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Dirichlet Forms and Markovian Semigroups on Standard Forms of von Neumann Algebras

Thesis submitted for the degree of
"Doctor Philosophiæ"

CANDIDATE
Fabio Cipriani

SUPERVISOR
Prof. Gianfausto Dell'Antonio

October 1992

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Dedicated to
my Family
and Antonella

Contents

Introduction, motivations and description of the main result.

In the present work a central role will be played by the notion of Standard Form $(\mathfrak{M}, \mathfrak{H}, \mathcal{J}, \mathcal{P})$ on a von Neumann algebra $\mathfrak{M}$. Here $\mathfrak{H}$ is a Hilbert space where $\mathfrak{M}$ acts, J is a unitary involution in $\mathfrak{H}$ and $\mathcal{P}$ is a selfdual closed convex cone in $\mathfrak{H}$. These objects are required to satisfy the following conditions:

i) $J\mathfrak{M}J = \mathfrak{M}'$

ii) $JxJ = x^* \quad \forall x \in \mathfrak{M} \cap \mathfrak{M}'$

iii) $J\xi = \xi \quad \forall \xi \in \mathfrak{H}$

iv) $xJxJ(\mathcal{P}) \subseteq \mathcal{P} \quad x \in \mathfrak{M}$

When $\mathfrak{M}$ is the commutative algebra L^∞ of some measure space, the above structure can be realized by considering the algebra as acting by multiplication in the Hilbert space L^2, then taking the complex conjugation as operator J and the positive elements in L^2 as cone $\mathcal{P}$.

The Standard forms of a von Neumann algebra were introduced in the study of operator algebras by H. Araki, A. Connes and U. Haagerup (see [Ara1], [Con1], [Haa1]) and it became a central tool in the area.

The relevance of the notion of Standard Form in our work, lies in the fact that it enable us to define, on $\mathfrak{H}$, Markovian semigroups and their associated Dirichlet forms; all these structures, arise naturally when one consider the following two kind of problems.

a) In the theory of Quantum Open System, in the theory of Quantum Diffusions and in Quantum Statistical Mechanics there are situations in which one considers a given von Neumann algebra $\mathfrak{M}$ representing observable of the system, and on it, a faithful normal state ϕ_0 and an operator δ. Then, one is interested in finding sufficient conditions on δ to guarantee that it generates a positivity preserving, ϕ_0-invariant semigroup on $\mathfrak{M}$, representing some dissipative evolution (these semigroups on $\mathfrak{M}$ are called Markovian).

Due to problems concerning the domain of δ, it is very difficult to verify dissipativeness and density conditions which ensure δ to be a generator (see [Eva1], [Bra3], [Fri], [Lin]). One recognizes that the same problems appears when one considers second order differential operators on a Riemannian manifold as generators of Markov Processes. In that particular (commutative) case, the Dirichlet form approach can be used to construct a well defined generator δ (and its associated

Markov semigroup on L^∞) from a Dirichlet form and its associated Markov semigroup on the Hilbert space L^2 (see [Dav3], [Fuk]). In the general non-commutative case described above, the Standard Form structure will enable us to perform the same program. In this sense Standard Forms permit a Dirichlet form approach to study generators of Markovian semigroups on von Neumann algebras.

b) The second kind of problems we have in mind, is the existence and uniqueness of an eigenvector associated to the greatest eigenvalue $\|A\|$ of a bounded, symmetric operator on $\mathfrak{H}$.

In case the Hilbert space coincide with a classical L^2 space one can try to exploit the fact that A preserves the cone of positive elements in L^2. In general, although one can represent $\mathfrak{H}$ as a classical L^2 space (for example diagonalizing A), this correspondence will define a cone $\mathcal{P}$ in $\mathfrak{H}$ which, in general, is not natural. We understand this by a fundamental result of A. Connes which has shown that a closed convex cone $\mathcal{P}$ in a Hilbert space $\mathfrak{H}$, is part of a Standard Form $(\mathfrak{M}, \mathfrak{H}, \mathcal{J}, \mathcal{P})$, if and only if it is *selfdual, homogeneous and oriented* (see [Con1]).

We will show that the Standard Form structure enable us to develop a Perron-Frobenius theory for operators which preserve the cone $\mathcal{P}$ and that this results apply naturally to the study of the Ground State problem for operators on $\mathfrak{H}$ associated to Dirichlet forms.

The study of Markov semigroups on Hilbert spaces acted on by a finite von Neumann algebra was initiated by L. Gross [Gro1]. In particular, he formulated a very clever version of the Perron-Frobenius theory and applied it to the study of the Ground State problem for Fermion systems. In this case the Standard Form is constructed representing the antisymmetric Fock-Hilbert space as the L^2 space associated to the Clifford algebra and a finite trace on it. W. Faris considered in [Far1],positivity preserving semigroups on ordered Hilbert spaces and gave applications to the uniqueness of the Ground State for Fermion system. Subsequently, L. Gross [Gro2] proved in connection with his Perron-Frobenius theory, the equivalence between hypercontractivity of Markov semigroups and logarithmic Sobolev inequalities for the associated Dirichlet forms; in particular he considered the Clifford-Dirichlet form, providing also the first example of non-commutative logarithmic Sobolev inequalities.

Some years later, S. Albeverio and R. Hoegh-Krohn [AHK] initiated the study of Dirichlet forms on finite von Neumann algebras, emphasizing the potential value of this approach in connection with problem a). In their work [Dav4], E.B. Davies and M. Lindsay, starting from [AHK], formulated a general theory for Markovian

semigroup and Dirichlet forms on finite von Neumann algebras, studying in particular their L^p-properties. E. Carlen and E. Lieb found in [Car] the best constant in the logarithmic Sobolev inequality for the Clifford-Dirichlet form introduced by L. Gross. Recently J. L. Sauvageot [Sau] has applied Dirichlet form techniques to foliations theory constructing the *transverse heat-semigroup* on the C^*-algebra associated to a Riemannian foliation.
In the present work we generalize part of the above results when $\mathfrak{M}$ is a general σ-finite von Neumann algebra.
In chapter 1 we define Markovian semigroups on Hilbert spaces $\mathfrak{H}$ of a Standard Forms and we prove they induce $\sigma(\mathfrak{M};\mathfrak{M}_*)$-continuous, positivity preserving, uniformly bounded semigroups on $\mathfrak{M}$ and strongly continuous semigroups on the predual $\mathfrak{M}_*$. We study also the L^p-contraction properties of these semigroups we give in §1.4 examples of constructions of semigroups.
In chapter 2 we study ergodic properties of Markovian semigroups introducing the properties of *indecomposability* and *hypercontractivity*, and proving also Perron-Frobenius Uniqueness and Existence type results.

In chapter 3 we introduce Dirichlet Forms and we prove that they are in one to one correspondence with Markovian semigroups. In §3.2 we provide some criteria which ensure that a form is markovian and closable. In §3.3, we apply this criteria to construct examples of Dirichlet forms. In §3.4 we apply the above results to problems of essential selfadjointness of perturbations of Dirichlet forms. In Appendix A and B we briefly review the theory of Standard Forms and non-commutative L^p-spaces.

Chapter 1.

Markovian Semigroups

Let $(\mathfrak{M}, \mathfrak{H}, J, \mathcal{P})$ be a *standard form* on the von Neumann algebra $\mathfrak{M}$, where $\mathfrak{H}$ is the Hilbert space where $\mathfrak{M}$ acts, J is the isometric involution in $\mathfrak{H}$ and $\mathcal{P}$ is the selfdual cone in $\mathfrak{H}$. Although some results (§1.1) can be obtained at this level of generality, we will consider in this chapter $\mathfrak{M}$ to be σ- finite, since this assumption guarantees the existence of a cyclic and separating vector in $\mathcal{P}$. For example if $\mathfrak{H}$ is separable then $\mathfrak{M}$ is σ-finite (see [Haa1] and [Bra1] §2.5.1). If one prefers, it is always possible to think the above standard form constructed from a faithful normal state ϕ_0 (see Appendix A).
Throughout this chapter we will consider a strongly continuous, symmetric, contraction semigroup $\{T_t\}_{t\geq 0}$ in $\mathfrak{H}$. In §§1.1, 1.2 we define J-real, Positivity Preserving and Markovian semigroup. In §1.3 we derive the L^p-properties of such semigroup and in the last section §1.4 we develop some methods of construction of Markovian semigroup.

1.1. J-real and Positivity Preserving semigroups.

Theorem 1.1.1. *The following conditions are equivalent:*
i) the semigroup $\{T_t\}_{t\geq 0}$ satisfies $T_t J = J T_t \quad \forall t \geq 0$;
ii) the resolvent $\{R_\lambda\}_{\lambda\geq 0}$ of $\{T_t\}_{t\geq 0}$ satisfies $R_\lambda J = J R_\lambda \quad \forall \lambda \geq 0$.

Proof. i) implies ii) since the formula

$$R_\lambda = \int_0^\infty e^{-t\lambda} T_t dt$$

ii) implies i) since the formula

$$T_t = \lim_{n\to+\infty} \left(\frac{n}{t} R_{n/t}\right)^n .$$

□

Definition 1.1.2. A bounded operator A in $\mathfrak{H}$ is J-real if

$$AJ = JA$$

and Positivity Preserving if

$$A(\mathcal{P}) \subseteq \mathcal{P}.$$

The semigroup $\{T_t\}_{t\geq 0}$ is J-real if T_t is J-real $\forall t \geq 0$ and it is Positivity Preserving if T_t is positivity preserving $\forall t \geq 0$.

Notice that if $\{T_t\}_{t\geq 0}$ is positivity preserving then it is also J- real, since we have the decomposition

$$\mathfrak{H} = \mathfrak{H}^{\natural} + i\mathfrak{H}^{\natural}$$

where $\mathfrak{H}^{\natural} \equiv \mathcal{P} - \mathcal{P} = \{\xi \in \mathfrak{H} : J\xi = \xi\}$ is the *real part* of $\mathfrak{H}$.

Theorem 1.1.3. *The semigroup $\{T_t\}_{t\geq 0}$ is Positivity Preserving if and only if its resolvent $\{R_\lambda\}_{\lambda>0}$ is such that R_λ is positivity preserving for sufficiently large λ.*

Proof. If $\xi \in \mathcal{P}$ and $\{T_t\}_{t\geq 0}$ is positivity preserving then $\{T_t\xi\}_{t>0} \subseteq \mathcal{P}$. Since $\lambda e^{-t\lambda} \cdot dt$ is a probability measure on R and $\mathcal{P}$ is a closed convex subset of $\mathfrak{H}$ we have $\lambda R_\lambda \xi = \lambda \int_0^\infty e^{-t\lambda} T_t \xi dt \in \mathcal{P}$.
The converse statement follows as ii) in Theorem 1.1.

□

Remark 1.1.4. The notion of positivity preserving operator can be introduced in the more general setting of real Hilbert spaces with selfdual pointed cones (see [Far1], [Cof], [Kre]). However if one adds the two regularity assumptions of *homogeneity* and *orientability* then the beautiful result of A. Connes [Con1] implies that there exists a von Neumann algebra and standard form of it with the same positive cone (at least if the cone is of *genre denombrable*)

(see Appendix A).

1.2. Markovian semigroups.

In this section we fix a normal faithful state $\phi_0 \in \mathfrak{M}_*^+$ or equivalently a cyclic and separating vector $\xi_0 \in \mathcal{P}$ which represents ϕ_0: $\phi_0(\cdot) = (\cdot\xi_0\,;\,\xi_0)$. Notice that in this case ξ_0 is also separating for $\mathfrak{M}$.
For $\xi_1, \xi_2 \in \mathcal{P}$ we denote by $[\xi_1; \xi_2]$ the *interval* $\{\xi \in \mathcal{P} : \xi_1 \leq \xi \leq \xi_2\}$ (the partial order in $\mathfrak{H}^\natural$ is induced by $\mathcal{P}$).

Definition 1.2.1. A bounded operator $A : \mathfrak{H} \to \mathfrak{H}$ is Markovian if it preserves the interval $[0; \xi_0]$:

$$0 \leq \xi \leq \xi_0 \qquad \textit{implies} \qquad 0 \leq A\xi \leq \xi_0.$$

A strongly continuous, symmetric, contraction semigroup $\{T_t\}_{t\geq 0}$ is said to be Markovian if T_t is markovian for every $t > 0$. A resolvent $\{R_\lambda\}_{\lambda>0}$ is said to be Markovian if λR_λ is markovian for every $\lambda > 0$.

Theorem 1.2.2. *An operator $A : \mathfrak{H} \to \mathfrak{H}$ is Markovian if A is Positivity Preserving and $A\xi_0 \leq \xi_0$.*

Proof. Assume that A is markovian. Then since ξ_0 is positive ($\xi_0 \in \mathcal{P}$), we have that $0 \leq \xi_0 \leq \xi_0$ implies $0 \leq A\xi_0 \leq \xi_0$. Now we recall that $\bigcup_{\lambda>0} \lambda[0; \xi_0]$ is dense in $\mathcal{P}$ in the norm topology (see [Bra1] Lemma 2.5.40). Therefore if $\xi \in \mathcal{P}$, there exists a sequence $\{\xi_n\}_{n=1}^\infty \subseteq \mathcal{P}$ converging to ξ, such that $0 \leq \xi_n/\|\xi_n\| \leq \xi_0$. Since A is markovian one has $0 \leq A\xi_n/\|\xi_n\| \leq \xi_0$. Also since A is bounded and $\mathcal{P}$ is closed it follows that A is positivity preserving. To prove the converse consider an element $\xi \in \mathcal{P}$ such that $0 \leq \xi \leq \xi_0$. Since A is positivity preserving we have $0 \leq A\xi \leq A\xi_0$. But $A\xi_0 \leq \xi_0$ by assumption and so A is markovian.

□

Theorem 1.2.3. *The following statement are equivalent:*

i) the semigroup $\{T_t\}_{t\geq 0}$ is Markovian

ii) the resolvent $\{R_\lambda\}_{\lambda>0}$ of $\{T_t\}_{t\geq 0}$ is Markovian.

Proof. i) $\Rightarrow$ ii): since $[0; \xi_0]$ is the intersection of two closed convex subset, $\mathcal{P}$ and $\xi_0 - \mathcal{P}$, it is closed and convex. The statement follows by the fact that $\lambda e^{-t\lambda}dt$ is a probability measure on R and by the formula

$$\lambda R_\lambda = \lambda \int_0^\infty e^{-t\lambda} T_t dt.$$

ii) $\Rightarrow i)$: it follows from the formula

$$T_t = \lim_{n\to+\infty} \left(\frac{n}{t} R_{n/t}\right)^n .$$

□

Remark 1.2.4. In the abelian case $\mathfrak{M} = \mathfrak{M}' = L^\infty(X, \mathcal{M}, \mu)$, where $(X, \mathcal{M})$ is a measurable space and μ is a positive measure. The standard form is

$$\left(L^\infty(X, \mathcal{M}, \mu); L^2(X, \mathcal{M}, \mu); J = \text{ complex conjugation } ; L^2_+(X, \mathcal{M}, \mu)\right)$$

and as a cyclic and separating vector ξ_0 one can take any μ-a.e. non zero function in $L^2(X, \mathcal{M}, \mu)$. In case μ is a probability and ξ_0 is the constant function 1, our definition of J-reality, Positivity Preserving and Markovianity of an operator on $L^2(X, \mathcal{M}, \mu)$ reduce to the well known ones (see [Fuk]).

Remark 1.2.5. Positivity Preserving and Markovian semigroups on a finite von Neumann algebra $\mathfrak{M}$ have been investigate by L. Gross [Gro1], S. Albeverio-R. Høegh-Krohn [AHK], W. Faris [Far1] and M. Lindsay-E. B. Davies [Dav4]. In our notation they consider the standard form

$$\left(\mathfrak{M}; L^2(\mathfrak{M}; \tau); J = \text{take the adjoint conjugation}; L^2_+(\mathfrak{M}; \tau)\right)$$

where $L^2(\mathfrak{M}; \tau)$ is the non-commutative L^2-space associated to a finite trace τ on $\mathfrak{M}$ (see [Seg1], [Nel2]), $L^2_+(\mathfrak{M}; \tau)$ is the subset of positive selfadjoint operator in $L^2(\mathfrak{M}; \tau)$ and $\xi_0 = I \in \mathfrak{M} \cap L^2(\mathfrak{M}; \tau)$ is the unity in $\mathfrak{M}$.

In general if $\mathfrak{M}$ is a finite von Neumann algebra in $\mathfrak{H}$ and $\xi_0 \in \mathfrak{H}$ is a cyclic and separating vector which represents the finite trace $\tau(x) = (\cdot\xi_0\, ;\, \xi_0)\ \ x \in \mathfrak{M}$, then one can choose as standard form

$$(\mathfrak{M}, \mathfrak{H}, \mathcal{J}_{\xi_0}, \mathcal{P}_{\xi_0} = (\mathfrak{M}_+\xi_0)^-)$$

where J_{ξ_0} is the isometric involution associated with ξ_0: $J_{\xi_0}(x\xi_0) \equiv x^*\xi_0\ \forall x \in \mathfrak{M}$.

1.3. L^p-properties of Markovian semigroups.

While the definition of Positivity Preserving semigroup involves only the structure of the standard form on which the semigroup is defined, the definition of Markovian semigroup involves the faithful normal state ϕ_0 or, equivalently, the unique cyclic and separating vector ξ_0 which represents ϕ_0 in $\mathcal{P}$. In the commutative case of Remark 1.9 after Theorem 1.8, it is well known (see [Dav3] Theorem 1.4.1) that is the distinctive property of markovian operators (among the positivity preserving ones) $T_t\xi_0 \leq \xi_0$, which allows to conclude that the semigroup is bounded in the L^∞-norm, and then, by duality and interpolation, on L^p-spaces. In the following we perform the same program in our non-commutative setting for (strongly continuous) Markovian semigroups. Among the different scales of non-commutative L^p-spaces associated to a faithful normal state ϕ_0 on $\mathfrak{M}$, we will use the spaces constructed by H. Kosaki [Kos] and studied also by M. Terp [Ter] (see Appendix B). We remark that for each fixed p, these spaces are isomorphic to the other realizations introduced by U. Haagerup [Haa2], A. Connes [Con2] and M. Hilsum [Hil], H. Araki and T. Masuda [Ara2] and C. Cecchini [Cec]. Moreover all of these constructions reduce to the usual ones in the commutative or tracial case. However, for technical reasons, one may prefer to express a particular property of some operator using a particular realization of L^p-spaces. For example we prefer to state the L^p-boundedness properties of semigroups in the spaces $L^p(\mathfrak{M};\phi_0)$ constructed by H. Kosaki, because they satisfy an inclusion property and because they make it easy to use the Riesz-Thorin interpolation theorem. However we want to emphasize that one should not disregard the other constructions mentioned above, because each of them permits to treat clearly a property that in an other realization appears complicated. In our investigations we shall assume that a semigroup $\{T_t\}_{t\geq 0}$ is given on the Hilbert space $\mathfrak{H}$ of a standard form and we will use other standard forms and scales of L^p-spaces, to express conditions which implies boundedness and compactness properties of our semigroup. In the non-trace case we are forced to use different type of L^p-spaces; we remark that this was already the case in the work of I. Segal [Seg2], E. Nelson [Nel1], L. Gross [Gro1], B. Simon and R. Høegh-Krohn [Høe] and W. Faris [Far1] in Quantum Field Theory. These authors represent the Fock-Hilbert spaces, and the operators of interest are typically defined as acting on an L^p-space over some infinite dimensional gaussian space (in Boson case), or on as L^p-space over the Clifford algebra $\mathcal{C}$, associated to a normal faithful trace on $\mathcal{C}$ (in the Fermion case).

Definition 1.3.1. A bounded operator $S : \mathfrak{M} \to \mathfrak{M}$ is said to be

- real if $S(x^*) = (Sx)^* \quad \forall x \in \mathfrak{M}$
- Positivity Preserving if $Sx \in \mathfrak{M}_+ \quad \forall x \in \mathfrak{M}_+$
- Markovian if $0 \leq Tx \leq \mathbb{I} \quad \forall 0 \leq x \leq \mathbb{I}$

Lemma 1.3.2.

i) S is Markovian if and only if it is Positivity Preserving and $S\mathbb{I} \leq \mathbb{I}$

ii) if S is Markovian then it is bounded with norm less or equal than 2 on $\mathfrak{M}$ and it is bounded with norm less or equal than 1 on the real Banach space $\mathfrak{M}_{sa} \equiv \{x \in \mathfrak{M} : x = x^*\}$.

Proof.

i): It is straightforward.

ii): If $x \in \mathfrak{M}_{sa}$ we have

$$
\begin{aligned}
-\|x\|_{\mathfrak{M}} &\leq x \leq +\|x\|_{\mathfrak{M}} \\
-S\|x\|_{\mathfrak{M}} &\leq Sx \leq +S\|x\|_{\mathfrak{M}} \\
-\|x\|_{\mathfrak{M}} &\leq Sx \leq +\|x\|_{\mathfrak{M}} \\
\|Sx\|_{\mathfrak{M}} &\leq +\|x\|_{\mathfrak{M}}.
\end{aligned}
$$

Moreover we can write a general element z of $\mathfrak{M}$ as $z = x + iy$ with $x, y \in \mathfrak{M}_{sa}$ and then

$$
\begin{aligned}
\|Sz\|_{fm} &= \|Sx + iSy\| \leq \|Sx\| + \|Sy\| \leq \|x\| + \|y\| \\
&= \frac{1}{2}\|z + z^*\| + \frac{1}{2}\|z - z^*\| \leq \|z\| + \|z^*\| = 2\|z\|.
\end{aligned}
$$

□

Our first result is the extension of a strongly continuous Markovian semigroup $\{T_t\}_{t\geq 0}$ to a $\sigma(\mathfrak{M};\mathfrak{M}_*)$-continuous Markovian semigroup on $\mathfrak{M}$ and to a strongly continuous semigroup on the predual $\mathfrak{M}_*$.
In the commutative case the restriction of $\{T_t\}_{t\geq 0}$ to a semigroup on $\mathfrak{M}$ is trivial because (at least on the probability space) the algebra $L^\infty(X, \mathcal{M}, \mu)$ is canonically embedded in the Hilbert space $L^2(X, \mathcal{M}, \mu)$. In our non-commutative setting there are several ways in which one can embed $\mathfrak{M}$ in $\mathfrak{H}$. We shall always use the *symmetric* embedding

$$
i_0 : \mathfrak{M} \to \mathfrak{H} \qquad i_0(x) \equiv \Delta_{\xi_0}^{1/4} x \xi_0 \quad \forall x \in \mathfrak{M}
$$

where Δ_{ξ_0} is the *modular* operator associated with the cyclic and separating vector ξ_0. We recall that we denote by $\Delta_{\xi_0}^{1/2}$ and J_{ξ_0}, the positive and the isometric part, respectively, of the *sharp* operator $\big(S_{\xi_0};\mathcal{D}(S_{\xi_0})\big)$

$$\mathcal{D}(S_{\xi_0}) \equiv \mathfrak{M}\xi_0 \qquad S_{\xi_0}(x\xi_0) \equiv x^*\xi_0 \qquad \forall x \in \mathfrak{M}$$

$$S_{\xi_0} = J_{\xi_0}\Delta_{\xi_0}^{1/2}$$

(see [Tak2]).

Theorem 1.3.3. *Let $(\mathfrak{M},\mathfrak{H},J,\mathcal{P})$ be a standard form of the von Neumann algebra $\mathfrak{M}$, ϕ_0 a faithful normal state on $\mathfrak{M}$ and ξ_0 the unique cyclic and separating vector in $\mathcal{P}$ which represents ϕ_0. Let $A:\mathfrak{H}\to\mathfrak{H}$ be a Markovian operator and let $\{T_t\}_{t\geq 0}$ be a Markovian semigroup on $\mathfrak{H}$. Define the operators $\widetilde{A}$ and $\widetilde{T}_t$ by:*

$$\begin{aligned} \widetilde{A}:\mathfrak{M}\to\mathfrak{M} \qquad & i_0\big(\widetilde{A}(x)\big) \equiv A\big(i_0(x)\big) \qquad \forall x\in\mathfrak{M} \\ \widetilde{T}_t:\mathfrak{M}\to\mathfrak{M} \qquad & i_0\big(\widetilde{T}_t(x)\big) \equiv T_t\big(i_0(x)\big) \qquad \forall x\in\mathfrak{M}. \end{aligned}$$

Then:

i) $\widetilde{A}$ is a well defined Markovian operator of norm less or equal than 2 on $\mathfrak{M}$ and less or equal than 1 on the real subspace $\mathfrak{M}_{sa}$.

ii) $\{\widetilde{T}_t\}_{t>0}$ is a well defined semigroup of Markovian operators with norms less than 2 on $\mathfrak{M}$ and norms less than 1 on $\mathfrak{M}_{sa}$.

iii) the maps defined by duality

$$\widetilde{T_t^*} \equiv \big(\widetilde{T}_t\big)^*:\mathfrak{M}^*\to\mathfrak{M}^*$$

induce on the predual $\mathfrak{M}_*$ a strongly continuous semigroup

iv) $\{\widetilde{T}_t\}_{t>0}$ is a $\sigma(\mathfrak{M};\mathfrak{M}_*)$-continuous (weakly*-continuous or C_0^*-semigroup) Markovian semigroup on $\mathfrak{M}$.

Proof. For all properties of the map i_0 we use , we refer to Proposition 2.5.40 in [Bra1]. *i)* Since

$$i_0(\mathfrak{M}_{sa}) = \bigcup_{\lambda>0}\lambda[-\xi_0,\xi_0]$$

and since A is Markovian, we have

$$A(i_0(\mathfrak{M}_{sa})) \subseteq \bigcup_{\lambda>0}\lambda A[-\xi_0,\xi_0] \subseteq \bigcup_{\lambda>0}\lambda[-\xi_0,\xi_0] = i_0(\mathfrak{M}_{sa}).$$

Then $\widetilde{A}$ is well defined, since i_0 is injective. Moreover $\widetilde{A}$ is Markovian because i_0 is an order preserving isomorphism between $\mathfrak{M}_{sa}$ and $i_0(\mathfrak{M}_{sa}) \subseteq \mathfrak{H}^{\natural}$, in fact $x \in \mathfrak{M}_+$ if and only if $i_0(x) \in \mathcal{P}$. Therefore this implies

$$\begin{aligned} 0 \leq x &\leq 1 \\ i_0(x) &\in [0, \xi_0] \\ A\big(i_0(x)\big) &\in [0, \xi_0] \\ \widetilde{A}x &\in [0, \mathbb{I}] \end{aligned}$$

To conclude this point we have just to apply Lemma 1.13. *ii)* The fact that $\{\widetilde{T}_t\}_{t>0}$ is a well defined family of markovian maps follows from i). The semigroup structure follows easily from that of $\{T_t\}_{t\geq 0}$. *iii)* First of all we have to prove that $\widetilde{T_t^*}$ maps, for each $t > 0$, the norm-closed subspaces of normal functionals in itself: $\widetilde{T_t^*}(\mathfrak{M}_*) \subseteq \mathfrak{M}_*$. We begin by showing that

$$\widetilde{\mathfrak{M}}_* \equiv \{\omega_\xi \in \mathfrak{M}_* : \xi \in \mathfrak{M}'\xi_0\}$$

is norm-dense in $\mathfrak{M}_*$ (ω_ξ being the vector state corresponding to the vector ξ: $\omega_\xi(\cdot) \equiv (\cdot\xi\,;\xi)$. Since ξ_0 is cyclic for $\mathfrak{M}$, it is separating for $\mathfrak{M}'$: $\mathfrak{H} = (\mathfrak{M}'\xi_0)^-$ (see [Bra1] Proposition 2.5.3). Each element ω in $\mathfrak{M}_*$ can be represented by an element ξ in $\mathfrak{H}$ (see [Bra1] Theorem 2.5.31 *a)*): $\omega = \omega_\xi$. Then there exist a sequence $\{\xi_n\} \subseteq \mathfrak{M}'\xi_0$ converging to ξ in the norm topology of $\mathfrak{H}$. We then have $\forall x \in \mathfrak{M}$:

$$\begin{aligned} &\big|\big(\omega_\xi - \omega_{\xi_n}\big)(x)\big| = |(x\xi;\xi) - (x\xi_n;\xi_n)| \\ =&|(x\xi;\xi) - (x\xi;\xi_n) + (x\xi;\xi_n) - (x\xi_n;\xi_n)| \\ =&|(x\xi;\xi-\xi_n) + (x(\xi-\xi_n);\xi_n)| \leq \|x\|\cdot\|\xi-\xi_n\|\cdot(\|\xi_n\|+\|\xi\|) \end{aligned}$$

and then

$$\|\omega_\xi - \omega_{\xi_n}\|_{\mathfrak{M}} \leq \|\xi - \xi_n\| \cdot \|\xi + \xi_n\|$$

tends to zero as $n \to \infty$. We proved that $\widetilde{\mathfrak{M}}_*$ is norm-dense in $\mathfrak{M}_*$. Since each map $\widetilde{T_t^*}$ is norm-continuous (being the dual map of a norm-continuous map), to prove that it preserves the subspace $\mathfrak{M}_*$ is sufficient to prove that $\widetilde{T_t^*}(\widetilde{\mathfrak{M}}_*) \subseteq \widetilde{\mathfrak{M}}_*$. In fact we have

$$\widetilde{T_t^*}(\mathfrak{M}_*) = \widetilde{T_t^*}(\widetilde{\mathfrak{M}}_*^-) = \big(\widetilde{T_t^*}(\widetilde{\mathfrak{M}}_*)\big)^- \subseteq (\widetilde{\mathfrak{M}}_*)^- = \mathfrak{M}_*$$

Let $\omega_\xi \in \widetilde{\mathfrak{M}}_*$ be the normal state corresponding to the vector $\xi \equiv y'\xi_0 \in \mathfrak{M}'\xi_0$ for some $y' \in \mathfrak{M}'$. From the definition of $\widetilde{T}_t$ we have that

$$i_0\big(\widetilde{T}_t(x)\big) = \Delta_{\xi_0}^{1/4}\widetilde{T}_t(x)\xi_0 = T_t\big(i_0(x)\big) \qquad \forall x \in \mathfrak{M}$$

$$\widetilde{T}_t(x)\xi_0 = \Delta_{\xi_0}^{-1/4}(T_t(i_0(x)) \qquad \forall x \in \mathfrak{M}$$

$\Delta_{\xi_0}^{1/4}$ being invertible. Moreover

$$\widetilde{T}_t(x)\xi = \widetilde{T}_t(x)y'\xi_0 = y'\widetilde{T}_t(x)\xi_0 = y'\Delta_{\xi_0}^{-1/4}(T_t(i_0(x))$$

$$\begin{aligned}&\big(\widetilde{T}_t^{\,*}(\omega_\xi)\big)(x) = \omega_\xi\big(\widetilde{T}_t(x)\big) = \big(\widetilde{T}_t(x)\xi;\xi\big) = \big(y'\Delta_{\xi_0}^{-1/4}T_t(i_0(x));\xi\big)\\ &= \big(\mathfrak{M}'\xi_0 \subseteq \mathcal{D}(\Delta_{\xi_0}^{-1/4})\big) = \big(T_t(i_0(x);\Delta_{\xi_0}^{-1/4}y'^*y'\xi_0\big) = \big(i_0(x);T_t\Delta_{\xi_0}^{-1/4}y'^*y'\xi_0\big)\\ &\big(\eta \equiv T_t\Delta_{\xi_0}^{-1/4}y'^*y'\xi_0 \in \mathfrak{H}\big) = (i_0(x);\eta) \equiv (\Lambda_\eta \circ i_0)(x)\end{aligned}$$

where we have defined $\Lambda_\eta : \mathfrak{H} \to \mathbb{C} \quad \forall \eta' \in \mathfrak{H}$:

$$\langle \Lambda_\eta;\eta'\rangle \equiv (\eta';\eta).$$

We have shown that for some $\eta \in \mathfrak{H}$ $\widetilde{T}_t^{\,*}(\omega_\xi) = \Lambda_\eta \circ i_0$. Now since Λ_η is norm-continuous, by the definition of the weak-topology on $\mathfrak{H}$, it is also weakly-continuous. Moreover $i_0 : \mathfrak{M} \to \mathfrak{H}$ is a $\sigma(\mathfrak{M};\mathfrak{M}_*) - \sigma(\mathfrak{H};\mathfrak{H})$-continuous map. Then we have that $\widetilde{T}_t^{\,*}(\omega_\xi)$ is a $\sigma(\mathfrak{M};\mathfrak{M}_*)$-continuous functional and so it is normal (see [Bra1] Theorem 2.4.21) $\widetilde{T}_t^{\,*}(\omega_\xi)$. We can now start to prove that $\{\widetilde{T}_t^{\,*}\}_{t>0}$ is a strongly continuous semigroup on $\mathfrak{M}_*$; for it, is it is sufficiently to prove that it is weakly-continuous (see [Bra1] Corollary 3.1.8). Notice since the map $\widetilde{T}_t^{\,*}$ is norm continuous, then it is also $\sigma(\mathfrak{M}_*;\mathfrak{M})$-continuous (i.e. weakly-continuous since $\mathfrak{M}$ is the dual of $\mathfrak{M}_*$). In fact, if $x \in \mathfrak{M}$, we have that the map

$$\mathfrak{M}_* \ni \omega \longmapsto \big(\widetilde{T}_t^{\,*}(\omega)\big)(x) \equiv \omega\big(\widetilde{T}_t^{\,*}(x)\big)$$

is obviously continuous in the $\sigma(\mathfrak{M}_*;\mathfrak{M})$-topology, and so $\widetilde{T}_t^{\,*} : \mathfrak{M}_* \to \mathfrak{M}_*$ is continuous when we consider on $\mathfrak{M}_*$ the $\sigma(\mathfrak{M}_*;\mathfrak{M})$-topology.

To conclude the proof of *iii)* we have now to prove that, for fixed $\omega \in \mathfrak{M}_*$, the map

$$f : R_+ \to \mathfrak{M}_* \qquad f(t) \equiv \widetilde{T}_t^{\,*}(\omega)$$

is continuous in in the weak-topology of $\mathfrak{M}_*$. By definition of weak-topology we have then to prove that for each fixed $x \in \mathfrak{M}$ the map

$$g_\omega : R_+ \to R \qquad g_\omega(t) \equiv \omega\big(\widetilde{T}_t^{\,*}(x)\big) \qquad t \in R_+$$

is continuous. Now we use again the approximation of elements of $\mathfrak{M}_*$ by elements $\widetilde{\mathfrak{M}}_*$ used above. If $\omega = \omega_\xi \in \widetilde{\mathfrak{M}}_*$ is easy to see that the continuity of

$$g_\omega(t) = \big(T_t i_0(x);\Delta_{\xi_0}^{-1/4}y'^*y'\xi_0\big)$$

follows from the strong continuity of the semigroup $\{T_t\}_{t\geq 0}$. For a generic element $\omega = \omega_\xi$ of $\mathfrak{M}_*$, with $\xi \in \mathfrak{H}$, we choose a sequence $\{\xi_n\} \subseteq \mathfrak{M}'\xi_o$ converging to ξ in the norm-topology of $\mathfrak{H}$. Then we have:

$$|g_\omega(t) - g_\omega(0)| \leq |g_\omega(t) - g_{\omega_{\xi_n}}(t)| + |g_{\omega_{\xi_n}}(t) - g_{\omega_{\xi_n}}(0)| + |g_{\omega_{\xi_n}}(0) - g_\omega(0)| \leq$$

$$\leq |(\omega - \omega_{\xi_n})(\widetilde{T}_t(x))| + |\omega_{\xi_n}(\widetilde{T}_t(x) - x)| + |(\omega - \omega_{\xi_n})(x)|$$

Therefore g_ω is continuous for $t = 0$ and then for each $t > 0$. This concludes the proof of *iii*).

iv) From the fact (proved in *iii*)) that $\widetilde{T}_t(\mathfrak{M}_*) \subseteq \mathfrak{M}_*$ it follows that each map $\widetilde{T}_t : \mathfrak{M} \to \mathfrak{M}$ is $\sigma(\mathfrak{M};\mathfrak{M}_*) - \sigma(\mathfrak{M};\mathfrak{M}_*)$-continuous. In fact this simply means that $\omega \circ \widetilde{T}_t$ is $\sigma(\mathfrak{M};\mathfrak{M}_*)$-continuous $\forall \omega \in \mathfrak{M}_*$. But we proved that $\omega \circ \widetilde{T}_t \equiv \widetilde{T}_t^*(\omega) \in \mathfrak{M}_*$. At the end of part *iii*) we also proved that, for fixed $x \in \mathfrak{M}$, the function $t \to \widetilde{T}_t(x)$ is continuous in the $\sigma(\mathfrak{M};\mathfrak{M}_*)$ topology.

□

Remark 1.3.4. In application one can consider maps A and semigroups $\{T_t\}_{t\geq 0}$ in $\mathfrak{H}$ which commute strongly with the modular operator Δ_{ξ_0} corresponding to ξ_0. In that case the map $\widetilde{A} : \mathfrak{M} \to \mathfrak{M}$ can be easily defined as $(\widetilde{A}x)\xi_0 \equiv A(x\xi_0)$ $\forall x \in \mathfrak{M}$. The same holds for semigroups.

Remark 1.3.5. In the commutative case one can prove that the maps $\widetilde{A}$ and $\widetilde{T}_t$ of Thm 5 are contractions on the entire space $\mathfrak{M}$ and not only on its selfadjoint part $\mathfrak{M}_{sa}$. In the non commutative case this conclusion cannot be improved in general. This is already true in the trace case, and one can give examples of this by considering 2×2 matrices with the usual definition of trace. In order to guarantee contraction on $\mathfrak{M}$ one has to add some requirements to the maps A and T_t on the Hilbert space. For example M.Lindsay and E.B.Davies requires 1 $\frac{1}{2}$-positivity (see [Dav4] after Lemma 2.3).

Among the non commutative L^p-spaces constructed by H.Kosaki [Kos], we consider those denoted by him as $C_{1/p}(\mathfrak{M}^{1/2};\mathfrak{M}_*)$. These spaces are constructed by the complex interpolation method (see [BeL]) using the following imbedding of $\mathfrak{M}$ into $\mathfrak{M}_*$:

$$j_0 : \mathfrak{M} \to \mathfrak{M}_* \;\; j_0(x)(y) \equiv (y\Delta_{\xi_0}^{1/2} x\xi_0|\xi_0) \;\; \forall x, y \in \mathfrak{M} \;\; (1)$$

(1) The scalar product is taken to be complex linear in its left argument.

(recall $\mathfrak{M}\xi_0$ is a core for $\Delta_{\xi_0}^{1/2}$).

We will use also another symbol for the injection j_0:

$$j_0(x) = \sigma_{-i/2}^{\phi_0}(x) \cdot \phi_0 \;\; \forall x \in \mathfrak{M}.$$

This can be justified as follows. Denote by $\sigma_t^{\phi_0} : \mathfrak{M} \to \mathfrak{M}$ the automorphisms group associated to ϕ_0:

$$\sigma_t^{\phi_0}(x) \equiv \Delta_{\xi_0}^{it} x \Delta_{\xi_0}^{-it} \qquad \forall x \in \mathfrak{M} \qquad \forall t \in \mathbb{R}.$$

Now consider the function $f_x : \mathbb{R} \to \mathfrak{M}_*$

$$f_x(t) \equiv \sigma_t^{\phi_0}(x) \cdot \phi_0 \equiv \phi_0(\cdot \sigma_t^{\phi_0}(x)) \qquad \forall t \in \mathbb{R}.$$

By the "predual version of the KMS condition" ([Kos] Theorem 2.5) the map f_x can be extended to a continuous map defined on the closure of the open strip $D_{-1} = \{z \in \mathbb{C} : -1 < Imz < 0\}$

$$f_x : \bar{D}_{-1} \to \mathfrak{M}_*$$

in such a way that f_x is analytic on the open strip D_{-1} and

$$f_x(t) = \sigma_t^{\phi_0}(x) \cdot \phi_0 \equiv \phi_0(\cdot \sigma_t^{\phi_0}(x))$$

$$f_x(-i + t) = \phi_0 \cdot \sigma_t^{\phi_0}(x) \equiv \phi_0(\sigma_t^{\phi_0}(x) \cdot).$$

Then the injection j_0 described above is just

$$f_x(-\frac{i}{2})$$

(see proof of Theorem 2.5 in [Kos]). We will denote $C_{1/p}(\mathfrak{M}^{1/2}; \mathfrak{M}_*)$ by

$$L^p(\mathfrak{M}; \phi_0) \qquad 1 < p < \infty$$

and we remark that all these spaces are subspaces of the predual $\mathfrak{M}_*$ (denoted by $L^1(\mathfrak{M}; \phi)$). The space $L^2(\mathfrak{M}; \phi)$ has a distinctive role being isomorphic to the standard Hilbert space $\mathfrak{H}$.

Theorem 1.3.6. *Let $(\mathfrak{M}, \mathfrak{H}, J, \mathcal{P})$ be a standard form of the von Neumann algebra $\mathfrak{M}$ and let ϕ_0 be a faithful normal state over $\mathfrak{M}$ whose corresponding representative in $\mathcal{P}$ is the vector ξ_0. Let us define the map*

$$i_0^* : \mathfrak{H} \longrightarrow \mathfrak{M}_*$$

$$i_0^*(\xi)(y) \equiv (\xi; \Delta_{\xi_0}^{1/4} y^* \xi_0) = (\xi; i_0(y^*)) \qquad y \in \mathfrak{M} \qquad \xi \in \mathfrak{H}.$$

Then i_0^* is an isometric isomorphism between $\mathfrak{H}$ and $L^2(\mathfrak{M}; \phi_0)$. Moreover the following diagram is commutative

$$\begin{array}{ccc} M & \xrightarrow{j_0} & L^\infty \\ \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathcal{H} & \xrightarrow{i_0^*} & L^2 \\ \downarrow{\scriptstyle i_0^*} & & \downarrow{\scriptstyle \mathrm{id}} \\ M_* & \xrightarrow{\mathrm{id}_{M_*}} & L^1 \end{array}$$

Proof. The first statement in the theorem follows from the non-commutative Stein-Weiss interpolation theorem (see [Kos] theorem 11.1). The second part follows from $j_0 = i_0^* \circ i_0$

□

We can now state the main theorem of this section:

Theorem 1.3.7. *Assume hypotheses of Theorem 1.14. Let us consider the operator $A^{(1)}$ (resp. $A^{(2)}$, $A^{(\infty)}$) which corresponds to the operator $\widetilde{A}^*$ (resp. A, $\widetilde{A}$) through the horizontal arrow $id_{\mathfrak{M}_*}$ (resp. i_0^*, j_0). Consider also the semigroup $T_t^{(1)}{}_{t>0}$ (resp. $T_t^{(2)}{}_{t>0}$, $T_t^{(\infty)}{}_{t>0}$) constructed in the same way on $L^1(\mathfrak{M}; \phi_0)$ (resp. $L^2(\mathfrak{M}; \phi_0)$, $L^\infty(\mathfrak{M}; \phi_0)$). Then*

- *i) $A^{(\infty)}$ is the restriction of $A^{(2)}$ to $L^\infty(\mathfrak{M}; \phi_0)$ and $A^{(2)}$ is the restriction of $A^{(1)}$ to $L^2(\mathfrak{M}; \phi_0)$*
- *ii) for each $t > 0$ $T_t^{(\infty)}$ is the restriction of $T_t^{(2)}$ to $L^\infty(\mathfrak{M}; \phi_0)$ and $T_t^{(2)}$ is the restriction of $T_t^{(1)}$ to $L^2(\mathfrak{M}; \phi_0)$*
- *iii) for each $p \geq 1$, $A^{(1)}(L^p(\mathfrak{M}; \phi_0)) \subseteq L^p(\mathfrak{M}; \phi_0)$, and the restriction $A^{(p)}$ of $A^{(1)}$ to $L^p(\mathfrak{M}; \phi_0)$ is bounded*
- *iv) for each $p \geq 1$, the semigroup $T_t^{(p)}{}_{t>0}$, obtained by restricting, (as in iii)), each $T_t^{(1)}$ to $L^p(\mathfrak{M}; \phi_0)$, is strongly continuous.*

Proof. i) and ii) follow from the diagram of Theorem 1.17 and the very definition of the map $A^{(\infty)}$, $A^{(2)}$, $A^{(1)}$, $T_t^{(1)}$, $T_t^{(\infty)}$.
iii) follows from *i)* and the non-commutative Riesz-Thorin interpolation theorem (see [Kos] theorem 1.2).
iv) applying *i)* we obtain that for each $t > 0$, $T_t^{(p)}$ is bounded on $L^p(\mathfrak{M};\phi_0)$. We have to verify that ${T_t^{(p)}}_{t>0}$ is strongly continuous on $L^p(\mathfrak{M};\phi_0)$. Since $\{\widetilde{T}_t^*\}_{t>0}$ and $\{T_t\}_{t>0}$ are strongly continuous the same is true for $\{T_t^{(1)}\}_{t>0}$ and $\{T_t^{(2)}\}_{t>0}$. We then have for $1 < p < 2$, $\xi \in L^p(\mathfrak{M};\phi_0)$:

$$\lim_{t\to 0} \|T_t^{(p)}\xi - \xi\|_p \leq \lim_{t\to 0} \|T_t^{(p)}\xi - \xi\|_2 = \lim_{t\to 0} \|T_t^{(2)}\xi - \xi\|_2 = 0$$

By duality $\{T_t^{(p)}\}_{t>0}$ is strongly continuous for $p > 2$.

□

Remark 1.3.8. If we know that $\{\widetilde{T}_t\}_{t>0}$ on $\mathfrak{M}$ is a contraction semigroup, then the same proof as for Theorem 1.18 shows that, for each $1 \leq p \leq \infty$, $\{T_t^{(p)}\}_{t>0}$ is a contraction semigroup.

1.4. Constructions of semigroups.

In this section we describe some methods for constructing Positivity Preserving and Markovian semigroups. Most of the constructions arise as combinations of some fundamental property of Standard Forms of von Neumann and a procedure called *subordination.* The latter is a well known tool in the theory of Markov Processes (see [Mey] §4), Potential Theory (see [BeF] §1.2) and Semigroup Theory (see [Dav2] chapter 2 §4). It is a construction which gives new semigroups from previous ones, through an averaging procedure.
The two basic examples we have in mind are the following. Take the standard form of Remark 1.2.4 associated to the Lebesgue space $(\mathbb{R}^n;\ M;\ m)$ and consider the regular representation U of the locally compact group $\mathbb{R}^n$ on $L^2(\mathbb{R}^n;\ M;\ m)$. Then the family of operators

$$\begin{aligned}(T_t f)(y) &= \frac{1}{(2\pi t)^{-n/2}} \cdot \int_{\mathbb{R}^n} e^{-\frac{x^2}{2t}} [U(y)f](x)dx \\ &= \frac{1}{(2\pi t)^{-n/2}} \cdot \int_{\mathbb{R}^n} e^{-\frac{x^2}{2t}} f(x+y)dx\end{aligned}$$

$y \in \mathbb{R}^n$, $f \in L^2(\mathbb{R}^n;\ M;\ m)$ is nothing but the *Heat-semigroup* with generator

$$-\frac{1}{2}\Delta \equiv -\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}.$$

The second-type is the *Symmetric Stable semigroup* S_t, whose generator is $(-\frac{1}{2}\Delta)^{1/2}$, constructed from the previous one in the following way:

$$S_t \equiv \int_0^\infty \frac{te^{-\frac{t^2}{4s}}}{2(\pi s^3)^{1/2}} T_s ds \qquad \forall t > 0.$$

These kind of constructions have for us two advantages. The first one is that, if the group (or the semigroup) from which one starts, are positivity preserving or markovian, also the new semigroups have the same properties (see Theorems 1.4.4 and 1.4.6). The second advantage is that we can use these constructions to exhibit , many Markovian semigroups on Standard Forms thanks to the following important result (see [Ara1], [Con1], [Haa1]):

Theorem 1.4.1. *Let $(\mathfrak{M}, \mathfrak{H}, J, \mathcal{P})$ be a standard form of the von Neumann algebra $\mathfrak{M}$. Then we have:*

*i) the group $Aut(\mathfrak{M})$ of all *automorphisms of $\mathfrak{M}$ has a unique unitary implementation $g \longmapsto U(g)$ in $\mathfrak{H}$ such that*

$$U(g)J = JU(g) \qquad \textit{for each} \quad g \in Aut(\mathfrak{M})$$
$$U(g)\mathcal{P} = \mathcal{P}U(g) \qquad \textit{for each} \quad g \in Aut(\mathfrak{M})$$

ii) if G is a locally compact group (or semigroup) and $\alpha : G \longrightarrow Aut(\mathfrak{M})$ is a σ-weakly continuous representation of G on $\mathfrak{M}$, then the canonical unitary implementation $g \longmapsto U(g)$ of G in$\mathfrak{H}$ is strongly continuous.

Definition 1.4.2. Let G be a locally compact abelian group. A family of probability measure $\{\mu_t\}_{t>0}$ on G is called a *convolution semigroup of probabilities on G*, if (δ_e be the Dirac measure at the origin $e \in G$):

i) $\mu_t * \mu_s = \mu_{t+s} \qquad \forall t, s > 0$

ii) $\lim_{t \to 0} \mu_t = \delta_e$ (the convergence being in the weak topology of the space of Radon measure on G). $\{\mu_t\}_{t>0}$ is called *symmetric* if each μ_t is a symmetric measure on G ($\mu_t(E^{-1}) = \mu_t(E) \quad E \subseteq G$ measurable; see [BeF] §8).

Example 1.4.3. Suppose $G = \mathbb{R}$.

i) Brownian (or Gaussian) semigroup

$$\mu_t(ds) = \frac{1}{(4\pi t)^{1/2}} \cdot e^{-\frac{s^2}{4t}} \cdot ds$$

ii) Cauchy semigroup

$$\mu_t(ds) = \frac{t}{\pi} \cdot \frac{1}{t^2 + s^2}^{-1} \cdot ds$$

iii)

$$\mu_t(E) = \frac{t}{(4\pi s^3)^{1/2}} \cdot \int_{E\cap[0,+\infty)} e^{-\frac{t^2}{4s}} \cdot ds \qquad E \subseteq \mathbb{R} \quad \text{measurable.}$$

Theorem 1.4.4. *Let $\{\mu_t\}_{t>0}$ be a convolution of probabilities measure on the locally compact group G and let $U : G \to B(\mathfrak{H})$ be a strongly continuous unitary representation of G in the Hilbert Space $\mathfrak{H}$ of a standard form $(\mathfrak{M}, \mathfrak{H}, J, \mathcal{P})$ of a von Neumann algebra $\mathfrak{M}$. Define for each $\xi \in \mathfrak{H}$, $t > 0$*

$$T_t : \mathfrak{H} \to \mathfrak{H} \qquad T_t\xi \equiv \int_G \mu_t(dg)U(g)\xi.$$

Then:

i) $\{T_t\}_{t\geq 0}$ is a strongly continuous, contraction semigroup in $\mathfrak{H}$

ii) if $\{\mu_t\}_{t>0}$ is symmetric then $\{T_t\}_{t\geq 0}$ is symmetric

iii) if for each $g \in G$, $U(g)$ is Positivity Preserving, then $\{T_t\}_{t\geq 0}$ is Positivity Preserving too

iv) if for each $g \in G$, $U(g)$ is Positivity Preserving and there exists a cyclic (hence separating) vector $\xi_0 \in \mathcal{P}$, such that, for each $g \in G$ one has $U(g)\xi_0 = \xi_0$, then $T_t\xi_0 = \xi_0$ and $\{T_t\}_{t\geq 0}$ is a Markovian semigroup

v) if $G = \mathbb{R}$, $\{\mu_t\}_{t>0}$ is the Gaussian convolution semigroup (see Example 1.4.2), H is the selfadjoint generator of the group $U(s) = \exp(isH)$ $s \in \mathbb{R}$, then the generator K of $\{T_t\}_{t\geq 0}$ ($T_t = \exp(-tK)$) is $K = \frac{1}{2}H^2$.

Proof. *i)* The fact that $\{T_t\}_{t\geq 0}$ is a contraction semigroup is easy to prove. To show the strong continuity it is sufficiently to prove the weak continuity: for each $\xi, \eta \in \mathfrak{H}$ we have

$$(\eta;\ T_t\xi) = \int_G \mu_t(dg)(\eta;\ U(g)\xi) \to (\eta;\ \xi)$$

as $t \to 0$, by assumption *ii)* in Definition 1.4.2 and the strongly continuity of U. *ii)* It is easy to prove. *iii)* This point follows from the fact that $\mathcal{P}$ is a closed convex cone and μ_t is a probability for each $t > 0$. *iv)* It is straightforward. *v)* It follows applying theorem 2.3.1 in [Dav2].

□

Definition 1.4.5. The semigroup $\{T_t\}_{t\geq 0}$ constructed in Theorem 1.4.4, from the unitary representation U and the convolution semigroup $\{\mu_t\}_{t>0}$, is called the semigroup subordinated to U by $\{\mu_t\}_{t>0}$.

The following theorem deals with the subordination with respect to semigroups. We omit its proof because it is completely analogous to that of Theorem 1.4.3.

Theorem 1.4.6. *Let $\{T_t\}_{t\geq 0}$ be a strongly continuous, symmetric, contraction semigroup on $\mathfrak{H}$ and let $\{\mu_t\}_{t>0}$ be a symmetric convolution semigroup of probability measures on $\mathbb{R}$ supported by $[0, +\infty) \subseteq \mathbb{R}$. Let us define*

$$S_t : \mathfrak{H} \to \mathfrak{H} \qquad S_t\xi \equiv \int_0^\infty \mu_t(ds)T_s\xi$$

for each $\xi \in \mathfrak{H}$ and $t > 0$. Then:

i) *$\{S_t\}_{t>0}$ is a strongly continuous, symmetric, contraction semigroup*
ii) *if $\{T_t\}_{t\geq 0}$ is Positivity Preserving so is $\{S_t\}_{t>0}$*
iii) *if there exists a cyclic (hence separating) vector $\xi_0 \in \mathcal{P}$ such that $\{T_t\}_{t\geq 0}$ is Markovian with respect to it, then $\{S_t\}_{t>0}$ is also Markovian with respect to ξ_0*
iv) *if $\{\mu_t\}_{t>0}$ is the family of the Example 1.4.3-iii), then the generator of $\{S_t\}_{t>0}$ is given by $K = H^{1/2}$ if H is the generator of $\{T_t\}_{t\geq 0}$.*

Now we show how to use the above theorems in some examples. In all of these examples $\{\mu_t\}_{t>0}$ will denote a fixed symmetric convolution semigroup on $\mathbb{R}$.

Example 1.4.7. Let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a σ-weakly continuous one parameter group of *automorphisms of $\mathfrak{M}$ and let $\{U(t)\}_{t\in\mathbb{R}}$ be its strongly continuous unitary implementation. By Theorems 1.4.1 and 1.4.3 we have that

$$T_t \equiv \int_{\mathbb{R}} \mu_t(ds)U(s)$$

defines a strongly continuous, Positivity Preserving semigroup on $\mathfrak{H}$. Moreover if $\phi_0 \in \mathfrak{M}_*^+$ is a $\{\alpha_t\}_{t\in\mathbb{R}}$-invariant state, then the unique vector $\xi_0 \in \mathcal{P}$ which represents ϕ_0, is $\{U(t)\}_{t\in\mathbb{R}}$ invariant and, by Theorem 1.4.3, $\{T_t\}_{t\geq 0}$ is a Markovian semigroup with respect to $\xi_0 \in \mathcal{P}$.

Example 1.4.8. A particular case of Example 1.4.7 is when one consider a faithful, semifinite, normal weight ϕ on $\mathfrak{M}$. The Tomita-Takesaki theory (see [Tak1], [Tak2]) associates to ϕ the cyclic representation $\pi_\phi : \mathfrak{M} \to B(\mathfrak{H}_\phi)$ into the Hilbert space $\mathfrak{H}_\phi$, the modular operators Δ_ϕ and J_ϕ and the standard form

$$\left(\pi_\phi(\mathfrak{M});\ \mathfrak{H}_\phi;\ J_\phi;\ \mathcal{P}_\phi\right).$$

Notice that since ϕ is faithful, $\pi_\phi(\mathfrak{M})$ is identified with$\mathfrak{M}$ through π_ϕ, which is faithful. A relevant result of the theory is that, for $x \in \pi_\phi$ and $t \in \mathbb{R}$,

$$\sigma_t^\phi(x) \equiv \Delta_\phi^{it} x \Delta_\phi^{-it}$$

defines a group of *automorphisms of $\pi_\phi(\mathfrak{M})$. Moreover, if ϕ is finite ($\phi(\mathbb{I}) < +\infty$), the finite weight $\phi \circ \pi_\phi^{-1}(\mathfrak{M})$ on $\pi_\phi(\mathfrak{M})$ is $\{\sigma_t^\phi\}_{t\in\mathbb{R}}$-invariant and the semigroup constructed in Example 1.4.7, with $\alpha_t \equiv \sigma_t^\phi$, is Markovian with respect to the unique vector $\xi_\phi \in \mathcal{P}_\phi$ which represents $\phi \circ \pi_\phi^{-1}(\mathfrak{M})$: $T_t\xi_\phi = \xi_\phi$.

Example 1.4.9. If in Example 1.4.8 we have a finite faithful normal state ψ on $\pi_\phi(\mathfrak{M})$ which is σ_t^ϕ-invariant, $\psi(x) = \psi(\sigma_t^{phi}(x))$ for each $t \in \mathbb{R}$, Then the semigroup $\{T_t\}_{t\geq 0}$ constructed above, is Markovian also with respect to the vector ξ_ψ which represents ψ in $\mathcal{P}_\phi$: $T_t\xi_\psi = \xi_\psi$ for each $t > 0$.

Notice that, if the algebra $\mathfrak{M}$ is already standardly realized in the Hilbert space $\mathfrak{H}$, in the sense that one considers a fixed standard form $(\mathfrak{M}, \mathfrak{H}, J, \mathcal{P})$ of $\mathfrak{M}$, and one consider a cyclic and separating vector $\xi_0 \in \mathcal{P}$, then construction in Examples 1.4.8 and 1.4.9 can be applied to the state $\phi_0(\cdot) \equiv (\cdot\xi_0; \xi_0)$ since, in this case, the semicyclic representation is trivial ($\pi_{\phi_0}(x) = x$), the standard form associated to ϕ_0 coincide with the given one and the modular operator are exactly those constructed from ξ_0 (recall that $\Delta_{\xi_0}^{1/2}$ and J_{ξ_0} are by definition the positive and the isometric part in the polar decomposition of the closure S_{ξ_0} of the operator defined by $\mathfrak{M}\xi_0 \ni x\xi_0 \longmapsto x^*\xi_0$; see [Tak2]).

Example 1.4.10. Let us consider a *symmetric spatial derivation* δ of $\mathfrak{M}$ implemented by a selfadjoint operator $(H,\ D(H))$ on $\mathfrak{H}$:

$$\delta(x) \equiv i[H;x] \qquad x \in D(\delta)$$

(see [Bra1] and [Bra2]). Suppose that $D(\delta)D(H) \subseteq D(H)$. Assume that there exists a cyclic and separating vector $\xi_0 \in \mathfrak{H}$ such that

$$\xi_0 \in D(H) \qquad H\xi_0 = 0$$

and consider the standard form associated to ξ_0:

$$(\mathfrak{M};\ \mathfrak{H};\ J_{\xi_0};\ \mathcal{P}_{\xi_0}).$$

Then, if $\{x\xi_0 :\ \delta(x) \in \mathfrak{M}\}$ is a core for H, the following conditions are equivalent:

i) $(\delta;\ D(\delta)$ generates a group $\{\tau_t\}_{t\in\mathbb{R}}$ of *automorphisms of $\mathfrak{M}$

ii) $e^{itH}\mathfrak{M}e^{-itH} = \mathfrak{M}$ (in this case $\tau_t(x) = e^{itH}xe^{-itH}$)

iii) $e^{itH}\mathfrak{M}_+\xi_0 \subseteq (\mathfrak{M}_+\xi_0)^-$

iv) H commutes strongly with Δ_{ξ_0} and

$$e^{itH}(\Delta_{\xi_0}^{1/4}\mathfrak{M}_+\xi_0) \subseteq (\Delta_{\xi_0}^{1/4}\mathfrak{M}_+\xi_0)^- \stackrel{\text{def}}{=} \mathcal{P}_{\xi_0}$$

(theorem 6 and corollary 2 in [Bra2]). If one of this conditions is satisfied, then applying the construction of Example 1.4.7, we obtain the following Markovian semigroup (with respect to ξ_0):

$$T_t \equiv \int_{\mathbb{R}} \mu_t(ds)e^{itH} \qquad t > 0.$$

Moreover $\{T_t\}_{t\geq 0}$ commutes strongly with Δ_{ξ_0}. Notice that condition *iii)* can be particularly useful since it does not involve the knowledge of Δ_{ξ_0}.

Example 1.4.11. Suppose $\{V(t)\}_{t\in\mathbb{R}}$ is a unitary group ,in $\mathfrak{H}$ such that $V(t) \in \mathfrak{M}$ for each $t \in \mathbb{R}$. Then automorphisms group

$$\alpha_t(x) \stackrel{\text{def}}{=} V(t)xV(t)^* \qquad t \in \mathbb{R}$$

is called *inner*. Now the group $U(t) \stackrel{\text{def}}{=} V(t)JV(t)J$ is Positivity Preserving since the cone $\mathcal{P}$ can be realized as $\mathcal{P} = \{xJxJ :\ \ x \in \mathfrak{M}\}^-$ (see [Con1]). Then if $\xi_0 \in \mathcal{P}$ is a cyclic and separating vector such that

$$V(t)\xi_0 = \xi_0$$

we have that

$$T_t = \int_{\mathbb{R}} \mu_t(ds)U(s) = T_t = \int_{\mathbb{R}} \mu_t(ds)V(s)JV(s)J$$

defines a Markovian semigroup with respect to ξ_0.

Example 1.4.12. Let us consider a $\sigma(\mathfrak{M} : \mathfrak{M}_*)$-continuous, strongly positive semigroup $\{\sigma_t\}_{t>0}$ on the von Neumann algebra $\mathfrak{M}$. This means simply that, for each $x \in \mathfrak{M}$, $t > 0$:

$$\sigma_t(x^*x) \geq \sigma_t(x)^*\sigma_t(x).$$

Suppose that $\phi_0 \in \mathfrak{M}_*^+$ is a faithful normal state on $\mathfrak{M}$ which is $\{\sigma_t\}_{t>0}$-invariant:

$$\phi_0(\sigma_t(x)) = \phi_0(x) \qquad \forall x \in \mathfrak{M} \quad \forall t > 0$$

and also symmetric with respect to $\{\sigma_t\}_{t>0}$. Now consider the cyclic representation (GNS-representation) associated to ξ_0: $(\pi;\ \mathfrak{H};\ \xi_0)$. Since ϕ_0 is faithful, ξ_0 is also separating for $\pi(\mathfrak{M})$. Now define, on $\mathfrak{H}$, the semigroup $\{T_t\}_{t\geq 0}$:

$$T_t(\pi(x)\xi_0) \stackrel{\text{def}}{=} \pi(\sigma_t(x))\xi_0 \qquad x \in \mathfrak{M} \quad t > 0.$$

Since $\{\sigma_t\}_{t>0}$ is strongly positive , and ϕ_0 is invariant, T_t is a contraction for each $t > 0$:

$$\begin{aligned}\|\pi(\sigma_t(x))\xi_0\|^2 &= \big(\xi_0;\ \pi(\sigma_t(x)^*\sigma_t(x))\xi_0\big) \leq \big(\xi_0;\ \pi(\sigma_t(x^*x))\xi_0\big)\\ &= \phi_0(\sigma_t(x^*x)) = \phi_0(x^*x) = \|\pi(x)\xi_0\|^2.\end{aligned}$$

Notice that $\mathfrak{M}\xi_0$ is dense in $\mathfrak{H}$ since ξ_0 is cyclic. Clearly $\{T_t\}_{t\geq 0}$ is a strongly continuous, symmetric, contraction semigroup on $\mathfrak{H}$. Moreover it preserves the cone $\pi(\mathfrak{M}_+)\xi_0$. In fact, $x \in \mathfrak{M}_+$ implies $\sigma_t(x) \in \mathfrak{M}_+$, and also

$$T_t\pi(x)\xi_0 = \pi(\sigma_t(x))\xi_0 \in \mathfrak{M}_+\xi_0.$$

Then by lemma 2 and lemma 3 in [Bra2], T_t preserves $\mathcal{P}\xi_0$ for each $t > 0$, and also, it commutes with the modular operators Δ_{ξ_0} and J_{ξ_0}. The semigroup $\{T_t\}_{t\geq 0}$ is then a Markovian semigroup with respect to ξ_0 (since clearly it preserves ξ_0).

Remark 1.4.13. Semigroups constructed as in Example 1.4.12 have been investigated by various authors, especially in the context of applications to Quantum Statistical Mechanics (see [Bat], [Maj], [Fri]). A difficult problem is considered, in general, the characterization of the generators of the strongly continuous semigroup $\{\sigma_t\}_{t>0}$ on $\mathfrak{M}$. A complete theory has been obtained only for bounded generators (*Lindblad generators*: see [Lin], [Eva2]). We note at this point that one could approach the problem of constructing strongly positive semigroups on $\mathfrak{M}$, by considering Markovian semigroups on $\mathfrak{H}$, using Theorem 1.3.3. One could expect that it could be, in this approach, easier to construct generators of Markovian semigroups on $\mathfrak{H}$, instead that generators of strongly positive, Markovian on the algebra $\mathfrak{M}$. In fact as we will show in Chapter 3, Markovian semigroups are characterized by Dirichlet Forms on $\mathfrak{H}$. Comparisons with the constructions of elliptic operators with non-smooth coefficients (see [Dav3]) by Dirichlet forms is a further hint in this direction.

Chapter 2.

Ergodic properties of Positivity Preserving semigroups

In this chapter we introduce some ergodic properties of positivity preserving maps, we compare these properties and we apply them to prove a Perron-Frobenius-Uniqueness type result for Positivity Preserving semigroups (§2.2.).
In §2.3. we introduce the notion called *hypercontractivity* and we use it to prove a Perron-Frobenius-Existence type result for Markovian semigroups.
The main tool is here the polar decomposition in the Hilbert space $\mathfrak{H}$ introduced and studied by H. Araki-T. Masuda and Connes (see [Ara2], [Con1]).

2.1. Ergodicity and Indecomposability

We start this section with some definition

Definition 2.1.1. A vector $\xi \in \mathcal{P}$ is called *strictly positive* if its support $S^{\mathfrak{M}}(\xi)$ is $I_{\mathfrak{H}}$ (the identity operator on $\mathfrak{H}$). A bounded operator $A : \mathfrak{H} \to \mathfrak{H}$ is said to be *positivity improving* if $\xi \in \mathcal{P}, \xi \neq 0$ imply $A\xi$ strictly positive. A is called *ergodic* if for all $\xi, \eta \in \mathcal{P}, \xi, \eta \neq 0$ there exists an integer n such that $(\xi; A^n\eta) > 0$. The positive operator A is said to be *indecomposable* if it leaves invariant no proper closed face [(2)] F of the natural cone $\mathcal{P}$: $F \subseteq \mathcal{P}$ closed face and $A(F) \subseteq F$ imply $F = 0$ or $F = \mathcal{P}$.

We recall that the support in $\mathfrak{M}$, $S^{\mathfrak{M}}(\xi)$, of a vector $\xi \in \mathcal{P}$ is the smallest projection in $\mathfrak{M}$ which has ξ in its range. Then it is not difficult to see that a vector $\xi \in \mathcal{P}$ is strictly positive if and only if it is cyclic and separating for $\mathfrak{M}$ (see Lemma 4.3. [Con1]). Hence we could avoid to introduce this notion but we like to use it, instead of cyclicity, for its geometrical appeal (at least in this section). We note that a positivity improving operator is in particular positivity preserving. We note also that ergodicity can be introduced in the more general setting of a real Hilbert

(2) A face of a convex set is a subset such that if a convex combination of elements of the set lies in the face, then each such element lies in the face.

space $\mathfrak{H}$ with a selfdual pointed cone $\mathfrak{H}^+$ (see [Far1]). However if one adds the two regularity assumptions of *homogeneity* and *orientability*, then the beautiful result of A. Connes in [Con1] implies that there exists a von Neumann algebra $\mathfrak{M}$ and a standard form for which $\mathfrak{H}^+$ is the natural cone $\mathcal{P}$ (at least if $\mathfrak{H}^+$ is *de genre denombrable*).
We will see later that for positivity preserving symmetric operators, ergodicity is equivalent to indecomposability. Nevertheless the latter properties is easier to verify in applications. In this fundamental work [Gro1] L.Gross introduced a notion of indecomposability in the special case of a *regular gage space* in the sense of I.E.Segal (see [Seg1]). After Theorem 1 we will see that our notion is an extension of the former.

In the following, when we will consider a standard form $(\mathfrak{M}, \mathfrak{H}, J, \mathcal{P})$ we will denote by with j the linear map

$$j : \mathfrak{M} \to \mathfrak{M}' \qquad j(x) \equiv Jx^*J \qquad \forall x \in M$$

induced by the modular conjugation J. If $e \in \mathbb{P}_{\mathfrak{M}}$ is a projection in $\mathfrak{M}$, $j(e)$ is a projection in $\mathfrak{M}'$ and since e and $j(e)$ commute, $ej(e)$ is a projection. The latter will be indicated by

$$P_e \equiv ej(e).$$

Theorem 2.1.2..

i) If e is a projection in $\mathfrak{M}$ then $e \neq 0$ if and only if $P_e \neq 0$

ii) If e is a projection in $\mathfrak{M}$ different from zero then there exists a positive vector $\xi \in \mathcal{P}$ in the range of P_e. Moreover

iii) A bounded positivity preserving operator $A : \mathfrak{H} \to \mathfrak{H}$ is indecomposable if and only if it commutes with no proper projection P_e with $e \in \mathbb{P}_{\mathfrak{M}}$.

Proof. i) Suppose $e \in \mathbb{P}_{\mathfrak{M}}$ and $P_e = 0$. Then take a cyclic and separating vector $\xi_0 \in \mathcal{P}$. If we denote with S_{ξ_0} the *sharp operator* $S_{\xi_0} = J_{\xi_0}\Delta_{\xi_0}^{1/2}$ we have ($J = J_{\xi_0}$ because ξ_0 is cyclic and separating in $\mathcal{P}$):

$$0 = \|P_e\xi_0\|^2 = (\xi_0; P_e\xi_0) = (\xi_0; eJeJ\xi_0) =$$

$$= (e\xi_0; Je\xi_0) = (e\xi_0; Je^*\xi_0) = (e\xi_0; JS_{\xi_0}(e\xi_0)) = (e\xi_0; \Delta_{\xi_0}^{1/2}e\xi_0) =$$

$$= \|\Delta_{\xi_0}^{1/4}e\xi_0\|^2 = \|i_0(e)\|^2.$$

The injectivity of the map i_0 (see Appendix A) implies $e = 0$.
ii) Let $e \in \mathbb{P}_{\mathfrak{M}}$ be a non zero projection. By i) $P_e \neq 0$ and since P_e is J-real (easy to verify) there exist a nontrivial J-real vector ξ in the range of P_e: $\xi \in \mathfrak{H}^\natural \equiv \mathcal{P} - \mathcal{P}$, $P_e\xi = \xi$. Let $\xi = \xi_+ - \xi_-$ the polar decomposition of ξ. Since P_e is positivity preserving (Lemma 2.9. in [Con1]) then $P_e\xi_\pm \in \mathcal{P}$. But since $\xi \neq 0$ at least one among $P_e\xi_+$ and $P_e\xi_-$ is different from zero. In either case

$$R_{P_e} = (R_{P_e} \cap \mathcal{P}) - (R_{P_e} \cap \mathcal{P})$$

iii) Suppose A is indecomposable. Let $e \in \mathbb{P}_{\mathfrak{M}}$, $e \neq 0, I$ and suppose that $P_eA = AP_e$. Let us consider the closed face $F_e \equiv P_e(\mathcal{P})$ (see Theorem 4.2. [Con1]). For each $\xi \in \mathcal{P}$, since A is positivity preserving, we have that $AP_e\xi = P_eA\xi \in P_e(\mathcal{P}) \equiv F_e$. The first part follows since $e \neq 0, I$ implies F_e non-trivial. The converse is also true because if F is a closed face of $\mathcal{P}$, then there exists a projection $e \in \mathbb{P}_{\mathfrak{M}}$ in $\mathfrak{M}$ such that $F = P_e(\mathcal{P})$ (Theorem 4.2. [Con1]) and F is non trivial if and only if $e \neq 0, I$. Now if A preserves F, taking $\xi \in R_{P_e}$ we have:

$$P_eA\xi = P_e(AP_e\xi_+ - AP_e\xi_-) =$$

$(P_e\xi_\pm \in \mathcal{P}$ implies $AP_e\xi_\pm \in F = P_e(\mathcal{P}))$

$$= AP_e\xi_+ - AP_e\xi_- = A\xi$$

and this concludes the proof.

□

Remark 2.1.3. If the von Neumann algebra $\mathfrak{M}$ is finite, then there exists a faithful normal finite trace τ on it (see theorem 2.7.17 [Bra1]). One can then construct the Hilbert space $L^2(\mathfrak{M};\tau)$ of all closed and densely defined operator x, affiliated to $\mathfrak{M}$ such that $\tau(|x|^2) < \infty$. The algebra $\mathfrak{M}$ acts by left and right multiplication:

$$L_a : L^2(\mathfrak{M};\tau) \to L^2(\mathfrak{M};\tau) \qquad L_a(x) \equiv ax \quad \forall x \in L^2(\mathfrak{M};\tau)$$

$$R_a : L^2(\mathfrak{M};\tau) \to L^2(\mathfrak{M};\tau) \qquad R_a(x) \equiv xa^* \quad \forall x \in L^2(\mathfrak{M};\tau)$$

If one defines as $\mathcal{P}$ the cone of positive operators in $L^2(\mathfrak{M};\tau)$, $L^2_+(\mathfrak{M};\tau)$, and as J the conjugation of operators in $L^2(\mathfrak{M};\tau)$ then one obtains a standard form for $\mathfrak{M}$:

$$\left(\mathfrak{M}, L^2(\mathfrak{M};\tau), J = \text{conjugation}, \ L^2_+(\mathfrak{M};\tau)\right).$$

Now, if $e \in \mathbb{P}_{\mathfrak{M}}$ is a projection, one can see that $P_e = ej(e) = L_e \circ R_e$. L. Gross called (in [Gro1]), the range of P_e, the *Pierce subspace associated to* e and defined a bounded, symmetric, positivity preserving operator A on $L^2(\mathfrak{M};\tau)$ as *indecomposable* if it leaves invariant no proper Pierce subspace. It is then evident that our definition of *indecomposable operator* is an extension of the one given by L. Gross. We refer to Remark 2.3 of L. Gross in [Gro1] for the argument (that works also in our setting) by which it does not seem possible to formulate indecomposability in terms of irreducibility of the operators $\{A\} \cup \mathfrak{M}$ as acting on $\mathfrak{H}$. However in the center of $\mathfrak{M}$, $\mathfrak{M} \cap \mathfrak{M}'$, the notion of indecomposability reduces to the usual one as soon as one realizes $\mathfrak{M} \cap \mathfrak{M}'$ as an algebra L^∞ on some measure space.

Theorem 2.1.4. *Let $A : \mathfrak{H} \to \mathfrak{H}$ be a bounded symmetric positivity preserving operator. Then A is ergodic if and only if it is indecomposable.*

Proof. Let us suppose that A is not indecomposable. Then by Theorem 2.1.2 *iii)* there exists a non trivial projection $e \in \mathbb{P}_{\mathfrak{M}}$ such that $AP_e = P_e A$. Since e is not trivial so are P_e and $P_{\mathrm{I}-e}$, and by Theorem 2.1.2 *ii)* there exist non zero positive vectors $\xi, \eta \in \mathcal{P}$ in the range of P_e and $P_{\mathrm{I}-e}$ respectively: $P_e\xi = \xi$, $P_{\mathrm{I}-e}\eta = \eta$. Then we have, for each $n > 0$:

$$\begin{aligned}(A^n\xi;\eta) &= (A^nP_e\xi;\eta) = (P_eA^n\xi;\eta)\\ &=(A^n\xi;P_e\eta) = (A^n\xi;P_eP_{\mathrm{I}-e}\eta) = 0.\end{aligned}$$

Thus A cannot be ergodic. Suppose now that A is not ergodic. Then there exist positive vectors $\xi, \eta \in \mathcal{P}$ such that $(A^n\xi;\eta) = 0 \quad \forall n > 0$. Let e_n be the projection in $\mathfrak{M}$ whose range is

$$\ker S^{\mathfrak{M}}(A^n\xi) = R^{\perp}_{S^{\mathfrak{M}}(A^n\xi)}.$$

Since $A^n\xi$ and η are positive orthogonal support:

$$S^{\mathfrak{M}}(A^n\xi) \cdot S^{\mathfrak{M}}(\eta) = 0.$$

Then $e_n \geq S^{\mathfrak{M}}(\eta) \neq 0 \quad \forall n \geq 0$ and also $e \equiv \bigwedge_{n\geq 1} e_n \geq S^{\mathfrak{M}}(\eta) \neq 0$. Again since e is a non zero projection in $\mathfrak{M}$, there exists a non zero vector $\xi \in \mathcal{P}$ in the range of P_e, $P_e\zeta = \zeta$, and moreover $\forall n \geq 0$:

$$\begin{aligned}(A^n\xi;A\zeta) &= (A^{n+1}\xi;\zeta) = (A^{n+1}\xi;P_e\zeta)\\ &=(P_eA^{n+1}\xi;\zeta) = (j(e)eA^{n+1}\xi;\zeta) = 0\end{aligned}$$

since $A^{n+1}\xi \in \ker e_n = R^{\perp}e_n = \left(\ker S^{\mathfrak{M}}(A^n\xi)\right)^{\perp} = R_{S^{\mathfrak{M}}(A^n\xi)}$. So we have $S^{\mathfrak{M}}(A^n\xi)\cdot S^{\mathfrak{M}}(A\xi) = 0$, $A\xi \in R_{e_n}$ for any $n \geq 0$ and $A\xi \in R_e$. Finally

$$\begin{aligned} P_e A\xi &= j(e)eA\xi = j(e)A\xi = JeJA\xi = (A\xi \in \mathcal{P}) \\ &= JeA\zeta = JA\zeta = A\zeta. \end{aligned}$$

Applying Theorem 2.1.2-*ii)* we have $P_eA = AP_e$ and (by Theorem 2.1.2-*iii)*) A cannot be indecomposable.

□

2.2. A Perron-Frobenius uniqueness-type result and its application to Positivity Preserving semigroups.

The following theorem contains the Perron-Frobenius Uniqueness type result for indecomposable, bounded, symmetric, positivity preserving operators mentioned above. It was proved by L. Gross in [Gro1] in case of a regular gage space. A version of it was proved by W. Faris in [Far1] with indecomposability replaced by ergodicity.

Theorem 2.2.1. *Let $A : \mathfrak{H} \to \mathfrak{H}$ be a bounded, symmetric, positivity preserving operator. Suppose that $\|A\|$ is an eigenvalue of A. Then $\|A\|$ has multiplicity one and there exists an associated eigenvector which is strictly positive (hence cyclic and separating) if and only if A is indecomposable.*

Proof. Suppose that A is indecomposable. Since $\mathfrak{H}$ splits as $\mathfrak{H}^{\natural} + i\mathfrak{H}^{\natural}$ and A is J-real, the eigenspace of $\|A\|$ is spanned by J-real eigenvectors. Let $\xi \in \mathfrak{H}^{\natural}$ be a J-real eigenvector of A corresponding to $\|A\|$. If $\xi = \xi_+ - \xi_-$ is the polar decomposition of ξ we want to prove firstly that ξ_+ and ξ_- are eigenvectors corresponding to$\|A\|$. We have, for $|\xi| = \xi_+ + \xi_-$:

$$\begin{aligned} \|A\|(|\xi|;|\xi|) &= \|A\|(\xi;\xi) \\ &= (A\xi_+;\xi_+) - (A\xi_-;\xi_+) - (A\xi_+;\xi_-) + (A\xi_+;\xi_-) \\ &\leq (A\xi_+;\xi_+) + (A\xi_-;\xi_+) + (A\xi_+;\xi_-) + (A\xi_+;\xi_-) \\ &= (A|\xi|;|\xi|) \leq \|A\|(|\xi|;|\xi|). \end{aligned}$$

Then

$$\begin{aligned}
\|A\|(|\xi|;|\xi|) &= (A|\xi|;|\xi|) \\
((\|A\|-A)|\xi|;|\xi|) &= \|(\|A\|-A)^{1/2}|\xi|\|^2 = 0 \\
(\|A\|-A)^{1/2}|\xi| &= 0 \\
(\|A\|-A)|\xi| &= 0 \\
A|\xi| &= \|A\||\xi|.
\end{aligned}$$

Then $|\xi|, \xi_\pm$ are eigenvectors of A corresponding to $\|A\|$. We have shown that each eigenvector of A corresponding to $\|A\|$ splits as

$$\xi = (\xi_+^r - \xi_-^r) + i(\xi_+^i - \xi_-^i)$$

where $\xi_\pm^\tau \in \mathcal{P} \quad \tau = i, r$, are mutually orthogonal and are all eigenvectors corresponding $\|A\|$. We will prove the first part of the theorem showing that if ξ is a positive, non zero eigenvector corresponding to $\|A\|$, then $S^{\mathfrak{M}}(\xi) = I_{\mathfrak{H}}$ (i.e. ξ is strictly positive). Let e be the projection $s^{\mathfrak{M}}(\xi)^\perp$. Then $e \in \mathbb{P}_{\mathfrak{M}}$, $e\xi = 0$ and since $\xi \neq 0$ we have $e \neq 0$. We want to show that A commutes with P_e. By Theorem 2.1.2 *iii)* it is sufficiently to show that they commute on $\mathcal{P}$. We have:

$$\begin{aligned}
(\xi; AP_e\eta) = (A\xi; P_e\eta) = \|A\|(A\xi; P_e\eta) = \\
\|A\|(P_e\xi;\eta) = \|A\|(j(e)e\xi;\eta) = 0.
\end{aligned}$$

Since $AP_e\eta \in \mathcal{P}$ (A, P_e are positivity preserving) then $AP_e\eta \in R_e$, i.e. $eAP_e\eta = AP_e\eta$ and also $P_e(AP_e\eta) = AP_e\eta$. Then A takes the range of P_e into itself and then A and P_e commute. The indecomposability of A implies $e = I_{\mathfrak{H}}$, i.e. $S^{\mathfrak{M}}(\xi) = I_{\mathfrak{H}}$. For proof that, if the multiplicity of $\|A\|$ is one and there is a strictly positive eigenvector, then A is indecomposable, we refer Theorem 1 and Theorem 2 in [Far1].

□

Remark 2.2.2. The original version of the classical Perron-Frobenius theorem can be recovered taking the standard form associated to the classical L^2 space over a set of finite cardinality with the counting measure.

Before applying Theorem 2.2.1 we have to adapt Definition 2.1.1 to semigroups.

Definition 2.2.3. Let $\{T_t\}_{t\geq 0}$ be a symmetric, strongly continuous, positivity preserving, contraction semigroup on $\mathfrak{H}$.

It is called *ergodic* if, for each $\xi\eta \in \mathcal{P}$, $\xi, \eta \neq= 0$, there exists $t > 0$ such that $(\xi; T_t\eta) > 0$. The semigroup $\{T_t\}_{t\geq 0}$ is said to be *indecomposable* if there exists $t > 0$ such that T_t is indecomposable.

Let H be the selfadjoint generator of $\{T_t\}_{t\geq 0}$: $T_t = e^{-tH}$. An eigenvector of H corresponding to the eigenvalue zero will be called a *ground state* for H (or $\{T_t\}_{t\geq 0}$).

In applications the operator H represents sometimes the hamiltonian of a physical system, and this justifies the name *ground state* given to an eigenvector with these properties.

We can now state the main result of this section:

Theorem 2.2.4. *Uniqueness of the Ground State.*

Let $\{T_t\}_{t\geq 0}$ be a symmetric, strongly continuous, positivity preserving, contraction semigroup. Assume that zero is an eigenvalue of the generator of $\{T_t\}_{t\geq 0}$ (i.e. there exists a ground state). Then the following conditions are equivalent:

i) zero has multiplicity one and there exists a strictly positive ground state

ii) $\{T_t\}_{t\geq 0}$ is ergodic

iii) $\{T_t\}_{t\geq 0}$ is indecomposable.

Proof. The equivalence of *i)* and *ii)* follows from Theorem 2.1.4. The equivalence of *i)* and *iii)* follows applying Theorem 2.2.1 $T_1 \equiv A$ (a ground state for $\{T_t\}_{t\geq 0}$ is eigenvector of A associated to the eigenvalue $1 = \|A\|$).

□

Remark 2.2.5. Definition 2.2.3 and Theorem 2.2.4 can be modified in the obvious way to include semigroups with lower semibounded generator.

Remark 2.2.6. Versions of the above theorem were proved by L. Gross [Gro1] and W. Faris [Far1] in the special case of a regular gage space and in the abstract case of a Hilbert space with a Hilbert cone, respectively. In [Gro] L. Gross applied the theorem to prove the uniqueness of the ground state of fermion systems. There, a hamiltonian is given on the antisymmetric Fock-Hilbert space, of some one-particle Hilbert space. The above mentioned version of the theorem is applied to a new operator, which is unitarily equivalent to the previous one, and is defined on the space $L^2(\mathcal{C})$ over the Clifford algebra $\mathcal{C}$. We have already seen in Remark 1.9 the corresponding standard form structure.

2.3. A Perron-Frobenius existence-type result and its application to Markovian semigroups.

As in the previous sections, let $(\mathfrak{M}, \mathfrak{H}, J, \mathcal{P})$ be a standard form for the von Neumann algebra $\mathfrak{M}$. Here we consider the problem to determine conditions on a symmetric Markovian semigroup $\{T_t\}_{t\geq 0}$, with generator H, which are sufficient in order that $\inf \sigma(H)$ be an eigenvalue of H associated to a strictly positive eigenvector. The result will follow from a general theorem about the existence of a strictly positive eigenvector, associated to the greatest eigenvalue $\|A\|$, of a bounded symmetric positivity preserving operator A on $\mathfrak{H}$. We will refer to this as a *Perron-Frobenius type* result because the theorem is a generalization of the well known theorem due to O. Perron, about the existence of an eigenvector in $\mathbb{R}^n$, with non negative coordinates, corresponding to the greatest eigenvalue of a matrix$\{a_{i,j}\}_{i,j=1}^n$ with non negative entries (see [Kre] page 201). The theorem of O. Perron was generalized by Jentsch in the case of positive kernels on $C[a,b]$ and by Krein and Rutman in a general Banach space setting, considering operators which leave invariant a cone possessing some geometrical property (for example in theorem 6.1 of [Kre], it is assumed that the closed linear hull of the cone is the whole space). However when one leaves the finite dimensional setting, some compactness property on the operator A has to be required to guarantee the existence of the Perron-eigenvector. In theorem 6.1 of [Kre], the authors require the operator to be compact. In his work [Gro1], L. Gross, treats the case of operators acting on the $L^2(\mathfrak{M};\tau)$ space of a finite von Neumann algebra $\mathfrak{M}$, corresponding to some finite faithful normal trace on $\mathfrak{M}$ (*regular gage space*). His radically new compactness assumption was the boundedness of the operator A from $L^2(\mathfrak{M};\tau)$ to $L^p(\mathfrak{M};\tau)$ for some $p > 2$. This assumption which is in general milder than the compactness of the operator, allowed L. Gross treat semigroups arising from Quantum Field Theory. Here we generalize the theorem of L. Gross to the case of operators A acting on the Hilbert space of a standard form $(\mathfrak{M}, \mathfrak{H}, J, \mathcal{P})$ of a σ-finite von Neumann algebra $\mathfrak{M}$, or in other words, to operators A acting on the Hilbert space of a cyclic representation, associated to a faithful normal state $\phi_0 \in \mathfrak{M}_*$. The generalization is carried out (following the spirit of the proof of L. Gross) using the scale of non-commutative L^p-spaces, $L^p(\mathfrak{M};\phi_0)$, constructed by H. Kosaki

([Kos]). As we noticed in the introduction of §1.3, these spaces are isometrically isomorphic to various other scales of spaces constructed in [Haa2], [Con2], [Hil], [Ter], [Ara2], [Cec]. This fact implies that the boundedness assumption from $L^2(\mathfrak{M};\tau)$ to $L^p(\mathfrak{M};\tau)$ for some $p>2$, can be restated equivalently on each different construction of the L^p-spaces.

Now we fix some notation.

We recall that we indicate with $L^p(\mathfrak{M};\phi_0)$, the interpolation space between $\mathfrak{M}$ and $\mathfrak{M}_*$, $C_{1/p}(\mathfrak{M}_{1/2};\mathfrak{M}_*)$, constructed by H. Kosaki ([Kos]). If A is an operator on $\mathfrak{H}$, A' will indicate the corresponding operator on $L^2(\mathfrak{M};\phi_0)$ and we will adopt the same assumption for vectors. To streamline the proof of the theorem, we state here the following lemmas.

Lemma 2.3.1. *Let $\mathfrak{H}$ be a Hilbert space and $\{A_i\}_{i\in I}$ a sequence of bounded symmetric operators in $\mathfrak{H}$, strongly convergent to a bounded symmetric operator A. Suppose also that for each $i\in I$, $\|A_i\|\le\|A\|$ and $\xi_i\in\mathfrak{H}$ is a normalized eigenvector of A_i corresponding to the eigenvalue $\|A_i\|$. Then each non zero weak-limit point of $\{\xi_i\}_{i\in I}$ is an eigenvector of A corresponding to the eigenvalue $\|A\|$.*

Proof. Since $\{\xi_i\}_{i\in I}$ is a norm bounded set in $\mathfrak{H}$, its closure is weakly-compact. Then there exists a weak limit $\xi\in\mathfrak{H}$ for the sequence. Fix ε. By the definition of $\|A\|$, there exists $\eta\in\mathfrak{H}$ such that

$$\|A\eta\|\ge(1-\varepsilon)\|A\|\cdot\|\eta\|.$$

But since $\{A_i\}_{i\in I}$ converges strongly to A, and $\|A_i\|\le\|A\|$ by hypothesis, we have

$$\begin{aligned}\|A\|\cdot\|\eta\| &\ge \lim_{i\in I}\|A_i\|\cdot\|\eta\|\ge\lim_{i\in I}\|A_i\eta\|\\ &=\|A\eta\|\ge(1-\varepsilon)\|A\|\cdot\|\eta\|.\end{aligned}$$

Hence $\|A\|=\lim_{i\in I}\|A_i\|$. By the continuity of the scalar product $(\cdot;\cdot)_{\mathfrak{H}}$ on $\mathfrak{H}\oplus\mathfrak{H}$ with the product topology *norm* $\times$ *weak*, we have that for each fixed $\eta\in\mathfrak{H}$:

$$\begin{aligned}(\eta;A\xi)=(A\eta;\xi)&=\lim_{i\in I}(A_i\eta;\xi_i)\\ &=\lim_{i\in I}(\eta;A_i\xi_i)=\lim i\in I\|A_i\|\cdot(\eta;\xi_i)\\ &\|A\|\cdot(\eta;\xi)\end{aligned}$$

and this concludes the proof.

□

Lemma 2.3.2. *Fix $p > 2$, then the following inequality holds for L^p-norms, for some $\vartheta \in (0,1)$:*

$$\|\eta\|_{L^2(\mathfrak{M};\phi_0)} \leq \|\eta\|^{\vartheta}_{L^1(\mathfrak{M};\phi_0)} \cdot \|\eta\|^{1-\vartheta}_{L^p(\mathfrak{M};\phi_0)}$$
$$\eta \in L^p(\mathfrak{M};\phi_0).$$

Proof. This inequality can be proved using some tools which are typical of the *complex interpolation method.* First of all we note that since $p > 2$, we have for the conjugate exponent $q < 2$. Now by the *reiteration theorem* (see [BeL] theorem 4.6.1), $L^2(\mathfrak{M};\phi_0)$ is an interpolation space of the couple $\left(L^\infty(\mathfrak{M};\phi_0); L^q(\mathfrak{M};\phi_0)\right)$ for some $\vartheta \in (0,1)$:

$$\left(L^\infty(\mathfrak{M};\phi_0); L^q(\mathfrak{M};\phi_0)\right)_\vartheta.$$

By the duality between $L^\infty(\mathfrak{M};\phi_0)$ and $L^1(\mathfrak{M};\phi_0)$, an element $\eta \in L^p(\mathfrak{M};\phi_0)$ gives rise to a linear functional on $L^1(\mathfrak{M};\phi_0)$, which, by Hölder inequality (see [Kos] proposition 5.2), is bounded with norm less or equal to $\|\eta\|_{L^1(\mathfrak{M};\phi_0)}$ on $L^\infty(\mathfrak{M};\phi_0)$ and with norm less or equal to $\|\eta\|_{L^p(\mathfrak{M};\phi_0)}$ on $L^q(\mathfrak{M};\phi_0)$. Interpolating this functional, we can evaluate the norm of its restriction to $L^2(\mathfrak{M};\phi_0)$ as:

$$\|\langle\eta;\cdot\rangle\|_{L^2\to\mathbb{C}} \leq \|\eta\|^{\vartheta}_{L^1} \cdot \|\eta\|^{1-\vartheta}_{L^p}.$$

But we already know this norm:

$$\|\langle\eta;\cdot\rangle\|_{L^2\to\mathbb{C}} = \|\eta\|_{L^2}.$$

Therefore we have:

$$\|\eta\|_{L^2(\mathfrak{M};\phi_0)} \leq \|\eta\|^{\vartheta}_{L^1(\mathfrak{M};\phi_0)} \cdot \|\eta\|^{1-\vartheta}_{L^p(\mathfrak{M};\phi_0)}$$
$$\eta \in L^p(\mathfrak{M};\phi_0).$$

□

Remark 2.3.3. The inequality proved above, in the commutative or tracial case, is a simple consequence of the Hölder inequality. In fact if $a \in L^2(\mathfrak{M};\tau)$ we have:

$$|a|^2 = |a|^{\frac{p-2}{p-1}} \cdot |a|^{\frac{p}{p-1}}$$

$$\|a\|_{L^2} \leq \|a\|^{\vartheta}_{L^1} \cdot \|a\|^{1-\vartheta}_{L^p} \qquad \vartheta = \frac{p-2}{2(p-1)}.$$

Now we state our main theorem:

Theorem 2.3.4. *let $(\mathfrak{M}, \mathfrak{H}, J, \mathcal{P})$ be a standard form for the von Neumann algebra $\mathfrak{M}$ and let ϕ_0 be a faithful normal state on $\mathfrak{M}$. Let A be a bounded, symmetric, positivity preserving operator on $\mathfrak{H}$. Assume that there exists $p > 2$ and $M < \infty$ such that, for each $\xi\prime \in L^p(\mathfrak{M}; \phi_0)$ we have:*

$$\|A'\xi'\|_{L^p(\mathfrak{M};\phi_0)} \leq m \cdot \|\xi'\|_{L^2(\mathfrak{M};\phi_0)}.$$

Then $\|A\|$ is an eigenvalue of A and there exists a positive eigenvector.

Proof. The first step is to approximate the operator A with a sequence $\{A_i\}_{i\in I}$ satisfying the assumption of Lemma 2.3.1. Let us consider the centralizer of the state ϕ_0:

$$\mathfrak{M}_{\phi_0} \equiv \{x \in \mathfrak{M} : \sigma_t^{\phi_0}(x) = x \qquad \forall t \in \mathbb{R}\}.$$

Then $\mathfrak{M}_{\phi_0}$ is a finite von Neumann subalgebra of $\mathfrak{M}$ and ϕ_0, restricted to $\mathfrak{M}_{\phi_0}$, is a faithful normal trace (see [Tak2] page 51). Then there exists an increasing sequence

$$\{e_i\}_{i\in I} \subseteq \mathbb{P}(\mathfrak{M}_{\phi_0})$$

of finite projections in $\mathfrak{M}_{\phi_0}$, converging to the identity of $\mathfrak{M}_{\phi_0}$ (and of $\mathfrak{M}$). Then, if we denote $p_i \equiv e_i j(e_i)$, we have that

$$\big(p_i\mathfrak{M}p_i;\ p_i(\mathfrak{H});\ p_iJp_i;\ p_i(\mathcal{P})\ \big)$$

are standard forms (see [Haa1]). Let us define the sequence of operators

$$A_i \equiv p_iAp_i \qquad i \in I.$$

Clearly since p_i's are projections: $\|A_i\| \leq \|A\|$. The A_i's are finite range, positivity preserving operators, since the p_i's are positivity preserving. Then we can apply theorem 6.1 in [Kre], to guarantee the existence of normalized positive eigenvectors $\xi_i \in p_i(\mathcal{P}) \subseteq \mathcal{P}$ corresponding the eigenvalues $\|A_i\|$:

$$A_i\xi_i = \|A_i\| \cdot \xi_i.$$

Now, the sequence $\{A_i\}_{i\in I}$ converges strongly to A since the p_i's converge to $\mathbb{1}$ (see [Haa] theorem 3). We can apply Lemma 2.3.1 and consider a weak-limit point $\xi \in \mathcal{P}$ such that $A\xi = \|A\|\xi$. We have just to prove that ξ is different from zero. To show this we consider the positivity preserving projections p_i's, and we prove first of all that they are Markovian with respect to the unique vector ξ_0 which represents ϕ_0 in $\mathcal{P}$. In fact we have:

$$
\begin{aligned}
&e_i \in \mathfrak{M}_{\phi_0}\\
&\Delta_{\xi_0}^{it} e_i = e_i \Delta_{\xi_0}^{it} \qquad \forall t \in \mathbb{R}\\
&(\Delta_{\xi_0}^{it}\xi_0 = \xi_0 \qquad \forall t \in \mathbb{R}\\
&\Delta_{\xi_0}^{it} e_i\xi_0 = e_i\xi_0 \qquad \forall t \in \mathbb{R}\\
&\Delta_{\xi_0}^{1/2} e_i\xi_0 = e_i\xi_0 \qquad (e_i\xi_0 \in \mathcal{D}(\Delta_{\xi_0}^{1/2})\\
&J\Delta_{\xi_0}^{1/2} e_i\xi_0 = Je_i\xi_0\\
&e_i\xi_0 = e_i^*\xi_0 = S_{\xi_0}e_i\xi_0 = J\Delta_{\xi_0}^{1/2}e_i\xi_0 = Je_i\xi_0 = Je_iJ\xi_0 = j(e_i)\\
&e_i\xi_0 = e_ie_i\xi_0 = e_ij(e_i)\xi_0 = p_i\xi_0.
\end{aligned}
$$

Moreover, since $e_i \in \mathfrak{M}_{\phi_0}$ implies $(\mathbb{I} - e_i) \in \mathfrak{M}_{\phi_0}$ and $(\mathbb{I} - e_i)$ is also a projection, we can repeat the above reasoning, substituting e_i with $(\mathbb{I} - e_i)$ and obtaining:

$$
\begin{aligned}
&0 \leq (\mathbb{I} - e_i)j(\mathbb{I} - e_i)\xi_0 = (\mathbb{I} - e_i)\xi_0\\
&e_i\xi_0 \leq \xi_0\\
&p_i\xi_0 = e_ij(e_i)\xi_0 = e_i\xi_0 \leq \xi_0.
\end{aligned}
$$

The markovianity of the p_i's implies, by Theorem 1.18-*iii)* of chapter 1, that the corresponding operators p_i''s on $L^2(\mathfrak{M};\phi_0)$ are bounded on $L^p(\mathfrak{M};\phi_0)$ for each $p \geq 1$ (with norm less or equal to 2).By the isomorphism between $\mathfrak{H}$ and $L^2(\mathfrak{M};\phi_0)$, we can now consider operators and vectors in $L^2(\mathfrak{M};\phi_0)$ corresponding to $A,\ A_i,\ \xi,\ \xi_i$. We denote this new objects as $A',\ A_i',\ \xi',\ \xi_i'$. Clearly $A_i' = p_i'A'p_i'$ and all properties of these objects which were related to the Hilbert space structure of $\mathfrak{H}$, remain true for the new objects defined in $L^2(\mathfrak{M};\phi_0)$. Therefore, to prove the theorem, we have just to prove that $\xi' \neq 0$. Notice we have:

$$
\begin{aligned}
\|A_i'\| \cdot \|\xi_i'\|_{L^p} &= \|\|A_i'\| \cdot \xi_i'\|_{L^p} = \|A_i'\xi_i'\|_{L^p}\\
&= \|A_i' = p_i'A'p_i'\xi_i'\|_{L^p} = \|A_i' = p_i'A'\xi_i'\|_{L^p} \leq 2 \cdot \|A_i'\xi_i'\|_{L^p}\\
&\text{(by the hypercontractivity assumption on } A' \text{)}\\
&\leq 2 \cdot M \cdot \|\xi_i'\|_{L^2} \leq 2M
\end{aligned}
$$

Moreover

$$
\begin{aligned}
1 = \|\xi_i'\|_{L^2} &\leq \|\xi_i'\|_{L^2(\mathfrak{M};\phi_0)} \leq \|\xi_i'\|_{L^1(\mathfrak{M};\phi_0)}^{\vartheta} \cdot \|\xi_i'\|_{L^p(\mathfrak{M};\phi_0)}^{1-\vartheta}\\
&\leq \|\xi_i'\|_{L^2(\mathfrak{M};\phi_0)} \cdot \left(\frac{2M}{\|A_i'\|}\right)^{(1-\vartheta)}
\end{aligned}
$$

hence

$$\|\xi_i'\|_{L^1} \geq \left(\frac{\|A_i'\|}{2M}\right)^{\frac{1-\vartheta}{\vartheta}}.$$

We recall now that $L^1 = \mathfrak{M}_*$ and that $\|\xi_i'\|_{L^1} = \langle \xi_i'; \mathbb{I}' \rangle$. Finally we compute ($\mathbb{I}' \in L^\infty \subseteq L^2$)

$$\begin{aligned} \|\xi'\|_{L^1} &= (\xi' \mid \mathbb{I}')_{L^2 \times L^2} = \lim_i (\xi_i' \mid \mathbb{I}') \\ &= \lim_i \|\xi_i'\|_{L^1} \geq \left(\frac{\|A_i'\|}{2M}\right)^{\frac{1-\vartheta}{\vartheta}} > 0 \end{aligned}$$

from which we deduce that $\xi' \neq 0$.

□

The following statement makes precise the result mentioned above about the existence of a ground state for the generator of a markovian semigroup. It states that under suitable assumption, there exists an eigenvector associated to the eigenvalue zero of the generator of a markovian semigroup. The result is a generalization to generic von Neumann algebras in standard form of the result of L. Gross in [Gro1], corresponding to the case of a finite von Neumann algebra.

Theorem 2.3.5. *Let $(\mathfrak{M}, \mathfrak{H}, J, \mathcal{P})$ be a standard form of the von Neumann algebra $\mathfrak{M}$ and let $\xi_0 \in \mathcal{P}$ be a cyclic and separating vector. Let $\{T_t\}_{t\geq 0}$ be a Markovian semigroup with respect to ξ_0 (cf. Definition 1.2.1); ϕ_0 will denote the state represented in $\mathcal{P}$ by ξ_0. Suppose that there exists $t_0 > 0$ such that*

$$T_{t_0}' \text{ is bounded from } L^2(\mathfrak{M}; \phi_0) \text{ to } L^p(\mathfrak{M}; \phi_0)$$

for some $p > 2$ (T_{t_0}' is the operator in $L^2(\mathfrak{M}; \phi_0)$ corresponding by Theorem 1.3.7 to T_{t_0} on $\mathfrak{H}$). Then there exists an eigenvector $\xi_G \in \mathcal{P}$ corresponding to the eigenvalue zero of the generator H of $\{T_t\}_{t\geq 0}$, $H\xi_G = 0$, which is strictly positive.

Proof. It is sufficient to apply Theorem 2.3.4 to the operator $A \equiv T_{t_0}$ to obtain the existence and the strictly positivity of ξ_G.

□

Remark 2.3.6. As the easy proof shows, Theorem 2.3.5 can be generalized to strongly continuous, symmetric semigroups which are uniformly bounded.

Chapter 3.

Dirichlet Forms

In previous chapters we introduced Markovian Semigroups on Standard Forms of von Neumann algebras. In the present chapter we address ourself the characterization of these semigroups from an infinitesimal point of view. We place emphasis on quadratic forms (called "Dirichlet Forms") associated with the generators of these semigroups, rather than of the generators themselves. The reasons for this choice are connected with the general problem of closability of forms and operators. Often one tries to construct a strongly continuous semigroup $\{T_t\}_{t\geq 0}$ from a symmetric operator L or its associated quadratic forms $\mathcal{E}$. These objects are initially defined on some nice domains D_L and $D_{\mathcal{E}}$ and most of the time they are not closed. This is because these domains are too small in some sense. The problem of closability is to prove that $(L\,;\,D_L)$ or $(\mathcal{E}\,;\,D_{\mathcal{E}})$ are closable, i,e, admit closed extensions. These extensions will be the generators of a well defined semigroup $\{T_t\}_{t\geq 0}$. Usually the domain D_L is much smaller than $D_{\mathcal{E}}$ and this is one reason why one could prefer to work with the form $\mathcal{E}$. From another point of view, this advantage can be seen if one tries to construct a generator depending on some coefficients. Dealing with forms, the coefficients can be assumed to be much less regular (*e.g.* as in the case of elliptic operators on regions of $\mathbb{R}^n$; cf. [D1]). This "Dirichlet Form approach" has been exploited in various directions. For example, Silverstein and Fukushima applied it to the construction of Markov Processes on locally compact spaces and Albeverio, Høegh-Krohn, Röckner and Ma extended this in the general case of Markov Processes on Topological vector spaces. B. Davies in [Dav3] used it to study elliptic operators and kernels with non smooth coefficients.

The study of Dirichlet forms in the non-commutative case dates back to L. Gross [Gro2] for the Clifford-Dirichlet form on the Clifford algebra and the first systematic account on this subject was given by S Albeverio and Høegh-Kröhn [AHK]. More recently, B Davies and M. Lindsay [Dav4] initiated a study of Dirichlet Forms and Markov semigroups on von Neumann, algebras especially addressed to the closability problem. All these authors deal with finite or semi-finite von Neumann algebras.

Another important potential application of non-commutative Dirichlet forms is the construction of Dynamical semigroups on C^*-algebras representing the dissipative time-evolution of some Open System [Dav1]. There, one is interested in constructing contraction semigroups on a C^*-algebra $\mathcal{A}$ which are completely positive and Markovian, in the sense that they preserve a given state on $\mathcal{A}$. The problem of classifying Dynamical semigroups on C^*-algebras is solved only for the class of norm-continuous semigroups (whose generators are bounded). Only on some special algebras (CCR or CAR algebras) non norm-continuous Dynamical semigroups are constructed.

3.1. Dirichlet Forms and Markovian Semigroups

In this section we consider a fixed standard form $(\mathfrak{M}, \mathfrak{H}, J, \mathcal{P})$ of the von Neumann algebra $\mathfrak{M}$. Throughout the section $\big(\mathcal{E}\,;\, D(\mathcal{E})\big)$ will denote a densely defined, positive ($\mathcal{E}(\xi,\xi) \geq 0$, for every $\xi \in D(\xi)$), symmetric bilinear form on $\mathfrak{H}$. If $\mathcal{E}$ is closed, $\big(L\,;\, D(L)\big)$ will denote the associated positive, self-adjoint operator. In this case $\{T_t\}_{t\geq 0}$ will denote the corresponding strongly continuous, symmetric, contraction semigroup and $\big(R_\lambda\big)_{\lambda>0}$ will denote the associated strongly continuous, symmetric, contraction resolvent; cf. *e.g.* Section 1.3 in [Fuk].

In this section we will study some contraction properties of forms and we will show how forms having these properties correspond to Positivity Preserving and Markovian semigroups.

Lemma 3.1.1. *Let $\big(\mathcal{E}\,;\, D(\mathcal{E})\big)$ be a closed form on $\mathfrak{H}$. Then the following statements are equivalent:*

i) $\big(\mathcal{E}\,;\, D(\mathcal{E})\big)$ commutes with the isometric involution J in the sense that

$$J(D(\mathcal{E})) \subseteq D(\mathcal{E}) \quad \textit{and} \quad \mathcal{E}(J\xi; J\zeta) = \mathcal{E}(\xi;\zeta)$$

for every $\xi, \zeta \in D(\mathcal{E})$;

ii) $\big(L\,;\, D(L)\big)$ commutes with J in the sense that

$$J(D(L)) \subseteq D(L) \quad \textit{and} \quad LJ\xi = JL\xi$$

for every $\xi \in D(L)$;

iii) $\big(R_\lambda\big)_{\lambda>0}$ is J-real;

iv) $\{T_t\}_{t\geq 0}$ is J-real.

Proof. The equivalence between *iii)* and *iv)* follows from Lemma 1.1.1.

i)$\Rightarrow$ *ii)* By hypothesis $J\Big(D(L^{1/2})\Big) \subseteq D(L^{1/2})$ and $JL\xi = LJ\xi$, for every $\xi \in D(L^{1/2})$. By spectral theorem, if $E^{L^{1/2}}(d\lambda)$ denotes the spectral measure associated to $L^{1/2}$, one has that $JE^{L^{1/2}}(\Omega)J = E^{L^{1/2}}(\Omega)$, for each Borel set Ω in $\mathbb{R}$. Representing $\Big(L\,;\,D(L)\Big)$ with the spectral measure $E^{L^{1/2}}(\cdot)$, one obtains the thesis.

ii)$\Rightarrow$ *iii)* Follows from the formula $R_\lambda = (\lambda + L)^{-1}$, for every $\lambda > 0$.

iii)$\Rightarrow$ *iv)* This implication follows from Lemma 1.1.1.

iv)$\Rightarrow$ *i)* This follows from the representation of $\mathcal{E}$ in terms of the semigroup $\{T_t\}_{t\geq 0}$; cf. Lemma 1.3.4 in [Fuk].

□

Definition 3.1.2. The form $\Big(\mathcal{E}\,;\,D(\mathcal{E})\Big)$ (resp. the operator $\Big(L\,;\,D(L)\Big)$) will be called *J-real* if the point *i)* (resp *ii)*) of Lemma 3.1.1 is verified.

Combining Definition 3.1.2 with Definition 1.1.2, we can say that the form $\mathcal{E}$ is J-real if and only if the operator L is J-real, if and only if the semigroup $\{T_t\}_{t\geq 0}$ is J-real and if and only if the resolvent $\big(R_\lambda\big)_{\lambda>0}$ is J-real.

In the abelian case of remark 1.2.4, our definition reduces to the usual notion of reality of forms and operators defined on an ordered $L^2(X, \mathcal{M}, \mu)$ space.

If the standard form is associated with a cyclic and separating vector $\xi_0 \in \mathfrak{H}$, which is a tracial vector, and the Hilbert space $\mathfrak{H}$ is the space $L^2(\mathfrak{M}, \tau)$, where τ is the finite trace associated with ξ_0, then the modular conjugation $J = J_{\xi_0}$ is the operator of taking the adjoint in $L^2(\mathfrak{M}, \tau)$; recall that the elements of this space are densely defined operators affiliated with $\mathfrak{M}$. In this situation, our definition reduces to those given in [Dav4] and [AHK].

Remark 3.1.3. We will indicate with $\Big(\mathcal{E}[\cdot]\,;\,D(\mathcal{E})\Big)$ the quadratic form associated to the bilinear form $\Big(\mathcal{E}\,;\,D(\mathcal{E})\Big)$:

$$\mathcal{E}[\xi] \stackrel{\text{def}}{=} \mathcal{E}(\xi, \xi) \qquad \forall \xi \in D(\mathcal{E}).$$

Now we start to compare the positivity preserving properties of bounded operators on $\mathfrak{H}$ with some "contraction" properties of associated forms.

Theorem 3.1.4. *Let $T : \mathfrak{H} \to \mathfrak{H}$ be a bounded, symmetric, positivity preserving operator. Let us define the bounded form $\mathcal{E}_T$ as the bilinear form associated to the operator $I - T$: $\mathcal{E}_T(\xi|\eta) \stackrel{\text{def}}{=} (\xi|\eta - T\eta)$, for every $\xi, \eta \in \mathfrak{H}$. Then:*
i) $\mathcal{E}_T[|\xi|] \leq \mathcal{E}_T[\xi]$, for every $\xi \in \mathfrak{H}$;
ii) if T is a contraction, we have

$$\mathcal{E}_T[\xi_\pm] \leq \mathcal{E}_T[\xi]$$

for every $\xi \in \mathfrak{H}$. Here $\xi = \xi_+ - \xi_-$ is the polar decomposition in $\mathfrak{H}$ given by the cone $\mathcal{P}$ and $|\xi| \stackrel{\text{def}}{=} \xi_+ + \xi_-$; see Appendix A or Lemma 1.3.3 in [Ara2].

Proof. i) We have the equality

$$\mathcal{E}_T[\xi_+ - \xi_-] = \mathcal{E}_T[\xi_+ + \xi_-] + 4(\xi_+|T\xi_-) - 4(\xi_+|\xi_-).$$

By properties of polar decomposition of ξ, ξ_- and ξ_+ are orthogonal, $(\xi_-|\xi_+) = 0$. Since T is positive preserving and $\mathcal{P}$ is selfdual, we have: $\xi_- \in \mathcal{P} \Rightarrow T\xi_- \in \mathcal{P} \Rightarrow (\xi_+|T\xi_-) \geq 0$. Then $\mathcal{E}_T[|\xi|] \leq \mathcal{E}_T[\xi]$.
ii) By the following inequality

$$\mathcal{E}_T[\xi_+ - \xi_-] = \mathcal{E}_T[\xi_+] + \mathcal{E}[\xi_-] - 2(\xi_+|\xi_-) + 2(\xi_+|T\xi_-)$$

Since T is a symmetric contraction, we have in general

$$1 \geq \|T\| = \sup_{\|\eta\|=1} |(\eta|T\eta)|$$

and also $(\eta;\eta) - (\eta|T\eta) \geq 0$, for every $\eta \in \mathfrak{H}$. Then T is Positive definite:$\mathcal{E}_T[\eta] \geq 0$, for every $\eta \in \mathfrak{H}$. Finally we have

$$\mathcal{E}_T[\xi] \geq \mathcal{E}_T[\xi_\pm]$$

□

In order to use Theorem 3.1.4 to compare Positivity Preserving semigroups with their quadratic forms, we exploit the following representation (see Lemma 1.3.4 in [Fuk])

$$D(\mathcal{E}) = \left\{\xi \in \mathfrak{H} : \exists \lim_{t\to 0} t^{-1}(\xi; \xi - T_t\xi) \in \mathfrak{H}\right\}$$

$$\mathcal{E}(\xi;\eta) = \lim_{t\to 0} \frac{(\xi; \xi - T_t\eta)}{t}, \qquad \forall \xi, \eta \in D(\mathcal{E}).$$

The following theorem is a generalization of a well-known criterion (in the commutative case) due to Beurling and Deny; cf. Theorem 1.3.2 in [Dav3].

Theorem 3.1.5. *Let $\{T_t\}_{t\geq 0}$ be a strongly continuous, symmetric, contraction semigroup on $\mathfrak{H}$ and let $\left(\mathcal{E}\,;\,D(\mathcal{E})\right)$ be its associated bilinear form. The following conditions are equivalent:*

i) $\{T_t\}_{t\geq 0}$ *is Positivity Preserving;*

ii) $\left(\mathcal{E}\,;\,D(\mathcal{E})\right)$ *is J-real,* $|\xi|\in D(\mathcal{E})\cap\mathfrak{H}^\natural$ *and*

a) $\mathcal{E}[|\xi|]\leq\mathcal{E}[\xi]$ *for each* $\xi\in D(\mathcal{E})\cap\mathfrak{H}^\natural$*, or equivalently*

b) $\mathcal{E}(\xi_+|\xi_-)\leq 0$*, for each* $\xi\in D(\mathcal{E})\cap\mathfrak{H}^\natural$*.*

Proof. i)$\Rightarrow$ *ii)* Since $\{T_t\}_{t\geq 0}$ is Positive Preserving, it is also J-real and then, applying Lemma 3.1.1, we obtain that also $\mathcal{E}$ is J-real. Let us consider the bounded operator $t^{-1}(I_{\mathfrak{H}}-T_t)$ and the bounded form

$$\mathcal{E}_t(\xi|\eta)\overset{\text{def}}{=}t^{-1}(\xi;(I_{\mathfrak{H}}-T_t)\eta),\qquad\forall\xi\eta\in\mathfrak{H}.$$

Then applying Theorem 3.1.3, we have for each $\xi\in\mathfrak{H}$

$$\begin{aligned}\mathcal{E}[|\xi|]&=\lim_{t\to 0}(|\xi|;|\xi|-T_t|\xi|)\\&=\lim_{t\to 0}\mathcal{E}_t[|\xi|]\leq\lim_{t\to 0}\mathcal{E}_t[\xi]\\&=\mathcal{E}[\xi].\end{aligned}$$

Now if $\xi\in D(\mathcal{E})\cap\mathfrak{H}^\natural$ one has $\mathcal{E}[\xi]<+\infty$ and then $\mathcal{E}[|\xi|]<+\infty$, which implies $|\xi|\in D(\mathcal{E})$. The equivalence between *a)* and *b)* is clear

ii)$\Rightarrow$ *i)* By Theorem 1.1.3 it is sufficient to prove that the resolvent $\left(R_\lambda\right)_{\lambda>0}$ is Positively Preserving, *i.e.* R_λ is positively preserving for each λ sufficiently large. Fixing $\xi\in\mathcal{P}$, we have to show that $R_\lambda\xi=(\lambda+L)^{-1}\xi\in\mathcal{P}$ for all $\lambda>0$ sufficiently large. Since R_λ is J-real (Lemma 1.1.1) we have $R_\lambda\xi\in\mathfrak{H}^\natural$, for every $\lambda>0$. Let us consider the following Hilbert space $\mathfrak{H}_\lambda$, $\lambda>0$

$$\begin{aligned}\mathfrak{H}_\lambda&\overset{\text{def}}{=}D(\mathcal{E})\\(\xi;\eta)_\lambda&\overset{\text{def}}{=}\mathcal{E}(\xi;\eta)+\lambda(\xi;\eta)\end{aligned}$$

for every $\eta,\xi\in D(\mathcal{E})$. $\mathfrak{H}_\lambda$ is complete since $\left(\mathcal{E}\,;\,D(\mathcal{E})\right)$ is closed. Let $I:\mathfrak{H}_\lambda\to\mathfrak{H}$ the canonical injection. Then we have $I^*=(\lambda+L)^{-1}=R_\lambda$. Clearly $\mathfrak{H}_\lambda^\natural\overset{\text{def}}{=}\mathfrak{H}_\lambda\cap\mathfrak{H}^\natural$ is a real Hilbert space. Let us consider the cone

$$\mathcal{C}\overset{\text{def}}{=}I^*(\mathcal{P})=R_\lambda(\mathcal{P})\subseteq\mathfrak{H}_\lambda^\natural.$$

We have to show that $\mathcal{C} \subseteq \mathcal{P}$. For each $\zeta \in \mathcal{C}$ there exists $\eta \in \mathcal{P}$ such that $\zeta = I^*\eta$ and, for each $\xi \in \mathfrak{H}^\natural_\lambda$, we have

$$\begin{aligned}(|\xi|;\zeta)_\lambda &= (|\xi|;I^*\eta)_\lambda = (I|\xi|;\eta) = (|\xi|;\eta)\\ &(\text{ since } \mathcal{P} \text{ is selfdual and } |\xi|,\eta \in \mathcal{P})\\ &\geq |(\xi;\eta)| = |(I\xi;\eta)| = (\xi;I^*\eta)|\\ &\qquad = |(\xi;\zeta)_\lambda|.\end{aligned}$$

By hypothesis we have $|\xi| \in \mathfrak{H}^\natural_\lambda$ and also:

$$\begin{aligned}\| \, |\xi| \, \|^2_\lambda &= \mathcal{E}[|\xi|] + \lambda \| \, |\xi| \, \|\\ &\quad (\text{by hypothesis } a))\\ &\leq \mathcal{E}[\xi] + \lambda \| \, |\xi| \, \| = \mathcal{E}[|\xi|] + \lambda \|\xi\|\\ &= \|\xi\|^2_\lambda.\end{aligned}$$

Applying Lemma 1.3.1 in [Dav1] to the real Hilbert space $\mathfrak{H}^\natural_\lambda$ and to the cone $\mathcal{C}$, with $\tilde{\xi} \stackrel{\text{def}}{=} |\xi|$, we have $\mathcal{C} \subseteq \mathcal{P}$. This concludes the proof of the theorem.

□

Remark 3.1.6. We remark that the proof of the above theorem exploits the selfduality and polar decomposition properties of the natural cone $\mathcal{P}$, in the spirit of the corresponding result in the commutative case; cf. Theorem 1.3.2 in [Dav3].

We begin now the characterization of forms associated to Markovian semigroups. To do this, we fix a cyclic and separating vector $\xi_0 \in \mathcal{P}$ or, equivalently, a faithful normal state $\varphi_0 \in \mathfrak{M}_*$. Recall that by one of the most fundamental properties of standard forms it is possible to represent each positive normal form φ on $\mathfrak{M}$ ($\varphi \in \mathfrak{M}^+_*$) uniquely by a vector $\xi_\varphi \in \mathcal{P}$:

$$\varphi(x) \stackrel{\text{def}}{=} (x\xi_\varphi;\xi_\varphi), \qquad \forall x \in \mathfrak{M}.$$

In the following, the Markov property of a semigroup $\{T_t\}_{t\geq 0}$ will be referred to a fixed vector $\xi_0 \in \mathcal{P}$ or, equivalently, to the state $\varphi_0 \in \mathfrak{M}^+_*$; see Definition 1.2.1.

Before to state the Markovian property of forms we are interested in, we analyze the situation in the commutative case. Consider a symmetric, positive, bilinear form $\big(\mathcal{E}\,;\,D(\mathcal{E})\big)$ on the space $L^2(X,\mathcal{M},\mu)$, where $(X,\mathcal{M},\mu)$ is a measure space. A criterion also due to Beurling and Deny states that the semigroup associated to $\mathcal{E}$ is Markovian if and only if the form has the following properties (see *e.g.* Theorems 1.3.2 and 1.3.3 in [Dav3]):

- $|f|$, $g \wedge 1 \in D(\mathcal{E})$, for all real valued $f \in D(\mathcal{E})$ and for all positive $g \in D(\mathcal{E})$;
- ; $\mathcal{E}[|f|] \leq \mathcal{E}[f]$, for all real valued $f \in D(\mathcal{E})$;
- $\mathcal{E}[g \wedge 1] \leq \mathcal{E}[g]$, for each positive $g \in D(\mathcal{E})$.

Here $g \wedge 1 = \inf(g, 1)$. Notice that $g \wedge 1$ is that element of the subset

$$\{h \in L^2(X, \mathcal{M}, \mu) : 0 \leq h(x) \leq 1, \forall x \in X\}$$

which is nearest to g.

In our non commutative situation we meaning to ξ_-, ξ_+ and $|\xi|$ by the polar decomposition in $\mathfrak{H}^\natural$ defined by the natural cone $\mathcal{P}$. By the above analogy with the commutative case, a possible candidate to substitute $g \wedge 1$ could be the element of $\mathcal{P} \cap (\xi_0 - \mathcal{P})$ which minimizes the distance from $\xi \in D(\mathcal{E}) \cap \mathcal{P}$. All this can be actually carried out, as we show in the next definition.

Definition 3.1.7 (Definition of Markovian and Dirichlet Forms) . Let $(\mathfrak{M}, \mathfrak{H}, J, \mathcal{P})$ be a standard form of the von Neumann algebra $\mathfrak{M}$. Let us fix a cyclic and separating vector $\xi_0 \in \mathcal{P}$. Let us denote by

$$Q_0 = \{\xi \in \mathfrak{H}^\natural : \xi \leq \xi_0\} = \xi_0 - \mathcal{P}.$$

Notice that Q_0 is a convex subset of $\mathfrak{H}^\natural$. Moreover, since $\mathcal{P}$ is closed both in the norm and in the weak topology of $\mathfrak{H}^\natural$, the same properties are shared by Q_0. by the fact that $\mathfrak{H}^\natural$ is a Hilbert space, we can define uniquely the "projection" of an element $\xi \in \mathfrak{H}^\natural$ onto Q_0; $Q_0(\xi)$ is the unique element of Q_0 which minimizes the distance from ξ:

$$\|Q_0(\xi) - \xi\| \stackrel{\text{def}}{=} \inf_{\eta \in Q_0} \|\eta - \xi\|.$$

We can now state the definition of Markovian form. Let $\left(\mathcal{E}\,;\, D(\mathcal{E})\right)$ be a densely defined, symmetric, positive definite, J-real form on $\mathfrak{H}$.

$\left(\mathcal{E}\,;\, D(\mathcal{E})\right)$ will be called *Markovian* if

a) $|\xi| \in D(\mathcal{E})$ for each $\xi \in D(\mathcal{E}) \cap \mathfrak{H}^\natural$ and

$$\mathcal{E}[|\xi|] \leq \mathcal{E}[\xi] \quad \text{for each}\, \xi \in D(\mathcal{E}) \cap \mathfrak{H}^\natural;$$

b) $Q_0(\xi) \in D(\mathcal{E})$ for each $\xi \in D(\mathcal{E}) \cap \mathcal{P}$ and

$$\mathcal{E}[Q_0(\xi)] \leq \mathcal{E}[\xi] \quad \text{for each}\, \xi \in D(\mathcal{E}) \cap \mathcal{P}.$$

The form $\left(\mathcal{E}\,;\, D(\mathcal{E})\right)$ will be called a *Dirichlet Form* if it is Markovian and closed.

Remark 3.1.8.. In the commutative case or when $\mathfrak{M}$ is a finite von Neumann algebra and ξ_0 is a traced state, our definition of Dirichlet Form is equivalent to the definition of Dirichlet Form in the sense of [Fuk] and also to the definition of Lipschitz form in the sense of [Dav4] and [AHK] (see Theorem 1. 4.1. in [Fuk], Theorems 1.3.2. and 1.3.3. in [Dav3] and Proposition 2.12. in [Dav4]). The definition of Dirichlet form in [Dav4] requires the form to be Lipschitz and also possessing some contraction property involving the complex structure of $D(\mathcal{E})$, which is useful to study further positivity preserving properties of semigroups. We note that all the above definitions of Dirichlet forms (in the commutative case) or Lipschitz forms (in the tracial case) are well suited to treat non-closed forms. For comparison in the commutative case, if a closable form is Markovian, its closure may not be a Dirichlet form (see [Fuk] discussion page 5).

We note that also the definition of *Lipschitz* form is tailored to treat (at least in the commutative case) *classical Dirichlet forms*:

$$\mathcal{E}[f] = \int_{\mathbb{R}^n} d\mu(x)|\nabla f(x)|^2$$
$$D(\mathcal{E}) = C_0^\infty(\mathbb{R}^n)$$

μ being a Borel measure.

In fact in this case it is easy to see that ξ is Markovian, in the sense of [AHK] and [Dav4], thanks to the presence of the gradient ∇ and by the Leibnitz rule for derivation.

In application, a closed form is defined initially only on a dense subdomain. It is then useful to have some criteria to decide when the closure of a given form is Dirichlet. We will treat this problem in section 3.2.

Before giving characterization of Markovian semigroups by Dirichlet forms we state the following lemma:

Lemma 3.1.9.. *Let $(T_t)_{t>0}$ be a Markovian semigroup and let $\Big(\mathcal{E}\,;\,D(\mathcal{E})\Big)$ be its associated form. Then*

$$\mathcal{E}(\xi;\xi_0) \geq 0$$

for each $\xi \in \mathcal{P}$. If $T_{\xi_0} = \xi_0$ for each $t > 0$ then $\mathcal{E}(\xi;\xi_0) = 0$ for each $\xi \in D(\xi)$.

Proof. If $\xi \in \mathcal{P}$ we have $(\xi;\xi_0 - T_t\xi_0) \geq 0$ for each $t > 0$ and then

$$\mathcal{E}(\xi;\xi_0) = \lim_{t\to 0} t^{-1}(\xi;\xi_0 - T_t\xi_0) \geq 0.$$

If $\xi_0 = T_t\xi_0 \ \ \forall t > 0$ the same argument gives $\mathcal{E}(\xi;\xi_0) = 0 \ \forall \xi \in D(\xi)$.

□

Theorem 3.1.10. *Let $(\mathfrak{M}, \mathfrak{H}, J, \mathcal{P})$ be a standard form of the von Neumann algebra $\mathfrak{M}$ and let $\xi_0 \in \mathcal{P}$ be a cyclic and separating vector. Let $(T_t)(t > 0)$ be a strongly continuous, symmetric, contraction semigroup and consider the associated symmetric, positive definite bilinear form $\left(\mathcal{E}\,;\,D(\mathcal{E})\right)$. The following statements are equivalent:*

i) $(T_t)_{t>0}$ is Markovian

ii) the resolvent $\left(R_\lambda\right)_{\lambda>0}$ of $(T_t)_{t>0}$ is Markovian

iii0 $\left(\mathcal{E}\,;\,D(\mathcal{E})\right)$ is a Dirichlet Form.

Proof. The equivalence between i) and ii) is in Theorem 1.2.3. Now we prove *i) ⇒ iii)*.

Property a) of Def. 3.1.7. follows applying Theorem 3.1.5. By the representation

$$\mathcal{E}[\xi] = \lim_{t\to 0} \mathcal{E}_t[\xi]$$

for each $\xi \in D(\xi)$ where $D(\mathcal{E}_t) = \mathfrak{H}$ and $\mathcal{E}_t[\xi] = t^{-1}(\xi;(I - T_t)\xi)$ for each $\xi \in \mathfrak{H}$ it is sufficient to prove property b) of Def 3.1.7. for the bounded forms $\mathcal{E}_t t > 0$. By the very definition of $Q_0(\xi)$ is easy to see that

$$Q_0(\xi) = \xi_0 - (\xi - \xi_0)_-.$$

Then we have for $\xi \in \mathcal{P}$:

$$\begin{aligned}
&\mathcal{E}_t[Q_0(\xi)] = \mathcal{E}_t[\xi_0 - (\xi - \xi_0)_-] = \\
&= \mathcal{E}_t[\xi_0] - 2\mathcal{E}_t(\xi_0;(\xi - \xi_0)_-) + \mathcal{E}_t[(\xi - \xi_0)_-] = \\
&\leq \text{ (by hypothesis and Theorem 3.1.4. ii)} \\
&\leq \mathcal{E}_t[\xi_0] - 2\mathcal{E}_t(\xi_0;(\xi - \xi_0)_-) + \mathcal{E}_t[(\xi - \xi_0)] \leq \\
&\leq \text{ (by Lemma 3.1.9.) } \leq \\
&\leq \mathcal{E}_t[\xi_0] - 2\mathcal{E}_t(\xi_0;(\xi - \xi_0)_-) + 2\mathcal{E}_t(\xi_0;(\xi - \xi_0)_+) + \mathcal{E}_t[(\xi - \xi_0)] = \\
&= \mathcal{E}_t[\xi_0] + 2\mathcal{E}_t(\xi_0;\xi - \xi_0) + \mathcal{E}_t[(\xi - \xi_0)] = \\
&= \mathcal{E}_t[\xi_0] - 2\mathcal{E}_t[\xi_0] + 2\mathcal{E}_t(\xi_0;\xi) + \mathcal{E}_t[\xi] - 2\mathcal{E}_t(\xi;\xi_0] + \mathcal{E}_t[\xi_0] = \\
&\mathcal{E}_t[\xi].
\end{aligned}$$

Now we prove $iii) \Rightarrow ii)$. Let $\xi \in \mathfrak{H}$ such that $0 \leq \xi \leq \xi_0$. To prove $\left(R_\lambda\right)_{\lambda>0}$ is Markovian we have to show that (L being the generator of $(T_t)_{t>0}$)

$$0 \leq (I_{\mathfrak{H}} + \alpha L)^{-1}\xi \leq \xi_0$$

for each $\alpha > 0$.

Fix $\eta = (I_{\mathfrak{H}} + \alpha L)^{-1}\xi$. Since ξ is J-real, $\left(R_\lambda\right)_{\lambda>0}$ is J- real and we have

$$\eta \in \mathfrak{H}^{\natural}.$$

Moreover, by Theorem 3.1.5. and Theorem 1.1.3. $\left(R_\lambda\right)_{\lambda>0}$ is positivity preserving, and we have $\eta \in \mathcal{P}$. By hypothesis $Q_0(\eta) \in D(\xi)$ (since $\eta \in D(L)$) and moreover:

$$\begin{aligned}
&\|(I_{\mathfrak{H}} + \alpha L)^{1/2}(Q_0(\eta) - \eta)\|^2 \\
&= ((I_{\mathfrak{H}} + \alpha L)^{-1}\xi;\xi) - 2(\xi;Q_0(\eta)) + \|(I_{\mathfrak{H}} + \alpha L)^{1/2}Q_0(\eta)\|^2 = \\
&= ((I_{\mathfrak{H}} + \alpha L)^{-1}\xi;\xi) - 2(\xi;Q_0(\eta)) + (Q_0(\eta);Q_0(\eta)) + \alpha\mathcal{E}[Q_0(\eta)] \leq \\
&\leq ((I_{\mathfrak{H}} + \alpha L)^{-1}\xi;\xi) - 2(\xi;Q_0(\eta)) + (Q_0(\eta);Q_0(\eta)) + \alpha\mathcal{E}[\eta] = \\
&= ((I_{\mathfrak{H}} + \alpha L)^{-1}\xi;\xi) + \|\xi - Q_0(\eta)\|^2 - \|\xi\|^2 + \alpha\mathcal{E}[\eta] \leq \\
&\leq ((I_{\mathfrak{H}} + \alpha L)^{-1}\xi;\xi) + \|\xi - \eta\|^2 - \|\xi\|^2 + \alpha\mathcal{E}[\eta] = \\
&= (\eta;\xi) - 2(\xi;\eta) + \|\eta\|^2 + \alpha\mathcal{E}[\eta] = \\
&= -(\xi;\eta) + (\eta;(I_{\mathfrak{H}} + \alpha L)\eta) = \\
&= -(\xi;\eta) + (\eta;(I_{\mathfrak{H}} + \alpha L)(I_{\mathfrak{H}} + \alpha L)^{-1}\xi) = \\
&= -(\xi;\eta) + (\eta;\xi) = 0
\end{aligned}$$

Then $Q_0(\eta) = \eta$ and we have: $\eta = \xi_0 - (\eta - \xi_0)_- \Rightarrow (\eta - \xi_0)_+ = 0 \Rightarrow \eta \leq \xi_0 \Rightarrow (I_{\mathfrak{H}} + \alpha L)^{-1}\xi \leq \xi_0$. This concludes the proof.

□

Remark 3.1.11.. The above proof follows the lines of the corresponding one in the commutative case as given for example in [Dav3] Theorem 1.3.3. We have exploited the geometrical properties of the projection $\xi \longmapsto Q_0(\xi)$, in particular we used the fact that, in a triangle, if a vertex P is opposed to the longest side λ, the perpendicular from λ from P intersects λ (i.e. "falls between the other two vertices".

Remark 3.1.12.. We remark at this point that, the equivalence we have proved between Dirichlet form and Markovian semigroup, gives a possible tool to construct Dynamical semigroup on the algebra $\mathfrak{M}$. In fact applying Theorem 1.3.3. ii) we can construct, from a Dirichlet form on $\mathfrak{H}$, a weakly*-continuous Markovian semigroup on the algebra M. Moreover we know that this semigroup is a contraction semigroup on the selfadjoint part of $\mathfrak{M}$ and that in general it has norm less that 2. Following the work of Lindsay and Davies, Albeverio and Høegh-Krohn in [Dav4] and [AHK] it seems promising to find properties of the Dirichlet form $\mathcal{E}$ on $\mathfrak{H}$ which guarantee that the semigroup on $\mathfrak{M}$ is Dynamical (i.e. completely positive).

3.2. Criteria of Markovianity and Closability.

The main problems that must be solved in constructing Dirichlet Forms are Markovianity and Closability. Here we exhibit some general sufficient criteria

The following extension result can be useful.

Proposition 3.2.1. *Let $\Big(\mathcal{E}\,;\,D(\mathcal{E})\Big)$ be a closable form on $\mathfrak{H}$. Then if $\mathcal{E}$ is Markovian, its closure is a Dirichlet form.*

Proof. We must to show that the closure of $\Big(\mathcal{E}\,;\,D(\mathcal{E})\Big)$, $(\overline{\mathcal{E}};D(\overline{\mathcal{E}}))$, is Markovian. Notice that the maps

$$f_1 : \mathfrak{H}^{\natural} \to \mathcal{P} \qquad f_1(\xi) \overset{\text{def}}{=} |\xi| \quad \xi \in \mathfrak{H}^{\natural}$$
$$f_2 : \mathfrak{H}^{\natural} \to \xi_0 - \mathcal{P} \qquad f_1(\xi) \overset{\text{def}}{=} Q_0(\xi) \quad \xi \in \mathfrak{H}^{\natural}$$

are continuous in the norm topology (see [Köt], [Goe]), since they are constructed through maps which minimize the distance from closed convex sets, in a Hilbert space, and these maps are continuous. $\overline{\mathcal{E}}$ is J-real because $\mathcal{E}$ has this property. Moreover, since $\overline{\mathcal{E}}$ is closed, its associated quadratic form is lower semicontinuous on $\mathfrak{H}$ (see theorem 1.2.1 in [Dav3]). To prove *a)* in Definition 3.1.7, we fix $\xi \in D(\mathcal{E}) \cap \mathfrak{H}^{\natural}$. Since $\overline{\mathcal{E}}$ is closed, there exists a sequence $\{\xi_n\}_{n=1}^{\infty} \subseteq D(\mathcal{E})$ such that $\xi_n \to \xi$ in the graph-norm (recall the proof of Theorem 3.1.5) and then $|\xi_n| \to |\xi|$

in the norm topology of $\mathfrak{H}$. Then, we have

$$
\begin{aligned}
\overline{\mathcal{E}}[|\xi|] &\leq \text{by lower semicontinuity} \\
\underline{\lim}_n \overline{\mathcal{E}}[|\xi_n|] &= (\xi_n \in D(\mathcal{E})) \\
\underline{\lim}_n \mathcal{E}[|\xi_n|] &\leq \text{(by hypothesis)} \\
\underline{\lim}_n \mathcal{E}[\xi_n] &= \text{(since } \xi_n \to \xi \text{ in the graph-norm)} \\
&= \mathcal{E}[\xi]
\end{aligned}
$$

The proof of point *b)* in Definition 3.1.7 is similar, exploiting now the fact that f_2 is continuous.

□

The above criterion can be used also in the following way. Suppose we have a closed form $\big(\mathcal{E}\,;\,D(\mathcal{E})\big)$ on $\mathfrak{H}$. Then, if D is a form core for $\mathcal{E}$ (i.e. a subspace dense in $D(\mathcal{E})$ in the graph-norm), $\mathcal{E}$ is the closure of its restriction to D. To prove $\big(\mathcal{E}\,;\,D(\mathcal{E})\big)$ is a Dirichlet form we can then apply Proposition 3.2.1 to $(\mathcal{E};\ D)$.

In Remark 3.1.8 we discussed briefly the Markovian property of forms in the commutative case, as introduced for example in [Fuk]. To be precise, a symmetric, positive definite, bilinear form $\big(\mathcal{E}\,;\,D(\mathcal{E})\big)$ on the space $L^2(X, M, \mu)$ is called Markovian if for each $\varepsilon > 0$ there exists a function $\phi_\varepsilon : \mathbb{R} \to \mathbb{R}$ with the properties

$$
\begin{aligned}
\phi_\varepsilon(t) &= t & t &\in [0,1] \\
-\varepsilon \leq \phi_\varepsilon(t) &\leq 1+\varepsilon & t &\in \mathbb{R} \\
0 \leq \phi_\varepsilon(t_1) - \phi_\varepsilon(t_2) &\leq t_1 - t_2 & t_1 &> t_2
\end{aligned}
$$

and also, for each $u \in D(\mathcal{E})$

$$
\phi_\varepsilon \circ u \in D(\mathcal{E})
$$

$$
\mathcal{E}[\phi_\varepsilon \circ u] \leq \mathcal{E}[u].
$$

As already noticed, this kind of definition is needed to treat cases in which the form is originally defined on some domain, which is "too small" in the sense that it is not carried into itself by the map $u \longmapsto |u|$, $u \longmapsto u \wedge 1$. The use of function ϕ_ε allows a regularization, in the domain of $D(\mathcal{E})$, of the elements $|u|$ and $u \wedge 1$. Nevertheless the above definition of Markovianity, is adapted to treat elliptic operators defined by classical Dirichlet forms. Since in our non-commutative situation we do not have a functional calculus in the domain $D(\mathcal{E})$, we have to replace the above definition-criterion by a more general. The following result is obtained in this spirit.

Theorem 3.2.2. *Suppose* $\left(\mathcal{E}\,;\, D(\mathcal{E})\right)$ *is a Dirichlet form on* $\mathfrak{H}$.
i) If for each $\xi \in D(\mathcal{E})$ *there a sequence* $\{\xi_n\}_{n=1}^{\infty} \subseteq D(\mathcal{E})$ *such that*

$$\xi_n \to |\xi| \text{ in } \mathfrak{H}$$

$$\sup_n \mathcal{E}[\xi_n] < \infty$$

then $|\xi| \in D(\underline{\mathcal{E}})$ *(the domain of the closure of* $\left(\mathcal{E}\,;\, D(\mathcal{E})\right)$*). If moreover*

$$\underline{\lim}_n \mathcal{E}[\xi_n] \leq \mathcal{E}[\xi]$$

then $(\overline{\mathcal{E}};\ D(\overline{\mathcal{E}}))$ *satisfies property a) in Definition 3.1.7:*

$$\overline{\mathcal{E}}[|\xi|] \leq \overline{\mathcal{E}}[\xi].$$

ii) If for each $\xi \in D(\mathcal{E}) \cap \mathcal{P}$ *there exists a sequence* $\{\xi_n\}_{n=1}^{\infty} \subseteq D(\mathcal{E})$ *such that*

$$\xi_n \to Q_0(\xi) \text{ in } \mathfrak{H}$$

$$\sup_n \mathcal{E}[\xi_n] < \infty$$

then $Q_0(\xi) \in D(\overline{\mathcal{E}})$. *If moreover*

$$\underline{\lim}_n \mathcal{E}[\xi_n] \leq \mathcal{E}[\xi]$$

then $(\overline{\mathcal{E}};\ D(\overline{\mathcal{E}}))$ *satisfies property b) in Definition 3.1.7:*

$$\overline{\mathcal{E}}[Q_0(\xi)] \leq \overline{\mathcal{E}}[\xi].$$

If the closable form $\left(\mathcal{E}\,;\, D(\mathcal{E})\right)$ *satisfies assumption i) and ii) then its closure* $(\overline{\mathcal{E}};\ D(\overline{\mathcal{E}}))$ *is a Dirichlet form.*

Proof. The proofs of *i)* and *ii)* are very similar. They are direct application of Lemma 2.18 in [Röc] to the closed form $(\overline{\mathcal{E}};\ D(\overline{\mathcal{E}}))$.

□

Here we state some well known criteria for closability of forms, which are of abstract nature (in the sense that no additional structure is required on the Hilbert space $\mathfrak{H}$).

Theorem 3.2.3. *Suppose $(\mathcal{E};\ D)$ is a symmetric, positive definite form on $\mathfrak{H}$. Then*

i) if $\{\xi_n\}_{n=1}^{\infty} \subseteq D$, $\xi_n \to 0$ in $\mathfrak{H}$ imply $\mathcal{E}(\xi_n;\eta) \to 0$ for each $\eta \in D$, then $(\mathcal{E};\ D)$ is closable

ii) if there exists a symmetric, positive operator $(S;\ D)$ on $\mathfrak{H}$ such that $\mathcal{E}(\xi;\eta) = (S\xi;\eta)$ for each $\xi,\eta \in D$ then $(\mathcal{E};\ D)$ is closable

iii) if there exists a symmetric, positive definite closable form $(\mathcal{E}';\ D)$ such that, for some $c > 1$, one has

$$c^{-1} \cdot \mathcal{E}_1'[\xi] \leq \mathcal{E}_1[\xi] \leq c \cdot \mathcal{E}_1'[\xi]$$

for all $\xi \in D$, then $(\mathcal{E};\ D)$ is closable ($\mathcal{E}_1[\cdot]$, ${\mathcal{E}'}_1[\cdot]$ denote the graph-norm on D). Let $((\mathcal{E}^{(k)};\ D^{(k)}))_{k=1}^{\infty}$ be a sequence of symmetric, positive definite forms, which are closable (resp. closed). Then

iv) if we define

$$D \stackrel{\text{def}}{=} \{\xi \in \bigcap_{k=1}^{\infty} D^{(k)} : \sum_{k=1}^{\infty} \mathcal{E}^{(k)}[\xi] < \infty\}$$

$$\mathcal{E}(\xi;\eta) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \mathcal{E}^{(k)}(\xi;\eta) \qquad \xi,\ \eta \in D$$

then $(\mathcal{E};\ D)$ is closable (resp. closed)

v) suppose $D^{(k+1)} \subseteq D^{(k)}$ and $\mathcal{E}^{(k)}[\xi] \leq \mathcal{E}^{(k+1)}[\xi]$ for each $\xi \in D^{(k+1)}$. Define

$$D \stackrel{\text{def}}{=} \{\xi \in \bigcap_{k=1}^{\infty} D^{(k)} : \sup_{k\geq 1} \mathcal{E}^{(k)}[\xi] < \infty\}$$

$$\mathcal{E}(\xi;\eta) \stackrel{\text{def}}{=} \sup_{k\geq 1} \mathcal{E}^{(k)}(\xi;\eta) \qquad \xi,\ \eta \in D$$

then $(\mathcal{E};\ D)$ is closable (resp. closed).

□

Remark 3.2.4. In all cases considered in Theorem 3.2.3 nothing is said about the size of the domain D. One can even have that $D = \{0\}$. The density of D in $\mathfrak{H}$ is a separate problem. In §3.4 we will consider this problem, giving sufficient conditions (see Theorem 3.4.1 and Remark 3.4.2).

Corollary 3.2.5. *In cases iv) and v) of Theorem 3.2.3, if each $(\mathcal{E}^{(k)};\ D^{(k)})$ satisfies a) andb) of Definition 3.1.7, then also $(\mathcal{E};\ D)$ satisfies the same conditions. In particular if each $(\mathcal{E}^{(k)};\ D^{(k)})$ is Markovian, then the closure of $(\mathcal{E};\ D)$ is a Dirichlet form.*

The proof is evident from the definition of $(\mathcal{E};\ D)$.

3.3. Constructions of Dirichlet Forms.

Here we show give methods to construct Markovian and Dirichlet forms based on result of §§1.4 and 3.2.. We remark that this constructions rely on general properties of Standard Forms of von Neumann algebras (*e.g.* Theorem 1.4.1) and on general properties of Dirichlet Form (*e.g.* Theorems 3.2.3 and 3.2.5). In particular most of our constructions are available for a general σ-finite von Neumann algebras.

Remark 3.3.1. In some of the following Examples we refer to Examples 1.4.7,...1.4.12 of §1.4. Although all of them can be considered for a general symmetric convolution semigroup on $\mathbb{R}$, or for a general symmetric convolution semigroup on $\mathbb{R}$ supported on $[0,\ \infty)$ (see Definition 1.4.2 and Theorems 1.4.4 and 1.4.6), we will consider in these two classes only the Gaussian semigroup (see Example 1.4.3 *i)* and the semigroup of Example *iii)*. Following notations of Theorems 1.4.4 and 1.4.6, we will indicate with $\{T_t\}_{t\geq 0}$ and $\{S_t\}_{t>0}$, the semigroups constructed by the two convolutions semigroups mentioned above respectively.

Example 3.3.2. Applying Theorem 3.1.10 to the semigroup $\{T_t\}_{t\geq 0}$ constructed in Example 1.4.7, we have that the corresponding Dirichlet form $\big(\mathcal{E}\,;\ D(\mathcal{E})\big)$ is given by

$$D(\mathcal{E}) = D(H)$$

$$\mathcal{E}[\xi] = \frac{1}{4}\|H\xi\|^2 \qquad \xi \in D(H)$$

where H is the generator of the unitary group that implements the group of *automorphisms $\{\tau_t\}_{t\in\mathbb{R}}$.

If we now subordinate $\{T_t\}_{t\geq 0}$ as indicated in Theorem 1.4.6, then the form associated to the new semigroup $\{S_t\}_{t>0}$ is given by

$$D(\mathcal{E}) = D(|H|^{1/2})$$

$$\mathcal{E}[\xi] = \frac{1}{2}\||H|^{1/2}\xi\|^2 \qquad \xi \in D(|H|^{1/2}).$$

To see this just apply the result of example 2.32 in [Dav].

Example 3.3.3. In case of Example 1.4.8 (and discussion after Example 1.4.9), where the group of *automorphisms was the *modular group* $\sigma_t^{\phi_0}(x) = \Delta_{\xi_0}^{it} x \Delta_{\xi_0}^{-it}$ associated to a cyclic and separating vector $\xi_0 \in \mathcal{P}$, the form associated to $\{T_t\}_{t\geq 0}$ is given by

$$D(\mathcal{E}) = D(\ln \Delta_{\xi_0})$$

$$\mathcal{E}[\xi] = \frac{1}{4}\| \ln \Delta_{\xi_0}\xi\|^2 \qquad \xi \in D(\ln \Delta_{\xi_0}).$$

and the form associated to $\{S_t\}_{t>0}$ is given by

$$D(\mathcal{E}) = D(|\ln \Delta_{\xi_0}|^{1/2})$$

$$\mathcal{E}[\xi] = \frac{1}{4}\||\ln \Delta_{\xi_0}|^{1/2}\xi\|^2 \qquad \xi \in D(|\ln \Delta_{\xi_0}|^{1/2}).$$

Example 3.3.4. Let $h = h^* \in \mathfrak{M}$ be a selfadjoint element of $\mathfrak{M}$, and consider the following bounded symmetric form

$$D(\mathcal{E}) \stackrel{\text{def}}{=} \mathfrak{H}$$

$$\mathcal{E}[\xi] \stackrel{\text{def}}{=} \|(h - j(h))\xi\|^2 \qquad \xi \in \mathfrak{H}$$

where $j(h) = JhJ$. Notice that by the modular properties of J, $j(h) = j(h)^* \in \mathfrak{M}'$. To show that $\mathcal{E}$ satisfies condition a of Definition 3.1.7, we have to show that

$$\mathcal{E}(\xi_+;\xi_-) \leq 0 \qquad \xi = \xi_+ - \xi_- \in \mathfrak{H}^\natural.$$

Recall that by the properties of the polar decomposition, we have that the supports $s_\pm$ of $\xi_\pm$ in $\mathfrak{M}$ and the supports $s'_\pm$ of $\xi_\pm$ in $\mathfrak{M}'$ satisfy

$$s_+ \cdot s_- = s_- \cdot s_+ = 0$$

$$s'_+ \cdot s'_- = s'_- \cdot s'_+ = 0$$

See [Ara2] Lemma 13.3; recall that the support of an element $\xi \in \mathfrak{H}^\natural$ in $\mathfrak{M}$ (resp. in $\mathfrak{M}'$) is the smallest projection $s \in \mathfrak{M}$ (resp. $s' \in \mathfrak{M}'$) such that $s\xi = \xi$ (resp.

$s'\xi = \xi$. We then have

$$\begin{aligned}\mathcal{E}(\xi_+;\xi_-) &= ((h-j(h))\xi_+;\ (h-j(h))\xi_-) \\ &= (h^2 s'_+\xi_+;\ s'_-\xi_-) + (j(h^2)s_+\xi_+;\ s_-\xi_-) - 2(hj(h)\xi_+;\ \xi_-) \\ &= (s'_+h^2\xi_+;\ s'_-\xi_-) + (s_+j(h^2)\xi_+;\ s_-\xi_-) - 2(hj(h)\xi_+;\ \xi_-) \\ &= -2(hj(h)\xi_+;\ \xi_-).\end{aligned}$$

Now we note that $hj(h)$ is positivity preserving (see lemma 2.5.26 in [Bra1]) and so, by selfduality of $\mathcal{P}$ we have:

$$\begin{gathered}\xi_\pm \in \mathcal{P} \\ hj(h)\xi_+ \in \mathcal{P} \\ -2(hj(h)\xi_+;\ \xi_-) \leq 0.\end{gathered}$$

Now we show that, if there exists a cyclic and separating vector $\xi_0 \in \mathcal{P}$ such that $h\xi_0$ is J-real

$$Jh\xi_0 = h\xi_0$$

then the form $\mathcal{E}$ is Markovian, hence a Dirichlet form. Notice that applying Theorem 3.1.5 we deduce that the semigroup generated by $\mathcal{E}$, is a strongly continuous, symmetric, Positivity Preserving contraction semigroup. Then applying Theorem 3.1.4 to the approximating forms $\mathcal{E}_t[\xi] \stackrel{\text{def}}{=} t^{-1}\cdot(\xi;\ (I-T_t)\xi)$, we have that

$$\mathcal{E}(\xi_0;\ \xi) = 0 \qquad \forall \xi \in \mathfrak{H}^{\natural}.$$

Collecting this fact, we calculate for $\xi \in \mathfrak{H}^{\natural}$:

$$\begin{aligned}\mathcal{E}[Q_0(\xi)] &= \mathcal{E}[\xi_0 - (\xi-\xi_0)_-] \\ &= \mathcal{E}[\xi_0] - 2\mathcal{E}(\xi_0;\ (\xi-\xi_0)_-) + \mathcal{E}[(\xi-\xi_0)_-] \\ &= \mathcal{E}[(\xi-\xi_0)_-] \leq \mathcal{E}[\xi-\xi_0] \\ &= \mathcal{E}[\xi] - 2\mathcal{E}(\xi;\ \xi_0) + \mathcal{E}[\xi_0] \\ &= \mathcal{E}[\xi].\end{aligned}$$

Then $(\mathcal{E},\ \mathfrak{H})$ is a Dirichlet form and its associated semigroup is Markovian.

Example 3.3.5. In this example we want to generalize the preceding result when the operator h is no more bounded. We assume h to be affiliated with $\mathfrak{M}$ ($h\eta\mathfrak{M}$), selfadjoint and such that

$$JD(h) \subseteq D(h).$$

This guarantee that we have a well densely defined, symmetric form

$$\begin{aligned} D(\mathcal{E}) &\stackrel{\text{def}}{=} D(h) \\ \mathcal{E}[\xi] &\stackrel{\text{def}}{=} \|(h - j(h))\xi\|^2 \qquad \xi \in D(h). \end{aligned}$$

We now use the following lemma proved by U. Haagerup in [Haa3] (Lemma 2.11).

Lemma 3.3.6. *Let $x, y \in \mathfrak{M}$ be two selfadjoint elements of $\mathfrak{M}$ and $\xi \in \mathfrak{H}$ be a fixed vector. Then there exists a positive bounded measure μ on $\mathbb{R}^2$, supported by spectrum$(x) \times$ spectrum(y), such that for any bounded Borel functions f and g on $\mathbb{R}$, we have the following integral representation:*

$$\|\big(f(x) - j(g(y))\big)\xi\|^2 = \int_{\mathbb{R}^2} |f(s) - g(t)|^2 \nu(ds, dt).$$

Now consider the bounded continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by ($n \in \mathbb{N}$):

$$f_n(s) = \begin{cases} s & \text{if } s \in [-n, n] \\ \\ n & \text{if } s \notin [-n, n]. \end{cases}$$

These functions are bounded, lipschitzian with Lipschitz constant 1. Moreover they tend pointwise to the identity function of $\mathbb{R}$. Let us the bounded forms

$$\begin{aligned} D(\mathcal{E}_n) &\stackrel{\text{def}}{=} \mathfrak{H} \\ \mathcal{E}_n[\xi] &\stackrel{\text{def}}{=} \|(f_n(h) - j(f_n(h)))\xi\|^2 \qquad \xi \in \mathfrak{H}. \end{aligned}$$

As shown in Example 3.3.4, these are Dirichlet forms. Moreover we calculate, with aid of Lemma 3.3.6:

$$\begin{aligned} \mathcal{E}_n[\xi] &= \int_{\mathbb{R}^2} |f_n(s) - f_n(t)|^2 \nu(ds, dt) \\ &\leq \int_{\mathbb{R}^2} |f_{n+1}(s) - f_{n+1}(t)|^2 \nu(ds, dt) = \mathcal{E}_{n+1}[\xi] \end{aligned}$$

for $\xi \in \mathfrak{H}$. Hence for each $\xi \in D(h)$ we have

$$\sup_n \mathcal{E}_n[\xi] = \mathcal{E}[\xi].$$

Applying Theorem 3.3.3-*v)* and corollary 3.3.5, we obtain that $\left(\mathcal{E}\,;\, D(\mathcal{E})\right)$ is a Dirichlet form with respect to ξ_0.

Remark 3.3.7. The Dirichlet form constructed in Example 3.3.4 is different from zero only if h does not in the center $\mathcal{Z}(\mathfrak{M}) \overset{\text{def}}{=} \mathfrak{M} \cap \mathfrak{M}'$ of the algebra. In fact, in that case $h = j(h)$ by the properties of J (see [Haa3]). Hence this kind of forms are typical of non abelian situations. This should be compared with the fact that, if $\mathfrak{M}$ is non abelian, then there exist non trivial inner (hence bounded), derivations (see [Sak]). Indeed, forms as in Example 3.3.4, come, as for inner derivations, from the fact that $\mathfrak{M}$ is not abelian. The difference is that, inner derivations on $\mathfrak{M}$ are constructed since $\mathfrak{M}$ is a non trivial bilateral $\mathfrak{M}$-module, while the forms considered above are, in general, non trivial because the Hilbert space $\mathfrak{H}$ is a non trivial $\mathfrak{M}$-module. The left action is simply given by that of $\mathfrak{M}$:

$$L_x : \mathfrak{H} \to \mathfrak{H} \qquad L_x = x \qquad x \in \mathfrak{M}$$

and the right action R_x is obtained combining the action of the commutant $\mathfrak{M}'$ and the modular operator J:

$$R_x : \mathfrak{H} \to \mathfrak{H} \qquad R_x = j(x) \qquad x \in \mathfrak{M}.$$

The fact that these two actions commute is at the basis of Lemma 3.3.6 (see discussion of U. Haagerup in [Haa] page 109). We like to consider the Dirichlet forms constructed in Examples 3.3.5 and 3.3.6 as the genuine forms constructed with spatial derivations

$$\delta(x) = i[h;\ x] \qquad x \in \mathfrak{M}.$$

Remark 3.3.8. We want to emphasize a little bit the fact that, applying Theorems 3.1.10 and 1.3.3-*i)* to the unbounded Dirichlet form of Example 3.3.5, one obtains a $\sigma(\mathfrak{M};\mathfrak{M}_*)$-continuous, positivity preserving, uniformly bounded semigroup $\{\widetilde{T}_t\}_{t>0}$ on the von Neumann algebra $\mathfrak{M}$, which is in general not uniformly continuous. This Example could be used to construct larger classes of non norm-continuous Dynamical Semigroups on von Neumann algebras (recall also Remark 1.4.13 and discussion in Introduction of [Dav4]).

3.4. Essential Selfadjointness of perturbations of Dirichlet Forms

In this section we fix two selfadjoint operators H_0 and H_1 on the Hilbert space $\mathfrak{H}$of a standard form $(\mathfrak{M}, \mathfrak{H}, J, \mathcal{P})$ of a von Neumann algebra $\mathfrak{M}$. we consider the problem of providing sufficient conditions to guarantee that the form sum $H = H_0 + H_1$ is a densely defined, self adjoint operator and that is essentially selfadjoint on the intersection $D(H_0) \cap D(H_1)$ of the domains $D(H_0)$ and $D(H_1)$ of the operators H_0 and H_1. This will give a positive answer to the problem quoted in Remark 3.3.4.

The basic assumptions on H_0, is that it be associated to a Dirichlet form on $\mathfrak{H}$ with respect to a cyclic and separating vector $\xi_0 \in \mathcal{P}$.

The results are obtained by a modification of the proofs of W. Faris in [Far2], in which are studied perturbations of positivity preserving semigroups on ordered Hilbert spaces.

We recall that i_0 denotes the injection of $\mathfrak{M}$ in $\mathfrak{H}$ given by $i_0(x) = \Delta_{\xi_0}^{1/4} x \xi_0$, $x \in \mathfrak{M}$, and that $\mathfrak{M}_{sa}$ denotes the selfadjoint part of $\mathfrak{M}$.

Theorem 3.4.1. *Let $(\mathfrak{M}, \mathfrak{H}, J, \mathcal{P})$ be a standard form of the von Neumann algebra $\mathfrak{M}$ and let $\xi_0 \in \mathcal{P}$ be a cyclic and separating vector. Suppose that H_0 is the positive, selfadjoint operator associated to a Dirichlet form on $\mathfrak{H}$ (with respect to ξ_0). Suppose that H_1 is the sum $V_1 + V_2$ of two commuting selfadjoint operator such that:*

i) $V_1 \geq 0$;

ii) $i_0(\mathfrak{M}) \subseteq D(\mathcal{E}_1)$, $\Big(\mathcal{E}_1\,;\, D(\mathcal{E}_1)\Big)$ *being the form associated to* V_1;

iii) $|V_2| \leq a(H_0 + b)$, *for some* $a < 1$ *and* b *real.*

Then the form sum defines a densely defined, selfadjoint operator which is bounded from below.

Proof. First we note that we can use the general framework of Faris in [Far2] with the following identifications. Taking as vector $e \in \mathcal{P}$ (in the notation of Sec. III in [Far2]) our $\xi_0 \in \mathcal{P}$, we have that the space $\mathcal{L}(e)$ is precisely $i_0(\mathfrak{M}_{sa})$ with the norm of $\mathfrak{M}_{sa}$. Moreover we have that $\mathcal{L}(e) = i_0(\mathfrak{M}_{sa})$ is dense in $\mathfrak{H}^{\natural}$. To prove the theorem we have then just to apply Theorem 4.11 in [Far2] pay attention to the following point. In its proof, Faris uses the stronger condition $H_0\xi_0 = 0$ just to make sure that the semigroup generated by H_0 sends $i_0(\mathfrak{M}_{sa})$ in itself; in our case, due to our assumption on H_0 this fact is guaranteed by Theorem 3.1.10 and Theorem 1.3.3.

□

Remark 3.4.2. The theorem above gives conditions which are sufficient to guarantee that the closure of the sum of two closable Markovian forms $\left(\mathcal{E}_0\,;\,D(\mathcal{E}_0)\right)$ and $\left(\mathcal{E}_1\,;\,D(\mathcal{E}_1)\right)$ defines a Dirichlet form: $i_0(\mathfrak{M}) \subseteq D(\mathcal{E})$ (see Remark 3.3.4).

Our second result refers the essential selfadjointness of the form sum $H_0 + H_1$ on the intersection of the domains of H_0 and H_1.

Theorem 3.4.3. *Let $(\mathfrak{M}, \mathfrak{H}, J, \mathcal{P})$ be a standard form of the von Neumann algebra $\mathfrak{M}$ and let $\xi_0 \in \mathcal{P}$ be a cyclic and separating vector. Assume that H_0 and H_1 are two positive, selfadjoint operators associated to the Dirichlet forms $\left(\mathcal{E}_0\,;\,D(\mathcal{E}_0)\right)$ and $\left(\mathcal{E}_1\,;\,D(\mathcal{E}_1)\right)$ (with respect to ξ_0). If*

$$i_0(\mathfrak{M}) \subseteq D(H_1),$$

then the operator H corresponding to the form sum $\mathcal{E}_0 + \mathcal{E}_1$ is densely defined, selfadjoint and essentially selfadjoint on $D(H_0) \cap D(H_1)$.

Proof. From Theorem 3.4.1 (with $V_2 = 0$), we know that H is densely defined and selfadjoint, since $i_0(H) \subseteq D(H_1) \subseteq D(\mathcal{E}_1)$. To prove the theorem, we apply Lemma 1 in [Far2], noting that the only assumption the author uses about U (H_1 in our notation) is that this operator generates a Markovian semigroup with respect to e (ξ_0 in our notation) and that this is true by our Theorem 3.1.10 and Theorem 1.3.3.

□

Appendix A.

Standard Forms of von Neumann algebras

In this section we briefly summarize some results of the theory of Standard Forms we used throughout this thesis. We refer to [Ara1], [Con1], [Haa1], [Tak2] and [Bra1] §2.5 for detailed expositions.
The theory of Standard Forms of von Neumann algebras was developed independently by H. Araki [Ara1], A. Connes [Con1] and U. Haagerup. The first two authors were concerned with σ-finite von Neumann algebras and U. Haagerup extended their works to general non necessarily σ-finite von Neumann algebras. Since in previous chapters we were mainly concerned with the σ-finite case, we recall the following characterization:

Theorem A.1. *Let $\mathfrak{M}$ be a von Neumann algebra acting on the Hilbert space $\mathfrak{H}$. Then the following conditions are equivalent:*

i) $\mathfrak{M}$ is σ-finite

ii) there exists a countable subset of $\mathfrak{H}$ which is separating for $\mathfrak{M}$

iii) there exists a faithful normal state on $\mathfrak{M}$

iv) $\mathfrak{M}$ is isomorphic with a von Neumann algebra $\pi(\mathfrak{M})$ which admits a separating and cyclic vector.

□

A selfdual cone $\mathcal{P}$ in the Hilbert space $\mathfrak{H}$ is a subset with the following property

$$\{\xi \in \mathfrak{H} : \ (\xi;\eta) \geq 0 \quad \forall \eta \in \mathcal{P}\} = \mathcal{P}.$$

A Standard Form $\mathfrak{M}$ of the von Neumann algebra $\mathfrak{M}$ (acting on the Hilbert space $\mathfrak{H}$) consists of a conjugate linear, isometric, involution J ($J^2 = J$) and a selfdual convex cone $\mathcal{P}$ fulfilling the following properties:

i) $J\mathfrak{M}J = \mathfrak{M}'$;

ii) $JxJ = x^*$, $\forall x \in \mathcal{Z}(\mathfrak{M}) \stackrel{\text{def}}{=} \mathfrak{M} \cap \mathfrak{M}'$;

iii) $J\xi = \xi$, $\forall \xi \in \mathcal{P}$;

iv) $xJxJ(\mathcal{P}) \subseteq \mathcal{P}$, $\forall x \in \mathfrak{M}$.

Theorem A.2. *Let $(\mathfrak{M}, \mathfrak{H}, J, \mathcal{P})$ be a Standard Form.*

i) Any von Neumann algebra is isomorphic to a von Neumann algebra in Standard Form

ii) if $(\widetilde{\mathfrak{M}}, \widetilde{\mathfrak{H}}, \widetilde{J}, \widetilde{\mathcal{P}})$ is an other Standard Form and $\Phi : \mathfrak{M} \longrightarrow \widetilde{\mathfrak{M}}$ is a *isomorphism then there exists a unique unitary $u : \mathfrak{H} \longrightarrow \widetilde{\mathfrak{H}}$ such that

$$a)\ \Phi(x) = uxu^{-1}$$
$$b)\ \widetilde{J} = uJU^{-1}$$
$$c)\ \widetilde{\mathcal{P}} = u(\mathcal{P})$$

For each $\xi \in \mathcal{P}$ define the normal positive functional $\phi_\xi(\cdot) = (\cdot\xi; \xi)$. Then the following properties hold:

iii) for any $\omega \in \mathfrak{M}_*^+$ there exists a unique $\xi \in \mathcal{P}$ such that $\omega = \phi_\xi$

iv) the mapping $\xi \longrightarrow \phi_\xi$ is a homeomorphism when both $\mathcal{P}$ and $\mathfrak{M}_*^+$ are equipped with the norm topology. Moreover the following estimates holds:

$$\|\xi - \eta\|^2 \le \|\phi_\xi - \phi_\eta\| \le \|\xi - \eta\| \cdot \|\xi + \eta\|$$

v) there exists a unique unitary representation

$$Aut(\mathfrak{M}) \ni \alpha \longrightarrow U(\alpha)$$

of the group $Aut(\mathfrak{M})$ of all *automorphism of $\mathfrak{M}$ on $\mathfrak{H}$ satisfying the following properties

a) $\alpha(x) = U(\alpha)xU(\alpha)^* \qquad x \in \mathfrak{M}$

b) $U(\alpha)\mathcal{P} \subseteq \mathcal{P}$

c) if $\xi_\psi \in \mathcal{P}$ is the vector which represents $\phi \in \mathfrak{M}_*^+$ we have

$$U(\alpha)\xi_\psi = \xi_{\alpha^{-1*}(\psi)}$$

d) $U(\alpha)J = JU(\alpha)$

The representation U is a homeomorphism when $Aut(\mathfrak{M})$ and $U\big(Aut(\mathfrak{M})\big)$ are equipped with their norm topologies. It is also an homeomorphism when $U\big(Aut(\mathfrak{M})\big)$ is equipped with the weak, strong, or strong *-topology and $Aut(\mathfrak{M})$ is equipped with the topology of strong convergence of $Aut(\mathfrak{M})^*$ on $\mathfrak{M}_*$ ($\alpha \to \beta$ in this topology if and only if $\alpha^*(\phi) \to \beta^*(\phi)$ in norm for each $\phi \in \mathfrak{M}_*$).

□

In case $\mathfrak{M}$ admits a cyclic and separating vector $\xi_0 \in \mathfrak{H}$, the above structure can be constructed by the Tomita-Takesaki theory. The starting point of this theory is the observation that the map

$$x\xi_0 \longrightarrow x^*\xi_0$$

defined on the dense subset $\mathfrak{M}\xi_0$ of $\mathfrak{H}$, is in general not bounded. This is the case only if $\mathfrak{M}$ is abelian or ξ_0 is a trace vector. However, the map is always closable. If S_{ξ_0} denotes its closure, we will indicate with $\Delta_{\xi_0}^{1/2}$ and J_{ξ_0}, the positive and isometric part in its polar decomposition

$$S_{\xi_0} = J_{\xi_0}\Delta_{\xi_0}^{1/2}.$$

The operator Δ_{ξ_0} allows us to construct the family of cones

$$\mathcal{P}_{\xi_0}^{\alpha} \stackrel{\text{def}}{=} \overline{(\Delta_{\xi_0}^{\alpha}\mathfrak{M}_{+}\xi_0)}.$$

The cone corresponding to $\alpha = 1/4$ will be denoted simply by $\mathcal{P}_{\xi_0}$ and it is sometimes called the *natural cone associated to* ξ_0 . It has the following relevant properties:

Theorem A.3.

i) $\mathcal{P}_{\xi_0} = \overline{\{xJ_{\xi_0}xJ_{\xi_0} : \ x \in \mathfrak{M}\}}$

ii) $\mathcal{P}_{\xi_0}$ *is a selfdual closed convex cone in* $\mathfrak{H}$ *and* $(\mathfrak{M}, \mathfrak{H}, J_{\xi_0}, \mathcal{P}_{\xi_0})$ *is Standard Form for* $\mathfrak{M}$

iii) $\Delta_{\xi_0}^{it}\mathcal{P}_{\xi_0} = \mathcal{P}_{\xi_0}$

iv) if $x \in \mathfrak{M}$, *then* $xj_0(x)\mathcal{P}_{\xi_0} \subseteq \mathcal{P}_{\xi_0}$ *where* $j_0(x) \stackrel{\text{def}}{=} J_{\xi_0}xJ_{\xi_0}$

v) if $\xi \in \mathcal{P}_{\xi_0}$, *then* ξ *is cyclic for* $\mathfrak{M}$ *if and only if* ξ *is separating for* $\mathfrak{M}$

vi) if $\xi \in \mathcal{P}_{\xi_0}$ *is a cyclic vector then*

$$J_\xi = J_{\xi_0} \qquad\qquad \mathcal{P}_\xi = \mathcal{P}_{\xi_0}$$

□

The *support* in $\mathfrak{M}$ (resp. in $\mathfrak{M}'$) of a vector $\xi \in \mathfrak{H}$ is by definition the projection on the closure of $\mathfrak{M}'\xi$ (resp. $\mathfrak{M}\xi$).

Theorem A.4. *Let $(\mathfrak{M}, \mathfrak{H}, J, \mathcal{P})$ be a Standard Form. Then each $\xi \in \{\eta \in \mathfrak{H} : J\eta = \eta\} \stackrel{\text{def}}{=} \mathfrak{H}^\natural$ has a unique decomposition*

$$\xi = \xi_+ - \xi_-$$

where the positive part, ξ_+, is the nearest element to ξ in $\mathcal{P}$. Moreover ξ_+ and ξ_- have orthogonal support both in $\mathfrak{M}$ and in $\mathfrak{M}'$.

If $\xi_0 \in \mathcal{P}$ is a cyclic vector then the map

$$i_0 : \mathfrak{M}_{sa} \longrightarrow \mathfrak{H}^\natural \qquad i_0(x) \stackrel{\text{def}}{=} \Delta_{\xi_0}^{1/4} x \xi_0$$

is injective, $\sigma(\mathfrak{M}; \mathfrak{M}_*) - \sigma(\mathfrak{H}; \mathfrak{H})$-continuous and it is an order isomorphism between $\mathfrak{M}_{sa}$ and the subset of elements $\xi \in \mathfrak{H}^\natural$ such that

$$-\lambda \xi_o \leq \xi \leq +\lambda \xi_0$$

for some constant $\lambda > 0$.

□

We conclude this short excursus stating the fundamental Theorem of A. Connes about the underlying structure of ordered vector space of a von Neumann algebra $\mathfrak{M}$: $(\mathfrak{M};\ \mathfrak{M}_+)$.

A face F of a convex cone $\mathcal{P}$ in the Hilbert space $\mathfrak{H}$, is a subset such that if it contains a convex combination of elements of $\mathcal{P}$, then each element of the convex combination lies in F. The *involutive Lie algebra*, $D(\mathcal{P})$, of the cone $\mathcal{P}$ is the set of all bounded operators δ on $\mathfrak{H}$ such that

$$e^{t\delta} \mathcal{P} = \mathcal{P}.$$

An *orientation* on $\mathcal{P}$ is a structure of complex linear space, on the quotient space of $D(\mathcal{P})$ by its center, which is compatible with its structure of involutive Lie algebra. The convex cone $\mathcal{P}$ is called *homogeneous* if for each face F of $\mathcal{P}$, the operator $P_F - P_{F^\perp}$ belong to the Lie algebra of $\mathcal{P}$ (here P_F is the projection onto the subspace generated by F and $F^\perp$ is the face orthogonal to F). Now we can state the Theorem of A. Connes in the following two equivalent formulation.

Theorem A.5.

i) An ordered complex vector space $(E;\ E^+)$, *is the underlying ordered vector space of a von Neumann algebra ofgenre denombrable, if and only if there exists on it a pre-Hilbert space structure on* E, *such that the completion of* E^+ *is a selfdual, homogeneous, cone.*

ii) An ordered Hilbert space $(\mathfrak{H},\ \mathcal{P})$ *is underlying a structure of Standard Form for some von Neumann algebra, if and only if the convex cone* $\mathcal{P}$ *is selfdual, homogeneous and orientable.*

□

The link between the von Neumann algebra $\mathfrak{M}$ and the ordered Hilbert space $(\mathfrak{H},\ \mathcal{P})$ is the one to one correspondence between the projections of $\mathfrak{M}$ and the closed face of $\mathcal{P}$.

Appendix B.

Non-commutative L^p-spaces

The theory of non commutative integration is very old and dates at least since the works of E. M. Stein and Kunze [Ste2] and I. Segal [Seg1]. These authors developed the theory of non-commutative L^p-spaces associated with a von Neumann algebra $\mathfrak{M}$ and a finite faithful trace on it. U. Haagerup generalized this theory when $\mathfrak{M}$ is no more finite, considering L^p-spaces associated to a faithful normal state ϕ_0 on $\mathfrak{M}$. His construction is based on the notion of *cross-product.* A. Connes and H. Hilsum studied in [Con2] and [Hil] other constructions of these spaces, based on the notions of *spatial derivative* of ϕ_0 and $\{\sigma_t^{\phi_0}\}_{t>0}$-homogeneous operator. M. Terp generalized the theory of A. Connes and M. Hilsum when $\phi_0$is a faithful normal weight on $\mathfrak{M}$. H. Araki and T. Masuda realized the spaces L^p associated with ϕ_0, as completion of $\mathfrak{H}$ with respect to certain norms for for $1 < p2$, and as subspaces of $\mathfrak{H}$ for $p > 2$. C. Cecchini [Cec] compared these constructions and gave also a new formulation of some of them. In the following we will show the construction of H. Kosaki because is this one we used in our work.
The idea of this construction comes from the classical result of E. M. Stein and G. Weiss [Ste1], about complex interpolation of classical L^p-spaces with changes of measure. h. Hosaki imbed the algebra $\mathfrak{M}$ into the predual of it $\mathfrak{M}_*$ with the aid of the modular automorphism group associated to ϕ_0:

$$j_o : \mathfrak{M} \longrightarrow \mathfrak{M}_* \qquad (j_0(x))(y) \stackrel{\text{def}}{=} (\sigma_{-i/2}^{\phi_0}(x) \cdot \phi_0)(y) \stackrel{\text{def}}{=} (y\Delta_{\xi_0}^{1/2} x\xi_0;\ \xi_0)$$

$x, y \in \mathfrak{M}$. Denoting $\mathfrak{M}^{1/2}$ the image, in $\mathfrak{M}^*$, of $\mathfrak{M}$, equipped with the norm topology of $\mathfrak{M}$, the author defines, for each p, the space $L^p(\mathfrak{M}; \phi_0)$, as the *complex interpolation space* of the couple $(\mathfrak{M}^{1/2};\ \mathfrak{M}_*)$ corresponding to the value $1/p$ of the parameter. These spaces satisfy the following properties.

Theorem B.1.

i) *$L^p(\mathfrak{M}; \phi_0)$ is isometrically isomorphic to the corresponding space constructed by the authors cited above;*

ii) *for each $1 < p < \infty$, $L^p(\mathfrak{M}; \phi_0)$ is uniformly convex and uniformly smooth;*

iii) *for $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$, $L^p(\mathfrak{M}; \phi_0)$ is the dual spaces of $L^q(\mathfrak{M}; \phi_0)$ and Hölder inequality holds;*

iv) the Riesz-Thorin interpolation theorem holds.

□

In case ϕ_0 is a trace, the constructions of U. Haagerup,A. Connes, H. Araki reproduce exactly the scales of spaces constructed by E. M. Stein, Kunze and I. Segal. In the trace case, the construction of H. Kosaki give spaces which are isomorphic to the previous ones, but they do not coincide. For example if we consider the abelian von Neumann algebra $L^\infty(\mathbb{R};\ dt)$ and the faithful normal state specified by a.e.-dt positive element h_0 in $L^1(\mathbb{R};\ dt)$, then the space L^p turns out to be

$$L^p(\mathbb{R};\ h_0(t)^{p-1}dt).$$

References

[AHK] S. Albeverio, R. Höegh-Krohn: Dirichlet Forms and Markovian semigroups on C^*-algebras. *Comm. Math. Phys.* **56** (1977), 173–187.

[Ara1] H. Araki: Some properties of modular conjugation operator of von Neumann algebras and non commutative Radon-Nikodym theorem with chain rule. *Pac. J. Math.* (2) **50** (1974), 309–354.

[Ara2] H. Araki, T. Masuda: Positive cones and L^p-spaces of von Neumann algebras. *Publ. R.I.M.S., Kyoto Univ.* **18** (1982), 339–411.

[Bat] C. J. K. Batty: Invariant states for strongly positive operators on C^*algebras. *Publ. R.I.M.S., Kyoto Unov.* **18** (1982), 1053–1066.

[BeF] C. Berg, G. Forst: *Potential Theory on Locally Compact Abelian Groups.* Ergebnisse der Mathematik und ihrer Grenzgebiete Band **87** (1975), Springer-Verlag Berlin Heidelbeg New York.

[BeL] J. Bergh, J. Löefström: *Interpolation spaces. An introduction.* Grundlehren der Mathematischen Wissenschften **223** (1976), Springer-Verlag Berlin Heidelbeg New York.

[Bra1] O. Bratteli, D. W. Robinson: *Operator Algebras and Quantum Statistical Mechanics 1.* Text and Monographs in Physics, second edition (1987), Springer-Verlag Berlin Heidelbeg New York.

[Bra2] O. Bratteli, D. W. Robinson: Unbounded derivations on von Neumann algebras. *Ann. Inst. Henri Poincaré* **25** (A) (1976), 139–164.

[Bra3] O. Bratteli, G. A. Elliott, D. E. Evans: Locality and Differential Operators on C^*-algebras. *J. Diff. Eq.* **64** (1986), 221–273.

[Car] E. Carlen, E. H. Lieb: Optimal Hypercontractivity for Fermi fields and related non commutative integration inequalities. *Preprint* Univ. Princeton (1992).

[Cof] C. V. Coffman, C. L. Glover: Obtuse cones in Hilbert spaces and applications to partial differential equations. *J. Funct. Anal.* **35** (1980), 369–396.

[Cec] C. Cecchini: Non commutative integration for states on von Neumann algebras. *J. Oper. Th.* **15** (1986), 217–237.

[Con1] A. Connes: Caracterisation des espaces vectoriel ordonnés sous-jacent aux algèbres de von Neumann. *Ann. Inst. Fourier Grenoble* (4) **24** (1974), 121–155.

[Con2] A. Connes: On the spatial theory of von Neumann algebras. *J. Funct.Anal.* **35** (1980), 153-164.

[Dav1] E. B. Davies: *Quantum Theory of Open Systems* (1976) Academic Press London New York San Francisco.

[Dav2] E. B. Davies: *One Parameter Semigroups* (1980) Academic Press London New York San Francisco.

[Dav3] E. B. Davies: *Heat-kernels and Spectral Theory* **92** (1989) Cambridge Univ. Press Academic Press London New York San Francisco.

[Dav4] E. B. Davies, M. Lindsay: Non commutative symmetric Markov Semigroups. to appear in *Math. Zeit.*.

[Eva1] D. E. Evans: A review on semigroups of completely positive maps. In "Mathematical Problems in Theoretical Physics" (K. Osterwalder ed.), 400–406.

[Eva2] D. E. Evans: Quantum Dynamical Semigroups, symmetry groups and Locality. *Acta. Appl. Math.* **2** (1984), 333–352.

[Far1] W. G. Faris: Invariant cones and Uniqurness of the Ground State for Fermion Systems. *J. Math. Phys.* (8) **13** (1972), 1285–1290.

[Far2] W. G. Faris: Essential Self-adjointness of Operators in Ordered Hilbert Spaces. *Comm. Math. Phys.* **30** (1973), 23–34.

[Fri] A. Frigerio, V. Gorini, A. Kossakowski, M. Verri: Quantum detailed Balance and KMS condition. *Comm. Math. Phys.* **57** (1977), 97–110.

[Fuk] M. Fukushima: *Dirichlet Forms and Markov Processes.* (1980) North-Holland Kodansha.

[Gro1] L. Gross: Existence and Uniqueness of Physical Ground States. *J. Funct. Anal.* **10** (1972), 52–109.

[Gro2] L. Gross: Hypercontractivity and Logarithmic Sobolev inequalities for the Clifford-Dirichlet form. *Duke Math. Jour.* (3) **42** (1975), 383–396.

[Goe] K. Goebel, S. Reich: *Uniform convexity, hyperbolic geometry, and non expan-*

sive mappings. Pure and applied mathematics. A series of Monographs and Textbooks vol. 83 Marcel Dekker inc. (1984).

[Haa1] U. Haagerup: The Standard Form of a von Neumann algebra. *Math. Scand.* **37** (1975), 271–283.

[Haa2] U. Haagerup: L^p-spaces associated with an arbitrary von Neumann algebra. *Colloques Internationaux C.N.R.S.* **274** (1977), 175–184.

[Haa3] U. Haagerup: Connes's bicentralizer problem and uniqueness of the ijective factor of type III_1. *Acta Math.* **158** (1987), 95–148.

[Hil] M. Hilsum: Les espaces L^p d'une algebre de von Neumann définies par la derivée spatiale. *J. Funct. AnaL.* **40** (1981), 151–169.

[Høe] R. Høegh-Krohn, B. Simon: Hypercontractive semigroups and two dimensional self-coupled Bose field. *J. Funct. Anal.* **9** (1972), 121–180.

[Kos] H. Kosaki: Application of the complex interpolation method to a vonNeumann algebra: Non commutative L^p-spaces. *J. Funct. Anal.* **56** (1984), 29–78.

[Köt] G. Köthe: *Topological Vector Spaces I.* Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen Band 159, Springer-Verlag New York 1969.

[Kre] M. G. Krein, M. A. Rutman: Linear operators leaving invariant a cone in a Banach space. *Amer. Math. Soc. Transl.* (1) **10** (1950), 199–325.

[Lin] G. Lindblad: On the generators of Quantum Dynamical Semigroups. *Comm. Math. Phys.* **48** (1976), 119–130.

[Maj] W. A. Majewski: Dynamical Semigroups in the algebraic formalism of Statistical Mechanics. *Fort. Phys.* (3) **32** (1984), 89–133.

[Mey] P. A. Meyer: *Processus de Markov.* Lectures Notes in Mathematics **26** (1967) Springer-Verlag Berlin Heidelbeg New York.

[Nel1] E. Nelson: The Free Markov Field. *J. Funct. Anal.* **12** (1973), 211–227.

[Nel2] E. Nelson: Notes on non commutative integration. *J. Funct. Anal.* **15** (1974), 103–116.

[Röc] M. Röckner, Z. M. Ma: General Theory of Dirichlet Forms and Applications.*Lectures held in Varenna at the C.I.M.E.-Summer School on "Dirichlet Forms"* June 1992.

[Sak] S. Sakai: *C^*-Algebras and W^*-Algebras.* Ergebnisse der Mathematik und ihrer Grenzgebiete Band **60** (1971), Springer-Verlag Berlin Heidelbeg New York.

[Sau] J.L. Sauvageot: Semi-groupe de la chaleur transverse sur la C^* algèbre d'un feuilletage riemannien. *Comptes Rendus Acad. Sci. Paris*, Série I **310**, (1990), 531–536.

[Segl] I. E. Segal: A non commutative extension of abstract integration. *Ann. Math.* **57** (1953), 401–457; correction **58** (1953), 595–596.

[Segl] I. E. Segal: Construction of non Linear local Quantum Processes II. *Inven. Math.* **14** (1971), 211–241.

[Ste1] E. M. Stein, G. Weiss: Interpolation of operators with change of measures. *Trans. Amer. Math. Soc.* **87** (1958),159–172.

[Ste2] E. M. Stein, R. Kunze: Uniformly bounded representations and harmonic analysis of the 2×2 real unimodular group. *Amer. J. Math.* **82** (1960), 1–62.

[Tak1] M. Takesaki: *Tomita's theory of Modular Hilbert Algebrasand its Applications.* Lectures Notes in Mathematics **128** (1970) Springer-Verlag Berlin Heidelbeg New York.

[Tak2] M. Takesaki: *Structure of Factors and automorphisms groups.* **51** (1982) C.B.M.S. Regional Conference Amer. Math. Soc. publisher.

[Ter] M. Terp: Interpolation spaces between a von Neumann algebra and its predual. *J. Oper. Th.* **8** (1982), 327–360.

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