VISIONS OF THE END OF INFLATION

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Abstract

We develop and extend the existing theory of reheating the universe after inflation. By exploiting elegant and powerful mathematical results we are able to find and classify new channels by which reheating may proceed. In particular we show that preheating via pure gravitational effects - *geometric reheating* - can be extremely efficient. We present new preheating classes and a spectral method for defining equivalence classes of reheating models. These results are then immediately applicable in the strong-coupling limit of realistic theories where reheating occurs via stimulated effects in many fields. This chaotic enhancement of the power of preheating leads to enhanced possibilities of unusual phenomena such as non-thermal symmetry restoration after inflation. Further we demonstrate that gravitational waves suffer a resonant amplification during oscillatory reheating in a manner dual to the scalar field case. Finally we exhibit an electromagnetic duality for classical General Relativity which allows us to map linearised Schwarzschild into linearised Taub-NUT, that is, linearised gravito-electric monopoles into linearised gravitomagnetic monopoles.

Contents

Dedicated to my parents

and

To the memory of Lando Caiani

Acknowledgments

Friendship is almost always the union of a part of one mind with a part of another; people are friends in spots.
George Santayana

I guess this section of a thesis should be like a 21st or wedding speech. It should be bitter, it should be sweet, it should make you cry, make you laugh, make you think deeply about life and romantically about the past, even if you don't know the author from the proverbial can of Campbell's soup. But let's be real, if I could do that I would be writing famous movie scripts.

Basically, writing acknowledgments is not the sort of thing one typically does very often, unless one has many friends who marry frequently, or one writes a lot of theses. This is partially true in my case I must confess and so I am faced with an additional problem: the problem of repetition. Afterall, what greater failing is there in a thesis than to be unoriginal, uninteresting or insincere in the space of the one page where it really matters ? So to avoid the repetition problem I will simply thank the people directly involved in my life during the last couple of years. If you are not thanked in my MSc then please forgive me or alternatively send hate mail to the address on page 245. So with this brief preamble perhaps now it is appropriate (at least for me) to undertake a retrospective wander in the field of past debts.

I thank first my parents who have supported me an entire life and to whom I owe almost all that I am now, at least intellectually. Defects in my personality are solely of my own doing and indeed I was forced to toil hard and long to undo the good foundation that they laid. Indeed, only after graduating did people start to call me "evil bruce".

My supervisor, Dennis Sciama, also deserves my sincerest thanks and commands my deep respect. It is to his credit that he never, despite the temptation, referred to me as "evil bruce", managing instead to stop at 'that bloody bruce' (although he also called me 'brian' for nearly six months !).

George Ellis, who referred to me as 'young Bassett', provided me a remarkable source of ideas and knowledge, as well as the opportunity to come to SISSA. Further the cosmology group in Cape Town has always been a place where I felt welcome and could find deep discussions about the real priorities in life.

Then there are my many friends who have given me so much more than I have given them. Joshua has been there for over a decade as a constant source of creative energy, Stefano and Paola who were always there when I needed them and who, almost by tradition now, provide me a bed to sleep in when I return to Trieste. I thank of course all my other friends from SISSA, Fin (and Tori, Iz and Joel), Vale, Laura and all the others. In addition there are the other friends I have made here in Italy who made my stay such a rewarding set of experiences. Too many to mention all but particularly I am

glad to have known the Martina's, Alessia and all the friends who have put me up (and permutations thereof) in the different countries in Europe.

I would also like to thank my collaborators of the last couple of years: Stefano, Roy, Fabrizio, David and Arjun, for making research painless and from whom I have gained and learned about the art and craft of research. To all the other people with whom I have worked and discussed life and physics go my sincere thanks.

Then I would like to express my warm feelings towards the people at the International Centre for Theoretical Physics (ICTP). Faheem, Martin, Tino, Hailu, Ramon and Alex and the other members of the Diploma course of '97 and '98, all of them have provided memories which will stay with me for ages and ages hence. I would also really like to thank Luisa Durrani for helping me prepare fashion-wise for Stefano & Paola's wedding !

In all the three years I have been here it has been a great pleasure to be within the gardens of the ICTP. Abdus Salam's legacy lives on in providing an exceptional environment where scientists, but primarily people, from different countries can meet. I have developed emotionally and learned about myself against this colourful backdrop of world cultures. In Salam's own words, "There is no instrument more potent in bringing an appreciation of different ... points of view than the atmosphere of an international university."

Finally, Lando Caiani introduced me to the theory of stochastic oscillations. He shared with me his deep insights into the nature of statistical physics and these led me to the results presented in Chapters (5) and (6), the core of the thesis.

Words to live by ... Not !

While looking for suitably inspiring quotations to add to my chapter title pages, I came across a source of rather remarkable quotes by that regal orator Dan Quayle, which provided me so much fun that I thought it appropriate to include in my thesis. Next to these, anything I have written will seem (I sincerely hope) intelligent.

In politics stupidity is not a handicap. – Napoleon

It isn't pollution that's harming the environment. It's the impurities in our air and water that are doing it.
– Vice President Dan Quayle

What a waste it is to lose one's mind. Or not to have a mind is being very wasteful. How true that is.
– Vice President Dan Quayle winning friends while speaking to the United Negro College Fund, 5/9/89 This gem has been added to Bartlett's 'Familiar Quotations'. (reported in Esquire, 8/92) (reported in the NY Times, 12/9/92)

Republicans understand the importance of bondage between a mother and child.
– Senator Dan Quayle, US News and World Report (10/10/88)

Welcome to President Bush, Mrs. Bush, and my fellow astronauts.
– Vice President Dan Quayle addressing the 20th anniversary celebration of the moon landing, 7/20/89 (reported in Esquire, 8/92)

Mars is essentially in the same orbit... Mars is somewhat the same distance from the Sun, which is very important. We have seen pictures where there are canals, we believe, and water. If there is water, that means there is oxygen. If oxygen, that means we can breathe.
– Vice President Dan Quayle, 8/11/89 (reported in Esquire, 8/92)

How about if we say when it's wet, it's wet?
– Vice President Dan Quayle when asked to define 'wetlands'. (from 'What a Waste it Is to Lose One's Mind' – the Unauthorized Autobiography)

And it was a very good book of Rasputin's involvement in that, which shows how people that are really very weird can get into sensitive positions and have a tremendous impact on history.
– Senator Dan Quayle gives his opinion of the book 'Nicholas, and Alexandra', to Hendrick Hertzberg of the New Republic, 9/28/88 (reported in Savvy Woman, 1/89, p. 56)

May our nation continue to be the beakon (sic.) of hope to the world.
– The Quayles' 1989 Christmas card.

Add one little bit on the end... Think of 'potatoe', how's it spelled? You're right phonetically, but what else...? There ya go... all right!
– Vice President Dan Quayle correcting a student's correct spelling of the word 'potato' during a spelling bee at an elementary school in Trenton.

I should have caught the mistake on that spelling bee card. But as Mark Twain once said, 'You should never trust a man who has only one way to spell a word'.
– Vice President Dan Quayle, actually quoting from President Andrew Jackson.

I should have remembered that was Andrew Jackson who said that, since he got his nickname 'Stonewall' by vetoing bills passed by Congress.
– Vice President Dan Quayle, confusing Andrew Jackson with Confederate General Thomas J. 'Stonewall' Jackson, who actually got his nickname at the first Battle of Bull Run.

I have made good judgments in the Past. I have made good judgments in the Future.
– Vice President Dan Quayle

Verbosity leads to unclear, inarticulate things.
– Senator Dan Quayle, 10/30/88 (reported in Esquire, 8/92 and the LA Times, 10/30/88)

My friends, no matter how rough the road may be, we can and we will never, never surrender to what is right.
– Vice President Dan Quayle speaking to the Christian Coalition about the need for abstinence to avoid AIDS, 11/15/91 (reported in Esquire, 8/92)

This isn't a man who is leaving with his head between his legs.
– Vice President Dan Quayle discussing John Sununu's resignation and apparent lack of flexibility, 12/6/91

The US has a vital interest in that area of the country.
– Vice President Dan Quayle referring to Latin America.

It's wonderful to be here in Latin America. It just makes me regret that I never studied more Latin at school.
– Vice President Dan Quayle.

We are leaders of the world of the space program. We have been the leaders of the world of our... of the space program and we're not going to continue where we're going to go, not withstanding the Soviet Union's demise and collapse – the former Soviet Union – we now have independent republics which used to be called the Soviet Union. Space is the next frontier to be explored. And we're going to explore. Think of all the things we rely upon in space today: communications from... Japan, detection of potential ballistic missile attacks. Ballistic missiles are still here. Other nations do have ballistic missiles. How do you think we were able to detect some of the Scud missiles and things like that? Space, reconnaissance, weather, communications – you name it. We use space a lot today.
– Vice President Dan Quayle (from 'What a Waste it Is to Lose One's Mind' – the Unauthorized Autobiography)

For NASA, space is still a high priority.
– Vice President Dan Quayle, talking to NASA employees, 9/5/90 (reported in Esquire, 8/92)

When I was a boy I was told that anybody could become President; I'm beginning to believe it.
– Clarence Darrow

Notation and Acronyms

The following acronyms are used in this thesis:

FLRW	Friedmann-Lemaître-Robertson-Walker spacetimes
GIC	Gauge-Invariant and Covariant
CMB	Cosmic Microwave Background radiation
CDM	Cold Dark Matter
HDM	Hot Dark Matter
MDM	Mixed Dark Matter
COBE	Cosmic Background Explorer satellite
DMR	Differential Microwave Radiometer
(I)SW	(Integrated) Sachs Wolfe effect

Introduction & motivation

In his 1937 Nobel Lecture, G. P. Thompson said "... The goddess of learning is fabled to have sprung full grown from the brain of Zeus, but it is seldom that a scientific conception is born in its final form, or owns a single parent. More often it is the product of a series of minds, each in turn modifying the ideas of those that came before, and providing material for those that come after."

Likewise it is rare in physics that conclusive and profound evidence is found in a single experiment rather than in a slow accumulation of knowledge at centres around the world. Perhaps cosmology and particle physics provide the only counterexamples to this. The discovery of the $(W^{\pm}, Z^0)$ bosons at CERN and the temperature anisotropies in the Cosmic Microwave Background (CMB) by the COBE satellite are two prime examples. Prior to and since the COBE discovery observational cosmology has largely been in stasis.

The main direction of this thesis, while obviously building on the work done by others, has been shaped by the potential of the future and lies on the interface between cosmology and particle physics. The CMB satellites MAP and PLANCK will, in principle at least, map the full-sky CMB anisotropies to a resolution of $10'$ by about 2010, also providing polarisation and cross-correlation data. These will be supplemented by ground-based and balloon-bourne experiments and the Sloan Digital Sky Survey (SDSS) and 2dF galaxy surveys. These will provide direct probes of the possible existence of topological defects, will test whether inflation ever occured and will help determine the nature of the dark matter. In addition the various gravitational wave telescopes, headed by LIGO, will become fully operational in the next 4-10 years and will provide a completely new view of the universe.

Each one of these 'big-science' projects, offers the possibility of completely altering our view of the universe. LIGO will test the existence and nature of gravitational radiation, the final outstanding test of General Relativity. The SDSS will provide an unprecedented three dimensional view of the observable universe which will provide amazing constraints on models put forward by theorists. More importantly, along with the CMB satellites, it will directly test, for the first time, the homogeneity of the observable universe - the Copernican principle - principally via the Sunyaev-Zel'dovich effect. The philosophical implications of this are huge. If the Copernican principle is verified, as most expect, the parameters of our Friedmann universe should theoretically be determined using all available data, to an accuracy of around $5 - 10\%$.

While this serves as a motivation for studying cosmology in general, it also holds a great threat to classical General Relativity, at least as a subject "relevant" to cosmology, and this is a pertinent influence on this thesis. If the above is true, the years around 2010 will act as a mirror. Research will either proceed to examination of smaller scales with fixed background cosmology and detailed modeling of non-linear hydrodynamical physics (cluster scales and below) or will proceed to much earlier phases in the universe's history where there is still freedom to create new ideas while having specific goals to aim at - the explanation of the state of the universe at and after the decoupling of photons from matter. This thesis has been coherently influenced by these issues and leans strongly towards the second path, and in particular towards the study of the end of inflation and the reheating of the universe.

The work in this thesis documents results which have been achieved in collaboration with several people and published in the following papers: Chapter 3: Stefano Liberati [11]. Chapter 5: Fabrizio Tamburini [98] and [154]. Chapter 7: Roy Maartens [210]. The work in Chapter 4 is based on the paper [148], while that in Chapter 6 reflects the articles [153, 149]. The papers [12, 13] are not included as they concern issues too distant from the main thrust of the thesis, *vis.*: multi-fluid galaxy formation and approximation theory for the error function, respectively. Other work published before completion of the thesis are the papers [14]-[17] and [93]

Guide for the reader

Here we make a brief tour of the thesis: Chapter (1) provides an introduction to the CMB, quantum field theory in curved spacetimes and the inflationary paradigm. Chapter (2) is dedicated to an extensive review of reheating and the explosive phase known now as preheating. This is important reading in the sense that it sets up the basic paradigm, conventions and notation of the field and provides a platform from which the rest of the thesis departs. Chapters (3,4) and (5) look at systematic attempts to understand preheating in realistic models of physics, each from a different point of view. Respectively they are concerned with the effects of running of coupling constants and non-minimal coupling to the curvature, with the classification of all preheating models when the expansion of the universe can be neglected and then, in Chapter (5), with preheating in strongly coupled theories with multiple fields. These three chapters are the heart of the thesis.

Chapter (6) examines gravitational wave evolution and amplification through preheating and in cosmologies which have large components of oscillating dark matter (e.g. axions) or global defects. Finally, Chapter (7) investigates the duality structure of General Relativity in the covariant formalism, showing that it processes a remarkable similarity to the duality of electromagnetism and other non-Abelian gauge theories which are currently providing so much excitement in theoretical physics.

Bruce A. Bassett

Trieste, Italy
August 1998

Chapter 1

Introduction and Setting

One never really understands mathematics...
one just gets used to it.
J. von Neumann

It is always the best policy to tell the truth,
unless, of course, you are an exceptionally good liar.
– Jerome K. Jerome

Only two things are infinite, the universe and human stupidity,
and I'm not sure about the former.
– Einstein

1.1 Introduction

As emphasized by many authors, notably by Geroch [18], the Einstein field equations are incomplete due to the intrinsic differences between the purely geometric left hand side - the Einstein tensor $G_{\mu\nu}$ - and the "physical" stress-energy tensor $T_{\mu\nu}$ on the right hand side.

Of course, the contracted Bianchi identities $G^{\mu\nu}{}_{;\nu} = 0$ link the geometrical "boundary of a boundary is zero" to the conservation of stress-energy, $T^{\mu\nu}{}_{;\nu} = 0$, but the freedom in constructing the stress tensor manifests itself in the myriad of approximations to the real universe.

We are particularly interested in field-theoretic and fluid definitions of the stress tensor and will often use the equivalence between a scalar field and its formulation as a perfect fluid, as is required for interfacing with standard results in general relativistic cosmological perturbation theory started by Landau and Lifschitz in 1946 and continued in a series of fundamental steps extensively laid out and reviewed in the works by Bardeen (1980) [19], Kodama & Sasaki (1984) [20], Ellis & Bruni (1989) [21] and Mukhanov, Feldman & Brandenberger (1992) [22] to name but a few.

At a deeper level, the differences between the fluid approach to cosmology and the field-theoretic approach of particle physicists evidence the fundamental issues and problems of quantum gravity: the problem of time, the problem of large quantum fluctuations etc... In the realm of inflation these issues have largely been skirted around.

The program of semi-classical quantum gravity developed on the notion that one could treat spacetime classically while quantising the matter, thereby further increasing the disparity in treatment of the left and right hand sides of the field equations and despite an extensive body of literature (see e.g. Birrel & Davies [23]) there is no guarantee that the limiting solutions to this ansatz will be solutions to the full quantum gravity theory. Indeed, there are strong indications in the recent work of Ashtekar (1997) [25] that many semi-classical solutions are spurious, or more precisely, that the domain of applicability of the semi-classical theory is very limited.

Semi-classical gravity becomes even more vulnerable once one allows the universe to have non-trivial topology. At the classical level this has no effect since GR is formulated purely in the tangent space and has no global "knowledge" of the topology of the underlying manifold [1]. Yet the renormalised stress tensor is sensitive to spacetime topology and contains global information such as the scale of compactification of the manifold (the topological Casimir effect). This immediately means that the Einstein field equations fail to hold [28, 29]. Quantum gravity cannot therefore simply be a quantization of General Relativity unless (a) non-trivial topologies are excluded, or (b) the true quantization procedure is not sensitive to topology. (a) implies that our current view of quantum

[1] In classical General Relativity the topology of spacetime is fixed by Geroch's theorem [26]. Thus gravity is played off on a manifold of constant, absolute topology much like Newtonian gravity was set in a constant curvature (flat) absolute space and time. Quantum effects are expected to allow topology change and imply that the metric becomes degenerate at the hyper-surface of topology change for Lorentzian signature metrics [27].

gravitational processes, founded on the early ideas of Wheeler, are completely wrong, while (b) implies that the renormalisation and regularisation techniques of standard quantum field theory, which have served us so well, are fundamentally flawed.

Given these unresolved issues, much of the work on inflation must be considered as dubious since it has relied heavily on the semi-classical and even purely classical theory of General Relativity. Nevertheless, current research in semi-classical quantum gravity is very active and is attacking very difficult issues such as non-equilibrium quantum field theory in expanding universes.

In this chapter we will cover the background needed for the remaining chapters on reheating after inflation. The reader is assumed to be familiar with General Relativity and basic quantum field theory both at the level of a simple post-graduate course. We start with the main observable and cornerstone of present cosmology: the cosmic microwave background (CMB), which will almost certainly provide the next major test for the inflationary paradigm.

1.2 The cosmic microwave background and inflation

The discovery of the CMB monopole radiation in 1965 and the positive detection of octopole and higher order anisotropies in the CMB by the COBE *DMR* instrument in 1994 are arguably the two most influential events in modern cosmology. The perfection of the first signalled the beginning of the end for the steady-state cosmology and marked the rise of the big-bang model, while the second, the flaws in the perfection, was a seemingly miraculous vindication of pure thought - the inflationary paradigm. Inflation was originally introduced by Guth, as we will discuss in a following section, to solve the problems of the standard cosmology that arose from the discovery of the CMB and its amazing isotropy.

Only later (but not much later) was it realized that quantum fluctuations during inflation could perhaps solve the other thorny observational cosmology issue: how to make galaxies and clusters of galaxies in accordance with observations, or in other words "how to get a scale-invariant (Harrison-Zel'dovich) power spectrum of energy density fluctuations ?" The near constancy of the Hubble constant over a long period needed to solve the horizon problem naturally supplied such scale invariance since the amplitude of the quantum fluctuations at a given time during inflation just turned out to be proportional to the Hawking temperature, i.e. $\propto H/2\pi$. where H is the Hubble constant. Inflation therefore seemed to provide a way to smooth out the universe while simultaneously providing the perfect green-house for seeding good galaxies.

The discovery by COBE of a spectrum of fluctuations (see figure 1.1) consistent with a scale-invariant spectrum ($n = 1.1 \pm 0.3$ from the 4-year data [30]) seemed therefore to vindicate the inflationary paradigm.

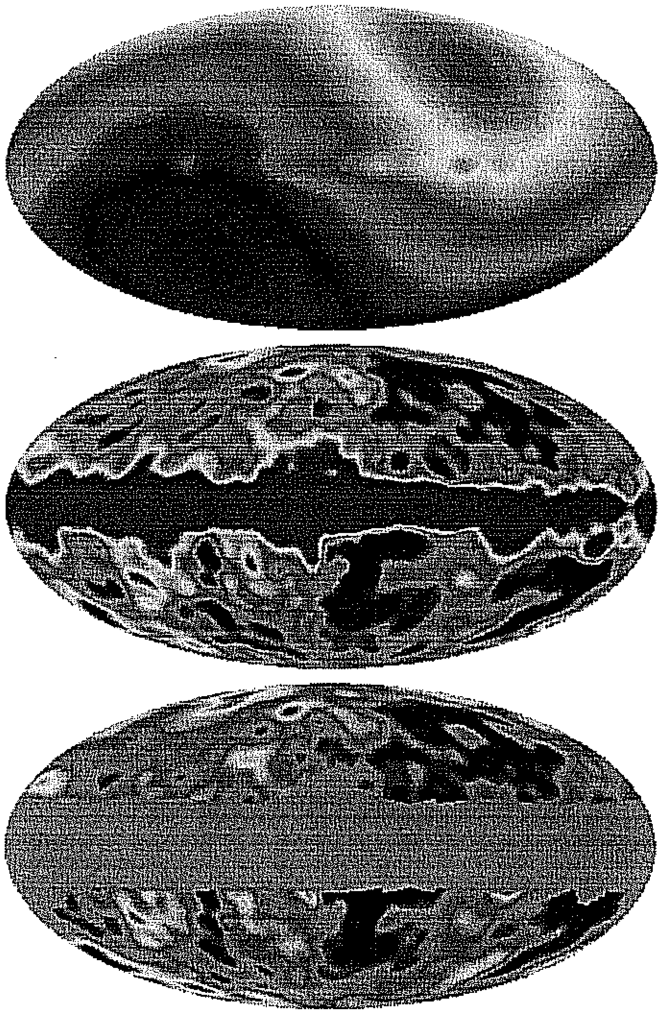


Figure 1.1: The four-year COBE DMR data. The full data including the dominant dipole (top), the data with the dipole removed (middle) and the full data after removal of the galactic plane contribution (bottom).

There are a number of weaknesses in this interpretation. Firstly, while inflationary models predict a nearly scale-invariant spectrum of fluctuations (a truly scale-free spectrum could only result from de Sitter spacetime from which there is no exit [2]), the natural amplitude of the fluctuations as controlled by the coupling constants of the theory, was wrong. For example, in one of the simplest models of the inflaton field ϕ, based on a quartic potential $V = \lambda\phi^4/4$, the temperature anisotropies predicted are roughly:

$$\frac{\Delta T}{T} \sim (10 - 100)\sqrt{\lambda} \tag{1.1}$$

which, given the COBE results, implied $\lambda \sim 10^{-12} - 10^{-14}$ for the self-interaction coupling. Such small values are found in other simple models too and are a severe embarrassment for a theory trying to solve fine-tuning problems. Secondly, scale-invariance is a natural result of scaling and fractal phenomena, and are typical in statistical physics and in particular in 2nd-order phase transitions where there are fluctuations on all scales and divergent correlation length of the field. Thirdly there are competitive models based on scaling networks of topological defects [36] which are ubiquitous in modern physics based on spontaneous symmetry breaking [37]. Finally there exist no truly convincing models of inflation that arise without prior thought or insertions by hand, from realistic models of particle physics.

Indeed, it is now believed [38], that it will be the next generation of microwave background satellites - MAP [3] and PLANCK [4] - that will provide the final tests of inflation, both in terms of the fluctuation spectra (temperature, polarisation & spectral distortions) and the parameters (and validity) of the FLRW background - $(\Omega, \Lambda, \Omega_b, H, ..)$.

So how is it possible to prove inflation or, what is more likely given the huge number of inflationary models, falsify its competitors ? This is a delicate task since the CMB angular spectra depend sensitively on the cosmological background and the fluctuation spectrum. The basic object of study are the Legendre transforms of the angular two point functions $C(\theta)$ of the temperature, polarisation etc... to yield the C_ℓ^T, C_ℓ^P respectively defined by (e.g. [39]):

$$C(\theta) = \sum_\ell (2\ell+1) C_\ell \; W_\ell \; P_\ell(\cos\theta) \tag{1.2}$$

and the *rms* anisotropy is:

$$\left(\frac{\Delta T}{T}\right)^2_{rms} = \frac{1}{4\pi}\sum_\ell (2\ell+1) C_\ell W_\ell \tag{1.3}$$

where P_ℓ are the usual Legendre polynomials, the C_ℓ are derived by angular averaging of the spherical harmonic coefficients of the distribution under study, *viz.* $C_\ell \equiv \langle |a_{\ell m}|^2 \rangle$ and the W_ℓ are the multipoles of the window function specific to the experiment (giving its resolution in ℓ-space).

Typical angular spectra for adiabatic inflationary models (with different dark matter) are shown as solid lines in figure (1.2). In contrast a typical topological defects spectrum (with non-zero Λ) is

[2] Except perhaps due to 2-loop quantum gravity effects which cause λ to decrease [31]. Then there is still the problem of how to reheat the universe however.

[3] Current planned launch date is late in 2000. Web site http://map.gsfc.nasa.gov

[4] Current planned launch date is 2007. Web site http://astro.estec.esa.nl/SA-general/Planck/

shown in the same figure as the dotted magenta line. The major difference lies in the nature of the Doppler, or acoustic, peaks at $\ell > 150$ - adiabatic inflationary models predict multiple peaks with the first at around $200/\sqrt{\Omega}$ while defect models typically predict a single peak shifted to larger ℓ (i.e. smaller angular scale).

Current and future large scale structure data is shown in figure (1.3) and plotted in figure (1.4) are the expected accuracies that are expected from the combined CMB satellites (MAP and PLANCK) and next generation galaxy surveys (2dF and SDSS).

One of the crucial parameters which has come under the spotlight is the spectral index n, defined via the power spectrum, $P_\Phi(k)$, of primordial fluctuations, the Fourier transform of the spatial two-point correlation function, defined via:

$$\langle \Phi(k)\Phi(k')\rangle = \delta^3(k-k')\frac{2\pi^2}{k^3}P_\Phi(k) \tag{1.4}$$

where we have already assumed statistical isotropy and homogeneity in dropping the vector dependence of k and Φ is a typical gauge-invariant scalar metric perturbation. For scale-invariant spectra, $P_\Phi(k) \propto k^n$ so that $d\ln P(k)/d\ln k = n$. A scale-free spectrum corresponds to $n = 1$.

1.3 Quantum field theory in General Relativity

Quantum field theory is elegantly described in the books by Birrel and Davies [23], Fulling [42] and Friedlander [43], so we will not cover such basic aspects as construction of the Hilbert space of Foch states, imposition of the commutation relations, Bogoluibov transformations, Greens functions (which may be familiar anyway from the theory of partial differential equations), time-ordering and the path-integral approach.

Neither shall we review the recent developments in non-equilibrium field theory since, while relevant to many aspects of reheating after inflation, involve extremely dry and technical details needed to develop machinery of sufficient power to handle the out of equilibrium situations and which so far have only been solved in situations of considerable simplicity relative to the full problem. A phenomenological discussion of the results of this sub-field is included in chapter (2).

Instead we will concentrate on lessons to be learned from QFT in an expanding universe: the conditions for particle production, the effects of renormalisation and regularisation in curved spacetime and an overview of de Sitter spacetime as a basis for the inflationary paradigm.

1.3.1 Archetypal examples in quantum field theory

The fundamental object in QFT is perhaps the Lagrangian $\mathcal{L}$, a scalar functional constructed from the tensor fields of the theory $\Psi^{a\dots b}{}_{c\dots d}$ their covariant derivatives and the metric g_{ab} of the background.

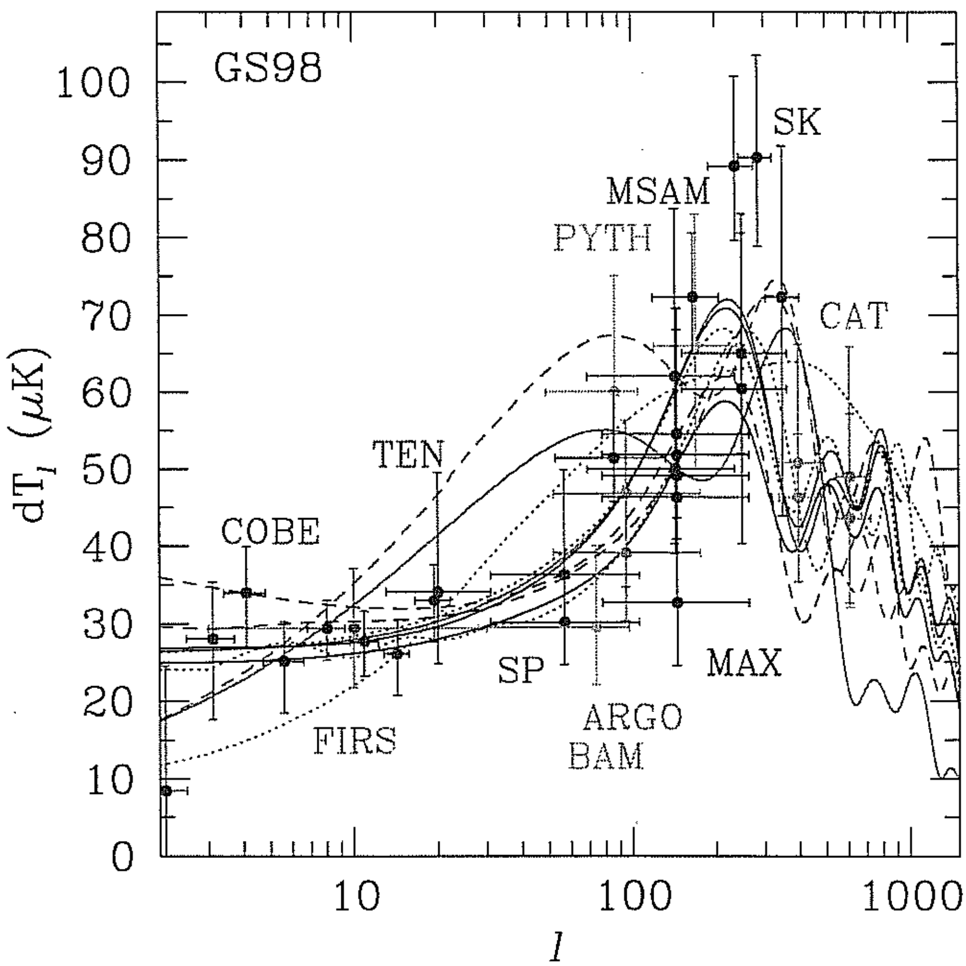


Figure 1.2: Compilation of CMB anisotropy results with horizontal error bars showing the full width at half maximum of each instrument's window function and vertical error bars showing the 68% confidence interval. The error bars include uncertainties due to instrument noise, calibration uncertainty, sample variance from observing only in one part of the sky, and cosmic variance from observing at only one location within the universe. The radiation power spectra are shown for each model with its best-fit normalization, including SCDM (solid black), TCDM (dashed black), CHDM (solid red), OCDM (dashed blue), LCDM (solid blue), PCDM (dotted black), BCDM (dotted blue), ICDM (dashed magenta), PBH BDM (solid magenta), and Strings+LCDM (dotted magenta). The ICDM, PBH BDM, and Strings+LCDM models disagree with the slope implied by COBE, SP, and BAM, which prefers the adiabatic models. Taken from [40].

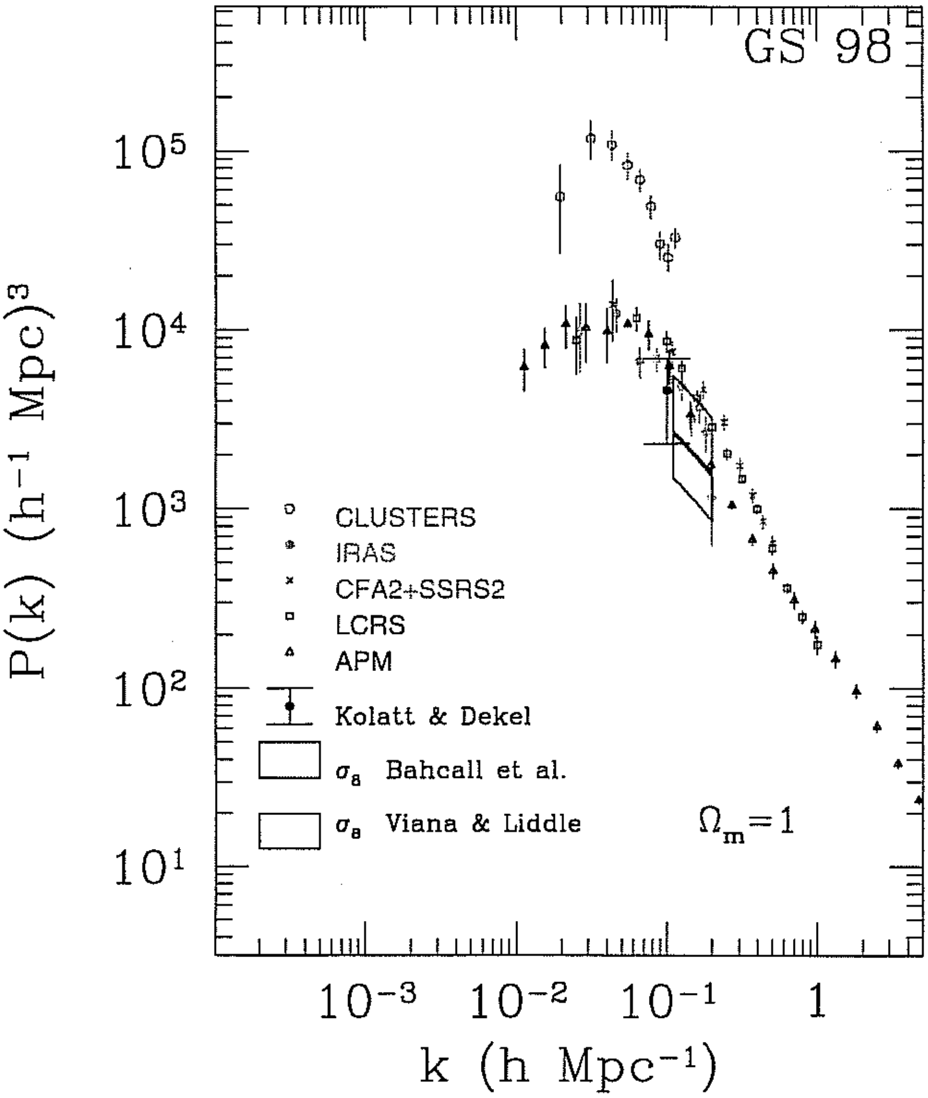


Figure 1.3: The compilation of large-scale structure observations by Gawsier and Silk [40] before making model-dependent corrections for bias, redshift distortions, and non-linear evolution. k is the wave number in comoving units of h/Mpc. The black box and black point with error bars scale with Ω_{matter}; $\Omega_{matter} = 1$ has been assumed here. The width of the σ_8 boxes represents the range of spatial scales probed by these observations, and the height shows the 68% confidence interval. From [40].

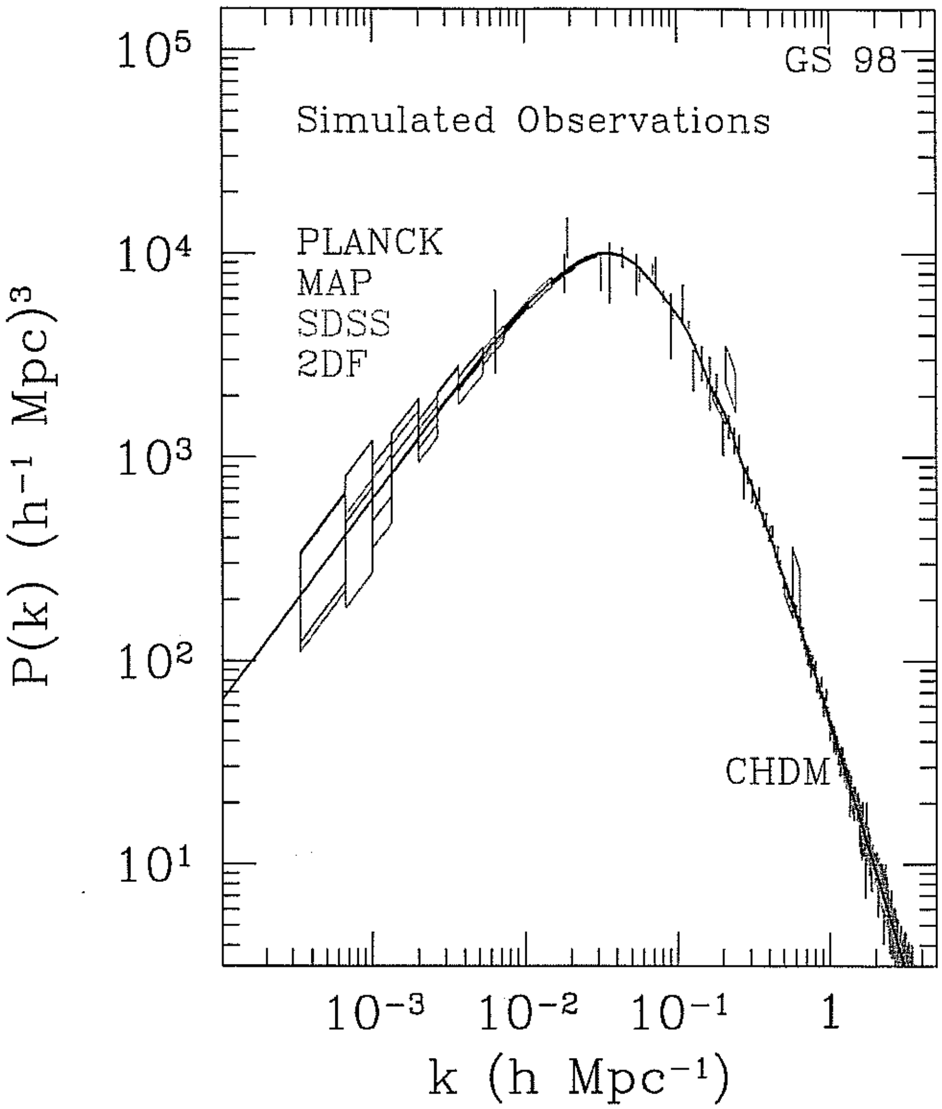


Figure 1.4: The accuracy of future CMB anisotropy observations by MAP (red boxes) and Planck (blue boxes) is simulated assuming that CHDM is the correct model . Green error bars show accuracy of the Sloan Digital Sky Survey and magenta error bars are for the 2 Degree Field Survey. This figure does not account for redshift distortions or non-linear evolution. The simulated data are indistinguishable from the underlying CHDM model for a wide range of k. From [40]

The equations of motion of the fields are then given by variations of the action to yield the Euler-Lagrange equations (e.g. [44]):

$$\frac{\partial \mathcal{L}}{\partial \Psi^{a\ldots b}{}_{c\ldots d}} - \left(\frac{\partial \mathcal{L}}{\partial \Psi^{a\ldots b}{}_{c\ldots d;e}}\right)_{;e} = 0 \quad (1.5)$$

where ; denotes covariant derivative, as per usual and these equations hold for all such fields $\Psi^{a\ldots b}{}_{c\ldots d}$ of the theory.

Example 1: the minimally coupled scalar field

By far the simplest and most relevant example for us, the Lagrangian of the minimally coupled ($\xi = 0$) scalar field with potential $V(\phi)$ is:

$$\mathcal{L} = \frac{1}{2} g^{ab} \phi_{;a} \phi_{;b} - V(\phi) \quad (1.6)$$

The equations of motion are then:

$$\phi_{;ab} g^{ab} + \frac{\partial V}{\partial \phi} = 0 \quad (1.7)$$

with the associated stress tensor:

$$T_{ab} = \phi_{;a}\phi_{;b} - g_{ab}\mathcal{L} \quad (1.8)$$

Example 2: the non-minimally coupled scalar field

The above Lagrangian was not completely general since there was no coupling to the curvature of spacetime. On dimensional grounds, such a coupling "must" be quadratic in the scalar field: [5] $-\frac{1}{2}\xi\phi^2 R$, where R is the Ricci scalar of spacetime. This leads to the modified equation of motion:

$$\phi_{;ab} g^{ab} + \frac{\partial V}{\partial \phi} + \xi R \phi = 0 \quad (1.9)$$

with stress tensor [45]:

$$T_{ab} = \phi_{;a}\phi_{;b} - g_{ab}(\tfrac{1}{2}\phi_{;m}\phi^{;m} - V) + \xi[g_{ab}(\phi^2)_{;m}{}^{;m} - (\phi^2)_{ab} - (R_{ab} - \tfrac{1}{2}g_{ab}R)\phi^2]\,. \quad (1.10)$$

R can be eliminated from Eq. (1.9) by using the trace of the unrenormalised Einstein field equations and Eq. (1.9) to get [45]:

$$R = \frac{8\pi G \phi_c^2}{\phi^2 - \phi_c^2}[(1 - 6\xi)\phi_{;m}\phi^{;m} - 4V + 6\xi\phi V']\,, \quad (1.11)$$

where $\phi_c^2 \equiv [8\pi G\xi(1-6\xi)]^{-1}$. In low curvature regions where $R \simeq 0$, such as the present day universe on average, even highly non-minimally coupled fields would be difficult to distinguish from minimally coupled scalar fields. In the early universe or near other curvature singularities, their evolution is significantly more complex than their minimally coupled counterparts and this can lead to interesting particle production effects, as we will discuss in chapter (3).

[5] Note that the choice of sign of ξ is highly inhomogeneous in the literature. e.g. Birrel and Davies choose ξ to have the same sign as the mass term so that conformal coupling corresponds to $\xi = \frac{1}{6}$, a convention which we follow.

Example 3: Maxwell electrodynamics

The electromagnetic field can be described perhaps most unambiguously by the Maxwell tensor F_{ab}. The Lagrangian is then:

$$L = -\frac{1}{16\pi} F_{ab}F^{ab} \tag{1.12}$$

where $F_{ab} = 2A_{[a;b]}$, A_b the four-potential. Varying the action w.r.t. the four potential yields the equations [44, 23]:

$$F^{ab}{}_{b} = 0 \tag{1.13}$$

which, together with the Bianchi identities $F_{[ab;c]} = 0$, fully specify the evolution of the source-free electromagnetic field.

Example 4: Charged, minimal, scalar field

One can choose to represent this as a single complex field or a pair of real scalar fields, ϕ_1 and ϕ_2. If we define $\phi = \phi_1 + i\phi_2$, then the Lagrangian of ϕ coupled to the electromagnetic field reads:

$$\mathcal{L} = -\frac{1}{2}(\phi_{;a} + ieA_a)g^{ab}(\overline{\phi}_{;b} - ieA_b) - V(\phi\overline{\phi}) - \frac{1}{16\pi}F_{ab}F^{ab} \tag{1.14}$$

where e is the gauge coupling, and $\overline{\phi}$ denotes the complex conjugate. The equations of motion are [44]:

$$\phi_{;a}{}^{;a} + V'(\phi) + ieA_a(2\phi^{;a} + ieA^a\phi) + ieA_a{}^{;a}\phi = 0 \tag{1.15}$$

and its complex conjugate, and the generalization of equation (1.13):

$$F_{ab}{}^{;b} - ie\phi(\overline{\phi}_{;a} - ieA_a\overline{\phi}) + ie\overline{\phi}(\phi_{;a} + ieA_a\phi) = 0 \tag{1.16}$$

with the corresponding stress-energy tensor:

$$T_{ab} = \phi_{(;a}\overline{\phi}_{;b)} + \frac{1}{2}(-\phi_{;a}ieA_b\overline{\phi} + \overline{\phi}_{;a}ieA_b\phi + \overline{\phi}_{;b}ieA_a\phi - \phi_{;b}ieA_a\overline{\phi}) + \frac{1}{4\pi}F_{ac}F_b{}^c + e^2A_aA_b\phi\overline{\phi} + g_{ab}\mathcal{L} \tag{1.17}$$

1.3.2 Conditions for particle creation in expanding spacetimes

To answer the question posed in this subsection: under what conditions is there particle creation in an curved, expanding spacetime, it is useful to consider what happens to curvature objects under a conformal transformation of Minkowski spacetime.

Conformal transformations preserve angles between vectors while simply scaling distances:

$$g_{ab} \rightarrow \tilde{g}_{ab} = \Omega^2(x^\mu)g_{ab} \tag{1.18}$$

where Ω is a continuous, non-zero and finite real-valued function of the spacetime coordinates. The geometric objects of the manifold are not invariant in general under (1.18) and are given, in four dimensions, by [23] (here the $\sim$ refers to the object w.r.t. the transformed metric $\tilde{g}_{ab}$):

$$\tilde{R}^a{}_b = \Omega^{-2}R^a{}_b - 2\Omega^{-1}(\Omega^{-1})_{;b}{}^{;a} + \tfrac{1}{2}\Omega^{-4}(\Omega^2)_{;c}{}^{;c}\delta^a{}_b \tag{1.19}$$

$$\tilde{R} = \Omega^{-2}R + 6\Omega^{-3}\Omega_{;a}{}^{;a} \tag{1.20}$$

$$\tilde{C}_{abcd} = C_{abcd} \tag{1.21}$$

The idea is quite simple. If we consider all metrics which are conformally flat (i.e. all $\tilde{g}_{ab}$ above such that $g_{ab} = \eta_{ab}$) then, by definition their Weyl tensor vanishes: $\tilde{C}_{abcd} = 0$. If we then consider a field (which might be any of the examples presented above) on the new manifold with metric $\tilde{g}_{ab}$, then there will be no particle production if the equation of motion of the field w.r.t $\tilde{g}_{ab}$ is the same (perhaps in new coordinates) as in Minkowski spacetime, i.e. if the equation is conformally invariant.

Two examples will suffice. The first comes from Example (2) above where we consider a massive scalar field with quartic self-interaction $\frac{\lambda}{4}\phi^4$ and non-minimal coupling $\xi = \frac{1}{6}$. The Klein Gordon equation before the conformal transformation is:

$$[\Delta + m^2 + \tfrac{1}{6}R]\phi + \lambda\phi^3 = 0 \tag{1.22}$$

while after the transformation it is (in four dimensions):

$$\begin{aligned}[\tilde{\Delta} + m^2 + \tfrac{1}{6}\tilde{R}]\tilde{\phi} + \lambda\tilde{\phi}^3 &= 0 \\ &= \Omega^{-2}[\Delta + \tfrac{1}{6}R]\tilde{\phi} + m^2\tilde{\phi} + \lambda\tilde{\phi}^3 \end{aligned} \tag{1.23}$$

where $\Delta = g^{ab}\nabla_a\nabla_b$ and $\tilde{\Delta}$ is the same object but taken w.r.t. the metric $\tilde{g}_{ab}$ and $\tilde{\phi}$ is the field transformed so as to try and insure invariance. From (1.23) we see that defining:

$$\tilde{\phi} = \Omega^{-1}\phi \tag{1.24}$$

we get:

$$\Omega^{-3}\left([\Delta + \tfrac{1}{6}R]\tilde{\phi} + \lambda\phi^3\right) + \Omega^{-1}m^2\phi \tag{1.25}$$

so that only in the massless $m = 0$, conformally coupled $\xi = \frac{1}{6}$ case, do we recover the Klein-Gordon equation (1.22) after the conformal transformation. Introducing a mass for the field generates a natural length scale for the problem and hence breaks conformal invariance. This is easily seen from the above equation (1.25), and results in particle production of massive scalar bosons in an expanding background.

The second example is Maxwell electrodynamics. Maxwell's equations are conformally invariant and hence there is no production of photons in FLRW backgrounds. Again giving the photon a mass breaks conformal invariance which leads us to the Proca equation which allows for production of particles due to the expansion (see chapter 3 for further discussion of this.)

1.4 The inflationary paradigm

1.4.1 Lackings of the standard cosmology

Here we will discuss the three main issues to which the standard FLRW cosmology can give no satisfactory response other than appealing to initial conditions. We then discuss in each case how inflation impacts on these issues and to what extent inflation can claim to solve them.

The horizon problem

Simply put, the horizon problem stems from the CMB with its near perfect isotropy (1 part in 10^3 including the dipole and 1 part in 10^5 without the dipole). Today, the horizon at the surface of last scattering subtends about 1° in the sky. This means that regions in opposite directions in the sky had never been in causal contact at a redshift of $z = 1200$, the approximate epoch for the start of photon decoupling. Of course, the intergalactic medium is reionised and the surface of last scattering may actually be much nearer in redshift, say $z \sim 50$, but the problem still remains, although it is less severe.

Inflation alleviates this problem by hugely increasing the distance between photon decoupling and the Planck epoch [46], thereby allowing much larger regions of the sky to come into causal contact. It can never, however, put the whole sky into causal contact with itself and hence there are limits to how large a temperature contrast across the sky inflation can 'deal' with. The question then becomes, "how likely is it that inflation would start with a temperature contrast small enough that it could solve the horizon problem ?"

The isotropy problem

Even if the universe had been in causal contact, there is no gaurrantee that it would have become isotropic. The set of models which isotropise permanently [6] in the set of all possible solutions is of measure zero [47]. This is easy to understand since it is the nature of gravity to cause inhomogeneity, it is the arrow of time associated with the increase of gravitational entropy. Hence the isotropy of the CMB, γ & X-ray background and galaxy distribution at large is a mystery.

While observations do not yet confirm that we are in an almost-FLRW phase, the only known alternatives are the spherically symmetric Tolman-Bondi solutions which violate the almost-Copernican principle. Either way we are left with fundamental questions without answer.

The flatness problem

Einstein sought a static universe but found that the only way to achieve this was to introduce a Λ-term. Even then however, it was apparent that this flat, static solution was unstable to perturbations

[6] Many Bianchi models go through an almost-FLRW phase at some stage but typically diverge away again after a finite amount of time.

because the density parameter $\Omega \rightarrow \infty$ or 0 with time.

The fact that our universe has lived for around $15Gyr$, i.e. 10^{60} units of the Planck time is a mystery given that the only dimensionless constant of quantum gravity yields the time scale $t_{pl} \sim 10^{-43}s$. Of course, any FLRW model with $\Omega < 1$ will have this property, but given that observations bound $\Omega_0 \in (0.1, 2)$ one can calculate that Ω at the Planck epoch would have had to satisfy [3]:

$$|\Omega - 1| \leq 10^{-59} \tag{1.26}$$

which is incredible fine-tuning w.r.t. a uniform measure. However, there are purely classical arguments that a uniform measure is not the correct one to use. Using a Bayesian interpretation of probability, the correct observational measure appears to be $\propto 1/|1-\Omega|$ [41]. Since this gives a dominant contribution around $\Omega = 1$, it is arguable that there is no flatness problem in the standard cosmology.

However, let us assume that the correct measure is the uniform measure, thereby bringing back the fine-tuning problem. Inflation alleviates the "flatness problem" of the standard cosmology by driving Ω towards unity at an exponential rate. Hence, if $\Omega < 10^{20}$ say at the Planck time, then 60 e-foldings of inflation will drive $\Omega \rightarrow 1$, thus explaining the current closeness of Ω to unity. But since we are using a uniform measure, all values of $\Omega \in [0, \infty)$ are equally likely at the Planck epoch. Therefore, the measure of the set of initial Ω which lead to models compatible with current observations after a *finite* amount of inflation, is zero, just as for the standard cosmology without inflation [49].

Clearly then, a uniform measure leaves us with no joy. However, this discussion is still salubrious since almost all discussions of the flatness problems implicitly assume a uniform measure based on "common sense" arguments. A more sophisticated approach, pioneered by Hawking and Page [50] is to try and construct a *natural* measure on the initial phase space. The obvious candidate is the Liouville measure which is volume (probability) preserving. The result of this construction, which is not without ambiguity [48], is that the flatness problem is *not* a problem, even without inflation. This is perhaps a boon since, as we will see later, new inflation cannot claim to solve the flatness problem anyway. Perhaps the only definite resolution to this issue will come when we have a full quantum gravity measure, however.

To end this discussion I think it is fair to say that a truly satisfactory resolution to the problems of the standard cosmology would start with a fully inhomogeneous and anisotropic quantum universe and show that a nearly flat (*i.e.*$|\Omega_0 - 1| < 0.8$) universe is the only attractor, e.g. a solution along the lines of Misner's chaotic cosmology programme [57]. Inflation has, with only two groups of exceptions, always started by writing down a (typically flat) FLRW line-element and working from there. The exceptions are the chaotic inflation model of Linde which still requires a nearly homogeneous background, and studies of scalar field dynamics in anisotropic or inhomogeneous backgrounds such as the Bianchi models or G_n $(n = 1, 2)$ cosmologies which find that inflation only initiates if inhomogeneity is not too large [51].

1.4.2 The monopole problem

Essentially all known GUT's predict monopoles due to the electromagnetic $U(1)$ sub-structure of the standard model. To see the reason why gives us a good reason to indulge in a little algebraic topology. Symmetry breaking implies going from a theory with a symmetry described by the group G to one with the smaller group H, $G \rightarrow H$. Topological defects in general (domain walls, cosmic strings, monopoles or textures) occur when the associated homotopy groups $\pi_n(G/H)$ [7] are non-trivial. Domain walls, strings, monopoles and textures respectively occur when $\pi_n(G/H)$, $n = 0, 1, 2, 3$ are non-trivial.

Now examine the beautiful exact homotopy sequence between different homotopy groups:

$$... \rightarrow \pi_n(G) \stackrel{\alpha_m}{\rightarrow} \pi_n(G/H) \stackrel{\alpha_{m+1}}{\rightarrow} \pi_{n-1}(H) \stackrel{\alpha_{m+2}}{\rightarrow} \pi_{n-1}(G) \rightarrow ... \tag{1.27}$$

where the homomorphisms α_i are such that:

$$\mathrm{Im}(\alpha_i) = \ker(\alpha_{i+1}) \tag{1.28}$$

where $\mathrm{Im}(\cdot)$ denotes the image of the map and $\mathrm{Ker}(\cdot)$ the kernel[8] of the mapping.

Now it is quite easy to show that if $\pi_n(G) = \mathbf{1} = \pi_{n-1}(G)$, i.e. are trivial, then the mapping α_{m+1} is both one-to-one and onto and hence is an isomorphism, so that $\rightarrow$ becomes equality above and we have:

$$\pi_n(G/H) = \pi_{n-1}(H) \tag{1.29}$$

The crucial point is that the group H must contain the $U(1)$ of electromagnetism and hence $\pi_1(H) = \mathbf{Z}$ since $\pi_1(U(1)) \simeq \mathbf{Z}$ and $U(1) \simeq \mathbf{S}^2$. Since the group G is usually taken to be simply connected and without monopoles, this implies, through Eq. (1.29), that $\pi_2(G/H) = \mathbf{Z}$, and we have shown fairly generically that GUT's predict monopoles in breaking down to the standard model.

This then is the monopole problem. Inflation might solve this if the symmetry breaking occurs at least 20 e-foldings *before* the end of inflation [129]. It solves the problems, not by bringing monopoles and anti-monopoles closer together so that they can annihilate, as in the Langacker-Pi mechanism (see section [3.6.1]) but by diluting their density to such a small amount that they no longer affect cosmic dynamics.

1.4.3 de Sitter spacetime

de Sitter spacetime provides the archetypal example of an inflationary model and illustrates beautifully some of the fundamental results common to most inflationary models. But there are some subtleties

[7] The n-th homotopy group $\pi_n(M)$ of a manifold M tells one about the "shrinkability" of closed n-dimensional loops in M. Thus e.g. $\pi_1(\mathbf{S}^1) = \mathbf{Z}$ - a closed loop around a circle cannot be shrunk to a point, in fact such loops are indexed by their winding number - a topological invariant. In fact we have $\pi_n(\mathbf{S}^n) = \mathbf{Z}$.

[8] The kernel of a mapping is the set of elements that are mapped to the identity element.

as well that should not be overlooked. Our aim is to present both of these aspects and then to move on to more current models based on scalar field dynamics.

Part of the confusion surrounding the de Sitter solution is that many similar objects are given that name. If for example, one imposes that the energy density $\mu > 0$ is a constant in space and time (equivalently that there is a cosmological constant), then the Friedmann equation:

$$H^2 + \frac{K}{a^2} = \frac{8\pi}{3}\mu \tag{1.30}$$

shows that $H^2 \to const.$ as the scale-factor expands. Indeed, the scale factor obeys [3]:

$$a(t) = H^{-1}\cosh Ht \quad K = +1 \tag{1.31}$$

$$a(t) = H^{-1}e^{Ht} \quad K = 0 \tag{1.32}$$

$$a(t) = H^{-1}\sinh Ht \quad K = -1 \tag{1.33}$$

so that asymptotically all solutions approach the $K = 0$ solution, with $H^2 = 8\pi\mu/3M_{pl}^2$. What is much less obvious is that all these solutions actually describe the same manifold in different coordinates.

The simplest interpretation found so far is to embed de Sitter, which is maximally symmetric (10 Killing vectors), into $\mathbf{R}^5$ with signature +3. It then takes the form of a hyperboloid (the symmetry group of the manifold is thus $SO(1,4)$) which can be given coordinates to make it look open, flat or closed. These different coordinate systems on the hyperboloid are crucial in defining a good vacuum [23]. Perhaps the most useful in this regard is the flat coordinate system which allows an adiabatic vacuum to be defined which is well behaved over the whole coordinate chart which, in this case, cover only half the hyperboloid.

Of course, invariants tell us about the true curvature of the manifold and we find that the Ricci scalar is:

$$R = 12H^2 \tag{1.34}$$

which is positive or negative in de Sitter and anti-de Sitter [9] respectively. de Sitter is one of the three constant curvature spacetimes - the others being Minkowski and anti-de Sitter - and has topology $S^1 \times \mathbf{R}^3$ [44]. We have thus been presented with some of the subtleties of general relativity (and have skipped over how they affect quantisation on de Sitter spacetime, see [23]). One more coordinate similarity will be useful. By changing to coordinates (t, r, θ, ϕ) defined in analogy with Schwarzschild coordinates the metric takes the interior Schwarzschild form, so that in static coordinates, de Sitter appears like the inside of a Schwarzschild black hole of radius H^{-1}.

This analogy is very useful in understanding the production of quantum fluctuations in de Sitter spacetime since with the above mapping onto Schwarzschild our problem is equivalent to that of Hawking radiation at a temperature:

$$T_H = \frac{M_{pl}^2}{8\pi M} = \frac{H}{2\pi} \tag{1.35}$$

[9] While anti-de Sitter spacetime plays an important rôle in recent advances in string theory [52] it will not be of any interest to us at this time.

where M is the mass of the black hole. The last step comes from a Wick rotation of the time which yields a Euclidean metric S^4. The finite-temperature Green's functions then become periodic (anti-periodic) for bosonic (fermionic) fields and as usual in Euclidean statistical field theory the period, $2\pi/T$, is simply inversely related to the temperature of the fields.

Naively we can understand the implications of this result for the spectrum of density perturbations. If we consider a scalar field with flat potential (equivalent to a cosmological constant) then the equation of motion for the Fourier modes, ϕ_k, of the field is just the Klein-Gordon equation:

$$\ddot{\phi}_k + 3H\dot{\phi}_k + \frac{k^2}{a^2}\phi_k = 0 \tag{1.36}$$

which has the general solution [23]:

$$\begin{aligned} \phi_k(t) &= \frac{\sqrt{\pi}}{2}H\eta^{3/2}\left[C_1(k)H^{(1)}_{3/2} + C_2(k)H^{(2)}_{3/2}\right] && (1.37)\\ H^{(2)}_{3/2}(x) &= \overline{H^{(1)}_{3/2}(x)} = -\sqrt{\frac{2}{\pi x}}e^{-ix}\left(1+\frac{1}{ix}\right) && (1.38)\end{aligned}$$

where $\eta = -H^{-1}e^{-Ht}$ is conformal time and the $H^{(i)}_{3/2}$ are Hankel functions. Clearly from Eq. (1.36) as $a(t) \to \infty$, $\phi(t)$ oscillates with longer and longer period and asymptotically is constant. By looking at the high-frequency limit (which must coincide with quantisation in Minkowski spacetime) we find $C_1 = 0, C_2 = -1 \ \forall\ k$, and one can show that [3]:

$$\phi_k \to iH/k\sqrt{2k}\,, \tag{1.39}$$

in the long-time limit.

Physically, this means that modes are excited with variances $\langle(\delta\phi)^2\rangle \propto T_H^2$ which are *independent of time*. This leads to our scale-invariant power spectrum, which we can calculate explicitly:

$$P_\phi(k) \equiv \frac{k^3}{2\pi^2}|\phi_k|^2 \simeq \frac{H^2}{4\pi^2} = T_H^2 \tag{1.40}$$

where we have used Eq. (1.4) and (1.39). Note that this is only valid asymptotically and that at small times, or for large momentum modes $k \gg 1$, the spectrum will have some extra k- and time-dependence, although it is typically weak because of the no-hair theorem. The no-hair theorem presents us with another problem: how does one exit the de Sitter state ? This is the graceful exit problem. Inflation is so powerful that it never ends and dominates the entire later history of the universe. There are two ways out of this: (i) either one asks that the de Sitter vacuum tunnels coherently to a radiation-dominated FLRW spacetime while preserving the isotropy and homogeneity produced during the de Sitter phase or (ii) we invoke scalar field dynamics with a potential which is not exactly flat, but rather allows a smooth transition to the FLRW state. This smooth transition to the FLRW state is exactly the phase known as "reheating" since the super-cooled vacuum phase is replaced with a very hot radiation FLRW model. Hence we will concentrate on option (ii) from here on.

1.4.4 Inflationary scenarios

Having realised that a cosmological constant, or equivalently a scalar field with potential $V(\phi) = \Lambda$, a constant, is not sufficient for our cosmological needs, we now find ourselves in the inflation model building department. Clearly, at the lowest level this becomes a game in which the shape of the potential is chosen in order to (a) allow the relevant cosmological equations to be solved, (e.g. power-law inflation where $V \propto e^{\kappa\phi}$ so $a(t) \propto t^p$) (b) match with potentials from particle physics (e.g. the Coleman-Weinberg potential, axion cosine potential, dual inflation [3, 137], etc..) (c) follow philosophical guidelines (e.g. chaotic inflation with quadratic or quartic potential).

What is even more disturbing perhaps is that a true theory of inflation will have to come from a post-standard model of high-energy physics and will almost certainly involve many fields, so that the inflaton effective potential will have a large number of possible degrees of freedom. This issue is particularly pertinent to our discussion of reheating in Grand Unified Theories (GUT's) as we will describe in chapter (5). Here however we will content ourselves with a brief overview of old inflation, new inflation and the chaotic inflationary scenarios.

Scenario 1: Old inflation

This is the model of inflation proposed by Guth [24]. In it the universe was in a state of restored symmetry $\phi = 0$ with positive vacuum energy density $V(0) > 0$, leading to exponential expansion which cooled the universe and reduced thermal contributions to the potential (see section 1.6) thereby creating a global minimum at some $\phi_c > 0$ with $V(\phi_c) = 0$. This super-cooled state $\phi = 0$ is unstable to fluctuations and after a certain time the field tunnels, via a first order phase transition, through the barrier to the global minimum at ϕ_c. At this time inflation ends and the universe is reheated via bubble wall collisions.

The problems with old inflation were mainly two-fold. Firstly, the bubble of true vacuum could not grow fast enough relative to the exponential background expansion to cause bubble collisions to be effective and the transition never occurred because of this. This is just another form of the graceful-exit problem and the no-hair theorem mentioned earlier. To picture it, one might think of water on the brink of going from the liquid to the gaseous phase, but that as the bubbles form and expand, the container holding the water expands exponentially (the density of water always being constant) so that more volume of water appears than of bubbles.

Secondly, even if the transition could be made successfully, collision of bubbles is a highly non-linear and violent process in which the much-prized isotropy and homogeneity of the super-cooled state would be lost. This can be understood further from the fact that each bubble would choose its four-velocity independently from all the others (since they are nucleated out of the maximally symmetric de Sitter spacetime) and hence, even neglecting bubble-wall collisions, one could not expect a homogeneous universe after merging all the bubbles.

Scenario 2: New inflation

The failings of the old inflationary scenario did not stop the model building long and soon after the new inflationary scenario was introduced [53, 55].

The idea of new inflation was to change the order of the phase transition from first to second order. Second order phase transitions are more well-behaved and less violent. To do this, the field must evolve continuously in a classically-allowed manner between $\phi = 0$ and ϕ_c, without the tunneling of a first-order transition. Many potentials allow this but the trick is to find one which will allow for sufficient inflation on the way down to the global minimum, i.e. a flat enough potential is required. This condition leads to the fine-tuning problems of new inflation but is philosophically close to the de Sitter paradigm which has an exactly flat potential.

The near-constancy of the Hubble constant on the way down leads to a nearly scale-invariant spectrum of density perturbations and the curvature of the potential at ϕ_c leads to inflaton oscillations which allow reheating to take place. The new inflationary scenario has its own share of (less disastrous) problems, however.

Firstly, the flatness of the potential requires fine-tuning. If one chooses at random a potential from the space of smooth potentials $C^{\infty}(\mathbf{R})$, what is the probability that it will have a region near the origin which is both sufficiently long and flat to give enough inflation and the right amplitude of density inhomogeneities ? Obviously the probability is very small w.r.t. any standard non-singular measure, so without motivation from particle physics, this is a severe fine-tuning problem which is, however, significantly eased with the introduction of supersymmetry.

Secondly, why should the field be in the $\phi = 0$ state ? Since the curvature of the potential is controlled by the inflaton self-coupling and its couplings to the other fields, the flatness of the potential implies very weak couplings which mean that the inflaton was almost certainly out of equilibrium at the beginning of inflation. So even though high-temperature corrections to the potential may have created a local minimum at $\phi = 0$, there is no reason to believe that the field would find itself there just after the Planck epoch t_{pl}.

Thirdly, in this scenario, the inflaton moved slowed towards $\phi = \phi_c$. However this is a classical picture and one should use the quantum viewpoint instead, in which case one is interested in $\langle\phi\rangle$. But as pointed out by Guth and Pi[56], if the potential has a discrete Z_2 symmetry (i.e. is invariant under $\phi \to -\phi$), then $\langle\phi\rangle = 0$ all the time, and so $\langle\phi\rangle$ never actually equals ϕ_c at any stage.

Finally, and perhaps most important historically [3], new inflation only begins when the temperature of the universe has dropped sufficiently: $T^4 < V(0)$ so that the vacuum energy starts to dominate the thermal energy. In the same way our universe may have a small cosmological constant which is undetectable at present but which would determine the future fate of our universe. In typical models of new inflation $V(0) \ll M_{pl}^4$ being related to e.g. the GUT phase-transition. This means that inflation only starts at $T^2 \ll M_{pl}^2$ or at a time $t \gg t_{pl}$. However, since in a naive measure, an average FLRW universe does not live past t_{pl}, or expands so rapidly that structures never form, new inflation cannot be said to solve the flatness problem because it already requires huge fine-tuning for the universe to live long enough for inflation to actually begin.

Scenario 3: Chaotic inflation

In response to these difficulties, Linde introduced the chaotic inflation scenario [3] which throws out the basic assumptions of the previous scenarios. In chaotic inflation, the initial value of ϕ is randomly distributed throughout the universe and curvature quantities have values consistent with those expected at the exit of the Planck epoch, $t = t_{pl}$:

$$(\phi_{;a}\phi^{;a}) \leq M_{pl}^4 \quad (1.41)$$

$$V(\phi) \leq M_{pl}^4 \quad (1.42)$$

$$R^2, R_{ab}R^{ab}, R_{abcd}R^{abcd} \leq M_{pl}^4 \quad (1.43)$$

For a theory with $V = \lambda\phi^4/4$, the second condition above implies with $\lambda \sim 10^{-12}$, that $\phi(t_{pl}) \sim 10^3 M_{pl}$, while the change in ϕ over the horizon scale at that time is only of order M_{pl} from Eq. (1.41). Linde [3] then discusses the possible onset of inflation in various special cases and argues that the probability of a region exiting the Planck epoch with 'small enough inhomogeneity' for inflation to occur, is not very small. At the end of the day it requires a weak form of the anthropic principle, not to mention ignoring the problems associated with a natural measure on the space of solutions at t_{pl} [48].

What casts further doubt on this scenario is that studies of inhomogeneous inflation find that quite strong constraints on the inhomogeneity or anisotropy of the spacetime must be imposed for successful inflation [51]. This should also be viewed in the context of the generic singularity question of general relativity. Latest evidence supports the mixmaster singularity as the most generic [58] and it seems highly doubtful that inflation would be generic at t_{pl} after one of those singularities due to the large shear and tidal terms.

However, there are also asymptotically velocity dominated singularities which might turn out to be the generic *classical* singularity [59]. Asymptotically velocity term dominated (AVTD) cosmologies are ones where the spatial gradients can be ignored relative to velocity and acceleration terms. If AVTD singularities are generic there would be much in favour of chaotic inflation since it has been shown [3] that if the energy in spatial gradients can be ignored relative to the kinetic energy - even if this completely dominates the potential energy - then assuming that the universe lives long enough, an inflationary stage will ensue.

This can be understood as follows: since $\dot{\phi}^2 \gg V(\phi)$, the equation of state of the field is $p \sim \rho$, i.e. that of stiff matter. If the potential is flat in the neibourhood of the field the potential energy drops very slowly (roughly logarithmically [3]) compared with the kinetic energy which typically falls off as a power-law like t^{-2}. Hence a region with $V > \dot{\phi}^2$ occurs at some time. Again the problem is that the universe is not allowed to recollapse before becoming potential dominated if inflation is to work. Hence again some fine-tuning is required.

Of course, an analysis based on classical general relativity can only be taken as a warning and nothing else. Two other arguments are in favour of chaotic inflation being successful. Penrose's Weyl curvature hypothesis states that the lowest gravitational entropy state should correspond to an initial singularity with zero Weyl tensor, i.e. a conformally flat metric. Since the Weyl tensor carries tidal forces, $C_{abcd} = 0$ places strict controls on the shear, which might be sufficient to allow inflation generically (the vorticity is zero by construction for a scalar field). Finally one might consider a spacetime-foam type model. If spacetime is discrete at the Planck scale, as seems plausible in reconciling gravities bad ultra-violet behaviour, it may be natural that there are no anisotropic or inhomogeneous degrees of freedom at scales smaller than M_{pl}^{-1}, so that spacetime is constant curvature at those scales. In this case the spatial inhomogeneities are zero and inflation is essentially gaurranteed. Clearly, however, this must wait for a full theory of quantum gravity for verification.

1.4.5 Inflation, slow-roll and the CMB

Since the CMB is so crucial in modern models of inflation, we will briefly discuss how to translate parameters of the effective potential into those of the CMB statistics for some simple, single field, models.

Perhaps the two most fundamental parameters in current large-angle CMB data are the normalisation and the spectral index, n. The spectral index determines the slope of the power spectrum on large scales, with $n = 1$ corresponding to the canonical scale-free Harrison-Zel'dovich spectrum. COBE yielded the constraint $n = 1.1 \pm 0.3$. In many models of inflation which have a long slow-roll phase, the value of n is almost independent of wavenumber k and is given by [35]:

$$n = 1 - 6\epsilon + 2\eta \tag{1.44}$$

where the so-called slow-roll parameters ϵ and η are defined in terms of the curvature of the potential, V, and are both zero for a cosmological constant (i.e. for de Sitter spacetime). They thus control by how much the Hubble 'constant' varies with time during inflation and are defined by:

$$\epsilon \equiv \frac{1}{2}M_{pl}^2\left(\frac{V'}{V}\right)^2 , \quad \eta \equiv M_{pl}^2\frac{V''}{V} \tag{1.45}$$

where here $()' \equiv d/d\phi$. The slow-roll approximation during inflation is to ignore the $\ddot{\phi}$ term in the Klein-Gordon equation for ϕ, Eq. (1.36), yielding:

$$\dot{\phi} \simeq -\frac{V'}{3H} \tag{1.46}$$

Successful inflation requires $\epsilon, |\eta| \ll 1$ and, in fact, that $V^{(n)}/V \ll 1$ [35], where $V^{(n)}$ denotes n-th derivative of V. To check the approximation, one can show [35]:

$$\frac{\ddot{\phi}}{H\dot{\phi}} = \epsilon - \eta \tag{1.47}$$

so that the scheme is internally consistent at least.

By accurate determination of the slow-roll parameters ϵ and η, one might hope to reconstruct the inflationary potential V. Indeed this is an active area of research [54] but one which will suffer from both systematic and instrumental errors. But in principle at least, one could place restrictions on the inflaton potential, and hence on particle physics at the GUT and higher scales.

A related issue is the gravitational wave contribution to the microwave background. Tensor perturbations will cause anisotropies in the CMB via the Sachs-Wolfe effect just like scalar perturbations. For any potential V, define r to be the ratio $C_T^{(2)}/C_S^{(2)}$ of the tensor quadrupole to the scalar quadrupole of the CMB temperature anisotropies. This is related to the deviation of the tensor spectral, n_t, from zero via [54]:

$$n_t = -\frac{r}{7} \tag{1.48}$$

If this relation was not respected in the CMB, this would be strong evidence against inflation. The problem is that typical potentials give very small gravitational wave signals and so extracting their signal from future CMB data will be extremely difficult and noisy. Nevertheless, the test remains powerful in principle.

The other COBE success was to fix the normalisation of the anisotropy spectrum, a result we have already used in Eq. (1.1). The error bars on the normalisation have two origins: one is due to instrument resolution limitations (COBE had an angular resolution around 10° [30]), the other is due to *cosmic variance*.

Simply put, cosmic variance is the unavoidable error that comes from only having one universe and one CMB to observe. Hence, statements about the parent statistical distribution cannot be trusted at any high level [10] At a slightly more precise level we see some added structure to the cosmic variance problem related to ergodicity. For an ergodic[11] random field, ensemble averaging and spatial averaging give the same results in the large volume limit.

A statistically isotropic and homogeneous Gaussian random field (the model for inflationary fluctuations) is ergodic. Thus assuming ergodicity of the fluctuation spectrum is very closely related, and perhaps one of the better formulations of, the Almost-Copernican principle [62]. What does this have to do with cosmic variance ? They are related by the fact that even if the spatial distribution of inhomogeneities is ergodic (e.g. in a spatially infinite $K = 0$ FLRW model), the temperature anisotropies in the CMB form part of a compact sub-manifold - our past light cone. Indeed, for all purposes it is

[10] This is why COBE places only weak limits on possible non-Gaussian signals, though see also [61].

[11] A random field can be thought of as a dynamical flow on a manifold $\mathcal{M}$. Hence ergodicity is inherited from the flow and implies coming arbitrarily close to all points of $\mathcal{M}$ in the large time limit.

just S^2, the celestial sphere. Hence it is impossible to take the large volume limit in the averaging and hence we can never get the same results that ensemble averaging would provide, no matter what the resolution of our instruments. Fortunately, cosmic variance drops off rapidly with ℓ so that only the lower multipoles, probed by COBE, are significantly affected.

Nevertheless, COBE was able to fix the normalisation at the level of around 10%, which gives the constraint (assuming no gravitational waves contributed to the CMB and $\Omega = 1$) [35]:

$$\frac{V^{1/4}}{\epsilon^{1/4}} = 6.7 \times 10^{16} GeV \tag{1.49}$$

where the numerical factor on the right is remarkably close to the best-guess GUT scale. Note however, that if we want to invoke chaotic inflation with $V \sim M_{pl}^4$, then this requires $\epsilon \sim 10^{+12}$! But slow-roll requires that $\epsilon \ll 1 \Rightarrow V^{1/4} < 10^{16} GeV$. Thus it appears that within the approximations made to find Eq. (1.49), that chaotic inflation is not compatible with COBE. Conversely, from the last problem discussed in the new inflationary scenario, these slow-roll calculations seem to imply inflation occurring at a scale much below the Planck scale. Hence we are left again with the flatness problem of the universe which realistically this model of (new) inflation cannot claim to solve [3].

1.5 More curved space quantum field theory

We now return to the foundations of inflation and indeed reheating. We will review some important aspects of renormalisation, regularisation, anomalies, corrections to the effective potential and will finish with a realistic supersymmetric model of inflation. By looking at the decay of the inflaton we will be able to constrain the masses of the heaviest neutrinos by constraining the reheating temperature to be below $10^{10} GeV$ required so as not to affect standard nucleosynthesis.

1.5.1 Renormalisation and regularisation

The divergence of the zero-point energy of the harmonic oscillator in Minkowski space is perhaps the most famous example of the bad ultra-violet (UV) behaviour of quantum-field theory which appears in all quantities such as $\langle 0|\phi^2|0\rangle$ and $\langle 0|T^{ab}|0\rangle$. These divergence can be removed simply by normal ordering [23].

By way of contrast, the same problem in a curved, expanding background such as a FLRW universe, is much more complex. For example, even in the simplest case of a massless, minimally coupled scalar field in a $K = 0$ FLRW background, the value of $\langle 0|T_{00}|0\rangle$, the expectation value of the energy density, contains extra quadratic and logarithmically divergent terms not found in the Minkowski expression.

The process by which these new infinities are removed is the subject of regularisation, and there exist several different regularisation methods, which we will not discuss in detail, but refer the reader to the literature for in-depth analyses [23, 42]. Instead we will give the physical results of regularisation.

Our starting point are the semi-classical Einstein field equations:

$$\begin{aligned} G_{ab} + \Lambda g_{ab} &= 8\pi G \langle T_{ab} \rangle \\ G_{ab} &\equiv R_{ab} - \tfrac{1}{2} R g_{ab} \end{aligned} \tag{1.50}$$

which axiomatically implies that we are treating spacetime geometry classically and the stress-tensor quantum mechanically. We have already expressed doubts about the range of validity of this scheme, but even at a practical level we are faced with the problem of computing $\langle T_{ab}\rangle$ which should be derived from the matter part of the action, just as in the classical field theory case. Thus we seek a functional W which, in analogy with the classical theory, gives $\langle T_{ab}\rangle$ via[23]:

$$\frac{2}{\sqrt{-g}} \frac{\delta W}{\delta g^{ab}} = \langle T_{ab} \rangle \tag{1.51}$$

where δ denotes functional differentiation. One can show that W is simply related to the Feynman propagator, $G_F(x, x')$, (the expectation value of the time-ordered product of the field), via [23]:

$$W = -\tfrac{i}{2} Tr[\ln(-G_F)] \tag{1.52}$$

$$\equiv -\tfrac{i}{2} \int d^4x \sqrt{-g} \langle x | \ln(-G_F) | x \rangle \tag{1.53}$$

$$= \tfrac{i}{2} \int_{m^2}^{\infty} dm^2 \int d^4x \sqrt{-g}\, G_F^{DS}(x, x) \tag{1.54}$$

where m^2 is the mass-squared of the particle, and $G_F^{DS}(x, x)$ is the DeWitt-Schwinger representation of G_F [23]. Notice that we have taken the limit $x \to x'$ in the integral and so the expression is completely local. The advantage of these manipulations is that the final spacetime integral above is just the expression for the one-loop Feynman diagram contribution to the vacuum energy density. Thus W is called the *one-loop effective action* and leads naturally to the effective Lagrangian, L_{eff}:

$$L_{eff} = \tfrac{i}{2} \lim_{x \to x'} \int_{m^2}^{\infty} dm^2 G_F^{DS}(x, x') \tag{1.55}$$

While this is all very neat, it is easy to show [23], that L_{eff} diverges as we take the limit $x \to x'$. In four dimensions the divergent terms in the DeWitt-Schwinger expansion of G_F are given by:

$$L_{div} \propto \lim_{x \to x'} \int_0^{\infty} \frac{ds}{s^3} e^{-i(m^2 s - \sigma/2s)} [a_0(x, x') + a_1(x, x') is + a_2(x, x')(is)^2] \tag{1.56}$$

where, in the limit $x \to x'$, the functions a_i above are given by (for a scalar field with non-minimal coupling ξ) :

$$a_0(x) = 1 \tag{1.57}$$

$$a_1(x) = (\tfrac{1}{6} - \xi) R \tag{1.58}$$

$$a_2(x) = \tfrac{1}{180} K^2 - \tfrac{1}{180} R_{ab} R^{ab} - \tfrac{1}{6}(\tfrac{1}{5} - \xi) \Delta R + \tfrac{1}{2}(\tfrac{1}{6} - \xi)^2 R^2 \tag{1.59}$$

where $K^2 \equiv R_{abcd}R^{abcd}$ is the Kretchmann scalar and, as before, Δ represents the covariant Laplacian. The crucial point for us is that all the terms a_0, a_1, a_2 are local geometric functions independent of the field itself. Since L_{eff} is only the matter part of the total Lagrangian, we can cancel L_{div} by adding suitable counterterms based on the a_i into the gravitational part of the action. This means that we no-longer have the Einstein-Hilbert action for gravity, since $a_2(x)$ involves fourth-order derivatives of the metric, but it means that we can remove the divergences.

Here we will discuss and use dimensional regularisation. Other methods such as zeta-function and point-splitting subtraction schemes give the same results [23]. The idea is that one allows the number of spacetime dimensions, n, to be a free parameter. In that case, we get a slight generalisation of Eq. (1.56) which has the first $n/2+1$ terms divergent as $\sigma \to 0$. The trick is then to analytically continue n throughout the complex plane, allowing the $x \to x'$ limit to be taken. This yields a series involving Γ functions, which can then be expanded around the point $n = 4$. In dimensions different from 4, L_{eff} has units other than $(\text{length})^{-4}$ unless one introduces an arbitrary mass scale μ which is then added appropriately into L_{eff}. This leads to renormalisation of the mass and other parameters of the theory, as we will see in chapter (3).

Let us now consider 4-dimensional FLRW models with flat spatial sections. The metric is then conformally flat:

$$ds^2 = a^2(\eta)ds^2_{mink} \tag{1.60}$$

where a is the scale factor of spacetime and $d\eta = dt/a(t)$ is conformal time (here t is proper time). The time evolution of the single degree of freedom $a(t)$, is governed by the "new" Einstein field equations:

$$G_{ab} + \alpha H^{(1)}_{ab} + \beta H^{(2)}_{ab} + \Lambda g_{ab} = -8\pi\langle T_{ab}\rangle \ , \tag{1.61}$$

where the extra terms $H^{(i)}$ come from the a_i of (1.59). The idea is that by correctly choosing the form of the $H^{(i)}$, they will cancel the divergent terms coming from L_{div} leaving only finite terms in the action. To do this we require [23]:

$$H^{(1)}_{ab} \equiv \frac{1}{(\sqrt{-g})}\frac{\delta}{\delta g^{ab}}\int(\sqrt{-g})R^2 d^4x \tag{1.62}$$

$$= 2R_{;ab} - 2g_{ab}\Delta R - \tfrac{1}{2}g_{ab}R^2 + 2RR_{ab} \tag{1.63}$$

and

$$H^{(2)}_{ab} \equiv \frac{1}{(\sqrt{-g})}\frac{\delta}{\delta g^{ab}}\int(\sqrt{-g})R^{ab}R_{ab}d^4x \tag{1.64}$$

$$= 2R_{;ab} - \tfrac{1}{2}g_{ab}\Delta R - \Delta R_{ab} - \tfrac{1}{2}g_{ab}R^{ab}R_{ab} + 2R^{cd}R_{cdab} \tag{1.65}$$

where:

$$\Delta \equiv g^{ab}\nabla_a\nabla_b = \sqrt{-g}\partial_a[\sqrt{-g}g^{ab}\partial_b] \tag{1.66}$$

is the covariant D'Alembertian. Now in $n \neq 4$ there is an additional term similar to the $H^{(1,2)}$ above involving the full Riemann tensor, but in four dimensions, it can be written in terms of the Euler characteristic via the generalised Gauss-Bonnet theorem which states that the following is a topological invariant:

$$\int \sqrt{-g}d^4x(R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2) \tag{1.67}$$

so that its variation with respect to the metric vanishes. The extra term then reduces to a linear combination of $H^{(1)}$ and $H^{(2)}$ in four dimensions and we do not include it. It is included in the arbitrary α and β of Eq. (1.61) already, which are to be determined experimentally (and which may be zero).

Further, if space-time is conformally flat, the $H^{(i)}$ terms are related by

$$H^{(2)}_{ab} = \frac{1}{3}\, H^{(1)}_{ab} \tag{1.68}$$

so that one can put $\beta = 0$ without loss of generality in, for example, FLRW backgrounds.

This discussion shows how consideration of semi-classical General Relativity naturally leads one to the study of higher order Lagrangians, even in a FLRW background. Similarly, and as we remarked earlier, the study of renormalisation and regularisation of General Relativity in topologically non-trivial spacetimes (which yields the topological Casimir effect [28, 29]), leads one to field equations which are sensitive to the topology of spacetime.

1.5.2 The conformal anomaly

One of the more interesting effects in quantum field theory is the anomaly. Simply put, an anomaly occurs when a classical symmetry (and hence conservation law) is violated at the quantum level. A famous example is the axial vector anomaly, discovered by Adler in QED. Our discussion of dimensional regularisation is particularly suited to studying the so-called conformal anomaly.

Now theories which are invariant under conformal transformations of the metric, Eq. (1.18) are particularly interesting. General relativity in 2 dimensions is a conformally invariant theory. Indeed, the general set of conformal field theories is most instructive in 2 dimensions since the algebra corresponding to the group of conformal transformations is infinite, unlike in other dimensions. This allows a huge amount of powerful, complex analysis and geometry to be brought to bear on the problem and many beautiful results have been obtained.

Now in general, by the definition of functional derivatives, the actions S, before and after the conformal transformation satisfy:

$$S[\bar{g}_{ab}] = S[g_{ab}] + \int \frac{\delta S[\bar{g}_{ab}]}{\delta \bar{g}^{cd}} \delta \bar{g}^{cd} d^n x \tag{1.69}$$

From Eq.(1.18) it follows that $\delta\bar{g}_{ab} = -2\bar{g}_{ab}\Omega^{-1}\delta\Omega(x)$ and by definition of the classical stress-energy tensor:

$$\frac{2}{\sqrt{-g}} \frac{\delta S[\bar{g}_{ab}]}{\delta \bar{g}_{ab}} = T_{ab}[\bar{g}_{ab}] \tag{1.70}$$

one finds that:

$$\frac{\delta S[\bar{g}_{ab}]}{\delta \bar{g}^{cd}} \delta \bar{g}^{cd} = -\sqrt{-g} T_a{}^a \Omega^{-1} \delta\Omega d^n x \tag{1.71}$$

so that if the theory is conformally invariant, $S[\tilde{g}_{ab}] = S[g_{ab}]$, it automatically follows that the trace of the classical stress tensor vanishes, as actually occurs for massless spin 1/2 fermions, the conformally-coupled, massless, scalar field and photons (in $n = 4$). The introduction of masses for fields will, as mentioned before, introduce a natural length scale and hence breaks conformal invariance. We are therefore interested in regularisation and renormalisation in the massless case which is significantly more subtle since all higher order terms in the expansion of Eq. (1.56) are infrared divergent in the massless limit, $m \to 0$.

Returning to the affective action W, Eq. (1.54), one may isolate the ultraviolet divergent piece, W_{div}, which *in four dimensions* may be written as a linear combination of a spacetime integral over $C_{abcd}C^{abcd}$ and the Euler characteristic, Eq. (1.67). Both of these are invariant under conformal transformations and hence W_{div} in four dimensions is *invariant* under conformal transformations. So one might suspect that quantum trace will vanish by the argument implicit in Eq. (1.71); i.e. that there would be no trace anomaly in four dimensions.

However, since we are using dimensional regularisation, as before we cannot set $n = 4$ until the very end. If we expand $\langle T_a{}^a \rangle_{div}$ in powers of $(n-4)$ we find that it *does not vanish* even at zero order ! This is understood from the fact that for $n \neq 4$, W_{div} is not conformally invariant and hence the price of regularising the stress tensor is that we pick up a conformal-breaking residue at the point $n = 4$ due to our analytical continuation to $n \neq 4$. It appears that the trace has a memory !

In general, the full expression for the renormalised trace is [23]:

$$\langle T_a{}^a \rangle_{ren} = -\frac{1}{2880\pi^2}[C_{abcd}C^{abcd} + R_{ab}R^{ab} - \tfrac{1}{3}R^2 - \Delta R] \tag{1.72}$$

which is particularly simple in FLRW spacetimes where $C_{abcd} = 0$ and R_{ab} is determined by R. We turn now to a less pernicious aspect of curved space quantum field theory, but one which of greater importance in model building.

1.6 Corrections to the effective potential

One of the most pervasive problems in inflationary cosmology comes under the title of this section heading: corrections to the effective potential. The very existence of inflation depends on having a potential whose curvature is small, so that slow-roll (see section 1.4.5) can occur. Secondly, the curvature controls the amplitude of density fluctuations and hence the size of CMB temperature anisotropies and finally also controls the scale-invariance, or rather breaking of scale-invariance, in the spectrum of inhomogeneities.

The curvature of the inflaton potential therefore holds the keys both to the observational consistency of inflation and specific signatures that might allow the distinguishing of one inflationary model from another.

Quantum corrections to the potential alter the curvature of the potential in a dynamic and often disastrous way. The basic problem is simple. Consider that one has a classical (tree-level) potential $V^{(0)}$ which has the desired property that its curvature is very small in some direction so that it predicts enough e-foldings of inflation and the right spectrum of density perturbations. Unfortunately, the effective potential will receive quantum corrections. In the language of Feynman diagrams these corrections will come from all one-particle irreducible vacuum diagrams (i.e. those diagrams which are still topologically connected after a single line is cut).

Thus it is conceivable (and in fact occurs often) that a classically 'nice' potential is spoilt by quantum corrections which lead to overproduction of anisotropies in the CMB. Here we give some field theoretic background to this problem by way of illustrative examples.

Example (1): Scalar field with broken symmetry potential

Let us consider the lagrangian:

$$\mathcal{L} = \frac{1}{2}(\phi_{;a}\phi^{;a}) + \frac{\mu^2}{2} - \frac{\lambda}{4}\phi^4 \tag{1.73}$$

which has as classical (tree-level) potential:

$$V_0 = \frac{1}{2}(\nabla\phi)^2 - \frac{\mu^2}{2} + \frac{\lambda}{4}\phi^4 \tag{1.74}$$

where ∇ denotes the usual 3-dimensional gradient. This potential has a local maximum at the origin $\phi = 0$ and a global minimum at $\phi_c = \pm\mu/\sqrt{\lambda}$. For this potential, expansion in the number of Feynman loops is equivalent to a perturbative expansion in λ [3]. In the one-loop approximation, the equilibrium correction to the effective potential is [3]:

$$V(\phi) = V_0 + \frac{1}{(2\pi)^3)}\int d^3k\sqrt{k^2 + m^2(\phi)} \tag{1.75}$$

where the effective mass^2 of ϕ is:

$$m^2(\phi) = 3\lambda\phi^2 - \mu^2\,. \tag{1.76}$$

Physically, the one-loop correction to the (renormalised) effective potential is given by adding a ϕ-dependent vacuum energy due to quantum fluctuations of ϕ. To simplify Eq. (1.75) we need to impose normalisation conditions on the potential (arising since counterterms have to be added to the bare Lagrangian for regularisation purposes, as we discussed in the previous section). If we choose them to be that the new potential has the global minimum at the same point, ϕ_c, and has the same curvature, V'', at ϕ_c as in the tree-level potential V_0, then we are lead to [3]:

$$V(\phi) = \left(\frac{21\lambda\mu^2}{64\pi^2} - \frac{\mu^2}{2}\right)\phi^2 + \left(\frac{\lambda}{4} - \frac{27\lambda^2}{128\pi^2}\right)\phi^4 + \frac{(3\lambda\phi^2 - \mu^2)^2}{64\pi^2}\ln\left(\frac{3\lambda\phi^2 - \mu^2}{2\mu^2}\right) \tag{1.77}$$

so that the coefficients of the quadratic and quartic terms are altered. Thus we need to impose that both μ^2 and λ are $\ll 1$ if the quantum corrections are not to spoil the shape of the potential. Since CMB anisotropies require this in this model, there is no problem since as long as $\phi \not\gg \mu^2$, the corrections are small. This happy state is not true in our next, and slightly more realistic, model.

Example (2): The (Abelian) Higgs model

Consider an Abelian vector field A_b (it might be the vector potential of electromagnetism) interacting with the complex scalar ϕ (c.f. Eq. 1.14). The Lagrangian is given by [3]:

$$\mathcal{L} = -\frac{1}{2}(A_{[a;b]})^2 + (\nabla_a + ieA_b)\overline{\phi}(\nabla_a + ieA_b)\phi + \mu^2\phi\overline{\phi} - \lambda(\phi\overline{\phi})^2 . \qquad (1.78)$$

This is the model Lagrangian for describing the Higgs mechanism: the spontaneous acquiring of mass by the vector particles, with $m_A = e\phi_c$, where $\phi_c = \mu/\sqrt{\lambda}$ again describes the global minimum of the field ϕ.

In the case where $e^2 \ll \lambda$ the quantum fluctuations of the vector particles are irrelevant, but since we are interested in the case where $\lambda \sim 10^{-14}$ due to the constraints from CMB temperature anisotropies, this would mean fine-tuning not only the scalar field self-interaction but also the gauge-couplings e^2.

Instead it is much more natural that $e^2 \gg \lambda$ (note that this doesn't mean that $e^2 > 1$; so perturbation theory can still be valid). In that case the scalar fluctuations are irrelevant and the vector quantum corrections give rise to the new potential [3]:

$$V(\phi) = \frac{\mu^2\phi^2}{2}\left(1 - \frac{3e^4}{16\pi^2\lambda}\right) + \frac{\lambda\phi^4}{4}\left(1 - \frac{9e^4}{32\pi^2\lambda}\right) + \frac{3e^4\phi^4}{64\pi^2}\ln\left(\frac{\lambda\phi^2}{\mu^2}\right) . \qquad (1.79)$$

which for $\lambda < 3e^4/32\pi^2$ has its global minimum at $\phi = 0$, so that the vector bosons become massless again. More important from an inflationary point of view, however, is that even if we fine-tune the tree-level potential so that it is very flat in the ϕ direction corresponding to inflation, quantum corrections will completely spoil the flatness at one-loop level (and in principle we should include all orders of loops !).

Thus we immediately see that (non-supersymmetric) inflation, which was introduced to solve severe fine-tuning problems in the standard model (which we discussed earlier in this chapter), not only involves its own fine-tuning (of the tree-level potential) but involves problems more fundamental and subtle than fine-tuning, such as forbidding the inflaton to couple more strongly to other fields than it does to itself.

Example (3): The general Coleman-Weinberg formula

Assuming that there is local thermodynamic equilibrium so that a temperature T, can be defined, the general form of the effective potential at high temperature was given (assuming weak couplings so that perturbation theory is valid) by Weinberg:

$$\Delta V(T) = \frac{T^2}{24}\left[\left(\frac{\partial V}{\partial\phi_i\partial\phi_j}\right) + 3(T_aT_a)_{ij}\phi^i\phi^j\right] \qquad (1.80)$$

where T_a are the gauge group generators and the ϕ_i are the real components of the fields. For $SU(5)$ for example, these corrections turn out to change the effective mass of the Higgs fields in a way that depends non-trivially on the self-, and mutual-, couplings of the Higgs fields. In fact, the possibility has been investigated [63] that the effective masses remain negative for all T - i.e. that the universe was never in a unbroken phase and hence that there was never symmetry breaking. In this case no monopoles would be formed.

Example (4): Gravitational corrections

Finally we arrive at the quantum gravity border - everything couples to gravity and hence there are always gravitational corrections to the potential even if the inflaton coupled to no other fields. Indeed, we will see later that this is a major headache in supergravity inspired models of inflation, the so-called η-problem of supergravity.

The gravitational corrections to the potential arise due to exchange of gravitons between vacuum fluctuations. Clearly this is expected to be sub-dominant except in extreme conditions. The quantum corrections to the potential are given by [3]:

$$\Delta V = C_1 \frac{d^2V(\phi)}{d\phi^2}\frac{V(\phi)}{M_{pl}^2}\ln\frac{\Lambda^2}{M_{pl}^2} + C_2\frac{V^2(\phi)}{M_{pl}^4}\ln\frac{\Lambda^2}{M_{pl}^2} \tag{1.81}$$

where the C_i are order unity and Λ is the UV cutoff scale. Note that the V^2 term will generally introduce non-renormalisable corrections (i.e. terms with powers higher than ϕ^4). But this is familiar to all attempts to renormalise gravity. If we argue that $\Lambda \sim M_{pl}$, then these quantum gravitational corrections to the potential will be order one or larger unless $V \leq M_{pl}^2$. Unfortunately we said in chapter (1) that the natural initial conditions for chaotic inflation were included $V \sim M_{pl}^4$! In this case quantum gravity corrections to the effective potential are expected to be very large.

Example (5): Supersymmetry and the non-renormalisation theorems

One of the beauties of supersymmetry is that it automatically requires the cosmological constant to be exactly zero. This occurs since it is a theory uniting fermions and bosons with a symmetry which makes them indistinguishable. For this to happen, it is easy to show that the true fermion groundstate must coincide with that of the bosons and that it must be at zero energy [35]. So let us imagine that our tree-level potential is supersymmetric and so has zero energy. What will happen when we include 1-loop, 2-loop etc... corrections ? Obviously if we have an exactly supersymmetric theory then the radiative corrections cannot change the zero point of the potential. In fact, all radiative corrections to an exactly supersymmetric theory turn out to be zero !

How can we understand this ? Well, because there are supersymmetric partners for all the known particles, the number of bosons equals the number of fermions. Because of this, when one calculates the Feynman diagrams giving the radiative corrections to the potential one finds that since the fermionic and bosonic degrees of freedom contribute with opposite sign and the number of fermionic fields equals the number of bosonic fields, the corrections cancel and give zero at each order in a supersymmetric theory.

Because of these non-renormalisation theorems it is gaurranteed that if one fine-tunes a parameter of the potential to be very small at tree-level then this small value is protected from getting any quantum corrections. Obviously this is a very desirable feature in inflation since we typically require very small coupling constants to ensure the potential is sufficiently flat. Indeed because of this, non-supersymmetric inflationary models looks significantly less natural than their supersymmetric counterparts.

1.7 A "realistic" inflationary model from Supersymmetry

Here we end this introductory chapter by discussing a model which contains many of the features expected of a complete theory of inflation. It is a theory based on supersymmetry which is experimentally unverified although it predicts a neat unification of the strong and electroweak coupling constants at around $M_{GUT} \sim 10^{16} GeV$ which does not occur in the known non-supersymmetric extensions of the standard model.

In the spirit of Dirac's large number hypothesis, it is then tempting to try to use the small dimensionless number $M_{GUT}/M_{pl} \sim 10^{-3}$ to explain cosmological data. Notably one might try and link this to the amplitude of the primordial density perturbation spectrum. Indeed, this appears rather naturally in the following scenario due to Lazarides, Schaefer and Shafi [65]. A nice review of this area is given in the work of Lyth and Riotto [35]. Cosmic strings are rather typically in these models and contribute both to structure formation and CMB anisotropies [129].

1.7.1 The superpotential

The idea behind supersymmetric inflation is very beautiful: when supersymmetry is unbroken the vacuum energy density is zero (the cosmological constant is forced to be zero) and there is no inflation. However, with chaotic initial conditions, a large inflaton value leads to a minimum of the potential which breaks supersymmetry but which causes all fields coupled to the inflaton to relax to zero. As these fields relax to zero, the curvature of the potential tends to zero and the potential of the inflaton becomes dominated by the constant term which is determined by the symmetry-breaking scale of the model. This leads to inflation and an almost constant Hubble constant (and therefore near scale-invariance of the density perturbation spectrum). However, since supersymmetry is broken, the potential will receive (typically small) radiative corrections which will give the potential a small amount of curvature and allow the inflaton to slow-roll to the global minimum of the theory and for reheating to occur. We will now present a specific model of this type.

Supersymmetric potentials V, are derived from the so-called superpotential, denoted W, by the following formula:

$$V = \sum_i \left| \frac{\partial W}{\partial \phi_i} \right|^2 + V_D \,, \tag{1.82}$$

where V_D are the D-terms which lead to spontaneous breaking of supersymmetry. The simplest renormalisable superpotential which leads to hybrid inflation is given by [12]:

$$W = \alpha\phi \; \sigma \; \overline{\sigma} - \mu^2 \; \phi \tag{1.83}$$

which via Eq. (1.82) yields the potential:

$$V(\phi, \sigma, \overline{\sigma}) = \alpha^2 |\phi|^2 (|\sigma|^2 + |\overline{\sigma}|^2) + |\alpha\sigma\overline{\sigma} - \mu^2|^2 + D - \text{terms}\,. \tag{1.84}$$

The D-terms vanish along the so-called D-flat direction $\sigma = \overline{\sigma}$ which contains the supersymmetric minimum defined by:

$$\langle\phi\rangle \;=\; 0 \tag{1.85}$$

$$\langle|\sigma|\rangle \;=\; \langle|\overline{\sigma}|\rangle = \frac{\mu}{\sqrt{\alpha}} \equiv M \tag{1.86}$$

As required of a supersymmetric minimum, $V = 0$ as can be easily checked by substituting these values into Eq. (1.84). However, this is not the minimum for all values of the inflaton field. In particular, for $\phi > M$, the potential is not minimised by the above values of $\langle|\sigma|\rangle$ & $\langle|\overline{\sigma}|\rangle$ since the first bracketed term in (1.84) dominates. Rather the lowest V value is given by the non-supersymmetric minimum:

$$\phi \;>\; M \tag{1.87}$$

$$\sigma \;=\; \overline{\sigma} = 0 \tag{1.88}$$

which makes the potential (1.84) very simple; it is just:

$$V = \mu^4 \tag{1.89}$$

If this were the end of the story we would be in exactly de Sitter spacetime and there would be no exit, the universe would inflate forever.

However, supersymmetry is broken by this non-zero energy density and hence all the terms which are allowed by renormalisbility in the corresponding non-supersymmetric theory should be included to Eq. (1.84). These terms break the flatness of the potential and cause the inflaton to evolve with time.

For $\phi \gg M$ the effective potential V has, at one-loop level, the following form [65]:

$$V_{eff}(\phi) = \mu^4 \left(1 + \frac{\alpha^2}{16\pi^2}\left[\ln\left(\frac{\alpha^2\phi^2}{\lambda^2}\right) + \frac{3}{2} - \frac{M^4}{12\phi^4} + ...\right]\right) \tag{1.90}$$

This determines the curvature of the potential during slow-roll and hence determines the amplitude and spectral characteristics of the CMB anisotropies.

[12] In supersymmetric theories of inflation the inflaton is often denoted by S to signify that it is a singlet under the gauge group. For continuity we continue to use ϕ to denote the inflaton. σ and $\overline{\sigma}$ denote standard model singlet components of a conjugate pair of $SU(2)_R \times U(1)_{B-L}$ doublet superfields [65].

1.7.2 Symmetry breaking and COBE

The simplest fits to the COBE and other CMB data involve two quantities: the normalisation of the spectrum and the deviation from a Harrison-Zel'dovich $n = 1$ slope power spectrum. The normalisation, or amplitude, is typically given in terms of the $rms - PS$ anisotropy, the value of $(\Delta T/T)^2$ at $10°$ or the quadrupole anisotropy, in decreasing order of complexity and sophistication. Lazarides *et al* choose the quadrupole anisotropy and show that for the effective potential given by Eq. (1.90) the quadrupole anisotropy is given by:

$$\left(\frac{\Delta T}{T}\right)_Q \simeq 8\pi \left(\frac{N_Q}{45}\right)^{1/2} \left(\frac{x_Q}{y_Q}\right) \left(\frac{M}{M_{pl}}\right)^2 \tag{1.91}$$

which, as desired, involves the dimensionless particle physics constant M/M_{pl}. Here $N_Q \simeq 55$ is the number of e-foldings between the time the scales giving the dominant contribution to the temperature quadrupole (i.e. the longest wavelength modes which contribute to the CMB anisotropy via the Sunyaev-Zel'dovich effect [66]) left the Hubble radius and the end of inflation. This is typically close to the number of e-folding for the whole period of inflation itself. $x_Q \equiv \phi_Q/M$ where ϕ_Q is the value of ϕ at the time when the present horizon scale crossed the Hubble scale for the first time (often incorrectly called "horizon exit").

Finally, y_Q is defined in terms of x_Q via the perturbative relation:

$$y_Q = x_Q(1 - \frac{7}{(12x_Q^2)} + ...) \tag{1.92}$$

The above potential also determines the slope of the primordial spectrum, which using Eq. (1.44) gives:

$$n \simeq 1 - \frac{1}{N_Q} \sim 0.98 \tag{1.93}$$

certainly consistent with measurements from COBE and small scale data.

1.7.3 Neutrino mass bounds and reheating

As an interesting example of how one can use the full theory in which the inflation is embedded we will consider constraints on the masses of the heaviest neutrinos that come from bounds on the allowed reheating temperature [65]. These calculations of the reheat temperature are not fully consistent since they ignore preheating and resonance effects (as is discussed in more detail in the next chapter) but are instructive in showing how tightly interwoven all of the features of a full theory are.

The essential idea is that the temperature after reheating, T_r, cannot be too high in supersymmetric theories since otherwise one would over-produce gravitinos - the supersymmetric partner of the graviton - which then decay around the time of nucleosynthesis and destroy the nice standard picture of those low-energy processes. Another constraint arises from the condition of having successful baryogenesis, which in this model occurs via a process involving leptons - so-called leptogenesis.

To estimate the reheating temperature, T_r in this model, (neglecting resonance or stimulated effects due to the coherent nature of the inflaton condensate), Lazarides *et al* [65] consider the evolution of the fields ϕ and $\sigma, \overline{\sigma}$ near the supersymmetric minimum $\phi = 0, \sigma = \overline{\sigma} = M$. Defining the new field $\theta \equiv (\delta\sigma + \delta\overline{\sigma})/\sqrt{2}$ where $\delta\sigma = \sigma - M$ and $\delta\overline{\sigma} = \overline{\sigma} - M$, its main decay channel is via the non-renormalisable superpotential term:

$$\frac{1}{2}\left(\frac{M_{\nu^c}}{M^2}\right)\overline{\sigma}\overline{\sigma}\nu^c\nu^c\,, \tag{1.94}$$

where M_{ν^c} denotes the Majorana mass of the right-handed neutrino ν^c. Since the coupling term is proportional to this mass squared, θ will decay predominantly into the heaviest neutrinos allowed. T_r is then determined by the decay rate, Γ_θ, of the field θ via [65]:

$$T_r \simeq \frac{1}{7}\sqrt{\Gamma_\theta M_{pl}} \tag{1.95}$$

$$\simeq \frac{1}{12}\left(\frac{56}{N_Q}\right)^{1/4}\sqrt{y_Q}M_{\nu^c} \tag{1.96}$$

so that T_r scales linearly with M_{ν^c}, which is very neat.

The constraint on the reheat temperature from the gravitino abundance is that $T_r \leq 10^{10}GeV$ unless a large amount of entropy is generated at a later stage (which is precisely the idea behind thermal inflation [158]). The subsequent calculations require a large amount of phenomenological input regarding the mass matrix of the left and right handed neutrinos, the masses of the top and charm quarks and of the tau and electron neutrinos. Combining Eq. (1.96) with generation of maximally allowed baryon asymmetry then yields the estimates of $\sim 2 \times 10^{13}$ and 6×10^9 GeV respectively for the masses of the heaviest and next-heaviest right-handed neutrinos.

1.7.4 Supergravity aches and the η-problem

In the above model it was argued that using a model in which supersymmetry is a global theory was sufficient due to the low scale at which inflation occurs. In general however, and certainly for chaotic inflation, one would not be able to do this and the correct theory would be supergravity where supersymmetry arises as a local, gauge symmetry. The problem that arises there is that the flatness of the potential, so nicely arranged in globally supersymmetric theories and protected by the non-renormalisation theorems (see section[1.6]), is spoiled by large gravitational corrections. This is the so-called η-problem of supergravity models of inflation, where η is the slow-roll parameter of inflation defined by Eq. (1.45).

Now in supergravity the tree-level potential is the sum of an F-term and a D-term (corresponding to the different possible ways of breaking supersymmetry) [35]:

$$V = V_F + V_D\,. \tag{1.97}$$

Since supergravity is not renormalisable, it is thought of as an effective theory - much like the Fermi model of weak interactions - since it is typically believed that nonrenormalisable theories are not generally exactly solvable [13]. An effective theory will then give valid predictions up to some scale M, which for supergravity is often taken to be the Planck scale, M_{pl}. Certainly we have $M \leq M_{pl}$. Since the theory is non-renormalisable, an infinite number of non-renormalisable terms (e.g. ϕ^n with $n > 4$) are expected to be present in the Lagrangian and these terms contribute to V_F generically giving [35]:

$$\begin{aligned} |V_F''| &\sim \frac{V_F}{M^2} \\ \Longrightarrow |\eta| &\sim \frac{M_{pl}^2}{M^2} > 1 \end{aligned} \tag{1.98}$$

while inflation requires $|\eta| \ll 1$. It was then proposed that the V_F vanish during inflation and that the D-term dominate, giving D-term inflation. However, even in this case, non-renormalisable terms typically contribute to V_D through the gauge kinetic function (which determines the kinetic terms of all gauge and gaugino fields) and yield the same estimate for η as in (1.98) [68].

In this case, however, it appears that these large contributions to V_D can be avoided, and hence the small value of $|\eta|$ protected, if one imposes a discrete symmetry on the potential [68]. Whether this discrete symmetry is actually present is, of course, something that that will only be answered from a more complete theory such as string theory.

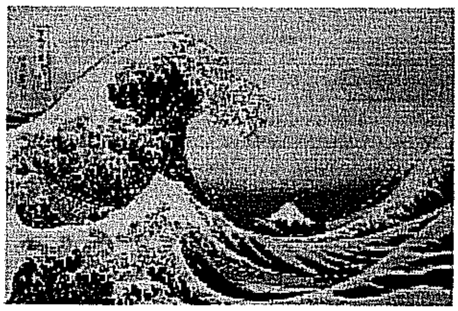
[13]This is known not to be a theorem since counterexample exist. The Gross-Nouveau model is not renormalisable but is exactly solvable [67].

Chapter 2

Reheating the Universe after Inflation

There is nothing permanent except change.
– Heraclitus

Nothing is built on stone; all is built on sand,
but we must build as if the sand were stone.
Jorge Luis Borges



2.1 Introduction

As the title "Visions of the end of inflation" suggests, the main thrust of this thesis is reheating after inflation. Having discussed in some detail the basic framework of current understanding of curved space quantum field theory, of how inflation will be tested with the CMB and of attempts to construct realistic models of inflation, we will now give an introduction and overview of the various aspects of reheating that will serve as a platform for the work presented in the following chapters.

This chapter begins by setting the problem of reheating in the greater inflationary context, then in (2.3) describes the old theory of reheating. From section (2.4) onwards we are concerned with *preheating* - the explosive and non-equilibrium production of particles that marks the end of inflation in many models. We begin the study of preheating with the classical theory of stimulated resonant decay of the inflaton in terms of the Mathieu equation and Floquet analysis [1] in general. We then go on to survey the more complex phases following this explosive "preheating" phase: backreaction, rescattering and finally thermalisation. These issues are non-linear and extremely complex, requiring the use of analytical approximations and numerical studies, both of which we review.

We then examine the impact of the expansion of the universe and the exciting possibility of non-thermal symmetry restoration whereby broken symmetries might be restored at preheating due to the large quantum corrections to the effective potential arising from preheating. Finally in section (2.13) we examine preheating in a realistic hybrid inflation model coming from the same superpotential that we studied at the end of chapter (1).

2.2 The onset of reheating

Before we describe how reheating proceeds, a natural question is "When exactly does reheating begin ?" Despite the lack of discussion of this point in the literature, the issue is in fact rather subtle. For example, one might argue that reheating begins:

(1a) when the couplings of other fields to the inflaton "switch on",

(1b) when the rate of energy transfer to other fields exceeds a certain critical value,

(1c) when the entropy of the universe starts to grow rapidly,

(2a) when the inflaton moves from the flat part of its potential to the highly curved part (if the potential has such a drastic change in curvature, as e.g. in dual inflation [137]),

(2b) when the slow roll parameters ϵ and η exceed a certain critical value (the end of slow-roll),

(2c) when the kinetic energy becomes of the same order as the potential energy (a virial-like

[1] Floquet analysis concerns itself with the solutions of linear differential equations with time-periodic coefficients.

epoch),

(3a) when the quantum-to-classical transition for metric fluctuations occurs.

These three groups of definitions differ significantly: (1a,b) rely on the actual transfer of energy to other fields, while those in group (2) depend only on the dynamics of the inflaton, and are concerned rather with the end of the quasi-de Sitter expansion of the universe with its near-constant expansion rate. Reheating had better start at this stage, if not before, or the universe ends up with a reheating temperature which is too low. In this sense definitions (2a-c) are wishful rather than prescriptive. Or perhaps to be fairer, they describe the onset of oscillations of ϕ, assuming that the effective potential has a local minimum, during which time, if the couplings to other fields are non-zero (1a), resonant production of quanta of the other fields may begin, and energy transferred ($\Rightarrow$ 1b). (3) is strongly related to (1c), although the exact nature of this connection is still rather controversial as we will discuss later.

Clearly, (1a) and (1b) are the most sensible, but perhaps also the most profound of definitions because they require complete knowledge of the β-functions of the theory within which the inflaton is embedded. The β-functions of a theory govern the "running" of the coupling constants, i.e. the changes of the couplings as the energy scale is altered [2]. If the couplings of the inflaton to other fields are large even during slow-roll one has warm-inflation or a variant thereof [203], while a wickedly asymptotically-free theory or one with a strong-weak coupling duality, will approximately give the standard isentropic new and chaotic inflationary models with almost no entropy production during the inflationary stage.

Now within the slow-roll approximation described above, what do (2b) and (2c) imply for the value of ϕ at the onset of reheating ?

(2b) For the potential $V = \lambda\phi^n/n$, the slow-roll parameters are:

$$\epsilon = \frac{n^2}{2}\frac{M_{pl}^2}{\phi^2} \quad , \quad \eta = n(n-1)\frac{M_{pl}^2}{\phi^2} \tag{2.1}$$

If we set $\epsilon = \eta < 1$ as the absolute limit at which the slow-roll approximation has completely broken down, then we find the corresponding values for ϕ are $\phi_{\epsilon=1} = nM_{pl}/\sqrt{2}$ and $\phi_{\eta=1} = \sqrt{n(n-1)}M_{pl}$. For $n = 2, 4$ these are $\phi_{\epsilon=1} = \sqrt{2}M_{pl}, 2\sqrt{2}M_{pl}$ and $\phi_{\eta=1} = \sqrt{2}M_{pl}, 2\sqrt{3}M_{pl}$ respectively. Thus for $n = 2$ the constraints coincide and give $\phi \simeq 1.3M_{pl}$ while for $n = 4$ the η constraint is stronger with inflation ending at around $3.4M_{pl}$. These should be contrasted with the initial values for oscillations taken by Linde *et al* [9, 70] of $\phi \sim 0.3M_{pl}$.

(2c) requires the equality of kinetic and potential energy, and hence in chaotic inflation models with $V = \lambda\phi^n/n$, we have:

$$\frac{\dot{\phi}^2}{2} = \lambda n \phi^n \tag{2.2}$$

[2] In chapter (3) we discuss the renormalisation group equations, β-functions and running of the couplings in some detail.

which leads to (using the expressions for $\dot{\phi}$ given by 1.46):

$$\phi_{virial} \approx \frac{\sqrt{n}}{10} M_{pl} \tag{2.3}$$

which for a quartic potential, $n = 4$, gives $\phi_{reheat} \simeq 0.2 M_{pl}$. To get this simple expression we have neglected the acceleration of the inflaton, $\ddot{\phi}$. Nevertheless, at this low value of ϕ the slow-roll condition would have been violated for a considerable amount of time and the virial estimate, which makes use of those same slow-roll conditions, cannot be trusted. Including the acceleration term would enhance the estimate of $\dot{\phi}$ and hence the virial equality would occur at a much earlier time, consistent with the estimates above where $\phi_{reheat} > M_{pl}$.

2.2.1 Entropy and the quantum to classical transition

Intimately related to the issue of reheating is the generation of the large entropy of the universe. This is a very tricky issue, since one must be careful what one means by "entropy" since there is still no accepted definition of gravitational entropy. Rather, we mean here the large entropy as normally ascribed in quantum field theory, and in the appropriate thermodynamic limit, to fluids.

Further, there exists at present a rather subtle discourse as to the true nature of the quantum-to-classical transition of metric perturbations during and after reheating and whether or not it is the huge generation of entropy that is responsible for the quantum-to-classical transition [72, 73].

We will not delve into these issues in detail here since they represent a seriously divergent course from that which we choose to follow. Nevertheless we present the two sides of the case more to highlight issues that will appear later in our discussions of reheating, since almost by definition reheating is the transition from a coherent condensate to a radiation dominated, large entropy and classical state.

First we present the proposal of Starobinsky, Polarski, Lesgourgues and Kiefer [72]. They examine metric perturbations (as opposed to quantum field perturbations) of scalar and tensor type and show that the corresponding states become highly squeezed on super-Hubble scales due to the expansion of the universe. They also find that the decaying mode decreases exponentially and that there is minimal information loss and entropy production with the metric perturbations becoming classical stochastic variables with Gaussian statistics.

The much more standard proposal for the quantum to classical transition leads to extensive entropy production, see e.g. [73, 74]. In this approach the field is split into an order parameter (given by the field coarse-grained or averaged over a certain spatial volume) and the high-frequency part which acts as an environment. The interaction of the environment with the order parameter implies that the order parameter describes an open system and decoherence occurs, with a significant growth in entropy. In this case, the off-diagonal terms in the density matrix describing the system oscillate rapidly in time on super-Hubble scales and so these off-diagonal terms time-average to zero, leaving an essentially diagonal and hence classical density matrix. The entropy generated is then associated

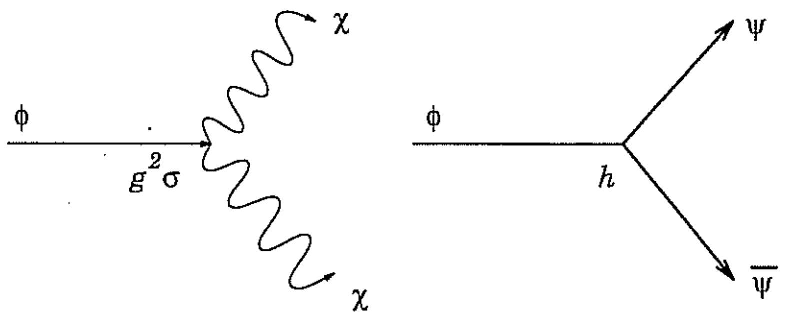


Figure 2.1: The Feynman diagrams corresponding to single body ϕ decay into scalar bosons (left) and fermions (right) with couplings $g^2\sigma$ and h respectively.

with the loss of information regarding the density matrix and to the implicit coarse graining of the density matrix or equivalent object implied by its temporal averaging.

For our later study of preheating in models with self-interaction, we note that the latter proposal, requires that the field either have self-interactions (so that the high-frequency part can actually couple to the order parameter) or be coupled to some other field(s) which then act as an environment bath. The first proposal, on the other hand, makes no such requirements and hence reheating does not follow necessarily. The metric perturbations could undergo the quantum to classical transition without the generation of much entropy (there will always be some decoherence due to the production of particles due to the expansion of the universe), with the temperature of the universe staying very near zero.

We now leave these side issues and begin an historical review of reheating.

2.3 Single-body and perturbative decays

The first models of reheating came out in 1982 [75] shortly after Guth's original paper on inflation [24]. There, perturbatively small couplings between the inflaton ϕ and other bosonic, χ, and fermionic, ψ, fields were considered with interaction terms given by $\nu^2\sigma\phi\chi^2$ and $h\overline{\psi}\phi\psi$ respectively [3]. Here ν, h are dimensionless coupling constants and σ is a mass parameter.

The single-body decay rates for the two channels are then given by ordinary one-loop calculations (assuming $h, g^2 \ll 1$, the associated Feynman diagrams are shown in figure [2.1]):

$$\Gamma(\phi \to \chi\chi) = \frac{\nu^4\sigma^2}{8\pi m_\phi} \tag{2.4}$$

$$\Gamma(\phi \to \psi\psi) = \frac{h^2 m_\phi}{8\pi} \tag{2.5}$$

In this old model of reheating, it was assumed that strong interactions between the decay products would ensure at least local thermodynamic equilibrium (strict equilibrium is not valid in an expanding universe and quarks and gluons go out of equilibrium at a temperature of above $10^{14}GeV$) so that one can define a reheating temperature T_r and the universe would take on a truly radiation-dominated character.

The condition for this local equilibrium is roughly that $\Gamma_{tot} \sim 3H$, H being the Hubble constant. At that time, in a flat FLRW model, the energy density is roughly [3]:

$$\begin{aligned} \rho_{eq} &\sim \frac{\Gamma_{tot}^2 M_{pl}^2}{24} \\ &\sim \frac{\pi N(T_r)}{30} T_r^4 \end{aligned} \tag{2.6}$$

where we have assumed radiation domination and thermalisation to T_r, in the second line and $N(T_r)$ is the effective number of degrees of freedom (see e.g. [37]) at the temperature T_r. Unlike in preheating, T_r depends only on Γ and not on the initial value of ϕ_0, where ϕ_0 denotes the vacuum expectation value of the inflaton. The above analysis leads to the estimate [3]:

$$T_r \sim 10^{-1}\sqrt{\Gamma_{tot} M_{pl}}\,, \tag{2.7}$$

Now, in this perturbative-style reheating, the evolution of gauge-invariant density perturbations is essentially unaffected by the inflaton oscillations. The standard requirements that metric perturbations not over-produce CMB anisotropies means that $m_\phi \sim 10^{-6} M_{pl}$ in the simple quadratic potential model. This is based on the curvature of the potential, which receives radiative quantum corrections from the couplings to ψ and χ [3]. These radiative corrections cannot be too large without endangering the anisotropy levels of the CMB, which places the constraints:

$$\begin{aligned} h^2 &\leq 8 m_\phi M_{pl}^{-1} \sim 10^{-5} \\ \nu\sigma &\leq 5 m_\phi \sim 10^{14} GeV \end{aligned} \tag{2.8}$$

on h^2 and $\nu\sigma$. Hence from Eq. (2.5) we see that Γ_χ dominates the contribution to T_r, leading to the constraint that $T_r \leq 10^{15} GeV$. This ensures that dangerous GUT-symmetries were unlikely to be restored after inflation and hence no GUT-scale topological defects would be produced after inflation. In section (2.12) we will show that this is no longer necessarily true in preheating.

It is also not necessarily true in supersymmetric models either, which may have very large h^2 and $\nu\sigma$ couplings while the effective potential is protected from radiative corrections by the powerful non-renormalisability theorems.

In this section we presented the theory of reheating based on treating the inflaton as a collection of bosons where the whole was just the sum of the parts. The revolution of preheating, from a physical point of view, is similar to the revolution caused by lasers. Instead of thinking of photons as always coming incoherently from individual atoms, laser physics concerns itself with coherent effects, where there is a strong dependence on the inverted population of atoms in excited states (leading to exponential growth) which then decay via stimulated emission. Preheating is a very similar phenomenon where inflaton decays are stimulated by the inflaton condensate which has very high occupation number and perfect coherence. Mathematically it is described, partially and at least classically, by Floquet theory - the theory of differential equations with periodic coefficients.

2.4 Floquet theory and the Mathieu equation

To illustrate the phenomenon of parametric resonance, consider the n-dimensional first order system:

$$\dot{\mathbf{y}} = P(t)\mathbf{y} \tag{2.9}$$

where P is any matrix with period T. Then Eq. (2.9) has n linearly independent normal solutions of the form:

$$\mathbf{y}_i = \mathbf{p}_i(t)e^{\mu_i t} \tag{2.10}$$

where the μ_i are the characteristic/Floquet exponents of the system and the $\mathbf{p}_i$ are functions of period T. Then the n characteristic numbers defined by $\rho_i = e^{\mu_i T}$ satisfy:

$$\rho_1\rho_2...\rho_n = \exp\left(\int_0^T TrP(s)ds\right) \tag{2.11}$$

with repeated characteristic numbers counted accordingly. The trace of $P(t)$ is thus the crucial factor determining the existence of exponentially amplified modes. If $\mathrm{Tr}P(t) > 0$ then eq. (2.11) implies that:

$$\Pi_{i=1}^{n}\rho_i > 1 \tag{2.12}$$

which implies that at least one of the $\rho_i > 1 \implies \mu_i > 0$ and hence by eq. (2.10) there is at least one unbounded, exponentially growing solution.

Now we derive the Mathieu equation. The evolution equation for the inflaton ϕ, with effective potential $V_{eff}(\phi)$ [3], is given by:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) + \Pi\phi = 0 \tag{2.13}$$

where $' \equiv \partial/\partial\phi$. Π in eq. (2.13) generically represents the backreaction of quantum fluctuations on the zero mode evolution via a change to the effective mass of the inflaton. It may be written specifically as the polarisation operator [3] or given explicitly in certain approximations, such as the Hartree-Fock approximation [69], or the large-N limit of $O(N)$ vector models [85] as we will discuss in the section on backreaction.

The geometry of space (if the background is FLRW) enters only through the expansion H, whose evolution is given by the Raychaudhuri equation [220], which in flat FLRW spacetime is:

$$\dot{H} = -4\pi G\dot{\phi}^2 \tag{2.14}$$

Here G is Newton's constant which we will set to unity throughout.

[3] From here on we drop the *eff* subscript from $V(\phi)$. It is implicit.

To illustrate preheating, assume that ϕ interacts with a light scalar field χ, which itself has no self-interaction, via the Lagrangian interaction term

$$\frac{1}{2}g^2\phi^2\chi^2 . \tag{2.15}$$

Consider the simplest effective potential for chaotic inflation:

$$V(\phi) = \frac{1}{2}m_\phi^2\phi^2 . \tag{2.16}$$

The solution of the inflaton equation of motion (2.13), when the frequency is an adiabatic invariant, is that of decaying sinusoidal oscillations, $\phi(t) = \Phi(t)\sin(m_\phi t)$. In the absence of particle production the amplitude varies roughly as $\Phi \sim \frac{1}{m_\phi t}$, due to the averaged expansion. The time evolution of the quantum fluctuations for each mode of the χ-field is given by [1]:

$$\ddot{\chi}_k + 3H\dot{\chi}_k + \left(\frac{k^2}{a(t)^2} + m_\chi^2 + g^2\Phi^2\sin^2(m_\phi t)\right)\chi_k = 0 \tag{2.17}$$

This can be put in canonical Mathieu form (the case $n = 2$ above with $P(t)$ particularly simple) if one neglects the expansion of the universe ($H = 0$):

$$\chi_k'' + [A(k) - 2q\cos(2z)]\,\chi_k = 0 \tag{2.18}$$

with dimensionless coefficients:

$$A_\chi(k) = \frac{k^2}{m_\phi^2 a^2} + \frac{m_\chi^2}{m_\phi^2} + 2q \quad , \qquad q_\chi = \frac{g^2\Phi^2}{4m_\phi^2} \tag{2.19}$$

and $z = m_\phi t$, $' \equiv d/dz$. Depending on the size of the coupling, g, and the mass m_ϕ, *certain* modes χ_k will thus be amplified exponentially: $\chi_k = p_k(m_\phi t)\exp({\mu_k}^{(n)} m_\phi t)$, where p_k are functions with the same period as the oscillations of the inflaton field and the positive ${\mu_k}^{(n)}$ are the Floquet indices corresponding to the n-th instability band.

In the first resonance band and for $1 \gg q > 0$, we have the explicit estimate for the Floquet index [9]:

$$\mu_k = ((q/2)^2 - (2k/m - 1)^2)^{1/2} \tag{2.20}$$

This can be extended to give μ_k^N in the N-th resonance band [82] as long as $A > 0$ and $2N^{3/2} \gg q$:

$$\mu_k^N = -\frac{1}{2N}\frac{\sin 2\delta}{[2^{N-1}(N-1)]^2}q^N , \tag{2.21}$$

where δ varies in the interval $[-\pi/2, 0]$ and $\mu_k^N \ll 1$.

The existence of resonance bands survives when the expansion of the universe is included [155, 69]. The parameter q divides the phase space into three broad classes with qualitatively different behaviour. The case $q \ll 1$ is well understood [180] and can be treated perturbatively; the effects of expansion being important. The broad resonance case is described roughly by $\pi^{-1} < q < q_*$, $q_* \sim 10^2$, is non-perturbative and requires consideration of the backreaction of created particles on the zero mode ϕ_0.

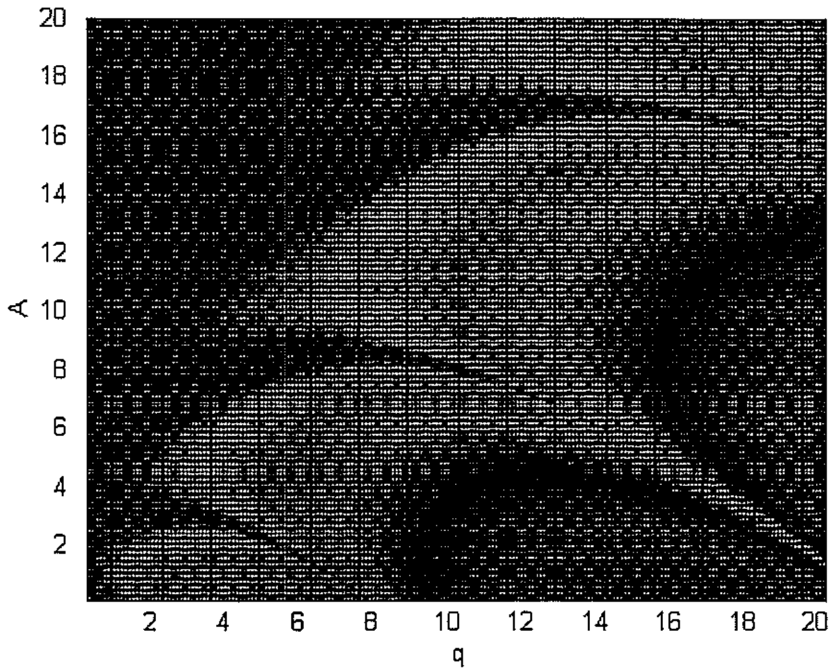


Figure 2.2: A density plot of the Floquet exponent on the A-q plane for the Mathieu equation. The broadening of the resonance bands as q increases is very evident.

The resonance bands are characterized by huge occupation numbers of produced particles, typically of order $n_k \sim \frac{1}{g^2}$. The upper limit, q_*, is for an expanding universe and is much lower in Minkowski spacetime ($H = 0$) [7]. The wide resonance, $q \gg q_*$, evolution is dominated by scattering effects which rapidly shut-off the exponential growth of the χ fluctuations [6, 7].

The growth of the χ_k modes during resonance is directly interpreted as particle production. The number density of created χ particles, n_k, in each mode can be estimated as the total energy in that mode, divided by the energy Ω_k of each particle (the χ are assumed massless):

$$n_k = \frac{\Omega_k}{2}\left(\frac{|\dot{\chi}_k|^2}{\Omega_k^2} + |\chi_k|^2\right) - \frac{1}{2} \tag{2.22}$$

which leads, under parametric growth to $n_k \simeq \chi_k^2 \simeq e^{2\mu_k m_\phi t}$.

2.5 The Lamé equation

In the previous section we considered a simple model of preheating with two fields and at most quadratic couplings. Here we consider a very interesting model, the highest polynomial power potential which is renormalisable in the old, Dyson power-counting, sense [76] - the quartic effective potential:

$$V(\phi) = \frac{\lambda}{4}\phi^4 \tag{2.23}$$

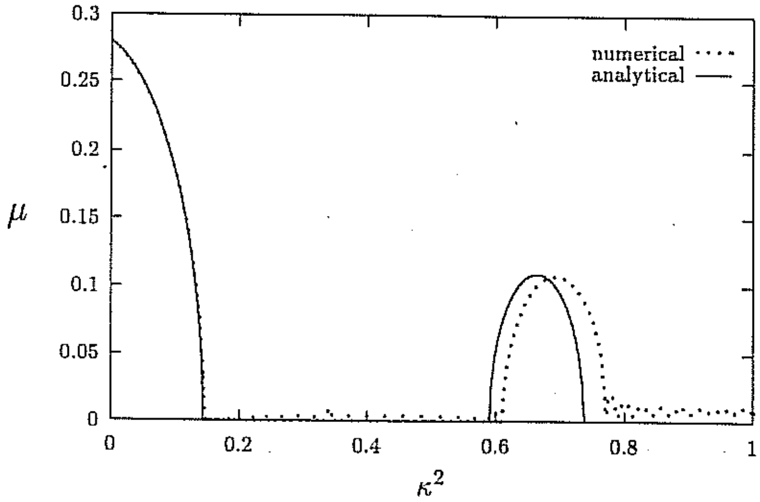


Figure 2.3: A slice through the Mathieu instability chart for $q = (32\pi)^2$. Notice the rapid decrease in μ_k as a function of the frequency $\kappa = k/(ma)$. From [9].

We will also assume that the field is conformally coupled $\xi = 1/6$. In this case the inflaton evolves according to:

$$\ddot{\phi} + 3H\dot{\phi} + \lambda\phi^3 + \tfrac{1}{6}R\phi = 0 \tag{2.24}$$

where R is the Ricci scalar. This equation, when we switch to conformal time $d\eta = dt/a(t)$ and the rescaled field variable $\varphi \equiv a(t)\phi(t)$ yields the equation ($' \equiv d/d\eta$):

$$\varphi'' + \lambda\varphi^3 = 0 \tag{2.25}$$

which is precisely the corresponding evolution equation in Minkowski spacetime, so that the effects of expansion are taken into account at the classical level. If we write the initial value of the inflaton as $\tilde{\varphi}$ then the energy density is simply $\mu_\varphi = \frac{\lambda}{4}\tilde{\varphi}^4 \propto a^{-4}$. Further, this is almost the canonical form of the Lamé equation which has solutions in terms of Jacobi elliptic functions. Indeed we can write [10]:

$$\phi = \frac{\tilde{\varphi}}{a(t)} f(x) \ , \quad x \equiv \sqrt{\lambda}\tilde{\varphi}\eta \tag{2.26}$$

where

$$\begin{aligned} f(x) &\equiv cn\left(x - x_0, \frac{1}{\sqrt{2}}\right) \\ &\approx \frac{8\pi\sqrt{2}}{T}\sum_{n=1}^{\infty} \frac{e^{-\pi(n-1/2)}}{1+e^{-\pi(n-1/2)}} \cos\frac{2\pi(2n-1)x}{T} \end{aligned} \tag{2.27}$$

where $T \simeq 7.416$ is the period of oscillations in units of x. The above Fourier expansion converges very rapidly with the first coefficient ≈ 0.9550 and the second 0.0431, so the oscillations are well approximated by $f(x) = 0.995\cos(2\pi x/T)$.

Since the field exhibits self-interactions, the quantum fluctuations $\delta\phi_k$ will exhibit resonant growth due to the oscillatory evolution of ϕ:

$$\delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k + \left(\frac{k^2}{a(t)^2} + 3\lambda\phi^2\right)\delta\phi_k = 0 \tag{2.28}$$

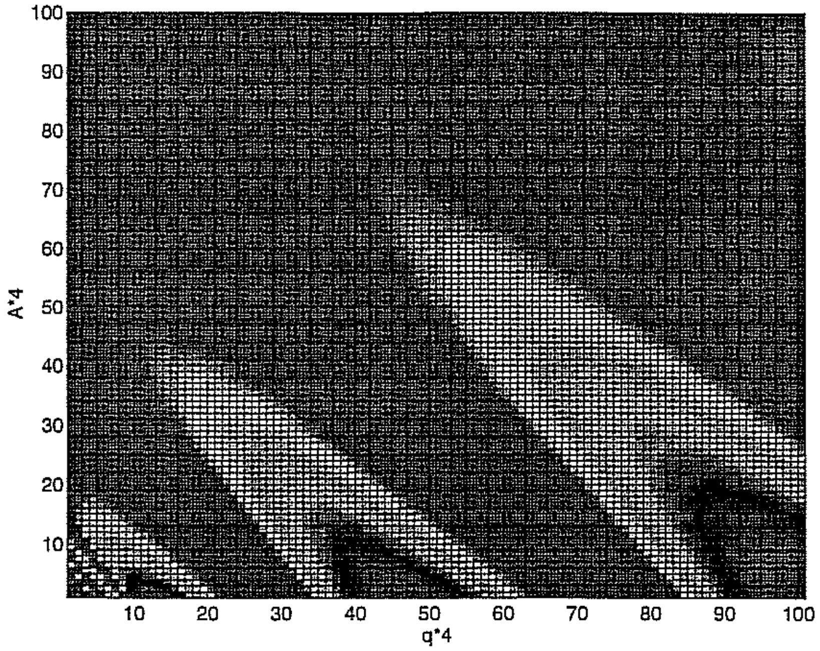


Figure 2.4: The density plot of the Floquet exponent on the A-q plane for the Lamé equation.

which again in terms of conformal time and a rescaled field $X_k \equiv a(t)\delta\phi_k$ takes the simplified, nearly conformally invariant, form:

$$X_k'' + \left(\frac{k^2}{\lambda\bar{\varphi}^2} + 3cn^2\left(x, \frac{1}{\sqrt{2}}\right)\right) X_k = 0 \tag{2.29}$$

We now see a beautiful example of non-perturbative corrections to the Mathieu picture of infinite instability bands. The above equation in fact, exhibits only one resonance band centred around $k^2 \sim 3\lambda a^2\Phi^2$ [1]. Thus the harmonic approximation to $cn(\cdot)$, while very accurate at the perturbative level, predicts the wrong number of instability bands (infinite instead of one !).

In the case that we consider the quartic potential and a coupling $g^2\phi^2\chi^2$ to a massless scalar field χ as before, the above equation for the fluctuations of the scaled field $Y_k \equiv a(t)\chi_k$ becomes modified by the simple transition $3 \rightarrow g^2/\lambda$ in the last term of eq. (2.29).

For general values of $q \equiv \frac{1}{2}\frac{g^2}{\lambda}$ eq. (2.29) has an infinite number of instability bands (see fig. 2.4) which compress to a single band when $q = \frac{3}{2}$ or $\frac{1}{2}$. The latter value arises in the large-N limit of an $O(N)$ vector model [183], which lead de Vega *et al* to emphasise the importance of the existence of only a single resonance band. From the general theory of Floquet theory, infinite numbers of instability bands are known to be generic, and hence the above emphasis was perhaps, in retrospect, unnecessary.

2.6 Negative coupling instability

When we wrote down the original model of reheating in section (2.4) we gave as interaction term $\frac{g^2}{2}\phi^2\chi^2$ so that the coupling g^2, and hence A and $q \propto g^2$, were automatically positive.

However, this is not forced upon one. Indeed, if one considers a potential, $V(\phi)$, with broken symmetry - such as the usual Mexican hat or wine-bottle shaped potential - then near the origin the potential has negative curvature, corresponding to a negative frequency of oscillation near the origin. Of course, as one approaches the global extrema of the potential, the quartic terms start to dominate and the curvature becomes positive again, leading to a positive mass $\sim V''$.

This type of behaviour with negative frequency - personified by the inverted harmonic oscillator - has been well studied but rejected because the potential is unbounded from below with exponentially growing solutions. Indeed, if we examine it:

$$\ddot{y} = m^2 y \tag{2.30}$$

we see that solutions are $y = c_1 \cosh(mt) + c_1 \sinh(mt)$. Thus, while for the Mathieu equation, μ_k was typically less than one even for very large values of q, the effective Floquet index in this case, played by m, can be much larger than 1.

Can we get this kind of behaviour in preheating ? The answer is yes and at present there exist two known mechanisms for full blown negative coupling instability, as this phenomenon has been dubbed in reheating. The first model derived showed [8] that it arises when one uses a coupling $g\phi^2\chi^2/2$ with the dimensionless coupling g replacing the g^2 that we had put in eq. (2.17). Since in the simple models $A = k^2/(m_\phi^2 a^2) + 2q$, negative g implies negative q which can yield negative A and hence gives the negative coupling instability.

This is a very interesting possibility since it means that the inflaton (which from naive CMB estimates must have a mass $m_\phi \sim 10^{-5} \sim \Delta T/T$) can produce much heavier particles $m_X \sim 10^{-3} M_{pl}$ as needed for GUT baryogenesis. This is extremely important since previously it was believed that GUT baryogenesis was almost impossible after inflation. Nevertheless, the model of the negative coupling instability outlined above is not completely efficient since the coupling between ϕ and χ gives χ a non-zero expectation value and this partially cripples the power of the effects.

The only other known model of a true negative coupling instability (the wine-bottle potential exhibits it near the origin, but this is perturbative) will be presented in the next chapter on geometric reheating [11]. There, the freedom in sign of the non-minimal coupling parameter ξ to the Ricci scalar R allows the negative coupling instability to exist in that model, with similar implications for GUT baryogenesis and the possibility of non-thermal symmetry restoration, which we will introduce later in this chapter.

To be more precise about the negative coupling instability, let us consider the potential:

$$V = \frac{m_\phi^2}{2}\phi^2 + \frac{m_\chi^2}{2}\chi^2 + \frac{g}{2}\phi^2\chi^2 + \frac{\lambda_\phi}{4}\phi^4 + \frac{\lambda_\chi}{4}\chi^4\,, \tag{2.31}$$

where g can be positive or negative. When it is positive we simply reproduce a generalised Lamé equation for the evolution of the modes χ_k in the absence of expansion, but when $g < 0$, we find that $A \approx k^2 + 2q$, $q \propto g$ and so A is negative for long-wavelength, $k \sim 0$ modes, and we have the negative coupling instability. Note that the quartic self-interaction terms are necessary in the absence of a non-minimal couplings to ensure that the potential is bounded below [4]. To do this they must satisfy the constraint that [8]:

$$\frac{\lambda_\phi \lambda_\chi}{g^2} > 1 \tag{2.32}$$

For $g < 0$ the physical region, ($k \geq 0$), is now given by we now have $A \geq -|2q|$ compared to the standard Mathieu case where $A \geq |2q|$. This extra resonance region leads to an enhanced Floquet index and for $2|q| \geq |A| \gg 1$ there is an explicit solution [8]:

$$\cosh 2\pi\mu_k = \cosh \mathrm{Im}\left[\int_0^{2\pi} dz\sqrt{A_0 - 2q_0\cos(2z)}\right] \times \cos \mathrm{Re}\left[\int_0^{2\pi} dz\sqrt{A_0 - 2q_0\cos(2z)}\right] \tag{2.33}$$

where the 0 subscript indicates the corresponding values of A, q with k set to zero. Along the physical separatrix $A_0 = -2|q_0|$, this evaluates to $\mu = (4/\pi)|q_0|^{1/2}$, which is slightly smaller than the naive estimate $\mu = (2|q_0|)^{1/2}$ we would get from treating this as a simple inverted harmonic oscillator (i.e. by averaging over the oscillations of the frequency).

However the situation is rather more subtle than we have made out since we implicitly assumed that $\chi_0 \equiv \langle\chi\rangle = 0$ in the above discussion. As we discussed in the first chapter, since typical couplings are rather small we cannot expect the fields to be at the origin or in general, at the minimum of the effective potential at the start of inflation. Nevertheless, by the end of inflation one would expect that χ_0 would have reached the minimum of the effective potential, but this does not correspond to $\chi_0 = 0$ in general due to the non-zero value of ϕ during inflation.

Minimising the potential[5] (2.31) w.r.t. χ we find that the expectation value χ_0, of χ is:

$$\chi_0^2 = -\frac{m_\chi^2 + g\phi_0^2}{\lambda_\chi} \quad \text{if } m_\chi^2 + g\phi^2 < 0 \tag{2.34}$$

$$= 0 \quad \text{otherwise} \tag{2.35}$$

In the absence of non-minimal coupling, the self-interaction terms $\lambda_{\phi,\chi}$ are required to make the potential bounded from below and it is they who cause χ_0 to be non-zero if g is sufficiently negative. The implication of this is that the Fourier equation for $\delta\chi$ modes is then [8]:

$$\frac{d^2\delta\chi_k}{d^2t} + \left(k^2 + m_\chi^2 + 3\lambda_\chi\chi^2 + g\phi^2\right)\delta\chi_k + 2g\chi\phi\delta\phi_k = 0 \tag{2.36}$$

[4] If V is not bounded below there is no global minimum and no sensible way to do perturbation theory or define a vacuum for the theory. As a result it would be possible to extract an infinite amount of energy from the system [42].

[5] i.e. setting $\partial V/\partial\chi = 0$.

so that for $\chi = \chi_0$ given by the non-zero value in (2.35) the effective value of q becomes positive and the negative coupling instability is lost – we return to the positive q case we had before.

In the massless case $m_\chi = 0$ the condition in (2.35) is, bar a set of measure zero, always satisfied and χ_0 is enslaved to follow ϕ. In the massive case this is not true since as ϕ oscillates, the condition that $m_\chi^2 + g\phi^2 < 0$ will be violated near every inflaton zero and χ will feel a restoring force towards the origin. If this force acts for long enough each cycle a phase mismatch between χ and ϕ will develop and then the $3\lambda_\chi \chi^2 + g\phi^2$ term in the frequency squared above can become negative as ϕ reaches its maximum each oscillation. During this part of the cycle the negative coupling instability is effective and leads to very powerful particle production.

The instabilities lead also to large field variances $\langle(\delta\chi)^2\rangle$ which alter the effective mass of the χ field. Even in the case $m_\chi = 0$, when the variance is so large that $\delta m_\chi \sim |q|^{1/4}\omega_\phi$ [8] (see later discussion on changes to the effective mass due to large field variances), the phase mismatch process becomes effective and the negative coupling instability can start to occur.

Even this partial effectiveness of the negative coupling instability leads to the possibility of producing extremely massive bosons with $m_\chi \gg m_\phi$. Kinematically the simple two-body decay $\phi\phi \rightarrow \chi\chi$ in this case would be impossible, and even with coherent effects described by the Mathieu equation with positive q resonance it is not possible if $m_\chi \geq 10m_\phi$ for reasonable coupling values. This puts the bound $m_\chi \leq 10^{14}GeV$ on the masses of produced particles in all but the most violent of preheating scenarios and hence GUT baryogenesis, which involves GUT bosons with masses around $10^{15-16}GeV$ are very difficult to produce. With the negative coupling instability however, producing particles of this mass is no problem [8] and leads to a model of baryogenesis at reheating which is consistent with the observed baryon to photon ratio.

2.7 Fermionic preheating

While it is true that reheating to bosonic fields is constrained *a priori* only by conservation of energy and momentum while production of fermionic fields is restricted due to the Pauli exclusion principle, rapid production of fermions is possible and yields very different dynamics to the corresponding perturbative case considered in section (2.3). This very recent development is the subject of this section.

The Pauli principle in the field theoretic context implies that the occupation number of each mode obeys $n_k \leq 1/2$. During preheating, some parameter values yield dynamics which periodically saturate this bound, in contrast to the perturbative result which gives $n_k \propto h^2 \ll 1/2$. The appropriate starting point for fermionic perheating is the Dirac equation for the Fermi field ψ [77]:

$$[i\gamma^\mu \nabla_\mu - h\phi(t)]\psi = 0 \tag{2.37}$$

which of course differs from the Klein-Gordon equation in that it is first order in time and space. Just as in bosonic preheating, it takes as input the background oscillating inflaton field, ϕ, whose evolution is determined at zero order by its potential.

If we are not to ignore the effects of expansion, the simplest model to consider is a conformally coupled inflaton ($\xi = 1/6$) with a quartic potential:

$$V(\phi,\psi) = \frac{\lambda}{4}\phi^4 + \frac{1}{6}\phi^2 R + h\overline{\psi}\phi\psi \tag{2.38}$$

with the last term a Yukawa interaction describing the single-inflaton decay into two fermions, or in our (non-perturbative) case, of fermions being produced due to their interaction with the *coherently oscillating field* $\phi(t)$.

The advantage of the above potential is that the inflaton evolution is, at zero order, conformally invariant (corrections to the effective mass and the backreaction of the fermionic fields may break this invariance but give second order effects) so that the evolution of ϕ can be reduced to that in Minkowski spacetime, as discussed in section (2.5). The appropriate conformal transformations are [10] $\varphi \equiv a\phi$ and $\Psi \equiv a^{3/2}\psi$ and conformal time defined by: $d\tau \equiv \sqrt{\lambda\tilde{\varphi}^2}dt/a(t)$ where $\tilde{\varphi}$ is the constant amplitude (when particle decay is neglected) of oscillation of φ. In full we have $\varphi(\tau) = \tilde{\varphi}cn(\tau, \frac{1}{\sqrt{2}})$ - oscillations with period $T \simeq 7.416$ and amplitude $\tilde{\varphi}$.

To make a connection with our previous study of bosonic reheating and the Mathieu or Lamé equations we follow [77] by introducing an auxiliary field X defined implicitly via the equation:

$$\Psi \equiv [i\gamma^\mu \nabla_\mu + h\varphi]X \tag{2.39}$$

which differs in structure from the Dirac equation (2.37) only in that it has a + sign which then gives a neat second order time equation for X in Fourier space:

$$\ddot{X}_k + \left(\frac{k^2}{\lambda\tilde{\varphi}^2} + qf^2 - i\sqrt{q}\dot{f}\right)X_k = 0 \tag{2.40}$$

where $q \equiv h^2/\lambda$ and in this case $f(\tau) = cn(\tau, \frac{1}{\sqrt{2}})$. The fermionic equation is thus more complex than its bosonic counterpart with the addition of a purely complex term in the frequency. The occupation number of fermionic particles in a given spin state is then given by [77]:

$$n_k = \frac{1}{2} - \frac{k^2}{\Omega_k\lambda\tilde{\varphi}^2}\mathrm{Im}(X_k\dot{X}_k^*) - \frac{\sqrt{q}f}{2\Omega_k} \tag{2.41}$$

where the real part of the frequency is $\Omega_k \equiv \kappa^2 + qf^2$ where the rescaled momentum is $\kappa \equiv k^2/\lambda\tilde{\varphi}^2$.

By numerically solving Eq. (2.40) for X_k one finds that the occupation number n_k cannot grow monotonically but oscillates on two distinct time scales - see figure (2.5). The first is associated with the underlying oscillations of the inflaton and so occur with period $\simeq T/2$ while the second oscillations typically occur on a much longer time scale determined by q and κ.

What is interesting is that, just as in the bosonic case, the occupation number n_k shows a resonance band structure as a function of k, as shown in figure (2.6). Further, as can be seen from figure (2.5),

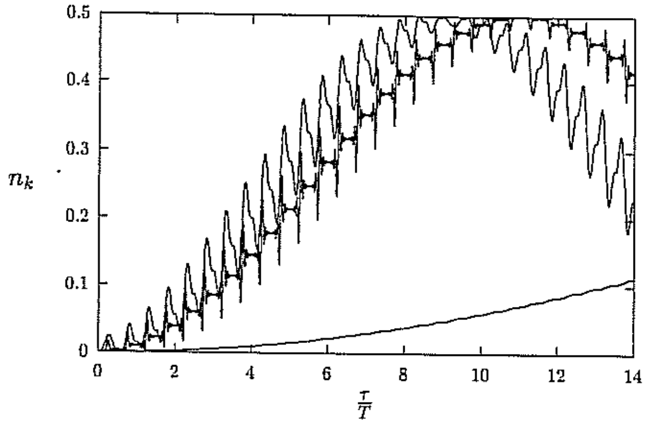


Figure 2.5: Numerical solutions showing the short ($\sim T/2$) and long period oscillations in the occupation number, n_k, of massless fermions for several parameter sets: $q \equiv h^2/\lambda = 10^{-4}$ (bottom), 1 (middle curve at right) and 100 (upper right) for normalised momenta $\kappa^2 = 0.18, 1.11$ and 11.9 respectively. From [77].

the occupation numbers reach $\sim 1/2$ after only a few tens of inflaton oscillations. This should be compared with the perturbative decays which require a time of order $\Gamma^{-1}_{\phi\rightarrow\psi\psi}$ corresponding to about 10^{14} oscillations. In the simple model presented above, the creation of fermions is essentially a reversible process and the number density oscillates back down to zero by the reverse process $\overline{\psi}\psi \rightarrow \phi$. In the real universe the decoherence due to the decay of the amplitude $\tilde{\varphi}$ and the backreaction due to the particle creation is likely to inhibit this reconversion process, leaving the fermion number evolution predominantly monotonic.

2.8 Backreaction

The resonances described in the earlier sections on bosonic preheating cannot last indefinitely, if through no other reason than energy conservation. If one considers two pendula coupled through a spring, one readily observes the resonant transfer of energy from one pendulum to the other *and back again*. In the quantum field theoretic case, with the existence of an infinite number of possible modes of χ to transfer energy to, there are selection rules deciding which χ modes are amplified (the resonance bands described earlier).

More complex is the nature of the backreaction. Energy is not simply transferred back to the inflaton condensate - indeed this would be an unhappy state of affairs since we are trying to rid the ϕ field of its energy - because the produced χ particles have a range of different momenta and hence frequencies. Thus the correct analogy is that of the inflaton coupled to an infinite number of oscillators, each oscillating with a different frequency. Once the amplitudes of these oscillators become significant, they do not resonantly transfer energy back to the inflaton because they interfere destructively, at least at the level where we neglect all the modes corresponding to the inflaton field.

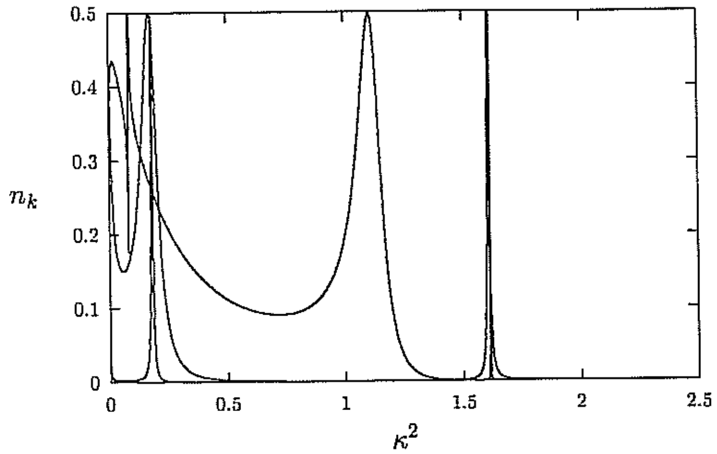


Figure 2.6: The resonance band structure as a function of the normalised momentum κ for the inflaton potential $\lambda\phi^4/4$ and $q \equiv h^2/\lambda = 10^{-4}, 10^{-2}$ and 1 respectively from narrowest to broadest bands. From [77].

So what happens ? When the oscillators have significant amplitude, they act at lowest order as an effective mass for the inflaton, thereby changing radically the frequency of the inflaton oscillations and decreasing the resonant transfer of energy to the coupled oscillators (leading to a significant reduction in q, see eq. 2.19).

2.8.1 The Hartree-Fock approximation

To study backreaction effects requires non-perturbative techniques. How can we proceed ? One of the simplest approximations is to model the backreaction using the mean-field or Hartree-Fock approximation [6]. In this approximation there is no scattering that does not involve the zero-momentum inflaton mode and energy is therefore only transferred via the condensate.

Let us consider a simple model with self-interaction to illustrate this:

$$V = \frac{\lambda}{4}\phi^4 \tag{2.42}$$

As we showed in section (2.5), this leads to a Lamé equation for $\delta\phi_k$. But in that discussion there was no attempt at describing the end of the resonance brought on either by the build-up of $\delta\phi_k$ fluctuations or by energy conservation. We now do this by defining the mean value of $\delta\phi_k$, $\langle(\delta\phi)^2\rangle$:

$$\langle(\delta\phi)^2\rangle = \frac{1}{(2\pi)^3}\int dk k^2 |\delta\phi_k|^2 \tag{2.43}$$

and then saying that the backreaction is completely encoded in this quantity. Hence the evolution equations for ϕ, the condensate, and $\delta\phi_k$ become (c.f. eq. 2.28):

$$\ddot{\phi} \quad + \quad 3H\dot{\phi} + \lambda\phi^3 + 3\lambda\langle(\delta\phi)^2\rangle\phi = 0$$

[6] There is a good deal of confusion regarding the naming of this, the mean-field approximation. Many authors refer to it simply as the Hartree approximation while others also attach the name of Fock to it as we have done. The main point is that they are one and the same approximation, up to small details.

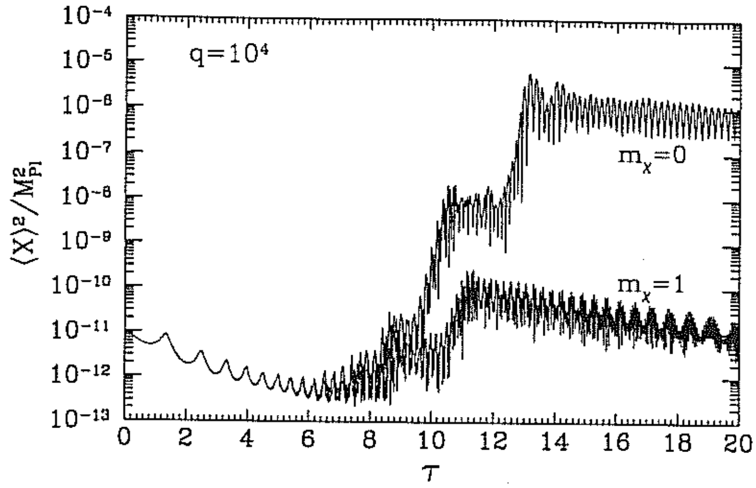


Figure 2.7: The backreaction in the Hartree approximation for a quadratic inflaton potential and quadratic coupling $\frac{g^2}{2}\phi^2\chi^2$. Two χ masses are shown. The significant decrease in $\langle(\delta\chi)^2\rangle$ in the massive-χ case can be simply understood from a Mathieu-type analysis - adding a χ mass simple increases the Mathieu parameter A, leading to a decrease in the Floquet index μ_k. From Khlebnikov and Tkachev [205].

$$\delta\ddot{\phi}_k \quad + \quad 3H\delta\dot{\phi}_k + \left(\frac{k^2}{a^2} + 3\lambda\phi^2 + 3\lambda\langle(\delta\phi)^2\rangle\right)\delta\phi_k = 0 \tag{2.44}$$

Equations (2.43,2.44) become a coupled set of ordinary integro-differential equations for the condensate and the quantum fluctuations which can be solved numerically [85, 77].

The results show that at a certain time the growth of quantum fluctuations is shut-off by the backreaction. See figure (2.7) which shows the Hartree-Fock approximation in the case where ϕ has a quadratic potential (2.16) and is coupled via the interaction term $\frac{g^2}{2}\phi^2\chi^2$ to the scalar field χ. In that case, the equation for ϕ in (2.44) above has $\lambda = 0$ and instead gains the term $g^2\langle(\delta\chi)^2\rangle\phi$ which again therefore leads to a change in the effective mass of the inflaton. We will see that the corresponding equation for $\delta\phi_k$ is best treated more completely than is possible with the Hartree-Fock approximation, as we will do later in the section on rescattering.

There are a couple of further points to add. Firstly, the above system of equations is classical. A full quantum study of these equations has been performed including regularisation and renormalisation of the bare quantities such as energy density and pressure [183]. However, qualitatively this does not really introduce new effects and the system is essentially classical after the first part of the resonance before backreaction. This is typically justified as due to the large occupation numbers $n_k \sim 1/\lambda$. One caveat to the previous statement is that in curved spacetime, renormalisation generically leads to a non-minimal coupling to the curvature $\xi \neq 0$, even if the bare coupling was minimal. This will serve as a justification for investigating *geometric reheating* in chapter (3).

2.8.2 The large-N expansion

The Hartree-Fock approximation is not significantly more complicated than our preliminary studies in section (2.4) and predicts the shut-off of the resonance. But how accurate is the approximation and can we derive it in some systematic expansion ? This might seem very unlikely since it contains non-perturbative effects, but in fact there is a way, based on a large-N expansion.

Let us consider a model for ϕ as an N-component vector with an $O(N)$-symmetric Lagrangian. We allow N to be a free variable. The Lagrangian density and potential for the O(N) vector model are given by:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi \cdot \phi) \tag{2.45}$$

and

$$V(\sigma, \phi) = \frac{1}{2}m^2 \phi \cdot \phi + \frac{\lambda}{8N}(\phi \cdot \phi)^2 \tag{2.46}$$

Here ϕ is an N dimensional $O(N)$ vector, $\phi = (\sigma, \pi)$ and (π) represents $N-1$ "pions". Notice the factor $\frac{\lambda}{8N}$ in the potential, which vanishes in the limit $N \to \infty$ at constant coupling λ. Hence we may study non-infinitesimal values of λ and still use perturbation theory, this time in the ratio $\lambda/(8N)$.

What is particularly elegant is that in this case, the large-N limit essentially reproduces the Hartree-Fock approximation, which becomes exact [183]. However, this is not always the case. When one couples ϕ to χ_i where now χ_i are N fields with an $O(N)$ symmetric Lagrangian, via the term (2.15), the χ_i gain a strongly ϕ-dependent mass given by $\frac{g^2}{N}\phi^2$ which is *not constant*, but oscillates during reheating. In this case, there are scattering terms which also survive the $N \to \infty$ limit, thus demonstrating the inadequacies of the Hartree-Fock approximation in non-equilibrium reheating. Indeed, rescattering, as these effects not included in the large-N limit are known, induces some very complex and interesting physics, which, at the classical level, bear strong similarities to nonlinear studies of turbulence, structure formation, magneto-hydrodynamics and so on [181, 9].

2.8.3 Non-equilibrium effects

One of the advantages of the above large-N and Hartree-Fock approximations is that they allow direct study of non-equilibrium effects. Clearly this is an issue of great interest since preheating leads to highly non-thermal spectra (see e.g. fig (2.3).

The general result typical of these non-equilibrium studies is that the quantum-backreaction cannot be described by a simple Markovian friction term [183, 4]. This means that the dissipation retains memory of the field value at earlier times. We will not describe in detail the techniques used to study out of equilibrium quantum fields (other uses for the same techniques include study of disordered chiral condensates and the quark-gluon plasma) but will attempt instead to carry over the bare essence only.

The most widely used technique is known as the closed-time-path (CTP) formalism and goes back

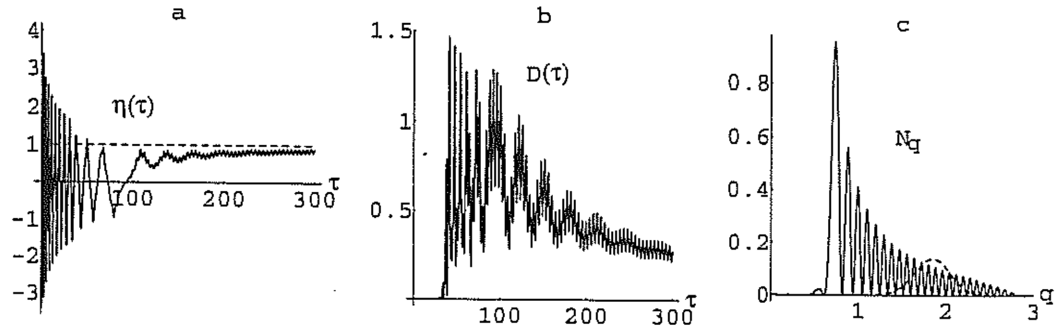


Figure 2.8: Non-equilibrium effects: (a) shows the evolution of the inflaton condensate (zero mode) in the large-N limit of an $O(N)$ symmetric scalar field. Note two non-equilibrium effects: (i) the drastic change to the oscillation frequency at around $\tau = 50$ and (ii) the sudden drop in amplitude at $\tau \sim 100$. (b) shows the growth of the variance of quantum field fluctuations while (c) gives the number of produced "pions" N_q vs $q = k/m$ at $\tau = 30$ (dashed line) and $\tau = 150$ (solid line). Adapted from [69].

to Julian Schwinger and Keldysh. The basic idea is to double the number of fields, and hence the number of Greens functions, and integrate along a closed time path in the complex plane [4].

This technique allows, in the large-N limit of an $O(N)$ symmetric model or in the Hartree-Fock approximation, to calculate the evolution of the inflaton zero mode. As mentioned above the non-perturbative effects give rise to a non-Markovian kernel in the evolution equation for the condensate. This yields two effects that can be seen in figure (2.8 a). Firstly the frequency of oscillation changes drastically due to the backreaction on the effective mass and secondly, the amplitude of oscillations exhibits near-discontinuous drops near $\tau = 50$ and $\tau = 100$. These sudden transitions are completely missed in Markovian models of dissipation based on adding the phenomenological term $\Gamma\dot{\phi}$ to the inflaton evolution equation, since that simply gives rise to an exponentially decaying envelope. This is a nice example of the power of non-equilibrium techniques since we see that most of the particle production - see part (b) of figure (2.8) - occurs in the narrow band between $\tau = 50$ and 100.

2.8.4 Changes to $m^2_{\phi,eff}$ and the Mathieu equation

Since one of effects of backreaction is to change the effective mass $m^2_{\phi,eff}$ of the inflaton, it is instructive to see how it affects the analyses based on the simple Mathieu equation. In the literature one typically finds that the mass term in the parameters A and q is replaced with $m^2_{\phi,eff}$. However, this leaves the ratio A/q invariant in the simplest models since both A and q are proportional to m_ϕ^{-2}, so that one might conclude that the Floquet index is not strongly dependent on $m^2_{\phi,eff}$ changes.

What has never been explicitly discussed in the literature is how the (A, q) parameters change due to having $\dot{m}^2_{\phi,eff} \neq 0$ during the backreaction phase. Here we remedy this. Again we start with the Klein-Gordon equation for the field χ and make the change of time variable $z = m_\phi t$, but now allowing for a time-dependent effective mass. As a result, the time derivative d^2/dt^2 now picks up $\dot{m}_\phi$

and $\ddot{m}_\phi$ terms which must be given by external inputs.

Crucially we find a term proportional to d/dz, i.e. a term associated with damping. We will neglect the expansion but since the effective mass typically changes in a very rapid fashion during violent preheating the small loss of accuracy in neglecting the expansion is more than made up for in clarity gained. The final, Mathieu-like, equation is (c.f. Eq. 2.18, $()' \equiv d/dz$):

$$\chi_k'' + \frac{m_\phi(\ddot{m}_\phi z + 2\dot{m}_\phi)}{(\dot{m}_\phi z + m_\phi)^2}\chi_k' + \frac{m_\phi^2}{(\dot{m}_\phi z + m_\phi)^2}\left(\frac{k^2}{a^2} + g^2\phi^2\right)\chi_k = 0 \tag{2.47}$$

Again it is evident that the ratio A/q is unchanged. The crucial development is the appearance of a χ_k' term typically associated with damping effects, assuming that the coefficient is positive.

In the Hartree approximation, it is possible to estimate the change to the effective mass in the simplest models [9]. We will briefly outline that calculation in order to discuss the effect on the resonant growth of χ_k fluctuations.

The effective mass for a quadratic inflaton potential is given in the Hartree approximation by:

$$m_{\phi,eff}^2 = m_\phi^2 + g^2\langle(\delta\chi)^2\rangle \tag{2.48}$$

$$\langle(\delta\chi)^2\rangle = \frac{1}{2\pi^3 a^3}\int dk k^2 a^3 |\chi_k(t)|^2 \tag{2.49}$$

Now, the simplest way to treat this is to neglect the oscillatory part of the $\chi_k(t)$ evolution and approximate it by its envelope. This captures the mean behaviour of the resonance and gives $|\chi_k(t)|^2 \propto \exp(2\mu m_\phi t) \simeq n_k/\omega_\chi$. The effective mass is then roughly:

$$m_{\phi,eff}^2 = m_\phi^2 + g\frac{n_\chi}{|\phi(t)|} \tag{2.50}$$

since $\omega_\chi \sim g|\phi(t)|$ due to the smallness of the momenta, k, of produced χ fluctuations, except when $\phi \sim 0$ when $\omega_\chi \sim k/a$. This means that $\dot{m}_{\phi,eff}$ is positive and oscillates.

Returning to Eq. (2.47), we see that the χ_k' coefficient has a complex behaviour even in this very simplified situation. To be consistent, we must neglect the oscillations in n_χ since we neglected the oscillations in Eq. (2.50). In that case $\dot{n}_\chi$ is strictly positive, and hence so is $\dot{m}_{\phi,eff}$, while $\ddot{m}_{\phi,eff}$ is negative at late times when the resonance starts to shut off.

We then see that the coefficient of χ_k' is positive almost everywhere and the mean effect of the change to the effective inflaton mass is to significantly damp the resonance. What happens when we reinsert the oscillatory behaviour of χ_k into the expression for the field variance $\langle(\delta\chi)^2\rangle$, Eq. (2.49) ? In that case the situation is significantly more complex and is partially described in Kofman *et al* [9].

Essentially the expression for $\langle(\delta\chi)^2\rangle$ contains an extra term $\mathrm{Re}(\alpha_k\beta_k^* \exp(-2i\int\omega_\chi dt))$ where α_k, β_k are the usual Bogoliubov coefficients [23]. The important point is that the exponential reduces approximately (when $\phi \neq 0$) to $\cos\left(\frac{2g\Phi}{m_\phi}\cos m_\phi t\right)$ which oscillates extremely rapidly with frequency $\sim 2g\Phi$. In the broad resonance regime this is $\gg m_\phi$, the underlying frequency of the inflaton oscillations. This high frequency modulation of the effective mass appears to have little effect on particle production [9] and can be understood from the numerical results we will present in chapter (4) for quasi-periodic potentials. In that chapter we consider potentials with two frequencies. When the frequencies are close together the change to preheating is large. When one of the frequencies is much higher than the other, the effect is a very small perturbation of the standard theory presented earlier in this chapter.

2.8.5 The effect of χ self-interaction

So far we have simply considered the reheated field, χ, to be a free field in the absence of the inflaton, with a mass m_χ. Now let us consider the χ field coupled to ϕ via eq. (2.15), and with χ exhibiting self-interaction:

$$V(\chi) = \frac{\lambda_\chi}{4}\chi^4 \tag{2.51}$$

The simple Mathieu-like analysis is altered because the equation of motion for χ_k now becomes, in the associated Hartree-Fock approximation:

$$\ddot{\chi}_k + 3H\dot{\chi}_k + \left(\frac{k^2}{a(t)^2} + 3\lambda_\chi\langle(\delta\chi)^2\rangle + g^2\Phi^2\sin^2(m_\phi t)\right)\chi_k = 0 \tag{2.52}$$

leading to the modified Mathieu-like parameters:

$$A_\chi(k) = \frac{k^2}{m_\phi^2 a^2} + \frac{3\lambda\langle(\delta\chi)^2\rangle}{m_\phi^2} + 2q \quad , \qquad q_\chi \sim \frac{g^2\Phi^2}{4m_\phi^2} \tag{2.53}$$

Self-interactions therefore cause the $\langle(\delta\chi)^2\rangle$ to appear directly in the equation of motion. This gives the χ an effective mass $\sim 3\lambda_\chi\langle(\delta\chi)^2\rangle^2$. As $\langle(\delta\chi)^2\rangle$ grows, this mass causes A to grow rapidly and hence there is vertical motion on the instability chart, which essentially shuts off the resonance since μ_k is a rapidly decreasing function of A.

The resonance effectively ends therefore, long before most of the inflaton energy is transferred to χ. In such a model, secondary scatterings prove to be very important in leading to large $\langle(\delta\chi)^2\rangle$ and $\langle(\delta\phi)^2\rangle$, as we will now describe. These *rescatterings* are not included in the Hartree-Fock, or Large-N approximations.

2.9 Rescattering effects

In the previous section we gave a simple pendulum-based analogy for the effect of backreaction, which tends to end the resonances well before all the condensate energy is transferred to χ particles. However, there is another effect that we have not discussed. In reality the inflaton corresponds to a bose condensate with all the non-zero momentum modes essentially empty. Nevertheless, we can think of an infinite hierarchy of ϕ_k-oscillators which are also coupled to the χ_k-oscillators. A natural effect, independent of the backreaction effect discussed above, is that the χ oscillators will transfer some of their energy to these non-zero momentum ϕ_k modes. In the particle picture, the stimulated, forward scattering effect occurs via "bose evaporation" when a χ particle interacts with a ϕ boson at zero momentum via:

$$\chi(nk_{res}) + \phi(k=0) \rightarrow \chi((n-m)k_{res}) + \delta\phi(mk_{res}) \tag{2.54}$$

thus shifting the χ particle to lower momentum and generally leading to a smoothing of the spectra of produced particles towards a thermal distribution.

This very interesting effect can be seen in fig. (2.9). Despite the fact that in this figure both the χ and ϕ fields have no self-interactions, and so a naive Mathieu-like analysis would insist that no $\delta\phi_k$ particles would be produced at $k \neq 0$, the $\langle(\delta\phi)^2\rangle$ actually grows to dominate over the $\langle(\delta\chi)^2\rangle$! This is due to scattering of the $\langle(\delta\chi)^2\rangle$ off the homogeneous inflaton mode $\phi(t)$. This efficient backscattering leads to strong $\delta\phi_k$ production.

There is another channel, "bose condensation", which is typically suppressed relative to (2.54), except at late times, in which a condensate boson is produced. This is very important since it leads to the production of higher and higher momentum modes (the "hard" modes).

$$\chi(nk_{res}) + \delta\phi(mk_{res}) \rightarrow \chi((n+m)k_{res}) + \phi(k=0) \tag{2.55}$$

Clearly the process given by (2.54) will dominate over the process (2.55) as long as the number density of the condensate is much larger than the effective number density at non-zero momentum [181]:

$$n_0 \gg \int \frac{d\mathbf{k} n_{\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{2.56}$$

where $\omega_{\mathbf{k}}$ is the effective frequency of the mode with momentum $\mathbf{k}$. At late times, when the n_k have large occupation numbers, the channel (2.55) becomes of comparable importance to (2.54) and the inflaton decay rate becomes very small [181]. We are then lead once more to the need for Yukawa-type couplings, such as $h\overline{\psi}\phi\psi$, involving only a single ϕ particle, if we wish for the inflaton field to decay completely. Otherwise we will be left with the inflaton as part or all of the dark matter of the universe [1, 2].

But why is $\langle(\delta\phi)^2\rangle > \langle(\delta\chi)^2\rangle$ in figure (2.9) ? This is due to the symmetric observation we made in the previous section. Due to the production of $\delta\phi_k$ particles, the χ_k equations of motion get a backreaction which manifests itself as a change in their effective mass. This has the effect of increasing A, i.e. causing a vertical motion on the instability chart of fig. (2.2) and leading to a rapidly decreasing μ and a rapid shut-off of the χ resonance, even if the previous backreaction mechanism (the alteration of $m_{\phi,eff}$) was not efficient.

To see this more precisely, consider the full equations of motion for the fields, including the polarisation operators, Π_ϕ and Π_χ. These can be given quite simply in the classical limit [9] by using the simple result that the Fourier transform of a product is the convolution of the individual Fourier

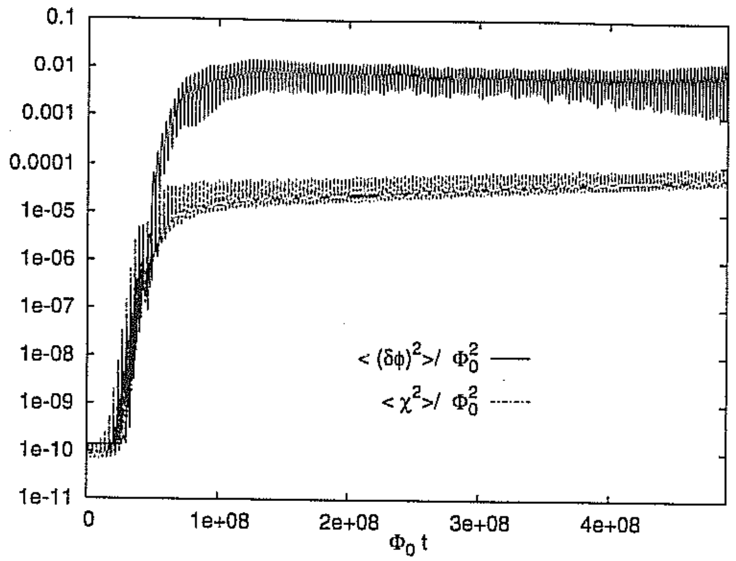


Figure 2.9: The variances of the fields $\langle(\delta\chi)^2\rangle$ and $\langle(\delta\phi)^2\rangle$ resolved using a full lattice simulation for the parameters $m = 10^{-6}\Phi$, $\lambda_\chi = \lambda_\phi = 0, g = 10^{-8}$ and $\Phi = 10^4$. From Prokopec and Roos [7].

transforms. Implicitly we have ignored this so far, as in e.g. Eq. (2.17). More precisely we have assumed that the inflaton field ϕ_k consisted only of a zero mode, i.e. $\phi_k = \phi(t)\delta(k)$, so that the convolution collapses to the term $g^2\phi^2(t)\chi_k$ on which our earlier studies were based.

As we have just outlined however, rescattering invalidates both this, and the better Hartree, approximation. Instead, for the quadratic potential and $g^2\phi^2\chi^2/2$ interaction term, the classical limit (i.e. ignoring quantum phases) yields the following equations for the rescaled variables [9]. The mode equation for $X_k = a^{3/2}\chi_k$ is

$$\begin{aligned}&\ddot{X}_k + \left(\frac{k^2}{a^2} + g^2\phi^2(t)\right) X_k \\ &= -\frac{g^2\phi(t)}{(2\pi)^3 a^{3/2}} \int d^3k' X_{\mathbf{k}-\mathbf{k}'}\varphi_{\mathbf{k}'} \\ &-\frac{g^2}{(2\pi a)^3} \int d^3k' d^3k'' X_{\mathbf{k}-\mathbf{k}'+\mathbf{k}''}\varphi_{\mathbf{k}'}\varphi_{\mathbf{k}''}. \end{aligned} \tag{2.57}$$

The mode equation for the non-zero momentum inflaton modes $\delta\phi_k(t) \equiv a^{-3/2}\varphi_k(t)$ is:

$$\begin{aligned}&\ddot{\varphi}_k + \left(\frac{k^2}{a^2} + m^2\right)\varphi_k = -\frac{g^2\phi(t)}{(2\pi)^3 a^{3/2}} \int d^3k' X_{\mathbf{k}-\mathbf{k}'} X_{\mathbf{k}'} \\ &-\frac{g^2}{(2\pi a)^3} \int d^3k' d^3k'' \; \varphi_{\mathbf{k}-\mathbf{k}'+\mathbf{k}''} \; X_{\mathbf{k}'} \; X_{\mathbf{k}''}. \end{aligned} \tag{2.58}$$

The equation for the oscillating background field $\phi(t)$ is finally given by:

$$\begin{aligned}&\ddot{\phi} + 3H\dot{\phi} + m^2\phi = -\frac{g^2\phi}{(2\pi)^3 a^3} \int d^3k' X_{\mathbf{k}'}^2 \\ &-\frac{g^2}{(2\pi)^3 a^{9/2}} \int d^3k' d^3k'' \; \varphi_{\mathbf{k}''-\mathbf{k}'} \; X_{\mathbf{k}'} \; X_{\mathbf{k}''}. \end{aligned} \tag{2.59}$$

At the first stage of the resonance, all the right hand sides of these three equations are zero. Eq. (2.57) then reduces to the Mathieu equation, Eq. (2.57) describes a free harmonic oscillator with no resonance and Eq. (2.57) is just the free evolution of the inflaton condensate under gravity.

The full set of equations describes two infinite sets of *classical* coupled harmonic oscillators and is therefore very similar to systems found in studies of turbulence and gravitational collapse in the nonlinear regime.

2.9.1 Parasitic growth of $\delta\phi_k$ fluctuations

In figure (2.9) from [7], it is evident that $\delta\phi_k$ fluctuations grow very quickly after initially being dormant and that they dominate the χ_k variance. Since this figure is for the case with no ϕ self-interaction, this cannot be due to parametric resonance, as we mentioned earlier.

By examining Eq. (2.58) we see that the first term on the r.h.s is the driving term $\propto \phi(t)X * X$ where $*$ denotes convolution. This term is almost $\phi(t)\langle(\delta\chi)^2\rangle$ of the Hartree approximation (which does appear on the RHS in the equation for $\phi(t)$, (2.59)). From these three equations we now know the form of the terms missing from the Hartree-Fock approximation.

While the complete set of equations is not possible to solve analytically, we can understand the effects in figure (2.9) with one simplification. Let us assume that the X_k are growing exponentially as they do in the first phase of the resonance, $X_k \propto e^{\mu_k mt}$ for k in a resonance band. Since the $\varphi_k \simeq 0$ initially, we can solve Eq. (2.58) iteratively via Green's functions to get the solution (neglecting the second term on the r.h.s. of Eq. 2.58):

$$\varphi_k \propto \int dk' k'^2 X_{k-k'} X_{k'} \tag{2.60}$$

$$\simeq \int dk' k'^2 e^{2\mu_k mt} \tag{2.61}$$

and hence the number density of $\delta\phi_k$ particles grows (see Eq. (2.22)) like $n_{\delta\phi} \propto n_\chi^2 \sim e^{4\mu mt}$! This is precisely what we see in figure (2.9) and is an example of an effect which was discovered in numerical lattice simulations first and then only understood analytically afterwards [9].

In chapter (5) we will study the rescattering equations in more detail and use them to study the qualitative differences between models of reheating presented in this chapter and the ones developed in the rest of the thesis.

2.10 Thermalisation and turbulence-like effects

At the end of preheating the occupation numbers are typically huge, but the individual energies and momenta of particles is typically very small, with k close to zero, i.e. corresponding to long-wavelength fluctuations with a non-thermal distribution. However, the strong couplings between the fields (needed for preheating to occur in the first place) ensure that the system relaxes quickly to local thermal equilibrium, leaving the universe in a radiation-dominated FLRW state. As we will show in chapter (6), this is important since it ends the oscillations in the curvature which resonantly amplify the stochastic background of gravitational waves. Further, as we discussed at the end of the previous chapter, the final reheating temperature is crucial in determining the gravitino abundance for which there are strict upper bounds coming from baryogenesis of $T_r \sim 10^8 - 10^{10} GeV$.

If preheating generically lead to temperatures significantly above this temperature, preheating would be in trouble, without some other mechanism to help reduce the gravitino abundance. Clearly then, the thermalisation of the large $\langle(\delta\chi)^2\rangle$ and $\langle(\delta\phi)^2\rangle$ after preheating is a crucial issue. The problem is that it is also highly non-trivial. At present there does not exist any way to estimate T_r in terms of the input parameters (such as the couplings and amplitude of ϕ oscillations) at the start of preheating.

However, one could make some naive attempts at estimating the temperature based on knowledge of the occupation number distribution at the end of the resonance. Let us assume that most particles lie around the narrow band at $k = k_*$. If we assumed (wrongly) that the particles were already part of a Bose-Einstein distribution, then we would have:

$$n_k = \frac{1}{e^{(E-\mu)/T} - 1} \qquad (2.62)$$

where μ here is a chemical potential. In the Wien (low frequency) part of the spectrum gives this gives $n_k \propto T_{eff}/k$. From this we would estimate a huge temperature T_{eff} which is of course completely fictitious because the rest of the distribution at higher frequencies is missing. One might then conjecture that rescattering would create higher momentum particles via bose condensation, but it is fairly easy to see that it is not possible to produce a Bose-Einstein spectrum while conserving particle number.

The closest that anyone has come so far to understanding thermalisation is through lattice simulations. Figure (2.10) shows some lattice results. We see that even when rescattering becomes effective and fills in the regions between the resonance bands, the particle number continues to increase without depleting the occupation number of the resonance bands. Further, even at the largest times, there is no real sign of a thermal spectrum. The final curve shows an exponential decay $\propto \exp(-k)$ but the turnover still occurs at k_*, the wavelength of the first resonance band.

At this stage we can imagine that this low-energy ensemble decays via single-body (number violating) decays via terms such as $h\overline{\psi}\phi\psi$ to much lighter particles, in which case the theory presented in section (2.3) may be valid.

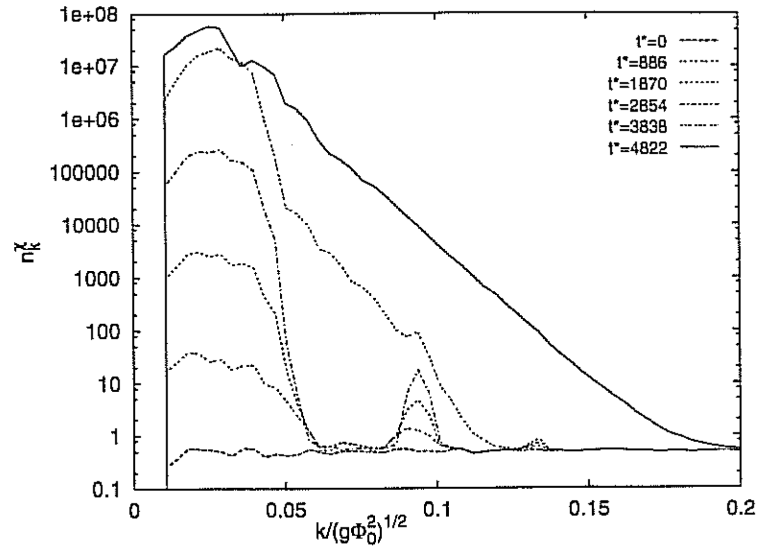


Figure 2.10: The occupation number n_χ as a function of k and time from 3-d numerical lattice simulations including rescattering effects. Initially three resonance bands are visible (bottom) while rescattering dominates at late times and causes a filling in of modes between the resonance bands. From Prokopec and Roos [7].

2.11 The effects of expansion on reheating

The effects of expansion on reheating are interesting and varied. Here we review results from the literature and delay until chapter (3) a discussion of geometric reheating which is driven by oscillations in the Ricci curvature.

2.11.1 Stochastic resonance

In section (2.4) we neglected the expansion of the universe, allowing us to study an exact Mathieu equation, with a given mode k in a fixed stability or instability band. However, when one includes the expansion of the universe, all modes are redshifted $k \to k/a(t)$ and the amplitude of inflaton oscillations decreases as $\Phi \sim t^{-1}$. This causes the parameters (A, q) to tend to $(0,0)$, and hence to remove the resonance, as we will discuss in the next section. However, an exciting effect preludes this damping.

At the epoch of the first oscillations, the amplitude ϕ decreases extremely rapidly. Indeed, one has approximately that $\Phi \sim 1/N$, where N is the number of oscillations undergone by the inflaton. We thus see that between $N = 1$ and $N = 2$, Φ drops by a factor 2, and hence q drops by a factor 4. If $q_{(N=1)} = 1000$, $q_{(N=2)} = 250$.

The crucial point about this is that this does not correspond to motion within a single band. Rather, it corresponds to rapid motion through many bands. Indeed, one can estimate the band number which is dominant at a value of q in the broad resonance regime. It is $n \simeq \sqrt{A} \geq \sqrt{2q}$.

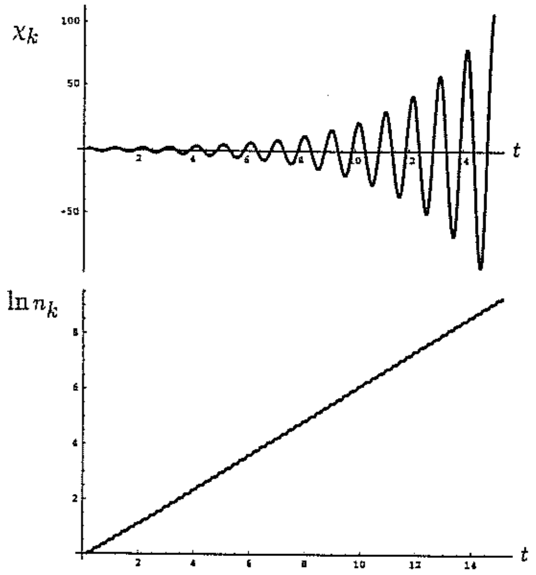


Figure 2.11: Narrow resonance, $q \sim 0.1$, showing the almost constant growth of the field during all of the inflaton oscillations. From [9].

Thus we see that the field moves through about $\sqrt{2q}/2$ resonance bands between the first and second inflaton oscillations ! If $q = 10^6$, for example, this means traversing around 700 resonance bands. Within ten oscillations, the field traverses 90% of all the possible bands.

This means that an adiabatic approach, based on the slow change of the parameters (A, q) relative to the inflaton oscillations, fails completely. Instead, one must use the sudden approximation, or as done by Kofman, Linde and Starobinsky [9], make some inspired simplifications. They proceed by noticing that in the broad resonance case ($q \gg 1$), χ particle production in the simple quadratic model (2.17) takes place not continuously, but rather in brief, highly non-adiabatic periods when $\phi(t) \approx 0$.

We can understand this behaviour in terms of the general power for particle production in an expanding spacetime which is strongly related to the dimensionless ratio $\dot{\Omega}_k/\Omega_k^2$, where Ω_k is the time-dependent frequency of the field under investigation. If this ratio is less than unity, then particle production is exponentially suppressed. On the other hand, if it is much greater than unity, then the adiabatic approximation is strongly violated and extensive particle production can occur. To understand the transition that occurs in going from the mild resonance (small q) to broad resonance ($q \gg 1$) cases that is illustrated in figures (2.11) and (2.12), let us write down the ratio $\dot{\Omega}_k/\Omega_k^2$ for the χ_k field:

$$\Omega_k = \frac{k^2}{a^2} + g^2\phi^2(t) \tag{2.63}$$

$$\frac{\dot{\Omega}_k}{\Omega_k^2} = \frac{-2k^2\dot{a}a^{-3}\phi^{-1}g^{-2} + \dot{\phi}}{\left(\frac{k^4}{g^2\phi a^4} + 2\frac{k^2}{a^2}\phi + g^2\phi^3\right)} \tag{2.64}$$

which is valid as long as $\phi \neq 0$. We can gain a wealth of information from this: (i) particle production is a decreasing function of k, it is more difficult to produce high-momentum (and hence energy)

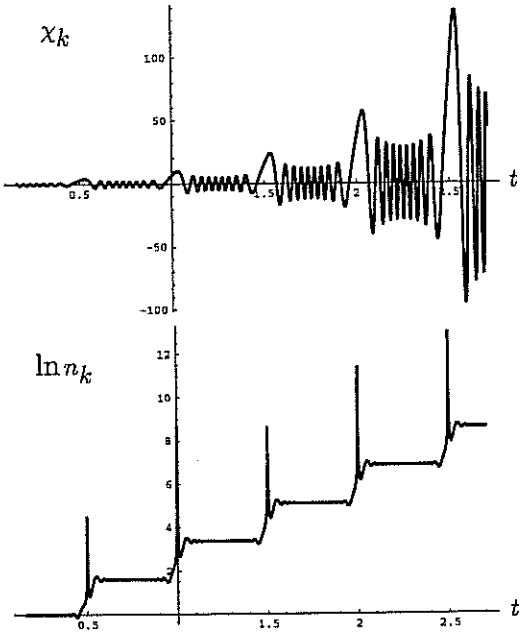


Figure 2.12: Broad resonance ($q \sim 200$) in Minkowski spacetime. Notice the sudden jumps in occupation number near the points where $\phi = 0$. From [9].

particles. (ii) the two terms in the numerator of Eq. (2.64) are maximised for different values of ϕ. The first term, $\propto \dot{a}$ is maximised at $\phi = 0$, while the second term is actually maximised at $\phi_m = \pm k/(3ga)$, so only for the infinite wavelength case $k = 0$ does most of the particle production actually occur when $\phi = 0$. This is shown in figure (2.13) where it is evident that most particle production will occur just before and after $\phi = 0$. More importantly perhaps, as g increases, the graphs of $\dot{\Omega}_k/\Omega_k^2$ become rapidly more and more peaked around the points ϕ_m which converge to $\phi = 0$. This explains why there is such a radical shift in the nature of particle production from the narrow-resonance to the broad resonance cases.

Having understood the reason for the change in the time evolution of n_k as a function of the parameter g, we can get a good quantitative grasp of the numerics by exploiting this insight. Near $\phi \approx 0$, the oscillating factor $\sin^2(m_\phi t)$ appearing in eq. (2.17) can be Taylor expanded to give $g^2\phi^2 \approx g^2\Phi^2 m_\phi^2 (t - t_j)^2$ where t_j label the zeros of $\phi(t)$. The Mathieu equation (2.19) is thus replaced with the problem of scattering in successive parabolic potentials, a problem for which the transmission and reflection coefficients are well known. This approach to wide resonance (without the above observation regarding the violation of the adiabaticity condition near $\phi = 0$ at large g) was first used by Yoshimura [5]. With the new insight presented above, we know that the particle number is essentially constant between the zeros of ϕ, and our differential equation collapses essentially to a discrete difference equation.

The full wavefunction for this problem consists of knowing the Bogoliubov coefficients α_j, β_j *and*

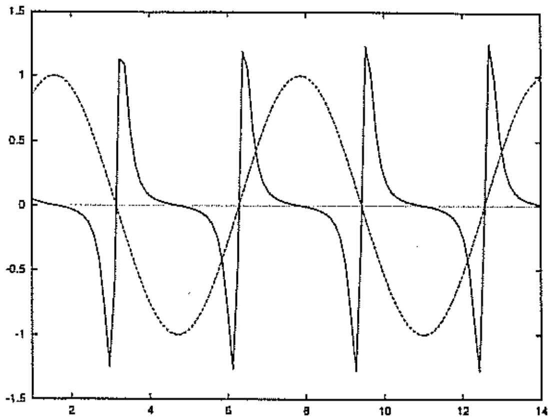


Figure 2.13: The ratio $\dot{\Omega}_k/\Omega_k^2$ as a function of $m_\phi t$ for the quadratic potential and $k = 1, g = 4$. Superimposed is the arbitrarily normalised inflaton evolution ($\phi \sim \sin m_\phi t$) with expansion effects not included. Note that $|\dot{\Omega}_k/\Omega_k^2|$ reaches its maximum near, but not at $\phi_k = 0$. The value of ϕ at this maximum varies as $\sim g^{-1}$ and the ratio becomes rapidly more peaked around these maxima as g increases, leading to the explosive jumps in occupation number.

the phase θ_j of the solution. The Floquet exponent for this problem can then be given as [9]:

$$\begin{aligned} \mu_k^j &= \frac{1}{2\pi}\ln\left(1+2e^{-\pi\kappa^2} - 2\sin\theta_{tot}^j e^{-\pi\kappa^2/2}\sqrt{1+e^{-\pi\kappa^2}}\right) \\ \kappa^2 &= (A_k - 2q)/2\sqrt{q} \end{aligned} \tag{2.65}$$

where θ_{tot}^j is the total phase accumulated at the time of the j-th zero. This explicit expression for the Floquet index is impressive since it is completely non-perturbative, $q \gg 1$. To be useful we must know θ_{tot}^j however, and this can be done explicitly in the case of no expansion.

When the expansion of the universe is included there are two interesting novelties. Firstly, $\theta_{tot}^j \in [0, 2\pi]$ so that it's contribution to μ in eq. (2.65) can cause an increase or reduction of the particle number, a purely quantum effect. More interesting perhaps, one can estimate the change in phase between successive zeros of the inflaton [9]:

$$\delta\theta_j \simeq \frac{gM_{pl}}{20m_\phi N^2} = \frac{\sqrt{q_0}}{2N^2} \tag{2.66}$$

where we have put q_0 to emphasise the *initial* value of q. The crucial observation, which leads to the name stochastic resonance, is that if we consider $q_0 \gg 1$, then $\delta\theta_j \gg 1$ during the first few inflaton oscillations. But being a phase, $\delta\theta_j \in [0, 2\pi]$. Therefore the change in phase between zeros is of the form $Z \ mod \ 2\pi$, $Z \gg 1$ which is the classic pseudo-random number generator used in most computer languages and simulations. The phase therefore behaves as a random variable, hence so does μ_k. The Floquet exponent ceases to resemble in any way that implied by the instability chart (fig. 2.2). This stochastic resonance lasts only for a few oscillations however until:

$$N_{stoch} \simeq \frac{q_0^{1/4}}{\sqrt{2\pi}} \tag{2.67}$$

which for $q_0 = 10^4$, is about five oscillations. Afterwards, the change in phase rapidly decreases $\sim 1/N^2$ and the Floquet exponent settles down to a more orderly existence, converging to a sedate,

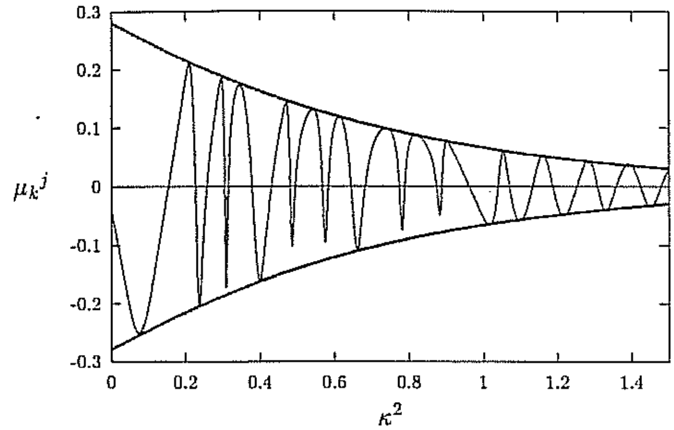


Figure 2.14: The characteristic exponent μ_k in the stochastic case as a function of $\kappa^2 = (A_k - 2q)/(2q)$ for the initial value of the parameter $q = (32\pi)^2 \approx 10^4$. The curve is obtained at the time after the first 5 oscillations, which corresponds to μ_k^j with $j = 10$. The envelope of the curve is obtained from Eq. (2.65) by taking there $\sin\theta = \pm 1$. The resonance is much broader, there are no distinguished stability/instability bands, and for certain values of momenta the function μ_k^j is negative. During the stochastic resonance regime, this function changes dramatically with every half period of the inflaton oscillations. Compare with fig. (2.3) for μ_k of the Mathieu equation. From Kofman *et al* [9].

adiabatic movement on the instability chart, fig. (2.2). The main effect of this is to damp the resonances, as we will now describe.

2.11.2 Expansion in the adiabatic realm

Now by introducing the variable $y = chi'$, Eq. (2.18) is converted into a $2-d$ *non-autonomous* dynamical system. The explicit time dependence of the system implies that trajectories can both self-intersect and intersect other trajectories, as can be seen in figures (2.15, 2.16). Nevertheless they are useful discriminators of evolution characteristics.

Typical Poincaré disks (compactified phase planes) are shown in figs. (2.15,2.16) for (A, q) values in the stable and unstable bands respectively. Fig. (2.16) is the trajectory of a single point as it converges to the boundary. Introducing the expansion of the universe alters the nature of the phase planes considerably. In Fig. (2.17) two trajectories for the same (A, q) values are shown. The resonance begins in the same way but at a certain stage the dissipative effects of the expansion start to win and the solution spirals into the origin.

In the case when the spatial curvature is zero, $K = 0$, the Raychaudhuri equation is:

$$\dot{H} = -\frac{1}{2}\dot{\phi}^2 . \tag{2.68}$$

The Friedmann equation can be solved perturbatively with the parameter $\epsilon = H/m_\phi$ to yield [150]:

$$H = \frac{2}{3t}\left[1 - \frac{\sin(m_\phi t)}{2m_\phi t}\right] \tag{2.69}$$

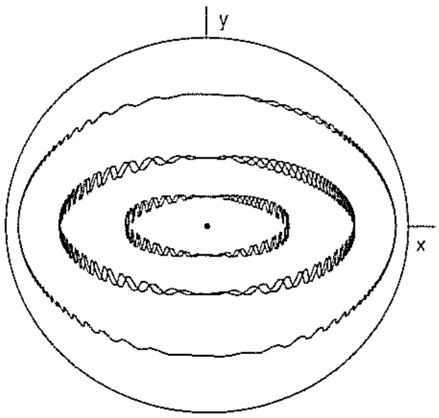


Figure 2.15: Poincaré disk for the Mathieu equation. The orbits of three typical trajectories within a stable band are shown. The bounded, periodic nature of the evolution is clear. $(A, q) = (0.13, 0.88)$. From [149].

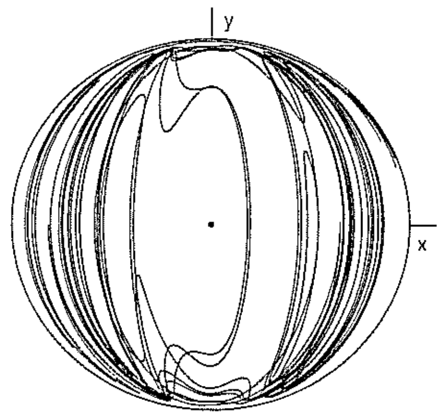


Figure 2.16: The evolution of a single trajectory of the Mathieu equation with $(A, q) = (3, 33/2)$, i.e. in the strongly unstable band. Notice the space-filling chaotic approach to the border at infinity in contrast to the well-behaved stable band motion. From [149].

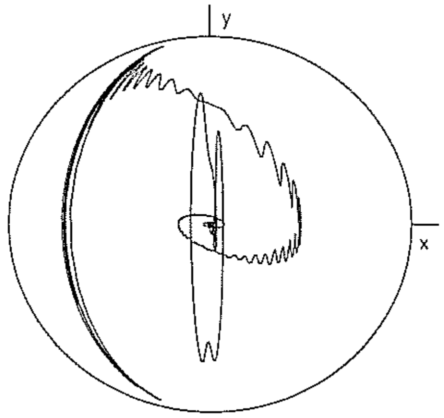


Figure 2.17: The effect of expansion. From [149].

This is only valid after the end of the preheating when the amplitude of oscillations is $\Phi < M_{pl}$. This equation can be solved perturbatively to give [11]:

$$a(t) \doteq \overline{a} \exp\left(\frac{\sin 2m_\phi t}{3m_\phi t} - \frac{2\mathrm{ci}(2m_\phi t)}{3}\right) \tag{2.70}$$

where $\overline{a} = t^{2/3}$ is the background EDS evolution, and $\mathrm{ci}(m_\phi t) = -\int_t^\infty \cos(m_\phi z)/z dz$. This example explicitly demonstrates how temporal averaging (which yields $\overline{a}$) removes the resonance.

In the special case where $V = \lambda\phi^4/4$, with non-minimal coupling $\xi = 1/6$ and $K = 0$, the whole system is conformally invariant and hence the expansion has no effect on the initial resonant decay of the inflaton [155].

2.12 Non-thermal symmetry restoration

Until now we have allowed for potentials with mass terms and self-interactions but not for potentials with broken symmetry. This seemingly small step has lead to the most controversial aspect of preheating to-date: the idea that symmetry can be restored by the large quantum corrections to the effective potential.

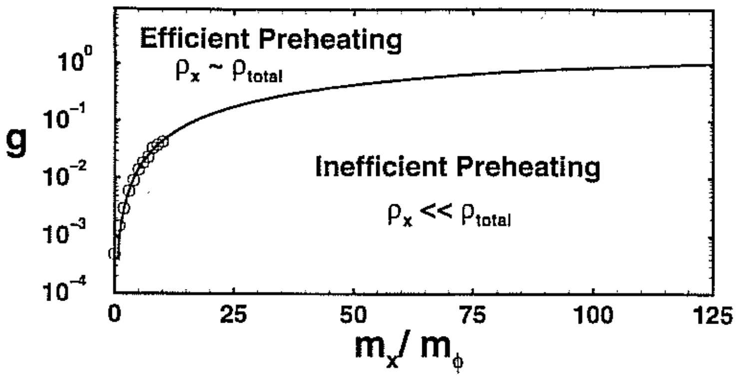


Figure 2.18: The coupling-mass parameter plane for the simple Mathieu equation in an expanding universe, showing the significant difficulty of reheating to massive bosons. From [156].

Let us consider the simple $\mathbf{Z}_2$ symmetric potential with broken symmetry:

$$V(\phi,\chi) = \frac{\lambda}{4}(\phi^2 - \phi_0^2)^2 + \frac{g^2}{2}\phi^2\chi^2 \,. \tag{2.71}$$

This gains the following quantum corrections to V due to the $\langle(\delta\phi)^2\rangle$ and $\langle(\delta\chi)^2\rangle$:

$$\Delta V = \frac{3\lambda}{2}\langle(\delta\phi)^2\rangle\phi^2 + \frac{g^2}{2}\langle(\delta\chi)^2\rangle\phi^2 + \frac{g^2}{2}\langle(\delta\phi)^2\rangle\chi^2 \tag{2.72}$$

[131]. In (equilibrium) thermal field theory, we have that $\langle(\delta\chi)^2\rangle, \langle(\delta\phi)^2\rangle \propto T^2$, the temperature squared. It is then easy to recover from eq. (2.72) that the symmetry is restored in the large-T limit. But what happens during preheating when the fluctuations in eq. (2.72) are completely non-equilibrium ?

Well, because the average energy of produced particles is very small (μ_k is greatest for small A and hence small k), the occupation numbers of modes is huge and is again dominated by the small-k part of the spectrum. This implies that the variances in (2.72) are much larger than one would naively expect. Indeed, from energy conservation, one has the bound [131] $\langle(\delta\phi)^2\rangle \leq Cg^{-1}\lambda M_{pl}\ln^{-2} g^{-2}$ with $C \sim 10^{-2} - 10^{-3}$. Saturating this bound implies that Eq. (2.71) can have a positive effective mass, $m^2_{\phi,eff} \equiv (V + \Delta V)''$ even for a GUT symmetry breaking scale $\phi_0 \sim 10^{16} GeV$.

As we saw from fig. (2.9), the variances tend to oscillate rapidly about a mean value which slowly decreases due to the expansion (gravitational evolution of the corresponding energy density perturbations having been completely ignored until now). Therefore it is plausible that symmetry could be restored for a considerable amount of time, certainly enough for the mean field ϕ to cross the origin and hence probe the alternative vacua of the theory. This naturally leads to the strong possibility of defect formation at reheating, including monopoles. We will see in chapter (5) that this is indeed the case in multi-field models ($n > 2$), especially at strong coupling. Further, the variances $\langle(\delta\chi)^2\rangle$ and $\langle(\delta\phi)^2\rangle$ estimated using the Hartree-Fock approximation are typically significantly smaller than those that are obtained from full $3-D$ lattice simulations due to the neglect of rescattering effects.

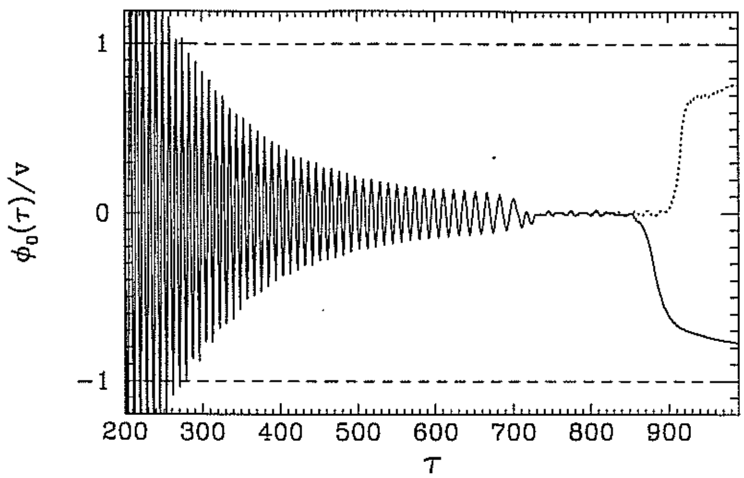


Figure 2.19: The evolution of the inflaton zero-mode as a function of time. Note that even once oscillations have amplitude less than $v \equiv \phi_0$, the oscillations are still centered on the origin $\phi_0 = 0$, rather than converging to one of the absolute minima at $\phi_0 = \pm v$, thus indicating the restoration of symmetry. At time ≈ 850 the symmetry is broken and the field starts to converge to $\phi_0 = \pm v$ in the two realisations shown. This random choosing of which vacuum to fall into will lead to defect formation via the Kibble mechanism. From [70].

2.12.1 First order non-thermal phase transitions

How does the non-thermal symmetry breaking occur ? Is it first order, weakly first-order or of second order ? Recently, Khlebnikov *et al* have claimed that typically the transition is of first order when the couplings in eq. (2.71) obey $g^2 \gg \lambda$. Presumably, as this condition is continuously weakened, the transition becomes weakly first order and then finally of second order.

Given the controversy over the existence of NTSR, the great advantage of a first order phase transition is that it occurs via nucleation of bubbles, which are relatively distinctive numerically. The bubbles of true vacuum then expand and collide, eventually leaving no regions of false vacuum. Such an event is shown in figure (2.20) resulting from a $3-D$ 128^3 lattice simulation of preheating.

The sufficient conditions that the phase transition be of first order transition are [70]:

(i) At the phase transition, $\phi = 0$ should be a local minimum of the potential. From Eq. (2.71) this means that $g^2\langle(\delta\chi)^2\rangle > \lambda\phi_0^2$.

(ii) The typical momentum of particles should be smaller than $g\phi_0$ to ensure that there is a potential barrier between $\phi = 0$ and the minima at $\pm\phi_0$.

Once the minima at $\pm\phi_0$ becomes deeper than that at the origin, fluctuations drive the system over the barrier creating an expanding bubble driven by a pressure due to the difference in potential at $\phi = 0$ and $\pm\phi_0$.

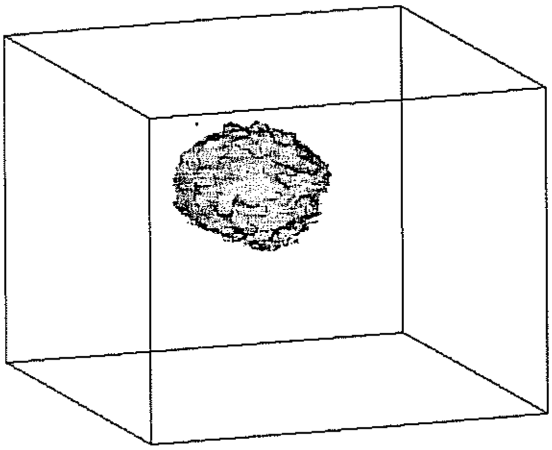


Figure 2.20: A numerically detected bubble of true vacuum nucleated via a first order phase transition after non-thermal symmetry restoration due to preheating. This is a surface of constant $\phi = -0.7\phi_0$. Inside the bubble $\phi < -0.7\phi_0$. From [70].

If these conditions are violated then the phase transition will most likely be of second order in which case there is no bubble nucleation. In either case there will be defect formation via the Kibble mechanism [78]. In the case of a first order phase transistion however, the collision of bubbles is expected to release a large amount of high-frequency gravitational radiation [207] which may be detectible with future gravitational wave experiments.

2.13 A "realistic" model of reheating from supergravity

Finally, to end this review chapter we study reheating in the simplest model of hybrid inflation based on supergravity using the same theory we studied in section (1.7) at the end of chapter (1). It is based on the superpotential:

$$W = \phi(\alpha\overline{\sigma}\sigma - \mu^2) \tag{2.73}$$

which, using Eq. (1.82), gives the tree-level hybrid inflation potential (we take $\sigma = \overline{\sigma}$):

$$V = 2\alpha^2|\phi|^2\sigma^2 + |\alpha\sigma^2 - \mu^2|^2 \tag{2.74}$$

Note that while supersymmetry is preserved there are no single body inflaton decays and if this were to remain true always there would almost surely be some amount of dark matter in the form of the inflaton field, since the concentration of inflaton particles drops rapidly with the expansion and the decay rate tends to zero just as happens during nucleosynthesis to protons and neutrons. Unbroken supersymmetry therefore naturally yields scalar-field models of dark matter now known as quintessense or Q-matter [71].

However, supersymmetry is broken during inflation by definition since the vacuum has non-zero energy density and so the above potential gains one-loop (and higher) radiative corrections. In section

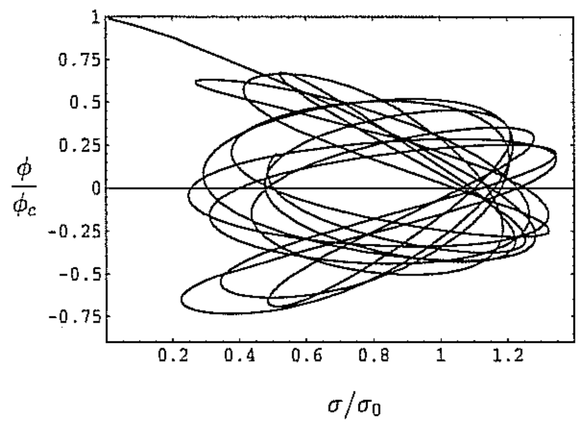


Figure 2.21: The $\phi - \sigma$ phase plane showing the rather chaotic evolution.From [138].

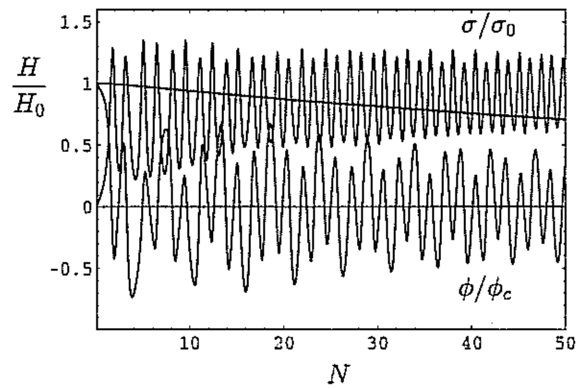


Figure 2.22: The time evolution of the fields ϕ and σ showing the very slow decrease in the amplitude as a function of time compared with the single field case. From [138].

(1.7) we studied the decay of the fields to neutrinos completely ignoring resonance effects. Now we are a position to understand what preheating looks like in this model.

The fact that there is only one coupling, λ, in the superpotential (2.73) means that both ϕ and σ are dynamically important and the evolution is typically chaotic (see figure 2.21) and even when the expansion of the universe is included the amplitudes of the oscillations decrease very slowly due to the fact that the evolution of $\dot{H}$ is highly non-monotonic [138]. The fact that a grows very slowly on average in this region of weakly chaotic oscillations is very interesting since the horizon size $\propto t$ grows much faster relative to a than it would in a single field model.

If we now couple ϕ and σ to our canonical massless scalar field χ via the interaction terms:

$$\tfrac{g^2}{2}\phi^2\chi^2 + \tfrac{h^2}{2}\sigma^2\chi^2 \tag{2.75}$$

where as usual g, h are dimensionless coupling constants, we are lead to the usual Klein-Gordon equation of motion for χ but with a mass term [138]:

$$m_\chi^2 = g^2\phi^2(t) + h^2\sigma^2(t)\,, \tag{2.76}$$

Also of interest is the evolution of $\delta\sigma_k$ fluctuations which obey the Klein-Gordon equation with mass term:

$$m_\sigma^2 = 12\alpha^2\sigma^2(t) + 2\alpha^2\phi^2(t) - 4\alpha^2\mu^2 \tag{2.77}$$

so that while the couplings determining the χ_k fluctuations are arbitrary, those in the $\delta\sigma_k$ equation are fixed by CMB anisotropy data.

Figure (2.23) shows the results from integration of the coupled equations for the occupation number for χ_k as a function of the number of oscillations of the σ field. Now to good approximation during reheating we have that:

$$\phi = \Phi \sin m_\phi t \quad , \qquad \sigma = \Sigma \sin m_\sigma t \tag{2.78}$$

so that the masses oscillate just as one would expect for a resonance system. But if one tries to convert the equations for χ_k or $\delta\sigma_k$ (the $\delta\phi_k$ case is similar to the $\delta\sigma_k$ case) we are faced with an immediate dilemma. A crucial step is to make the equations dimensionless which we did in section (2.4) and (2.5) by switching to a new time such as $z = m_\phi t$. If we do that then we find terms of the form $\sin^2(\frac{m_\sigma}{m_\phi}z)$ in the equations and this cannot be reduced to Mathieu form except for very special values of the ratio $\frac{m_\sigma}{m_\phi}$. In particular a minimum requirement is that the ratio be an integer.

So the natural question is, "what happens when the ratio is not an integer ?" What happens when it is some other fraction, or as turns out to be much more interesting, what happens when the *ratio is irrational ?* This is the partly the subject of chapter (4) and actually defines a new reheating class, missed in [138]. In [138] they further claim that the chaotic phase of oscillations does not lead to any particle production. In chapters (4) and (5) we reexamine this issue more closely with analytical and numerical results which point strongly to the opposite conclusion - particle production is generically enhanced in the case of stochastic field evolution.

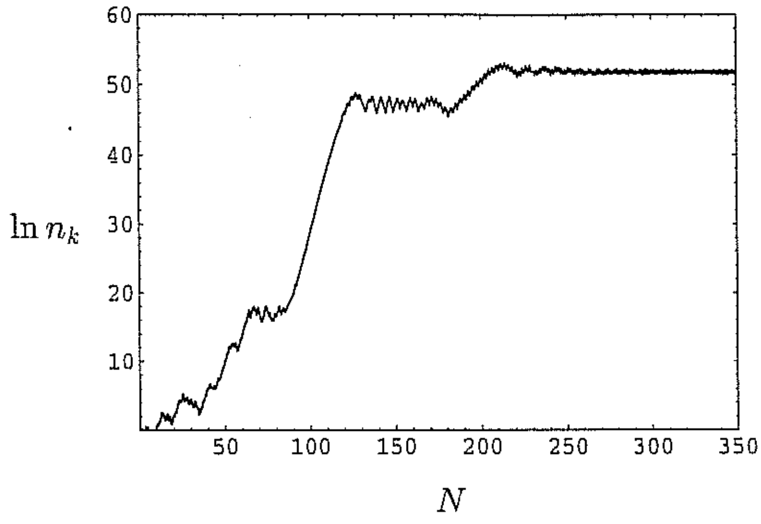


Figure 2.23: The exponential growth of the occupation number versus the number of σ oscillations N for the mode $k = \mu$. From [138].

2.14 Where the thesis fits into the reheating hierarchy

In this chapter we have reviewed the stages of reheating as they are now known. Almost certainly there are major changes in store but at present it is safe to make the decomposition into four conceptual, but not chronological, phases. The first phase is the initial resonance. This is very strong in preheating and weak in perturbative reheating. This is also the phase on which the work in this thesis concentrates most since it is the phase which lays the foundation for the rest of reheating.

Let us make an analogy. If reheating were a car, then the resonance is like the engine. With a big V12 engine, acceleration can be very large, but drag goes up as the square of the velocity and soon backreaction becomes important. If aerodynamics of the car are good then backreaction need not cause a large breaking effect, if the aerodynamics are bad (corresponding to large self-interaction in the χ field; section 2.8.5) the maximum speed of the car is low. Depending on the situation and design of the car, eddies and essentially nonlinear effects could lead to unexpected instabilities – and back-propagation on the ambient medium – this is like rescattering. Finally, the return to rest of the car and the surrounding air once the engine is turned off and the car free-wheels to a stand-still is like thermalisation. How the turbulent eddies dissipate their energy and return to laminar flow is almost precisely the classical analog of thermalisation after preheating.

The engine is thus the focus of most of this thesis, although we will consider the other aspects, especially in chapter (5). In the next chapter we look at geometric reheating where the driving engine for the resonance comes directly from the geometry. In chapter (4) we use elegant spectral methods to classify all qualitatively different types of engines for resonance. In chapter (5) we put everything together and examine what effects multiple fields have on reheating. In our analogy, the question is how several engines in the same car affect performance. Naturally adding several engines into a car body on a single chassis affects aerodynamics and drag too. In the same way we examine the implications of multiple fields, arising naturally in GUT and supergravity theories, for preheating, backreaction and rescattering. Finally in chapter (6) we examine the effect of reheating on the evolution of gravitational waves and find that they too are amplified, leaving a signal that may be detectable with future experiments such as LIGO.

Chapter 3

Geometric Reheating After Inflation

Time is nature's way of keeping everything from happening at once.
Woody Allen

The first sign of a nervous breakdown is when you start thinking your work is terribly important.
Milo Bloom

3.1 Introduction

Having reviewed the current status of reheating [1] in the previous chapter, we now move on to a reheating based purely on General Relativity. However, its motivation comes from particle physics and renormalisation in curved spacetime. In fact, this chapter is largely based on giving the answer to the question "if all couplings of the inflaton to other fields are weak, must reheating be of the perturbative form described in section (2.3) or is preheating in some form still possible ?"

The answer to the last question is yes and its implementation is the substance of *geometric reheating*. The idea of geometric reheating is quite simple. At the end of chaotic inflation the Ricci curvature follows the inflaton in executing coherent oscillations. If fields other than the inflaton are non-minimally coupled to the curvature then this leads to a gravitational preheating.

3.2 Renormalisation in curved spacetimes

As we discussed in chapter (1) (see also [136, 81, 85]), renormalisation in curved space time generically leads fields to be non-minimally coupled even if the bare coupling was minimal. Further, all the couplings in a theory typically run with energy, except in a few very special cases. Indeed, this is the very foundation of the GUT concept and the basic ingredient of almost all unified theories: that the coupling constants of the various forces unify at some high energy. This running of the couplings, masses and non-minimal couplings of a theory is controlled by the β-functions of its renormalisation group equations [2].

Typical modern incarnations of inflation arise within supergravity, string or GUT theories where the inflaton, ϕ, is only one of many fields. Studies of inflation including couplings to these other fields, yield extremely complex dynamics [140] and are little investigated beyond two-field hybrid models [130]. However, since we are particularly interested in the preheating realm which occurs when inflation ends near the Planck scale, we are near the high-energy ultra-violet (UV) fixed points of the renormalisation group equations, so that even if there are many fields involved in realistic theories, the dynamics typically simplify greatly and the various fields usual decouple, either partially or completely [3]

Now, while the UV fixed points may correspond to a conformally invariant field ($m = 0$, $\xi = \frac{1}{6}$), in different GUT models the coupling may also diverge, $|\xi| \to \infty$, in the UV limit [136, 84] as we will now discuss in some detail.

However to do this we must first continue our discussion of renormalisation in curved spacetime,

[1] As of August 1998.

[2] For a thorough and extensive introduction to renormalisation in the presence of gravity and renormalisation group equations for different theories, see the book by Buchbinder *et al* [136].

[3] Alternative approaches to the problem of multiply coupled fields will be presented in the following two chapters.

started in chapter (1). The details in the following two subsections are not vital however, though very useful for putting the results in context, for the rest of the chapter.

3.2.1 Renormalisation of the non-minimal coupling

Let us consider a general theory in curved spacetime with $\Phi^A = (\phi^i, \psi^i, A^i_\mu)$ where i numbers the field; ϕ^i are scalar fields, ψ^i are spinor fields and the A^i_μ are vector fields. Let the scalar and fermionic Lagrangian mass terms [4] be of the form $\frac{1}{2}m_{ij}\phi^i\phi^j + M_{rs}\overline{\psi}^r\psi^s$ and let the non-minimal couplings be $\frac{1}{2}\xi_{ij}\phi^2\phi^j$.

One finds [136] that the renormalisation of all fields Φ^A and parameters (which throughout this chapter will mean the masses and the Yukawa-, gauge-, self- and non-minimal couplings of the fields) other than ξ_{ij} is carried out with the same renormalisation constants as in flat spacetime. Dimensional and gauge-invariance constraints force the relation:

$$\xi_{0ij} = Z_{2ij}{}^{kl}\xi_{kl} + Z_{3ij} \tag{3.1}$$

where the subscript 0 denotes the bare (unrenormalised) value of a field or parameter and the Z^{ij}_n are the renormalisation constants, which map the unrenormalised action into the renormalised one. The fact that this equation is inhomogeneous provides the reason for why $\xi \neq 0$ in general, even if $\xi_0 = 0$. Now at the one-loop level, the constants Z_2 and Z_3 can be related, as first shown in general in [87]. Z_2 is connected with the renormalisation of the mass and the relation between Z_2 and Z_3 shows how *all* renormalised fields and parameters can be calculated via the corresponding theory in flat spacetime.

Interestingly, for the renormalisation of the ξ_{ij} it is sufficient to consider the massless theory only. In four dimensions it turns out that conformal coupling $\xi = \frac{1}{6}$ is a fixed point and so $\xi = \xi_0$ if $\xi_0 = \frac{1}{6}$. Using this in equation (3.1) we find that at one loop the desired relationship between Z_2 and the purely curved-spacetime-constant Z_3 is:

$$Z_3^{(1)} = -\tfrac{1}{6}(Z_2^{(1)} - 1) \tag{3.2}$$

where the superscript (1) reminds us that it is a one-loop result and we have suppressed the ij indices for notational clarity.

3.2.2 Renormalisation group equations

The effective action turns out to be invariant under renormalisation [5] in the sense that:

$$W_0[h_{\mu\nu}, \phi_0, p_0, n] = W[g_{\mu\nu}, \phi, p, \mu, n] \tag{3.3}$$

[4] Vector fields are not given mass explicitly so as to preserve gauge invariance.
[5] Technical remark: this follows from multiplicative renormalisability of the theory [136].

where again a subscript 0 denotes a bare quantity, ϕ, p and n denote the fields, parameters and dimension of spacetime respectively and μ is the mass scale that was introduced in subsection (1.5.1) to ensure that the effective Lagrangian retained the correct dimensions when $n \neq 4$. The above expression (3.3) can be restated differentially as [136]:

$$\mu \frac{d}{d\mu} W[g_{\mu\nu}, \phi, p, \mu, n] = 0 \tag{3.4}$$

which, expanding the total functional derivative $d/d\mu$, gives:

$$\left[\mu \frac{\partial}{\partial \mu} + \beta_p \frac{\partial}{\partial p} + \gamma(n) \int d^n x \; \phi(x) \frac{\delta}{\delta \phi(x)}\right] W[g_{\mu\nu}, \phi, p, \mu, n] = 0\,, \tag{3.5}$$

where $\delta/\delta\phi$ denotes functional differentiation w.r.t the field $\phi(x)$ and we have defined:

$$\beta_p \equiv \mu \frac{dp}{d\mu} \tag{3.6}$$

$$\gamma(n)\phi(x) \equiv \mu \frac{d\phi(x)}{d\mu} \tag{3.7}$$

The β-functions β_p, which depend explicitly on the parameters p, control the evolution of the parameters under changes of the mass scale μ. But what we would really like to know is how the parameters and fields change with energy, or via the uncertainty principle, with length scale. Now for the simple case of a massive, non-minimally coupled scalar field with quartic self-interaction, W satisfies a simple scaling relation under a constant conformal transformation:

$$g_{\mu\nu} \rightarrow e^{-2\tau} g_{\mu\nu}\,, \tag{3.8}$$

where τ is a constant. This allows the $d/d\mu$ term in Eq. (3.5) to be replaced by a term involving $d/d\tau$, which is what we wanted.

This leads to a solution $W[g_{\mu\nu}, \phi(\tau), p(\tau), \mu]$ where the τ dependent ϕ, p satisfy the *renormalisation group equations* [6]:

$$\frac{d\phi}{d\tau} = (\gamma(\tau) - d_\phi)\phi(\tau) \tag{3.9}$$

$$\frac{dp}{d\tau} = \beta_p(\tau) - d_p)p(\tau) \tag{3.10}$$

where $d_{\phi,p}$ are constants, the so-called canonical dimensions of ϕ, p at $n = 4$ [136], and γ, β_p depend implicitly on τ through p.

3.2.3 Asymptotic freedom, conformal invariance and finite models

The Einstein field equations are invariant under the constant conformal scaling in Eq. (3.8) - see chapter [7] - since under this transformation $R_{\mu\nu}$ is invariant while $R \rightarrow e^{2\tau} R$. Similarly the Kretchmann scalar ("Riemann tensor squared") scales as $e^{4\tau}$ so that as $\tau \rightarrow \infty$, the curvature invariants diverge

[6] The constant scaling of the metric $g_{\mu\nu} \rightarrow e^{-2\tau} g_{\mu\nu}$ defines a group of transformations (technically a semi-group).

and $g_{\mu\nu} \to 0$. The high energy, short-distance (UV), behaviour of the theory is then controlled by this limit.

We have a dilemma however. How can we look at the $\tau \to \infty$ limit using only the one-loop approximation, as we are doing here ? The answer is that it only makes sense in theories which are *asymptotically free*, i.e. those theories whose coupling constants g^2 are such that $g^2 \to 0$ as $\tau \to \infty$. QCD is a relevant example. As it turns out, the $\tau \to \infty$ limit only exists in a theory with scalars, spinors and non-Abelian gauge fields if that theory is asymptotically free [90].

Now the equation for ξ coming from (3.10) can be solved [136] and has the general structure (we suppress unnecessary detail):

$$\xi(\tau) = \tfrac{1}{6} + \textstyle\sum_s (\xi - \tfrac{1}{6}) \left(g^2(\tau)\right)^{-\lambda_s} \tag{3.11}$$

where the λ_s are the eigenvalues of a certain constant matrix. Clearly we can decern three possibilities as $\tau \to \infty$:

(1) All $\lambda_s < 0$

In this case $\xi \to \frac{1}{6}$ independently of the initial, bare value ξ_0. Since as $\tau \to \infty$, the theory becomes massless ($m^2 \to 0$) the theory exhibits what is known as *asymptotic conformal invariance.* This means that particle production due to the expansion of the universe will progressively shut-off as a conformally flat singularity is approached.

This is a nice example showing how, even though the classical Einstein field equations are invariant under the constant conformal scaling, quantum effects are not.

(2) At least one $\lambda_s > 0$

In this case, $|\xi| \to \infty$ irrespective of ξ_0. This result is of central interest to this chapter since it shows that ξ may be the only interesting coupling in a theory at high energy. All fields can be described in the above limit as free but highly non-minimally coupled fields, i.e. it is not necessary to consider their masses, self-interactions or couplings to other fields such as the inflaton.

(3) All $\lambda_s = 0$

$\xi(\tau) = \frac{1}{6}$ and the value of ξ is not renormalised in the one-loop approximation.

Finally we arrive at *finite* theories. These are theories where all the β-functions controlling the renormalisation of fields, masses and couplings (other than for ξ) vanish. The couplings g^2 are truly constants, invariant under τ. Perhaps the classic example is $N = 4$ supersymmetric Yang-Mills theory where the β-functions vanish *at all orders in perturbation theory*, and not just at one-loop.

In these finite theories the above three situations for $\xi(\tau)$ are again possible. As an example consider $SU(2)$ gauge theory with an $SU(4)$ global invariance and action containing Weyl spinors and scalars in the adjoint representation of $SU(2)$ [136]. In this theory, $|\xi| \to \infty$ as $\tau \to \infty$, while at low energies it tends towards conformal coupling as preferred by the equivalence principle of General Relativity [80]. Indeed, since these one loop calculations typically yield $(\frac{1}{6}, \infty)$ as the fixed points of the renormalisation group equations as $\tau \to \pm\infty$, and since the equivalence principle strongly favours conformal coupling at low energies (i.e. $\tau \to -\infty$), it appears much more likely and natural that $|\xi| \to \infty$ in the high energy limit than the other way around. This is precisely the motivation for this chapter where we consider preheating to non-minimal scalar fields. The strength of the effect is due to the congruence of two facts: (1) the Ricci curvature oscillates during preheating and (2) the non-minimal coupling, $|\xi|$, is a free parameter. The first ensures that there is resonance, the second that the effect is non-perturbative.

Further on in the chapter we shall study the gravitational production of spin 0,1 and 2 particles due to the expansion of the universe during preheating, and will show that a unified treatment in terms of parametric resonance exists. This is shown by reducing the evolution equations to generalized Mathieu form:

$$x'' + [A(k) - 2q\cos 2z]x = 0\,, \tag{3.12}$$

with time-dependent parameters [7].

When $A(k) < 0$ a qualitative change occurs and the dominant effect comes from the fact that one effectively has an inverted harmonic oscillator yielding the *negative coupling instability* [8], as discussed in section (2.6). In this case the Floquet index can be as large as $\mu_k \sim |q|^{1/2}$, there are no stability bands to speak of and the typical variances are larger by a factor $|q|^{1/2}$ than in the $A(k) > 0$ case.

3.3 The specific setting for the model of geometric reheating

To be concrete, consider the case of a scalar field in a FLRW universe ($g_{\mu\nu} = diag(-1, a^2(t)/(1 - Kr^2), a^2(t)r^2, a^2(t)r^2 sin^2\theta)$, $K = \pm 1, 0$ is the curvature constant). We shall restrict[8] ourselves to the quadratic potential,

$$V(\phi) = \frac{m_\phi^2}{2}\phi^2\,. \tag{3.13}$$

For $K = 0$, the latter potential gives $\phi = \Phi\sin(m_\phi t)$, with $\Phi \sim 1/m_\phi t$. In what follows we shall try to preserve maximal generality; we denote with $\doteq$ results which are derived specifically for the potential (3.13).

The energy density and pressure for a minimally coupled scalar field, treated as a perfect fluid,

[7] We can then use all of the extensive formalism developed in chapter (2) regarding the instability chart of the Mathieu equation, effect of expansion, estimates for the Floquet index etc...

[8] We note that application of stochastic resonance methods to the vector, tensor and non-minimal scalar fields of this chapter for the potential $V = \frac{\lambda}{4}\phi^4$ requires an extension of the existing theory to scattering in a quartic potential as opposed to the standard quadratic potential [9, 10].

are:

$$\mu = \kappa(\frac{1}{2}\dot{\phi}^2 + V(\phi)) \quad , \quad p = \kappa(\frac{1}{2}\dot{\phi}^2 - V(\phi)) \,. \tag{3.14}$$

These equations breaks down if the field is non-minimally coupled (an imperfect fluid treatment must be used), if the effective potential is not adequate [183], or if large density gradients exist [37]. The FLRW Ricci tensor is [20]

$$\begin{aligned} R^0{}_0 &= 3\frac{\ddot{a}}{a} \\ R^i{}_j &= \left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2K}{a^2}\right]\delta^i{}_j \,, \end{aligned} \tag{3.15}$$

where $i, j = 1..3$. The Ricci scalar is:

$$R = 6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2}\right) \,. \tag{3.16}$$

The Raychaudhuri equation for the evolution of the expansion $\Theta = 3H = 3\dot{a}/a$ is [9] given by:

$$\dot{\Theta} = -\frac{3\kappa}{2}\dot{\phi}^2 + \frac{3K}{a^2} \,, \tag{3.17}$$

while the Friedmann equation is

$$\Theta^2 + \frac{9K}{a^2} = 3\kappa\mu = 3\kappa\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \,. \tag{3.18}$$

As an example, when $K = 0$ and $\dot{a}/(am_\phi) \ll 1$, one may solve Eq. (3.18) perturbatively [150], to obtain:

$$\Theta \doteq \frac{2}{t}\left[1 - \frac{\sin 2m_\phi t}{2m_\phi t}\right] \,, \tag{3.19}$$

to first order in $\dot{a}/(am_\phi)$. This is only valid after preheating when $\Phi \ll 1$ but shows that the expansion oscillates about the mean Einstein-de Sitter (EDS) pressure-free solution. Eq. (3.19) can be integrated to give the scale factor:

$$a(t) \doteq \overline{a}\exp\left(\frac{\sin 2m_\phi t}{3m_\phi t} - \frac{2\mathrm{ci}(2m_\phi t)}{3}\right) \tag{3.20}$$

where $\overline{a} = t^{2/3}$ is the background EDS evolution, and $\mathrm{ci}(m_\phi t) = -\int_t^\infty \cos(m_\phi z)/z dz$. This example explicitly demonstrates how temporal averaging (which yields $\overline{a}$) removes the resonance.

Via Eq.'s (3.17,3.18) one can systematically replace all factors of $\dot{a}, \ddot{a}$ with factors of $\dot{\phi}$ and $V(\phi)$ terms[10]. In this way one can show that the vector and tensor wave equations take the form of Mathieu equations during reheating [148].

3.4 Scalar fields

Consider now the effective potential:

$$V(\phi, \chi_\nu) = V(\phi) + \frac{1}{2}\sum_\nu^N m_\nu^2\chi_\nu^2 + \frac{1}{2}\sum_\nu^N \xi_\nu\chi_\nu^2 R \,, \tag{3.21}$$

[9] The expansion is generally defined as $\Theta \equiv u^a{}_{;a}$ where u^a is the 4-velocity and ; denotes covariant derivative [220].
[10] Indeed, a useful combination is $2\dot{\Theta} + \Theta^2 = -3\kappa p \doteq \frac{3}{2}\kappa m_\phi^2\Phi^2\cos(2m_\phi t)$.

describing the inflaton with potential $V(\phi)$ coupled only via gravity to N scalar fields, χ_ν, which have masses m_ν, no self-interactions and non-minimal couplings ξ_ν. The equation of motion for modes of the ν-th field is:

$$\ddot{\chi}_k^\nu + \Theta\dot{\chi}_k^\nu + \left(\frac{k^2}{a^2} + m_\nu^2 + \xi_\nu R\right)\chi_k^\nu = 0\,, \tag{3.22}$$

From Eq (3.16,3.17,3.18) the Ricci scalar is given by [11]:

$$R = -\kappa\dot{\phi}^2 + 4\kappa V(\phi) \quad (K=0)\,. \tag{3.23}$$

3.4.1 The minimally coupled case

Consider $\xi_\nu = 0$. Then (3.22) reduces, on using Eq's (3.17,3.18), to:

$$\begin{aligned}\frac{d^2(a^{3/2}\chi_k)}{dt^2} &+ \left(\frac{k^2}{a(t)^2} + m_\nu^2 + \kappa\frac{3}{8}\dot{\phi}^2\right. \\ &- \left.\kappa\frac{3}{4}V(\phi) + \frac{3}{4}\frac{K}{a^2}\right)(a^{3/2}\chi_k) = 0\,.\end{aligned} \tag{3.24}$$

There exists parametric resonance because the expansion Θ oscillates. The potential (3.13) yields equation (3.12) ($K=0$) with time-dependent parameters:

$$A(k,t) \doteq \frac{k^2}{a^2 m_\phi^2} + \frac{m_\nu^2}{m_\phi^2} \quad , \quad q \doteq \frac{3}{16}\kappa\Phi^2 \tag{3.25}$$

From this we see that the production of particles is reduced as m_ν increases. Indeed, since $A \to m_\nu^2/m_\phi^2$, $q \to 0$ due to the expansion, production of minimally coupled bosons is rather weak and shuts off quickly due to horizontal motion on the stability chart. We stress that the production is, however, stronger than that obtained in previous studies where the scalar factor evolves monotonically [152]. This mild situation changes dramatically when a non-minimal coupling is introduced.

3.4.2 Non-minimal preheating

Now include the *arbitrary* non-minimal coupling ξ_ν. Using Eq. (3.23) one can reduce Eq. (3.22) to ($K=0$):

$$\begin{aligned}\frac{d^2(a^{3/2}\chi_k)}{dt^2} &+ \left(\frac{k^2}{a(t)^2} + m_\nu^2 + \kappa\left(\frac{3}{8} - \xi\right)\dot{\phi}^2\right. \\ &- \left.\kappa\left(\frac{3}{4} - 4\xi\right)V(\phi)\right)(a^{3/2}\chi_k) = 0\,.\end{aligned} \tag{3.26}$$

Defining a new variable $z = m_\phi t + \pi/2$, Eq. (3.26) takes the form of equation (3.12) with time-dependent parameters:

$$\begin{aligned}A(k,t) &\doteq \frac{k^2}{a^2 m_\phi^2} + \frac{m_\nu^2}{m_\phi^2} + \frac{\kappa\xi}{2}\Phi^2 \\ q(t) &\doteq \frac{3}{4}\kappa\left(\frac{1}{4} - \xi\right)\Phi^2\end{aligned} \tag{3.27}$$

[11] Assuming that at the start of reheating the inflaton is the dominant contributor to the energy density of the universe.

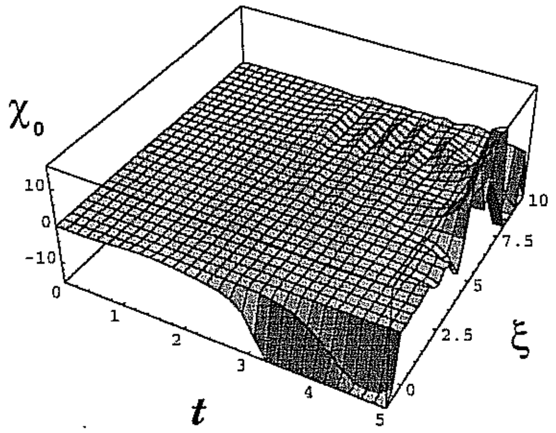


Figure 3.1: The evolution of the $k = 0$ mode ($m_\nu^2/m_\phi^2 \simeq 1$), as a function of time and the non-minimal coupling parameter ξ. For positive ξ the evolution is qualitatively that of the standard preheating with resonance bands. However, for negative A (negative ξ) the solution changes qualitatively and there is a negative coupling instability. There are generically no stable bands and the Floquet index corresponding to $-|\xi|$ is much larger, scaling as $\mu_k \sim |\xi|^{1/2}$.

The crucial observation is that since ξ is initially free to take on any value [12], $A(k)$ is neither restricted to be positive nor small in magnitude.

From Eq. (3.27) it is clear that $A(k) < 0$ for sufficiently negative ξ. The possibility of negative A was the thesis of the work by Greene *et al* [8]. However, in their model, this powerful negative coupling instability was only partially effective due to the non-zero vacuum expectation value acquired by the χ field due to its coupling, g, with the inflaton. Here we only have gravitational couplings and the same constraint is removed.

Negative A (induced when $q < 0$) implies that the physical region of the (A, q) plane is altered. Instead of $A \geq 2|q|$ we have $A \geq \pi\Phi^2 - 2|q|/3$. Now when $2|q|/3 > |A| \gg 1$ we have $\mu_k \sim |q|^{1/2} \simeq (6\pi|\xi|)^{1/2}\Phi$ along the physical separatrix $A = \pi\Phi^2 - 2|q|/3$. Since the renormalised $|\xi|$ may have very large values, this opens the way to exceptionally efficient reheating - see Figs. (3.1,3.2) - via resonant production of highly non-minimally coupled fields with important consequences for GUT baryogenesis [8] and non-thermal symmetry restoration.

For example, let us consider $m_\phi \simeq 2 \times 10^{13} GeV$ as required to match large angle CMB anisotropies $\Delta T/T \approx 10^{-5} \sim m_\phi/M_{pl}$. Then GUT baryogenesis with massive bosons χ with $m_\chi > 10^{14} GeV$ simply requires $\xi < -(\pi\Phi^2)^{-1}$, with Φ in units of the Planck energy. Instead if one requires the production of GUT-scale gauge bosons with masses $m_{gb} \sim 10^{16} GeV$ this is still possible if the associated non-

[12] The only constraints that one might impose are that the effective potential should be bounded from below and that the strong energy condition, $R_{ab}u^a u^b > 0 \Leftrightarrow T_{ab}u^a u^b > -T/2$, be satisfied. The first is difficult to impose since R oscillates and the second since one should use the renormalised stress-tensor, $\langle T_{ab}\rangle$, and even in this case there is no bound on ξ as we discussed in subsection (3.2.3).

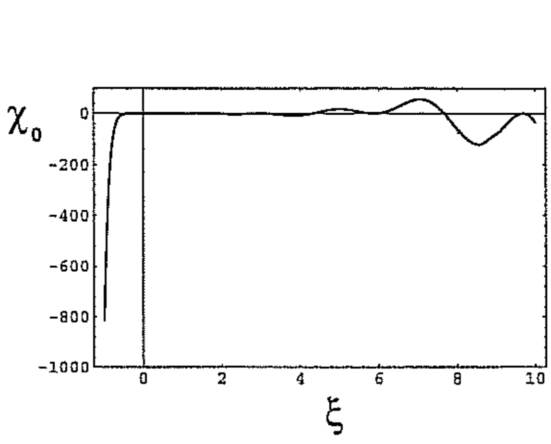


Figure 3.2: A slice of the spectrum in fig. (1) at $t = 5$ as a function of the non-minimal coupling ξ. The qualitative differences between $\xi < 0$ and $\xi > 0$ are clear.

minimal coupling is of order $\xi \sim -10^3$. Such coupling values have been considered in e.g. [91]. The massive bosons with $m_\chi \sim 10^{14} GeV$ can be produced in the usual manner via parametric resonance if $\xi > 0$, but this process is weaker (c.f. [135]).

Since the coupling between ϕ and χ is purely gravitational, backreaction effects in the standard sense (see [9, 183]) cannot shut off the resonance. The inflaton continues to oscillate and produce non-minimally coupled particles, receiving no corrections to $m^2_{\phi,eff}$ from $\langle\delta\chi^2\rangle$.

To estimate the maximum variance $\langle\delta\chi^2\rangle$ is therefore rather difficult. The standard method is to establish the time when the resonance is shut off by the growth of $A(k)$ which pushes the $k = 0$ mode out of the dominant first resonance band. For this we must understand how $A(k)$ changes as the χ-field gains energy and alters the Ricci curvature. If we assume that most of the energy goes into the χ_0 mode, justified in the $\xi < 0$ case [13], then the change to the Ricci curvature is $\delta R_\chi = 8\pi(E - S)$, where [92]:

$$E = G_{eff}\left[\frac{\dot{\chi}_0^2}{2} + \frac{m_\chi^2\chi_0^2}{2} - 12\xi\chi_0\dot{\chi}_0\frac{\dot{a}}{a}\right] \tag{3.28}$$

is the T^{00} component of the χ stress tensor,

$$\begin{aligned} S &= \frac{3G_{eff}}{1+192\pi G_{eff}\xi^2\chi_0^2}\left[\frac{\dot{\chi}_0^2}{2} - \frac{m_\chi^2\chi_0^2}{2}\right. \\ &+ \left. 4\xi\left(\frac{\dot{a}^2}{a^2} - \chi_0\dot{\chi}_0\frac{\dot{a}}{a} - m_\chi^2\chi_0^2\right) + 64\pi\xi^2\chi_0^2 E\right], \end{aligned} \tag{3.29}$$

is the spatial trace of the stress tensor $T^i{}_i$ corresponding to $3p$ in the perfect fluid case and $G_{eff} = (1+16\pi\xi\chi^2)^{-1}$ is the effective gravitational constant. Since χ_0 is rapidly growing, the major contribution of δR_χ will be to $A(k)$, causing a rapid vertical movement on the instability chart. Once $\delta A + A >$

[13] In the case $\xi \gg 1$ one needs to use $\langle\delta\chi^2\rangle$ instead.

$2|q| + |q|^{1/2}$, the resonance is shut-off. If $\xi < 0$, most of the decaying ϕ energy is pumped into the small k modes (see fig. 3.2). Subsequently we expect the oscillations in χ_0 to produce a secondary resonance due to the self-interaction and non-linearity of Eq's (3.28,3.29).

The case of a $\lambda_\chi \chi^4/4$ self-interaction provides another mechanism that may be dominant in ending the resonance: namely $m^2_{\phi,eff}$, and hence $A(k)$, gains corrections proportional to $\lambda_\chi \langle \delta\chi^2 \rangle$ which shuts off the resonance [8] leaving a peak variance of order $\langle \delta\chi^2 \rangle \simeq m^2_\phi (4|q| - m^2_\chi)/\lambda_\chi$ (assuming that $\xi < 0$). If $\xi > 0$ the variance is smaller by a factor $|q|^{1/2}$.

3.5 Non-thermal symmetry restoration

Let us now consider a modification of the effective potential to include symmetry-breaking:

$$V(\phi,\chi) = \frac{1}{2}m^2_\phi \phi^2 + \frac{g^2}{2}\phi^2\chi^2 + \frac{\lambda}{4}(\chi^2 - \chi_0^2)^2 \,, \tag{3.30}$$

The effective mass of χ is now given by the curvature of the effective potential including non-thermal corrections due to preheating:

$$m^2_{\chi,eff} = -\lambda\chi_0^2 + 3\lambda\chi^2 + \xi R + \lambda\langle(\delta\chi)^2\rangle + g^2\langle(\delta\phi)^2\rangle \,, \tag{3.31}$$

When $0 < \xi \ll 1$ but $g^2 \gg m^2_\phi$ we return to the standard preheating theory and non-thermal symmetry restoration discussed in sections (2.4) and (2.12) [131] where there is the difficult task of understanding the effects the highly non-equilibrium fluctuations have on the effective potential. Here we rather concentrate on the $g = 0$ case. During the early phase of preheating the backreaction term $\propto \langle(\delta\chi)^2\rangle$ in (3.31) is relatively small and we neglect it.

It is clear from (3.31) that when $|\xi| \gg 1$, $m^2_{\chi,eff}$ will change sign periodically as the curvature oscillates, with $R = \kappa\Phi^2(1 + 3\cos 2m_\phi t)/2$, valid as long as ϕ is still the dominant contributor to the energy density of the universe.

Since the period of the oscillations of R is $\sim (2m_\phi)^{-1}$, while the χ field initially has a period of oscillation given approximately by $(\xi R - \lambda\chi_0^2 + 3\lambda\chi^2)^{1/2}$, we see that as long as $\xi \gg \lambda\chi_0^2$, the oscillations of χ will be much faster than those in the Ricci curvature and hence, during the time(s) that $m^2_{\chi,eff} > 0$, the χ field will effectively probe the alternative vacuum many times and hence it is extremely likely that topological defects will form (in this case domain walls) once $m^2_{\chi,eff} < 0$ again, with the universe cut into roughly equal numbers of regions with $-\chi_0$ and χ_0 due to the thermal Kibble mechanism.

In the case when $\xi < 0$ the restoration of symmetry via the ξR term is still available, except for small values of ξ in which case symmetry restoration must occur through the fluctuations $\langle(\delta\chi)^2\rangle$. This is not a problem, however, given the power of the negative coupling instability. Indeed, one simply requires a small value of $|\xi| \geq \lambda\chi_0^2/(3\kappa m^2_\phi \Phi^2) \sim \Phi^{-2}$ for GUT-scale symmetry breaking and coupling $\lambda \sim 10^{-3}$. Hence obtaining $m^2_{\chi,eff} > 0$ is very easy. Further, while $\langle(\delta\chi)^2\rangle$ typically oscillates extremely rapidly (see section [2.8.1] and figure [2.7]), the time average $\overline{\langle(\delta\chi)^2\rangle} > 0$ so again defects can form as in the usual theory.

There are two important differences between the two cases described above however. in the $|\xi| \gg 1$ case, defects are expected every time R oscillates from its maximum to its minimum. In that case the fluctuations $\langle(\delta\chi)^2\rangle$ and $\langle(\delta\phi)^2\rangle$ will be small and the field will evolve coherently. The defect density will thus be determined by the instantaneous horizon size. In the second case, and in the first case as $\langle(\delta\chi)^2\rangle$ becomes very large, the fluctuations are highly non-thermal and the correlation length Ξ of the χ field will be much smaller than the horizon scale in general. The defect density will then be much larger, determined by Ξ^{-n} where $n = 1, 2, 3$ for domain walls, cosmic strings and monopoles respectively [97]. Further non-thermal issues will be discussed in chapter (5).

3.6 A solution to the monopole problem ?

In the previous section we have seen that we may restore symmetries which may have been broken before or during inflation. In the case that the potential has a discrete symmetry, domain walls will be formed and if in addition true GUT restoration occurs, monopoles will also be formed. These are cosmologically lethal if allowed to survive very far into the matter-dominated epoch, since in the case of a low-Ω universe, the transition from radiation to matter domination occurs after decoupling of photons and recombination of hydrogen ions with the ambient electrons. In short, after the CMB was formed. Excellent reviews of defects in cosmology can be found in the book [79] and the review by Zurek [97].

Here we present a recent proposal for solving the monopole problem that has at least two new implementations at reheating. One we will present in this chapter and the other in chapter (5).

3.6.1 Alternative solutions to the monopole problem

Here we briefly review the known solutions to the monopole problem which are not directly related to inflation. Further, as we have just seen and shall discuss in the following chapters, reheating after inflation can quite effectively restore GUT-scale symmetries in several ways and hence regenerate monopoles and domain walls.

Before 1997, there were only three known alternatives to inflation regarding the monopole problem: (a) symmetry non-restoration at high energy, (b) the Langacker-Pi mechanism and (c) a non-trivial topology for the universe. Symmetry non-restoration relies on the idea that symmetry is not necessarily restored at high temperatures, and that it might have been broken for the whole history of the universe. This has a basis in condensed matter physics with the prototypical example given by Rochelle salts which roughly speaking, involve increased symmetry breaking at higher temperatures [32]. Within this framework there is no symmetry breaking at all and thus domain walls and monopoles would never have been produced.

In the Langacker-Pi scenario [33], the idea is to do the opposite to inflation: instead of diluting the monopole density, the aim is to bring couples of monopoles and anti-monopoles together efficiently so that they can annihilate. To do this, one needs to break the $U(1)$ electromagnetic symmetry for a time. During this phase, photons gain a mass, and the monopoles and anti-monopoles become joined by strings which cause them to be drawn together facilitating annihilation [14]. Finally, if the universe has a non-trivial topology, monopoles and domain walls may be excluded automatically due to topological arguments.

None of these mechanisms however, fits into what might be called the mainstream vision of high energy cosmo-physics. A scenario which is more in the standard vein has recently been proposed, and at least two implementations of this new mechanism appropriate to the end of reheating are possible. Here we will discuss one, leaving the other for chapter (5).

This new mechanism, proposed by Dvali, Liu and Vachaspati (DLV) [160], is quite simple and works within the simplest GUT model, namely $SU(5)$. The idea runs as follows.

The $SU(5)$ GUT model has a Higg's field Φ, responsible for the breaking of $SU(5)$ to the standard model (see Eq. [3.33] below) with potential:

$$V(\Phi) = -\frac{1}{2}m^2 Tr\Phi^2 + \frac{h}{4}(Tr\Phi^2)^2 + \frac{\lambda}{4}Tr\Phi^4 + \frac{\gamma}{3}Tr\Phi^3 \qquad (3.32)$$

Here Tr denotes a trace over the multi-component Higgs field. In the usual case where $\gamma = 0$, this potential has a $\Phi \rightarrow -\Phi$ symmetry - a $\mathbf{Z}_2$ symmetry. In this case the symmetry is broken down to that of the full standard model via the usual Higgs mechanism:

$$SU(5) \times \mathbf{Z}_2 \rightarrow [SU(3) \times SU(2) \times U(1)]/\mathbf{Z}_6 \qquad (3.33)$$

when Φ relaxes to the non-zero value (the vacuum expectation value - vev)

$$\Phi_0 = v \times diag(2,2,2,-3,-3)/\sqrt{30} \qquad (3.34)$$

where $v = m/\lambda'$ and $\lambda' \equiv h + 7\lambda/30$ [160]. The two vev's $\pm\Phi_0$ are degenerate due to the exact $\mathbf{Z}_2$ symmetry. This discrete symmetry leads to domain walls due to the Kibble mechanism [78] whose energy density, like that of monopoles, decays much more slowly than normal matter and radiation and hence come to dominate the energy density of the universe causing unacceptable anisotropies in the CMB.

This degeneracy between the two vev's is broken if $\gamma \neq 0$. In this case, there is so-called biased domain wall formation with regions of space preferentially relaxing into the lower energy vev, denoted

[14] Indeed, this is very similar to the Seiberg-Witten mechanism for quark-confinement in Supersymmetric $SU(2)$ Yang-Mills theory. For an introduction to this beautiful work see [34].

Φ_+ (the higher energy vev we denote Φ_-). Then the probability, P_- of a domain falling into the Φ_- vacuum is given by [160]:

$$P_- = P_+ \exp(-\Delta F_V / T_c) \tag{3.35}$$

where P_+ is the probability of going into the P_+ vacuum. Here ΔF_V is the free energy difference between two domains of volume V in the two different vacua and T_c is the so-called freeze-out temperature of the domains during the phase transition [97].

The walls so produced are not stable (as they are in the $\gamma = 0$ case). This is because the breaking in degeneracy creates a pressure difference across the walls of order $2\gamma m\ Tr\Phi_0^3/3$ which causes the walls to shrink around regions in the Φ_- vacuum and eventually collapse. If this occurs on a time-scale which is not too long, then they will disappear before dominating the evolution of the universe. This constraint gives a lower-bound on γ of [160]:

$$\begin{aligned} \gamma &> \frac{10}{\sqrt{\lambda'}}\left(\frac{M_G}{M_{pl}}\right)^2 \\ &\sim (10^{-9} - 10^{-5})\lambda'^{-1/2} \end{aligned} \tag{3.36}$$

where M_G is the GUT symmetry breaking scale.

On the other hand, if γ is too large, the pressure difference will cause the initially plane walls to curve and collapse on short timescales. Why is this undesirable ? The whole purpose of this discussion is to remove monopoles, and so far we have not discussed them at all. The idea of Dvali *et al* is that if the domain walls percolate to form infinite walls which move relativistically, they will sweep through the entire horizon volume in a time which is short compared with cosmological time scales. When monopoles collide with the walls, one of two things may happen: (i) either the monopoles scatter from, or pass through, the wall or (ii) the monopoles are captured on the wall. In this second case, since the domain wall is by definition a region where the full ($SU(5)$) symmetry is restored, the monopole has no topological barrier to its decay and will preferentially decay by unwinding, dissipating its energy as waves on the wall surface. Evidence in favour of the second possibility comes from numerical simulations and simple phase coherence arguments [160], although we note that it is not certain yet that this is what actually would happen.

However, accepting the scenario that monopoles are predominantly trapped on the walls, the idea is then that the walls move throughout space sweeping up most of the monopoles, which then unwind on the walls. The domain walls then become pressure dominated and decay themselves, thus removing both problems simultaneously, while dumping a large amount of energy in the form of gravitational waves and radiation into the ambient arena [96].

The first constraint is then that the walls actually percolate to form infinite walls. This means that the pressure difference across the walls must not dominate the surface tension on the wall given by σ/r, where r is the radius of curvature and σ is the energy per unit area. This places an upper bound on γ given by [160]:

$$\begin{aligned} \gamma &\leq \frac{10\sqrt{\lambda'}}{rm} \\ &\sim 10\lambda'^{1/2} \end{aligned} \tag{3.37}$$

Given the constraints (3.36) and (3.37), the walls slowly become pressure dominated and collapse. A fortunate result is that these two constraints allow a large range of values for γ for reasonable values of λ' in this model of $SU(5)$.

Having shown that this mechanism can work for a large range of $\gamma \neq 0$, a natural question to ask is "what process generates a non-zero γ ?". Dvali *et al* do not consider this problem in detail, noting rather that a cubic term naturally exists in the standard model. They also outline a way of breaking the discrete $\mathbf{Z}_2$ symmetry via quantum effects: the symmetry would be anomalous (i.e. not respected by quantum aspects of the theory) and explicitly broken by instanton effects. We will not discuss this possibility here any further, but look at the implication of reheating for the DLV solution.

3.6.2 A modified solution for geometric reheating

The crucial ingredient of the DLV mechanism is the odd term in the potential which breaks the $\mathbf{Z}_n$ symmetry. Now if one includes a linear term then quantum effects will automatically generate a cubic term at 1-loop. However, in chapter (1) and at the start of this chapter we saw that in a curved space-time, renormalisation generically leads to fields being non-minimally coupled with a coupling which runs with energy.

The work of Hosotani [45] is very interesting in this respect. It examined the classical stability of maximally symmetric spacetimes (de Sitter, Minkowski and Anti-de Sitter) as stable vacua and hence suitable backgrounds to do perturbation theory around.

He found that if $\xi \neq 0$ then the coefficient of the cubic term Φ^3 of a Higgs field *must vanish* since otherwise the Klein-Gordon equation has no absolute minimum, it is unbounded below and quantum fluctuations grow without bound since we essentially have an untamed negative coupling instability [15]. While his results were concerned with classical stability, we immediately see that a linear term is not allowed either, since that would immediately give rise to a cubic term, as mentioned previously. Thus at this level it appears that in an expanding FLRW universe, and particularly in the context of geometric reheating, we have lost the basic ingredient of the DLV mechanism.

But happily this is not true since the basic ingredient is not the cubic term but the destabilisation of domain walls by breaking of discrete symmetries of the effective potential. Indeed, the DLV mechanism is sufficiently flexible to find an implementation in geometric reheating. Up until now we have presented the simplest possible model of geometric reheating by considering a non-minimal

[15] This is instructive since at first glance one might think that as $\Phi \to \pm\infty$, the Φ^4 term would always dominate the Φ^3 term. This is true in the minimally coupled case but is not true when $\xi \neq 0$.

coupling of the form $-\xi\chi^2 R$. This preserves any discrete symmetries including the archetypical $\mathbf{Z}_2$ symmetries $\chi \rightarrow -\chi$ or $\phi \rightarrow -\phi$. Thus, at this level it appears that there does not exist a way of implementing the DLV mechanism in geometric reheating.

However, even this is premature since our previous choice of the non-minimal coupling term was not unique. Any quadratic combination of fields multiplying the Ricci curvature would be acceptable. Indeed, if one believes the adage that everything in particle physics occurs if it is not explicitly forbidden, then couplings of the form:

$$\tfrac{1}{2}\sum_{i,j}^{N} \xi_{ij}\chi_i\varphi_j R$$

can be consistently expected, where χ_i and φ_j represent all relevant fields at reheating. In this case, we will typically break around half of the $\mathbf{Z}_n$ symmetries of the potential and thus destabilise half of the domain walls. Unfortunately the rest are left untouched and are cosmologically unacceptable.

The problem again lies in our choice of coupling to the curvature. Indeed we chose a quadratic coupling of the form $\xi\phi^2 R$ because it leaves ξ dimensionless. However, in more general theories arising from string theory or supergravity, the coupling will be of the much more general form: $f(\phi/M_{pl})R$ where f is the gauge kinetic function. In string theory one also has the coupling e^{ϕ} to the curvature where ϕ is the dilaton. Under the inversion $\phi \rightarrow -\phi$ we have $e^{\phi} \rightarrow 1/e^{\phi}$ (the basis of S-duality in string theory since f is the coupling constant in perturbative expansions [238, 239]).

Returning to the DLV mechanism, we see that all discrete symmetries will be broken if $f = f(\phi, \chi, ...)$ is an *odd* function of *all the fields* in the problem. This provides a sufficient condition for destabilising all domain walls and hence allows implemention of the general DLV proposal at reheating [16].

We leave to future work the explicit implementation of this mechanism for specific gauge kinetic functions to get numerical results regarding the size of parameter space for which effective solutions to the monopole problem exist.

Next we consider the resonant production of vector bosons during reheating.

3.7 The vector case

Until now, reheating studies have been limited to minimally-coupled scalar fields, fermions and gauge bosons [94]. In the case of vector fields the minimum one can do to preserve gauge-invariance is to couple to a complex scalar field via the current since real scalar fields carry no quantum numbers. We consider here only vacuum vector resonances, however.

[16] We note that in general we also expect non-renormalisable, Planck-suppressed terms of the form ϕ^n/M_{pl}^{n-4} where $n > 4$ to come into the effective theory that gives rise to inflation. For n odd these will also give rise to explicit breaking of the discrete symmetries of the potential [88].

A massive spin–1 vector field in curved spacetime satisfies the equations:

$$-\nabla_a\nabla^a\mathcal{A}^b + m_{\mathcal{A}}^2\mathcal{A}^b + R^b{}_a\mathcal{A}^a = 0 \tag{3.38}$$

These equations are equivalent to the Maxwell-Proca equations for the vector potential $\mathcal{A}_a$ only after an appropriate gauge choice which removes one unphysical polarization (the massive case has an extra longitudinal degree of freedom). In our case we shall use the so-called tridimensional transversal gauge condition:

$$\mathcal{A}_0 = 0, \quad \nabla^i\mathcal{A}_i = 0 \tag{3.39}$$

This set is equivalent to the Lorentz gauge, although it doesn't conserve the covariant form of the latter. Nonetheless, in either case, gauge-invariant quantities such as the radiation energy density, are unaffected.

In a FLRW background, the Ricci tensor is diagonal, which together with the gauge choice (3.39) and expansion over eigenfunctions, ensures the decoupling of the set of equations (3.38). We can reduce the system to a set of decoupled Mathieu equations. The Ricci tensor is (see Eq. 3.15):

$$R^a{}_b = \kappa V(\phi)\delta^a{}_b - \kappa\dot{\phi}^2\delta^a{}_0\delta^0{}_b\,, \tag{3.40}$$

which leads to the Mathieu parameters for for the spatial components $(a^{3/2}\mathcal{A}^i)$:

$$A(k) \doteq \frac{k^2}{a^2m_\phi^2} + \frac{m_{\mathcal{A}}^2}{m_\phi^2} + 2q \quad , \quad q \doteq \frac{\kappa\Phi^2}{8} \tag{3.41}$$

showing that vector fields are also parametrically amplified, albeit weakly, during reheating as in the scalar case.

3.8 The non-minimal vector case

The proof of a resonance in the minimally coupled cases for spin 0,1 (and as we shall see in section [3.9] spin 2) particles is pleasing for its unified nature but the effect is rather weak in all cases. The non-minimally coupled scalar case on the other hand provided an appropriate opposite limit in which purely cosmological effects could cause non-perturbatively large particle production. In this section we examine this possibility in the spin 1 case. Non-minimal coupling of the electromagnetic field to spacetime curvature has been considered previously as a mechanism for producing the large-scale magnetic fields which are known to persist in galaxies and the intergalactic medium.

As done in section (3.7), the typical assumption is that the electromagnetic field is governed by the minimal Lagrangian:

$$L = \tfrac{1}{2}\sqrt{-g}F^{\mu\nu}F_{\mu\nu} \tag{3.42}$$

where $g = \det[g_{\mu\nu}]$, which leads to the very simple equations of motion in vacuum:

$$F^{\mu\nu}_{;\nu} = 0 \tag{3.43}$$

which together with the Bianchi identities give the complete evolution of the field.

Breaking the hypothesis of minimal coupling involves the addition of terms to eq. (3.42) which couple the electromagnetic field directly to the curvature in one form or another, in analogy to the scalar field case. There are seven possible ways to do this [89] and these seven possibilities fall into two groups: those which are electromagnetically gauge invariant and those that are not [89]:

Gauge-dependent

$$L_1 = RA_\mu A^\mu \tag{3.44}$$

$$L_2 = R_{\mu\nu} A^\mu A^\mu \tag{3.45}$$

Gauge-invariant

$$L_3 = RF_{\mu\nu} F^{\mu\mu} \tag{3.46}$$

$$L_4 = RF_{\mu\nu} F^{*\mu\mu} \tag{3.47}$$

$$L_5 = R_{\mu\nu} F_{\mu c} F^{c\mu} \tag{3.48}$$

$$L_6 = R_{\alpha\beta\mu\nu} F_{\alpha\beta} F^{\mu\nu} \tag{3.49}$$

$$L_7 = R_{\alpha\beta\mu\nu} F_{\alpha\beta} F^{*\mu\nu} \tag{3.50}$$

where $*$ indicates the Hodge dual[17].

Now L_1 and L_2 are gauge-dependent but they have the correct dimension in the sense that in the action:

$$S = \int \sqrt{-g}(\lambda_1 L_1 + \lambda_2 L_2) \tag{3.51}$$

the arbitrary non-minimal couplings λ_1 and λ_2 are *dimensionless.* The second class of Lagrangian terms $L_3 ... L_7$ are gauge-invariant but require coupling constants λ_i which have dimensions of $(\text{length})^2$, related to the introduction of curvature scales in the background manifold.

Since we wish to consider only the possibility of strong particle production in the non-minimal vector case we wish to choose the strongest candidate among $L_1 ... L_7$. We therefore exclude L_1 and L_2 because they are not gauge-invariant. L_4 and L_7 are eliminated since they will lead to violation of parity, involving as they do the dual of $F_{\mu\nu}$. Since we consider the background to be flat FLRW, the Weyl tensor vanishes and the curvature is completely determined by the scale factor, so that L_3, L_5 and L_6 become essentially equivalent for our purposes. Hence we will consider L_3 as the simplest physically motivated example, yielding the new complete Lagrangian density for the electromagnetic field:

$$L_{NM} = \tfrac{1}{2}\sqrt{-g}\left((\lambda R - 1) F_{\mu\nu} F^{\mu\nu}\right) \tag{3.52}$$

[17] See chapter (7) for the definition and in-depth discussion of the dual operator.

and modified equations of motion [89]:

$$\begin{aligned} F^{\mu\nu}_{;\nu} &= \lambda(RF^{\mu\nu})_{;\nu} \\ &= \frac{\lambda\dot{R}\delta^0_\nu F^{\mu\nu}}{(1-\lambda R)} \end{aligned} \tag{3.53}$$

where the second equivalence in Eq. (3.53) comes due to the FLRW background where R depends only on time.

Note that when $\lambda \neq 0$, photons still follow null geodesics of the background geometry, but that the number of photons is not conserved in general in the geometric optics approximation. One might hope to place limits on λ via optics experiments or other observational evidence, however, this is extremely difficult due to the appearance of the $\dot{R}$ factor on the RHS of Eq. (3.53) which is very small, being of the order of $\ddot{H} + \dot{H}H$. This offers the possibility that λ might be large, and hence the non-minimal effect might be very important in high curvature regions such as reheating.

3.9 The graviton case

We will show in chapter (6) using the electric and magnetic parts of the Weyl tensor [148] that there exists a formal analogy between the scalar field and graviton cases during resonant reheating. Here we will show that the correspondence also holds in the Bardeen formalism. The gauge-invariant (at first order) transverse-traceless (TT) metric perturbations h_{ij} describe gravitational waves in the classical limit. In the Heisenberg picture one expands over eigenfunctions, Y_{ab} of the *tensor* Laplace-Beltrami operator [20] with scalar mode functions h_k, which satisfy the equation of motion:

$$\ddot{h}_k + \Theta\dot{h}_k + \left(\frac{k^2+2K}{a^2}\right)h_k = 0\,, \tag{3.54}$$

or equivalently

$$(a^{3/2}h_k)\ddot{} + \left(\frac{k^2+2K}{a^2} + \tfrac{3}{4}p\right)(a^{3/2}h_k) = 0\,, \tag{3.55}$$

where $p = \kappa(\dot{\phi}^2/2 - V)$ is the pressure. This gives a time-dependent Mathieu equation (c.f. Eq. 3.25) with parameters:

$$A(k) \doteq \frac{k^2}{a^2m_\phi^2}\ ,\ q \doteq -\frac{3\kappa\Phi^2}{16}\,. \tag{3.56}$$

In this case, a negative coupling instability is impossible and only for $\Phi \sim M_{pl}$ is there significant graviton production. Note, however, that if temporal averaging is used, the average equation of state is that of dust, $\overline{p} = 0$. Eq. (3.55) then predicts (falsely) that there is no resonant amplification of gravitational waves since the value of q corresponding to the temporarily averaged evolution vanishes.

What happens to the gravitational wave evolution once most of the energy is transfered into the χ field ? This is a highly non-trivial question for two reasons since the expansion (eq. 3.18) and curvature (eq. 3.16) now gain non-negligible contributions from (i) the backreaction due to the large fluctuations $\langle(\delta\chi)^2\rangle$ [111] and (ii) the anisotropic stress of χ coming from the fact that $\xi \neq 0$.

We will not look at (i) since the metric perturbations are assumed to remain small enough not to violate CMB anisotropy levels. In the models in which this is true, the backreaction will be small. Secondly, at least in the $\xi < 0$ models, the mode most amplified is the $k = 0$, homogeneous mode, so that neglecting the short-wavelength fluctuations is certainly justified.

In the presence of an anisotropic stress, the RHS of eq. (3.54) is modified to $\kappa a^2 p\Pi$, instead of being zero. The pressure p also gains nonlinear terms proportional to ξ [18] [20]:

$$p = \frac{1}{2}\dot{\chi}^2 - V(\chi) - \xi\left[2H\chi^2 + \frac{1}{3}\Theta^2\chi^2 + \ddot{(\chi^2)} + \frac{\Theta}{3}\dot{(\chi^2)}\right] . \tag{3.57}$$

Π is the anisotropic stress and is given by:

$$p\Pi = -\xi\frac{k^2}{a^2}[2\chi X + (\Phi + \Psi)\chi^2] , \tag{3.58}$$

where X, Φ, Ψ represent the various scalar metric perturbations associated with χ and the inflaton [20]. We see that in the case where $|\xi|$ is finite, the $k = 0$ mode is still conserved: $h_0 = const$ remains a solution of eq. (3.54), although the other modes will evolve differently due to the change in $a(t)$ through eq. (3.57). A crucial difference compared with the minimally coupled case however, is the fact that eq. (3.17) is no longer *monotonically decreasing* in time when $|\xi|$ is large (χ can violate the weak energy condition; see eq. 3.28). This means that a can actually decrease for short times which turns the $k = 0$ gravitational wave decaying mode $h_0 \sim a^{-3}$ into a growing mode, something not possible in the simple, minimally coupled one-field models usually considered.

In the $k \neq 0$ the situation is much more complex as gravitational waves couple to the density perturbations via eq. (3.58), something which in the minimally coupled case only happens at second order. Moreover, for sub-horizon modes and $|\xi| \sim 10^5$, the driving term given by (3.58) grows to be of order unity and hence very important for gravitational wave evolution. Nevertheless an analysis of these issues is beyond the scope of this chapter and is left for future work.

3.10 Conclusions

We have described a new – geometric – reheating channel after inflation, one which occurs solely due to gravitational couplings. While this is not very strong in the gravitational wave and minimally coupled scalar field cases, it can be very powerful in the non-minimally coupled case, either due to broad-resonance ($\xi \gg 1$) or negative coupling ($\xi < 0$) instabilities. Particularly in the latter case, it is possible to produce large numbers of bosons which are significantly more massive than the inflaton, as required for GUT baryogenesis. Non-thermal symmetry restoration may also be very effective in geometric reheating, leading to the resurrection of the monopole problem after inflation although an implementation of the Dvali-Liu-Vachaspati mechanism for its solution is possible.

[18] This time we neglect the energy that may remain in the field ϕ; justified if reheating has been successful.

Geometric reheating further gives rise to the possibility that the post-inflationary universe may be dominated by non-minimally coupled fields. These must be treated as imperfect fluids which would thus alter both density perturbation and background spacetime evolution, which are known to be significantly different [95] compared with the simple perfect fluid case. We have further presented a unified approach to resonant production of vector and tensor fields during reheating in analogy to the scalar case.

Chapter 4

First Light - a Classification via Spectral Methods

Come forth into the light of things
let nature be your teacher
Wordsworth

Man wants to move the stars to tears with his words,
while all the while he beats out a broken tune for bears to dance to.

I have hardly ever known a mathematician who was capable of reasoning.
Plato

Pure logical thinking cannot yield us any knowledge of the empirical world;
all knowledge of reality starts from experience and ends in it.
Einstein

4.1 Introduction

As we discussed in chapter (2), reheating in standard models of inflation can largely be split into four phases: the initial resonance ("first light"), the phase when rescattering becomes important, the phase when backreaction becomes important, and thermalisation. The middle two phases can occur simultaneously in some models, or may well be completely absent in others, such as in reheating where couplings are very weak.

This chapter[1] will present a method for a systematic classification of the first phase where backreaction and rescattering can be neglected. This phase typically starts with the universe energy density completely dominated by the inflaton energy with the temperature of the universe near absolute zero. The initial phase of reheating therefore introduces the first radiation back into the universe and hence we label it the "first light" epoch.

The techniques used in this chapter arise in the spectral theory of linear operators on Hilbert spaces and are functional analytical in nature. Because the Schrödinger equation is one of the most important examples these techniques have been applied since the beginning of quantum mechanics and indeed, the stability bands of the Mathieu equation will be seen to correspond simply to the conduction bands of metals described by a $1-D$ Schrödinger equation with periodic (in space) potential. The more complex potentials we will study were also first used in condensed matter physics in the study of correlated electron systems and phenomena such as Anderson localisation.

The mathematics itself is extremely beautiful with connections to many branches of mathematics - solitons, inverse scattering techniques, geodesics on the ellipsoid and so on. For a beautiful exposition of some of these aspects the reader is referred to the small monograph by Moser [119] which strikes a balance between clarity and rigor. Much more complete, and perhaps necessarily dry, discussions of related issues can be found in the following books: "*Spectra of Random and Almost-Periodic Operators*" [106], "*Functional Analysis I*" [104] and "*Theory of Solitons*" [110].

4.1.1 Our *modis operandi*

Given that the simple models of two-field reheating have been thoroughly studied in the last few years, it is natural to ask how these studies an be extended to the case of reheating based on realistic models of inflation coming from GUT, supergravity or even string-inspired theories, some of which we considered in chapter (2). This question raises several non-trivial but fascinating issues:

•How does the symmetry group of the theory and the symmetry-breaking pattern affect reheating and non-thermal defect production ?

[1] This chapter is based on the papers [148] and [154].

•The particle content of the theory - what couples to the inflaton [2] and what spin fields are they ?

•Quantum corrections to the potential [3] and the effects of renormalisation and regularisation.

•Non-equilibrium corrections to the equations of motion and to symmetry breaking/restoration.

•The beta-functions of the theory governing the running of the coupling constants as a function of the energy. In particular, for chaotic reheating which initiates at field values a bit below the Planck scale, the high-energy, ultra-violet (UV) fixed points of the theory are very interesting. In gauge theories one often has different phases at different energies (e.g. SUSY QCD [105]) where the theory may be asymptotically free, where chiral symmetry is broken, where the theory is described by a σ-model etc... These different phases and the existence of supersymmetry are responsible for many of the exciting features such as strong-weak coupling (S) duality [238, 239].

•The nature of field evolution at strong coupling and the existence of quantum chaos. What is the relationship with the classical evolution ?

These issues lie within the broad reach of some of the most difficult and exciting frontiers of much of modern theoretical physics. As such, it is extremely difficult to answer these questions in realistic theories, based for example on the standard model or minimal extensions thereof.

Thus to answer the question about GUT reheating to any satisfactory degree one is faced with a bifurcation in approaches, which yield rather different results. Firstly, one may choose a *specific* model. Such a specification requires (i) a choice of gauge group, e.g. $SO(10)$ or $SU(5)$. (ii) the imposition or not of supersymmetry, and/or the consideration of supergravity corrections; (iii) the choice of what fields will play the role of the inflaton and which couplings to the inflaton to include. (iv) Calculation of the β-functions of the resulting model and the quantum corrections to the effective potential.

After these issues have been taken care of, one could examine if there is successful reheating in the correct vacuum corresponding to the standard model $SU(3) \times SU(2) \times U(1)$. Even if the field reaches this correct vacuum, which is not gaurranteed since the inflaton may end up in the wrong local minimum of the effective potential [86], reheating might allow one to rule out certain regions of parameter space. In conjunction with experimental data (coming from nucleosynthesis constraints or accelerator data) this might even allow the ruling out of certain models due to reheating predicting too high a gravitino (the spin 3/2 superpartner of the graviton) abundance or the over-production of moduli/Polonyi fields [35].

We have already discussed in chapter (2) the constraint on the reheating temperature of $T_r \leq$

[2] For example, if one has only fermions coupled to the inflaton then preheating is qualitatively very different to the bosonic case due to the exclusions principle, as discussed in chapter (2).

[3] And in general the quantum moduli space compared with the classical moduli space [109].

$10^{10}GeV$ coming from the gravitino abundance and how it can be used to bound masses of neutrinos in a minimal supersymmetric model. Nevertheless, this constraint is only valid if entropy remained conserved after reheating, an observation which significantly weakens the power of the neutrino mass bound.

This approach is plagued by the sheer complexity of the problem and while leading to precise predictions, these predictions are typically only valid under certain strong assumptions and in small ranges of parameter space (e.g. such as the perturbative coupling region) and are, as in the example above, almost always subject to strong caveats.

The alternative approach [148] is to attempt a broad classification of reheating classes - a panoramic portrait or landscape of possible models. Here a reheating class is defined to consist of those reheating models which give qualitatively similar results. But how can we make a sensible definition of reheating class ? After all, physical quantities such as the reheating temperature, size of the quantum fluctuations, gravitino abundance etc... are all continuous variables and there appears no elegant way to create these classes [4]. It seems that we can take our landscape and blur and deform it into any other landscape.

Fortunately, this is not true, at least for the first-light phase of reheating, since there exists a rather beautiful equivalence between the Klein-Gordon equation in Fourier space in a Minkowski background and the 1-D Schrödinger operator-eigenvalue problem [148].

Preheating is largely controlled by the nature of the Floquet indices, as we discussed in depth in chapter (2). The above mentioned equivalence maps the set of Fourier modes onto the possible real eigenvalues of the Schrödinger operator. The spectrum consists of those eigenvalues (and hence modes) for which there is not exponential growth. A subtlety arises in the structure of the spectrum as we will discuss later, but it is true that the set of modes with positive Floquet exponent, contains (often properly in the set theoretic sense), the complement of the spectrum of the corresponding Schrödinger operator.

Thus by studying how the spectra flow as one varies parameters and potentials (corresponding to inflaton evolution), one naturally achieves a division of reheating into inequivalent classes classified by the very different nature of the spectrum in the different cases. For example, periodic operators (Floquet theory), which include the Hill, Lamé and Mathieu equations, always have band spectra. Quasi-periodic operators on the other hand, typically have Cantor-set spectra which are no-where dense. This implies that the number of exponentially growing modes is usually much larger in the case of quasi-periodic evolution, than in the case of simple periodic evolution. As we will show in the next chapter, these potentials arise naturally in multi-field models with incommensurate mass spectra.

[4] One might set excursion levels for these quantities and build classes around these, but they are completely arbitrary, not diffeomorphism invariant and thus unsatisfactory.

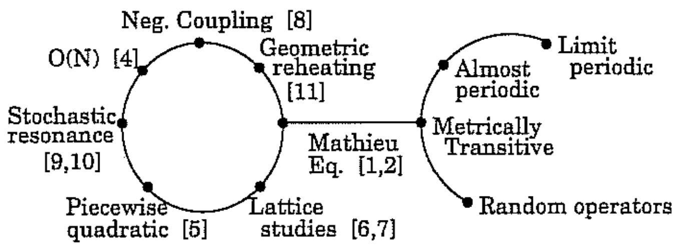


Figure 4.1: A schematic map of models and approximations in preheating. Minimal references for techniques in the exactly periodic case are shown in brackets. The right-hand branch corresponds to paradigms developed in this chapter.

Thus our aim is to classify all the interesting types of spectra that arise and therefore create a table of equivalence classes of preheating. The exit to thermalisation where backreaction and rescattering are important, is a secondary and much more intricate problem requiring a more nuts-and-bolts approach [9].

Even when limited to the first-light phase, this equivalence has some lackings and is only really useful in its present form in a Minkowski background or perturbatively when the the expansion of the universe can be neglected ($m \gg H$). This simple equivalence should, in the future, be extended to include the expansion of the universe and the backreaction of the produced particles, which can, nevertheless be done in principle, by extending the equivalence to the Sturm-Liouville operator.

4.2 Classes of reheating during "first light"

We will now present in greater detail the program we outlined above. First we lay down the equivalence more precisely followed by some spectral theory that we will need. This presents the necessary background to appreciate the fundamental theorems of the chapter that are tested numerically in the second half of the chapter.

4.2.1 The 1-D Schrödinger – Klein-Gordon equivalence

Now the Klein-Gordon equation for the field χ in Minkowski spacetime is:

$$(\partial_\nu \partial^\nu + V')\chi(t, \mathbf{x}) = 0 \tag{4.1}$$

where $\nu = 1, 2, 3, 4$ and V is the total potential describing the mass, self-interaction, coupling to the curvature and couplings of χ to other fields and $V' \equiv \partial V/\partial\chi$.

If we Fourier transform this equation we find:

$$\ddot{\chi}_k + \left(\frac{k^2}{a_0^2} + V'\right)\chi_k = 0 \tag{4.2}$$

where a_0 is an arbitrary normalisation. Now if we consider that χ is essentially a test field in its vacuum state, then the potential term V' can in principle be given explicitly as a function of time $V' = V'(t)$, as we did all through chapter (2). Further, if V' is a periodic function, then the Fourier-space Klein-Gordon equation above takes the form of the Hill equation:

$$\ddot{y} + [A - 2qP(2t)]y = 0 \tag{4.3}$$

where $P(2t)$ is any periodic function of the independent variable t. When $P(\cdot) \propto \cos(\cdot)$, we have the Mathieu equation, while $P(\cdot) = \mathrm{cn}^2(\cdot)$ yields the Lamé equation, where $\mathrm{cn}(\cdot)$ is the elliptic cosine function. Now Eq. (4.3) is equivalent to the one-dimensional Schrödinger operator-eigenvalue problem:

$$\mathcal{L}(y) \equiv -\frac{d^2y}{dx^2} + Q(x)y = \lambda y \tag{4.4}$$

under the transformations $x \leftrightarrow t$, $Q(x) \leftrightarrow 2qP(2t)$, $\lambda \leftrightarrow A$. We denote the spectrum of $\mathcal{L}$ by $\sigma(\mathcal{L})$, its complement by $\overline{\sigma(\mathcal{L})}$. The crucial point to notice is that the set of modes with positive Floquet index of Eq. (4.3) exactly coincides with $\overline{\sigma(\mathcal{L})}$ of Eq. (4.4) for the case of Periodic $P(2t)$.

Indeed, this equivalence can be extended, if suitable care is taken, to the case where $P(2t)$ is a metrically transitive potential, as will be discussed in coming section.

4.3 Spectral theory

Here we will present the minimum amount of spectral theory required for our classification. Rigorous details (and the inevitable caveats and lemmas) can be found in many books, such as Pastur and Figotin [106].

4.3.1 Metrically transitive potentials

The potential $Q(x)$ in the Schroödinger equation above can, *a priori* take any form. However, for our purposes it is convenient to restrict ourselves here to a class of potentials known as *Metrically transitive operators*, that are appropriate for studying first light in preheating in Minkowski spacetime. We state here that inclusion of the expansion in the present classification program can be achieved in principle by extending the equivalence of subsection (4.2.1) to a Stürm-Liouville – Klein-Gordon equivalence. However, the number and power of spectral results in this case are much more limited.

Metrically transitive potentials are those which, simply stated, show a specific form of statistical homogeneity in the x-coordinate (or in the reheating case, under the equivalence of Eq. (4.4), t). At one end of this class therefore, lie the exactly periodic potentials which have a discrete translation group invariance determined by the minimal period, T. Thus, we call already see that all of the standard theory of preheating based on periodic inflaton oscillations (leading to the Mathieu eq., the Lamé eq. and in general the Hill eq.) is contained as a special sub-group in the set of 1-D metrically transitive potentials. At the extreme opposite end lie the completely random potentials. So what, exactly unifies these potentials since random potentials seem at first sight very different from periodic potentials ?

We have to clarify our earlier statement and say that a metrically transitive potential is one such that the *probability distribution* induced by the potential is homogeneous (or stationary in the case of time-processes). Secondly, to be metrically transitive, the potential should have zero correlations at infinite spatial separation [106], the weakest formulation of which is to require *ergodicity*.

The reader is referred to [106] for the gory details. Here we present some examples of metrically transitive functions:

Examples of metrically transitive functions

(1) Independent, identically distributed random variables.

(2) Random Markov processes (processes without memory).

(3) Gaussian random fields.

(4) Periodic, quasi-periodic and almost-periodic functions.

Examples (2) - (4) will prove useful later in extending the set of known reheating classes. We will not require a more precise definition of metrically transitive potentials and go on to discuss more relevant details.

4.3.2 The spectrum

Here we present a brief overview of spectral results in infinite dimensional vector spaces (Banach or in our case, typically Hilbert). See for example Reed and Simon, [104].

Definition (1) - The spectrum

Let $\mathcal{L}$ be a bounded operator on a Hilbert space $\mathcal{H}$. Then λ is in the resolvant set $\rho(\mathcal{L})$ of $\mathcal{L}$ if $\lambda I - \mathcal{L}$ is a bijection with a bounded inverse (here I is the identity). Then $R_\lambda(\mathcal{L}) = (\lambda I - \mathcal{L})^{-1}$ is called the *resolvant* of $\mathcal{L}$ at λ. If $\lambda \notin \rho(\mathcal{L})$ then λ is said to be in the *spectrum* of $\mathcal{L}$, denoted $\sigma(\mathcal{L})$.

Definition (2) - The point spectrum

An $x \neq 0$ which satisfies $\mathcal{L}x = \lambda x$ for some $\lambda \in C$ is called an eigenvector of $\mathcal{L}$, λ an eigenvalue of $\mathcal{L}$ which implies that $\lambda I - \mathcal{L}$ is not injective so $\lambda \in \sigma(\mathcal{L})$. The set of all eigenvalues of $\mathcal{L}$ is called the *point spectrum* of $\mathcal{L}$.

4.3.3 Dissecting the spectrum further

In the case of a separable Hilbert space as we have for the 1-D Schrödinger operator, we may decompose the spectrum with respect to an abstract measure $d\mu$ [5] as:

$$\sigma = \sigma_{AC} \cup \sigma_{SC} \cup \sigma_P \tag{4.5}$$

That is, we split the spectrum into three pieces: an *absolutely continuous* (AC) part, a *singular continuous* (SC) part and the pure *point* (P) component. At this stage this decomposition is formal and indeed it turns out that we do not need to use the continuity properties of the spectrum for our work. Rather we can simply use high-level results regarding the Floquet index depending on whether λ is a member of σ_{AC}, σ_{SC} or σ_P.

Before we go any further, however, let us consider an example, p the canonical example for reheating. Let $Q(x)$ be exactly periodic. In fact, let it be $\cos(2x)$. Then $\mathcal{L}$ is simply the Mathieu eq. in disguise. In this case, for fixed q, $\sigma(\mathcal{L})$ is just a union of bands, which are just the *stable* bands of the Mathieu equation (when converted into frequency space k - slices though the red region of fig. 2.2). The obvious step then is to observe that the instability bands - where resonant particle production occurs - simply correspond to $\overline{\sigma(\mathcal{L})}$ at any fixed q.

But we have oversimplified a little. Here we have spoken about $\sigma(\mathcal{L})$ and $\overline{\sigma(\mathcal{L})}$, but what about the decomposition above in Eq. (4.5) ? Why did we introduce this refinement of the spectrum if it was unnecessary ? Well, it turns out that in this case (and in general for any periodic $Q(x)$), the spectrum *only has an absolutely continuous part*:

$$\sigma(\mathcal{L}) = \sigma_{AC} \qquad \text{(Periodic operators)} \tag{4.6}$$

and we begin to see the importance of σ_{AC} for reheating.

4.3.4 Reheating and σ_{AC}

Within the set of periodic potentials, one might try to search for a $Q(x)$ which lead to the smallest σ_{AC} in an attempt to make as many modes grow resonantly as possible, i.e. to find the most efficient reheating model. Indeed, if we look back at the history of preheating, this sort of argument caused quite some controversy, as we touched on in chapter (2).

[5]The measure is not to be confused with the Floquet index μ_k. Since we only use the measure in formal definitions, this should not occur however.

The original paper by Kofman, Linde and Starobinsky in 1994 [1] looked at the Mathieu equation, with its infinite number of stable and unstable bands. Later, Boyanovsky *et al* [183] claimed that the correct equation, suitable for a quartic potential, is the Lamé equation (2.29). The corresponding Schödinger operator has a spectrum of the form:

$$\sigma = [0, \lambda_0] \cup [\lambda_1, \infty) \tag{4.7}$$

in contrast to the Mathieu case. The lamé equation thus only *one* instability band corresponding to $\overline{\sigma(\mathcal{L})} = (\lambda_0, \lambda_1)$. This, Boyanovsky *et al* claimed, made a big difference to the nature of preheating. Unfortunately, their claim was unstable since inclusion of even perturbatively weak couplings to other fields breaks up the spectrum in Eq. (4.7) into an infinite disjoint sequence of bands, again resembling the Mathieu case.

Nevertheless, it is an interesting and unanswered question as to which periodic potential gives rise to the largest (in the set theoretic sense) $\overline{\sigma(\mathcal{L})}$ and hence the most efficient preheating. Although perhaps a caveat is in order here. One could easily imagine a perverse situation in which there were two models, one with a very large $\overline{\sigma(\mathcal{L})}$ but very small Floquet index μ_k, and one with small $\overline{\sigma(\mathcal{L})}$ but much larger μ_k for modes in $\overline{\sigma(\mathcal{L})}$. The question as to which leads to the more effective preheating is a difficult one. Indeed, at present there is little understanding about how μ_k and $\overline{\sigma(\mathcal{L})}$ are related. In the case of the Mathieu equation, however, we have at least a partial answer since $\overline{\sigma(\mathcal{L})}$ is an increasing function of q and we can use the equation for μ in the first resonance band:

$$\mu_k = \sqrt{\frac{q^2}{4} - \left(\frac{2k}{m} - 1\right)^2} \tag{4.8}$$

so as we increase q, we increase $\overline{\sigma(\mathcal{L})}$ as is evident since the width Δk of modes for which μ_k is real from Eq. (4.8) increases with q. And as a result the maximum value of μ_k increases. Hence for the Mathieu equation, increasing $\overline{\sigma(\mathcal{L})}$ implies an increase in the maximum, and average value, of μ. In other cases, however, we cannot be sure that this will be the case, although it seems reasonable.

With this broad preamble we come to the next obvious generalisation. Is it possible that outside the class of periodic potentials, there exist classes of potentials which are more efficient than any periodic potential ? and how would we even go about quantifying this issue ?

It turns out that σ_{AC} plays the key rôle in the study and answering of this issue. We have the constraint [106] that in general for a metrically transitive potential (and not just the periodic potentials considered above), bar a set of λ with Lebesgue measure zero, if $\lambda \in \sigma_{AC}$ then

$$\sigma_{AC} = \{\lambda \in \mathbf{R} | \mu_\lambda = 0\} \,. \tag{4.9}$$

i.e. σ_{AC} coincides with the set of stable modes, exactly as it did in the periodic case.

This gives us a weapon of great power, since we also know that on the complement $\overline{\sigma(\mathcal{L})}$, $\mu_k > 0$ [118]. Hence, we may search for potentials which have very thin $\overline{\sigma(\mathcal{L})}$ and be assured that the resulting reheating models are good candidates for explosive particle production. Before we examine our results in this direction, let us first make a slight, but enlightening detour into the theory of isospectral deformations of the Schödinger equation.

4.3.5 Isospectral deformations

The spectral theory of the Schrödinger equation abounds with beauty, but one of the most elegant aspects must be the theory of iso-spectral deformations, namely deformations of the potential $Q(x)$ which yield the same spectrum $\sigma(\mathcal{L})$ for the operator $\mathcal{L}$. Naturally one would expect that very few types of deformations would leave the spectrum unchanged - one cannot just run around the space $C^\infty(\mathbf{R})$, of smooth potentials expecting the spectrum to stay the same. Rather one expects to have to have to pick a well chosen path, depending on a parameter, say τ, for which $\tau = 0$ corresponds to the starting potential, $Q(x) \equiv Q(x, \tau = 0)$. Indeed, *a priori* it is not obvious that such a curve exists or that the solution will depend continuously on τ.

The answer to this question, for the 1-D Schrödinger equation (4.4), turns out [119] to be that if $Q(x,\tau)$ is our one-parameter family of potentials depending on the space dimension x, then the spectrum σ will be independent of the parameter $\tau \in \mathbf{R}$ *iff* $Q(x,\tau)$ is a solution of the KdV equation:

$$Q_\tau - 6QQ_x + Q_{xxx} = 0 \tag{4.10}$$

Suddenly our linear Schrödinger equation is linked to the famous nonlinear KdV equation with its solitonic solutions and infinite hierarchy of conserved quantities. Hence we may take a solution of the KdV equation, e.g. a solitonic solution, allow it to evolve in time (which is our parameter τ), and know that $\sigma(\mathcal{L})$ will be unchanged.

Using this we can generate, in principle, potentials which give the exactly the same spectra, although we are not sure how μ_k will depend on the parameter τ above. Let us examine this in more detail. The potential $Q(x)$ in the Schrödinger equation comes, under equivalence with the Klein-Gordon equation, from the coupling between the inflaton ϕ and the reheated field χ_k. Indeed, the equivalence with the linear Schrödinger equation only makes sense for couplings of the form $g^2\mathcal{F}(\phi)\chi^2$, in which case $Q(x) = g^2\mathcal{F}(\phi(x))$. To find $Q(x)$ we thus need to find the evolution of ϕ as a function of time (which then becomes x under the equivalence between Schrödinger and Klein-Gordon equations).

To solve for $\phi(t)$ is typically rather simple if we ignore backreaction from the produced particles and the expansion, as we are doing here. The evolution equation is:

$$\ddot{\phi} + V' = 0 \tag{4.11}$$

where $V(\phi)$ is the inflaton effective potential. In the limit where the expansion is unimportant we recover the energy as a conserved quantity (we get a timelike Killing vector):

$$E = V + \frac{\dot{\phi}^2}{2}. \tag{4.12}$$

Now in our case, $Q = g^2\phi^2$, which means that we can write the KdV equation (4.10) in terms of ϕ and $V = V(\phi, \tau)$.

In principal at least, it should be possible to obtain a partial differential equation for $V(\phi, \tau)$ whose solution would yield a reheating with precisely the same spectrum of amplified modes as the potential $V(\phi, 0)$. So far I have not succeeded in deriving this closed partial differential equation.

4.4 Spectral results for almost-periodic potentials

Here we present some spectral results for potentials which form equivalence classes for reheating. The almost-periodic potentials have as sub-classes the quasi- and limit-periodic potentials. Here we concentrate on the mathematics and delay giving real examples of these potentials to the section on numerical studies .

Definition (3a) - almost periodic potentials [106]

A bounded, continuous potential $Q(x)$ on $\mathbf{R}$ is called almost-periodic if *any* subset of the set of all its possible shifts $Q(x+\xi)$, $\xi \in \mathbf{R}$, contains a sequence that converges uniformly on the entire real line[6].

Perhaps this is not very helpful to the reader. Another, more physical definition is as follows:

Definition (3b) - almost periodic potentials [113]

The almost-periodic potentials are those for which their Fourier transform consists of a frequency basis $\{\omega_i\}$ where the smallest vector space[7] containing this basis, $\mathcal{M}$, is dense in $\mathbf{R}$.

As a simple example, one can immediately see that $\cos(\Omega x)$ with Ω a constant, is *not* almost-periodic (fortunately !) since its frequency basis is just $\{\Omega_1\} = \Omega$ and the module $\omega_1\mathbf{Z}$ is a *proper subset* of $\mathbf{Z}$, $\mathbf{Q}$ or $\not{\mathbf{Q}}$ in the cases where $\Omega \in \mathbf{Z}$, $\mathbf{Q}$, or $\not{\mathbf{Q}}$ respectively[8]. it is of course never equal to any of these and hence is not dense in $\mathbf{R}$ for any Ω.

The spectrum of the Schrödinger equation associated with almost-periodic potentials can be pure-point - the spectrum is made up completely by eigenvalues - and is *believed* to be generically a nowhere dense Cantor set[9]. However this remains unproven in any rigorous sense [114].

[6] i.e. every subset is precompact in the topology $C(\mathbf{R})$.
[7] Technically a module.
[8] We denote the rationals by $\mathbf{Q}$ and the irrationals by $\not{\mathbf{Q}}$.
[9] A nowhere dense set $\mathcal{A}$ has dense complement $\overline{\mathcal{A}}$.

4.4.1 Quasi-periodic potentials

Using definition (3b) above for almost-periodic potentials, we arrive at the following definition of quasi-periodic potentials:

Definition (4) - quasi periodic potentials [113]

If the module $\mathcal{M}$ in definition (3a) is generated by *finitely many* ω_i, then the potential, $Q(t)$, is quasi-periodic.

If $Q(x)$ is quasi-periodic then $Q(x) = f(\omega_1 x, ..., \omega_n x)$ with $f(x_1 + m_1, ..., x_n + m_n) = f(x_1, ..., x_n)$ with $m_i \in \mathbf{Z}$ and the ω_i pairwise incommensurate [113].

One can visualise quasi-periodic potentials very simply as flows on an n-dimensional torus, T^n. If the flow repeats after a certain time, then the flow is exactly periodic. However, if the flow moves over the torus coming arbitrarily close to any point then the flow is ergodic and quasi-periodic. The almost-periodic case corresponds to taking the $n \to \infty$ limit in this example.

From this it is easy to construct physical models which lead to quasi-periodic frequencies. All one requires (c.f. section 2.13) is two periodic functions whose periods of oscillation are irrationally related. The corresponding module $\mathcal{M}$ will then be dense in $\mathbf{R}$.

4.4.2 Limit-periodic potentials

Definition (5) - limit periodic potentials[106]

A limit periodic potential $Q(x)$ is an almost-periodic potential which is also a *uniform limit of periodic potentials.*

Limit periodic potentials are therefore very well approximated by Fourier series. A typical example is provided by:

$$Q(t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi t}{2^n}\right) \; ; \; \sum |a_n| < \infty \tag{4.13}$$

More rigorous results (instead of the previous hand-waving !) exist in the case that Q is limit-periodic. For these potentials we have the theorem:

Theorem 1 [103] - $\sigma(\mathcal{L})$ of Eq. (4.4) is generically[10] *a nowhere dense Cantor set for Q an element of the space of limit periodic potentials. Hence $\overline{\sigma(\mathcal{L})}$ is dense in* $\mathbf{R}$.

Thus the set of k for which $\mu_k \neq 0$, is dense in $\mathbf{R}^+$ despite the fact that σ_{AC} is not empty. This existence of only a Cantor set of stable modes leads us to call this cantor reheating. An associated

[10] The space of limit periodic potentials is a complete metric space and hence *generic* means here "for a dense G_δ" e.g. [114].

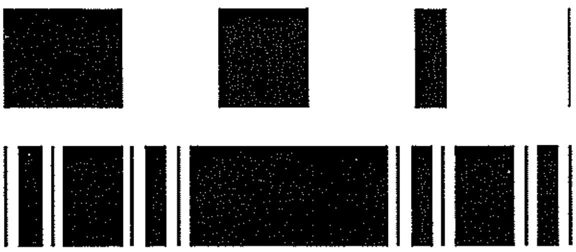


Figure 4.2: A schematic diagram comparing the spectrum of the Mathieu equation (upper) and that of a generic limit-periodic operator with fractal Cantor spectrum (lower). The black shading represents the complement of the spectrum, i.e. modes with $\mu_k \neq 0$. The Cantor spectrum is only shown at second level due to resolution limitations. The same pattern repeats itself in each white gap and hence the number of stable modes is much smaller than in the periodic case.

issue is what happens as the coupling to the potential (analogous to q in Eq. (1)) is increased. While in Eq. (1) this simply changes the breadth of the instability bands and magnitude of μ_k while always leaving $\sigma = \sigma_{AC}$, in the almost-periodic case this is not the case. Indeed, as in the example below, the nature of the spectrum (w.r.t. the splitting into $\sigma_P, \sigma_{SC}, \sigma_{AC}$) can change suddenly with q.

4.4.3 The discrete almost-Mathieu equation

Consider the discretized Schrödinger eq. (4.4), which is also used (in higher dimensions) in numerical preheating studies [205, 7]. A special case is the almost-Mathieu equation with exhibits a rich variety of effects [11]:

$$-y(x+1) - y(x-1) + 2q\cos(\alpha x + \omega)y(x) = \lambda y(x) \tag{4.14}$$

When α is rational the potential is periodic and one has the band spectrum of the Mathieu equation. However, for irrational α, the spectrum is a Cantor set generically for pairs $(q, \alpha) \in \mathbf{R}^2$ [122]. This shows that the nowhere dense nature of the spectrum is not lost as one increases q and hence moves from perturbative reheating to broad-resonance preheating. Further, when α is irrational, the Floquet index has the lower bound [124]:

$$\mu_k(q) \geq \ln|q| \tag{4.15}$$

so that for $|q| > 1$, $\mu_k > 0$ and hence the absolutely continuous part of the spectrum, σ_{AC}, becomes empty. In this case, if α is a Liouville number [12], then the spectrum is purely singular continuous, $\sigma = \sigma_{SC}$ [116]. Conversely if $|q| < 1$, the point part of the spectrum is absent for irrational α [123].

4.4.4 Finite-band spectra and the Bargmann potentials

In subsection (4.3.5) we presented a beautiful counter-example to the intuitively sensible idea that if one deforms the potential $Q(x)$, then the corresponding spectrum $\sigma(\mathcal{L})$ must change too.

[11] For example, Eq. (4.14) also exhibits the beautiful property of duality, similar to the S-duality of string theory, since under Fourier transform $q \to 1/q$ and $\lambda \to -\lambda/q$ [124].

[12] A Liouville number, α, is irrational but well approximated by rationals so that there exist integers $p_n, q_n \to \infty$ and a number C with $|\alpha - p_n/q_n| \leq Cn^{-q_n}$.

In this subsection we do the same for the following (incorrect) assertions:

"All almost-periodic potentials have strange, non band-like spectra"

or 'equivalently':

"Only periodic potentials have band-spectra"

Now a natural enough question to ask is "give me all potentials which lead to spectra which consist of finitely many bands". How does one solve this problem ? The answer is given by the inverse scattering method[13] and while we will not describe the results here (see e.g. [119]) it turns out that potentials having finite-band spectra are actually quasi-periodic in general [119][14] and the periodic potentials are quite special corresponding to a frequency basis which is made up of only one frequency.

As an nice example, one can ask what potential corresponds to the case when one lets the bands shrink down to single points so that one obtains a discrete spectrum which is pure point. The corresponding potentials are known as the Bargmann potentials and can be obtained as limiting cases of quasi-periodic potentials.

4.5 Stochastic potentials

In the introduction to this chapter, we gave perhaps the archetypical physical example of the band structure arising from a periodic potential - the conduction bands in perfect metals with their regular electron lattice structure. What happens if we add random impurities to the otherwise periodic (1-D) electron lattice ?

This is modeled by adding a stochastic element to our periodic potential. The resulting spectrum splits into different colours in the sense that $\sigma_{AC} \supset \sigma$ and $\sigma_P, \sigma_{SC} \neq \emptyset$. The impurities act like a spectral prism. Physically this typically gives rise to Anderson localisation, as it is known in condensed matter physics. The eigenfunctions corresponding to allowed eigenvalues become exponentially decreasing about some point x, in space, and the wave function is confined to the region around x. By our equivalence however, it is the values outside the spectrum that we are interested in.

Stochastic potentials are at the opposite end of the metrically transitive 'world' to periodic potentials since they generically have empty σ_{AC}. In fact we have the following theorem:

Theorem 2 [120, 121] - If $Q(x)$ is a sufficiently random potential for Eq. (4.4), then $\mu_\lambda > 0$ for almost all $\lambda \in \mathbf{R}$ and σ_{AC} of $\mathcal{L}$ is empty with probability one.

Indeed we see that a positive Floquet exponent is guaranteed for all modes bar a set of measure

[13] The inverse scattering technique allows one to reconstruct the potential from observations of its spectrum [110].

[14] This does not, of course, imply that most quasi-periodic potentials have band spectra !

zero, and again reheating is significantly different from the periodic case. Here "sufficiently random" means nondeterministic [106] and is typically a requirement that correlations decay sufficiently rapidly. An example is a Gaussian random field whose correlation function has compact support. To proceed to obtain quantitative estimates of the Floquet indites, we exploit the fact that Eq.s (4.25,4.26) have the form of stochastic harmonic oscillators.

4.5.1 Explicit estimates for the Floquet indices

Consider the stochastic harmonic oscillator, with frequency given by $\omega^2 = \kappa^2 + q\xi(t)$, where q is a dimensionless coupling to the mean-zero coloured stochastic process $\xi(t)$ and $\kappa^2 = k^2/a^2$ in our case. The Floquet index has been shown to be strictly positive for all q [117]. It has also been related to the spectral density of fluctuations via averaging over the second moments [126] of the random process ξ [127]:

$$\mu_k = \frac{\int_0^\infty \langle \xi(t)\xi(t-t')\rangle \cos[2\langle\omega^2\rangle t']dt'}{2|\langle\omega^2\rangle|}\,. \tag{4.16}$$

This has the form of a fluctuation-dissipation theorem since fluctuations in the inflaton field determine the dissipation rate into other fields. In the case that $\xi(t)$ is a mean zero ergodic Markov process, μ_k can be explicitly estimated as [125]:

$$\mu_k = \frac{\pi}{4}\frac{q^2}{\kappa^2}\hat{f}(2\kappa) + O(q^3) \tag{4.17}$$

where $\hat{f}$ is the Fourier transform of the expectation value of the two-time correlation function $\langle\xi(t)\xi(t-t')\rangle$. Note that $\mu_k \propto k^{-2}$, so that although all modes grow exponentially, the Floquet index is, as in the periodic and stochastic resonance [9] cases, a rapidly decreasing function of k. In the broad-resonance limit, $q \to \infty$, we write $\kappa^2 = \kappa_0^2 + q\kappa_1^2$, which gives (assuming $max\ \xi < \kappa_1^2$) [125]:

$$\begin{aligned}\mu_k &= \frac{\kappa_1}{4\pi}\int_0^{2\pi} d\theta \int d\xi \frac{\sqrt{\kappa_1^2 - \xi}}{\kappa_1^2 - \xi\cos^2\theta} G(\ln(\kappa_1^2 - \xi\cos^2\theta)) \\ &+ O(1/\sqrt{q})\end{aligned} \tag{4.18}$$

where G is the infinitesimal generator of $\xi(t)$ defined by the limit of the operator sequence: $G = lim_{t\to 0}(U_t - \mathbf{I})/t$. Here $\mathbf{I}$ is the identity operator and U_t is a family of operators on the space of bounded continuous functions f, defined by $U_t f(x) = E[f(\xi(t))|\xi(0) = x]$, where E denotes the expectation operator [107]. Again $\mu_k > 0$ or all k, and given a model of ϕ evolution, one can explicitly estimate μ_k, and hence the numbers of produced particles n_k and variances $\langle\delta\phi^2\rangle$ and $\langle\chi^2\rangle$.

4.6 Numerical results

Here we present numerical simulations[15] to test the theorems we have given regarding the behaviour of μ for almost-periodic and stochastic potentials. We will also provide reheating examples for why these potential are physically interesting, a task taken significantly further in the next chapter.

[15]These simulations appear in the papers [98, 154] done in collaboration with F. Tamburini.

In the next section we show how μ compares in the purely periodic Mathieu case and the almost-periodic and stochastic cases respectively.

4.6.1 Quasi-periodic potentials - Cantor reheating

As we discussed in section (2.13), reheating with two or more fields leads naturally to quasi-periodic evolution of the frequencies of the fields undergoing particle production.

We limit our discussion in this section to Minkowski spacetime [16]. Consider the inflaton, ϕ, and the two minimally coupled scalar fields φ, χ with the natural mass hierarchy $m_\phi \gg m_\varphi \gg m_\chi$. In this section we consider the effective potential:

$$\begin{aligned} V(\phi,\varphi,\chi) &= \frac{m_\phi^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 + \frac{m_\varphi^2}{2}\varphi^2 + \frac{m_\chi^2}{2}\chi^2 \\ &+ \frac{g^2}{2}F(\phi)\chi^2 + \frac{h^2}{2}J(\varphi)\chi^2 \end{aligned} \tag{4.19}$$

$F(\cdot), J(\cdot)$ are assumed to be analytic in their respective arguments. The evolution of the χ_k modes is then [1]:

$$\ddot{\chi}_k + \left(\frac{k^2}{a^2} + m_\chi^2 + g^2F(\phi) + h^2J(\varphi)\right)\chi_k = 0 \tag{4.20}$$

Instead the inflaton zero-mode evolves according to:

$$\ddot{\phi} + m_{\phi,eff}^2\phi + \lambda\phi^3 = 0 \tag{4.21}$$

where the frequency of oscillation is partially controlled by the effective mass:

$$m_{\phi,eff}^2 = m_\phi^2 + g^2\frac{F'}{\phi}\langle\chi^2\rangle + 3\lambda\langle\delta\phi^2\rangle \tag{4.22}$$

In the case that $\langle\chi^2\rangle = \langle\delta\phi^2\rangle = 0$, the inflaton simply oscillates with constant period. However, when there are multiple-fields, or the mass acquires corrections due to quantum fluctuations, the period is no-longer constant, but may increase monotonically, oscillate or exhibit random fluctuations. As an example, in the special case of a Yukawa interaction, $F(\phi) = \phi$, Eq. (4.21) has a pure driving term $\propto g^2\langle\chi^2\rangle$.

As a simple example consider Eq. (4.20) with $F(\phi) = \phi^2$, $J(\varphi) = \varphi^2$ and $\lambda = 0$. Then the equations for the quantum fluctuations of χ_k are:

$$\ddot{\chi}_k + \left(\frac{k^2}{a^2} + m_\chi^2 + g^2\phi^2 + h^2\varphi^2\right)\chi_k = 0\,. \tag{4.23}$$

with $\phi \sim \sin(m_\phi t)$ and $\varphi \sim \sin(m_\varphi t)$. When m_ϕ/m_φ is irrational, the potential is quasi-periodic. The spectrum of Eq. (4.4) for almost-periodic potentials can be pure point and is, in a non-rigorous way, generally a nowhere dense Cantor set [114]. Hence, in the example above, for infinitely many irrational values of m_ϕ/m_φ, we may expect that the spectrum of χ_k will be nowhere dense, and consequently that almost all modes will grow exponentially.

[16] The general expanding FLRW case is contained within the spectral theory of Sturm-Liouville operators.

However, unlike the random potentials considered in section (4.5), the Lebesgue measure of the spectrum, even if a Cantor set, need not be zero [113] and σ_{AC} may be non-empty. Indeed it is possible to have $\sigma = \sigma_{AC}$ [103].

4.6.2 Stochastic inflationary reheating

We now consider the case where the potential Q in Eq. (4.4) is random. This models for example, the classical limit of stochastic inflation, where the dynamics of the local order parameter in a FLRW background, are described by [108]:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = \frac{H^2}{8\pi^3} V'''(\phi)\xi(t)\,, \tag{4.24}$$

where $H \equiv \dot{a}/a$ is the stochastically evolving Hubble constant [112], ξ is a coloured Gaussian noise of unit amplitude with a correlation time of order H^{-1}, so that the inflaton evolves stochastically. The origin of the noise is the backreaction of quantum fluctuations with wavelengths shorter than the coarse-graining scale [111, 112, 100].

Here we will consider the potential $V(\phi) = \lambda\phi^4/4$ and as before an interaction term $g^2\phi^2\chi^2/2$. The quantum fluctuations of the fields ϕ and χ are then given by [17]:

$$(a^{3/2}\delta\phi_k)\ddot{} + \left(\frac{k^2}{a^2} + 3\lambda\phi^2 + 6\pi G p - \frac{3\lambda H^2}{4\pi^3}\xi(t)\right)(a^{3/2}\delta\phi_k) = 0\,, \tag{4.25}$$

$$(a^{3/2}\chi_k)\ddot{} + \left(\frac{k^2}{a^2} + m_\chi^2 + 6\pi G p + g^2\phi^2\right)(a^{3/2}\chi_k) = 0\,. \tag{4.26}$$

where $p = \kappa(\frac{1}{2}\dot{\phi}^2 - V(\phi))$ is the pressure and $\kappa = 8\pi G$. These equations are again equivalent to Eq. (4.4), but this time with stochastic potentials, which are the opposite extreme to periodic potentials since they generically have empty σ_{AC}.

Note one important point. From Eq. (4.24) above we see that the inflaton evolution is driven by a coloured noise driving term which generically ensures that ϕ itself will not be a white noise in stochastic inflation. Therefore the numerical simulations that we will present in the next section - which are for white noise - cannot be taken as truly reflective of reheating in stochastic inflation. The white noise results will, however, be extremely useful in the next chapter where we attempt to study reheating at the strong-coupling limit of realistic multi-field theories.

4.7 Comparison of the Floquet index for the various potentials

Here we present Floquet spectra at various q and g to show the relative power of noise and quasi-periodic potentials over the pure Mathieu potential.

[17] Here we neglect the backreaction of the produced field χ on the expansion; valid during the first phase of reheating before backreaction terminates the resonance.

Below we show cross-sections of the Floquet index at constant q. Physically this gives μ_k versus frequency (or equivalently, momentum) k.

The quasi-periodic spectra are all shown for the case $m_\phi/m_\chi = \pi$. The fact that the magnitude of the ratio is close to one is important, as we described earlier, for the increase in μ. Thus, we should interpret our earlier spectral results, and our numerical results, with caution, since:

(1) the spectral theory results may be valid for all irrational m_ϕ/m_χ, with $\mu > 0$ for almost all k, but μ is so small that it does not show in out numerical simulations for large mass ratios.

(2) The case of large mass ratios corresponds to Cantor sets with large measure (where $\mu = 0$).

(3) The nature of the "genericness" depends on the mass ratio.

Of these, it seems most likely that (1) and (2) are at play. From our studies of the white noise case, μ appears to vanish on non-zero measure sets even though theory states that the contrary is true. Therefore, (1) is almost certainly true (we are also dealing essentially with chaotic dynamical systems for which computer errors are significantly amplified). Issues (2) and (3) are beyond our reach to study unfortunately, but an experiment in the periodic case would be interesting. One may parametrise the band spectra by a continuous variable, say r, such that $r = 0$ implies that the spectrum fills the whole real line, while increasing r uniformly decreases the bands of the spectrum with $r = \infty$ making the spectrum empty. By the inverse scattering technique, one may hope to obtain the corresponding potentials as a function of r, though this might not be unique. In the Mathieu case, one might choose $q \equiv r$.

To summarise the findings in the following graphics, it is evident for all values of q that (i) the Floquet indices are strongly enhanced both by adding a white noise component and by going to a quasi-periodic potential. (ii) Secondly the regions where $\mu = 0$ are strongly diminished for these two cases compared with the purely periodic Mathieu case. Both of these findings are in line with what we expected from our theoretical considerations based on spectral theory in the earlier part of the chapter.

Spectra for $q = 5$

Here we compare the pure periodic, quasi-periodic and white noise Floquet spectra for fixed $q = 5$ and with the ratio of the frequencies in the quasi-periodic case always equal to π.

Spectra for $q = 10$

Here and in the case $q = 20$, the development of edges where μ jumps almost discontinuously is evident (here at $q \simeq 2.$). We have no explanation for these "non-thermal edges" at present but are

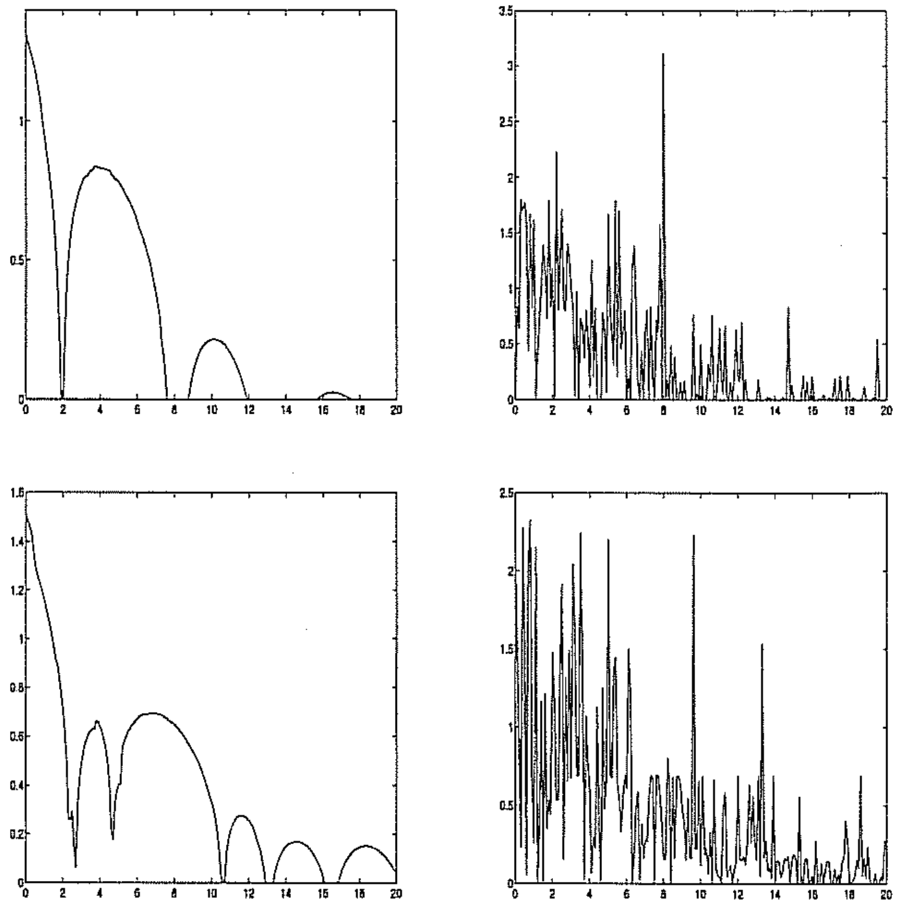


Figure 4.3: $\mathbf{q = 5}$. Top: the Mathieu (left) and stochastic Mathieu ($g = 5$) spectra - μ vs k. The quasi-periodic (left) case and the quasi-periodic case with noise ($g = 5$) are shown as the bottom left and right figures respectively. .

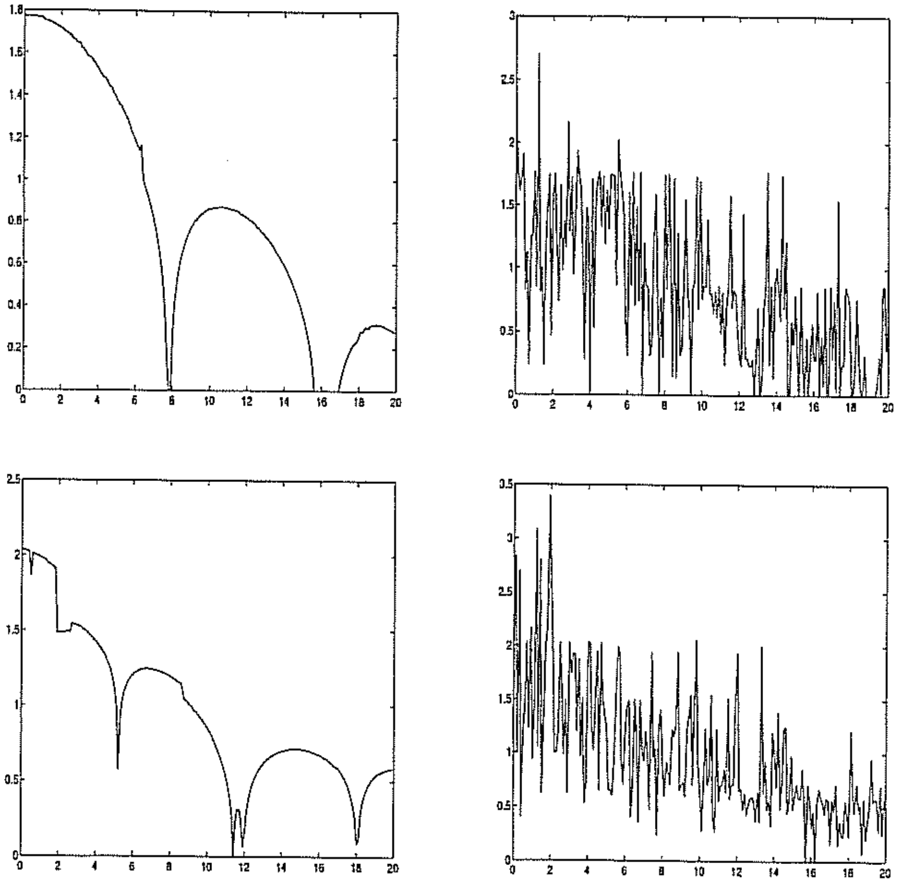


Figure 4.4: $\mathbf{q} = \mathbf{10}$. Top: the Mathieu (left) and stochastic Mathieu ($g = 5$) spectra (right) - μ vs k. The quasi-periodic (left) case and the quasi-periodic case with noise ($g = 5$) are shown as the bottom left and right figures respectively. .

confident that they are not numerical abberations.

Spectra for $q = 20$

Here we present the case for $q = 20$. Again we see the same patterns as before plus a new one: the stochastic case appears to develop a flat "table-top" part.

4.7.1 The incommensurate sub-space at large ratios

We have shown that for a ratio of masses equal to π, the increase in the μ_k is be drastic. What happens when we allow the mass ratio to become much larger or smaller ?

One might expect that the high frequency component would become less important and this is

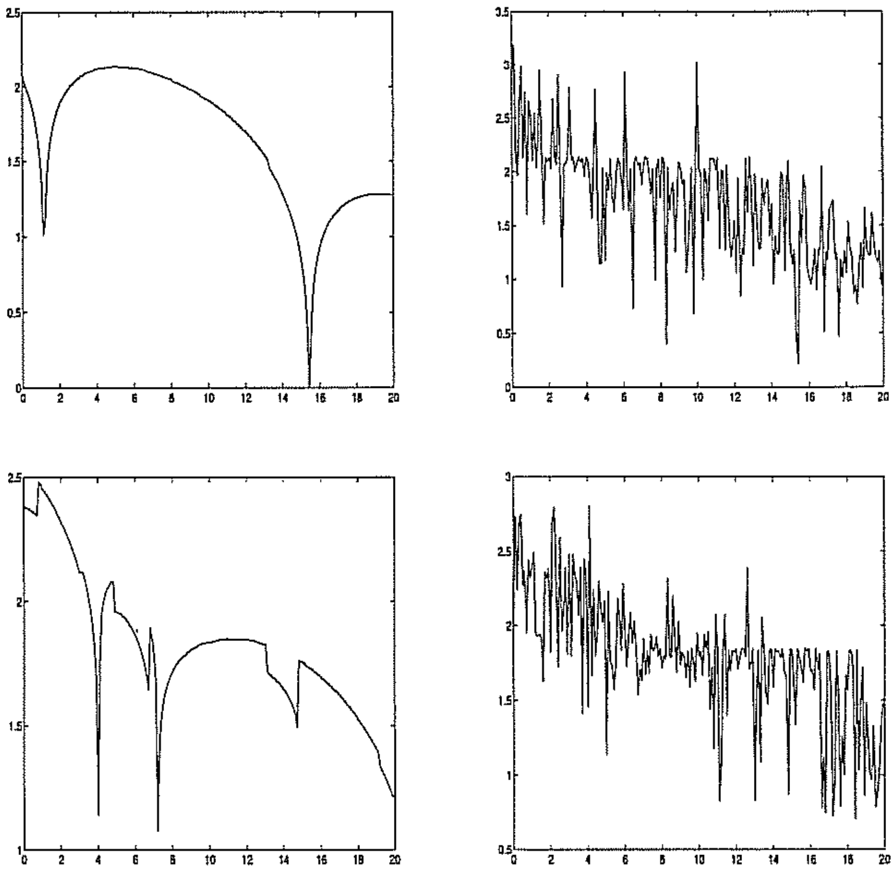


Figure 4.5: $\mathbf{q = 20}$. Top: the Mathieu (left) and stochastic Mathieu ($g = 5$) spectra (right) - μ vs k. The quasi-periodic (left) case and the quasi-periodic case with noise ($g = 5$) are shown as the bottom left and right figures respectively. .

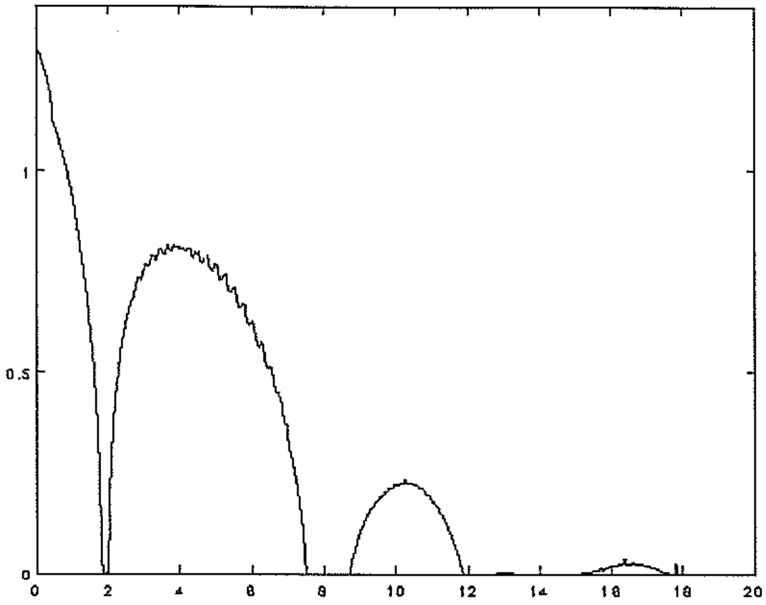


Figure 4.6: The Floquet index vs k in the quasi-periodic case but this time with a ratio of frequencies $= 100\pi$, instead of π. The effect of the extra frequency is a very small perturbation of the single-frequency case. In this case one can do perturbation theory in the dimensionless frequency ratio and the distinction between the rational and irrational nature of the ratio is much less important.

indeed seen in figure (4.6). One way to understand this is that in the case where the ratio m_ϕ/m_φ is very small (it is of order 1/300 in the figure), one can use this dimensionless ratio to reduce the problem to a perturbation of the single-frequency Mathieu equation and the second frequency is expected to make only a small impact at small times, the distinction between the rational and irrational nature of the ratio is then much less important than before.

4.8 Including self-interaction: the Gross-Pitaevski and NLS equations

In chapter (2) we saw that the introduction of self-interaction in the reheated field χ, at least in the Hartree approximation to a rapid shut-off of the resonance. In the case that that we have a potential:

$$V(\phi,\chi) = \frac{m_\phi^2}{2}\phi^2 + \frac{g^2}{2}\phi^2\chi^2 + \frac{\lambda_\chi}{4}\chi^4 \qquad (4.27)$$

the equation for the quantum fluctuations χ_k has the extra term $\lambda\chi^3$ which makes it nonlinear, and the equivalence with the 1-D Schrödinger equation breaks down.

This represents no theoretical difficulty however, since the problem is now equivalent to a different operator: the Gross-Pitaevski and non-linear Schrödinger operator equations. Our only problem is to find spectral results for these equations.

4.9 From Minkowski to FLRW

As in the previous section where we allowed self-interaction, inclusion of the expansion of the universe causes problems with the equivalence procedure that we have successfully employed so far in this chapter. Indeed in full generality this is a serious problem since our equations become a coupled, autonomous[18] *set* of equations including the Hamiltonian constraint (the Friedmann equation).

Nevertheless, in the case where we consider the growth of fluctuations in an expanding background determined by the inflaton evolution. This splits the set of equations and breaks the autonomy of the equation for the fluctuations so that we can hope to reestablish our link with spectral theory. Now we have an equation (again assuming a simple $g^2\phi^2\chi^2/2$ interaction term):

$$\ddot{\chi}_k + 3H\dot{\chi}_k + \left(\frac{k^2}{a^2} + g^2\phi^2(t)\right)\chi_k = 0 \tag{4.28}$$

which can be cast into the form of a general Stürm-Liouville problem which is the natural generalisation of the 1-D Schrödinger problem and is usually written:

$$-(s(x)\psi'(x))' + q(x)\psi(x) = \lambda r(x)\psi(x) \tag{4.29}$$

where the functions $s(x), q(x), r(x)$ are arbitrary and $s(x) = 1 = r(x)$ gives us back the Schrödinger equation.

To include the time-dependence of the scale-factor $a(t)$ one can then simply make the following identifications between (4.28) and (4.29):

$$\begin{aligned} x &\to t \\ \lambda &= k^2 \\ s(x) &= a(t) \\ r(x) &= \frac{1}{a(t)} \\ q(x) &= -ag^2\phi^2(t)\,. \end{aligned} \tag{4.30}$$

Again the usefulness of this equivalence is bounded by how much is known about the spectral theory of such operators for relevant $a(t), \phi(t)$.

4.10 Conclusions

In this chapter we have used spectral theory both to shed light on the deeper origin of many of the results in reheating and to illustrate the connections reheating shares with many parts of condensed matter physics. Further we have found an invariant way, based on the spectra of the associated

[18]'Autonomous' implies that the time variable appears in none of the equations *explicitly*. This is simply a consequence of the covariance of GR and relativistic field theory.

$1-D$ Schrödinger equation, to classify the initial phases of qualitatively different reheating classes. Simultaneously we have been able to enlarge the number of known classes to include potentials of interest in the modeling of such complex phenomena as quantum backreaction on the metric and effective masses of fields.

The equivalence between the unstable modes of the Klein-Gordon equation and the complement of the spectrum of the associated Schrödinger equation seems to be completely new and holds much promise, especially if new developments in condensed matter or the spectral theory of linear operators occur. In the next chapter we shall find that many of the results can be put to new use.

The most powerful method of advance ... is to employ all the resources of pure mathematics in attempts to perfect and generalise the mathematical formalism that forms the existing basis of theoretical physics ... and to try and interpret the new mathematical features in terms of physical entities.

P. A. M. Dirac

Chapter 5

Inflationary Reheating in Realistic Theories

An expert is a man who has made all the mistakes which can be made in a narrow field.
Neils Bohr

Time is a great teacher, but unfortunately it kills all its pupils.
Hector Berlioz

5.1 Introduction

What would preheating look like in realistic models of inflation embedded in a complete and consistent theory of particle physics ? Clearly in answering this question one immediately faces two problems: (i) firstly, we do not have such a theory and (ii) even if we did, practical computational problems might be prohibit us in obtaining precise answers, due to lack of resources or ingenuity. With regard to (i), even if one restricts oneself to Grand Unified Theories (GUT's), the choice is not unique, or rather there is no obviously 'best' candidate, although currently $SO(10)$ leads the pack.

As a result of (i) and (ii) above the situation appears rather bleak. While in the rest of the literature on reheating the standard policy has been to either (a) restrict attention to two-field models or (b) choose a specific model (such as $SU(5)$ or $SO(10)$) but make many simplifying approximations, we have in the past two chapters taken a different route. In chapter (3) we asked about the high-energy fixed points of the renormalisation group equations and found that in many cases $|\xi| \to \infty$ was the appropriate result with all other couplings very small. This lead to geometric reheating and seems to cover many interesting and even realistic models.

In chapter (4) we tried to classify all possible reheating classes. We used spectral methods to set up strong boundaries between the different classes. This allows us to understand the reheating "moduli space", the space of qualitatively different types of "first light" preheating. In that chapter we took little specific input from particle physics and were guided principally by mathematical results on the spectral theory of the $1-D$ Schrödinger equation.

Finally, in this chapter we try a different, but related approach, in an attempt to close some of the gaps left in our understanding of realistic reheating theories. Here we examine what characteristic features one is almost sure to find in the promised complete and consistent next-generation theories of physics, be they GUT, supersymmetric or string based. Other than the running of the coupling constants, perhaps the most fundamental feature we can expect is the existence of multiple fields. Indeed, if we look back at the history of physics in the last century, the number of known fields has monotonically increased with time, and supersymmetry, if correct, implies many more, as yet undiscovered, super-partner fields.

The *leit-motif* for this chapter then, is to understand what reheating in the presence of many fields looks like. What is its shape, its form ? How does it differ from the reheating models presented in the review in chapter (2) ? Nowhere in the literature does there exist another investigation of reheating with more than 2 fields [1]. This is a very relevant question: for example the **126** representation of $SO(10)$ involves 252 real Higgs fields, not to mention the rest of the fields corresponding to the

[1] The hybrid preheating paper [138] examines three fields but treats the third as a test field, and is therefore not consistently a three-field study. The $SO(10)$ study [145] started with four fields but then restricted the field fluctuations to coincide, thereby reducing the problem to the two field case. The work of Levin and Cornish [140] studied several fields ($n = 3, 4$) but treated only the homogeneous $k = 0$ modes, so that reheating was actually impossible since they had only 3 or 4 degrees of freedom (classical mechanics as opposed to field theory). Nevertheless their discovery of chaos formed part of the inspiration for the work in this chapter which is based on the papers [98, 154].

standard model and super-partners. Since it is not realistic to numerically solve for the evolution of all of these fields, yet alone investigate the parameter space associated with all of their couplings and masses (the number of which grow quadratically with the number of fields), it seems that our study is doomed from the start.

But fortunately this is not the case. We are, however, forced to take an alternative approach. Instead of choosing a realistic model and then approximating the dynamics by excluding almost all of the degrees of freedom (which thereby makes the results valid only for a restricted parameter set of a single theory), we try to capture statistical properties of the complete ensemble set of degrees of freedom. Indeed, our approximation is statistical in spirit.

Our guiding observation is that in General Relativity, as in other gauge theories, the introduction of $n > 3$ degrees of freedom typically leads to chaos at the classical level. We will discuss the issues associated with this later but first we add another condition: strong coupling. Without strong couplings between the fields the evolution is not necessarily chaotic. In the case that only one coupling is strong we regain the theory of preheating to a single field, with its simple description in terms of the Mathieu equation. Thus we are interested in discussing preheating when there are two or more strong couplings between fields.

But what happens when the inflaton evolution is not chaotic but there exist couplings (not necessarily weak) to many fields ? i.e. when we find ourselves both out of the chaotic range and the range covered by the simple perturbative single-body decay theory presented in section (2.3). In this case, what we find is that the mass spectrum - how the masses of fields are arranged hierarchically - of the fields which couple of the inflaton, is crucial. This issue, and how it affects preheating, will be discussed in section (5.6).

5.2 The strong-coupling limit of GUT reheating

In this section our aim is to justify why a strongly coupled phase is interesting both from the new reheating effects that may arise and from a more general phenomenological/theoretical point of view. A selection of unrelated scenarios is presented where strong coupling is required or implicit, but first we make a cautionary note that strong coupling during inflation is not compatible with the COBE data in non-supersymmetric theories since large couplings will cause large radiative corrections to the potential, spoiling its flatness and over-producing temperature anisotropies. Nevertheless, this is also true of the ordinary preheating reviewed in section (2.4) and so is not specific to the things we will discuss now. Indeed, one could imagine a model where the couplings run and become large only near the end of inflation, i.e. exactly at reheating. A perfect example of this is *Dual* inflation, as we will now discuss.

5.2.1 Seiberg-Witten models and dual inflation

The work of Seiberg & Witten (1994) [240] was perhaps one of the most influential events of the last 10 years in theoretical physics. It proved quark confinement in $N = 2$ supersymmetric Yang-Mills theory via monopole confinement by exploiting a strong-weak coupling duality under which electric monopoles were mapped into magnetic monopoles and excited a large part of the physics community to search for duality aspects in other areas. Indeed, that work was the inspiration for the study we will present in chapter (7).

It was then realised [137] that the associated low-energy $N = 0$ (i.e. broken supersymmetry) scalar potential in the moduli space was very flat along one direction without the need for any fine-tuning, even with the inclusion of all non-perturbative quantum corrections. Inflation then proceeds along this flat direction which corresponds to the weakly coupled (or Higgs) phase of the theory. There is then a transition to a highly curved part of the potential which corresponds to the strongly-coupled, confining phase where the monopole gains a non-zero vacuum expectation value and the condensate (playing the rôle of the inflaton) oscillates. The crucial thing for us is that since oscillations occur in the strong-coupling region, and there are many fields to which the inflaton couples, we have found a specific implementation of our earlier general discussion.

To amplify on this point, the single-field Mathieu parameter $q \sim g^2\Phi^2/m^2$ in this case has $g \geq 1$ (strong coupling), $\Phi \sim \Lambda$ where Λ is the only free parameter of the Seiberg-Witten solution and $m \sim f_0$, where f_0 is the supersymmetry breaking scale of the theory. Natural values both from the CMB and abstract concerns favour $f_0 \sim 10^{-3}\Lambda$. This gives $q \geq 10^6$, deep in the broad resonance regime. Note however that ϕ will couple strongly to all members of the supermultiplet and so we have multiple fields at strong coupling.

5.2.2 Symmetry non-restoration

Another example requiring strong-coupling is of a different type. It is provided by the solution of the monopole problem discussed in section (3.6.1) based on symmetry non-restoration at high temperature [63], due to the effective mass of the Higgs fields remaining negative at all temperatures. The effective mass receives finite temperature corrections dependent on the gauge coupling g^2, and dimension of the representation; see section (1.6). For example, in the case of $SU(5)$ in the five-dimensional representation, symmetry non-restoration imposes a constraint on the quartic self-interaction coupling, λ_Φ, of $\lambda_\Phi \geq 39g^2/10$, which implies that $\lambda_\Phi \geq 1$ for typical gauge couplings $g^2/(4\pi) \sim 1/50$ [64]. Clearly this mechanism was seen as an alternative to inflation and is therefore not directly related to reheating, but it serves as an example of a completely separate reason for studying the strong-coupling limit.

5.3 Chaotic reheating

Setting aside the subtle issue of the quantum behaviour of gauge theories at strong coupling [139], one finds that in the two and three scalar-field cases studied so far, (classical) chaotic motion is typical [140], especially at reheating. This chaotic evolution parallels results in the Einstein-Yang-Mills equations [141], semi-classical QCD and lattice gauge theory [142]. Thus a natural question is "what is the nature of chaotic reheating ?"

The correct setting for this question is therefore to consider n fields [2] Ψ which are in homogeneous condensate states, coupled to the inflaton ϕ via the $n(n-1)/2$ couplings λ_Ψ and all evolving chaotically in time. We then couple a scalar field, χ, to the inflaton via our canonical interaction term $\frac{\tilde{g}^2}{2}\phi^2\chi^2$. Equally we may think of $\chi_k = \delta\Psi_k$ as the non-zero momentum modes ($k \neq 0$) of the fields Ψ with self-interaction.

The modes of χ then obey: [11]:

$$\frac{d^2(a^{3/2}\chi_k)}{dt^2} + \left(\frac{k^2}{a^2} + m_\chi^2 + 6\pi G p + \tilde{g}^2\phi^2\right)(a^{3/2}\chi_k) = 0 \tag{5.1}$$

where $a(t)$ is the scale factor of the universe obeying the Hamiltonian (Friedmann) constraint $H^2 \equiv (\dot{a}/a)^2 = \mu_{tot}/3$ with μ_{tot} the total energy density and p the total pressure of the universe.

Now increase the couplings λ_Ψ of the fields Ψ, thus moving into the strongly coupled, chaotic region of the parameter space. While we are unable to study chaotic reheating in full generality, we have full control over the region in which the chaotic fluctuations are extremely rapid, due to the following powerful statistical result.

5.3.1 Taylor's theorem and the stochastic limit

In the strong coupling limit it is fair to say that many systems show an increase in chaoticity, namely an increase in Lyapunov exponents (equivalent to our Floquet indices) and a decrease in correlation times. In these cases we can apply Taylor's theorem to replace the chaotic flow by a white noise random process.

More precisely, let ϕ obey Smale's Axiom A type dynamics with chaotic flow Φ_t, and non-periodic attractor Σ. This ensures that the flow is sufficiently chaotic. For an open, dense set of functions f with $E(f) = 0$, $\exists\sigma^2, \epsilon > 0$ and a Brownian motion $B(t)$ such that for almost all initial $\phi_0 \in \Sigma$ [146]:

$$|\int_0^t \sqrt{\lambda} f(\Phi_{\lambda\tau}\phi_0)d\tau - \frac{\sigma}{\sqrt{\lambda}}B(\lambda t, \phi_0)| = \mathcal{O}\left(\frac{1}{\lambda^\epsilon}\right). \tag{5.2}$$

here $E(f)$ denoting the expectation value of f.

[2] Here we represent the n fields as a vector Ψ suppressing associated indices. Similarly we write the couplings as the vector λ_Ψ.

Since white noise is the derivative of Brownian motion, this implies that chaotic motion is, in the rapid fluctuation limit $\lambda \to \infty$, indistinguishable from white noise.

The beauty of this result is that while we have had to restrict ourselves to the rapid fluctuation limit, which will only be an approximation to true chaotic reheating, we have reduced our problem to one we have studied in the previous chapter: namely one within the stochastic reheating class.

5.4 Reheating using white noise

In the limit where Taylor's theorem is applicable, Eq. (5.1) reduces to quasi-Mathieu form:

$$\frac{d^2(a^{3/2}\chi_k)}{dt^2} + \left(\frac{k^2}{a^2} + F(\phi,t)\right)(a^{3/2}\chi_k) = 0 \tag{5.3}$$

where the potential $F(\phi,t)$ is given by:

$$F(\phi,t) = g^2\xi(t)\,. \tag{5.4}$$

which corresponds to λ_Ψ-independent stochastic evolution - $\xi(t)$ is a Gaussian white noise - describing the effect of multiple fields in the strong coupling limit. Actually there is a slight subtlety here. We have "neglected" the total pressure (all fields are assumed to be minimally coupled):

$$p = \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}\textstyle\sum_i^n \dot{\Psi}_i{}^2 - V(\phi,\Psi) \tag{5.5}$$

which involves time-derivatives of the fields ϕ, $\boldsymbol{\Psi}$. Time-derivatives of a white noise process are not themselves white noise, and so one might feel that Eq. (5.4) is much too simplistic. However, this is the wrong way to go. For each set of couplings $\lambda_\Psi > \lambda_{chaos}$ we have a chaotic flow Φ_t and potential $F(\Phi_t)_{\lambda_\Psi}$. Now if a function ϕ is chaotic, then the time-derivative and powers of ϕ are also generally chaotic (although this may not be true in all possible cases). In that case we can take the rapid fluctuation limit for all components of p and hence, by Taylor's theorem - which can be extended to vector flows [146] - find that p tends to a white noise process and that in the limit $F(\Phi_t)_{\lambda_\Psi} = F(\phi,t)$ as given by Eq. (5.4), with the appropriate coupling g^2 including the various contributions from the pressure [3] and inflaton.

As an aside we can generalise Eq. (5.4) slightly to include a periodic component:

$$F = -2q\cos(2t) + g^2\xi(t)\,. \tag{5.6}$$

$q = 0$ corresponds to the strong-coupling limit above and $g = 0$ gives us back the simple Mathieu equation and leads us to the standard theory of preheating. This generalised potential is therefore a realistic way of examining the effect of small stochastic fluctuations in the effective mass of the field, as was done, for example, in [147]. In our graphics we will present results for both parameters (q, g). This will enable us to examine what happens to physical quantities as we vary both q and g.

[3] Which is, after-all, a weak effect - see chapter (3).

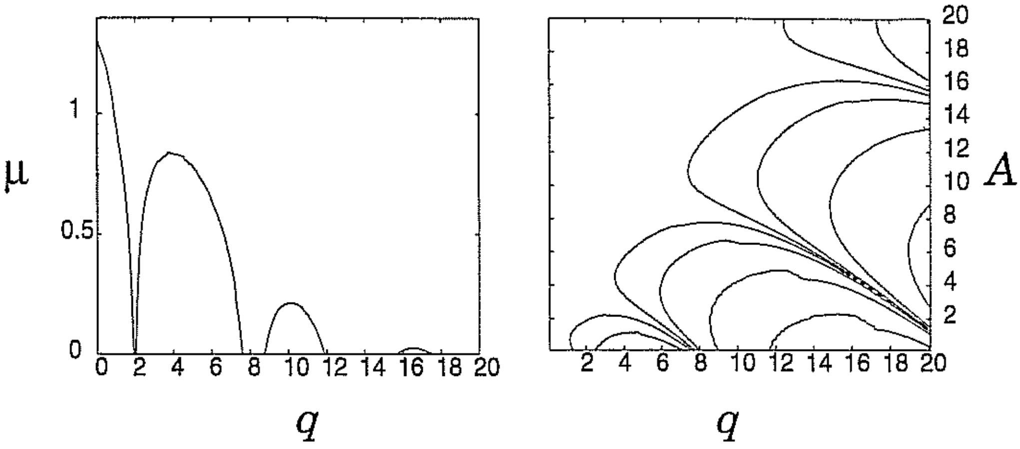


Figure 5.1: Mathieu eq. (a) Floquet index μ vs q along $A = 2q$. (b) Contour plot of μ on the instability chart (A, q).

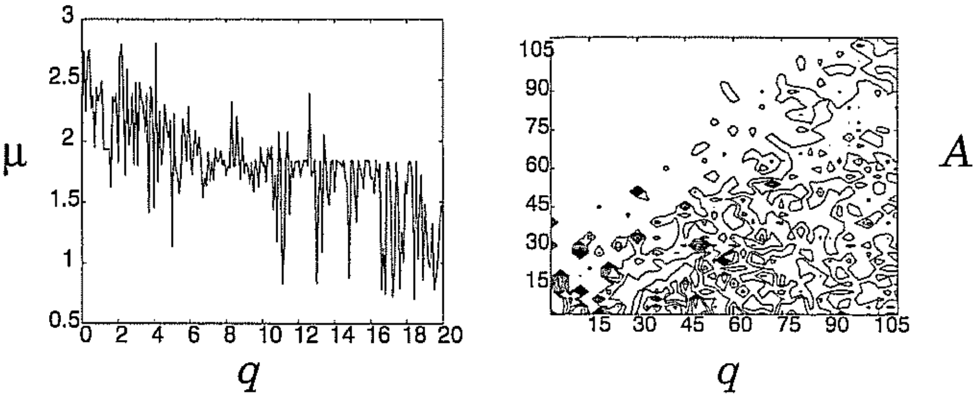


Figure 5.2: Stochastic eq. (2), $g = 5$: (a) Floquet index μ vs q along $A = 2q$. (b) Contour plot of instability chart - note the breakup of Mathieu bands and the very large peaks in μ.

5.4.1 The Floquet index

In chapter (4) we presented very generic theorems about the existence of instability in the modes of the Klein-Gordon equation (5.1) for the case of a stochastic potential. In particular, theorem (2) gaurranteed that with measure one, all modes grow exponentially, i.e. $\mu_k > 0$. But we had no real analytical control over the magnitude of the μ_k relative to the Mathieu case.

In figures (5.1), (5.2) we present a side-by-side comparison of the instability charts for the pure Mathieu and mixed stochastic-periodic (see Eq. 5.6) cases along the physical separatrix $A = 2q$. The band structure of the Mathieu equation is clearly visible while the complete destruction of these bands is visible in figure (5.2). The other effect clearly evident is the significant growth at all wavelengths in the Floquet exponent over the purely periodic case ($g = 0$).

5.4.2 Backreaction in the Hartree-Fock approximation

A crucial result of the large Floquet indices μ_k is the possibility of non-thermal symmetry restoration (NTSR) [131], as we described in section (2.12). Consider the simplest 2-field effective potential with symmetry breaking:

$$V(\phi,\chi) = \frac{\lambda}{4}(\phi^2 - \phi_0^2)^2 + \frac{\tilde{g}^2}{2}\phi^2\chi^2 \; . \tag{5.7}$$

This gains the following quantum corrections:

$$\Delta V = \tfrac{3\lambda}{2}\langle(\delta\phi)^2\rangle\phi^2 + \tfrac{\tilde{g}^2}{2}\langle(\delta\chi)^2\rangle\phi^2 + \tfrac{\tilde{g}^2}{2}\langle(\delta\phi)^2\rangle\chi^2 \tag{5.8}$$

[131], where the dominant variance is given in the Hartree approximation by [9]

$$\begin{aligned} \langle(\delta\phi)^2\rangle &= (2\pi^2a^3)^{-1}\int dkk^2|a^{3/2}\delta\phi_k|^2 \\ &\sim \int dkk^2e^{4\mu m_\phi t} \end{aligned} \tag{5.9}$$

where the last step comes from assuming effective rescattering.

From energy conservation, one has the bound [131]:

$$\langle(\delta\phi)^2\rangle \le C\tilde{g}^{-1}\lambda M_{pl}\ln^{-2}\tilde{g}^{-2} \tag{5.10}$$

with $C \sim 10^{-2}$. Saturating this bound implies that Eq. (5.7) can have a positive effective mass, $m^2_{\phi,eff} \equiv (V + \Delta V)''$ even for $\phi_0 \sim 10^{16}GeV$.

Nevertheless, exhausting the available energy and realising these large variances in the simple Mathieu equations is highly non-trivial [134]:

(i) sufficiently large μ are required since $\langle(\delta\phi)^2\rangle \sim \int dkk^2e^{2\mu t}$.

(ii) The breadth (and very existence of) the first instability band controls how long the resonance continues before backreaction shuts it off (see figure 5.1).

(iii) The expansion of the universe is forced to be monotonically decreasing: $\dot{H} = -\kappa\dot{\phi}^2/2$, which drives $(A,q) \to (0,0)$, damping the resonance and reducing the variances [155].

(iv) Finally, for defect production we require $m^2_{\phi,eff} > 0$ for time scales $\delta t > \omega_\phi^{-1}$, the period of ϕ oscillations.

These factors conspire against NTSR and defect production in the simplest models of preheating [134]. However, in the strong-coupling limit of GUT's considered here we know that:

(i) we can easily achieve very large μ (see figure 5.2).

(ii) since the stability bands are completely destroyed [147, 148], modes never stop growing, $\mu > 0$, even when backreaction becomes important.

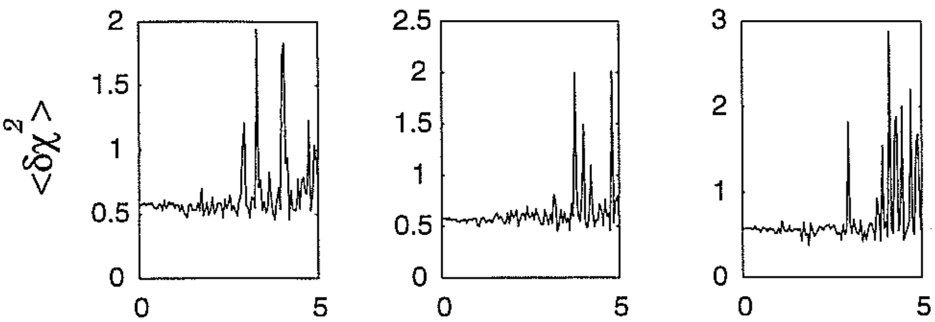


Figure 5.3: Three realisations of $\langle(\delta\chi)^2\rangle$ vs g at $t = 2m_\phi^{-1}$ for $q = 5$. At larger times, $\langle(\delta\chi)^2\rangle$ becomes completely dominated by a single peak (c.f. fig. 2(b)).

(iii) Finally, and very importantly, when there are multiple fields, the expansion need not be monotonic and indeed, in our case will increase and decrease stochastically [140] until reheating and thermalisation are completed. This means that the expansion need not only have a damping effect, but can also act as a pump, when $\dot{H} > 0$ [140], enhancing the fluctuations $\langle(\delta\phi)^2\rangle$ and $\langle(\delta\chi)^2\rangle$.

From these robust considerations, we expect NTSR to be much more effective at strong-coupling, allowing us to approach the limiting variances set by energy conservation. Indeed this is borne out by direct simulation of $m^2_{\phi,eff} \propto \lambda\langle(\delta\phi)^2\rangle + \tilde{g}^2\langle(\delta\chi)^2\rangle$ (see fig. 5.3).

Examining the statistics of $m^2_{\phi,eff}$ [98] we see that while $m^2_{\phi,eff}$ fluctuates very rapidly, very large values of $\langle(\delta\chi)^2\rangle$ occur, with $m^2_{\phi,eff} \in [-\lambda\phi_0^2, g^2M^2_{pl})$. Assuming $\lambda{\phi_0}^2 \ll g^2M^2_{pl}$, the statistics of $m^2_{\phi,eff}$ are highly skewed, yielding a temporal average:

$$\overline{m^2_{\phi,eff}} > 0\,. \tag{5.11}$$

. This means that most of the time symmetry is restored and ϕ is likely to diffuse across the origin, leading to defect production including monopoles in a full model including $SU(3)_c \times SU(2)_L \times U(1)_Y$.

5.5 Backreaction and rescattering – beyond Hartree-Fock

Since the produced spectra in the white noise case are very different from the simple band spectra of the Mathieu equation, we have a much smoother distribution function $f(p)$ of produced particles instead of the rather sharp *k – space* shells obtained from the Mathieu equation. Clearly this difference is likely to have a strong influence on both backreaction and rescattering.

Rescattering typically leads to a strong enhancement of the field variances in simple periodic models over the estimates provided by the Hartree-Fock approximation which includes no scattering effects and energy is transfered only via the mean field. Is the same true in the cases we have studied in this chapter ?

Here we consider the simplest model exhibiting rescattering:

$$V(\phi,\chi) = \frac{m_\phi^2}{2}\phi^2 + \frac{g^2}{2}\phi^2\chi^2 \tag{5.12}$$

although we will also make comments regarding the case with ϕ self-interaction. Fluctuations of bose fields generated from vacuum by an external field in the large occupation number limit can be considered as classical waves with Gaussian statistics [9]. Therefore in the first approximation all fields χ, $\delta\phi$ can be treated as interacting classical waves. We can then use the equations (2.57-2.59), which we give again for completeness.

The Fourier decomposition of the Klein-Gordon equations of the interacting fields can be reduced to mode equations. The mode equation for $X_k = a^{3/2}\chi_k$ is

$$\begin{aligned}\ddot{X}_k + \left(\frac{k^2}{a^2} + g^2\phi^2(t)\right) X_k &= -\frac{g^2\phi(t)}{(2\pi)^3 a^{3/2}} \int d^3k' X_{\mathbf{k}-\mathbf{k}'}\varphi_{\mathbf{k}'} \\ &- \frac{g^2}{(2\pi a)^3}\int d^3k' d^3k'' X_{\mathbf{k}-\mathbf{k}'+\mathbf{k}''}\varphi_{\mathbf{k}'}\varphi_{\mathbf{k}''}, \end{aligned} \tag{5.13}$$

where we label the first and second integrals I_X and II_X respectively.

The mode equation for $\varphi_k \equiv \delta\phi_k(t)a^{3/2}$ is

$$\begin{aligned}\ddot{\varphi}_k + \left(\frac{k^2}{a^2} + m^2\right)\varphi_k = -\frac{g^2\phi(t)}{(2\pi)^3 a^{3/2}} \int d^3k' X_{\mathbf{k}-\mathbf{k}'} X_{\mathbf{k}'} \\ -\frac{g^2}{(2\pi a)^3}\int d^3k' d^3k''\ \varphi_{\mathbf{k}-\mathbf{k}'+\mathbf{k}''}\ X_{\mathbf{k}'}\ X_{\mathbf{k}''}. \end{aligned} \tag{5.14}$$

The first term on the right of this equation, I_φ, describes rescattering of χ-particles on the classical field $\phi(t)$, which leads to ϕ-particle production. The second term, II_φ describes scattering of higher-momentum ϕ-particles and χ-particles. Corrections to the effective mass of the modes ϕ_k appear as a result of the iterative solution of the system of equations which we now present.

The equation for the oscillating background field $\phi(t)$ takes the following form:

$$\begin{aligned}\ddot{\phi} + 3H\dot{\phi} + m^2\phi = -\frac{g^2\phi}{(2\pi)^3 a^3}\int d^3k' X_{\mathbf{k}'}^2 \\ -\frac{g^2}{(2\pi)^3 a^{9/2}}\int d^3k' d^3k''\ \varphi_{\mathbf{k}''-\mathbf{k}'}\ X_{\mathbf{k}'}\ X_{\mathbf{k}''}. \end{aligned} \tag{5.15}$$

The first term, I_ϕ on the r.h.s. of this equation is proportional to the polarization operator Π_ϕ^1. The second term, II_ϕ describes rescattering, which is related to the imaginary part of the polarization operator Π_ϕ^2.

5.5.1 Rescattering in a simple δ-approximation

To obtain a simple analytical understanding of rescattering and backreaction, we will limit ourselves to Minkowski spacetime (modes are then not redshifted). This means that broadening of the produced particle spectrum is solely due to rescattering. Further, since particle production due to periodic parametric resonance is peaked around the center of the first resonance band, we will make the approximation that the produced spectrum for X_k is a δ-function at the center of the band, $k = k_*$, with a time dependent coefficient which describes the development of the spectrum due to the resonance, an effect determined by the Floquet index μ_{k_*}.

Thus we write:

$$X_k = X^0(t)\delta(k - k_*)\,, \tag{5.16}$$

where $X^0(t) \propto e^{\mu_{k_*} t}$. One might argue that the higher order bands should be included in the approximation, but as we shall see, this is not necessary to achieve transfer of energy to higher momentum modes and simply increases the complexity of the system whereas our main aim is pedagogical clarity. In this approximation, integrals over the spectrum of X_k collapse immediately due to the δ-function in Eq. (5.16).

Neglecting four-particle scatterings

To being with, we will make another approximation: *we will neglect all the second integrals labeled* $II_{X,\varphi,\phi}$ in the full equations. This is justified when the number density of the condensate is much larger than the total occupation number at non-zero momentum, which is valid in many cases (see e.g. [181]), including the initial phase in which rescattering becomes important.

The Eq. (5.13) for X_k now becomes, :

$$\ddot{X}_k + \left(\frac{k^2}{a^2} + g^2\phi^2(t)\right) X_k = -\frac{g^2\phi(t)X^0(t)}{(2\pi)^3 a^{3/2}}(k - k_*)^2\varphi_{k-k_*}\,, \tag{5.17}$$

and we see that I_X describes scattering of $\delta\phi$ particles at frequency $|k - k_*|$ with X_k particles. Initially in the potential given by eq. (5.12), φ_k is in a vacuum state with number density near zero consistent with the uncertainty principle (in the expanding case there would also be weak production due to the expansion of the universe). In the case of ϕ self-interaction, this is not true since there is also the usual φ resonance - see section (2.5) [10].

So, initially, I_X has no effect on the evolution of X_k. Similarly, we can see that II_X will have no effect since it is proportional to $\varphi_{k'}\varphi_{k''}$.

Changing our focus to equation (5.14) we find under our approximations that:

$$\ddot{\varphi}_k + \left(\frac{k^2}{a^2} + m^2\right)\varphi_k = -\frac{g^2\phi(t)X^0(t)^2}{(2\pi)^3 a^{3/2}}k_*^2\delta(k - 2k_*) \tag{5.18}$$

This equation is very interesting since we see that there is φ_k particle production only at the frequency $k = 2k_*$. This is our first rescattering effect. The production of a φ_k boson at this frequency is simply an issue of energy and momentum conservation in the process:

$$X_{k_*} + X_{k_*} \rightarrow \phi + \varphi_{2k_*} \tag{5.19}$$

This process implies that the condensate number density increases (bose condensation) while the resonant production of X_k particles is counteracted.

We can gain an accurate understanding of the early development of the rescattering effect on φ_k via a Green's function solution of eq. (5.18) neglecting the backreaction of the φ_k on the eq. for X_k.

What we obtain is the equation for the forced oscillations of $\delta\phi_k(t)$. The homogeneous part of this inhomogeneous linear differential equation has a simple Green function $\propto \sin\Omega_{\mathbf{k}}(t-t')$, where $\Omega_{\mathbf{k}}^2 = \mathbf{k}^2 + m_\phi^2$. Then the solution of Eq. (5.14) with only the first integral term is

$$\begin{aligned}\delta\phi_k(t) &= -\frac{g^2}{(2\pi)^3\Omega_{\mathbf{k}}}\int_0^t dt' \sin\Omega_{\mathbf{k}}(t-t')\phi(t') \\ &\times \int dk' k'^2 \chi_{\mathbf{k}-\mathbf{k}'}(t')\chi_{\mathbf{k}'}(t') + h.c. \,.\end{aligned} \tag{5.20}$$

It is immediately evident that φ_k grows very rapidly in time, with an envelope approximately given by $e^{(\mu_{k-k'}+\mu_{k'})m_\phi t} \approx e^{2\mu_k m_\phi t}$. This parasitic behaviour of the φ_k fluctuations was already discussed in section (2.9). Note that once the φ_k occupation numbers $n_\varphi \sim e^{4\mu_k m_\phi t}$ grow sizable, the backreaction on the X_k growth cannot be neglected and the X_k-resonance is shut-off. However, even with $X_k = const$, we see from eq. (5.20) above that the φ_k grow linearly with time, showing that rescattering is effective even then.

Of course, the true situation (at the classical level) is one of coupled oscillations with energy transfered non-resonantly between the X_k, φ_k and ϕ.

So far we have shown that production of χ_k particles at $k = k_*$ leads to production of $\delta\phi_k$ quanta at $k = 2k_*$. Is this where the story stops (i.e. these frequencies form a closed system) or is there more redistribution of energy ? Returning to the I_X term in eq. (5.17), if we now include the φ_{2k_*} quanta we see that it is the X_k mode with $k = 3k_*$ which gains a driving term. So the system is not closed at two frequencies. Nor is it closed at any finite number of frequencies: one can easily show that if the spectrum of X_k and φ_k contain the terms $\delta(k - nk_*)$ and $\delta(k - mk_*)$ respectively, then the term I_X will generate X_k quanta at the frequency $(n+m)k_*$.

Thus by induction we have shown that starting from our initial approximation in eq. (5.16), power is cascaded to larger momenta. The distribution of produced quanta is the module $k_*\mathbf{Z}$, i.e. only integer multiples of the original resonance frequency are produced by the three-particle scatterings. We shall see that this is not changed by including the four-particle scatterings but that it does changes drastically in the case of stochastic reheating.

Including four-particle scatterings

Now consider the term II_ϕ. If we allow X_k to have non-zero values at any $k = nk_*$, then II_ϕ will consist of a double sum for all values of n. However, they have a closed structure. There is no way to produce quanta at non-integer multiples of k_*.

The net result of this is that the produced spectrum is peaked around the integer values of the initial resonance frequency k_*. Clearly both the expansion of the universe and the finite k-space width of the resonance shells will cause these peaks to be smeared. In some sense they are complementary effects: at small q, the resonance bands are very narrow but expansion is effective in redshifting modes through the bands. In the broad resonance case, the particle production is too rapid for the expansion to have much effect but then the resonance bands are much broader. Nevertheless, the basic structure of the rescattering effects are those presented above.

5.5.2 Rescattering in the stochastic case

The crucial effect of noise is to destroy the stable bands. This means that all modes, bar a set of measure zero, experience resonant growth (at least classically)[4]. The Floquet exponent is now larger and has a completely different distribution in k-space. Of course, there is still a trend for μ to decrease with increasing k. To model the spectrum we take X_k to have a flat spectrum out to $k = \overline{k}$ and to be zero at larger momentum:

$$\begin{aligned} X_k(t) &= X^0(t) \quad \text{for} \quad k \leq \overline{k} \\ &= 0 \qquad\quad k > \overline{k} \end{aligned} \tag{5.21}$$

The difference this has is that all momenta less than the cutoff can be produced and hence the spectrum is much closer to thermal than before. In particular, any frequency k can be amplified or de-amplified, so that noise is a necessary requirement for full thermalisation.

Neglecting four-particle scatterings

Consider the I_X term for our new approximation eq. (5.21):

$$I_X \propto X^0(t) \int_{-|k-\overline{k}|}^{|k+\overline{k}|} dk' dk'^2 \varphi_{k'} \tag{5.22}$$

Again initially $\varphi_k = 0$, so this term is inoperative. The I_φ term on the other hand is:

$$\begin{aligned} I_\varphi &\propto \int_{-|k-\overline{k}|}^{|k+\overline{k}|} dk' dk'^2 X_{k-k'} X_{k'} \\ &\propto \frac{X^0(t)^2}{3} k^3 \Big|_{-|k-\overline{k}|}^{|k+\overline{k}|} \quad \text{with} \quad k \leq 2\overline{k} \end{aligned} \tag{5.23}$$

so that we see that *all* modes with $k \leq 2\overline{k}$ are amplified, in strong contrast to the situation described for the simple preheating model in the previous section which could only produce integer multiple of k_*, the resonant frequency.

[4] Quantum effect may cause local changes to the growth rate but not to the general picture [9].

The Hartree-Fock term I_ϕ is given by:

$$\begin{aligned} I_\phi &\propto \int_{-|\overline{k}|}^{|\overline{k}|} dk' dk'^2 X_{k'}^2, \\ &\propto \frac{2X^0(t)^2}{3}\overline{k}^3 \end{aligned} \tag{5.24}$$

This is much larger than the corresponding term in the δ-function approximation for preheating. This is because of the distributed nature of the spectrum and the increased μ_k. The effective mass will thus change more rapidly and strongly, causing q and g to decrease very rapidly, as well as contributing $\dot{m}_\phi$ terms to the effective mass of the χ field. All of these effects will help to shut-off the resonance. However, since μ_k is not zero at any k, q or $g \neq 0$, the modes will continue to grow, though more slowly, and are never pushed into a region where $\mu_k = 0$, as happens in the pure-Mathieu case.

By returning to I_X above, we see that $k \leq 3\overline{k}$ modes can now be produced and in analogy with the previous section one can show easily by induction that a cascade happens to ever larger $k \in \mathbf{R}$.

The inclusion of four-particle scatterings again seems to have no qualitative impact on rescattering, although it obviously affects the transition amplitudes for bose condensation and evaporation and interchange between $k \neq 0$ particles.

5.5.3 Realistic modeling of the stochastic spectrum

What spurious effects does our approximation (5.21) bring in ? From figure (5.2) we see two effects: (1) there is a decrease in μ_k with k and (2) μ_k is a random function of time and k, with fluctuations about a mean $\overline{\mu}_k$.

A more realistic model to address issue (1) would be to give X_k a monotonically decreasing behaviour with k, whether it be linear, quadratic etc... However, the first term in such a polynomial expansion will always be the constant term that we have just studied. Thus our calculations act as a limit of strong coupling where we can neglect the decay of μ_k. Indeed, the two approximations we have presented represent the two limiting cases for preheating.

Issue (2) is more subtle and is expected to be much more important for studies of the actual time development of the spectra of X_k and φ_k, something which is beyond the scope of our present study and is left as a challenge for the future.

5.5.4 Conjecture regarding backreaction in the stochastic case

The destruction of the stability bands and the knowledge that $\mu_k > 0$ for all k in the white-noise case leads naturally to the question "is there a backreaction mechanism that stops the stochastic resonance or does it simply continue until energy conservation forces a halt ?"

In the pure Mathieu case backreaction causes corrections to the effective mass and hence to the parameters (A, q), causing a flow on the instability chart. Once the $k = 0$ mode is forced out of the lowest resonance band because of these corrections the whole spectrum lies in stability regions or secondary, and hence very weak, resonance bands. The resonance is therefore ended effectively. But because of the destruction of the stability bands in the stochastic case it appears that this exit is missing.

Of course, backreaction will cause the coupling g^2 to the white noise to be reduced but that is not expected to be very effective in shutting off the resonance. Instead here we *conjecture* that even if Taylor's theorem assured us that the inflaton evolution was (nearly) white noise at the start of reheating, finite coupling effects will cause the colour of the inflaton noise to evolve in time.

White noise is colourless, that is it has a flat spectrum in frequency space much like the Harrison-Zel'dovich spectrum in cosmology. This means that the two-time correlation function is a δ-function - white noise has no memory. Generalised stochastic processes have colour and memory and the coloured noises may be described as α-indexed random Brownian processes where $\alpha \in [0, 1]$ and $\alpha = \frac{1}{2}$ corresponds to white noise.

Memory effects appear to reduce the Floquet index quite significantly [154] and it is quite natural to expect that backreaction will cause α to diverge away from $\frac{1}{2}$ in the finite coupling realm (finite λ in Eq. 5.2) and hence lead to a strong damping of μ_k.

5.6 Sensitivity to mass spectrum deformations

We now study the effect of mass spectrum deformations on reheating [98]. A simplified toy model to study this issue is given by the 3-field effective potential:

$$V(\phi, \varphi, \chi) = \frac{m_\phi^2}{2}\phi^2 + \frac{m_\varphi^2}{2}\varphi^2 + \frac{\bar{g}^2}{2}\phi^2\chi^2 + \frac{\lambda_1^2}{2}\varphi^2\chi^2 \,. \tag{5.25}$$

Eq. (5.1) is modified and leads to a new F [148]:

$$F_{qp} = -q(\cos 2t + \cos 2\pi t) \,. \tag{5.26}$$

where we have chosen $m_\phi/m_\varphi = \pi$, which implies that (5.26) is a quasi-periodic function [148, 98] since the masses are irrationally related. Such issues are known to be crucial in the integrability of conformal field theories [161] and in discussions of periodic orbits *vs* ergodic flows in dynamical systems.

In deriving (5.26) we have used the fact that $\phi = \Phi \sin(m_\phi t)$, $\varphi = \Upsilon \sin(m_\varphi t)$ and we neglect the small contribution from p, which also oscillates. Further we choose $\bar{g}^2\Phi^2 = \lambda_1^2\Upsilon^2$. Relaxing this condition adds an extra parameter to eq. (5.26) but it remains quasi-periodic, the crucial condition.

In our case, spectral theory results guarantee that the stable bands generically form a Cantor set

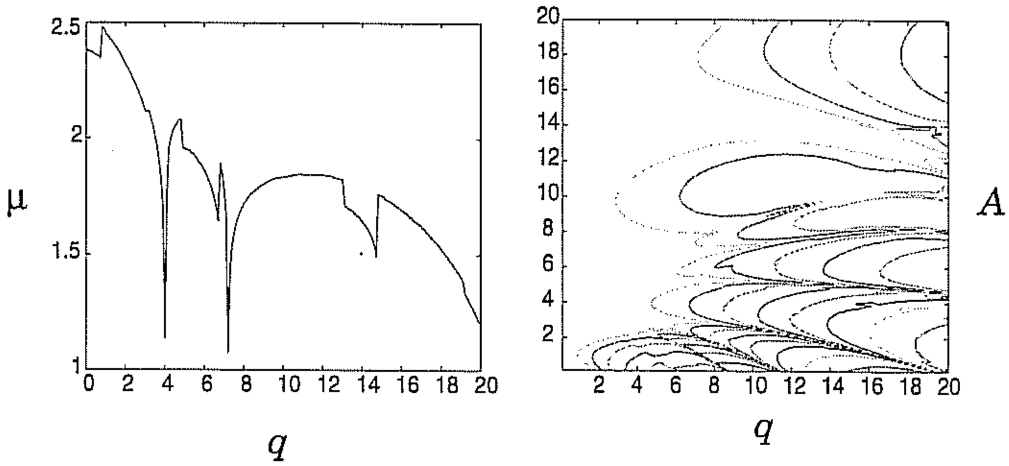


Figure 5.4: Quasi-periodic case, eq: (4) (a) μ vs q on $A = 2q$. (b) Instability chart (A, q). Note the larger μ and proliferation of instability bands compared with figs. (1 a,b).

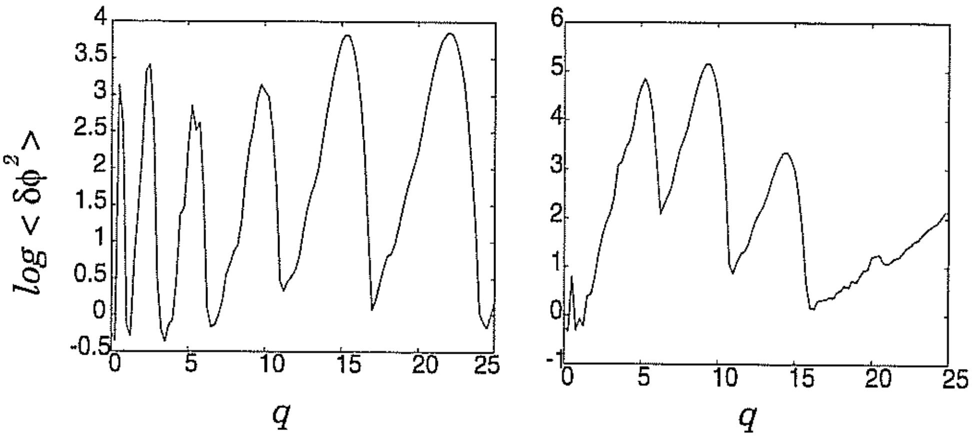


Figure 5.5: $\log\langle(\delta\phi)^2\rangle$ vs q for (a) the Mathieu eq., and (b) the quasi-periodic eq., both calculated at $t = 23m_\phi^{-1}$. The maximum value of $\langle(\delta\phi)^2\rangle$ is about 20 times larger in (b).

[148], which are often of very small measure - hence physically unimportant - mimicking the stochastic case presented earlier. Again however, no estimates for μ are available analytically. Our numerical results show the growth of μ in this case and the significant widening and steepening of the instability bands relative to the Mathieu case. Also interesting is the development of "non-thermal edges" in μ (figure 5.4 and [98]).

From our discussion of the stochastic case, we expect $\langle(\delta\phi)^2\rangle$ to increase and again this is borne out numerically. In figure (5.5) we plot $\log\langle(\delta\phi)^2\rangle$ for the pure Mathieu and quasi-periodic cases as functions of q. The maximum variance is ~ 1.5 orders of magnitude larger in the quasi-periodic case after the short time $t = 23m_\phi^{-1}$. The quasi-periodic model therefore is significantly more efficient at restoring symmetry. In general the NTSR strength of GUT theories will depend sensitively on the mass spectrum of the theory, here encoded by m_ϕ/m_φ.

5.7 A non-thermal resolution to the monopole problem ?

The monopole problem has now again become a major concern in large ("chaotic or incommensurate") regions of the coupling/mass parameter space, even with the expansion of the universe included. Due to the non-thermal, quench-like, nature of the symmetry breaking, the correlation length Ξ of the fields will be much smaller than in the equilibrium case, and therefore the defect density $\propto \Xi^{-n}$ ($n = 1$ for domain walls, $n = 3$ for monopoles) will be correspondingly larger than the equilibrium Kibble prediction [157].

If NTSR succeeds, a second stage of inflation will occur [131, 158] while the vacuum energy $V(0)$ dominates over the energy of the $\langle(\delta\chi)^2\rangle$. During this time $a(t)$ increases by a factor $\sim (\tilde{g}^2/\lambda)^{1/4}$ for the potential (5.7), which cannot, therefore, supply the needed ~ 20 e-foldings to dilute the monopole density sufficiently [129]. In $SO(10)$ or $SU(6)$, however, the monopole transition is separated from the lower transitions and this may allow enough secondary inflation to dilute the monopole abundance sufficiently.

However, this is rather model dependent and the large corrections to the effective potential offer us an alternative escape route via defect-defect interactions. The full corrections to the effective potential, include not only the quadratic contributions affecting $m^2_{\phi,eff}$, but also odd powers which do not respect any previously existing discrete symmetries (required for example for successful D-term inflation [68]). The odd-power corrections to eq. (5.7), with $\tilde{g} = 0$, are:

$$\Delta V_{odd} = \lambda\phi(\delta\phi^3 - \phi_0^2\delta\phi) + \lambda\delta\phi\phi^3 \tag{5.27}$$

These terms softly break the $\mathbf{Z}_2$ $\phi \rightarrow -\phi$ symmetry of eq. (5.7). This allows an implementation of the Dvali-Liu-Vachaspati mechanism [160] for solving the monopole problem as follows: imagine that NTSR was successful enough to produce monopoles and domain walls during reheating. The $\delta\phi$ terms in eq. (5.27) automatically cause the domain walls to be unstable [159] since the minima at $\pm\phi_0$ are no longer degenerate, causing a pressure difference $\sim 2\Delta V_{odd}(\phi_0)$ across the walls.

Since after preheating $\phi \ll \langle(\delta\phi)^2\rangle$, the first term of (5.27) dominates. The constraint that the walls percolate gives us $\langle\delta\phi\rangle \leq 10^{-1}M_{pl}$. Monopoles are then swept up on the walls and dissipate because the full symmetry (e.g. $SU(5)$) is restored there [160]. Requiring that the pressure difference drives the domain walls to collapse and decay before dominating the energy density of the universe yields [98] $\langle\delta\phi\rangle \geq 10^{-4}\lambda^{-1/3}M_{pl} \sim 10^{-3}M_{pl}$ if $\lambda \sim 10^{-3}$. These bounds are exactly in the range of values expected if NTSR is successful. Note that we cannot simply use the same self-coupling λ for this process and the process of CMB anisotropy formation since the above bounds are violated if we put $\lambda \sim 10^{-14}$. This issue can, as we have discussed before, easily be accommodated if we allow for strong running of coupling constants or supersymmetry.

The advantage of this formulation is that it provides a specific implementation of the general Dvali-Liu-Vachaspati mechanism - one that is perhaps preferable to the one we presented in section (3.6.2) - and uses the same large quantum fluctuations which produced the monopoles, to remove them.

Chapter 6

Reheating and the Evolution of Gravitational Waves

I don't want to achieve immortality through my work.
I want to achieve it through not dying.
– Woody Allen

6.1 Introduction

The appearance of Grishchuk's 1993-1996 papers [196]-[198] caused a controversy regarding perturbation evolution through cosmological phase transitions such as reheating after inflation, the electroweak phase transition and quark confinement.

Here we will only study the evolution of gravitational waves[1] through second order phase transitions [2]. We will not look at density perturbations because, while very interesting, the evolution equations typically become singular periodically during a second order phase transition due to the oscillations of the order parameter (terms of the form (ϕ^{-1}) occur in the evolution equations), and hence require rather subtle analysis. On the other hand, first order phase transitions proceed via bubble nucleation and collision and hence are nonlinear phenomena best studied numerically with nonlinear models [207].

However the controversy surrounding the work of Grishchuk is particularly focussed on gravitational wave evolution through reheating, which in new inflationary models occurs via a second order phase transition and in chaotic inflation via field oscillations. The part of his argument relating to classical evolution of gravitational waves through reheating is easy to state: (1) the evolution equation for tensor perturbations can be cast in the form of a harmonic oscillator equation with a time dependent frequency. Depending on the exact nature of this time-dependence, the tensor perturbations can be amplified or not. (2) He then treats reheating as an instantaneous surface matching a de Sitter universe to radiation-dominated FLRW, and imposes the Darmois junction conditions across the surface [3]

However, the implications of oscillatory reheating (through a second order phase transition) and the resulting large quantum fluctuations for local curvature (metric) fluctuations has been largely unexplored until now, and limited to scalar perturbations [150, 151]. The evolution of the tensor (gravitational wave) spectrum has essentially been completely ignored, apart from a study of gravitational bremsstrahlung generated through the interactions of the large quantum fluctuations in the inflaton and decay-product fields [184]. Gravitational wave production during the bubble-wall collisions of a first order phase transition have been rather more studied [207]. However, the gravitational waves produced in these mechanisms are backreaction phenomena, since they are due to the scalar field fluctuations, rather than the background zero mode evolution itself.

In this chapter we wish to demonstrate that there exists such an amplification of gravitational

[1] The work in this chapter is partially based on the papers [153] and [149].

[2] A phase transition is said to be of $n-th$ order ($n = 1, 2$) if the $n-th$ derivative of the potential is the first to be discontinuous at the phase transition. Thus first order phase transitions proceed by nucleation and growth of bubbles of true vacuum, such as occurs with boiling water. At a second order phase transition, the correlation length of the field diverges and there are fluctuations on all scales (neglecting causality and critical slowing down) leading to conformal invariance and critical behaviour near the phase transition.

[3] The Darmois conditions require matching of the first and second fundamental forms of the metric, i.e. h_{ab} and the extrinsic curvature θ_{ab} across the junction hypersurface.

waves, essentially due to the oscillation of the zero mode of the inflaton during reheating. This is in addition to any gravitational bremsstrahlung that may be produced by the associated scalar fluctuations. Further, we show that a direct analogy exists in the treatment of preheating and gravitational wave evolution at the end of inflation if this occurs via a second order phase transition, as we shall assume here. Indeed, both are governed by (approximately) Floquet systems [4]. In the case of the quadratic potential, $V(\phi) = \frac{1}{2}m_\phi^2\phi^2$, both preheating and gravitational wave amplification are initially well approximated by the Mathieu equation. This "duality" is exhibited using the covariant Maxwell-Weyl form of the Einstein field equations (see section 6.2 and e.g. [220]), and is partially hidden in the Bardeen formalism [11]. This demonstration of gravitational wave amplification should be considered in the light of the aforementioned controversy surrounding tensor amplification during instantaneous phase transitions.

6.1.1 The Electric and Magnetic Weyl Tensors

A purely tensor description of gravitational waves is still partially lacking in the covariant approach [246], but a sufficient description was given first by Hawking (1966) [193] in terms of the electric (E_{ab}) and magnetic (H_{ab}) parts of the Weyl tensor, which for a given four-velocity u^a are defined by:

$$E_{ab} = C_{acbd}u^c u^d \ , \quad H_{ab} = {}^*C_{acbd}u^c u^d \, , \tag{6.1}$$

Now in general H_{ab} and E_{ab} contain information about all types of matter and metric fluctuation. For purely tensor perturbations about FLRW, they are related to the h_k via [168, 169]:

$$E_{ab} = -\frac{1}{2}h_k'' - (k^2 + K)h_k Y_{ab} \tag{6.2}$$

and

$$H_{ab} = S^{-2}h_k' Y_{(a}^{c|d}\eta_{b)0cd} \tag{6.3}$$

where $|$ denotes the covariant derivative with respect to the spatial metric $h_{ab} = g_{ab} + u^a u^b$, $()'$ conformal time derivative and Y^{ab} are the tensor eigenfunctions of the Laplace-Beltrami operator in the constant curvature spacelike slices [20].

The evolution equations for E_{ab} and H_{ab} come from the cyclic Bianchi identities, which when written out give the nonlinear Maxwell-like equations for the free gravitational field. For brevity we will write them out in terms of the complex variable [210] $\mathcal{I}_{ab} \equiv E_{ab} + iH_{ab}$, yielding:

$$\begin{aligned} \mathrm{D}^b\mathcal{I}_{ab} &= 3i\omega^b\mathcal{I}_{ab} - i[\sigma,\mathcal{I}]_a + \Psi_a \, , \\ \dot{\mathcal{I}}_{(ab)} + i\,\mathrm{curl}\,\mathcal{I}_{ab} &= -\Theta\mathcal{I}_{ab} + 3\sigma_{c(a}\mathcal{I}_{b)}{}^c - \omega^c\epsilon_{cd(a}\mathcal{I}_{b)}{}^d \\ &\quad -2i\,\dot{u}^c\epsilon_{cd(a}\mathcal{I}_{b)}{}^d - \frac{1}{2}(\mu+p)\sigma_{ab} \end{aligned} \tag{6.4}$$

$$\Psi_a = \frac{1}{3}\mathrm{D}_a\mu + i(\mu+p)\omega_a \, . \tag{6.5}$$

[4] A Floquet system is any set of linear ODE's with periodic coefficients. The solutions of such systems are characterized by resonance bands of exponentially growing modes, indexed by the momentum k.

where the spatial gradient D^a, curl and commutator [,] are defined via:

$$\begin{aligned} D^a C_d &= h^a{}_b \nabla^b C_d \\ \mathrm{curl}\, A_{ab} &= \eta_{cd(ae} u^e D^c A_{b)}{}^d \end{aligned} \tag{6.6}$$

and

$$[A, B]_a = \eta_{abcd} u^d A^b{}_e B^{ce} \,, \tag{6.7}$$

where η_{abcd} is the totally antisymmetric volume element. These extend the usual vector operators to second rank tensors. Further, the projected trace-free and symmetric part of a tensor A_{ab} is denoted by $A_{\langle ab \rangle}$ and defined by:

$$A_{\langle ab \rangle} = h^c_{(a} h^d_{b)} A_{cd} - \tfrac{1}{3} h_{cd} A^{cd} h_{ab} \,. \tag{6.8}$$

In the case of purely gravitational wave perturbations we must set $\Psi_a = 0$ to get a fully consistent characterisation. One of the main outstanding problems in the covariant approach is a unique characterisation of tensor perturbations in the presence of matter and vorticity perturbations.

Once the specification $\Psi_a = 0$ is made, the Ricci identities imply the following relevant constraint [246]:

$$H_{ab} = -\mathrm{curl}\, \sigma_{ab} \tag{6.9}$$

together with the Raychaudhuri equation for the expansion:

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 + 2(\sigma^2) + \tfrac{1}{2}(\mu + 3p) = 0 \tag{6.10}$$

where $\sigma^2 = \frac{1}{2}\sigma_{ab}\sigma^{ab}$. In FLRW spacetimes $\Theta = 3\dot{S}/S = 3H$ where H is the Hubble constant. The shear is governed by the fully nonlinear equation:

$$\dot{\sigma}_{ab} + \tfrac{2}{3}\Theta\sigma_{ab} + \sigma_{c<a}\sigma_{b>}{}^c = -E_{ab} \tag{6.11}$$

which shows that the shear is a "potential" for both E_{ab} and H_{ab}. Given σ_{ab}, both E_{ab}, H_{ab} follow immediately ($\Psi_a = 0$). So far, all the equations have been fully nonlinear, a great advantage of the covariant formalism over the Bardeen approach.

A natural interpretation of the linearised electric Weyl part is given by the geodesic deviation equations, which for the special case of a plane gravitational wave propagating along the x^1 direction are: [128]

$$\ddot{\xi}^\alpha = E^\alpha{}_\beta \xi^\beta \quad , \quad \alpha, \beta = 2, 3 \tag{6.12}$$

where ξ^α is the connecting vector between orthonormal tetrads associated with the congruence of null geodesics ruled by gravitons. This means that we can directly attribute the physical effects of linear tensor perturbations, on e.g. a gravitational wave detector, to the electric part of the Weyl tensor.

Now in general, for a perfect fluid, the linearisation of the system (6.5) yields equations that are coupled to energy density gradients and the vorticity. Setting $\Psi_a = 0$ ensures that $\mathrm{D}^a \mathcal{I}_{ab} = 0$.

Note that $\mathcal{I}^a{}_a = 0$ by construction and hence one has the exact analogue of the transverse-traceless conditions in the metric approach. In this case, one finds the coupled wave equations for the modes of $H_{ab}, E_{ab}, \sigma_{ab}$ after expansion in eigenfunctions of the tensor Helmholtz equation [93]:

$$\ddot{H}_k + \frac{7}{3}\Theta \dot{H}_k + \left[\frac{k^2}{S^2} + \dot{\Theta} + \Theta^2 + \frac{1}{2}(\mu - p)\right] H_k = 0 \ , \tag{6.13}$$

where μ and p are the relativistic energy density and pressure respectively and $\Theta = \dot{S}/S$. This is a simple decoupled equation while the shear satisfies:

$$\ddot{\sigma}_k + \frac{5}{3}\Theta\dot{\sigma}_k + \left[\frac{k^2}{S^2} + \frac{\Theta^2}{9} + \frac{1}{6}\mu - \frac{3}{2}p\right]\sigma_k = 0 \ . \tag{6.14}$$

The modes of the electric part obey an equation which is coupled to the shear:

$$\begin{aligned} \ddot{E}_k &+ \frac{7}{3}\Theta\dot{E}_k + \left[\frac{k^2}{S^2} + \dot{\Theta} + \Theta^2 + \frac{1}{2}(\mu - p)\right] E_k \\ &= -\left[\frac{1}{3}\Theta(\mu + p) + \frac{1}{2}(\dot{\mu} + \dot{p})\right]\sigma_k \end{aligned} \tag{6.15}$$

6.1.2 A 2nd order evolution equation for E_{ab}

Previously it was believed [93, 227] that an evolution equation for E_k containing only zero order quantities must necessarily contain third order time derivatives. It is easy to see why. Eq. (6.15) has a driving term due to the shear which appears impossible to remove in terms solely of E_{ab}.

Infact there are (at least) two different ways to proceed to find a second order evolution equation for the modes of E_{ab} - the E_k. The simplest is as follows: linearise Eq. (6.11) and take its time derivative. Eliminate the $\dot{\sigma}_{ab}$ terms in $\frac{d}{dt}$(6.11) and Eq. (6.14) via Eq. (6.11) to get:

$$\ddot{\sigma}_{ab} = (\frac{4\Theta^2}{9} - \frac{2\dot{\Theta}}{3})\sigma_{ab} + \frac{2\Theta}{3}E_{ab} - \dot{E}_{ab} \tag{6.16}$$

Similarly, remove the $\dot{\sigma}_k$ term from Eq. (6.14) and equate (6.14) with Eq. (6.16). This gives one σ_k in terms of $\dot{E}_k$ and E_k:

$$\sigma_k = \mathcal{B}(\Theta E_k + \dot{E}_k) \tag{6.17}$$

where

$$\mathcal{B}^{-1} = \left(\frac{k^2}{S^2} - \frac{5\Theta^2}{9} - \frac{2\dot{\Theta}}{3} + \frac{1}{6}\mu - \frac{3}{2}p\right) . \tag{6.18}$$

This leads to the linear, second order equation for E_k:

$$\begin{aligned} \ddot{E}_k &+ \left(\frac{7\Theta}{3} + \mathcal{B}\ \mathcal{C}\right)\dot{E}_k \\ &+ \left(\frac{k^2}{S^2} + \mathcal{B}\mathcal{C}\Theta + \Theta^2 + \dot{\Theta} + \frac{1}{2}(\mu + p)\right) E_k = 0 \, , \end{aligned} \tag{6.19}$$

where

$$\mathcal{C} = \frac{1}{3}\Theta(\mu + p) + \frac{1}{2}(\dot{\mu} + \dot{p}) \, . \tag{6.20}$$

Alternatively, the linearised, real part of Eq. (6.4) relates the shear to $E_{ab}, \dot{E}_{ab}$ and $\mathrm{curl}\, H_{ab}$. Taking the curl of both sides of Eq. (6.9) gives $\mathrm{curl}\, H_{ab}$ in terms of $\mathrm{curl}\,\mathrm{curl}\, \sigma_{ab}$ which can be written solely in terms of E_{ab} and derivatives. Using this in the linearised real part of Eq. (6.4) and then in the RHS of Eq. (6.15) gives an equation for E_{ab} which is second order in time but contains higher order spatial derivative terms. The resulting equation is again irrelevant for actual calculations and we do not give it explicitly.

6.1.3 Oscillatory dynamics in reheating

Now let us specialise equations (6.13-6.15) to the case of classical scalar field dynamics. Treating here only the case of a single scalar field, we have, using the equivalence of ϕ with a perfect fluid:

$$\mu = \kappa \left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right] \ , \ \ p = \kappa \left[\frac{1}{2}\dot{\phi}^2 - V(\phi)\right] \tag{6.21}$$

so that

$$\frac{1}{2}(\mu - p) = \kappa V(\phi), \ \ (\mu + p) = \kappa\dot{\phi}^2 \ , \ \ (\dot{\mu} + \dot{p}) = 2\kappa\dot{\phi}\ddot{\phi} \ . \tag{6.22}$$

Here $V(\phi)$ is the effective potential of the scalar field. Note that if scalar perturbations become important then these relations will gather terms proportional to $(\nabla\phi)^2$ [37]. This is precisely the case if one wishes to consistently study the production of tensor perturbations from gravitational bremsstrahlung [184]. However, it brings with it a host of complexities, since for example, eq.s (6.13 - 6.15) must be re-derived in the presence of the backreaction of matter fluctuations, a highly non-trivial problem. We will not consider this issue further here.

The important point is that with the above identifications, the equations (6.13), (6.14), (6.15) become generalizations of eq.(2.18). There thus exists a strong connection between the evolution governing reheating under a given potential and the equations governing the evolution H_k, E_k and σ_k particularly when $V(\phi)$ is an even polynomial in ϕ, as in chaotic inflation.

6.2 Chaotic inflation and Duality

Consider again the quadratic chaotic potential, Eq. (2.16). In addition to chaotic inflation, this describes the dynamics of the invisible axion, and the Polonyi and moduli fields of supergravity and string theory, with appropriate changes to the values of the masses, $m^2 \equiv V''(\phi)$.

Here we will neglect the expansion of the universe as before to delineate the duality [155]. This would of course not be adequate for the study of long-time oscillatory amplification, but is justified if the period of oscillations of ϕ is small compared with typical averaged expansion times, i.e. $m_\phi \gg \overline{\Theta}$. However, the oscillations of the expansion rate should also be included in a full description [5][11] which will act as an additional source of resonance. This will be important in obtaining the tensor spectrum

[5]That the expansion also oscillates can be seen from eq. (6.10).

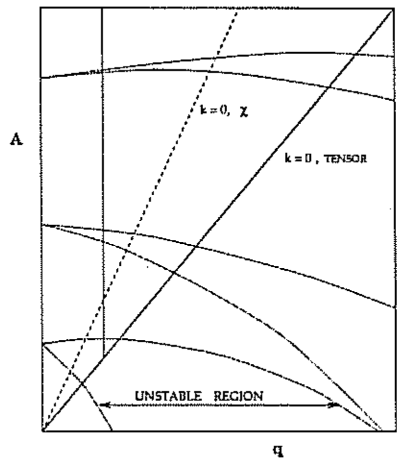


Figure 6.1: A schematic of the instability chart for the Mathieu equation. The two diagonal lines correspond respectively to the zero-modes ($k = 0$) of the gravitational wave spectrum and the χ-spectrum of eq. (4), for the case $g^2 < m_\phi^2$. The vertical line near the left border of the chart is a sample spectrum for a given model (i.e. value of q).

from oscillating cold dark matter (CDM) relics like the invisible axion [175], as we will discuss in section (6.4). The equation (6.13) for the modes, H_k, of the magnetic part H_{ab} now becomes:

$$\ddot{H}_k + \left(\frac{k^2}{S^2} + \frac{\kappa m_\phi^2}{2}\Phi^2 \sin^2(m_\phi t)\right) H_k = 0 \tag{6.23}$$

This is the Mathieu equation and is precisely the same as equation (2.18) for the evolution of the quantum fluctuations χ_k, with the replacement $g^2 \to \kappa\frac{m_\phi^2}{2} = 4\pi m_\phi^2$. Hence $q_H = \pi\Phi$. The requirement that production of χ bosons is more efficient than H_{ab} amplification is then $g^2 > 4\pi m_\phi^2$. If $m_\phi \sim 10^{-6}$, as required to match CMB observations [209], then this is a weak constraint on the coupling g, namely $g^2 > 10^{-11}$. Nevertheless, it is a constraint independent of Φ and hence applies to both chaotic and new inflationary models (which have quadratic potentials near the global minima). If the constraint is not met, it implies that reheating occurs preferentially via production of gravitons rather than the χ-channel [6].

In figure (6.1) this situation is depicted schematically on the instability chart of the Mathieu equation: namely, the situation in which $q_H > q_\chi$. The two diagonal lines corresponding to the mode $k = 0$, delimit the physical (i.e. upper) region of the chart. The single vertical line corresponds to a sample spectra for a given value of q. Note that for this value of q, the tensor spectrum has modes in the first fundamental resonance band, while there are no χ modes which lie in this band. Figure (6.2) shows a numerical integration of the spectrum as a function of time for modes in both stable and unstable bands. Figure (6.3) zooms in on the spectrum within a stable band.

[6]This is not the only constraint to be met however. If the χ field has moderate or strong self-interactions, the χ-resonance is strongly suppressed [7]. This is also the case if $q_\chi \gg 10^3$, due to rescattering effects [7, 6]. In these cases, the graviton decay-channel could still be important and perhaps even dominant.

Can H_{ab} be amplified in the broad resonance regime ? This is necessary if gravitational wave enhancement is to be really effective over the expansion. This requires $q_H > \pi^{-1}$. Since $q_H = \pi\Phi^2$ this implies $\Phi > \pi^{-1}$ in units of the Planck energy. In the case of chaotic inflation, the amplitude of oscillations goes as $\Phi \sim 1/N$, where N is the number of oscillations of ϕ, neglecting the non-equilibrium backreaction at the end of preheating which often leads to a sudden decrease in Φ [183]. Thus during preheating proper, $q_H \sim \pi/N^2$, and H_{ab} is initially amplified in the broad resonance regime, but moves rapidly towards narrow resonance.

	χ_k	H_k	σ_k
A	$\frac{k^2}{S^2 m_\phi^2} + 2q$	$\frac{k^2}{S^2 m_\phi^2} + 2q$	$\frac{k^2}{S^2 m_\phi^2} + \frac{2}{19}q$
q	$\frac{g^2\Phi^2}{4m_\phi^2}$	$\pi\Phi^2$	$\frac{19}{6}\pi\Phi^2$

Consider now the equations for the electric Weyl field, E_{ab}, and the shear σ_{ab}. They form a partially decoupled linear system with time-dependent coefficients that form an approximately Floquet system. Here we ignore the decay of the amplitude. This is appropriate during preheating while the non-equilibrium effects and backreaction of graviton production are not too strong and the oscillation period of ϕ, (controlled by m_ϕ) is short compared with the expansion time scale. This gives:

$$\ddot{E}_k + \left(\frac{k^2}{S^2} + \frac{m_\phi}{2}\Phi^2 \sin(m_\phi t)\right) E_k = \frac{1}{8}\left[\Phi^2 m_\phi^3 \sin(2m_\phi t)\right]\sigma_k \tag{6.24}$$

with

$$\ddot{\sigma}_k + \left(\frac{k^2}{S^2} - \kappa\frac{3}{4}\dot{\phi}^2 + \kappa\frac{5}{6}m_\phi^2\phi^2\right)\sigma_k = 0 \tag{6.25}$$

The equation for the shear can again be cast in the form of the Mathieu equation with

$$A(k)_\sigma = \frac{k^2}{m_\phi^2 S^2} + \frac{2}{19}q_\sigma \ , \quad q_\sigma = \kappa\frac{19}{48}\Phi^2 \ . \tag{6.26}$$

A comparison of parameters is given in table 1.

Note that the shear *always* lies in a region of broader resonance than the magnetic part of the Weyl tensor because $q_\sigma > q_H$ and $A_\sigma(k) < A_H(k)$. Now $q_\sigma = \frac{19\pi}{6}\Phi^2$ with the requirement of broad resonance amplification of the shear $\Phi > \sqrt{\frac{6}{19}}\pi^{-1} \sim 0.56\pi^{-1} \sim 0.18$.

Since it is the shear which directly determines the tensor contribution to the anisotropy of the CMB, this may allow one to place constraints on large-amplitude oscillatory reheating. Since power-law inflation is known to produce one of the strongest tensor signals during slow-roll [54] the signal to noise (due to cosmic variance and instrument) ratio for the tensor component in the CMB may be significantly larger than previously hoped [164]. The CMB anisotropy from a tensor signal is [208]:

$$\left(\frac{\delta T}{T}\right)^* = -\int_E^R S\ \sigma_{ab}\ k^a k^b d\lambda \tag{6.27}$$

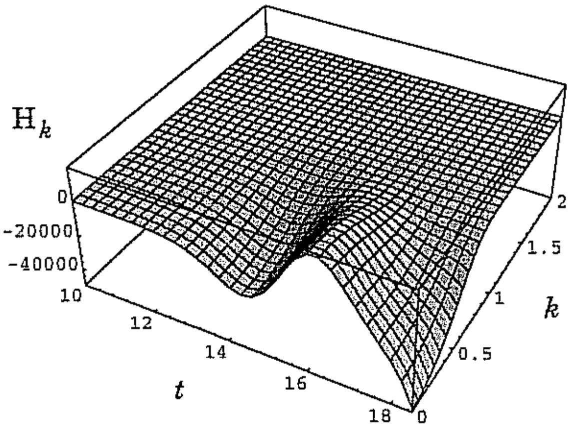


Figure 6.2: The gravitational wave spectrum, H_k as a function of k and t, with the initial condition $H(0) = 0, \dot{H}(0) = 10^{-4}$ and starting in the broad resonance regime. As can be seen, the power spectrum becomes highly k-dependent, breaking the usual scale-invariance of the envelope predicted by inflation. Further, the amplitude of the spectrum is exponentially enhanced by resonant reheating over its value during inflation.

The left hand side is a gauge-invariant measure of the anisotropy in the CMB, and k^a is the wave vector ruling our past null cone.

Now for purely tensor perturbations, σ_{ab} and E_{ab} are related by [93]:

$$\dot{\sigma}_{ab} = -\frac{2}{3}\Theta\sigma_{ab} - E_{ab} \tag{6.28}$$

we see that exponential growth of the shear implies exponential growth of the electric Weyl field, and hence the energy in gravitational waves, Ω_{GW}, increases exponentially, as can be seen from the time-like component of the Bel-Robinson tensor which is naturally interpreted as the super-energy of the gravitational field (see chapter 7):

$$E_{GW} \propto E_{ab}E^{ab} + H_{ab}H^{ab} \, . \tag{6.29}$$

However, a counter example to the above situation occurs if there is non-thermal symmetry restoration [131, 4]: start with $\phi \gg M_{pl}$, as in the chaotic inflation scenario, but with a Coleman-Weinberg type new-inflationary potential, which is flat near the origin. If preheating restores symmetry, a phase of new inflation begins with $\phi = 0$. This redshifts away the resonantly amplified tensor spectrum. The second stage of reheating will be much less effective in amplifying the tensor spectrum than the first if $V(0) \ll M_{pl}^4$. This is the generic case for non-thermal symmetry restoration and hence amplification will occur in the very narrow resonance region where expansion effects are expected to dominate, leaving an almost untouched, scale-invariant tensor spectrum.

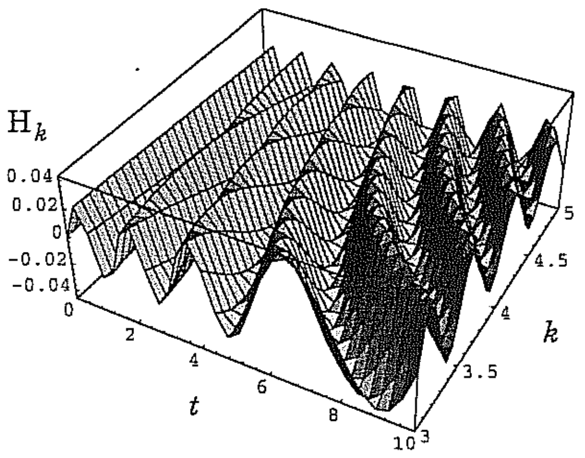


Figure 6.3: A zoom of the stable band region $3 \leq k \leq 5$. The solution is the usual bounded oscillatory one. Note the scale on the z-axis (c.f. fig. 2).

6.3 Evolution in the case of a quartic potential

A natural generalisation of the quadratic potential is the effective potential $V(\phi) = \frac{\lambda}{4}\phi^4$. For this potential there is an added freedom in the parameters controlling the resonance compared with the corresponding $\delta\phi_k$ growth. Neglecting again the expansion of the universe for clarity, the modes of the magnetic part satisfy the equation:

$$\ddot{H}_k + \left(\frac{k^2}{a^2} + \frac{\lambda}{4}\phi^4\right) H_k = 0 \tag{6.30}$$

when the expansion is neglected. Using the sinusoidal approximation to the Jacobi elliptic functions for the evolution of ϕ and neglecting the contribution of particle backreaction the equation for quantum fluctuations $\delta\phi_k$ in the inflaton field can be written in the form of the Mathieu equation [1] with $A = k^2/(c^2\lambda a^2\Phi^2) + 3/(2c^2)$, $q = 3/(4c^2) \simeq 1.04$. Thus resonant particle production only occurs in the second resonance band with wavelengths $k^2 \simeq 3\lambda a^2\Phi^2$ [1]. There is no dependence on the amplitude of inflaton oscillations Φ as there is in the quadratic potential case.

Within the same above approximation, and transforming to conformal time $d\eta = dt/S(t)$, eq. (6.30) is the simplest generalisation of Mathieu's equation and lies within the class of Hill equations:

$$\frac{d^2H_k}{d\eta^2} + [\theta_0 + 2\theta_1\cos(2z) + 2\theta_2\cos(4z)]H_k = 0 \tag{6.31}$$

where:

$$\theta_0 = \frac{k^2}{\lambda c^2 a^2 \Phi^2} + \frac{3\Phi^2}{32c^2} \tag{6.32}$$

$$\theta_1 = -\frac{\Phi^2}{16c^2} \tag{6.33}$$

$$\theta_2 = \frac{\Phi^2}{64c^2} \tag{6.34}$$

and $c \simeq 0.85$. The μ_k are given by the solution of the equation:

$$\sin^2(\frac{i\pi\mu_k}{2}) = \Delta(0)\sin^2(\pi\sqrt{\theta_0}) \tag{6.35}$$

where Δ is the Hill determinant which depends on all θ_i. The interesting difference that arises in the gravitational wave case is that the second term in θ_0 and both θ_1 and θ_2 depend on Φ^2 while $q = 1.04$ is fixed in the case of $\delta\phi_k$ fluctuations. Thus if Φ could vary arbitrarily then broad resonance gravitational wave amplification could occur in all forbidden bands and not just the second one and graviton production might be much stronger than $\delta\phi_k$ production. However in practise, $\theta_1 \sim -0.87\Phi^2$ so that if $\Phi \leq 1$ then graviton production occurs in the realm of narrow resonances and is less effective that $\delta\phi$ amplification. Nevertheless, the extra freedom accorded in the tensor case is very interesting.

The sinusoidal approximation used above should be compared with the exact solution for ϕ which is given in terms of elliptic functions as we described in section (2.5) [10]. In this case both the modes of the Magnetic Weyl tensor eq. (6.30) and the shear obey a natural generalisation of the Lamé equation, namely the ellipsoidal wave equation, which has solutions in terms of products of elliptic functions and a convergent power series with argument $\mathrm{sn}^2(\eta)$, the square of the Jacobi sine function [195].

6.4 Axion oscillations and the gravity-wave background

Next we focus instead on another physical mechanism for distortion and amplification of any existing gravitational wave background - damped parametric resonance due to long time oscillatory phases that the universe may have undergone. The best example are the coherent axion or moduli oscillations if they form a significant portion of the dark matter. Indeed, models of quintessense [7] based on convex potentials will give similar effects.

For completeness we choose to discuss the amplification within the gauge-invariant Bardeen formalism. The conversion to the covariant approach presented earlier is straightforward. The evolution of the transverse-traceless (TT) metric perturbations h_{ab} are naturally described by the Fourier mode functions $h_{\epsilon,k}$, where $\epsilon = \{+, \times\}$ are the polarisation states. The h_k [8] satisfy:

$$\ddot{h}_k + 3\frac{\dot{a}}{a}\dot{h}_k + \frac{k^2}{a^2}h_k = 0\,. \tag{6.36}$$

Here $a(t)$ is the scale factor of the universe which obeys the Friedmann Eq.:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\mu\,, \tag{6.37}$$

[7] Q-matter or quintessense is the old idea with a new name that part of the matter in the universe might be in the form of a dynamical scalar field - inflation applied to large scale structure as it were ! Axions are one example.

[8] From now on we suppress the polarisation label. We also restrict ourselves to the case of flat spatial sections.

where μ is the relativistic energy density which again we have specified to be in the form of a scalar field ϕ, with potential $V(\phi)$. This gives us enough freedom to model the oscillations of the axion condensate. Using Eq. (6.37), we can rewrite Eq. (6.36) as [11] (see also chapter [3]):

$$\frac{d^2(a^{3/2}h_k)}{dt^2} + \left(\frac{k^2}{a^2} + \frac{3}{4}\kappa p\right)(a^{3/2}h_k) = 0\,, \tag{6.38}$$

where $p = \dot{\phi}^2/2 - V(\phi)$ is the pressure.

The archetypal example of resonance, as we discussed in chapter (2), is provided by the Mathieu equation. Defining $\epsilon = A/(2q) - 1$, we plot μ vs. (ϵ, q) in figure (6.4) using *the piecewise quadratic approximation* [171, 172]. In this approximation the sinusoidal part of the Mathieu frequency is replaced piecewise with parabola's. We have already come across this approximation in section (2.11.1) which is particularly accurate for $q \gg 1$. When $\epsilon < 0$ the resonance is particularly strong due either to the ratio $A/(2q)$ being very small or because $A < 0$, the negative coupling instability.

For the quadratic potential the pressure is (neglecting the adiabatic expansion):

$$p = -\frac{m_\phi^2}{2}\Phi^2\cos(2m_\phi t) \tag{6.39}$$

yielding a Mathieu Eq. with parameters:

$$A(k) = \frac{k^2}{a^2 m_\phi^2}\,,\quad q = \frac{3\kappa\Phi^2}{16}\,,\quad \epsilon = \frac{32k^2}{3\kappa a^2 m_\phi^2\Phi^2} - 1 \tag{6.40}$$

showing that in this case, unlike in the case of standard reheating where $A = k^2/(a^2m_\phi^2) + 2q$ [1], $\epsilon < 0$ is possible and gravitational wave amplification can be significant if $\Phi \sim M_{pl}$.

The effect of the expansion of the universe decreases Φ and redshifts k, causing a decrease of both A and q, though ϵ remains roughly constant. The decrease of q to below unity is particularly important in stopping the resonance, and there is thus a competition between the damping effect of the expansion, and the amplification due to resonance.

6.4.1 The axion and massive moduli

The axion[173] is an oscillating scalar field and a natural cold dark matter candidate. Unlike reheating, which lasts a very short time, the axion oscillations would last a large proportion of the universe's history, and hence might cause significant tensor amplification. The axion potential is given by [174, 175]:

$$V(\phi) = \Lambda^4\left[1 - \cos\left(\frac{\phi}{f_a}\right)\right] \tag{6.41}$$

with f_a the axion decay constant and $\Lambda = f_a m_a$, where m_a is the axion mass. The standard QCD axion has $\Lambda = \Lambda_{QCD} \sim 200$ MeV, $f_a \sim 10^{12}$ GeV and gains a non-zero mass due to instanton effects at an energy around Λ_{QCD} [175]. There also exist massive moduli in supergravity and superstring

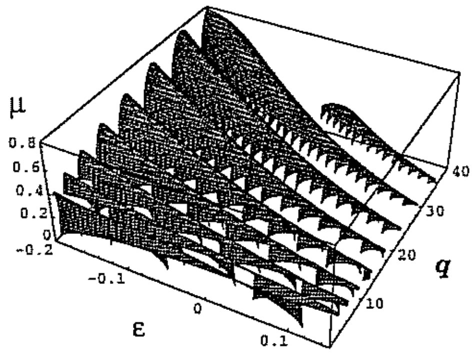


Figure 6.4: The Floquet index μ on the stability-instability chart for Eq. (8) using the numerically cheap piecewise quadratic approximation compared with the full numerical solution shown in figure (2.2). Notice the rapid increase of μ for decreasing ϵ and increasing q.

theories with much less constrained parameters, for example one may take $\Lambda \sim 10^{16}$ GeV and $f_a \sim M_{pl}$ [174], with the moduli generically gaining mass at the epoch of supersymmetry breaking.

To understand the implications of axion oscillations, let us approximate Eq. (6.41) by the first, quadratic, term in the Taylor series. We can then use the results of Eq. (6.40) with the replacement $m_\phi^2 \rightarrow \Lambda^4/f_a^2$ so that roughly we have $A \simeq k^2 f_a^2/(\Lambda^4 a^2)$ and $q \propto \Phi^2$. For the values given above, this yields

$$A_{QCD} \sim 10^{27}\frac{k^2}{a^2}\,, \quad A_{moduli} \sim 10^{-26}\frac{k^2}{a^2} \tag{6.42}$$

This implies that massive moduli are more likely to lead to large amplifications of the background gravity wave spectrum since $\epsilon = A/(2q) - 1 < 0$ for a huge range of modes, while in the case of the QCD axion, only a tiny fraction of the modes, near $k = 0$, have negative ϵ.

On the other hand, only if the moduli or axions started with near-Planck expectation values, $\Phi \sim M_{pl}$, will there be significant amplification in either case.

6.5 A nonlinear $O(N)$ σ-model

Another interesting issue concerns the effect of nonlinearity on the gravitational wave amplification. Here we give a necessarily very brief discussion via a simple example: the large-N limit of a nonlinear σ model. This is the appropriate model for discussing topological defects ($N = 1, ...4$ corresponding respectively to domain walls, strings, monopoles and textures) since it exhibits broken global symmetries where $O(N)$ is broken to $O(N-1)$. For $N > 4$ the dynamics simply corresponds to that of non-linearly coupled Goldstone boson modes.

The model consists of N scalar fields assembled into a vector field σ. The potential we consider is $V(\sigma) = \lambda\eta^2(\beta^2 - 1)$ where η is the mass scale. After the phase transition and for $N > 2$ the field is constrained to wander about the vacuum manifold visualised as S^{N-1} imposing the constraint $\sigma_\mu\sigma^\mu = 1$. The equation of motion is [167]:

$$\Box\sigma - (\sigma\cdot\Box)\sigma = 0 \tag{6.43}$$

The nonlinear coupling comes from the second term $-(\sigma\cdot\Box)\sigma = (\partial_\mu\sigma\cdot\partial^\mu\sigma)\sigma$ and in the large-N limit, this sum can be replaced by ensemble average resulting in a linear equation. This amounts to replacing the trace of the stress-energy tensor, $\mu - 3p$, by its spatial average. Under this condition the solution to Eq. (6.43) is given by [167]:

$$\sigma(\mathbf{k},t) = At^{3/2}\frac{J_\nu(kt)}{(kt)^\nu}\sigma(\mathbf{k},0)\,, \tag{6.44}$$

where $\nu = 2, 3$ for radiation and dust-dominated backgrounds respectively. Note however, that the oscillations in the expansion where not included. The energy density and pressure are given by:

$$\mu = \tfrac{1}{2}(\dot\sigma^2 + (\nabla\sigma)^2)\ \ ,\ \ p = \tfrac{1}{2}(\dot\sigma^2 + \tfrac{1}{3}(\nabla\sigma)^2)\,, \tag{6.45}$$

so that the evolution of h_{ab} is governed by the equation ($K = 0$):

$$(a^{3/2}h_k)\ddot{} + \left(\frac{k^2}{a^2} + \tfrac{3}{8}(\dot\sigma^2 + \tfrac{1}{3}(\nabla\sigma)^2)\right)(a^{3/2}h_k) = 0 \tag{6.46}$$

For super-horizon modes ($kt \ll 1$), the solution scales as:

$$\sigma \rightarrow \frac{\sqrt{A}t^{3/2}}{2^\nu\Gamma(\nu+1)}\sigma(\mathbf{k},0)\,, \tag{6.47}$$

so that the $k = 0$ mode grows, unlike the linear scalar field model considered in the earlier parts of this chapter. Here most of the nonlinearity is on horizon scales. This should be clearly reflected in the cosmic microwave background (CMB) anisotropy spectra of the theories [166]. The $k \neq 0$ modes tend to grow at first according to (6.47) and then begin oscillating with rapidly decaying envelope given by Eq. (6.44). Thus from Eq. (6.46) we expect there initially to be resonance but that the strong time-dependence of the oscillating terms to damp any resonance, particularly at late times.

6.6 Conclusions

The main result of this chapter is that gravitational wave perturbations can be naturally amplified during a second order phase transition. Thus, in addition to the standard creation of a scale-invariant stochastic gravitational wave spectrum due to quantum fluctuations during inflation, there may be amplification of this spectrum during reheating which *breaks* the scale invariance and enhances the *rms* amplitude of the tensor perturbation spectrum with no *a priori* limit on the maximum wavelength affected. However the exact pattern and size of this "symmetry breaking" is highly model dependent, and is hence relevant to inflationary potential reconstruction attempts [54, 205]. This amplification is qualitatively different from gravitational bremsstrahlung [184] since it is due to the coherent oscillation of the *mean* energy density and pressure of the inflaton, and is largely insensitive to the nature of the scalar field fluctuations.

We have further examined chaotic inflation with a quadratic potential in detail and found a duality between the equations describing the growth of χ fluctuations and those of the magnetic part of the Weyl tensor and the shear, the latter being the variable which determines the gravitational wave signal in the CMB; eq (6.27). Finally, the resonant enhancement of the tensor spectrum is completely missing in those models of inflation which involve a continuous production of entropy during inflation, such as in the "warm" inflation models [203].

We have shown that damped parametric resonance is important in understanding gravitational wave evolution during phases where a significant component of the energy density of the universe oscillates, such as during a second order phase transition or if the dark matter lies in an oscillating scalar field.

This parametric resonance amplifies the resident stochastic background, changing the frequency dependence of the spectrum and enhancing the *rms* amplitude. This implies that the possibilities of detecting the stochastic background of gravitational waves may be better than previously thought. In addition there is the intriguing possibility of indirect detection of the axion or moduli via their fingerprints on the gravitational wave spectrum, although this seems unlikely in the near future unless the fields are highly non-minimally coupled.

Finally we have derived a homogeneous second order evolution equation for the modes of E_{ab}, something that previously was thought impossible due to the appearance of a driving term proportional to the shear.

Chapter 7

Duality in the Vacuum Einstein Field Equations

To be interested in the changing seasons is a happier state of mind
than to be hopelessly in love with spring.
– George Santayana

Kingfisher.
There stand humble white tombs.

Kakio Tomizawa

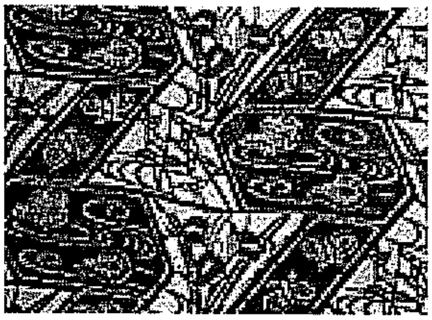


Figure 7.1: A Japanese wallpaper design: one of the simplest of the 17 possible group-theoretic motifs.

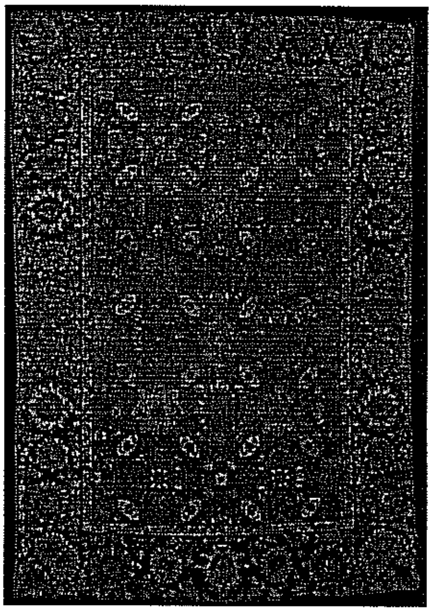


Figure 7.2: A 19th century Indian carpet involving all four symmetry operations on the plane.

7.1 Introduction

In the previous chapter we discussed the evolution of linear gravitational waves in both the covariant and gauge-invariant approach and the Bardeen formalism. What was hopefully apparent in the covariant equations was the amazing similarity of their structure to those for freely propagating electromagnetic radiation.

Indeed, there is a surprisingly rich and detailed correspondence between electromagnetism and General Relativity, uncovered in a series of fundamental papers by Bel [212], Penrose [213] and others [214, 215, 216, 217, 218, 219, 220] (see [220, 221] for more references), and further developed recently (see, e.g., [222, 223, 224, 225, 226, 227, 228, 93, 229]). This correspondence is reflected in the Maxwell-like form of the gravitational field tensor (the Weyl tensor), the super-energy-momentum tensor (the Bel-Robinson tensor) and the dynamical equations (the Bianchi identities). Another form of the correspondence arises in the search for geons (localized, non-singular, topological solutions of Einstein's field equations with mass and angular momentum): in the known (approximate) solutions, the geometry of the electromagnetic geon is identical to that of the gravitational geon [230, 231].

In this chapter we pursue the 'electromagnetic' properties of gravity [1] in a more formal and abstract study. We are particularly interested in areas which have already proved useful for extensions of electromagnetism to non-Abelian gauge theories and string theory. Our emphasis is on a $1+3$ covariant, physically transparent, and non-perturbative approach, with the gravito-electric and gravito-magnetic spatial tensor fields as the fundamental physical variables. Using an improved covariant formalism, including a covariant generalization to spatial tensors of spatial vector algebra and calculus, we show in detailed and transparent form the correspondence between the electric/ magnetic parts of the gravitational field and of the Maxwell field. We identify gravitational source terms, couplings and potentials with and without electromagnetic analogues, thus providing further physical insight into the role of the kinematic quantities shear, vorticity and four-acceleration.

In the vacuum case, we show that the nonlinear (non-perturbative) Bianchi equations for the gravito-electric and gravito-magnetic fields are invariant under covariant spatial duality rotations, in exact analogy with the source-free Maxwell equations for the electric and magnetic fields. The analogy is of course limited by the fact that the Maxwell field propagates on a given spacetime, whereas the gravitational field itself generates the spacetime. The electromagnetic vectors fully characterize a Maxwell solution, and duality maps Maxwell solutions into Maxwell solutions. The gravito-electric/magnetic tensors are not sufficient to characterize covariantly a solution of Einstein's equations – one needs also the kinematic quantities which are subject to the Ricci identities [220]. Duality is an invariance only of the Bianchi identities, and not the Ricci identities, so that it does not map Einstein solutions into Einstein solutions. Nevertheless, the covariant gravito-electric/magnetic duality reveals important properties of the gravitational field.

[1] The work in this chapter was done in collaboration with Roy Maartens and is partially reflected in [210]

The covariant $1+3$ duality has not to our knowledge been given before. Although duality invariance follows implicitly from Penrose's spinor formalism [213, 232], this is in terms of the 4-dimensional Weyl spinor, rather than its 1+3 electric and magnetic tensor parts. Four-dimensional covariant tensor approaches to the electromagnetic analogy have been developed (see e.g. [223]), and non-covariant linearized Maxwell-type equations are well established, both in terms of gravito-electromagnetic vectors (see e.g. [233, 234]) and tensors (see e.g. [219]). In [235], a covariant and nonlinear vector approach is developed for stationary spacetimes. Our approach is fully covariant and non-perturbative, and in addition is centred on the gravito-electromagnetic spatial tensor fields, allowing for a more direct and transparent interpretation based on the Maxwell vector analogy. This approach is a development of the work by Trümper [216], Hawking [218] and Ellis [220], and is related to recent work on a covariant approach to gravitational waves [227, 228, 93, 229] and to local freedom in the gravitational field [229]. A shadow of our general duality result arises in linearized gravitational wave theory, where for vacuum or de Sitter spacetime, there is an interchange symmetry between the gravito-electric and -magnetic tensors [93].

Duality invariance has important implications in field theory in general. It was essentially this symmetry of the Abelian theory, and attempts to extend it to include matter, which led to the Montonen-Olive electromagnetic duality conjecture that there exists a group transformation mapping electric monopoles into magnetic monopoles within the framework of a non-Abelian (specifically $SU(2)$) gauge theory [236]. This conjecture has proved particularly fruitful, stimulating work on S, T and U dualities in string theory (see e.g. [237, 238, 239]), the extension of the electromagnetic duality to magnetically charged black holes and nonlinear electrodynamics [241, 242] and leading to the Seiberg-Witten proof of quark confinement in supersymmetric Yang-Mills theory via monopole condensation [240].

We use the covariant spatial duality to find the gravitational super-energy density and super-Poynting vector as natural group invariants, and derive a new covariant super-energy conservation equation. Finally, we discuss gravito-electric/magnetic monopoles, providing a covariant characterization, in contrast to previous non-covariant treatments [243, 244, 234]. In the linearized case, we show that the Taub-NUT gravito-magnetic monopole given in [243] is related to the Schwarzschild gravito-electric monopole by a spatial duality rotation and an interchange of four-acceleration and vorticity. This provides a covariant form of the relation previously given in non-covariant approaches [18, 245]. It is well-known that the NUT metrics may be obtained from the Schwarzschild metric via the Ehlers-Geroch transformation [18]. This transformation is in fact the generator of T-duality in string theory, but it is not a duality transformation in the sense described here, since it maps Einstein solutions to Einstein solutions and thus necessarily involves kinematic and geometric conditions in addition to a duality rotation. Furthermore, the Ehlers-Geroch transformation requires the existence of a Killing vector field, whereas the general duality that we present does not require any spacetime symmetry.

The relationship between the free gravitational field, and in particular the Bianchi identities, and Maxwell's electromagnetism is strongly evident in the covariant approach to General Relativity .

In this chapter we will present the discovery of a rotation duality in the vacuum Bianchi identities which closely parallels that of electromagnetism. This will allow us to discuss gravitational monopoles from a unified point of view and to understand better to which level General Relativity supports the type of duality which is playing such a prominent rôle in Yang-Mills and string theory at present.

7.2 The streamlined covariant approach

We introduced several notational devices in the previous chapter without providing a detailed background and setting. Here we remedy this.

The covariant approach to General Relativity , based on the choice of a fundamental time-like four velocity, u^a, to perform a 3+1 splitting of spacetime, has been elegantly discussed in several major reviews, see e.g. [220]. Here we present a supplementary review designed to introduce the streamlined covariant formalism. This introduces new notation which neatens up the old formulae and allows the presentation of results in a semi-coordinate-free way.

Among the new developments is the extension of standard vector calculus to second rank tensors (div, curl, commutator) and the derivation of a number of new identities in both the fully nonlinear and linearised field equations.

To elaborate the electromagnetic properties of the free gravitational field in General Relativity, we first present the required covariant formalism, which is based on [246], a streamlined and extended version of the Ehlers-Ellis $1+3$ formalism [220]. Then we give the covariant form of the Maxwell spatial duality in a general curved spacetime. In the following section we extend the treatment to the gravitational field.

Given a congruence of observers with four-velocity field u^a, then $h_{ab} = g_{ab} + u_a u_b$ projects into the local rest spaces, where g_{ab} is the spacetime metric.[2] The spatially projected part of a vector is

$$V_{\langle a\rangle} = h_a{}^b V_b \,,$$

and the spatially projected, symmetric and tracefree part of a rank-2 tensor is

$$A_{\langle ab\rangle} = h_{(a}{}^c h_{b)}{}^d A_{cd} - \tfrac{1}{3} h_{cd} A^{cd} h_{ab} \,.$$

The spatial alternating tensor is

$$\epsilon_{abc} = \eta_{abcd} u^d = \epsilon_{[abc]} \,,$$

[2]We follow the notation and conventions of [220, 246]. (Square) round brackets enclosing indices denote (anti-) symmetrization, while angled brackets denote the spatially projected, symmetric and tracefree part; $a, b, \cdots$ are spacetime indices.

where $\eta_{abcd} = \eta_{[abcd]}$ is the spacetime alternating tensor. Any spatial rank-2 tensor has the covariant irreducible decomposition:

$$A_{ab} = \tfrac{1}{3}h_{cd}A^{cd}h_{ab} + A_{\langle ab\rangle} + \epsilon_{abc}A^c ,$$

where

$$A_a = \tfrac{1}{2}\epsilon_{abc}A^{[bc]}$$

is the vector that is the spatial dual to the skew part. Thus the skew part of a spatial tensor is vectorial, and the irreducibly tensor part is symmetric. In the 1 + 3 covariant approach [220, 229], all physical and geometric variables split into scalars, spatial vectors or spatial tensors that satisfy $A_{ab} = A_{\langle ab\rangle}$. From now on, all rank-2 spatial tensors will be assumed to satisfy this condition.

The covariant spatial vector product is

$$[V, W]_a = \epsilon_{abc}V^bW^c ,$$

and the covariant generalization to spatial tensors is

$$[A, B]_a = \epsilon_{abc}A^b{}_dB^{cd} ,$$

which is the vector that is spatially dual to the covariant tensor commutator.

The covariant time derivative is

$$\dot{A}^{a\cdots}{}_{b\cdots} = u^c\nabla_cA^{a\cdots}{}_{b\cdots} ,$$

and the covariant spatial derivative is

$$\mathrm{D}_aA^{b\cdots}{}_{c\cdots} = h_a{}^ph^b{}_q\cdots h_c{}^r\cdots\nabla_pA^{q\cdots}{}_{r\cdots} .$$

Then the covariant spatial divergence and curl of vectors and rank-2 tensors are defined by [246, 229]:

$$\operatorname{div}V = \mathrm{D}^aV_a , \qquad \operatorname{curl}V_a = \epsilon_{abc}\mathrm{D}^bV^c , \tag{7.1}$$

$$(\operatorname{div}A)_a = \mathrm{D}^bA_{ab} , \qquad \operatorname{curl}A_{ab} = \epsilon_{cd(a}\mathrm{D}^cA_{b)}{}^d , \tag{7.2}$$

where $\operatorname{curl}A_{ab}$ is tracefree if $A_{ab} = A_{(ab)}$. The tensor curl and divergence are related by

$$\epsilon_{abc}\mathrm{D}^bA_d{}^c = \operatorname{curl}A_{ad} + \tfrac{1}{2}\epsilon_{adc}\mathrm{D}_bA^{bc} .$$

The kinematics of the u^a-congruence are described by the expansion $\Theta = \mathrm{D}^au_a$, the shear $\sigma_{ab} = \mathrm{D}_{\langle a}u_{b\rangle}$, the vorticity $\omega_a = -\frac{1}{2}\operatorname{curl}u_a$, and the four-acceleration $\dot{u}_a = \dot{u}_{\langle a\rangle}$.

The above operators obey the covariant identities

$$(\mathrm{D}_af)^{\cdot} = \mathrm{D}_a\dot{f} - \tfrac{1}{3}\Theta\mathrm{D}_af + \dot{u}_a\dot{f} - \sigma_a{}^b\mathrm{D}_bf - [\omega, \mathrm{D}f]_a + u_a\dot{u}^b\mathrm{D}_bf , \tag{7.3}$$

$$\operatorname{curl}\mathrm{D}_af = -2\dot{f}\omega_a , \tag{7.4}$$

$$\mathrm{D}^a[V, W]_a = W^a\operatorname{curl}V_a - V^a\operatorname{curl}W_a , \tag{7.5}$$

$$\mathrm{D}^a[A, B]_a = B^{ab}\operatorname{curl}A_{ab} - A^{ab}\operatorname{curl}B_{ab} , \tag{7.6}$$

together with far more complicated identities [246, 229]. In the case where spacetime is almost spatially isotropic and homogeneous, i.e. a linearized perturbation of a Friedmann-Lemaitre-Robertson-Walker (FLRW) background, some of the main further identities take the linearized form [21, 247]

$$
\begin{aligned}
(\mathrm{D}^a V_a)^{\cdot} &\approx \mathrm{D}^a \dot{V}_a - H\mathrm{D}^a V_a\,, &(7.7)\\
\left(\mathrm{D}^b A_{ab}\right)^{\cdot} &\approx \mathrm{D}^b \dot{A}_{ab} - H\mathrm{D}^b A_{ab}\,, &(7.8)\\
(\mathrm{curl}\, V_a)^{\cdot} &\approx \mathrm{curl}\, \dot{V}_a - H\mathrm{curl}\, V_a\,, &(7.9)\\
(\mathrm{curl}\, A_{ab})^{\cdot} &\approx \mathrm{curl}\, \dot{A}_{ab} - H\mathrm{curl}\, A_{ab}\,, &(7.10)\\
\mathrm{D}^a \mathrm{curl}\, V_a &\approx 0\,, &(7.11)\\
\mathrm{D}^b \mathrm{curl}\, A_{ab} &\approx \tfrac{1}{2}\mathrm{curl}\left(\mathrm{D}^b A_{ab}\right)\,, &(7.12)\\
\mathrm{curl}\,\mathrm{curl}\, V_a &\approx -\mathrm{D}^2 V_a + \mathrm{D}_a\left(\mathrm{D}^b V_b\right) + \tfrac{2}{3}\left(\rho - 3H^2\right) V_a\,, &(7.13)\\
\mathrm{curl}\,\mathrm{curl}\, A_{ab} &\approx -\mathrm{D}^2 A_{ab} + \tfrac{3}{2}\mathrm{D}_{\langle a}\mathrm{D}^c A_{b\rangle c} + \left(\rho - 3H^2\right) A_{ab}\,, &(7.14)
\end{aligned}
$$

where H is the background Hubble rate, ρ is the background energy density and $\mathrm{D}^2 = \mathrm{D}^a\mathrm{D}_a$ is the covariant Laplacian.

7.3 The free gravitational field

In this section we will review the vacuum Einstein field equations. In vacuum, $T_{\mu\nu} = 0$ and hence we are restricted to so-called Einstein-manifolds, where:

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \tag{7.15}$$

and Λ corresponds as usual, to the cosmological constant. In the case $\Lambda \neq 0$, the Riemann tensor is not fully determined by the Weyl tensor, C_{abcd} since the Ricci tensor is non-zero. In this chapter, however, we will assume that $\Lambda = 0$ and hence that spacetime is Ricci-flat.

The governing equations then become (vacuum, $\Lambda = 0$):

$$
\begin{aligned}
R_{ab} &= 0 && \text{The field equations}\\
\nabla_{[a}\nabla_{b]}u_c &= 2C_{abcd}u^d && \text{The Ricci identities}\\
\nabla^d C_{abcd} &= 0 && \text{The Bianchi Identities} \qquad (7.16)
\end{aligned}
$$

where the Bianchi identities have been given with the Riemann tensor decomposed into Weyl and Ricci parts. We now develop a covariant version of electrodynamics in preparation for the gravitational case and as a way of further exhibiting the remarkable similarity of electromagnetism and General Relativity.

7.3.1 Covariant Electromagnetism

The electric and magnetic fields measured by u^a observers are defined via the Maxwell tensor F_{ab} by

$$E_a = F_{ab}u^b = E_{\langle a\rangle}\,,\ H_a = \tfrac{1}{2}\epsilon_{abc}F^{bc} \equiv {}^*F_{ab}u^b = H_{\langle a\rangle}\,, \tag{7.17}$$

where * denotes the dual. These spatial physically measurable vectors are equivalent to the spacetime Maxwell tensor, since

$$F_{ab} = 2u_{[a}E_{b]} + \epsilon_{abc}H^c\,. \tag{7.18}$$

Maxwell's equations $\nabla_{[a}F_{bc]} = 0$ and $\nabla^b F_{ab} = J_a$ are given in $1+3$ covariant form for E_a and H_a by Ellis [248]. In the streamlined formalism, these equations take the simplified form

$$\mathrm{D}^a E_a = -2\omega^a H_a + \varrho\,, \tag{7.19}$$

$$\mathrm{D}^a H_a = 2\omega^a E_a\,, \tag{7.20}$$

$$\dot{E}_{\langle a\rangle} - \mathrm{curl}\, H_a = -\tfrac{2}{3}\Theta E_a + \sigma_{ab}E^b - [\omega, E]_a + [\dot{u}, H]_a - j_a\,, \tag{7.21}$$

$$\dot{H}_{\langle a\rangle} + \mathrm{curl}\, E_a = -\tfrac{2}{3}\Theta H_a + \sigma_{ab}H^b - [\omega, H]_a - [\dot{u}, E]_a\,, \tag{7.22}$$

where $\varrho = -J_a u^a$ is the electric charge density and $j_a = J_{\langle a\rangle}$ is the electric current. In flat spacetime, relative to an inertial congruence ($\Theta = \dot{u}_a = \omega_a = \sigma_{ab} = 0$), these equations take their familiar non-covariant form.

Introducing the complex electromagnetic spatial vector field $\mathcal{I}_a = E_a + iH_a$, we see that in the source-free case ($J_a = 0$) Maxwell's equations become

$$\mathrm{D}^a \mathcal{I}_a = 2i\,\omega^a \mathcal{I}_a\,, \tag{7.23}$$

$$\dot{\mathcal{I}}_{\langle a\rangle} + i\,\mathrm{curl}\,\mathcal{I}_a = -\tfrac{2}{3}\Theta\mathcal{I}_a + \sigma_{ab}\mathcal{I}^b - [\omega, \mathcal{I}]_a - i\,[\dot{u}, \mathcal{I}]_a\,. \tag{7.24}$$

It follows that the source-free Maxwell equations in an arbitrary curved spacetime, relative to an arbitrary congruence of observers, are invariant under the covariant global spatial duality rotation $\mathcal{I}_a \to e^{i\phi}\mathcal{I}_a$, where ϕ is constant. The energy density and Poynting vector

$$U = \tfrac{1}{2}\mathcal{I}^a\overline{\mathcal{I}}_a = \tfrac{1}{2}(E_aE^a + H_aH^a)\,, \tag{7.25}$$

$$P_a = \frac{1}{2i}[\overline{\mathcal{I}}, \mathcal{I}]_a = [E, H]_a\,, \tag{7.26}$$

are natural group invariants. Their invariance also follows from the duality invariance of the energy-momentum tensor [232, 248]

$$M_a{}^b = \tfrac{1}{2}\left(F_{ac}F^{bc} + {}^*F_{ac}{}^*F^{bc}\right)\,, \tag{7.27}$$

since $U = M_{ab}u^a u^b$ and $P_a = -M_{\langle a\rangle b}u^b$. Using the identity (7.5), and the propagation equations (7.21) and (7.22), we find a covariant energy conservation equation:

$$\dot{U} + \mathrm{D}^a P_a = -\tfrac{4}{3}\Theta U - 2\dot{u}^a P_a + \sigma_{ab}\left(E^aE^b + H^aH^b\right)\,. \tag{7.28}$$

This reduces in flat spacetime for inertial observers to the well-known form $\partial_t U + \operatorname{div} \vec{P} = 0$.

A further natural group invariant is

$$\pi_{ab} = -\mathcal{I}_{\langle a}\overline{\mathcal{I}}_{b\rangle} = -E_{\langle a}E_{b\rangle} - H_{\langle a}H_{b\rangle}\,, \tag{7.29}$$

which is just the anisotropic electromagnetic pressure [248]. It occurs in the last term of the conservation equation (7.28), i.e. $-\sigma_{ab}\pi^{ab}$.

For later comparison with the gravitational case, we conclude this section by considering the propagation of source-free electromagnetic waves on an FLRW background, assuming that $E_a = 0 = H_a$ in the background. We linearize and take the curl of equation (7.21), evaluating curl curl H_a by the identity (7.13) and equation (7.20). We eliminate curl $\dot{E}_a$ by linearizing equation (7.22), taking its time derivative, and using identity (7.9). The result is the wave equation

$$\Box^2 H_a \equiv -\ddot{H}_a + \mathrm{D}^2 H_a \approx 5H\dot{H}_a + \left(2H^2 + \tfrac{1}{3}\rho - p\right) H_a\,, \tag{7.30}$$

where p is the background pressure, and we used the FLRW field equation $3\dot{H} = -3H^2 - \frac{1}{2}(\rho + 3p)$. A similar wave equation may be derived for E_a.

7.4 The Bianchi identities and nonlinear duality

The Maxwell analogy in General Relativity is based on the the correspondence $C_{abcd} \leftrightarrow F_{ab}$, where the Weyl tensor C_{abcd} is the free gravitational field (see [229]). For a given u^a, it splits irreducibly and covariantly into

$$E_{ab} = C_{acbd}u^c u^d = E_{(ab)}\,,\ H_{ab} = {}^*C_{acbd}u^c u^d = H_{(ab)}\,, \tag{7.31}$$

as we described in the chapter (6). These gravito-electric/magnetic spatial tensors are in principle physically measurable in the frames of comoving observers, and together they are equivalent to the spacetime Weyl tensor, since [246]

$$C_{ab}{}^{cd} = 4\left\{u_{[a}u^{[c} + h_{[a}{}^{[c}\right\} E_{b]}{}^{d]} + 2\epsilon_{abe}u^{[c}H^{d]e} + 2u_{[a}H_{b]e}\epsilon^{cde}\,. \tag{7.32}$$

This is the gravito-electromagnetic version of the expression (7.18). The electromagnetic interpretation of E_{ab} and H_{ab} is reinforced by the fact that these fields covariantly (and gauge-invariantly) describe gravitational waves on an FLRW background (including the special case of a flat vacuum background) [218, 21].

In the $3+1$ covariant approach to General Relativity [220], the fundamental quantities are not the metric (which in itself does not provide a covariant description), but the kinematic quantities of the fluid, its energy density ρ and pressure p, and the gravito-electric/magnetic tensors. The fundamental equations governing these quantities are the Bianchi identities and the Ricci identities for u^a, with

Einstein's equations incorporated via the algebraic definition of the Ricci tensor R_{ab} in terms of the energy-momentum tensor T_{ab}. We assume that the source of the gravitational field is a perfect fluid (the generalization to imperfect fluids is straightforward). The Bianchi identities are

$$\nabla^d C_{abcd} = \nabla_{[a}\left(-R_{b]c} + \tfrac{1}{6}Rg_{b]c}\right) , \tag{7.33}$$

where $R = R_a{}^a$ and $R_{ab} = T_{ab} - \frac{1}{2}T_c{}^c g_{ab}$. The contraction of (7.33) implies the conservation equations. The tracefree part of (7.33) gives the gravitational equivalents of the Maxwell equations (7.19)–(7.22), via a covariant $1+3$ decomposition [216, 220]. In our notation, these take the simplified form:

$$\mathrm{D}^b E_{ab} = -3\omega^b H_{ab} + \tfrac{1}{3}\mathrm{D}_a\rho + [\sigma, H]_a , \tag{7.34}$$

$$\mathrm{D}^b H_{ab} = 3\omega^b E_{ab} + (\rho+p)\omega_a - [\sigma, E]_a , \tag{7.35}$$

$$\begin{aligned}\dot{E}_{\langle ab\rangle} - \mathrm{curl}\, H_{ab} &= -\Theta E_{ab} + 3\sigma_{c\langle a}E_{b\rangle}{}^c - \omega^c \epsilon_{cd(a}E_{b)}{}^d \\ &\quad +2\dot{u}^c \epsilon_{cd(a}H_{b)}{}^d - \tfrac{1}{2}(\rho+p)\sigma_{ab} ,\end{aligned} \tag{7.36}$$

$$\begin{aligned}\dot{H}_{\langle ab\rangle} + \mathrm{curl}\, E_{ab} &= -\Theta H_{ab} + 3\sigma_{c\langle a}H_{b\rangle}{}^c - \omega^c \epsilon_{cd(a}H_{b)}{}^d \\ &\quad -2\dot{u}^c \epsilon_{cd(a}E_{b)}{}^d .\end{aligned} \tag{7.37}$$

These are the fully nonlinear equations in covariant form, and the analogy with the Maxwell equations (7.19)–(7.22) is made strikingly apparent in our formalism.

Vorticity couples to the fields to produce source terms in both cases, but gravity has additional sources from a *tensor coupling of the shear* to the field. The analogue of the charge density ϱ as a source for the electric field, is the energy density spatial gradient $\mathrm{D}_a\rho$ as a source for the gravito-electric field. Since $\mathrm{D}_a\rho$ covariantly describes *inhomogeneity* in the fluid, this is consistent with the fact that the gravito-electric field is the generalization of the Newtonian tidal tensor [220].

There is no magnetic charge source for H_a, but the gravito-magnetic field H_{ab} has the source $(\rho+p)\omega_a$. Since $\rho+p$ is the relativistic inertial mass-energy density [220], $(\rho+p)\omega_a$ is the *'angular momentum density'*, which we identify as a gravito-magnetic 'charge' density. Note however that angular momentum density does not always generate a gravito-magnetic field. The Gödel solution [220] provides a counter-example, where $H_{ab} = 0$ and the non-zero angular momentum density is exactly balanced by the vorticity/ gravito-electric coupling in equation (7.35), with $\sigma_{ab} = 0$.

For both electromagnetism and gravity, the propagation of the fields is determined by the spatial curls, together with a coupling of the expansion, shear, vorticity, and acceleration to the fields. The analogue of the electric current j_a is the gravito-electric 'current' $(\rho+p)\sigma_{ab}$, which is the *'density of the rate-of-distortion energy'* of the fluid. There is no magnetic current in either case.

If the Maxwell field is source-free, i.e. $\varrho = 0 = j_a$, and the gravitational field is source-free, i.e. $\rho = 0 = p$, then the similarity of the two sets of equations is even more apparent, and only the tensor shear coupling in the case of gravity lacks a direct electromagnetic analogue. (Note that these shear coupling terms govern the possibility of simultaneous diagonalization of the shear and E_{ab}, H_{ab} in tetrad formulations of general relativity [249, 250].)

To obtain the gravitational analogue of the complex equations (7.23) and (7.24), which lead to the Maxwell duality invariance, we consider the vacuum case $\rho = 0 = p$. In general, u^a is no longer uniquely defined in vacuum, although in particular cases (such as stationary spacetimes), there may be a physically unique choice. However, our results hold for an *arbitrary* covariant choice of u^a, without any special conditions on the congruence. By analogy with the complex electromagnetic spatial vector $\mathcal{I}_a$, we define the complex gravito-electromagnetic spatial tensor

$$\mathcal{I}_{ab} = E_{ab} + i\, H_{ab} \,. \tag{7.38}$$

Then equations (7.34)–(7.37) reduce to:

$$\begin{aligned} \mathrm{D}^b \mathcal{I}_{ab} &= 3i\,\omega^b \mathcal{I}_{ab} - i\,[\sigma, \mathcal{I}]_a \,, && (7.39)\\ \dot{\mathcal{I}}_{\langle ab \rangle} + i \,\mathrm{curl}\, \mathcal{I}_{ab} &= -\Theta \mathcal{I}_{ab} + 3\sigma_{c\langle a} \mathcal{I}_{b\rangle}{}^c \\ &\quad -\omega^c \epsilon_{cd(a} \mathcal{I}_{b)}{}^d - 2i\,\dot{u}^c \epsilon_{cd(a} \mathcal{I}_{b)}{}^d \,. && (7.40) \end{aligned}$$

Apart from the increased economy, the system is now clearly seen to be invariant under the global $U(1)$ transformation:

$$\mathcal{I}_{ab} \;\rightarrow\; e^{i\phi} \mathcal{I}_{ab} \,, \tag{7.41}$$

which is precisely the tensor (spin-2) version of the vector symmetry of the source-free Maxwell equations. We have thus established the existence of the covariant spatial duality at the level of the physically relevant gravito-electric/magnetic fields, in the general (non-perturbative, arbitrary observer congruence) vacuum case. (As with electromagnetism, duality invariance breaks down in the presence of sources.)

A covariant super-energy density and super-Poynting vector arise naturally as invariants under spatial duality rotation, in direct analogy with the Maxwell invariants of equations (7.25) and (7.26):

$$\begin{aligned} U &= \tfrac{1}{2} \mathcal{I}^{ab} \overline{\mathcal{I}}_{ab} = \tfrac{1}{2} \left(E_{ab} E^{ab} + H_{ab} H^{ab} \right) \,, && (7.42)\\ P_a &= \frac{1}{2i} [\overline{\mathcal{I}}, \mathcal{I}]_a = [E, H]_a \equiv \epsilon_{abc} E^b{}_d H^{cd} \,. && (7.43) \end{aligned}$$

This reflects the duality invariance of the Bel-Robinson tensor [212]

$$M_{ab}{}^{cd} = \tfrac{1}{2} \left(C_{aebf} C^{cedf} + {}^*C_{aebf}\, {}^*C^{cedf} \right) \,, \tag{7.44}$$

which is the natural covariant definition of the super-energy-momentum tensor for the free gravitational field, since [212, 221]

$$\begin{aligned} U &= M_{abcd} u^a u^b u^c u^d \,, && (7.45)\\ P_a &= -M_{\langle a \rangle bcd} u^b u^c u^d \,. && (7.46) \end{aligned}$$

The agreement between equations (7.45) and (7.42) follows obviously from equation (7.44) on using equation (7.31). However, it is not obvious that equation (7.46) agrees with our equation (7.43) for the super-Poynting vector, and one requires the identity (7.32) to show the agreement.

Our expression (7.42) for the gravitational super-energy density gives a direct and clear analogy with the electromagnetic energy density (7.25). Our expression (7.43) for the gravitational super-Poynting vector, in terms of the tensor generalization of the vector product, provides a clearer analogy with the electromagnetic Poynting vector.(7.26). The analogy is reinforced by the fact that U and P_a obey a super-energy conservation equation which is the tensor version of the electromagnetic energy conservation equation (7.28). To show this, we need the new covariant identity (7.6). Using this and the Bianchi propagation equations (7.36) and (7.37), we find that

$$\dot{U} + \mathrm{D}^a P_a = -2\Theta U - 4\dot{u}^a P_a + 3\sigma^c{}_{(a}\left[E_{b)c}E^{ab} + H_{b)c}H^{ab}\right] . \tag{7.47}$$

This is the non-perturbative and covariant generalization of Bel's linearized conservation equation [212, 221]: $\partial_t U = -\mathrm{div}\,\vec{P}$.

The last term in the conservation equation (7.47) contains another natural group invariant

$$\pi_{ab} = -\mathcal{I}_{c\langle a}\overline{\mathcal{I}}_{b\rangle}{}^c = -E_{c\langle a}E_{b\rangle}{}^c - H_{c\langle a}H_{b\rangle}{}^c , \tag{7.48}$$

which we interpret as the anisotropic super-pressure of the gravito-electromagnetic field.

As pointed out in the introduction, duality rotations preserve the Bianchi identities in vacuum, but not the Ricci identities for u^a. This is clearly apparent from the spatial tensor parts of the Ricci identities [220], which in our formalism have the simplified form

$$E_{ab} = \mathrm{D}_{\langle a}\dot{u}_{b\rangle} - \dot{\sigma}_{\langle ab\rangle} - \tfrac{2}{3}\Theta\sigma_{ab} - \sigma_{c\langle a}\sigma_{b\rangle}{}^c - \omega_{\langle a}\omega_{b\rangle} + \dot{u}_{\langle a}\dot{u}_{b\rangle} , \tag{7.49}$$

$$H_{ab} = \mathrm{curl}\,\sigma_{ab} + \mathrm{D}_{\langle a}\omega_{b\rangle} + 2\dot{u}_{\langle a}\omega_{b\rangle} . \tag{7.50}$$

In order to preserve the Ricci identities, and map Einstein solutions to Einstein solutions, one needs to perform kinematic transformations in addition to the duality rotation. An example is presented in the following section.

The electromagnetic analogy suggests a further interesting interpretation of the kinematic quantities arising from the Ricci equations (7.49) and (7.50).[3] In flat spacetime, relative to inertial observers, the electric and magnetic vectors may be written as

$$\vec{E} = \vec{\nabla}V - \partial_t\vec{\alpha} , \ \vec{H} = \mathrm{curl}\,\vec{\alpha} ,$$

where V is the electric scalar potential and $\vec{\alpha}$ is the magnetic vector potential.[4] Comparing now with the Ricci equations (7.49) and (7.50), we see that the four-acceleration is a covariant gravito-electric vector potential and the shear is a covariant gravito-magnetic tensor potential. The vorticity derivative in (7.50) has no electromagnetic analogue, and vorticity appears to be an additional gravito-magnetic vector potential. Furthermore, the gauge freedom in the electromagnetic potentials does not have a direct gravitational analogue in the Ricci gravito-potential equations (7.49) and (7.50), since

[3]Note that these Ricci equations have the same form in the non-vacuum case.

[4]The covariant form of these potentials is $V = u^a A_a$, $\alpha_a = A_{\langle a\rangle}$, where A_a is the four-potential.

the gravito-electric/magnetic potentials are invariantly defined kinematic quantities. (Note that the Lanczos potential for the Weyl tensor does have a gauge freedom analogous to that in the Maxwell four-potential [223].)

The remaining Ricci equations in $1+3$ covariant form are [229]

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 = -\tfrac{1}{2}(\rho + 3p) + \mathrm{D}^a\dot{u}_a + \dot{u}^a\dot{u}_a + 2\omega^a\omega_a - \sigma^{ab}\sigma_{ab}\,, \tag{7.51}$$

$$\dot{\omega}_{\langle a\rangle} + \tfrac{2}{3}\Theta\omega_a = -\tfrac{1}{2}\mathrm{curl}\,\dot{u}_a + \sigma_{ab}\omega^b\,, \tag{7.52}$$

$$\tfrac{2}{3}\mathrm{D}_a\Theta = -\mathrm{curl}\,\omega_a + \mathrm{D}^b\sigma_{ab} + 2[\omega,\dot{u}]_a\,, \tag{7.53}$$

$$\mathrm{D}^a\omega_a = \dot{u}^a\omega_a\,, \tag{7.54}$$

and do not involve the gravito-electromagnetic field.

Finally in this section, we extend the analogy to wave propagation. The magnetic wave equation (7.30) has a simple gravito-magnetic analogue. In order to isolate the purely tensor perturbations of an FLRW background in a covariant (and gauge-invariant) way, one imposes $\omega_a = 0$ [228]. We linearize and take the curl of equation (7.36), using the linearisations of equations (7.37) and (7.35), and identities (7.10) and (7.14). This does not directly produce a wave equation, since the curl of the shear term in (7.36) has to be eliminated. (In the Maxwell case this feature did not arise, since we set $j_a = 0$.) The elimination is achieved via the Ricci equation (7.50), and we find that

$$\Box^2 H_{ab} \equiv -\ddot{H}_{ab} + \mathrm{D}^2 H_{ab} \approx 7H\dot{H}_{ab} + 2\left(3H^2 - p\right)H_{ab}\,, \tag{7.55}$$

in agreement with [228, 93], and in striking analogy with the magnetic wave equation (7.30). Further discussion of covariant gravitational wave theory may be found in [228, 93, 229, 251].

7.4.1 topological invariants

A further covariant quantity that may be naturally constructed from Eq. (7.38) is

$$\begin{aligned} \mathcal{I} \equiv \mathcal{I}_{ab}\mathcal{I}^{ab} &= (E_{ab}E^{ab} - H_{ab}H^{ab}) + 2i\,E_{ab}H^{ab} \\ &= \tfrac{1}{8}(C_{abcd}C^{abcd} + iC_{abcd}{}^{*}C^{abcd})\,, \end{aligned} \tag{7.56}$$

which is *not* invariant under Eq. (7.41). It vanishes in Petrov type-III and type-N spacetimes [224] (supporting the existence of gravitational waves in these spacetimes,since the analogous quantities vanish for purely radiativeelectromagnetic fields). Further, the electromagnetic analogue of the real part of Eq. (7.56), namely $E_aE^a - H_aH^a = -\frac{1}{2}F_{ab}F^{ab}$, is just the Lagrangian density. The analogue of the imaginary part is $E_aH^a = \frac{1}{4}F_{ab}\,{}^{*}F^{ab}$ whose integral in non-Abelian gauge theories is proportional to the topological instanton number.

Indeed, the imaginary part is just proportional to the Pontryargin density if the spacetime is compact. The four dimensional Pontryargin class is defined by [211]:

$$P_4 = \frac{1}{8\pi^2}\int_{M_4} R^{abcd}\,{}^{*}R_{abcd}d^4x \tag{7.57}$$

For a vacuum spacetime this is just:

$$P_4 = \frac{2}{\pi^2}\int_{M_4} E^{ab}H_{ab}\sqrt{-g}d^4x \tag{7.58}$$

where $g = det(g_{ab})$. This takes on integer values that label topologically distinct four-geometries. Further, within vacuum spacetimes, it is the Pontryargin density which determines the chiral anomaly:

$$D_\mu J^{5\mu} = \frac{2}{\pi^2}E^{ab}H_{ab} \tag{7.59}$$

As discussed earlier, $E_{ab}H^{ab}$ vanishes in Petrov type-III and N spacetimes and hence so does P_4 and the chiral anomaly.

In addition there is another topological invariant in four dimensions, the Euler characteristic, χ, which is related to the curvature via the four dimensional Gauss-Bonnet theorem:

$$\chi = \int (R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2)\sqrt{-g}d^4x \tag{7.60}$$

where χ takes on integral values. Now using the fact that:

$$C_{abcd}C^{abcd} = R_{abcd}R^{abcd} - 2R_{ab}R^{ab} + \frac{1}{3}R^2 \tag{7.61}$$

we may rewrite Eq. (7.60), using Eq. (7.56) as:

$$\chi = \int (8E_{ab}E^{ab} - 8H_{ab}H^{ab} - 2R_{ab}R^{ab} + \frac{2}{3}R^2)\sqrt{-g}d^4x \tag{7.62}$$

From which we see that if the spacetime is vacuum, and type-III or type N, the Euler characteristic is also zero.

7.4.2 local transformations

A natural generalisation of the global transformation is to consider the duality rotation $\mathcal{I}_{ab} \to e^{i\phi}\mathcal{I}_{ab}$ but with ϕ a spacetime dependent, local transformation. The hope might be that the extra terms that would be generated would take the form of some matter fields.

The Bianchi identities are then transformed into:

$$\mathrm{D}^b\mathcal{I}_{ab} = 3i\,\omega^b\mathcal{I}_{ab} - i\,[\sigma,\mathcal{I}]_a - i\mathrm{D}^b\phi\mathcal{I}_{ab}\,, \tag{7.63}$$

$$\begin{aligned}\dot{\mathcal{I}}_{\langle ab\rangle} + i\,\mathrm{curl}\,\mathcal{I}_{ab} &= -\Theta\mathcal{I}_{ab} + 3\sigma_{c(a}\mathcal{I}_{b)}{}^c \\ &- \omega^c\epsilon_{cd(a}\mathcal{I}_{b)}{}^d - 2i\,\dot{u}^c\epsilon_{cd(a}\mathcal{I}_{b)}{}^d - i\dot{\phi}\mathcal{I}_{ab} \\ &+ \epsilon_{ac(d}\mathrm{D}^c\phi\mathcal{I}_{b)}{}^d\,.\end{aligned} \tag{7.64}$$

Thus the extra terms proportional to $\mathrm{D}^c\phi$ and $\dot{\phi}$ couple directly to $\mathcal{I}_{ab}$ and hence cannot be written as the effect of extra matter terms since in the Bianchi identities the stress tensor does not couple to $\mathcal{I}_{ab}$.

7.5 Gravitational monopoles

The electromagnetic correspondence we have developed suggests a covariant characterization of gravito-electric (magnetic) *monopoles,* as stationary vacuum spacetimes outside isolated sources, with purely electric (magnetic) free gravitational field, i.e., $H_{ab} = 0$ ($E_{ab} = 0$). This is reinforced by the fact that monopoles do not radiate, and gravitational radiation necessarily involves both E_{ab} and H_{ab} nonzero (see [227, 93, 229], consistent with Bel's criterion $P_a \neq 0$ [212, 221]). Our identification in the previous section of density inhomogeneity and angular momentum density as sources of, respectively, gravito-electric and gravito-magnetic fields, suggests that the monopole sources will be respectively mass and angular momentum. However, as pointed out previously, it is possible that non-zero angular momentum is compatible with a purely gravito-electric field, as illustrated by the Gödel solution.

The four-velocity field u^a is not defined by a fluid, but is defined as the normalization of the stationary Killing vector field $\xi^a = \xi u^a$. As a consequence of Killing's equations, we have $\Theta = 0 = \sigma_{ab}$ [252], so that

$$\nabla_b u_a = \epsilon_{abc}\omega^c - \dot{u}_a u_b\,.$$

The covariant equations governing non-perturbative monopoles are complicated. Some simplification arises from the Killing symmetry, which implies

$$\begin{aligned} \mathcal{L}_\xi \omega_a &= \xi\dot{\omega}_a + u_a\omega^b \mathrm{D}_b\xi = 0\,, \\ \mathcal{L}_\xi H_{ab} &= \xi\dot{H}_{ab} + 2\xi\omega^c\epsilon_{cd(a}H_{b)}{}^d - 2\xi u_{(a}H_{b)c}\dot{u}^c = 0\,, \end{aligned}$$

and a similar equation for E_{ab}. Then it follows that

$$\dot{\omega}_{\langle a\rangle} = 0\,, \tag{7.65}$$

$$\dot{H}_{\langle ab\rangle} = -2\omega^c\epsilon_{cd(a}H_{b)}{}^d\,, \tag{7.66}$$

$$\dot{E}_{\langle ab\rangle} = -2\omega^c\epsilon_{cd(a}E_{b)}{}^d\,. \tag{7.67}$$

Now equations (7.65)–(7.67), together with the basic monopole conditions, are applied to the Bianchi equations (7.34)–(7.37) and Ricci equations (7.49)–(7.54). We obtain:

Gravito-electric and -magnetic monopoles:

$$\mathrm{D}^a\dot{u}_a = -\dot{u}^a\dot{u}_a - 2\omega^a\omega_a\,, \tag{7.68}$$

$$\dot{\omega}_{\langle a\rangle} = 0\,, \tag{7.69}$$

$$\mathrm{curl}\,\dot{u}_a = 0\,, \tag{7.70}$$

$$\mathrm{curl}\,\omega_a = -2[\dot{u},\omega]_a\,, \tag{7.71}$$

$$\mathrm{D}^a\omega_a = \dot{u}^a\omega_a\,. \tag{7.72}$$

Gravito-electric monopole:

$$\begin{aligned}
\mathrm{D}^b E_{ab} &= 0\,, &(7.73)\\
0 &= E_{ab}\omega^b\,, &(7.74)\\
\dot{E}_{\langle ab\rangle} &= 0\,, &(7.75)\\
0 &= \omega^c\epsilon_{cd(a}E_{b)}{}^d\,, &(7.76)\\
\mathrm{curl}\,E_{ab} &= -2\dot{u}^c\epsilon_{cd(a}E_{b)}{}^d\,, &(7.77)\\
E_{ab}-\mathrm{D}_{\langle a}\dot{u}_{b\rangle} &= \dot{u}_{\langle a}\dot{u}_{b\rangle}-\omega_{\langle a}\omega_{b\rangle}\,, &(7.78)\\
\mathrm{D}_{\langle a}\omega_{b\rangle} &= -2\dot{u}_{\langle a}\omega_{b\rangle}\,. &(7.79)
\end{aligned}$$

Gravito-magnetic monopole:

$$\begin{aligned}
\mathrm{D}^b H_{ab} &= 0\,, &(7.80)\\
0 &= H_{ab}\omega^b\,, &(7.81)\\
\dot{H}_{\langle ab\rangle} &= 0\,, &(7.82)\\
0 &= \omega^c\epsilon_{cd(a}H_{b)}{}^d\,, &(7.83)\\
\mathrm{curl}\,H_{ab} &= -2\dot{u}^c\epsilon_{cd(a}H_{b)}{}^d\,, &(7.84)\\
\mathrm{D}_{\langle a}\dot{u}_{b\rangle} &= -\dot{u}_{\langle a}\dot{u}_{b\rangle}+\omega_{\langle a}\omega_{b\rangle}\,, &(7.85)\\
H_{ab}-\mathrm{D}_{\langle a}\omega_{b\rangle} &= 2\dot{u}_{\langle a}\omega_{b\rangle}\,. &(7.86)
\end{aligned}$$

Equation (7.70) implies that there exists an acceleration potential:

$$\dot{u}_a = \mathrm{D}_a\Phi\,. \tag{7.87}$$

This holds even when $\omega_a \neq 0$, despite the identity (7.4), since Φ is invariant under ξ^a, so that $\dot{\Phi}=0$. Equation (7.71) shows that $\mathrm{curl}\,\omega_a$ is orthogonal to the vorticity and four-acceleration:

$$\omega^a\mathrm{curl}\,\omega_a = 0 = \dot{u}^a\mathrm{curl}\,\omega_a\,.$$

Schwarzschild spacetime, where also $\omega_a = 0$ (since staticity implies u^a is hyper-surface orthogonal), is clearly a non-perturbative gravito-electric monopole according to our covariant definition: it is a static vacuum spacetime satisfying $H_{ab}=0$, by virtue of the Ricci equation (7.50). Equations (7.87) and (7.68) imply

$$\mathrm{D}^2\Phi + \mathrm{D}^a\Phi\mathrm{D}_a\Phi = 0\,. \tag{7.88}$$

The solution Φ determines $\dot{u}_a$ and E_{ab}, and equation (7.88) ensures that the monopole conditions (7.68)–(7.79) are identically satisfied.

It is not clear whether there exist consistent non-perturbative gravito-magnetic monopoles, i.e. spacetimes satisfying the covariant equations (7.68)–(7.72) and (7.80)–(7.86).[5] However, linearized

[5]In [253] it is shown that non-flat vacuum solutions with purely magnetic Weyl tensor are a very restricted class, and it is suggested that there may be no such solutions.

gravito-magnetic monopoles have been found, for example the Demianski-Newman solution [243] (see below). It is also not clear whether there exist gravito-electric monopoles with angular momentum (i.e. $\omega_a \neq 0$).

In the case of linearisation about a flat Minkowski spacetime, the right-hand sides of equations (7.68)–(7.86) may all be set to zero. In particular, equation (7.71) implies that there is a vorticity potential:

$$\omega_a \approx \mathrm{D}_a \Psi \,. \tag{7.89}$$

The linearisation of equations (7.78) and (7.86), together with the scalar potential equations (7.87) and (7.89), then imply that the curls vanish to linear order. Thus the linearized gravito-electric monopole is covariantly characterized by equations (7.87), (7.89) and

$$\mathrm{D}^2\Phi \approx 0 \,, \quad E_{ab} \approx \mathrm{D}_a\mathrm{D}_b\Phi \,, \quad \mathrm{D}_{(a}\mathrm{D}_{b)}\Psi \approx 0 \,, \tag{7.90}$$

while for the linearized gravito-magnetic monopole

$$\mathrm{D}^2\Psi \approx 0 \,, \quad H_{ab} \approx \mathrm{D}_a\mathrm{D}_b\Psi \,, \quad \mathrm{D}_{(a}\mathrm{D}_{b)}\Phi \approx 0 \,. \tag{7.91}$$

It follows in particular that a linearized non-rotating gravito-electric monopole is mapped to a linearized non-accelerating gravito-magnetic monopole via

$$\mathcal{I}_{ab} \to i\mathcal{I}_{ab} \,, \quad \omega_a \to \dot{u}_a \,, \quad \dot{u}_a \to -\omega_a \,. \tag{7.92}$$

Linearized Schwarzschild spacetime is readily seen to satisfy equation (7.90) with $\Phi = -M/r$, where M is the mass and r the area coordinate. Using the spatial duality rotation and kinematic interchange described by equation (7.92), this monopole is mapped to a linearized non-accelerating gravito-magnetic monopole with potential $\Psi = -M/r$. In comoving stationary coordinates, the metric of the linearized magnetic monopole follows from $\dot{u}_a = 0$ and $\omega_a = \mathrm{D}_a\Psi$, using a theorem in [248] (p 24):

$$ds^2 \;=\; -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) + 4M\cos\theta\, d\varphi\, dt \,. \tag{7.93}$$

This is a Taub-NUT solution with $m = 0$, $\ell = -M$ and linearized in ℓ ([252], p.133; see also [245]). In fact, this is precisely the linearized solution found in [243], so that we have a covariant characterization of that solution in the framework of gravitational duality. Clearly the magnetic 'charge' M is an angular momentum parameter, not a mass parameter, and the metric in equation (7.93) describes an isolated source with angular momentum but no mass.

7.6 Concluding remarks

A covariant $1+3$ approach, based on [220] and its extension [246, 229], is ideally suited to an analysis of the free gravitational field that is based on observable physical and geometric quantities, with a clear and transparent analogy in well-established electromagnetic theory. We have used such an approach, including in particular the generalization of covariant spatial vector analysis to spatial tensor analysis, which involves developing a consistent covariant definition of the tensor curl and its properties. Via this approach, we showed the remarkably close analogy between the Maxwell equations for the electric/magnetic fields and the Bianchi identities for the gravito-electric/magnetic fields. Although this analogy has long been known in general terms, our approach reveals its properties at a physically transparent level, with a detailed accounting for each physical and geometric quantity. We found new interpretations of the role of the kinematic quantities – expansion, acceleration, vorticity and shear – in the source and coupling terms of gravito-electromagnetism. The tracefree part of the Ricci identities also reveals the role of the kinematic quantities as gravito-electric/magnetic potentials.

The analogy provides a simple interpretation of the super-energy density and super-Poynting vector as natural $U(1)$ invariants, and we derived the exact nonlinear conservation equation that governs these quantities, and which involves a further natural invariant, i.e. the anisotropic super-pressure. We also used the analogy to show that a covariant spatial duality invariance exists in vacuum gravito-electromagnetism, precisely as in source-free electromagnetism. Duality invariance has been important in some recent developments in field and string theory, and the gravito-electromagnetic invariance in the form found here may also facilitate new insights into gravity. A crucial feature in the gravitational case, arising from its intrinsic nonlinearity, is that the duality invariance does not map Einstein solutions to Einstein solutions, since the Ricci identities are not invariant. Further work is needed to investigate whether a simultaneous geometric or kinematic transformation can be found, so that the Bianchi and Ricci equations are invariant under the combined transformation.

We showed that in linearized vacuum gravity, there is a simple combined duality/ kinematic transformation that maps the Schwarzschild gravito-electric monopole to the Demianski-Newman gravito-magnetic monopole. This covariant characterization of the relation between these linearized solutions was based on our covariant definition of gravito-monopoles in the general nonlinear theory. Further work is needed on the governing equations for these monopoles, in particular to see whether nonlinear gravito-magnetic monopole solutions may be found. A better understanding of the relation between nonlinear gravito-electric/magnetic monopoles could, as in field theory, open up new approaches and insights.

Chapter 8

Conclusions and Reheating Issues past 2001

This thesis has discussed one of the most violent epochs in the universe's history ever proposed. The explosive particle production that is the basic nature of preheating causes inflation to end in a manner that is currently beyond a full theoretical understanding due to its complex quantum, non-equilibrium and non-perturbative essense. Nevertheless, the understanding that we have achieved is remarkable perhaps principally for the stimulus it has given basic research in non-equilibrium and non-perturbative quantum field theory in curved backgrounds. Preheating is a virtual laboratory which has inspired many new insights and the application of complex techniques to an arena that previously was rather barren of critical investigation.

It is against this background that the work in this thesis has been undertaken. We have attempted to understand gravitational aspects of preheating, which has lead to the idea of geometric reheating due to the oscillations of the Ricci curvature. Indeed, the very idea that the curvature oscillates is a foreign one, even to many Relativists, so that the insights are not limited to field theorists. We have investigated the mathematical foundations upon which the basic paradigm of preheating was initially built - the Mathieu equation - and found that it is but a small spring that is sourced by a much deeper well. Preheating exists in models with a much broader base and can be classified into distinct classes using the beautiful mathematics of spectral theory of bounded linear operators on Hilbert spaces.

Further these spectral theory insights have enabled us to study aspects of preheating in more realistic models of inflation with many fields. Without having to resort to the study of specific models, the nature of preheating in the strong-coupling limit of chaotic theories has been analysed and found to be even stronger than in the case where the inflaton oscillates alone, interacting with only one other field. Returning to the implications for General Relativity, we then saw that the oscillations in the curvature are sources for resonant amplification of gravitational waves - this is the gravitational analog of populating higher-momentum modes via self-interaction.

Finally we have investigated the mathematical and symmetry properties of vacuum General Relativity itself - the most basic framework for the study of nonlinear gravitational waves and exterior solutions. In response to the duality conjectures in supersymmetric Yang-Mills and string theory we have shown that General Relativity exhibits essentially the same rotation duality as vacuum electromagnetism. In string theory the S-duality maps magnetic monopoles into electric monopoles and we have found a linearised equivalent of this for gravitational monopoles which maps Schwarzschild into Taub-NUT, exchanging mass with angular momentum. This suggests that there may be deeper dualities at work in General Relativity.

To end with, we list problems left to be resolved regarding the nature of reheating and its implications for inflation and observational cosmology in general. Given the rapid nature with which reheating is currently evolving, this list is likely to be out of date soon, the issues answered and with new questions replacing the old.

Non-equilibrium Issues

Problem

How bad is the equilibrium effective potential at describing the true dynamics in reheating and what are the qualitative differences ? In particular, how is non-thermal symmetry restoration and defect formation different when the full non-equilibrium theory is used ?

Problem

How does thermalisation of the large quantum fluctuations to a reheating temperature T_r actually proceed ?

Perturbation Evolution through reheating

Studies of reheating have, until now, mainly ignored the evolution of metric perturbations through reheating, based on the opinion that reheating cannot affect super-Hubble scale modes, and hence has little implications for the CMB or large scale structure. Given the explosive nature of preheating, is this true ?

Problem

How do metric perturbations evolve in the large q, strong coupling, region and how does this affect the CMB ?

Problem

How do the large variances $\langle(\delta\phi)^2\rangle$ evolve under gravity ? How do the variances affect metric perturbation evolution ? What, if any, are the effects on the SAD (Sakharov, acoustic, Doppler) peaks of the CMB ? Can one make a meaningful Hartree-Fock approximation for gravity in this context ?

Problem

How do metric perturbations affect reheating. In particular, how do the constraints of General Relativity impact on the exponential resonances of preheating ? How does multi-fluid perturbation theory affect preheating ?

Problem

Can one make a realistic multi-fluid theory of reheating based on non-equilibrium, dissipative and causal thermodynamics ?

The issues dealing with metric perturbation evolution above have partially been resolved recently [257][1], although more questions remain than are answered. In particular the constraint equations coming from the Bianchi and Ricci identities force scalar metric fluctuations to grow with the field fluctuations, with the variance of the metric perturbations saturating their linear bound before the end of preheating. This implies that linear perturbation theory breaks down and opens up a Pandora's box of problems and challenges. Only time will tell whether the box can be sealed safetly or whether preheating will be excluded on observational grounds.

[1] See also the web page http://www.sissa.it/~bassett/reheating/.

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University politics are vicious precisely because the stakes are so low.
– Henry Kissinger

No science is immune to the infection of politics and the corruption of power.
– Jacob Bronowski

Who knows for what we live, struggle and die?... Wise men write many books,
in words too hard to understand. But this, the purpose of our lives,
the end of all our struggle, is beyond all human wisdom.
Alan Paton

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