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A Maximum Principle for Conjugate BVPs

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INTRODUCTION

The aim of this thesis is to prove a general maximum principle for conjugate boundary value problems and deduce a variety of applications to linear and nonlinear problems.

This work started from a problem raised by Prof. Vidossich during a class. He pointed out the following. Erbe and Wang [15;14] published some existence and multiplicity results for positive solutions to singular two-point BVP whose proofs are centered on a tricky inequality on the interval $[1/4, 3/4]$ for the Green function of the two-point BVP on $[0, 1]$. Prof. Vidossich asked for a generalization to multipoint BVPs. The difficulty in the extension lies on the fact that Erbe-Wang's inequality depends heavily on the simplicity of the Green function in the two-point case. It is not clear even what should be the counterpart in the multipoint case.

Working on this question, I proved the following maximum principle for conjugate BVPs, inspired by an inequality established by Pokornyi [20] in a special case:

$$\left.\begin{array}{l} Ly \ \geq \ 0 \\ y^{(j)}(a_i) = 0 \ , \quad 1 \leq i \leq m \ , \quad 0 \leq j \leq k_i - 1 \end{array}\right\} \quad \Longrightarrow \quad S_P(t)y(t) \ \geq \ \varphi(t)||y||_\infty$$

where S_P indicates the sign of the Levin polynomial $P(t) = \prod_{i=1}^{m}(t - a_i)^{k_i}$ and φ is a continuous function with the property

$$\frac{\varphi(t)}{|P(t)|} \ \geq \ c_0 \ > \ 0$$

for a suitable c_0. The proof is somehow involved, it is based on Levin's theorem about the sign of the Green function, and on Polya's factorization of disconjugate linear differential operators. Polya's theorem claims that $Lx = x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x$ is disconjugate if and only if L admits the factorization

$$Lx = v_1 \cdots v_n D \frac{1}{v_n} D \cdots D \frac{1}{v_1} x \ ,$$

where $D = \frac{d}{dt}$, $v_i \in C^{n-i+1}([a, b])$ and $v_i > 0$. It follows that every "suboperator"

$$L_k x = \frac{1}{v_k} D \cdots D \frac{1}{v_1} x$$

is disconjugate. We use this idea repeatedly in order to get a successive lowering of the order of the equation under consideration, as a fundamental ingredient in the proof of the maximum principle.

The maximum principle suggested the definition of two new ordered Banach spaces:

(i) a vector subspace of $\mathcal{C}^0$

$$X = \left\{ x \in \mathcal{C}^0 : \ |x(t)| \leq c|P(t)| \ , \ a \leq t \leq b \ ; \ \text{ with } c = c(x) \right\}$$

endowed with the norm

$$||x||_P = ||x||_\infty + \sup_{t \neq a_1, \cdots, a_m} \frac{|x(t)|}{|P(t)|}$$

and the cone

$$\mathcal{K} = \left\{ x \in X : S_P(t)x(t) \geq 0 \ , \ \ a \leq t \leq b \right\} .$$

We show that int($\mathcal{K}$) is nonempty and that the linear operator T associated to the Green function is strongly positive. In this way, we can apply the strongest version of the Krein-Rutman theorem [avoiding the use of Krasnosel'ski's order unit as well as of any differentiability property of the Green function] in order to study the existence and the property of the principal eigenvalue $\lambda_1(q)$ of

$$\begin{aligned} &Ly = \lambda q(t) y \\ &y^{(j)}(a_i) = 0 \ , \ \ 1 \leq i \leq m \ , \ \ 0 \leq j \leq k_i - 1 \end{aligned}$$

where $q \in \mathbf{L}^1$. It is found that $\lambda_1(q)$ is the inverse of the spectral radius of T and has an eigenfunction in int($\mathcal{K}$). Several applications to nonlinear BVPs are obtained.

(ii) $\mathcal{C}^0$ ordered by the cone

$$\mathcal{K} = \left\{ x \in \mathcal{C}^0 : \ S_P(t)x(t) \geq \varphi(t)||x||_\infty \ , \ \ a \leq t \leq b \right\} .$$

This allows the use of Krasnoselski's theorem about the compression/expansion of the cone, in order to study singular problems in analogy to the mentioned work of Erbe and Wang.

The detailed content of the thesis is the following.

In §0 we recall the notations, definitions and main well-known results needed in the sequel, in order to make the thesis self-contained. Our terminology and notations are the most traditional ones, being based on the book Elias [9].

In §1 we prove the above mentioned maximum principle and we show that it includes several known inequalities.

§2 provides the detailed study of the ordered Banach space and the eigenvalue problem mentioned in (i). We prove existence and comparison results for the principal eigenvalue when $q \in \mathbf{L}^1$. Besides the key role of the new Banach space, the argument is based on a particular approximation of q by continuous functions q_k obtained by suitably glueing together the approximations of $q|_{[a_i, a_{i+1}]}$ by continuous functions with compact support in $]a_i, a_{i+1}[$; the maximum principle is applied to each q_k and we pass to the limit to reach the desired conclusions.

§3 is devoted to the study of the existence and multiplicity of positive solutions to conjugate BVPs for nonlinear equations $Ly = f(t, y)$ when f is singular [roughly speaking, this means that of the two limits

$$f_0 = \lim_{y \downarrow 0} \frac{f(t,y)}{y} \quad \text{and} \quad f_\infty = \lim_{y \uparrow \infty} \frac{f(t,y)}{y}$$

one is finite and the other is infinite]. There are two types of results:

(a) those concerned with solutions y that are positive in the traditional sense, i.e. $y(t) \geq 0$ for all t. They are obtained by applying the maximum principle of §1 in the context of the ordered Banach space mentioned in (ii);

(b) those concerned with solutions y which belong to the cone introduced in (i). Their proofs are based on the comparison of the principal eigenvalues stated in §2.
In both cases, the nonlinearity enters through an application of Krasnosel'skii theoren on the compression/expansion of the cone.

In §4 we prove some uniqueness and existence theorem similar to the well-known results for nonresonant elliptic problems with the nonlinearity below the eigenvalue.

In §5 we apply the theorems of §3 to extend to arbitrary conjugate BVP the bifurcation results of Agarwal-Bohner-Wong [2]. Our assumptions are less general and the proofs are quite different leading to a kind of Ambrosetti-Prodi alternative for the parameter λ.

To finish, we remark that

(i) The Theorems stated in §3, 4 and 5 have immediate corollaries on the solvability for differential equations on annular domains since the problem

$$\begin{aligned} -\Delta u &= g(|x|)f(u) \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

where

$$\Omega = \{x \in \mathbf{R}^N : r_1 < |x| < r_2\} \ ; \quad \text{with } 0 < r_1 < r_2 \text{ and } N \geq 2 \ ,$$

has a radially symmetric solution

$$u(x) = v(|x|)$$

if and only if v is a solution of the BVP

$$\begin{aligned} &v'' + \tfrac{N-1}{t} v' = -g(t)f(v) \\ &v(r_1) = 0 \\ &v(r_2) = 0 \ , \end{aligned}$$

with the observation that the second order differential operator

$$Lz = z'' + \frac{N-1}{t} z'$$

is always disconjugate on $[r_1, r_2] \subset]0, \infty[$ (cf. Proposition 0.4-(1)).

(ii) The content of this thesis produced the papers [7], [8] and [25].

CONTENT

STANDING NOTATIONS

- When we consider a conjugate BVP [cf. §0 for the definition];

$$\begin{cases} Ly = f \\ y^{(j)}(a_i) = 0\ , \quad 1 \le i \le m\ , \quad 0 \le j \le k_i - 1 \end{cases}$$

  it is understood that $a = a_1 < \cdots < a_m = b$, $2 \le m \le n$ and $\sum_{i=1}^{m} k_i = n$.

- P denotes the Levin's polynomial $P(t) = \prod_{i=1}^{m}(t - a_i)^{k_i}$ associated to the given conjugate BVP.

- G denotes the Green function associated to the given linear conjugate BVP.

- T_q denotes the linear operator defined by

$$T_q x(t) = \int_a^b G(t,s)q(s)x(s)\ ds\ .$$

- S_f denotes the sign of the function f: $S_f(t) = \begin{cases} \frac{f(t)}{|f(t)|} & \text{if } f(t) \ne 0 \\ 0 & \text{otherwise} \end{cases}$.

- $\preceq$ denotes the order relation defined by the cone under consideration.

- When a notation is used several times in a section, then it is mentioned in the beginning of the same section.

§0. PRELIMINARIES: Notations, Definitions and fundamental results

In this section we state the notations, definitions and the main well-known results needed in the sequel. We shall not provide the proofs of these results since they are long and involved, above in the case of the results related to ordinary differential equations in §0.A, 0.B and 0.C [The reason why these proofs are long is because they are based on a detailed algebraic analysis of the *generalized Wronskian*; i.e., the Wronskian evaluated at the boundary points with the corresponding order of derivation].

Our terminology and notations are based on the book Elias [9].

Let $a < b$ be two real numbers and donote by $C([a,b])$ the set of continuous functions on the compact interval $[a,b]$. Given any positive integer k, $C^k([a,b])$ will hold for the set of functions k times continuously differentialble.

We shall deal with n^{th}-order linear differential operators of the form

$$Lx = x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x \quad ;$$

where n is a positive integer and the coefficients $p_1, \cdots, p_n$ are continuous functions.

0.A Disconjugacy

Definition 0.1 An n^{th}-order differential linear operator L defined by

$$Lx = x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x$$

is *disconjugate* on an interval $I \subset [a,b]$, if every nontrivial solutions has less than n zeros counting their multiplicities.

Theorem 0.1 (Polya Factorization) *An n^{th}-order differential linear operator L;*

$$Lx \equiv x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x$$

with $p_1, \cdots, p_n \in C([a,b])$, is disconjugate on $[a,b]$ if and only if L has a Polya factorization; that is, there exist n smooth positive functions $v_i \in C^{n-i+1}([a,b])$, $1 \le i \le n$, such that

$$Lx = v_1 \cdots v_n D \frac{1}{v_n} D \cdots D \frac{1}{v_1} x \quad \textit{for every } x \in C^n([a,b]) \quad ,$$

where $D = \frac{d}{dt}$.

Proof:
Cf. [6, Theorem 2 at page 93 and Theorem 3 at page 94]. □

Remark 0.1 Note that any Polya factorization can be performed into a Mammana factorization (and vice versa). In fact, given smooth positive functions $v_i \in C^{n-i+1}([a,b])$, $1 \leq i \leq n$,

$$v_1 \cdots v_n D \frac{1}{v_n} D \cdots D \frac{1}{v_1} x \equiv (D - u_n) \cdots (D - u_1) x$$

with

$$u_i = \frac{v_1'}{v_1} + \cdots + \frac{v_i'}{v_i} \in C^{n-i}([a,b]) \ , \quad 1 \leq i \leq n \ ,$$

(vice versa $v_i(t) = \exp(\int^t (u_i(s) - u_{i-1}(s))\, ds)$, $1 \leq i \leq n$, putting $u_0 \equiv 0$).

The simplest example of disconjugate operator is $Lx = x^{(n)}$.

The following two propositions provide general examples of linear differential operators as regards disconjugacy.

Proposition 0.2 *Let*

$$Lx \equiv x^{(n)} + p_1(t) x^{(n-1)} + \cdots + p_n(t) x$$

be an n^{th}-order differential linear operator with continuous coefficients $p_1, \cdots, p_n$. Then there exists a positive real number $\ell \geq \min\{1, \frac{1}{nM}\}$; where $M = \max_{1 \leq i \leq n} ||p_i||_\infty$, such that L is disconjugate on any subinterval of $[a,b]$ with length less than ℓ.

Proof:
Cf. [6, Proposition 1 at p. 81] and see [9, p. 1]. □

Proposition 0.3 *L is disconjugate on $[a,b]$ if $\chi(\frac{b-a}{2}) \leq 1$ where χ is defined by*

$$\chi(h) = \sum_{i=1}^{n} \frac{h^i ||p_i||_\infty}{k[\frac{k-1}{2}]![\frac{k}{2}]!} \ .$$

Proof:
Cf. [6, Theorem 1 at p. 86]. □

In the second order case, we have the following supplementary criterias.

Proposition 0.4
(1) *The second order linear differential operator* $x'' + p(t)x' + q(t)x;\ x \in C^2([a,b])$, $p, q \in C([a,b])$, *is disconjugate whenever* q *fulfills*

$$q(t) \leq 0\ ,\quad a \leq t \leq b\ .$$

(2) *The second order linear differential operator* $x'' + q(t)x;\ x \in C^2([a,b])$, *with* $q \in C([a,b])$, *is disconjugate whenever* q *fulfills*

$$q(t) \geq 0\ ,\quad a \leq t \leq b\ ,\quad \text{and}\quad \int_a^b q(s)\ ds\ \leq\ \frac{4}{b-a}\ .$$

Proof:
For the statement (1) cf. [21, Theorem 3 at p. 6], and for (2) cf. [6, Theorem 13 at p. 21]. □

The following assertion gives a structure of the set of disconjugate linear differential operators, showing that there are enough of them.

Proposition 0.5 *Let* $\mathcal{L}$ *be the set of all* n^{th}*-order differential linear operators of the form*

$$Lx \equiv x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x$$

with coefficients $p_1, \cdots, p_n$ *continuous on* $[a,b]$. *Let define the distance in* $\mathcal{L}$ *from* L_1 *to* L_2 *with*

$$L_i x \equiv x^{(n)} + p_{i,1}(t)x^{(n-1)} + \cdots + p_{i,n}(t)x\quad ,\quad i = 1,2\quad ,\quad by$$

$$d(L_1, L_2) = \max_{a \leq t \leq b} \sum_{j=1}^{n} |p_{1,j}(t) - p_{2,j}(t)|\ .$$

Then the subset consisting of all disconjugate n^{th}*-order differential linear operators of* $\mathcal{L}$ *is connected and open in the metric space* $(\mathcal{L}, d)$.

Proof:
Cf. [6, Proposition 9 at p. 95]. □

0.B Conjugate boundary value problems

Definition 0.2 Let n, m and $k_1, \cdots, k_m$ be positive integers such that $2 \leq m \leq \sum_{i=1}^{m} k_i = n$ and let $a = a_1 < \cdots < a_m = b$ be m ordered real numbers. We call

$(a_1, \cdots, a_m;\ k_1, \cdots, k_m)$-conjugate boundary value problem, shortly *Conjugate BVP*, the problem of finding a solution to

$$\begin{cases} Lx = f(t,x) \ , \\ x^{(j)}(a_i) = 0 \ , \quad 1 \le i \le m \ , \quad 0 \le j \le k_i - 1 \end{cases}$$

where f is a given function.
We are particularly interested in the eigenvalue problem for conjugate BVP

$$\begin{cases} Lx = \lambda p(t) x \ , \\ x^{(j)}(a_i) = 0 \ , \quad 1 \le i \le m \ , \quad 0 \le j \le k_i - 1 \end{cases}$$

that will be studied in §2.

With this, we can rephrase the definition of disconjugacy: L is disconjugate if and only if $\lambda = 0$ is not an eigenvalue of any conjugate BVP for L.

Definition 0.3 Let n, m and $k_1, \cdots, k_m$ be positive integers such that $2 \le m \le \sum_{i=1}^{m} k_i = n$ and let $a = a_1 < \cdots < a_m = b$ be m ordered real numbers. Then the *Levin polynomial* P of the conjugate BVP

$$\begin{cases} Lx = f(t,x) \ , \\ x^{(j)}(a_i) = 0 \ , \quad 1 \le i \le m \ , \quad 0 \le j \le k_i - 1 \end{cases}$$

is defined by

$$P(t) = \prod_{i=1}^{m} (t - a_i)^{k_i} \ .$$

0.C Properties of the Green function of a disconjugate linear differential equation associated with a conjugate boundary condition

Assume that L is disconjugate on $[a,b]$. Let n, m and $k_1, \cdots, k_m$ be positive integers such that $2 \le m \le \sum_{i=1}^{m} k_i = n$ and let moreover $a = a_1 < \cdots < a_m = b$. Then the Green function of the equation

$$Lx = 0$$

associated with the $(a_1, \cdots, a_m;\ k_1, \cdots, k_m)$-conjugate boundary condition is the unique real function G defined on $[a,b] \times [a,b]$ and satisfying the following conditions:

(i) as a function of t, $G(t,s)$ satisfies the equation $Lx = 0$ on the two intervals $[a, s[$ and $]s, b]$ with the boundary condition

$$\frac{\partial^j G}{\partial t^j}(a_i, s) = 0 \ , \quad 1 \le i \le m \ , \quad 0 \le j \le k_i - 1$$

(ii) as a function of t, $G(t,s)$ and its first $n-2$ derivatives are continuous at $t=s$, while

$$\frac{\partial^{n-1}G}{\partial t^j}(s+0,s) - \frac{\partial^{n-1}G}{\partial t^j}(s-0,s) = 1 \ .$$

As a result, given $f \in \mathcal{C}([a,b])$, $x \in \mathcal{C}^n([a,b])$ is the solution to

$$\begin{array}{l} Lx = f \ , \\ x^{(j)}(a_i) = 0 \ , \quad 1 \le i \le m \ , \quad 0 \le j \le k_i - 1 \end{array}$$

if and only if

$$x(t) = \int_a^b G(t,s) f(s) \, ds \ , \quad a \le t \le b \ .$$

Theorem 0.2 **(Levin)** *Assume that L is disconjugate on $[a,b]$. Let n, m and $k_1, \cdots, k_m$ be positive integers such that $2 \le m \le \sum_{i=1}^m k_i = n$ and let moreover $a = a_1 < \cdots < a_m = b$. Furthermore, let P be the Levin polynomial. Then*

(i) *the Green function G of the boundary value problem*

$$\begin{array}{l} Lx = 0 \ , \\ x^{(j)}(a_i) = 0 \ , \quad 1 \le i \le m \ , \quad 0 \le j \le k_i - 1 \end{array}$$

has the sign property

$$G(t,s)P(t) \ge 0 \ , \quad a \le t, s \le b \ .$$

(ii) *the quotient map $(t,s) \mapsto \frac{G(t,s)}{P(t)}$ has a continuous extension to $[a,b] \times [a,b]$ with positive infimum.*

Proof:
For (i) cf. [6, Lemma 14 at p. 106] and for (ii) cf. [6, Theorem 11 (and its proof) at p. 108]. □

0.D Cones and the Krein-Rutman Theorem

Definition 0.4 Let X be a Banach space and K a nonempty subset of X. Then we say that K is a *cone* in X if

(i) $(\alpha K + \beta K) \subset K$, for all $\alpha, \beta \ge 0$

(ii) $K \cap (-K) = \{0\}$ and

(iii) K is closed.

Remark 0.2 If K is a cone in a Banach space X, then the relation $\preceq_K$ in X defined by

$$x \preceq_K y \iff y - x \in K$$

is an order relation in X compatible with the addition, the multiplication by nonnegative scalar as well as the convergence in X. Hence we say that (X, K) is an ordered Banach space.

Vice-versa, if $\preceq$ is an order relation compatible with the addition, the multiplication by nonnegative scalar and the convergence in X, so that $(X, \preceq)$ is an ordered set, then the set

$$K = \{x \in X : 0 \preceq x\}$$

is a cone in the Banach space X.

Definition 0.5 A cone K of a Banach space X is said to be:

(1) *reproducing* if the linear subspace $K - K$ coincides with X; that is, $X = K - K$,

(2) *total* if $K - K$ is dense in X; that is, $X = \overline{K - K}$.

Remark 0.3 If K is a cone with nonempty interior in a Banach space X, then K is reproducing since $K - K$ holds then for the only linear subspace with nonempty interior in X.

Definition 0.6 Let K be a cone in a Banach space $(X, \|\ \|)$. Then we say that the cone K is *normal* if the norm $\|\ \|$ is (monotone) increasing in K; that is,

$$0 \preceq x \preceq y \implies \|x\| \leq \|y\| .$$

Definition 0.7 Let X be a Banach space and K a cone in X. Then a bounded linear operator $T : X \to X$ is said to be:

(1) *positive* with respect to K if

$$T(K) \subset K$$

(2) *strongly positive* with respect to K if (K has nonempty interior $\operatorname{int}(K)$ and)

$$T(K) \subset \operatorname{int}(K) .$$

Follow now two famous versions of the Krein-Rutman theorem.

Theorem 0.3 *Let K be a cone with nonempty interior in a Banach space X, and let $T : X \to X$ be a strongly positive linear compact operator on K.*
Then the spectral radius of T, $r(T)$, is an algebraically simple eigenvalue of T and T^* having associated eigenvectors (unique up to normalization) in $\text{int}(K)$ *and* $\text{int}(K^*)$ *respectively, where T^* is the adjoint of T and K^* is the dual cone of K; that is,*

$$K^* = \left\{ f^* \in X^* : < f^*, u > \geq 0 \;\; \textit{for all} \;\; u \in K \right\} .$$

Proof:
Cf. [24, Theorem 7.C at p. 290]. □

Theorem 0.4 *Let K be a total and normal cone in a Banach space X. Suppose that $T : X \to X$ is a positive linear operator (with respect to K) and some iterate of T is compact. Assume moreover that for some $x \in X \setminus (-K)$, natural number k, and some positive real number α we have*

$$T^k x \succeq \alpha x .$$

Then the spectral radius of T, $r(T)$, is greater than or equal to α and is an eigenvalue of T with an eigenvector in K.

Proof:
Cf. [17, Theorem 9.4 at p. 89] and [19, Corollary 2.5 at p. 62]. □

0.E Krasnosel'skii compression and expansion of the cones

Definition 0.8 Let (X, K) be an ordered Banach space and A a completely continuous operator on X, mapping K into itself. Then we say that

(1) A is a *compression* of the cone K if there exist two positive real numbers $r < R$ such that

$$0 < ||x|| \leq r , \;\; x \in K \implies x - Ax \notin K , \;\; \text{and}$$

$$||x|| \geq R , \;\; x \in K \implies Ax - x \notin K .$$

(2) A is an *expansion* of the cone K if there are two positive real numbers $r < R$ such that

$$0 < ||x|| \leq r , \;\; x \in K \implies Ax - x \notin K , \;\; \text{and}$$

$$||x|| \geq R , \;\; x \in K \implies x - Ax \notin K .$$

Theorem 0.5 *Let (X, K) be an ordered Banach space and $\varrho_1 \neq \varrho_2$ be two positive real numbers. Let moreover $A : K \to K$ be a completely continuous map such that*

(i) *for any $x \in K$ with $||x|| = \varrho_1$, we have $Ax - x \notin K$, while*

(ii) *for any $x \in K$ with $||x|| = \varrho_2$, we have $x - Ax \notin K$.*

Then A has a fixed point $u \in K$ such that

$$\min\{\varrho_1, \varrho_2\} \leq ||u|| \leq \max\{\varrho_1, \varrho_2\} .$$

Proof:
Cf. [16, Theorem 2.3.3 at p. 93]. □

§1. THE MAXIMUM PRINCIPLE

In this section we prove a maximum principle for conjugate boundary value problems aiming to unify the known results. Concerning maximum principles for multipoint boundary value problems, the direction goes back to Pokornyi [20] who established some inequalities in 1968, on the basis of the differential properties of the Green's function for a multipoint boundary problem proved by Levin, and applied them conjointly with Krasnosel'skii [18]. Subsequently, in 1996 R.P. Agarwal [1] and [2] appealed elegantly on Hermite interpolations to prove a generalization of a series of maximum principles due to Chow-Dunninger-Lasota, Dunninger, Kuttler and Seda, with some application. Meanwhile, L.H. Erbe and H. Wang [15] proved an existence result for singular two-point boundary value problems whose proof is based on a variant of Pokornyi's inequality. This has attracted the interest of many mathematicians, who managed to extend to multipoint boundary value problems, the results of Erbe and Wang, cf §16 of [3] for the pertinent and up-to-date references.

Special cases of our theorem include the inequalities of Pokornyi [20], partially the maximum principles of Agarwal [1] and Eloe-Henderson [13], and imply the inequalities of [11], [12] and [23].

The proof is based on a repeated use of the Levin's Theorem 0.2 about the sign of the Green's function, and on a successive lowering of the order of the equation by using Polya factorization for disconjugate differential operators.

We shall consider n^{th} order differential operators of the form

$$Lx := x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x$$

where the coefficients $p_1, \cdots, p_n$ are given continuous functions on $[a, b]$. Let P denote the Levin polynomial.

Besides, given $f \in \mathcal{C}([a, b])$, we set $||f||_\infty = \sup_{a \le t \le b} |f(t)|$ and

$$S_f(t) = \begin{cases} \frac{f(t)}{|f(t)|} & if \quad f(t) \neq 0 \\ 0 & if \quad f(t) = 0 \end{cases} .$$

Our main result is the following theorem

Theorem 1.1 *If L is disconjugate there exists a continuous function φ, strictly positive on $\bigcup_{i=1}^{m-1}]a_i, a_{i+1}[$ with $\varphi/|P|$ having a positive infimum on $\bigcup_{i=1}^{m-1}]a_i, a_{i+1}[$, such that*

$$S_P(t)y(t) \;\geq\; \varphi(t)||y||_\infty \;, \quad a \leq t \leq b \;,$$

for every $y \in C^n([a,b])$ satisfying the differential inequality

$$Ly \geq 0$$

and the homogeneous Hermite m-point conditions

$$y^{(j)}(a_i) = 0 \;, \quad 1 \leq i \leq m \;, \quad 0 \leq j \leq k_i - 1 \;.$$

Remark: An explicit evaluation of φ is given at the end of the proof. In fact the function φ may be taken as $\varphi(t) = \min\{|\alpha(t)|/||\alpha||_\infty, \; |\beta(t)|/||\beta||_\infty\}$; where α is any nontrivial solution (which is unique up to a multiplicative constant) of the boundary value problem

$$\begin{aligned} &Lx = 0 \;, \\ &x^{(j)}(a_1) = 0 \quad \text{if } \; 0 \leq j \leq k_1 - 2 \;, \\ &x^{(j)}(a_i) = 0 \quad \text{for } \; 2 \leq i \leq m \;; \; \text{ and } \; 0 \leq j \leq k_i - 1 \;, \end{aligned}$$

while β is any nontrivial solution (also unique up to a multiplicative constant) of the boundary value problem

$$\begin{aligned} &Lx = 0 \;, \\ &x^{(j)}(a_i) = 0 \quad \text{for } \; 1 \leq i \leq m-1 \;; \; \text{ and } \; 0 \leq j \leq k_i - 1 \;, \\ &x^{(j)}(a_m) = 0 \quad \text{if } \; 0 \leq j \leq k_m - 2 \;, \end{aligned}$$

in agreement with the results of [11], [12] and [23] for special cases.

Proof of Theorem 1.1:
Since the equation $Lx = 0$ is disconjugate on $[a,b]$, L has a Polya factorization

$$Lx = v_{n+1} D \frac{1}{v_n} D \cdots D \frac{1}{v_1} x$$

where $D = \frac{d}{dt}$, $v_i \in C^{n-i+1}[a_1, a_m]$ with $v_i > 0$, $i = 1, \cdots, n$, and $v_{n+1} = v_1 \cdots v_n$, by Theorem 0.1.
Now let $y \in C^n([a,b])$ satisfy $Ly \geq 0$ and $y^{(j)}(a_i) = 0$, $1 \leq i \leq m$, $0 \leq j \leq k_i - 1$. Then there exits $\tau \in \bigcup_{i=1}^{m-1}]a_i, a_{i+1}[$ (that we fix at once) such that $|y(\tau)| = ||y||_\infty$, and we see from Levin's Theorem 0.2 and $Ly \geq 0$ that $y(t)P(t) \geq 0$. This implies

$$|y(t)| = S_P(t) y(t) \;, \quad a \leq t \leq b \;. \tag{1.1}$$

Let P_i denote the polynomial $(t - \tau)\frac{P(t)}{(t - a_i)}$.

To achieve the conclusion, at first we deduce a series of inequalities.

Case 1: Suppose that for $l = 1$ or $l = m$ we have $k_l \geq 2$. Then there exists a unique function Q_l satisfying

$$\begin{aligned} &\tfrac{1}{v_n} D \tfrac{1}{v_{n-1}} \cdots D \tfrac{1}{v_1} Q_l = 1 \quad \Longleftrightarrow \quad v_{n+1} \tfrac{1}{v_n} D \tfrac{1}{v_{n-1}} \cdots D \tfrac{1}{v_1} Q_l = v_{n+1} \\ &Q_l^{(j)}(a_l) = 0 \; \text{ for } \; 0 \leq j \leq k_l - 2 \;, \\ &Q_l^{(j)}(a_i) = 0 \; \text{ for } \; 1 \leq i \leq m \text{ with } i \neq l \;, \; \text{ and } \; 0 \leq j \leq k_i - 1 \end{aligned}$$

because of the disconjugacy on $[a,b]$ of the $(n-1)^{th}$ order differential operator

$$v_1 \cdots v_{n-1} D \frac{1}{v_{n-1}} D \cdots D \frac{1}{v_1} x \ ,$$

by Theorem 0.1. If g_l is the Green's function of the related homogeneous problem (i.e., the Green's function associated to the right handside problem according to §0.C), then from the relation

$$Q_l(t) = \int_a^b g_l(t,s) v_{n+1}(s) \ ds$$

and the fact that $P(t)/(t-a_l)$ is the corresponding Levin's polynomial, it follows by applying Levin's Theorem 0.2 that $Q_l(t)\frac{P(t)}{(t-a_l)} \geq 0$, hence

$$S_{Q_l}(t) = S_P(t) S_{(\cdot\ -a_l)}(t)$$

so that

$$S_P(\tau) = S_{(\cdot\ -a_l)}(\tau) S_{Q_l}(\tau) \quad \text{and} \quad S_{P_l}(t) = S_{(\cdot\ -\tau)}(t) S_{Q_l}(t) \ , \quad a \leq t \leq b \ . \tag{1.2}$$

We have moreover $Q_l \not\equiv 0$ and

$$\begin{aligned}
&LQ_l = 0 \ , \\
&Q_l^{(j)}(a_l) = 0 \ \text{ for } \ 0 \leq j \leq k_l - 2 \ , \\
&Q_l^{(j)}(a_i) = 0 \ \text{ for } \ 1 \leq i \leq m \text{ with } i \neq l \ , \ \text{ and } \ 0 \leq j \leq k_i - 1 \ ,
\end{aligned}$$

whereas

$$Q_l^{k_l - 1}(a_l) \neq 0 \neq Q_l(\tau)$$

otherwise we would have a non-vanishing solution of $Lx = 0$ with n zeros in $[a,b]$, counting multiplicities, against the disconjugacy of L.
Since y is the unique solution of

$$\begin{aligned}
&Lx = Ly \ , \\
&x(\tau) = y(\tau) \ , \\
&x^{(j)}(a_l) = 0 \ \text{ for } \ 0 \leq j \leq k_l - 2 \ , \\
&x^{(j)}(a_i) = 0 \ \text{ for } \ 1 \leq i \leq m \text{ with } i \neq l \ , \ \text{ and } \ 0 \leq j \leq k_i - 1 \ ,
\end{aligned}$$

from the classical representation by solutions of homogeneous and non-homogeneous problems, we have

$$y(t) = \frac{Q_l(t)}{Q_l(\tau)} y(\tau) + \int_a^b G_1(t,s) Ly(s) \ ds \ , \qquad a \leq t \leq b \ ;$$

where G_1 denotes the Green's function of the conjugate boundary value problem:

$$\begin{aligned}
&Lx = 0 \ , \\
&x(\tau) = 0 \ , \\
&x^{(j)}(a_l) = 0 \ \text{ for } \ 0 \leq j \leq k_l - 2 \ , \\
&x^{(j)}(a_i) = 0 \ \text{ for } \ 1 \leq i \leq m \ \text{ with } i \neq l \ , \ \text{ and } \ 0 \leq j \leq k_i - 1 \ .
\end{aligned}$$

Noting now that $S_{P_l}(t)G_1(t,s) \geq 0$ and recalling that $Ly \geq 0$, we have for every $t \in [a,b]$,

$$S_{P_l}(t)y(t) \;\geq\; \frac{S_{P_l}(t)Q_l(t)}{Q_l(\tau)}y(\tau) \;.$$

Hence, from (1.1), (1.2) and the definition of $S_f(t)$, we have

$$S_{P_l}(t)y(t) \;\geq\; \frac{|Q_l(t)|}{|Q_l(\tau)|}||y||_\infty S_{(\cdot\;-\tau)}(t)S_{(\cdot\;-a_l)}(\tau) \;,\quad a \leq t \leq b \qquad (1.3)$$

Case 2: Suppose that $k_1 = 1$. Then there exists a unique function q_1 such that

$$\begin{aligned} &Lq_1 = 0 \;, \\ &q_1(a_1) = 1 \;, \\ &q_1^{(j)}(a_i) = 0 \;,\quad 2 \leq i \leq m \;,\quad 0 \leq j \leq k_i - 1 \end{aligned}$$

since the equation $Lx = 0$ is disconjugate on $[a,b]$ (see [6, Proposition 1 at p. 1] for the evident existence, while noting that the uniqueness follows from the disconjugacy). Observe (again by disconjugacy of $Lx = 0$ on $[a,b]$) that q_1 cannot have other zeros, counting multiplicities. Therefore q_1 has the same zeros with the same respective multiplicities as $t \mapsto P(t)/(t-a_1)$, so that for either $\varepsilon = 1$ or $\varepsilon = -1$, we have $S_{q_1}(t) = \varepsilon S_P(t)$ for every $t \in]a,b]$. Then we have

$$y(t) = \frac{q_1(t)}{q_1(\tau)}y(\tau) + \int_a^b G_2(t,s)Ly(s)\;ds \;,\quad a \leq t \leq b \;;$$

where G_2 denotes the Green's function of the conjugate boundary value problem:

$$\begin{aligned} &Lx = 0 \;, \\ &x(\tau) = 0 \;, \\ &x^{(j)}(a_i) = 0 \;,\quad 2 \leq i \leq m \;,\quad 0 \leq j \leq k_i - 1 \;. \end{aligned}$$

Applying again Levin's Theorem 0.2, we get

$$S_{P_1}(t)y(t) \;\geq\; \frac{S_{P_1}(t)q_1(t)}{q_1(\tau)}y(\tau) \;=\; \frac{|q_1(t)|}{|q_1(\tau)|}||y||_\infty S_{(\cdot\;-\tau)}(t) \;,\quad a < t \leq b \;. \qquad (1.4)$$

Case 3: Suppose that $k_m = 1$. Proceeding as in the previous case, we consider the unique function q_m such that

$$\begin{aligned} &Lq_m = 0 \;, \\ &q_m(a_m) = 1 \;, \\ &q_m^{(j)}(a_i) = 0 \;,\quad 1 \leq i \leq m-1 \;,\quad 0 \leq j \leq k_i - 1 \end{aligned}$$

and we have $S_{q_m}(t) \equiv \varepsilon S_P(t)$, $a \leq t < b$, with either $\varepsilon = 1$ or $\varepsilon = -1$, and

$$y(t) = \frac{q_m(t)}{q_m(\tau)}y(\tau) + \int_a^b G_3(t,s)Ly(s)\;ds \;,\quad a \leq t \leq b \;;$$

where G_3 denotes the Green's function of the disconjugate boundary value problem:

$$\begin{aligned} &Lx = 0 \;, \\ &x(\tau) = 0 \;, \\ &x^{(j)}(a_i) = 0 \;,\quad 1 \leq i \leq m-1 \;,\quad 0 \leq j \leq k_i - 1 \;. \end{aligned}$$

Again from Levin's Theorem 0.2 we see that

$$S_{P_m}(t)y(t) \ \geq \ \frac{S_{P_m}(t)q_m(t)}{q_m(\tau)}y(\tau) \ = \ -\frac{|q_m(t)|}{|q_m(\tau)|}||y||_\infty S_{(\cdot\ -\tau)}(t) \ , \quad a \leq t < b \ . \tag{1.5}$$

Consequently we have:

* if $k_1 = 1$ and $k_m = 1$, using (1.4) when $t > \tau$ and (1.5) when $t < \tau$ we see that

$$|y(t)| \ \geq \ \min\left\{\frac{|q_1(t)|}{||q_1||_\infty}, \ \frac{|q_m(t)|}{||q_m||_\infty}\right\}||y||_\infty \ , \quad a \leq t \leq b \ ;$$

* if $k_1 = 1$ and $k_m \geq 2$, using (1.3) when $t < \tau$ and (1.4) when $t > \tau$ we see that

$$|y(t)| \ \geq \ \min\left\{\frac{|q_1(t)|}{||q_1||_\infty}, \ \frac{|Q_m(t)|}{||Q_m||_\infty}\right\}||y||_\infty \ , \quad a \leq t \leq b \ ;$$

* if $k_1 \geq 2$ and $k_m = 1$, using (1.3) when $t > \tau$ and (1.5) when $t < \tau$ we have

$$|y(t)| \ \geq \ \min\left\{\frac{|Q_1(t)|}{||Q_1||_\infty}, \ \frac{|q_m(t)|}{||q_m||_\infty}\right\}||y||_\infty \ , \quad a \leq t \leq b \ ;$$

* if $k_1 \geq 2$ and $k_m \geq 2$, then (1.3) with $l = m$ if $t < \tau$ and $l = 1$ if $t > \tau$ show that

$$|y(t)| \ \geq \ \min\left\{\frac{|Q_1(t)|}{||Q_1||_\infty}, \ \frac{|Q_m(t)|}{||Q_m||_\infty}\right\}||y||_\infty \ , \quad a \leq t \leq b \ ,$$

and these inequalities imply the exact definition of φ in each case.

To show that $\varphi/|P|$ has a positive infimum on $\bigcup_{i=1}^{m-1}]a_i, a_{i+1}[$, it is enough to use repeatedly de l'Hopital rule. □

§2. THE PRINCIPAL EIGENVALUE WITH L^1-COEFFICIENTS

This section is devoted to the study of the principal eigenvalue problem for the conjugate multipoint BVP

$$\begin{aligned} &Ly = \lambda_1 q(t)y \ , \\ &y^{(j)}(a_i) = 0 \ , \quad 1 \le i \le m \ , \quad 0 \le j \le k_i - 1 \end{aligned} \tag{P}$$

with $q \in \mathbf{L}^1$.

The maximum principle of §1 will be used several times in this section. In §2.1 it is used to introduce an ordered Banach space structure on a suitable vector subspace of C^0 in such a way that the cone has interior points and the integral operator associated to the Green function turns out to be strongly positive. As a consequence, in §2.2 it is shown that $||q||_{\mathbf{L}^1} > 0$ implies that (P) has a positive eigenvalue which is the inverse of the spectral radius of the integral operator associated to the Green function (it turns out to be the unique positive eigenvalue with a positive eigenfunction belonging to the interior of the cone), a fact that provides comparison results for the principal eigenvalues related to different q's.

§2.1 An ordered Banach space of continuous functions

Following an intuition suggested by the above maximum principle, in this section we introduce a new ordered Banach space of continuous functions and we study its properties, mainly with respect to the linear operator associated to Green funtions.

We shall consider disconjugate n^{th} order differential operators of the form

$$Lx := x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x$$

where the coefficients $p_1, \cdots, p_n$ are given continuous functions on $[a, b]$.

Let G be the Green's function of the boundary value problem

$$\begin{aligned} &Ly = 0 \ , \\ &y^{(j)}(a_i) = 0 \ , \quad 1 \le i \le m \ , \quad 0 \le j \le k_i - 1 \ . \end{aligned}$$

Therefore

$$\begin{aligned} &Ly = \lambda q(t)y \ , \quad y \not\equiv 0 \ , \\ &y^{(j)}(a_i) = 0 \ , \quad 1 \le i \le m \ , \quad 0 \le j \le k_i - 1 \end{aligned} \tag{P}$$

if and only if

$$\lambda \neq 0 \quad \text{and} \quad 0 \not\equiv \int_a^b G(t,s)q(s)y(s)\, ds = \frac{1}{\lambda}y(t) \ , \quad a \le t \le b \ .$$

We shall use the following inequality that follows from Levin's Theorem 0.2-(ii) :

$$0 < \frac{G(t,s)}{P(t)} \leq c_1 < \infty \ . \tag{2.1}$$

From this, we have the following property for eigenfunctions of (P):

$$\begin{aligned} |y(t)| &\leq |\lambda| \int_a^b |G(t,s)| \cdot |q(s)| \cdot |y(s)| \, ds \\ &\leq |\lambda| \cdot ||y||_\infty \int_a^b c_1 |P(t)| \cdot |q(s)| \, ds \\ &\leq c_1 |\lambda| \cdot ||y||_\infty |P(t)| \cdot ||q||_{\mathrm{L}^1} \\ &= \text{const.} ||y||_\infty |P(t)| \ . \end{aligned}$$

Motivated by this, we introduce the following definitions :

$$X = \left\{ x \in \mathcal{C}([a,b]) : \ |x(t)| \leq c|P(t)| \ , \ a \leq t \leq b \ ; \ \text{for a positive constante } c = c(x) \right\} \ ,$$

$$||x||_P = ||x||_\infty + \sup_{t \neq a_1, \cdots, a_m} \frac{|x(t)|}{|P(t)|} \ ,$$

$$\mathcal{K} = \left\{ x \in X : \ S_P(t) x(t) \geq 0 \ , \ a \leq t \leq b \right\} \ .$$

We shall denote by $\prec$ the order defined by $\mathcal{K}$ on X, i.e.

$$x \preceq y \iff y - x \in \mathcal{K} \ .$$

Lemma 2.1 *$(X, \ || \ ||_P)$ is an ordered Banach space with $\mathcal{K}$ as the cone of positive elements.*
The cone $\mathcal{K}$ has nonempty interior. In fact

$$x \in \operatorname{int}(\mathcal{K}) \iff \inf_{t \neq a_1, \cdots, a_m} \frac{x(t)}{P(t)} > 0$$

so that at least $P \in \operatorname{int}(\mathcal{K})$.

Proof:
It is quite clear that X is a linear space, that $|| \ ||_P$ is a norm on X and that $\mathcal{K}$ is a closed cone in X.
To show that X is complete, consider a Cauchy sequence $(x_k)_k$. Since it is also a Cauchy sequence in $\mathcal{C}^0 := \mathcal{C}([a,b])$, we have $x_k \to x_0$ uniformly for a suitable $x_0 \in \mathcal{C}^0$. Now, to every $\epsilon > 0$, there corresponds k_ϵ such that

$$\sup_{t \neq a_1, \cdots, a_m} \left| \frac{x_k(t) - x_l(t)}{P(t)} \right| \leq ||x_k - x_l||_P < \epsilon$$

for $k, l \geq k_\epsilon$. Here we fix $t \neq a_1, \cdots, a_m$ and k, and take $\lim_l$ obtaining

$$\left|\frac{x_k(t) - x_0(t)}{P(t)}\right| \leq \epsilon$$

for $k \geq k_\epsilon$. It follows that $x_0 \in X$ and $x_k \to x_0$ in the norm $\|\ \|_P$. Thus X is complete. To prove the statement about $\mathrm{int}(\mathcal{K})$, define

$$\mathcal{U} = \left\{x \in \mathcal{K} : \inf_{t \neq a_1, \cdots, a_m} \frac{x(t)}{P(t)} > 0\right\}.$$

We desire to show that $\mathcal{U}$ is an open subset of $(X, \|\ \|_P)$ and $\mathcal{K} \setminus \mathcal{U}$ coincides with the boundary of $\mathcal{K}$. For this end, let $u \in \mathcal{U}$, then there exists $\epsilon > 0$ such that

$$\frac{u(t)}{P(t)} \geq 2\epsilon \quad \text{for} \quad t \neq a_1, \cdots, a_m$$

so that given any $x \in X$ satisfying $\|x - u\|_P < \epsilon$, we have

$$\begin{aligned} \frac{x(t)}{P(t)} &= \frac{x(t)-u(t)}{P(t)} + \frac{u(t)}{P(t)} \\ &\geq -\|x - u\|_P + \frac{u(t)}{P(t)} \\ &> -\epsilon + 2\epsilon \\ &= \epsilon \end{aligned}$$

for $t \neq a_1, \cdots, a_m$, which implies that

$$\inf_{t \neq a_1, \cdots, a_m} \frac{x(t)}{P(t)} \geq \epsilon > 0$$

showing that $x \in \mathcal{U}$. Thus $\mathcal{U}$ is open. It follows that $\mathcal{K} \setminus \mathcal{U}$ is closed and then to complete the proof it is enough to prove that

$$u \in \mathcal{K} \setminus \mathcal{U} \implies u \in \partial\mathcal{K}$$

where $\partial\mathcal{K}$ is the boundary of $\mathcal{K}$. Now, $u \in \mathcal{K} \setminus \mathcal{U}$ implies $\inf_{t \neq a_1, \cdots, a_m} \frac{u(t)}{P(t)} = 0$, hence there exists $t_l \neq a_1, \cdots, a_m$ such that $\frac{u(t_l)}{P(t_l)} \to 0$. Fix $\epsilon > 0$. We have

$$\begin{aligned} S_P(t_l)\Big(u(t_l) - \epsilon P(t_l)\Big) &= S_P(t_l) \cdot P(t_l) \cdot \left\{\frac{u(t_l)}{P(t_l)} - \epsilon\right\} \\ &= |P(t_l)| \left\{\frac{u(t_l)}{P(t_l)} - \epsilon\right\} \\ &< 0 \end{aligned}$$

for l large enough. Therefore $u - \epsilon P \notin \mathcal{K}$. Letting $\epsilon \downarrow 0$ we deduce $u \in \partial\mathcal{K}$ as desired. □

Lemma 2.2 *Let $q \in \mathrm{L}^1([a,b])$. Then the operator*

$$T_q \; : \; x \mapsto T_q x$$

defined by

$$T_q x(t) = \int_a^b G(t,s)q(s)x(s)\ ds$$

is a compact linear operator on $(X, \ || \ ||_P)$.

In the sequel T_q will always denote the above operator for a given q.

Proof:
It follows from (2.1) that T_q maps X into X. To state its compactness, fix a sequence $(x_l)_l$ in X with $||x_l||_P = 1$ for all l. We need to show the existence of a convergent subsequence of $\left(T_q x_l\right)_l$. Standard arguments based on Ascoli theorem and the uniform continuity of G guarantee the existence of a subsequence such that

$$T_q x_{l_k} \quad \longrightarrow \quad y \quad \text{uniformly} \ . \tag{2.2}$$

for a suitable $y \in C^0$. To simplify notations, we set $T := T_q$ and $x_k := x_{l_k}$. So we have done if we show that

$$\left| \frac{Tx_k(t) - y(t)}{P(t)} \right| \quad \longrightarrow \quad 0$$

uniformly on $t \neq a_1, \cdots, a_m$. We have

$$\left| \frac{Tx_k(t) - y(t)}{P(t)} \right| \; = \; \left| \int_a^b \frac{G(t,s)}{P(t)} q(s) x_k(s)\ ds \; - \; \frac{y(t)}{P(t)} \right| \ .$$

By Coppel's result mentioned for (2.1), $\frac{G(t,s)}{P(t)}$ is uniformly continuous. Thus we can apply again Ascoli theorem and obtain a subsequence such as

$$\int_a^b \frac{G(t,s)}{P(t)} q(s) x_{k_j}(s)\ ds \quad \longrightarrow \quad z(t)$$

uniformly on t for a suitable $z \in C^0$. For $t \neq a_1, \cdots, a_m$, we deduce from (2.2) that

$$\int_a^b \frac{G(t,s)}{P(t)} q(s) x_{k_j}(s)\ ds \quad \longrightarrow \quad \frac{y(t)}{P(t)} \ .$$

Thus $z(t) = \frac{y(t)}{P(t)}$ for $t \neq a_1, \cdots, a_m$, and we have done. □

Lemma 2.3 *If $q \in \mathrm{L}^1([a,b])$ satisfies*

$$q \neq 0 \textit{ on a set of positive measure, and } S_P(t)q(t) \; \geq \; 0 \;\; \textit{for a.e. } t \in [a,b] \ ,$$

then the operator T_q is positive on the ordered Banach space X. In fact

$$x \in \mathcal{K} \quad \Longrightarrow \quad \textit{either} \ \ T_q x = 0 \ \ \textit{or} \ \ T_q x \in \operatorname{int}(\mathcal{K}) \ ,$$

hence particularly

$$x \in \operatorname{int}(\mathcal{K}) \quad \Longrightarrow \quad T_q x \in \operatorname{int}(\mathcal{K}) \ .$$

In case q does not vanish identically in any subinterval of $[a,b]$, T_q is strongly positive, i.e.

$$x \in \mathcal{K} \setminus \{0\} \quad \Longrightarrow \quad T_q x \in \operatorname{int}(\mathcal{K}) \ .$$

Proof:
The set of continuous functions with compact support contained in $[a_i, a_{i+1}]$ is dense in $\mathbf{L}^1([a_i, a_{i+1}])$. Glueing together functions chosen on each interval $[a_i, a_{i+1}]$, $1 \le i \le m-1$, we deduce the existence of a sequence $(g_k)_k$ of continuous functions vanishing at $a_1, \cdots, a_m$ and converging to q in $\mathbf{L}^1([a,b])$. Set

$$q_k \ = \ |g_k| S_P \quad \text{ for } \ k = 1, 2, \ \cdots \ .$$

Since g_k vanishes at $a_1, \cdots, a_m$, q_k is continuous. Let us show that $q_k \to q$ in $\mathbf{L}^1([a,b])$. We have

$$\begin{aligned} \int_a^b |q_k(s) - q(s)| \, ds \ &= \ \int_a^b | \ S_P(s)|g_k(s)| - S_P(s)|q(s)| \ | \, ds \\ &\le \ \int_a^b | \ |g_k(s)| - |q(s)| \ | \, ds \\ &\le \ ||g_k - q||_{\mathbf{L}^1} \end{aligned}$$

from which the desired conclusion follows.

Now choose any $x \in \mathcal{K}$ and set

$$y_k = T_{q_k} x \ \text{ and } \ y = T_q x \ .$$

From $q_k \to q$ in $\mathbf{L}^1([a,b])$, it follows

$$y_k \longrightarrow y \ \text{ uniformly } \ .$$

For every k, we have

$$\begin{aligned} & L y_k = q_k(t) x = S_P(t) q_k(t) \cdot S_P(t) x \ \ge \ 0 \\ & y_k^{(j)}(a_i) = 0 \ , \quad 1 \le i \le m \ , \quad 0 \le j \le k_i - 1 \ . \end{aligned}$$

Then Theorem 1.1 implies

$$\begin{aligned} S_P(t) y_k(t) \ &\ge \ \varphi(t) ||y_k||_\infty \\ &\ge \ c_0 |P(t)| \cdot ||y_k||_\infty \ . \end{aligned}$$

Taking limits, we deduce:

$$S_P(t) y(t) \ge \ c_0 |P(t)| \cdot ||y||_\infty$$

showing that either $y = T_q x = 0$, or $y = T_q x \in \text{int}(\mathcal{K})$ by virtue of Lemma 2.1. It also follows from the above inequality and from Lemma 2.1 that T_q maps $\text{int}(\mathcal{K})$ into itself, while in case q does not vanish identically in any subinterval of $[a, b]$, we have

$$x \succ 0 \implies T_q x \in \text{int}(\mathcal{K})$$

since

$$x \succ 0 \implies T_q x \neq 0$$

by (2.1). □

§2.2 Existence and properties of the eigenvalues

Here we apply the lemmas of the previous sections in order to study the eigenvalue problem. We need, however, also the following lemma:

Lemma 2.4 *Let Y be a Banach space. Denote by $L(Y)$ its space of bounded linear operators and by $K(Y)$ the subspace of $L(Y)$ consisting of compact linear operators. Then the spectral radius $r(T)$ depends upper semi-continuously on $T \in L(Y)$ and continuously on $T \in K(Y)$ in the operator topology.*

Proof:
We need only to show that $r : L(Y) \to [0, \infty[$ is upper semi-continuous in view of the proof given to the version of Nussbaum [19]. For this end, fix arbitrarily $T \in L(Y)$ and let $(T_i)_i$, $i = 1, 2, \cdots$, be any sequence of $L(Y)$ converging to T in the operator topology. We show that

$$\limsup_{i\to\infty} r(T_i) \leq r(T) .$$

For any positive integer $k \geq 2$ and any $H \in L(Y)$, we have

$$\begin{aligned} \|(T+H)^k\| &\leq \|T^k\| + \textstyle\sum_{j=0}^{k-1} C_k^j \|T\|^j \|H\|^{k-j} \\ &\leq \|T^k\| + \tfrac{k}{k-j}\|H\| \textstyle\sum_{j=0}^{k-1} C_{k-1}^j \|T\|^j \|H\|^{k-1-j} \\ &\leq \|T^k\| + k\|H\|(\|T\| + \|H\|)^{k-1} . \end{aligned}$$

Thus

$$\|(T+H)^k\|^{\frac{1}{k}} \leq 2^{\frac{1}{k}} \left(\|T^k\|^{\frac{1}{k}} + k^{\frac{1}{k}} \|H\|^{\frac{1}{k}} (\|T\| + \|H\|)^{\frac{k-1}{k}} \right)$$

which implies

$$\|(T+H)^k\|^{\frac{1}{k}} \leq 2^{\frac{1}{k}} \left(\|T^k\|^{\frac{1}{k}} + e\|H\|^{\frac{1}{k}} (\|T\| + \|H\| + 1) \right) . \tag{2.3}$$

Suppose now by contradiction that

$$r(T) < \limsup_{i\to\infty} r(T_i) .$$

Therefore, there exists a subsequence (T_{i_l}) of (T_i) such that

$$\lim_{l\to\infty} r(T_{i_l}) \;=\; \limsup_{i\to\infty} r(T_i) \;>\; r(T)\ .$$

Now fix $\epsilon > 0$ such that

$$r(T) + \epsilon \;<\; \lim_{l\to\infty} r(T_{i_l})\ .$$

Thus there exists l_ϵ such that for every $l \geq l_\epsilon$,

$$r(T) + \epsilon \;<\; r(T_{i_l}) \;=\; \inf_k ||T_{i_l}^k||^{\frac{1}{k}} \;=\; \lim_k ||T_{i_l}^k||^{\frac{1}{k}}\ .$$

Hence, for every positive integer k, by taking an integer $l_k \geq l_\epsilon$ such that

$$||T_{i_{l_k}} - T|| \;<\; e^{-k^2}$$

and using (2.3), we get

$$r(T) + \epsilon \;<\; ||T_{i_{l_k}}^k||^{\frac{1}{k}} \;\leq\; 2^{\frac{1}{k}}\left(||T^k||^{\frac{1}{k}} + e^{1-k}(||T|| + e^{-k^2} + 1)\right)$$

Consequently we have

$$r(T) + \epsilon \;\leq\; \inf_k ||T_{i_{l_k}}^k||^{\frac{1}{k}} \;\leq\; r(T)$$

which is absurd. □

Theorem 2.1 *If $q \in \mathrm{L}^1([a,b])$ satisfies*

$$q \neq 0 \textit{ on a set of positive measure, and } S_P(t)q(t) \;\geq\; 0 \;\; \textit{for a.e. } t \in [a,b]\ ,$$

then (P) has a positive eigenvalue $\lambda_1(q)$ with an eigenfunction $y \in \mathrm{int}(\mathcal{K})$ (which is unique up to normalization). Moreover, $\lambda_1(q)$ is the inverse of the spectral radius of T_q and $\lambda_1(q)$ depends continuously on q with respect to the L^1-norm.

Proof:

At first, we prove the existence part of the eigenvalues, dividing the argument in two steps.

Step 1: $S_P(t)q(t) > 0$ for a.e. t. In this case, Lemmas 2.2 and 2.3 guarantee that T_q is a strongly positive compact linear operator on the ordered Banach space X. Therefore the classical version of the Krein-Rutman theorem, cf. Amann [4], implies that the spectral radius of T_q is the inverse of the principal eigenvalue $\lambda_1(q)$ of (P) and that it admits an eigenfunction $y \in \mathrm{int}(\mathcal{K})$.

Step 2: The general case $S_P(t)q(t) \geq 0$ a.e. with $||q||_{\mathrm{L}^1} > 0$. Refering to the proof of Lemma 2.3 we know the existence of a sequence $(g_k)_k$ of continuous functions vanishing at $a_1, \cdots, a_m$ and converging to q in $\mathrm{L}^1([a,b])$. Set

$$q_k \;=\; |g_k|S_P \;+\; \frac{P}{k} \quad \text{for } k = 1, 2, \cdots$$

so that q_k is continuous (since g_k vanishes at $a_1, \cdots, a_m$) and $q_k \to q$ in $\mathrm{L}^1([a,b])$ with

$$S_P(t)q_k(t) \;>\; 0 \quad \text{for a.e. } t\ .$$

Therefore, Step 1 implies that the spectral radius of T_{q_k} is the inverse of the principal eigenvalue $\lambda_1(q_k)$ of (P) with $q = q_k$ and that there is an eigenfunction $y_k \in \text{int}(\mathcal{K})$ such that

$$y_k(t) \;=\; \lambda_1(q_k) \int_a^b G(t,s)q_k(s)y_k(s)\; ds \;\; , \quad a \le t \le b \;\; . \tag{2.4}$$

We assume $||y_k||_\infty = 1$. From the maximum principle of Degla [7] we have

$$S_P(t)y_k(t) \;\ge\; \varphi(t) \;\; , \quad a \le t \le b \;\; . \tag{2.5}$$

Multiplying (2.4) by S_P, we get

$$\begin{aligned} |y_k(t)| \;=\; S_P(t)y_k(t) \;&=\; \lambda_1(q_k) \textstyle\int_a^b S_P(t)G(t,s) \cdot S_P(s)q_k(s) \cdot S_P(s)y_k(s)\; ds \\ &\le\; \lambda_1(q_k) \textstyle\int_a^b |G(t,s)| \cdot |q_k(s)|\; ds \end{aligned}$$

and then from (2.5)

$$\varphi(t) \;\le\; \lambda_1(q_k) \int_a^b |G(t,s)| \cdot |q_k(s)|\; ds \;\; , \quad a \le t \le b \;\; . \tag{2.6}$$

Clearly

$$\int_a^b |G(t,s)| \cdot |q_k(s)|\; ds \;\longrightarrow\; \int_a^b |G(t,s)| \cdot |q(s)|\; ds \quad \text{uniformly on } t \;\; ,$$

so that

$$\max_{a \le t \le b} \int_a^b |G(t,s)| \cdot |q_k(s)|\; ds \;\longrightarrow\; \max_{a \le t \le b} \int_a^b |G(t,s)| \cdot |q(s)|\; ds \;\; .$$

Since the last quantity is positive, taking $\sup_t$ in (2.6) we get

$$\lambda_1(q_k) \;\ge\; \text{const.} \;>\; 0 \;\; . \tag{2.7}$$

Using again (2.4) and (2.5) we have

$$1 \;=\; ||y_k||_\infty \;\ge\; \lambda_1(q_k) \int_a^b |G(t,s)| \cdot |q_k(s)| \varphi(s)\; ds \;\; , \quad a \le t \le b \;\; ,$$

and from here we can repeat the argument leading to (2.7) in order to deduce

$$\lambda_1(q_k) \;\le\; \text{const.} \;\; . \tag{2.8}$$

On the other hand, $q_k \to q$ in $\mathbf{L}^1$ implies $T_{q_k} \longrightarrow T$ in the operator norm [since its topology corresponds to uniform convergence on the unit ball], hence from Lemma 2.4 it follows

$$r(T_{q_k}) = \frac{1}{\lambda_1(q_k)} \quad \longrightarrow \quad r(T_q) =: \frac{1}{\lambda_0} \;\; . \tag{2.9}$$

By this and (2.7), (2.8), we deduce that λ_0 is a positive real number. Besides, set for every k,

$$z_k(t) \;=\; \int_a^b G(t,s)q_k(s)y_k(s)\; ds \;\; , \quad a \le t \le b \;\; .$$

It is easily seen from $||q_k||_{L^1} \le$ const. and $||y_k||_\infty = 1$, that $(z_k)_k$ is equicontinuous and pointwise bounded. Therefore by Ascoli theorem, there exists a subsequence and a suitable $z_0 \in C^0$ such that

$$z_{k_l} \longrightarrow z_0 \quad \text{uniformly} \ .$$

Combining with (2.9), we take limits in (2.4) and we find that $y_{k_l} \to y$ uniformly, y being a suitable continuous function such that $||y||_\infty = 1$, and that

$$y(t) = \lambda_0 \int_a^b G(t,s)q(s)y(s)\,ds \ , \quad a \le t \le b \ .$$

Since $y_k \in \mathcal{K}$, also $y \in \mathcal{K}$. Moreover, (2.5) implies that $S_P(t)y(t) \ge \varphi(t)$ for $a \le t \le b$. Thus $y \in \text{int}(\mathcal{K})$ by Lemma 2.1. We conclude that y is an eigenfunction of (P) corresponding to the eigenvalue $\lambda_0 = \frac{1}{r(T_q)} =: \lambda_1(q)$.

It remains to state the continuous dependence. To this aim, assume that $q_k \to q$ in $L^1([a,b])$ with $q_k(t) \ne 0$ on a set of positive measure, and $S_P(t)q_k(t) \ge 0$ for a.e. $t \in [a,b]$. By the above, $\lambda_1(q_k) = \frac{1}{r(T_{q_k})}$ and $r(T_{q_k}) = \frac{1}{\lambda_1(q_k)}$. Therefore, since $q_k \to q$ implies $T_{q_k} \longrightarrow T_q$ in the operator norm, we need only to apply Lemma 2.4. □

The following result implies a type of variational characterization of the first eigenvalue:

Corollary. *If $S_P(t)q(t) > 0$ for a.e. t , then:*

(i) $\lambda_1(q) = \min\left\{\lambda > 0 : \ x \preceq \lambda T_q x \ \textit{for some} \ x \succ 0\right\}$;

(ii) $0 \prec x \prec \lambda T_q x \implies \lambda_1(q) < \lambda$.

Proof:
(i) From $0 \prec x \preceq \lambda T_q x$, it follows $\frac{1}{\lambda}x \preceq T_q x$. This implies $r(T_q) \ge \frac{1}{\lambda}$ by the normality of the ordered Banach space $(X, || \ ||_P)$, and the conclusion follows.
(ii) Suppose $0 \prec x \prec \lambda T_q x$ and set $y = T_q x$. Since $x \succ 0$, we have $y \succ 0$. Since $\lambda y - x = \lambda T_q x - x \succ 0$, we have $T_q(\lambda y - x) \in \text{int}(\mathcal{K})$ by Lemma 2.3. By the continuity of $\phi(t) = tT_q y - y$, there exists $0 < \lambda_0 < \lambda$ such that $\lambda_0 T_q y - y \in \mathcal{K}$, hence $y \preceq \lambda_0 T_q y$. Then (i) implies $\lambda_1(q) \le \lambda_0 < \lambda$. □

Theorem 2.2 *The following comparison results hold.*
(a) *If L is disconjugate and $q_1, q_2 \in L^1([a,b])$ do not vanish on a set of positive measure, are different on a set of positive measure and satisfy*

$$0 \le S_P(t)q_1(t) \le S_P(t)q_2(t) \quad \textit{for a.e. } t \in [a,b] \ ,$$

then

$$0 < \lambda_1(q_2) < \lambda_1(q_1) \ .$$

(b) *If L is disconjugate and $q \in L^1([a,b])$ does not vanish on a set of positive measure and* *satisfies* $S_P(t)q(t) \geq 0$ *for a.e.* t *, with* $L - q$ *as a disconjugate operator, then*

$$\lambda_1(q) > 1 .$$

Proof:

(a) For simplicity of notations, set $T_i = T_{q_i}$ for $i = 1, 2$. On the basis of the assumptions on q_1 and q_2, we see from Lemma 2.3 that the compact linear operators T_1, T_2 and $T_2 - T_1$ are positive with respect to $\mathcal{K}$ and

$$(T_2 - T_1)(\text{int}(\mathcal{K})) \subset \text{int}(\mathcal{K}) .$$

Therefore, letting $y_1 \in \text{int}(\mathcal{K})$ be an eigenvector of T_1 corresponding to $r(T_1) > 0$ (cf. Theorem 2.1), we have

$$T_2 y_1 - T_1 y_1 = T_2 y_1 - r(T_1) y_1 \in \text{int}(\mathcal{K}) ,$$

which implies that

$$T_2 y_1 - (r(T_1) + \epsilon) y_1 \in \mathcal{K}$$

for sufficiently small $\epsilon > 0$, and so $r(T_2) > r(T_1)$ by Theorem 0.4. That is

$$0 < \lambda_1(q_2) < \lambda_1(q_1) .$$

(b) To show that $\lambda_1(q) > 1$, it suffices to prove that $\lambda_1(q)$ is not less than 1 due to the disconjugacy of $L - q$. For each $\alpha \in]a_l, a_{l+1}]$, $1 \leq l \leq m - 1$, define the operator $T_\alpha : C([a, b]) \to C([a, b])$ by

$$T_\alpha x(t) = \begin{cases} \int_a^\alpha G_\alpha(t,s) q(s) x(s)\, ds & if \ \ a \leq t \leq \alpha \\ 0 & if \ \ \alpha < t \leq b \end{cases} ,$$

where $G_\alpha(t, s)$ is the Green's function of the boundary value problem

$$\begin{aligned} & Ly = 0, \\ & y^{(j)}(a_i) = 0 , \quad 1 \leq i \leq l , \quad 0 \leq j \leq k_i - 1 , \\ & y^{(j)}(\alpha) = 0 , \quad 0 \leq j \leq k_{l+1} + \cdots + k_m - 1 . \end{aligned}$$

Note that the mapping $\alpha \mapsto T_\alpha$ is continuous, and then by continuity of the spectral-radius mapping on compact linear operators (cf. Lemma 2.4), the mapping $\alpha \mapsto r(T_\alpha)$ is also continuous. Since $T_\alpha \to 0$ as $\alpha \to a$, if $r(T_b) = \frac{1}{\lambda_1(q)}$ were not less than 1, there would exist $\alpha \in]a, b]$ and $l \in \{1, \cdots, m-1\}$ such that $r(T_\alpha) = 1$ which would imply that the linear problem

$$\begin{aligned} & Ly = q(t) y , \\ & y^{(j)}(a_i) = 0 , \quad 1 \leq i \leq l , \quad 0 \leq j \leq k_i - 1 , \\ & y^{(j)}(\alpha) = 0 , \quad 0 \leq j \leq k_{l+1} + \cdots + k_m - 1 \end{aligned}$$

has a non trivial solution by Theorem 2.1, contradicting the disconjugacy of $L - q$. □

As a consequence we have

Corollary. *Let $q \in \mathrm{L}^1([a,b])$ be such that $||q||_{\mathrm{L}^1} > 0$ and $S_P(t)q(t) \geq 0$ for a.e. $t \in [a,b]$, and assume that L and the perturbed differential operator $L - q$ are both disconjugate on $[a,b]$. Moreover let $f \in \mathrm{L}^1([a,b])$ satisfy $||f||_{\mathrm{L}^1} > 0$ with $f \geq 0$ for a.e. $t \in [a,b]$. Then there exists a unique function y such that*

$$\begin{aligned} &Ly - q(t)y = f(t) \ , \\ &y^{(j)}(a_i) = 0 \ , \quad 1 \leq i \leq m \ , \quad 0 \leq j \leq k_i - 1 \ , \end{aligned}$$

and the function $t \mapsto y(t)/P(t)$, $t \neq a_1, \cdots, a_m$ has a continuous extension to $[a,b]$ with positive infimum. Furthermore, given any $y_0 \in C([a,b])$ with $S_P(t)y_0(t) \geq 0$, the sequence $(y_l)_l$; $l = 1, 2, \cdots$, defined recursively by

$$\begin{aligned} &Ly_{l+1} = q(t)y_l + f(t) \ , \quad for \ \ a.e. \ \ t \in [a,b] \ , \\ &y_{l+1}^{(j)}(a_i) = 0 \ , \quad 1 \leq i \leq m \ , \quad 0 \leq j \leq k_i - 1 \ , \end{aligned}$$

converges uniformly (and thus with respect to the C^{n-1}-norm) to y.

Proof:
The nonhomogeneous problem

$$\begin{aligned} &Ly - q(t)y = f(t) \ , \quad \text{for a.e. } t \in [a,b] \ , \\ &y^{(j)}(a_i) = 0 \ , \quad 1 \leq i \leq m \ , \quad 0 \leq j \leq k_i - 1 \ , \end{aligned}$$

is equivalent to

$$(I - T_q)y = \tilde{f} \ , \quad \tilde{f} \in \mathcal{K} \ , \quad \text{where } \tilde{f}(t) = \int_a^b G(t,s)f(s) \ ds \ .$$

And since T_q is positive and $r(T_q) < 1$ by Theorem 2.1, the inverse operator $(I - T_q)^{-1}$ is well-defined and can be represented as the series $I + T_q + T_q^2 + \cdots$ and so is positive with respect to $\mathcal{K}$. Therefore the conclusions of the Proposition follow from [17, Theorem 15.1 at p. 158]. □

§3. Application to the existence and multiplicity of strictly positive solutions to singular BVPs

In this section, we apply the previous results in order to deduce the existence of positive solutions to singular BVPs. There are two types of results leading respectively to:

- solutions y that are positive in the traditional sense, i.e. $y(t) \geq 0$ with $y \neq 0$, obtained by using the maximum principle of §1. These are found in §3.1.
- solutions y that belong to the cone introduced in §2.1. There are found in §3.2.

Another important ingredient of our proofs is Krasnosel'ski's Theorem 0.5 about the compression/expansion of the cone.

§3.1 From the maximum principle

We start with

Theorem 3.1 *Let $F : [a,b] \times \mathbf{R} \to [0,\infty[$ be a nonnegative continuous function which admits continuous functions $q : [a,b] \to [0,\infty[$ and $f : [a,b] \times \mathbf{R} \to [0,\infty[$ such that:*
(1) $q \neq 0$,
(2) $F(t,y) \geq q(t)f(t,y)$ *for all* $(t,y) \in [a,b] \times \mathbf{R}$, *and*
(3) *either*

$$\text{(i)} \quad F_0 = \limsup_{y\to 0} \frac{F(t,y)}{|y|} \equiv 0 \ \text{ and } \ f_\infty = \liminf_{|y|\to\infty} \frac{f(t,y)}{|y|} \equiv \infty \ \text{ uniformly on } t \ , \ \text{ or}$$

$$\text{(ii)} \quad f_0 = \liminf_{y\to 0} \frac{f(t,y)}{|y|} \equiv \infty \ \text{ and } \ F_\infty = \limsup_{|y|\to\infty} \frac{F(t,y)}{|y|} \equiv 0 \ \text{ uniformly on } t \ .$$

If L is disconjugate, then the conjugate problem

$$\begin{aligned} &Ly = F(t,y) \ , \\ &y^{(j)}(a_i) = 0 \ , \ \ 1 \leq i \leq m \ , \ \ 0 \leq j \leq k_i - 1 \ , \end{aligned}$$

has at least one nontrivial solution y such that the function $t \mapsto y(t)/P(t)$, $t \notin \{a_1, \ldots, a_m\}$, has a continuous extention to $[a,b]$ with a positive infimum.

Proof:
Let φ be as in Theorem 1.1. Since F is a nonnegative continuous function, Theorem 1.1 implies that every solution of the problem

$$\begin{array}{l} Ly = F(t,y) \ , \\ y^{(j)}(a_i) = 0 \ , \quad 1 \leq i \leq m \ , \quad 0 \leq j \leq k_i - 1 \ , \end{array}$$

belongs to the subset

$$\mathcal{K} = \{x \in \mathcal{C}([a,b]) : \ S_P(t)x(t) \geq \varphi(t)||x||_\infty, \ a \leq t \leq b\} \ .$$

The set $\mathcal{K}$ is clearly a cone in the Banach space $\left(\mathcal{C}([a,b]), \ ||\cdot||_\infty\right)$ and has the following property:

$$\text{if} \ \ x \in \mathcal{K} \setminus \{0\}, \ \text{then} \ \ x(t) \neq 0 \ \ \text{for} \ \ t \notin \{a_1, \ldots, a_m\} \ .$$

Consider the map $T : \mathcal{K} \to \mathcal{K}$ defined for every $y \in \mathcal{K}$ by

$$Ty : \ t \mapsto \int_a^b G(t,s)F(s,y(s)) \ ds$$

where G is the Green's function of the conjugate boundary value problem

$$\begin{array}{l} Lx = 0 \ , \\ x^{(j)}(a_i) = 0 \ , \quad 1 \leq i \leq m \ , \quad 0 \leq j \leq k_i - 1 \ . \end{array}$$

We note that T maps indeed $\mathcal{K}$ into $\mathcal{K}$ by Theorem 1.1. Furthermore, a standard argument based on the Ascoli theorem shows that T is completely continuous (i.e., T is continuous and maps bounded subsets into compact sets).

Now to prove that $T : \mathcal{K} \to \mathcal{K}$ has a fixed point with a positive norm, it suffices to establish the existence of two positive real numbers $r_1 < r_2$ satisfying one of the two conditions:

(a) $||Ty||_\infty < ||y||_\infty$ if $y \in \mathcal{K}$ and $||y||_\infty = r_1$, and $||Ty||_\infty > ||y||_\infty$ if $y \in \mathcal{K}$ and $||y||_\infty = r_2$

(b) $||Ty||_\infty > ||y||_\infty$ if $y \in \mathcal{K}$ and $||y||_\infty = r_1$, and $||Ty||_\infty < ||y||_\infty$ if $y \in \mathcal{K}$ and $||y||_\infty = r_2$;

as shown by the Krasnosel'skii theorem on the expansion and compression of cones, in the recent version of [3, Theorem 11.2], as a result of Theorem 0.5. To this aim, fix $\epsilon > 0$, $\alpha > 0$, $\delta \notin \{a_1, \ldots, a_m\}$, and a compact set $A \subset \bigcup_{i=1}^{m-1}]a_i, a_{i+1}[$ with a positive measure, such that

$$\epsilon \max_{a \leq t \leq b} \int_a^b |G(t,s)| \ ds \ < \ 1 \ < \ \alpha \int_A |G(\delta,s)| q(s) \varphi(s) \ ds \ .$$

Case 1: Suppose that the assumption (3)-(i) holds, and let us show the existence of $0 < r_1 < r_2 < \infty$ such that:

$$\begin{array}{l} \text{if} \ \ y \in \mathcal{K} \ \ \text{satisfies} \ \ ||y||_\infty = r_1 \ , \quad \text{then} \ \ ||Ty||_\infty < ||y||_\infty \ , \\ \text{whereas if} \ y \in \mathcal{K} \ \ \text{satisfies} \ \ ||y||_\infty = r_2 \ , \quad \text{then} \ \ ||Ty||_\infty > ||y||_\infty \ . \end{array}$$

Indeed, on one hand we have the existence of $r_1 > 0$ such that $F(t,y) \leq \epsilon|y|$ for all $t \in [a,b]$ and all $y \in [-r_1, r_1]$. Therefore, given $y \in \mathcal{K}$ with $||y||_\infty = r_1$, we have

$$|Ty(t)| = \int_a^b |G(t,s)|F(s,y(s)) \ ds \ \leq \ \epsilon||y||_\infty \int_a^b |G(t,s)| \ ds \ < \ ||y||_\infty \ , \quad a \leq t \leq b \ ,$$

and thus $||Ty||_\infty < ||y||_\infty$. On the other hand, there exists $r_0 > 0$ such that $f(t,y) \geq \alpha|y|$ for $a \leq t \leq b$ and $|y| \geq r_0$. Set $r_2 = r_1 + (r_0/\min\{\varphi(s) : s \in A\})$ so that $r_2 > r_1$ and $r_2 \min\{\varphi(s) : s \in A\} \geq r_0$. Therefore, given $y \in \mathcal{K}$ with $||y||_\infty = r_2$, we have

$$S_P(t)y(t) \ \geq \ \varphi(t)||y||_\infty \ \geq \ r_0 \quad \text{for} \quad t \in A \ ,$$

which implies $f(t,y(t)) \geq \alpha|y(t)|$, $t \in A$, and so

$$\begin{aligned} |Ty(\delta)| = S_P(\delta)Ty(\delta) \ &= \ \textstyle\int_a^b |G(\delta,s)|F(s,y(s)) \ ds \qquad &&\geq \ \textstyle\int_A |G(\delta,s)|F(s,y(s)) \ ds \\ &\geq \ \textstyle\int_A |G(\delta,s)|q(s)f(s,y(s)) \ ds \qquad &&\geq \ \alpha \textstyle\int_A |G(\delta,s)|q(s)|y(s)| \ ds \\ &\geq \ \alpha||y||_\infty \textstyle\int_A |G(\delta,s)|q(s)\varphi(s) \ ds \ && > \ ||y||_\infty \end{aligned}$$

i.e. $||Ty||_\infty > ||y||_\infty$.
Now we apply the above mentioned theorem of Krasnosel'skii and we see that T has a fixed point $y \in \mathcal{K}$ such that $r_1 < ||y||_\infty < r_2$.
Case 2: Suppose that the assumption (3)-(ii) holds, then let us prove the existence of $0 < \eta_1 < \eta_2 < \infty$ such that:

$$\begin{aligned} &\text{if} \ \ y \in \mathcal{K} \ \ \text{satisfies} \ \ ||y||_\infty = \eta_1 \ , \quad \text{then} \ \ ||Ty||_\infty > ||y||_\infty \ , \\ &\text{whereas if} \ y \in \mathcal{K} \ \ \text{satisfies} \ \ ||y||_\infty = \eta_2 \ , \quad \text{then} \ \ ||Ty||_\infty < ||y||_\infty \ . \end{aligned}$$

In fact, on the one hand there exists $\eta_1 > 0$ such that $f(t,y) \geq \alpha|y|$ for $a \leq t \leq b$ and $|y| \leq \eta_1$. Therefore, given $y \in \mathcal{K}$ with $||y||_\infty = \eta_1$, we repeat the series of inequalities above to obtain

$$|Ty(\delta)| \ > \ ||y||_\infty$$

and so $||Ty||_\infty > ||y||_\infty$. On the other hand, there exists $\eta_0 > 0$ such that $F(t,y) \leq \epsilon|y|$ for $a \leq t \leq b$ and $|y| \geq \eta_0$. Setting

$$\eta_2 = \eta_0 + \eta_1 + \left(\max\{F(t,y) : a \leq t \leq b, \ |y| \leq \eta_0\}/\epsilon\right) \ ,$$

we have $\eta_2 > \eta_1$ and $F(t,y) \leq \epsilon\eta_2$ for all $t \in [a,b]$ and $y \in [-\eta_2, \eta_2]$. Therefore, given $y \in \mathcal{K}$ with $||y||_\infty = \eta_2$ we see again that $||Ty||_\infty < ||y||_\infty$. Thus also in this second case T has a fixed point $y \in \mathcal{K}$ with $\eta_1 < ||y||_\infty < \eta_2$.

Applying the remark after Theorem 1.1, we have

$$|y(t)| \ \geq \ \varphi(t)||y||_\infty \ \geq \ c|P(t)| \ ||y||_\infty$$

for a suitable positive constant c. If follows that y/P has a strictly positive continuous extension from $[a,b] \setminus \{a_1, \ldots, a_m\}$ to $[a,b]$. □

Remark: It follows from the proof that the assumption (3) can be refined to other assumptions which make either F_0 sufficiently small and f_∞ sufficiently large, or f_0 sufficiently large but F_∞ sufficiently small. For instance letting φ be as in Theorem 1.1, (3) can be replaced by:
(3′) *There exits some $\delta \notin \{a_1, \ldots, a_m\}$ such that either*

(i) $F_0 \max\limits_{a \leq t \leq b} \int_a^b |G(t,s)| \ ds \ < \ 1 \ < \ f_\infty \int_a^b |G(\delta,s)|q(s)\varphi(s) \ ds$, *or*

(ii) $f_0 \int_a^b |G(\delta,s)|q(s)\varphi(s)\, ds \;>\; 1 \;>\; F_\infty \max_{a\le t\le b} \int_a^b |G(t,s)|\, ds$.

As consequences of Theorem 3.1, we have the following corollaries:

Corollary 3.1 *Let $q:[a,b]\to[0,\infty[$ be continuous with $q\neq 0$, and also let $f:\mathbf{R}\to[0,\infty[$ be continuous and such that either*

(i) $\lim_{y\to 0}\frac{f(y)}{|y|}=0$ *and* $\lim_{|y|\to\infty}\frac{f(y)}{|y|}=\infty$, *or*

(ii) $\lim_{y\to 0}\frac{f(y)}{|y|}=\infty$ *and* $\lim_{|y|\to\infty}\frac{f(y)}{|y|}=0$.

Then the nonlinear problem with homogeneous Hermite m-point conditions

$$Ly=q(t)f(y)\ ,$$
$$y^{(j)}(a_i)=0\ ,\quad 1\le i\le m\ ;\quad 0\le j\le k_i-1\ ,$$

has at least one nontrivial solution y such that the function $t\mapsto y(t)/P(t)$, $t\notin\{a_1,\dots,a_m\}$, has a continuous extention to $[a,b]$ with a positive infimum.

Corollary 3.2 *Let k be a positive integer such that $1\le k\le n-1$ and suppose that $F:[a_1,a_2]\times[0,\infty[\to[0,\infty[$ is a nonnegative continuous function which admits continuous functions $q:[a_1,a_2]\to[0,\infty[$ and $f:[a_1,a_2]\times[0,\infty[\to[0,\infty[$ such that:*

(1) $q\neq 0$,

(2) $F(t,y)\ge q(t)f(t,y)$ *for* $a_1\le t\le a_2$ *and* $y\ge 0$, *and*

(3) *either*

(i) $F_0=\lim_{y\to 0^+}\frac{F(t,y)}{y}\equiv 0$ *and* $f_\infty=\lim_{y\to\infty}\frac{f(t,y)}{y}\equiv\infty$ *uniformly on t* , *or*

(ii) $f_0=\lim_{y\to 0^+}\frac{f(t,y)}{y}\equiv\infty$ *and* $F_\infty=\lim_{y\to\infty}\frac{F(t,y)}{y}\equiv 0$ *uniformly on t* .

Then the $(k,n-k)$-conjugate boundary value problem

$$y^{(n)}=(-1)^{n-k}F(t,y)\ ,$$
$$y^{(j)}(a_1)=0\ ,\quad 0\le j\le k-1\ ,$$
$$y^{(j)}(a_2)=0\ ,\quad 0\le j\le n-k-1\ ,$$

has at least one solution y which is positive on $]a_1,a_2[$.

Proof:

Since the equation $y^{(n)}=0$ is disconjugate on $[a_1,a_2]$, Theorem 3.1 implies the exitence of a function $x\in C^n([a_1,a_2])$, with $\frac{x(t)}{(t-a_1)^k(t-a_2)^{n-k}}>0$ on $]a_1,a_2[$, satisfying the following $(k,n-k)$-conjugate boundary value problem

$$x^{(n)}=F(t,|x|)\ ,$$
$$x^{(j)}(a_1)=0\ ,\quad 1\le i\le k-1\ ,$$
$$x^{(j)}(a_2)=0\ ,\quad 0\le j\le n-k-1\ .$$

Letting now $y\equiv(-1)^{n-k}x$, we have $|x|\equiv y$, $y(t)>0$; $a_1<t<a_2$, and

$$y^{(n)}=(-1)^{n-k}F(t,y)\ ,$$
$$y^{(j)}(a_1)=0\ ,\quad 0\le j\le k-1\ ,$$
$$y^{(j)}(a_2)=0\ ,\quad 0\le j\le n-k-1\ .\ \square$$

In particular, Corollary 3.2 implies that for a positive integer k such that $1 \leq k \leq n-1$ and for given continuous functions $q : [a_1, a_2] \to [0, \infty[$, $q \neq 0$, and $f : [0, \infty[\to [0, \infty[$ satisfying either
(i) $\lim_{y \to 0+} \frac{f(y)}{y} = 0$ and $\lim_{y \to \infty} \frac{f(y)}{y} = \infty$, or (ii) $\lim_{y \to 0+} \frac{f(y)}{y} = \infty$ and $\lim_{y \to \infty} \frac{f(y)}{y} = 0$,
the $(k, n-k)$-conjugate boundary value problem

$$\begin{aligned} &y^{(n)} = (-1)^{n-k} q(t) f(y) , \\ &y^{(j)}(a_1) = 0 , \quad 0 \leq j \leq k-1 , \\ &y^{(j)}(a_2) = 0 , \quad 0 \leq j \leq \ n-k-1 \quad . \end{aligned}$$

has at least one solution y which is positive on $]a_1, a_2[$.

Theorem 3.2 *Let $F : [a, b] \times \mathbf{R} \to [0, \infty[$ be a nonnegative continuous function which admits continuous functions $q : [a, b] \to [0, \infty[$ and $f : [a, b] \times \mathbf{R} \to [0, \infty[$ such that:*
(1) $q \neq 0$,
(2) $F(t, y) \geq q(t) f(t, y)$ *for all* $(t, y) \in [a, b] \times \mathbf{R}$, *and*
(3)

$$F_0 = \limsup_{y \to 0} \frac{F(t,y)}{|y|} \ \equiv \ 0 \ \equiv \ F_\infty = \limsup_{|y| \to \infty} \frac{F(t,y)}{|y|} \quad uniformly \ on \ t$$

(4) *For some $\delta \in \bigcup_{i=1}^{m-1}]a_i, a_{i+1}[$, $r > 0$ and a compact set $A \subset \bigcup_{i=1}^{m-1}]a_i, a_{i+1}[$; with positive measure, on which $q \neq 0$ and $\varphi \neq 1$ (which hold whenever $\mathrm{dist}(A, \{a_1, \ldots, a_m\})$ is sufficiently small), we have*

$$f(t, y) > \alpha r \quad for \ \ \sigma r \leq |y| \leq r \ \ and \ \ t \in A \subset [a, b] \ \ ;$$

where

$$\alpha = \Big(\int_A |G(\delta, s)| q(s) \, ds \Big)^{-1} \quad and \quad \sigma = \min_{t \in A} \varphi(t)$$

with G as the Green's function of the conjugate boundary value problem

$$\begin{aligned} &Lx = 0 , \\ &x^{(j)}(a_i) = 0 , \quad 1 \leq i \leq m , \quad 0 \leq j \leq k_i - 1 . \end{aligned}$$

If L is disconjugate, then the conjugate problem

$$\begin{aligned} &Ly = F(t, y) , \\ &y^{(j)}(a_i) = 0 , \quad 1 \leq i \leq m , \quad 0 \leq j \leq k_i - 1 , \end{aligned}$$

has at least two nontrivial solutions y_1 and y_2 with $||y_1||_\infty < r < ||y_2||_\infty$ such that each of the functions $t \mapsto y_l(t)/P(t)$, $t \notin \{a_1, \ldots, a_m\}$, $l = 1, 2$, has a continuous extention to $[a, b]$ with a positive infimum.

Proof:
Let consider (again) the cone $\mathcal{K}$ of $\mathcal{C}([a, b])$ defined by

$$\mathcal{K} = \{x \in \mathcal{C}([a, b]) : \ S_P(t) x(t) \geq \varphi(t) ||x||_\infty, \ a \leq t \leq b\}$$

and the completely continuous map $T : \mathcal{K} \to \mathcal{K}$ well defined for every $y \in \mathcal{K}$ by

$$Ty : \ t \mapsto \int_a^b G(t,s)F(s,y(s)) \ ds \ .$$

To complete the proof it suffices to show that there exist two positive real numbers r_1 and r_2 satisfying $r_1 < r < r_2$ and such that T is an expansion of $\{y \in \mathcal{K} : r_1 \le ||y||_\infty \le r\}$; i.e., T satisfies our condition (a) quoted in the proof of Theorem 3.1, and T is a compression of $\{y \in \mathcal{K} : r \le ||y||_\infty \le r_2\}$; i.e., T satisfies our condition (b) quoted in the proof of Theorem 3.1, according to the Krasnosel'skii theorem on the expansion and compression of cones. Indeed, fix $\epsilon > 0$ such that

$$\epsilon \max_{a \le t \le b} \int_a^b |G(t,s)| \ ds \ < \ 1 \ .$$

Therefore by assumption (3), as $F_0 = 0$, there exists $r_1 \in]0, r[$ such that $F(t,y) \le \epsilon |y|$ for all $t \in [a,b]$ and all $y \in [-r_1, r_1]$ so that

$$||Ty||_\infty < ||y||_\infty \quad \text{for} \ \ y \in \mathcal{K} \ \ \text{with} \ \ ||y||_\infty = r_1 \ .$$

Moreover, as $F_\infty = 0$, there exists $r_0 > r$ such that $F(t,y) \le \epsilon |y|$ for $a \le t \le b$ and $|y| \ge r_0$. Setting now

$$r_2 = r_0 + \big(\max\{F(t,y) : a \le t \le b, \ |y| \le r_0\}/\epsilon\big) \ ,$$

we have $F(t,y) \le \epsilon r_2$ for $t \in [a,b]$ and $y \in [-r_2, r_2]$ implying

$$||Ty||_\infty < ||y||_\infty \quad \text{for} \ \ y \in \mathcal{K} \ \ \text{with} \ \ ||y||_\infty = r_2 \ .$$

Furthermore, by assumption (4), we have for every $y \in \mathcal{K}$ with $||y||_\infty = r$,

$$\sigma r \le |y(t)| \le r \quad \text{for} \ \ t \in A$$

and

$$\begin{aligned} |Ty(\delta)| \ &= \textstyle\int_a^b |G(\delta,s)| F(s,y(s)) \ ds \\ &\ge \textstyle\int_A |G(\delta,s)| F(s,y(s)) \ ds \\ &\ge \textstyle\int_A |G(\delta,s)| q(s) f(s,y(s)) \ ds \\ &> \alpha r \textstyle\int_A |G(\delta,s)| q(s) \ ds \ = \ r \ . \end{aligned}$$

yielding $||Ty||_\infty > ||y||_\infty$. □

Example: Let $A \subset]a_1, a_2[$ be a closed nondegenerate interval such that $\sigma = \min_{t \in A} \varphi(t) < 1$ and set

$$\alpha = \Big(\max_{a \le t \le b} \int_A |G(t,s)| \ ds\Big)^{-1} \ .$$

Let also $\lambda \in]1, 2[$ and then consider the function

$$F(t,y) := \mu \frac{1+\sigma}{\sigma^\lambda} \cdot \frac{|y|^\lambda}{1+|y|} \quad \text{with} \ \ \mu > \alpha \ .$$

If L is disconjugate, then the conjugate problem

$$Ly = F(t,y) \ ,$$
$$y^{(j)}(a_i) = 0 \ , \quad 1 \le i \le m \ , \quad 0 \le j \le k_i - 1 \ ,$$

has at least two nontrivial solutions y_1 and y_2 with $||y_1||_\infty < 1 < ||y_2||_\infty$ such that each of the functions $t \mapsto y_l(t)/P(t)$, $t \notin \{a_1, \ldots, a_m\}$, $l = 1, 2$, has a continuous extension to $[a, b]$ with a positive infimum.

Theorem 3.3 *Let $F : [a,b] \times \mathbf{R} \to [0, \infty[$ be a nonnegative continuous function which admits continuous functions $q : [a,b] \to [0,\infty[$ and $f : [a,b] \times \mathbf{R} \to [0,\infty[$ such that:*
(1) $q \neq 0$,
(2) $F(t,y) \ge q(t) f(t,y)$ *for all* $(t,y) \in [a,b] \times \mathbf{R}$, *and*
(3)

$$f_0 = \limsup_{y \to 0} \frac{f(t,y)}{|y|} \ \equiv \ \infty \ \equiv \ f_\infty = \limsup_{|y| \to \infty} \frac{f(t,y)}{|y|} \qquad uniformly \ \ on \ \ t$$

(4) *There exists a positive real number η such that*

$$F(t,y) \ < \ \epsilon\eta \quad for \ \ |y| \le \eta \ \ and \ \ a \le t \le b \ \ ;$$

where

$$\epsilon = \Big(\max_{a \le t \le b} \int_a^b |G(t,s)| \ ds \Big)^{-1}$$

with G as the Green's function of the conjugate boundary value problem

$$Lx = 0 \ ,$$
$$x^{(j)}(a_i) = 0 \ , \quad 1 \le i \le m \ , \quad 0 \le j \le k_i - 1 \ .$$

If L is disconjugate, then the conjugate boundary value problem

$$Ly = F(t,y) \ ,$$
$$y^{(j)}(a_i) = 0 \ , \quad 1 \le i \le m \ , \quad 0 \le j \le k_i - 1 \ ,$$

has at least two nontrivial solutions y_1 and y_2 with $||y_1||_\infty < \eta < ||y_2||_\infty$ such that each of the functions $t \mapsto y_l(t)/P(t)$, $t \notin \{a_1, \ldots, a_m\}$, $l = 1, 2$, has a continuous extention to $[a, b]$ with a positive infimum.

Proof:
Consider the cone $\mathcal{K}$ of $C([a,b])$ defined by

$$\mathcal{K} = \{x \in C([a,b]) : \ S_P(t)x(t) \ge \varphi(t)||x||_\infty, \ a \le t \le b\}$$

and the completely continuous map $T : \mathcal{K} \to \mathcal{K}$ well defined for every $y \in \mathcal{K}$ by

$$Ty : \ t \mapsto \int_a^b G(t,s) F(s, y(s)) \ \ ds$$

We shall complete the proof by showing that there exist two positive real numbers η_1 and η_2 satisfying $\eta_1 < \eta < \eta_2$ and such that T is a compression of $\{y \in \mathcal{K} : \eta_1 \le ||y||_\infty \le \eta\}$; i.e.,

T satisfies our condition (b) quoted in the proof of Theorem 3.1, and T is an expansion of $\{y \in \mathcal{K} : \eta \leq ||y||_\infty \leq \eta_2\}$; i.e., T satisfies our condition (a) quoted in the proof of Theorem 3.1, according to the Krasnosel'skii theorem on the expansion and compression of cones. For this end, fix $\alpha > 0$, $\delta \notin \{a_1, \ldots, a_m\}$, and a compact set $A \subset \bigcup_{i=1}^{m-1}]a_i, a_{i+1}[$ with a positive measure, such that

$$\alpha \int_A |G(\delta, s)| q(s) \varphi(s) \, ds > 1 .$$

By assumption (3), as $f_0 = \infty$ on the one hand, there exists $\eta_1 \in]0, \eta[$ such that $f(t, y) \geq \alpha |y|$ for $t \in A$ and $y \in [-\eta_1, \eta_1]$. Therefore, given $y \in \mathcal{K}$ with $||y||_\infty = \eta_1$, we have

$$\begin{aligned} |Ty(\delta)| = \textstyle\int_a^b |G(\delta, s)| F(s, y(s)) \, ds \;&\geq\; \alpha \textstyle\int_A |G(\delta, s)| q(s) |y(s)| \, ds \\ &\geq\; \alpha ||y||_\infty \textstyle\int_A |G(\delta, s)| q(s) \varphi(s) \, ds \end{aligned}$$

which implies that $||Ty||_\infty > ||y||_\infty$; that is,

$$||Ty||_\infty > ||y||_\infty \quad \text{for} \quad y \in \mathcal{K} \quad \text{with} \quad ||y||_\infty = \eta_1 .$$

As $f_\infty = \infty$ on the other hand, there exists $\eta_0 > 0$ such that $f(t, y) \geq \alpha |y|$ for $t \in A$ and $|y| \geq \eta_0$. Set $\eta_2 = \eta + (\eta_0 / \min\{\varphi(s) : s \in A\})$ so that $\eta_2 > \eta$ and $\eta_2 \min\{\varphi(s) : s \in A\} \geq \eta_0$. Thus, given $y \in \mathcal{K}$ with $||y||_\infty = \eta_2$, we have

$$|y(t)| \geq \varphi(t) ||y||_\infty \geq \eta_0 \quad \text{for} \quad t \in A ,$$

that implies $f(t, y(t)) \geq \alpha |y(t)|$, $t \in A$, and so

$$|Ty(\delta)| = \int_a^b |G(\delta, s)| F(s, y(s)) \, ds \geq \alpha ||y||_\infty \int_A |G(\delta, s)| q(s) \varphi(s) \, ds > ||y||_\infty$$

yielding

$$||Ty||_\infty > ||y||_\infty \quad \text{for} \quad y \in \mathcal{K} \quad \text{with} \quad ||y||_\infty = \eta_2 .$$

Finally, it is clear by assumption (4) that

$$||Ty||_\infty < ||y||_\infty \quad \text{for} \quad y \in \mathcal{K} \quad \text{with} \quad ||y||_\infty = \eta . \; \square$$

Example: Let $\lambda > 1$, set

$$\epsilon = \Big(\max_{a \leq t \leq b} \int_a^b |G(t, s)| \, ds \Big)^{-1} ,$$

and consider

$$F(t, y) := \mu |1 - |y||^\lambda \quad \text{with} \quad 0 < \mu < \epsilon .$$

If L is disconjugate, then the conjugate problem

$$\begin{aligned} &Ly = F(t, y) , \\ &y^{(j)}(a_i) = 0 , \quad 1 \leq i \leq m , \quad 0 \leq j \leq k_i - 1 , \end{aligned}$$

has at least two nontrivial solutions y_1 and y_2 with $||y_1||_\infty < 1 < ||y_2||_\infty$ such that each of the functions $t \mapsto y_l(t)/P(t)$, $t \notin \{a_1, \ldots, a_m\}$, $l = 1, 2$, has a continuous extension to $[a, b]$

with a positive infimum.

As an immediate consequence of Theorem 3.3 combined with its proof (above), we have

Corollary 3.3 *Let $q : [a,b] \to [0,\infty[$ and $f : \mathbf{R} \to [0,\infty[$ be nonnegative continuous functions such that:*

(1) $q \neq 0$,

(2)

$$f_0 = \limsup_{y \to 0} \frac{f(y)}{|y|} \;=\; \infty \;=\; f_\infty = \limsup_{|y| \to \infty} \frac{f(y)}{|y|} \quad , \quad \textit{and}$$

(3) *there exists a positive real number η such that*

$$f(y) < \epsilon\eta \quad \textit{for} \;\; |y| \leq \eta \;\; ;$$

where

$$\epsilon = \Big(\max_{a \leq t \leq b} \int_a^b |G(t,s)| q(s) \; ds \Big)^{-1}$$

with G as the Green's function of the conjugate boundary value problem

$$\begin{aligned} &Lx = 0 \; , \\ &x^{(j)}(a_i) = 0 \; , \quad 1 \leq i \leq m \; , \quad 0 \leq j \leq k_i - 1 \; . \end{aligned}$$

If L is disconjugate, then the conjugate problem

$$\begin{aligned} &Ly = q(t) f(y) \; , \\ &y^{(j)}(a_i) = 0 \; , \quad 1 \leq i \leq m \; , \quad 0 \leq j \leq k_i - 1 \; , \end{aligned}$$

has at least two nontrivial solutions y_1 and y_2 with $||y_1||_\infty < \eta < ||y_2||_\infty$ such that each of the functions $t \mapsto y_l(t)/P(t)$, $t \notin \{a_1, \ldots, a_m\}$, $l = 1, 2$, has a continuous extension to $[a,b]$ with a positive infimum.

Corollary 3.4 *Let $f : \mathbf{R} \to [0,\infty[$ be a nonnegative continuous function which is nondecreasing on $[0,\infty[$ (e.g., $y \mapsto e^{\lambda|y|}$ with $\lambda > 0$, $y \mapsto |y|^\gamma + |y|^\mu$ with $0 < \gamma < 1 < \mu$) and assume that $q : [a,b] \to [0,\infty[$ is a nonnegative continuous function, such that*

(1) $q \neq 0$,

(2) $f(y) \leq f(|y|)$ *for all* $y \in \mathbf{R}$,

(3)

$$f_0 = \limsup_{y \to 0} \frac{f(y)}{|y|} \;=\; \infty \;=\; f_\infty = \limsup_{|y| \to \infty} \frac{f(y)}{|y|} \quad , \quad \textit{and}$$

(4)

$$\min_{y > 0} \frac{f(y)}{y} \; < \; \Big(\max_{a \leq t \leq b} \int_a^b |G(t,s)| q(s) \; ds \Big)^{-1} \;\; ;$$

where G is the Green's function of the conjugate boundary value problem

$$\begin{aligned} &Lx = 0 \; , \\ &x^{(j)}(a_i) = 0 \; , \quad 1 \leq i \leq m \; , \quad 0 \leq j \leq k_i - 1 \; . \end{aligned}$$

If L is disconjugate, then the conjugate problem

$$\begin{aligned} &Ly = q(t)f(y)\ , \\ &y^{(j)}(a_i) = 0\ ,\quad 1 \le i \le m\ ,\quad 0 \le j \le k_i - 1\ , \end{aligned}$$

has at least two nontrivial solutions y_1 and y_2 such that the functions $t \mapsto y_l(t)/P(t)$, $t \notin \{a_1, \ldots, a_m\}$, $l = 1, 2$, have each a continuous extension to $[a,b]$ with a positive infimum.

Proof:
On the basis of the assumptions on f, the function

$$y \ \mapsto\ \frac{f(y)}{y}$$

defined from $]0,\infty[$ into $[0,\infty[$ is continuous and admits a (minimum) point $\eta > 0$ such that

$$\frac{f(\eta)}{\eta} < \epsilon = \Big(\max_{a\le t\le b} \int_a^b |G(t,s)| q(s)\ ds\Big)^{-1}\ .$$

Thus

$$f(y) \le f(|y|) \le f(\eta) < \ \epsilon\eta \quad for\ \ |y| \le \eta\ .$$

Therefore Corollary 3.3 yields the result. □

In particular, Corollary 3.4 implies that for a positive interger k such that $1 \le k \le n-1$ and for given continuous functions $q : [a_1, a_2] \to [0,\infty[$ with $q \neq 0$, and $f : [0,\infty[\to [0,\infty[$ with f nondecreasing (e.g., $y \mapsto e^{\lambda y}$ with $\lambda > 0$, $y \mapsto y^\gamma + y^\mu$ with $0 < \gamma < 1 < \mu$), such that

$$f_0 = \limsup_{y\to 0^+} \frac{f(y)}{y} \ = \infty \ = \ f_\infty = \limsup_{y\to\infty} \frac{f(y)}{y} \quad , \ \ \mathit{and}$$

$$\min_{y>0} \frac{f(y)}{y} \ < \ \Big(\max_{a_1\le t\le a_2} \int_{a_1}^{a_2} |g(t,s)| q(s)\ ds\Big)^{-1}\ ;$$

where g is the Green's function of the conjugate boundary value problem

$$\begin{aligned} &x^{(n)} = 0\ , \\ &x^{(j)}(a_1) = 0\ ,\quad 0 \le j \le k-1\ , \\ &x^{(j)}(a_2) = 0\ ,\quad 0 \le j \le n-k-1\ , \end{aligned}$$

the $(k, n-k)$-conjugate boundary value problem

$$\begin{aligned} &y^{(n)} = (-1)^{n-k} q(t) f(y)\ , \\ &y^{(j)}(a_1) = 0\ ,\quad 0 \le j \le k-1\ , \\ &y^{(j)}(a_2) = 0\ ,\quad 0 \le j \le n-k-1\ , \end{aligned}$$

has at least two solutions which are positive on $]a_1, a_2[$, in agreement with [2] and [5, Example 3.2].

3.2. From the comparison of the eigenvalues

In this section we apply the previous results to the search of strictly positive solutions, in the order defined by $\mathcal{K}$, of nonlinear BVPs.

Theorem 3.4 *Assume that L is disconjugate and that $F : [a,b] \times \mathbf{R} \to \mathbf{R}$ is a continuous function such that*
(1) $F(t, S_P(t)y) \geq 0 \ , \ a \leq t \leq b \ , \ y \geq 0 \ ;$
(2) $F(t,0) = 0 \ , \ a \leq t \leq b \ ;$
(3) *The function $y \mapsto F(t,y)$ is differentiable at 0 with*

$$\lim_{y\to 0} \frac{F(t,y)}{y} = F_y(t,0) =: q(t) \quad uniformly \ for \ \ t \in [a,b]$$

and is such that $L - q$ is disconjugate;
(4)

$$\lim_{y\to+\infty} \frac{F(t, S_P(t)y)}{y} \equiv +\infty \ \ uniformly \ for \ t \ in \ a \ subset \ of \ [a,b] \ with \ positive \ measure.$$

Then the conjugate BVP

$$\begin{aligned} &Ly = F(t,y) \ , \\ &y^{(j)}(a_i) = 0 \ , \quad 1 \leq i \leq m \ , \quad 0 \leq j \leq k_i - 1 \ , \end{aligned}$$

has at least one nontrivial solution such that the quotient $y(t)/P(t)$ has a continuous extention to $[a,b]$ with positive infimum.

Proof:
The nonlinear problem

$$\begin{aligned} &Ly = F(t,y) \ , \\ &y^{(j)}(a_i) = 0 \ , \quad 1 \leq i \leq m \ , \quad 0 \leq j \leq k_i - 1 \ , \end{aligned}$$

is equivalent to

$$y(t) = \int_a^b G(t,s) F(s, y(s)) \, ds \ , \quad a \leq t \leq b \ ,$$

and we plan to seek for a solution y such that $S_P(t)y(t) \geq 0$ which will yield the result. Recall, for later use, that by virtue of the maximum principle for multipoint boundary value problems, Theorem 1.1, there exists $\varphi \in \mathcal{C}([a,b])$ positive on $\bigcup_{i=1}^{m-1}]a_i, a_{i+1}[$ and such that for every $y \in \mathcal{C}^n([a,b])$ satisfying the differential inequality $Ly \geq 0$ and the homogeneous Hermite boundary condition

$$y^{(j)}(a_i) = 0 \ , \quad 1 \leq i \leq m \ , \quad 0 \leq j \leq k_i - 1 \ ,$$

we have

$$S_P(t)y(t) \geq \varphi(t) \|y\|_\infty \ , \quad a \leq t \leq b \ .$$

Now let

$$Y = \{y \in \mathcal{C}[a,b] : \ y(a_1) = \cdots = y(a_m) = 0\}$$

$$\mathcal{K}_\varphi = \{y \in Y : \ S_P(t)y(t) \geq \varphi(t)||y||_\infty \ , \quad a \leq t \leq b\}$$
$$\mathcal{K}_0 = \{y \in Y : \ S_P(t)y(t) \geq 0 \ , \quad a \leq t \leq b\}$$

and note at once that the sup-norm $|| \ ||_\infty$ is monotone (increasing) in $\mathcal{K}_\varphi$ and $\mathcal{K}_0$ with respect to their corresponding orders.
Consider moreover the integral operator A defined for every $y \in \mathcal{K}_0$ by

$$Ay : \ t \mapsto \int_a^b G(t,s)F(s,y(s)) \ ds \ .$$

Then A maps $\mathcal{K}_0$ into $\mathcal{K}_\varphi$ since given any $y \in \mathcal{K}_0$ and putting $z = Ay$, we have via the property of the Green's function G that

$$Lz = F(t,y(t)) = F(t, S_P(t)|y(t)|) \ \geq 0 \ ,$$
$$z^{(j)}(a_i) = 0 \ , \quad 1 \leq i \leq m \ , \quad 0 \leq j \leq k_i - 1 \ ,$$

which implies that

$$S_P(t)z(t) \geq \varphi(t)||z||_\infty \ , \quad a \leq t \leq b \ .$$

In particular A maps $\mathcal{K}_\varphi$ into itself. Since furthermore A is completely continuous, to conclude that $A : \mathcal{K}_\varphi \to \mathcal{K}_\varphi$ has a fixed point with a positive norm, it suffices to show the existence of two positive real numbers $\epsilon < R$ such that:

(i) if $y \in \mathcal{K}_\varphi$ and $||y||_\infty = \epsilon$, then $Ay - y \notin \mathcal{K}_\varphi$, while
(ii) if $y \in \mathcal{K}_\varphi$ and $||y||_\infty = R$, then $y - Ay \notin \mathcal{K}_\varphi$

as shown by the Krasnosel'skii theorem on the expansion and compression of cones in the version of Theorem 0.5.
Indeed, on the one hand, by assumption (3) above, A is Frechet differentiable at 0 with $A'(0) = T_q =: T$ (for simplicity). If $q = 0$, then $r(T) = 0$. If otherwise $q \neq 0$, then using the above assumptions (1) and (3), we have $r(T) = \frac{1}{\lambda_1(q)} < 1$ by Theorem 2-1 and Theorem 2.2-b. Thus $r(T) < 1$ in any case, hence $I - T$ is invertible. Now suppose by the way of contradiction that (i) is not satisfied for any $\epsilon > 0$. Then there would exist sequences $\epsilon_l \downarrow 0$ and $y_l \in \mathcal{K}_\varphi$ such that $||y_l||_\infty = \epsilon_l$ whereas $Ay_l - y_l \in \mathcal{K}_\varphi$. Putting $z_l = y_l/\epsilon_l$ (so that $||z_l|| = 1$ for all l) and $My_l = (Ay_l - Ty_l)/\epsilon_l$, we would then have

$$My_l - (I-T)z_l \ \in \ \mathcal{K}_\varphi \subset \mathcal{K}_0 \ ,$$

which, by the positivity of $(I-T)^{-1} = I + T + T^2 + \cdots$, would imply

$$(I-T)^{-1}My_l - z_l \in \mathcal{K}_0 \ , \quad \text{already with} \ \ z_l \in \mathcal{K}_0 \ \ ,$$

and so for all l we would get, by the monotonicity of the sup norm,

$$||(I-T)^{-1}|| \cdot ||My_l||_\infty \ \geq \ ||(I-T)^{-1}My_l||_\infty \ \geq \ ||z_l||_\infty \ = \ 1$$

contradicting the fact that $\lim_l ||My_l||_\infty = 0$ since $T = A'(0)$. Hence (i) holds for some $\epsilon > 0$ that we fix for the sequel.
On the other hand, by assumption (4) above, there exists a compact set $\Omega \subset [a,b]$ with positive measure such that

$$\lim_{y \to +\infty} \frac{F(t, S_P(t)y)}{y} \equiv +\infty \ \text{ uniformly for } \ t \in \Omega$$

and moreover, $a_j \notin \Omega$ for $j = 1, \cdots, m$. Now choose $\delta \notin \{a_1, \cdots, a_m\}$ and $\alpha > 0$ such that

$$\alpha \int_\Omega |G(\delta, s)|\varphi(s)\, ds \ > \ 1$$

(this is possible since $|G(\delta, s)|\varphi(s) > 0$ for $s \in \Omega$). Therefore, there exists $R_0 > 0$ such that

$$F(t, S_P(t)y) \ \geq \ \alpha y \ , \ \ t \in \Omega \ , \ \ y \geq R_0 \ .$$

Set now

$$R = \epsilon + \frac{R_0}{\inf_\Omega \varphi} \ .$$

Hence, if $y \in \mathcal{K}_\varphi$ and $||y||_\infty = R$, then

$$|y(t)| \ \geq \ \varphi(t)||y||_\infty \ \geq \ R_0 \ , \ \ t \in \Omega$$

which implies

$$F(t, y(t)) \ \geq \ \alpha\varphi(t)||y||_\infty \ , \ \ t \in \Omega \ .$$

Therefore

$$\begin{aligned} |Ay(\delta)| \ = \ S_P(\delta)Ay(\delta) \ &= \ \int_a^b |G(\delta, s)|F(s, y(s))\, ds \\ &\geq \ \int_\Omega |G(\delta, s)|F(s, y(s))\, ds \\ &\geq \ \left(\alpha \int_\Omega |G(\delta, s)|\varphi(s)\, ds \right) ||y||_\infty \end{aligned}$$

and so

$$||Ay||_\infty \ > \ ||y||_\infty$$

implying in its turn that $y - Ay \notin \mathcal{K}_\varphi$ and showing (ii). $\square$

As a consequence we have

Corollary. *Assume that L is disconjugate and let $q : [a, b] \to \mathbf{R}$ and $h : \mathbf{R} \to [0, \infty[$ be continuous functions such that*

(1) $q \neq 0$, $S_P(t)q(t) \geq 0$ *and* $L - q$ *is disconjugate;*

(2) $h(0) = 1$ *and* $\lim_{|y| \to +\infty} h(y) = +\infty$.

Then the nonlinear problem

$$\begin{aligned} &Ly = q(t)h(y)y \ , \\ &y^{(j)}(a_i) = 0 \ , \ \ 1 \leq i \leq m \ , \ \ 0 \leq j \leq k_i - 1 \ , \end{aligned}$$

has at least one nontrivial solution y such that the function $t \mapsto y(t)/P(t)$, $t \notin \{a_1, \cdots, a_m\}$, has a continuous extention to $[a, b]$ with positive infimum.

Theorem 3.5 *Assume that L is disconjugate and let $F : [a,b] \times \mathbf{R} \to \mathbf{R}$ be a continuous function such that*
(1) $F(t, S_P(t)y) \geq 0 \ , \quad a \leq t \leq b \ , \ y \geq 0 \ ;$
(2) *there exists the limit*

$$\lim_{|y| \to +\infty} \frac{F(t,y)}{y} =: q(t) \ \ uniformly \ for \ t \in [a,b]$$

with $L - q$ disconjugate;
(3) *we have*

$$\lim_{y \to 0^+} \frac{F(t, S_P(t)y)}{y} \equiv +\infty \ \ uniformly \ for \ t \ in \ a \ subset \ of \ [a,b] \ with \ positive \ measure.$$

Then the nonlinear BVP

$$\begin{aligned} &Ly = F(t,y) \ , \\ &y^{(j)}(a_i) = 0 \ , \quad 1 \leq i \leq m \ , \quad 0 \leq j \leq k_i - 1 \ , \end{aligned}$$

has a least one nontrivial solution y such that the quotient $y(t)/P(t)$ has a continuous extention to $[a,b]$ with positive infimum.

Proof:
Let $\mathcal{K}_\varphi$ be as in the proof of Theorem 3.1 and consider the operator A assigning to every $y \in \mathcal{K}_\varphi$, the function Ay defined by

$$Ay : \ t \mapsto \int_a^b G(t,s) F(s, y(s)) \ ds$$

which belongs to $\mathcal{K}_\varphi$. It suffices to show that this completely continuous operator $A : \mathcal{K}_\varphi \to \mathcal{K}_\varphi$ has a nonzero fixed point y yielding that the corresponding quotient $y(t)/P(t)$ has a continuous extension to $[a,b]$ with positive infimum by the property of the Green's function G.
For this end, we prove that there exist two positive real numbers $\epsilon < R$ such that:

(i) if $y \in \mathcal{K}_\varphi$ and $||y||_\infty = \epsilon$, then $y - Ay \notin \mathcal{K}_\varphi$, while
(ii) if $y \in \mathcal{K}_\varphi$ and $||y||_\infty = R$, then $Ay - y \notin \mathcal{K}_\varphi$

and we apply the Krasnosel'skii theorem on the expansion and compression of cones in the version of Theorem 0.5.
(i) By assumption (3), there exists a compact set $\Omega \subset \cup_{i=1}^{m-1}]a_i, a_{i+1}[$ with positive measure such that

$$\lim_{y \to 0^+} \frac{F(t, S_P(t)y)}{y} \equiv +\infty \ \text{ uniformly for } \ t \in \Omega \ .$$

Now choose $\delta \notin \{a_1, \cdots, a_m\}$ and $\alpha > 0$ such that

$$\alpha \int_\Omega |G(\delta, s)| \varphi(s) \ ds \ > 1 \ .$$

Therefore, there exists $\epsilon > 0$ such that

$$F(t, S_P(t)y) \geq \ \alpha y \ , \quad t \in \Omega \ , \quad 0 \leq y \leq \epsilon \ .$$

Thus, if $y \in \mathcal{K}_\varphi$ with $||y||_\infty = \epsilon$ we have:

$$\begin{aligned} |Ay(\delta)| &= S_P(\delta) \int_a^b G(\delta,s)F(s,y(s))\ ds &= \int_a^b |G(\delta,s)|F(s,S_P(s)|y(s)|)\ ds \\ & &\geq \alpha \int_a^b |G(\delta,s)| \cdot |y(s)|\ ds \\ & &\geq \alpha \int_a^b |G(\delta,s)|\varphi(s)||y||_\infty\ ds \\ & &> ||y||_\infty \end{aligned}$$

which implies $||Ay||_\infty > ||y||_\infty$, and then $y - Ay \notin \mathcal{K}_\varphi$.
(ii) Suppose by contradiction that there does not exist any positive real number $R > \epsilon$ such that $Ay - y \notin \mathcal{K}_\varphi$ for every $y \in \mathcal{K}_\varphi$ satisfying $||y||_\infty = R$. Then, there would exist a sequence $(y_l)_l$ of elements of $\mathcal{K}_\varphi \setminus \{0\}$ such that $||y_l|| \uparrow \infty$ and $Ay_l - y_l \in \mathcal{K}_\varphi$.
Considering $A'(\infty) = T_q =: T$ that exists and is defined (according to assumption (2)) by

$$Tx(t) = \int_a^b G(t,s)q(s)x(s)\ ds\ , \quad a \leq t \leq b\ ,$$

we observe that if $q = 0$, then $r(T) = 0$, while if otherwise $q \neq 0$, then using the assumptions (1) and (2), we have $r(T) = \frac{1}{\lambda_1(q)} < 1$ as a result of Theorem 2.2-b. Thus $r(T) < 1$ in any case so that the inverse of $I - T$, $(I-T)^{-1}$, exists and is positive. It follows that

$$(Ay_l - Ty_l) + (Ty_l - y_l) = Ay_l - y_l\ \in\ \mathcal{K}_\varphi$$

and so

$$My_l + (Tz_l - z_l)\ \in\ \mathcal{K}_\varphi\ ;$$

where

$$z_l = \frac{y_l}{||y_l||_\infty} \quad \text{and} \quad My_l = \frac{(Ay_l - Ty_l)}{||y_l||_\infty}$$

which implies that

$$(I-T)^{-1}My_l - z_l\ \in\ \mathcal{K}_\varphi\ , \quad \text{already with}\ \ z_l \in \mathcal{K}_\varphi\ .$$

Consequently, by the monotonicity of the sup norm, we have

$$||(I-T)^{-1}|| \cdot ||My_l||_\infty\ \geq\ ||(I-T)^{-1}My_l||_\infty\ \geq\ ||z_l||_\infty\ =\ 1\ , \quad \text{for all}\ \ l\ ,$$

on contrast to the fact that $\lim_l ||My_l||_\infty = 0$ since $T = A'(\infty)$. □

§4. Application to uniqueness and existence for nonlinear problem below the principal eigenvalue

From the comparison result in §2, we deduce in this section a uniqueness and an existence theorem for nonlinear BVPs.

Theorem 4.1 *Assume that $f : [a,b] \times \mathbf{R} \to \mathbf{R}$ satisfies the generalized Caratheodory conditions and that $f_x = \frac{\partial}{\partial x} f$ exists and satisfies the generalized Carathedory conditions. If there exist $p, q \in \mathbf{L}^1$ such that*

(a) *p and q do not vanish in sets of positive measures;*

(b)

$$0 \le S_P(t) f_x(t,x) \le S_P(t)p(t) \le S_P(t)q(t)\lambda_1(q)$$

for a.e. t and all x, $\lambda_1(q)$ being the principal eigenvalue of (P) ;

(c) *$p \ne \lambda_1(q)q$ in a set of positive measure;*

then the conjugate BVP

$$\begin{aligned} &Lx = f(t,x) \ , \\ &x^{(j)}(a_i) = A_{ij} \ , \quad 1 \le i \le m \ , \quad 0 \le j \le k_i - 1 \end{aligned}$$

has at most one solution whenever $A_{ij} \in \mathbf{R}$.

Proof:
Assume the existence of two different solutions x and y of the same conjugate BVP, and argue for a contradiction. Setting

$$Q(t) = \int_0^1 f_x(t, y(t) + \xi(x(t) - y(t)))\, d\xi \ ,$$

we have

$$f(t,x(t)) - f(t,y(t)) = Q(t) \cdot (x(t) - y(t)) \ .$$

It follows that the function $u = x - y$ is an eigenfunction corresponding to $\lambda = 1$ of the eigenvalue problem

$$\begin{aligned} &Lu = \lambda Q(t) u \ , \\ &u^{(j)}(a_i) = 0 \quad \text{for } 1 \le i \le m \ , \quad 0 \le j \le k_i - 1 \ . \end{aligned} \tag{4.1}$$

Clearly we have $S_P(t)Q(t) \le S_P(t)q(t)\lambda_1(q)$ a.e. and $Q(t) \ne q(t)\lambda_1(q)$ in a set of positive measure. Then an application of Theorem 2.2-a to the eigenvalue problems (4.1) and to

$$\begin{aligned} &Ly = \lambda[\lambda_1(q)q(t)]y \ , \\ &y^{(j)}(a_i) = 0 \quad \text{for } 1 \le i \le m \ , \quad 0 \le j \le k_i - 1 \end{aligned}$$

provides

$$\lambda_1(Q) > \lambda_1(\lambda_1(q)q) = 1 \ . \tag{4.2}$$

But then we have a contradiction: since the principal eigenvalue is less than or equal to the absolute value of any other eigenvalue of the same BVP, and since $\lambda = 1$ is an eigenvalue of (4.1), we should have $\lambda_1(Q) \leq 1$, contradicting (4.2). □

Theorem 4.2 *Assume that $f : [a,b] \times \mathbf{R} \to \mathbf{R}$ satisfies the generalized Caratheodory conditions and that $f_x = \frac{\partial}{\partial x} f$ exists and satisfies the generalized Carathedory conditions. If there exist $p, q \in \mathbf{L}^1$ and $R > 0$ such that*

(a) *p and q do not vanish in sets of positive measures;*

(b)
$$0 \;\leq\; S_P(t) f_x(t,x) \;\leq\; S_P(t) p(t) \;\leq\; S_P(t) q(t) \lambda_1(q)$$
for a.e. t and $|x| \geq R$, $\lambda_1(q)$ being the principal eigenvalue of (P) ;

(c) *$p \neq \lambda_1(q) q$ in a set of positive measure;*

then for every $g : [a,b] \times \mathbf{R} \to \mathbf{R}$ satisfying the generalized Caratheodory conditions and

$$\lim_{|x| \to \infty} \frac{g(t,x)}{|x|} = 0 \ ,$$

the conjugate BVP

$$\begin{array}{l} Lx = f(t,x) + g(t,x) \ , \\ x^{(j)}(a_i) = 0 \ , \quad 1 \leq i \leq m \ , \quad 0 \leq j \leq k_i - 1 \end{array}$$

has at least one solution.

Proof:
Applying Theorem 2.2-b to the eigenvalue problems

$$\left\{ \begin{array}{l} Ly = \lambda p(t) y \\ y^{(j)}(a_i) = 0 \quad \text{for } \ 1 \leq i \leq m \ , \quad 0 \leq j \leq k_i - 1 \end{array} \right. ,$$

$$\left\{ \begin{array}{l} Ly = \lambda [\lambda_1(q) q(t)] y \\ y^{(j)}(a_i) = 0 \quad \text{for } \ 1 \leq i \leq m \ , \quad 0 \leq j \leq k_i - 1 \end{array} \right.$$

we see that

$$\lambda_1(p) \;>\; \lambda_1(\lambda_1(q) q) \;=\; 1 \ . \tag{4.3}$$

Let $p_k = p + \frac{P}{k}$. We claim the existence of $k_0 \geq 1$ such that

$$\lambda_1(p_{k_0}) \;>\; 1 \ . \tag{4.4}$$

Otherwise, we have $\lambda_1(p_k) \leq 1$ for all k. Since $\lambda_1(p_k)$ is an increasing sequence by Theorem 2.2-a, we have

$$\lim_k \lambda_1(p_k) \;=\; \lambda_\infty \;\leq\; 1 \ .$$

Let $x_k \in \mathcal{K}$ be an eigenfunction corresponding to $\lambda_1(p_k)$ with norm 1. From

$$x_k(t) \;=\; \lambda_1(p_k) \int_a^b G(t,s)p_k(s)x_k(s)\;ds \quad , \quad a \le t \le b \tag{4.5}$$

and Ascoli Theorem, we get the existence of a subsequence $x_{k_l} \to x_\infty$ uniformly. Taking limits in (4.5) for $k = k_l$ we obtain

$$x_\infty(t) \;=\; \lambda_\infty \int_a^b G(t,s)p(s)x_\infty(s)\;ds \quad , \quad a \le t \le b \;.$$

Therefore x_∞ is an eigenfunction in $\text{int}(\mathcal{K})$ of

$$\begin{aligned} &Lx = \lambda_\infty p(t)x \\ &x^{(j)}(a_i) = 0 \quad \text{for} \;\; 1 \le i \le m \;, \;\; 0 \le j \le k_i - 1 \end{aligned}$$

Claim: The principal eigenvalue $\lambda_1(p)$ of (P) with $q = p$ is the only positive eigenvalue having eigenfunctions in $\text{int}(\mathcal{K})$ despite the fact that T_p need not be strongly positive.
Assume for a moment that this claim holds. Then $\lambda_\infty = \lambda_1(p)$ contradicting (4.3). Thus (4.4) holds.
Now we define

$$q(t,x) = \int_0^1 f_x(t,\xi x)\;d\xi$$

and the given equation can be rewritten as

$$Lx \;=\; q(t,x)x + f(t,0) + g(t,x) \;\;.$$

Clearly $S_P(t)q(t,x) \ge 0$ for a.e. t and all x. Let $|x| \ge R$. From

$$q(t,x) \;=\; \int_0^{R/|x|} f_x(t,\xi x)\;d\xi \;+\; \int_{R/|x|}^1 f_x(t,\xi x)\;d\xi$$

and from (b), we deduce the existence of $r \ge R$ such that

$$0 \;\le\; S_P(t)q(t,x) \;\le\; S_P(t)p_{k_0}(t)$$

for a.e. t and $|x| \ge r$. Now fix any $h \in \mathbf{L}^1$ such that $|h(t)| \ge r$ a.e. By the above, we have

$$0 \;\le\; S_P(t)q(t,h(t)) \;\le\; S_P(t)p_{k_0}(t)$$

a.e., hence Theorem 2.2-a implies

$$\lambda_1(q(\,\cdot\,,h(\,\cdot\,))) \;\ge\; \lambda_1(p_{k_0}) \;>\; 1 \;\;.$$

Then

$$\begin{aligned} &Lx = q(t,h(t))x \\ &x^{(j)}(a_i) = 0 \quad \text{for} \;\; 1 \le i \le m \;, \;\; 0 \le j \le k_i - 1 \end{aligned}$$

has only the trivial solution. It follows that we can apply Theorem 1 and Example 2 of Vidossich [22] by considering the first order system in $\mathbf{R}^n$ equivalent to the given equation $Lx = f(t,x) + g(t,x)$, letting X be the set corresponding in this equivalence to

$$Y = \left\{ h \in \mathbf{L}^1 \,:\; 0 \le S_P(t)h(t) \le S_P(t)p_{k_0} \;\text{a.e.} \right\} \;,$$

and obtain the existence of at least one solution to the given BVP.
Proof of the claim: Suppose that $\alpha > 0$ is an eigenvalue of T_p having an eigenfunction $x \in \text{int}(\mathcal{K})$. We show that $\alpha = r(T_p)$. Since it is evident that $r(T_p) \geq \alpha$ by the monotonicity of the sup norm in $\mathcal{K}$ (see Theorem 0.4), we need to prove that $r(T_p) \leq \alpha$. To this end, let $y \in \text{int}(\mathcal{K})$ be an eigenfunction corresponding to $r(T_p)$ provided by Theorem 2.1. Now set

$$t_0 = \max\{t \geq 0 : \ ty \preceq x\} \tag{4.6}$$

noting at once that

$$0 \ < \ t_0 \ \leq \ \frac{||x||_\infty}{||y||_\infty} \quad .$$

since $x \in \text{int}(\mathcal{K})$ and $\mathcal{K}$ is normal, and put $z = t_0 y - x \preceq 0$. Therefore

$$Tz \ = \ t_0 r(T_p)y - \alpha x \ \preceq \ 0$$

which implies $\frac{r(T_p)}{\alpha} \leq 1$ by (4.6). □

§5. Some bifurcation problem

In this section we make use of the results of §3 to show how our theorems may enable to solve bifurcation problems. The assumptions about f in the next theorem are fulfilled by $y \mapsto e^{\sigma|y|}$ with $\sigma > 0$, and $y \mapsto |y|^\gamma + |y|^\mu$ with $0 < \gamma < 1 < \mu$.

Theorem 5.1 *Let $f : \mathbf{R} \to [0, \infty[$ be a nonnegative continuous function which is nondecreasing on $[0, \infty[$ and assume that $q : [a, b] \to [0, \infty[$ is a nonnegative continuous function, such that*
(1) $q \neq 0$,
(2) $f(y) \le f(|y|)$ *for all* $y \in \mathbf{R}$,
(3)

$$f_0 = \limsup_{y \to 0} \frac{f(y)}{|y|} = \infty = f_\infty = \limsup_{|y| \to \infty} \frac{f(y)}{|y|} .$$

Then there are

$$\lambda^* \ge \lambda_* > 0$$

such that the boundary value problem

$$Ly = \lambda q(t) f(y) ,$$
$$y^{(j)}(a_i) = 0 , \quad 1 \le i \le m , \quad 0 \le j \le k_i - 1$$

has:

(i) *at least two nontrivial solutions y_1 and y_2 with*

$$S_P(t) y_2(t) \ge \varphi(t) S_P(t) y_1(t) \ge \varphi^2(t) ||y_1||_\infty , \quad a \le t \le b ,$$

for $0 < \lambda < \lambda_*$,

(ii) *at least one nontrivial solution y with*

$$S_P(t) y(t) \ge \varphi(t) ||y||_\infty , \quad a \le t \le b ,$$

for $\lambda_* \le \lambda \le \lambda^*$, *and*

(iii) *no solution for* $\lambda > \lambda^*$.

Proof:
We first highlight that such a function f must be positive on $\mathbf{R}$.
Let define for each $\lambda > 0$, the function $f_\lambda : \mathbf{R} \to [0, \infty[$ by

$$f_\lambda : y \mapsto \lambda f(y) .$$

Let $\eta_0 > 0$ be a minimum point of f, whose existence is guaranted by the continuity of f and the assumption (3). Therefore

$$\frac{f(\eta_0)}{\eta_0} \ge \min \left\{ \frac{f(y)}{|y|} : y \neq 0 \right\} > 0 .$$

Set

$$\lambda_0 \;=\; \frac{\eta_0}{f(\eta_0)} \cdot \Big(\max_{a \le t \le b} \int_a^b |G(t,s)| q(s) \; ds \Big)^{-1} \quad .$$

Then for every $\lambda \in]0, \lambda_0[$, the BVPs

$$\begin{aligned} & Ly = \lambda q(t) f(y) \;, \\ & y^{(j)}(a_i) = 0 \;, \quad 1 \le i \le m \;, \quad 0 \le j \le k_i - 1 \end{aligned} \qquad (P_\lambda)$$

has at least two nontrivial solutions $y_{\lambda,1}$ and $y_{\lambda,2}$ such that

$$||y_{\lambda,1}||_\infty \;<\; \eta_0 \;<\; ||y_{\lambda,2}||_\infty$$

by Theorem 3.3 (cf. also Corollary 3.4).
Now set

$$\Gamma_* \;=\; \{\lambda > 0 : \; (P_\lambda) \text{ has at least two nontrivial solutions}\}$$

and

$$\Gamma^* \;=\; \{\lambda > 0 : \; (P_\lambda) \text{ has a nontrivial solution}\} \quad .$$

Claim 1: The nonempty set $\Gamma^* \supset \Gamma_*$ is bounded.
Indeed let $\lambda \in \Gamma^*$, then there exists y_λ satisfying (P_λ) (as well as $Ly_\lambda \ge 0$) and so

$$\begin{aligned} |y_\lambda(t)| \;&=\; \lambda \textstyle\int_a^b |G(t,s)| q(s) f(y_\lambda(s)) \; ds \\ &\ge\; \lambda \cdot \min\{\tfrac{f(y)}{|y|} : \; y \ne 0\} \cdot \textstyle\int_a^b |G(t,s)| q(s) |y_\lambda(s)| \; ds \\ &\ge\; \lambda \cdot \min\{\tfrac{f(y)}{|y|} : \; y \ne 0\} \cdot \textstyle\int_a^b |G(t,s)| q(s) \varphi(s) ||y_\lambda||_\infty \; ds \end{aligned}$$

by Theorem 1.1, implying

$$1 \;\ge\; \lambda \cdot \min\left\{ \frac{f(y)}{|y|} : \; y \ne 0 \right\} \cdot \max_{a \le t \le b} \int_a^b |G(t,s)| q(s) \varphi(s) \; ds \;>\; 0 \;.$$

Claim 2: $\lambda_0 \in \Gamma^*$.
In fact, this is evident since the operator defined by

$$y \;\mapsto\; \lambda_0 \int_a^b G(\cdot\;, s) q(s) f(y(s)) \; ds$$

maps compactly the closed ball of C^0 with radius η_0 into itself and thus has a fixed point by Schauder's fixed point theorem. This fixed point is different from 0 by assumption (3) and the monotonicity of f. Another way to realize that (P_{λ_0}) has a solution is to observe that there exists a sequence $\lambda_l \uparrow \lambda_0$ such that $y_{\lambda_l,1}$ converges.
Hence there does exist

$$\lambda_* \;\ge\; \lambda_0 \quad .$$

Put

$$\lambda^* \;:=\; \sup \Gamma^* \;\le\; \left(\min\left\{ \frac{f(y)}{|y|} : \; y \ne 0 \right\} \cdot \max_{a \le t \le b} \int_a^b |G(t,s)| q(s) \varphi(s) \; ds \right)^{-1} \quad .$$

To complete the proof of the theorem, it suffices to prove that $\lambda^* \in \Gamma^*$ since the rest will concern the sign property of the solutions which follow from Theorem 1.1.
By definition of λ^* there exists a sequence $\lambda_l \in \Gamma^*$ converging to λ^*. Let now z_l be a solution of (P_{λ_l}). We need only to prove that $(z_l)_l$ is bounded in $\mathcal{C}^0$ since the operator defined by

$$y \mapsto \int_a^b G(\cdot\,,s)q(s)f(y(s))\;ds$$

is completely continuous from $\mathcal{C}^0$ into itself. Suppose by contradiction that

$$\lim_{l\to\infty} ||z_l||_\infty \;=\; \infty \;.$$

Then, put $\bar{\lambda} = \min_l\{\lambda_l\}$, let $A \subset \bigcup_{i=1}^{m-1}]a_i, a_{i+1}[$ be a compact set with positive measure containing a number δ such that $q(\delta) > 0$, and fix moreover $\alpha > 0$ with

$$\alpha\bar{\lambda}\int_A |G(\delta,s)|q(s)\varphi(s)\;ds \;>\; 1 \;.$$

Therefore, by assumption (3), there exists $\eta > 0$ such that

$$f(y) \;\geq\; \alpha|y| \quad \text{for } |y| \geq \eta \;.$$

If

$$||z_l||_\infty \;>\; \frac{\eta}{\min_A \varphi} \;,$$

then by the maximum principle proved in Theorem 1.1, we have

$$S_P(s)z_l(s) \;\geq\; \varphi(s)||z_l||_\infty \;\geq\; \eta \;, \quad s \in A \;,$$

so that

$$f(z_l(s)) \;\geq\; \alpha|z_l(s)| \;\geq\; \alpha\varphi(s)||z_l||_\infty \;, \quad s \in A \;,$$

yielding

$$\begin{aligned} |z_l(\delta)| \;=\; \lambda_l \textstyle\int_a^b |G(\delta,s)q(s)f(z_l(s))\;ds \;&\geq\; \bar{\lambda}\textstyle\int_A |G(\delta,s)q(s)f(z_l(s))\;ds \\ &\geq\; \alpha\cdot\bar{\lambda}\textstyle\int_A |G(\delta,s)|q(s)\varphi(s)||z_l||_\infty\;ds \\ &>\; ||z_l||_\infty \end{aligned}$$

which is absurd. □

Theorem 5.2 *Let consider continuous functions $H \leq Q : [a,b] \times \mathbf{R} \to \mathbf{R}$. Assume that there exist $h_0, h_1, q_0, q_1 \in \mathbf{L}^\infty([a,b])$ and a positive continuous function $f : \mathbf{R} \to]0,\infty[$ which is nondecreasing on $[0,\infty[$ such that*

(1) $f(y) \leq f(|y|)$ *for all* $y \in \mathbf{R}$,
(2) $q_0 - h_1 \geq 0$ *with* $q_0 - h_1 \neq 0$ *on a set of positive measure,*
(3)

$$h_0(t) \;\leq\; \frac{H(t,y)}{f(y)} \;\leq\; h_1(t) \quad \textit{for a.e. } t \textit{ and all } y \;,$$

(4)

$$q_0(t) \leq \frac{Q(t,y)}{f(y)} \leq q_1(t) \quad \textit{for a.e. } t \textit{ and all } y \ .$$

Then for every $\varrho > 0$, *there exists*

$$\lambda_*(\varrho) \in \,]0,\infty] \ ;$$

$$\lambda_*(\varrho) > \lambda_0 := \frac{\varrho}{f(\varrho)} \cdot \Big(\max_{a\leq t\leq b} \int_a^b |G(t,s)|(q_1(s) - h_0(s)) \ ds \Big)^{-1} \ ,$$

such that the BVPs

$$Ly + \lambda H(t,y) = \lambda Q(t,y) \ ,$$
$$y^{(j)}(a_i) = 0 \ , \quad 1 \leq i \leq m \ , \quad 0 \leq j \leq k_i - 1 \ ,$$

has a least one nontrivial solution y *with*

$$S_P(t)y(t) \geq \varphi(t)||y||_\infty \ , \quad a \leq t \leq b \ ,$$

such that

$$||y||_\infty \leq \varrho \ ,$$

given any

$$\lambda \in \,]0, \lambda_*(\varrho)[\ .$$

Proof:
For each $\lambda \in \,]0, \lambda_0]$, it is not hard to check that the operator defined by

$$y \mapsto \lambda \int_a^b G(\cdot\,,s)(Q(s,y(s)) - H(s,y(s)) \ ds$$

maps compactly the closed ball of C^0 with radius ϱ into itself and thus has a fixed point by Schauder's fixed point theorem. The sign property of this solution follows from Theorem 1.1. □

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