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Seiberg-Witten Theory for Four Manifolds Split Along Three Manifolds of Positive Scalar Curvature

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Prof. Stefano Demichelis

Thesis submitted for the degree of "Doctor Philosophiæ"

Academic Year 1997/98

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Il presente lavoro costituisce la tesi presentata da Stefano Vidussi, sotto la direzione del Prof. Stefano Demichelis, al fine di ottenere il diploma di *"Doctor Philosophiæ"* presso la Scuola Internazionale Superiore di Studi Avanzati, Classe di Matematica, Settore di Analisi Funzionale ed Applicazioni.

Ai sensi del Decreto del Ministero della Pubblica Istruzione n. 419 del 24. 04.1987, tale diploma di ricerca post-universitaria è equipollente al titolo di *"Dottore di Ricerca in Matematica"*

E je passade unevóre di timp (salacôr masse) de prime volte ch'o ài pensât ce scrivij, te prefazion a mê tesi di dotorât. E cumò, tal scrivi, dut chel ch'o vevi pensât di metij, i ricuarts, i sintiments, mi semein masse fûr di lûc par lassâju achi.
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Contents

Main Results

The aim of this thesis is to discuss some results concerning Seiberg-Witten invariants of a closed four manifold M having $b_2^+(M) > 1$, which decomposes in two factors along a three manifold of positive scalar curvature.

A typical problem which arises in the study of the differential topology of smooth four manifolds is to determine under which conditions a given four manifold admits a decomposition along three manifolds with some topological or geometrical properties.

The prototype of such a kind of question has been investigated by S.Donaldson, in the 80's, when he has proven that a closed four manifold with non trivial Donaldson invariants (e.g. an algebraic surface) does not admit a decomposition along S^3, i.e. is not a connected sum, with two factors having both $b_2^+ > 0$. In the framework of Donaldson theory this kind of investigation has produced some partial or conjectural results extending the class of three manifolds for which a result as the one mentioned holds true. The difficulty that such a kind of study have to face up is related to the study of Donaldson-Floer theory for a general three manifold, which is technically very complicated, and although in the beginning of the 90's many steps had been done, mainly thanks to the contribution of J.Morgan, T.Mrowka, C.Taubes et al., the route seemed still long to go.

With the appearance of Seiberg-Witten theory, in Fall '94, also this type of questions has quite simplified, thanks to the usual patterns of mild nonlinearity of the equations and compactness properties of the spaces of solutions which are well known for this theory.

In this thesis we will use Seiberg-Witten theory to prove some results concerning the decomposition of four manifolds having non trivial Seiberg-Witten invariants along three manifolds of positive scalar curvature, class which generalizes in a way which is very natural, in view of Seiberg-Witten-Floer theory, the case of S^3. To prove these results, we will study the moduli space of finite energy solutions of Seiberg-Witten equations on a riemannian cylindrical end four manifold $(\hat{X}, g_{\hat{X}})$, whose end is isometric to $N \times [0, \infty)$ with its product metric, where (N, g_N) is a three manifold of positive scalar curvature. We will suppose that $\hat{X}$ is built starting from a compact open manifold (X, g_X) with boundary $\partial X = N$ in such a way that

$$\hat{X} = X \cup_N N \times [0, \infty);$$

with the quotient metric, this manifold is complete, and Fredholm theory for differential operators in such a setting is well established, see [APS], [T1].

One of the major ingredients of our study is the fact that in the case of spin^c structures on $\hat{X}$ whose determinant bundles restrict to torsion bundles on the end, the curvature condition on N compels the solutions of the SW equations to decay exponentially fast, along the cylinder, to static solutions which correspond to $U(1)$ flat connections on the three manifold. This result allows to apply in a fairly easy way standard results for studying the moduli spaces on the cylindrical end manifold, and in particular to compute their dimension, which depends on the geometry of X. In suitable cases, as illustrated below, the study of this class of spin^c structures on $\hat{X}$ and the analysis of the possible perturbations for SW equations will be sufficient to prove the following results:

Theorem 1 *Let M be a closed four manifold with $b_2^+(M) > 1$ which decomposes as $M = M^+ \cup_N M^-$ along a three manifold of positive scalar curvature in two factors and which satisfies the relation*

$$b_2^+(M) > b_2^+(M^{\pm});$$

then the Seiberg-Witten invariants

$$SW_M : Spin^c(M) \to \mathbb{Z}$$

are identically zero.

Here $b_2^+(M^{\pm})$ refers to the number of positive eigenvalues for the (possibly degenerate) pairing induced by Lefschetz duality on $H_2(M^{\pm})$ or, which is the same, for the non degenerate pairing on $Im(H^2(M^{\pm}, \partial M^{\pm}) \to H^2(M^{\pm}))$.

The statement of Theorem 1 is clarified by recalling that, under the above decomposition, the positive eigenspaces of the pairing are related by the formula

$$H_2^+(M^+) \oplus H_2^+(M^-) \rightarrowtail H_2^+(M) \twoheadrightarrow Im(H_2(M) \to H_1(N)).$$

Theorem 1 defines whether, according to the form of the previous short exact sequence, the Seiberg-Witten invariants have to vanish. The statement of Theorem 1 is just a reformulation of the results of two vanishing theorems we shall proof:

Theorem 2 *Let M be a closed four manifold which decomposes as $M = M^+ \cup_N M^-$ along a three manifold of positive scalar curvature in two parts both having $b_2^+(M^\pm) > 0$; then the Seiberg-Witten invariants $SW_M : Spin^c(M) \to \mathbb{Z}$ are identically zero.*

The proof of Theorem 2 will follow from the moduli dimension formula we will discuss in Chapter 3. This theorem has an obvious consequence for the topology of some classes of four manifolds:

Corollary 1 *Let M be a four manifold with non vanishing Seiberg-Witten invariants which decomposes as $M = M^+ \cup_N M^-$; then one of the factors has non positive definite intersection form.*

The typical examples of such a kind of manifolds are symplectic ones and more general examples are listed in Chapter 1. Theorem 2 generalizes the well known case of S^3, proven within the framework of Yang-Mills theory by Donaldson (see [DK]). There are interesting examples of allowed decompositions, apart from the obvious S^3 case of the blow ups, for the case of some lens spaces in [FS1].
The statement of Theorem 1 is completed, when $b_1(N) > 0$, with the following

Theorem 3 *Let M be a closed four manifold with $b_2^+(M) > 1$ which decomposes as $M = M^+ \cup_N M^-$ along a three manifold of positive scalar curvature and $b_1(N) > 0$ in such a way that $b_1(M) > b_1(M^+) + b_1(M^-) - b_1(N)$; then the Seiberg-Witten invariants $SW_M : Spin^c(M) \to \mathbb{Z}$ are identically zero.*

In this case the proof follows from a perturbation of the SW equation. The request on the first Betti numbers of the manifolds involved, which corresponds to the possibility of such a perturbation, will appear clearly from the proof.
As a corollary of this result, we obtain the natural non-decomposition theorem for the manifolds with non-vanishing SW invariants analogous to Corollary 1. We note that, in Theorem 3, no conditions on $b_2^+(M^\pm)$ are involved: as the case of [MST], Lemma 10.2 shows, this is not a necessary condition.

We will prove the above Theorem 2 under the hypothesis that M is simple type, but this can easily be removed, along the lines of [Sa].

The existence, in some form, of a vanishing theorem of the type above, in the case of S^3, is already suggested in the original paper by Witten [Wi], and in other cases in a survey paper by Donaldson [Do], but a complete proof is present, as far as we know, only for S^3 [Sa], with a technique which appears not to be extendible to other cases.

In Chapter 4 we will analyse some topological consequences of these results; in particular we will prove the following:

Proposition 1 *Let M_1 and M_2 be two closed four manifolds diffeomorphic in the complement of a point or in the complement of a wedge of circles: then $SW_{M_1} = SW_{M_2}$.*

Proposition 2 *Let M be a closed four manifold with $b_2^+(M) > 1$ which contains a two sphere S of self intersection $S \cdot S \geq 0$ and infinite order; then the Seiberg-Witten invariants vanish.*

Proposition 1 parallels the result, in Donaldson theory, of [De], while Proposition 2 is a "classical" result and has been obtained with different techniques by [MST].
In Chapter 4 we shall show as well that the proof of Theorem 3 can be in fact reduced to the particular case of $S^1 \times S^2$ and, starting from this observation, we will discuss some generalizations of Theorem 3 to a class of three manifolds that we define as $\mathbb{Q}$-reducible, i.e. containing a sphere which does not bound any rational homology disk. $\mathbb{Q}$-reducible manifolds admit a decomposition of the form

$$N = (\#_{i=1}^p Y_i) \# (\#_{j=1}^q S^1 \times S^2) \# (\#_{k=1}^r K(\pi_k, 1))$$

with Y_i a rational homology sphere and $q + r > 1$ or $q = 1$. We prove the following

Theorem 4 *Let M be a closed four manifold with $b_2^+(M) > 1$ which decomposes as $M = M^+ \cup_N M^-$ along a $\mathbb{Q}$-reducible three manifold in such a way that the map*

$$H_2(M, \mathbb{Q}) \to H_1(N, \mathbb{Q}) = (\oplus_1^q H_1(S^1 \times S^2, \mathbb{Q})) \oplus (\oplus_1^r H_1(K(\pi_k, 1), \mathbb{Q}))$$

has rank at least one on the first q factors or is nontrivial on at least two factors; then the Seiberg-Witten invariants $SW_M : Spin^c(M) \to \mathbb{Z}$ are identically zero.

This result appears as a consequence, in many cases, of Theorem 3, and in general from a discussion of three dimensional Seiberg-Witten theory for $\mathbb{Q}$-reducible manifolds, that we assume to have the same patterns of the four dimensional one.

This thesis is organized as follows. Chapter 1 contains a quick review of Seiberg-Witten theory on a closed four manifold, focusing mainly on the results and techniques which will be used in the other Chapters. These results, when not explicitly appeared in literature, are more or less common folklore, eventually with the exception of Section 1.2. Chapter 2 is dedicated to SW theory on a cylinder $N \times [0, T]$ with N of positive scalar curvature and it is proven that, up to a gauge transformation, a finite energy solution of SW equation on the infinite cylinder decays exponentially fast to a static one: this is a technical ingredient which is used in the proof of Theorem 2. Chapter 3 contains a discussion of geometric limits of solutions of SW equations on a closed four manifold after "stretching the neck" and a description of the moduli spaces of solutions of SW equations for a spin^c structure on a cylindrical end four manifold; with these results we give then a proof of Theorem 2; then we prove Theorem 3, independently, with the study of some perturbations to SW equations. Chapter 4 is devoted to some applications of these Theorems and to the proof of Theorem 4. The Appendix contains some technical results which are useful in the proof of the decay result of Chapter 2.
Concerning the notations, we have tried to follow, with minor modifications, the one of the book [Mo].

The content of this thesis is, to the best of our knowledge, original, except when otherwise indicated; when we used or adapted others' results, we tried to make explicit our contribution.

Chapter 1

Seiberg-Witten Invariants

In this Chapter we will outline the basic facts concerning the definition of Seiberg-Witten invariants, introduced by Edward Witten in late 1994 in [Wi] following previous work with Nathan Seiberg. A rigorous construction of the invariants is in some sense routine, since it follows closely the definition of Donaldson polynomials in Yang-Mills theory [DK] or of Gromov-Witten invariants in the case of pseudoholomorphic curves in symplectic manifolds [MS]. Apart from Section 1.2, where the approach is quite original, the purpose of this Chapter is more or less just filling the details of the results outlined in [KM1]. A standard reference on the subject is, by now, the book [Mo], although our treatment will follow more closely the usual approach for Yang-Mills theory.

1.1 Seiberg-Witten Equations.

In this Section we will introduce a couple of equations for the pair $(M, \tilde{P}_M)$ given by a smooth compact closed riemannian four manifold M endowed with a spin^c structure $\tilde{P}_M$ whose determinant bundle will be denoted by $\mathcal{L}_M$. The study of these equations will bring to the definition of some differential invariants for M. These equations will appear as God-given (in fact, as Yang-Mills equations do), but their origin comes from physical considerations that are beyond the scope of our exposition.

When we consider the fundamental complex representation of $Spin(4)$, which has $dim_{\mathbb{C}} = 4$, we can canonically define an associated spin^c vector bundle $S(\tilde{P}_M)$ over M, which decomposes as $S^+(\tilde{P}_M) \oplus S^-(\tilde{P}_M)$: the factors are $U(2)$ vector bundles with common determinant bundle $\mathcal{L}_M$. As follows from the definition of spin^c structures, the set of

spin^c structures $Spin^c(M)$ is a 2-torsor over the set of line bundles (which admits as well known a canonical identification with $H^2(M,\mathbb{Z})$) and therefore, for a given $\mathcal{L}_M$, the choice of $\tilde{P}_M$ depends on the 2-torsion part of $H^2(M,\mathbb{Z})$.
Once we endow M with a metric, we have a canonical connection on the tangent bundle, the Levi-Civita connection. The sections of the complexified cotangent bundle of M act on spinor sections, as elements of the complex Clifford bundle on M, via Clifford multiplication; we can define a Dirac operator, acting on the spinor bundle, by choosing an abelian connection on $\mathcal{L}_M$. With this construction we have a family of Dirac operators $\partial\!\!\!/_A$ parameterized by $\mathcal{A}(\mathcal{L}_M)$, the space of connections on $\mathcal{L}_M$, which act as

$$\partial\!\!\!/_A : \Gamma(S^+(\tilde{P}_M)) \longrightarrow \Gamma(S^-(\tilde{P}_M)). \tag{1.1}$$

As $dimM$ is even the action of forms on the spinor bundles induces an isomorphism

$$\Omega^*(M;\mathbb{C}) = End(S(\tilde{P}_M)); \tag{1.2}$$

if we restrict to the even forms this isomorphism specializes to

$$\Omega^{2*}(M;\mathbb{C}) = End(S^+(\tilde{P}_M)) \oplus End(S^-(\tilde{P}_M)) \tag{1.3}$$

and the direct sum decomposition corresponds, on the left hand side, to positive, resp. negative eigensections of the complex unit $\omega_{\mathbb{C}}$ of the Clifford bundle. It is not difficult verifying, by direct computation, that in four dimensions the positive eigensections are those of the form

$$\mathbb{C}(\frac{1+\omega_{\mathbb{C}}}{2}) \oplus \Omega^2_+(M;\mathbb{C}) \tag{1.4}$$

(in fact, on two forms, $\omega_{\mathbb{C}}$ coincides with the Hodge star operator) and that traceless endomorphism correspond to the second factor.
Summing up, we have an identification

$$\Omega^2_+(M;\mathbb{C}) = End_0(S^+(\tilde{P}_M)). \tag{1.5}$$

Now we give a way to relate self dual forms and spinors: given a couple of sections ψ, ϕ in $\Gamma(S^+(\tilde{P}_M))$ we can construct a bilinear map

$$\begin{aligned} q(\cdot,\cdot) : \Gamma(S^+(\tilde{P}_M)) \otimes \Gamma(S^+(\tilde{P}_M)) &\longrightarrow End_0(S^+(\tilde{P}_M)) \\ (\psi,\phi) \mapsto q(\psi,\phi) &= \tfrac{1}{2}(\psi\otimes\phi^* + \phi\otimes\psi^* - Re <\psi,\phi> Id). \end{aligned} \tag{1.6}$$

Once for all, this has the matrix form

$$q(\psi,\phi)=\frac{1}{4}\begin{pmatrix}\psi_1\bar{\phi}_1+\bar{\psi}_1\phi_1-\psi_2\bar{\phi}_2-\bar{\psi}_2\phi_2 & 2\psi_1\bar{\phi}_2+2\phi_1\bar{\psi}_2\\ 2\psi_2\bar{\phi}_1+2\phi_2\bar{\psi}_1 & -\psi_1\bar{\phi}_1-\bar{\psi}_1\phi_1+\psi_2\bar{\phi}_2+\bar{\psi}_2\phi_2\end{pmatrix}. \tag{1.7}$$

This clearly represents a traceless endomorphism; moreover, as $q(\psi,\phi)^\dagger = q(\psi,\phi)$, this endomorphism corresponds to purely imaginary elements of the Clifford algebra, and then to $\Omega^2_+(M;i\mathbb{R})$. In terms of forms, via the inverse of the identification map of equation 1.5, a straightforward explicit calculation shows that $q(\psi)$ is identified, as element of $\Omega^2_+(M,i\mathbb{R})$, with

$$q(\psi)=\frac{1}{16}\sum_{ijkl}\epsilon_{ijkl}e^i\wedge e^j < e^k\cdot e^l\cdot\psi,\psi > . \tag{1.8}$$

We are now in position to write the Seiberg-Witten equations: they are given by the couple

$$\begin{cases} \not{\partial}_A\psi=0\\ F_A^+=q(\psi)\end{cases} \tag{1.9}$$

for $\psi\in\Gamma(S^+(\tilde{P}_M))$ and $A\in\mathcal{A}(\mathcal{L}_M)$. We will call a solution of equations 1.9 reducible if it has the form $(A,0)$, and irreducible otherwise. Reducible solutions correspond evidently to anti self dual abelian instantons.

An equivalent alternative procedure to introduce SW equations is to work with a connection on $\tilde{P}_M$ which induces the Levi-Civita on the tangent bundle, and define the second of equations 1.9 with the induced connection in $\mathcal{A}(\mathcal{L}_M)$. The difference with our definition is immaterial but it implies some minor change in the formulae involving the action of group of automorphisms of $\tilde{P}_M$.

1.2 Topology of the Orbit Space.

Now we want to study the topology of the space on which Seiberg-Witten equations are defined; of course both the space of connections on $\mathcal{L}_M$ and the space of sections $\Gamma(S^+(\tilde{P}_M))$, being respectively an affine space and a vector space, are contractible infinite dimensional manifolds, but the equations 1.9 admit a gauge symmetry. It is immediate to see that the equations are invariant with respect to the action of the gauge group of those vertical automorphism of $\tilde{P}_M$ which project to the identity automorphism of the frame bundle, i.e. those which act trivially on the forms. This group corresponds, as

follows from the defining group exact sequence of $Spin^c(4)$, to transformations having value in the central $U(1)$ subgroup of $Spin^c(4)$. The gauge group, that we will denote by $\mathcal{G}(\tilde{P}_M)$, corresponds therefore to the space of $Spin^c(4)$-invariant maps $\gamma : \tilde{P}_M \to U(1) \subset Spin^c(4)$, which coincides, of course, with the space of maps $Map(M, U(1))$. The subgroup of base point fixing automorphisms is given by the space of maps of pointed spaces $Map_*(M, U(1))$.
The gauge group acts by projection on $\mathcal{L}_M$: its action, on the space of connections, is related to the usual action of the group of gauge transformations of the line bundle, $\mathcal{G}(\mathcal{L}_M)$, by push forward of the covering map of $U(1)$ to itself given, pointwise, by the determinant (that on the $U(2)$ factors correspond really to the algebraic determinant), while on spinors it just acts by multiplication. The explicit form of the action of $\mathcal{G}(\tilde{P}_M)$ is given by

$$\begin{gathered} \mathcal{G}(\tilde{P}_M) \times (\mathcal{A}(\mathcal{L}_M) \times \Gamma(S^+(\tilde{P}_M))) \to (\mathcal{A}(\mathcal{L}_M) \times \Gamma(S^+(\tilde{P}_M))) \\ g \cdot (A, \psi) \mapsto (A + (detg)^{-1} d(detg), g^{-1}\psi) \end{gathered} \tag{1.10}$$

(note: the form of the action brings often to "factors 2" in the formulae involving gauge transformations; unfortunately very often, in the literature, there are mistakes on this point because of the confusion between gauge transformations on the $Spin^c(4)$ bundle and their projection on $\mathcal{L}_M$; we hope we will be precise on this point). Because of gauge invariance of equations 1.9 we are in fact interested in the quotient of the product of the spaces of connections and spinors by the action of the gauge group,

$$\mathcal{B}(\tilde{P}_M) := (\mathcal{A}(\mathcal{L}_M) \times \Gamma(S^+(\tilde{P}_M)))/\mathcal{G}(\tilde{P}_M). \tag{1.11}$$

It is a standard fact in gauge theory that once completed in suitable Sobolev norms, respectively $\mathcal{L}^2_{k+1}, \mathcal{L}^2_k$ with $k \geq 2$, the gauge group is a Banach-Lie group and acts on the space $\mathcal{A}(\mathcal{L}_M) \times \Gamma(S^+(\tilde{P}_M))$. A slice for the action is provided by a suitable generalization of the standard one of Yang-Mills theory; the linearization of the gauge group action is given by

$$\delta(\chi)_{(A,\psi)} = (2d\chi, -\chi\psi) \tag{1.12}$$

and the slice orthogonal to the gauge action is contained in the plane

$$ker\delta^*_{(A,\psi)} = \{(a, \phi) \in \Omega^1(M, i\mathbb{R}) \times \Gamma(S^+(\tilde{P}_M)) | d^*a + \frac{i}{2} Im < \psi, \phi >= 0\}, \tag{1.13}$$

as follows from a straightforward explicit computation.

The gauge group acts freely outside the space of reducible couples $(A, 0)$, where it has isotropy group isomorphic to $U(1)$. There are two ways to make this action free: the first one is to consider only base point fixing automorphisms, $\mathcal{G}^o(\tilde{P}_M)$, defining the orbit space of based couples

$$\mathcal{B}^o(\tilde{P}_M) := (\mathcal{A}(\mathcal{L}_M) \times \Gamma(S^+(\tilde{P}_M)))/\mathcal{G}^o(\tilde{P}_M). \tag{1.14}$$

The second one is to restrict to the space of irreducible couples, defining the irreducible orbit space

$$\bar{\mathcal{B}}(\tilde{P}_M) := (\mathcal{A}(\mathcal{L}_M) \times (\Gamma(S^+(\tilde{P}_M)) \setminus 0))/\mathcal{G}(\tilde{P}_M). \tag{1.15}$$

Naturally the space of irreducible based couples $\bar{\mathcal{B}}^o(\tilde{P}_M)$ fibers over $\bar{\mathcal{B}}(\tilde{P}_M)$ with fiber $U(1)$, via base point fibration. Note that the homotopy type of $\bar{\mathcal{B}}^o(\tilde{P}_M)$ is the same as that of $\mathcal{B}^o(\tilde{P}_M)$, as reducible couples have infinite codimension in the latter space.

As the action of $\mathcal{G}^o(\tilde{P}_M)$ (respectively $\mathcal{G}(\tilde{P}_M)$) is free over $\mathcal{A}(\mathcal{L}_M) \times \Gamma(S^+(\tilde{P}_M))$ (respectively $\mathcal{A}(\mathcal{L}_M) \times (\Gamma(S^+(\tilde{P}_M)) \setminus 0)$) the orbit spaces, for a given spin^c structure, appear as classifying spaces $B\mathcal{G}^o(\tilde{P}_M)$ and $B\mathcal{G}(\tilde{P}_M)$. We must study therefore the topology of the classifying space of $Map(M, U(1))$.

Our aim is to prove the following

Claim 1.2.1 *The gauge orbit space $\bar{\mathcal{B}}(\tilde{P}_M)$ has the weak homotopy type of a product of Eilenberg-MacLane spaces*

$$\bar{\mathcal{B}}(\tilde{P}_M) \cong K(H^1(M, \mathbb{Z}), 1) \times K(\mathbb{Z}, 2). \tag{1.16}$$

To prove this claim we will start by studying the topology of the based orbit space $\mathcal{B}^o(\tilde{P}_M)$. First of all we study its homotopy groups.

As $\mathcal{B}^o(\tilde{P}_M)$ is the base space of a $\mathcal{G}^o(\tilde{P}_M)$-fibration on a contractible space we can compute its homotopy groups from those of $\mathcal{G}^o(\tilde{P}_M) = Map_*(M, U(1))$. These homotopy groups appear as

$$\pi_k(\mathcal{G}^o(\tilde{P}_M)) = \pi_0 Map_*(S^k, Map_*(M, U(1))). \tag{1.17}$$

We can then apply the exponential law for spaces of maps of pointed spaces, i.e.

$$Map_*(X, Map_*(Y, Z)) = Map((X \times Y, X \vee Y), (Z, *))). \tag{1.18}$$

We get the equivalence

$$Map_*(S^k, \mathcal{G}(\tilde{P}_M)) = Map((S^k \times M, S^k \vee M), (U(1), 1)). \tag{1.19}$$

We now note that $U(1)$ is the Eilenberg-Maclane space $K(\mathbb{Z},1)$; this allows to compute easily the homotopy type of the indicated space of maps. Summing up we have

$$\pi_k(\mathcal{G}^o(\tilde{P}_M)) = H^1(S^k \times M, S^k \vee M, \mathbb{Z}). \tag{1.20}$$

This latter cohomology group can be computed noting that it is by definition the first cohomology group of the smash product $S^k \wedge M$, i.e. the $k-th$ suspension $\Sigma^k M$ of M. The suspension isomorphism

$$\tilde{H}^p(\Sigma^k M, \mathbb{Z}) = \tilde{H}^{p-k}(M, \mathbb{Z}) \tag{1.21}$$

brings immediately to the proof of the

Proposition 1.2.2 *The orbit space $\mathcal{B}^o(\tilde{P}_M)$ has the weak homotopy type of the Eilenberg-Maclane space $K(H^1(M,\mathbb{Z}),1)$.*

In fact, from the previous computation and the exact homotopy sequence of a fibration we have that

$$\pi_1(\mathcal{B}^o(\tilde{P}_M)) = \pi_0(\mathcal{G}^o(\tilde{P}_M)) = H^1(M,\mathbb{Z}), \;\; \pi_n(\mathcal{B}^o(\tilde{P}_M)) = \pi_{n-1}(\mathcal{G}^o(\tilde{P}_M)) = 0 \;\; for \;\; n \neq 1, \tag{1.22}$$

as claimed.

Note that the isomorphism of homotopy groups is not enough, in general, to prove the weak homotopy equivalence of two spaces, but in the case of Eilenberg-Maclane spaces this holds true. Note, moreover, that the based gauge group retracts to the set of its components $\pi_*(M, U(1))$. The definition of cohomology classes in terms of obstruction theory shows that this retraction is canonically identified with the correspondence between a map $g : M \to K(\mathbb{Z},1)$ and the element $g^*k \in H^1(M,\mathbb{Z})$ where $k \in H^1(K(\mathbb{Z},1),\mathbb{Z})$ is the characteristic class.

We can pass now to the study of the topology of the orbit space $\mathcal{B}(\tilde{P}_M)$, and to do this we look for the topology of the unbased gauge group $\mathcal{G}(\tilde{P}_M)$. First, the inclusion of $\mathcal{G}^o(\tilde{P}_M)$ in $\mathcal{G}(\tilde{P}_M)$ identifies the homotopy fiber for the fibration

$$Map_*(M, U(1)) \overset{i}{\hookrightarrow} Map(M, U(1)) \to U(1) \tag{1.23}$$

which induces the exact sequence of homotopy groups

$$\pi_1(\mathcal{G}(\tilde{P}_M)) \rightarrowtail \pi_1(U(1)) \to \pi_0(\mathcal{G}^o(\tilde{P}_M)) \overset{i_*}{\twoheadrightarrow} \pi_0(\mathcal{G}(\tilde{P}_M)). \tag{1.24}$$

We claim that the map i_* is also injective, i.e. the based and unbased gauge group have the same number of components: in fact, let $g_0, g_1 \in Map_*(M, U(1))$ be two maps which are homotopic in $Map(M, U(1))$: this means that there exist an homotopy

$$\Phi : I \times M \to U(1), \qquad \Phi(j, m) = g_j(m); \tag{1.25}$$

it is quite clear now that if $p \in M$ is the base point of M the map

$$\Phi' : (I, \emptyset) \times (M, p) \to (U(1), 1), \qquad \Phi'(t, m) := \Phi(t, p)^{-1} \cdot \Phi(t, m) \tag{1.26}$$

defines a based homotopy between g_0 and g_1 (put it in another way, this is nothing but the fact that as $U(1)$ is an H-space, its fundamental group acts trivially on the set of components). It follows from this that the set of components of $\mathcal{G}(\tilde{P}_M), \mathcal{G}^o(\tilde{P}_M)$ coincide and, restricting to one component, the fibration 1.23 has contractible fiber and each connected component of $\mathcal{G}(\tilde{P}_M)$ has therefore the homotopy type of $U(1)$. The unbased gauge group has therefore the homotopy type of $H^1(M, \mathbb{Z}) \times K(\mathbb{Z}, 1)$.
In order to identify the homotopy type of the orbit space we recall that the latter is defined as quotient of the principal action of the gauge group over the contractible space of (irreducible) couples, see equation 1.15, and thus has the weak homotopy type of the classifying space of $H^1(M, \mathbb{Z}) \times K(\mathbb{Z}, 1)$. This gives the weak homotopy equivalence

$$\bar{\mathcal{B}}(\tilde{P}_M) = K(H^1(M, \mathbb{Z}), 1) \times K(\mathbb{Z}, 2). \tag{1.27}$$

This result proves Claim 1.2.1. The $U(1)$ base point fibration corresponds exactly to the $K(\mathbb{Z}, 2)$ factor of the quotient.
We note that, as a consequence of this result, the torsion part of the fundamental group of M plays no role in the topology of the orbit space.

It is interesting to analyse the relation between the orbit space $\mathcal{A}(\mathcal{L}_M)/\mathcal{G}^o(\tilde{P}_M)$, defined with the use of the gauge group $\mathcal{G}^o(\tilde{P}_M)$, and the usual Yang-Mills (or better to say, Maxwell) orbit space $\mathcal{B}^o(\mathcal{L}_M) := \mathcal{A}(\mathcal{L}_M)/\mathcal{G}^o(\mathcal{L}_M)$, as in the first case $\mathcal{G}^o(\tilde{P}_M)$ acts on $\mathcal{A}(\mathcal{L}_M)$ via the square of the usual action.
We must compare the different actions of two groups, which are of course homeomorphic, but act in a different way on the same contractible space. We have the following

Proposition 1.2.3 *There is a natural covering map $\mathcal{A}(\mathcal{L}_M)/\mathcal{G}^o(\tilde{P}_M) \to \mathcal{B}^o(\mathcal{L}_M)$, whose fiber is given by $Hom(H^1(M, \mathbb{Z}), \mathbb{Z}_2)$.*

Proof: we observe first of all that the maps of $\mathcal{G}^o(\tilde{P}_M)$ appear, with respect to their action on connections, as the subset of the maps of $\mathcal{G}^o(\mathcal{L}_M)$ which are squares of other maps. Consider now the subgroup of $\mathcal{G}^o(\mathcal{L}_M)$ given by those elements which lie in a component which is in the image of the natural homomorphism σ, given by multiplication by 2:

$$H^1(M,\mathbb{Z}) \overset{\sigma}{\rightarrowtail} H^1(M,\mathbb{Z}) \twoheadrightarrow Hom(H^1(M,\mathbb{Z}),\mathbb{Z}_2); \tag{1.28}$$

we claim that this subgroup corresponds to elements which are in fact image, via the square, of a map of $\mathcal{G}^o(\tilde{P}_M)$, and moreover there is an homeomorphism between components which correspond under the map σ. Proving this is means essentially to analyse whether a map admits a "square root" and whether this square root is unique.
Now it is obvious that a necessary condition for a map $g \in \mathcal{G}^o(\mathcal{L}_M)$ to have a square root is that $g^*k \in H^1(M,\mathbb{Z})$ lies in $Im\sigma$, i.e. that g lives in an "even" connected component. This condition is also sufficient: we take any map $s = t^2$, in that component, which is already a square (the existence of such a map is obvious); by the previous results s is connected with g via an homotopy that we denote by Φ. Find a square root of g now simply amounts to solve the homotopy lifting problem

$$\begin{array}{ccc} \{0\}\times M \cup [0,1]\times\{p\} & \xrightarrow{(t,*)} & U(1) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \mathbb{Z}_2} \\ [0,1]\times M & \xrightarrow{\Phi} & U(1) \end{array} \tag{1.29}$$

The uniqueness of the solution follows from the fact that $Map_*(M,\mathbb{Z}_2)$, the space which acts freely and transitively on the set of square roots of a map, is a point.
From this results, the statement follows: the only difference between the action of $\mathcal{G}^o(\tilde{P}_M)$ and $\mathcal{G}^o(\mathcal{L}_M)$ on $\mathcal{A}(\mathcal{L}_M)$ amounts to the different action of components.

From Claim 1.2.1, the structure of the integral cohomology ring of $\bar{\mathcal{B}}(\tilde{P}_M)$ follows immediately: if we set $n = rkH^1(M,\mathbb{Z})$ and we let $e_1,...,e_n$ be the generators of $H^1(\bar{\mathcal{B}}(\tilde{P}_M),\mathbb{Z})$ and μ the generator of $H^2(\bar{\mathcal{B}}(\tilde{P}_M),\mathbb{Z})$ the cohomology ring of $\bar{\mathcal{B}}(\tilde{P}_M)$ is given by

$$H^*(\bar{\mathcal{B}}(\tilde{P}_M),\mathbb{Z}) = \mathbb{Z}(e_1,...,e_n)\otimes\mathbb{Z}[\mu]. \tag{1.30}$$

These cohomology generators have a geometrical meaning, which naturally is very similar to the one that analogous cohomology generators have in the Yang-Mills case. In fact these can be realized as images, under the slant product on the homology classes of M, of the first Chern class of the universal line bundle $\mathcal{E}$ over $M \times \bar{\mathcal{B}}(\tilde{P}_M)$; this is just the abelian case of Donaldson μ map, defined as

$$\begin{aligned} \mu_{c_1(\mathcal{E})} &: H_i(M, \mathbb{Z}) \longrightarrow H^{2-i}(\bar{\mathcal{B}}(\tilde{P}_M), \mathbb{Z}) \\ \mu_{c_1(\mathcal{E})}([\gamma]) &= c_1(\mathcal{E})/[\gamma]. \end{aligned} \tag{1.31}$$

In detail the generators of degree 1 can be interpreted as pull-backs of the fundamental class k of $U(1)$ under the holonomy representation around loops of M (in the Yang-Mills case, this happens for the generators of degree 3): for a given loop $[\gamma_i] \in H_1(M, \mathbb{Z})$ we consider the holonomy around γ_i

$$h_{\gamma_i} : \bar{\mathcal{B}}(\tilde{P}_M) \longrightarrow U(1). \tag{1.32}$$

We then have that $e_i = h^*_{\gamma_i}(K)$ as follows from the fact that for any 1-cycle $T \in H_1(\bar{\mathcal{B}}(\tilde{P}_M), \mathbb{Z})$

$$< h^*_{\gamma_i}(k), T >=< c_1(\mathcal{E}), [\gamma_i] \times T > . \tag{1.33}$$

(both represent in fact the degree of the holonomy map, as follows from the definition of the Euler class via obstruction theory).

The generator of degree 2 of $H^*(\bar{\mathcal{B}}(\tilde{P}_M), \mathbb{Z})$ appears, in the construction of the μ map, as image of an element $[p] \in H_0(M, \mathbb{Z})$. It is interesting to relate this generator with the determinant of the index bundle of the Dirac operator $\partial\!\!\!/_A$. In fact we can represent μ as

$$\mu_{c_1(\mathcal{E})}([p]) = c_1(\mathcal{E})/[p] = \int_M c_1(\mathcal{E}) \wedge [p]^{P.D.}. \tag{1.34}$$

If the manifold M has non vanishing $\hat{\mathbf{A}}$-genus we can take as Poincaré dual of a point, up to a multiplicative constant, the $\hat{\mathbf{A}}$-class and therefore

$$\mu \propto \int_M c_1(\mathcal{E}) \wedge \hat{\mathbf{A}}(M) = c_1(detind\partial\!\!\!/_A), \tag{1.35}$$

as follows from Atiyah-Singer index theorem, and thus the determinant index bundle of $\partial\!\!\!/_A$ is nontrivial: otherwise

$$c_1(detind\partial\!\!\!/_A) = c_1(\mathcal{E})/[\hat{\mathbf{A}}(M)^{P.D.}] = 0 \tag{1.36}$$

and that line bundle is trivial.

The identification of the generators of the cohomology ring of $\mathcal{B}(\tilde{P}_M)$ will allow later on to define differential invariants associated to the solutions of Seiberg-Witten equations, mimicking the construction in Donaldson theory.

1.3 Properties of the Moduli Spaces.

In this Section the object of our analysis will be the space of solutions of the equations 1.9 modulo the action of the gauge group.

Definition 1.3.1 *The moduli space of solutions of Seiberg-Witten equations for a spin^c structure $\tilde{P}_M$ on M is defined as*

$$\mathcal{M}(\tilde{P}_M) := \{(A,\psi) \in \mathcal{A}(\mathcal{L}_M) \times \Gamma(S^+(\tilde{P}_M)) | F_A^+ = q(\psi), \not{\partial}_A \psi = 0\}/\mathcal{G}(\tilde{P}_M) \subset \mathcal{B}(\tilde{P}_N); \tag{1.37}$$

We will denote by $\mathcal{M}^*(\tilde{P}_M)$ the subset given by the gauge equivalence classes of irreducible solutions.

Let's consider the properties of these solutions. The more important fact, concerning the solutions of SW equations, is that the norm of the spinors is bounded by above in terms of the scalar curvature of the metric. As we will see this property is in some sense the basic feature of these equations and will bring important consequences.

Proposition 1.3.2 *Let (A,ψ) be a solution of Seiberg-Witten equations and consider the function $|\psi|^2 : M \to \mathbb{R}$; at a point of maximum $p \in M$ such a function satisfies the inequality*

$$|\psi|^2 \leq max(0,-s). \tag{1.38}$$

Proof: if (A,ψ) is a reducible solution the statement is trivially true. Let's suppose it is irreducible; the standard Bochner-Weitzenböck formula, applied to the case of a spin^c structure, for a couple (A,ψ), takes the form

$$\not{\partial}_A^2 \psi = \nabla_A^* \nabla_A \psi + \frac{s}{4}\psi + \frac{1}{2} F_A^+ \cdot \psi, \tag{1.39}$$

where s denotes the scalar curvature of M. This formula follows from the usual one making explicit the Clifford multiplication and noting that $F_A^- \cdot \psi = 0$ for positive spinors. Consider now a solution (A,ψ) of SW equations: working on a compact manifold we can consider a point where $|\psi|^2$, the point square norm of ψ, attains its maximum. At this point, the Bochner-Weitzenböck formula and SW equations imply

$$\begin{gathered} 0 \leq d^*d|\psi|^2 = 2 < \nabla_A^* \nabla_A \psi, \psi > -2 < \nabla_A \psi, \nabla_A \psi > \leq \\ \leq 2 < \nabla_A^* \nabla_A \psi, \psi >= -\tfrac{s}{2} < \psi, \psi > - < q(\psi)\psi, \psi >= -\tfrac{s}{2}|\psi|^2 - \tfrac{1}{2}|\psi|^4 \end{gathered} \tag{1.40}$$

(the last term comes from the explicit formula 1.6 for $q(\psi)$). If we suppose now that $|\psi|$ does not vanish at its maximum, i.e. that ψ is not identically zero, we can divide by $|\psi|^2$ and we obtain exactly what we stated in the Proposition.

This proposition has two fundamental corollaries, the first of which follows immediately from the proposition:

Corollary 1.3.3 *On a manifold M with positive scalar curvature the SW equation admit no irreducible solutions.*

Corollary 1.3.4 *The moduli space of solutions of SW equations $\mathcal{M}(\tilde{P}_M)$ is sequentially compact.*

Proof: we must show that for any sequence of solutions $(A_i, \psi_i)_{i\in I}$ in $\mathcal{A}(\mathcal{L}_M)\times\Gamma(S^+(\tilde{P}_N))$ we can extract a subsequence indexed by $J \subset I$ and gauge transformations $(g_i)_{i\in J}$ such that $g_i \cdot (A_i, \psi_i)$ converges smoothly to a solution in $\mathcal{A}(\mathcal{L}_M) \times \Gamma(S^+(\tilde{P}_N)$. First of all we will prove a global gauge fixing theorem for a connection A, following the same line of reasoning of the non abelian case.
Start by considering a smooth reference connection $A_0 \in \mathcal{A}(\mathcal{L}_M)$: putting a connection A in Coulomb gauge with respect to A_0 corresponds to finding a gauge transformation $g \in \mathcal{G}(\tilde{P}_M)$ such that

$$g \cdot A - A_0 \in ker d^*. \tag{1.41}$$

Let's see that this can be achieved using an element in the connected component of the gauge group: we can write any such element as $g = exp(\chi)$ for $\chi \in Lie\mathcal{G}(\tilde{P}_M)$, the Lie algebra of the gauge group. The gauge fixing condition has the form

$$d^*(A + 2d\chi - A_0) = 0 \tag{1.42}$$

and we look thus for a χ satisfying that equation. This corresponds to solving the elliptic equation

$$2\Delta\chi = d^*(A_0 - A). \tag{1.43}$$

The latter equation can be solved once is known that Δ maps onto imd^*: but this follows from Hodge decomposition of 0-forms, as these decompose, on a closed manifold, as $imd^* \oplus ker\Delta$. Once such a χ is found, $g = exp(\chi) \in \mathcal{G}^e(\tilde{P}_M)$ will define the suitable gauge transformation. The hypothesis 1.41 allows us to write

$$g \cdot A - A_0 = d^*\gamma + \omega, \tag{1.44}$$

where ω is an harmonic 1-form, as in general $Harm^1(M) = H^1_{DR}(M, \mathbb{R}) \neq 0$. The presence of this harmonic term does not create any problem, as we can control its norm taking gauge transformations in a suitable component of the gauge group, whose component group is, as shown in Section 1.2,

$$\pi_0 Map_*(M, U(1)) = \pi_0 Map_*(M, K(\mathbb{Z}, 1)) = H^1(M, \mathbb{Z}) \tag{1.45}$$

and thus ω is defined in the Jacobian torus $H^1(M, \mathbb{R})/H^1(M, \mathbb{Z})$.
Now, for each p, we can control the $\mathcal{L}^p_1$ norm of $g_i \cdot A_i - A_0$, with (A_i, ψ_i) solution of SW equations, through the fundamental inequality for the elliptic operator $d^+ + d^*$ (for notational simplicity, we will not indicate no more the gauge transformations and will assume the A_i as gauge fixed w.r.t. A_0):

$$\begin{aligned} \|A_i - A_0\|_{\mathcal{L}^p_1} &\leq C(\|(d^+ + d^*)(A_i - A_0)\|_{\mathcal{L}^p} + \|\omega_i\|_{\mathcal{L}^p}) = C(\|d^+(A_i - A_0)\|_{\mathcal{L}^p} + \|\omega_i\|_{\mathcal{L}^p}) \leq \\ &\leq C(\|q(\psi_i)\|_{\mathcal{L}^p} + \|d^+ A_0\|_{\mathcal{L}^p} + \|\omega_i\|_{\mathcal{L}^p}) = C(\tfrac{1}{2}\|\psi_i\|^2_{\mathcal{L}^{2p}} + \|d^+ A_0\|_{\mathcal{L}^p} + \|\omega_i\|_{\mathcal{L}^p}). \end{aligned} \tag{1.46}$$

Note that in the estimate it does not appear the $\mathcal{L}^p$ norm of the term $d^*\gamma_i$, as it is $\mathcal{L}^2$-orthogonal to the kernel of $d^+ + d^*$, which is $ker(d^+ + d^*) = kerd|_{kerd^*} = Harm^1(M)$. Now ψ_i satisfies a $\mathcal{C}^0$ uniform bound, as proven previously, and then $\|\psi_i\|^2_{\mathcal{L}^{2p}}$ is bounded; the previous remark on ω_i, plus the hypothesis on A_0, brings an uniform (in i) bound, depending only on A_0, for $\|A_i - A_0\|_{\mathcal{L}^p_1}$. If we put $p > 4$, in particular, $\mathcal{L}^p_1 \subset \mathcal{C}^0$ and $(A_i - A_0)$ has a $\mathcal{C}^0$ bound. From this bound we get an $\mathcal{L}^p_1$ bound on ψ_i, as the elliptic inequality applied to the Dirac operator $\partial\!\!\!/_{A_0}$ brings

$$\|\psi_i\|_{\mathcal{L}^p_1} \leq C'(\|\partial\!\!\!/_{A_0}\psi_i\|_{\mathcal{L}^p} + \|\psi_i\|_{\mathcal{L}^p}) \tag{1.47}$$

(the reason why we choose an estimate w.r.t. $\partial\!\!\!/_{A_0}$ is that we look for a bound not depending on i) and this inequality becomes

$$\|\psi_i\|_{\mathcal{L}^p_1} \leq C'(\|\partial\!\!\!/_{A_i}\psi_i\|_{\mathcal{L}^p} + \frac{1}{2}\|(A_i - A_0)\psi_i\|_{\mathcal{L}^p} + \|\psi_i\|_{\mathcal{L}^p}). \tag{1.48}$$

Now SW equations and the $\mathcal{C}^0$ bound on $(A_i - A_0)\psi_i$ bring to the required uniform $\mathcal{L}^p_1$ bound on ψ_i. We can continue this process; with our choice of $p > 4$, $\mathcal{L}^p_1$ is an algebra; this provides a control on the $\mathcal{L}^p_1$ norm of $q(\psi_i)$ and the fundamental elliptic inequality for $d^+ + d^*$ gives a bound on the $\mathcal{L}^p_2$ norm of $(A_i - A_0)$; analogously we have $(A_i - A_0)\psi_i \in \mathcal{L}^p_1$ and the fundamental elliptic inequality for $\partial\!\!\!/_{A_0}$ gives a bound on the $\mathcal{L}^p_2$ norm of ψ_i. We can continue this bootstrapping process to gain regularity in higher derivatives: this allows to take converging subsequences in each Sobolev space, as the sequence lives in a bounded and complete subset of a function space; Sobolev embedding theorem brings then smooth convergence and thereafter sequential compactness. We note that as the moduli space can easily be given a metric, this result implies compactness, as well.

Once proven the compactness of the moduli space, our aim is to study its structure. As it is usual in similar situations, we will not study directly the solutions of SW equations, but the solutions of a parameterized perturbation of them, in such a way to study a family of moduli spaces parameterized by the perturbation parameters. This will allow, by use of the implicit function theorem, to deduce results of regularity for generic values of the parameter and, eventually, for generic paths of parameters. We will perturb the equations adding, in the curvature part, an imaginary self dual term contained in a suitable Sobolev space: consider thus a perturbation $\eta \in \Omega^2_+(M, i\mathbb{R})$ and define the η-SW equations as

$$\begin{cases} \not{\partial}_A \psi = 0 \\ F_A^+ = q(\psi) + \eta. \end{cases} \tag{1.49}$$

We call $\mathcal{M}(\tilde{P}_M, \eta)$ the moduli space of solutions of the η-SW equations. It is easy to prove that this moduli space, once η is contained in a suitable Sobolev space (e.g. $\mathcal{L}^2_3$, in such a way that such a perturbation is continuous) has the same compactness property proven in the case $\eta = 0$. We define now the parameterized moduli space as

$$\mathcal{PM}(\tilde{P}_M) = \bigcup_{\eta \in \Omega^2_+(M, i\mathbb{R})} \mathcal{M}(\tilde{P}_M, \eta) \tag{1.50}$$

with obvious notation. If we complete the space of connections and the gauge group in the suitable Sobolev spaces the irreducible parts of the moduli spaces live in a Banach space, allowing the use of the implicit function theorem for Banach spaces. We are in position to give a regularity theorem for the parameterized moduli space:

Theorem 1.3.5 *$\mathcal{PM}^*(\tilde{P}_M)$ is a smooth manifold, the projection map $\pi : \mathcal{PM}^*(\tilde{P}_M) \to \Omega^2_+(M, i\mathbb{R})$ is proper Fredholm with index*

$$d(\tilde{P}_M) = \frac{1}{4}(c_1(\mathcal{L}_M)^2 - (2\chi(M) + 3\sigma(M))) \tag{1.51}$$

and thus, for a second category subset of the parameter space, the fiber $\mathcal{M}^*(\tilde{P}_M, \eta)$ is smooth of dimension $d = d(\tilde{P}_M)$.

Proof: consider the solutions of the parameterized equations, with nonvanishing ψ. We can consider this set of solutions as the zero locus of a $\mathcal{G}(\tilde{P}_M)$-equivariant map

$$SW : \mathcal{A}(\mathcal{L}_M) \times \Gamma^*(S^+(\tilde{P}_M)) \times \Omega^2_+(M, i\mathbb{R}) \longrightarrow \Omega^2_+(M, i\mathbb{R}) \times \Gamma(S^-(\tilde{P}_M))$$
$$SW(A, \psi, \eta) = (F_A^+ - q(\psi) - \eta, \partial\!\!\!/_A \psi); \tag{1.52}$$

this map is evidently smooth. In order to obtain transversality results we must first of all prove that the linearization of the map is surjective, as we restrict to irreducible solutions: the linearization of SW, computed in correspondence of a solution (A, ψ, η), is the operator

$$DSW_{(A,\psi,\eta)} : \Omega^1(M, i\mathbb{R}) \times \Gamma(S^+(\tilde{P}_M)) \times \Omega^2_+(M, i\mathbb{R}) \longrightarrow \Omega^2_+(M, i\mathbb{R}) \times \Gamma(S^-(\tilde{P}_M))$$
$$DSW_{(A,\psi,\eta)}(a, \phi, \epsilon) = (d^+ a - 2q(\phi, \psi) - \epsilon, \partial\!\!\!/_A \phi + \tfrac{1}{2} a \cdot \psi) \tag{1.53}$$

(recall that working with a connection on $\mathcal{L}_M$ we have that $\partial\!\!\!/_{A+a}\psi = \partial\!\!\!/_A \psi + \frac{1}{2} a \cdot \psi$). We must prove that this operator DSW is surjective, to use the implicit function theorem. If an element (b, φ) is $\mathcal{L}^2$-orthogonal to $ImDSW$, we must have, $\forall (a, \phi, \epsilon)$,

$$< d^+ a - 2q(\phi, \psi) - \epsilon, b > + < \partial\!\!\!/_A \phi + \frac{1}{2} a \cdot \psi, \varphi >= 0. \tag{1.54}$$

First, we show that $(b, 0)$ is in $ImDSW$: if there exist a $b \in \Omega^2_+(M, i\mathbb{R})$ such that $(b, 0) \notin ImDSW$, by varying a we see that it must live in *cokerd*$^+$ (in particular it must satisfy unique continuation theorem because of surjective ellipticity of d^+, and so it cannot vanish on an open set without vanishing everywhere); by varying ϵ then (even within the Banach subspace of forms supported in a ball) we see that it must vanish everywhere. Now let's prove that $(0, \varphi) \in ImDSW$: if it where orthogonal to $ImDSW$ then it would satisfy, for any $(a, \phi, 0)$

$$< \partial\!\!\!/_A \phi + \frac{1}{2} a \cdot \psi, \varphi >= 0. \tag{1.55}$$

Putting a equal to zero we see that φ solves a Dirac equation, and so cannot vanish on an open set without vanishing globally. Using the fact that ψ as well solve Dirac equation and is not identically zero, and choosing $\phi = 0$ and a bumped around a point of the (nonempty) common support of ψ and φ we get $\varphi = 0$ too. This implies that DSW is surjective. From the construction it is clear the reason of the choice of $\Omega^2_+(M, i\mathbb{R})$ as parameter space. From the previous discussion, moreover, we see that we can also choose,

as perturbation space, forms supported in a ball.
By equivariance, the map SW descends to a map over the gauge orbit space, and its zero locus is the moduli space of irreducible solutions.
We can now complete the prove that the zero locus of this map is regular: first of all we note that this map is Fredholm: its linearization up to zero order term is the sum of two elliptic operators, namely the selfduality Hodge operator and the Dirac operator and thus is Fredholm; this linearization, as seen above, is surjective; standard results of transversality theory bring thereafter the regularity of the parameterized moduli space. The restriction of the projection map π to the zero locus of SW is a Fredholm map of index given the sum of the real index of Dirac and selfduality Hodge operators, and is proper, because of $\mathcal{C}^0$ bounds on the solutions. Sard-Smale theorem guarantees then the existence of a Baire second category (i.e. countable intersection of open and dense) subset of regular values for π in $\Omega^2_+(M, i\mathbb{R})$ for which π provides fiberwise local charts in such a way that $\mathcal{M}(\tilde{P}_M, \eta)$ is a compact smooth manifold. Moreover, applying density arguments it is also possible to show that we can assume the perturbation to be of class $\mathcal{C}^\infty$, see [Mo].

To compute the dimension of a generic fiber $\mathcal{M}^*(\tilde{P}_M, \eta)$ of π, after Coulomb gauge fixing, we note that we have to compute the index of an operator which is the compact perturbation of the sum of the operators

$$\begin{gathered} \not{\partial}_A : \Gamma(S^+(\tilde{P}_M)) \longrightarrow \Gamma(S^-(\tilde{P}_M)) \\ d^* + d^+ : \Omega^1(M, i\mathbb{R}) \longrightarrow \Omega^0(M, i\mathbb{R}) \oplus \Omega^2_+(M, i\mathbb{R}). \end{gathered} \tag{1.56}$$

The first index is in fact the index of the spin^c Dirac operator that we can compute via the Atiyah-Singer index theorem which gives, in general,

$$ind_{\mathbb{C}} \not{\partial}_A = \int_M e^{\frac{1}{2}c_1(\mathcal{L}_M)} \cdot \hat{\mathbf{A}}(M). \tag{1.57}$$

We develop the computation on a four manifold: the $\hat{\mathbf{A}}$-genus satisfies

$$\hat{\mathbf{A}}(M) = 1 - \frac{1}{24}p_1(TM) \tag{1.58}$$

and thus

$$\int_M [1 + \frac{1}{2}c_1(\mathcal{L}_M) + \frac{1}{8}c_1^2(\mathcal{L}_M)][1 - \frac{1}{24}p_1(TM)] = \frac{1}{8}(c_1^2(\mathcal{L}_M) - \sigma(M)) \tag{1.59}$$

as Hirzebruch signature theorem gives $p_1(TM) \cap [M] = 3\sigma(M)$. The second index to compute is the Euler characteristic of the complex

$$0 \to \Omega^0(M, i\mathbb{R}) \to \Omega^1(M, i\mathbb{R}) \to \Omega^2_+(M, i\mathbb{R}) \to 0 \tag{1.60}$$

which has index equal to $-\frac{1}{2}(\chi(M) + \sigma(M))$ as can be readily computed.
The sum of these indexes gives immediately the formula stated in the proposition.

Now we have proven that, in correspondence of the generic fiber, the moduli space $\mathcal{M}^*(\tilde{P}_M, \eta)$ is smooth and compact; in order to associate it a fundamental homology class, which will allow the definition of the invariants, we have to prove that it is orientable.
To prove orientability, let's see that a local model of the moduli space of irreducible solutions is provided by the zero set of a map between finite dimensional vector spaces, according to the decomposition of deformation complex associated to the differential of a Fredholm map as

$$H^1(M, i\mathbb{R}) \rightarrowtail \Omega^1(M, i\mathbb{R}) \stackrel{d^+ + d^*}{\to} \Omega^2_+(M, i\mathbb{R}) \oplus \Omega^0(M, i\mathbb{R}) \twoheadrightarrow H^2_+(M, i\mathbb{R}) \oplus H^0(M, i\mathbb{R}),$$

$$ker \not{\partial}_A \rightarrowtail \Gamma(S^+(\tilde{P}_M)) \stackrel{\not{\partial}_A}{\to} \Gamma(S^-(\tilde{P}_M)) \twoheadrightarrow coker \not{\partial}_A. \tag{1.61}$$

It is clear that this identifies the tangent space to the moduli space as the zero set of a map

$$f : H^1(M, i\mathbb{R}) \oplus \ker \not{\partial}_A \longrightarrow H^2_+(M, i\mathbb{R}) \oplus H^0(M, i\mathbb{R}) \oplus coker \not{\partial}_A \tag{1.62}$$

(for a discussion of this point in Yang-Mills theory see [DK]). An orientation bundle will be therefore defined by the tensor product of the real determinant bundles of the index bundles of the Fredholm operators $d^+ + d^*$ and $\not{\partial}_A$. Clearly the first bundle is trivial on the orbit space, and the second, although the complex determinant bundle of the Dirac operator is in general not trivial, as we have seen in Section 1.2, it is trivial as real line bundle. This proves not only the orientability, but gives a canonical orientation bundle: the choice of an orientation is provided by a choice of sign in the maximal external product in

$$H^0(M, \mathbb{R}) \oplus H^1(M, \mathbb{R}) \oplus H^2_+(M, \mathbb{R}), \tag{1.63}$$

as the Dirac determinant bundle has a given canonical orientation provided by its complex structure.

1.4 Definition of the Invariants.

We are now in position to define, for a given metric on M, an invariant (respect to the choice of the parameter η) associated to each spin^c structure and then discuss its dependence on the metric. The basic problem is that we would like to work with irreducible solutions, to maintain smoothness of moduli space, while changing the parameters (which could be the perturbation parameter η or the metric), we could encounter reducible solutions. Let's see how to treat this problem.

First of all we see that, for any metric, reducible solutions for the unperturbed equations correspond to connections A with anti self dual curvature: this is quite a strong requirement, as $\frac{i}{2\pi}F$ is the Chern class of the line bundle $\mathcal{L}_M$ and thus it represents a point in the integral lattice of $H^2(M,\mathbb{R})$ which has zero projection on the self dual subspace $H^2_+(M,\mathbb{R})$; more generally, a reducible solution to η-SW equations corresponds to a point of $H^2(M,\mathbb{Z})$ whose projection to $H^2_+(M,\mathbb{R})$ coincides with the harmonic self dual part of $\frac{2\pi}{i}\eta$; it is clear (applying an easy Hodge decomposition argument) that this condition is not generic w.r.t. the choice of the perturbation, once $b^2_+(M) > 0$, and therefore in that case, outside a closed nowhere dense subset of the parameter space, we have not reducible solutions. Note that this is true also for spin^c structures which have torsion determinant bundle. In the case that $b^2_+(M) > 1$, more is true: if we take a generic couple of perturbation parameters η_0 and η_1 and a generic path η_t transverse to the projection map π in $\Omega^2_+(M, i\mathbb{R})$ connecting them, then the moduli spaces associated to η_0 and η_1 are compactly cobordant: in fact, as long as $b^2_+ > 1$, the self dual harmonic component of the path $\frac{2\pi}{i}\eta_t$ will not meet any value which is self dual part of a point in the integral lattice $H^2(M,\mathbb{Z})$; the counterimage of the path η_t w.r.t. π is a smooth, compact, oriented manifold of dimension $dim\mathcal{M}(\tilde{P}_M) + 1$ with boundary $\mathcal{M}(\tilde{P}_M, \eta_0) \coprod \mathcal{M}(\tilde{P}_M, \eta_1)$ (the proofs of these statements follow exactly like in the case of fixed perturbation) and thus provides a compact oriented cobordism between the two moduli spaces: this is nothing but an easy consequence of transversality.

With this approach we can also deal with the problem of the dependence from the metric: the same line of reasoning as above allows to show that taking any two metrics g_0, g_1 on M and two generic perturbations η_0, η_1 self dual w.r.t. the corresponding metric, we obtain cobordism between the moduli spaces, considering a path of metrics connecting g_0 and g_1 and a generic path η_t of g_t self dual forms connecting η_0 and η_1, see [Mo].

In the rest of this work we will always assume that $b^2_+(M)$ is greater than 1 and so all moduli spaces will be generically cobordant.

We can now define, regardless of the perturbation and the metric chosen an invariant which depends only on the smooth structure of M.
Consider first the case of $d(\tilde{P}_M) > 0$: the space of based solutions $\mathcal{M}^o(\tilde{P}_M)$ fibers over $\mathcal{M}(\tilde{P}_M)$ with fiber $U(1)$ via the base point fixing fibration. Denote by μ the Euler class of this fibration; as we have seen in Section 1.2 this is the restriction of the two dimensional generator of the cohomology ring of the orbit space $\mathcal{B}(\tilde{P}_M)$:

Definition 1.4.1 *The Seiberg-Witten invariant associated to a spin^c structure $\tilde{P}_M$ with $d(\tilde{P}_M) > 0$ is defined as*

$$SW(\tilde{P}_M) := \mu^{[\frac{d}{2}]} \cap [\mathcal{M}(\tilde{P}_M)]. \tag{1.64}$$

This vanishes trivially if d is odd and is a well defined integer if d is even, which corresponds to the case of $b^1(M) + b^2_+(M)$ odd.
The Euler class μ is the restriction of one generator of the cohomology ring of the orbit space; it is quite clear that using other generators we can similarly define other invariants, which are polynomial in $H_1(M,\mathbb{Z})$ and $H_0(M,\mathbb{Z})$; these will not concern the sequel of this work.
Consider now the case of zero dimensional moduli space: in that case $\mathcal{M}(\tilde{P}_M)$ is a smooth compact zero-dimensional oriented manifold and thus a finite set of oriented points:

Definition 1.4.2 *The Seiberg-Witten invariant associated to a spin^c structure $\tilde{P}_M$ with $d(\tilde{P}_M) = 0$ is defined as*

$$SW(\tilde{P}_M) = \#\mathcal{M}(\tilde{P}_M). \tag{1.65}$$

These invariants, for what we have shown, are well defined and depend only on the smooth structure of M and the spin^c structure which is considered, as long as $b^2_+(M) > 1$. It is clear from the dimension formula that the dimension of these moduli spaces, for a fixed manifold, depends only on the Chern class of the determinant bundle of the spin^c structure. There's a fundamental conjecture that states that, under the hypothesis of $b^2_+(M) > 1$, the only spin^c structures that can give non zero Seiberg-Witten invariants are those who have a zero dimensional moduli space (simple type conjecture). Up to now there's not a precise strategy of how to prove this conjecture in generality, apart from specific classes of four manifolds.

Using the previous definition we can associate to the manifold M a function, denoted by SW_M, from the set of spin^c structures on it, an affine $H^2(M,\mathbb{Z})$, to the integer numbers, i.e.

$$SW_M : Spin^c(M) \longrightarrow \mathbb{Z} \tag{1.66}$$

defined associating to any spin^c structure $\tilde{P}_M$ the value of $SW(\tilde{P}_M)$. The determinant bundle of a spin^c structure with non vanishing SW invariant is called a basic class; an important fact that comes quite directly from the definition is that the set of basic classes is finite, which amounts to say that the function SW_M is non zero only on a finite set of spin^c structures.

Now that we have defined these invariants it is relevant to know some classes of four manifolds for which the function SW_M is non zero. This is, of course, interesting *per se* and moreover this group of manifolds (which extends from time to time) will naturally be a field of application of the results of the sequel. We limit ourselves to quote some classes (eventually overlapping) of four manifold with $b^2_+(M) > 1$ sharing this property:

- Kähler surfaces ([Wi],[FM] *et al.*);
- Symplectic manifolds ([T3]);
- Non Kähler complex surfaces ([Bi]);
- Manifolds obtained from those in the first two classes which contain *c-embedded tori* (e.g. algebraic elliptic surfaces), removing their neighborhoods and then gluing the open manifold with the complement of a link $L \subset S^3$ times S^1 ([Sz],[FS2]).

The computation of the Seiberg-Witten invariants in the quoted cases is based on explicit analysis of the space of solutions, using the Kähler or symplectic structure, or via gluing formulae along T^3 ([MMS]).

Chapter 2

Seiberg-Witten Theory on $N \times [0, T]$

In this Chapter, which is rather technical, we will discuss the behavior of a finite energy solution of Seiberg-Witten equations on a cylinder $N \times [0, T]$, where N is a three dimensional compact closed three manifold admitting a metric of positive scalar curvature. Following a pattern that is common with Yang-Mills theory, the Seiberg-Witten equations on a cylinder appear as gradient flow equations for the Chern-Simons-Dirac functional, defined over the three dimensional orbit space ([KM1]). We prove here that these gradient flows converge exponentially, as the cylinder becomes infinite, to the critical points of the CSD functional, which correspond to the solutions of a three dimensional version of Seiberg-Witten equations on N i.e., up to a covering, to a component of the $U(1)$ character variety of N.

2.1 Gradient Flow Equations.

Let M be a closed four manifold which decomposes along a three manifold N. The Seiberg-Witten theory on M is clearly related to the Seiberg-Witten theory on the two factors of the decomposition, which appear as open manifolds with boundary N, so we address ourselves to the study of it.

Let X be a four manifold whose boundary $\partial X = N$ admits a metric of positive scalar curvature. Over X we can consider a spin^c structure $\tilde{P}_X$ and we can define, in the very same way of Chapter 1, the Seiberg-Witten equations associated to this spin^c structure. Instead of studying directly these equations, a fruitful approach to the problem of manifolds with boundary is to consider the complete manifold, with cylindrical end, defined as

$$\hat{X} = X \cup_N N \times [0, \infty). \tag{2.1}$$

This manifold, from the topological point of view, is equivalent to X, as it retracts onto it, but has different geometrical properties. Our aim is to study the moduli space of solutions of SW equations for a spin^c structure $\tilde{P}_{\hat{X}}$ defined on $\hat{X}$; these are induced from spin^c structures on the compact part X. Let $\tilde{P}_X$ be a spin^c structure on X, with determinant line bundle $\mathcal{L}_X$; this spin^c structure can be extended to a spin^c structure on $\hat{X}$. We denote by $\tilde{P}_N$ the restriction of $\tilde{P}_X$ to $\partial X = N$, and L its determinant line bundle, that we identify with its topological Chern class in $H^2(N, \mathbb{Z})$. On the end of $\hat{X}$ we have a bundle isomorphism $[0, \infty) \times L \cong \mathcal{L}_{\hat{X}}|_{N\times[0,\infty)}$, and a correspondingly an embedding $[0, \infty) \times \tilde{P}_N \hookrightarrow \tilde{P}_{\hat{X}}|_{N\times[0,\infty)}$ which covers the embedding of frame bundles. With a slight abuse of notation we will say that, on the end, the spin^c structure $\tilde{P}_{\hat{X}}$ is the pull-back of $\tilde{P}_N$; although this is evidently false in the sense of the bundles, it is true from the point of view of their cohomological representatives.

A major ingredient in the study of the structure of the moduli spaces of solutions of the Seiberg-Witten equations defined for a spin^c structure on a cylindrical end manifold is the knowledge of the behavior of the solutions on the end of the manifold. We want to analyse the asymptotic behavior of finite energy solutions of SW equations for $\tilde{P}_{\hat{X}}$ on the end of $\hat{X}$; we need to study therefore the Seiberg-Witten-Floer theory on the cylinder $N \times [0, \infty)$.

The study of the form of Seiberg-Witten equations on a cylinder (SWF equations now on), already began in [KM1], parallels strictly the analogous one in Yang-Mills theory, in [MMR], [T2], so we limit ourselves to discuss the points of major interest.
Consider a spin^c structure $\tilde{P}$ with determinant bundle $\mathcal{L}$ on a cylinder $N \times [0, T]$ (the results of this Section include as well the case of $T = \infty$).
We denote as usual with $\mathcal{A}(\mathcal{L})$ the space of $U(1)$ connections on the line bundle $\mathcal{L}$ and with $\mathcal{A}(L)$ the space of $U(1)$ connections on the line bundle L, while we denote by $S(\cdot)$ the bundle of spinors associated to a spin^c structure.
When we work with an open manifold, we suppose that connections and spinors are locally in some Sobolev space, so they live, say, in $\mathcal{L}^2_{k,loc}$, $k \geq 2$ and the gauge group is defined in such a way to act on them, so in $\mathcal{L}^2_{k+1,loc}$.
The embedding of the pull back of the spin^c structure on N in the spin^c structure $\tilde{P}$

induces isomorphisms of spinor bundles $[0,T] \times S(\tilde{P}_N) \cong S^{\pm}(\tilde{P})$. As a consequence of this a section $\psi \in \Gamma(S^{\pm}(\tilde{P}))$ defines a path in $\Gamma(S(\tilde{P}_N))$. We can obtain a similar result also for the connections: a connection $A \in \mathcal{A}(\mathcal{L})$ is said to be in temporal gauge if the $[0,T]$ direction is parallel w.r.t. the connection, i.e. A has no temporal component. It is clear that there exist a gauge transformation, connected to the identity, which puts a connection in temporal gauge. Therefore, up to a gauge transformation, we can assume that A as well defines a path in $\mathcal{A}(L)$.

Under the aforementioned assumptions, elements of $\mathcal{A}(\mathcal{L}) \times \Gamma(S^{+}(\tilde{P}))$ appear therefore as paths

$$(A, \psi) : [0,T] \longrightarrow \mathcal{A}(L) \times \Gamma(S(\tilde{P}_N)), \tag{2.2}$$

The four dimensional equations, in temporal gauge, for a couple connection-spinor (A,ψ) in $\mathcal{A}(\mathcal{L}) \times \Gamma(S^{+}(\tilde{P}))$ on $N \times [0,T]$, appear as a pair of equations for a path of couples $(A(t), \psi(t))$ in $\mathcal{A}(L) \times \Gamma(S(\tilde{P}_N))$ on N of the form

$$\begin{cases} \frac{\partial}{\partial t}\psi(t) = \not{\partial}_{A(t)}\psi(t), \\ \frac{\partial}{\partial t}A(t) = *(q(\psi(t)) - F_{A(t)}). \end{cases} \tag{2.3}$$

These equations can be obtained from equations 1.9 by direct computation, specializing to the cylindrical case. The bilinear term $q(\cdot)$ is defined, with the suitable modifications, like the one in equation 1.6. After endowing $\mathcal{A}(L) \times \Gamma(S(\tilde{P}_N))$ of an inner product (slightly different from the standard one induced by the pointwise product of equation A.2, as on the spinor part we take twice the real part of the hermitean product), equations 2.3 are just the gradient flow equations for a suitable generalization of the Chern-Simons functional, whose critical points are given by the static solutions ([MST]). This Chern-Simons-Dirac functional is defined, on $\mathcal{A}(L) \times \Gamma(S(\tilde{P}_N))$, as

$$\begin{gathered} C : \mathcal{A}(L) \times \Gamma(S(\tilde{P}_N)) \longrightarrow \mathbb{R}, \\ C(A,\psi) = \frac{1}{2}\int_N (F_A + F_{A_0}) \wedge (A - A_0) + \int_N < \psi, \not{\partial}_A \psi >, \end{gathered} \tag{2.4}$$

where A_0 is a fixed reference connection in $\mathcal{A}(L)$. As happens with the Chern-Simons functional in Yang-Mills theory, the CSD functional is not in general invariant under the action of the three dimensional gauge group, but only under the component connected to the identity.

Acting with the set of components of the three dimensional gauge group, $C(A,\psi)$ changes by the cup product with a term proportional to $c_1(L)$:

$$C(A^g, \psi^g) = C(A, \psi) + 8\pi^2 c_1(L) \cup [g], \tag{2.5}$$

where $[g] \in H^1(N, \mathbb{Z})$ is the class corresponding to the set of components: an application of the results of Section 1.2 to the three manifold N identifies in fact $\pi_0(\mathcal{G}(\tilde{P}_N))$ with $H^1(N, \mathbb{Z})$. In the case where L is a torsion bundle (i.e. $c_1(L) = 0$ in rational cohomology), therefore, C is invariant under the full gauge group.
Equations 2.3 have therefore the form

$$\frac{\partial(A(t), \psi(t))}{\partial t} = \nabla C(A(t), \psi(t)). \tag{2.6}$$

It is not difficult to verify, using various explicit formulae computed in the appendix, that the gradient flow is orthogonal (w.r.t. the inner product used to define it) to the gauge orbits. Following what has become an habit, we will not keep track of the different inner product, all the difference amounting, for what will concern us, to an irrelevant scaling between connection and spinor part. For example, using the standard inner product, it is not ∇C but $(*(q(\psi) - F_A), 2\partial\!\!\!/_A \psi)$ which is orthogonal to the gauge orbits.

2.2 Seiberg-Witten Equations in Dimension 3.

The critical points of the Chern-Simons-Dirac functional defined in equation 2.4 correspond to the static solutions of the SWF equations, i.e. to couples $(A, \psi) \in \mathcal{A}(L) \times \Gamma(S(\tilde{P}_N))$ such that

$$\begin{cases} \partial\!\!\!/_A \psi = 0, \\ F_A = q(\psi). \end{cases} \tag{2.7}$$

These equations are the natural three dimensional version of SW equations. Adopting the notation of [KM3] we introduce the following

Definition 2.2.1 *An integral cohomology class represented by the determinant bundle L of a spin^c structure $\tilde{P}_N$ for which equations 2.7 admit solutions for any metric is called a monopole class.*

It is quite easy to construct a general theory for the moduli space of solutions of equations 2.7 by adapting, *mutatis mutandis*, the results of Chapter 1. In the case under analysis, anyhow, we will not need such a kind of results, as it is possible to recognize explicitly the solutions of equations 2.7:

Proposition 2.2.2 *For a three manifold admitting a metric of positive scalar curvature the monopole classes are the torsion classes; with that metric the only possible solutions of the three dimensional SW equations $\partial\!\!\!/_A\psi = 0$, $F_A = q(\psi)$ are the reducible couples $(A,0)$ where A is a flat connection.*

Proof: we first note that, for any metric on N, reducible couples $(A,0)$ with A a flat connection on a torsion bundle, are solutions to equations 2.7. If we now endow N with a metric with positive scalar curvature s, these are the only possible solution: the proof of this is a standard application of three dimensional Bochner-Weitzenböck formula, which in our case reads as

$$\partial\!\!\!/_A^2\psi = \nabla_A^*\nabla_A\psi + \frac{1}{2}F_A\cdot\psi + \frac{s}{4}\psi. \tag{2.8}$$

For solutions of the equations 2.7 this becomes

$$0 = \nabla_A^*\nabla_A\psi + \frac{1}{2}q(\psi)\cdot\psi + \frac{s}{4}\psi. \tag{2.9}$$

Taking the hermitean product with ψ and integrating over the manifold we get the integral formula

$$\int_N |\nabla_A\psi|^2 + \frac{1}{4}|\psi|^4 + \frac{s}{4}|\psi|^2 = 0. \tag{2.10}$$

In correspondence of positively curved N, we get $\psi = 0$. This means that A must be flat, and L torsion: the only monopole classes are therefore torsion elements in $H^2(N,\mathbb{Z})$.

The moduli space of solutions, that we denote by $\chi(N)$, is defined as quotient of the space of flat connections on L by the action of the gauge group $\mathcal{G}^o(\tilde{P}_N)$. This space appears as a covering, with fiber $Hom(H^1(N,\mathbb{Z}),\mathbb{Z}_2)$, of a component of the $U(1)$ character variety of N, i.e. $Hom(\pi_1(N),U(1))$, which is identified with the space of flat connections modulo $\mathcal{G}^o(L)$: in fact, the only difference amounts to the different action of the gauge group, as follows from the proof of Proposition 1.2.3, applied to the three dimensional case.
When N is a rational homology sphere, all line bundles are torsion and the only gauge equivalence class of flat connections is the one defining the torsion bundle L in

$$H^2(N,\mathbb{Z}) = Hom(\pi_1(N),U(1)) = \chi(N), \tag{2.11}$$

where the last equivalence (with the "honest" character variety) is implied by the fact that $H^1(N,\mathbb{Z}) = 0$. In the case where $b_1(N) > 0$, instead, this proposition tells us that static solutions of SWF equations appear only if L is a torsion bundle and that for each such L they correspond to a torus $T^{b_1(N)}$, which covers a component of $Hom(\pi_1(N), U(1))$ with fiber $\mathbb{Z}_2^{b_1(N)}$. By abuse of notation we will often refer to $\chi(N)$ as character variety.

We note that, of course, the proof of Proposition 2.2.2 applies also for proving that the kernel of a Dirac operator twisted by a flat connection vanishes on a three manifold of positive scalar curvature, result that will be used in the sequel.

Along the same lines of Proposition 2.2.2 we can prove as well that there are not irreducible solutions of equations 2.7 after a perturbation of the curvature equation of the form

$$F_A = q(\psi) + i\delta, \tag{2.12}$$

as long as the perturbation $\delta \in \Omega^2(N, \mathbb{R})$ is small enough, in the sense that the C^0 norm of the endomorphism $i\delta$ satisfies

$$\int_N \frac{s}{4}|\psi|^2 + < i\delta \cdot \psi, \psi > \geq (\frac{min(s)}{4} - ||i\delta||_{C^0}) \int_N |\psi|^2 > 0, \tag{2.13}$$

which we will always suppose later on.

We recall that, although in all formulas we will suppose that the manifold N has a scalar curvature which depends on the point, it is a general result that any manifold which admits a metric with nonnegative scalar curvature, which does not vanish identically, admits also a metric with constant positive scalar curvature ([Be]) so in fact we could even assume that s is constant.

We now prove that, in the case of rational homology sphere, the reducible connections are non degenerate solutions of Seiberg-Witten equations, while for $b_1(N) > 0$ they are nondegenerate in the sense of Bott, i.e. the hessian of our functional is non degenerate on the normal bundle to the critical set. We will use, like in Chapter 1, an argument based on transversality of a gauge equivariant map defining the equations; we can interpret this map as a section of the tangent bundle of $\mathcal{B}^o(\tilde{P}_N)$: it has the form

$$\xi(A, \psi) = (*(q(\psi) - F_A), \partial\!\!\!/_A \psi) \tag{2.14}$$

and it linearizes, around a static solution, to

$$D\xi_{(A_0,0)}(a, \phi) = (- * da, \partial\!\!\!/_{A_0} \phi). \tag{2.15}$$

The tangent space to the orbit space at $(A_0, 0)$ is given, by the gauge fixing condition, by

$$T_{(A_0,0)}\mathcal{B}^o(\tilde{P}_N) = ker d^* \oplus \Gamma((S(\tilde{P}_N)) = d^*\Omega^2(N, i\mathbb{R}) \oplus Harm^1(N, i\mathbb{R}) \oplus \Gamma(S(\tilde{P}_N)). \tag{2.16}$$

The map in equation 2.15 is therefore defined as an endomorphism

$$D\xi_{(A_0,0)} : ker d^* \oplus \Gamma(S(\tilde{P}_N)) \to ker d^* \oplus \Gamma(S(\tilde{P}_N)). \tag{2.17}$$

If an element (b,φ) is $\mathcal{L}^2$ orthogonal to $Im D\xi_{(A_0,0)}$ then it must satisfy both

$$< *da, b >= 0, \quad < \partial\!\!\!/_{A_0}\phi, \varphi >= 0, \quad \forall a, \phi; \tag{2.18}$$

by the selfadjointness of $\partial\!\!\!/_{A_0}$ we get, from the second equation,

$$\partial\!\!\!/_{A_0}\varphi = 0, \tag{2.19}$$

which implies $\varphi = 0$, as A_0 is flat (see the note at the end of Proposition 2.2.2). In the case where N is a real homology sphere, Hodge decomposition of one forms gives $b = d^*c$ and thus the first condition of equations 2.18 (putting $a = *c$) implies b to vanish as well. This result of non degeneracy is related to the fact that these solutions, although being reducibles, are not removable with a perturbation of the equation, as we will analyse later on.

In the case that $b_1(N) > 0$, instead, the computation above says that a couple $(b, 0)$ is $\mathcal{L}^2$ orthogonal to $Im D\xi_{(A_0,0)}$ when $b \in Harm^1(N, i\mathbb{R})$, i.e. that the hessian of the Chern-Simons-Dirac functional is non degenerate on the normal bundle to the critical set. This tells us that the critical set, which is smooth, is non degenerate in the sense of Bott. In fact in this case a perturbation of the equations, as in 2.12, with a closed non exact form, can remove all the solutions. Summing up, we have the

Proposition 2.2.3 *The solutions of three dimensional Seiberg-Witten equations on a three manifold N of positive scalar curvature are non degenerate if $b_1(N) = 0$ and non degenerate in the sense of Bott if $b_1(N) > 0$.*

It is the right moment to point out a basic difference between the case of positive scalar curvature and the case of zero scalar curvature. It is easy to verify that the proof of Proposition 2.2.2, applied to the case of a manifold N with $s = 0$ brings as well to a vanishing result for the spinor part of the solutions of SW equations, i.e. the moduli space of solutions, also in that case, corresponds to the $U(1)$ character variety of the three manifold. What does not hold true anymore, instead, is that having a connection A flat implies $\ker \partial\!\!\!/_A = 0$; the Bochner-Weitzenböck formula implies just, in that case, that an element in the kernel is ∇_A-covariantly constant. If there exist covariantly constant spinors, for some flat connection, this connection is a non smooth point of the space of solutions: the proof of Proposition 2.2.3 gives up (see equation 2.19 and its consequences). In the sequel of this work it will be quite clear how such a situation would affect all the discussion of the problem of four manifolds split along a three manifold of zero scalar curvature; for an analysis of this situation, the only reference, as far as we know, is [MMS], where the case of T^3 is analysed.

Another remark that is worthy is that the result we have obtained, for three dimensional manifolds, is related to the scalar curvature and not only on the homotopy type. In Yang-Mills theory the set of critical points of Chern-Simons functional is the $SU(2)$ character variety of the three manifold and the theory depends therefore only on the fundamental group of the three manifold. In particular S^3 and fake spheres, in Donaldson theory, behave in the same way with respect to the problem of decomposition of four manifolds. In Seiberg-Witten theory, instead, the critical set depends also on the curvature and there is no reason why homotopic three manifold should have the same critical set. For example, as we don't know whether fake spheres admit a metric of positive scalar curvature, we don't know any way to deduce any information on the decomposition problem for fake spheres using directly Seiberg-Witten theory.

2.3 Exponential Decay Along the Cylinder.

In this Section we will prove that a finite energy solution of SW equations on an infinite cylinder decays exponentially to a static solution and therefore such a kind of solutions differ from a static one by a term contained in a weighted Sobolev space on the end.

First of all the energy of a couple $(A,\psi) \in \mathcal{A}(\mathcal{L}) \times \Gamma(S^+(\tilde{P}))$ on the cylinder $N \times [0,T]$ is defined as

$$E(0,T) := \int_0^T dt \int_N |q(\psi(t)) - F_{A(t)}|^2 + |\partial\!\!\!/_{A(t)}\psi(t)|^2 \qquad (2.20)$$

and finite energy condition on an infinite cylinder means that this term remains bounded as T goes to infinity (observe that the definition of energy does not depend on the choice of gauge). The finite energy condition for a solution is the natural transposition, in Seiberg-Witten theory, of the $\mathcal{L}^2$ condition that is imposed to the curvature, in Yang-Mills theory, to study similar problems.

To study the moduli space of finite energy solutions of SW equations on the cylinder we must obtain some information on the way such solutions decay along the t coordinate.

In the rest of this Section we will often use the fact that $s > 0$, as this simplifies in some points the discussion, but this is not a necessary hypothesis for decay properties like the ones we are going to discuss.
As our interest is in those spin^c structures, on the cylinder, which arise as restriction of a spin^c structure on a compact closed manifold $M = M^+ \cup_N M^-$ which has non trivial SW invariant, we will focus on those solutions, on the cylinder, that can glue to global solutions on M. This imposes some limits on the classes of spin^c structures, on $N \times [0,T]$, that we need to analyse; in fact we will now prove that we just need to focus to the case where $\mathcal{L}$ restricts to a torsion bundle on N. We have already seen in Proposition 2.2.2 that this class of bundles is the only one which admits static solutions to the gradient flow equation: we claim that, under finite energy condition, this is the only class of spin^c structures which gives non trivial Seiberg-Witten invariants for the manifold M. This is the content of the following

Claim 2.3.1 *Let $M, M^\pm, N$ be as above; let $\tilde{P}_M$ be a spin^c structure with determinant bundle $\mathcal{L}_M$ which restricts to a line bundle L on N; if $SW(\tilde{P}_M)$ is non zero then L is torsion.*

Proof: this claim is simply an application of Prop.8 of [KM1], where it is proven that if the moduli space $\mathcal{M}(\tilde{P}_M)$ is non empty for any metric, condition which is satisfied under the hypothesis that $SW(\tilde{P}_M)$ is non zero, then, by constructing a family of metrics g_T on M which identify isometrically a tubular neighborhood of N with the cylinder $N \times [-T,T]$ and then taking the limit $T \mapsto \infty$, there must exist at least a translation invariant solution on the cylinder, which corresponds of course to a static solution on N. In other words the determinant bundle $\mathcal{L}_M$ must restrict to a line bundle L which represents a monopole class on N. In our case, as the only spin^c structures on N which correspond to a monopole class are those with $c_1(L)$ torsion, the Claim follows.

We will therefore consider, now on, only the case of L torsion, without explicitly mentioning it.

From a general viewpoint, which has been investigated, at least at a sketchy level, in Donaldson-Floer theory, there are two cases where we expect an exponential decay condition for solutions of Seiberg-Witten (or Yang-Mills) equations on a cylinder: the case of isolated nondegenerate static solutions, which corresponds to the Chern-Simons-Dirac (or Chern-Simons) functional being a Morse function, and the case where the static solutions, although appearing in families, define a smooth variety, with tangent space everywhere coinciding with the first cohomology group of the equation complex, that we implicitly analyzed in Section 2.2 (and which in our case is given by $H^1(N, i\mathbb{R})$), which corresponds to Morse-Bott case. This general viewpoint, although being suggestive, is not too helpful from the practical viewpoint, in face of the difficulty of Morse theory on infinite dimensional manifolds (and in fact often results on these topics are extracted from results on finite dimensional models), so we will approach directly the problem of convergence. It will not escape from an analysis of the proof, anyhow, that the deep reasons of convergence are related to the aforementioned regularity properties of the character variety (about this point, recall the remarks at the end of Section 2.2, on the T^3 case: there, the first cohomology group of the deformation complex, in correspondence of the connection θ with $ker\nabla_\theta \neq \emptyset$, has an extra $ker\not{\partial}_\theta$ subspace).

We start with a study of the behavior of a solution of SWF on a finite cylinder under a condition which is more restrictive than just having finite energy, i.e. we suppose that a solution $(A(t), \psi(t))$ on $N \times [0, T]$ has distance from a static solution $(\Gamma, 0)$, measured in the $\mathcal{L}^2_1$ norm on the based orbit space, which is less than a given ϵ.
We need now some comments on the choice of an $\mathcal{L}^2_1$ norm on the space of orbits. In the case of $b_1(N) > 0$ the space of static solution is not composed of an isolated point (when $b_1(N) = 0$ some of the results proven in this Section are in fact unnecessary). At a reducible point $(\Gamma, 0) \in \mathcal{A}(L) \times \Gamma(S(\tilde{P}_N))$, a slice for the action of the gauge group $\mathcal{G}^o(\tilde{P}_N)$ is given by

$$ker\delta^*_{(\Gamma,0)} = \{(a, \phi) \in \Omega^1(N, i\mathbb{R}) \times \Gamma(S(\tilde{P}_N))|d^*a = 0\}. \tag{2.21}$$

In that slice an $\mathcal{L}^2_1$ norm is defined by

$$\begin{aligned} ||a||^2_{\mathcal{L}^2_1} &= ||a||^2_{\mathcal{L}^2} + ||da||^2_{\mathcal{L}^2} \\ ||\phi||^2_{\mathcal{L}^2_1(\Gamma)} &= ||\phi||^2_{\mathcal{L}^2} + ||\nabla_\Gamma\phi||^2_{\mathcal{L}^2}. \end{aligned} \tag{2.22}$$

A ball in the slice provides a chart, around $(\Gamma, 0)$, for the orbit space $\mathcal{B}^o(\tilde{P}_N)$.
Note that the definition of the spinor norm requires the choice of a connection on L. When we work in a neighborhood of the space of flat connections $\chi(N)$ it is important to be able to compare the Sobolev norms defined with respect to different flat connections. We represent elements in $\chi(N)$ in Coulomb gauge, in such a way that the difference of

two flat connections $(\Gamma' - \Gamma)$ is represented by a purely imaginary harmonic one-form. In particular, the sup norm of $(\Gamma' - \Gamma)$ on $\chi(N)$ is well defined and uniformly bounded, as $\chi(N)$ is compact. Now we claim the following result:

Claim 2.3.2 *The* $\mathcal{L}_1^2$ *norms defined w.r.t. any two flat connections are commensurate, in the sense that there exist a finite positive constant* C*, independent of* Γ, Γ'*, s.t. for any* $\psi \in \Gamma(S(\tilde{P}_N))$ *then*

$$||\psi||_{\mathcal{L}_1^2(\Gamma)} \leq C||\psi||_{\mathcal{L}_1^2(\Gamma')}. \tag{2.23}$$

Proof: the proof is as follows: first, by elliptic estimates, as $ker \partial\!\!\!/_{\Gamma'} = 0$, we have

$$||\psi||_{\mathcal{L}^2} \leq \tilde{C}(\Gamma')||\partial\!\!\!/_{\Gamma'}\psi||_{\mathcal{L}^2}, \tag{2.24}$$

where in general the constant $\tilde{C}(\Gamma')$ depends on its argument; anyhow, if $\partial\!\!\!/_{\Gamma'}\psi$ is in $\mathcal{L}^2$, so is ψ, and an eigenfunctions expansion of ψ w.r.t. the selfadjoint $\partial\!\!\!/_{\Gamma'}$ tells that $\tilde{C}(\Gamma') \leq (\lambda_1(\Gamma'))^{-1}$ where $\lambda_1(\Gamma')$ is the the first eigenvalue of $\partial\!\!\!/_{\Gamma'}$. Now, because of positive curvature condition and compactness of the character variety, there exist a $\tilde{C} := max_{\chi(N)}(\lambda_1(\Gamma))^{-1}$ and this provides an uniform bound on the elliptic constant. We claim that we can obtain a similar result also for $\partial\!\!\!/_{\Gamma}\psi$, as

$$\begin{aligned} ||\partial\!\!\!/_{\Gamma}\psi||_{\mathcal{L}^2} = ||\partial\!\!\!/_{\Gamma'}\psi + \tfrac{1}{2}(\Gamma - \Gamma')\psi||^2_{\mathcal{L}} \leq ||\partial\!\!\!/_{\Gamma'}\psi||_{\mathcal{L}^2} + \tfrac{1}{2}||(\Gamma' - \Gamma)\psi||_{\mathcal{L}^2} \leq \\ \leq ||\partial\!\!\!/_{\Gamma'}\psi||_{\mathcal{L}^2} + \tfrac{1}{2}||\Gamma' - \Gamma||_{\mathcal{C}^0}||\psi||_{\mathcal{L}^2} \leq C||\partial\!\!\!/_{\Gamma'}\psi||_{\mathcal{L}^2} \end{aligned} \tag{2.25}$$

where C depends only on the manifold N. By symmetry, the same result holds interchanging Γ, Γ'. It is now easy to verify, using Bochner-Weitzenböck formula and $s > 0$, that we can in fact use $||\partial\!\!\!/_{\Gamma}\psi||_{\mathcal{L}^2}$ to define the $\mathcal{L}_1^2(\Gamma)$ norm on $\Gamma(S(\tilde{P}_N))$ and it follows therefore, from the formula 2.25, that all $\mathcal{L}_1^2(\Gamma)$ norms are uniformly compatible.

This allows us to use any flat reference connection to compute spinor norms, all the difference amounting to multiplying by finite constants. In the sequel we will assume this uniformity, which in practice corresponds to uniformity in the Sobolev constants, without explicitly mentioning it. Moreover we will often simply denote by $\mathcal{L}_1^2$ any $\mathcal{L}_1^2(\Gamma)$ norm, on spinors, defined with the choice of a flat connection in Coulomb gauge.

Under the condition of small distance we can prove the following

Lemma 2.3.3 *There exist positive constants* ϵ*,* c*,* δ*, such that if* $(A(t), \psi(t))$ *is a solution of SWF on* $N \times [0,T]$ *which is in an* ϵ*-neighborhood of a static solution* $(\Gamma, 0)$ *in the* $\mathcal{L}_1^2$

norm, in the based orbit space, then its $\mathcal{L}^2_1$ distance from a static solution $(\Gamma_t, 0)$ (which is parameterized by t, and is given by the harmonic part of $A(t)$), satisfies the relation

$$d^2(t) := d^2_{\mathcal{L}^2_1}((A(t), \psi(t)), (\Gamma_t, 0)) < cd^2(0)exp[-2\delta t] + cd^2(T)exp[-2\delta(T-t)]; \tag{2.26}$$

moreover all the values of the constants can be chosen to depend only on N.

Proof: we claim that under the above hypothesis, the square norm of the Chern-Simons-Dirac gradient

$$f(t) = f(A(t), \psi(t)) := ||\partial\!\!\!/_{A(t)}\psi(t)||^2 + ||q(t) - F(t)||^2 \tag{2.27}$$

(when no risk of confusion arises we denote by $||\cdot||$ the $\mathcal{L}^2$ norm, reserving $||\cdot||_{\mathcal{L}^p_k}$ to other Sobolev norm) satisfies an inequality of type $f'' \geq 4\delta^2 f$, from which we have the inequality

$$f(t) < f(0)exp[-2\delta t] + f(T)exp[-2\delta(T-t)]. \tag{2.28}$$

The proof of this claim, which is quite technical, is contained in the appendix.
Now note that the CSD gradient, which is orthogonal to the gauge orbits, can be interpreted as a section $\xi(A, \psi) = (*(q(\psi) - F_A), \partial\!\!\!/_A\psi))$ of the tangent bundle of $\mathcal{B}^o(\tilde{P}_N)$. As we have seen in detail in Section 2.2, ξ is transversal to the normal direction to the harmonic subspace of $\Omega^1(N, i\mathbb{R})$; according whether $b_1(N)$ vanishes or not its $\mathcal{L}^2$ norm gives a different control on $(A(t), \psi(t))$: in the first case, in a neighborhood of the critical point $(\Gamma, 0)$ the norm of the gradient bounds, up to a constant, the $\mathcal{L}^2_1$ distance of $(A(t), \psi(t))$ from $(\Gamma, 0)$, i.e.

$$d^2_{\mathcal{L}^2_1}((A(t), \psi(t)), (\Gamma, 0)) \leq c_{12} f(A(t), \psi(t)) \tag{2.29}$$

(we continue to index the constants following the enumeration of the Appendix); in the second case it just bounds the distance from the critical set, i.e.

$$d^2(t) := d^2_{\mathcal{L}^2_1}((A(t), \psi(t), (\Gamma_t, 0)) \leq c_{12} f(t), \tag{2.30}$$

where Γ_t is the flat connection such that $(A(t) - \Gamma_t)$ has no harmonic component (the other Hodge components are independent of the choice of Γ_t). The size of the neighborhood where this holds true depends ultimately only on N.
Therefore, for some constant c_{12}, that we can choose to depend only on the three manifold, by compactness of $\chi(N)$, we have in all cases

$$d^2(t) < c_{12} f(0)exp[-2\delta t] + c_{12} f(T)exp[-2\delta(T-t)]. \tag{2.31}$$

We can get, as well, a control of $f(t)$ in terms of $d^2(t)$; with a suitable choice of the representative, we have a formula similar to the one obtained in equation A.38 in the appendix for the distance of $(A(t), \psi(t))$ from $(\Gamma, 0)$; here we can write, in the gauge where $(A(t) - \Gamma_t, \psi(t))$ has small $\mathcal{L}^2_1$ norm,

$$f(t) \leq \|F_{A(t)}\|^2 + \tfrac{1}{2}\|\psi(t)\|^4_{\mathcal{L}^4} + 2\|\nabla_{A(t)}\psi(t)\|^2 + \tfrac{max(s)}{2}\|\psi(t)\|^2 \leq$$

$$\leq \|F_{A(t)}\|^2 + \tfrac{1}{2}\|\psi(t)\|^4_{\mathcal{L}^4} + 2\|\nabla_{\Gamma_t}\psi(t)\|^2 + c_1\|A(t) - \Gamma_t\|^2_{\mathcal{L}^2_1}\|\psi(t)\|^2_{\mathcal{L}^2_1} + \tfrac{max(s)}{2}\|\psi(t)\|^2 \tag{2.32}$$

and the latter term is bounded in terms of the distance $d^2(t)$.
Applying these relations at $t = 0, T$ and rearranging the constants we obtain the formula 2.26 in the statement of the lemma, for a suitable value of the constant c. Concerning the constants involved, these depend only on N, as follows from Proposition A.1.2 and Claim 2.3.2.

So far we have obtained, in the orbit space, a good exponential control on distances under the hypothesis of small distance: now we want to pass from a condition of small distance to a condition of small energy, which is the one suitable for our purposes. To do so we have two viable alternatives: the first is to analyse carefully the weak compactness theorems of [KM1] to deduce that finite energy implies the existence of a limit point for the path in the orbit space; this is the approach outlined, e.g., in [MOY]. This allows to use directly Lemma 2.3.3 and then pass to Lemma 2.3.6 to deduce the decay result. The second, which is more transparent and we will follow here, uses the content of Lemma 2.3.3 to deduce a control of distance from the control on energy. This method has been used in [MST] to deal with a case of isolated static solution; here we make also use of Simon's estimates on lengths in terms of energy to deal with the case of $b_1(N) > 0$.

We claim first of all that, working on the based orbit space, the statement of Lemma 6.10 of [MST] holds true, namely we have

Lemma 2.3.4 *For any $\eta > 0$ there exist a $\lambda > 0$ such that if $(A, \psi) \in \mathcal{B}^o(\tilde{P}_N)$ has $\mathcal{L}^2_1$ distance from the critical set greater than η, then $f(A, \psi) \geq \lambda$.*

The proof is a straightforward modification of the original one.

If we suppose that the energy of a cylinder $N \times [a, b]$ satisfies

$$\int_a^b f(t)dt \leq E_0 \tag{2.33}$$

with E_0 sufficiently small, we deduce that there must exist a value t_1 in the interval for which the distance $d(t_1)$ is smaller than a given value, that we fix, for reasons that will appear clearly in the rest of the Section, equal to $\frac{\epsilon}{n}$, where the ϵ is the one determined by the previous lemma and we take an $n \gg 1$.
If we are considering the problem of the decay of a finite energy solution on the cylinder $N \times [0, \infty)$ this small energy hypothesis (and the possibility of the choice of a t_1 as above) is satisfied under the condition of working on an interval whose infimum is greater or equal than a value which depends on the way the energy is distributed over the cylinder, i.e. on the particular solution. In that case the decay results we are going to prove in the rest of the Section, will hold, for each solution, starting from an initial value which depends on the solution itself.
Keeping memory of this observation, we reparameterize the cylinder in such a way to have $[a, b] = [-1, T+1]$, $t_1 \in [0, T]$ and we put $\Gamma_{t_1} =: \Gamma$. Then the following holds:

Lemma 2.3.5 *There exist a value of n, E_0 such that $(A(t), \psi(t))$ remains in an ϵ-neighborhood of $(\Gamma, 0)$ in the $\mathcal{L}^2_1$ metric for all the interval $[0, T]$.*

Proof: the proof is by contradiction. If this did not hold true we could find a value $t_2 \in [0, T]$ that by symmetry we suppose greater than t_1 and a $1 < p' \ll n$ such that

$$d((A(t_2), \psi(t_2)), (\Gamma, 0)) = \frac{\epsilon}{p'} \tag{2.34}$$

($d(\cdot)$ without any subscript refers to the $\mathcal{L}^2_1$ distance); we can deduce from this that there exist a p with $1 < p \ll n$ s.t. $d(t_2) = \frac{\epsilon}{p}$; in fact we have

$$d(t_2) \geq d((A(t_2), \psi(t_2)), (\Gamma, 0)) - d_{\mathcal{L}^2_1}((\Gamma_{t_2}, 0), (\Gamma, 0)) = \frac{\epsilon}{p'} - d_{\mathcal{L}^2}((\Gamma_{t_2}, 0), (\Gamma, 0)); \tag{2.35}$$

but

$$d_{\mathcal{L}^2}((\Gamma_t, 0), (\Gamma, 0)) \leq d_{\mathcal{L}^2}((A(t_2), \psi(t_2)), (A(t_1), \psi(t_1))). \tag{2.36}$$

Now we look for a control on the term on the r.h.s.: as we are in a neighborhood of Γ we can apply Simon's estimates for the $\mathcal{L}^2$ length of paths in terms of the energy, see [MMR], which guarantee that there exist an $0 < \theta \leq \frac{1}{2}$ for which

$$\int_{-1}^{T+1} ||\frac{\partial[(A, \psi)]}{\partial t}||_{\mathcal{L}^2} dt \leq \frac{4}{\theta} E_0^{\theta}; \tag{2.37}$$

choosing a suitable E_0 we can suppose that $d_{\mathcal{L}^2}((A(t_2),\psi(t_2)),(A(t_1),\psi(t_1))) \leq \frac{\epsilon}{n}$ (this choice is purely conventional, we just need a bound smaller than $\frac{\epsilon}{p'}$) and therefore, rearranging the various terms,

$$d(t_2) := \frac{\epsilon}{p} \geq \frac{\epsilon}{p'} - \frac{\epsilon}{n} \tag{2.38}$$

(note that this term is evidently bounded by above by $\frac{\epsilon}{p'}$; the relevant fact is the bound by below, which guarantees that if $(A(t_2),\psi(t_2))$ is far from Γ, then it is far also from Γ_{t_2}).

We denote now by $u \in (t_1,t_2)$ the point in which $d(u) = \frac{\epsilon}{2p}$; as in the interval $[t_1,t_2]$ we remain in the ϵ neighborhood of Γ we can use the results obtained under the small distance hypothesis, and so apply Lemma 2.3.3 to the interval $[t_1,t_2]$ to deduce that

$$d^2(u) = \frac{\epsilon^2}{4p^2} < c\frac{\epsilon^2}{n^2}exp[-2\delta(u-t_1)] + c\frac{\epsilon^2}{p^2}exp[-2\delta(t_2-u)]; \tag{2.39}$$

the utility of this formula is that it allows to obtain a finiteness result on $t_2 - u$: as $p \ll n$ we deduce that, say,

$$\frac{\epsilon^2}{5cp^2} < \frac{\epsilon^2}{p^2}exp[-\delta(t_2-u)] \tag{2.40}$$

from which we obtain that $0 < t_2 - u < \infty$. This finiteness result will be very important, as we will see.

Now we want measure the $\mathcal{L}^2_1$ distance of $(A(t_2),\psi(t_2))$ and $(A(u),\psi(u))$; we have

$$d((A(t_2),\psi(t_2)),(A(u),\psi(u))) \geq d((A(t_2),\psi(t_2)),(\Gamma_u,0)) - d((A(u),\psi(u)),(\Gamma_u,0)) \geq$$

$$\geq \tfrac{\epsilon}{p} - \tfrac{\epsilon}{2p} = \tfrac{\epsilon}{2p}; \tag{2.41}$$

and so the $\mathcal{L}^2_1$ length of the path within u and t_2 has a bound by below, i.e. we have at least

$$\int_u^{t_2} ||\frac{\partial(A,\psi)}{\partial t}||_{\mathcal{L}^2_1} dt \geq \frac{\epsilon}{2p}. \tag{2.42}$$

We claim that if the energy in the interval is small enough this brings a contradiction: in fact, the previous bound on the length implies, by Cauchy-Schwarz, that

$$\int_0^T ||\frac{\partial(A,\psi)}{\partial t}||^2_{\mathcal{L}^2_1} dt \geq \int_u^{t_2} ||\frac{\partial(A,\psi)}{\partial t}||^2_{\mathcal{L}^2_1} dt \geq (volN\cdot(t_2-u))^{-1}\frac{\epsilon^2}{4p^2}; \tag{2.43}$$

the regularity result proven in [MST], Lemma 6.14, tells that we can bound the first term in terms of the energy, as

$$E_0 \geq \int_{-1}^{T+1} ||\frac{\partial(A,\psi)}{\partial t}||^2_{\mathcal{L}^2} dt \geq K\int_0^T ||\frac{\partial(A,\psi)}{\partial t}||^2_{\mathcal{L}^2_1} dt \tag{2.44}$$

where K is a constant depending only on the geometry of N; we deduce that choosing E_0 small enough we obtain a contradiction and consequently we prove the Lemma.

We want to point out that the method we used here, that of working in a neighborhood of a fixed flat connection, (which, as we have seen, is guaranteed by the application of Simon's estimate) is not the only possible one. In fact it is not difficult to modify the proofs of Propositions 2.3.3 and 2.3.5, and part of the Appendix, to obtain the same decay result with the condition of small distance from the critical set, without appealing to any condition on the behavior of the harmonic part of $A(t)$. This latter problem, as in the method we have followed, can be then dealt to with the approach we will analyse in the rest of the Section.

Let's discuss the consequences of Lemma 2.3.5; this Lemma tells us that the $\mathcal{L}^2_1$ distance of $(A(t), \psi(t))$ from $(\Gamma, 0)$, in the orbit space, is bounded by above, for sufficiently small energy, and this brings an uniform bound as well on the value of $f(t)$, call it λ, which does not depend on the particular solution. Together with inequality 2.28, this implies that $\forall t \in [0, T]$

$$f(t) < \lambda exp[-2\delta t] + \lambda exp[-2\delta(T - t)]. \tag{2.45}$$

When we work on an infinite cylinder which has energy sufficiently small, as previously remarked, this equation proves the desired exponential decay, as λ does not depend on time, and therefore

$$f(t) < \lambda exp[-2\delta t]. \tag{2.46}$$

We concentrate now on on this result, which controls the behavior of $f(t)$, to deduce a decay result to a limit point for the solution $(A(t), \psi(t))$ in the space $\mathcal{A}(L) \times \Gamma(S(\tilde{P}_N))$. We can state now our decay lemma:

Lemma 2.3.6 *There exist two positive constants C, $\bar{\delta}$, depending only on N, such that a finite energy solution of the Seiberg-Witten equations in temporal gauge on a cylinder $N \times [0, \infty)$ is equivalent, up to a time-independent gauge transformation, to a solution $(A(t), \psi(t))$ which converges exponentially fast with weight $\bar{\delta}$ in any $\mathcal{L}^p_{k,\bar{\delta}}(N \times [0, \infty))$ norm in $\mathcal{A}(\mathcal{L}) \times \Gamma(S^+(\tilde{P})$ to a static solution $(A_0, 0)$, i.e.*

$$||A - A_0||_{\mathcal{L}^p_{k,\bar{\delta}}} \leq C, \tag{2.47}$$

$$||\psi||_{\mathcal{L}^p_{k,\bar{\delta}}} \leq C. \tag{2.48}$$

The convergence is moreover smooth on compact subsets.

Proof: our aim is to use the previous result on $f(t)$ to deduce an exponential decay result on the $\mathcal{L}^2_1(N)$ norm of a solution in temporal gauge, as we already have a exponential control on the decay behavior of the time derivative. As we mentioned, general lemmas would guarantee us, after application of some compactness arguments, convergence to a static solution in the orbit space (as it can be proven that Γ_t converges to some limit value), but we want to prove that in fact $(A(t), \psi(t))$ itself converges exponentially to a limit, and in order to do so we will use the decay properties of $f(t)$.
We start with the following observation: for any solution $(A(t), \psi(t))$ in temporal gauge there exist a time independent gauge transformation g_∞ and a flat connection A_0 such that, $\forall t \in [0, \infty)$,

$$||g_\infty \cdot A(t) - A_0||_{\mathcal{L}^2_1} \leq Kexp[-\delta t]; \tag{2.49}$$

we want obtain such an estimate by application of Sobolev inequality to the elliptic operator $d + d^*$; it is immediate to verify, using the definition and Weitzenböck formula, that

$$\frac{1}{2}||F_{A(t)}||^2_{\mathcal{L}^2} + \frac{1}{4}||\psi(t)||^4_{\mathcal{L}^4} + ||\nabla_{A(t)}\psi(t)||^2_{\mathcal{L}^2} + \frac{min(s)}{4}||\psi(t)||^2_{\mathcal{L}^2} \leq f(t) \tag{2.50}$$

from which, as $s > 0$, $||F_A||^2_{\mathcal{L}^2} \leq 2f(t)$ and so $d(A(t) - \bar{A}_0)$, for any flat $\bar{A}_0$, is already controlled in $\mathcal{L}^2$ norm by $\sqrt{2\lambda}exp[-\delta t]$. We would like to obtain a similar control with $d^*(A(t) - \bar{A}_0)$.
We can try to analyse the gauge fixing condition: in order not to compromise the temporal gauge condition, we look for a time independent gauge transformation $g = exp[\chi]$ s.t. $g \cdot A(t) - \bar{A}_0 \in kerd^*$: this amounts to solve the elliptic equation

$$2\Delta\chi = d^*(\bar{A}_0 - A(t)) \tag{2.51}$$

and this has a solution once we know that Δ surjects onto Imd^*. This follows from Hodge decomposition of 0 forms, as these split as $Imd^* \oplus ker\Delta$.
Now, from SWF,

$$\frac{\partial}{\partial t}d^*(\bar{A}_0 - A(t)) = *d(q(\psi(t)) - F_{A(t)}) = iIm < \partial\!\!\!/_A\psi, \psi >, \tag{2.52}$$

as follows from the formula for $*dq(\psi)$ proved in the appendix.
From this result we see that we can not apply a constant three dimensional gauge transformation to $A(t)$ to gauge fix it w.r.t. $\bar{A}_0$. This represents a difference w.r.t. Yang-Mills

theory, where the gauge fixing is available and the δ-decay of a connection in temporal gauge to a flat one can be proved directly by elliptic techniques starting from the result corresponding to equation 2.46 for the Chern-Simons functional (see [JRS]).
Anyhow we can try to gauge fix a connection at infinity: we want to estimate the $\mathcal{L}^2$ length of the path of $d^*(\bar{A}_0 - A(t))$: we have, from equation 2.52,

$$\int_t^\infty ds \|\frac{\partial}{\partial s} d^*(\bar{A}_0 - A(s))\|_{\mathcal{L}^2} \leq \int_t^\infty ds \| < \psi(s), \not{\partial}_{A(s)}\psi(s) > \|_{\mathcal{L}^2}. \tag{2.53}$$

Choosing, as usual, a suitable gauge to make the computation, we have

$$\begin{aligned} \| < \psi, \not{\partial}_A \psi > \|_{\mathcal{L}^2} = \| < \psi^g, \not{\partial}_{A^g}\psi^g > \|_{\mathcal{L}^2} \leq c_{13} \|\not{\partial}_{A^g}\psi^g\|_{\mathcal{L}^2_1} \|\psi^g\|_{\mathcal{L}^2_1} \leq \\ \leq c_{14} \|\not{\partial}^2_{A^g}\psi^g\|^2_{\mathcal{L}^2} = c_{14} \|\not{\partial}^2_A \psi\|^2_{\mathcal{L}^2}. \end{aligned} \tag{2.54}$$

So, the $\mathcal{L}^2$ length of the path of $d^*(\bar{A}_0 - A(s))$ within the interval $[t, \infty)$ is bounded, as follows from Proposition A.1.4, by $c_{15} exp[-\delta t]$ (we choose $\omega = \delta$), which is finite; we have therefore convergence to a value, say ξ_∞, in such a way that

$$\|d^*(\bar{A}_0 - A(t)) - \xi_\infty\|_{\mathcal{L}^2} \leq c_{15} exp[-\delta t] \tag{2.55}$$

(note, from the remarks at the end of Proposition A.1.4 that, eventually at the price of reparameterizing the t axis, the constant c_{15} is independent of any choice, in view of the uniform bound on $f(t)$). Now just define χ_∞ to be the (time independent) solution of the equation

$$2\Delta\chi_\infty = \xi_\infty. \tag{2.56}$$

We gauge transform now $(A(t), \psi(t))$ by the action of $g_\infty = exp[\chi_\infty]$; this transformation does not affect, of course, any result concerning the behavior of the solution w.r.t. time. Now the gauge transformed solution will satisfy

$$\|d^*(g_\infty \cdot A(t) - \bar{A}_0)\|_{\mathcal{L}^2} \leq c_{15} exp[-\delta t]. \tag{2.57}$$

In the case of a rational homology sphere, this is the "missing half" of the elliptic inequality. If harmonic one forms are present, a little more work is in order: first we asymptotically gauge fix $A(t)$ as we did in the homology sphere case (note that $d^*(A(t) - \bar{A}_0)$ does not depend on the harmonic part of $(A(t) - \bar{A}_0)$), and then decompose $(g_\infty \cdot A(t) - \bar{A}_0)$ in its Hodge components:

$$g_\infty \cdot A(t) - \bar{A}_0 = d^*\rho(t) + a(t), \tag{2.58}$$

where $a(t)$ contains both the harmonic and the exact part; SWF equations tell us now that $\frac{\partial}{\partial t}a(t)$ is related only to $*q(\psi(t))$. Arguing as above, we have that

$$||\frac{\partial}{\partial t}a(t)||_{\mathcal{L}^2} \leq \frac{1}{2}||\psi(t)||^2_{\mathcal{L}^4}; \tag{2.59}$$

the right hand side term has decay conditions which are dictated from its bound in terms of $f(t)$, equation 2.50; the $\mathcal{L}^2$ length of the path of $a(t)$ is therefore bounded by $\frac{2\sqrt{\lambda}}{\delta}exp[-\delta t]$ and we have $\mathcal{L}^2$ convergence to a limit a, which is in fact harmonic because of the asymptotic gauge fixing conditions, for which

$$||a(t) - a||_{\mathcal{L}^2} \leq \frac{2\sqrt{\lambda}}{\delta}exp[-\delta t]. \tag{2.60}$$

For notational simplicity, now on, we will chose, as reference connection, $(\bar{A}_0{+}a)$ as defined above, denoting it as A_0, and will consider A as already asymptotically gauge fixed w.r.t. it. If we consider now $(A(t) - A_0)$ and we put together everything, by application of elliptic inequality, we obtain the result claimed in equation 2.49.
To obtain a similar result for the spinor part presents no difficulty, using directly the standard elliptic inequality applied to $\partial\!\!\!/_{A(t)}$; inequality 2.49 guarantees, in asymptotic gauge and with the right choice of the reference connection, that $(A(t) - A_0)$ is small, in $\mathcal{A}(L)$, in $\mathcal{L}^2_1$ norm; we can therefore use directly the fundamental elliptic inequality for the Dirac operator $\partial\!\!\!/_{A_0}$, as we did in the Appendix with equation A.50, to obtain the inequality

$$||\psi(t)||_{\mathcal{L}^2_1} \leq c_4||\partial\!\!\!/_{A(t)}\psi(t)||_{\mathcal{L}^2} \tag{2.61}$$

with the constant depending only on the geometry of N.
Now we have an exponential $\mathcal{L}^2(N)$ control on $(A(t) - A_0, \psi(t))$ and on its derivatives, both in the N direction (equations 2.49 and 2.61) and in the cylinder direction (equation 2.46) and this gives, by definition, an $\mathcal{L}^2_{1,\bar{\delta}}(N \times [0,\infty))$ control on $(A - A_0, \psi)$, for a constant $0 < \bar{\delta} < \delta$.
We can now bootstrap these results, using Sobolev embedding theorems for weighted spaces (at the price of eventually decreasing again the weight) to gain the convergence in higher Sobolev norms.

Remember that the initial value of t, that we denoted for simplicity as $t = 0$, from which the decay result holds, depends (at least at this stage) on the particular solution, with the decay coefficient and constants which depend instead only on N.

Chapter 3

Cylindrical End Moduli Spaces

In this Chapter we will study the moduli spaces of finite energy solutions of Seiberg-Witten equations on a cylindrical end four manifold, and the way it is connected to the moduli space of solutions on a closed manifold. We will start with an analysis of the problem of geometric limits of solutions of SW equations on closed manifolds and their relations with solutions on cylindrical end manifolds. We will proceed then to deduce regularity results, for the moduli space of finite energy solutions on a cylindrical end manifold, similar to those obtained in Chapter 1 for the case of a closed manifold. With these results we will be able to relate some properties of these moduli spaces, in particular their dimension, and this will allow to deduce Theorem 2. Without appealing to these regularity results, by a direct analysis of some classes of perturbations to the SW equations, we will then obtain, in the case of $b_1(N) > 0$, Theorem 3, whose proof just requires a modification of some results of [KM1]. We finish this Chapter by exhibiting a gluing formula which relates moduli spaces on a closed four manifold, which decomposes along N in a way admitted by the previous theorems, with the moduli on the two factors. In this Chapter we will use often an index for four dimensional objects (connections, spinors, and so on) to distinguish them from their three dimensional counterpart. This is slightly incoherent with the notation of Chapter 1 but is notationally much clearer.

3.1 Definition of the Moduli Spaces.

In Section 2.3 we have shown that finite energy solutions of Seiberg-Witten equations on a cylinder $N \times [0, \infty)$ have nice decay properties to some static solution. We will apply this result to analyse the moduli spaces of finite energy solutions of SW equations on $\hat{X}$,

as these solutions, on the end, satisfy the aforementioned decay properties.
As one of the aims of this Chapter is to obtain regularity results for the moduli spaces, we will introduce, as usual, a compactly supported self dual perturbation term in the four dimensional SW equations, i.e.

$$\begin{cases} \not{\partial}_{A_4}\psi_4 = 0 \\ F_{A_4}^+ = q(\psi_4) + \eta_4. \end{cases} \tag{3.1}$$

with $\eta_4 \in \Omega^2_{c,+}(X, i\mathbb{R})$ compactly supported. The hypothesis on the support of the perturbation guarantees us that the solutions of η_4-SW equations share the same decay properties, on the end of $\hat{X}$, with the unperturbed case. These equations, as in the closed case, are invariant under the gauge group $\mathcal{G}(\tilde{P}_{\hat{X}})$ of vertical automorphisms which project to the identity on the frame bundle.
The natural constraint on energy, on $\hat{X}$, is to consider finite energy, on the end, as defined in equation 2.20: if we have a solution $(A_4, \psi_4) \in \mathcal{A}(\mathcal{L}_{\hat{X}}) \times \Gamma(S^+(\tilde{P}_{\hat{X}}))$, we consider its restriction to the end $N \times [0, \infty)$; we can find a gauge transformation, connected to the identity (which extends to $\hat{X}$ as it is unobstructed, see later) which puts it in temporal gauge. We denote the gauge transformed pair, on the end, as $(A(t), \psi(t))$ (we will usually omit any reference to the gauge transformation, for sake of notation). This solutions is said therefore to have finite energy if it satisfies

$$\lim_{T\to\infty} \int_0^T dt(||q - F||^2_{\mathcal{L}^2(N)} + ||\not{\partial}_A\psi||^2_{\mathcal{L}^2(N)}) < \infty. \tag{3.2}$$

Restricting now on to this class of solutions we introduce the following

Definition 3.1.1 *The moduli space of finite energy solutions of η_4-Seiberg-Witten equations for a spin^c structure $\tilde{P}_{\hat{X}}$ on $\hat{X}$ is defined as*

$$\mathcal{M}(\tilde{P}_{\hat{X}}, \eta_4) := \{(A_4, \psi_4) \in \mathcal{A}(\mathcal{L}_{\hat{X}}) \times \Gamma(S^+(\tilde{P}_{\hat{X}}))|F^+_{A_4} = q(\psi_4) + \eta_4, \not{\partial}_{A_4}\psi_4 = 0\}/\mathcal{G}(\tilde{P}_{\hat{X}}). \tag{3.3}$$

We will reserve the notation $\mathcal{M}^*(\tilde{P}_{\hat{X}}, \eta_4)$ to the gauge equivalence classes of irreducible solutions.

We now define the Chern integral of a solution (A_4, ψ_4) on $\hat{X}$ as

$$c_X(A_4, \psi_4) := -\frac{1}{4\pi^2}\int_{\hat{X}} F_{A_4} \wedge F_{A_4}; \tag{3.4}$$

it is a well defined, finite term, as there are not limit terms due to the decay conditions, and it is a locally constant function on the moduli space; we denote by $\mathcal{M}_{c_X}(\tilde{P}_{\hat{X}}, \eta_4)$ the union of the components of the perturbed moduli space with Chern integral c_X. This moduli space can be endowed of a natural topology as in [MST], par. 8.

The definitions above are valid for any three manifold N; in the case when N has positive scalar curvature the results of the previous Chapter have relevant consequences on the structure of these moduli spaces. When we take a solution (A_4, ψ_4) of Seiberg-Witten equations for a spin^c structure $\tilde{P}_{\hat{X}}$ on $\hat{X}$ and we apply to its (gauge transformed) restriction on the end the results of Lemma 2.3.6 we deduce the existence of a limit map

$$\partial_\infty : \mathcal{M}(\tilde{P}_{\hat{X}}, \eta_4) \longrightarrow \hat{\chi}(N), \tag{3.5}$$

which sends the $\mathcal{G}(\tilde{P}_{\hat{X}})$-gauge equivalence class of finite energy solution $(A(t), \psi(t))$ on the end to the $\mathcal{G}(\tilde{P}_{\hat{X}})$-gauge equivalence class of its limit flat connection A_0 on the bundle L on N. The space $\hat{\chi}(N)$ is, by definition, the space of gauge equivalence classes of flat connections on N under the action of those gauge transformations of $\mathcal{G}(\tilde{P}_N)$ which extend to $\tilde{P}_{\hat{X}}$. Concerning this space, the following holds:

Proposition 3.1.2 *$\hat{\chi}(N)$ is a covering of $\chi(N)$ with fiber $H^1(N, \mathbb{Z})/H^1(X, \mathbb{Z})$.*

Proof: proving the statement amounts to identify the obstructions for the extension problem, for a gauge transformation $u \in \mathcal{G}(\tilde{P}_N) = Map(N, U(1))$, represented by the diagram

$$\begin{array}{ccc} X & \xrightarrow{?} & U(1) \\ {\scriptstyle i}\uparrow & & \downarrow{\scriptstyle \cong} \\ N & \xrightarrow{u} & U(1). \end{array} \tag{3.6}$$

Standard homotopy theory (Eilenberg extension theorem, see e.g. [Wh], Chapter 5) identifies as only obstruction class, in $H^2(X, N, \pi_1(U(1)))$, the class $\delta^* u^* k$, where k is the characteristic class of $U(1)$ and δ^* is the coboundary operator of the exact sequence

$$H^1(X, \mathbb{Z}) \to H^1(N, \mathbb{Z}) \xrightarrow{\delta^*} H^2(X, N, \mathbb{Z}). \tag{3.7}$$

Therefore u extends if and only if it lies in a component of $\mathcal{G}(\tilde{P}_N)$ which is labeled, in $H^1(N, \mathbb{Z})$, by an element of the image of $H^1(X, \mathbb{Z})$. From this the statement follows.

Note that although the decay results on the cylinder have been proven for the orbit space

$\mathcal{B}^o(\tilde{P}_N)$, as they correspond to a control on the length of a path, they hold in fact true in any covering of the orbit space, so we don't need to care, concerning this point, about the relation between $\hat{\chi}(N)$ and $\chi(N)$.

A relevant fact, when N has positive scalar curvature, is that the decay of the connection part of a solution to a flat connection implies that the value of the Chern integral of a solution is determined by the spin^c structure $\tilde{P}_{\hat{X}}$; in fact, if A_4, A_4' are the connection parts of two solutions, we will have $A_4' = A_4 + \xi_4$ for some $\xi_4 \in \Omega^1(\hat{X}, i\mathbb{R})$ which decays to the difference of the limit values of A_4' and A_4, i.e. a closed one form; in particular, $d\xi_4$ decays to zero at infinity. With this in mind we have, with selfexplaining notation,

$$\begin{gathered}\int_{\hat{X}} F_{A_4'} \wedge F_{A_4'} = lim_T \int_{X_T} F_{A_4} \wedge F_{A_4} + d(\xi_4 \wedge (2F_{A_4} + d\xi_4)) = \\ = lim_T \int_{X_T} F_{A_4} \wedge F_{A_4} + \int_{N\times\{T\}} \xi_4 \wedge (2F_{A_4} + d\xi_4) = \int_{\hat{X}} F_{A_4} \wedge F_{A_4}.\end{gathered} \tag{3.8}$$

In the sequel, for sake of clarity, we will often keep track anyhow of the value of the Chern integral of a solution.

The map ∂_∞ of equation 3.5 which sends each solution (A_4, ψ_4) to its limit value on the cylinder $(A_0, 0)$ is continuous, by construction, and *by fiat* smooth if $b_1(N) = 0$. In the other case, we have the following

Claim 3.1.3 *the limit map $\partial_\infty : \mathcal{M}(\tilde{P}_{\hat{X}}) \longrightarrow \hat{\chi}(N)$ is smooth.*

Proof: the proof follows by analysing the way a solution $(A(t), \psi(t))$ on the cylinder relates to its limit value. Roughly speaking, in gauge theory, the moduli space "fibers" over the limit set, and has its same regularity; in our case the non degeneracy in the sense of Bott of the critical manifold tells that we can consider the three dimensional orbit space, defined as the space $\mathcal{A}(L) \times \Gamma(S(\tilde{P}_N))$ modulo the gauge transformation on N which extend to M, as fibering over the critical manifold, identified for a given spin^c structure with $H^1(N, \mathbb{R})/Im(H^1(X, \mathbb{Z}))$, with a smooth projection map. Now to each point (A_4, ψ_4) in the moduli space we associate the harmonic components of the Hodge decomposition of $A(0)$ and $*q(\psi(t))$; these are smooth maps, and in the first case, up to the action of the group of gauge components, it corresponds to the smooth projection map to the critical manifold. We have seen in Section 2.3 how the harmonic component $A^h(0)$ relates to the limit value: we have

$$A^h(0) - A_0 = \int_\infty^0 (*q(\psi(t)))^h dt; \tag{3.9}$$

all the operations are smooth w.r.t. the topology of the orbit space, and therefore the limit map is smooth.

3.2 Geometric Limits.

The aim of this Section is to discuss the definition of geometric limit in Seiberg-Witten context and to study its consequences. For sake of notation in this Section we will omit the index which denotes four dimensional objects. First we recall some definitions, suitably adapted, from Chapter 6 of [MMR].

Definition 3.2.1 *Let (M_n, g_n, p_n^i) be a sequence of complete riemannian four manifolds with k base points $p_n^i \in M_n$ satisfying the condition that $lim_n d(p_n^i, p_n^j) = \infty$ for $i \neq j$. We say that a complete riemannian four manifold $(\hat{M}, g, b^i)$ with k base points $b^i \in \hat{M}$ is the geometric limit of the sequence if $\hat{M}$ consists of k connected components $\hat{M}^i \ni b^i$ such that for each $1 \leq i \leq k$ the following holds: $\forall t \geq 0$ and all $n \geq n(t)$ there exist compact submanifolds with boundary $A_i \subset \hat{M}^i$ (respectively $B_{n,i} \subset M_n$) containing the ball of radius t around b^i (respectively p_n^i) and k diffeomorphisms*

$$\phi_{n,i} : (A_i, b^i) \to (B_{n,i}, p_n^i) \tag{3.10}$$

such that the induced metrics satisfy

$$\lim_{n\to\infty} \phi_{n,i}^*(g_n|_{B_{n,i}}) = g|_{A_i}. \tag{3.11}$$

Once a sequence of manifolds has a well defined geometric limit we can study the relations between differential geometric objects on the elements of the sequence and on the limit. In particular we are interested to the case of spin^c structures and couples connection-spinor associated to them. The following definition arises naturally:

Definition 3.2.2 *The couple $(\tilde{P}_{\hat{M}}, (A, \psi))$, with $(A, \psi) \in \mathcal{A}(\mathcal{L}_{\hat{M}}) \times \Gamma(S^+(\tilde{P}_{\hat{M}}))$, is the geometric limit of a sequence $(\tilde{P}_{M_n}, (A_n, \psi_n))$, with $(A_n, \psi_n) \in \mathcal{A}(\mathcal{L}_{M_n}) \times \Gamma(S^+(\tilde{P}_{M_n}))$, if $(\hat{M}, g, b^i)$ is the geometric limit of (M_n, g_n, p_n^i) with diffeomorphisms $\phi_{n,i}$, these diffeomorphisms are covered by spin^c bundle isomorphisms*

$$\tilde{\phi}_{n,i}^* : \tilde{P}_{B_{n,i}} \to \tilde{P}_{A_i} \tag{3.12}$$

and, for each i, $\tilde{\phi}_{n,i}^* \cdot (A_n, \psi_n)|_{B_{n,i}}$ converges in the smooth topology, as n goes to infinity, to $(A, \psi)|_{A_i}$.

Note that, unlike the Yang-Mills case, as there are no bubbling phenomena, we are not forced to extend the definition of limit to cover something analogous to generalized connections.

We continue the discussion on geometric limits considering the particular case we have to deal with.
Consider a four manifold $(M = M^+ \cup_N M^-, g)$ and define the sequence of manifolds $(M_n, g_n, p^\pm)$ as $M_n = M^+ \cup_N N \times [-n, n] \cup_N M^-$ with the product metric $g = dt^2 + g_N$ on the internal cylinder glued in a smooth way to the metric of $M \setminus [-\epsilon, \epsilon] \times N$ (as this kind of process is largely standard in Donaldson theory we will not bother with the details) and two base points $p^\pm \in intM^\pm$. The geometric limit of this sequence is given by the manifold $(\hat{M}, g, p^\pm)$ which has two components $\hat{M}^\pm = M^\pm \cup_N N \times [0, \infty)$, with the product metric on the cylindrical end and with the same base points $p^\pm$. In the very same way we can identify the geometric limit of a sequence of spin^c structures $\tilde{P}_{M_n}$ defined, starting from a spin^c structure $\tilde{P}_M$ on M, as $\tilde{P}_{M^\pm}$ on $M^\pm$ while on the finite cylinders $[-n, n] \times N$ they are pull-back of $\tilde{P}_N$; the geometric limit $\tilde{P}_{\hat{M}}$ will coincide, on the two components $\hat{M}^\pm$, with the union of the spin^c structures $\tilde{P}_{\hat{M}^\pm}$ on a cylindrical end manifold we have been considering in Chapter 2.

In this context we can consider the problem of the geometric limit of sequences of couples connection-spinor which solve SW equations.
The main result on convergence is the following compactness lemma, which parallels Theorem 6.1.1 of [MMR] in Seiberg-Witten set up:

Lemma 3.2.3 *Let $(M_n, g_n, p^\pm)$ be the sequence of four manifolds defined above, with geometric limit $(\hat{M}, g, p^\pm)$, and $\tilde{P}_{M_n}$ the sequence of spin^c structures on M_n with geometric limit $\tilde{P}_{\hat{M}}$. Consider a sequence (A_n, ψ_n) of solutions of SW equations associated to $\tilde{P}_{M_n}$. Then there exist a finite energy solution (A, ψ) of SW equations for $\tilde{P}_{\hat{M}}$ which is the geometric limit of a subsequence of (A_n, ψ_n).*

Proof: the proof requires essentially an adaptation of Kronheimer-Mrowka compactness Lemma for manifolds with boundary, Lemma 4 of [KM1]. The content of this Lemma is that a sequence (A_m, ψ_m) of solutions of SW equations on a spin^c structure $\tilde{P}_Z$ on a compact manifold with boundary Z with uniform C^0 bound on ψ_m, admits a subsequence $(m') \subset (m)$ and a family of gauge transformations $g_{m'} \in \mathcal{G}(\tilde{P}_Z)$ such that $g_{m'} \cdot (A_{m'}, \psi_{m'})$ converges smoothly to (A, ψ). Now consider the manifolds (M_n, g_n): these are closed manifold with a bound on scalar curvature which does not depend on n and therefore Proposition 1.3.2 applies to prove that the $\mathcal{C}^0$ norm of ψ_n has a uniform bound. Now for any $t \geq 0$, $n \geq n(t)$ consider the manifolds with boundary $A^{\pm} \subset \hat{M}^{\pm}$, $B_n^{\pm} \subset M_n$ which contain the ball of radius t around the base point $p^{\pm}$; by definition of geometric limit we will have a sequence of isometries $A^{\pm} = B_n^{\pm}$ and of spin^c bundle isomorphisms of spin^c structure covering it; we can identify therefore the sequence $(A_n, \psi_n)|_{B^{\pm}}$, up to gauge transformation, with a sequence of solutions on the spin^c structure $\tilde{P}_{A^{\pm}} := \tilde{P}_{\hat{M}}|_{A^{\pm}}$ with bounded $\mathcal{C}^0$ spinor norm. We apply to this sequence the Kronheimer-Mrowka Lemma to obtain smooth converge, up to gauge transformation, of a subsequence $(n') \subset (n)$. Direct use of the definition of geometric limit of (A_n, ψ_n) and a standard diagonal argument brings therefore to the identification of the geometric limit, up to gauge transformation, of a subsequence $(n'') \subset (n)$ which coincides, on each compact subset of $\hat{M}$, with the limit described above. This geometric limit has finite energy by virtue of the fact that each solution of the sequence has energy bounded by a constant depending only on M and $\tilde{P}_M$ (see [KM1] or [MST]).

Note that, until now, we put no conditions on the geometry of N and in fact the previous result holds without any further hypothesis on N. There is another aspect, instead, that depends strictly on the three manifold considered, namely the convergence without variation of the Chern integral. Normally in Yang-Mills case the condition analysed is that of convergence without variation of energy, where the energy of a solution is given by the Yang-Mills functional. This functional, on the space of solutions, is a fixed multiple of the Pontrjagin number and is therefore determined by the geometry of the problem. In Seiberg-Witten case a complete analogy is not so immediately available (or, at least, we have not been able to develop it in a satisfactory way), due to the slightly more involved form of the functional which defines, via variational methods, SW solutions as absolute minima. We have anyhow at our disposal the integral of the square norm of the first Chern class of $\mathcal{L}_{M_n}$ which, via Chern-Weil theory, takes the form

$$c_{M_n} = c_1^2(\mathcal{L}_{M_n}) \cap [M_n] = -\frac{1}{4\pi^2} \int_{M_n} F_{A_n} \wedge F_{A_n}, \tag{3.13}$$

and therefore coincides, on a closed four manifold, with the Chern integral of a solution as defined in equation 3.4. It is clear that c_{M_n} is in fact independent from n and is determined only by the spin^c structure $\tilde{P}_M$. We can naturally study the question whether the Chern integral of the geometric limit of a sequence of SW solutions has the same value

of the Chern integral of the elements of the sequence.
The condition of convergence without variation of the Chern integral depends strictly on N or, better to say, on the interplay between the equations and the geometry of N. At this point, to make the discussion nontrivial, we will keep a point of view which is intermediate between full generality, i.e. N whatsoever, and our specific case (otherwise, as we will see soon, the whole argument would collapse). In particular we will make no assumption on the connectedness of the critical set of the CSD functional, that we continue denoting by $\chi(N)$. Instead we will assume that the CSD functional is gauge invariant, as happens in our case; in the general case this could require a passage to the universal cover of the orbit space.

So let's consider the geometric limit (A, ψ) on $\hat{M}$ of a sequence (A_n, ψ_n). Because of smooth convergence over compact subsets of $\hat{M}$ problems of Chern integral variation arise, roughly speaking, "within the cylinder". Here the arguments, for any kind of gradient flow equations (Yang-Mills, Seiberg-Witten), are analogous, so we follow the discussion of [MMR] (in a slightly different context, see also [MST]). We claim that for any η smaller than an η_0, eventually passing to a subsequence, there exist a partition of the internal cylinder of M_n, $[-n, n] \times N$, composed of the following pieces: first, a collection of disjoint cylinders $T_n^1, ..., T_n^r$, which have energy smaller than a fixed value η, whose length goes to infinity as n goes to infinity; second, the complement $C_n = [-n, n] \times N \setminus \cup_i T_n^i$ which has the property that each cylinder having energy less than η has length uniformly bounded. The length of C_n, moreover, is uniformly bounded.
Let's briefly discuss why this hold true. First we notice that, for any η suitably small, there exist a neighborhood ν_i of any component of the critical set $\chi_i(N)$, such that if (A_1, ψ_1) and (A_2, ψ_2) are both contained in ν_i then

$$|C(A_1, \psi_1) - C(A_2, \psi_2)| < \eta \tag{3.14}$$

(here and in what follows, when we talk about distances, neighborhoods and so on we always refer to the based orbit space). In the case when the two points belong to neighborhoods of different components, two cases can appear, depending whether the two components have the same value of the CSD functional or not; in the first case, the relation 3.14 again holds true, while in the second case we have

$$|C(A_1, \psi_1) - C(A_2, \psi_2)| > \frac{1}{2}\Delta C >> \eta \tag{3.15}$$

where ΔC denotes the minimal nonzero gap in the CSD functional between components of the critical set.

Now we can identify, within $[-n, n]$, a set of jumping intervals, i.e. minimal intervals whose endpoints have the property that (A_n, ψ_n) lies in the boundary of neighborhoods of components of the critical set having different value of CSD functional. We observe that the number N_n of such jumping intervals is finite, as it must satisfy the relation $N_n \cdot \frac{1}{2}\Delta C < E$ where E is a bound to the total energy of the cylinder. Moreover the length of these jumping intervals must be finite, as for each internal point of the intervals Lemma 2.3.4 applies and finite energy condition imposes a bound on the length. The complement to the set of jumping intervals is composed by a finite number of intervals. By eventually cutting out at the two extremes an interval, whose length is again finite by Lemma 2.3.4, we can suppose that these intervals connect neighborhoods of components of $\chi(N)$ which coincide or have the same value of CSD functional; the energy of these intervals is given by the variation of the CSD functional between the endpoints and therefore its value is bounded by η, by virtue of equation 3.14. These intervals, as n goes to infinity, can assume or not infinite length. We label the cylinders identified by intervals of the latter type as T_n^s. The statement on the finite length of C_n is at this point quite evident. Passing eventually to subsequences we might assume that the number of cylinders T_n^s is constant and that the label respect an ascending order.

Now if we consider the sequence (A_n, ψ_n) restricted to the cylinder T_n^s, the output of Lemma 3.2.3 is that we can extract a subsequence converging to a geometric limit (A_s, ψ_s) on the infinite cylinder $N \times \mathbb{R}$ which is a solution of SW equations having energy bounded by η. The results of Section 2.3 apply to deduce that its limit values as $t \mapsto \pm\infty$ are static solutions. The small energy condition, once η is chosen small enough, dictates that the only possibility is that these static solutions have the same value of Chern-Simons functional, and therefore the solution, over the cylinder, must be a static one.

Coming back to the cylinder $[-n, n] \times N$ we see therefore that the possibility of variations of the energy is due to components of C_n; moreover, because of convergence over compacts, the eventual external components are excluded form this phenomenon, and the only available remaining possibilities are related to components which are comprised between two cylinders of type T_n^s. If we apply the same arguments as above, moreover, we see that this component must contain at least a jumping cylinder. The geometric limit of the sequence (A_n, ψ_n) restricted to this component together with its neighboring cylinders $T_n's$ is a finite energy solution on $N \times \mathbb{R}$ with energy bounded by below by ΔC, as on a compact interval of $\mathbb{R}$ each element of the sequence has energy bounded by below by equation 3.15 and connects components of $\chi(N)$ having different CSD functional, i.e. represents a nonstatic flowline between static solution. The variation of energy due to the presence of such a cylinder implies a variation of Chern integral, according to the formula which relates energy and Chern integral,

$$C(b)-C(a) = \int_{N\times[a,b]} F_A \wedge F_A + \int_N < \psi(b), \partial\!\!\!/_{A(b)}\psi(b) > - \int_N < \psi(a), \partial\!\!\!/_{A(a)}\psi(a) >, \quad (3.16)$$

formula which can be obtained from Stokes' Lemma.

Now let's come again to the specific case of a four manifold M with $b_+^2(M) > 1$ decomposed along a three manifold of positive scalar curvature, having a spin^c structure $\tilde{P}_M$ with nonzero SW invariant. In this case we can obtain strong results on the variation of Chern integral: we have the following "ungluing" property, which refines Lemma 3.2.3:

Proposition 3.2.4 *Let M,$M^{\pm}$,N as above, and suppose that $SW(\tilde{P}_M)$ is different from zero: then there exist a couple $c_{M^\pm}$ with $c_{M^+} + c_{M^-} = c_M$ such that $\mathcal{M}_{c_{M^\pm}}(\tilde{P}_{\hat{M}^\pm})$ are non empty.*

Proof: the proof of this proposition is an obvious consequence of the previous discussion. As $b_+^2(M) > 1$ the Seiberg-Witten invariant is independent of the metric, and the fact that $SW(\tilde{P}_M)$ is non zero implies that, for any n, $\mathcal{M}(\tilde{P}_{M_n})$ is non empty. We can therefore consider a sequence of solutions (A_n, ψ_n) on M_n like in Lemma 3.2.3 and the geometric limit of this sequence will be a finite energy solution of Seiberg-Witten equations on $\hat{M} = \hat{M}^+ \coprod \hat{M}^-$, i.e. defines two points in $\mathcal{M}(\tilde{P}_{\hat{M}^\pm})$. In the case where the critical set of CSD functional is connected, as happens when N has positive scalar curvature, the critical set has the same value of CSD functional and, *by fiat*, there can not be flow between static solutions of different value of CSD functional (in practice, we can identify a single cylinder T_n^1) and thus no variation for the Chern integral of the geometric limit. Consequently the Chern integral c_M will get shared in some way (that will generally depend on the geometry and topology of $M^{\pm}$) between the solutions on $\hat{M}^{\pm}$; from this the part of the statement on the Chern integral of the moduli spaces follows.

In the case when $b_1(N) > 0$ one can investigate the relation between the limit values of the two solutions, on M^+ and M^-, identified in Proposition 3.2.4. We can prove that these limits are compatible, i.e. the following holds:

Proposition 3.2.5 *The solutions constructed in Proposition 3.2.4 have the same limit point under the limit map* $\partial_\infty^\pm : \mathcal{M}_{c_{\hat{M}^\pm}}(\tilde{P}_{\hat{M}^\pm}) \to \chi(N)$.

Proof: this is as well a consequence of the absence of nonstatic flowlines between static solutions: in fact for n high enough we can assume that the energy on the cylinder T_n^1 is arbitrarily small and, by virtue of Lemma 2.3.5, all values $(A(t), \psi(t))$ on the cylinder lie inside a neighborhood, which can be made arbitrarily small with η, of a static solution (this can be seen as a consequence of equation 2.37). This is compatible only with the condition that the limit points under the map ∂_∞ do coincide.

We finish with the obvious remark that all the discussion of this Section continues to hold true when we considered solutions of η-Seiberg-Witten equations with η compactly supported.

3.3 Regularity of the Moduli Spaces.

We now come to the problem of regularity of the moduli spaces for cylindrical end manifolds. The first thing we are interested in is the study of smoothness and compactness property of the moduli spaces of finite energy solutions of SW equations for a spin^c structure $\tilde{P}_{\hat{X}}$ on $\hat{X} = X \cup_N N \times [0, \infty)$. Regarding the compactness, the proof will be, similarly to what we have done in Section 3.2, an application of the compactness lemma of [KM1], for the case of manifolds with boundary, while to obtain a smoothness result we must appeal to genericity theorems which follow from perturbation of the equations, with the condition of having $b_+^2(X) > 0$. We will see that the knowledge of the decay properties of the solutions allows the construction of an infinitesimal theory, for the moduli space, which is essentially the same we found for the case of closed manifolds. We can state the main

Proposition 3.3.1 *There exist a second category subset of the perturbation parameter space such that the moduli space of solution* $\mathcal{M}_{c_X}(\tilde{P}_{\hat{X}}, \eta_4)$ *of* η_4*-Seiberg-Witten equations on a cylindrical end manifold* $\hat{X} = X \cup_N N \times [0, \infty)$ *with* $b_+^2(X) > 0$ *is a compact smooth oriented manifold of the expected dimension.*

Proof: the proof of compactness follows almost word by word part of the discussion of geometric limits of Section 3.2. The basic ingredient of the proof is as well the connectedness of the critical set of the CSD functional, so we will provide a proof under this hypothesis, more general than having N with positive scalar curvature. We have first to verify that the hypothesis of compactness lemma of [KM1] are satisfied: in fact, we simply need a $\mathcal{C}^0$ bound on $\psi_{4,i}$ for any sequence of solutions, and this follows immediately from the maximum principle and the decay conditions on the spinors along the cylinder; the compactness lemma of [KM1] implies then smooth convergence of a subsequence of the sequence of solutions over a compact subset of $\hat{X}$ to a solution (A_4, ψ_4) of Seiberg-Witten equations for $\tilde{P}_{\hat{X}}$; applying a diagonal argument we obtain convergence over any compact subset of $\hat{X}$. Similarly to the closed case, when we consider a sequence of solutions with fixed Chern integral the energy of the solutions on the cylinder is uniformly (w.r.t. the index i) bounded. The limit solution (A_4, ψ_4), therefore, is as well finite energy, it defines a point in $\mathcal{M}(\tilde{P}_{\hat{X}}, \eta_4)$ and has all the decay properties discussed in the previous Sections. We have to check, now, that our hypothesis guarantee strong convergence to the limit point, i.e. convergence in the topology of the moduli space, and in particular that there are no variation in the Chern integral of the limit (fact not obvious in the general case). These two aspects are in fact related and depend both on the uniform decay for the elements of the sequence that we will now establish. This uniform decay follows in fact directly from an application of Lemma 2.3.4 and the observation that the energy is uniformly bounded (for fixed Chern integral), but analysing this in detail we can better understand the role of connectedness of the critical set.

Assume that there's not uniformity, in i, for the decay along the t-coordinate: this implies that, for some finite K, there is a sequence $T_i \mapsto \infty$ such that the energy of the i^{th} solution, on the cylinder $N \times [T_i, \infty)$, is greater that K:

$$C(\chi(N), 0) - C(A_{4,i}(T_i), \psi_{4,i}(T_i)) > K \tag{3.17}$$

(we will implicitly pass to subsequences and relabel whenever necessary); this means that the energy of the solutions of the sequence walks to the end. Application of Lemma 2.3.4 guarantees that, for a positive $\eta < \frac{K}{2}$ there exist a finite $T_{\eta,i}$ such that $(A_{4,i}(T_i + T_{\eta,i}), \psi_{4,i}(T_i + T_{\eta,i}))$ is sufficiently near $\chi(N)$ to have

$$C(\chi(N), 0) - C(A_{4,i}(T_i + T_{\eta,i}), \psi_{4,i}(T_i + T_{\eta,i})) < \eta \tag{3.18}$$

Moreover, as the energy of the sequence has an uniform bound, there is a upper bound T (depending on η, but this has no relevance for us) to these $T_{\eta,i}$. This implies that the sequence $(A_{4,i}, \psi_{4,i})$ satisfies the condition

$$C(A_{4,i}(T_i+T), \psi_{4,i}(T_i+T)) - C(A_{4,i}(T_i), \psi_{4,i}(T_i)) > \frac{K}{2}. \tag{3.19}$$

If we now translate the solution $(A_{4,i}, \psi_{4,i})|_{[0,2T_i]\times N}$ by $-T_i$ we obtain a sequence of finite energy solutions of SW equations on $[-T_i, T_i] \times N$, with energy over $[0,T] \times N$ uniformly bounded by below; the geometric limit is a finite energy solution on $N \times \mathbb{R}$, nonstatic as $C(A_4(T), \psi_4(T)) - C(A_4(0), \psi_4(0))$ is bounded from below by $\frac{K}{2}$; this condition is excluded in the case of N having connected critical set, and in particular having positive scalar curvature, as all flows between static solutions must be static.
Convergence over compacts and uniform t-decay imply now the strong convergence and absence of variation in the Chern integral. In fact, the same argument as above, with minor modification, can be applied by controlling, instead that the energy of the i^{th} solution over $N \times [T_i, \infty)$, the integral $\int_{T_i}^{\infty} F_{A_{4,i}} \wedge F_{A_{4,i}}$; equation 3.16 suggests how to relate the two approaches. In particular, when N has positive or zero scalar curvature, and thus c_X is fixed, convergence over compacts and connectedness of the critical set imply directly compactness of the moduli space.
As remarkable consequence of the previous discussion we note therefore the uniformity, in the moduli space, for the exponential decay to the limit set, i.e. the initial value of Lemma 2.3.6 can be chosen independent of the particular solution.
Another point that worths noticing is that this compactness result, differently from the one proved in the closed case, is not intrinsic of Seiberg-Witten theory. As well known, in Yang-Mills case, there are two non compactness phenomena that arise dealing with moduli spaces on manifolds with cylindrical end: the first one, shared with the closed manifold case, is due to the bubbling phenomenon; in SW theory, the absence of such phenomenon (due to the "$\mathcal{C}^0$ bound" on spinors, strictly speaking), forbids, as seen above, such a kind of problem. The second one, instead, has to do with non static flows on cylinders, with appear, in the compactification process, as solutions that "walk off the end". There's no *a priori* reason why such phenomenon should not arise in SW theory, and in general, as seen in Section 3.2, it does. The absence of this, that we have proven before, is related to the geometry of the three manifold under analysis.
Concerning smoothness, we apply Fredholm theory for manifolds with cylindrical end ([APS], [T1]) to obtain a smoothness result, for generic parameter, away from reducibles. In order to do so we will need to perturb SW equations using a parameter space that fits

in Sard-Smale scheme, i.e. a Banach manifold: compactly supported self dual forms, in particular, do not satisfy this requirement as they do not form a Banach space. There are several possible approaches, at this point: the first one is to consider the fact that compact supported perturbations (say of class $\mathcal{C}^r$) are dense in any Sobolev space of δ-decaying self dual forms. Once extended the class of perturbation to this new space, at the price of recover some of the previous results (explicitly those concerning decay of any solution of perturbed SW equations) in this new set up, results that in fact are often given for granted, we use the fact that the Sard-Smale theorem will provide a second category subset of parameter space where the moduli space is smooth.
In fact we prefer to use another approach, which has the advantage of limiting ourselves to the usual set up of perturbations supported on a compact subset of $\hat{X}$. The idea is to use the fact, well known in Yang-Mills theory, that it is enough, to obtain smoothness, to perturb equations (or metrics, in Yang-Mills case), in any open subset of the manifold. The reason of this is that the operator which linearizes unperturbed equations is surjective elliptic, and its cokernel (which is what we want to "kill" by perturbing the equations) satisfies unique continuation properties: in particular, if it vanishes on an open set, it vanishes everywhere. We already gave some detail of this, in the closed case, in Section 1.3. We will consider therefore as perturbation class the Banach space of $\mathcal{C}^r$ self dual forms supported on a fixed compact subset of $\hat{X}$ (with nonempty interior), say $K = X \cup_N N \times [0,1]$ (but we will see that also any ball in $\hat{X}$ would do the job); with this class of perturbations, all decay results are automatically satisfied.
Recall now that the domain of Seiberg-Witten equations is given by $\mathcal{A}(\mathcal{L}_{\hat{X}}) \times \Gamma(S^+(\tilde{P}_{\hat{X}}))$, which is defined by forms and spinors which are locally in some Sobolev space; with this space it is not possible to construct a Fredholm theory, so we are interested in redefining the domain of SW equations. The guidelines of this construction are the results of [LM], concerning Fredholm theory for manifolds with cylindrical ends, and the idea of extended solutions contained in [APS]. In light of the results of Chapter 2 we introduce the space $\mathcal{A}_\delta(\mathcal{L}_{\hat{X}}) \times \Gamma_\delta(S^+(\tilde{P}_{\hat{X}}))$ (see Chapter 7 of [MMR] for the analogous definition in Yang-Mills case) which is defined as the space of couples which δ-decay, in some Sobolev norm, to (the pull-back of) a gauge fixed solution on N, which is conveniently represented by an imaginary harmonic one form on N; in particular, once a cut-off function β equal to zero

(say) on X and equal to one on $\hat{X} \setminus K$ is defined, we have a (noncanonical) isomorphism

$$\begin{aligned} Harm^1(N, i\mathbb{R}) \times \Omega^1_\delta(\hat{X}, i\mathbb{R}) &\longrightarrow \mathcal{A}_\delta(\mathcal{L}_{\hat{X}}) \\ (A_0, \alpha_4) &\mapsto \beta\pi^* A_0 + \alpha_4; \end{aligned} \tag{3.20}$$

in the sequel, for sake of simplicity, we will omit to mention the pull-back and A_0 will be interpreted as a form on $\hat{X}$, constant on the end.

To make contact with the results of [MMR], the space $Harm^1(N, i\mathbb{R})$ plays the role of the center manifold, the fact that the flow is constant on it corresponds to the fact that all the points of the critical set are smooth (see the introduction to Part 1 of [MMR]), and this witnesses the property, shared with the analogous case in Yang-Mills theory, that the solutions exponentially decay to a static one (and not only to the center manifold, as happens in the general case). An example of center manifold which gives rise to nonstatic flow is discussed in [MMS]; also in that case, anyway, outside the nonsmooth point of the critical set, the situation is identical to ours.

We consider now the perturbed Seiberg-Witten equations as a map

$$\begin{aligned} SW : \mathcal{A}_\delta(\mathcal{L}_{\hat{X}}) \times \Gamma_\delta(S^+(\tilde{P}_{\hat{X}})) \times \Omega^2_{o,+}(K, i\mathbb{R}) &\longrightarrow \Omega^2_{\delta,+}(\hat{X}, i\mathbb{R}) \times \Gamma_\delta(S^-(\tilde{P}_{\hat{X}})) \\ SW(A_4, \psi_4, \eta_4) &= (F^+_{A_4} - q_4(\psi_4) - \eta_4, \not{\partial}_{A_4}\psi_4). \end{aligned} \tag{3.21}$$

We will consider also the projection from the domain of SW to the parameter space, that we will denote by π.

The map SW is clearly smooth, by construction, and $\mathcal{G}_\delta(\tilde{P}_{\hat{X}})$-equivariant, where $\mathcal{G}_\delta(\tilde{P}_{\hat{X}})$ is the group of the automorphisms of $\tilde{P}_{\hat{X}}$ which project to the identity automorphism on the frame bundle and act on $\mathcal{A}_\delta(\mathcal{L}_{\hat{X}}) \times \Gamma_\delta(S^+(\tilde{P}_{\hat{X}}))$. In order to identify this group, which is a subgroup of the gauge group of $\tilde{P}_{\hat{X}}$, we need to take care, roughly speaking, of both the components of $\mathcal{A}_\delta(\mathcal{L}_{\hat{X}})$ which appear in equation 3.20. In the case of a rational homology sphere, or more generally whenever the action of the group of components of $\mathcal{G}(\tilde{P}_{\hat{X}})$ (which is given by $H^1(X, \mathbb{Z})$) is trivial on $Harm^1(N, i\mathbb{R})$, i.e. $H^1(X, \mathbb{Z}) \to H^1(N, \mathbb{Z})$ is the zero map, the situation is a bit simpler than the general case; as we have chosen to already gauge fix the extended solutions on N, representing them as harmonic forms, the action of $\mathcal{G}_\delta(\tilde{P}_{\hat{X}})$ must be trivial at infinity (recall the proof of Proposition 3.1.2) up to the constant action of the stabilizer of the limit point, and the work of Taubes [T1] identifies $\mathcal{G}_\delta(\tilde{P}_{\hat{X}})$ as

$$\mathcal{G}_\delta(\tilde{P}_{\hat{X}}) = \{g \in \mathcal{G}(\tilde{P}_{\hat{X}}) | g^{-1} d_{\hat{X}} g \in \mathcal{L}^2_{k,\delta}\}, \tag{3.22}$$

as the asymptotic values are reducible. Its Lie algebra is identified, via Lemma 5.12 of [T1], with $\Omega^0_\delta(\hat{X}, i\mathbb{R}) \oplus i\mathbb{R}$. Each element of $\mathcal{G}_\delta(\tilde{P}_{\hat{X}})$ has a limit map r with values in the stabilizer of a flat solution on N, i.e. $U(1)$.

In the general case, the group defined as above describes only some of the components of the whole gauge group (those which are labeled by $H^1(X, N, \mathbb{Z})$), as we must also deal with those elements of $\mathcal{G}(\tilde{P}_{\hat{X}})$ which act, at infinity, via a translation on $Harm^1(N, i\mathbb{R})$ along its lattice $ImH^1(X, i\mathbb{Z}) \subset H^1(N, i\mathbb{Z})$, which labels the components of $\mathcal{G}(\tilde{P}_N)$ which extend to $\hat{X}$, and whose action on $\Omega^1_\delta(\hat{X}, i\mathbb{R})$, w.r.t. to the identification of equation 3.20, is the same as in the previous case. These elements of $\mathcal{G}(\tilde{P}_{\hat{X}})$ define as well an action on $\mathcal{A}_\delta(\mathcal{L}_{\hat{X}}) \times \Gamma_\delta(S^+(\tilde{P}_{\hat{X}}))$ and together with the elements identified by the right hand side of equation 3.22 define the whole subgroup of $\mathcal{G}(\tilde{P}_{\hat{X}})$ acting on $\mathcal{A}_\delta(\mathcal{L}_{\hat{X}}) \times \Gamma_\delta(S^+(\tilde{P}_{\hat{X}}))$. We now have, as expected, $\pi_0(\mathcal{G}_\delta(\tilde{P}_{\hat{X}})) = H^1(X, \mathbb{Z})$. Applying componentwise the results of [T1] we still have a limit map, that we denote again by r, for any element of $\mathcal{G}_\delta(\tilde{P}_{\hat{X}})$, which identifies (apart from the lattice translation) a constant $U(1)$ transformation on the end.

The action of the gauge group is free on $\mathcal{A}_\delta(\mathcal{L}_{\hat{X}}) \times \Gamma_\delta(S^+(\tilde{P}_{\hat{X}}))$ outside the reducible; we will make use also of the based gauge group defined as the subgroup of $\mathcal{G}_\delta(\tilde{P}_{\hat{X}})$ whose action at infinity is (eventually) just given by translations, i.e. whose limit element in the stabilizer of a flat solution is the identity:

$$\mathcal{G}^o_\delta(\tilde{P}_{\hat{X}}) := r^{-1}(1) \subset \mathcal{G}_\delta(\tilde{P}_{\hat{X}}); \tag{3.23}$$

its Lie algebra $Lie\mathcal{G}^o_\delta(\tilde{P}_{\hat{X}})$ is given by $\Omega^0_\delta(\hat{X}, i\mathbb{R})$. This based gauge group acts freely on $\mathcal{A}_\delta(\mathcal{L}_{\hat{X}}) \times \Gamma_\delta(S^+(\tilde{P}_{\hat{X}}))$.

Note that the approach we followed i.e. that of fixing the gauge at infinity and take the full gauge group $\mathcal{G}_\delta(\tilde{P}_{\hat{X}})$, is not the only possible one when one deals with extended solutions; slight differences are possible, for example the approach of [KM2], Section 5.3, differs from our by the choice of the gauge group (only the components labeled by $H^1(X, N, \mathbb{Z})$ are considered) and gauge equivalent (on N) static solutions are identified *a posteriori*, in order to define the correct moduli space.

The results of Chapter 2 tell us that, in correspondence of compactly supported perturbations, the parameterized moduli space fits in the above scheme as

$$\mathcal{PM}(\tilde{P}_X) = SW^{-1}(0)/\mathcal{G}_\delta(\tilde{P}_{\hat{X}}). \tag{3.24}$$

The relation is constructed by taking, for the $\mathcal{G}(\tilde{P}_{\hat{X}})$-class of a solution, a representative which δ-decays to a flat connection on the end; we can always suppose, up to the action of the connected component of $\mathcal{G}(\tilde{P}_{\hat{X}})$, that this connection decays to an element of $Harm^1(N, i\mathbb{R})$; the action of the group of components identifies then an element of $\hat{\chi}(N)$. We associate then to this representative, which lies in $\mathcal{A}_\delta(\mathcal{L}_{\hat{X}}) \times \Gamma^*_\delta(S^+(\tilde{P}_{\hat{X}}))$, its $\mathcal{G}_\delta(\tilde{P}_{\hat{X}})$-equivalence class. This shows that the choice of $\mathcal{A}_\delta(\mathcal{L}_{\hat{X}})$, built as indicated, in the domain of SW, is the correct one to obtain the relation 3.24.

Now we want to prove that, away from reducible solutions, the moduli space is a smooth manifold. In order to prove this we need to show that SW has zero as regular value, i.e. its linearization, which is the map

$$DSW_{(A_4,\psi_4,\eta_4)} : T\mathcal{A}_\delta(\mathcal{L}_{\hat{X}}) \times \Gamma_\delta(S^+(\tilde{P}_{\hat{X}})) \times \Omega^2_{o,+}(K, i\mathbb{R}) \longrightarrow \Omega^2_{\delta,+}(\hat{X}, i\mathbb{R}) \times \Gamma_\delta(S^-(\tilde{P}_{\hat{X}})), \tag{3.25}$$

is surjective whenever the solution is not reducible. By definition, there exist an equivariant limit map that we denote as well ∂_∞:

$$\begin{aligned} \partial_\infty : \mathcal{A}_\delta(\mathcal{L}_{\hat{X}}) \times \Gamma_\delta(S^+(\tilde{P}_{\hat{X}})) \times \Omega^2_{o,+}(K, i\mathbb{R}) &\longrightarrow Harm^1(N, i\mathbb{R}) \\ (A_4, \psi_4, \eta_4) &\mapsto A_0, \end{aligned} \tag{3.26}$$

(here, the action of $\mathcal{G}_\delta(\tilde{P}_{\hat{X}})$ on the domain is the one described above, while on the codomain $Harm^1(N, i\mathbb{R})$ it is the action of the group of components $\pi_0(\mathcal{G}_\delta(\tilde{P}_{\hat{X}}))$, whose quotient, by definition, is $\hat{\chi}(N)$).

We claim that, in correspondence of each fiber $\partial_\infty^{-1}(A_0)$ of the limit map, zero is a regular value of SW: this means, obviously, that zero is a regular value and that ∂_∞ is a submersion, transverse to any finite set of submanifolds of $Harm^1(N, i\mathbb{R})$. To prove that, we must show that the component of the linearization DSW on a fiber, that we will denote by D_fSW, is a surjective operator: the linearized equations around an irreducible solution have the form

$$\begin{gathered} D_fSW_{(A_4,\psi_4,\eta_4)} : \Omega^1_\delta(\hat{X}, i\mathbb{R}) \times \Gamma_\delta(S^+(\tilde{P}_{\hat{X}})) \times \Omega^2_{o,+}(K, i\mathbb{R}) \longrightarrow \Omega^2_{\delta,+}(\hat{X}, i\mathbb{R}) \times \Gamma_\delta(S^-(\tilde{P}_{\hat{X}})) \\ D_fSW_{(A_4,\psi_4,\eta_4)}(a_4, \phi_4, \epsilon_4) = (d^+_{\hat{X}} a_4 - 2q_4(\psi_4, \phi_4) - \epsilon_4, \not{\partial}_{A_4}\phi_4 + \tfrac{1}{2} a_4 \cdot \psi_4); \end{gathered} \tag{3.27}$$

to prove surjectivity of D_fSW we proceed like in the closed case. First, any element of the type $(b_4, 0)$ lies in ImD_fSW: by contradiction, if $(b_4, 0) \notin ImD_fSW$, then varying a_4 we see that $b_4 \in coker d^+_{\hat{X}}$, so that b_4 must satisfy unique continuation theorem; by varying ϵ_4 we deduce that it must vanish in any open set contained in K and so it has to vanish everywhere. For what concerns the spinor part, the proof that any $(0, \varphi_4)$ belongs to ImD_fSW goes as in the closed case, by application of unique continuation theorem, using irreducibility of (A_4, ψ_4).

It follows from the above property of SW that $SW^{-1}(0)$, restricted to irreducibles, is a smooth infinite dimensional manifold; its quotient by the (say) based gauge group $\mathcal{PM}^{o,*}(\tilde{P}_X)$ is a parameterized moduli space which has a map π to the space of parameters and a submersion ∂_∞ to the limit set $\hat{\chi}(N)$, in such a way that its tangent space fits in the exact sequence

$$ker\partial_{\infty *} \rightarrowtail T\mathcal{PM}^{o,*}(\tilde{P}_X) \twoheadrightarrow \partial^*_\infty T\hat{\chi}(N); \tag{3.28}$$

the first term of the sequence, as seen above, already surjects to the codomain of DSW, and the index of π is given by the sum of the index of D_fSW plus the dimension of $T\hat{\chi}(N) = b_1(N)$. To prove that the the generic fiber of π is smooth we want to apply the Sard-Smale theorem, which can be applied when π is Fredholm. As in the closed case, this property follows by the analysis of the deformation complex associated to a fixed perturbation, namely

$$0 \to \Omega^0_\delta(\hat{X}, i\mathbb{R}) \to \Omega^1_\delta(\hat{X}, i\mathbb{R}) \times \Gamma_\delta(S^+(\tilde{P}_{\hat{X}})) \to \Omega^2_{\delta,+}(\hat{X}, i\mathbb{R}) \times \Gamma_\delta(S^-(\tilde{P}_{\hat{X}})) \to 0; \tag{3.29}$$

it is a standard fact that for all but a discrete set of δ without accumulation points this is a Fredholm complex (the first map is just the Lie algebra action, while the second map is D_fSW for a fixed value of the perturbation). Moreover the map π is proper, as we have already proven the compactness of the fibers, unaffected by the perturbation we have considered. The application of Sard-Smale theorem guarantees that there is a second category subset of $\Omega^2_{o,+}(K, i\mathbb{R})$ such that the fibers are smooth.

Note that the vertical complex of equation 3.29 is a subcomplex of the one corresponding to the full linearization DSW, with quotient the trivial complex having single nonzero term $Harm^1(N, i\mathbb{R})$.

Now let's deal with the issue of reducible solutions. Reducible solutions of η_4-SW equations correspond to solutions of

$$F^+_{A_4} = \eta_4, \tag{3.30}$$

where η_4 is a purely imaginary self dual form, chosen within our perturbations space. Our aim is to find an open and dense subset of the perturbation space such that, in correspondence of the intersection of this subset with the second category subset for which the moduli space $\mathcal{M}^{o,*}(\tilde{P}_{\hat{X}}, \eta_4)$ is smooth, the full moduli space $\mathcal{M}^o(\tilde{P}_{\hat{X}}, \eta_4)$ will be smooth and composed of irreducible points. The intersection of the aforementioned subsets of the parameter space will still be, of course, second category.

It is immediate to see, by an Hodge decomposition argument, that if we take a perturbation which is in the image of d^+, this can be reabsorbed by adding to the (eventual) solution $(A_4, 0)$ a term of the type $(\alpha_4, 0) \in \Omega^1_\delta(\hat{X}, i\mathbb{R}) \times \Gamma_\delta(S^+(\tilde{P}_{\hat{X}}))$; "efficient" perturbation must have, therefore, nontrivial projection on $coker d^+$. So we look for perturbations, supported in K, which have this property. Now take a term $\eta_4 \in \Omega^2_{o,+}(K, i\mathbb{R})$ and decompose it, according to Hodge decomposition of $\Omega^2_{\delta,+}(\hat{X}, i\mathbb{R})$, as $\eta_4 = d^+\alpha_4 + \beta_4$, with $\alpha_4 \in \Omega^1_\delta(\hat{X}, i\mathbb{R})$ and $\beta_4 \in coker d^+ \subset \Omega^2_{\delta,+}(\hat{X}, i\mathbb{R})$; the subspace of $\Omega^2_{o,+}(K, i\mathbb{R})$ which has nontrivial projection on $coker d^+$ corresponds therefore to the subspace of those $\beta_4 \in coker d^+ \subset \Omega^2_{\delta,+}(\hat{X}, i\mathbb{R})$ which can be represented (up to a term in $Im d^+$) by terms supported in K. Now it follows, by unique continuation (much as above) that any element in $coker d^+$ is representable in such a way (otherwise we could construct, by contradiction, a nonzero term in $coker d^+$ which has to vanish on any open subset of K). We will see in Section 3.4 that the dimension of the cokernel of d^+, in the complex 3.29 with vanishing spinor part (i.e. the complex 3.35) is equal to $b^2_+(X) + dim(H^1(N, \mathbb{R})/Im H^1(X, \mathbb{R}))$. But in the case where $H^1(N, \mathbb{R})$ is not zero, not all the perturbations corresponding to the cokernel of d^+ are necessarily efficient, as they could be reabsorbed also by a shift the limit value of a given abelian instanton, i.e. using an horizontal transformation in $\mathcal{A}_\delta(\mathcal{L}_{\hat{X}})$, whose tangent space decomposes according to

$$\Omega^1_\delta(\hat{X}, i\mathbb{R}) \rightarrowtail T\mathcal{A}_\delta(\mathcal{L}_{\hat{X}}) \twoheadrightarrow Harm^1(N, i\mathbb{R}). \tag{3.31}$$

The correct dimension to compute is therefore the dimension of the second cohomology group of the complex

$$0 \to \Omega^0_\delta(\hat{X}, i\mathbb{R}) \to T\mathcal{A}_\delta(\mathcal{L}_{\hat{X}}) \to \Omega^2_{\delta,+}(\hat{X}, i\mathbb{R}) \to 0; \tag{3.32}$$

It is not difficult to verify (see [MMR], Section 8.7) that the second cohomology group of the complex 3.32 is isomorphic to $Im[H^2(X, N, \mathbb{R}) \to H^2_+(X, \mathbb{R})]$ (our complex coincides, up to the term of degree zero and abelianization, with the complex E_δ of [MMR]). From this we deduce, applying the standard arguments regarding reducibles (see Section 1.4) that, up to a closed nowhere dense subset of codimension $b^2_+(X)$ in $\Omega^2_{o,+}(K, i\mathbb{R})$, equation 3.30 has no solutions.

The based irreducible moduli space $\mathcal{M}^{o,*}(\tilde{P}_{\hat{X}}, \eta_4)$ carries a free $U(1)$ limit action; if

$b_+^2(X) > 0$ we can suppose, by the previous discussion, that there are no reducible solutions for a second category subsets of perturbations, given by the intersection of the second category subset of regular values and the open and dense subset of perturbations which give no reducible solutions. In correspondence of this second category subset (which in particular is dense by Baire theorem) the full moduli space $\mathcal{M}(\tilde{P}_{\hat{X}}, \eta_4)$ is composed by smooth irreducible points.
As a consequence of this regularity result we have, in particular, that if $b_2^+(X) > 0$ and the dimension of $\mathcal{M}_{c_X}(\tilde{P}_{\hat{X}}, \eta_4)$ is negative, then, for generic η_4, $\mathcal{M}_{c_X}(\tilde{P}_{\hat{X}}, \eta_4) = \emptyset$.
Finally, the proof of orientability follows *verbatim* the proof for the closed case, see Section 1.3, applied to the complex of equation 3.29 plus an $\mathbb{R}$ factor, at the level of Lie algebra, to keep track of the $U(1)$ limit point action; the choice of an orientation corresponds to an orientation of the character variety and the choice of a sign in the maximal external product in

$$\mathbb{R} \oplus H_\delta^0(\hat{X}) \oplus H_\delta^1(\hat{X}) \oplus H_{\delta,+}^2(\hat{X}) \tag{3.33}$$

where the $H_\delta^i(\hat{X})$ are the cohomology groups of the Hodge-DeRahm part of the complex of equation 3.29, that we will compute in the next Section.

3.4 Dimension of the Moduli Spaces.

In order to compute the dimension of the moduli space $\mathcal{M}_{c_X}(\tilde{P}_{\hat{X}}, \eta_4)$, under the hypothesis of smoothness, in particular $b_+^2(X) > 0$, we need to to compute the index of the complex in equation 3.29, which gives the dimension of a fiber of the limit map ∂_∞, add the dimension of $\hat{\chi}(N)$, and then subtract 1 because of the free $U(1)$ action.
To get the index of the complex, we can homotopize it in such a way to decouple the connection and the spinor part, as in the closed case, and then we compute the sum of twice the (complex) index of the four dimensional Dirac operator coupled with a connection A_4 on the spin^c determinant line bundle $\mathcal{L}_{\hat{X}}$ on the four manifold $\hat{X}$, decaying to a flat connection A_0 on N, plus the Euler characteristic of the δ-half DeRahm complex.

The part concerning the Dirac operator gives, with the use of Atiyah-Patodi-Singer index theorem ([APS]) and excision formula along the lines of [MMR], Section 8.4, the following result:

$$ind_{\mathbb{C}} \partial\!\!\!/_{A_4} = -\frac{1}{32\pi^2}\int_{\hat{X}} F_{A_4} \wedge F_{A_4} - \frac{1}{8}\sigma(X) - \frac{\eta_{A_0}(0)}{2}, \tag{3.34}$$

as in our case there are no harmonic spinor for the boundary operator $\not\partial_{A_0}$ according to the fact (Proposition 2.2.2) that on a positively curved three manifold flat connections do not admit harmonic spinors.

Concerning the second term, we just pick the analogous result for the Yang-Mills case (see e.g. [R]). We consider the following complex of exponentially decaying functions and forms:

$$0 \to \Omega^0_\delta(\hat{X}, i\mathbb{R}) \to \Omega^1_\delta(\hat{X}, i\mathbb{R}) \to \Omega^2_{\delta,+}(\hat{X}, i\mathbb{R}) \to 0; \tag{3.35}$$

in [R] it is proven that the cohomology groups of this complex are given by

$$\begin{aligned} H^0_\delta(\hat{X}) &= 0, \\ H^1_\delta(\hat{X}) &= ker[H^1(X,\mathbb{R}) \to H^1(N,\mathbb{R})], \\ H^2_{\delta,+}(\hat{X}) &= Im[H^2(X,N,\mathbb{R}) \to H^2_+(X,\mathbb{R})] \oplus H^1(N,\mathbb{R})/Im(H^1(X,\mathbb{R})). \end{aligned} \tag{3.36}$$

Looking at the moduli space and its limit map,

$$\partial_\infty : \mathcal{M}_{c_X}(\tilde{P}_{\hat{X}}, \eta_4) \longrightarrow \hat{\chi}(N), \tag{3.37}$$

we get that the fiber has dimension

$$\begin{aligned} dim\mathcal{M}_{c_X}(\tilde{P}_{\hat{X}}, \eta_4, A_0) &= \tfrac{1}{4}c_X - \tfrac{1}{4}\sigma(X) - \eta_{A_0}(0) - 1 + b^1_\delta(\hat{X}) - b^2_{\delta,+}(\hat{X}) = \\ &= \tfrac{1}{4}c_X - \tfrac{1}{4}\sigma(X) - \eta_{A_0}(0) - 1 + b^1(X) - b^2_+(X) - b^1(N). \end{aligned} \tag{3.38}$$

To get the dimension of the moduli space we have still to add the dimension of the character variety, and we obtain the

Proposition 3.4.1 *The dimension of the moduli space of solution of Chern integral c_X for a spin^c structure $\tilde{P}_{\hat{X}}$ on a manifold $\hat{X} = X \cup_N N \times [0,\infty)$ is given by*

$$dim\mathcal{M}_{c_X}(\tilde{P}_{\hat{X}}, \eta_4) = \frac{1}{4}c_X - \frac{1}{4}\sigma(X) - \eta_{A_0}(0) - 1 + b^1(X) - b^2_+(X). \tag{3.39}$$

It is possible to rewrite this expression in a form which is more similar to the dimension formula for the closed case: we have

$$dim\mathcal{M}_{c_X}(\tilde{P}_{\hat{X}}, \eta_4) = \frac{1}{4}(c_X - 2\chi(X) - 3\sigma(X)) - \frac{1 - b_1(N)}{2} - \eta_{A_0}(0), \tag{3.40}$$

although we will use the previous form, more suitable for calculations.
The dimension formula is only apparently depending on A_0, via the η invariant: in effect η is constant in the space of flat connections on a bundle L on N. To give a quick proof of this, let Γ_0, Γ_1 be two flat connections on L and let $\Gamma(\cdot)$ be a path connecting them, suitably chosen in order to be constant in a neighborhood of the endpoints; define, on the finite cylinder $[0,1] \times N$ (where the coordinate on the interval is denoted by s), the continuous family of Dirac operators, parameterized by t, given by

$$\not{D}_t := \frac{\partial}{\partial s} + \not{\partial}_{\Gamma(st)}. \tag{3.41}$$

Due to the absence of harmonic spinors on N, for flat connections, because of the curvature condition, the index of $\not{D}_t$ for $t = 0, 1$ is given, using [APS], by

$$ind_{\mathbb{C}}(\not{D}_t) = -\frac{1}{32\pi^2} \int_{[0,1]\times N} F_{\Gamma(st)} \wedge F_{\Gamma(st)} - \frac{\eta_{\Gamma(t)} - \eta_{\Gamma(0)}}{2} \tag{3.42}$$

and as $ind_{\mathbb{C}}(\not{D}_1) = ind_{\mathbb{C}}(\not{D}_0)$ we have $\eta_{\Gamma_1} = \eta_{\Gamma_0}$.
In many cases, as there's an orientation reversing diffeomorphism of N, the η invariant vanishes.

3.5 Proof of Theorem 2.

In this Section, using the results on cylindrical end moduli spaces obtained in the previous Sections, we will give a proof of Theorem 2. We will consider how the moduli spaces of solutions of SW equations for a spin^c structure $\tilde{P}_M$ on a closed four manifold M with $b_2^+ > 1$, split along a positively curved N in two factors, say

$$M = M^+ \cup_N M^-, \tag{3.43}$$

relates to the moduli spaces for the spin^c structures $\tilde{P}_{\hat{M}^\pm}$ on $\hat{M}^\pm := M^\pm \cup_{\partial M^\pm} N \times [0, \infty)$ induced by $\tilde{P}_M$ (clearly the two manifolds with boundary $M^\pm$ correspond to the manifold that in the previous Sections was denoted by X). In particular we will analyse the consequences of the dimension formulae obtained in Section 3.4.
Let $\tilde{P}_M$ be a spin^c structure on M which restricts to $\tilde{P}_{M^+}$ and $\tilde{P}_{M^-}$ on the two factors $M^\pm$; extend the spin^c structures $\tilde{P}_{M^\pm}$ to $\tilde{P}_{\hat{M}^\pm}$ and suppose that the Seiberg-Witten invariants of M do not vanish for $\tilde{P}_M$: under this hypothesis we have the following

Proposition 3.5.1 *Let M, $M^{\pm}$, N be defined as in 3.43 and suppose that there exist a spin^c structure $\tilde{P}_M$ such that $SW(\tilde{P}_M) \neq 0$; then one of the two factors must have $b^2_+ = 0$.*

Proof: denote by $c_M = c_1^2(\mathcal{L}_M) \cap [M]$ and let $c_{M^\pm}$ be a couple of real numbers which satisfy the relation $c_M = c_{M^+} + c_{M^-}$; consider the (eventually empty) moduli spaces of solutions of SW equations for the spin^c structures $\tilde{P}_{\hat{M}^\pm}$ having Chern integral $c_{M^\pm}$. Suppose by contradiction that both sides have $b^2_+(M^\pm) > 0$: then, by Proposition 3.3.1, the moduli spaces $\mathcal{M}_{c_{M^\pm}}(\tilde{P}_{\hat{M}^\pm})$ are generically composed of irreducible solutions, smooth and of the expected dimension. We will analyse the constraints imposed by the dimension formulae.

For more clearness, we start by considering the easier case, that of a rational homology sphere, as the mechanism of the proof is more transparent in this case. The computation of the dimension for these moduli spaces is particularly simple, in that case: we have $b^1_\delta(\hat{M}^\pm) = b^1(M^\pm)$ and $b^2_{\delta,+}(\hat{M}^\pm) = b^2_+(M^\pm)$; moreover the splitting of the cohomology groups of M, over the reals, is just direct sum; therefore we have:

$$\begin{aligned} dim\mathcal{M}_{c_{M^+}}(\tilde{P}_{\hat{M}^+}) &= \tfrac{1}{4}c_{M^+} - \tfrac{1}{4}\sigma(M^+) - \eta_{A_0}(0) - 1 + b^1(M^+) - b^2_+(M^+), \\ dim\mathcal{M}_{c_{M^-}}(\tilde{P}_{\hat{M}^-}) &= \tfrac{1}{4}c_{M^-} - \tfrac{1}{4}\sigma(M^-) - \bar{\eta}_{A_0}(0) - 1 + b^1(M^-) - b^2_+(M^-) \end{aligned} \tag{3.44}$$

and then, using Mayer-Vietoris, Novikov additivity of signature and the relation $\bar{\eta} = -\eta$

$$\begin{aligned} dim\mathcal{M}_{c_{M^+}}(\tilde{P}_{\hat{M}^+}) + dim\mathcal{M}_{c_{M^-}}(\tilde{P}_{\hat{M}^-}) &= \tfrac{1}{4}(c_{M^+} + c_{M^-}) - \tfrac{1}{4}\sigma(M) + b^1(M) - b^2_+(M) - 2 = \\ &= \tfrac{1}{4}(c_M - 2\chi(M) - 3\sigma(M)) - 1 = dim\mathcal{M}_{c_M}(\tilde{P}_M) - 1. \end{aligned} \tag{3.45}$$

This formula holds for any couple of Chern integrals $c_{M^\pm}$ such that $c_M = c_{M^+} + c_{M^-}$. If we consider simple type manifold, as supposed, the only interesting terms for SW invariant come from zero dimensional moduli spaces, and therefore we have the formula

$$dim\mathcal{M}_{c_M}(\tilde{P}_M) = 0 = dim\mathcal{M}_{c_{M^+}}(\tilde{P}_{\hat{M}^+}) + dim\mathcal{M}_{c_{M^-}}(\tilde{P}_{\hat{M}^-}) + 1. \tag{3.46}$$

This requires one of the moduli space, say that corresponding to M^-, to have negative dimension. It follows from Proposition 3.3.1 that such a moduli space, in our contradictory hypothesis that $b^2_+(M^-) > 0$, is generically empty.

The "mismatch" of dimension, in some sense, comes from the fact that the base point action, on the character variety of N, is trivial, as points in $\hat{\chi}(N)$ are reducible solutions of the three dimensional Seiberg-Witten equations (the "+1" in formula 3.46 is often referred to as "gluing factor").

In order to prove the proposition, we must now show that the absence of solutions on one side is incompatible with the hypothesis that $SW(\tilde{P}_M) \neq 0$. But this is exactly the content of Proposition 3.2.4 of Section 3.2: this Proposition shows that if $SW(\tilde{P}_M)$ is non zero, then both sides must have a solutions. As on M^- there are no irreducible solutions, as the irreducible part of the moduli space is empty, nor reducible ones because of genericity assumptions, we obtain a contradiction. It follows that M^- must have $b^2_+(M^-) = 0$, as claimed in the proposition; in that case the solution identified by Proposition 3.2.4 on $\tilde{P}_{M^-}$ is a reducible one.

This result is of course equivalent to Theorem 2 in the case of $b_1(N) = 0$.

Now we consider the case of a manifold split along a three manifold with $b_1(N) > 0$. In this case we have the relations

$$\begin{aligned} b^1(M) &= b^1(M^+) + b^1(M^-) - b_1(N) + rkIm(H^1(N) \to H^2(M)), \\ b^2_+(M) &= b^2_+(M^+) + b^2_+(M^-) + rkIm(H^2(M) \to H^2(N)). \end{aligned} \tag{3.47}$$

Using these formulae and Proposition 3.4.1 we obtain the following relation for the dimension of the moduli spaces:

$$dim\mathcal{M}_{c_{M^+}}(\tilde{P}_{\hat{M}^+}) + dim\mathcal{M}_{c_{M^-}}(\tilde{P}_{\hat{M}^-}) = dim\mathcal{M}_{c_M}(\tilde{P}_M) + b_1(N) - 1. \tag{3.48}$$

In the hypothesis of simple type M, if we suppose that the moduli space of $\tilde{P}_M$ is non empty, we get the relation

$$dim\mathcal{M}_{c_{M^+}}(\tilde{P}_{\hat{M}^+}) + dim\mathcal{M}_{c_{M^-}}(\tilde{P}_{\hat{M}^-}) = b_1(N) - 1 \tag{3.49}$$

(again, the factor 1 in the formula is the gluing factor and has the same origin as discussed above). As we know that the limit maps $\partial^\pm_\infty$ are transversal, and $\chi(N)$ has dimension $b_1(N)$, we deduce that, for generic compactly supported perturbations $\eta^\pm_4$ on both $\hat{M}^\pm$, chosen as in Section 3.3, the images of the moduli spaces w.r.t. the maps $\partial^\pm_\infty$ (after the natural projection over $\chi(N)$, that by abuse of notation we call as well limit values) do not intersect. We can see this in full detail: the submersion property of the limit maps $\partial^\pm_\infty$ on both parameterized moduli spaces, and the consequent parameterized transversality, guarantees that we can assume, for generic regular values of $\pi^\pm$, that the product limit map

$$(\partial^+_\infty, \partial^-_\infty) : \mathcal{M}_{c_{M^+}}(\tilde{P}_{\hat{M}^+}, \eta^+_4) \times \mathcal{M}_{c_{M^-}}(\tilde{P}_{\hat{M}^-}, \eta^-_4) \longrightarrow \chi(N) \times \chi(N) \tag{3.50}$$

is transversal to the diagonal $\Delta \subset \chi(N) \times \chi(N)$, which represents common limit values of solutions on the two sides (in fact, more is true: an index calculation and the results of Section 3.3 show that the limit maps are immersions). The fiber product

$$\mathcal{M}_{c_{M^+}}(\tilde{P}_{\hat{M}^+}, \eta_4^+) \times_{\chi(N)} \mathcal{M}_{c_{M^-}}(\tilde{P}_{\hat{M}^-}, \eta_4^-) = (\partial_\infty^+, \partial_\infty^-)^{-1}\Delta \tag{3.51}$$

is therefore a well defined submanifold of the cartesian product of the moduli spaces, and has codimension

$$codim\mathcal{M}_{c_{M^+}}(\tilde{P}_{\hat{M}^+}, \eta_4^+) \times_{\chi(N)} \mathcal{M}_{c_{M^-}}(\tilde{P}_{\hat{M}^-}, \eta_4^-) = b_1(N). \tag{3.52}$$

In order to be non empty, i.e. to have an intersection point on the diagonal Δ, we must have

$$dim(\mathcal{M}_{c_{M^+}}(\tilde{P}_{\hat{M}^+}, \eta_4^+)) + dim(\mathcal{M}_{c_{M^-}}(\tilde{P}_{\hat{M}^-}, \eta_4^-)) - b_1(N) \geq 0, \tag{3.53}$$

condition which is excluded by equation 3.49.
The hypothesis that $SW(\tilde{P}_M) \neq 0$ implies that the moduli spaces associated to $\tilde{P}_{M_n}$, with perturbation $\eta_4^+ + \eta_4^-$ (which is sufficient to satisfy any genericity requirement) are non empty; the existence of a solution in such moduli spaces, for every n, entails the existence of two points in the moduli spaces of $\hat{M}^{\pm}$, by Proposition 3.2.4. Proposition 3.2.5, moreover, ensures that these two solutions have the same limit point. This limit point would represent a point in the intersection of the two limit sets, and as we have seen this contradicts, assuming irreducibility of both moduli spaces, the dimensional relations of equation 3.49.
We deduce from this, as above, that one of the two moduli spaces cannot be composed only of irreducible points, and therefore one of the factors must have $b_+^2(M^-) = 0$. This statement is equivalent to Theorem 2 for the case of $b_1(N) > 0$, and thus completes the proof of Theorem 2.

3.6 Proof of Theorem 3.

In this Section, using a perturbation of Seiberg-Witten equations, we will prove Theorem 3. This Theorem covers some cases where the previous vanishing theorem already holds, and might therefore provide an alternative proof of the same results, but, as we will discuss in Chapter 4, it applies also to cases excluded from Theorem 2, and in fact we will see that the mechanism of the proof is of completely different nature. In particular, the value of b_+^2 of the two sides plays no role.

We start by asking whether, by perturbing the equations on M, we can eliminate static solutions on the cylinder. We will see that in order of this being possible we must have $b_1(N) > 0$.
When this holds we have to analyse whether the absence of static solutions for these perturbed equations on the cylinder entails the absence of any solution to the equations, in the very same way we discussed, in the unperturbed case, along the proof of Claim 2.3.1. If this still hold true, we can deduce a vanishing theorem. But let's proceed in order.
First, we can consider, as in the proof of Claim 2.3.1, the manifold M_T obtained by replacing an open tubular neighborhood of N with the cylinder $N\times[-T,T]$; a perturbation of the four dimensional SW equations on M_T takes the form

$$F^+_{A_4} = q_4(\psi_4) + \eta_4. \tag{3.54}$$

Now we consider the form that the perturbation takes on a cylinder $N \times [-T,T]$:

$$\eta_4|_{N\times[-T,T]} = \eta - *_3\eta \wedge dt, \quad \eta \in \Omega^2(N, i\mathbb{R}). \tag{3.55}$$

For our purposes it will be useful to consider, as perturbation, self dual terms which induce a perturbation, on the cylinder, given by a closed two form on N. We are constructing now perturbations which have different features w.r.t. the compactly supported ones of Sections 3.3 and 3.5: let's call h_T a positive smooth function on $N \times [-T,T]$ s.t.

$$\begin{gathered} supp(h) \subset N \times [-T+1, T-1], \quad supp(dh) \subset N \times [-T, -T+1) \cup N \times (T-1, T], \\ h|_{N\times[-T+1,T-1]} = 1; \end{gathered} \tag{3.56}$$

we can consider, as perturbation classes for the equations,

$$\mathcal{N}_\delta := \{\eta_{4,T} \in \Omega^2_+(M_T, i\mathbb{R}) | \eta_{4,T} = ih_T(\delta - *\delta \wedge dt)\}, \quad \delta \in \Omega^2(N, \mathbb{R}) \cap ker d \tag{3.57}$$

where δ is taken suitably small in a C^0 norm on N. If we consider the effect of a generic perturbation on the two factors, there's no reason why the solutions induced on the cylindrical ends of the two elongated factors by the $\eta_{4,T}$-SW equations will preserve, as $T \mapsto \infty$, the same decay properties to static solutions we proved for the unperturbed case. This is of course true if η_4 is compactly supported, and the class of compactly supported

perturbation was sufficient to prove the smoothness properties of a generic moduli space, but does not hold in general.
In any case, on the cylinder, $\eta_{4,T}$-SW equations correspond to the gradient flow equations for a perturbed Chern-Simons-Dirac functional, which has the form, when $t \in [-T+1, T-1]$,

$$C(A,\psi,\delta) = \frac{1}{2}\int_N (F_A - i\delta) \wedge (A - A_0) + \int_N < \psi, \partial\!\!\!/_A \psi > . \tag{3.58}$$

These gradient flow equations have the form

$$\begin{cases} \frac{\partial}{\partial t}\psi(t) = \partial\!\!\!/_{A(t)}\psi(t), \\ \frac{\partial}{\partial t}A(t) = *(q(\psi(t)) - F_{A(t)} + i\delta). \end{cases} \tag{3.59}$$

The static solutions of equations 3.59 are the couples $(A,\psi) \in \mathcal{A}(L) \times \Gamma(S(\tilde{P}_N))$ which satisfy

$$\begin{cases} \partial\!\!\!/_A \psi = 0, \\ F_A = q(\psi) + i\delta. \end{cases} \tag{3.60}$$

It is quite clear from this equation the reason to choose $\delta \in kerd$, as both F_A and $q(\psi)$, when $\psi \in ker\partial\!\!\!/_A$, are closed, the curvature because of Bianchi identity, while $q(\psi)$ as consequence of equation A.13.
Due to the positive scalar curvature conditions on N, the proof of Proposition 2.2.2 applies to show that, once δ is taken with a suitably small norm, there are not solutions to equations 3.60 but reducible ones.

Let's consider the consequence of this: if we consider a rational homology sphere, as the second DeRahm cohomology group of N vanishes, $i\delta$ is exact and then cohomologous to F; by an easy Hodge decomposition argument, the equations 3.60 admit still reducible solutions.
In the case of a three manifold with $b_1(N) > 0$, instead, it is immediate that the choice of a closed perturbation δ such that $i[\delta] \neq [F_A]$ forbids reducible solutions to appear.
We analyse now this latter case. What has to be studied, now, is whether the absence of static solutions for equations 3.59 on the cylinder constraints the absence of any solution on the glued manifold. We need to study whether, with M defined as above and with a perturbation term in some class $\mathcal{N}_\delta$ the proof of Proposition 8 of [KM1] continues to hold true. We assume the reader has familiarity with that paper. We can prove the following

Proposition 3.6.1 *Let $M, M^{\pm}, N$ be defined as in 3.43 and $\tilde{P}_M$ be a spin^c structure with determinant bundle $\mathcal{L}_M$ which restricts to a torsion line bundle L on N, and suppose that $b_1(M) > b_1(M^+) + b_1(M^-) - b_1(N)$. Then if $SW(\tilde{P}_M)$ is different from zero there exist a perturbation δ with $[\delta] \in H^2(N, \mathbb{R})$ different from zero s.t. the SWF equations on the cylinder with perturbation of class $\mathcal{N}_\delta$ must admit a static solution.*

Proof: we show that we can adapt the proof of [KM1] to the perturbed case for at least one perturbation class (this is quite important!). As the Seiberg-Witten invariants of M are not zero we know that for any T the moduli space $\mathcal{M}(\tilde{P}_{M_T}, \eta_{4,T})$ is non empty; the core of the proof of [KM1] bases on the fact that invariance for gauge transformation of the CSD functional of a generic solution brings to the existence of a static solution; in our case, with the perturbation term, the variation of the perturbed CSD functional, after a gauge transformation, has the form

$$C(A^g, \psi^g) = C(A, \psi) + 4\pi^2 \delta \cup [g] \tag{3.61}$$

(recall that L is torsion); we know that if the evaluation of g is zero on the Poincaré dual of δ, the proof of [KM1] holds true: now we just need to find a single δ which satisfies this, and this corresponds to a non trivial kernel for the Mayer-Vietoris map mv

$$H_1(N, \mathbb{Q}) \overset{mv}{\to} H_1(M^+, \mathbb{Q}) \oplus H_1(M^-, \mathbb{Q}) \twoheadrightarrow H_1(M, \mathbb{Q}). \tag{3.62}$$

It is easy to see that this corresponds to the hypothesis on the Betti numbers claimed in the statement.
It is immediate to see that Proposition 3.6.1, together with Proposition 2.2.2, implies Theorem 3.

It is important to see the role played by the perturbation, in considering the CSD invariance; for the unperturbed functional the gauge invariance always holds true, and this gives a definite relation between the existence of solutions and the presence of static solutions, while here we can not deduce the same relation for any $\eta_{4,T}$-SW equation without the previous assumption on b_1's. This illustrates us that when we consider the problem of decomposing a four manifold along a three manifold (not necessarily of positive scalar curvature) the single datum of three dimensional Seiberg-Witten theory, when dealing with torsion spin^c structures, is not the only ingredient of the theory and we could be obliged, in general, to take care of reducible solutions if we look for the definition of some

kind of relative (à la Donaldson-Floer) invariants.
Note that the result proven is, in some sense, sharp: there are cases where the condition on the first Betti number is not satisfied and no vanishing theorem appears because of the lack of an adaptation of the [KM1] argument: the case where M is a symplectic manifold, $M^+ = M \setminus S^1 \times D^3$ and

$$M = M^+ \cup_{S^1 \times S^2} S^1 \times D^3, \tag{3.63}$$

with S^1 a nullhomologous circle in M, is a natural example, as

$$H_1(S^1 \times S^2, \mathbb{Q}) \overset{mv}{\rightarrowtail} H_1(M^+, \mathbb{Q}) \oplus H_1(S^1 \times D^3, \mathbb{Q}). \tag{3.64}$$

We note that the Proposition 3.6.1 follows essentially from the fact that, after an admissible perturbation, there are not monopole classes on N, and does not involve strictly curvature conditions on N; we can conclude therefore that a vanishing theorem for $SW(\tilde{P}_M)$ holds in each case where the determinant line bundle $\mathcal{L}_M$ restricts to a bundle L which is not a monopole class for three dimensional SW equations (as discussed in the proof of Claim 2.3.1) or is not a monopole class for three dimensional SW equations perturbed with an admissible perturbation as above.
We remark that, for $b_1(N) > 0$, of course, there is plenty of manifold for which both the vanishing results stated in Theorem 2 and 3 apply; it is possible anyway exhibit examples of manifolds for which only one of the statements applies.

3.7 A Gluing Formula.

In this Section, that will be rather sketchy, we will provide a gluing formula, which comes out on the nose, for cylindrical end moduli spaces associated to the usual decomposition 3.43, in some cases allowed by the previous vanishing theorems (i.e. cases for which the previous vanishing theorems do not apply), in particular with the factor M^- having non-positive definite intersection form. We will not provide details of the proof, which follows in virtue of the regularity and compactness properties shown in the previous Sections and the need, for a solution on $\tilde{P}_M$, to decompose in two solutions, one of which reducible, having compatible limit values, as shown in Propositions 3.2.4 and 3.2.5.
We will restrict ourselves to the case, that will interest us in Chapter 4, where $H_2(M^-, \mathbb{Z}) = 0$, $H_1(N, \mathbb{Z})$ is free on k generators and the inclusion $N = \partial M^- \hookrightarrow M^-$ induces an isomorphism on the inclusion $H_1(N, \mathbb{Z}) \to H_1(M^-, \mathbb{Z})$. We assume, moreover, that the reducible

solutions on M^- define a smooth (based) moduli space, cut transversely by SW equations. This requires the vanishing of the kernel of the Dirac operator for any reducible solution. With these hypothesis the result becomes almost tautological.

Given M, $M^{\pm}$, N as in decomposition 3.43 we want to discuss how the moduli space of Seiberg-Witten equations for a spin^c structure $\tilde{P}_M$ relates to the cylindrical end moduli spaces on $\hat{M}^{\pm}$; as our discussion involves reducible solutions, we need to work with moduli spaces based on a point of N, in order to have all moduli spaces embedded in the respective orbit spaces as Banach submanifolds. First, at level of bundles, as $H^2(M^-, \mathbb{Z}) = 0$, the Mayer-Vietoris sequence in cohomology tells that

$$H^1(M^+, \mathbb{Z}) \oplus H^1(M^-, \mathbb{Z}) \overset{\rho}{\twoheadrightarrow} H^1(N, \mathbb{Z}) \to H^2(M, \mathbb{Z}) \rightarrowtail H^2(M^+, \mathbb{Z}) \to H^2(N, \mathbb{Z}), \tag{3.65}$$

where the map ρ is a surjection because of our hypothesis on the first homology groups of N and M^-. From the analysis of the previous sequence we see that each spin^c structure $\tilde{P}_M$ on M defines uniquely a spin^c structure $\tilde{P}_{M^+}$ on M^+, and therefore one on $\hat{M}^+$, and a spin^c structure on M^+ uniquely extends to M under the hypothesis that its restriction to N is trivial (a trivial spin^c structure is the only compatible with Claim 2.3.1, and the only interesting for us). With the previous hypothesis on M^-, we have two limit maps

$$\partial_\infty^+ : \mathcal{M}^o_{c_{M^+}}(\tilde{P}_{\hat{M}^+}) \to \hat{\chi}(N), \qquad \partial_\infty^- : \chi(M^-) \overset{\cong}{\to} \chi(N) \tag{3.66}$$

which enter in play: note that we used the notation $\hat{\chi}(N)$ to distinguish the two (eventually) different character varieties, as these are defined via different gauge groups; $\hat{\chi}(N)$ covers $\chi(N)$. The formula which relates the moduli spaces is

$$\mathcal{M}^o(\tilde{P}_M) = \mathcal{M}^o_c(\tilde{P}_{\hat{M}^+}) \times_{\chi(N)} \chi(M^-) = \mathcal{M}^o_c(\tilde{P}_{\hat{M}^+}); \tag{3.67}$$

in fact each solution, on $\hat{M}^+$, decays to a flat limit value on N and defines a solution on M by gluing with the flat extension of its limit value at N. The value of c is defined by the requirement that $c = c_1^2(\mathcal{L}_M) \cap [M]$, as the flat solution on $\hat{M}^-$ does not contribute to the Chern integral.

A note on the gauge groups involved: the gauge group acting on a glued bundle $\tilde{P}_M$ is given, in general (see, e.g. [Br]), by the fiber product

$$\mathcal{G}^o_M = \mathcal{G}^o_{M^+} \times_{\mathcal{G}^o_N, \phi} \mathcal{G}^o_{M^-} = \{(g^+, g^-) \in \mathcal{G}^o_{M^+} \times \mathcal{G}^o_{M^-} | r(g^+) = \phi r(g^-)\} \tag{3.68}$$

where ϕ is the automorphism of $\tilde{P}_N$ which defines $\tilde{P}_M$ and r is the restriction to the common boundary N. The set of isomorphism classes of bundles glued from $\tilde{P}_{M+}$ and $\tilde{P}_{M-}$ is given by the set of components of the double coset

$$r(\mathcal{G}^o_{M+})\backslash\mathcal{G}^o_N/r(\mathcal{G}^o_{M-}) \tag{3.69}$$

(remember that we are always consider automorphisms which act trivially on the frame bundle, as the way the frame bundles glue is given). As, in our case, any ϕ extends to an automorphism of the whole $\tilde{P}_{M-}$, the gauge group is simply given by

$$\mathcal{G}^o_M = \mathcal{G}^o_{M+} \times_{\mathcal{G}^o_N} \mathcal{G}^o_{M-} = \{(g^+, g^-) \in \mathcal{G}^o_{M+} \times \mathcal{G}^o_{M-} | r(g^+) = r(g^-)\} \tag{3.70}$$

and the set of isomorphism classes of glued bundles is, as it should be clear after the analysis of sequence 3.65, just a point. The same results, for the gauge groups, hold when we consider the action of the full groups, forgetting the base point.

If we want to consider the unbased moduli space, the $U(1)$ base point fixing action is trivial over reducible solutions on $\hat{M}^-$ (this is what makes possible, at level of dimension, the gluing with reducibles) and we have the isomorphism

$$\mathcal{M}(\tilde{P}_M) = \mathcal{M}_c(\tilde{P}_{\hat{M}+}). \tag{3.71}$$

We will make use of this formula in the next Chapter.

Chapter 4

Topological applications

In this chapter we will discuss some consequences and applications of the results proven in Chapter 3. First we analyse the relations between the Seiberg-Witten invariants of two four manifolds which are diffeomorphic outside a point or a wedge of circles, and prove Proposition 1. We will then use Theorems 2 and 3 to prove that a closed four manifold containing an embedded two sphere of infinite order and nonnegative selfintersection has vanishing Seiberg-Witten invariants (Proposition 2). We will then prove that Theorem 3 can be reduced to the case of $N = S^1 \times S^2$. A discussion of this result and of three dimensional Seiberg-Witten theory will bring us to a partial generalization of Theorem 3 to $\mathbb{Q}$-reducible manifolds, that we state as Theorem 4.

4.1 Manifolds Diffeomorphic Outside a Set.

In this Section we will study the relations occurring between the Seiberg-Witten invariants of two four manifolds M_1, M_2 which are diffeomorphic *a)* on the complement of a point or *b)* on the complement of a wedge of k circles. The case *a)* has been discussed, in the framework of Donaldson theory, in [De], where it is proven that the Donaldson polynomials of two such manifolds coincide.

It is natural to test the conjectured relation of Donaldson and Seiberg-Witten theory by proving the same result for the Seiberg-Witten invariants.

As noted in [De], in the case of Yang-Mills theory two approaches are viable: the first is a direct study of the relations of the two Yang-Mills moduli spaces, the second is to observe that two such manifold admit a quasiconformal homeomorphism and have therefore coinciding Donaldson polynomials, as comes from the work of [DS].

We will follow here the first approach, which is really elementary for the Seiberg-Witten case.

If M_1, M_2 are two manifolds diffeomorphic on the complement of a point, denote the diffeomorphism as

$$\varphi : M_1 \setminus p_1 \longrightarrow M_2 \setminus p_2; \tag{4.1}$$

we take neighborhoods U_i of p_i in such a way that the image via φ of a standard three-sphere $S^3 = \partial D^4 \subset D^4 \subset U_1$, where D^4 contains p_1, is contained in U_2; now $\varphi(S^3)$ separates M_2 in two components, one of which, that we denote by D^4_Γ, is homeomorphic (but not necessarily diffeomorphic, in absence of the generalized Schönfliess conjecture) to the four dimensional disc; calling $M^+ = M_1 \setminus D^4$, we can suppose that the two manifolds decompose as

$$M_1 = M^+ \cup_{S^3} D^4, \quad M_2 = \varphi(M^+) \cup_{\varphi(S^3)} D^4_\Gamma \tag{4.2}$$

and $\varphi(M^+) \cong M^+$; although we have no information on the smooth structure of D^4_Γ, the knowledge of its homotopy type is sufficient for our purposes. From the proof of Theorem 2 and the gluing theory, for exponentially decaying solutions, that we outlined in Section 3.7, we can observe that the moduli space $\mathcal{M}(\tilde{P}_M)$ corresponding to a manifold $M^+ = Z \cup_{S^3} M^-$ where M^- is an homotopy four disc coincides with $\mathcal{M}_{c_{M^+}}(\tilde{P}_{\hat{M}^+})$, where $\tilde{P}_{\hat{M}^+}$ is the spin^c structure induced on $\hat{M}^+ = M^+ \cup_N N \times [0, \infty)$ by $\tilde{P}_M$ (which is moreover its only extension to M) and c_{M^+} is determined by the requirement that it coincides with $c_1^2(\mathcal{L}_M) \cap [M]$, as points in this moduli space give rise to a solution on M by the extension with the only reducible solution on M^-; this follows from the fact that we can assume that, generically, this reducible solution is a smooth point, as stated in the next

Lemma 4.1.1 *For a second category subset of perturbations $\eta = d^+\nu$, $\nu \in \Omega^1_\delta(\hat{M}^-, i\mathbb{R})$, the reducible solutions of $d^+\nu$-SW equations have vanishing kernel.*

Proof: note first that, up to gauge equivalence, the reducible solution of $d^+\nu$-SW equations is given by $(A_0 + \nu, 0)$ where A_0 is the trivial connection. We have to check that, for a generic choice of ν, $ker \partial\!\!\!/_{A_0+\nu}$ vanishes. In order to do so we consider Dirac equation as defining a section of the $\Gamma_\delta(S^-(\tilde{P}_{\hat{M}^-}))$ bundle over $\{A_0\} \times \Gamma^*_\delta(S^+(\tilde{P}_{\hat{M}^-})) \times \Omega^1_\delta(\hat{M}^-, i\mathbb{R})$ (we can assume we are working in a fixed based gauge). It is easy to verify, in the way we did several times, that the section $s(\psi, \nu) = \partial\!\!\!/_{A_0+\nu}\psi$ is transverse to the zero set and defines therefore a parameterized space of harmonic spinor which maps to the parameter space with a Fredholm map whose index coincides with the index of the Dirac operator.

Application of Atiyah-Singer index theorem shows that this index is zero, as the signature of the manifolds under analysis is zero and the η-invariant of the boundary vanishes. This implies that for generic ν the kernel of $\not{\partial}_{A_0+\nu}$ has dimension zero. Now, as $\not{\partial}_{A_0+\nu}$ is $\mathbb{C}$-linear, this kernel must have complex dimension greater or equal than one or vanishes. This proves the Lemma.

Note that the previous Lemma makes no hypothesis on the metric of M^- and works both for D^4 and an homotopy D^4; in fact we can not assume, in our case, that D^4_Γ has metric of positive scalar curvature, assumption which will automatically imply the statement. Moreover we have not to care of the existence on $\hat{D}^4_\Gamma$ of irreducible solution (anyhow decaying exponentially to a flat one), as any extension of a solution on $\hat{M}^+$ by a non reducible solution would imply in fact a contradiction in the dimension formula. It follows that two manifolds decomposed along S^3 having diffeomorphic positive definite factors and homotopic second factors have diffeomorphic moduli spaces and same value of Seiberg-Witten invariants.

An equivalent proof of this result comes by observing, as in [De] that we can consider $M_2 = M_1 \# \Sigma$, for Σ an homotopy sphere, and then applying the previous idea to the related "blow up" formula for Seiberg-Witten invariants.

Now we prove the analogous result for *b)*, the case of two manifolds diffeomorphic outside a wedge of k circles $\bigvee_k S^1$: we have

$$\varphi : M_1 \setminus \bigvee_k S^1 \longrightarrow M_2 \setminus \bigvee_k S^1. \tag{4.3}$$

In this case we consider, in the neighborhood $U_1 \subset M_1$ of the wedge, a standard $\#_k S^1 \times S^2$ which bounds the boundary connected sum $\#_{\partial,k} S^1 \times D^3$; this time the two manifold are split according to

$$M_1 = M^+ \cup_{\#_k S^1 \times S^2} (\#_{\partial,k} S^1 \times D^3), \quad M_2 = \varphi(M^+) \cup_{\varphi(\#_k S^1 \times S^2)} [\#_{\partial,k} S^1 \times D^3]_\Gamma \tag{4.4}$$

with again $\varphi(M^+) \cong M^+$; note that in that case, for $k \geq 2$, we just know the homotopy type of $[\#_{\partial,k} S^1 \times D^3]_\Gamma$, as its fundamental group is free on k generators and therefore we cannot conclude that it has the same homeomorphism type of the standard $\#_{\partial,k} S^1 \times D^3$. Once again we can prove the result by observing that the moduli space $\mathcal{M}_{c_M}(\tilde{P}_M)$ corresponding to a manifold $M = M^+ \cup_{\#_k S^1 \times S^2} M^-$ having $H_1(M^-, \mathbb{Z}) = H_1(\#_{\partial,k} S^1 \times D^3, \mathbb{Z})$ and $H_2(M^-, \mathbb{Z}) = 0$ coincides with $\mathcal{M}_{c_{M^+}}(\tilde{P}_{\hat{M}^+})$. The idea of the proof is the same as above, but this time we need a little extra care to keep control of the regularity of the reducible connections on M^-. We start with a result which generalizes Lemma 4.1.1; we observe first of all that for a perturbation $\eta = d^+\nu$ the set of reducible solutions of $d^+\nu$-SW equations is an "affine" $\chi(M^-)$, i.e. solutions have the form $(A+\nu, 0)$ for $A \in \chi(M^-)$; we will denote such a set as $\chi_\nu(M^-)$ (which coincides, using the previous notation, with the reducible solutions of $\mathcal{M}_0(M^-, U(2) \times M^-, d^+\nu)$).

Lemma 4.1.2 *For a second category subset of perturbations $\eta = d^+\nu$, $\nu \in \Omega^1_\delta(\hat{M}^-, i\mathbb{R})$ the set of reducible solutions $s_\nu^{-1}(0)$ of $d^+\nu$-SW equations, is a smooth submanifold of $\chi(M^-) \times \Gamma^*_\delta(S^+(\tilde{P}_{\hat{M}^-}))$ of dimension $b_1(M^-)$; the natural projection map*

$$\pi : s_\nu^{-1}(0) \longrightarrow \chi(M^-) \tag{4.5}$$

can be assumed to be transverse to any finite set of submanifolds and its fiber has real dimension at least 2.

Proof: as we did above in the case of a single flat connection, we have to study the zero set of the section $s(A, \psi, \nu) = \not{\partial}_{A+\nu}\psi$ of the $\Gamma_\delta(S^-(\tilde{P}_{\hat{M}^-}))$ bundle over $\chi(M^-) \times \Gamma^*_\delta(S^+(\tilde{P}_{\hat{M}^-})) \times \Omega^1_\delta(\hat{M}^-, i\mathbb{R})$. On any fiber $\pi^{-1}(A_0)$, zero is a regular value, so that $s^{-1}(0)$ is a smooth submanifold, $\pi : s^{-1}(0) \to \chi(M^-)$ (naturally identified with the limit map) is a submersion and has a Fredholm map to the parameter space of real index $index\not{\partial}_A + b_1(M^-) = b_1(M^-)$; for a generic choice of ν it is therefore a $b_1(M^-)$ dimensional smooth manifold. Now, as observed above, the $\mathbb{C}$-linearity of Dirac equation ensures that in correspondence of any connection in $\chi(M^-)$ such that $(A+\nu)$ has nonvanishing kernel this kernel has real dimension at least two. This proves the statement on the dimension of the fiber.

It follows from the previous lemma that the image under π of $s_\nu^{-1}(0)$ (which is the natural analog, in this context, of the *theta* divisor of a Jacobian torus) has zero measure in $\chi(M^-)$ and in particular it misses generically a finite set of points in $\chi(M^-)$. This means that a generic flat connection on the boundary extends, on M^-, to a regular reducible solution. We have one more step to do to prove the equality of moduli spaces on M_1 and M_2, namely to prove that $\partial_\infty(\mathcal{M}_{c_{M^+}}(\tilde{P}_{\hat{M}^+}))$ has dimension at most zero (and so it is at most a finite set of points): but this follows just from equation 3.39, as a quick check shows that

$$dim(\mathcal{M}_{c_{M^+}}(\tilde{P}_{\hat{M}^+})) = dim(\mathcal{M}_{c_{M_i}}(\tilde{P}_{\hat{M}_i})) = 0, \quad i = 1, 2. \tag{4.6}$$

This result tells that we have not to care about the eventual presence of non regular reducible connections on $(\#_{\partial,k}S^1 \times D^3)_\Gamma$, as $\partial_\infty(\mathcal{M}_{c_{M^+}}(\tilde{P}_{\hat{M}^+}))$ avoids generically the flat connections, in $\chi(\#_k S^1 \times S^2)$, which extend to non regular reducible connections. The gluing therefore is the same both for the standard $\#_{\partial,k}S^1 \times D^3$ (which has just regular reducibles because of curvature condition) and the fake ones, for which, as in the previous case, we have no information on the curvature. Summing up, we have the following

Proposition 4.1.3 *Let M_1 and M_2 be two closed four manifolds diffeomorphic in the complement of a point or in the complement of a wedge of circles: then $SW_{M_1} = SW_{M_2}$.*

4.2 Spheres in Three and Four Manifolds.

Another consequence of the two vanishing theorems of Chapter 3 is a new proof of proposition 10.1 of [MST]:

Proposition 4.2.1 *Let M be a symplectic four manifold (in fact, in any manifold with non trivial SW_M) with $b_2^+(M) > 1$; then M can not contain a sphere S of infinite order in $H_2(M, \mathbb{Z})$ and self intersection $S \cdot S \geq 0$.*

Proof: our proof of the proposition, in the case $S \cdot S = k > 0$, is based on the fact that if such a sphere exist, it has a tubular neighborhood ν_S which is a disk bundle over S and whose boundary is a lens space $L(-k, 1)$; as such a neighborhood has $b_+(\nu_S) = 1$ and $b_+(M \setminus \nu_S) \geq 1$ the vanishing of the invariants follows from Theorem 2. In the case that $S \cdot S = 0$ (and S of infinite order), the tubular neighborhood is $\nu_S = D^2 \times S$ and it is bounded by $S^1 \times S$ in such a way that the circle S^1 is the generator of the kernel of the Mayer-Vietoris map mv of the sequence 3.62, (i.e. it must go to zero over the rationals both in $D^2 \times S$ and in its complement), as

$$H_2(M, \mathbb{Q}) \twoheadrightarrow H_1(S^1 \times S, \mathbb{Q}) \overset{mv}{\to} H_1(M^+, \mathbb{Q}) \oplus H_1(D^2 \times S, \mathbb{Q}). \tag{4.7}$$

The vanishing of the invariants follows then from Theorem 3. As noted in [MST], the case of $k > 0$ can be reduced to $k = 0$ by blowing up. We remark that the proof of [MST] of this latter statement, which follows from a reasonable gluing formula for the moduli spaces, makes as well use, although in quite a different way, of the fact that the circle represented by S^1 is torsion in M^+.

This last vanishing result, for the case of an embedded two-sphere S of infinite order and

zero self intersection, appears as a particular case of Theorem 3 and in fact the proof of that theorem can be specifically adapted to this case.
We claim now that this vanishing result is in fact equivalent to Theorem 3, i.e. that we can reduce the situation of a three manifold N of positive scalar curvature decomposing M with the usual matching of first Betti numbers to the case of an embedded sphere of zero self intersection. But first we need to point out a classical result on the representability by spherical classes of the homology of a manifold of positive scalar curvature, result which follows immediately from Theorem 6.1 of [SY];

Proposition 4.2.2 *Let N a three manifold of positive scalar curvature and let $\alpha \in H_2(N,\mathbb{Z})$ be a nontrivial class: then there exist a finite set of embedded two spheres $i_k : S_k \to N$ such that we can represent α as*

$$\alpha = \sum_k m_k (i_k)_*[S_k], \quad m_k \in \mathbb{Z}. \tag{4.8}$$

We can now state a proposition which shows the equivalence of the vanishing results:

Proposition 4.2.3 *Let M be a four manifold with $b_2^+(M) > 1$ which decomposes as $M = M^+ \cup_N M^-$ along a three manifold of positive scalar curvature and $b_1(N) > 0$ in such a way that $b_1(M) > b_1(M^+) + b_1(M^-) - b_1(N)$; then there exist an embedded two sphere $S \subset N$, of infinite order in M and zero self intersection.*

Proof: in the case under analysis $Im(H_2(M,\mathbb{Z}) \to H_1(N,\mathbb{Z}))$ has rank at least one. This group is naturally isomorphic to $Im(H_2(N,\mathbb{Z}) \to H_2(M,\mathbb{Z}))$, where the arrow is induced by the inclusion map. By Proposition 4.2.2 each two cycle of N is representable by a sum of embedded spheres; we can therefore identify at least one class of infinite order, in $H_2(M,\mathbb{Z})$, which is represented by an embedded sphere; this class has necessarily zero self intersection, being in the image of $H_2(N,\mathbb{Z})$, and is exactly the sphere required in the statement.

From this proposition we see that it is in fact enough to prove Theorem 3 in the particular case of $N = S^1 \times S^2$.

We can now ask, in the light of the previous results, if we can extend Theorem 3 to other classes of three manifold which have not positive scalar curvature.
As a first step we can immediately observe that a three manifold which does not admit positive scalar curvature, but whose second homology group is represented by spherical

classes, satisfies, via Proposition 4.2.3, a vanishing theorem under the same hypothesis of Theorem 3. For the same reasons there exists a slightly more involved version of Theorem 3 for a three manifold N in which only some of the generators of $H_2(N,\mathbb{Q})$ are representable by spheres: a careful check of the proof of Proposition 4.2.3 shows that we must require the natural map $i_* : H_2(N,\mathbb{Q}) \to H_2(M,\mathbb{Q})$ to be nontrivial at least on one of these spherical classes.

In order to have a better understanding of this kind of results, we recall (see [Mi]) that any three manifold N admits a decomposition of the form

$$N = (\#_{i=1}^{p} Y_i)\#(\#_{j=1}^{q} S^1 \times S^2)\#(\#_{k=1}^{r} K(\pi_k,1)), \tag{4.9}$$

where the factors Y_i are rational homology spheres and $b_1(K(\pi_k,1)) > 0$. It is known, by the results of [GL], that manifolds of type

$$(\#_{i=1}^{m}(S^3/\Gamma_i))\#(\#_{j=1}^{q} S^1 \times S^2) \tag{4.10}$$

admit a metric of positive scalar curvature and we can conjecture that this is a sharp class of manifolds having such property, the only possible exceptions corresponding to manifolds admitting, as factor, finite group quotients of fake three spheres. Moreover, the generators of the second homology group of three dimensional Eilenberg-Maclane spaces cannot be represented by spheres.

In these terms, the class of manifolds for which Theorem 3 holds true without any modification is given by manifolds of type

$$N = (\#_{i=1}^{p} Y_i)\#(\#_{j=1}^{q} S^1 \times S^2) \tag{4.11}$$

where Y_i are rational homology spheres of curvature whatsoever; apart from the case where all their rational homology spheres have positive scalar curvature, these manifolds do not admit positive scalar curvature.

For manifolds of type

$$N = (\#_{i=1}^{p} Y_i)\#(\#_{j=1}^{q} S^1 \times S^2)\#(\#_{k=1}^{s} K(\pi_k,1)) \tag{4.12}$$

with $b_1(K(\pi_k,1)) > 0$, instead, a vanishing result will hold true under the aforementioned condition of nontriviality of i_* on $H_2(\#_{j=1}^{q} S^1 \times S^2,\mathbb{Q}) \subset H_2(N,\mathbb{Q})$.

4.3 Monopole Classes on $\mathbb{Q}$-Reducible Three Manifolds.

In the previous Section we have seen that the existence, in M, of a three manifold N, whose second homology group has generators which can be represented by embedded two spheres, induces, under hypothesis that can be reconduced to Proposition 4.2.1, a vanishing theorem for Seiberg-Witten invariants on a four manifold M which splits along N. All the manifold of this type are, by definition, reducible three manifolds, i.e. contain a two sphere which does not bound a three disk. For sake of notation, we allow ourselves to introduce the following definition, which extends the concept of irreducibility:

Definition 4.3.1 *A three manifold N is called to be $\mathbb{Q}$-irreducible if any sphere bounds a rational homology disk.*

With this definition, connected sum with rational homology spheres does not change $\mathbb{Q}$-irreducibility of a manifold.
The aim of this section is to discuss, in terms of monopole classes, the existence of a vanishing theorem, in the spirit of Theorem 3, for $\mathbb{Q}$-reducible manifolds.

In the case where N has positive scalar curvature the original proof of Theorem 3, contained in Section 3.6, follows from the fact that there can not be nontorsion monopole classes, for unperturbed SW equations, and that torsion monopole classes, which correspond to reducible solutions, can be removed, in the appropriate cases, after perturbation of the equations with a nonexact closed two form. In the case where N has not positive scalar curvature, but some of its two cycles are represented by spheres, we are naturally lead to suppose that a similar procedure holds, i.e. that information on monopole classes of N, eventually after perturbation of the equations, should give us the input to prove (some version of) Theorem 3 along the same lines of Section 3.6. We will now discuss this point, in all the cases analysed above, and we will see that these arguments allow us to extend a little further Theorem 3, to the whole class of $\mathbb{Q}$-reducible three manifolds. But let's proceed in order: we have the following

Claim 4.3.2 *Let N be a manifold which admits a decomposition of the form*

$$N = (\#_{i=1}^{p} Y_i)\#(\#_{j=1}^{q} S^1 \times S^2)\#(\#_{k=1}^{r} K(\pi_k, 1)) \tag{4.13}$$

with $b_1(K(\pi_k, 1)) > 0$ and $q + r > 1$ or $q = 1$; then, for all spin^c structures, the three dimensional SW equations, eventually after generic perturbation with some closed nonexact two form, do not admit any solution.

Proof: in order to prove this Claim we will make the assumption that three dimensional Seiberg-Witten theory has the same behavior, concerning decomposition of a three manifold along a sphere, that we have discussed in the previous Chapters in dimension four. It is straightforward to see, in this context, that the role played by b_2^+ in dimension four, concerning genericity of moduli spaces, is played, in dimension three, by b_1.
First we consider the case where all two cycles are representable by spheres: in that case $q = b_1(N) > 0$ and a solution must restrict, because of curvature condition, to a reducible one on each factor $S^1 \times S^2$. This solution can be perturbed away after addition, on any of the factors, of a closed nonexact two form.
We can clearly treat similarly the case when the spherical classes do not span the whole $H_2(N, \mathbb{Q})$; in that case we are ensured of the vanishing of any solution after perturbation on any of the $S^1 \times S^2$ factors. But in that case, and also in the case where q is zero, there's another possible way to remove the solutions; as in dimension four, independently on the curvature of the two factors of a connected sum, a solution of SW equations on the three manifold must restrict, on one of the two factors, to a reducible one. In order to perturb away a solution on a multiple connected sum, in general, we can, as before, add to three dimensional SW equations a closed nonexact two form for any one of the $S^1 \times S^2$, as discussed above, or for any two of the other factors (a perturbation on just one factor would not be enough, as it would not remove solutions which are irreducible on that factor and reducible on the others)

From the proof of the previous Claim we can not only extract information on the monopole classes of $\mathbb{Q}$-reducible manifolds, but we know also how to obtain a result in the spirit of Proposition 3.6.1 for deducing a vanishing theorem which generalizes Theorem 3; removing away the solutions in a way which does not affect the proof of [KM1] on the existence of static solutions on a cylindrical neck of a four manifold is possible under the condition that the map $H_2(M, \mathbb{Q}) \to H_1(N, \mathbb{Q})$ has the correct rank behavior: precisely, for the manifold N of equation 4.13 we need to require the map

$$H_2(M, \mathbb{Q}) \to (\oplus_1^q H_1(S^1 \times S^2, \mathbb{Q})) \oplus (\oplus_1^r H_1(K(\pi_k, 1), \mathbb{Q})) \tag{4.14}$$

to have rank at least one on the first q factors or to be nontrivial on at least two factors of the direct sum decomposition. These results are coherent, in the case where spherical classes are present, with the discussion on embedded spheres in four manifolds contained in the previous Section.

Summing up, we see that a generalization of Theorem 3 holds true for any $\mathbb{Q}$-reducible three manifold:

Theorem 4.3.3 *Let M be a closed four manifold with $b_2^+(M) > 1$ which decomposes as $M = M^+ \cup_N M^-$ along a $\mathbb{Q}$-reducible three manifold in such a way that the map*

$$H_2(M, \mathbb{Q}) \to H_1(N, \mathbb{Q}) = (\oplus_1^q H_1(S^1 \times S^2, \mathbb{Q})) \oplus (\oplus_1^r H_1(K(\pi_k, 1), \mathbb{Q})) \tag{4.15}$$

has rank at least one on the first q factors or is nontrivial on at least two factors; then the Seiberg-Witten invariants $SW_M : Spin^c(M) \to \mathbb{Z}$ are identically zero.

A weaker form of this theorem, eventually more practical, requires to have $b_1(M) = b_1(M^+) + b_1(M^-)$; this is a sufficient condition but, in many case, not a necessary one.

We point out that, excluding the case where the $\mathbb{Q}$-reducible manifold has positive scalar curvature, there is no any obvious reason why a generalization, this time, of Theorem 2 should hold true; in fact the proof of Theorem 2 depends on the presence of the "gluing" factor, i.e. from the fact that, without any perturbation, the only static solutions are reducible ones. In the case under analysis, even in the case where all monopole classes are torsion (e.g. the connected sum of $S^1 \times S^2$ with an Y_i having nonpositive scalar curvature) there is no reason why a solutions should restrict to a reducible one on all factors.

We finish by making some almost obvious comparative considerations on the case of three manifolds N with $b_1(N) > 0$ of zero scalar curvature and manifolds whose second homology group is representable with tori. We will focus on the possible extension of the vanishing theorems to cover these cases. In the case of zero scalar curvature, we know from the proof of Proposition 2.2.2 that the only monopole classes, for the unperturbed equations, are torsion ones. As argued before with the embedded spheres, it is possible to extend this result also for the case of a manifold whose homology classes are represented by embedded tori. It follows from this, and from Claim 2.3.1, whose statement applies word by word to this case, that a spin^c structure $\tilde{P}_M$, on a four manifold M decomposing along an N with the properties above, and Seiberg-Witten invariants different from zero must define a torsion line bundle L on N. For what concerns the torsion classes, instead, nor in the case of zero scalar curvature, nor in the other cases, we can obtain any result after perturbation of three dimensional Seiberg-Witten equations, as nor Weitzenböck formula, nor decomposition argument, allow to remove torsion classes from the set of monopole classes.

Appendix A

Appendix

A.1

The purpose of this appendix is to study the decay conditions for the square norm of the gradient of the Chern-Simons-Dirac functional, i.e.

$$f(t) := ||\partial\!\!\!/_{A(t)}\psi(t)||^2 + ||q(\psi(t)) - F_{A(t)}||^2 \tag{A.1}$$

for a solution $(A(t), \psi(t))$ sufficiently near to a static one in the based orbit space.

To give the complete proofs, which are quite long, of the results we will discuss we need some preliminary work to simplify the direct computation.
We will denote with $<\cdot|\cdot>$ the $\mathcal{L}^2$ product on both spinors and forms, and with $<\cdot,\cdot>$ the pointwise product given by

$$< a, b >= -a \wedge *b, \quad < \psi, \phi >= (\psi, \phi)d^3N \tag{A.2}$$

on forms and spinors respectively, where $(\cdot,\cdot)$ is the ordinary hermitean product, $\mathbb{C}$-linear on the second variable.

To fix the notation, the bilinear term appearing in three dimensional SW equation has the same form we have seen in the case of the four dimensional theory (equations 1.6), i.e.

$$q(\psi, \phi) = \frac{1}{2}(\psi \otimes \phi^* + \phi \otimes \psi^* - Re < \psi, \phi > Id). \tag{A.3}$$

From this explicit formula it is easy verify that, for a solution of SWF, the equality

$$\frac{\partial}{\partial t}q(\psi) = 2q(\psi, \partial\!\!\!/_A\psi) \tag{A.4}$$

holds. We recall, moreover, the formula that relates Clifford product and hermitean product, which says, for an imaginary one-form $\eta \in \Omega^1(N, i\mathbb{R})$ that

$$< *\eta, q(\psi) >= \frac{1}{2} < \eta \cdot \psi, \psi >, \tag{A.5}$$

while for a $\rho \in \Omega^2(N, i\mathbb{R})$, similarly,

$$< \rho, q(\psi) >= \frac{1}{2} < \rho \cdot \psi, \psi > . \tag{A.6}$$

A more delicate equality we will need is the following:

$$\frac{\partial}{\partial t} q(\psi) - d * q(\psi) = i * Im < \psi, \nabla_A \psi > . \tag{A.7}$$

We prove this from the expression of q in terms of differential forms. As element of $\Omega^2(N, i\mathbb{R})$, $q(\psi)$ has the form

$$q(\psi) = \frac{1}{4} \sum_{ijk} \epsilon_{ijk} e^i \wedge e^j < e^k \cdot \psi, \psi > \tag{A.8}$$

(the change of dimension brings to a different normalization term w.r.t. equation 1.8). It follows that

$$\begin{aligned} d * q(\psi) &= \tfrac{1}{2} \textstyle\sum_k (d < e^k \cdot \psi, \psi >) \wedge e^k = \\ &= (\tfrac{1}{2} \tfrac{\partial}{\partial e_1} < e^2 \cdot \psi, \psi > - \tfrac{1}{2} \tfrac{\partial}{\partial e_2} < e^1 \cdot \psi, \psi >) e^1 \wedge e^2 + \\ &+ (\tfrac{1}{2} \tfrac{\partial}{\partial e_2} < e^3 \cdot \psi, \psi > - \tfrac{1}{2} \tfrac{\partial}{\partial e_3} < e^2 \cdot \psi, \psi >) e^2 \wedge e^3 + \\ &+ (\tfrac{1}{2} \tfrac{\partial}{\partial e_3} < e^1 \cdot \psi, \psi > - \tfrac{1}{2} \tfrac{\partial}{\partial e_1} < e^3 \cdot \psi, \psi >) e^3 \wedge e^1 . \end{aligned} \tag{A.9}$$

On the same vein, we can give an expression for the time derivative of $q(\psi)$, as

$$\begin{aligned} \tfrac{\partial}{\partial t} q(\psi) &= \tfrac{1}{4} \textstyle\sum_{ijk} \epsilon_{ijk} e^i \wedge e^j [< e^k \cdot \partial\!\!\!/_A \psi, \psi > + < e^k \cdot \psi, \partial\!\!\!/_A \psi >] = \\ &= (\tfrac{1}{2} < e^3 \cdot \textstyle\sum_k e^k \cdot \nabla_{e_k} \psi, \psi > + \tfrac{1}{2} < e^3 \cdot \psi, \textstyle\sum_k e^k \cdot \nabla_{e_k} \psi >) e^1 \wedge e^2 + perm. \end{aligned} \tag{A.10}$$

We can therefore compute $\dot{q} - d * q$; the term in $e^1 \wedge e^2$, say, gives, after some algebra,

$$\frac{1}{2} < \psi, \nabla_{e_3} \psi > - \frac{1}{2} < \nabla_{e_3} \psi, \psi > \tag{A.11}$$

and we get the equality

$$\frac{\partial}{\partial t} q(\psi) - d * q(\psi) = i * Im < \psi, \nabla_A \psi >, \tag{A.12}$$

which is what we wanted to obtain.
Last, we compute the value of $d^* * (q(\psi) - F)$; applying Bianchi identity we have, from the definition of adjoint differential, that

$$d^* * (q(\psi) - F) = - * dq(\psi) = \tag{A.13}$$
$$= -\frac{1}{2}\sum_i (< e^i \cdot \nabla_{e_i}\psi, \psi > + < e^i \cdot \psi, \nabla_{e_i}\psi >) = iIm < \psi, \partial\!\!\!/_A \psi > .$$

Now we quote a result which will be the core of the proofs of the following Propositions.

Claim A.1.1 *Let f be a function in $C^2([0,T],\mathbb{R})$, strictly positive and satisfying*

$$f'' \geq 4\delta^2 f. \tag{A.14}$$

Then, $\forall t \in [0,T]$,

$$f(t) < f(0)exp[-2\delta t] + f(T)exp[-2\delta(T-t)]. \tag{A.15}$$

If $T = \infty$ and f is non negative and does not diverge at infinity, then

$$f(t) \leq f(0)exp[-2\delta t]. \tag{A.16}$$

Moreover, if f satisfies the inequality

$$f'' + 4\delta f' + 4\delta^2 f \geq 0 \tag{A.17}$$

then

$$\int_0^\infty dtexp[\omega t]f''(t) = C < \infty \quad \forall 0 \leq \omega < 2\delta. \tag{A.18}$$

Proof: although we guess that a proof of this Claim should exist in literature, we have not been able to find a complete reference; we start by a direct proof of inequality A.15 (an alternative proof can be given with the use of the maximum principle). Define the new function

$$y(t) := \frac{f'(t)}{2\delta f(t)}. \tag{A.19}$$

This function is well defined, as $f(t) > 0$. As $f'' \geq 4\delta^2 f$, $y(t)$ satisfies the following differential inequality:

$$y'(t) = \frac{f''(t)}{2\delta f(t)} - \frac{(f'(t))^2}{2\delta f(t)^2} \geq 2\delta(1 - y(t)^2). \tag{A.20}$$

We deduce from this that $y(t)$ has the following behavior: if there exist a $t_0 \in [0,T]$ s.t. $y(t_0) = 1$ then, $\forall t \in [t_0,T]$, $y(t) \geq 1$. We can therefore divide the interval $[0,T]$ in two disjoint subsets (one of which can be eventually empty) such that

$$\forall t \in [0,t_0) \;\; f'(t) < 2\delta f(t); \qquad \forall t \in [t_0,T] \;\; f'(t) \geq 2\delta f(t). \tag{A.21}$$

Let's consider this second interval. As $f'(t) \geq 2\delta f(t)$, by integrating the differential inequality between t and T, we have

$$f(t) \leq f(T)exp[-2\delta(T-t)] \tag{A.22}$$

and, *a fortiori*, $\forall t \in [t_0,T]$

$$f(t) < f(0)exp[-2\delta t] + f(T)exp[-2\delta(T-t)]. \tag{A.23}$$

For the first interval we need some more work. Let's define the auxiliary function

$$g(t) := f(t)exp[2\delta t]; \tag{A.24}$$

it is easy to check that the inequality $f''(t) \geq 4\delta^2 f(t)$ implies the differential relation

$$g'(s)exp[-4\delta s] \leq g'(t_0)exp[-4\delta t_0] \;\; \forall s \in [0,t_0]; \tag{A.25}$$

let's integrate this inequality, w.r.t. s, between 0 and $t < t_0$. We obtain

$$g(t) - g(0) \leq g'(t_0)(\frac{1}{4\delta}exp[4\delta t] - \frac{1}{4\delta})exp[-4\delta t_0]. \tag{A.26}$$

As f, at the value t_0, satisfies $f'(t_0) = 2\delta f(t_0)$, g satisfies the equation $g'(t_0) = 4\delta g(t_0)$ from which we obtain the inequality

$$g(t) - g(0) \leq g(t_0)(exp[4\delta t] - 1)exp[-4\delta t_0] < g(t_0)exp[-4\delta(t_0 - t)]. \tag{A.27}$$

This implies, for f, the relation, $\forall t \in [0,t_0)$

$$f(t) < f(0)exp[-2\delta t] + f(t_0)exp[-2\delta(t_0-t)] < f(0)exp[-2\delta t] + f(T)exp[-2\delta(T-t)], \tag{A.28}$$

where in the last equation we have used equation A.22 to estimate $f(t_0)$. Putting together equations A.23 and A.28 we get the desired inequality A.15. Note that in the case where there does not exist a t_0 s.t. $f'(t_0) = 2\delta f(t_0)$, the proof of the inequality A.15 for the

whole interval simply follows from the integration of equation A.25, with the extremum T instead of t_0 and the use of $f'(T) < 2\delta f(T)$.
For what concerns the part of the statement (equation A.16) which covers the case with $T = \infty$ the prove can be given with an easy modification of the previous arguments (but note that, in that case, the inequality is no more a strict one, although this has no relevance for us). Moreover it holds also under the weaker hypothesis that $f \geq 0$; in fact we used a stronger hypothesis only to deal with the case where there exist a t_0 with $f'(t_0) \geq 0$ and, if f does not vanish identically for $t \geq t_0$, this is incompatible with the condition that f does not diverge at infinity (if $f'(t_0) \geq 0$ there must exists an $s \in (t_0, \infty)$ with $f'(t) \geq f'(s) > f'(t_0) \geq 0 \ \forall t \geq s$, as an analysis of f'' shows). For a specific proof of this case, anyhow, see [JRS], where it is treated in detail. We give instead, as we will need them, the details of the proof of equation A.18; $f(t)$ now is supposed to satisfy equation A.16. Let's consider again $g(t)$; we have, with some straightforward calculations which follow from our hypothesis,

$$g'(t) = (2\delta f(t) + f'(t))exp[2\delta t] \leq 0, \tag{A.29}$$

$$g''(t) = (f''(t) + 4\delta f'(t) + 4\delta^2 f(t))exp[2\delta t] \geq 0$$

and, as g' is non decreasing and integrable over $[0, \infty)$, $\lim_{t \to \infty} g'(t) = 0$ and $g''(t)$ is integrable over an interval $[\bar{t}, \infty)$ with integral equal to $\int_{\bar{t}}^{\infty} dt g''(t) = -g'(\bar{t})$.
We remark that, eventually increasing the value of $\bar{t}$, we can choose the integral of $g''(t)$ over the interval $[\bar{t}, \infty)$ equal to a fixed constant independent of any choice.
We have, integrating over $[0, \infty)$,

$$-g'(0) = \int_0^{\infty} dt g''(t) \geq \int_0^{\infty} dt exp[\omega t](f''(t) + 4\delta f'(t) + 4\delta^2 f(t)) \ \ \forall 0 \leq \omega < 2\delta. \tag{A.30}$$

We need some more information on the second term on the r.h.s., as the third one is finite and positive: let's define $h(t) = f(t)exp[\omega t]$; we have

$$\int_0^{\infty} dt h'(t) = -h(0) = -f(0) = \omega \int_0^{\infty} dt exp[\omega t] f(t) + \int_0^{\infty} dt exp[\omega t] f'(t), \tag{A.31}$$

and thus

$$-4\delta \int_0^{\infty} dt exp[\omega t] f'(t) = 4\delta f(0) + \frac{\omega}{\delta} 4\delta^2 \int_0^{\infty} dt exp[\omega t] f(t) < \infty. \tag{A.32}$$

We can write

$$\int_0^{\infty} dt exp[\omega t] f''(t) \leq -g'(0) - 4\delta \int_0^{\infty} dt exp[\omega t] f'(t) - 4\delta^2 \int_0^{\infty} dt exp[\omega t] f(t) \tag{A.33}$$

which becomes, using equation A.32

$$0 \leq \int_0^\infty dtexp[\omega t]f''(t) \leq -g'(0) + 4\delta f(0) - (1 - \frac{\omega}{\delta})4\delta^2 \int_0^\infty dtexp[\omega t]f(t) = C < \infty. \tag{A.34}$$

In particular, note that choosing $\omega = \delta$ we get

$$\int_0^\infty dtexp[\omega t]f''(t) \leq -g'(0) + 4\delta f(0). \tag{A.35}$$

The previous remarks on the choice of the starting value on the interval tell us, in particular, that restricting eventually the domain to the interval $[\bar{t}, \infty)$ we can even bound the integral $\int_{\bar{t}}^{\infty} dtexp[\omega t]f''(t)$ in terms of the upper value of $f(t)$ only, property we will use later on. This concludes the proof of the Claim.

We can state now our first result:

Proposition A.1.2 *There exist positive constants ϵ, δ, such that if $(A(t), \psi(t))$ is a solution of SWF equations on a cylinder $N \times [0,T]$, which is contained in a ball of radius ϵ, in the $\mathcal{L}_1^2$ norm in the orbit space, around a static solution $(\Gamma, 0)$, then the square norm of the gradient of the Chern-Simons-Dirac functional satisfies the inequality*

$$f(t) < f(0)exp[-2\delta t] + f(T)exp[-2\delta(T-t)]. \tag{A.36}$$

Moreover, the values of ϵ, δ can be taken to be independent of the chosen solution.

Proof: our strategy will be to show that the square norm of the Chern-Simons-Dirac gradient satisfies an inequality of the type discussed in the previous claim, and then to deduce the decay conditions. As first observation we note that as the solutions of SW equation on the cylinder are gauge equivalent to smooth solutions, by usual elliptic techniques, the CSD functional is a smooth function of the t variable, and therefore so is $f(t)$, which is the time derivative of $C(t)$ once its argument $(A(t), \psi(t))$ is a solution of SWF equations. Second, we can assume that $f(t)$ is strictly positive, as if $\exists t_0$ with $f(t_0) = 0$, then we are at a static solution and f is identically zero.
It is practical to treat separately, first, the connection and the spinor part; we denote therefore

$$h(t) := ||\partial\!\!\!/_{A(t)}\psi(t)||^2, \quad g(t) := ||q(\psi(t)) - F_{A(t)}||^2, \quad f(t) = g(t) + h(t). \tag{A.37}$$

Our working hypothesis guarantee us that the $\mathcal{L}^2_1$ distance, in the based orbit space, of $(A(t),\psi(t))$ from $(\Gamma,0)$ is small; this means that we can determine, for any value of t, a gauge transformation $g\in\mathcal{G}^o(\tilde{P}_N)$ s.t. $||A^g-\Gamma||_{\mathcal{L}^2_1}$ and $||\psi^g||_{\mathcal{L}^2_1}$ are suitably small. Now we use the gauge invariance of $f(A,\psi)$ to write it as

$$f(A,\psi)=f(A^g,\psi^g)=||q(\psi^g)-F_{A^g}||^2+||\partial\!\!\!/_{A^g}\psi^g||^2\leq||q(\psi^g)-F_{A^g}||^2+2||\partial\!\!\!/_{A^g}\psi^g||^2\leq$$

$$\leq||F_{A^g}||^2+\tfrac{1}{2}||\psi^g||^4_{\mathcal{L}^4}+2||\nabla_{A^g}\psi^g||^2+\tfrac{max(s)}{2}||\psi^g||^2\leq$$

$$\leq||F_{A^g}||^2+\tfrac{1}{2}||\psi^g||^4_{\mathcal{L}^4}+2||\nabla_\Gamma\psi^g||^2+c_1||A^g-\Gamma||^2_{\mathcal{L}^2_1}||\psi^g||^2_{\mathcal{L}^2_1}+\tfrac{max(s)}{2}||\psi^g||^2 \tag{A.38}$$

where we have used Bochner-Weitzenböck formula for the Dirac laplacian. From this expression we see that we can suppose that

$$f(t)\leq c_2\epsilon^2=:\eta^2 \tag{A.39}$$

with η that can be made small with ϵ (now on we use the c_i's to denote "universal" constants; note that all the constant coming from Sobolev multiplication theorems and Sobolev embeddings can depend only on Γ and thus, in view of Claim 2.3.2, only on N). An important remark is in order: in general, the gauge transformation $g\in\mathcal{G}^o(\tilde{P}_N)$ which makes $||A^g-\Gamma||_{\mathcal{L}^2_1}$ and $||\psi^g||_{\mathcal{L}^2_1}$ small depends on (A,ψ) and therefore, for the gradient flow, on t. A time dependent gauge transformation would naturally affect the gradient flow equations: but what we have done, before, is not to study the behavior of the gauge transformed pair (A^g,ψ^g), we have just chosen a gauge suitable to make the computations easier.

We will often apply this procedure, so it is better to keep in mind this remark.

We are now ready for the proof of the proposition: we want apply Claim A.1.1 to $h(t)+g(t)$; let's compute, separately, their second time derivative; we start by analysing the spinor part of the Chern-Simons-Dirac functional; we have

$$h(t)=\int_N<\partial\!\!\!/_{A(t)}\psi(t),\partial\!\!\!/_{A(t)}\psi(t)>; \tag{A.40}$$

the first time derivative is

$$\begin{aligned}\tfrac{\partial}{\partial t}h(t)&=\textstyle\int_N<\tfrac{\partial}{\partial t}(\partial\!\!\!/_A\psi),\partial\!\!\!/_A\psi>+<\partial\!\!\!/_A\psi,\tfrac{\partial}{\partial t}(\partial\!\!\!/_A\psi)>=\\&=2Re\textstyle\int_N<\partial\!\!\!/^2_A\psi+\tfrac{1}{2}*(q-F)\cdot\psi,\partial\!\!\!/_A\psi>.\end{aligned} \tag{A.41}$$

We derive once again to get

$$\begin{aligned}
&\tfrac{\partial^2}{\partial t^2}h(t) = 2||\partial\!\!\!/^2_A\psi + \tfrac{1}{2} * (q-F)\cdot\psi||^2 + 2Re[\textstyle\int_N < \tfrac{\partial}{\partial t}(\partial\!\!\!/^2_A\psi), \partial\!\!\!/_A\psi > + \\
&+\tfrac{1}{2}\textstyle\int_N < *(\dot{q} - d*(q-F))\cdot\psi, \partial\!\!\!/_A\psi > +\tfrac{1}{2}\int_N < *(q-F)\cdot\partial\!\!\!/_A\psi, \partial\!\!\!/_A\psi >] = \\
&= 2||\partial\!\!\!/^2_A\psi + \tfrac{1}{2} * (q-F)\cdot\psi||^2 + 2Re[\textstyle\int_N < \partial\!\!\!/^3_A\psi, \partial\!\!\!/_A\psi > + \\
&+\tfrac{1}{2}\textstyle\int_N < d*(q-F)\cdot\psi, \partial\!\!\!/_A\psi > -\tfrac{1}{2}\int_N < *dq\cdot\psi, \partial\!\!\!/_A\psi > + \\
&-\textstyle\int_N < \sum_i[*(q-F)]_i\nabla_{A_i}\psi, \partial\!\!\!/_A\psi > +\tfrac{1}{2}\int_N < *(\dot{q} - d*(q-F))\cdot\psi, \partial\!\!\!/_A\psi > + \\
&+\tfrac{1}{2}\textstyle\int_N < *(q-F)\partial\!\!\!/_A\psi, \partial\!\!\!/_A\psi >] = 2||\partial\!\!\!/^2_A\psi + \tfrac{1}{2} * (q-F)\cdot\psi||^2 + \\
&+2||\partial\!\!\!/^2_A\psi||^2 + Re\textstyle\int_N < *\dot{q}\cdot\psi, \partial\!\!\!/_A\psi > -Re\int_N < *dq\cdot\psi, \partial\!\!\!/_A\psi > + \\
&+Re\textstyle\int_N < *(q-F)\cdot\partial\!\!\!/_A\psi, \partial\!\!\!/_A\psi > -2Re\int_N < \sum_i[*(q-F)]_i\nabla_{A_i}\psi, \partial\!\!\!/_A\psi >
\end{aligned} \tag{A.42}$$

(we used the equality

$$\partial\!\!\!/_A(*(q-F)\cdot\psi) = d*(q-F)\cdot\psi - *dq\cdot\psi - 2\sum_i[*(q-F)]_i\nabla_{A_i}\psi - *(q-F)\cdot\partial\!\!\!/_A\psi \tag{A.43}$$

which can be derived from the compatibility property of covariant derivative w.r.t. the Clifford product). The terms under integral need some additional work.
The first two terms can be explicitly computed and give

$$Re\int_N < *\dot{q}\cdot\psi, \partial\!\!\!/_A\psi >= \int_N |\psi|^2|\partial\!\!\!/_A\psi|^2 - (Im < \psi, \partial\!\!\!/_A\psi >)^2, \tag{A.44}$$

$$-Re\int_N < *dq\cdot\psi, \partial\!\!\!/_A\psi >= \int_N (Im < \psi, \partial\!\!\!/_A\psi >)^2. \tag{A.45}$$

These two terms give therefore a positive contribution, and require no other treatment.
The third one is

$$2Re < (q-F)|q(\partial\!\!\!/_A\psi) >; \tag{A.46}$$

our treatment of this term will be a bit tricky: our will is to apply Hölder inequality, with coefficients $(p,q) = (2,2)$ and to do so we need some kind of $\mathcal{L}^4$ control on $\partial\!\!\!/_A\psi$; we will do this by controlling its $\mathcal{L}^2_1$ norm and then using Sobolev embedding theorem. If we use elliptic inequalities applied the elliptic operator $\partial\!\!\!/_A$ we would obtain estimates which depend on the operator, and therefore on time, while we look for estimates which are uniform in t. To circumvent this difficulty we must compare estimates for $\not{\partial}_A$ and for $\not{\partial}_\Gamma$; to do this, we use the fact that $\mathcal{L}_1^2$ distance of A and Γ in the orbit space has a value suitably small once we take a small value of ϵ. As first step we note that the term of equation A.46 is gauge invariant and so we can write it, choosing a gauge transformation g such that

$$||A^g - \Gamma||_{\mathcal{L}_1^2} \leq \epsilon, \tag{A.47}$$

in the form

$$2Re < (q(\psi^g) - F_{A^g})|q(\not{\partial}_{A^g}\psi^g) >; \tag{A.48}$$

we will obtain, in such a gauge, an estimate which does not depend both of g and t that we will apply then to the term in equation A.46. We point out again the role of the gauge transformation g: we are using it here simply as choice of gauge where make the computation, at the instant t, of gauge invariant quantities and their (gauge independent) relations.

We use equation A.47 to compare estimates for $\not{\partial}_{A^g}$ and for $\not{\partial}_\Gamma$: we have, by definition,

$$||\phi^g||_{\mathcal{L}_1^2} \leq c_3||\not{\partial}_\Gamma\phi^g||_{\mathcal{L}^2} \leq c_3(||\not{\partial}_{A^g}\phi^g||_{\mathcal{L}^2}+\frac{1}{2}||(A^g-\Gamma)\cdot\phi^g||_{\mathcal{L}^2}) \leq c_3(||\not{\partial}_{A^g}\phi^g||_{\mathcal{L}^2}+\frac{1}{2}\epsilon c_1||\phi^g||_{\mathcal{L}_1^2}). \tag{A.49}$$

Rearranging the terms, we get

$$||\phi^g||_{\mathcal{L}_1^2} \leq (1-\frac{1}{2}\epsilon c_3 c_1)^{-1}c_3||\not{\partial}_{A^g}\phi^g||_{\mathcal{L}^2} =: c_4||\not{\partial}_{A^g}\phi^g||_{\mathcal{L}^2} \tag{A.50}$$

which is exactly the result we looked for, as now the elliptic constant c_4 does not depend on t, nor on the gauge transformation, as it is related only to the operator $\not{\partial}_\Gamma$. Note the need of an $\mathcal{L}_1^2$ control on $(A^g - \Gamma)$ in order to apply Sobolev multiplication theorem $\mathcal{L}_1^2 \otimes \mathcal{L}_1^2 \subset \mathcal{L}^2$. This result is, for our purposes, very good, as it allows to use elliptic estimates in terms of $\not{\partial}_{A^g}$ without the loss of uniformity in t. Moreover the estimates can also be taken uniformly w.r.t. the point Γ in the space of flat connection, as shown in Claim 2.3.2; for this very same reason, the value of ϵ can be chosen not depending on Γ. Now we can proceed with the $\mathcal{L}_1^2$ estimate on $\not{\partial}_{A^g}\psi^g$; from the formula A.50, putting $\phi = \not{\partial}_A\psi$, we have

$$||\not{\partial}_{A^g}\psi^g||_{\mathcal{L}_1^2} \leq c_4||\not{\partial}^2_{A^g}\psi^g||_{\mathcal{L}^2} \tag{A.51}$$

and this is allows to estimate the full term A.48 as

$$| < (q(\psi^g) - F_{A^g})|q(\not{\partial}_{A^g}\psi^g) > | \leq ||q(\psi^g) - F_{A^g}||_{\mathcal{L}^2}||q(\not{\partial}_{A^g}\psi^g)||_{\mathcal{L}^2} \leq \eta c_5||\not{\partial}^2_{A^g}\psi^g||^2_{\mathcal{L}^2} \tag{A.52}$$

which is enough for our purposes, as we will see later: it is clear that all the terms of this equation are gauge invariant, and so the estimate holds, with constant independent of time and gauge transformation, and we can use it directly to estimate the term of equation A.46.

The moral of the previous result is the following: each time we need an estimate of two gauge invariant quantities, depending on $(A(t), \psi(t))$, we can obtain such an estimate working with any choice of gauge, and then apply to the original term. We will often use this procedure, so we will omit sometimes to mention the various steps.

The remaining term to control in equation A.42 is

$$< \sum_i [*(q-F)]_i \nabla_{A_i}\psi | \partial\!\!\!/_A \psi > \tag{A.53}$$

and admits a similar treatment: we need again control on the $\mathcal{L}^4$ norm of $\partial\!\!\!/_A\psi$ as above, plus a control on the $\mathcal{L}^4$ norm of $\nabla_A\psi$; this is obtained by the following formulae, which have the same nature of those proven above: making the computations with a suitable choice of gauge, that we omit to mention,

$$||\nabla_A\psi||_{\mathcal{L}^2_1} \leq ||\nabla_\Gamma\psi||_{\mathcal{L}^2_1} + \frac{1}{2}\epsilon c_1||\psi||_{\mathcal{L}^2_2} \leq (1+\frac{1}{2}\epsilon c_1)||\psi||_{\mathcal{L}^2_2} \tag{A.54}$$

and concerning this latter term we have

$$||\psi||_{\mathcal{L}^2_2} \leq c_3||\partial\!\!\!/_\Gamma\psi||_{\mathcal{L}^2_1} \leq c_3(||\partial\!\!\!/_A\psi||_{\mathcal{L}^2_1} + \frac{1}{2}\epsilon c_1||\psi||_{\mathcal{L}^2_2}). \tag{A.55}$$

Rearranging carefully the terms we obtain

$$||\nabla_A\psi||_{\mathcal{L}^2_1} \leq (1+\frac{1}{2}\epsilon c_1)c_4||\partial\!\!\!/^2_A\psi||_{\mathcal{L}^2} =: c_6||\partial\!\!\!/^2_A\psi||_{\mathcal{L}^2}; \tag{A.56}$$

from this we estimate the full term, as the one we treated previously, as

$$| < \sum_i [*(q-F)]_i \nabla_{A_i}\psi | \partial\!\!\!/_A \psi > | \leq ||q-F||_{\mathcal{L}^2}||\nabla_A\psi||_{\mathcal{L}^2_1}||\partial\!\!\!/_A\psi||_{\mathcal{L}^2_1} \leq \eta c_7||\partial\!\!\!/^2_A\psi||^2_{\mathcal{L}^2}. \tag{A.57}$$

We see, therefore, that the two terms with undetermined sign are proportional, with a multiplicative constant arbitrarily small, to the square of the $\mathcal{L}^2$ norm of $\partial\!\!\!/^2_A\psi$ and are therefore smaller than the leading term of h''. Summing up, we have

$$h'' \geq (2 - 2\eta c_5 - 2\eta c_7)||\partial\!\!\!/^2_A\psi||^2_{\mathcal{L}^2}. \tag{A.58}$$

We compute now the second derivative of the curvature part of Chern-Simons-Dirac functional. We have

$$g(t) = \int_N < q - F, q - F >, \tag{A.59}$$

and its first time derivative is

$$\begin{gathered} \tfrac{\partial}{\partial t} g(t) = 2 < \dot{q} - d * (q - F) | q - F >= \\ = 2 \int_N < i * Im < \psi, \nabla_A \psi >, q - F > +2 < d * F | q - F > . \end{gathered} \tag{A.60}$$

The second derivative gives then

$$\begin{gathered} \tfrac{\partial^2}{\partial t^2} g(t) = 2 || \dot{q} - d * (q - F) ||^2 + 2 < d * d * (q - F) | q - F > + \\ +2 \int_N < i * \tfrac{\partial}{\partial t} Im < \psi, \nabla_A \psi >, q - F >= \\ = 2 || \dot{q} - d * (q - F) ||^2 + 2 || d^* (q - F) ||^2 + 2 \int_N < i * \tfrac{\partial}{\partial t} Im < \psi, \nabla_A \psi >, q - F > . \end{gathered} \tag{A.61}$$

We now must gain some control on the latter term: note, first of all, that

$$\begin{gathered} < i \tfrac{\partial}{\partial t} (Im < \psi, \nabla_A \psi >), *(q - F) >= \tfrac{1}{2} < *(q - F) |\psi|^2, *(q - F) > + \\ + < i Im < \partial\!\!\!/_A \psi, \nabla_A \psi > + i Im < \psi, \nabla_A \partial\!\!\!/_A \psi >, *(q - F) >= \\ = \tfrac{1}{2} |q - F|^2 |\psi|^2 - < i d Im < \partial\!\!\!/_A \psi, \psi >, *(q - F) > + \\ + < 2 i Im < \partial\!\!\!/_A \psi, \nabla_A \psi >, *(q - F) > . \end{gathered} \tag{A.62}$$

Applying the formula A.13 for $dq(\psi)$ we see that the integral equals

$$\begin{gathered} \int_N |q - F|^2 |\psi|^2 + 2 \int_N (Im < \psi, \partial\!\!\!/_A \psi >)^2 + \\ +4 \int_N < i Im < \partial\!\!\!/_A \psi, \nabla_A \psi >, *(q - F) > . \end{gathered} \tag{A.63}$$

For the last term we have in fact the following identity:

$$\begin{gathered} \int_N < i Im < \partial\!\!\!/_A \psi, \nabla_A \psi >, *(q - F) >= - \int_N i \textstyle\sum_i [*(q - F)]_i Im < \partial\!\!\!/_A \psi, \nabla_{A_i} \psi >= \\ = - Re \int_N < \textstyle\sum_i [*(q - F)]_i \nabla_{A_i} \psi, \partial\!\!\!/_A \psi > \end{gathered} \tag{A.64}$$

(we used the fact that $*(q - F)$ is a purely imaginary form; recall moreover, the minus sign in the definition of the product of forms); this term, therefore, has the same shape

of the one of equation A.53, and is controlled in the same way.
We have, therefore,

$$g'' \geq 2||d^*(q-F)||^2_{\mathcal{L}^2} - 4\eta c_7 ||\partial\!\!\!/^2_A \psi||^2_{\mathcal{L}^2}. \tag{A.65}$$

What we have to do now is to show that $f''(t)$ bounds, up to a constant, $f(t) = h(t) + g(t)$; concerning $h(t)$, it is bounded, in light of the arguments which brought to equation A.50, by a multiple of $||\partial\!\!\!/^2_A \psi||_{\mathcal{L}^2}$, as

$$||\partial\!\!\!/_A \psi||_{\mathcal{L}^2} = ||\partial\!\!\!/_{A^g} \psi^g||_{\mathcal{L}^2} \leq ||\partial\!\!\!/_{A^g} \psi^g||_{\mathcal{L}^2_1} \leq c_4 ||\partial\!\!\!/^2_{A^g} \psi^g||_{\mathcal{L}^2} = c_4 ||\partial\!\!\!/^2_A \psi||_{\mathcal{L}^2}, \tag{A.66}$$

and so the spinor part of the second derivative on CSD functional satisfies our requests.
We try as well to apply the Sobolev inequality for the operator $d+d^*$ to control $||q-F||^2_{\mathcal{L}^2}$; decompose $(q-F)$ in its Hodge components:

$$q - F = d\mu + d^*\nu + \gamma; \tag{A.67}$$

evidently only $q(\psi)$ contributes to the last two terms in the r.h.s.: we deduce, therefore, that

$$||q-F||^2 = ||d\mu||^2 + ||d^*\nu + \gamma||^2 \leq ||d\mu||^2 + ||q(\psi)||^2. \tag{A.68}$$

We can now apply Sobolev inequality to the exact part $d\mu$: there exist a positive constant $c_8 \in \mathbb{R}$ such that

$$||d\mu|| \leq c_7 ||d^*(q-F)||; \tag{A.69}$$

we can write therefore, working in the gauge where $(A - \Gamma)$ and ψ have small $\mathcal{L}^2_1$ norm,

$$c_8^2 ||d^*(q-F)||^2_{\mathcal{L}^2} \geq ||q-F||^2_{\mathcal{L}^2} - ||q(\psi)||^2_{\mathcal{L}^2} = ||q-F||^2_{\mathcal{L}^2} - \frac{1}{2}||\psi||^4_{\mathcal{L}^4} \geq ||q-F||^2_{\mathcal{L}^2} - \eta^2 c_9 h, \tag{A.70}$$

where we have estimated the $\mathcal{L}^4$ norm of ψ in terms of its $\mathcal{L}^2_1$ norm, and used again inequality A.50.
We now put together the results of equations A.66, A.70 on the two derivatives, and we see that, taking now ϵ, and consequently η small enough, we can cook out a δ s.t.

$$f'' \geq 4\delta^2 f. \tag{A.71}$$

The various Sobolev constants do not depend on the particular solution but only on Γ and thus, definitely, on N and therefore so do the constants ϵ, δ, as stated in the Proposition. Application of Claim A.1.1 gives then

$$f(t) < f(0)exp[-2\delta t] + f(T)exp[-2\delta(T-t)], \tag{A.72}$$

and this concludes the proof of the proposition.

The second result we will discuss is strictly related to the previous one. It concerns the application of the second part of Claim A.1.1 to $h(t) = ||\partial\!\!\!/_{A(t)}\psi(t)||^2$. It is clear the $h(t)$ is non negative and, for some δ (that we can always assume coinciding with one for which equation A.71 continues to hold true), it satisfies $h'' \geq 4\delta^2 h$, as show equations A.58 and A.66. To apply the second part of Claim A.1.1 we need one more step, which is contained in

Lemma A.1.3 *Under the conditions of Proposition A.1.2, the square norm of $\partial\!\!\!/_A\psi$ satisfies the inequality*

$$h'' + 4\delta h' + 4\delta^2 h \geq 0. \tag{A.73}$$

Proof: first observe from equation A.58 that we can tune ϵ in such a way that $h'' \geq ||\partial\!\!\!/_A^2\psi||^2$. Now we need some control on h': from equation A.41 we deduce, in the suitable gauge, that

$$|h'| \leq c_{10}(||\partial\!\!\!/_A^2\psi||_{\mathcal{L}^2}||\partial\!\!\!/_A\psi||_{\mathcal{L}^2} + ||q-F||_{\mathcal{L}^2}||\partial\!\!\!/_A^2\psi||^2_{\mathcal{L}^2_{\frac{3}{2}}}) \leq c_{11}||\partial\!\!\!/_A^2\psi||^2_{\mathcal{L}^2} \tag{A.74}$$

as comes from iterated elliptic inequalities. It is straightforward to see now that, eventually decreasing the value of δ, we have

$$h'' + 4\delta h' \geq ||\partial\!\!\!/_A^2\psi||^2_{\mathcal{L}^2} - 4c_{11}\delta||\partial\!\!\!/_A^2\psi||^2_{\mathcal{L}^2} \geq 0 \tag{A.75}$$

from which the inequality claimed in the lemma follows.

Application of formula A.18 of Claim A.1.1 guarantees therefore the following

Proposition A.1.4 *Under the condition of Proposition A.1.2 the square norm of $\partial\!\!\!/_A^2\psi$ satisfies the following inequality:*

$$\int_t^\infty ds||\partial\!\!\!/_A^2\psi||^2_{\mathcal{L}^2} \leq C exp[-\omega t] \qquad \forall 0 \leq \omega < 2\delta. \tag{A.76}$$

Proof: as we have checked $h(t)$ satisfies the condition imposed from Claim A.1.1 and so its second derivative can be integrated with an exponential measure of weight $0 \leq \omega < 2\delta$. Equation A.58 tells us that so does $||\partial\!\!\!/_A^2\psi||_{\mathcal{L}^2}$ and thus we have, applying equation A.18,

$$exp[\omega t]\int_t^\infty ds||\partial\!\!\!/_{A(s)}^2\psi(s)||^2_{\mathcal{L}^2} \leq \int_t^\infty ds exp[\omega s]||\partial\!\!\!/_{A(s)}^2\psi(s)||^2_{\mathcal{L}^2} \leq C, \tag{A.77}$$

which is what we wanted to prove.

Note, from the various remarks along the proof of Claim A.1.1 that, eventually restricting the interval $[t, \infty)$, we can make C independent of any choice.

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