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Some Lower Semicontinuity and Relaxation Results for Functionals Defined on BV(Ω)

Candidate:
Virginia De Cicco

Supervisor:
Prof. Gianni Dal Maso

Thesis submitted for the degree of "Doctor Philosophiæ"
Academic Year 1991-92

TRIESTE

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Il presente lavoro costituisce la tesi presentata dalla dott.ssa Virginia De Cicco, sotto la direzione del Prof. Gianni Dal Maso, al fine di ottenere l'attestato di ricerca post-universitaria "Doctor Philosophiæ" presso la S.I.S.-S.A., Classe di Matematica, Settore di Analisi Funzionale ed Applicazioni. Ai sensi del Decreto del Ministro della Pubblica Istruzione 24-4-1987, n. 419, tale diploma è equipollente al titolo di "Dottore di Ricerca in Matematica".

Trieste, anno accademico 1991-92.

In ottemperanza a quanto previsto dall'art.1 del decreto legislativo luogotenenziale 31-8-1945, n.660, le prescritte copie della presente pubblicazione sono state depositate presso la Procura della Repubblica di Trieste e il Commissariato del Governo della Regione Friuli Venezia Giulia.

Ai miei genitori

Acknowledgements

I would like to thank first Prof. Gianni Dal Maso, who suggested the subject of this thesis to me; he introduced me to this research field, he followed and guided my progress with deep insight, giving me continuous and valuable suggestions.

I also wish to express all my thanks to him for supporting me during these years with patient encouragement and precious advice.

Furthermore I wish to thank Micol Amar and Andrea Braides. Part of the work presented in this thesis is the result of a helpful collaboration with them. Working with them has been deeply gratifying.

I am also very grateful to all the professors and students of the Class of Mathematics, at the S.I.S.S.A., for the frequent friendly discussions, both scientific and not, and also to my "physicist" friends at the S.I.S.S.A., for creating a nice and productive atmosphere of work.

Finally I take pleasure in expressing heartfelt thanks to my family, and especially to my parents, to whom I dedicate this thesis, for the moral and material support and encouragement through all the years of student life. Their devotion and affection gave me constant determination and inspiration.

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Introduction

This thesis deals with some problems in Calculus of Variations concerning integral functionals with linear growth. The "natural" function space where these problems admit a solution is the space BV of the functions of bounded variation.

A classical problem of the Calculus of Variations is to find, among all functions with prescribed boundary condition, those which minimize a given functional. In a general framework, one considers integral functionals of the form

$$F(u) = \int_\Omega f(x, u(x), Du(x))dx \tag{1}$$

where u is a real-valued (or vector-valued) function defined on a bounded open subset Ω of $\mathbb{R}^n$ and the integrand f satisfies some appropriate conditions.

The classical approach looks for necessary or sufficient conditions that the solution of the "minimum problem" must satisfy (*e.g.* Euler equation, Weierstrass condition).

At the beginning of the 20th Century, in order to prove the existence of minimizers, the so-called "Direct Methods" are introduced and developed by Tonelli and other mathematicians. They consist in proving the lower semicontinuity and the coerciveness of the functional with respect to a suitable topology. This latter requirement guarantees the compactness of the minimizing sequences, i.e. it allows to extract a convergent subsequence; then, thanks to the lower semicontinuity, the limit of this subsequence achieves the minimum value.

The compactness of the minimizing sequence is usually ensured if one considers a weak topology in a Banach reflexive space and assumes an estimate from below of the functional. In general, one considers functionals for the type (1) satisfying the condition

$$f(x, s, p) \geq a(x) + b|z|^p \tag{2}$$

where a is an integrable function and b is a positive constant, and one assumes suitable "qualitative" properties of the integrand f (Carathéodory condition and convexity or quasi-convexity with respect to the last variable), which imply the lower semicontinuity. Under these hypotheses, the Direct Methods work if one studies the problem in the Sobolev space $W^{1,p}(\Omega)$ with the same exponent p involved in the estimate (2). Hence the existence of minima in this space is ensured. Actually the Sobolev spaces are reflexive for $1 < p < +\infty$. When $p = 1$, (*e.g.* in the classical non-parametric Plateau problem of the minimal surfaces), since $W^{1,1}(\Omega)$ is not reflexive, the existence of minimizer of a minimum problem for an integral functional having linear growth cannot be guaranteed in this Sobolev space. Actually, this class of problems admits a solution in a larger space of integrable functions, which is also not reflexive, but where a compactness result holds: the space of functions of bounded variation.

The functions of bounded variation of one variable, first employed at the beginning of this century by Vitali and Lebesgue in the development of measure and integration theory, have been utilized by Tonelli and Cesari in several fields such as area theory and Calculus of Variations. This notion has been generalized to the case of several variables in the fifties by Caccioppoli and De Giorgi and had a considerable development during the past 30 years through contributions of many authors.

In the fifties, in order to solve the classical Plateau problem (i.e. to find minimal surfaces among those bounded by a given curve), Caccioppoli and De Giorgi took again into account the notion of function of bounded variation and the related notion of set of finite perimeter. The definition proposed (for functions depending on several variables) can be seen in the context of the distribution theory. In this environment a function of bounded variation (or BV function) is an integrable function whose distributional derivatives are measures of bounded total variation.

An important aspect of the theory of BV functions is the analysis of the measurable sets in $\mathbb{R}^n$ whose characteristic functions belong to the space BV (the so-called sets of finite perimeter or "Caccioppoli sets"). This class of sets (which includes the class of the domains with Lipschitz boundary) has the important property that the Gauss-Green Theorem holds. De Giorgi introduced the notion of reduced boundary of a Caccioppoli set and of generalized exterior normal; he established the rectifiability of this boundary and the existence of this normal in "almost" every point of the boundary and he also proved the isoperimetric inequality for sets of finite perimeter. In this class the Plateau problem admits a solution which is almost everywhere regular, except on a possible singular set, having very lower Hausdorff dimension.

Since the BV functions can be regarded as integral currents in $\mathbb{R}^n$, De Giorgi's theory can be examined in the context of geometric measure theory. Such theory had an important development in the sixties by many contributors, in particular Federer and Fleming. The normal and the boundary introduced by De Giorgi are equivalent to the normal and the boundary of the geometric measure theory.

Subsequently, pointwise behavior of BV functions has been studied by Vol'pert and other authors. They applied the BV functions to the theory of hyperbolic non-linear equations.

Recent developments of the BV functions are due to De Giorgi and Ambrosio who considered a class of "special" BV functions, where some variational problems involving discontinuities, such as the image segmentation in Computer Vision theory, the phase transitions and the liquid crystals, find a solution.

Since on $BV(\Omega)$ the usual distributional gradient is not a function, but only a measure, the integral functionals of the type (1) are not well defined. They are well defined on the space of regular functions and need a suitable extension to a wider space. Then for this extended functional one uses the Direct Methods and finds a minimizer on $BV(\Omega)$. As mentioned above, these methods work on the space BV, because a Rellich's compactness theorem holds. Whereas, in general, the coerciveness easily follows from proper "quantitative" hypotheses (growth estimates), the lower semicontinuity involves "qualitative" properties of the integrand function, such as the lower semicontinuity and the convexity with respect to some variables. One tries to prove the lower semicontinuity requiring as little as possible of the integrand function. In order to treat the situation, where the functional is not lower semicontinuous, the Relaxation Methods have been introduced. They consist in defining a new functional as the lower semicontinuous envelope. This relaxed functional has the interesting property that its minimizers are the limit points of a minimizing sequence of the given functional and its minimum value equals the "infimum" of the functional. Actually the Relaxation Methods allow to extend to a larger space, preserving the lower semicontinuity, a functional which is well defined only on a subspace. In some sense the largest space where a variational problem can be "naturally" studied is the space where the relaxed functional is finite. For instance, by relaxation of the functional

$$F(u) = \begin{cases} \int_\Omega |Du|^p dx + \int_\Omega |u|^p dx & \text{if } u \in C^1(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

it is possible to construct the space $W^{1,p}(\Omega)$, if $1 < p < +\infty$, and $BV(\Omega)$, if $p = 1$. This argument clearly justifies the introduction of the space BV in order to study problems with linear growth.

However a shortcoming of the relaxation approach is that the definition (which implies directly the lower semicontinuity) is given in an abstract way. Then it is interesting to check whether this relaxed functional is an integral functional, i.e. it admits an integral representation formula with a suitable integrand, and also to check the cases where this integrand can be exactly characterized. It is worth noticing that if the functional is coercive, then the relaxed functional satisfies also the coerciveness condition; hence it admits a minimum point.

The purpose of this thesis is to collect several results concerning the lower semicontinuity and the relaxation of integral functionals with linear growth.

In chapter 1 we recall some preliminary notions which will be useful in the next chapters; in particular we establish the notation, we list some definitions and we recall some general properties, which we often use in this thesis.

In chapter 2, we expose a semicontinuity theorem for the following extension to the space $BV(\Omega)$ of the functional (1)

$$(3) \qquad \widetilde{F}(u) = \int_\Omega f(x, u, D_a u)dx + \int_\Omega \left[\fint_{u_{(x)}}^{u_+(x)} f^\infty \left(x, t, \frac{D_s u}{|D_s u|} \right) dt \right] |D_s u|$$

where $Du = D_a u \; dx + D_s u$ is the decomposition of the measure Du in its absolutely continuous and singular part with respect to the Lebesgue measure, $u_-(x)$ and $u_+(x)$ denote the approximate limits at x, and f^∞ indicate the recession function of f.

For f dependent only on Du, the functional (3) has been introduced by Goffman and Serrin (see [59]), and for functional dependent also on x by Giaquinta, Modica and Soucek (see [57]). In the general case the functional has been proposed by Dal Maso in [33], considering the mean value in the singular part of the functional. Dal Maso proved that $\widetilde{F}$ is a lower semicontinuous extension of F to $BV(\Omega)$ under suitable hypotheses on the integrand $f(x,s,p)$; in particular, the lower semicontinuity is assumed with respect to the variable s.

Chapter 2 of this thesis is devoted to establish that the lower semicontinuity of $\widetilde{F}$ holds without any semicontinuity assumption on s. An analogous result has been proved in [39] where f does not depend on the variable x. Similarly, in [41] and [7], the same hypothesis has been dropped for functionals of the type (1) on $W^{1,1}(\Omega)$ for integrand $f(s,p)$ and $f(x,s,p)$ respectively. The idea of the proof introduced in [41] consists of two steps: first, one approximates from below the functional F, having integrand $f(x,s,p)$ (convex in p), with affine functionals whose integrand is of the type $a(x,s) + \langle b(x,s), p\rangle$, where a, b are Carathéodory functions; then the proof is reduced to establish the lower semicontinuity of these linear functionals. This is obtained, in the second step of the proof, by using the chain rule for BV functions. In order to apply this idea to the functional $\widetilde{F}$, there are some difficulties to treat the singular part of the functional. To overcome this complication, we use a result of Miranda (see [69]) which allows to write again $\widetilde{F}$ as an integral on $\Omega \times \mathbb{R}$ with respect to a suitable measure on $\Omega \times \mathbb{R}$. Moreover we need a sharper approximation of this functional and the chain rule for BV functions proven by Ambrosio and Dal Maso in [9].

In chapter 3 we give an integral representation result concerning the lower semicontinuous envelope of an integral functional, defined on vector functions and depending on the higher order derivatives. More precisely, we consider a functional of the type

$$F(u) = \int_\Omega f(\nabla^k u)dx, \tag{4}$$

where f is a function with linear growth, $k \in \mathbb{N}$, $u \in W^{k,1}(\Omega;\mathbb{R}^m)$ and $\nabla^k u$ is the k-th derivative of u. We study the lower semicontinuous envelope $\overline{F}$ of F on the space BV^k of the integrable functions, whose k-th derivative in the sense of distributions is a measure with bounded total variation. We state that $\overline{F}$ can be represented in the following way

$$\overline{F}(u) = \int_\Omega g(\nabla^k u)dx + \int_\Omega g^\infty\left(\frac{D^k_s u}{|D^k_s u|}\right)|D^k_s u|, \tag{5}$$

where g is the quasi-convex envelope of f, g^∞ is the recession function of g and

$D^k u = \nabla^k u \; dx + D^k_s u$ is the decomposition of the measure $D^k u$ in its absolutely continuous and singular parts with respect to the Lebesgue measure.

This result generalizes to the case of higher order derivatives a theorem of Ambrosio and Dal Maso (see [10]), which deals with the case $k = 1$. The notion we use of quasi-convexity (for functions depending on the k-th derivative) has been given by Meyers and it extends the usual definition of Morrey (which concerns the case $k = 1$). The difference between the notions of quasi-convexity in the sense of Meyers and of Morrey implies that our result cannot be obtained directly from the case $k = 1$. However we follow the outline of the proof in [10]. In order to obtain the result, first we assume a coerciveness condition and we adapt to the case $k > 1$ a blow up technique. Moreover we use a rank-one property for the gradient of BV functions, proven by Alberti and true also for BV^k functions. Finally the coerciveness hypothesis can be dropped using a perturbation technique.

In chapter 4 we study the lower semicontinuous envelope of degenerate quadratic integral functionals on the space $BV(I;\mathbb{R}^n)$ with I a real interval. For these functionals, which are still quadratic functionals, we give an integral representation formula involving a new "relaxed" matrix and some linear constraints on the derivative measure.

We consider a functional $F : L^1(I;\mathbb{R}^n) \to [0, +\infty]$ of the type

$$F(u) = \begin{cases} \displaystyle\int_I \sum_{i,j=1}^{n} a_{ij}(t)\dot{u}_i\dot{u}_j \, \mathrm{dt} & \text{if } u \in \mathrm{W}^{1,1}(I;\mathbb{R}^n) \\ +\infty & \text{otherwise} \end{cases} \tag{6}$$

where $A(t) = (a_{ij}(t))$ is a symmetric matrix of Borel functions satisfying the only assumption

$$0 \le \sum_{i,j=1}^{k} a_{ij}(t) z_i z_j. \tag{7}$$

We remark that, if some coerciveness hypothesis is assumed, then the functional F is lower semicontinuous in a (possibly "weighted") Sobolev space. Under the condition (7) F may not be lower semicontinuous; hence we consider the semicontinuous envelope $\overline{F}$ in the space $BV(I;\mathbb{R}^n)$ and we find a characterization in an integral form. The main result of chapter 4 is an integral representation theorem of $\overline{F}$; we state that

$$(8) \quad \overline{F}(u) = \begin{cases} \displaystyle\int_I \sum_{i,j=1}^{n} \widetilde{a}_{ij}(t)(\dot{u}_a)_i(t)(\dot{u}_a)_j(t)\mathrm{d}t & \text{if } \frac{\dot{u}_s}{|\dot{u}_s|}(t) \in E(t) \quad |\dot{u}_s|\text{-a.e. on I} \\ +\infty & \text{elsewhere in } BV(I;\mathbb{R}^n), \end{cases}$$

where $\widetilde{A}(t) = (\widetilde{a}_{ij}(t))$ is a new matrix of Borel functions, $E(t)$ is a linear subspace of $\mathbb{R}^n$, and $\dot{u} = \dot{u}_a \mathrm{d}t + \dot{u}_s$ is the decomposition of the Radon measure $\dot{u}$ in its absolutely continuous part ($\dot{u}_a$) and singular part ($\dot{u}_s$) with respect to the Lebesgue measure.

In the same chapter, we consider a sequence of functionals of this type and we prove a stability property with respect to a suitable variational convergence, known as Γ-convergence. In fact we prove that the limit functional admits a representation similar to (8), but involving a new Radon measure.

In the proof of these theorems we use some representation results proven by Bouchitté for convex functionals on the space of measures. Eventually we discuss some examples in order to describe the matrix $\widetilde{A}(t)$ and the linear subspace $E(t)$ of $\mathbb{R}^n$ which appear in the formula (8).

In chapter 5 we give an uniqueness result concerning the minimizers of the functional proposed by Mumford and Shah in order to study the problem of the image segmentation in Computer Vision Theory (see [76] and [77]). This problem admits a solution in the class of the functions of bounded variation, in particular in a subclass of "special" BV function, called SBV. This notion has been introduced in the last years by De Giorgi and Ambrosio. The functional proposed, in order to give a mathematical description of the segmentation problem, is the following

$$(9) \qquad F_\gamma^g(u) = \int_\Omega |\nabla u|^2 dx + \int_\Omega |u-g|^2 dx + \gamma H^{n-1}(S_u)$$

where S_u is the jumping set of the function u, H^{n-1} is the $n-1$ Hausdorff measure on $\mathbf{R}^n$, g is a given function, called "grey-level", and γ is a real parameter.

For the corresponding minimum problem, Ambrosio proved in [8] an existence result on the space SBV. Moreover some regularity results about the singular set has been established. In general, for this problem the uniqueness of the minimizers does not hold. Very simple examples show that the problem may admit more than one solution.

In chapter 5 we concern the model case in dimension 1 of the functional (9). We state that the uniqueness is a generic property in the sense that for "almost all" the grey-level functions g and the parameters γ of the problem, the minimum point is unique. More precisely, we prove two theorems. First of all, for every $\gamma \in \mathbb{R}^+$ the uniqueness of the problem is a generic property of $g \in L^2$; this means that for a G_δ-subset of L^2 the uniqueness property holds. Secondly, for a generic $g \in L^2$, the uniqueness is guaranteed if γ belongs to $\mathbb{R}^+ \setminus N^g$ and N^g is a countable set depending on g. The proof of these results is constructive; in order to find the G_δ-set, we study in detail the properties of the solution, and in particular its form and its discontinuities, when g is piecewise constant.

The content of this thesis, which is published in the papers [4], [5], [19] and [38], is the result of a research activity carried on by the Author during her graduate studies at the International School for Advanced Studies in Trieste, under the guidance of Prof. Gianni Dal Maso and in collaboration with Dr. Micol Amar and Prof. Andrea Braides.

Chapter 1

Notations and Preliminaries

1.1 Notations and definitions

First of all, we list the basic notations frequently used in this thesis.

Let $n \geq 1$ be an integer and let Ω be an open bounded set in $\mathbb{R}^n$. We denote by

$L(\mathbb{R}^n)$ the class of Lebesgue-measurable sets in $\mathbb{R}^n$;

$\mathcal{L}^n(D)$ the Lebesgue measure of a set $D \in L(\mathbb{R}^n)$ (sometime we use also the notation meas(D));

$B(\Omega)$ the class of Borel subsets of Ω;

$H^{n-1}(B)$ the Hausdorff $(n-1)$-dimensional measure of any Borel set $B \subseteq \mathbb{R}^n$;

$\mathbf{1}_A$ the characteristic function of any set $A \subseteq \mathbb{R}^n$;

$supp\ \phi$ the support of any function $\phi \in C_0(\Omega)$;

$\langle x, y\rangle$ the scalar product of x and y in $\mathbb{R}^n$;

$|x|$ the norm of $x \in \mathbb{R}^n$;

$B_\rho(x)$ the open ball of radius ρ at x;

t^+ the positive part of $t \in \mathbb{R}$, i.e. $\max\{0, t\}$;

$f \otimes g$ the tensorial product of two functions $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$, defined by $(f\otimes g)(x, y) = f(x)g(y)$ for all $(x, y) \in A \times B$ (A, B contained in $\mathbb{R}^n$).

Now we recall some preliminary definitions. Let $f : \mathbb{R}^n \to [0, +\infty]$ be a function. We denote by f^* the conjugate of f and by f^{**} the conjugate of f^*.

Moreover we denote by $f\nabla g$ the *infimal convolution* of the two functions f and g, which is defined by

$$(f\nabla g)(z) = \inf_{y\in\mathbb{R}^n}[f(z-y)+g(y)].$$

If f is convex, we define also the so-called *recession function* $f^\infty : \mathbb{R}^n \to [0,+\infty]$ by

$$f^\infty(z) = \lim_{t\to+\infty}\frac{f(tz)}{t};$$

we remark that f^∞ is a Borel function, which is convex and positively homogeneous of degree 1. It is easy to see that

$$f^\infty(z) = \sup_{z^*\in A^*}\langle z, z^*\rangle,$$

where A^* is the *effective domain* of the conjugate function f^* of f; *i.e.*, the set of the $z^*\in\mathbb{R}^n$ such that $f^*(z^*) < +\infty$.

1.2 Radon measures

A Borel measure μ on $\mathbb{R}^n$ is called a "Radon measure", if every subset of $\mathbb{R}^n$ is contained within a Borel set of equal μ measure, and $\mu(K) < +\infty$ for every compact set $K\subset\mathbb{R}^n$.

Let $\mu : B(\Omega)\to\mathbb{R}^n$ a vector valued Radon measure on Ω. We adopt the notation $|\mu|$ for its total variation, which is defined by

$$(1.1)\quad |\mu|(A) = \left\{\sup\sum_{i\in\mathbb{N}}|\mu(A_i)| : \bigcup_{i\in\mathbb{N}} A_i, A_i\in B(\Omega),\ A_i \quad \text{mutually} \quad \text{disjoint}\right\}$$

for every $A\in B(\Omega)$. We say that μ is a measure with finite (or bounded) total variation if $|\mu|(\Omega) < +\infty$.

Let $\lambda : B(\Omega)\to[0,+\infty[$ and $\mu : B(\Omega)\to\mathbb{R}^n$ be measures on Ω with bounded total variation; we say that λ is absolutely continuous with respect to μ, and we denote by $\lambda << \mu$, if $\mu(E) = 0$ implies that $\lambda(E) = 0$ for every Borel set E contained in Ω.

If $\lambda << \mu$, then by the Radon-Nikodym Theorem, there exists a density $\frac{\lambda}{\mu}$ such that $\lambda(E) = \int_E \frac{\lambda}{\mu} d\mu$ (we denote it also with $\frac{d\lambda}{d\mu}$ or λ^μ). This function is usually called the Radon-Nikodym derivative of μ with respect to λ. We remark that, since $\mu << |\mu|$, always there exists the Radon-Nikodym derivative $\frac{\mu}{|\mu|}$.

We denote by $\mathcal{M}(\Omega; \mathbb{R}^m)$ the set of the $\mathbb{R}^m$-valued Radon measures on Ω with bounded total variation, and by $\mathcal{M}^+(\Omega)$ the space of the (scalar) positive Radon measures on Ω with bounded total variation.

1.3 Sets of finite perimeter

Let E a Borel subset of $\mathbb{R}^n$. We define the perimeter of E in Ω by

$$P(E, \Omega) = \sup \left\{ \int_E \mathrm{div} g dx : g \in C_0^1(\Omega; \mathbb{R}^n), |g| \leq 1 \right\}.$$

If $P(E, \Omega) < +\infty$, then E is said a set of finite perimeter. If $P(E, \Omega') < +\infty$, then E is called a set of locally finite perimeter in Ω or "Caccioppoli set".

We remark that if E is a set of finite perimeter, then there exists a vector valued Radon measure μ in $B(\Omega)$ with finite total variation such that

$$\int_E \mathrm{div} g dx = - \int_\Omega g d\mu$$

for every $g \in C_0^1(\Omega; \mathbb{R}^n)$, i.e. $\mu = D\mathbf{1}_E$, and

$$P(E, \Omega) = |D\mathbf{1}_E|(\Omega).$$

1.4 Functions of bounded variation

The space $BV(\Omega; \mathbb{R}^m)$ of the *functions of bounded variation* is the space of all functions $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ whose distributional gradient Du belongs to $\mathcal{M}(\Omega; \mathbb{R}^m)$ (if $m = 1$ we use also the notations $BV(\Omega)$, as well as $\mathcal{C}_c(\Omega)$, $\mathcal{C}_o(\Omega)$ and $\mathrm{L}^1(\Omega)$). Moreover the space of all functions which belong to $BV(\Omega'; \mathbb{R}^m)$ for every open set $\Omega' \subset\subset \Omega$, is denoted by $BV(\Omega'; \mathbb{R}^m)$.

The total variation $|Du|$ of the measure Du on a Borel set B will be denoted by $\int_B |Du|$. It is well known that for every $u \in BV(\Omega)$ for every open set $A \subseteq \Omega$ the integral of the measure $|Du|$ over A is given by

$$\int_A |Du| = \sup\left\{ \int_\Omega u \operatorname{div} g dx : g \in C_0^\infty(A; \mathbb{R}^n), |g| \leq 1 \right\}.$$

Moreover a function $u \in L^1(\Omega)$ belongs to $BV(\Omega)$ if and only if the quantity $\int_\Omega |Du|$ defined by (1.1) is finite. The integral of a Borel function f with respect to the measure $|\mu|$ will be denoted by $\int_B f|Du|$. Moreover a Borel set E contained in $\mathbb{R}^n$ is a set of (locally) finite perimeter in Ω if and only if $\mathbf{1}_E \in BV(\Omega)$ ($BV_{\mathrm{loc}}(\Omega)$ respectively) and in this case $P(E, \Omega) = \int_\Omega |D\mathbf{1}_E|$.

Fixed a measure $\lambda \in \mathcal{M}^+(\Omega)$, for every function $u \in BV(\Omega; \mathbb{R}^m)$ we consider the decomposition $Du = D_{\mathrm{a}}^\lambda u d\lambda + D_{\mathrm{s}}^\lambda u$ of the measure Du in its absolutely continuous part and singular part with respect to the measure λ. When λ is the Lebesgue measure we decompose Du as $Du = D_{\mathrm{a}} u dx + D_{\mathrm{s}} u$; moreover if $u \in \mathrm{W}^{1,1}(\Omega; \mathbb{R}^m)$, with abuse of notations, we use also Du in order to note the Radon-Nikodym derivative $D_{\mathrm{a}} u$. On $BV(\Omega; \mathbb{R}^m)$ we shall consider the norm

$$\|u\|_{BV} = \|u\|_{\mathrm{L}^1} + \int_\Omega |Du|.$$

It can be seen that $|Du|(C) = 0$ for every $C \in B(\Omega)$ with $H^{n-1}(C) = 0$.

Given a Borel function $u : \Omega \to \mathbb{R}$, we define the approximate upper and lower limits $u_+, u_- : \Omega \to [-\infty, +\infty]$ as

$$u_+(x) = \inf\left\{ t \in [-\infty, +\infty] : \lim_{\rho \to 0+} \frac{\operatorname{meas}(B_\rho(x) \cap \{y \in \Omega : u(y) > t\})}{\operatorname{meas}(B_\rho(x))} = 0 \right\}$$

and

$$u_-(x) = \inf\left\{ t \in [\infty, +\infty] : \lim_{\rho \to 0+} \frac{\operatorname{meas}(B_\rho(x) \cap \{y \in \Omega : u(y) < t\})}{\operatorname{meas}(B\rho(x))} = 0 \right\}$$

and we observe that u_+ and u_- are Borel functions.

Let $u \in BV(\Omega)$; we say that $x \in \Omega$ is a *jump point* of u, if $u_-(x) < u_+(x)$. The set of the jump points of u has Lebesgue measure zero. Moreover, it can be almost covered with C^1 hypersurfaces; in fact, it is countable $(H^{n-1}, n-1)$-rectifiable in the sense of Federer, i.e. it can be written as

$$\bigcup_{h\in\mathbb{N}} K_h \bigcup N,$$

where (K_h) is a sequence of compact sets, each contained in a C^1 hypersurface and $H^{n-1}(N) = 0$.

For the general properties of the functions of bounded variation we refer to [50], [58], [88], [89] and [92].

1.5 Relaxation

We recall the notion of *relaxed functional.*

Let (X, τ) be a topological space and $F : X \to \overline{\mathbb{R}}$; the relaxed functional $\overline{F}$ of F (in the τ-topology) is the lower τ-semicontinuous envelope of F; *i.e.,*

$$\overline{F}(u) = \sup_{U\in\mathcal{N}(u)} \inf_{v\in U} F(v), \tag{1.2}$$

where $\mathcal{N}(u)$ is the set of all open neighbourhoods of u in the τ-topology.

It is easy to check that $\overline{F}$ is the greatest lower semicontinuous functional majorized by F. Moreover, if X satisfies the first axiom of countability, then $\overline{F}$ may be characterized in terms of sequences by the following conditions:

a) for every sequence (x_h) converging to x in X we have

$$\overline{F}(x) \leq \liminf_{h\to\infty} F(x_h);$$

b) there exists a sequence (x_h) converging to x in X such that

$$\overline{F}(x) \geq \limsup_{h\to\infty} F(x_h).$$

An important property of $\overline{F}$ is the equality

$$\min_X \overline{F}(x) = \inf_X F(x).$$

We remark also that the minimum point of $\overline{F}$ can be characterized as the limit point of the minimizing sequence for F.

For a general survey on the relaxation theory we refer to [20] and [32].

Chapter 2

A Lower Semicontinuity Result

2.1 Introduction

The aim of this chapter is to show some lower semicontinuity results for an extension to the space $BV(\Omega)$ of an integral functional of the type

$$u \mapsto \int_\Omega f(x, u, Du)dx \qquad u \in W^{1,1}(\Omega). \tag{1.1}$$

In many papers the lower semicontinuity on $W^{1,1}(\Omega)$ for this functional has been studied, assuming some lower semicontinuity and convexity hypothesis on the integrand f (see, for instance, [27], [61], [78] and [83]).

For f independent of x, a lower semicontinuity result with respect to the convergence $L^1(\Omega)$ is proved in [41]; this result differs from the previous ones since a weaker assumption is made on the regularity of the integrand $f(s,p)$ with respect to the variable s, i.e. measurability instead of semicontinuity. The technique employed consists in approximating the given functional with simpler ones (whose integrands are affine functions) and in proving the lower semicontinuity of the approximating functionals. Using a suitable approximation technique, in [7] this theorem is generalized to the case of a function f depending on x too.

However, since $W^{1,1}(\Omega)$ is not reflexive, many integral functionals with linear growth do not admit any minimizer on $W^{1,1}(\Omega)$; hence one considers the wider space $BV(\Omega)$ of the functions $u \in L^1(\Omega)$, whose generalized derivatives are Radon measures of finite variation, and one finds a suitable extension of the functional (1.1) to this space (see [51], [57], [59], [83]); then, if the extension considered is lower semicontinuous, one can apply the direct methods of the Calculus of Variations and finds a minimizer in $BV(\Omega)$ for the extended problem.

In this chapter we consider the following extension to $BV(\Omega)$, given in [33], of the functional (1.1):

$$F(u) = \int_\Omega f(x,u,D_a u)dx + \int_\Omega \left[\fint_{u_-(x)}^{u_+(x)} f^\infty \left(x,t,\frac{D_s u}{|D_s u|}(x) \right) dt \right] |D_s u|(x),$$

where $D_a u dx$ and $D_s u$ are the absolutely continuous and the singular part of the measure Du, $\frac{D_s u}{|D_s u|}$ is the Radon-Nikodym derivative of the measure $D_s u$ with respect to its total variation, f^∞ denotes the recession function of $f(x,s,p)$ with respect to p and $u_+(x)$ and $u_-(x)$ are the upper and lower approximate limits of u at x.

In [33] it is proved that this functional is lower semicontinuous on $BV(\Omega)$ for the $L^1(\Omega)$ topology, under suitable assumptions on $f(x,s,p)$, including the lower semicontinuity with respect to s.

Using a similar approach as in [41], a more general theorem for functionals F, whose integrands do not depend on x, is obtained in [39] without requiring the lower semicontinuity of $f(s,p)$ in s. The idea of the proof is to represent the functional $F(u)$ as an integral over the reduced boundary of the subgraph of the function u in the space $\Omega \times \mathbb{R}$. This representation allows to approximate F with functionals whose integrands are affine functions; the lower semicontinuity for these functionals is obtained using the chain rule for $BV(\Omega)$ functions proved in [9] and [35].

In the present chapter we extend this result to the general case when f depends on x too, without assuming the lower semicontinuity of the map $s \mapsto f(x,s,p)$, except for $p = 0$. Putting together the above technique of [39] and the approximation technique of [7], we establish a lower semicontinuity result for the functional F on $BV(\Omega)$. The basic idea of the proof consists in writing again F as an integral on $\Omega \times \mathbb{R}$ with respect a suitable measure on $\Omega \times \mathbb{R}$ and so in reducing the problem to prove the lower semicontinuity for functionals with integrands of the type

$$f(x,s,p) = \psi(s)[a(x,s) + \langle b(x,s), p \rangle]^+,$$

where $\psi \in C_0^\infty(\Omega)$ and $a(x,s)$, $b(x,s)$ are continuous in x and measurable in s. This proof is based on further approximations and on a projection result which permits finally to write the approximating functional again as an integral on Ω

with respect to the measure Du; the proof is concluded using the chain rule for $BV(\Omega)$ functions.

2.2 Some definitions and preliminary lemmas

2.1. – For every Borel function $u : \Omega \to \mathbb{R}$ we define two functions $u_+, u_- : \Omega \to [-\infty, +\infty]$ by

(2.1)
$$u_+(x) = \inf \left\{ t \in [-\infty, +\infty] : \lim_{\rho \to 0+} \frac{\text{meas}(B_\rho(x) \cap \{y \in \Omega : u(y) > t\})}{\text{meas}(B_\rho(x))} = 0 \right\}$$

and

$$(2.2) \quad u_-(x) = \inf \left\{ t \in [\infty, +\infty] : \lim_{\rho \to 0+} \frac{\text{meas}(B_\rho(x) \cap \{y \in \Omega : u(y) < t\})}{\text{meas}(B\rho(x))} = 0 \right\},$$

the functions u_+ and u_-, which are called the approximate upper and lower limits, are Borel functions.

2.2. – For every $u \in BV(\Omega)$ we consider $N(u) = \{x \in \Omega : u_-(x) < u_+(x)\}$; a point $x \in N(u)$ is said a jump point of u. It is known that $N(u)$ is a Borel set and

$$(2.3) \qquad \mathcal{L}^n(N(u)) = 0.$$

Moreover we recall that we have (see [50], Theorem 3.2.23)

$$(2.4) \qquad H^n = H^{n-1} \times L^1 \quad \text{on} \quad N(u) \times \mathbb{R}.$$

2.3. – A point $x \in \Omega$ is said a Lebesgue point of u if there exists a number $\overline{u}(x)$ such that

$$\lim_{\rho \to 0} \rho^{-n} \int_{B_\rho(x)} |u(y) - \overline{u}(x)| dy = 0;$$

we indicate by $\Omega(u)$ the set of the Lebesgue points of u and we recall that (see [50], Theorem 4.5.9 (21))

$$(2.5) \qquad H^{n-1}(\Omega \setminus (\Omega(u) \cup N(u))) = 0.$$

2.4. – For every $u \in BV(\Omega)$ we define

$$S(u) = \{(x,s) \in \Omega \times \mathbb{R} : s < u_+(x)\} \tag{2.6}$$

and

$$C(u) = \{(x,s) \in \Omega \times \mathbb{R} : u_-(x) \leq s \leq u_+(x)\}; \tag{2.7}$$

we recall that $S(u)$ is a set with locally finite perimeter in $\Omega \times \mathbb{R}$, i.e. the distributional derivative of $\mathbf{1}_{S(u)}$ is a Radon measure on $\Omega \times \mathbb{R}$ (see [69], Theorem 1.10). We indicate by $\alpha(u) = (\alpha_1(u), \ldots, \alpha_{n+1}(u))$ the derivative measure of $\mathbf{1}_{S(u)}$ and we remark that

$$|\alpha(u)|(B) = H^n(C(u) \cap B) \tag{2.8}$$

for every Borel set B in $\Omega \times \mathbb{R}$ (see [50], Theorem 4.5.9 (5)). Let $\nu(u)$ be the $\mathbb{R}^{n+1}$-valued measure on Ω defined by

$$\nu(u) = (D_1 u, \ldots, D_n u, -\mathcal{L}^n).$$

We remark (see [50], Theorem 4.5.9 (15)) that

$$|\nu(u)| = |Du| = (u_+ - u_-)H^{n-1} \qquad \text{on} \quad N(u); \tag{2.9}$$

moreover $|\nu(u)|$ and, for $i : 1, \ldots, n$, $D_i u$ are the images of $|\alpha(u)|$ and $\alpha_i(u)$ respectively under the canonical projection of $\Omega \times \mathbb{R}$ onto Ω (see [69], Theorem 1.10) i.e.

$$\int_{D \times \mathbb{R}} |\alpha(u)| = \int_D |\nu(u)| \quad \text{and} \quad \int_{D \times \mathbb{R}} \alpha_i(u) = \int_D D_i u \tag{2.10}$$

for every Borel set D in Ω. Finally we observe that for $|\alpha(u)|$ almost all $(x,s) \in \Omega \times \mathbb{R}$ we have (see [49], par. 5)

$$n[(x,s); S(u)] = \frac{\alpha(u)}{|\alpha(u)|}(x,s), \tag{2.11}$$

where $n[(x,s); S(u)]$ is the interior normal of $S(u)$ at (x,s), if it exists, and is 0 otherwise, and $\frac{\alpha(u)}{|\alpha(u)|}$ is the Radon-Nikodym derivative of the measure $\alpha(u)$ with respect to its total variation.

2.5. – We adopt the following conventions:

for every Borel set B in Ω $\int_B g|Du|$ is the integral on B of a Borel function $g : \Omega \to \mathbf{R}$ with respect to the Radon measure $|Du|$ and $\int_B \langle f, Du\rangle$ is the integral on B of a vector valued Borel function $f : \Omega \to \mathbb{R}^n$ with respect to the vector valued Radon measure Du. Analogously for every $\phi \in C_0^\infty(\Omega)$ $\int_\Omega \langle f, Du\rangle\, \phi$ is the integral on Ω of ϕ with respect to the measure $\langle f, Du\rangle$.
Moreover we denote by $\langle f, Du\rangle^+$ the positive part of this measure; we note that

$$\int_\Omega \langle f, Du\rangle^+ = \sup\left\{\int_\Omega \langle f, Du\rangle\, \phi : \phi \in C_0^\infty(\Omega), 0 \leq \phi \leq 1\right\}$$

and, using the decomposition of the measure Du,

$$\int_\Omega \langle f, Du\rangle^+ = \int_\Omega \langle f, D_a u\rangle^+ \, dx + \int_\Omega \left\langle f, \frac{D_s u}{|D_s u|}\right\rangle^+ |D_s u|. \tag{2.12}$$

2.6. – Given $u \in BV(\Omega)$ and $a \in L^1_{loc}(\mathbb{R})$ we denote by $\widehat{a}(u)$ the Vol'pert's averaged superposition of a and u, defined in the following way:

for H^{n-1} almost all $x \in \Omega$

$$\widehat{a}(u)(x) = \fint_{u_-(x)}^{u_+(x)} a(t)dt, \tag{2.13}$$

where, for every $t_1, t_2 \in \mathbb{R}$, $t_1 \leq t_2$, we set

$$\fint_{t_1}^{t_2} a(t)dt = \begin{cases} \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} a(t)dt & \text{if } t_1 < t_2, \\ a(t_1) & \text{if } t_1 = t_2. \end{cases}$$

We remark that $\widehat{a}(u)(x) = a(\overline{u}(x))$ for every $x \in \Omega(u)$ and by (2.12) we obtain
(2.14)

$$\int_\Omega \langle \widehat{a}(u), Du\rangle^+ = \int_\Omega \langle a(u), D_a u\rangle^+ \, dx + \int_\Omega \left[\fint_{u_-(x)}^{u_+(x)} \left\langle a(t), \frac{D_s u}{|D_s u|}(x)\right\rangle dt\right]^+ |D_s u|.$$

2.7. – Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, +\infty]$ be a Borel function such that the map $p \mapsto f(x, s, p)$ is convex on $\mathbb{R}^n$ for each $(x, s) \in \Omega \times \mathbb{R}$. We can define the so-called recession function $f^\infty : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, +\infty]$ by

$$f^\infty(x, s, p) = \lim_{t \to 0+} f(x, s, p/t)t \tag{2.15}$$

and a new function $\widetilde{f} : \Omega \times \mathbb{R} \times \mathbb{R}^n \times]\infty, 0] \to [0, +\infty]$ by

$$\widetilde{f}(x,s,p,t) = \begin{cases} -f(x,s,-p/t) & \text{if } t < 0, \\ \\ f^\infty(x,s,p) & \text{if t=0.} \end{cases} \tag{2.16}$$

We note that $\widetilde{f}$ is a Borel function too and for each $(x,s) \in \Omega \times \mathbb{R}$ the map $(p,t) \mapsto \widetilde{f}(x,s,p,t)$ is convex and positively homogeneous of degree 1 on $\mathbb{R}^n \times]-\infty, 0]$. Now we define the functional $F : BV(\Omega) \to [0, +\infty]$ by

$$F(u) = \int_\Omega f(x,u,D_a u)dx + \int_\Omega \left[\fint_{u_-}^{u_+} f^\infty \left(x, t, \frac{D_s u}{|D_s u|} \right) dt \right] |D_s u|, \tag{2.17}$$

where, in the last term, we mean that the function

$$x \mapsto \fint_{u_-(x)}^{u_+(x)} f^\infty \left(x, t, \frac{D_s u}{|D_s u|}(x) \right) dt$$

is integrated with respect to the measure $|D_s u|$. Moreover we consider a more general functional: for every $\phi \in C^o(\Omega)$, $\phi \geq 0$, we define

$$I(f,u,\phi) = \int_\Omega f(x,u,D_a u)\phi dx + \int_\Omega \left[\fint_{u_-}^{u_+} f^\infty \left(x, t, \frac{D_s u}{|D_s u|} \right) dt \right] \phi |D_s u|. \tag{2.18}$$

In the sequel, we will prove that these functionals (now defined for Borel functions f) can be defined also with weaker measurability assumptions.

Finally we say that a functional $G : BV(\Omega) \to [-\infty, +\infty]$ is $L^1(\Omega)$ lower semicontinuous along sequences bounded in $BV(\Omega)$, if for every sequence (u_h) in $BV(\Omega)$ and for every $u \in BV(\Omega)$ such that u_h converges to u in $L^1(\Omega)$ and

$$\limsup_h \int_\Omega |Du_h| < +\infty,$$

the following inequality holds

$$G(u) \leq \liminf_h G(u_h).$$

Now we recall some basic results which will play a fundamental rôle in the following proofs.

Lemma 2.1. *Let $u \in BV(\Omega)$, let N a Borel subset of $\mathbb{R}$ with Lebesgue measure zero and set $E = \{x \in \Omega(u) : \overline{u}(x) \in N\}$, where $\overline{u}$ is defined in par. 2.3. Then the measure Du vanishes on E, i.e. $|Du|(E) = 0$.*

Proof. See Appendix of [35]. $\square$

Lemma 2.2. *Let $h : \mathbb{R} \to \mathbb{R}$ be a globally Lipschitz continuous function. Then for every $u \in BV(\Omega)$ the function $h \circ u$ belongs to $BV(\Omega)$ and the chain rule*

$$D(h \circ u) = \widehat{h^*}(u)Du, \quad \text{as} \quad \text{measures} \quad \text{on} \quad \Omega,$$

holds, where $h^* : \mathbb{R} \to \mathbb{R}$ is any Borel function such that $h^*(t) = h'(t)$ a.e. on $\mathbb{R}$ and $\widehat{h^*}(u)$ is defined by (2.13).

Proof. See [9] and [35]. $\square$

Lemma 2.3. *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, +\infty]$ be a Borel function such that for each $x \in \Omega$ and $s \in \mathbb{R}$ the map $p \mapsto f(x, s, p)$ is convex on $\mathbb{R}^n$. Then*

$$F(u) = \int_{\Omega \times \mathbb{R}} \tilde{f}\left((x,s), \frac{\alpha(u)}{|\alpha(u)|}(x,s)\right) |\alpha(u)|(x,s),$$

where F is defined by (2.17), *$\tilde{f}$ by* (2.16) *and $\alpha(u)$ as in par. 2.4.*

Proof. See [33], Lemma 2.2. $\square$

Proposition 2.4. *Let μ be a positive Radon measure on $\Omega \times \mathbb{R}$ and let (f_k) be a sequence of functions of $L^1(\mu)$ such that $f := \sup\limits_{k \in \mathbb{N}} f_k \geq 0$. Then for every open subset A of $\Omega \times \mathbb{R}$ we have*

$$\int_A f d\mu = \sup_{k \in \mathbb{N}} \sup \left\{ \sum_{i=1}^{k} \int_A f_i \eta_i d\mu : \eta_i \in C_0^\infty(A), \eta_i \geq 0, \sum_{i=1}^{k} \eta_i \leq 1 \right\}.$$

Proof. For every $k \in \mathbb{N}$ let $g_k = \sup\{f_i^+ : i = 1, \dots, k\}$; from Beppo Levi's Theorem we have

$$\int_A f d\mu = \sup_{k \in \mathbb{N}} \int_A g_k d\mu.$$

Fixed $k \in \mathbb{N}$, there exist measurable pairwise disjoint subsets $B_0, B_1, \dots, B_k$ of A such that

$$A = \bigcup_{j=0}^{k} B_j,$$

$$g_k | B_0 = 0,$$

$$g_k | B_i = f_i \qquad \text{and} \qquad g_k|_{B_i} \geq 0 \qquad \text{for} \quad i = 1, \dots, k;$$

since μ is a regular measure, we have

$$\begin{aligned}
\int_A g_k d\mu &= \sum_{i=1}^{k} \int_{B_i} f_i d\mu = \sup\left\{ \sum_{i=1}^{k} \int_{K_i} f_i d\mu : K_i \subseteq B_i, K_i \quad \text{compact} \right\} \leq \\
&\leq \sup\left\{ \sum_{i=1}^{k} \int_A f_i \eta_i d\mu : \eta_i \in C_0^\infty(A), \eta_i \geq 0, \sum_{i=1}^{k} \eta_i \leq 1 \right\} \leq \\
&\leq \sup\left\{ \sum_{i=1}^{k} \int_A g_k \eta_i d\mu : \eta_i \in C_0^\infty(A), \eta_i \geq 0, \sum_{i=1}^{k} \eta_i \leq 1 \right\} = \\
&= \int_A g_k d\mu.
\end{aligned}$$

Then

$$\int_A g_k d\mu = \sup\left\{ \sum_{i=1}^{k} \int_A f_i \eta_i d\mu : \eta_i \in C_0^\infty(A), \eta_i \geq 0, \sum_{i=1}^{k} \eta_i \leq 1 \right\}.$$

□

Lemma 2.5. *Let μ be a positive Radon measure on $\Omega \times \mathbb{R}$ and let (f_k) be a sequence of functions of $L^1(\mu)$ such that $f := \sup_{k\in\mathbb{N}} f_k \geq 0$. Then for every open subset A of $\Omega \times \mathbb{R}$ we have*

$$\int_A f d\mu = \sup_{\mathcal{B}} \sum_{i\in I} \int_A f_{k_i}(x,s)\phi_i(x)\psi_i(s)d\mu(x,s),$$

where $\mathcal{B}$ is the set of all families $(k_i, \phi_i, \psi_i)_{i\in I}$ with I a finite set, $k_i \in \mathbb{N}$, $\phi_i \in C_0^\infty(\Omega)$, $\psi_i \in C_0^\infty(\mathbb{R})$, $\phi_i \geq 0$, $\psi_i \geq 0$, $\sum_{i\in I} \phi_i \otimes \psi_i \leq 1$, and $(supp\ \phi_i \times supp\ \psi_i) \subseteq A$.

Proof. By Proposition 2.4 it is enough to prove that for every $\eta \in C_0^\infty(A)$ with $0 \leq \eta \leq 1$ and for every $\varepsilon > 0$ there exist two finite families $(\phi_j)_{j\in J}$, $(\psi_j)_{j\in J}$, where

$$\phi_j \in C_0^\infty(\Omega), \quad \psi_j \in C_0^\infty(\mathbb{R}), \quad \phi_j \geq 0, \quad \psi_j \geq 0,$$
$$(supp\ \phi_j) \times (supp\ \psi_j) \subseteq A, \quad \sum_{j\in J} \phi_j \otimes \psi_j \leq 1,$$

and

$$\left|\eta(x,t) - \sum_{j\in J}(\phi_j \otimes \psi_j)(x,t)\right| < \varepsilon \quad \text{for} \quad \text{every} \quad (x,t) \in A.$$

We set

$$Q_o = \{z \in \Omega \times \mathbb{R} : |z_k| \leq 1 \quad \text{for} \quad k = 1, \dots, n+1\}$$

and for every $j \in \mathbf{Z}^{n+1}$ and $\delta > 0$ we set

$$Q_j(\delta) = \delta(j + Q_o);$$

moreover we fix $(x_j, t_j) \in Q_j(\delta)$. Now, for every $j \in \mathbf{Z}^{n+1}$ we denote by $R_j(\delta)$ the union of $Q_j(\delta)$ and of its adjacent cubes and by K_j the finite subset of $\mathbf{Z}^{n+1}$ such that

$$R_j(\delta) = \bigcup_{s\in K_j} Q_s(\delta).$$

For every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\eta(x,t) - \eta(x_j,t_j)| < \varepsilon \quad \text{for} \quad \text{every} \quad (x,t) \in R_j(\delta);$$

moreover there exists a finite subset J of $\mathbf{Z}^{n+1}$ such that

$$supp\ \eta \subseteq \bigcup_{j\in J} Q_j(\delta).$$

We indicate by $\mathcal{R}$ the family $(R_j^0(\delta))_{j\in J}$, where $R_j^0(\delta)$ denotes the interior of $R_j(\delta)$. It is easy to see that there exists a partition of unity subordinate to the cover $\mathcal{R}$ consisting of tensor products $\chi_j \otimes \psi_j$ with $j \in J$, where

$$\chi_j \in C_0^\infty(\Omega), \quad \psi_j \in C_0^\infty(\mathbb{R}), \quad 0 \le \chi_j \le 1, \quad 0 \le \psi_j \le 1,$$

$$\chi_j \otimes \psi_j = 1 \quad \text{on} \quad Q_j(\delta), \quad \sum_{j\in J} \chi_j \otimes \psi_j = 1,$$

and

$$(supp\ \chi_j) \times (supp\ \psi_j) \subseteq R_j^0(\delta).$$

For each $j \in J$ we define

$$\phi_j = \eta(x_j, t_j)\chi_j;$$

fixed $(x,t) \in supp\ \eta$ there exists $j_o \in J$ such that $(x,t) \in Q_{j_o}(\delta)$; then we have

$$\begin{aligned} |\eta(x,t) - \sum_{j\in J}(\phi_j \otimes \psi_j)(x,t)| &= |\eta(x,t) - \sum_{j\in J} \eta(x_j,t_j)(\chi_j \otimes \psi_j)(x,t)| = \\ &= |\eta(x,t) - \sum_{s\in K_{j_o}} \eta(x_s,t_s)(\chi_s \otimes \psi_s)(x,t)| = \\ &= \sum_{s\in K_{j_o}} (\chi_s \otimes \chi_s)(x,t)|\eta(x,t) - \eta(x_s,t_s)| < \varepsilon. \end{aligned}$$

□

Now we deal with the problem that the functional F is well defined.

Definition 1. We call "integrand" a function $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [\infty, +\infty]$ such that

(i) f is $B(\Omega) \otimes L(\mathbb{R}) \otimes B(\mathbb{R}^n)$-measurable;

(ii) the map $(x,p) \mapsto f(x,s,p)$ is lower semicontinuous on Ω for each $s \in \mathbb{R}$;

(iii) the map $(x,s) \mapsto f(x,s,0)$ is $B(\Omega) \otimes B(\mathbf{R})$-measurable.

Definition 2. We say that two functions $f, g : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [\infty, +\infty]$ are "equivalent" if

(a) there exists a Borel subset N of $\mathbb{R}$ such that $\text{meas}(N) = 0$ and $f(x,s,p) = g(x,s,p)$ for each $x \in \Omega$, $s \in \mathbb{R} \setminus N$ and $p \in \mathbb{R}^n$;

(b) $f(x,s,0) = g(x,s,0)$ for each $(x,s) \in \Omega \times \mathbb{R}$.

Remark 1. For every $B(\Omega) \otimes L(\mathbb{R}) \otimes B(\mathbb{R}^n)$-measurable function f there exist a $B(\Omega) \otimes B(\mathbb{R}) \otimes B(\mathbb{R}^n)$ – *measurable* function g and a subset N of $\mathbb{R}$ with $\text{meas}(N) = 0$ such that $f(x,s,p) = g(x,s,p)$ for every $x \in \Omega$, $s \in \mathbb{R} \setminus N$ and $p \in \mathbb{R}^n$. Indeed, for every $A \in B(\Omega) \otimes L(\mathbb{R}) \otimes B(\mathbb{R}^n)$ there exist a set $B \in B(\Omega) \otimes B(\mathbb{R}) \otimes B(\mathbb{R}^n)$ and a subset N of $\mathbb{R}$ with $\text{meas}(N) = 0$ such that for every $s \in \mathbb{R} \setminus N$

$$\{(x,p) \in \Omega \times \mathbb{R}^n : (x,s,p) \in A\} = \{(x,p) \in \Omega \times \mathbb{R}^n : (x,s,p) \in B\}.$$

This easily follows from the fact that the σ-algebra of the sets $A \subseteq \Omega \times \mathbb{R} \times \mathbb{R}^n$ satisfying this property contains the sets of the form $A_1 \times A_2 \times A_3$, with $A_1 \in B(\Omega)$, $A_2 \in L(\mathbb{R})$, $A_3 \in B(\mathbb{R}^n)$. Moreover we remark that for every integrand f, convex in p, there exists a $B(\Omega) \otimes B(\mathbb{R}) \otimes B(\mathbb{R}^n)$-measurable function g, convex in p, such that f and g are equivalent integrands. In fact from the previous part of this remark there exist a Borel function $h : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [-\infty, +\infty]$, convex in p, and a Borel set $N \subseteq \mathbb{R}$ such that $\text{meas}(N) = 0$ and $f(x,s,p) = h(x,s,p)$ for every $x \in \Omega$, $s \in \mathbb{R} \setminus N$ and $p \in \mathbb{R}^n$. Now we define a new function $g : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [-\infty, +\infty]$ by

$$g(x,s,p) = \begin{cases} h(x,s,p) & \text{if } s \in \mathbb{R} \setminus N, \\ \\ f(x,s,0) & \text{if } s \in N. \end{cases}$$

We note that g is a Borel function convex in p and f and g are equivalent integrands.

Lemma 2.6. *Let f, g be two equivalent integrands, convex in the variable p. If $u \in BV(\Omega)$, then*

$$f(x,u(x),D_a u(x)) = g(x,u(x),D_a u(x)) \quad \text{a.e.} \quad \text{on} \quad \Omega \tag{2.19}$$

and

(2.20)
$$\int_{u_-}^{u_+} f^\infty\left(x,t,\frac{D_s u}{|D_s u|}(x)\right)dt = \int_{u_-}^{u_+} g^\infty\left(x,t,\frac{D_s u}{|D_s u|}(x)\right)dt \quad |D_s u|-a.e. \quad on \quad \Omega.$$

Proof. Let B be a Borel set in $\mathbb{R}$ such that $\operatorname{meas}(B) = 0$ and $f(x,s,p) = g(x,s,p)$ for each $x \in \Omega$, $s \in \mathbb{R} \setminus N$ and $p \in \mathbb{R}^n$. We consider the set $E = \{x \in \Omega(u) : \overline{u}(x) \in N\}$. From Lemma 2.1 we obtain that Du vanishes on E; hence, from (b) of Definition 2, the formula (2.19) holds. On the other hand, f^∞ and g^∞ are also equivalent functions and $f^\infty(x,s,p) = g^\infty(x,s,p)$ for each $x \in \Omega, s \in \mathbb{R} \setminus N$ and $p \in \mathbb{R}^n$. Then we have

(2.21)
$$f^\infty\left(x,u(x),\frac{D_s u}{|D_s u|}(x)\right) = g^\infty\left(x,u(x),\frac{D_s u}{|D_s u|}(x)\right) \quad |D_s u| - a.e. \quad \text{on} \quad \Omega \setminus N(u);$$

moreover, since $\operatorname{meas}(N) = 0$, we have

$$\int_{u_-(x)}^{u_+(x)} f^\infty\left(x,t,\frac{D_s u}{|D_s u|}(x)\right)dt = \int_{u_-(x)}^{u_+(x)} g^\infty\left(x,t,\frac{D_s u}{|D_s u|}(x)\right)dt \tag{2.22}$$

for every $x \in N(u)$. Therefore (2.20) follows from (2.21) and (2.22). □

Lemma 2.7. *If f is an integrand convex in p and $u \in BV(\Omega)$, then the function $x \mapsto f(x,u(x),D_a u(x))$ is measurable on Ω and the function*

$$x \mapsto \int_{u_-(x)}^{u_+(x)} f^\infty\left(x,t,\frac{D_s u}{|D_s u|}(x)\right)dt$$

is $|D_s u|$-measurable on Ω.

Proof. By Remark 1 there exists a Borel function g, convex in p, such that f and g are equivalent integrands. Then from Lemma 2.6 the assertion follows. □

2.3 Semicontinuity of approximating functionals

In this section we prove the lower semicontinuity of the functional (2.18) in the particular case in which the integrand function f is of the form

$$f(x,s,p) = \psi(s)[a(x,s) + \langle b(x,s), p\rangle]^+,$$

where $\psi \in C_0^\infty(\mathbb{R})$ and $a(x,s)$, $b(x,s)$ are continuous in x and measurable in s. Since the lower semicontinuity of the first term of (2.18) follows easily from Scorza-Dragoni's Theorem and Fatou's Lemma, it is enough to consider the case where

$$f(x,s,p) = y(s)\,\langle b(x,s), p\rangle^+ .$$

The idea of the proof consists in representing the functional as an integral over the reduced boundary of the subgraph of u in $\Omega \times \mathbb{R}$ and in approximating this integral with simpler ones, whose integrands are of the type

$$f(x,s,p) = y(s)\mathbf{1}_E(s)\,\langle q\mathbf{1}_U(x), p\rangle^+ ,$$

where $E \subseteq \mathbb{R}$, $U \subseteq \Omega$ and $q \in \mathbf{Q}^n$. The result for this functional is then obtained using the chain rule for $BV(\Omega)$ functions and the following technical lemma which allows to write the above functional as an integral on Ω with respect to the measure Du.

Lemma 3.1. *Let $\psi : \mathbb{R} \to \mathbb{R}^n$ be a measurable and bounded function. Then for every $u \in BV(\Omega)$, $\chi \in C_0^\infty(\Omega)$, $\chi \geq 0$, and $i : 1, \ldots, n$ we have*

$$\int_{\Omega\times\mathbb{R}} \psi(s)\chi(x)\alpha_i(u)(x,s) = \int_\Omega \widehat{\psi}(u)(x)\chi(x)D_i u(x),$$

where $\widehat{\psi}(u)$ denotes the Vol'pert averaged superposition.

Proof. First of all we note that from (2.8) and (2.11) we have

$$\begin{aligned} &\int_{\Omega\times\mathbb{R}} \psi(s)\chi(x)\frac{\alpha_i(u)}{|\alpha(u)|}(x,s)|\alpha(u)|(x,s) = \\ &\int_{(\Omega\times\mathbb{R})\cap C(u)} \psi(s)\chi(x)n_i[(x,s);S(u)]H^n(x,s), \end{aligned} \tag{3.1}$$

where $C(u)$ is defined by (2.7). Now, since

$$[(\Omega \setminus N(u)) \times \mathbb{R}] \cap C(u) = \{(x,s) \in \Omega \times \mathbb{R} : u_-(x) = s = u_+(x)\},$$

from (2.8) and (2.10) we have

$$\begin{aligned}
&\int_{[(\Omega\setminus N(u))\times\mathbb{R}]\cap C(u)} \psi(s)\chi(x)n_i[(x,s);S(u)]H^n(x,s) = \\
&= \int_{[(\Omega\setminus N(u))\times\mathbb{R}]\cap C(u)} \widehat{\psi}(u)(x)\chi(x)n_i[(x,s);S(u)]H^n(x,s) = \\
&= \int_{(\Omega\setminus N(u))\times\mathbb{R}} \widehat{\psi}(u)(x)\chi(x)\alpha_i(u)(x,s) = \\
&= \int_{\Omega\setminus N(u)} \widehat{\psi}(x)\chi(x)D_i u(x).
\end{aligned} \tag{3.2}$$

On the other hand from (2.4) we obtain

$$\begin{aligned}
&\int_{[N(u)\times\mathbb{R}]\cap C(u)} \psi(s)\chi(x)n_i[(x,s);S(u)]H^n(x,s) = \\
&\int_{[N(u)\times\mathbb{R}]\cap C(u)} \psi(s)\chi(x)n_i[(x,s);S(u)]H^{n-1}(x) \times \mathcal{L}^1(s);
\end{aligned}$$

now since for H^{n-1} almost all $x \in N(u)$ there exists $n(x) = (n_1(x), \ldots, n_n(x)) \in \mathbb{R}^n$ such that $n[(x,s);S(u)] = (n(x), 0)$, whenever $s \in]u_-(x), u_+(x)[$ (see [50], Theorem 4.5.9 (17)), and by (2.9), (2.10) and (2.11) we have

$$\begin{aligned}
&\int_{[N(u)\times\mathbb{R}]\cap C(u)} \psi(s)\chi(x)n_i[(x,s);S(u)]H^n(x,s) = \\
&= \int_{N(u)} \left[\int_{u_-(x)}^{u_+(x)} \psi(s)ds\right] \chi(x)n_i(x)H^{n-1}(x) = \\
&= \int_{N(u)} \left[⨍_{u_-(x)}^{u_+(x)} \psi(s)ds\right] \chi(x)n_i(x)|\nu(u)|(x) = \\
(3.3) \qquad &= \int_{N(u)} \widehat{\psi}(u)(x)\chi(x)n_i(x)|\nu(u)|(x) = \\
&= \int_{N(u)\times\mathbb{R}} \widehat{\psi}(u)(x)\chi(x)n_i[(x,s);S(u)]|\alpha(u)|(x,s) = \\
&= \int_{N(u)\times\mathbb{R}} \widehat{\psi}(u)(x)\chi(x)\frac{\alpha_i(u)}{|\alpha(u)|}(x,s)|\alpha(u)|(x,s) = \\
&= \int_{N(u)\times\mathbb{R}} \widehat{\psi}(u)(x)\chi(x)D_iu(x).
\end{aligned}$$

Therefore from (3.1), (3.2), and (3.3) the assertion follows. □

For every $u \in BV(\Omega)$ we denote by $\alpha_{1,\dots,n}(u)$ the vector measure $(\alpha_1(u),\dots,\alpha_n(u))$ formed by the first n components of the measure $\alpha(u)$ introduced in par. 2.4 .

Lemma 3.2. *Let E be a measurable and bounded subset of $\mathbb{R}$ and let $q \in \mathbb{R}^n$. Then for every $\chi \in C_0^\infty(\Omega)$, $\chi \geq 0$, and for every $\psi \in C_0^\infty(\mathbb{R})$, $\psi \geq 0$, the functional $L : BV(\Omega) \to \mathbb{R}$ defined by*

$$L(u) = \int_{\Omega\times\mathbb{R}} 1_E(s)\psi(s)\left\langle q, \frac{\alpha_{1,\dots,n}(u)}{|\alpha(u)|}(x,s)\right\rangle \chi(x)|\alpha(u)|(x,s),$$

is continuous on $BV(\Omega)$ with respect to the topology induced by $L^1(\Omega)$.

Proof. By Lemma 3.1 we have

$$L(u) = \int_\Omega (\widehat{1_E\psi}(u)(x)\,\langle q, Du\rangle\,\chi(x).$$

Now we consider $\Psi : \mathbb{R} \to \mathbb{R}$ defined by

$$\Psi(t) = \int_0^t \mathbf{1}_E(s)\psi(s)ds$$

and, since Ψ is a Lipschitz continuous function, using the chain rule for $BV(\Omega)$ functions, (see Lemma 2.2), we obtain

$$L(u) = \int_\Omega \langle q, D(\Psi \circ u)\rangle \chi(x) = -\int_\Omega \langle q, D\chi\rangle \Psi(u)dx.$$

Therefore the continuity of L immediately follows. □

Lemma 3.3. *Let E be a measurable bounded subset of $\mathbb{R}$ and let $q \in \mathbb{R}^n$. Then for every $\chi \in C_0^\infty(\Omega)$, $\chi \geq 0$, and for every $\psi \in C_0^\infty(\mathbb{R})$, $\psi \geq 0$, the functional $H : BV(\Omega) \to [0, +\infty[$ defined by*

$$H(u) = \int_{\Omega\times\mathbb{R}} \psi(s)\mathbf{1}_E(s)\left\langle q, \frac{\alpha_{1,\dots,n}}{|\alpha(u)|}(x,s)\right\rangle^+ \chi(x)|\alpha(u)|(x,s),$$

is $L^1(\Omega)$ lower semicontinuous along sequences bounded in $BV(\Omega)$.

Proof. First we observe that

$$H(u) = \sup\{\int_{\Omega\times\mathbb{R}} \psi(x)\mathbf{1}_E(s)\left\langle q, \frac{\alpha_{1,\dots,n}(u)}{|\alpha(u)|}(x,s)\right\rangle \chi(x)\eta(x,s)|\alpha(u)|(x,s) :$$
$$\eta \in C_0^\infty(\Omega\times\mathbb{R}), 0 \leq \eta \leq 1\}.$$

Now arguing as in the proof of Lemma 2.5 (see [39]) for every $\eta \in C_0^\infty(\mathbb{R}^{n+1})$, with $0 \leq \eta \leq 1$, and for every $\varepsilon > 0$ there exist two finite families $(\phi_i)_{i\in I}$ and $(\omega_i)_{i\in I}$, where $\phi_i \in C_0^\infty(\Omega)$, $\omega_i \in C_0^\infty(\mathbb{R})$, $\phi_i \geq 0$, $\omega_i \geq 0$, $(supp\ \phi_i) \times (supp\ \omega_i) \subseteq \Omega \times \mathbb{R}$, $\sum_{i\in I} \phi_i \otimes \omega_i \leq 1$,

$$|\eta(x,s) - \sum_{i\in I}(\phi_i \otimes \omega_i)(x,s)| < \varepsilon \quad \text{for} \quad \text{every} \quad (x,s) \in \Omega\times\mathbb{R}.$$

Then

$$H(u) = \sup_{\mathcal{D}} \sum_{i\in I} \int_{\Omega\times\mathbb{R}} \psi(s)\mathbf{1}_E(s)\left\langle q, \frac{\alpha_{1,\dots,n}(u)}{|\alpha(u)|}(x,s)\right\rangle \chi(x)\phi_i(x)\omega_i(x)|\alpha(u)|(x,s),$$

where $\mathcal{D}$ is the set of all families $(\phi_i, \omega_i)_{i\in I}$ with I a finite set, $\phi_i \in C_0^\infty(\Omega)$, $\omega_i \in C_0^\infty(\mathbb{R})$, $\phi_i \geq 0$, $\omega_i \geq 0$, $\sum_{i\in I} \phi_i \otimes \omega_i \leq 1$ and $(supp\ \phi_i) \times (supp\ \omega_i) \subseteq \Omega \times \mathbb{R}$.

Therefore the thesis follows from Lemma 3.2. □

Lemma 3.4. *Let U be an open subset of Ω, let E be a measurable bounded subset of $\mathbb{R}$ and let $q \in \mathbb{R}^n$. Then for every $\chi \in C_0^\infty(\Omega)$, $\chi \geq 0$, and for every $\psi \in C_0^\infty(\mathbb{R})$, $\psi \geq 0$, the functional $\Phi : BV(\Omega) \to [0, +\infty[$, defined by*

$$\Phi(u) = \int_{\Omega\times\mathbb{R}} \psi(s)\mathbf{1}_E(s)\left\langle q, \frac{\alpha_{1,\ldots,n}(u)}{|\alpha(u)|}(x,s)\right\rangle^+ \chi(x)\mathbf{1}_U(x)|\alpha(u)|(x,s),$$

is $L^1(\Omega)$ lower semicontinuous along sequences bounded in $BV(\Omega)$.

Proof. It is easy to see that there exists an increasing sequence (χ_h), $\chi_h \in C_0^\infty(\Omega)$, $0 \leq \chi_h \leq 1$ converging to $\mathbf{1}_U$ almost everywhere. Then from Beppo Levi's Theorem we have

$$\Phi(u) = \sup_{h\in\mathbb{N}} \int_{\Omega\times\mathbb{R}} \psi\mathbf{1}_E(s)\left\langle q, \frac{\alpha_{1,\ldots,n}(u)}{|\alpha(u)|}(x,s)\right\rangle^+ \chi(x)\chi_h(x)|\alpha(u)|(x,s).$$

Therefore from Lemma 3.3 the functional Φ is lower semicontinuous. $\square$

Proposition 3.5. *Let $b : \Omega\times\mathbb{R} \to \mathbb{R}^n$ be a Borel function satisfying the following properties:*

(i) for every $s \in \mathbb{R}$ the map $x \mapsto b(x,s)$ is continuous in Ω;

(ii) there exists $M \in \mathbb{R}^+$ such that $|b(x,s)| \leq M$ for every $x \in \Omega$ and $s \in \mathbb{R}$. Then for every $\chi \in C_0^\infty(\Omega)$, $\chi \geq 0$, and for every $\psi \in C_0^\infty(\mathbb{R})$, $\psi \geq 0$, the functional $F : BV(\Omega) \to [0, +\infty[$, defined by

$$\mathcal{F}(u) = \int_\Omega \psi(u)\langle b(x,u), D_a u\rangle^+ \chi dx + \int_\Omega \left[\fint_{u_-(x)}^{u_+(x)} \psi(t)\left\langle b(x,t)\frac{D_s u}{|D_s u|}\right\rangle^+ dt \right] \chi |D_s u|,$$

is $L^1(\Omega)$ lower semicontinuous along sequences bounded in $BV(\Omega)$.

Proof. First of all, we note that from Lemma 2.3 we have

$$\mathcal{F}(u) = \int_{\Omega\times\mathbb{R}} \psi \left\langle b(x,s)\frac{\alpha_{1,\ldots,n}(u)}{|\alpha(u)|}(x,s)\right\rangle^+ \chi(x)|\alpha(u)|(x,s).$$

Following the lines of the proof of Lemma 2.6 in [7], we define

$$S = \{(q, U) \in \mathbf{Q}^n \times \Delta : |q| \leq M\},$$

where Δ is a countable base of open sets in Ω such that each element of Δ is relatively compact in Ω. Let $S = \{(q_k, U_k)\}_{k \in \mathbb{N}}$; given $\varepsilon > 0$, for every $k \in \mathbb{N}$ we define

$$E_k = \{s \in \mathbb{R} : \langle b(x,s), p\rangle^+ + \varepsilon|p| \geq \langle q_k \mathbf{1}_{U_k}(x), p\rangle^+ \quad \forall x \in \Omega, \quad \forall p \in \mathbb{R}^n\}.$$

From the projection theorem (see [26], Theorem III.23) for every $k \in \mathbb{N}$ E_k is a measurable set; now we define for each $k \in \mathbb{N}$

$$g_k : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, +\infty[$$

by

$$g_k(x, s, p) = \mathbf{1}_{E_k}(s) \langle q_k \mathbf{1}_{U_k}(x), p\rangle^+$$

and $g : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, +\infty[$ by

$$g(x, s, p) = \sup_{k \in \mathbb{N}} g_k(x, s, p).$$

From the definition of g_k and E_k the function g satisfies the following properties: for every $x \in \Omega$, $s \in \mathbb{R}$ and $p \in \mathbb{R}^n$

$$0 \leq g(x, s, p) \leq \langle b(x,s), p\rangle^+ + \varepsilon|p|,$$

$$0 \leq g(x, s, p) \leq M|p|$$

and, by using the continuity of the map $x \mapsto b(x, s)$,

$$\langle b(x,s), p\rangle^+ \leq g(x, s, p).$$

Now we define the functional $G : BV(\Omega) \to [0, +\infty[$ by

$$G(u) = \int_{\Omega \times \mathbb{R}} \psi(s) g\left((x,s), \frac{\alpha_{1,\ldots,n}(u)}{|\alpha(u)|}(x,s)\right) \chi(x) |\alpha(u)|(x,s)$$

and we prove that the functional G is lower semicontinuous on $BV(\Omega)$ with respect to the topology induced by $L^1(\Omega)$. From (2.10) follows that for each $k \in \mathbb{N}$

$$|\alpha(u)|(U_k \times \mathbb{R}) = |\nu(u)|(U_k) \leq \int_{U_k} |Du| + \mathcal{L}^n(U_k) < +\infty;$$

then the function $(x,s) \mapsto g_k\left((x,s), \frac{\alpha_{1,\dots,n}(u)}{|\alpha(u)|}(x,s)\right)$ belongs to $L^1(|\alpha(u)|)$. Hence from lemma 2.5 we have

$$G(u) = \sup_{\mathcal{B}} \sum_{i\in I} \int_{\Omega\times\mathbb{R}} \left[\psi(s) g_{k_i}\left((x,s), \frac{\alpha_{1,\dots,n}(u)}{|\alpha(u)|}(x,s)\right)\chi(x)\right] \phi_i(x)\omega_i(s)|\alpha(u)|(x,s),$$

where $\mathcal{B}$ is the set of all families $(k_i, \phi_i, \omega_i)_{i\in I}$ with I a finite set, $k_i \in \mathbb{N}$, $\phi_i \in C_0^\infty(\Omega)$, $\phi_i \geq 0$, $\omega_i \in C_0^\infty(\mathbb{R})$, $\omega_i \geq 0$, $\sum_{i\in I} \phi_i \otimes \omega_i \leq 1$ and $(supp\ \phi_i) \times (supp\ \omega_i) \subseteq \Omega \times \mathbb{R}$.

Therefore the lower semicontinuity of G follows from Lemma 3.4. Finally we prove the lower semicontinuity of $\mathcal{F}$. Let (u_h) be a sequence in $BV(\Omega)$ and $u \in BV(\Omega)$ such that (u_h) converges to u in $L^1(\Omega)$ and $\limsup_h \int_\Omega |Du_h| < +\infty$. From the previous properties of g and from the lower semicontinuity of G we have:

$$\begin{aligned}\mathcal{F}(u) &\leq G(u) \leq \liminf_h G(u_h) \leq \\ &\leq \liminf_h \mathcal{F}(u_h) + \varepsilon \limsup_h \int_{\Omega\times\mathbb{R}} \psi(s)\chi(x)|\alpha_{1,\dots,n}(u_h)|(x,s).\end{aligned}$$

Since u_h converges weakly on $BV(\Omega)$ to u and so $\limsup_h \int_B |Du_h| < +\infty$, and since by (2.10)

$$\int_B |Du_h| = \int_{B\times\mathbb{R}} |\alpha_{1,\dots,n}(u_h)|$$

for every $B \in B(\Omega)$, we have

$$\limsup_h \int_{B\times\mathbb{R}} |\alpha_{1,\dots,n}(u_h)| < +\infty.$$

Therefore the lower semicontinuity of F follows from the arbitrariness of ε. $\square$

Corollary 3.6. *Let $b : \Omega \times \mathbb{R} \to \mathbb{R}^n$ be as in Proposition 3.5 and let $a : \Omega \times \mathbb{R} \to]-\infty, 0]$ be a Borel function such that:*

(i) for every $s \in \mathbb{R}$ the map $x \mapsto a(x,s)$ is continuous in Ω;

(ii) there exists $M \in \mathbb{R}^+$ such that $-M \leq a(x,s)$ for every $x \in \Omega$ and $s \in \mathbb{R}$. Then for every $\chi \in C_0^\infty(\Omega)$, $\chi \geq 0$, and for every $\psi \in C_0^\infty(\mathbb{R})$, $\psi \geq 0$, the functional $G : BV(\Omega) \to [0, +\infty[$ defined by

$$G(u) = \int_\Omega \psi(u)[a(x,u) + \langle b(x,u), D_a u\rangle]^+ \chi(x)dx + \int_\Omega \left[\fint_{u_-(x)}^{u_+(x)} \psi(t) \left\langle b(x,t), \frac{D_s u}{|D_s u|} \right\rangle^+ dt \right] \chi(x)|D_s u|,$$

is $L^1(\Omega)$ lower semicontinuous along sequences bounded in $BV(\Omega)$.

Proof. By Scorza-Dragoni's Theorem there exists an increasing sequence (K_h) of compact subsets of $\mathbb{R}$ such that $\text{meas}(\mathbb{R} \setminus E) = 0$, where $E = \bigcup_{k \in \mathbb{N}} K_h$, and the function $a : \Omega \times K_h \to \mathbb{R}$ is continuous (see [47], page 235). From Lemma 2.6 we obtain

$$\mathcal{G}(u) = \int_\Omega \mathbf{1}_E(u)\psi(u)[a(x,u) + \langle b(x,u), D_a u\rangle]^+ \chi(x)dx + \int_\Omega \left[\fint_{u_-(x)}^{u_+(x)} \mathbf{1}_E(t)\psi(t) \left\langle b(x,t), \frac{D_s u}{|D_s u|} \right\rangle^+ dt \right] \chi(x)|D_s u|.$$

From Beppo Levi's Theorem we have

$$\mathcal{G}(u) = \sup_{h \in \mathbb{N}} \left\{ \int_\Omega \mathbf{1}_{K_h}(u)\psi(u)[a(x,u) + \langle b(x,u), D_a u\rangle]^+ \chi(x)dx + \int_\Omega \left[\fint_{u_-(x)}^{u_+(x)} \mathbf{1}_{K_h}(t)\psi(t) \left\langle b(x,t), \frac{D_s u}{|D_s u|} \right\rangle^+ dt \right] \chi(x)|D_s u| \right\};$$

now, since $a(x,s) \leq 0$, we have $[a(x,s) + \langle b(x,s), p\rangle]^+ = [a(x,s) + \langle b(x,s), p\rangle^+]^+$;

hence

$$\mathcal{G}(u) = \sup_{h\in\mathbb{N}} \sup_{\substack{\eta\in C_0^\infty(\Omega)\\ 0\le\eta\le 1}} \left\{ \int_\Omega \mathbf{1}_{K_h}(u)\psi(u)a(x,u)\eta(x)\chi(x)dx + \right.$$

$$+ \int_\Omega \mathbf{1}_{K_h}(u)\psi(u)\left\langle b(x,u), D_a u\right\rangle^+ \eta(x)\chi(x)dx +$$

$$\left. + \int_\Omega \left[\fint_{u_-(x)}^{u_+(x)} \mathbf{1}_{K_h}(t)\psi(t)\left\langle b(x,t), \frac{D_s u}{|D_s u|}\right\rangle^+ dt \right] \eta(x)\chi(x)|D_s u| \right\}.$$

Now we note that $\psi(s)a(x,s) \le 0$ for each $x \in \Omega$ and $s \in \mathbb{R}$ and so the map $s \mapsto \mathbf{1}_{K_h}(s)\psi(s)a(x,s)$ is lower semicontinuous on $\mathbb{R}$. Therefore the lower semicontinuity of $\mathcal{G}$ follows from Fatou's Lemma and Proposition 3.5. $\square$

2.4 The main semicontinuity result

In this section we prove some lower semicontinuity results for the functionals

$$F(u) = \int_\Omega f(x,u,D_a u)dx + \int_\Omega \left[\fint_{u_-}^{u_+} f^\infty\left(x,t,\frac{D_s u}{|D_s u|}\right) dt \right] |D_s u|$$

and

$$I(f,u,\phi) = \int_\Omega f(x,u,D_a u)\phi dx + \int_\Omega \left[\fint_{u_-}^{u_+} f^\infty\left(x,t,\frac{D_s u}{|D_s u|}\right) dt \right] \phi|D_s u|,$$

defined in par. 2.7.

The technique we used in [39] allows us to prove the first theorem in which we assume that $f(x,s,0) = 0$.

Theorem 4.1. *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0,+\infty]$ be a $B(\Omega) \otimes L(\mathbb{R}) \otimes B(\mathbb{R}^n)$-measurable function such that:*

(a) for every $s \in \mathbb{R}$ the map $(x,p) \mapsto f(x,s,p)$ is lower semicontinuous on $\Omega \times \mathbb{R}^n$;

(b) for every $x \in \Omega$ and $s \in \mathbb{R}$ the map $p \mapsto f(x,s,p)$ is convex on $\mathbb{R}^n$;

(c) for every $x \in \Omega$ and $s \in \mathbb{R}$ we have $f(x,s,0) = 0$.

Then for every $\phi \in C^o(\Omega)$, $\phi \geq 0$, the functional $u \mapsto I(f,u,\phi)$, is well defined on $BV(\Omega)$ and $L^1(\Omega)$ lower semicontinuous along sequences bounded in $BV(\Omega)$.

Proof. First of all, Lemma 2.7 assures that for every $\phi \in C^o(\Omega)$ the functional $u \mapsto I(f,u,\phi)$ is well defined. Moreover we observe that it is enough to prove the theorem for the case where $f \in C_0^\infty(\Omega)$; in fact, in the general case, we can approximate a non negative function $\phi \in C^o(\Omega)$ with an increasing sequence (ϕ_h) of non negative functions in $C_0^\infty(\Omega)$ and using Beppo Levi's Theorem we obtain

$$I(f,u,\phi) = \sup_h I(f,u,\phi_h).$$

Hence we assume that $\phi \in C_0^\infty(\Omega)$. By Lemma 1.7 and Remark 1.8 of [7] there exist two sequences $a_h : \Omega \times \mathbb{R} \to \mathbb{R}$ and $b_h : \Omega \times \mathbb{R} \to \mathbb{R}^n$ of bounded functions such that for every $h \in \mathbb{N}$ the maps $x \mapsto a_h(x,s)$ and $x \mapsto b_h(x,s)$ are continuous on Ω for each $s \in \mathbb{R}$, the maps $s \mapsto a_h(x,s)$ and $x \mapsto b_h(x,s)$ are measurable on $\mathbb{R}$ for each $x \in \Omega$ and

$$f(x,s,p) = \sup_{h \in \mathbb{N}} [a_h(x,s) + \langle b_h(x,s), p\rangle]^+.$$

By the hypothesis (c) we have $a_h(x,s) \leq 0$ for every $x \in \Omega$ and $s \in \mathbb{R}$. Let us define

$$f_h(x,s,p) = [a_h(x,s) + \langle b_h(x,s), p\rangle]^+.$$

It is easy to verify that

$$\widetilde{f}(x,s,p,t)\phi(x) = \sup_{h \in \mathbb{N}} \widetilde{f}_h(x,s,p,t)\phi(x)$$

for every $x \in \Omega$, $s \in \mathbb{R}$, $p \in \mathbb{R}^n$ and $t \in]-\infty, 0]$, where the function $\widetilde{f}$ and $\widetilde{f}_h$ are defined as in (2.16). Using Lemma 2.3 applied to the integrand $f(x,s,p)\phi(x)$ we can represent the functional $I(f,u,\phi)$ as

$$I(f,u,\phi) = \int_{\Omega \times \mathbb{R}} \widetilde{f}\left((x,s), \frac{\alpha(u)}{|\alpha(u)|}(x,s)\right) \phi(x)|\alpha(u)|(x,s).$$

From Lemma 2.5

$$I(f,u,\phi)=\sup_{\mathcal{B}}\sum_{i\in I}\int_{\Omega\times\mathbb{R}}\widetilde{f}_{k_i}\left((x,s),\frac{\alpha(u)}{|\alpha(u)|}(x,s)\right)\phi_i(x)\psi_i(s)|\alpha(u)|(x,s),$$

where $\mathcal{B}$ is the set of all families $(k_i,\phi_i,\psi_i)_{i\in I}$ with I a finite set, $k_i\in\mathbb{N}$, $\phi_i\in C_0^\infty(\Omega)$, $\phi_i\geq 0$, $\psi_i\in C_0^\infty(\mathbb{R})$, $\psi_i\geq 0$, $\sum_{i\in I}\phi_i\otimes\psi_i\leq 1$ and $(supp\ \phi_i)\times(supp\ \psi_i)\subseteq\Omega\times\mathbb{R}$. Now by Lemma 2.3 again

$$I(f,u,\phi)=\sup_{\mathcal{B}}\sum_{i\in I}G_i(u),$$

where

$$\begin{aligned}G_i(u)=&\int_\Omega\psi_i(u)f_{k_i}(x,u,D_au)\phi_i(x)dx+\\&+\int_\Omega\left[\fint_{u_-(x)}^{u_+(x)}\psi_i(t)f_{k_i}^\infty\left(x,t,\frac{D_su}{|D_su|}\right)dt\right]\phi_i(x)|D_su|=\\=&\int_\Omega\psi_i(u)[a_{k_i}(x,u)+\langle b_{k_i}(x,u),D_au\rangle]^+\phi_i(x)dx+\\&+\int_\Omega\left[\fint_{u_-(x)}^{u_+(x)}\psi_i(t)\left\langle b_{k_i}(x,t),\frac{D_su}{|D_su|}\right\rangle^+dt\right]\phi_i(x)|D_su|.\end{aligned}$$

Therefore the lower semicontinuity follows from Corollary 3.6. □

The hypothesis (c) in the previous theorem is too restrictive. In the next theorem we look for a more general hypothesis, which still assures the lower semicontinuity of the functional F. Suppose that f satisfies all hypotheses of Theorem 4.1 except (c). Arguing as in [AM] and [DBD], we will introduce a new function

$$g(x,s,p)=f(x,s,p)-f(x,s,0)-\langle\lambda(x,s),p\rangle\,,$$

where $\lambda(x,s)\in\partial f(x,s,0)$ for each $(x,s)\in\Omega\times\mathbb{R}$, i.e.

$$f(x,s,p)\geq f(x,s,0)+\langle\lambda(x,s),p\rangle$$

for each $p\in\mathbb{R}^n$. The function g satisfies the hypotheses of Theorem 4.1; hence the functional $u\mapsto I(g,u,\phi)$ is lower semicontinuous. Then, in order to prove the lower semicontinuity of the functional $u \mapsto I(f, u, \phi)$, it is enough to prove it for the functionals

$$u \mapsto \int_\Omega f(x, u, 0)dx \qquad \text{and} \qquad u \mapsto \Lambda(u, \phi),$$

where

$$\Lambda(u,\phi) = \int_\Omega \langle \lambda(x,u), D_a u\rangle\, \phi dx + \int_\Omega \left[\int_{u_-(x)}^{u_+(x)} \left\langle \lambda(x,t), \frac{D_s u}{|D_s u|} \right\rangle dt \right] \phi |D_s u|. \tag{4.1}$$

The semicontinuity of $u \mapsto \int_\Omega f(x, u, 0)dx$ follows from Fatou's Lemma, provided that the function $(x, s) \mapsto f(x, s, 0)$ is lower semicontinuous.

The lower semicontinuity of the functional Λ will be obtained, under suitable hypotheses on λ, by the same technique introduced in [7], i.e. by approximating the functional Λ with simpler ones whose integrands are functions of the type

$$h(x, s, p) = \left\langle \sum_{i=1}^{\infty} \mathbf{1}_{E_i}(s) f_i(x), p \right\rangle,$$

where E_i are measurable disjoint sets in $\mathbb{R}$ and $f_i \in C^\infty(\Omega; \mathbb{R}^n)$, and by proving the lower semicontinuity of the approximating functionals. In order to prove this property, we need, as in [7], the following lemma which gives a formula of integration by parts and which is obtained, in our context, by applying the chain rule for $BV(\Omega)$ functions.

Lemma 4.2. *Let $(E_i)_{i\in\mathbb{N}}$ be a sequence of measurable pairwise disjoint sets in $\mathbb{R}$ such that* $\sum_{i=1}^{\infty} \operatorname{meas}(E_i) < +\infty$. *Let $(f_i)_{i\in\mathbb{N}} = (f_i^1, \dots, f_i^n)_{i\in\mathbb{N}}$ be a sequence in $C^\infty(\Omega; \mathbb{R}^n)$ such that* $\sup\{|\operatorname{div} f_i(x)| : x \in \Omega, i \in \mathbb{N}\} = M < +\infty$. *Let $b : \Omega \times \mathbb{R} \to \mathbb{R}^n$ be the function defined by*

$$b(x, s) = \sum_{i=1}^{\infty} \mathbf{1}_{E_i}(s) f_i(x).$$

Let us assume that there exists $g \in L^1(\mathbb{R})$ such that

$$|b(x, s)| \le g(s) \tag{4.2}$$

for every $x \in \Omega$ and $s \in \mathbb{R}$. Then for every $u \in BV(\Omega)$ such that

$$\int_\Omega \langle b(x,u), D_a u\rangle^+ \, dx < +\infty \tag{4.3}$$

and

$$\int_\Omega \left[\fint_{u_-}^{u_+} \left\langle b(x,t), \frac{D_s u}{|D_s u|} \right\rangle^+ dt \right] |D_s u| < +\infty \tag{4.4}$$

and for every $\phi \in C_0^\infty(\Omega)$, $\phi \geq 0$, we have

$$\begin{aligned} &\int_\Omega \langle b(x,u), D_a u\rangle \, \phi dx + \int_\Omega \left[\fint_{u_-}^{u_+} \left\langle b(x,t), \frac{D_s u}{|D_s u|} \right\rangle dt \right] \phi |D_s u| = \\ &= - \int_\Omega \langle \sigma_u, D\phi \rangle \, dx - \int_\Omega \left[\int_0^{u(x)} \sum_{i=1}^\infty \operatorname{div} f_i(x) \mathbf{1}_{E_i}(s) ds \right] \phi(x) dx, \end{aligned} \tag{4.5}$$

where $\sigma_u(x) = \displaystyle\int_0^{u(x)} b(x,s)ds$.

Proof. First we assume that $g \in L^\infty(\mathbb{R})$. Let us define for each $i \in \mathbb{N}$ and $j = 1, \dots, n$

$$\sigma_i^j(x) = f_i^j(x) \int_0^{u(x)} \mathbf{1}_{E_i}(s)ds.$$

We note that from Lemma 2.2 the functions σ_i^j belong to $BV(\Omega)$ and

$$D_j \sigma_i^j = D_j f_i^j(x) \int_0^{u(x)} \mathbf{1}_{E_i}(s)ds + f_i^j(x) \widehat{\mathbf{1}_{E_i}}(u) D_j u \quad \text{as measures on } \Omega.$$

Hence, if we set $\sigma_i = (\sigma_i^1, \dots, \sigma_i^n)$, we have

$$\operatorname{div} \sigma_i = \operatorname{div} f_i \int_0^{u(x)} \mathbf{1}_{E_i}(s)ds + \left\langle f_i(x) \widehat{\mathbf{1}_{E_i}}(u), Du \right\rangle \quad \text{as measures on } \Omega.$$

Then for every $\phi \in C_0^\infty(\Omega)$, $\phi \geq 0$, using the decomposition of the measure Du and the definition (2.13), we have

$$\begin{aligned} &\int_\Omega \langle f_i(x) \mathbf{1}_{E_i}(u), D_a u \rangle \, \phi dx + \int_\Omega \left[\fint_{u_-}^{u_+} \left\langle f_i(x) \mathbf{1}_{E_i}(t), \frac{D_s u}{|D_s u|} \right\rangle dt \right] \phi |D_s u| = \\ &= - \int_\Omega \langle \sigma_i, D\phi \rangle \, dx - \int_\Omega \left[\int_0^{u(x)} \operatorname{div} f_i(x) \mathbf{1}_{E_i}(s) ds \right] \phi(x) dx. \end{aligned} \tag{4.6}$$

Now we observe that from (4.2)

$$\sigma_u = \sum_{i=1}^{\infty} \sigma_i$$

and for every $k \in \mathbb{N}$ the following inequalities hold:

$$\left| \sum_{i=1}^{k} f_i(x) \mathbf{1}_{E_i}(s) \right| \leq \|g\|_{L^\infty}, \tag{4.7}$$

$$\left| \int_{u_-}^{u_+} \sum_{i=1}^{k} f_i(x) \mathbf{1}_{E_i}(t) dt \right| \leq \|g\|_{L^\infty}, \tag{4.8}$$

$$\left| \sum_{i=1}^{k} \sigma_i(x) \right| \leq c \|g\|_{L^\infty}, \tag{4.9}$$

$$\left| \sum_{i=1}^{k} \operatorname{div} f_i(x) \mathbf{1}_{E_i}(s) \right| \leq M \tag{4.10}$$

and

$$\left| \int_{0}^{u(x)} \sum_{i=1}^{k} \operatorname{div} f_i(x) \mathbf{1}_{E_i}(s) ds \right| \leq cM, \tag{4.11}$$

where $c = \sum_{i=1}^{\infty} \operatorname{meas}(E_i) < +\infty$.

Then from (4.6), (4.7), (4.8), (4.9), (4.10) and (4.11), using Lebesgue's dominated convergence Theorem the formula (4.5) holds. Now we consider the general case where $g \in L^1(\mathbb{R})$. For each $h \in \mathbb{N}$ we set

$$A_h = \{s \in \mathbb{R} : g(s) \leq h\}$$

and from Lemma 2.1 we can assume that $\bigcup_{h \in \mathbb{N}} A_h = \mathbb{R}$; moreover we define

$$b_h(x, s) = b(x, s) \mathbf{1}_{A_h}(s)$$

and

$$\sigma_{h,u}(x) = \int_0^{u(x)} b_h(x,s)ds;$$

hence we have

$$(4.12)\quad \begin{aligned} &\int_\Omega \langle b_h(x,u), D_a u\rangle \phi dx + \int_\Omega \left[⨍_{u_-}^{u_+} \left\langle b_h(x,t), \frac{D_s u}{|D_s u|}\right\rangle dt\right] \phi |D_s u| = \\ &= -\int_\Omega \langle \sigma_{h,u}, D\phi\rangle\, dx - \int_\Omega \left[\int_0^{u(x)} \sum_{i=1}^\infty \operatorname{div} f_i(x) \mathbf{1}_{E_i\cap A_h}(s) ds\right] \phi dx. \end{aligned}$$

Since

$$|\sigma_{h,u}(x)| \le \|g\| L^1,$$

$$|\sum_{i=1}^\infty \operatorname{div} f_i(x)\mathbf{1}_{E_i\cap A_h}(s)| \le M$$

and

$$\left| \left[\int_0^{u(x)} \sum_{i=1}^\infty \operatorname{div} f_i(x)\mathbf{1}_{E_i\cap A_h}(s)ds\right]\phi\right| \le cM\|f\|_{L^\infty},$$

we obtain

$$(4.13)\quad \begin{aligned} &\lim_{h\to\infty} \left\{ \int_\Omega \langle \sigma_{h,u}, D\phi\rangle\, dx + \int_\Omega \left[\int_0^{u(x)} \sum_{i=1}^\infty \operatorname{div} f_i(x)\mathbf{1}_{E_i\cap A_h}(s)ds\right]\phi dx\right\} = \\ &= \int_\Omega \langle \sigma_u, D\phi\rangle\, dx + \int_\Omega \left[\int_0^{u(x)} \sum_{i=1}^\infty \operatorname{div} f_i(x)\mathbf{1}_{E_i}(s)ds\right]\phi dx. \end{aligned}$$

Since $\langle b_h(x,s), p\rangle^+$ and $\langle b_h(x,s), p\rangle^-$ are increasing sequences which converge to $\langle b(x,s), p\rangle^+$ and $\langle b(x,s), p\rangle^-$ respectively, from Beppo Levi's Theorem we obtain that

$$\lim_{h\to\infty} \int_\Omega \langle b_h(x,u), D_a u\rangle^+ \phi dx = \int_\Omega \langle b(x,u), D_a u\rangle^+ \phi dx$$

and

$$\lim_{h\to\infty} \int_\Omega \langle b_h(x,u), D_a u\rangle^- \phi dx = \int_\Omega \langle b(x,u), D_a u\rangle^- \phi dx$$

so that from (4.3) we have

(4.14)

$$\lim_{h\to\infty} \int_\Omega \langle b_h(x,u), D_a u\rangle \phi dx = \int_\Omega \langle b(x,u), D_a u\rangle^+ \phi dx - \int_\Omega \langle b(x,u), D_a u\rangle^- \phi dx =$$

$$= \int_\Omega \langle b(x,u), D_a u \rangle \, \phi dx.$$

Analogously using (4.4) we have

$$\text{(4.15)} \qquad \begin{aligned} &\lim_{h\to\infty} \int_\Omega \left[\fint_{u_-}^{u_+} \left\langle b_h(x,t), \frac{D_s u}{|D_s u|} \right\rangle dt \right] \phi |D_s u| = \\ &\int_\Omega \left[\fint_{u_-}^{u_+} \left\langle b(x,t), \frac{D_s u}{|D_s u|} \right\rangle dt \right] \phi |D_s u|. \end{aligned}$$

Therefore the assertion follows from (4.12), (4.13), (4.14) and (4.15). □

Lemma 4.3. *Let $\lambda : \Omega \times \mathbb{R} \to \mathbb{R}^n$ be a function such that*

a) there exists a constant $C > 0$ such that $\lambda(x,s) = 0$ for each $x \in \Omega$ and $s \in \mathbb{R}$, with $|s| \geq C$;

b) for every $x \in \Omega$ the map $s \mapsto \lambda(x,s)$ is measurable;

c) the functions $\{x \mapsto \lambda(x,s) : s \in \mathbb{R}\}$ are equicontinuous on Ω;

d) the function $g : \mathbb{R} \to [-\infty, +\infty]$ defined by

$$g(s) = \sup_{x\in\Omega} |\lambda(x,s)|$$

belongs to $L^1(\mathbb{R})$. Let (u_h) be a sequence in $BV(\Omega)$ and $u_\infty \in BV(\Omega)$ such that u_h converges to u_∞ in $L^1(\Omega)$,

$$\limsup_h \int_\Omega |Du_h| < +\infty,$$

$$\text{(4.16)} \qquad \sup_{h\in\mathbb{N}} \int_\Omega \langle \lambda(x,u_h), D_a u_h \rangle^+ \, dx < +\infty$$

and

$$\text{(4.17)} \qquad \sup_{h\in\mathbb{N}} \int_\Omega \left[\fint_{u_{h_-}}^{u_{h_+}} \left\langle \lambda(x,t), \frac{D_s u_h}{|D_s u_h|} \right\rangle^+ dt \right] |D_s u_h| < +\infty.$$

Then for every $\phi \in C_0^\infty(\Omega)$, $\phi \geq 0$, *we have*

$$\lim_{h \to \infty} \Lambda(u_h, \phi) = \Lambda(u_\infty, \phi),$$

where Λ *is defined by* (4.1).

Proof. Of course we may suppose λ to be a Borel function (see Lemma 2.6). Since the function $h(x, s, p) = \langle \lambda(x, s), p \rangle^+$ satisfies the hypotheses of Theorem 4.1, from (4.16) and (4.17) we obtain
(4.18)

$$\int_\Omega \langle \lambda(x, u_\infty), Du_\infty \rangle^+ \, dx + \int_\Omega \left[\fint_{u_{\infty -}}^{u_{\infty +}} \left\langle \lambda(x, t), \frac{D_s u_\infty}{|D_s u_\infty|} \right\rangle^+ dt \right] |D_s u_\infty| < +\infty.$$

From Lemma 4.11 of [7], fixed $\varepsilon > 0$, there exists a sequence (f_i) in $C^\infty(\Omega; \mathbb{R}^n)$ such that

$$\text{(4.19)} \qquad \begin{array}{c} \text{for every } s \in \mathbb{R} \text{ there exists } i \in \mathbb{N} \\ \text{such that } \|f_i - \lambda(\cdot, s)\|_{L^\infty(\Omega; \mathbb{R}^n)} < \varepsilon \end{array}$$

and

$$\text{(4.20)} \qquad R = \sup\{\|\partial f_i / \partial x_j\|_{L^\infty(\Omega; \mathbb{R}^n)} : i \in \mathbb{N}, 1 \leq j \leq n\} < +\infty.$$

For each $i \in \mathbb{N}$ we set

$$B_i = \{s \in]-C, C[: \|f_i - \lambda(\cdot, s)\|_{L^\infty(\Omega; \mathbb{R}^n)} < \varepsilon\}$$

and

$$E_i = B_i \setminus \bigcup_{j<i} B_j;$$

we note that (E_i) is a sequence of measurable and disjoint sets. Now we define $b : \Omega \times \mathbb{R} \to \mathbb{R}^n$ by

$$b(x, s) = \sum_{i=1}^\infty \mathbf{1}_{E_i}(s) f_i(x).$$

From the hypothesis *d)* we have

$$\text{(4.21)} \qquad \begin{aligned} |b(x, s)| &\leq \sum_{i=1}^\infty \mathbf{1}_{E_i}(s) |f_i(x) - \lambda(x, s)| + |\lambda(x, s)| \leq \\ &\leq \varepsilon \mathbf{1}_{]-C, C[}(s) + |\lambda(x, s)| \leq \varepsilon \mathbf{1}_{]-C, C[}(s) + g(s) \end{aligned}$$

and for each $h \in \mathbb{N} \cup \{+\infty\}$

$$\begin{aligned}
&\int_\Omega \langle b(x,u_h), D_a u_h\rangle^+ \, dx + \int_\Omega \left[⨍_{u_{h_-}}^{u_{h_+}} \left\langle b(x,t), \frac{D_s u_h}{|D_s u_h|} \right\rangle^+ dt \right] |D_s u_h| \le \\
(4.22) \quad &\le \int_\Omega \langle \lambda(x,u_h), D_a u_h\rangle^+ \, dx + \int_\Omega \left[⨍_{u_{h_-}}^{u_{h_+}} \left\langle \lambda(x,t), \frac{D_s u_h}{|D_s u_h|} \right\rangle^+ dt \right] |D_s u_h| + \\
&2\varepsilon \int_\Omega |Du_h|.
\end{aligned}$$

It is easy to see that from (4.16), (4.17), (4.18), (4.20) and (4.22) the hypotheses of Lemma 4.2 are satisfied; therefore for each $\phi \in C_0^\infty(\Omega)$, $\phi \ge 0$, and for every $h \in \mathbb{N} \cup \{+\infty\}$ we obtain

$$\begin{aligned}
&\int_\Omega \langle b(x,u_h), D_a u_h\rangle \, \phi dx + \int_\Omega \left[⨍_{u_{h_-}}^{u_{h_+}} \left\langle b(x,t), \frac{D_s u_h}{|D_s u_h|} \right\rangle dt) \right] \phi |D_s u_h| = \\
(4.23) \quad &= - \int_\Omega \langle \sigma_{u_h}, D\phi\rangle \, dx - \int_\Omega \left[\int_0^{u_h(x)} \sum_{i=1}^\infty \operatorname{div} f_i(x) \mathbf{1}_{E_i}(s) ds \right] \phi dx,
\end{aligned}$$

where $\sigma_{u_h}(x) = \displaystyle\int_0^{u_h(x)} b(x,s)ds$. Since from the hypothesis $a)$ and (4.21) for each $h \in \mathbb{N}$ we have

$$|\sigma_{u_h}(x)| \le 2\varepsilon C + \|g\|_{L^1}$$

and from (4.20)

$$\left| \int_0^{u_h(x)} \sum_{i=1}^\infty \operatorname{div} f_i(x) \mathbf{1}_{E_i}(s) ds \right| \le 2CnR,$$

applying the Lebesgue's dominated Convergence Theorem, from (4.23) we have

$$\begin{aligned}
&\lim_{h \to \infty} \left\{ \int_\Omega \langle b(x,u_h), D_a u_h\rangle \, \phi dx + \int_\Omega \left[⨍_{u_{h_-}}^{u_{h_+}} \left\langle b(x,t), \frac{D_s u_h}{|D_s u_h|} \right\rangle dt \right] \phi |D_s u_h| \right\} = \\
&= \int_\Omega \langle b(x,u_\infty), D_a u_\infty\rangle \, \phi dx + \int_\Omega \left[⨍_{u_{\infty_-}}^{u_{\infty_+}} \left\langle b(x,t), \frac{D_s u_\infty}{|D_s u_\infty|} \right\rangle dt \right] \phi |D_s u_\infty|.
\end{aligned}$$

Now, since by (4.19) for every $x \in \Omega$ and $s \in \mathbb{R}$

$$|b(x,s) - \lambda(x,s)| < \varepsilon,$$

we obtain, for each $h \in \mathbb{N} \cup \{+\infty\}$

$$
\begin{aligned}
&\Big| \int_\Omega \langle b(x,u_h), D_a u_h\rangle \,\phi dx + \int_\Omega \left[\fint_{u_{h_-}}^{u_{h_+}} \left\langle b(x,t), \frac{D_s u_h}{|D_s u_h|} \right\rangle dt \right] \phi |D_s u_h| - \\
&\Big| \int_\Omega \langle \lambda(x,u_h), D_a u_h\rangle \,\phi dx - \int_\Omega \left[\fint_{u_{h_-}}^{u_{h_+}} \left\langle \lambda(x,t), \frac{D_s u_h}{|D_s u_h|} \right\rangle dt \right] \phi |D_s u_h| \\
&\qquad\qquad \leq 2\varepsilon \|f\|_{L^\infty} \int_\Omega |Du_h|.
\end{aligned}
$$

Since (u_h) converges to u_∞ in $L^1(\Omega)$ and $\limsup_h \int_\Omega |Du_h| < +\infty$, the assertion follows from the arbitrariness of ε. □

Theorem 4.4. *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0,+\infty]$ be a $\mathcal{B}(\Omega) \otimes \mathcal{L}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^n)$-measurable function such that*

(a) for every $x \in \Omega$ and $s \in \mathbb{R}$ the map $p \mapsto f(x,s,p)$ is convex on $\mathbb{R}^n$;

(b) for every $s \in \mathbb{R}$ the map $(x,p) \mapsto f(x,s,p) - f(x,s,0)$ is lower semicontinuous on $\Omega \times \mathbb{R}^n$;

(c) the map $(x,s) \mapsto f(x,s,0)$ is $\mathcal{B}(\Omega) \otimes \mathcal{B}(\mathbb{R})$-measurable and for a.e. $x \in \Omega$ the map $s \mapsto f(x,s,0)$ is lower semicontinuous on $\mathbb{R}$;

(d) there exists a function $\lambda : \Omega \times \mathbb{R} \to \mathbb{R}^n$ such that

i) $\lambda(x,s)$ is continuous in x and measurable in s;

ii) $\lambda(x,s) \in \partial f(x,s,0)$ for every $x \in \Omega$ and $s \in \mathbb{R}$, where ∂f denotes the subdifferential of $f(x,s,p)$ with respect to the variable p;

iii) for every open set $A \subset\subset \Omega$ the function $g_A(s) = \sup\{|\lambda(x,s)| : x \in A\}$ belongs to $L^1_{loc}(\mathbb{R})$;

iv) for every open set $A \subset\subset \Omega$ and for every set $B \subset\subset \mathbb{R}$ the functions $\{x \mapsto \lambda(x,s) : s \in B\}$ are equicontinuous on A.

Then the functional F, introduced in (2.17), is well defined on $BV(\Omega)$ and $L^1(\Omega)$ lower semicontinuous along sequences bounded in $BV(\Omega)$.

Proof. We may suppose that there exists a constant $C > 0$ such that $f(x,s,p) = 0$ for every $x \in \Omega$, $p \in \mathbb{R}^n$ and $s \in \mathbb{R}$, with $|s| \geq C$, since in the general case f can be approximated from below by an increasing sequence of functions having this property: in fact we can write

$$f(x,s,p) = \sup_{k\in\mathbb{N}} f(x,s,p)\mathbf{1}_{]-k,k[}(s).$$

Moreover we may suppose that $\lambda(x,s) = 0$ for every $x \in \Omega$ and $s \in \mathbb{R}$, with $|s| \geq C$. Let (u_h) be a sequence in $BV(\Omega)$ and let u_∞ in $BV(\Omega)$. Suppose that (u_h) converges to u_∞ in $L^1(\Omega)$ and $\limsup_h \int_\Omega |Du_h| < +\infty$. We may assume that $F(u_h) \leq M < +\infty$ for each $h \in \mathbb{N}$. Now, we define a new function $g : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0,+\infty]$ by

$$g(x,s,p) = f(x,s,p) - f(x,s,0) - \langle \lambda(x,s), p \rangle .$$

Then for every $\phi \in C_0^\infty(\Omega)$, $\phi \geq 0$, and for every open set $A \subset\subset \Omega$ we have

$$I_A(f,u,\phi) = I_A(g,u,\phi) + H_A(u,\phi) + \Lambda_A(u,\phi), \tag{4.24}$$

where

$$I_A(f,u,\phi) = \int_A f(x,u,D_a u)\phi dx + \int_A \left[\fint_{u_-}^{u_+} f^\infty \left(x, t, \frac{D_s u}{|D_s u|} \right) dt \right] \phi |D_s u|,$$

$$H_A(u,\phi) = \int_A f(x,u,0)\phi dx$$

and

$$\Lambda_A(u,\phi) = \int_A \langle \lambda(x,u), D_a u \rangle \phi dx + \int_A \left[\fint_{u_-}^{u_+} \left\langle \lambda(x,t), \frac{D_s u}{|D_s u|} \right\rangle dt \right] \phi |D_s u|.$$

Since the function g satisfies the hypotheses of Theorem 4.1, we obtain that

$$I_A(g,u_\infty,\phi) \leq \liminf_h I_A(g,u_h,\phi). \tag{4.25}$$

Moreover, from hypothesis (c) and Fatou's Lemma we get

$$H_A(u_\infty,\phi) \leq \liminf_h H_A(u_h,\phi). \tag{4.26}$$

Furthermore, since $\lambda(x,s) \in \partial f(x,s,0)$, i.e. for each $p \in \mathbb{R}^n$ $f(x,s,p) \geq f(x,s,0)+\langle \lambda(x,s), p\rangle$, and since $f(x,s,p) \geq 0$, we have $f(x,s,p) \geq \langle \lambda(x,s), p\rangle^+$ so that for every $h \in \mathbb{N} \cup \{+\infty\}$

$$\int_A \langle \lambda(x,u_h), D_a u_h\rangle^+ \, dx \leq F(u_h) \leq M$$

and

$$\int_A \left[\fint_{u_{h_-}}^{u_{h_+}} \left\langle \lambda(x,t), \frac{D_s u_h}{|D_s u_h|} \right\rangle^+ dt \right] |D_s u_h| \leq F(u_h) \leq M;$$

then using Lemma 4.3 we obtain

$$\Lambda_A(u_\infty, \phi) = \lim_h \Lambda_A(u_h, \phi). \tag{4.27}$$

Therefore from (4.24), (4.25), (4.26) and (4.27) we have

$$I_A(f, u_\infty, \phi) \leq \liminf_h I_A(f, u_h, \phi) \leq \liminf_h I(f, u_h, \phi);$$

then, since A is arbitrary, the functional $u \mapsto I(f,u,\phi)$ is lower semicontinuous. The conclusion follows immediately, because

$$F(u) = \sup\{I(f,u,\phi) : \phi \in C_0^\infty(\Omega), 0 \leq \phi \leq 1\}.$$

$\square$

Remark 2. The hypothesis (d) in Theorem 4.4 can be weakened; in fact if the following condition is satisfied, then the same result holds:

(d') for every $\varepsilon > 0$ there exists a function $\lambda_\varepsilon : \Omega \times \mathbb{R} \to \mathbb{R}^n$ such that

i) $\lambda_\varepsilon(x,s)$ is continuous in x and measurable in s;

ii) $\lambda_\varepsilon(x,s) \in \partial_\varepsilon f(x,s,0)$ for every $x \in \Omega$ and $s \in \mathbb{R}$, i.e. $f(x,s,p) \geq f(x,s,0)+\langle \lambda_\varepsilon(x,s), p\rangle - \varepsilon$ for each $p \in \mathbb{R}^n$;

iii) for every open set $A \subset\subset \Omega$ the function $g_{\varepsilon,A}(s) = \sup\{|\lambda_\varepsilon(x,s)| : x \in A\}$ belongs to $L^1_{loc}(\mathbb{R})$;

iv) for every open set $A \subset\subset \Omega$ and for every set $B \subset\subset \mathbb{R}$ the functions $\{x \mapsto \lambda_\varepsilon(x,s) : s \in B\}$ are equicontinuous in $C(A;\mathbb{R}^n)$.
In fact for each $\varepsilon > 0$, if we define

$$g_\varepsilon(x,s,p) = f(x,s,p) - f(x,s,0) - \langle \lambda_\varepsilon(x,s), p\rangle$$

and

$$h_\varepsilon(x,s,p) = g_\varepsilon^+(x,s,p) + f(x,s,0) + \langle \lambda_\varepsilon(x,s), p\rangle$$

for every $x \in \Omega$, $s \in \mathbb{R}$ and $p \in \mathbb{R}^n$, then we have

$$f(x,s,p) + \varepsilon \geq h_\varepsilon(x,s,p) \geq f(x,s,p), \tag{4.28}$$

$$f^\infty(x,s,p) = h_\varepsilon^\infty(x,s,p) \tag{4.29}$$

and $\lambda_\varepsilon(x,s) \in \partial h_\varepsilon(x,s,0)$. Now applying Theorem 4.4 to the function h_ε, we get the lower semicontinuity of the functional

$$H_\varepsilon(u) = \int_\Omega h_\varepsilon(x,u,D_a u)dx + \int_\Omega \left[\fint_{u_-}^{u_+} h_\varepsilon^\infty\left(x,t,\frac{D_s u}{|D_s u|}\right) dt \right] |D_s u|;$$

then from (4.28) and (4.29), for every sequence (u_h) in $BV(\Omega)$ converging in $L^1(\Omega)$ to $u_\infty \in BV(\Omega)$ and $\limsup_h \int_\Omega |Du_h| < +\infty$

$$F(u_\infty) \leq H_\varepsilon(u_\infty) \leq \liminf_h H_\varepsilon(u_h) \leq \liminf_h F(u_h) + \varepsilon\mathcal{L}^n(\Omega).$$

Therefore the same conclusion holds.

Remark 3. In [7] is proved that if f satisfies the hypotheses (a), (b) and (c) of Theorem 4.4, then for each $\varepsilon > 0$ there exists a function $\lambda_\varepsilon(x,s)$, continuous in x and measurable in s, such that $\lambda_\varepsilon(x,s) \in \partial_\varepsilon f(x,s,0)$ for each $x \in \Omega$ and $s \in \mathbb{R}$. Then the conditions $i)$ and $ii)$ in (d') are always satisfied. On the other hand, the hypothesis $iii)$ cannot be disregarded (see Examples in [7] and in [41]), but it can be replaced by suitable estimates from above of the integrand function. We remark that the hypothesis $iv)$ also cannot be dropped (see [7], Example 6).

Finally we note that in the particular case where f does not depend on x (see [8], Theorem 1 and [7], Theorem 2), the existence of a measurable selection $\lambda(s)$ of $\partial f(s,0)$ follows from the other assumptions of the theorem; while the condition $\lambda \in L^1_{\mathrm{loc}}(\mathbb{R})$ is obtained by assuming that the function

$$\alpha_f(s) = \limsup_{p \to 0} \frac{[f(s,0) - f(s,p)]^+}{|p|}$$

belongs to $L^1_{\mathrm{loc}}(\mathbb{R})$.

Chapter 3

Relaxation of Quasi-Convex Integrals of Arbitrary Order

3.1 Introduction

In this chapter we give an integral representation result for the lower semicontinuous envelope of the functional $\int_\Omega f(\nabla^k u)dx$ on the space $BV^k(\Omega;\mathbb{R}^m)$ of the integrable functions, whose the k-th derivative in the sense of distributions is a Radon measure with bounded total variation.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary, let f be a function with p-growth (with $p \geq 1$) and let us consider the functional

$$\mathcal{F}(u) = \int_\Omega f(\nabla u)dx$$

defined on the space $C^1(\Omega;\mathbb{R}^m)$.

In [1], [29] and [31], there was considered the relaxed functional $\overline{\mathcal{F}}$ defined on the space $W^{1,p}(\Omega;\mathbb{R}^m)$ and there was proved that it admits an integral representation of the form

$$\overline{\mathcal{F}}(u) = \int_\Omega g(\nabla u)\ dx$$

where g is the quasi-convex envelope of f.

The quasi-convexity, introduced by Morrey in [72] and [73], is the appropriate condition in order to deal with functionals defined on vector valued functions.

We note that a convex function is also quasi-convex. On the contrary, it is well known that, for $p > 1$, a quasi-convex function is not necessarily convex. Some recent examples (see [75], [86], [91]) show that also in the case $p = 1$ there exist quasi-convex functions which are not convex.

If $p > 1$ the minimum problem associated to $\overline{\mathcal{F}}$ on $W^{1,p}(\Omega;\mathbb{R}^m)$ admits at least one solution, thanks to the reflexivity of this functions space. When $p = 1$, the existence of a minimum in $W^{1,1}(\Omega;\mathbb{R}^m)$ is not guaranteed, since the direct methods of the Calculus of Variations fail. In this case, it is well known that the appropriate space in which the minimization problem must be considered is $BV(\Omega;\mathbb{R}^m)$.

Recently, Ambrosio and Dal Maso in [10] proved an integral representation result on $BV(\Omega;\mathbb{R}^m)$ for integral functionals with quasi-convex integrands having linear growth, where the relaxed functional is considered with respect to the L^1-topology. Previous results concerning the integral representation on $W^{1,1}(\Omega;\mathbb{R}^m)$ of the same functional can be found in [52] and [53].

In this chapter, we consider the same problem for the functional

$$F(u) = \int_\Omega f(\nabla^k u)dx \tag{1.1}$$

where f is a function with linear growth, $k \in \mathbb{N}$, $u \in \mathrm{W}^{k,1}(\Omega;\mathbb{R}^m)$ and $\nabla^k u$ is the derivative of order k.

We recall that there exists a notion of quasi-convexity for functions depending on higher order derivatives (given by Meyers in [67]): a function f is said to be quasi-convex if

$$\int_\Omega f(\xi + \nabla^k z)dx \geq f(\xi)\ \mathrm{meas}(\Omega) \tag{1.2}$$

for every open bounded subset Ω of $\mathbb{R}^n$, for every constant ξ and for every $z \in C_0^k(\Omega;\mathbb{R}^m)$. Since ξ can be considered as the k-th derivative of a polynomial w of degree equal to k, the previous definition means that each polynomial w realizes the minimum of the integral functional in the class of functions $C^k(\Omega;\mathbb{R}^m)$ assuming the same datum on $\partial\Omega$. When $k = 1$, this notion coincides with the usual quasi-convexity.

In [67] and, for a more general case, in [55], it is proved that, in the case of p-growth, the condition (1.2) is necessary and sufficient in order to obtain the lower semicontinuity of (1.1) on $W^{k,p}(\Omega;\mathbb{R}^m)$ for $p \geq 1$. Further results for functionals depending on higher order derivatives are contained in [13], [14], [15] et al.

In the case $p = 1$, the direct methods of the Calculus of Variations work if we relax the functional (1.1) on the space $BV^k(\Omega;\mathbb{R}^m)$, of those functions u belonging to $L^1(\Omega;\mathbb{R}^m)$, whose k-th derivative in the sense of distributions is a measure with bounded total variation.

In what follows, we also assume that f has linear growth and satisfies a coerciveness hypothesis.

We state an integral representation result in $BV^k(\Omega;\mathbb{R}^m)$ for the relaxed functional $\overline{F}$ of F, with respect to the L^1-topology; we prove the following formula:

$$\overline{F}(u) = \int_\Omega g(\nabla^k u)dx + \int_\Omega g^\infty\left(\frac{D_s^k u}{|D_s^k u|}\right)|D_s^k u| \tag{1.3}$$

where g is the quasi-convex envelope of f, g^∞ is the so called recession function of g, defined by

$$g^\infty(\xi) = \limsup_{t\to+\infty}\frac{g(t\xi)}{t}$$

and $D^k u = \nabla^k u\ dx + D_s^k u$ is the Lebesgue decomposition of the measure $D^k u$ in its absolutely continuous part $\nabla^k u\ dx$ and its singular part $D_s^k u$.

We want to point out that this result cannot be obtained by applying the result in [10] to those functions v of the type $v = \nabla^{k-1}u$, since the notion of quasi-convexity for functions depending on the k-th derivative ($k > 1$) does not imply the usual notion of quasi-convexity.

The proof is obtained following the outline of [10] and introducing a blow up technique for the functions belonging to $BV^k(\Omega;\mathbb{R}^m)$, similar to the one in [53]. A crucial tool is the rank-one property for the higher order derivatives of a function in $BV^k(\Omega;\mathbb{R}^m)$, proved by Alberti in [3]. Finally, using a perturbation technique, we obtain the same representation formula (without assuming the coerciveness hypothesis) for the relaxed functional with respect to the weak convergence on $BV^k(\Omega;\mathbb{R}^m)$.

3.2 Some definitions

2.1 Let n, m and k be positive integers; let us denote by $\mathbf{T}^k(\mathbb{R}^n)$ the space of the k-covariant tensors on $\mathbb{R}^n$. Now let us define the space $\mathbf{T}_{\mathrm{m}}^{\mathrm{n,k}}$ by

$$\mathbf{T}_{\mathrm{m}}^{\mathrm{n,k}} = \mathbb{R}^m \otimes \mathbf{T}^k(\mathbb{R}^n);$$

it is (canonically) isomorphic to the space $\mathcal{L}^k(\mathbb{R}^n;\mathbb{R}^m)$ of the k-linear functions defined on $\mathbb{R}^n$ with values in $\mathbb{R}^m$. Let $e_1,\dots,e_n$ be a basis of $\mathbb{R}^n$; let $e^1,\dots,e^n$ be the dual basis and let $\varepsilon_1,\dots,\varepsilon_m$ be a basis of $\mathbb{R}^m$; then a basis of $\mathrm{T}^{\mathrm{n,k}}_{\mathrm{m}}$ is given by the tensors $\varepsilon_j\otimes e^{i_1}\otimes\cdots\otimes e^{i_k}$, with $j=1,\dots,m$ and $i_1,\dots,i_k=1,\dots,n$. Hence a tensor $\xi\in\mathrm{T}^{\mathrm{n,k}}_{\mathrm{m}}$ can be written as

$$\xi=\sum_{\substack{j=1,\dots,m\\ i_1,\dots,i_k=1,\dots,n}}\xi^j_{i_1,\dots,i_k}\varepsilon_j\otimes e^{i_1}\otimes\cdots\otimes e^{i_k}. \tag{2.1}$$

We endow the space $\mathrm{T}^{\mathrm{n,k}}_{\mathrm{m}}$ with the Euclidean norm

$$|\xi|^2=\sum_{\substack{j=1,\dots,m\\ i_1,\dots,i_k=1,\dots,n}}(\xi^j_{i_1,\dots,i_k})^2.$$

In the sequel, we will deal only with k-covariant symmetric tensors, which are characterized by the invariance of the coefficients $\xi^j_{i_1,\dots,i_k}$ in (2.1), under permutations of the indices $i_1,\dots,i_k$.

The subspace of the k-covariant symmetric tensors is isomorphic to the space $\mathcal{L}^k_{sym}(\mathbb{R}^n;\mathbb{R}^m)$ of the k-linear and symmetric functions defined on $\mathbb{R}^n$ with values in $\mathbb{R}^m$. This space is also canonically isomorphic to the space $\mathcal{L}_{sym}\left(\mathbb{R}^n;\mathcal{L}^{k-1}_{sym}(\mathbb{R}^n;\mathbb{R}^m)\right)$.

We will say that a k-covariant symmetric tensor has rank one, if the range of the corresponding linear and symmetric function belonging to $\mathcal{L}_{sym}\left(\mathbb{R}^n;\mathcal{L}^{k-1}_{sym}(\mathbb{R}^n;\mathbb{R}^m)\right)$ has dimension one. In this case, it is easy to see that the k-covariant symmetric tensor has the following representation

$$\xi=|\xi|\eta\otimes\underbrace{\nu\otimes\ldots\otimes\nu}_{k-times}$$

with $\eta\in\mathbb{R}^m$, $\nu\in\mathbb{R}^n$ and $|\eta|=|\nu|=1$.

2.2 Let $f:\mathrm{T}^{\mathrm{n,k}}_{\mathrm{m}}\to[0,+\infty[$ be a Borel measurable function. We say that f is quasi-convex if

$$\int_\Omega f(\xi+D^kz)dx\geq f(\xi)\ \mathrm{meas}(\Omega) \tag{2.2}$$

for every bounded open set Ω contained in $\mathbb{R}^n$, every $\xi \in \mathbf{T}^{n,k}_m$ and every $z \in \mathcal{C}^k_0(\Omega;\mathbb{R}^m)$. Since every $\xi \in \mathbf{T}^{n,k}_m$ is the k-th derivative of a polynomial w of degree equal to k, the previous definition means that each polynomial w realizes the minimum of the integral functional in the class of functions $\mathcal{C}^k(\Omega;\mathbb{R}^m)$ assuming the same datum on $\partial\Omega$. This notion of quasi-convexity, introduced by Meyers in [67], generalizes to the higher order the notion of quasi-convexity due to Morrey (see [72] and [73]). It is easy to see that any convex function is quasi-convex and in the case $m = 1$ and $k = 1$ the two notions coincide. Moreover every quasi-convex function is rank-one convex ; *i.e.*, the map $t \mapsto f(\xi + t\zeta)$ is convex, for every $\xi, \zeta \in \mathbf{T}^{n,k}_m$ with $rank(\zeta) = 1$.

2.3 A $\mathbf{T}^{n,k}_m$-valued Radon measure will be a set function, defined on the σ-algebra of the Borel sets, with values in the space $\mathbf{T}^{n,k}_m$, whose components are scalar Radon measures on Ω. Given a $\mathbf{T}^{n,k}_m$-valued Radon measure μ on Ω, we use the notation $|\mu|$ for its total variation, which is the scalar non-negative measure on Ω defined for every Borel set $B \subset \Omega$ by

$$|\mu|(B) = \sup \sum_{i \in \mathbb{N}} |\mu(B_i)|,$$

where the *supremum* is taken over all the countable families $(B_i)_{i\in\mathbb{N}}$ of mutually disjoint Borel subsets contained in B and relatively compact in Ω; the number $|\mu|(\Omega)$ is said the total variation of μ (it is denoted also by $\int_\Omega |\mu|$). Let μ be any scalar or vector valued Radon measure; the integral of a Borel scalar function g (defined on Ω) with respect to the measure μ will be denoted by $\int_\Omega g d\mu$ or $\int_\Omega g\mu$ and for every scalar non-negative Radon measure λ on Ω, we indicate by μ^λ_a and by μ^λ_s respectively the absolutely continuous and the singular part of μ with respect to the measure λ; when λ is the Lebesgue measure we prefer write μ_a and μ_s. The density of μ^λ_a with respect to μ will be denoted by $\frac{d\mu}{d\lambda}$ or by $\frac{\mu}{\lambda}$; then we have $\mu^\lambda_a(B) = \int_B \frac{d\mu}{d\lambda} d\lambda$ for every Borel set B contained in Ω. The *support* of a scalar non-negative Radon measure μ on Ω is the set

$$supp(\mu) = \{x \in \Omega : \mu(\Omega \cap B_\rho(x)) > 0 \quad \forall \rho > 0\}.$$

2.4 Fixed a positive integer k, we say that a function $u \in L^1(\Omega;\mathbb{R}^m)$ belongs to $BV^k(\Omega;\mathbb{R}^m)$ if its k-th derivative in the sense of distributions is a $\mathbf{T}^{n,k}_m$-valued

Radon measure with bounded total variation; more precisely, the k-th derivative $D^k u$ takes its values in the space of symmetric $\mathrm{T}^{n,k}_{m}$-tensors. The k-th derivative $D^k u$ of u will be decomposed as $\nabla^k u\ dx + D^k_s u$. In the case $k = 1$ these functions are the BV functions. Using Theorem 1.8 of [68] and [66] section 6.1.7, given $u \in BV^k(\Omega; \mathbb{R}^m)$ we have that $\nabla^\alpha u$ is a summable function for every $\alpha = 1, \dots, k-1$; then $u \in BV^k(\Omega; \mathbb{R}^m)$ if and only if u belongs to $\mathrm{W}^{k-1,1}(\Omega; \mathbb{R}^m)$ and $D^k u$ is a $\mathrm{T}^{n,k}_{m}$-valued measure. Moreover for every $u \in BV^k(\Omega; \mathbb{R}^m)$ the $(k-1)$-th derivative $\nabla^{k-1} u$ belongs to $BV(\Omega; \mathrm{T}^{n,k-1}_{m})$. It is easy to see that $BV^k(\Omega; \mathbb{R}^m)$ is a Banach space endowed with the norm

$$\|u\|_{BV^k} = \sum_{0 \leq \alpha < k} \int_\Omega |\nabla^\alpha u| dx + |D^k u|(\Omega).$$

We consider in $BV^k(\Omega; \mathbb{R}^m)$ the weak convergence BV^k-w defined in the following way: a sequence $(u_h)_{h \in \mathbb{N}}$ belonging to $BV^k(\Omega; \mathbb{R}^m)$ weakly converges to a function u belonging to $BV^k(\Omega; \mathbb{R}^m)$ (and we use the notation $u_h \rightharpoonup u$) if u_h strongly converges in $W^{k-1,1}(\Omega; \mathbb{R}^m)$ and the sequence of the $\mathrm{T}^{n,k}_{m}$-measures $(D^k u_h)_{h \in \mathbb{N}}$ weakly converges to $D^k u$ in the sense of measures; *i.e.*,

$$\int_\Omega \varphi D^k u_h \to \int_\Omega \varphi D^k u$$

for every continuous function φ with compact support.

In the following proposition we state a compactness result in the space $BV^k(\Omega; \mathbb{R}^m)$ with respect to the BV^k-w convergence.

Proposition 2.1. *Let $(u_h)_{h \in \mathbb{N}}$ be a sequence contained in* $\mathrm{W}^{k,1}(\Omega; \mathbb{R}^m)$.

i) If $\|u_h\|_{BV^k} \leq C$, then there exists a subsequence $(u_{h_l})_{l \in \mathbb{N}}$ BV^k-w converging to some function u of $BV^k(\Omega; \mathbb{R}^m)$.

ii) If for every $j = 0, \dots, k-1$ we have $\int_\Omega D^j u_h dx = 0$ and if $\int_\Omega |\nabla^k u_h| \leq C$, then there exists a subsequence $(u_{h_l})_{l \in \mathbb{N}}$ BV^k-w converging to some function u of $BV^k(\Omega; \mathbb{R}^m)$.

Proof. $i)$ It is enough to apply the compactness theorem of BV (see, for instance, [58]) to $(\nabla^j u_h)_{h \in \mathbb{N}} \subseteq BV$ for every $j = 0, \dots, k-1$.

ii) It is enough to note that, since for every $j = 1, \dots, k-1$ $\nabla^j u_h$ has mean value zero, then there exists a positive constant c_j such that

$$\int_\Omega |\nabla^j u_h| dx \leq c_j \int_\Omega |\nabla^{j+1} u_h| dx.$$

Then the assertion follows by *i*). □

In the following proposition we prove a Taylor's formula for the BV^k functions.

Proposition 2.2. *Let $u \in BV^k(\Omega; \mathrm{IR}^m)$. Then for a.e. $x_0 \in \Omega$*

$$\lim_{\rho \to 0^+} \rho^{-n} \int_{B_\rho(x_0)} \frac{|u(x) - P_k(x_0, x)|}{|x - x_0|^k} dx = 0 \tag{2.3}$$

where $P_k(x_0, x) = \sum_{0 \leq |\alpha| \leq k} \frac{1}{\alpha!} \nabla^\alpha u(x_0)(x - x_0)^\alpha$ is the Taylor polynomial of degree k of u with initial point in x_0.

Proof. When $k = 1$ the assertion is proved in [50] Th. 4.5.9 (26). The general case can be proved using the analogous arguments as in Chapter 6 of [48] and using the Taylor's formula for the C^k or $\mathrm{W}^{k,1}$ functions on IR^n (see [92], Th. 3.4.1, page 126). □

2.5 Let $f : \mathbf{T}_{\mathrm{m}}^{\mathrm{n,k}} \to [0, +\infty[$ be a Borel function; we will assume that there exists a constant $M > 0$ such that

$$0 \leq f(\xi) \leq M(1 + |\xi|). \tag{2.4}$$

Associated to f, we consider the so-called *recession* function $f^\infty : \mathbf{T}_{\mathrm{m}}^{\mathrm{n,k}} \to [0, +\infty]$ defined by

$$f^\infty(\xi) = \limsup_{t \to +\infty} \frac{f(t\xi)}{t}. \tag{2.5}$$

We remark that, if f is quasi-convex, then it is also rank-one convex; hence, it is possible to prove that f is a Lipschitz function (with Lipschitz constant L, which depends only on M, n and m) and, when ξ is a tensor with $rank(\xi) = 1$, f^∞ is actually a limit.

Let $F : BV^k(\Omega;\mathbb{R}^m) \to [0,+\infty[$ be the functional defined by

$$F(u,\Omega) = \begin{cases} \int_\Omega f(\nabla^k(u))dx & \text{if } u \in W^{k,1}(\Omega;\mathbb{R}^m), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.6}$$

We consider the lower semicontinuous envelope (or *relaxed functional*) $\overline{F}$ of F with respect to the L^1-topology, which is defined by

$$\overline{F}(u,\Omega) = \inf \liminf_{h\to+\infty} \int_\Omega f(\nabla^k u_h)dx, \tag{2.7}$$

where the *infimum* is taken on the sequences $(u_h)_{h\in\mathbb{N}}$ belonging to $C^k(\Omega;\mathbb{R}^m)$ converging to u in the L^1-topology. Moreover, we consider the functional $\widetilde{F}$, which is the greatest sequentially lower semicontinuous (with respect to the BV^k-w convergence) functional not greater than F.

For the main properties and a general survey of the relaxation theory we refer to the books [20], [30] and [31].

Now, we list some invariance properties of the functional $\overline{F}$, which will be useful in the following and which can be directly proved using the definition of the relaxed functional:

i) for every $z \in \mathbb{R}^n$

$$\overline{F}(\tau_z u, \tau_z\Omega) = \overline{F}(u,\Omega),$$

where $(\tau_z u)(x) = u(x-z)$ and $\tau_z(\Omega) = z + \Omega$;

ii) for every polynomial P^{k-1} of degree $k-1$ with values in $\mathbb{R}^m$

$$\overline{F}(u + P^{k-1},\Omega) = \overline{F}(u,\Omega);$$

iii) for every $\rho > 0$

$$\overline{F}(\theta_\rho u, \theta_\rho\Omega) = \rho^{-n}\overline{F}(u,\Omega),$$

where $(\theta_\rho u)(x) = \rho^{-k}u(\rho x)$ and $\theta_\rho(\Omega) = \rho^{-1}\Omega$.

3.3 Preliminary results

The first result of this section is a continuity theorem for integral functional on $BV^k(\Omega;\mathbb{R}^m)$.

Lemma 3.1. *Let Ω be an open bounded set with Lipschitz boundary. Let $(u_h)_{h\in\mathbb{N}}$ be a sequence contained in $\mathrm{W}^{k,1}(\Omega;\mathbb{R}^m)$ and let $u \in BV^k(\Omega;\mathbb{R}^m)$ such that u_h converges to u in the L^1-topology. Let us assume that*

$$\lim_{h\to+\infty} |\nabla^k u_h|(\Omega) = |D^k u|(\Omega). \tag{3.1}$$

Then for every continuous function $g : \mathbf{T}^{\mathrm{n,k}}_{\mathrm{m}} \to \mathbb{R}$ we have

$$\lim_{h\to+\infty} \int_\Omega g\left(\frac{\nabla^k u_h}{|\nabla^k u_h|}\right) |\nabla^k u_h| = \int_\Omega g\left(\frac{D^k u}{|D^k u|}\right) |D^k u|. \tag{3.2}$$

Proof. Repeating k-times an integration by parts, it is easy to see that the measure $\nabla^k u_h\, dx$ converges weakly in the sense of measures to $D^k u$. Then the thesis follows from the Reshetnyak continuity theorem (see [63], Appendix and [80], Theorem 3). □

Now we introduce the appropriate notation in order to apply the blow up technique to BV^k functions. Let $u \in BV^k(\Omega;\mathbb{R}^m)$ and let C be a convex open subset of $\mathbb{R}^n$; for every $x_0 \in C$ and every ρ sufficiently small, we consider the function $u_\rho : C \to \mathbb{R}^m$ defined by

$$u_\rho(y) = \rho^{-k} u(x_0 + \rho y). \tag{3.3}$$

For every $s > 0$ set

$$C_s(x_0) = \{sy + x_0 : y \in C\} \quad \text{and} \quad C_s = C_s(0).$$

Then for each $0 < \sigma \leq 1$

$$D^k u_\rho(C_\sigma) = \rho^{-n} D^k u(C_{\sigma\rho}(x_0)) \quad \text{and} \quad |D^k u_\rho|(C_\sigma) = \rho^{-n} |D^k u|(C_{\sigma\rho}(x_0)). \tag{3.4}$$

Theorem 3.2. *Let $u \in BV^k(\Omega;\mathbb{R}^m)$ and let $\xi : \Omega \to \mathrm{T}^{n,k}_{m}$ be the density of $D^k u$ with respect to $|D^k u|$; i.e., $\xi := \frac{D^k u}{|D^k u|}$. Then, for $|D^k_s u|$-a.e. $x_0 \in \Omega$ we have $|\xi(x_0)| = 1$, $rank(\xi(x_0)) = 1$, and for every C convex bounded open subset of $\mathbb{R}^n$ containing the origin we obtain*

$$(3.5) \qquad \lim_{\rho\to 0+} \frac{D^k u(C_\rho(x_0))}{|D^k u|(C_\rho(x_0))} = \xi(x_0) \quad \text{and} \quad \lim_{\rho\to 0+} \frac{|D^k u|(C_\rho(x_0))}{\rho^n} = +\infty.$$

Let $x_0 \in supp(|D^k u|)$ such that $\xi(x_0)$ can be written as

$$\xi(x_0) = \eta \otimes \underbrace{\nu \otimes \ldots \otimes \nu}_{k-times},$$

with $\eta \in \mathbb{R}^m$, $\nu \in \mathbb{R}^n$ and $|\eta| = |\nu| = 1$. Let u_ρ be as in (3.3) and let

$$v_\rho(y) = \frac{\rho^n}{|D^k u|(C_\rho(x_0))}(u_\rho(y) - m_\rho(y)),$$

where m_ρ is a polynomial of degree $k-1$ with values in $\mathbb{R}^m$ such that

$$(3.6) \qquad \int_C \nabla^j v_\rho(y)dy = 0$$

for every $j = 0,\ldots,k-1$.

Then for every $0 < \rho < 1$ and every $0 < \sigma \leq 1$ we obtain

$$(3.7) \qquad |D^k v_\rho|(C_\sigma) = \frac{|D^k u|(C_{\sigma\rho}(x_0))}{|D^k u|(C_\rho(x_0))} \leq 1.$$

Moreover for every $0 < \sigma < 1$ there exist a sequence $(\rho_h)_{h\in\mathbb{N}}$ and a non-decreasing function $\psi :]a,b[\to \mathbb{R}$, where $a = \inf_{y\in C}\langle y,\nu\rangle$ and $b = \sup_{y\in C}\langle y,\nu\rangle$, such that

a) ρ_h converges to zero, when h goes to $+\infty$,

b) v_{ρ_h} converges in L^1 to a function v belonging to $BV^k(C;\mathbb{R}^m)$,

c) $|D^k v|(\bar{C}_\sigma) \geq \sigma^n$,

d) $D^{k-1}v(y) = \psi(\langle y,\nu\rangle)\eta \otimes \underbrace{\nu \otimes \ldots \otimes \nu}_{(k-1)-times}$.

Proof. The Rank One Property of higher order derivatives has been proved by G. Alberti (see [3], Corollary 4.14). The equalities in (3.5) are a consequence of a strong version of the Besicovitch Covering Theorem contained in [10] (Proposition 2.2). In order to prove the second part of the theorem, we state that

$$\limsup_{\rho \to 0+} \frac{|D^k u|(C_{\sigma\rho}(x_0))}{|D^k u|(C_\rho(x_0))} > \sigma^n. \tag{3.8}$$

By contradiction, we suppose that there exists $\rho_0 > 0$ such that (setting $\omega(\rho) = |D^k u|(C_\rho)$)

$$\frac{\omega(\sigma\rho)}{\omega(\rho)} \leq \sigma^n \qquad \text{for} \quad \text{every} \quad 0 < \rho \leq \rho_0.$$

Then for every $h \in \mathbb{N}$

$$\frac{\omega(\sigma^h \rho_0)}{(\sigma^h \rho_0)^n} \leq \frac{\omega(\rho_0)}{\rho_0^n};$$

this is a contradiction, since, when $h \to +\infty$, $\sigma^h \rho_0 \to 0$ and by (3.5)

$$\lim_{h \to +\infty} \frac{\omega(\sigma^h \rho_0)}{(\sigma^h \rho_0)^n} = +\infty.$$

Then (3.8) is proved. Now by the definition of v_ρ and since $0 < \sigma \leq 1$, we have that

$$|D^k v_\rho|(C_\sigma) = \frac{\rho^n}{|D^k u|(C_\rho(x_0))} |D^k u_\rho|(C_\sigma) = \frac{|D^k u|(C_{\sigma\rho}(x_0))}{|D^k u|(C_\rho(x_0))} \leq 1.$$

Then (3.7) holds and so by (3.8) there exists a sequence $(\rho_h)_{h \in \mathbb{N}}$ converging to 0 such that

$$\lim_{h \to +\infty} |D^k v_{\rho_h}|(C_\sigma) > \sigma^n. \tag{3.9}$$

Setting $v_h := v_{\rho_h}$, we note that by (3.7) and (3.6) the sequence $(v_h)_{h \in \mathbb{N}}$ satisfies the conditions of the Proposition 2.1 ii). Then (passing, if necessary, to a subsequence) v_h strongly converges in $W^{k-1,1}(\Omega; \mathbb{R}^m)$ to some function $v \in BV^k(C; \mathbb{R}^m)$ and $D^k v_h$ weakly converges in the sense of measures to $D^k v$. By the compactness theorem on the space of measures (passing to some new subsequence) we assume that the total variations $|D^k v_h|$ converge weakly in the sense of measures to a Radon measure μ on C. We will prove that $\mu = |D^k v|$ on C. The lower semicontinuity of the total variation implies that $|D^k v| \leq \mu$. For every $0 < s < 1$ such that $\mu(\partial C_s) = 0$ we have that

$$D^k v_h(C_s) \to D^k v(C_s) \quad \text{and} \quad |D^k v_h|(C_s) \to \mu(C_s).$$

Then, for every $\sigma < s < 1$ such that $\mu(\partial C_s) = 0$, we have by (3.9)

$$\mu(\overline{C_s}) > \sigma^n. \tag{3.10}$$

We remark that by (3.4), for every $0 < s < 1$

$$\frac{D^k v_h(C_s)}{|D^k v_h|(C_s)} = \frac{D^k u_{\rho_h}(C_s)}{|D^k u_{\rho_h}|(C_s)} = \frac{D^k u(C_{\rho_h s}(x_0))}{|D^k u|(C_{\rho_h s}(x_0))} \to \xi(x_0).$$

This implies that $D^k v(C_s) = \xi(x_0)\mu(C_s)$ for any $\sigma < s < 1$ with $\mu(\partial C_s) = 0$. When $s \to 1$, recalling that $|\xi(x_0)| = 1$, we have

$$|D^k v|(C) \leq \mu(C) = |D^k v(C)| \leq |D^k v|(C),$$

i.e. $\mu(C) = |D^k v|(C)$. Since the other inequality holds, we get $|D^k v| = \mu$ on C. In particular, by (3.10), we have $|D^k v|(C_s) > \sigma^n$. Setting $\gamma := \frac{D^k v}{|D^k v|}$, we obtain

$$\int_C \left|\gamma - \frac{D^k v(C)}{|D^k v|(C)}\right|^2 |D^k v| = 2\left[|D^k v|(C) - \frac{|D^k v(C)|^2}{|D^k v|(C)}\right] = 0.$$

As $D^k v(C) = \xi(x_0)\mu(C)$, this implies that

$$\gamma(x) = \frac{D^k v(C)}{|D^k v|(C)} = \eta \otimes \underbrace{\nu \otimes \ldots \otimes \nu}_{k-times} \tag{3.11}$$

for $|D^k v|$-a.e. $x \in C$. We claim that

$$\nabla^{k-1} v(y) = \psi(\langle y, \nu\rangle)\eta \otimes \underbrace{\nu \otimes \ldots \otimes \nu}_{(k-1)-times}, \tag{3.12}$$

with $\psi :]a, b[\to \mathrm{I\!R}$ a non-decreasing function, $a = \inf\limits_{y \in C}\langle y, \nu\rangle$ and $b = \sup\limits_{y \in C}\langle y, \nu\rangle$. In fact, if we denote by $\phi(y) = \nabla^{k-1} v(y)$, by (3.11) we get

$$\frac{D\phi}{|D\phi|} = \eta \otimes \underbrace{\nu \otimes \ldots \otimes \nu}_{k-times} = (\eta \otimes \underbrace{\nu \otimes \ldots \otimes \nu}_{(k-1)-times}) \otimes \nu$$

for $|D\phi|$-a.e. $x \in C$. This implies that ϕ satisfies the relation (2.9) of [10] with η replaced by $\eta \otimes \underbrace{\nu \otimes \ldots \otimes \nu}_{(k-1)-times}$, hence $\frac{D\phi}{|D\phi|}$ admits a representation as in (3.12). $\square$

In the following lemma, we state the so called "*fundamental estimate*" (see, for instance, [32] chapter 18, [24], [36]).

Lemma 3.3. *Let* $f : \mathrm{T}^{\mathrm{n,k}}_{\mathrm{m}} \to [0, +\infty[$ *satisfying the condition*

$$M_1|\xi| \le f(\xi) \le M_2(1 + |\xi|) \tag{3.14}$$

for every $\xi \in \mathrm{T}^{\mathrm{n,k}}_{\mathrm{m}}$, *for some positive constants* M_1 *and* M_2. *Let* A_1, A_2, C_1, C_2 *be open bounded subsets of* $\mathbb{R}^n$ *such that* $C_1 \subset\subset A_1$ *and* $C_2 \subset A_2$. *Let* $(u_h)_{h\in\mathbb{N}}$ *and* $(v_h)_{h\in\mathbb{N}}$ *be two sequences of* C^k *functions such that*

$$u_h \to u \qquad v_h \to u \qquad \text{strongly in } L^1(\Omega)$$

and

$$\limsup_{h\to+\infty} \int_\Omega f(\nabla^k u_h)dx \le C, \quad \limsup_{h\to+\infty} \int_\Omega f(\nabla^k v_h)dx \le C$$

for a suitable positive constant C. *Then there exists a sequence of functions* $(\phi_h)_{h\in\mathbb{N}} \subset C^k(\mathbb{R}^n;[0,1])$, *which are 0 in a neighbourhood of* $\mathbb{R}^n \setminus A_1$ *and such that the functions* $w_h = \phi_h u_h + (1-\phi_h)v_h$ *satisfy*

$$\limsup_{h\to+\infty} \int_{C_1\cap C_2} f(\nabla^k w_h)dx \le \limsup_{h\to+\infty} \int_{A_1} f(\nabla^k u_h)dx + \limsup_{h\to+\infty} \int_{A_2} f(\nabla^k v_h)dx.$$

Proof. First of all, we note that it is enough to prove that, for every $\varepsilon > 0$, there exists a sequence of functions $(\phi_h^\varepsilon)_{h\in\mathbb{N}}$ belonging to $C^k(\mathbb{R}^n;[0,1])$ such that, setting $w_h^\varepsilon = \phi_h^\varepsilon u_h + (1-\phi_h^\varepsilon)v_h$, we have

$$\limsup_{h\to+\infty} \int_{C_1\cap C_2} f(\nabla^k w_h^\varepsilon)dx \le \limsup_{h\to+\infty} \int_{A_1} f(\nabla^k u_h)dx + \limsup_{h\to+\infty} \int_{A_2} f(\nabla^k v_h)dx + \varepsilon. \tag{3.15}$$

In fact, using a standard diagonal procedure, it is possible to construct a sequence of functions $(w_h^{\varepsilon_h})_{h\in\mathbb{N}}$ such that

$$\limsup_{h\to+\infty} \int_{C_1\cap C_2} f(\nabla^k w_h^{\varepsilon_h})dx \le \limsup_{h\to+\infty} \int_{A_1} f(\nabla^k u_h)dx + \limsup_{h\to+\infty} \int_{A_2} f(\nabla^k v_h)dx.$$

For the sake of simplicity, in what follows we omit to write explicitly the dependence on ε.

In order to prove (3.15), let $\delta < \text{dist}(C_1, \partial A_1)$ and let

$$S = C_2 \cap \left\{ x \in A_1 : \frac{\delta}{3} < \text{dist}(x, C_1) < \frac{2}{3}\delta \right\}.$$

Let us assume that $S \subset\subset S' \subset\subset A_1 \cup A_2$ with $\partial S'$ Lipschitz.

Fix $\varepsilon > 0$, since $\limsup_{h\to+\infty} \int_\Omega f(\nabla^k u_h)dx$ and $\limsup_{h\to+\infty} \int_\Omega f(\nabla^k v_h)dx$ are bounded, by the coerciveness follows that there exists a positive constant M (which depends only upon C) such that

$$\int_\Omega (1 + |\nabla^k u_h| + |\nabla^k v_h|)dx \leq M.$$

Now, let $l \in \mathbb{N}$ be a constant sufficiently large such that

$$M_2 \int_S (1 + |\nabla^k u_h| + |\nabla^k v_h|)dx \leq \varepsilon l$$

(for instance $l = [\frac{M_2 M}{\varepsilon}] + 1$). Now for every $i = 1, \ldots, l$, we set

$$S_i = \left\{ x \in \mathbb{R}^n : \frac{l+i-1}{3l}\delta < \text{dist}(x, C_1) \leq \frac{l+i}{3l}\delta \right\} \cap C_2$$

and we consider $\phi_i : \mathbb{R}^n \to [0,1]$ belonging to $C^k(\mathbb{R}^n)$ such that $\phi_i(x) = 1$ if $\text{dist}(x, C_1) \leq \frac{l+i-1}{3l}\delta$, $\phi_i(x) = 0$ if $\text{dist}(x, C_1) \geq \frac{l+i}{3l}\delta$ and for every $m = 0, \ldots, k$

$$\|\nabla^m \phi_i\|_{L^\infty(\mathbb{R}^n)} \leq \left(\frac{4l}{\delta}\right)^m.$$

Then, if we set $w_h^i := \phi_i u_h + (1-\phi_i)v_h$, we obtain

$$\int_{C_1\cup C_2} f(\nabla^k w_h^i)dx \le \int_{A_1} f(\nabla^k u_h)dx + \int_{A_2} f(\nabla^k v_h)dx +$$

$$+ M_2 \int_{S_i} (1+|\nabla^k w_h^i|)dx \le$$

$$\le \int_{A_1} f(\nabla^k u_h)dx + \int_{A_2} f(\nabla^k v_h)dx +$$

$$+ M_2 \int_{S_i} (1 + |\sum_{m=0}^{k} \binom{k}{m} [\nabla^m \phi_i \nabla^{k-m} u_h + \nabla^m (1-\phi_i)\nabla^{k-m} v_h]|)dx \le$$

$$\le \int_{A_1} f(\nabla^k u_h)dx + \int_{A_2} f(\nabla^k v_h)dx + M_2 \int_{S_i} (1+|\nabla^k u_h| + |\nabla^k v_h|)dx +$$

$$+ \widetilde{M} \sum_{m=1}^{k} \int_{S\cup S_i} |\nabla^m \phi_i||\nabla^{k-m} u_h - \nabla^{k-m} v_h|dx \le$$

$$\le \int_{A_1} f(\nabla^k u_h)dx + \int_{A_2} f(\nabla^k v_h)dx + M_2 \int_{S_i} (1+|\nabla^k u_h| + |\nabla^k v_h|)dx +$$

$$+ \sum_{m=1}^{k} \widetilde{M} \left(\frac{4l}{\delta}\right)^m \int_{S_i} |\nabla^{k-m} u_h - \nabla^{k-m} v_h|dx.$$

For every $h \in \mathbb{N}$, there exists an index $i_h \in \{1,\dots,l\}$ such that, setting $w_h = \phi_{i_h} u_h + (1-\phi_{i_h})v_h$, we have

$$\int_{C_1\cap C_2} f(\nabla^k w_h)dx \le \frac{1}{l}\sum_{i=1}^{l} \int_{C_1\cap C_2} f(\nabla^k w_h^i)dx \le$$

$$\le \int_{A_1} f(\nabla^k u_h)dx + \int_{A_2} f(\nabla^k v_h)dx + \varepsilon +$$

$$+ \widetilde{C} \sum_{m=1}^{k} \left(\frac{l}{\delta}\right)^m \int_{S} |\nabla^{k-m} u_h - \nabla^{k-m} v_h|dx \le$$

$$\le \int_{A_1} f(\nabla^k u_h)dx + \int_{A_2} f(\nabla^k v_h)dx + \varepsilon +$$

$$+ \widetilde{C} \sum_{m=1}^{k} \left(\frac{l}{\delta}\right)^m \int_{S'} |\nabla^{k-m} u_h - \nabla^{k-m} v_h|dx.$$

Since S' is regular, since $u_h - v_h \to 0$ strongly in $L^1(\Omega)$, which contains S, and since $\int_\Omega |\nabla^k(u_h - v_h)|dx \le 2C$, by the interpolation inequality (see, for instance

[2]) and by the Proposition 2.1 we obtain that $\nabla^j u_h - \nabla^j v_h \to 0$ strongly in $L^1(S')$ for every $j = 1, \dots, k-1$. Taking the upper limit in the previous chain of inequalities, we get

$$\limsup_{h\to+\infty} \int_{C_1\cap C_2} f(\nabla^k w_h)dx \le \limsup_{h\to+\infty} \int_{A_1} f(\nabla^k u_h)dx + \limsup_{h\to+\infty} \int_{A_2} f(\nabla^k v_h)dx + \varepsilon,$$

hence the thesis follows. □

Using the previous lemma, we can state that $\overline{F}(u,\cdot)$ is a measure.

Theorem 3.4. *Let $f : \mathrm{T}^{\mathrm{n,k}}_{\mathrm{m}} \to [0,+\infty[$ be a function which satisfies the condition* (3.14) *of Lemma 3.3. Let us consider the relaxed functional $\overline{F}$ defined in* (2.7). *Then, for every $u \in BV^k(\Omega;\mathrm{I\!R}^m)$ and for every open subset A of Ω, we have*

$$M_1|D^k u|(A) \le \overline{F}(u,A) \le M_2(\text{meas}(A) + |D^k u|(A)). \tag{3.16}$$

Moreover, for every $u \in BV^k(\Omega;\mathrm{I\!R}^m)$, the set function $\overline{F}(u,\cdot)$ is the restriction to the family of the open sets contained in Ω of a σ-additive measure on the σ-algebra of the Borel subsets of Ω.

Proof. First we note that, for every $u \in BV^k(\Omega;\mathrm{I\!R}^m)$, there exists a sequence $(u_h)_{h\in\mathrm{I\!N}}$ of C^k functions converging to u strongly in L^1 and such that

$$\lim_{h\to+\infty} \int_A |\nabla^k u_h|dx = |D^k u|(A).$$

In fact, it is sufficient to repeat the proof of the Theorem 1.17 of [58], with minor modifications. Then $\overline{F}(u,A) \le \liminf_{h\to+\infty} \int_A f(\nabla^k u_h)dx \le M_2(\text{meas}(A) + |D^k u|(A))$.

On the other hand, by the definition of $\overline{F}$, there exists a sequence $(v_h)_{h\in\mathrm{I\!N}}$ in $C^k(A;\mathrm{I\!R}^m)$ converging to u in the L^1-topology such that

$$\overline{F}(u,A) = \lim_{h\to+\infty} \int_A f(\nabla^k v_h)dx \ge M_1 \lim_{h\to+\infty} \int_A |\nabla^k v_h|dx.$$

Hence, by the semicontinuity of the total variation

$$\overline{F}(u,A) \ge M_1|D^k u|(A).$$

Now, let $u \in BV^k(\Omega;\mathbb{R}^m)$ and set $\mu(A) = \overline{F}(u,A)$. In order to prove the second part of the theorem, it is enough to show (see [44]) that for all bounded open subsets A and A' of Ω we have
(4.a) if $A \subset A'$, then $\mu(A) \leq \mu(A')$;
(4.b) if $A \cap A' = \emptyset$, then $\mu(A \cup A') \geq \mu(A) + \mu(A')$;
(4.c) $\mu(A) = \sup\{\mu(A') : A' \subset\subset A\}$;
(4.d) $\mu(A \cup A') \leq \mu(A) + \mu(A')$.

(4.a) and (4.b) follow easily by the definition of μ; (4.c) and (4.d) can be obtained (in a similar way as in Theorem 3.1 of [10]) as a consequence of the fundamental estimate proven in Lemma 3.3. □

3.4 Integral representation of the relaxed functional

In this section, we will give the integral representation of the relaxed functional $\overline{F}$ defined in (2.7) and of the relaxed functional $\widetilde{F}$ defined in section 2.5. We begin by proving the inequality from above for $\overline{F}$; in Lemma 4.3, we will prove the inequality from below for $\overline{F}$. Lemma 4.2 is a technical lemma, which is used in order to prove the inequality from below. Finally, in Theorem 4.4 we state the integral representation for $\overline{F}$, as a consequence of Lemma 4.1 and 4.2, and in Theorem 4.5 we state the integral representation for $\widetilde{F}$, applying a perturbation technique to $\overline{F}$.

Lemma 4.1. *Let $f : \mathbf{T}^{n,k}_m \to [0,+\infty[$ be a quasi-convex function and let M_1, M_2 be two positive constants such that*

$$M_1|\xi| \leq f(\xi) \leq M_2(1+|\xi|) \quad \forall \xi \in \mathbf{T}^{n,k}_m. \tag{4.1}$$

Then

$$\overline{F}(u,\Omega) \leq \int_\Omega f(\nabla^k u)dx + \int_\Omega f^\infty\left(\frac{D^k_s u}{|D^k_s u|}\right)|D^k_s u| \tag{4.2}$$

for every open and bounded subset Ω of $\mathbb{R}^n$ with Lipschitz boundary and for every $u \in BV^k(\Omega;\mathbb{R}^m)$, where $\overline{F}$ is the relaxed functional with respect to the strong L^1-topology, defined in (2.7).

Proof. The thesis follows by Lemma 3.1 and by [10] Proposition 4.2, where ∇ and D_s are replaced by ∇^k and D_s^k. $\square$

Lemma 4.2. *Let $f : \mathrm{T}_{\mathrm{m}}^{\mathrm{n},\mathrm{k}} \to [0, +\infty[$ be a quasi-convex function satisfying* (4.1), *let Ω be an open bounded subset of $\mathbb{R}^n$ and let $u \in BV^k(\Omega; \mathbb{R}^m)$.*

(i) Let u be a homogenous polynomial of degree k on $\mathbb{R}^n$ with values in $\mathbb{R}^m$; i.e., there exists $\xi \in \mathrm{T}_{\mathrm{m}}^{\mathrm{n},\mathrm{k}}$ such that

$$u^j(x) = \sum_{i_1,\ldots,i_k=1}^{n} \xi^j_{i_1\ldots i_k} x_{i_1} \ldots x_{i_k} \qquad j = 1, \ldots, m.$$

Then

$$\overline{F}(u, \Omega) \geq \int_\Omega f(\nabla^k u) dx = f(\xi) \mathrm{meas}(\Omega).$$

(ii) Let $\Omega = Q$ be a unit n-cube contained in $\mathbb{R}^n$, whose sides are orthogonal or parallel to the unit vector $\nu \in \mathbb{R}^n$. Let $v \in BV^k(\Omega; \mathbb{R}^m)$ be a function such that

$$\nabla^{k-1} v(y) = \psi(\langle y, \nu \rangle) \eta \otimes \underbrace{\nu \otimes \ldots \otimes \nu}_{(k-1)-times}$$

as in Theorem 3.2. Then, if $\mathrm{supp}(v - u) \subset\subset Q$, *we have*

$$\overline{F}(u, \Omega) \geq f(D^k u(Q)).$$

Proof. *(i)* Let Ω_1, Ω_2 and Ω_3 be three open sets such that $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset \Omega$. Let $(u_h)_{h\in\mathbb{N}} \subseteq C^k(\Omega, \mathbb{R}^m)$ be a sequence such that $u_h \to u$ strongly in L^1 and

$$\lim_{h\to+\infty} \int_\Omega f(\nabla^k u_h) dx = \overline{F}(u, \Omega).$$

By Lemma 3.3 with $C_1 = \Omega_2$, $A_1 = \Omega_3$, $C_2 = A_2 = \Omega \setminus \overline{\Omega}_1$ and $v_h \equiv u$, we obtain a sequence $(w_h)_{h\in\mathbb{N}} \subseteq C^k(\Omega, \mathbb{R}^m)$ converging to u in L^1 such that $\mathrm{supp}(w_h - u) \subset\subset \Omega$ and

$$\overline{F}(u, \Omega) + \int_{\Omega\setminus\Omega_1} f(\nabla^k u) dx \geq \limsup_{h\to+\infty} \int_\Omega f(\nabla^k w_h) dx \geq f(\xi) \mathrm{meas}(\Omega),$$

where the last inequality is due to the quasi-convexity of f.

By letting $\Omega_1 \nearrow \Omega$ the thesis follows.

(ii) Without loss of generality, we may assume that $\nu = e_1$ and $Q = [0,1]^n$. Hence

$$\nabla^{k-1}v(y) = \psi(y_1)\eta \otimes \underbrace{e_1 \otimes \ldots \otimes e_1}_{(k-1)-times}.$$

Since ψ is a non decreasing function, we may write

$$\alpha := \lim_{t\to 1^-}\psi(t) - \lim_{t\to 0^+}\psi(t) = |\psi|(]0,1[) = |D^k v|(Q) < +\infty.$$

Let us consider the function $w \in BV^k(]0,+\infty[^n;\mathbb{R}^m)$ defined by

$$w(y) = u(y-[y]) + \frac{\alpha}{k!}[y_1^k]\eta$$

where $[y_i]$ denotes the integer part of y_i and $[y] = ([y_1],\ldots,[y_n])$. We observe that, when $u_h(y) = \frac{1}{h^k}w(hy)$, we have

$$\begin{aligned}
&\|u_h\|_{BV^k(\Omega;\mathbb{R}^m)} \le C\\
&u_h(y) = \frac{1}{h^k}w(hy) = \frac{u(hy-[hy])}{h^k} + \frac{\alpha}{k!}\frac{[h^k y_1^k]}{h^k}\eta \to \frac{\alpha}{k!}y_1^k\eta =: u_0(y_1)
\end{aligned}$$

strongly in L^1; in fact

$$\begin{aligned}
\int_Q \left|\frac{u(hy-[hy])}{h^k}\right|dy &= \frac{1}{h^{n+k}}\int_{]0,h[^n}|u(y-[y])|dy =\\
&= \frac{h^n}{h^{n+k}}\int_Q |u(y-[y])|dy = \frac{1}{h^k}\int_Q |u(y)|dy \to 0.
\end{aligned}$$

Let us decompose Q in h^n congruent cubes Q_i, in a standard way. Clearly, $|D^k u_h|(Q\cap\partial Q_i) = |D^k w|(Q) = 0$, since $D^k w$ does not charge any hyperplane of the form $y_j = l$ with $l \in \mathbb{N}$ and $j = 1,\ldots,n$. By the properties $i)$, $ii)$ and $iii)$ listed in section 2.5, we obtain

$$\begin{aligned}
&\overline{F}(u_h, Q\cap\partial Q_i) = 0\\
&\overline{F}\left(u_h, \left]0,\frac{1}{h}\right[^n\right) = \overline{F}\left(\frac{1}{h^k}w(hy), \frac{1}{h}]0,1[^n\right) = h^{-n}\overline{F}(w,Q) = h^{-n}\overline{F}(u,Q)\\
&\overline{F}\left(u_h, \left]0,\frac{1}{h}\right[^n\right) = \overline{F}(u_h, Q_i)
\end{aligned}$$

and hence

$$\overline{F}(u_h, Q) = \sum_{i=1}^{h^n} \overline{F}(u_h, Q_i) = h^n \overline{F}\left(u_h, \left]0, \frac{1}{h}\right[^n\right) = \overline{F}(u, Q).$$

By *(i)* and by the lower semicontinuity of $\overline{F}(\cdot, Q)$, we have

$$\begin{aligned}\overline{F}(u, Q) &= \lim_{h \to +\infty} \overline{F}(u_h, Q) \geq \overline{F}(u_0, Q) = \\ &= \overline{F}\left(\frac{\alpha}{k!} y_1^k \eta, Q\right) \geq f(\alpha\eta \otimes \underbrace{e_1 \otimes \ldots \otimes e_1}_{k-times}) = f(D^k u(Q))\end{aligned}$$

since $D^k u(Q) = D^k v(Q) = \alpha\eta \otimes \underbrace{e_1 \otimes \ldots \otimes e_1}_{k-times}$, and the proof is complete. □

Lemma 4.3. *Let* $f : \mathrm{T}_{\mathrm{m}}^{\mathrm{n,k}} \to [0, +\infty[$ *be a quasi-convex function satisfying* (4.1). *Then*

$$(a) \qquad \overline{F}_a(u, \Omega) \geq \int_\Omega f(\nabla^k u) dx$$

$$(b) \qquad \overline{F}_s(u, \Omega) \geq \int_\Omega f^\infty\left(\frac{D_s^k u}{|D_s^k u|}\right) |D_s^k u|$$

for every open and bounded subset Ω *of* $\mathbb{R}^n$ *with Lipschitz boundary and every* $u \in BV^k(\Omega; \mathbb{R}^m)$.

Proof. We begin by proving (a). By Proposition 2.2, it follows that

$$\lim_{\rho \to 0^+} \rho^{-n} \int_{B_\rho(x_0)} \frac{|u(x) - P_{k-1}(x_0, x) - Q_k(x_0, x)|}{|x - x_0|^k} dx = 0 \tag{4.3}$$

for a.e. $x_0 \in \Omega$, where

$$P_{k-1}(x_0, x) = \sum_{0 \leq |\alpha| \leq k-1} \frac{1}{\alpha!} \nabla^\alpha u(x_0)(x - x_0)^\alpha,$$

$$Q_k(x_0, x) = \sum_{|\alpha| = k} \frac{1}{\alpha!} \nabla^\alpha u(x_0)(x - x_0)^\alpha$$

are, respectively, the polynomial of degree $k-1$ and the homogenous polynomial of degree k with initial point x_0 associated to u by the Taylor formula.

Let us fix such an x_0 and set $u_0(y) = Q_k(x_0, \rho y)$. Let $y \in B_1 := B_1(0)$, $0 < \rho < dist(x_0, \partial\Omega)$ and

$$u_\rho(y) = \frac{u(x_0 + \rho y) - P_{k-1}(x_0, \rho y)}{\rho^k}.$$

Since

$$\int_{B_1} |u_\rho(y) - u_0(y)| dy = \rho^{-n} \int_{B_\rho(x_0)} \frac{|u(x) - P_{k-1}(x_0, x) - Q_k(x_0, x)|}{|x - x_0|^k} dx,$$

when ρ goes to zero, it follows by (4.3) that u_ρ converges to u_0 strongly in $L^1(B_1, \mathbb{R}^m)$.

By *iii)* of section 2.5, by Lemma 4.2 *(i)* and by the lower semicontinuity of $\overline{F}$, we have

$$\liminf_{\rho \to 0+} \rho^{-n} \overline{F}(u, B_\rho(x_0)) = \liminf_{\rho \to 0+} \overline{F}(u_\rho, B_1) \geq \overline{F}(u_0, B_1) \geq f(\nabla^k u(x_0)) \text{meas}(B_1).$$

Finally

$$\liminf_{\rho \to 0+} \frac{\overline{F}(u, B_\rho(x_0))}{\text{meas}(B_\rho(x_0))} \geq f(\nabla^k u(x_0))$$

and hence $\overline{F}_a(u, \Omega) \geq \int_\Omega f(\nabla^k u) dx$. This proves (a).

In order to prove (b), we will previously show the following claim.

CLAIM: *Let* $(v_h)_{h \in \mathbb{N}} \subseteq BV^k(Q, \mathbb{R}^m)$ *defined by*

$$v_h(y) = \frac{\rho_h^n}{|D^k u|(C_{\rho_h}(x_0))} (u_{\rho_h}(y) - m_{\rho_h}(y))$$

with

$$v_h \rightharpoonup v \in BV^k(Q, \mathbb{R}^m) \qquad \text{weakly in } BV^k(Q, \mathbb{R}^m),$$

$$\nabla^{k-1} v(y) = \psi(\langle y, \nu \rangle) \eta \otimes \underbrace{\nu \otimes \ldots \otimes \nu}_{(k-1)-times} \qquad (\psi \text{ non decreasing}),$$

$$\sigma^n \leq |D^k v|(\overline{Q}_\sigma) \leq |D^k v|(Q) \leq 1 \qquad (Q_\sigma = \{\sigma y : y \in Q\}),$$

$$\limsup_{h \to +\infty} |D^k v_h|(Q_\sigma) \geq \sigma^n$$

as in Theorem 3.2. Let $w_h = \phi v_h + (1-\phi)v$ *with* $\phi \in C_0^k(Q)$, $0 \le \phi \le 1$ *and* $\phi \equiv 1$ *in a neighborhood of* $\overline{Q}_\sigma$. *Then*

$$(i) \qquad \limsup_{h\to+\infty} |D^k(w_h - v_h)|(Q) \le 2\omega_\sigma$$

$$(ii) \qquad \limsup_{h\to+\infty} |D^k w_h|(S_\sigma) \le 2\omega_\sigma$$

where $S_\sigma = Q \setminus \overline{Q}_\sigma$ *and* $\omega_\sigma = 1 - \sigma^n$.

Proof of the claim. Since $w_h - v_h = (1-\phi)(v - v_h)$, we have

$$\begin{aligned}
&|D^k(w_h - v_h)|(Q) = |D^k(1-\phi)(v - v_h)|(Q) \le \\
&\le C(\sigma)\sum_{j=0}^{k-1}\int_Q |\nabla^j v - \nabla^j v_h| + |D^k(v - v_h)|(Q\setminus\overline{Q}_\sigma) \le \\
&\le C(\sigma)\sum_{j=0}^{k-1}\int_Q |\nabla^j v - \nabla^j v_h| + |D^k v|(Q\setminus\overline{Q}_\sigma) + |D^k v_h|(Q\setminus\overline{Q}_\sigma).
\end{aligned}$$

Recalling that $v_h \rightharpoonup v$ weakly in $BV^k(Q, \mathbb{R}^m)$ and hence $\nabla^j v_h \to \nabla^j v$ strongly in $L^1(Q, \mathbb{R}^m)$ for $j = 1, \ldots, k-1$, we have

$$\limsup_{h\to+\infty} |D^k(w_h - v_h)|(Q) \le \limsup_{h\to+\infty} |D^k v_h|(Q\setminus\overline{Q}_\sigma) + |D^k v|(Q\setminus\overline{Q}_\sigma) \le$$

$$\le \limsup_{h\to+\infty} \left(|D^k v_h|(Q) - |D^k v_h|(\overline{Q}_\sigma)\right) + |D^k v|(Q) - |D^k v|(\overline{Q}_\sigma) \le 2(1-\sigma^n) = 2\omega_\sigma.$$

This proves (i). The proof of (ii) is carried on in a similar way, hence the claim is done.

Now the proof of (b) can be obtained as in [10] Proposition 4.5, where D is replaced by D^k. □

We are now in a position to give the main result of our chapter. In order to state the following theorem, we need the notion of quasi-convex envelope of a given function f, which is the greatest quasi-convex function less than or equal to f.

Theorem 4.4. *Let $f : \mathrm{T}^{\mathrm{n,k}}_{\mathrm{m}} \to [0, +\infty[$ be a Borel function and let M_1, M_2 be two positive constants such that*

$$M_1|\xi| \leq f(\xi) \leq M_2(1 + |\xi|) \quad \forall \xi \in \mathrm{T}^{\mathrm{n,k}}_{\mathrm{m}}.$$

Let us consider the integral functional F defined in (2.6); then the corresponding relaxed functional in the strong L^1-topology is given by

$$\overline{F}(u, \Omega) = \int_\Omega g(\nabla^k u)dx + \int_\Omega g^\infty \left(\frac{D^k_s u}{|D^k_s u|}\right) |D^k_s u| \tag{4.4}$$

for every open and bounded subset Ω of $\mathbb{R}^n$ with Lipschitz boundary and for every $u \in BV^k(\Omega; \mathbb{R}^m)$, where g is the quasi-convex envelope of the function f.

Proof. If we consider the integral functional $F : C^k(\Omega; \mathbb{R}^m) \to [0, +\infty[$ of the form $F(u, \Omega) = \int_\Omega f(\nabla^k u)\ dx$ where f has linear growth, the result in [1] assures that its relaxed functional in $\mathrm{W}^{k,1}(\Omega; \mathbb{R}^m)$ is given by the functional

$$G(u, \Omega) = \begin{cases} \int_\Omega g(\nabla^k(u))dx & \text{if } u \in \mathrm{W}^{k,1}(\Omega; \mathbb{R}^m), \\ +\infty & \text{otherwise} \end{cases}$$

where g is the quasi-convex envelope of f. Moreover, if we relax G and F in the space $BV^k(\Omega; \mathbb{R}^m)$ with respect to the L^1-topology, it is easy to see that the two relaxed functionals do coincide. Hence there is no loss of generality, assuming that the function f is itself quasi-convex. Then the proof of the theorem follows by Lemmas 4.1, 4.2 and 4.3. □

It is possible to obtain an integral representation of the relaxed functional even if the function f is not coercive, and in this case the relaxation takes place in the weak convergence of $BV^k(\Omega; \mathbb{R}^m)$.

Theorem 4.5. *Let $f : \mathrm{T}^{\mathrm{n,k}}_{\mathrm{m}} \to [0, +\infty[$ be a Borel function and let M be a positive constant such that*

$$0 \leq f(\xi) \leq M(1 + |\xi|) \quad \forall \xi \in \mathrm{T}^{\mathrm{n,k}}_{\mathrm{m}}. \tag{4.5}$$

Let us consider the integral functional F defined in (2.6); then, for every open and bounded subset Ω of $\mathbb{R}^n$ with Lipschitz boundary and for every $u \in BV^k(\Omega;\mathbb{R}^m)$, the corresponding relaxed functional $\widetilde{F}$ with respect to the weak convergence of $BV^k(\Omega;\mathbb{R}^m)$ is given by

$$\widetilde{F}(u,\Omega) = \int_\Omega g(\nabla^k u)dx + \int_\Omega g^\infty\left(\frac{D_s^k u}{|D_s^k u|}\right)|D_s^k u| \tag{4.6}$$

where g is the quasi-convex envelope of the function f.

Proof. Without loss of generality, we can assume, as in the proof of the previous theorem, that f itself is quasi-convex. Hence (4.6) will be proved with g and g^∞ replaced by f and f^∞.

Let us consider the functional

$$G(u,\Omega) := \int_\Omega f(\nabla^k u)dx + \int_\Omega f^\infty\left(\frac{D_s^k u}{|D_s^k u|}\right)|D_s^k u|$$

defined on $BV^k(\Omega;\mathbb{R}^m)$. It is our purpose to show that $G(u,\Omega) = \widetilde{F}(u,\Omega)$.

Let $f_\varepsilon(\xi) = f(\xi) + \varepsilon|\xi|$ and

$$F_\varepsilon(u,\Omega) = \begin{cases} \int_\Omega f_\varepsilon(\nabla^k(u))dx & \text{if } u \in \mathrm{W}^{k,1}(\Omega;\mathbb{R}^m), \\ \\ +\infty & \text{otherwise.} \end{cases}$$

Clearly, for every $\varepsilon > 0$, f_ε is coercive; *i.e.*, it satisfies (4.1) for a proper choice of $M_1(\varepsilon)$ and $M_2(\varepsilon)$; hence, by Theorem 4.4, it follows that

$$\overline{F}_\varepsilon(u,\Omega) = \int_\Omega f_\varepsilon(\nabla^k u)dx + \int_\Omega f_\varepsilon^\infty\left(\frac{D_s^k u}{|D_s^k u|}\right)|D_s^k u|.$$

We observe that

$$f_\varepsilon^\infty(\xi) = \limsup_{t\to+\infty}\frac{f_\varepsilon(t\xi)}{t} = \limsup_{t\to+\infty}\frac{f(t\xi)}{t} + \varepsilon|\xi| = f^\infty(\xi) + \varepsilon|\xi|,$$

hence $\overline{F}_\varepsilon(u,\Omega)$ converges to $G(u,\Omega)$ when ε goes to zero. Since F_ε is coercive, we have that $\overline{F}_\varepsilon = \widetilde{F}_\varepsilon$ and hence, from $F \leq F_\varepsilon$, it follows that $\widetilde{F} \leq \overline{F}_\varepsilon$ for every $\varepsilon > 0$. Passing to the limit when ε goes to zero, we obtain $\widetilde{F}(u,\Omega) \leq G(u,\Omega)$.

Let $(u_h)_{h\in\mathbb{N}}$ be a sequence in $BV^k(\Omega;\mathbb{R}^m)$ such that $u_h \rightharpoonup u$ weakly in $BV^k(\Omega;\mathbb{R}^m)$; then, since $\int_\Omega |\nabla^k u_h| dx \leq C$, it follows

$$G(u,\Omega) \leq \overline{F}_\varepsilon(u,\Omega) \leq \liminf_{h\to+\infty} \overline{F}_\varepsilon(u_h,\Omega) \leq \liminf_{h\to+\infty} G(u_h,\Omega) + \varepsilon C.$$

Passing to the limit when ε goes to zero, we obtain the lower semicontinuity of G. Moreover, it is clear that, by definition, $G(u,\Omega) \leq F(u,\Omega)$, hence $G(u,\Omega) \leq \widetilde{F}(u,\Omega)$. This implies

$$\widetilde{F}(u,\Omega) = \int_\Omega f(\nabla^k u)dx + \int_\Omega f^\infty\left(\frac{D_s^k u}{|D_s^k u|}\right)|D_s^k u|$$

and the theorem is proved. $\square$

Chapter 4

Relaxation of Quadratic Forms

4.1 Introduction

The study of quadratic forms, and of linear elliptic PDEs, has always been a guideline for the understanding of many phenomena related to more general functionals of the Calculus of Variations. In the framework of the so-called Direct Method, fundamental issues are the lower semicontinuity with respect to suitable topologies and the possible extension to larger spaces of a functional of the form

$$F(u) = \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x) D_i u D_j u \,\mathrm{dx}, \tag{1.1}$$

which is naturally defined for every open set $\Omega \subseteq \mathbb{R}^n$ and for every $u \in C^1(\Omega)$, when $A = (a_{ij})$ is a symmetric matrix of Borel functions.

A "classical" situation is when, beside the usual boundedness and symmetry condition, F is coercive, *i.e.* if the matrix (a_{ij}) satisfies the estimate

$$b|z|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) z_i z_j,$$

for a suitable constant $b > 0$. In this case the lower semicontinuity in the space $\mathrm{H}^1(\Omega)$ with respect to the L^2-topology is ensured, together with stability with respect to variational convergence of sequences of such form (see [45] and [85]).

Nevertheless, in some problems, quadratic degenerate forms may occur, *e.g.*, in elasticity and in the theory of composite media and homogenization. When a weaker coerciveness condition of the type

$$b(x)|z|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) z_i z_j,$$

is satisfied, with b belonging to a proper class of weights, the space where the problem has a solution is the corresponding weighted Sobolev space. If also this hypothesis fails, the functional F as defined in (1.1) may not be lower semicontinuous in some natural space, hence the problem arises of the characterization of its semicontinuous envelope.

When Ω is an open bounded interval ($\Omega \subseteq \mathbb{R}$), Marcellini in [64] gave an explicit formula for the integrand which represents the lower semicontinuous envelope in H^1, with respect to the L^2-topology, without assuming any coerciveness hypothesis. More precisely, the lower semicontinuous envelope of

$$F(u) = \int_\Omega a(x)|\dot{u}(x)|^2\,\mathrm{d}x, \tag{1.2}$$

is given by

$$\overline{F}(u) = \int_\Omega b(x)|\dot{u}(x)|^2\,\mathrm{d}x,$$

where

$$b(x) = \lim_{\varepsilon\to 0}\left(\frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon}(a(t))^{-1}\,\mathrm{d}t\right)^{-1}$$

if a^{-1} is integrable in a neighbourhood of x, and $b(x) = 0$ otherwise. This formula has been generalized in [56] for $\Omega \subseteq \mathbb{R}^n$ (for a survey of known results see also [81] II.3).

If we consider a sequence of functionals of the type (1.2) (defined using a sequence of functions a_h), an example, given in [65], shows that their variational limit may be representable in an integral form, but the integration is performed with respect to a Radon measure different from the Lebesgue measure. A theorem is given in [23] for the Γ-limit of a sequence of general functionals, under proper conditions on the functions a_h, which assures a representation formula involving a new Radon measure.

In this article we deal with the case of quadratic forms defined on vector-valued functions of one real variable; *i.e.*, $I \subset \mathbb{R}$, $u : I \to \mathbb{R}^k$, and we consider the functional

$$F(u) = \begin{cases} \int_\Omega \sum_{i,j=1}^{n} a_{ij}(t)\dot{u}_i\dot{u}_j\,\mathrm{d}t & \text{if } u \in \mathrm{W}^{1,1}(I) \\ +\infty & \text{otherwise} \end{cases} \tag{1.3}$$

under the only assumption

$$0 \leq \sum_{i,j=1}^{k} a_{ij}(t) z_i z_j.$$

In this case we have no *a priori* coerciveness in any (weighted) Sobolev space. A natural setting on which to consider the lower semicontinuous envelope $\overline{F}$ of the quadratic form F is then the space $BV(I; \mathbb{R}^k)$ of functions of bounded variation. We give an integral representation theorem for $\overline{F}$ (Theorem 3.6) on $BV(I; \mathbb{R}^k)$:

$$(1.4)\quad \overline{F}(u) = \begin{cases} \displaystyle\int_I \sum_{i,j=1}^{k} \widetilde{a}_{ij}(t)(\dot{u}_a)_i(t)(\dot{u}_a)_j(t)\mathrm{d}t & \text{if } \frac{\dot{u}_s}{|\dot{u}_s|}(t) \in E(t) \quad |\dot{u}_s|\text{-a.e. on I} \\ +\infty & \text{elsewhere in } BV(I; \mathbb{R}^k), \end{cases}$$

where $(\widetilde{a}_{ij})$ is a new matrix of Borel functions, $E(t)$ is a linear subspace of $\mathbb{R}^k$, and $\dot{u} = \dot{u}_a \mathrm{d}t + \dot{u}_s$ is the decomposition of the Radon measure $\dot{u}$ in its absolutely continuous part $(\dot{u}_a)$ and singular part $(\dot{u}_s)$ with respect to the Lebesgue measure. An analogous result holds if the Lebesgue measure is replaced by an arbitrary positive Radon measure in (1.3) and (1.4).

Moreover, we study the stability properties of functionals of the form (1.4) with respect to Γ-convergence, and we prove an integral representation result for the Γ-limit F (Theorem 5.1) of a sequence of functionals F_h of the form (1.3) defined using a sequence of matrices a_{ij}^h. Under the condition that

$$\sum_{i,j=1}^{n} a_{ij}^h(t) z_i z_j \geq b_h(t)|z|^2$$

for a sequence of scalar Radon measures (μ_h) and a sequence of positive integrable functions (b_h) such that $\int_I (b_h(t))^{-1} d\mu_h \leq b$ with $b > 0$, there exist a measure μ, a matrix $(\widetilde{a}_{ij})$ of Borel functions and a family of linear subspaces $G(t)$ of $\mathbb{R}^n$, such that

$$(1.5)\qquad F(u) = \begin{cases} \displaystyle\int_I \sum_{i,j=1}^{n} \widetilde{a}_{ij}(t)(\dot{u}_{\mathrm{a}}^{\mu})_i(t)(\dot{u}_{\mathrm{a}}^{\mu})_j(t)\mathrm{d}\mu & \text{if } \dot{u} \in \mathcal{M}_\mu^G \\ +\infty & \text{otherwise,} \end{cases}$$

where $\mathcal{M}_\mu^G$ is the subset of $\mathcal{M}(I;\mathbb{R}^n)$ of the measures λ such that $|\lambda| << \mu$ and $\mathrm{d}\lambda_{\mathrm{a}}^\mu/\mathrm{d}\mu(t) \in G(t)$ for μ-a.e. $t \in I$.

The proof utilizes the representation results of Bouchitté (see [16] and [17]) for convex functionals on the space of measures.

Note that we obtain an integral representation for the Γ-limit without any boundedness condition. These result can not be extend when Ω is contained in $\mathbb{R}^n$ with $n > 1$, as shown by a counterexample given in [21], further developed in [74].

The plan of the chapter is the following. In Section 1 we fix some notations and we discuss some preliminaries. Since $BV(I;\mathbb{R}^n)$ is a dual of a Banach space we shall consider the weak* topology (noted by BV-w^*) with respect to this duality. Hence we can use the representation theorems of Bouchitté for the lower semicontinuous envelope and the Γ-limit of convex functionals on the space of measures to obtain representation theorems on $BV(I;\mathbb{R}^n)$ with respect to the BV-w^* topology.

In Section 2 we consider quadratic degenerate functionals (no coerciveness condition is supposed) and we look for the lower semicontinuous envelope, which is still quadratic. By using the results in Section 1, we establish an integral representation formula, involving a new matrix and for every $t \in I$ a linear subspace $E(t)$ of $\mathbb{R}^n$. More precisely, the relaxed functional on a function u is finite (and representable with this new matrix) if the singular part of the gradient of u belongs to $E(t)$ for a.e. $t \in I$. Loosely speaking, $E(t)$ is the set of the directions along which u can jump in the point $t \in I$.

Section 3 contains some examples which illustrate that, while in the case of semicontinuous dependence of the matrix on the point it is possible to give explicitly $E(t)$ and $A(t)$, in the general case it is not possible to give an easy characterization of the matrix and of the subspace.

Finally Section 4 is devoted to state an integral representation formula for the Γ-limit of quadratic functionals on $BV(I;\mathbb{R}^n)$. This representation involves a new Radon measure. Moreover, we prove that the functionals of the form (1.3) are dense in the class of functionals which can be represented as in (1.5).

4.2 Preliminaries

The letter I will denote in the sequel a fixed bounded open interval of $\mathbb{R}$. We denote by $\mathcal{C}_o(I;\mathbb{R}^n)$ (resp. $\mathcal{C}_o^\infty(I;\mathbb{R}^n)$) and by $\mathcal{C}_c(I;\mathbb{R}^n)$ (resp. $\mathcal{C}_c^\infty(I;\mathbb{R}^n)$) the space of the $\mathbb{R}^n$-valued continuous (resp. $\mathcal{C}^\infty$) functions defined on I which vanish at the endpoints of I, and with compact support respectively.

The space $BV(I;\mathbb{R}^n)$ of the *functions of bounded variation* is the space of all functions $u \in \mathrm{L}^\infty_{\mathrm{loc}}(I;\mathbb{R}^n)$ whose distributional gradient $\dot{u}$ belongs to $\mathcal{M}(I;\mathbb{R}^n)$ (if $n = 1$ we use also the notations $BV(I)$, as well as $\mathcal{C}_c(I)$, $\mathcal{C}_o(I)$ and $\mathrm{L}^1(I)$). The total variation $|\dot{u}|$ of the measure $\dot{u}$ on a Borel set B will be denoted by $\int_B |\dot{u}|$. Moreover the integral of a Borel function f with respect to this measure will be denoted by $\int_B f|\dot{u}|$. Fixed a measure $\lambda \in \mathcal{M}^+(\Omega)$, for every function $u \in BV(I;\mathbb{R}^n)$ we consider the decomposition $\dot{u} = \dot{u}_{\mathrm{a}}^\lambda \mathrm{d}\lambda + \dot{u}_{\mathrm{s}}^\lambda$ of the measure $\dot{u}$ in its absolutely continuous part and singular part with respect to the measure λ. When λ is the Lebesgue measure we use the notation $\dot{u}$ as $\dot{u} = \dot{u}_{\mathrm{a}}\mathrm{d}t + \dot{u}_{\mathrm{s}}$; moreover if $u \in \mathrm{W}^{1,1}(I;\mathbb{R}^n)$, with abuse of notations, we use also $\dot{u}$ in order to note the Radon-Nikodym derivative $\dot{u}_{\mathrm{a}}$. On $BV(I;\mathbb{R}^n)$ we shall consider the norm

$$\|u\|_{BV} = \|u\|_{\mathrm{L}^\infty} + \int_I |\dot{u}|.$$

On the space $BV(I;\mathbb{R}^n)$ we can consider a weak* topology (in some text referred to simply as the "weak" topology of BV); in fact, the space $BV(I;\mathbb{R}^n) \sim (BV(I))^n$ is (isometric to) a dual of a Banach space, as shown in the following lemma.

Lemma 2.1. *Let $Y = \mathrm{L}^1(I) \times \mathcal{C}_o(I)$ endowed with the norm $\|(f,\phi)\|_Y = \max\{\|f\|_{\mathrm{L}^1}, \|\phi\|_{\mathcal{C}_o}\}$ and let Z be the closure of the set $\{(\dot{\chi},\chi) : \chi \in \mathcal{C}_c^\infty\}$. Then $BV(I)$ is (isometric to) the dual of the Banach space Y/Z.*

Proof. (Dal Maso [34]) It is easy to check that $Z^\perp = \{(u,\dot{u}) : u \in BV(I)\} \subset \mathrm{L}^\infty(I) \times \mathcal{M}(I) = Y^*$ (endowed with the dual norm $\|(u,\mu)\|_{Y^*} = \|u\|_{\mathrm{L}^\infty} + |\mu|(I)$), and that $Z^\perp$ is isometric to $(Y/Z)^*$ (see [46], Exercise II.4.18(b)). Denoting the Z-equivalence class of $(f,\phi) \in Y$ by $[f,\phi]$, the isometry $u \mapsto \Phi_u$ between $BV(I)$ and $(Y/Z)^*$ is defined by

$$\Phi_u([f,\phi]) = \int_I fu\,\mathrm{d}x + \int_I \phi\dot{u}$$

for every $u \in BV(I), f \in \mathrm{L}^1(I)$, and $\phi \in \mathcal{C}_o(I)$. □

The *weak* topology* on $BV(I)$, that we shall denote by $BV\text{-}w^*$, is then defined as the weakest topology on $BV(I)$ for which the maps $u \mapsto \Phi_u([f,\phi])$ are continuous for every $f \in L^1(I)$, and for every $\phi \in \mathcal{C}_o(I)$.

With a similar notation as above, if $\psi \in \mathcal{C}_o(I)$ and $\mu \in \mathcal{M}(I)$, we set $\Psi_\mu(\psi) = \int_I \psi\mu$. The usual *weak* topology* on $\mathcal{M}(I)$ is then defined as the weakest topology on $\mathcal{M}(I)$ for which the maps $\mu \mapsto \Psi_\mu(\psi)$ are continuous for every $\psi \in \mathcal{C}_o(I)$.

Let us define the subspace

$$BV_*(I;\mathbb{R}^n) = \left\{ u \in BV(I;\mathbb{R}^n) : \int_I u(t)\mathrm{dt} = 0 \right\}.$$

Let us remark that in order to restrict the $BV\text{-}w^*$ topology on $BV_*(I)$ it is sufficient to consider the maps $\Phi_u([f,\phi])$ when $\int_I f(t)\mathrm{dt} = 0$, since $\Phi_u([f,\phi]) = \Phi_u([f+c,\phi])$ (c any arbitrary constant).

Lemma 2.2. *Endowed with the restriction of the $BV\text{-}w^*$ topology, the space $BV_*(I)$ is linearly and topologically isomorphic to the space $\mathcal{M}(I)$ with the weak* topology.*

Proof. The map $T : BV_*(I) \to \mathcal{M}(I)$, $u \mapsto Tu = \dot{u}$, is a bijection. Notice also that we have the 1-1 correspondence between $\{[f,\phi] \in Y/Z : \int_I f(t)\mathrm{dt} = 0\}$ and $\mathcal{C}_o(I)$ given by $[f,\phi] \mapsto \phi - \int f$ (where $\int f$ is the primitive of the function f which vanishes at the endpoints of I); in fact, $[f,\phi] = [0, \phi - \int f]$, and hence the inverse map is given by $\psi \mapsto [0,\psi]$. In order to prove that T is a linear and topological isomorphism is then sufficient to remark that

$$T(\{v \in BV_*(I) : |\Phi_v([f,\phi])| < \varepsilon\}) = \{\mu \in \mathcal{M}(I) : |\Psi_\mu(\psi)| < \varepsilon\},$$

whenever $[f,\phi] \in Y/Z$ with $\int_I f\mathrm{dx} = 0$ and $\psi \in \mathcal{C}_o(I)$ such that $\psi = \phi - \int f$, and that from this equality we obtain a correspondence between the neighbourhoods of 0 (and hence of every point in $BV_*(I)$) in the weak* topologies. $\square$

On $BV(I;\mathrm{I\!R}^n)$ we shall consider the product topology BV-w^* of the weak* topologies on $BV(I)$.

Given $\lambda \in \mathcal{M}^+(I)$, let us consider the functional $F : BV(I;\mathrm{I\!R}^n) \to [0,+\infty]$ defined by

$$(2.1) \qquad F(u) = \begin{cases} \displaystyle\int_I f\left(t, \frac{d\dot{u}}{d\lambda}(t)\right) d\lambda & \text{if } |\dot{u}| << \lambda \\ +\infty & \text{otherwise.} \end{cases}$$

When λ is the Lebesgue measure we have

$$(2.2) \qquad F(u) = \begin{cases} \displaystyle\int_I f(t,\dot{u}(t))\mathrm{d}t & \text{if } u \in \mathrm{W}^{1,1}(I;\mathrm{I\!R}^n) \\ +\infty & \text{otherwise.} \end{cases}$$

The theorems in the rest of this section will be obtained as a direct consequence of the Lemmas above, and of the representation results contained in [16] and [17].

Theorem 2.3. *Let $f : I \times \mathrm{I\!R}^n \to [0,+\infty]$ be a Borel function and $\lambda \in \mathcal{M}^+(\Omega)$ such that*

a) for every $t \in I$ the map $z \mapsto f(t,z)$ is convex and lower semicontinuous on $\mathrm{I\!R}^n$;

b) there exists $v \in \mathrm{L}^1_{\mathrm{loc}}(I;\mathrm{I\!R}^n)$ such that $\int_I f(t,v(t))\mathrm{d}\lambda < +\infty$.

Let

$$E = \{\phi \in C_c(I;\mathrm{I\!R}^n) : \int_I f^*(t,\phi(t))\mathrm{d}\lambda < +\infty\}$$

and let $h : I \times \mathrm{I\!R}^n \to [0,+\infty]$ be the function, which is lower semicontinuous and positively homogeneous of degree 1 in the second variable, defined by

$$h(t,z) = \sup_{\phi \in E} \langle \phi(t), z\rangle.$$

Let us consider the function $g : I \times \mathrm{I\!R}^n \to [0,+\infty]$ defined, for every $t \in I$, by

$$g(t,\cdot) = [f(t,\cdot)\nabla h(t,\cdot)]^{**},$$

let F be defined as in (2.1), and let $\overline{F}$ be the relaxed functional of F in the BV-w^* topology. Then for every $u \in BV(I;\mathbb{R}^n)$ we have

$$\overline{F}(u) = \int_I g(t, \dot{u}_a^\lambda(t))\mathrm{d}\lambda + \int_I h\left(t, \frac{\dot{u}_s^\lambda}{|\dot{u}_s^\lambda|}(t)\right) |\dot{u}_s^\lambda|, \tag{2.3}$$

where $\dot{u} = \dot{u}_a^\lambda d\lambda + \dot{u}_s^\lambda$ is the decomposition of the Radon measure $\dot{u}$ in its absolutely continuous part and singular part with respect to the measure λ, and $\dot{u}_s^\lambda/|\dot{u}_s^\lambda|$ denotes the Radon-Nikodym derivative of the measure $\dot{u}_s^\lambda$ with respect to its total variation. Moreover

$$h(t,\cdot) = g^\infty(t,\cdot)$$

for λ-a.e. $t \in I$.

Proof. By Lemma 2.2 we can identify $\mathcal{M}(I;\mathbb{R}^n)$ with $(BV_*(I))^n \sim BV_*(I;\mathbb{R}^n)$. We can consider the restriction of F to $BV_*(I;\mathbb{R}^n)$. From Theorem 4 in [16] (which gives the analogous representation formula on $\mathcal{M}(I;\mathbb{R}^n)$), we obtain then formula (2.3) for the relaxed functional on $BV_*(I;\mathbb{R}^n)$. Noticing that $\overline{F}(u) = \overline{F}(u - \int_I u)$, we eventually obtain the representation on the whole $BV(I;\mathbb{R}^n)$. $\square$

Now we recall the definition of the Γ-*limit* of a sequence of functionals. Let (X,σ) be a topological space, and let $F_h : X \to [0,+\infty]$ be a sequence of functionals on X.

Let us define the Γ-*lower limit* and the Γ-*upper limit* respectively by

$$(\Gamma\text{-}\liminf_{h\to\infty} F_h)(x) = \sup_{U\in\mathcal{N}(x)} \liminf_{h\to\infty} \inf_{y\in U} F_h(y)$$

and

$$(\Gamma\text{-}\limsup_{h\to\infty} F_h)(x) = \sup_{U\in\mathcal{N}(x)} \limsup_{h\to\infty} \inf_{y\in U} F_h(y).$$

If we have

$$\Gamma\text{-}\liminf_{h\to\infty} F_h(x) = \Gamma\text{-}\limsup_{h\to\infty} F_h(x),$$

then we say that the sequence (F_h) Γ-converges at x and that the value $F(x)$ of the $\Gamma\text{-}\liminf_{h\to\infty} F_h(x)$ is the Γ-*limit* of the sequence (F_h) at x.

We say that a functional $F : X \to [0,+\infty]$ is the *sequential* Γ-*limit* of the sequence (F_h) (or F_h sequentially Γ-converges to F) if the following conditions are satisfied for every $x \in X$:
a) for every sequence (x_h) such that $x_h \to x$ we have $F(x) \leq \liminf\limits_{h\to\infty} F_h(x_h)$;
b) there exists a sequence $(\widetilde{x}_h)$ such that $\widetilde{x}_h \to x$ and $F(x) = \lim\limits_{h\to\infty} F_h(\widetilde{x}_h)$. If (X,σ) is a metric space, then the Γ-limit coincides with the sequential Γ-limit.

The Γ-convergence has been studied by many authors (for an introduction and an extensive bibliography we refer to [32]). Since it coincides with the convergence (in the sense of Kuratowski) of the epigraphs of F_h, it is called sometimes also *epi-convergence* (see [12]).

In the following, we will consider the relaxation and the Γ-convergence on the space $BV(I;\mathbb{R}^n)$ with respect the BV-w^* topology. We observe that when the sequence of functionals (F_h) satisfies a condition of the type

$$\int_I |Du| - c \leq F_h(u),$$

then their Γ-limit (or the relaxation) with respect to the BV-w^* topology coincides with the corresponding sequential Γ-limit (see Remark 2.3 in [23).

Theorem 2.4. *Let $F_h : BV(I;\mathbb{R}^n) \to [0,+\infty]$ be a sequence of convex functionals such that:*

a) for every $h \in \mathbb{N}$, $F_h(0) = 0$,

b) for every $h \in \mathbb{N}$, F_h is an additive functional; i.e., for every $u_1, u_2 \in BV(I;\mathbb{R}^n)$ such that the measure Du_1 is orthogonal to the measure Du_2 we have

$$F_h(u_1 + u_2) = F_h(u_1) + F_h(u_2),$$

c) there exists a constant $b > 0$ such that

$$\int_I |Du| - b \leq F_h(u)$$

for every $u \in BV(I;\mathbb{R}^n)$ and $h \in \mathbb{N}$.

Then there exists a convex lower semicontinuous functional F and a subsequence (h_k) such that F equals both the sequential Γ-limit and the Γ-limit of F_{h_k}. Moreover *there exist two functions* $g, h : I \times \mathbb{R}^n \to [0, +\infty]$ *and a measure* $\mu \in \mathcal{M}^+(\Omega)$ *such that*

i) g *is lower semicontinuous and the map* $z \mapsto g(t, z)$ *is convex,*

ii) h *is lower semicontinuous and the map* $z \mapsto h(t, z)$ *is convex and positively homogeneous of degree 1,*

iii) for every $z \in \mathbb{R}^n$ *and* μ*-a.e.* $t \in I$ *we have* $h(t, z) = g^\infty(t, z)$*,*

iv) F *admits the following representation formula:*

$$F(u) = \int_I g(t, \dot{u}_a^\mu(t)) d\mu + \int_I h\left(t, \frac{\dot{u}_s^\mu}{|\dot{u}_s^\mu|}(t)\right) |\dot{u}_s^\mu|, \tag{2.4}$$

where $\dot{u} = \dot{u}_a^\mu d\mu + \dot{u}_s^\mu$ *is the decomposition of the Radon measure* $\dot{u}$ *in its absolutely continuous part and singular part with respect to the measure* μ*, and* $\dot{u}_s^\mu / |\dot{u}_s^\mu|$ *denotes the Radon-Nikodym derivative of the measure* $\dot{u}_s^\mu$ *with respect to its total variation. Moreover we have*

$$\int_I |Du| - b \leq F(u).$$

Proof. As in the proof of the previous theorem, the assertion follows from Theorem 3.11 in [3] which gives a representation formula on the space of measures. □

4.3 Integral representation of the relaxed functional

In this section we give an integral representation theorem for the lower semicontinuous envelope of a quadratic functional on $BV(I; \mathbb{R}^n)$, with respect to the BV-w^* topology.

We begin by recalling the definition, and some properties of quadratic forms. A *non-negative quadratic form* f on a topological vector space X is a function

$f : X \to [0, +\infty]$ such that there exist a linear subspace E of X and a symmetric bilinear form $f_o : E \times E \to [0, +\infty]$ such that

$$f(x) = \begin{cases} f_o(x, x) & \text{if } x \in E \\ +\infty & \text{otherwise.} \end{cases}$$

We remark that every non-negative quadratic form is a convex function. It is possible to give an algebraic characterization of quadratic forms by means of the parallelogram identity as in the following proposition.

Proposition 3.1. *Let $f : X \to [0, +\infty]$ be a lower semicontinuous function. Then f is a quadratic form iff $f(0) = 0$ and $f(z_1 + z_2) + f(z_1 - z_2) = 2f(z_1) + 2f(z_2)$ for every $z_1, z_2 \in X$.*

Proof. The statement is a direct consequence of standard algebraic manipulations similar to those customarily used to show that any norm satisfying the parallelogram identity can be obtained from a scalar product (see *e.g.* [90] Ch.1, Sect.5. Theorem 1). □

Moreover quadraticity is preserved on passing to the relaxed functional and to the Γ-limit, as explained in the following proposition.

Proposition 3.2.

a) Let F be a non-negative quadratic form. Then the lower semicontinuous envelope is still a non-negative quadratic form.

b) Let (F_h) be a sequence of non-negative quadratic forms Γ-converging to a functional F. Then F is a non-negative quadratic form.

Proof. See [82]. □

Remark 3.3. Let $f : \mathbb{R}^n \to [0, +\infty]$ be a quadratic form positively homogeneous of degree 1. Then there exists a subspace E of $\mathbb{R}^n$ such that $f(z) = \chi_E(z)$. In fact, since $tf(x) = f(tx) = t^2 f(x)$ for every $t > 0$, we have that $f(z) = 0$, whenever

$f(z) < +\infty$. If we define $E = \{z \in \mathbb{R}^n : f(z) < +\infty\}$, then E is a subspace of $\mathbb{R}^n$; in fact, since f is convex,

$$\frac{1}{2}f(z_1+z_2) = f(\frac{z_1+z_2}{2}) \leq \frac{1}{2}f(z_1) + \frac{1}{2}f(z_2)$$

for every $z_1, z_2 \in E$; hence $z_1 + z_2 \in E$; moreover, for every $t \in \mathbb{R}$, $f(tz_1) = tf(z_1)$ implies $tz_1 \in E$.

Remark 3.4. We observe that the recession function f^∞ of a quadratic form $f : \mathbb{R}^n \to [0, +\infty]$ is quadratic. In fact, $f^\infty(0) = 0$ and, since $f(tz) = t^2 f(z)$, for every $z_1, z_2 \in \mathbb{R}^n$ we have

$$\begin{aligned} f^\infty(z_1+z_2) + f^\infty(z_1-z_2) &= \lim_{t\to+\infty} t[f(z_1+z_2) + f(z_1-z_2)] \\ &= \lim_{t\to+\infty} t[2f(z_1) + 2f(z_2)] = 2f^\infty(z_1) + 2f^\infty(z_2). \end{aligned}$$

Then by Remark 3.3, there exists a subspace E of $\mathbb{R}^n$ such that $f^\infty(z) = \chi_E(z)$.

We can turn our attention now to the relaxation of a quadratic form in $BV(I;\mathbb{R}^n)$. Let $\lambda \in \mathcal{M}^+(\Omega)$ and let $A(t)$ be a $n \times n$ symmetric matrix of measurable functions $(a_{ij}(t))$ defined on I such that for every $z \in \mathbb{R}^n$

$$\sum_{i,j=1}^{n} a_{ij}(t) z_i z_j \geq 0. \tag{3.1}$$

Let $F : BV(I;\mathbb{R}^n) \to [0, +\infty]$ be the quadratic functional defined by:

$$F(u) = \begin{cases} \displaystyle\int_I \sum_{i,j=1}^{n} a_{ij}(t) \frac{d\dot{u}_i}{d\lambda}(t) \frac{d\dot{u}_j}{d\lambda}(t) d\lambda & \text{if } |\dot{u}| << \lambda \\ +\infty & \text{otherwise.} \end{cases} \tag{3.2}$$

When λ is the Lebesgue measure we have

$$F(u) = \begin{cases} \displaystyle\int_I \sum_{i,j=1}^{n} a_{ij}(t) \dot{u}_i(t) \dot{u}_j(t) \mathrm{dt} & \text{if } u \in \mathrm{W}^{1,1}(I;\mathbb{R}^n) \\ +\infty & \text{otherwise.} \end{cases}$$

We shall consider the relaxed functional of F with respect to the BV-w^* topology.

We remark that if λ is the Lebesgue measure, then we obtain the same result if we consider the functional which coincides with F on $C^1(I;\mathbb{R}^n)$ and equals $+\infty$ on $BV(I;\mathbb{R}^n)\setminus C^1(I;\mathbb{R}^n)$. In fact, it is easy to see that a function $u \in W^{1,1}(I;\mathbb{R}^n)$ can be approximated (using convolution and truncation arguments) by a sequence of C^1 functions (u_h) such that u_h converges to u in BV-w^* and $F(u_h)$ converges to $F(u)$.

The following lemma will be useful in the proof of Theorem 3.6.

Lemma 3.5. *Let $\lambda \in \mathcal{M}^+(\Omega)$, let B be a Borel subset of I and let $f, h : I \times \mathbb{R}^n \to [0,+\infty]$ be two Borel functions such that*

$$\int_I f(t, v(t))\mathrm{d}\lambda = \int_I h(t, v(t))\mathrm{d}\lambda$$

for every integrable function v on I with respect to the measure λ. Then $f(t,z) = h(t,z)$ for λ-a.e. $t \in B$ and for every $z \in \mathbb{R}^n$.

Proof. See for instance [22], Corollary 2.3. $\square$

Theorem 3.6. *Let $A(t)$ be a $n \times n$ symmetric matrix of measurable functions $(a_{ij}(t))$ satisfying (3.1). Let $\lambda \in \mathcal{M}^+(I)$ and let F be defined as in (3.2). Then there exist*

i) a $n \times n$ symmetric matrix $\widetilde{A}(t)$ of measurable functions $(\widetilde{a}_{ij}(t))$ such that for every $z \in \mathbb{R}^n$

$$\sum_{i,j=1}^{n} a_{ij}(t)z_iz_j \geq \sum_{i,j=1}^{n} \widetilde{a}_{ij}(t)z_iz_j \geq 0,$$

ii) a Borel multivalued function $t \mapsto E(t)$ with values linear subspaces of $\mathbb{R}^n$, such that the relaxed functional $\overline{F}$ of F in the BV-w^* topology admits the following representation formula for $u \in BV(I;\mathbb{R}^n)$:

(3.3)

$$\overline{F}(u) = \begin{cases} \displaystyle\int_I \sum_{i,j=1}^{n} \widetilde{a}_{ij}(t)(\dot{u}_a^\lambda)_i(t)(\dot{u}_a^\lambda)_j(t)\mathrm{d}\lambda & \text{if } \frac{\dot{u}_s^\lambda}{|\dot{u}_s^\lambda|}(t) \in E(t) \quad |\dot{u}_s^\lambda|\text{-a.e. on } I \\ +\infty & \text{otherwise,} \end{cases}$$

where $\dot{u} = \dot{u}_a^\lambda d\lambda + \dot{u}_s^\lambda$ *is the decomposition of the Radon measure* $\dot{u}$ *in its absolutely continuous part and singular part with respect to the measure* λ, *and* $\dot{u}_s^\lambda / |\dot{u}_s^\lambda|$ *denotes the Radon-Nikodym derivative of the measure* $\dot{u}_s^\lambda$ *with respect to its total variation.*

Proof. First we observe that by proposition 3.2 the relaxed functional $\overline{F}$ is a quadratic functional and so it is convex; moreover by proposition 3.1

$$\overline{F}(0) = 0 \tag{3.4}$$

and

$$\overline{F}(u+v) + \overline{F}(u-v) = 2\overline{F}(u) + 2\overline{F}(v) \tag{3.5}$$

for every $u, v \in BV(I; \mathbb{R}^n)$. We can apply Theorem 2.3 with $f(t,z) = \sum_{i,j=1}^{n} a_{ij}(t) z_i z_j$; then

$$\overline{F}(u) = \int_I g(t, \dot{u}_a(t)) d\lambda + \int_I h\left(t, \frac{\dot{u}_s^\lambda}{|\dot{u}_s^\lambda|}(t)\right) |\dot{u}_s^\lambda|, \tag{3.6}$$

where g and h are defined in Theorem 2.3. Let $u_1, u_2 \in BV(I; \mathbb{R}^n)$ such that the measures Du_1 and Du_2 are absolutely continuous with respect to λ; the formulas (3.5) and (3.6) imply that

$$\int_I g(t, \dot{u}_1(t) + \dot{u}_2(t)) d\lambda + \int_I g(t, \dot{u}_1(t) - \dot{u}_2(t)) d\lambda =$$
$$2 \int_I g(t, \dot{u}_1(t)) d\lambda + 2 \int_I g(t, \dot{u}_2(t)) d\lambda.$$

By using Lemma 3.5 we get for λ-a.e. $t \in I$ and for every $z_1, z_2 \in \mathbb{R}^n$

$$g(t, z_1 + z_2) + g(t, z_1 - z_2) = 2g(t, z_1) + 2g(t, z_2).$$

On the other hand, by (3.4) for λ-a.e. $t \in I$ we have $g(t, 0) = 0$. Therefore for λ-a.e. $t \in I$ the map $z \mapsto g(t, z)$ is quadratic.

Let us call A the negligible set of the t for which $h(t, \cdot) \neq g^\infty(t, \cdot)$. Using Remark 3.3, the map $z \mapsto h(t, z)$ is quadratic and homogeneous of degree 1 for every $t \in I \setminus A$. Moreover for every $t_o \in A$ and for every $z \in \mathbb{R}^n$ let us consider the function

$$u_0(t) = \begin{cases} 0 & \text{if } t < t_o \\ z & \text{if } t \geq t_o. \end{cases} \tag{3.7}$$

We have $\overline{F}(u_0) = h(t_o, z)$; hence, by the quadraticity of $\overline{F}$, we obtain the quadraticity of h in z. Therefore by Remark 3.3, for every $t \in I$ there exists a linear subspace $E(t)$ of $\mathbb{R}^n$ such that

$$h(t,z) = \chi_{E(t)}(z). \tag{3.8}$$

Moreover

$$g(t,z) = [\inf_{y \in \mathbb{R}^n} \{f(t, z-y) + h(t,y)\}]^{**}$$

$$= \inf_{y \in E(t)} \sum_{i,j=1}^{n} a_{ij}(t)(z_i - y_i)(z_j - y_j) \leq \sum_{i,j=1}^{n} a_{ij}(t) z_i z_j < +\infty.$$

Since for λ-a.e. $t \in I$ the map $t \mapsto g(t,z)$ is quadratic and finite, there exists a positive symmetric bilinear form Q_t on $\mathbb{R}^n$ such that $g(t,z) = Q_t(z,z)$ for all $z \in \mathbb{R}^n$. Fixed $e_1, \ldots, e_n$ a base of $\mathbb{R}^n$, we have that for every $z, w \in \mathbb{R}^n$

$$Q_t(z,w) = \sum_{i,j=1}^{n} z_i w_j Q_t(e_i, e_j).$$

Hence we obtain

$$g(t,z) = \sum_{i,j=1}^{n} \widetilde{a}_{ij}(t) z_i z_j, \tag{3.9}$$

where $\widetilde{a}_{ij}(t) = Q_t(e_i, e_j)$. Therefore by (3.6), (3.8) and (3.9) $\overline{F}$ admits the representation formula (3.3). $\square$

Remark 3.7. If, in addition to the hypotheses of Theorem 3.6, we assume that

$$F(u) \geq \int_I |Du| - c, \tag{3.10}$$

we can take $E(t_0) = \{0\}$ for all $t_0 \in I$. Hence

$$\overline{F}(u) = \begin{cases} \displaystyle\int_I \sum_{i,j=1}^{n} \widetilde{a}_{ij}(t)(\dot{u}_a^\lambda)_i(t)(\dot{u}_a^\lambda)_j(t)\mathrm{d}\lambda & \text{if } |\dot{u}| << \lambda \\ +\infty & \text{otherwise.} \end{cases}$$

In fact, if t_0 is an atom of λ ; *i.e.*, $\lambda(\{t_0\}) > 0$, then the singular part $\dot{u}_s^\lambda$ of $\dot{u}$ with respect to the measure λ does not weigh the point t_0. On the other hand, if $\lambda(\{t_0\}) = 0$, let us consider $z \in E(t_0)$ and the function u_0 defined in (3.7). By (3.3) for every $k \in \mathbb{N}$ we have $\overline{F}(ku_0) = 0$; then we obtain

$$0 = \overline{F}(ku_0) \geq k \int_I |Du_0| - c = kz - c.$$

If $z \neq 0$ we obtain a contradiction for large k; hence $E(t) = \{0\}$.

Proposition 3.8. *Let (a_{ij}) and λ as in the previous theorem. Let $t \mapsto G(t)$ be a Borel multivalued function, with values linear subspaces of $\mathbb{R}^n$. Let $F : BV(I;\mathbb{R}^n) \to [0,+\infty]$ be the quadratic functional defined by:*

(3.11)

$$F(u) = \begin{cases} \displaystyle\int_I \sum_{i,j=1}^{n} a_{ij}(t)\frac{d\dot{u}_i}{d\lambda}(t)\frac{d\dot{u}_j}{d\lambda}(t)\mathrm{d}\lambda & \textit{if } |\dot{u}| << \lambda \textit{ and } \frac{d\dot{u}}{d\lambda}(t) \in G(t) \quad \lambda\textit{-a.e. on } I \\ +\infty & \textit{otherwise.} \end{cases}$$

Then

i) there exists an $n \times n$ symmetric matrix $\widetilde{A}(t)$ of measurable functions $(\widetilde{a}_{ij}(t))$ such that for every $z \in \mathbb{R}^n$ we have

$$\sum_{i,j=1}^{n} a_{ij}(t)z_iz_j \geq \sum_{i,j=1}^{n} \widetilde{a}_{ij}(t)z_iz_j \geq 0,$$

ii) there exists a Borel multivalued function $t \mapsto E(t)$ with values linear subspaces of $\mathbb{R}^n$, *such that the relaxed functional $\overline{F}$ in the BV-w^* topology can be represented for every $u \in BV(I;\mathbb{R}^n)$ as:*

(3.12)

$$\overline{F}(u) = \begin{cases} \displaystyle\int_I \sum_{i,j=1}^{n} \widetilde{a}_{ij}(t)(\dot{u}_a^\lambda)_i(t)(\dot{u}_a^\lambda)_j(t)\mathrm{d}\lambda & \textit{if } \dot{u}_a^\lambda(t) \in G(t)\ \lambda\textit{-a.e. on } I \\ & \textit{and } \frac{\dot{u}_s^\lambda}{|\dot{u}_s^\lambda|}(t) \in E(t) \quad |\dot{u}_s^\lambda|\textit{-a.e.} \\ +\infty & \textit{otherwise.} \end{cases}$$

Proof. As in the proof of Theorem 3.6, $\overline{F}$ admits the representation formula (3.6) with $h(t,z) = \chi_{E(t)}(z)$ and g such that for λ-a.e. $t \in I$ the map $t \mapsto g(t,z)$ is quadratic. Since now

$$f(t,z) = \begin{cases} \displaystyle\sum_{i,j=1}^{n} a_{ij}(t)z_i z_j & \text{if } z \in G(t) \\ +\infty & \text{otherwise.} \end{cases}$$

Then

$$g(t,z) = \Big[\inf_{y\in\mathbb{R}^n}\{f(t,z-y)+h(t,y)\}\Big]^{**} =$$

$$= \begin{cases} \displaystyle\inf_{y\in E(t)\cap(z+G(t))} \sum_{i,j=1}^{n} a_{ij}(t)(z_i-y_i)(z_j-y_j) & \text{if } z \in G(t) \\ +\infty & \text{otherwise.} \end{cases}$$

Since for λ-a.e. $t \in I$ the map $z \mapsto g(t,z)$ is quadratic and finite on $G(t)$, there exists a symmetric matrix of measurable functions $\widetilde{a}_{ij}(t)$ such that

$$g(t,z) = \begin{cases} \displaystyle\sum_{i,j=1}^{n} \widetilde{a}_{ij}(t)z_i z_j & \text{if } z \in G(t) \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover for λ-a.e. $t \in I$ and $z \in G(t)$ we get

$$\sum_{i,j=1}^{n} \widetilde{a}_{ij}(t)z_i z_j = g(t,z) \le f(t,z) = \sum_{i,j=1}^{n} a_{ij}(t)z_i z_j.$$

$\square$

Corollary 3.9. *Let (a_{ij}), λ, $G(t)$ and F as in the previous proposition. We assume also that*

$$\sum_{i,j=1}^{n} a_{ij}(t) z_i z_j \geq b(t)|z|^2 \tag{3.13}$$

with $\int_I (b(t))^{-1} d\lambda < +\infty$. Then the functional F is lower semicontinuous with respect to the L^1 topology.

Proof. We consider the relaxed functional $\overline{F}(u)$. By Proposition 3.8 it can be represented as in (3.12). As in Remark 3.7, since (3.13) implies (3.10), we can assume that $E(t) = \{0\}$ for all $t \in I$. Moreover

$$\sum_{i,j=1}^{n} \widetilde{a}_{ij}(t) z_i z_j = \inf_{y \in E(t)} \sum_{i,j=1}^{n} a_{ij}(t)(z_i - y_i)(z_j - y_j) = \sum_{i,j=1}^{n} a_{ij}(t) z_i z_j.$$

Therefore $\overline{F} = F$. □

4.4 Study of the set $E(t)$ and of the matrix $\widetilde{A}(t)$

The set $E(t)$ is the vector subspace of $\mathbb{R}^n$ of the directions along which u "may jump" in t. We want to give a description of the set $E(t)$. We shall restrict to the case when λ is the Lebesgue measure, the general case being treated in the same way.

Remark 4.1. When the dependence of the matrix A on the variable t is lower semicontinuous, it is possible to give a characterization of the subspaces $E(t)$. Let us define

$$\widehat{E}(t) = \{z \in \mathbb{R}^n : A(t)z = 0\}; \tag{4.1}$$

it is easy to see that for a.e. $t \in I$ the recession function of the map $z \mapsto \langle A(t)z, z \rangle$ is given by $\chi_{\widehat{E}(t)}(z)$. Hence by Proposition 10 of [16] we obtain

$$E(t) = \widehat{E}(t) \text{ for a.e. } t \in I.$$

Moreover we have

$$\widetilde{A}(t) = A(t) \text{ for a.e. } t \in I. \tag{4.2}$$

In fact, by Theorem 2.3 we get

$$\langle \widetilde{A}(t)z, z\rangle = \inf_{y \in E(t)} \langle A(t)(z-y), z-y\rangle. \tag{4.3}$$

Then (4.2) follows from (4.3). In the general case, $E(t) \supseteq \widehat{E}(t)$ for a.e. $t \in I$ (see Proposition 7 and Proposition 8 of [16]).

Proposition 4.2. *Let $t \in I$; then we have*

(4.4)

i) $E(t) \supset \{\nu \in \mathbb{R}^n : \exists (u_\varepsilon); u_\varepsilon \in C^1((t-\varepsilon, t+\varepsilon); \mathbb{R}^n), u_\varepsilon(t+\varepsilon) - u_\varepsilon(t-\varepsilon) \to \nu$

$$\int_{t-\varepsilon}^{t+\varepsilon} \sum_{i,j=1}^n a_{ij}(t)(\dot{u}_\varepsilon)_i(t)(\dot{u}_\varepsilon)_j(t)\mathrm{dt} \le c, \int_{t-\varepsilon}^{t+\varepsilon} |\dot{u}_\varepsilon(t)|\mathrm{dt} \le c\}.$$

(4.5)

ii) $E(t) \subset \{\nu \in \mathbb{R}^n : \exists (u_\varepsilon); u_\varepsilon \in C^1((t-\varepsilon, t+\varepsilon); \mathbb{R}^n), u_\varepsilon(t+\varepsilon) - u_\varepsilon(t-\varepsilon) \to \nu$

$$\int_{t-\varepsilon}^{t+\varepsilon} \sum_{i,j=1}^n a_{ij}(t)(\dot{u}_\varepsilon)_i(t)(\dot{u}_\varepsilon)_j(t)\mathrm{dt} \le c\}.$$

Proof. *i)* Suppose that there exists a sequence $u_h \in C^1((t-\frac{1}{h}, t+\frac{1}{h}); \mathbb{R}^n)$ such that

$$\int_{t-\frac{1}{h}}^{t+\frac{1}{h}} \sum_{i,j=1}^n a_{ij}(t)(\dot{u}_h)_i(t)(\dot{u}_h)_j(t)\mathrm{dt} \le c, \qquad \int_{t-\frac{1}{h}}^{t+\frac{1}{h}} |\dot{u}_h(t)|\mathrm{dt} \le c,$$

and

$$u_h(t-\frac{1}{h}) - u_h(t+\frac{1}{h}) \to \nu \in \mathbb{R}^n.$$

Then it is easy to extend (a translation of) u_h to $C^1(I; \mathbb{R}^n)$ in such a way that

$$F(u_h) \le c, \quad \int_{-1}^{1} |\dot{u}_h(t)|\mathrm{dt} \le c,$$

and $u_h \to u_\nu$ in $L^1(I)$. Since the sequence (u_h) is bounded in BV, we have (passing possibly to a subsequence) that $u_h \rightharpoonup u_\nu$ in BV. This implies that $\overline{F}(u_\nu) < +\infty$; hence $\frac{\nu}{|\nu|} \in E(t)$; *i.e.*, $\nu \in E(t)$.

ii) Fix $t \in I$; if $\nu \in E(t)$, $|\nu| = 1$, and we define

$$u_\nu(\tau) = \begin{cases} 0 & \text{if } \tau \le t \\ \nu & \text{if } \tau > t, \end{cases}$$

then we have

$$\dot{u}_\nu = (\dot{u}_\nu)_s = \nu\delta_t,$$

hence

$$\frac{(\dot{u}_\nu)_s}{|(\dot{u}_\nu)_s|}(t) = \nu \in E(t).$$

By Theorem 3.6 we have $\overline{F}(u_\nu) = 0$. Remark that the relaxed functional of F in L^1 is less than $\overline{F}(u_\nu)$, and hence it needs equals zero. We can find a sequence $u_h \in C^1(I; \mathbb{R}^n)$ such that $u_h \to u_\nu$ in L^1 and a.e., and $\lim_h F(u_h) = 0$. In particular there exists $c > 0$ such that

$$\int_{t-(1/h)}^{t+(1/h)} \sum_{i,j=1}^{n} a_{ij}(t)(\dot{u}_h)_i(t)(\dot{u}_h)_j(t)\mathrm{dt} \le c. \tag{4.6}$$

Moreover, we can suppose (renumbering if necessary the sequence) that $u_h(t - \frac{1}{h}) \to 0$, $u_h(t + \frac{1}{h}) \to \nu$. □

Remark 4.3. If $n = 1$ (hence $F(u) = \int_{-1}^{1} a(t)|\dot{u}(t)|^2\mathrm{dt}$), then either $E(t) = \{0\}$ or $E(t) = \mathbb{R}$. It is easy to see (*e.g.* in [16] Example 5) that $E(t) = \{0\}$ *iff* t belongs to the set I', where

(4.7)

$$I' = \{t \in I : \exists\varepsilon > 0 \text{ such that } a(t) > 0 \text{ a.e. on } (t-\varepsilon, t+\varepsilon), \text{ and } \frac{1}{a} \in L^1(t-\varepsilon, t+\varepsilon)\}.$$

We "may have a jump" in t then only if $1/a$ is "not integrable near t". Such a simple description is not possible any more in general if $n \ge 2$, because we have more freedom of choice in the direction of the jump.

We discuss here some simple examples which will illustrate the situation. From now on $n = 2$ and $I =]-1, 1[$.

Example 1. Let $a_{11} = 1$, $a_{12} = a_{22} = a_{21} = 0$. Then $E(t) = \langle e_2 \rangle$ for all $t \in I$(*). Hence jumps on the direction of e_1 are forbidden, while we can jump in the orthogonal direction e_2 at every point of I. □

In Example 1 the fact that a_{11}^{-1} was integrable guaranteed the impossibility of jumps in the e_1-direction. This may suggest that we could obtain a description of $E(t)$ studying for every $\nu \in \mathbb{R}^n$ the real function

$$a_\nu(t) = \Big(\sum_{i,j=1}^{n} a_{ij}(t)\nu_i\nu_j \Big)^{-1}. \tag{4.8}$$

This is not the case, as the following example shows.

Example 2. Let $a_{12} = a_{21} = 0$,

$$a_{11}(t) = \begin{cases} 1 & \text{if } t \le 0 \\ 0 & \text{if } t > 0, \end{cases}$$

$a_{22}(t) = 1 - a_{11}(t)$. Then we have

$$E(t) = \begin{cases} \langle e_2 \rangle & \text{if } t < 0 \\ \mathbb{R}^2 & \text{if } t = 0 \\ \langle e_1 \rangle & \text{if } t > 0. \end{cases}$$

If $\nu \in \mathbb{R}^2$ and $\nu_1\nu_2 \neq 0$ then

$$a_\nu(t) = \begin{cases} (\nu_1)^{-2} & \text{if } t < 0 \\ (\nu_2)^{-2} & \text{if } t > 0, \end{cases}$$

which is clearly in L^1. The only two directions ν for which a_ν is not integrable near 0 are e_1 and e_2. □

(*) We denote by $\langle \nu \rangle$ the subspace of $\mathbb{R}^n$ spanned by ν.

The previous example suggests that $E(t)$ could be *spanned* by the directions for which a_ν is not integrable. Again, we can give a counterexample.

Example 3. Let us define

$$a_{11}(t) = |\sin t|^{1/2}, \qquad a_{22} = (\cos t)^{1/2},$$

$$a_{12} = a_{21} = -\mathrm{sign}\, t|\cos t \sin t|^{1/4}.$$

If $\nu \in \mathbb{R}^2 \setminus \{0\}$ the function

$$\begin{aligned} a_\nu(t) &= \Big(\nu_1^2|\sin t|^{1/2} - 2\nu_1\nu_2 \mathrm{sign}\, t|\cos t \sin t|^{1/4} + \nu_2^2(\cos t)^{1/2}\Big)^{-1} \\ &= \Big(\nu_1|\sin t|^{1/4} - \nu_2 \mathrm{sign}\, t|\cos t|^{1/4}\Big)^{-2} \end{aligned}$$

is L^1 near 0. In fact, if $\nu_2 \neq 0$ it is bounded, while if $\nu_2 = 0$ then it behaves like $|t|^{-1/2}$.

Let us consider the functions $u_\varepsilon \in C^1((-\varepsilon,\varepsilon);\mathbb{R}^2)$ defined by

$$u_{\varepsilon 1}(t) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{t} |\cos s|^{1/2}\mathrm{d}s \qquad u_{\varepsilon 2}(t) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{t} \mathrm{sign}\, s|\cos s \sin s|^{1/4}\mathrm{d}s.$$

We have $u_\varepsilon(-\varepsilon) = (0,0)$, $u_\varepsilon(\varepsilon) \to (1,0)$, $\int_{-\varepsilon}^{\varepsilon} |\dot u_\varepsilon|\mathrm{d}t \le 2$, and

$$\int_{-\varepsilon}^{\varepsilon} \sum_{i,j=1}^{2} a_{ij}(t)(\dot u_\varepsilon)_i(t)(\dot u_\varepsilon)_j(t)\mathrm{d}t = 0.$$

Hence by (4.4) we have $(1,0) \in E(0)$, despite a_ν being integrable for all non-zero $\nu \in \mathbb{R}^2$. □

Example 4. Let $a : I \to \mathbb{R}^n$ be a Borel function whose direction is a piecewise continuous function. We consider the functional

$$F(u) = \begin{cases} \displaystyle\int_I \sum_{i=1}^n (a_i(t)\dot{u}_i(t))^2 \mathrm{dt} & \text{if } u \in \mathrm{W}^{1,1}(I;\mathbb{R}^n) \\ +\infty & \text{otherwise.} \end{cases}$$

Then the relaxed functional $\overline{F}$ can be represented as:

$$\overline{F}(u) = \begin{cases} \displaystyle\int_A \sum_{i=1}^n (a_i(t)\dot{u}_i(t))^2 \mathrm{dt} & \text{if } \displaystyle\sum_{i=1}^n \frac{(\dot{u}_s)_i}{|\dot{u}_s|} a_i(t) = 0 \quad |\dot{u}_s|\text{-a.e. on } A \\ +\infty & \text{otherwise,} \end{cases}$$

where A is the set of all t such that the left-hand and right-hand side limits of a at t have the same direction. In fact, it can be seen that

$$E(t) = \begin{cases} \langle a(t) \rangle^{\perp} & \text{if } t \in A \\ \mathbb{R}^n & \text{otherwise} \end{cases}$$

and the integrand function values $\displaystyle\sum_{i=1}^n (a_i(t)z_i)^2$ if $t \in A$ and 0 otherwise. $\square$

Now we will discuss some examples in order to describe the matrix $\widetilde{A}(t)$. In the 1-dimensional case (see [17] and [65]) we have that

$$\widetilde{a}(t) = \begin{cases} a(t) & \text{if } t \in I' \\ 0 & \text{otherwise,} \end{cases}$$

where I' is defined by (4.7). In general, as we will see in the following examples, it is not possible to characterize in this way the matrix $\widetilde{A}(t)$.

Example 5. Let C be a *Cantor-like set* contained in I with strictly positive Lebesgue measure (see for instance [60], Definition 6.62, page 70). Let $a_{11}(t) = \mathbf{1}_C(t)$, $a_{12}(t) = a_{21}(t) = 0$ and $a_{22}(t) = 1$. If we relax the associated functional

$$F(u) = \int_C [(\dot{u}_1(t))^2 + (\dot{u}_2(t))^2]\mathrm{dt} + \int_{I \setminus C} (\dot{u}_2(t))^2 \mathrm{dt},$$

then we obtain

$$\overline{F}(u) = \begin{cases} \int_I (\dot{u}_2(t))^2 \mathrm{dt} & \text{if } u_2 \in \mathrm{W}^{1,1}(I) \\ +\infty & \text{otherwise.} \end{cases}$$

In fact, for every $t \in I$ we have that $E(t) = \langle e_1 \rangle$, $\widetilde{a}_{11}(t) = \widetilde{a}_{12}(t) = \widetilde{a}_{21}(t) = 0$ and $\widetilde{a}_{22}(t) = 1$. We remark that in this example $E(t) \neq \widehat{E}(t)$ (see definition (4.1)) and $A(t) \neq \widetilde{A}(t)$ on the set C which has strictly positive Lebesgue measure. □

In this example $A(t)$ can be describe in the following way:

$$\widetilde{A}(t)z = \begin{cases} A(t)z & \text{if } z \in (E(t))^{\perp} \\ 0 & \text{if } z \in E(t). \end{cases}$$

In general this simple description does not hold, as it is shown in the following example.

Example 6. Let C be as in Example 5. Let $a_{11}(t) = 2\mathbf{1}_C(t)$, $a_{12}(t) = a_{21}(t) = \mathbf{1}_C(t)$ and $a_{22}(t) = 1$. The lower semicontinuous envelope of the functional (associated to this matrix)

$$F(u) = \int_C [2(\dot{u}_1(t))^2 + 2(\dot{u}_1(t))(\dot{u}_2(t)) + (\dot{u}_2(t))^2]\mathrm{dt} + \int_{I\setminus C} (\dot{u}_2(t))^2 \mathrm{dt}$$

can be represented as

$$\overline{F}(u) = \begin{cases} \frac{1}{2}\int_C (\dot{u}_2(t))^2 \mathrm{dt} + \int_{I\setminus C} (\dot{u}_2(t))^2 \mathrm{dt} & \text{if } u_2 \in \mathrm{W}^{1,1}(I) \\ +\infty & \text{otherwise.} \end{cases}$$

In fact, again $E(t) = \langle e_1 \rangle$ for every $t \in I$ and $\widetilde{a}_{11}(t) = \widetilde{a}_{12}(t) = \widetilde{a}_{21}(t) = 0$, but $\widetilde{a}_{22}(t) = 1 - \frac{1}{2}\mathbf{1}_C(t)$. The proof of Theorem 3.6 suggests a construction of the "minimizing sequences". It is clear that it is sufficient to deal with piecewise affine functions and hence it is enough to show the construction just for a linear function $u(t) = (\xi_1 t, \xi_2 t)$. For every $h \in \mathbb{N}$ we can find $\varepsilon_h > 0$, $\delta_h > 0$ and 2^h points $t_1^h, \dots, t_{2^h}^h$ of I such that

$$\varepsilon_h + \delta_h \leq t_{i+1}^h - t_i^h \quad \forall i : 1, \dots, 2^h$$

and

$$C \subseteq \bigcup_{i=1}^{2^h} [t_i^h, t_i^h + \varepsilon_h].$$

We can construct a function u_h such that

$$u_h(0) = 0, \quad \dot{u}_h^2(t) = \xi_2 \quad \text{for every } h \in \mathbb{N}$$

and

$$\dot{u}_h^1(t) = \begin{cases} -\frac{1}{2}\xi_2 & \text{if } t \in [t_i^h, t_i^h + \varepsilon_h] \\ \xi_1 + \frac{\varepsilon_h}{\delta_h}(\xi_1 + \frac{1}{2}\xi_2) & \text{if } t \in [t_i^h + \varepsilon_h, t_i^h + \varepsilon_h + \delta_h] \\ \xi_1 & \text{otherwise.} \end{cases}$$

The functions u_h converge to u in BV-w^* and we can check that $\overline{F}(u) = \lim_h F(u_h)$. □

4.5 Integral representation of the Γ-limit

Now we give an integral representation of the Γ-limit of quadratic functionals on $BV(I;\mathbb{R}^n)$.

Theorem 5.1. *Let $(A^h(t))$ be a sequence of $n \times n$ symmetric matrices of measurable functions $a_{ij}^h(t)$, let $b > 0$, let (μ_h) be a sequence of measures belonging to $\mathcal{M}^+(\Omega)$ and let (b_h) be a sequence of positive functions in $L^1(I)$ such that $\int_I (b_h(t))^{-1} d\mu_h \leq b$ and*

$$b_h(t)|z|^2 \leq \sum_{i,j=1}^{n} a_{ij}^h(t) z_i z_j \tag{5.1}$$

for every $h \in \mathbb{N}$, $z \in \mathbb{R}^n$ and for a.e. $t \in I$. For every $h \in \mathbb{N}$ let $F_h : BV(I;\mathbb{R}^n) \to [0, +\infty]$ be the quadratic functional defined by

$$F_h(u) = \begin{cases} \displaystyle\int_I \sum_{i,j=1}^{n} a_{ij}^h(t)(\dot{u}_a^{\mu_h})_i(t)(\dot{u}_a^{\mu_h})_j(t)\, d\mu_h & \text{if } |\dot{u}| << \mu_h \\ +\infty & \text{otherwise.} \end{cases} \tag{5.2}$$

Then there exists a quadratic lower semicontinuous functional F and a subsequence (h_k) such that F equals both the sequential Γ-limit and the Γ-limit of F_{h_k}. Moreover

i) there exists a measure $\mu \in \mathcal{M}^+(\Omega)$,

ii) there exists a $n \times n$ symmetric matrix $\widetilde{A}(t)$ of measurable functions $(\widetilde{a}_{ij}(t))$ such that

$$\sum_{i,j=1}^{n} \widetilde{a}_{ij}(t) z_i z_j \geq 0 \text{ for } \mu\text{-a.e. } t \in I \text{ and for each } z \in \mathbb{R}^n,$$

iii) for μ-a.e. $t \in I$ there exists a linear subspace $G(t)$ of $\mathbb{R}^n$ such that the map $t \mapsto G(t)$ is a Borel multivalued function,

such that F can be represented in the following way:

$$F(u) = \begin{cases} \displaystyle\int_I \sum_{i,j=1}^{n} \widetilde{a}_{ij}(t)(\dot{u}_{\mathrm{a}}^{\mu})_i(t)(\dot{u}_{\mathrm{a}}^{\mu})_j(t)\,\mathrm{d}\mu & \text{if } \dot{u} \in \mathcal{M}_{\mu}^{G} \\ +\infty & \text{otherwise,} \end{cases} \tag{5.3}$$

where $\mathcal{M}_{\mu}^{G}$ is the subset of $\mathcal{M}(I;\mathbb{R}^n)$ of the measures λ such that $|\lambda| << \mu$ and $\mathrm{d}\lambda_{\mathrm{a}}^{\mu}/\mathrm{d}\mu(t) \in G(t)$ for μ-a.e. $t \in I$. Moreover there exists a constant $c > 0$ such that

$$F(u) \geq \int_I |Du| - c. \tag{5.4}$$

Proof. The sequence of the functionals defined by (5.2) satisfies all the hypotheses of Theorem 2.4: the hypotheses a) and b) are straightforward, while c) follows from the integrability conditions on $(\frac{1}{b_h})$, remarking that we have

$$|z| - \frac{1}{4b_h(t)} \leq b_h(t)|z|^2,$$

and so all functionals F_h verify

$$F_h(u) \geq \int_I |Du| - \overline{b},$$

with $\overline{b} = \frac{1}{4}b$. Hence (F_h) admits a subsequence (F_{h_k}) Γ-converging to a functional F, which can be represented as in (2.4) with a measure μ and suitable functions

h and g. By Proposition 3.2 F is quadratic; hence by using similar arguments as in the proof of Theorem 3.6 it is possible to prove that for every $t \in I$ there exists a linear subspace $E(t)$ of $\mathbb{R}^n$ such that

$$h(t,z) = \chi_{E(t)}(z). \tag{5.5}$$

As in Remark 3.7, since $F(u) \geq \int_I |Du| - \bar{b}$ we can assume that $E(t) = \{0\}$ for every $t \in I$. Note that for every $t \in I$ the map $z \mapsto g(t,z)$ is quadratic. Hence for every $t \in I$ there exists a linear subspace $G(t)$ of $\mathbb{R}^n$ and a non-negative symmetric bilinear form $B_t : G(t) \times G(t) \to [0, +\infty[$ such that

$$g(t,z) = \begin{cases} B_t(z,z) & \text{if } z \in G(t) \\ +\infty & \text{otherwise.} \end{cases}$$

Now we consider the form $Q_t : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty[$ which is defined for each $z, w \in \mathbb{R}^n$ by

$$Q_t(z,w) = B_t(P_{G(t)}z, P_{G(t)}w),$$

where $P_{G(t)}$ denotes the canonical projection on $G(t)$. The function Q_t is a non-negative symmetric bilinear form on $\mathbb{R}^n$ and

$$g(t,z) = \begin{cases} Q_t(z,z) & \text{if } z \in G(t) \\ +\infty & \text{otherwise.} \end{cases}$$

Fixed $e_1, \ldots, e_n$ be a base of $\mathbb{R}^n$, for every $z, w \in G(t)$ we have

$$Q_t(z,w) = \sum_{i,j=1}^{n} z_i w_j Q_t(e_i, e_j).$$

Therefore we obtain

$$g(t,z) = \begin{cases} \displaystyle\sum_{i,j=1}^{n} \tilde{a}_{ij}(t) z_i z_j & \text{if } z \in G(t) \\ +\infty. & \text{otherwise,} \end{cases} \tag{5.6}$$

where $\tilde{a}_{ij}(t) = Q_t(e_i, e_j)$. The conclusion follows by (2.4), (5.5) and (5.6). $\square$

Remark 5.2. We note that in the case $n = 1$ we can take $\widetilde{a}(t) = 1$, and the measure μ the weak limit in the sense of measures of the sequence $(a_h(t))^{-1}\mathrm{d}t$ (see [23]).

In the following theorem we prove that the functionals which admit a representation with the Lebesgue measure are in some sense dense, with respect to the Γ-convergence, in the class of functionals representable as in the previous theorem.

Theorem 5.3. *Let $\mu \in \mathcal{M}^+(\Omega)$, let $A(t)$ and $\Psi(t)$ be $n \times n$ symmetric matrices of measurable functions $a_{ij}(t)$ and continuous functions $\psi_{ij}(t)$ respectively and let b be an integrable function on I with respect to the measure μ such that $\int_I (b(t))^{-1} d\mu < +\infty$. Let us suppose that*

$$a_{ij}(t) = \psi_{ij}(t) b(t)$$

for every $i, j = 1, \ldots, n$, and

$$\sum_{i,j=1}^{n} \psi_{ij}(t) z_i z_j \geq |z|^2.$$

Moreover let us suppose that for μ-a.e. $t \in I$ there exists a linear subspace $G(t)$ of $\mathbb{R}^n$ such that the map $t \mapsto G(t)$ is a continuous multivalued function. Let us consider the functional $F : BV(I; \mathbb{R}^n) \to [0, +\infty]$ defined by

$$F(u) = \begin{cases} \displaystyle\int_I \sum_{i,j=1}^{n} a_{ij}(t) (\dot{u}^{\mu}_{\mathrm{a}})_i(t) (\dot{u}^{\mu}_{\mathrm{a}})_j(t) \mathrm{d}\mu & \text{if } |\dot{u}| << \mu \quad \text{and } \dot{u}^{\mu}(t) \in G(t) \ \mu-\text{a.e.} \\ +\infty & \text{otherwise} \end{cases}$$

and assume that F is lower semicontinuous with respect to the BV-w^* topology. Then there exists a sequence $(A^h(t))$ of $n \times n$ symmetric matrices of measurable functions $a^h_{ij}(t)$ such that the sequence of functionals defined for every $h \in \mathbb{N}$ by

$$F_h(u) = \begin{cases} \displaystyle\int_I \sum_{i,j=1}^{n} a^h_{ij}(t) \dot{u}_i(t) \dot{u}_j(t) \mathrm{d}t & \text{if } u \in \mathrm{W}^{1,1}(I; \mathbb{R}^n) \\ +\infty & \text{otherwise} \end{cases}$$

Γ-converges to the functional F.

Proof. Let

$$f(t,z)=\begin{cases}\sum_{i,j=1}^{n} a_{ij}(t)z_iz_j & \text{if } z\in G(t)\\ +\infty & \text{otherwise,}\end{cases}$$

and for every $k\in\mathbb{N}$ let

$$f_h(t,z)=h\sum_{i,j=1}^{n} a_{ij}(t)(z-P_{G(t)}z)_i(z-P_{G(t)}z)_j+\sum_{i,j=1}^{n} a_{ij}(t)(P_{G(t)}z)_i(P_{G(t)}z)_j,$$

where $P_{G(t)}$ denotes the projection onto the subspace $G(t)$. We remark that for every $t\in I$ and $z\in\mathbb{R}^n$ $(f_h(t,z))$ is an increasing sequence which converges to $f(t,z)$ for a.e. $t\in I$. For every $h\in\mathbb{N}$ f_h is finite and quadratic; then for every $h\in\mathbb{N}$ there exists a $n\times n$ symmetric matrix $\Psi_h(t)$ of continuous functions $\psi^h_{ij}(t)$ such that

$$f_h(t,z)=\sum_{i,j=1}^{n}\psi^h_{ij}(t)b(t)z_iz_j\geq b(t)|z|^2.$$

Let us consider the associated functionals

$$\mathcal{F}_h(u)=\begin{cases}\int_I\sum_{i,j=1}^{n}\psi^h_{ij}(t)b(t)(\dot u^\mu_{\mathrm{a}})_i(t)(\dot u^\mu_{\mathrm{a}})_j(t)\mathrm{d}\mu & \text{if } |\dot u|<<\mu\\ +\infty & \text{otherwise.}\end{cases}$$

Corollary 3.9 implies that for every $h\in\mathbb{N}$ $\mathcal{F}_h$ is lower semicontinuous with respect to the L^1-topology. On the other hand by Beppo Levi's Theorem for every $u\in BV(I;\mathbb{R}^n)$ the sequence $\mathcal{F}_h(u)$ converges increasing to $F(u)$; hence $\mathcal{F}_h$ Γ-converges to F (see for instance Proposition 5.4 of [8]). We remark that we can write

$$\mathcal{F}_h(u)=\begin{cases}\int_I\sum_{i,j=1}^{n}\psi^h_{ij}(t)\frac{d\dot u_i}{d\widetilde\mu}(t)\frac{d\dot u_j}{d\widetilde\mu}(t)\mathrm{d}\widetilde\mu & \text{if } |\dot u|<<\widetilde\mu\\ +\infty & \text{otherwise,}\end{cases}$$

where $\widetilde\mu=(b(t))^{-1/2}\mu$.

Now fix $k \in \mathbb{N}$ and let $\widetilde{\mu}_k = (\widetilde{\mu} * \rho_k)\mathrm{dt}$, where ρ_k is a positive symmetric mollifier. For every $k, h \in \mathbb{N}$ let us define

$$\mathcal{F}_{hk}(u) = \begin{cases} \displaystyle\int_I \sum_{i,j=1}^n \psi_{ij}(t)\frac{d\dot{u}_i}{d\widetilde{\mu}_k}(t)\frac{d\dot{u}_j}{d\widetilde{\mu}_k}(t)\mathrm{d}\widetilde{\mu}_k & \text{if } |\dot{u}| << \widetilde{\mu}_k \\ +\infty & \text{otherwise} \end{cases}$$

$$= \begin{cases} \displaystyle\int_I \sum_{i,j=1}^n \psi_{ij}(t)\dot{u}_i(t)\dot{u}_j(t)((\widetilde{\mu} * \rho_k)(t))^{-1}\mathrm{dt} & \text{if } u \in \mathrm{W}^{1,1}(I;\mathbb{R}^n) \\ +\infty & \text{otherwise.} \end{cases}$$

Let us remark that for every $h \in \mathbb{N}$ $\widetilde{\mu}_k$ weakly converges in the sense of measures to $\widetilde{\mu}$, when $k \to +\infty$. Since

$$\sum_{i,j=1}^n \psi_{ij}^h(t)z_i z_j \geq |z|^2 \geq |z| - c,$$

the recession function of the map $z \mapsto \sum_{i,j=1}^n \psi_{ij}^h(t)z_i z_j$ is the function $\chi_{\widehat{E}_h(t)}(z)$, where

$$\widehat{E}_h(t) = \{z \in \mathbb{R}^n : \sum_{i,j=1}^n \psi_{ij}^h(t)z_i z_j = 0\},$$

and by Remark 3.7 we can take $\widehat{E}_h(t) = \{0\}$ for all t. Therefore by Theorem 2.2 of [23] for every $h \in \mathbb{N}$ $\mathcal{F}_{hk}$ Γ-converges to the functional $\mathcal{F}_h(u)$. Hence the sequence (F_{hk}) Γ-converges to $\mathcal{F}_h$, when $k \to +\infty$. Now we remark that the sequence $(\mathcal{F}_h)$ satisfies the condition

$$\mathcal{F}_h(u) \geq \int_I |\dot{u}| - \mu(I) - \int_I (b(t))^{-1}\mathrm{d}\mu = \int_I |\dot{u}| - c;$$

moreover, since μ_h weakly converges to μ in the sense of measures, the analogous condition

$$\mathcal{F}_{hk}(u) \geq \int_I |\dot{u}| - \frac{1}{4}\mu_h(I) \geq \int_I |\dot{u}| - c$$

for the sequence $(\mathcal{F}_{hk})$ holds. At this point, we recall that the Γ-convergence with respect to the BV-w^* -topology on the class $\mathcal{E}$ of lower semicontinuous functionals

$H : BV(I;\mathbb{R}^n) \to [0,+\infty]$ such that $H(u) \geq \int_I |\dot{u}| - c$, is equivalent to the Γ-convergence with respect to the L^1-topology, then it is metrizable. Then we can use a general result of the Γ-convergence theory (see Theorem 10.22 of [8]), and we can conclude that the Γ-convergence on $\mathcal{E}$ is induced by a distance. Hence there exists a sequence $(k(h))$ such that the functionals $F_{hk(h)}$ Γ-converges to F for $h \to +\infty$. The conclusion follows by taking

$$F_h(u) = \mathcal{F}_{hk(h)}(u)$$

and

$$a_{ij}^h(t) = \psi_{ij}(t)((\mu_h * \rho_{k(h)})(t))^{-1}.$$

$\square$

Chapter 5

Uniqueness in Segmentation Problems

5.1 Introduction

Given a function $g \in L^2(\Omega)$, with Ω an open bounded subset of $\mathbb{R}^n$, and three real numbers $\alpha, \beta, \gamma \in (0, +\infty]$, let us consider the functional

$$(1.1) \qquad F^g_{\alpha,\beta,\gamma}(u) = \alpha \int_\Omega |\nabla u|^2 dx + \beta \int_\Omega |u - g|^2 dx + \gamma H^{n-1}(S_u)$$

where S_u is the jumping set of the function u and H^{n-1} is the $n-1$ Hausdorff measure on $\mathbb{R}^n$.

We can associate to $F^g_{\alpha,\beta,\gamma}$ the following minimizing problem

$$(1.2) \qquad \min_u F^g_{\alpha,\beta,\gamma}(u)$$

where the minimum is taken on a suitable class of functions.

In the case $n = 2$, the functional defined in (1.1) was proposed by Mumford and Shah in [77], in order to give a mathematical description to a problem of image segmentation in Computer Vision Theory.

In [76] and [77], Mumford and Shah conjectured that $F^g_{\alpha,\beta,\gamma}$ has minimizers, whose discontinuity set S_u is piecewise smooth. In [8] Ambrosio proved the existence of the solution of (1.2) in the general case of the space dimension $n \geq 1$. Some results about the regularity of S_u can be found in [42].

In [37] Dal Maso, Morel and Solimini studied the particular case $n = 2$, giving a constructive proof of the existence.

Further results about this problem can be found in [6], [11] and [42].

Moreover, we recall also that in [84], the one-dimensional case has been considered; in particular the smoothing properties given by the formulation (1.1) of the segmentation problem have been studied.

It is possible also to consider the functional

$$\tilde{F}^g_\gamma(u) = \int_\Omega |u-g|^2 dx + \gamma H^{n-1}(S_u) \tag{1.3}$$

and the associated problem

$$\min_u \tilde{F}^g_\gamma(u). \tag{1.4}$$

We point out that (1.3) can be considered a particular case of (1.1), in which we restrict our attention to the piecewise constant functions or equivalently in which we put $\alpha = +\infty$ and $\beta = 1$. In the case of $n = 2$, a constructive method provides the existence of minimizers for problem (1.4), as proved in [70] and [71]. The general case $n \geq 1$ is studied in [28] by Congedo and Tamanini.

However it is not possible, in general, to say that the minimizers for these problems are unique. To this purpose, let us consider the simple case $n = 1$, $\Omega = [0,1]$ and the function $g : [0,1] \to \mathbb{R}$, $g \in L^\infty([0,1])$ defined by $g(x) = \chi_{[\frac{1}{2},1]}(x)$, where χ_E is the characteristic function of the set E; then the minimum problem (1.4) for that g has, as unique solution, the function $u_1 = \chi_{[\frac{1}{2},1]}$ for $0 < \gamma < \frac{1}{4}$ and the function $u_2 = \frac{1}{2}$ for $\gamma > \frac{1}{4}$, but for $\gamma = \frac{1}{4}$ both functions u_1 and u_2 are solutions.

From these arguments, one could expect that given a function $g \in L^2(\Omega)$, there is uniqueness for these minimum problems except for a "small" set (possibly countable) of values of the parameter γ.

Unfortunately, this is not the case in general, as the following counterexample shows.

Let $g(x) = \chi_{[\frac{1}{3},\frac{2}{3}]} + 2\chi_{(\frac{2}{3},1]}$ and consider again the problem (1.4) associated to this function g; then it is easy to prove that for $\gamma > \frac{1}{2}$ the unique solution is $u_1 = 1$ and for $0 < \gamma < \frac{1}{6}$ the unique solution is $u_2 = g$, but for all the interval $\frac{1}{6} < \gamma < \frac{1}{2}$ we have two solutions $u_3 = \frac{3}{2}\chi_{[\frac{1}{3},1]}$ and $u_4 = \frac{1}{2}\chi_{[0,\frac{2}{3}]} + 2\chi_{(\frac{2}{3},1]}$ and finally for $\gamma = \frac{1}{6}$ the functions u_2, u_3 and u_4 are solutions and for $\gamma = \frac{1}{2}$ the functions u_1, u_3 and u_4 are solutions.

Actually, we will see that for every non constant function $g \in L^2([0,1])$ it will be possible to find $\gamma \in (0,+\infty)$ such that the problems (1.2) and (1.4) have more than one solution.

On the other hand, fixed $\gamma \in (0,+\infty)$, we can find $g \in L^2([0,1])$ such that (1.4) has more minimizers. In fact it is enough to take, for instance, $g = (1 + 2\sqrt{\gamma})\chi_{(0,\frac{1}{2})} + \chi_{(\frac{1}{2},1)}$, and to observe that $\tilde{F}^g_\gamma(g) = \tilde{F}^g_\gamma(\overline{g})$, where $\overline{g}$ is the mean value of g on $[0,1]$. The same property holds also for problem (1.2) (see Corollary 3.5 and Remark 3.6).

These arguments lead us to observe that the best we can hope is the uniqueness for these minimum problems only if we restrict the functions g or the values of the parameter γ to suitable "large" subsets of $L^2(\Omega)$ and $\mathbb{R}^+$ respectively.

The aim of this chapter is, indeed, to give a rigorous proof of this fact for problems (1.2) and (1.4), in dimension $n = 1$.

The main result is, in fact, that for every γ belonging to $\mathbb{R}^+$ uniqueness for (1.2) and (1.4) is a generic property of $g \in L^2([0,1])$.

Moreover, for a generic g belonging to $L^2([0,1])$, uniqueness for (1.2) and (1.4) is a generic property of $\gamma \in \mathbb{R}^+$.

To prove these results, we adapt an argument of G. Vidossich in [87] to our situation, following the outline of Carriero and Pascali in [25].

More precisely, given $\alpha > 0$ and $\beta > 0$, we construct a countable subset $\mathcal{M}^0$ of $L^2([0,1])$, dense in $L^2([0,1])$ and a countable subset Γ of $\mathbb{R}^+$, such that for every $g \in \mathcal{M}^0$ and for every $\gamma \in \mathbb{R}^+ \setminus \Gamma$, problem (1.2) relative to g has a unique solution. Then, by means of $\mathcal{M}^0$, for every $\gamma \in \mathbb{R}^+$, we can construct a dense G_δ-subset $\mathcal{M}^*_\gamma$ of $L^2([0,1])$, such that when the datum g is chosen in $\mathcal{M}^*_\gamma$ the corresponding problem (1.2) has only one minimizer. Really this result can be improved by constructing a dense G_δ-subset which works for all the parameters γ of a countable subset contained in $\mathbb{R}^+$. On the other hand, we can construct a dense G_δ-subset of $L^2([0,1])$ such that when g belongs to this set, problems (1.2) is uniquely solvable if γ belongs to the complement of a countable subset Γ^g in $\mathbb{R}^+$ depending on g.

Similar arguments are used to obtain analogous results for problem (1.4).

Since the complement of a G_δ-subset of $L^2([0,1])$ is a set of first category and Γ^g is countable it is clear now what we meant by "large" or "generic" in the previous informal discussion. We observe that, from this point of view, our results are in line with the genericity results of [25], [62], [79] and [87].

In particular, the set $\mathcal{M}^0$ will be constructed by means of a suitable class of piecewise constant functions. In order to find this class, we will study in detail the properties of the solution, and in particular its form and its discontinuities, when g is piecewise constant.

The chapter is organized as follows: in the second section we reformulate the problem in a suitable way to the one-dimensional case, which permits us to reduce (1.2) and (1.4) to the study of simpler problems, with fixed jump term; in the third section we state some preliminary results about the form of the solutions of (1.2) and (1.4) and their continuous dependence on the datum g; finally section 4 contains the main theorems.

5.2 Formulation of the problem

We will write L^2, L^∞ instead of $L^2([0,1])$, $L^\infty([0,1])$.

In the following, for $j \in \mathbb{N}$, a partition $\mathcal{Q} = (b_s)_{s=0}^{j+1}$ of $[0,1]$ will be identified with a subset $\{b_0, \dots, b_{j+1}\}$ of $[0,1]$ such that $0 = b_0 < b_1 < \dots < b_{j+1} = 1$.

Fixed $j \in \mathbb{N}$, we denote by $\mathcal{H}^1_j$ the space of all the functions u on $[0,1]$ such that there exists a partition $\mathcal{Q} = (b_s)_{s=0}^{j+1}$ of $[0,1]$ such that the restriction of u to (b_s, b_{s+1}) belongs to $H^1((b_s, b_{s+1}))$, for every $s = 0, \dots, j$. Therefore, we define $\mathcal{H}^1 = \bigcup_{j \in \mathbb{N}} \mathcal{H}^1_j$. For every $j \in \mathbb{N}$ we consider also the subset $\mathcal{K}^1_j$ of $\mathcal{H}^1_j$ composed by the functions which have exactly j jumps.

Moreover, we denote by $\mathcal{S}$ the space of all the piecewise constant functions on $[0,1]$. It is easy to see that each function $u \in \mathcal{S}$ can be written in the form $u = \sum_{s=0}^{j} \beta_s \chi_{(b_s, b_{s+1})}$ with $\beta_s \in \mathbb{R}$ for $s = 0, \dots, j$ and $j \in \mathbb{N}$.

Finally S_u is the set of jump points of the function u belonging to $\mathcal{H}^1$ or $\mathcal{S}$ and $\#$ is the counting measure on $\mathbb{R}$.

Given $g \in L^2$ and $\gamma \in \mathbb{R}^+$, we consider the following functional $F^g_\gamma : \mathcal{H}^1 \to$

$[0, +\infty]$

$$F_\gamma^g(u) = \sum_{s=0}^{l} \int_{b_s}^{b_{s+1}} (u')^2 dx + \int_0^1 (u-g)^2 dx + \gamma \#(S_u) \tag{2.1}$$

where $l = \#(S_u)$; it is easy to see that the functional depends only on u and not on its representation. Moreover we consider the functional $\tilde{F}_\gamma^g : \mathcal{S} \to [0, +\infty]$

$$\tilde{F}_\gamma^g(u) = \int_0^1 (u-g)^2 dx + \gamma \#(S_u)$$

and the associated problems:

$$\min\{F_\gamma^g(u) : u \in \mathcal{H}^1\} \tag{2.2}$$

and

$$\min\{\tilde{F}_\gamma^g(u) : u \in \mathcal{S}\}; \tag{2.3}$$

we note that (2.1) is obtained by (1.1) with $\alpha = \beta = 1$.

We note that all the results we are going to prove still hold for finite α and β different from 1, because it is possible to reduce the general functional to our case.

We observe that the existence for this problems will be discussed in the following.

We point out that the results we are going to obtain for problem (2.3) cannot be derived directly from those for problem (2.2), but since the method is the same in both cases, we treat explicitly only problem (2.2), remarking, when it is necessary, the differences and the analogies with problem (2.3).

Given $g \in L^2$, for every $\gamma \in \mathbb{R}^+$ we define

$$m^g(\gamma) = \min\{F_\gamma^g(u) : u \in \mathcal{H}^1\}; \tag{2.4}$$

we shall see later that the minimum is achieved.

Moreover, we consider the functional $G^g : \mathcal{H}^1 \longrightarrow [0, +\infty]$ defined by

$$G^g(u) = \sum_{s=0}^{j} \int_{b_s}^{b_{s+1}} (u')^2 dx + \int_0^1 (u - g)^2 dx. \tag{2.5}$$

For every $j \in \mathbb{N}$ we consider the problem

$$M_j^g = \min\{G^g(u) : u \in \mathcal{H}_j^1\}. \tag*{$(2.6)_j$}$$

The existence for this minimum problem follows by the usual compactness property of the sequences of partitions and by the standard direct methods of calculus of variation applied on each subinterval of $[0, 1]$.

It is clear that

$$M_j^g = \inf\{G^g(u) : u \in \mathcal{K}_j^1\},$$

but, in this case, the minimum is not always achieved.

Let us define the non empty subset N^g of $\mathbb{N}$ of the integers j for which the value M_j^g is attained on at least a function which has exactly j jumps.

Moreover, it can be easily seen that $j \in \mathrm{N}^g$ if and only if the minimum of G on $\mathcal{K}_j^1$ is achieved and, in this case,

$$\min_{\mathcal{H}_j^1} G^g(u) = \min_{\mathcal{K}_j^1} G^g(u). \tag{2.7}$$

For every $j \in \mathbb{N}$ and for every $\gamma \in \mathbb{R}^+$, let us define now

$$m_j^g(\gamma) = M_j^g + \gamma j. \tag*{$(2.8)_j$}$$

Since $m_j^g(\gamma) \geq \gamma j$ and $\gamma > 0$, it follows that for every $\gamma \in \mathbb{R}^+$ there exists the $\min\limits_{j \in \mathbb{N}} m_j^g(\gamma)$. Moreover we are going to prove that

$$m^g(\gamma) = \min_{j \in \mathbb{N}} m_j^g(\gamma). \tag{2.9}$$

In fact, given $u \in \mathcal{H}^1$ with $\#(S_u) = j$, we have

$$F_\gamma^g(u) = G^g(u) + \gamma j \geq M_j^g + \gamma j \geq \min_{j \in \mathbb{N}} m_j^g(\gamma)$$

and taking the infimum with respect to $u \in \mathcal{H}^1$, it follows that

$$\inf_{u \in \mathcal{H}^1} F_\gamma^g(u) \geq \min_{j \in \mathbb{N}} m_j^g(\gamma).$$

The opposite inequality is trivial.

In order to prove that such an infimum is attained, we fix $\gamma \in \mathbb{R}^+$ and we choose $j_o \in \mathbb{N}$ such that $\min_{j \in \mathbb{N}} m_j^g(\gamma) = m_{j_0}^g(\gamma)$; this implies that there exists $u_0 \in \mathcal{H}_{j_0}^1$ such that

$$\inf_{u \in \mathcal{H}^1} F_\gamma^g(u) = G^g(u_0) + \gamma j_0 \geq F_\gamma^g(u_0), \tag{2.10}$$

hence the infimum in (2.10) is attained on u_0 and actually it is a minimum, moreover (2.9) holds.

We note that $\#(S_{u_0}) = j_0$; in fact if $\#(S_{u_0}) = l < j_0$ we have $F_\gamma^g(u_0) = G^g(u_0) + \gamma l < G^g(u_0) + \gamma j_0$ and this contradicts (2.10). Therefore $u_0 \in \mathcal{K}_{j_0}^1$ and $j_0 \in \mathrm{N}^g$. This proves that

$$m^g(\gamma) = \min_{j \in \mathrm{N}^g} m_j^g(\gamma), \tag{2.11}$$

for every $\gamma \in \mathbb{R}^+$, and that, if $m_{j_0}^g(\gamma) = \min_{j \in \mathbb{N}} m_j^g(\gamma)$ for some $\gamma \in \mathbb{R}^+$, then every minimizer u_0 of problem $(2.6)_{j_0}$ has exactly j_0 jumps, i.e. $u \in \mathcal{K}_{j_0}^1$.

This leads us to define the subset J^g of N^g in the following way:

$$\mathrm{J}^g = \{j \in \mathrm{N}^g : \exists \gamma \in \mathbb{R}^+ \text{ s.t. } m_j^g(\gamma) = \min_{s \in \mathrm{N}^g} m_s^g(\gamma)\}$$

(see figure 1).

Remark 2.1. It is clear that (2.11) can be rewritten as

$$m^g(\gamma) = \min_{j \in \mathrm{J}^g} m_j^g(\gamma). \tag{2.12}$$

Moreover, if $j \in \mathrm{J}^g$, every minimizer u of problem $(2.6)_j$ has exactly j jumps, i.e. $u \in \mathcal{K}_j^1$. We observe that, by $(2.8)_j$, $m_j^g(\gamma)$ has a linear dependence on γ, hence by (2.12) $m^g(\gamma)$ is a concave function (see figure 1). Finally we point out that the sequence $(M_j^g)_{j\in\mathbb{N}}$ is decreasing since $\mathcal{H}_j^1 \subseteq \mathcal{H}_k^1$ for $j < k$. In particular, if $j \in \mathrm{J}^g$, it is strictly decreasing; in fact if by contradiction $j, k \in \mathrm{J}^g$ with $j < k$ and $M_j^g = M_k^g$, then for every $\gamma \in \mathbb{R}^+$

$$m_j^g(\gamma) = M_j^g + \gamma j < M_k^g + \gamma k = m_k^g(\gamma)$$

which implies $k \notin \mathrm{J}^g$.

We introduce, for every $j \in \mathrm{J}^g$, the non empty subsets of $\mathbb{R}$

$$\Gamma_j^g = \{\gamma \in \mathbb{R}^+ : m^g(\gamma) = m_j^g(\gamma)\}$$

and

$$\Gamma^g = \{\gamma \in \mathbb{R}^+ : \exists j, j' \in \mathrm{J}^g, \text{ consecutive in } J^g, \text{ s.t. } \gamma \in \Gamma_j^g \cap \Gamma_{j'}^g\} \tag{2.13}$$

(see figure 1).

Remark 2.2. It is clear that Γ_j^g is a (possibly degenerate) interval of $\mathbb{R}^+$, since it can be rewritten as

$$\Gamma_j^g = \left(m^g - m_j^g\right)^{-1}([0, +\infty))$$

and $m^g - m_j^g$ is a concave function. Moreover the intervals Γ_j^g with $j \in \mathrm{J}^g$ are non overlapping, since the angular coefficient of m_j^g is strictly increasing with j, and Γ_0^g is unbounded. Hence for every $i \neq j$ and every γ belonging to the interior of Γ_j^g we have $m^g(\gamma) = m_j^g(\gamma) < m_i^g(\gamma)$. Given two consecutive elements j and j' of J^g, the equality $m_j^g(\gamma) = m_{j'}^g(\gamma)$ is satisfied for at most one $\gamma \in \mathbb{R}^+$; finally we note that Γ^g is the set of all the endpoints of the intervals Γ_h^g, hence it is a discrete countable subset of $\mathbb{R}^+$ and the only possible accumulation point is the point $\gamma = 0$ (see figure 1).

Proposition 2.3. *Fixed $\gamma \in \mathbb{R}^+$, we consider the non empty subset of* J^g

$$\mathrm{J}^g_\gamma = \{j \in \mathrm{J}^g : m^g_j(\gamma) = m^g(\gamma)\}.$$

Then $m^g(\gamma)$ is attained on $u \in \mathcal{H}^1$ if and only if there exists $j \in \mathrm{J}^g_\gamma$ such that u is a minimum point of the problem which defines M^g_j. In particular, if there exists a unique $j \in \mathrm{J}^g_\gamma$ and if the problem $(2.6)_j$ *has uniqueness, then also the problem* (2.2) *has uniqueness.*

Proof. If $j = \#(S_u)$ then $u \in \mathcal{H}^1_j$. Hence, if $m^g(\gamma)$ is attained on u, then

$$m^g(\gamma) = \sum_{s=0}^{j} \int_{b_s}^{b_{s+1}} (u')^2\,dx + \int_0^1 (u-g)^2\,dx + \gamma j \geq m^g_j(\gamma) \geq m^g(\gamma)$$

which implies that $j \in \mathrm{J}^g_\gamma$ and that M^g_j is attained on u.

Viceversa, if $j \in \mathrm{J}^g_\gamma$ and M^g_j is attained on u, then $j = \#(S_u)$ (see Remark 2.1). The conclusion follows by

$$m^g(\gamma) = m^g_j(\gamma) = M^g_j + \gamma j = G^g_\gamma(u) + \gamma j = F^g_\gamma(u).$$

□

Corollary 2.4.

(i) If $\overline{\gamma} \in \Gamma^g$, then problem (2.2) *has more than one solution.*

(ii) If, for a $\overline{\gamma}$ belonging to the interior of Γ^g_j with $j \in \mathrm{J}^g$, problem (2.2) *has more than one solution, then for all γ belonging to the interior of Γ^g_j problem* (2.2) *has not uniqueness.*

Proof. (i) follows by the definition of Γ^g and Γ^g_j.
(ii) If u_1, u_2 are minimizers of $F^g_{\overline{\gamma}}$, then by Proposition 2.3 they are minimizers of G^g, that is for every $s = 1,2$ $G^g(u_s) = M^g_j$. Since for every γ belonging to the interior of Γ^g_j we have

$$M^g_j + \gamma j = m^g(\gamma),$$

with j fixed, it follows that u_1, u_2 are minimizers of F^g_γ. □

Proposition 2.5. *Let $Q = (b_s)_{s=0}^{j+1}$ be a fixed partition of $[0,1]$ and $\mathcal{H}^1_Q$ be the subset of $\mathcal{H}^1_j$ constituted by the functions $u = \sum_{s=0}^{j} \beta_s(x)\chi_{(b_s,b_{s+1})}(x)$ with $\beta_s \in H^1((b_s,b_{s+1}))$ for every $s = 0,1,\ldots,j$. Then the functional G^g defined in (2.5) has exactly one minimizer on $\mathcal{H}^1_Q$.*

Proof. The existence is standard. For the uniqueness it is sufficient to observe that $\mathcal{H}^1_Q$ is a linear subspace of $\mathcal{H}^1$ and the functional G^g is strictly convex on $\mathcal{H}^1_Q$. □

Proposition 2.6. *Fixed a partition $(a_i)_{i=0}^{k+1}$ of $[0,1]$; if g is a function of the type $g = \sum_{i=0}^{k} \alpha_i\chi_{(a_i,a_{i+1})}$, where $\alpha_i \in \mathbb{R}$ for every $i = 0,\ldots,k$, then fixed $j > k$ we have that $m^g(\gamma) < m^g_j(\gamma)$ for every $\gamma \in \mathbb{R}^+$, $\mathrm{J}^g \subseteq \{1,\ldots,k\}$ and Γ^g is finite.*

Proof. Since for every $j \geq k$ $g \in \mathcal{H}^1_j$, it follows that $M^g_j = 0$. Assume by contradiction that, given $j > k$ there exists $\overline{\gamma} \in \mathbb{R}^+$ such that $m^g(\overline{\gamma}) = m^g_j(\overline{\gamma})$, then

$$m^g(\overline{\gamma}) = \overline{\gamma}j > \overline{\gamma}k.$$

But, if we take $u = g$, we have that $F^g_{\overline{\gamma}}(g) \leq \overline{\gamma}k$ which is a value strictly less than $m^g(\overline{\gamma})$ and this is not possible.

This implies also that for every $j > k$ we obtain that $j \notin \mathrm{J}^g$; then J^g is finite and is contained in $\{1,\ldots,k\}$ and, by (2.13) and Remark 2.2, Γ^g is the set of points $\gamma \in \mathbb{R}^+$ such that $m^g(\gamma) = m^g_j(\gamma) = m^g_{j'}(\gamma)$, where j and j' are two consecutive elements of J^g. Therefore it follows that Γ^g is finite and contains at most k points. □

Remark 2.7. We may analogously define the minimum problem

$$\tilde{m}^g(\gamma) = \min\{\tilde{F}^g_\gamma(u) : u \in \mathcal{S}\};$$

moreover we can consider the functional $\tilde{G}^g : \mathcal{S} \to [0,+\infty]$ defined by

$$\tilde{G}^g(u) = \int_0^1 (u-g)^2 dx$$

and the associated problem

$$\tilde{M}_j^g = \min\{\tilde{G}^g(u) : u \in \mathcal{H}_j^1 \cap \mathcal{S}\}.$$

With suitable modifications, we can also introduce the definitions of $\tilde{m}_j^g$, $\tilde{N}^g$, $\tilde{\mathrm{J}}^g$, $\tilde{\Gamma}^g$ and $\tilde{\Gamma}_j^g$ relative to the problem (2.3).

Remark 2.8. In order to explain better the previous definitions, we give an easy example, relative to the functional F_γ^g, in which we emphasize those concepts. Let $g(x) = \chi_{(\frac{1}{3},\frac{2}{3})}(x)$. An easy calculation shows that G^g has one minimizer u_1 on $\mathcal{K}_0^1$ and one minimizer u_4 on $\mathcal{K}_2^1$, and that G^g has two minimizers u_2 and u_3 on $\mathcal{K}_1^1$. Moreover, u_1 is the minimizer of F_γ^g on $\mathcal{H}^1$ for $0 \leq \gamma \leq \overline{\gamma}$ and u_4 is the minimizer of F_γ^g on $\mathcal{H}^1$ for $\gamma \geq \overline{\gamma}$, where $\overline{\gamma} \simeq 0,11$. This allows us to construct the graph in Fig. 1.

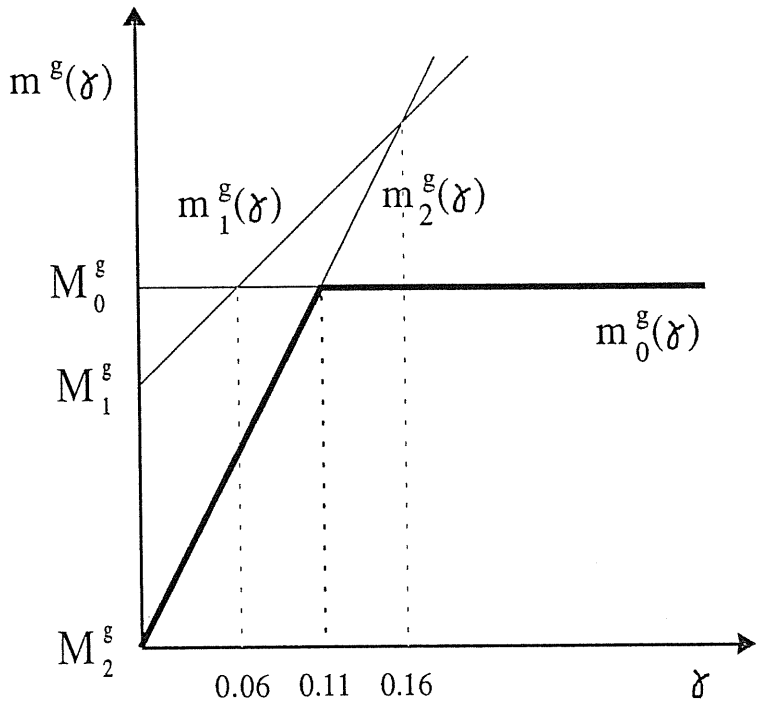

Fig. 1

$N^g = \{0,1,2\}$, $J^g = \{0,2\}$

$M_0^g \simeq 0.22$, $M_1^g \simeq 0.16$, $M_2^g = 0$

$\overline{\gamma} \simeq 0.11$: $\Gamma^g = \{\overline{\gamma}\}$, $\Gamma_0^g = [\overline{\gamma}, +\infty)$, $\Gamma_2^g = [0, \overline{\gamma}]$.

5.3 Preliminary results

In this section we state some results concerning the form of a solution of the minimum problem. In particular, we give an explicit formula, in terms of g, for a minimum point $u \in \mathcal{H}^1$ of the problem $(2.6)_j$, and for a minimum point $\tilde{u} \in \mathcal{S}$ of the analogous problem for the functional without the derivative term, and we study where such a minimum point can jump and the continuous dependence of it on the datum g. Moreover we investigate the non uniqueness: in particular we show how it is possible, fixed $g \in L^2$ (or $\gamma \in \mathbb{R}^+$), to construct $\gamma \in \mathbb{R}^+$ (or $g \in L^2$ respectively) for whose minimum problems have non uniqueness.

Remark 3.1. It is easy to see that, when $j \in \mathrm{J}^g$, a minimizer u of problem $(2.6)_j$ must be of the form

$$u(x) = \sum_{s=0}^{j} \overline{\beta}_s(x)\chi_{(b_s,b_{s+1})}(x)$$

where $\mathcal{Q} = (b_s)_{s=0}^{j+1}$ is a partition of $[0,1]$ and for every $s = 0,\ldots,j$ $\overline{\beta}_s$ is the solution of the Euler equation in the subinterval (b_s, b_{s+1}) of $[0,1]$, i.e.

$$\overline{\beta}_s(x) = c_s \ \cosh(x - b_s) + d_s(x) \tag{3.1}$$

where

$$c_s = \frac{e^{b_s}}{e^{2b_{s+1}} - e^{2b_s}} \left[e^{2b_{s+1}} \int_{b_s}^{b_{s+1}} g(t)e^{-t}dt + \int_{b_s}^{b_{s+1}} g(t)e^{t}dt \right],$$

$$d_s(x) = \frac{e^{-x}}{2} \int_{b_s}^{x} g(t)e^{t}dt - \frac{e^{x}}{2} \int_{b_s}^{x} g(t)e^{-t}dt$$

and cosh is the hyperbolic cosine. Finally, recalling the definition of J^g and Remark 2.1, for every $s = 0,\ldots,j-1$ $\overline{\beta}_s(b_{s+1}) \neq \overline{\beta}_{s+1}(b_{s+1})$.

Remark 3.2. In the case of problem (2.3), the situation is even much easier, so that for $j \in \tilde{J}^g$ a minimizer function has the form

$$\tilde{u}(x) = \sum_{s=0}^{j} \overline{\beta}_s \chi_{(b_s,b_{s+1})}(x)$$

where $Q = (b_s)_{s=0}^{j+1}$ is a partition of $[0,1]$ and for $s = 0,\dots,j$

$$\overline{\beta}_s = \frac{1}{b_{s+1} - b_s} \int_{b_s}^{b_{s+1}} g(t)dt;$$

finally we have that $\overline{\beta}_s \neq \overline{\beta}_{s+1}$ for every $s = 0,\dots,j-1$.

Remark 3.3. Fixed $\gamma \in \mathbb{R}^+$, $j \in J^g_\gamma$ and the solution $u(x) = \sum_{s=0}^{j} \overline{\beta}_s(x)\chi_{(b_s,b_{s+1})}(x)$ of problem $(2.6)_j$, we have that for every $s = 0,\dots,j$ and every function $\beta_s \in H^1((b_s,b_{s+1}))$

$$F^g_\gamma\left(\sum_{s=0}^{j} \beta_s(x)\chi_{(b_s,b_{s+1})}(x)\right) > F^g_\gamma\left(\sum_{s=0}^{j} \overline{\beta}_s(x)\chi_{(b_s,b_{s+1})}(x)\right)$$

if for some $s \in \{0,\dots,j\}$ we have that $\beta_s \neq \overline{\beta}_s$. In fact, by Proposition 2.5 the functional G^g defined in (2.5) is strictly convex on $\mathcal{H}^1_Q$, where $Q = (b_s)_{s=0}^{j+1}$, hence G^g has a unique minimizer on $\mathcal{H}^1_Q$.

In the following two corollaries we will show that, arbitrarily fixed the non constant datum $g \in L^2$ or the parameter $\gamma \in \mathbb{R}^+$, it is possible to choose the parameter $\gamma \in \mathbb{R}^+$ or the datum $g \in L^2$ respectively such that problems (2.2) and (2.3) have non uniqueness.

Corollary 3.4. *For every non constant function $g \in L^2$*

(i) there exists $\overline{\gamma} \in \mathbb{R}^+$ such that $F^g_{\overline{\gamma}}$ has more than one minimizer;

(ii) there exists $\tilde{\gamma} \in \mathbb{R}^+$ such that $\tilde{F}^g_{\tilde{\gamma}}$ has more than one minimizer.

Proof. Let us fix a non constant function $g \in L^2$.

(i) Let u_0 be the unique solution of the equation

$$-u'' + u = g \tag{3.2}$$

with the Neumann conditions $u'(0) = 0 = u'(1)$; then by definition $M_0^g = \int_0^1 (u_0')^2 dx + \int_0^1 (u_0 - g)^2 dx$. By (3.2), since g is not constant, also u_0 is not constant; moreover $u_0 \in C^1([0,1])$, hence there exists $b \in (0,1)$ such that $u_0'(b) \neq 0$. Let now $v(x) = \beta_0(x)\chi_{(0,b)}(x) + \beta_1(x)\chi_{(b,1)}(x)$, where β_0 and β_1 are the solutions of the equation (3.2) with the Neumann conditions $\beta_0'(0) = 0 = \beta_0'(b)$ and $\beta_1'(b) = 0 = \beta_1'(1)$ respectively.

We observe that clearly u_0 does not satisfy the Neumann conditions in $[0,b]$ and in $[b,1]$, hence it follows that $G^g(v) < G^g(u_0)$, i.e.

$$M_0^g = G^g(u_0) > G^g(v) \geq M_1^g.$$

This implies that, if $\overline{\gamma} = \min \Gamma_0^g$, then $\overline{\gamma} > 0$; since $\overline{\gamma} \in \Gamma^g$, by Corollary 2.4 (i) we obtain that $F_{\overline{\gamma}}^g$ has at least two minimizers.

(ii) Let $u_0 = \int_0^1 g(t)dt$, then $\tilde{M}_0^g = \int_0^1 (u_0 - g)^2 dx$. Let $v(x) = \beta_0\chi_{(0,b)}(x) + \beta_1\chi_{(b,1)}(x)$, where $\beta_0 = \frac{1}{b}\int_0^b g(t)dt$ and $\beta_1 = \frac{1}{1-b}\int_b^1 g(t)dt$. Since g is not constant, then for a proper choice of b we have

$$\left(\int_0^b g(t)dt - b\int_0^1 g(t)dt\right)^2 > 0$$

which implies, after some calculation,

$$\tilde{M}_0^g = \tilde{G}^g(u_0) > \tilde{G}^g(v) = \tilde{M}_1^g.$$

Now, the same arguments used in (i) give that $\tilde{\gamma} = \min \tilde{\Gamma}_0^g$ is strictly positive and $\tilde{F}_{\tilde{\gamma}}^g$ has at least two minimizers. □

Corollary 3.5. *For every $\gamma \in \mathbb{R}^+$ there exists $g \in L^2$ such that F_γ^g has more than one minimizer.*

Proof. Let us fix $\overline{g} \in L^2$; by Corollary 3.4 (i) there exists $\overline{\gamma} \in \mathbb{R}^+$ such that $F_{\overline{\gamma}}^{\overline{g}}$ has more than one minimizer. We can find $\alpha \in \mathbb{R}^+$ such that $\overline{\gamma}\alpha = \gamma$; then defining $v = \sqrt{\alpha}u$ and $g = \sqrt{\alpha}\overline{g}$, it follows that $\alpha F_{\overline{\gamma}}^{\overline{g}}(u) = F_\gamma^g(v)$. This implies that, if $u_1, \ldots, u_l$ are minimizers for $F_{\overline{\gamma}}^{\overline{g}}$, then $\sqrt{\alpha}u_1, \ldots, \sqrt{\alpha}u_l$ are minimizers for F_γ^g. □

Remark 3.6. It is clear that Corollary 3.5 can be proved with the same rescaling technique also for the functional $\tilde{F}^g_\gamma$. Moreover the previous proof shows that there exists $g \in C^\infty$ (or g piecewise constant) such that F^g_γ has more than one minimizer.

We want to study now the continuous dependence of the solution u of problem (2.2) on the datum g. We will prove that this dependence holds when problem (2.2) has uniqueness, and in this case it is a direct consequence of the one-dimensional case of the results of compactness and lower semicontinuity of Ambrosio in [6].

Lemma 3.7. *Let (g_n) be a sequence of functions in L^2 such that $g_n \to g$ strongly in L^2. Fix $\gamma \in \mathbb{R}^+$ and assume that problem (2.2) for F^g_γ is uniquely solvable by $\tilde{u} \in \mathcal{H}^1$. Let $(\tilde{u}_n)$ be a sequence of functions in $\mathcal{H}^1$ such that for every $n \in \mathbb{N}$ $F^{g_n}_\gamma(\tilde{u}_n)$ takes the minimum value. Then $\tilde{u}_n \to \tilde{u}$ strongly in L^1.*

Proof. By the convergence of g_n to g in L^2, it follows that $\|g_n\|_{L^2} \le C_1$.

Since $\tilde{u}_n$ is a minimizer, it is easy to verify that $F^{g_n}_\gamma(\tilde{u}_n) \le F^{g_n}_\gamma(\tilde{u}) \le C_2$, where C_2 depends on C_1 and the H^1-norm of $\tilde{u}$.

This implies that there exists a constant C_3 depending on C_1 and C_2 such that $\|\tilde{u}'_n\|_{L^2} + \|\tilde{u}_n\|_{L^2} \le C_3$; moreover $\#(S_{\tilde{u}_n}) \le C_2$, hence by a compactness result due to Ambrosio (see [6] Theorem 2.1), there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $\tilde{u}_{n_k} \to \overline{u}$ strongly in L^1, with $\overline{u} \in \mathcal{H}^1$.

To show that $\overline{u}$ coincide with $\tilde{u}$, we apply again Theorem 2.1 in [6] obtaining, after some calculation,

$$
\begin{aligned}
&F^g_\gamma(\overline{u}) \le \liminf_{k\to+\infty} F^g_\gamma(\tilde{u}_{n_k}) \le \\
&\le \liminf_{k\to+\infty} \left[F^{g_{n_k}}_\gamma(\tilde{u}_{n_k}) + \int_0^1 (g_{n_k} - g)^2 dx + 2\int_0^1 (\tilde{u}_{n_k} - g_{n_k})(g_{n_k} - g) dx\right] \le \\
&\le \lim_{k\to+\infty} F^{g_{n_k}}_\gamma(v) = F^g_\gamma(v) \qquad \forall v \in \mathcal{H}^1.
\end{aligned}
$$

This shows that $\overline{u}$ is a solution of (2.2) for F^g_γ, hence, by uniqueness, $\overline{u} = \tilde{u}$ and all the sequence $\tilde{u}_n$ converges to $\tilde{u}$. □

To conclude this section, we want to show that, when the datum $g \in L^2$ is piecewise constant, a solution of problem $(2.6)_j$ (and hence a solution of problem (2.2)) can jump only where g jumps.

We note that this fact had already appeared in the examples reported in the introduction; in general this kind of behaviour is a feature of the minimum points of F_γ^g, independently of the choice of g, if g is piecewise constant.

Given $k \in \mathbb{N}$, for every partition $\mathcal{P} = (a_i)_{i=0}^{k+1}$ of [0,1] we consider the set $\mathcal{M}_\mathcal{P}$ of the functions g of the type

$$g(x) = \sum_{i=0}^{k} \alpha_i \chi_{(a_i,a_{i+1})}(x)$$

with $\alpha_i \in \mathbb{R}$ for every $i = 0, \ldots, k$; we remark that $\mathcal{M}_\mathcal{P}$ is a linear subspace of L^2.

Lemma 3.8. *Let* $g(x) = \sum_{i=0}^{k} \alpha_i \chi_{(a_i,a_{i+1})}(x)$ *be a function belonging to* $\mathcal{M}_\mathcal{P}$. *Fixed* $j \in \mathrm{J}^g$ *, and let* u *be a solution of* $(2.6)_j$ *of the type*

$$u(x) = \sum_{s=0}^{j} \beta_s(x) \chi_{(b_s,b_{s+1})}(x)$$

where $\mathcal{Q} = (b_s)_{s=0}^{j+1}$ *is a partition of* $[0,1]$. *Then*

$$\{b_1, \ldots, b_j\} \subseteq \{a_1, \ldots, a_k\}.$$

Proof. We recall that by Proposition 2.6 J^g is contained in $\{0, \ldots, k\}$.

We argue by contradiction. Suppose that there exist $s \in \{0, \ldots, j\}$ and $i \in \{1, \ldots, k\}$ such that $b_s \in (a_i, a_{i+1})$. First of all we observe that, since $j \in \mathrm{J}^g$, $\beta_{s-1}(b_s) \neq \beta_s(b_s)$, hence we may assume that $\beta_{s-1}(b_s) > \beta_s(b_s)$ (the other case follows by analogous arguments) and we can consider the following two cases:

(1) $\beta_{s-1}(b_s) > \beta_s(b_s) \geq \alpha_i$,

(2) $\beta_{s-1}(b_s) \geq \alpha_i \geq \beta_s(b_s)$.

(The third case $\alpha_i \geq \beta_{s-1}(b_s) > \beta_s(b_s)$ is similar to the first one).

For every $0 < \epsilon < b_s - a_i$ we define a function $u_\epsilon : [0,1] \to \mathbb{R}$ by

$$u_\epsilon(x) = \begin{cases} u(x) & \text{if } x \in (0, b_s - \epsilon) \cup (b_s, 1) \\ \beta_s(b_s) & \text{if } x \in (b_s - \epsilon, b_s). \end{cases}$$

In the case (1) we note that there exists $\delta > 0$ such that for every x such that $b_s - \delta < x < b_s$ we have

$$\beta_{s-1}(x) > \beta_s(b_s).$$

Hence when $\epsilon < \delta$ we obtain

$$G^g(u_\epsilon) - G^g(u) \leq \int_{b_s - \epsilon}^{b_s} [(\beta_s(b_s) - \alpha_i)^2 - (\beta_{s-1}(x) - \alpha_i)^2] dx =$$
$$= \int_{b_s - \epsilon}^{b_s} [\beta_s(b_s) - \beta_{s-1}(x)][(\beta_s(b_s) - \alpha_i) + (\beta_{s-1}(x) - \alpha_i)] dx < 0.$$

This contradicts the hypothesis that u is a minimizer of G^g.

In the case (2) we may consider the following two subcases:

$(2)_a$ $\beta_{s-1}(b_s) - \alpha_i > \alpha_i - \beta_s(b_s)$,

$(2)_b$ $\beta_{s-1}(b_s) - \alpha_i = \alpha_i - \beta_s(b_s) > 0$.

(The last case $\beta_{s-1}(b_s) - \alpha_i < \alpha_i - \beta_s(b_s)$ can be studied similarly to the $(2)_a$).

If $(2)_a$ is satisfied, then there exists $\delta > 0$ such that for every x with $0 < b_s - \delta < x < b_s$ we have $\beta_{s-1}(x) - \alpha_i > \alpha_i - \beta_s(b_s)$ and $\beta_{s-1}(x) > \beta_s(b_s)$. Hence for every $\epsilon < \delta$ we obtain again $G^g(u_\epsilon) - G^g(u) < 0$. Now we consider the case where $(2)_b$ is satisfied. First we remark that from (3.1) for every $x \in (b_s, a_{i+1})$ we have

$$\beta_s(x) = (c_s - \alpha_i) \cosh(x - b_s) + \alpha_i$$

and so

$$\beta_s'(x) = (c_s - \alpha_i) \sinh(x - b_s);$$

then since $c_s = \beta_s(b_s)$, we can conclude that $\beta_s'(x) < 0$ for every $x \in (b_s, a_{i+1})$. Therefore β_s is a strictly decreasing function on (b_s, a_{i+1}). Now for every $0 < \eta < a_{i+1} - b_s$ we define a function $v_\eta : [0,1] \to \mathbb{R}$ by

$$v_\eta(x) = \begin{cases} u(x) & \text{if } x \in (0, b_s) \cup (b_s + \eta, 1) \\ \beta_{s-1}(b_s) & \text{if } x \in (b_s, b_s + \eta). \end{cases}$$

We point out that there exists $\delta > 0$ such that for every x with $b_s < x < b_s + \delta$ we have $\beta_{s-1}(b_s) > \beta_s(x)$; hence for every $\eta < \delta$, using $(2)_b$ and the fact that

$\beta_s'(x) < 0$ implies that $\beta_s(x) < \beta_s(b_s)$, we obtain

$$G^g(u_\eta) - G^g(u) \le \int_{b_s}^{b_s+\eta} [\beta_{s-1}(b_s) - \beta_s(x)][(\beta_{s-1}(b_s) - \alpha_i) - (\alpha_i - \beta_s(x))]dx =$$

$$= \int_{b_s}^{b_s+\eta} [\beta_{s-1}(b_s) - \beta_s(x)][\beta_s(x) - \beta_s(b_s)]dx < 0.$$

This contradiction concludes the proof. □

Corollary 3.9. *Let g, k and j as in Lemma 3.8 and $\gamma \in \mathbb{R}^+$. Then the minimizers of the functional F_γ^g with j jumps are at most $\binom{k}{j}$. Moreover the minimizers of F_γ^g are at most 2^k.*

Proof. By Proposition 2.5, for a fixed partition $\mathcal{Q} = (b_s)_{s=0}^{j+1}$ of $[0,1]$, G^g has exactly one solution $u \in \mathcal{H}_{\mathcal{Q}}^1$; by the preceeding lemma the partitions $\mathcal{Q} = (b_s)_{s=0}^{j+1}$ corresponding to a minimizer of $(2.6)_j$ must be contained in the partition $\mathcal{P} = (a_i)_{i=0}^{k+1}$ and hence they can be at most $\binom{k}{j}$. The conclusion follows since $J^g \subseteq \{0, \dots, k\}$ (see Proposition 2.6) and $\sum_{j=0}^{k} \binom{k}{j} = 2^k$. □

Remark 3.10. It is clear that if we repeat step by step the arguments used in Remark 3.3, Lemma 3.7 and Lemma 3.8 cancelling out the term with the derivative in the functional F_γ^g, we obtain the same results also for $\tilde{F}_\gamma^g$.

5.4 Some genericity results

In Theorem 4.3 we will prove that for "almost all" $g \in \mathcal{M}_{\mathcal{P}}$ we have uniqueness for problem $(2.6)_j$, for each $0 \le j \le k$; but to obtain this result we need before the following lemmas.

Lemma 4.1. *Assume that* $0 \le a_{i_0} < a_{i_1} \le a_{i_2} < a_{i_3} \le 1$ *and that* $g = \chi_{[a_{i_0}, a_{i_1}]}$. *Let us consider for* $m = 2, 3$ *the following functionals*

$$(4.1)_m \qquad \int_{a_{i_0}}^{a_{i_m}} \left[|u'|^2 + |u(x) - g(x)|^2\right] dx.$$

Assume that for $m = 2, 3$ u_m *are minimum points for* $(4.1)_m$ *on* $H^1([a_{i_0}, a_{i_m}])$, *then*

$$(4.2) \qquad \int_{a_{i_0}}^{a_{i_2}} \left[|u_2'|^2 + |u_2(x) - g(x)|^2\right] dx < \int_{a_{i_0}}^{a_{i_3}} \left[|u_3'|^2 + |u_3(x) - g(x)|^2\right] dx.$$

Proof. Since u_2 is a minimum point for $(4.1)_2$, it follows that

$$(4.3) \qquad \int_{a_{i_0}}^{a_{i_2}} \left[|u_2'|^2 + |u_2(x) - g(x)|^2\right] dx \le \int_{a_{i_0}}^{a_{i_2}} \left[|u_3'|^2 + |u_3(x) - g(x)|^2\right] dx.$$

Moreover, if we had that

$$\int_{a_{i_0}}^{a_{i_2}} \left[|u_3'|^2 + |u_3(x) - g(x)|^2\right] dx = \int_{a_{i_0}}^{a_{i_3}} \left[|u_3'|^2 + |u_3(x) - g(x)|^2\right] dx$$

then it should be $u_3 = g$ on (a_{i_2}, a_{i_3}), that means $u_3 \equiv 0$ on (a_{i_2}, a_{i_3}); but by Remark 3.1 it is not possible, since u_3 is a minimum point for $(4.1)_3$. Hence it is clear that

$$\int_{a_{i_0}}^{a_{i_2}} \left[|u_3'|^2 + |u_3(x) - g(x)|^2\right] dx < \int_{a_{i_0}}^{a_{i_3}} \left[|u_3'|^2 + |u_3(x) - g(x)|^2\right] dx$$

and this inequality together with (4.3) gives (4.2). This concludes the proof. □

Lemma 4.2. *Let us fix a partition* $\mathcal{P} = (a_i)_{i=0}^{k+1}$ *of* $[0,1]$ *and for any choice of* $(\alpha_0, \ldots, \alpha_k) \in \mathbb{R}^{k+1}$ *let us define a function* $g = \sum_{i=0}^{k} \alpha_i \chi_{(a_i, a_{i+1})}$ *belonging to* $\mathcal{M}_{\mathcal{P}}$.

Let $j \in \{0,\dots,k\}$ and let $\mathcal{Q} = (q_s)_{s=0}^{j+1}$, $\mathcal{R} = (r_s)_{s=0}^{j+1}$ be two different partitions of $[0,1]$ such that $\mathcal{Q},\mathcal{R} \subseteq \mathcal{P}$. Let us define two functions $Q, R : \mathbb{R}^{k+1} \to \mathbb{R}$ by

$$Q(\alpha_0,\dots,\alpha_k) = \min_{u\in\mathcal{H}^1_{\mathcal{Q}}} \left[\sum_{s=0}^{j}\int_{q_s}^{q_{s+1}} |u'|^2 dx + \int_0^1 |u-g|^2 dx\right]$$

and

$$R(\alpha_0,\dots,\alpha_k) = \min_{v\in\mathcal{H}^1_{\mathcal{R}}} \left[\sum_{s=0}^{j}\int_{r_s}^{r_{s+1}} |v'|^2 dx + \int_0^1 |v-g|^2 dx\right],$$

where $\mathcal{H}^1_{\mathcal{Q}}$ and $\mathcal{H}^1_{\mathcal{R}}$ are defined as in Proposition 2.5. Then Q and R are two different polynomial functions and the set

$$\mathcal{A}_{\mathcal{Q}\mathcal{R}} = \{(\alpha_0,\dots,\alpha_k) \in \mathbb{R}^{k+1} : Q(\alpha_0,\dots,\alpha_k) \neq R(\alpha_0,\dots,\alpha_k)\} \tag{4.4}$$

is an open set dense in $\mathbb{R}^{k+1}$.

Proof. Clearly Q and R are polynomial functions of degree 2 in the $k+1$ variables $\alpha_0,\dots,\alpha_k$. The proof is accomplished if we prove that the equality $Q(\alpha_0,\dots,\alpha_k) = R(\alpha_0,\dots,\alpha_k)$ is not identically satisfied. Since $\mathcal{Q}$ is different from $\mathcal{R}$ there exists l belonging to $\{0,\dots,j\}$ such that $q_m = r_m$ for every $m \in \{0,\dots,l\}$ and $q_{l+1} \neq r_{l+1}$; we suppose that $q_{l+1} < r_{l+1}$. By hypothesis there exist $i_0, i_2, i_3 \in \{0,\dots,k\}$ such that $q_l = r_l = a_{i_0}$, $q_{l+1} = a_{i_2}$ and $r_{l+1} = a_{i_3}$. Let us take now $\alpha_0,\dots,\alpha_k$, where $\alpha_{i_0} = 1$ and $\alpha_m = 0$ for $m \neq i_0$. Then by Lemma 4.1 with $a_{i_1} = a_{i_0+1}$ we have that $Q(\alpha_0,\dots,\alpha_k) < R(\alpha_0,\dots,\alpha_k)$. This concludes the proof.

□

Theorem 4.3. *Given a partition $\mathcal{P} = (a_i)_{i=0}^{k+1}$ of $[0,1]$, there exists a subset $\mathcal{M}^0_{\mathcal{P}}$ of $\mathcal{M}_{\mathcal{P}}$, which is dense in $\mathcal{M}_{\mathcal{P}}$ with respect to the L^2-topology, and such that for every $g \in \mathcal{M}^0_{\mathcal{P}}$ problem $(2.6)_j$ has a unique solution, for every $j \in \mathrm{J}^g$.*

Proof. Let $\mathcal{P} = (a_i)_{i=0}^{k+1}$ be a partition of $[0,1]$ and let $g \in \mathcal{M}_{\mathcal{P}}$. Let $j \in \mathrm{J}^g$; by Proposition 2.6 we have that $0 \leq j \leq k$. If $j = k$, then for every $g \in \mathcal{M}_{\mathcal{P}}$ the problem $(2.6)_k$ has the unique solution g.

Now we define

$$\mathcal{A}_{\mathcal{P}} = \bigcap_{0\le j<k} \bigcap_{\substack{\mathcal{Q},\mathcal{R}\subseteq\mathcal{P}\\ \#\mathcal{Q}=\#\mathcal{R}=j+2}} \mathcal{A}_{\mathcal{Q}\mathcal{R}},$$

where the set $\mathcal{A}_{\mathcal{Q}\mathcal{R}}$ is defined by (4.4), and

$$\mathcal{M}^0_{\mathcal{P}} = \{g \in \mathcal{M}_{\mathcal{P}} : g = \sum_{i=0}^{k} \alpha_i \chi_{(a_i,a_{i+1})} \text{ with } (\alpha_0,\dots,\alpha_k) \in \mathcal{A}_{\mathcal{P}}\};$$

by Lemma 4.2 and by Baire's Theorem, $\mathcal{A}_{\mathcal{P}}$ is an open set dense in $\mathbb{R}^{k+1}$ and hence $\mathcal{M}^0_{\mathcal{P}}$ is dense in $\mathcal{M}_{\mathcal{P}}$ with respect to the L^2-topology. Moreover for every $g \in \mathcal{M}^0_{\mathcal{P}}$ the problem $(2.6)_j$ has uniqueness, for every $j \in \mathrm{J}^g \setminus \{k\}$. In fact let $g = \sum_{i=0}^{k} \alpha_i \chi_{(a_i,a_{i+1})}$ be a function belonging to $\mathcal{M}^0_{\mathcal{P}}$. Then $(\alpha_0,\dots,\alpha_k) \in \mathcal{A}_{\mathcal{Q}\mathcal{R}}$ for every $\mathcal{Q}$, $\mathcal{R} \subseteq \mathcal{P}$, with $\#\mathcal{Q} = \#\mathcal{R} = j+2$ and for every $0 \le j < k$. We suppose by contradiction that there exist $j_0 \in \mathrm{J}^g$ and two different solutions $u, v \in \mathcal{K}^1_{j_0}$ of the problem $(2.6)_{j_0}$ (see Remark 2.1). Let $\mathcal{Q}$ and $\mathcal{R}$ the partitions associated to u and v; by Proposition 2.5 $\mathcal{Q}$ and $\mathcal{R}$ are different and by Lemma 3.8 $\mathcal{Q}$ and $\mathcal{R}$ are contained in $\mathcal{P}$. Hence from the definition of $\mathcal{A}_{\mathcal{Q}\mathcal{R}}$ $Q(\alpha_0,\dots,\alpha_k)$ must be different from $R(\alpha_0,\dots,\alpha_k)$, where Q and R are defined as in Lemma 4.2. But since u and v are minimizers of the problem $(2.6)_{j_0}$, we have that $Q(\alpha_0,\dots,\alpha_k) = M^g_{j_0} = R(\alpha_0,\dots,\alpha_k)$; this contradiction concludes the proof. □

Theorem 4.4. *There exists a countable set $\mathcal{M}_0$ dense in L^2 and a countable set Γ in $\mathbb{R}^+$ such that for every $g \in \mathcal{M}^0$ and $\gamma \in \mathbb{R}^+ \setminus \Gamma$ problem* (2.2) *admits a unique solution.*

Proof. For every $k \in \mathbb{N}$ we consider the partition $\mathcal{P}_k = \{0, \frac{1}{k}, \frac{2}{k}, \dots, 1\}$ of $[0,1]$; hence by Theorem 4.3 there exists a set $\mathcal{M}^0_k$ dense in $\mathcal{M}_{\mathcal{P}_k}$ such that for every $g \in \mathcal{M}^0_k$ problem $(2.6)_j$ has a unique solution for every $j \in \mathrm{J}^g$. By the density of characteristic functions in L^2, the set

$$\mathcal{M} = \bigcup_{k\in\mathbb{N}} \mathcal{M}^0_k;$$

is dense in L^2. Moreover by the separability of L^2, there exists a countable set $\mathcal{M}^0 \subseteq \mathcal{M}$, which is dense in L^2. Let us consider $\Gamma = \bigcup_{g \in \mathcal{M}^0} \Gamma^g$, where Γ^g is defined by (2.13). Since, by Remark 2.2, Γ^g is a countable set and $\mathcal{M}^0$ is countable, then Γ is a countable set.

Now, fixed $g \in \mathcal{M}^0$ and $\gamma \in \mathbb{R}^+ \setminus \Gamma$ the uniqueness for the problem (2.2) follows by the uniqueness for problem $(2.6)_j$, by Proposition 2.3 and by the definition of Γ. $\square$

In the following theorem we give a "genericity" result: we establish that the uniqueness of the solution to problem (2.2) is a generic property.

Theorem 4.5. *Let us assume that there exists a countable set $\mathcal{M}^0$, which is dense in L^2, and a countable set Γ in $\mathbb{R}^+$ such that for every $g \in \mathcal{M}^0$ and for every $\gamma \in \mathbb{R}^+ \setminus \Gamma$ problem (2.2) has a unique solution. Then for every $\gamma \in \mathbb{R}^+$ there exists a G_δ-set $\mathcal{M}^*_\gamma$ dense in L^2 such that for each $g \in \mathcal{M}^*_\gamma$ the solution of problem (2.2) is unique.*

Proof. Let $\mathcal{M}^0$ be as in the statement of the theorem. We fix $\gamma \in \mathbb{R}^+ \setminus \Gamma$ and $g \in L^2$ and define

$$S(g) = \{u \in \mathcal{H}^1 : u \text{ is a solution of } (2.2)\}.$$

We observe that $S(g) \neq \emptyset$, since as we have seen, there exists at least one solution of (2.2). Let us define $D : L^2 \to [0, +\infty]$ by

$$D(g) = \sup_{u,v \in S(g)} \|u - v\|_{L^1}.$$

This definition implies that (2.2) has a unique solution if and only if $D(g) = 0$.

Now, we are going to prove that the function D is continuous in the points of the set $\mathcal{M}^0$.

Let us fix $\bar{f} \in \mathcal{M}^0$ and suppose that there exist $\bar{n} \in \mathbb{N}$ and a sequence (f_k) in L^2 such that f_k converges to $\bar{f}$ in the L^2-topology and

$$D(f_k) \geq \frac{1}{\bar{n}}, \quad \text{for every } \; k \in \mathbb{N}.$$

This implies that there are two sequences (v_k) and (u_k) in $S(f_k)$ such that

$$\|v_k - u_k\|_{L^1} \geq \frac{1}{n}, \quad \text{for every} \quad k \in \mathbb{N}. \tag{4.5}$$

On the other hand, since $\bar{f} \in \mathcal{M}^0$, by hypothesis there exists a unique solution $u_{\bar{f}}$ of the problem

$$\min \{F_\gamma^{\bar{f}}(u) \ : \ u \in \mathcal{H}^1\}.$$

Therefore from Lemma 3.7 we can conclude that v_k and u_k converge to $u_{\bar{f}}$ strongly in L^1; but this contradicts (4.5). Hence for every $f \in \mathcal{M}^0$ and $n \in \mathbb{N}$ there exists an open neighborhood U_f^n of f in the L^2-topology such that $D(g) < \frac{1}{n}$ for all $g \in U_f^n$.

At this point, if we denote $U^n = \bigcup_{f \in \mathcal{M}^0} U_f^n$, we have that U^n is an open subset of L^2 with respect to the L^2-topology. Then let us define $\mathcal{M}_\gamma^* = \bigcap_{n \in \mathbb{N}} U^n$; $\mathcal{M}_\gamma^*$ is a G_δ-set in L^2 and, by construction, contains $\mathcal{M}^0$; so, by hypothesis, $\mathcal{M}_\gamma^*$ is dense in L^2. Moreover we observe that, fixed $g \in \mathcal{M}_\gamma^*$, for each $n \in \mathbb{N}$, g belongs to U^n and so $D(g) = 0$.

This proves the theorem when $\gamma \in \mathbb{R}^+ \setminus \Gamma$. Let now $\gamma \in \Gamma$ and fix $\gamma_0 \in \mathbb{R}^+ \setminus \Gamma$. Then there exists $\alpha > 0$ such that $\alpha\gamma_0 = \gamma$. By the first part of the theorem, we know that for every $g \in \mathcal{M}_{\gamma_0}^*$ the problem

$$\min_{u \in \mathcal{H}^1} \left[\sum_{s=0}^{l} \int_{b_s}^{b_{s+1}} |u'|^2 dx + \int_0^1 |u - g|^2 dx + \gamma_0 \#(S_u) \right]$$

has only one solution. Multiplying this expression by α, defining $v = \sqrt{\alpha} u$ and taking into account that $\#(S_u) = \#(S_v)$ we obtain that, if $f \in \sqrt{\alpha}\mathcal{M}_{\gamma_0}^* = \mathcal{M}_\gamma^*$, then the problem

$$\min_{v \in \mathcal{H}^1} \left[\sum_{s=0}^{l} \int_{b_s}^{b_{s+1}} |v'|^2 dx + \int_0^1 |v - f|^2 dx + \gamma \#(S_v) \right]$$

has only one minimizer. Since $\sqrt{\alpha}\mathcal{M}_{\gamma_0}^*$ is clearly a dense G_δ-set the proof is accomplished. □

Corollary 4.6. *If Γ_0 is a countable subset of $\mathbb{R}^+$, then there exists a dense G_δ-set $\mathcal{M}^*_{\Gamma_0}$ such that for every $g \in \mathcal{M}^*_{\Gamma_0}$ and for every $\gamma \in \Gamma_0$ problem* (2.2) *has uniqueness.*

Proof. It is enough to define $\mathcal{M}^*_{\Gamma_0} = \bigcap_{\gamma \in \Gamma_0} \mathcal{M}^*_\gamma$ and to observe that by Baire's Lemma the countable intersection of dense G_δ-set is still a dense G_δ-set. □

In the following theorem, we shall construct a dense G_δ-subset of $L^2([0,1])$ such that when g belongs to this set, problems (1.2) is uniquely solvable if γ belongs to the complement of a countable subset Γ^g in $\mathbb{R}^+$ depending on g.

Theorem 4.7. *There exists a G_δ-set $\mathcal{M}^*$ dense in L^2 such that for every $g \in \mathcal{M}^*$ and $\gamma \in \mathbb{R}^+ \setminus \Gamma^g$, where Γ^g is countable, problem* (2.2) *has uniqueness.*

Proof. In the previous corollary we may choose in particular $\Gamma_0 = \mathbb{Q}^+$, where $\mathbb{Q}^+$ denotes the set of the positive rational numbers and we can define $\mathcal{M}^* := \mathcal{M}^*_{\Gamma_0}$. Let us take now $g \in \mathcal{M}^*$. Since Γ_0 is dense in $\mathbb{R}^+$, we have that its intersection with the interior of Γ^g_h is non empty, for every interval Γ^g_h. Moreover, when γ is a rational number belonging to the interior of Γ^g_h, by Corollary 4.6 problem (2.2) relative to g has uniqueness. Hence by Corollary 2.4, we have uniqueness for every γ belonging to the interior of Γ^g_h and this is true for every $h \in J^g$. The proof follows, recalling that $\mathbb{R}^+ = \Gamma^g \cup (\bigcup_{h \in J^g} \text{int } \Gamma^g_h)$, where int Γ^g_h denotes the interior of Γ^g_h. □

Remark 4.8. It is clear that there is nothing difference in the proof if we substitute the functional F^g_γ with $\tilde{F}^g_\gamma$, hence the preceeding results continue to hold.

Remark 4.9. We observe that Theorem 4.7 cannot be improved, that is we cannot expect, fixed $g \in L^2$, to have a unique solution for problem (2.2) for every γ belonging to the complement in $\mathbb{R}^+$ of a countable set depending on g. In fact, as we saw in the second example of the introduction, there are functions $g \in L^2$ for which we have to remove a whole interval of $\mathbb{R}^+$ in order to have uniqueness.

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