b\rangle = 0$. The relation between the momentum basis and the new oscillator basis is given by (for each string)

$$|\{p^\alpha\}\rangle = (2\pi/b)^{-k/4}\exp\Big[-\frac{b}{4}p^\alpha p^\alpha + \sqrt{b}a_0^{\alpha\dagger}p^\alpha - \frac{1}{2}a_0^{\alpha\dagger}a_0^{\alpha\dagger}\Big]|\Omega_b\rangle\,. \qquad \text{(C.5)}$$

In the above equation $\{p^\alpha\}$ label momentum eigenvalues. Substituting eq.(C.5) into eq.(C.1), and integrating over $p^\alpha_{(i)}$, we can express the three string vertex as

$$\begin{aligned}|V_3\rangle \;=\; & \int d^{26-k}p_{(1)}d^{26-k}p_{(2)}d^{26-k}p_{(3)}\delta^{(26-k)}(p_{(1)}+p_{(2)}+p_{(3)}) \\ & \exp\Big(-\frac{1}{2}\sum_{\substack{r,s\\ m,n\geq 1}}\eta_{\bar\mu\bar\nu}a_m^{(r)\bar\mu\dagger}V^{rs}_{mn}a_n^{(s)\bar\nu\dagger} - \sum_{\substack{r,s\\ n\geq 1}}\eta_{\bar\mu\bar\nu}p^{\bar\mu}_{(r)}V^{rs}_{0n}a_n^{(s)\bar\nu\dagger} - \frac{1}{2}\sum_{r}\eta_{\bar\mu\bar\nu}p^{\bar\mu}_{(r)}V^{rr}_{00}p^{\bar\nu}_{(r)}\Big)|0,p\rangle_{123} \\ & \otimes\Big(\frac{\sqrt{3}}{(2\pi b^3)^{1/4}}(V^{rr}_{00}+\frac{b}{2})\Big)^{-k}\exp\Big(-\frac{1}{2}\sum_{\substack{r,s\\ M,N\geq 0}}a_M^{(r)\alpha\dagger}V'^{rs}_{MN}a_N^{(s)\alpha\dagger}\Big)|\Omega_b\rangle_{123}\,. \qquad \text{(C.6)}\end{aligned}$$

In this expression the sums over $\bar\mu,\bar\nu$ run from 0 to $(25-k)$, and the sum over α runs from $(26-k)$ to 25. Note that in the last line the sums over M, N run over 0, 1, 2 .... The new b-dependent V' coefficients are given in terms of the V coefficients by

$$\begin{aligned}V'^{rs}_{mn}(b) &= V^{rs}_{mn} - \frac{1}{V^{rr}_{00}+\frac{b}{2}}\sum_{t=1}^{3}V^{tr}_{0m}V^{ts}_{0n}\,, \qquad m,n\geq 1\,,\\ V'^{rs}_{0n}(b) &= V'^{sr}_{n0} = \frac{1}{V^{rr}_{00}+\frac{b}{2}}\sqrt{b}\,V^{rs}_{0n}\,, \qquad n\geq 1\,,\\ V'^{rs}_{00}(b) &= \frac{1}{3}\frac{b}{V^{rr}_{00}+\frac{b}{2}}\,, \qquad r\neq s\,,\\ V'^{rr}_{00}(b) &= 1-\frac{2}{3}\frac{b}{V^{rr}_{00}+\frac{b}{2}}\,. \qquad \text{(C.7)}\end{aligned}$$

In deriving the above relations we have used eq.(C.3). These relations can be readily inverted to find

$$\begin{aligned}V^{rs}_{mn} &= V'^{rs}_{mn}(b) + \frac{2}{3}\frac{1}{1-V'^{rr}_{00}(b)}\sum_{t=1}^{3}V'^{rt}_{m0}(b)V'^{ts}_{0n}(b)\,, \qquad m,n\geq 1\,,\\ V^{rs}_{0n} &= \frac{2}{3}\frac{1}{1-V'^{rr}_{00}(b)}\sqrt{b}\,V'^{rs}_{0n}(b)\,, \qquad n\geq 1\,,\\ V^{rr}_{00} &= \frac{b}{6}\frac{1+3V'^{rr}_{00}(b)}{1-V'^{rr}_{00}(b)}\,. \qquad \text{(C.8)}\end{aligned}$$

We shall now describe how our variables V^{rs}_{mn} and V'^{rs}_{mn} are related to the variables introduced in ref.[68]. For this we begin by comparing the variables in the oscillator representation. Since ref.[68] uses the $\alpha' = 1/2$ convention rather than the $\alpha' = 1$ convention used here, every factor of p (x) in [68] should be multiplied (divided) by $\sqrt{2\alpha'}$, and then α' should be set equal to one in order to compare with our equations. With this prescription eqs.(2.5b) of [68] giving $a_0 = \frac{1}{2}\hat{p} - i\hat{x}$ becomes $a_0 = \frac{1}{\sqrt{2}}\hat{p} - \frac{i}{\sqrt{2}}\hat{x}$, which corresponds to (C.4) for $b = 2$. Thus, we can directly compare our variables with those of [68] for the case $b = 2$.

Ref.[68] introduced a matrix U which appears, for example, in their eq.(2.47). We shall denote this matrix by U^{gj}. This matrix appears in the construction of the vertex in the oscillator basis ([68], eqn.(2.52) and (2.53)). This implies that the V' coefficients for $b = 2$ can be expressed in terms of U^{gj} using their results. In particular, defining V'^{rs} to be the matrices V'^{rs}_{mn} with m, n now running from 0 to ∞, we have (see [68], eqn.(2.53)):[1]

$$V'^{rs}(2) = \frac{1}{3}(C' + \omega^{s-r} U^{gj} + \omega^{r-s} \bar{U}^{gj}) , \tag{C.9}$$

where $\omega = \exp(2\pi i/3)$, $C'_{mn} = (-1)^m \delta_{mn}$ with $m, n \geq 0$, and the matrix U^{gj} satisfies the relations (eq.(2.51) of [68]):

$$U^{gj\dagger} = U^{gj} , \quad \bar{U}^{gj} \equiv (U^{gj})^* = C' U^{gj} C' , \quad U^{gj} U^{gj} = 1 . \tag{C.10}$$

Eq.(C.9) gives us, $V'^{rr}_{00}(2) = \frac{1}{3}(1 + 2U^{gj}_{00})$. With this result, the last equation in (C.8) can be used with $b = 2$ to find

$$V^{rr}_{00} = \frac{1 + U^{gj}_{00}}{1 - U^{gj}_{00}} . \tag{C.11}$$

Similarly, the second equation in (C.8) gives:

$$V^{rs}_{0n} = \frac{1}{1 - U^{gj}_{00}} \sqrt{2}\, V'^{rs}_{0n}(2) , \qquad \text{for} \quad n \geq 1 . \tag{C.12}$$

Making use of (C.9) and $\bar{U}^{gj}_{0n} = (U^{gj}_{0n})^*$ we find that we can write, for $n \geq 1$:

$$V^{rs}_{0n} = \frac{1}{3}(\omega^{s-r} W_n + \omega^{r-s} W_n^*) , \tag{C.13}$$

where

$$W_n = \frac{\sqrt{2} U^{gj}_{0n}}{1 - U^{gj}_{00}} . \tag{C.14}$$

[1]As explained at the end of appendix E.36, U^{gj} should really be identified with $\bar{U}$ of ref.[68].

The first equation in (C.8) together with (C.9) gives us [70]

$$V^{r\,s} = \frac{1}{3}(C + \omega^{s-r}U + \omega^{r-s}\bar{U})\,, \tag{C.15}$$

where V^{rs}, U and C are regarded as matrices with indices running over $m, n \geq 1$, $C_{mn} = (-1)^m \delta_{mn}$ and U is given as

$$U_{mn} = U^{gj}_{mn} + \frac{U^{gj}_{m0}U^{gj}_{0n}}{1 - U^{gj}_{00}}\,. \tag{C.16}$$

By virtue of this relation, and the identities in (C.10) we have that the matrix U satisfies

$$\bar{U} \equiv U^* = CUC, \qquad U^2 = \bar{U}^2 = 1, \qquad U^\dagger = U\,, \quad \bar{U}^\dagger = \bar{U}\,. \tag{C.17}$$

It follows from (C.10) and (C.14) that W_n satisfies the relations:

$$W^*_n = (-1)^n W_n, \qquad \sum_{n\geq 1} W_n U_{np} = W_p, \qquad \sum_{m\geq 1} W^*_m W_m = 2V^{rr}_{00}\,. \tag{C.18}$$

Appendix D

The coefficients $\mathcal{V}_{MN}^{\alpha\beta,rs}$

In this Appendix we derive the properties of the coefficients $\mathcal{V}_{MN}^{\alpha\beta,rs}$ which has been essential for the definition of lump solutions with B field. These properties are parallel to those enjoyed by the ordinary coefficients, reported in Appendix B, and first found in [68, 69, 70, 73].

Let us quote first two straightforward properties of $\mathcal{V}_{MN}^{\alpha\beta,rs}$:

- (i) they are symmetric under the simultaneous exchange of all the three couples of indices;
- (ii) they are endowed with the property of cyclicity in the r,s indices, i.e. $\mathcal{V}^{rs} = \mathcal{V}^{r+1,s+1}$, where $r,s=4$ is identified with $r,s=1$ and we have dropped the other indices.

The first property is immediate. The second can also be proven directly from eqs.(6.25). However, since it will be an easy consequence of eq.(D.11) below, we pass immediately to the derivation of the latter.

To this end we need the following representation of the coefficients V_{0n}^{rs}, derived from [68]:

$$V_{0n}^{rs} = \begin{cases} Z_n \chi^{rs}, & n \quad \text{odd} \\ -\frac{2}{\sqrt{3}} Z_n \phi^{rs}, & n \quad \text{even} \end{cases} \tag{D.1}$$

where

$$Z_n = \sqrt{\frac{2}{3n}}\, B_0 A_n \tag{D.2}$$

The numbers B_0 and A_n were defined in ref.[68]. Notice that, since we have assumed $Z_n^{rs} = Z_n^{sr}$, we must have, by definition, $V_{0n}^{rs} = V_{n0}^{rs}$ for n even and $V_{0n}^{rs} = -V_{n0}^{rs}$ for n odd. Finally, for convenience, we introduce $Z_0 = \sqrt{\frac{b}{3}}$.

Substituting (D.1) into eqs.(6.25) and using (6.27), we obtain

$$\mathcal{V}_{NM}^{\alpha\beta,rs} = \begin{cases} \mathcal{V}_{NM}^{\alpha\beta,rs}(\infty) - \frac{6A^{-1}}{4a^2+3} K_\infty^{\alpha\beta,rs} Z_N Z_M, & N+M \quad \text{even} \\ \mathcal{V}_{NM}^{\alpha\beta,rs}(\infty) + \frac{\sqrt{3}A^{-1}}{4a^2+3} H_\infty^{\alpha\beta,rs} (-1)^N Z_N Z_M, & N+M \quad \text{odd} \end{cases} \tag{D.3}$$

In these equations

$$K_\infty^{\alpha\beta,rs} = G^{\alpha\beta}\phi^{rs} - ia\hat{\epsilon}^{\alpha\beta}\chi^{rs} \tag{D.4}$$

$$H_\infty^{\alpha\beta,rs} = 3G^{\alpha\beta}\chi^{rs} + 4ia\hat{\epsilon}^{\alpha\beta}\phi^{rs} \tag{D.5}$$

and $\mathcal{V}_{NM}^{\alpha\beta,rs}(\infty)$ is

$$\begin{aligned} \mathcal{V}_{00}^{\alpha\beta,rs}(\infty) &= G^{\alpha\beta}\delta^{rs} \\ \mathcal{V}_{0m}^{\alpha\beta,rs}(\infty) &= 0 \\ \mathcal{V}_{nm}^{\alpha\beta,rs}(\infty) &= G^{\alpha\beta}V_{nm}^{rs} \end{aligned} \tag{D.6}$$

The coefficients V_{nm}^{rs} are the same as in ref.[73] for $n, m \geq 1$.

We can also express the $\mathcal{V}_{NM}^{\alpha\beta,rs}$ in the following way

$$\mathcal{V}_{NM}^{\alpha\beta,rs} = \begin{cases} \mathcal{V}_{NM}^{\alpha\beta,rs}(0) + \frac{6A^{-1}}{4a^2+3} K_0^{\alpha\beta,rs} Z_N Z_M, & N+M \quad \text{even} \\ \mathcal{V}_{NM}^{\alpha\beta,rs}(0) + \frac{\sqrt{3}A^{-1}}{4a^2+3} H_0^{\alpha\beta,rs} (-1)^N Z_N Z_M, & N+M \quad \text{odd} \end{cases} \tag{D.7}$$

where

$$K_0^{\alpha\beta,rs} = \frac{4}{3}a^2 G^{\alpha\beta}\phi^{rs} + ia\hat{\epsilon}^{\alpha\beta}\chi^{rs} \tag{D.8}$$

$$H_0^{\alpha\beta,rs} = -4a^2 G^{\alpha\beta}\chi^{rs} + 4ia\hat{\epsilon}^{\alpha\beta}\phi^{rs} \tag{D.9}$$

and $\mathcal{V}_{NM}^{\alpha\beta,rs}(0) = G^{\alpha\beta}V_{NM}^{'rs}$ are the values taken by $\mathcal{V}_{NM}^{\alpha\beta,rs}$ for $B = 0$. As expected, the symbols $V_{NM}^{'rs}$ are the same as the coefficients $V_{nm}^{'rs}(b)$ with $n, m \geq 0$, used in [73].

Next we introduce the third root of unity $\omega = e^{i\frac{2\pi}{3}}$ and notice that

$$\phi^{rs} = \frac{1}{2}(\omega^{r-s} + \omega^{s-r}), \qquad \chi^{rs} = \frac{i}{\sqrt{3}}(\omega^{r-s} - \omega^{s-r}), \tag{D.10}$$

Inserting these relations into (D.3,D.7) and rearranging the terms we find the basic relation

$$\mathcal{V}_{NM}^{\alpha\beta,rs} = \frac{1}{3}\left(C'_{NM}G^{\alpha\beta} + \omega^{s-r}\mathcal{U}_{NM}^{\alpha\beta} + \omega^{r-s}\bar{\mathcal{U}}_{NM}^{\alpha\beta}\right) \tag{D.11}$$

where

$$\mathcal{U}_{NM}^{\alpha\beta} = \begin{cases} G^{\alpha\beta}\mathcal{U}_{NM}(\infty) + R^{\alpha\beta} Z_N Z_M, & N+M \quad \text{even} \\ G^{\alpha\beta}\mathcal{U}_{NM}(\infty) + iR^{\alpha\beta}(-1)^N Z_N Z_M, & N+M \quad \text{odd} \end{cases} \tag{D.12}$$

Moreover

$$\bar{\mathfrak{U}}^{\alpha\beta} = (\mathfrak{U}^{\beta\alpha})^* \tag{D.13}$$

where * denotes complex conjugation. In (D.11) $C'_{NM} = (-1)^N \delta_{NM}$ and

$$R^{\alpha\beta} = \frac{6A^{-1}}{4a^2+3}\left(-\frac{3}{2}G^{\alpha\beta} + \sqrt{3}a\hat{\epsilon}^{\alpha\beta}\right) \tag{D.14}$$

Moreover

$$\begin{aligned} \mathfrak{U}^{\alpha\beta}_{00}(\infty) &= G^{\alpha\beta}, \qquad \mathfrak{U}^{\alpha\beta}_{0n} = 0 \\ \mathfrak{U}^{\alpha\beta}_{nm}(\infty) &= G^{\alpha\beta}U_{nm} \end{aligned} \tag{D.15}$$

In the last equation U_{nm} coincides with the same symbol used in [73] (see eq.(B.15) in that reference).

Alternatively one can split $\mathfrak{U}$ into the $B = 0$ part and the rest. Then

$$\mathfrak{U}^{\alpha\beta}_{NM} = \begin{cases} G^{\alpha\beta}\mathfrak{U}_{NM}(0) + T^{\alpha\beta}Z_N Z_M, & N+M \quad \text{even} \\ G^{\alpha\beta}\mathfrak{U}_{NM}(0) + iT^{\alpha\beta}(-1)^N Z_N Z_M, & N+M \quad \text{odd} \end{cases} \tag{D.16}$$

where

$$T^{\alpha\beta} = \frac{12A^{-1}}{4a^2+3}\left(a^2 G^{\alpha\beta} + \frac{\sqrt{3}}{2}a\hat{\epsilon}^{\alpha\beta}\right) \tag{D.17}$$

and $\mathfrak{U}^{\alpha\beta}_{NM} = G^{\alpha\beta}U'_{NM}$. The coefficients $U'_{nm}, U'_{0n}, U'_{00}$ are the same as in ref.[73] (see eq.(B.19) therein).

Let us discuss the properties of $\mathfrak{U}$. Since

$$(\mathfrak{U}^{\alpha\beta}_{NM})^* = \begin{cases} \mathfrak{U}^{\alpha\beta}_{NM}, & N+M \quad \text{even} \\ -\mathfrak{U}^{\alpha\beta}_{NM}, & N+M \quad \text{odd} \end{cases}$$

it is easy to prove the following properties (where we use the matrix notation for the indices N, M)

$$(\mathfrak{U}^{\alpha\beta})^* = C'\mathfrak{U}^{\alpha\beta}C' \tag{D.18}$$

and

$$(\mathfrak{U}^{\alpha\beta})^\dagger = (\mathfrak{U}^{\alpha\beta})^{*T} = (C'\mathfrak{U}^{\alpha\beta}C')^T = \mathfrak{U}^{\alpha\beta} \tag{D.19}$$

Finally, if tilde denotes transposition in the indices α, β, it is possible to prove that (the proof is rather technical and deferred to the end of this Appendix)

$$(\mathcal{U}\tilde{\mathcal{U}})_{NM}^{\alpha\beta} = (\tilde{\mathcal{U}}\mathcal{U})_{NM}^{\alpha\beta} = G^{\alpha\beta}\delta_{NM} + \left(RG + G\tilde{R} + \frac{2}{3}AR\tilde{R}\right) Z_N Z_M \qquad \text{(D.20)}$$

Now, remembering that $\hat{\epsilon}^{\alpha\gamma}\hat{\epsilon}_{\gamma}{}^{\beta} = -G^{\alpha\beta}$, it is elementary to prove that

$$RG + G\tilde{R} + \frac{2}{3}AR\tilde{R} = 0 \qquad \text{(D.21)}$$

Therefore, finally,

$$(\mathcal{U}\tilde{\mathcal{U}})_{NM}^{\alpha\beta} = (\tilde{\mathcal{U}}\mathcal{U})_{NM}^{\alpha\beta} = G^{\alpha\beta}\delta_{NM} \qquad \text{(D.22)}$$

Eqs.(D.18, D.19, D.22) are the generalization of the analogous ones in [68, 69, 70, 73]. Using in particular (D.22), it is easy to prove that

$$[C'\mathcal{V}^{rs}, C'\mathcal{V}^{r's'}] = 0. \qquad \text{(D.23)}$$

This follows from

$$9[C'\mathcal{V}^{rs}, C'\mathcal{V}^{r's'}] = \omega^{s-r+r'-s'}(C'\mathcal{U}\tilde{\mathcal{U}}C' - \tilde{\mathcal{U}}\mathcal{U}) + \omega^{s-r+s'-r'}(\tilde{\mathcal{U}}\mathcal{U} - C'\mathcal{U}\tilde{\mathcal{U}}C')$$

and from eq.(D.22). In the two previous equations matrix multiplication is understood both in the indices M, N and α, β. In the same sense, on the wake of [70, 73], we can also write down the following identities

$$C'\mathcal{V}^{12}C'\mathcal{V}^{21} = C'\mathcal{V}^{21}C'\mathcal{V}^{12} = (C'\mathcal{V}^{11})^2 - C'\mathcal{V}^{11} \qquad \text{(D.24)}$$
$$(C'\mathcal{V}^{12})^3 + (C'\mathcal{V}^{21})^3 = 2(C'\mathcal{V}^{11})^3 - 3(C'\mathcal{V}^{11})^2 + G \qquad \text{(D.25)}$$

which will be needed in the next section.

Notice however that, unlike refs.[68, 69, 70, 73], we have

$$C'\mathcal{V}^{rs} = \tilde{\mathcal{V}}^{sr}C' \qquad C'\mathcal{X}^{rs} = \tilde{\mathcal{X}}^{sr}C' \qquad \text{(D.26)}$$

where tilde denotes transposition with respect to the α, β indices. Finally one can prove that

$$\begin{aligned}
&\mathcal{X} + \mathcal{X}^{12} + \mathcal{X}^{21} = \mathbb{I}\\
&\mathcal{X}^{12}\mathcal{X}^{21} = \mathcal{X}^2 - \mathcal{X}\\
&(\mathcal{X}^{12})^2 + (\mathcal{X}^{21})^2 = \mathbb{I} - \mathcal{X}^2\\
&(\mathcal{X}^{12})^3 + (\mathcal{X}^{21})^3 = 2\mathcal{X}^3 - 3\mathcal{X}^2 + \mathbb{I}
\end{aligned} \qquad \text{(D.27)}$$

In the matrix products of these identities, as well as throughout the paper, the indices α, β must be understood in alternating up/down position: $\mathcal{X}^{\alpha}{}_{\beta}$. For instance, in (D.27) $\mathbb{I}$ stands for $\delta^{\alpha}{}_{\beta}\, \delta_{MN}$.

Derivation of $(\mathcal{U}\tilde{\mathcal{U}})^{\alpha\beta}_{NM}$

We derive now eq.(D.20). This can be done starting both from the representation (D.12) and from (D.16). In the first case we need the following identities taken from the Appendix B of [73].

$$\sum_{n\geq 1} W_n U_{nm} = W_m, \qquad \sum_{n\geq 1} W_n^* W_n = 2V_{00} \tag{D.28}$$

The numbers W_n are defined via the equation

$$V_{0n}^{rs} = \frac{1}{3}(\omega^{s-r} W_n + \omega^{r-s} W_n^*) \tag{D.29}$$

On the other hand we have

$$\begin{aligned} V_{0n}^{rs} &= \frac{i}{\sqrt{3}}(\omega^{r-s} - \omega^{s-r}) Z_n, \qquad n \quad \text{odd} \\ V_{0n}^{rs} &= -\frac{1}{\sqrt{3}}(\omega^{r-s} + \omega^{s-r}) Z_n, \qquad n \quad \text{even} \end{aligned} \tag{D.30}$$

This allows us to identify W_n and Z_n as follows:

$$\begin{aligned} W_n &= -i\sqrt{3} Z_n, \qquad n \quad \text{odd} \\ W_n &= -\sqrt{3} Z_n, \qquad n \quad \text{even} \end{aligned} \tag{D.31}$$

In particular, from the second equation in (D.28), we get

$$\sum_{n\geq 1} Z_n^2 = \frac{2}{3} V_{00} \tag{D.32}$$

Next one has to consider $(\mathcal{U}\tilde{\mathcal{U}})_{NM}$ case by case according to the various possibilities for N, M. As a sample, let us consider $N = n$ odd and $M = m$ odd. Then

$$(\mathcal{U}\tilde{\mathcal{U}})_{nm} = \mathcal{U}_{n0}\tilde{\mathcal{U}}_{0m} + \sum_{k\,\text{odd}} \mathcal{U}_{nk}\tilde{\mathcal{U}}_{km} + \sum_{k\,\text{even}} \mathcal{U}_{nk}\tilde{\mathcal{U}}_{km}$$

Now we replace on the RHS the values extracted from eq.(D.12). After rearranging the terms we get

$$(\mathcal{U}\tilde{\mathcal{U}})_{nm} \;=\; G\delta_{nm} + \frac{b}{3} R\tilde{R} Z_n Z_m + R\tilde{R} Z_n Z_m \sum_{k\geq 1} Z_k^2$$

$$-\frac{i}{\sqrt{3}}G\tilde{R}\sum_{k\geq 1}U_{nk}W_k^*Z_m+\frac{i}{\sqrt{3}}RGZ_n\sum_{k\geq 1}W_kU_{km}$$
$$= G\delta_{nm}\left(RG+G\tilde{R}+\frac{2}{3}(V_{00}+\frac{b}{2})R\tilde{R}\right)Z_nZ_m \qquad \text{(D.33)}$$

where use has been made of (D.28) and (D.32). In the same way all other cases of the identity (D.20) can be proved.

Alternatively one can prove (D.20) by means of the representation (D.16). The procedure is the same, but the matrix involved is U' instead of U. For this reason we need, instead of the second eq.(D.28), the identity

$$\sum_{n\geq 1}W_nU'_{nm}=\frac{\frac{b}{2}-V_{00}}{\frac{b}{2}+V_{00}}W_m \qquad \text{(D.34)}$$

Appendix E

Some proofs

In this Appendix we collect some proofs that otherwise would have uselessy made heavy the treatment of VSFT with B field. First, we explicitly show that the ratio $\mathcal{R}$ defined in (6.45)

$$\mathcal{R} = \frac{A^4(3+4a^2)^2}{2\pi b^3(\mathrm{Det}G)^{1/4}} \frac{\mathrm{Det}(\mathbb{I}-\mathcal{X})^{3/4}\mathrm{Det}(\mathbb{I}+3\mathcal{X})^{1/4}}{\det(1-X)^{3/2}\det(1+3X)^{1/2}},$$

is indeed equal to 1. Second, we prove the fundamental properties (6.89) and (6.90) of the states $|\Lambda_n\rangle$, i.e.

$$|\Lambda_n\rangle * |\Lambda_m\rangle = \delta_{n,m}|\Lambda_n\rangle$$
$$\langle\Lambda_n|\Lambda_m\rangle = \delta_{n,m}\langle\Lambda_0|\Lambda_0\rangle$$

E.1 Proof that $\mathcal{R} = 1$

This section is devoted to the proof of

$$\mathcal{R} = 1 \tag{E.1}$$

What we need is compute the ratio of $\mathrm{Det}(\mathbb{I}-\mathcal{X})$ and $\mathrm{Det}(\mathbb{I}+3\mathcal{X})$ with respect to the squares of $\mathrm{Det}(1-X)$ and $\mathrm{Det}(1+3X)$, respectively. To this end we follow the lines of ref.[88]. To start with we rewrite $\mathcal{V}^{11} \equiv \mathcal{V}$ in a more convenient form. Following [88], we introduce the vector notation $|v_e\rangle$ and $|v_0\rangle$ by means of

$$|v_e\rangle_n = \frac{1+(-1)^n}{2}\frac{A_n}{\sqrt{n}}, \qquad |v_o\rangle_n = \frac{1-(-1)^n}{2}\frac{A_n}{\sqrt{n}},$$

The constants A_n are as in [68]. Now we can write

$$\begin{aligned}
\mathcal{V}_{00} &= \left(1 - \frac{2A^{-1}b}{4a^2+3}\right)\mathbf{1} \\
\mathcal{V}_{0n} &= -\frac{2A^{-1}\sqrt{2b}}{4a^2+3}\,\mathbf{1}\,\langle v_e|_n + i\sqrt{\frac{2b}{3}}\frac{4aA^{-1}}{4a^2+3}\,\mathbf{e}\,\langle v_o|_n, \qquad \mathcal{V}_{0n} = (-1)^n \mathcal{V}_{n0} \\
\mathcal{V}_{nm} &= \left(V_{nm} - \frac{4A^{-1}}{4a^2+3}(|v_e\rangle\langle v_e| + |v_o\rangle\langle v_o|)_{nm}\right)\mathbf{1} + i\frac{8}{\sqrt{3}}\frac{aA^{-1}}{4a^2+3}(|v_e\rangle\langle v_o| - |v_o\rangle\langle v_e|)_{nm}\,\mathbf{e}
\end{aligned} \tag{E.2}$$

where we have understood the indices α, β. They can be reinserted using

$$\mathbf{1}^\alpha{}_\beta = \delta^\alpha{}_\beta, \qquad \mathbf{e}^\alpha{}_\beta = \epsilon^\alpha{}_\beta$$

Now $\mathcal{X} = C'\mathcal{V}$ can be written in the following block matrix form

$$\mathcal{X} = \begin{pmatrix} (1-2Kb)\mathbf{1} & -2K\sqrt{2b}\,\mathbf{1}\,\langle v_e| + 4iaK\sqrt{\frac{2b}{3}}\,\mathbf{e}\,\langle v_o| \\ -2K\sqrt{2b}|v_e\rangle\,\mathbf{1} & X\mathbf{1} - 4K\,\mathbf{1}\,(|v_e\rangle\langle v_e| - |v_o\rangle\langle v_o|) \\ +4iaK\sqrt{\frac{2b}{3}}|v_o\rangle\,\mathbf{e} & +\frac{8}{\sqrt{3}}iaK\,\mathbf{e}\,(|v_e\rangle\langle v_o| + |v_o\rangle\langle v_e|) \end{pmatrix} \tag{E.3}$$

where all m, n as well as all α, β indices are understood, $K = \frac{A^{-1}}{4a^2+3}$.

The first determinant we have to compute is the one of the matrix $\mathbb{I} - \mathcal{X}$. Using (E.3) we extract from $\mathbb{I} - \mathcal{X}$ the factor $2bK$ and represent the rest in the block form

$$\frac{1}{2bK}(\mathbb{I} - \mathcal{X}) = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}$$

By a standard formula, the determinant of the RHS is given by the determinant of $\mathcal{D} - \mathcal{C}\mathcal{A}^{-1}\mathcal{B}$. After some algebra and using the obvious identity $\langle v_o|v_e\rangle = 0$, one gets

$$\begin{aligned}
\mathcal{D} - \mathcal{C}\mathcal{A}^{-1}\mathcal{B} &= \begin{pmatrix} 1 - X - \frac{4}{3}A^{-1}|v_o\rangle\langle v_o| & 0 \\ 0 & 1 - X - \frac{4}{3}A^{-1}|v_o\rangle\langle v_o| \end{pmatrix} \\
&= \left(1 - X - \frac{4}{3}A^{-1}|v_o\rangle\langle v_o|\right)\mathbb{I}
\end{aligned}$$

The rest of the computation is straightforward,

$$\begin{aligned}
\mathrm{Det}(\mathbb{I} - \mathcal{X}) &= (2bK)^2\left(\mathrm{Det}\left(1 - X - \frac{4}{3}A^{-1}|v_o\rangle\langle v_o|\right)\right)^2 \\
&= (2bK)^2\,(\mathrm{Det}(1-X))^2\left(\mathrm{Det}\left(1 - \frac{4}{3}A^{-1}\frac{1}{1-X}|v_o\rangle\langle v_o|\right)\right)^2 \\
&= \left(\frac{b}{A}\right)^4\left(\frac{1}{4a^2+3}\right)^2(\mathrm{Det}(1-X))^2
\end{aligned} \tag{E.4}$$

In the last step we have used the identities, see [88],

$$\mathrm{Det}\left(1-\frac{4}{3}A^{-1}\frac{1}{1-X}|v_o\rangle\langle v_o|\right)=1-\frac{4}{3}A^{-1}\langle v_o|\frac{1}{1-X}|v_o\rangle \tag{E.5}$$

and

$$\langle v_o|\frac{1}{1-X}|v_o\rangle=\frac{3}{4}V_{00} \tag{E.6}$$

The treatment of $\mathrm{Det}(\mathbb{I}+3\mathfrak{X})$ is less trivial. We start again by writing $(\mathbb{I}+3\mathfrak{X})$ in block matrix form

$$\mathbb{I}+3\mathfrak{X}=\begin{pmatrix} (4-6Kb)\mathbf{1} & -6K\sqrt{2b}\,\mathbf{1}\,\langle v_e|+4iaK\sqrt{6b}\,\mathsf{e}\,\langle v_o| \\ -6K\sqrt{2b}\,|v_e\rangle\,\mathbf{1} & (1+3X)\mathbf{1}-12K\,\mathbf{1}\,(|v_e\rangle\langle v_e|-|v_o\rangle\langle v_o|) \\ +4iaK\sqrt{6b}\,|v_o\rangle\,\mathsf{e} & +8\sqrt{3}iaK\,\mathsf{e}\,(|v_e\rangle\langle v_o|+|v_o\rangle\langle v_e|) \end{pmatrix} \tag{E.7}$$

and set

$$\mathbb{I}+3\mathfrak{X}\equiv(4-6bK)\begin{pmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{pmatrix} \tag{E.8}$$

Therefore

$$\begin{aligned}\mathrm{Det}(\mathbb{I}+3\mathfrak{X}) &= (4-6bK)^2\mathrm{det}\left(\mathcal{D}-\mathcal{C}\mathcal{A}^{-1}\mathcal{B}\right)\\ &= (4-6bK)^2\,(\mathrm{det}(1+3X))^2\,\mathrm{det}\left(\frac{1}{1+3X}\left(\mathcal{D}-\mathcal{C}\mathcal{A}^{-1}\mathcal{B}\right)\right)\end{aligned} \tag{E.9}$$

The last expression is formal. In fact X has an eigenvalue $-\frac{1}{3}$ which renders the RHS of (E.9) ill–defined. To avoid this we follow [88] and introduce the regularized inverse

$$Y_\varepsilon=\frac{1}{1+3X-\varepsilon^2X} \tag{E.10}$$

where ε is a small parameter, and replace it into (E.9). After some algebra we find

$$Y_\varepsilon\left(\mathcal{D}-\mathcal{C}\mathcal{A}^{-1}\mathcal{B}\right)=\mathbb{A}\cdot\mathbb{B} \tag{E.11}$$

The matrices in the RHS are given by

$$\mathbb{A}=\begin{pmatrix}1+\alpha Y_\varepsilon|v_e\rangle\langle v_e|+\beta Y_\varepsilon|v_o\rangle\langle v_o| & 0\\ 0 & 1+\alpha Y_\varepsilon|v_e\rangle\langle v_e|+\beta Y_\varepsilon|v_o\rangle\langle v_o|\end{pmatrix} \tag{E.12}$$

where

$$\alpha=-\frac{24K}{2-3bK},\qquad \beta=12K\frac{2-A^{-1}}{2-3bK}, \tag{E.13}$$

and

$$\mathcal{B} = \begin{pmatrix} 1 & \lambda Y_\varepsilon |v_2\rangle\langle v_o| + \mu Y_\varepsilon |v_o\rangle\langle v_e| \\ -\lambda Y_\varepsilon |v_e\rangle\langle v_o| - \mu Y_\varepsilon |v_o\rangle\langle v_e| & 1 \end{pmatrix} \tag{E.14}$$

where,

$$\lambda = \frac{\gamma}{1+\alpha\langle v_e|Y_\varepsilon|v_e\rangle}, \qquad \mu = \frac{\gamma}{1+\beta\langle v_0|Y_\varepsilon|v_0\rangle}, \qquad \gamma^2 + \alpha\beta = -\frac{4}{V_{00}}\beta \tag{E.15}$$

Now, after some computation,

$$\det\mathcal{A} = (1+\alpha\langle v_e|Y_\varepsilon|v_e\rangle)^2\,(1+\beta\langle v_o|Y_\varepsilon|v_o\rangle)^2 \tag{E.16}$$

and

$$\det\mathcal{B} = \left(1 + \frac{\gamma^2\langle v_e|Y_\varepsilon|v_e\rangle\langle v_o|Y_\varepsilon|v_0\rangle}{(1+\alpha\langle v_e|Y_\varepsilon|v_e\rangle)\,(1+\beta\langle v_o|Y_\varepsilon|v_o\rangle)}\right)^2 \tag{E.17}$$

As a consequence

$$\det\mathcal{A}\,\det\mathcal{B} = \left(1+\alpha\langle v_e|Y_\varepsilon|v_e\rangle + \beta\langle v_o|Y_\varepsilon|v_0\rangle\left(1-\frac{4}{V_{00}}\langle v_e|Y_\varepsilon|v_e\rangle\right)\right)^2 \tag{E.18}$$

Now we can remove the regulator ε by using the basic result of [88]:

$$\lim_{\varepsilon\to 0}\left(1-\frac{4}{V_{00}}\langle v_e|Y_\varepsilon|v_e\rangle\right)\,\langle v_o|Y_\varepsilon|v_o\rangle = \frac{\pi^2}{12V_{00}} \tag{E.19}$$

and

$$\langle v_e|\frac{1}{1+3X}|v_e\rangle = \frac{V_{00}}{4}.$$

Inserting this result in (E.18) we find

$$\det\mathcal{A}\,\det\mathcal{B} = \frac{A^2}{(8a^2A+6V_{00})^2}\left(8a^2+\frac{2\pi^2}{A^2}\right)^2 \tag{E.20}$$

As a consequence of eqs.(E.9,E.11,E.18,E.20) we find

$$\frac{\mathrm{Det}(\mathbb{I}+3\mathcal{X})}{(\mathrm{Det}(1+3X))^2} = \frac{4}{(4a^2+3)^2}\left(8a^2+\frac{2\pi^2}{A^2}\right)^2 \tag{E.21}$$

Finally, substituting this and (E.4) into $\mathcal{R}$, we get

$$\mathcal{R} \;=\; \frac{A^4(3+4a^2)^2}{2\pi b^3(\mathrm{Det}G)^{1/4}}\frac{\mathrm{Det}(\mathbb{I}-\mathcal{X})^{3/4}\mathrm{Det}(\mathbb{I}+3\mathcal{X})^{1/4}}{\det(1-X)^{3/2}\mathrm{Det}(1+3X)^{1/2}} = 1 \tag{E.22}$$

This is what we wanted to show. It implies

$$\frac{\mathfrak{e}_{23}}{\mathfrak{e}_{25}} = \frac{(2\pi)^2}{(\mathrm{Det}G)^{1/4}} \tag{E.23}$$

which corresponds to the expected result for this ratio, as explained in [108]. We remark that (E.21) implies that the eigenvalue $-\frac{1}{3}$ is also contained in the spectrum of $\mathcal{X}$ with double multiplicity with respect to X.

E.2 Proofs of eqs.(6.89) and (6.90)

The star product $|\Lambda_n\rangle * |\Lambda_{n'}\rangle$ can be evaluated by using the explicit expression of the Laguerre polynomials

$$|\Lambda_n\rangle * |\Lambda_{n'}\rangle = \Big((-\kappa)^n \sum_{k=0}^{n} \binom{n}{k} \frac{(-\mathrm{x}/\kappa)^k}{k!}|\mathcal{S}_\perp\rangle\Big) * \Big((-\kappa)^{n'} \sum_{p=0}^{n'} \binom{n'}{p} \frac{(-\mathrm{x}/\kappa)^p}{p!}|\mathcal{S}_\perp\rangle\Big) \tag{E.24}$$

Therefore we need to compute $(\mathrm{x}^k|\mathcal{S}_\perp\rangle) * (\mathrm{x}^p|\mathcal{S}_\perp\rangle)$. According to [74], this is given by

$$\begin{aligned}(\mathrm{x}^k|\mathcal{S}_\perp\rangle) * (\mathrm{x}^p|\mathcal{S}_\perp\rangle) = \; & (\xi\tau C')^{\alpha_1}_{l_1}\dots(\xi\tau C')^{\alpha_k}_{l_k}\zeta^{\beta_1}_{j_1}\dots\zeta^{\beta_k}_{j_k}\frac{\partial}{\partial\mu^{\alpha_1}_{l_1}}\cdots\frac{\partial}{\partial\mu^{\alpha_k}_{l_k}}\frac{\partial}{\partial\mu^{\beta_1}_{j_1}}\cdots\frac{\partial}{\partial\mu^{\beta_k}_{j_k}}\\ \cdot\; & (\xi\tau C')^{\bar\alpha_1}_{\bar l_1}\dots(\xi\tau C')^{\bar\alpha_p}_{\bar l_p}\zeta^{\bar\beta_1}_{\bar j_1}\dots\zeta^{\bar\beta_p}_{\bar j_p}\frac{\partial}{\partial\bar\mu^{\bar\alpha_1}_{\bar l_1}}\cdots\frac{\partial}{\partial\bar\mu^{\bar\alpha_p}_{\bar l_p}}\frac{\partial}{\partial\bar\mu^{\bar\beta_1}_{\bar j_1}}\cdots\frac{\partial}{\partial\bar\mu^{\bar\beta_p}_{\bar j_p}}\\ \cdot\; & \exp\Big(-\chi^T\mathcal{K}^{-1}M - \frac{1}{2}M^T\mathcal{V}\mathcal{K}^{-1}M\Big)|\mathcal{S}_\perp\rangle\Big|_{\mu=\bar\mu=0}\end{aligned} \tag{E.25}$$

where

$$\mathcal{K} = \mathbb{I} - \mathcal{T}\mathcal{X}, \qquad \mathcal{V} = \begin{pmatrix} \mathcal{V}^{11} & \mathcal{V}^{12} \\ \mathcal{V}^{21} & \mathcal{V}^{22}\end{pmatrix} \tag{E.26}$$

and

$$M = \begin{pmatrix}\mu \\ \bar\mu\end{pmatrix}, \qquad \chi^T = (a^\dagger\mathcal{V}^{12}, a^\dagger\mathcal{V}^{21}), \qquad \chi^T\mathcal{K}^{-1}M = a^\dagger C'(\rho_1\mu + \rho_2\bar\mu) \tag{E.27}$$

The explicit computation, at first sight, looks daunting. However, we may avail ourselves of the following identities

$$\begin{aligned}\xi^T(\mathcal{V}\mathcal{K}^{-1})_{\alpha\alpha}\zeta &= \xi^T\tau C'(\mathcal{V}\mathcal{K}^{-1})_{\alpha\alpha}\tau C'\zeta = \xi^T C'\frac{\mathcal{T}}{\mathbb{I}-\mathcal{T}^2}\zeta = 0\\ \xi^T\tau C'(\mathcal{V}\mathcal{K}^{-1})_{\alpha\alpha}\zeta &= \xi^T(\mathcal{V}\mathcal{K}^{-1})_{\alpha\alpha}\tau C'\zeta = \xi^T\tau\frac{\mathcal{T}}{\mathbb{I}-\mathcal{T}^2}\zeta = -\kappa\end{aligned} \tag{E.28}$$

for $\alpha = 1, 2$, and

$$\begin{aligned}
\xi^T(\mathcal{V}\mathcal{K}^{-1})_{12}\zeta &= \xi^T\tau C'(\mathcal{V}\mathcal{K}^{-1})_{21}\tau C'\zeta = -\xi^T C'\frac{\mathcal{T}}{\mathbb{I}-\mathcal{T}^2}\zeta = 0\\
\xi^T(\mathcal{V}\mathcal{K}^{-1})_{21}\zeta &= \xi^T\tau C'(\mathcal{V}\mathcal{K}^{-1})_{12}\tau C'\zeta = \xi^T C'\frac{1}{\mathbb{I}-\mathcal{T}^2}\zeta = 0\\
\xi^T(\mathcal{V}\mathcal{K}^{-1})_{12}\tau C'\zeta &= \xi^T\tau C'(\mathcal{V}\mathcal{K}^{-1})_{21}\zeta = \xi^T\tau\frac{1}{\mathbb{I}-\mathcal{T}^2}\zeta = -1\\
\xi^T\tau C'(\mathcal{V}\mathcal{K}^{-1})_{12}\zeta &= \xi^T(\mathcal{V}\mathcal{K}^{-1})_{21}\tau C'\zeta = -\xi^T\tau\frac{\mathcal{T}}{\mathbb{I}-\mathcal{T}^2}\zeta = \kappa
\end{aligned} \tag{E.29}$$

Moreover

$$\begin{aligned}
(\chi^T\mathcal{K}^{-1})_1\xi = 0, &\qquad (\chi^T\mathcal{K}^{-1})_1\tau C'\xi = a^\dagger\tau\xi\\
(\chi^T\mathcal{K}^{-1})_2\xi = a^\dagger C'\xi, &\qquad (\chi^T\mathcal{K}^{-1})_1\tau C'\xi = 0
\end{aligned} \tag{E.30}$$

with analogous equations for ζ.

In evaluating (E.28, E.29, E.30) we have used the methods of ref.[74] (see also [109]), together with eqs.(6.91, 6.95). These results are all we need to explicitly compute (E.25). In fact it is easy to verify that the latter can be mapped to a rather simple combinatorial problem. To show this we introduce generic variables $x, y, \bar{x}, \bar{y}$, and make the following formal replacements:

$$\begin{aligned}
A \equiv \chi^T\mathcal{K}^{-1}M &\longrightarrow x(a^\dagger\tau\xi) + \bar{y}(a^\dagger C'\zeta),\\
B \equiv M^T\mathcal{V}\mathcal{K}^{-1}M &\longrightarrow (-\kappa xy + \kappa x\bar{y} - \bar{x}y - \kappa\bar{x}\bar{y})
\end{aligned} \tag{E.31}$$

and

$$(\tau C'\xi)_l^\alpha\frac{\partial}{\partial\mu_l^\alpha} = \partial_x, \quad \zeta_j^\beta\frac{\partial}{\partial\mu_j^\beta} = \partial_y, \quad (\tau C'\xi)_{\bar{l}}^{\bar{\alpha}}\frac{\partial}{\partial\bar{\mu}_{\bar{l}}^{\bar{\alpha}}} = \partial_{\bar{x}}, \quad \zeta_{\bar{j}}^{\bar{\beta}}\frac{\partial}{\partial\bar{\mu}_{\bar{j}}^{\bar{\beta}}} = \partial_{\bar{y}}, \tag{E.32}$$

Then (E.25) is equivalent to

$$\partial_x^k\partial_y^k\partial_{\bar{x}}^p\partial_{\bar{y}}^p\; e^{-A-\frac{1}{2}B}\Big|_{x=\bar{x}=y=\bar{y}=0} \tag{E.33}$$

This in turn can be easily calculated and gives

$$\sum_{m=0}^{[p,k]}\mathrm{x}^m\frac{k!p!}{m!}\sum_{l=m}^{[p,k]}(-1)^{l+m}\binom{k}{l}\binom{p}{l}\binom{l}{m}\kappa^{p+k-l-2m} \tag{E.34}$$

where $[n, m]$ stands for the minimum between n and m. Now we insert this back into the original equation (E.24), we find

$$|\Lambda_n\rangle * |\Lambda_{n'}\rangle = \sum_{k=0}^{n}\sum_{p=0}^{n'}\sum_{m=0}^{[p,k]}\sum_{l=0}^{[p-m,k-m]} \frac{(-1)^{p+k+l}}{m!} \cdot \kappa^{n+n'-l-2m} \binom{n}{k}\binom{n'}{p}\binom{k}{m}\binom{k-m}{l}\binom{p}{l+m} \mathrm{x}^m|\mathcal{S}_\perp\rangle \tag{E.35}$$

In order to evaluate these summations we split them as follows

$$\sum_{k=0}^{n}\sum_{p=0}^{n'}\sum_{m=0}^{[p,k]}\sum_{l=0}^{[p-m,k-m]}(\ldots) = \sum_{k=0}^{n}\left(\sum_{p=k+1}^{n'}\sum_{m=0}^{k}\sum_{l=0}^{k-m} + \sum_{p=0}^{k}\sum_{m=0}^{p}\sum_{l=0}^{p-m}\right)(\ldots) \tag{E.36}$$

Next we replace $l \to l+m$ and (E.36) becomes

$$\sum_{k=0}^{n}\left(\sum_{m=0}^{k}\sum_{l=m}^{k}\sum_{p=k+1}^{n'} + \sum_{m=0}^{k}\sum_{p=m}^{k}\sum_{l=m}^{p}\right)(\ldots) =$$

$$= \sum_{k=0}^{n}\left(\sum_{m=0}^{k}\sum_{l=m}^{k}\sum_{p=k+1}^{n'} + \sum_{m=0}^{k}\sum_{l=m}^{k}\sum_{p=l}^{k}\right)(\ldots) = \sum_{k=0}^{n}\sum_{m=0}^{k}\sum_{l=m}^{k}\sum_{p=l}^{n'}(\ldots) \tag{E.37}$$

Summarizing, we have now to calculate

$$|\Lambda_n\rangle * |\Lambda_{n'}\rangle = \sum_{k=0}^{n}\sum_{m=0}^{k}\sum_{l=m}^{k}\sum_{p=l}^{n'} \frac{(-1)^{p+k+l+m}}{m!} \cdot \kappa^{n+n'-l-m} \binom{n}{k}\binom{n'}{p}\binom{k}{m}\binom{k-m}{l-m}\binom{p}{l} \mathrm{x}^m|\mathcal{S}_\perp\rangle \tag{E.38}$$

Now

$$\sum_{p=l}^{n'}(-1)^{p+l}\binom{n'}{p}\binom{p}{l} = \binom{n'}{l}\sum_{p=0}^{n'-l}(-1)^p\binom{n'-l}{p} = \binom{n'}{l}(1-1)^{n'-l} \tag{E.39}$$

This vanishes unless $l = n'$. In the case $n' > n$, $l < n'$. Inserting this into (E.38), for $n' > n$ we get 0.

In the case $n = n'$, l can take the value n'. This corresponds to the case $k = p = l = n = n'$ in eq.(E.38). The result is easily derived

$$|\Lambda_n\rangle * |\Lambda_n\rangle = \sum_{m=0}^{n} \frac{(-1)^{n+m}}{m!}\binom{n}{m}\kappa^{n-m}\mathrm{x}^m|\mathcal{S}_\perp\rangle = (-\kappa)^n L_n\left(\frac{\mathrm{x}}{\kappa}\right)|\mathcal{S}_\perp\rangle = |\Lambda_n\rangle \tag{E.40}$$

This proves eq.(6.89).

One could as well derive these results numerically. For instance, in order to obtain (E.40) one could proceed, alternatively, as follows. After setting $n = n'$ in (E.35), one realizes that $|\Lambda_n\rangle * |\Lambda_n\rangle$ has the form

$$|\Lambda_n\rangle * |\Lambda_n\rangle = \sum_{m=0}^{n} F_m^{(n)} \left(\frac{\mathbf{x}}{\kappa}\right)^m |\mathcal{S}_\perp\rangle \tag{E.41}$$

where

$$\begin{aligned} F_m^{(n)} &= 2\sum_{p=0}^{n-m}\sum_{k=0}^{p}\sum_{l=0}^{k} \frac{(-1)^{p+k+l}\kappa^{2n-l-m}(n!)^2}{(m!)^2(n-k-m)!(n-p-m)!l!(l+m)!(k-l)!(p-l)!} \\ &- \sum_{p=0}^{n-m}\sum_{l=0}^{p} \frac{(-1)^l \kappa^{2n-l-m}(n!)^2}{[m!(n-p-m)!(p-l)!]^2 l!l+m)!} \end{aligned} \tag{E.42}$$

This corresponds to the desired result if

$$F_m^{(n)} = \frac{(-1)^{n+m}}{m!}\kappa^n \binom{n}{m} \tag{E.43}$$

Using *Mathematica* one can prove (numerically) that this is true for any value of n and m a computer is able to calculate in a reasonable time.

The value of the SFT action for any solution $|\Lambda_n\rangle$ is given by

$$\mathcal{S}(\Lambda_n) = \mathcal{K}\langle\Lambda_n|\Lambda_n\rangle \tag{E.44}$$

where $\mathcal{K}$ contains the ghost contribution. As shown in [75], $\mathcal{K}$ is infinite unless it is suitably regularized. Nevertheless, as argued there, $|\Lambda_n\rangle$, together with the corresponding ghost solution, can be taken as a representative of a corresponding class of smooth solutions.

Our task now is to calculate $\langle\Lambda_n|\Lambda_n\rangle$. However it may be important to consider states which are linear combinations of $|\Lambda_n\rangle$. In order to evaluate their action we have to be able to compute $\langle\Lambda_n|\Lambda_{n'}\rangle$. Without loss of generality we can assume $n' > n$. By defining $\tilde{\mathbf{x}} = (a^\dagger\tau C'\xi)\,(a^\dagger\zeta)$ we get

$$\begin{aligned} \langle\Lambda_n|\Lambda_{n'}\rangle &= (-\kappa)^{n+n'}\langle\tilde{0}|L_n(\tilde{\mathbf{x}}/\kappa)e^{-\frac{1}{2}a\tilde{\mathcal{S}}a}L_{n'}(\mathbf{x}/\kappa)e^{\frac{1}{2}a^\dagger\mathcal{S}a^\dagger}|\tilde{0}\rangle \\ &= L_n\left(\frac{1}{\kappa}(\tau C'\xi)_l^\alpha\zeta_j^\beta\frac{\partial}{\partial\lambda_l^\alpha}\frac{\partial}{\partial\lambda_j^\beta}\right) L_{n'}\left(\frac{1}{\kappa}(\tau\xi)_l^\alpha(C'\zeta)_j^\beta\frac{\partial}{\partial\mu_l^\alpha}\frac{\partial}{\partial\mu_j^\beta}\right) \\ &\cdot \frac{1}{\sqrt{\det(\mathbb{I}-\mathcal{T}^2)}}\, e^{\lambda C'\frac{1}{\mathbb{I}-\mathcal{T}^2}C'\mu-\frac{1}{2}\lambda C'\frac{\mathcal{T}}{\mathbb{I}-\mathcal{T}^2}\lambda-\frac{1}{2}\mu\frac{\mathcal{T}}{\mathbb{I}-\mathcal{T}^2}C'\mu}\Big|_{\lambda=\mu=0} \end{aligned} \tag{E.45}$$

For the derivation of this equation, see [73, 70, 74]. Now, let us set

$$A = \lambda C' \frac{1}{\mathbb{I} - \mathcal{T}^2} C' \mu, \qquad B = \lambda C' \frac{\mathcal{T}}{\mathbb{I} - \mathcal{T}^2} \lambda, \qquad C = \mu \frac{\mathcal{T}}{\mathbb{I} - \mathcal{T}^2} C' \mu$$

and introduce the symbolic notation

$$(\tau C' \xi)_l^\alpha \frac{\partial}{\partial \lambda_l^\alpha} = \partial_x, \quad \zeta_j^\beta \frac{\partial}{\partial \lambda_j^\beta} = \partial_y, \qquad (\tau \xi)_l^\alpha \frac{\partial}{\partial \mu_l^\alpha} = \partial_{\bar{x}}, \qquad (C' \zeta)_j^\beta \frac{\partial}{\partial \mu_j^\beta} = \partial_{\bar{y}}, \tag{E.46}$$

Then, using (6.95) and (E.28, E.29), we find

$$\begin{aligned} &\partial_x \partial_{\bar{x}} A = 0, \quad &&\partial_x \partial_{\bar{y}} A = -1, \quad &&\partial_y \partial_{\bar{x}} A = -1, \quad &&\partial_y \partial_{\bar{y}} A = 0 \\ &\partial_x \partial_x B = 0, &&\partial_x \partial_y B = -2\kappa, &&\partial_y \partial_y B = 0, && \\ &\partial_{\bar{x}} \partial_{\bar{x}} C = 0, &&\partial_{\bar{x}} \partial_{\bar{y}} C = -2\kappa, &&\partial_{\bar{y}} \partial_{\bar{y}} C = 0 && \end{aligned} \tag{E.47}$$

We can therefore make the replacement

$$A - \frac{1}{2} B - \frac{1}{2} C \to \kappa x y + \kappa \bar{x} \bar{y} - x \bar{y} - \bar{x} y \tag{E.48}$$

In (E.45) we have to evaluate such terms as

$$\partial_x^k \partial_y^k \partial_{\bar{x}}^p \partial_{\bar{y}}^p (\kappa x y + \kappa \bar{x} \bar{y} - x \bar{y} - \bar{x} y)^{k+p}$$

for any two natural numbers k and p. It is easy to obtain

$$\frac{1}{(p+k)!} \partial_x^k \partial_y^k \partial_{\bar{x}}^p \partial_{\bar{y}}^p (\kappa x y + \kappa \bar{x} \bar{y} - x \bar{y} - \bar{x} y)^{k+p} = \sum_{s=0}^{[p,k]} \binom{k}{s} \binom{p}{s} k!\, p!\, \kappa^{p+k-2s} \tag{E.49}$$

Therefore we have

$$\begin{aligned} &\langle \Lambda_n | \Lambda_{n'} \rangle \\ &= \sum_{k=0}^{n} \sum_{p=0}^{n'} \frac{(-1)^{k+p} \kappa^{n+n'-p-k}}{k!p!} \binom{n}{k} \binom{n'}{p} \partial_x^k \partial_y^k \partial_{\bar{x}}^p \partial_{\bar{y}}^p \, e^{A - \frac{1}{2}B - \frac{1}{2}C} \Big|_{x=y=\bar{x}=\bar{y}=0} \langle \mathcal{S}_\perp | \mathcal{S}_\perp \rangle \\ &= \sum_{k=0}^{n} \sum_{p=0}^{n'} \frac{(-1)^{k+p}}{k!p!} \binom{n}{k} \binom{n'}{p} \sum_{s=0}^{[p,k]} \binom{k}{s} \binom{p}{s} k!p! \kappa^{n+n'-2s} \langle \mathcal{S}_\perp | \mathcal{S}_\perp \rangle \end{aligned} \tag{E.50}$$

As in the previous subsection, we can rearrange the summations as follows,

$$\begin{aligned} \sum_{k=0}^{n} \sum_{p=0}^{n'} \sum_{p=0}^{[p,k]} (\dots) &= \sum_{k=0}^{n} \left(\sum_{p=0}^{k} \sum_{s=0}^{p} + \sum_{p=k+1}^{n'} \sum_{s=0}^{k} \right) (\dots) \\ &= \sum_{k=0}^{n} \left(\sum_{s=0}^{k} \sum_{p=s}^{k} + \sum_{s=0}^{k} \sum_{p=k+1}^{n'} \right) (\dots) = \sum_{k=0}^{n} \sum_{s=0}^{k} \sum_{p=s}^{n'} (\dots) \end{aligned} \tag{E.51}$$

In conclusion we have to compute

$$\langle \Lambda_n | \Lambda_{n'} \rangle = \sum_{k=0}^{n} \sum_{s=0}^{k} \sum_{p=s}^{n'} (-1)^{p+k} \frac{n! n'!}{(n-k)!(n'-p)!(k-s)!(p-s)!(s!)^2} \kappa^{n+n'-2s} \quad (E.52)$$

Now,

$$\sum_{p=s}^{n'} (-1)^p \frac{1}{(n'-p)!(p-s)!} = \sum_{p=0}^{n'-s} \frac{(-1)^{p+s}}{(n'-s)!} \binom{n'-s}{p} = \frac{(-1)^s}{(n'-s)!} (1-1)^{n'-s} \quad (E.53)$$

The right end side vanishes if $n' \neq s$, which is certainly true if $n' > n$. Therefore in such a case, inserting (E.53) into (E.52) we get $\langle \Lambda_n | \Lambda_{n'} \rangle = 0$. When $s = n'$, eq.(E.53) is ambiguous. But this corresponds to $p = k = s = n = n'$ in (E.52). The relevant contribution is elementary to compute, and one gets

$$\langle \Lambda_n | \Lambda_n \rangle = \langle \Lambda_0 | \Lambda_0 \rangle \quad (E.54)$$

This completes the proof of (6.90).

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