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COMPLEX GEODESICS IN REINHARDT DOMAINS

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CONTENTS

INTRODUCTION

A holomorphic map from the unit disc $\Delta \subseteq \mathbb{C}$ into a domain $D \subseteq \mathbb{C}^n$ is a complex geodesic for the Carathéodory (or for the Kobayashi) pseudodistance if it is an isometry with respect to the Poincaré distance ω on Δ and the Carathéodory (or the Kobayashi) pseudodistance on D. The notion of complex geodesic was introduced by Vesentini in 1979 to study the automorphism group of the unit ball of $L^1(M, \mu)$, where (M, μ) is a measure space ([Vesentini 1979]).

Complex geodesics turned out to be a useful tool not only in the investigation of the automorphism group of complex domains, but also in several other questions concerning complex analysis and complex geometry. At this concern, a few significant results will be now mentioned. It has been proved that the image of a complex geodesic of a convex bounded domain $D \subseteq \mathbb{C}^n$ is the fixed point set of a holomorphic endomorphism of D ([Vigué 1984 b]). Complex geodesics are a main tool in the proof of some results which generalize the classical Schwarz Lemma to the case of holomorphic mappings between domains in $\mathbb{C}^n$, which are isometries for the Kobayashi or for the Carathéodory metric at one point (see, for example, [Vigué 1985 b] and [Graham 1989]). Also, some characterizations of the unit ball of $\mathbb{C}^n$ by its automorphism group involve complex geodesics (see, for example, [Rosay 1979]). Abate used complex geodesics to generalize Schields' Theorem and Julia-Wolff-Carathéodory Theorem to strongly convex domains ([Abate 1989 a]). In [Lempert 1981], a relation between complex geodesics and the complex Monge-Ampère equation on a convex bounded domain has been found and investigated.

As far as existence of complex geodesics on a given domain $D \subseteq \mathbb{C}^n$ is concerned, it may well happen that there does not exist any holomorphic isometry $\varphi : \Delta \to D$ with respect to ω and the Carathéodory pseudodistance c_D. For example, this is the case for the annulus $A = \{ z \in \mathbb{C} \mid 1 < |z| < r \}$, $r > 1$ ([Vesentini 1981]). On the other hand, Lempert proved that in a convex bounded domain $D \subseteq \mathbb{C}^n$, for any two distinct points, there exists a complex geodesic for the Kobayashi distance on D whose image contains the two points ([Lempert 1981]). The method used by Lempert consists in characterizing complex geodesics for the Kobayashi distance as "stationary maps" and in proving existence and uniqueness of stationary maps in strongly convex domains with C^3 boundary.

We recall that a stationary map on a domain $D \subseteq \mathbb{C}^n$ is a holomorphic map $\varphi : \Delta \to D$ such that:

(a) $\varphi \in C^{1/2}(\overline{\Delta}, \overline{D})$;

(b) $\varphi(\partial\Delta) \subseteq \partial D$;

(c) there exists a function $p \in C^{1/2}(\partial\Delta, \mathbb{R}^+)$ such that the map

$$\begin{aligned} \partial\Delta &\to \mathbb{C}^n \\ \xi &\mapsto \xi\, p(\xi)\, \overline{\nu(\varphi(\xi))} \end{aligned}$$

extends to a map $\widetilde{\varphi} \in Hol(\Delta, \mathbb{C}^n) \cap C^{1/2}(\overline{\Delta}, \mathbb{C}^n)$, where $\nu(z)$ is the unit outer normal vector to ∂D at $z \in \partial D$ and $C^{1/2}(B_1, B_2)$ denotes the set of 1/2-Hölder maps from the metric space B_1 to the metric space B_2.

Thus complex geodesics of a domain $D \subseteq \mathbb{C}^n$ are naturally related to the "shape" of the boundary ∂D.

The characterization of complex geodesics for the Kobayashi distance as stationary maps allows a different description: a subset of a strongly convex domain is the image of a complex geodesic for the Kobayashi distance if, and only if, it is a one-dimensional holomorphic retract of the domain ([Lempert 1982]).

This result has a remarkable consequence; in fact, it has been used to prove that on a convex domain the Kobayashi and the Carathéodory distances coincide. Therefore, on a convex domain we may speak about complex geodesics *tout court*.

A different approach led Royden and Wong to state that, on any convex domain, complex geodesics for the Kobayashi distance are exactly the stationary maps, regardless of regularity hypotheses on the boundary ([Royden-Wong 1983]).

In [Pang 1993], the techniques used by Lempert have been developed to prove that on a strongly pseudoconvex domain a complex geodesic for the Kobayashi distance is necessarily a stationary map. Examples show that the converse of this statement does not hold.

The explicit computation of complex geodesics of a given domain has been performed only in some classes of convex domains, e.g. convex complex ellipsoids $\mathcal{E}(p_1, \ldots, p_n) = \{ z \in \mathbb{C}^n \mid |z_1|^{2p_1} + \ldots + |z_n|^{2p_n} < 1 \}$, where $p_j \geq 1/2$ for all $j = 1, \ldots, n$ ([Gentili 1986 b], [Blank et alia 1992], [Jarnicki-Pflug-Zeinstra 1993], [Jarnicki-Pflug 1995]) and some of their possible generalizations ([Zwonek 1995 a], [Visintin 1995]).

The technique used in the cases quoted above consists in determining explicitly the stationary maps. The fact that the domains under investigation are Reinhardt domains allows to find the outer factors and the inner factors of the components of a stationary map.

In this dissertation, we investigate the question of determining the complex geodesics of an arbitrary convex bounded Reinhardt domain, with the aim of giving a characterization of complex geodesics suitable for their explicit computation. A closely related problem we deal with concerns the comparison between the balls defined on a domain $D \subseteq \mathbb{C}^n$ by the Minkowski functional of D and the balls defined by the Carathéodory distance c_D. In order to solve this problem, which has already been studied in some cases (see, for example, [Schwarz-Srebro 1996] and [Zwonek 1996]), the knowledge of explicit formulas for complex geodesics seems to be crucial.

A closer inspection of procedures used in [Gentili 1986 b], [Jarnicki-Pflug 1995] and in [Visintin 1995] shows that, on any convex bounded Reinhardt domain, a simpler characterization of stationary maps holds, namely, in this dissertation we have proved the following (see [Visintin 1998])

Theorem 1 *Let $D \subseteq \mathbb{C}^n$ be a convex bounded Reinhardt domain. Let μ_D be the Minkowski functional of D. Let $\varphi = (\varphi_1, \dots, \varphi_n) : \Delta \to D$ be a non-constant holomorphic map such that $\varphi_j \not\equiv 0$ for all $j = 1, \dots, n$ and that the boundary values $\varphi^*(\xi)$ belong to ∂D for a.a. $\xi \in \partial\Delta$. Let M_j be the inner factor of φ_j. Moreover, suppose that the unit outer normal vector to ∂D is defined at $\varphi^*(\xi)$ for a.a. $\xi \in \partial\Delta$.*

Then φ is a complex geodesic if, and only if, there exist $\alpha_\circ, \alpha_1, \dots, \alpha_n \in \overline{\Delta}$, $r_1, \dots, r_n \geq 0$ such that

$$(1.a) \qquad r_j \frac{|1-\overline{\alpha}_j\xi|^2}{|1-\overline{\alpha}_\circ\xi|^2} = \frac{\partial \mu_D}{\partial |z_j|}(|\varphi_1^*(\xi)|, \dots, |\varphi_n^*(\xi)|)|\varphi_j^*(\xi)| \qquad \text{for a.a. } \xi \in \partial\Delta$$

for all $j = 1, \dots, n$;

$$(1.b) \qquad \alpha_\circ = \sum_{j=1}^{n} r_j \alpha_j \qquad 1 + |\alpha_\circ|^2 = \sum_{j=1}^{n} r_j(1+|\alpha_j|^2).$$

If $r_j > 0$, then there exists $\theta_j \in \mathbb{R}$ such that

$$(1.c) \qquad M_j(\lambda) = e^{i\theta_j}\left(\frac{\lambda-\alpha_j}{1-\overline{\alpha}_j\lambda}\right)^{s_j} \qquad \lambda \in \Delta,$$

where $s_j \in \{\, 0, 1 \,\}$ and $s_j = 1$ implies $\alpha_j \in \Delta$.

This result makes completely clear the influence of the "shape" of ∂D on the structure of the family of complex geodesics in the case of a convex bounded Reinhardt domain D.

The "core" of a complex geodesic on a Reinhardt domain lies in the outer factors of its components; in fact, the inner factor of a component of a complex geodesic is either a Möbius transformation or whatever inner function.

Remark that Theorem 1 can be used not only to find the family of all complex geodesics of a given convex bounded Reinhardt domain explicitly, but also to study its general structure. In [Kaup-Umpeier 1976], by using Lie group theory, it has been proved that if two bounded balanced pseudoconvex domains D_1, $D_2 \subseteq \mathbb{C}^n$ are biholomorphic, then there exists a biholomorphic map $\psi : D_1 \to D_2$ such that $\psi(0) = 0$. The Carathéodory metric can be exploited to prove that this fact implies that D_1 and D_2 are linearly equivalent. It would be interesting to find a proof of the Kaup-Upmeier Theorem involving complex analysis, and we believe that complex geodesics and Theorem 1 are suitable for this purpose. Complex geodesics and Theorem 1 again seem to be, a powerful tool to study the group of the automorphisms of a given convex bounded Reinhardt domain.
At this concern, we would like to point out that recent results on the geometry of Teichmüller spaces (see [Abate Patrizio 1997]) suggest the use of the family of complex geodesics to obtain a deeper knowledge of the geometric structure of the Teichmüller spaces themself.

As we already pointed out, the knowledge of explicit formulas of complex geodesics is useful in studying a geometric property of the Carathéodory distance on a domain $D \subseteq \mathbb{C}^n$, namely the natural question of the comparison between the balls defined on D by the Minkowski functional μ_D of D and those defined by the Carathéodory distance c_D. The Carathéodory ball with centre at $\widetilde{z} \in D$ and radius $\operatorname{arctanh} r$, for $r \in (0,1)$, is defined as follows

$$B^*_{c_D}(\widetilde{z}, r) = \{\ z \in D \mid c_D(z, \widetilde{z}) < \operatorname{arctanh} r\ \}.$$

If $D \subseteq \mathbb{C}^n$ is a bounded, convex and balanced domain, then the norm ball with centre at $\widetilde{w} \in \mathbb{C}^n$ and radius $s > 0$ is defined as

$$B_D(\widetilde{w}, s) = \{\ z \in \mathbb{C}^n \mid \mu_D(z - \widetilde{w}) < s\ \}.$$

The interest on this question follows from the fact that on the unit disc $\Delta \subseteq \mathbb{C}$, every Carathéodory ball is an Euclidean ball. Moreover, on any convex bounded and balanced domain Carathéodory balls centred at the origin and norm balls centred at the origin coincide.
The question of comparing Carathéodory balls and norm balls not centred at the origin has been discussed for some homogeneous domains, such as, for example, the Euclidean ball $B_n \subseteq \mathbb{C}^n$ (the only Carathéodory balls which are also norm balls are those centred at the origin, [Rudin 1980]) and the polydisc $\Delta^n \subseteq \mathbb{C}^n$ (the only Carathéodory balls which are also norm balls are those centred at a point $\widetilde{z} = (\widetilde{z}_1, \ldots, \widetilde{z}_n)$ such that $|\widetilde{z}_1| = \ldots = |\widetilde{z}_n|$).

As far as non-homogeneous domains are concerned, the convex complex ellipsoids have been considered. It has been proved that the only Carathéodory balls which are also norm balls are those centred at the origin if the complex ellipsoid under discussion is $\mathcal{E}(p_1, \ldots, p_n)$, where $p_j > 1/2$ and $p_j \neq 1$ for all $j = 1, \ldots, n$ ([Schwarz-Srebro 1996], [Zwonek 1996]). Moreover, it has been found that $B^*_{c_{\mathcal{E}(1/2,1)}}((0, w_2), r)$ is a norm ball for any $(0, w_2) \in \mathcal{E}(1/2, 1)$ and for any $r \in (0, 1)$ ([Zwonek 1996]).
Since it seems hard to generalize the methods used in the works quoted above to compare Carathéodory balls and norm balls on an arbitrary Reinhardt domain, then we used a different approach to prove the following general result (see [Visintin 1997])

Theorem 2 *Let $D \subseteq \mathbb{C}^n$ be a strictly convex bounded Reinhardt domain with C^1 boundary. Let $\widetilde{z} \in D$ be such that at least two of its components are non zero. Then*

$$B^*_{c_D}(\widetilde{z}, r) \neq B_D(\widetilde{w}, s)$$

for all $\widetilde{w} \in D$, $r \in (0, 1)$, $s \in \mathbb{R}^+$.

In order to determine whether a Carathéodory ball $B^*_{c_D}(\widetilde{z}, r)$ centred on some axis of a domain D is a norm ball, it seems necessary to know the formulas for the family of all complex geodesics through the centre $\widetilde{z} \in D$. By using such formulas, in this dissertation, the following result on a class of convex bounded Reinhardt domains, which naturally generalizes complex ellipsoids, has been proved (see [Visintin 1997])

Theorem 3 *Let $0 \leq a \leq 1$ and $\mathbf{p} = (p_1, \ldots p_n)$ be such that $p_1 \geq 1$, $p_2 \geq 1$, $p_j \geq 1/2$ for all $j = 3, \ldots, n$. Let*

$$D_{a,\mathbf{p}} = \left\{ z \in \mathbb{C}^n \mid |z_1|^{2p_1} + 2a|z_1|^{p_1}|z_2|^{p_2} + |z_2|^{2p_2} + \sum_{j=3}^{n} |z_j|^{2p_j} < 1 \right\}.$$

Let $\widetilde{z}, \widetilde{w} \in D_{a,\mathbf{p}}$, $\widetilde{z} \neq 0$, $r \in (0, 1)$ and $s \in \mathbb{R}^+$. Then

$$B^*_{c_{D_{a,\mathbf{p}}}}(\widetilde{z}, r) = B_{D_{a,\mathbf{p}}}(\widetilde{w}, s)$$

if, and only if,

$$D_{a,\mathbf{p}} = \mathcal{E}(p_1, \ldots, p_n)$$

where $p_k = 1$ for exactly one $k \in \{1, \ldots, n\}$, $p_j = 1/2$ for all $j \neq k$, $\widetilde{z}_j = 0$ for all $j \neq k$, and

$$\widetilde{w} = \frac{1 - r^2}{1 - r^2|\widetilde{z}_k|^2}\widetilde{z} \qquad \text{and} \qquad s = \frac{1 - |\widetilde{z}_k|^2}{1 - r^2|\widetilde{z}_k|^2} r.$$

In particular, this result completely solves the question in the case of convex complex ellipsoids, because $D_{0,\mathbf{p}} = \mathcal{E}(\mathbf{p})$.

In what follows, we briefly describe the content of this thesis.

In the First Chapter, we present the main properties of the Carathéodory and Kobayashi pseudodistances and pseudometrics, which are among the most important invariant pseudodistances and pseudometrics defined on complex domains.

The Second Chapter is devoted to complex geodesics. After the definition and the first properties, we survey the work of Lempert, Royden and Wong on this matter ([Lempert 1981, 1982], [Royden-Wong 1983]).
By generalizing a result proved in [Gentili 1986 b], we give a better insight on the function p appearing in the definition of stationary map. In particular, we get the following

Theorem 4 *Let $D \subseteq \mathbb{C}^n$ be a convex, bounded and balanced domain. Let μ_D be the Minkowski functional of D. Let $\varphi : \Delta \to D$ be a holomorphic map such that $\varphi^*(\xi) \in \partial D$ for a.a. $\xi \in \partial\Delta$ and that the unit outer normal vector to ∂D is defined at $\varphi^*(\xi) \in \partial D$ for a.a. $\xi \in \partial\Delta$. Then $\varphi : \Delta \to D$ is a complex geodesic if, and only if, there exist $h \in H^1(\Delta, \mathbb{C}^n)$, $h \not\equiv 0$, $r_\circ > 0$ and $\alpha_\circ \in \overline{\Delta}$ such that*

$$\frac{1}{\xi} h_j^*(\xi) = r_\circ |1 - \overline{\alpha}_\circ \xi|^2 \frac{\partial \mu_D}{\partial z_j}(\varphi^*(\xi)) \qquad \textit{for a.a. } \xi \in \partial\Delta,$$

$j = 1, \ldots, n$.

In the last paragraph of the Second Chapter, we give a brief account of known results concerning uniqueness of complex geodesics, which is important in many applications.

In the Third Chapter, we prove Theorem 1 concerning complex geodesics on convex bounded Reinhardt domains and present some of its consequences.
In particular, we prove that each component of a complex geodesic in a strictly convex bounded Reinhardt domain may have at most one zero in Δ (Corollary 3.1.4). We also prove that if one considers some particular families of convex bounded Reinhardt domains, then the knowledge of the complex geodesics of one particular domain of the family suffices to determine the complex geodesics of all other domains belonging to that family (Proposition 3.1.6).

In the Fourth Chapter, as an application of Theorem 1, we compute the complex geodesics of some classes of domains explicitly. The first example concerns the complex geodesics of convex complex ellipsoids, which were already known. In the second example, which is taken from [Visintin 1995], we determine the complex geodesics of the generalization of complex ellipsoids defined in Theorem 3. The last example concerns the complex geodesics of another class of convex bounded Reinhardt domains in $\mathbb{C}^2$.

The Fifth Chapter is devoted to the discussion and the proof of Theorem 2 and Theorem 3 on the comparison between Carathéodory balls and norm balls.

In the Sixth Chapter we present some examples and some results on complex geodesics on pseudoconvex domains. We remark that on such domains existence and uniqueness of complex geodesics may well fail.
As a consequence of Theorem 1 and of the forementioned result obtained by Pang, we get that a complex geodesic on a strongly pseudoconvex Reinhardt domain necessarily satisfies conditions $(1.a)$, $(1.b)$, and $(1.c)$. An explicit example showing that the converse of the above statement fails to hold is exhibited.

Finally, in the Appendix we have collected some classical results and definitions which are used throughout this work, namely the Factorization Theorem for H^p functions on Δ, some results on regularity of Minkowski functionals, the various notions of convexity and pseudoconvexity and the Maximum Principle for subharmonic functions.

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First Chapter

PRELIMINARY RESULTS ON INVARIANT DISTANCES AND METRICS

A classical general problem in complex analysis is to decide whether two given complex domains are biholomorphic. One of the major results concerning this question is the Riemann mapping theorem, which says that any proper simply connected domain in $\mathbb{C}$ is biholomorphic to the unit disc Δ (this result was stated in [Riemann 1851] and proved in [Osgood 1900]). Therefore a purely topological property suffices to determine the holomorphic structure of a planar domain. It is hopeless to try to extend this result to higher dimensions, as the unit ball and the bidisc in $\mathbb{C}^2$, though topologically equivalent simply connected domains, are not biholomorphic ([Poincaré 1907]). So, since topology seems to have a poor influence on the holomrphic structure of an arbitrary complex domain, we may ask how geometry is involved in such questions.

An idea is to define distances and metrics on complex domains which are invariant under biholomorphic mappings and then to examine how geometric properties of domains reflect on the holomorphic structure.

Recall that a pseudodistance on a domain $D \subseteq \mathbb{C}^n$ is a function $d : D \times D \to [0, \infty)$ such that

- $d(z, z) = 0$;
- $d(z, w) = d(w, z)$;
- $d(z, w) \leq d(z, v) + d(v, w)$

for all $z,\ w,\ v \in D$.

A pseudodistance is a distance if $d(z, w) = 0$ implies that $z = w$, for all $z,\ w \in D$.

A pseudometric on a domain $D \subseteq \mathbb{C}^n$ is a function $\delta : D \times \mathbb{C}^n \to [0, \infty)$ such that

- $\delta(z; \lambda v) = |\lambda|\, \delta(z; v)$

for all $z \in D$, $v \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$.
A pseudometric is a metric if $\delta(z;v) = 0$ implies that $v = 0$, for all $z \in D$.

1. The Poincaré metric

Historically, the first invariant metric to be defined was the Poincaré metric on the unit disc Δ, which is given by (see, for example, [Abate 1989 a], [Jarnicki-Pflug 1993], [Vesentini 1984])

$$\langle v \rangle_\xi = \frac{|v|}{1 - |\xi|^2}$$

for all $v \in \mathbb{C}$ and all $\xi \in \Delta$. This metric was investigated in [Riemann 1854] as an example of what we now call a metric of constant Gaussian curvature (-4). Then in [Beltrami 1868 a, b] it was used to devise the disc model of the Lobačevski hyperbolic plane. Last, but not least, Poincaré began to deal with "his" metric in 1882 in connection with the study of Fuchsian groups ([Poincaré 1882]).
The Poincaré distance is, by definition, the integrated form of the Poincaré metric, i.e.

$$\omega(\xi_1, \xi_2) = \inf \left\{ \int_0^1 \langle f'(t) \rangle_{f(t)} dt \mid f \in A(\xi_1, \xi_2) \right\}$$

where $A(\xi_1, \xi_2) = \{ f : [0,1] \to \Delta$ piecewise C^1 curve $: f(0) = \xi_1, \quad f(1) = \xi_2 \}$. It turns out that ([Abate 1989 a], [Jarnicki-Pflug 1993], [Vesentini 1984])

$$\omega(\xi_1, \xi_2) = \operatorname{arctanh} \left| \frac{\xi_1 - \xi_2}{1 - \bar{\xi}_1 \xi_2} \right| = \frac{1}{2} \log \frac{1 + \left| \frac{\xi_1 - \xi_2}{1 - \bar{\xi}_1 \xi_2} \right|}{1 - \left| \frac{\xi_1 - \xi_2}{1 - \bar{\xi}_1 \xi_2} \right|}$$

The classical Schwarz-Pick Lemma implies that the Poincaré metric and distance satisfy the "contraction properties" we are looking for, namely, we have the following

Proposition 1.1.1 *Let $f : \Delta \to \Delta$ be a holomorphic mapping. Then*

$$\langle f'(\xi) \rangle_{f(\xi)} \leq \langle 1 \rangle_\xi$$

$$\omega(f(\xi_1), f(\xi_2)) \leq \omega(\xi_1, \xi_2)$$

for all ξ, ξ_1, $\xi_2 \in \Delta$. If, moreover, either there exists $\xi \in \Delta$ such that

$$\langle f'(\xi) \rangle_{f(\xi)} = \langle 1 \rangle_\xi,$$

or there exist $\xi_1, \xi_2 \in \Delta$ *such that*

$$\omega(f(\xi_1), f(\xi_2)) = \omega(\xi_1, \xi_2),$$

then f *is an automorphism of* Δ.

Therefore, in particular, every automorphism of Δ is a ω-isometry, i.e. an isometry for the Poincaré distance ω. Indeed, it can be proved that the Poincaré distance is the only distance on Δ (satisfying some natural regularity conditions) invariant under $Aut(\Delta)$ ([Vesentini 1984], [Dineen 1989], [Jarnicki-Pflug 1993]). Moreover, the group of all isometries for the Poincaré metric consists of all holomorphic and antiholomorphic automorphisms of Δ ([Vesentini 1984], [Jarnicki-Pflug 1993]).
In [Vesentini 1982 a] it has been proved that the Poincaré distance is a logarithmically plurisubharmonic function.
Let us look closer at the geometry determined on Δ by the Poincaré distance. An open Poincaré disc of centre $\xi_o \in \Delta$ and radius $r > 0$ is defined by

$$B_\omega(\xi_o, r) = \{ \xi \in \Delta \mid \omega(\xi, \xi_o) < r \}.$$

It has been proved that a Poincaré disc coincides with an Euclidean disc. Precisely, we have the following ([Vesentini 1984], [Abate 1989 a], [Jarnicki-Pflug 1993])

Lemma 1.1.2 *Let* $\xi_o \in \Delta$ *and* $r > 0$; *then*

$$B_\omega(\xi_o, r) = \left\{ \xi \in \Delta \mid \left| \xi - \frac{1 - R^2}{1 - R^2|\xi_o|^2}\xi_o \right| < \frac{R}{1 - R^2|\xi_o|^2}(1 - |\xi_o|^2) \right\},$$

where $R = \tanh r$.

Therefore, in particular, the Poincaré distance induces on Δ the standard topology and (Δ, ω) is a complete metric space.
In the sequel, we will consider analogous questions for the invariant pseudodistances we are going to define on arbitrary complex domains and in that case the answers will be not so straightforward.

2. The Carathéodory and the Kobayashi pseudodistances

The first generalization of the Poincaré distance is due to Carathéodory and dates back to the Twenties ([Carathéodory 1926, 1927, 1928]). Given a bounded domain $G \subseteq \mathbb{C}^2$,

Carathéodory used the set of holomorphic functions from G into Δ and the Poincaré distance to define a new pseudodistance.
Let $D \subseteq \mathbb{C}^n$ be an arbitrary domain, i.e. a simply conncted open subset of $\mathbb{C}^n$. The Carathéodory pseudodistance is defined as follows (see, for example, [Jarnicki-Pflug 1993], [Dineen 1989], [Abate 1989 a])

$$c_D(z, w) = \sup\{\ \omega(f(z), f(w)) \mid f \in Hol(D, \Delta)\}$$

for all z, $w \in D$. It is checked that c_D is a pseudodistance.
Remark that the definition of the Carathéodory pseudodistance is meaningful for domains in complex normed spaces of infinite dimension ([Franzoni-Vesentini 1980]) and for complex manifolds ([Kobayashi 1970, 1976]).
The simplest example we can produce to show that the Carathéodory pseudodistance between two distinct points in a domain can be zero is $\mathbb{C}$: in fact, Liouville Theorem implies that $c_{\mathbb{C}} \equiv 0$. However, it can be proved that if $D \subseteq \mathbb{C}^n$ is (biholomorphic to) a bounded domain, then c_D is a distance ([Jarnicki-Pflug 1993], pg. 28).
A domain D such that c_D is a distance is called c-hyperbolic.

Proposition 1.2.1 *Let $D_1 \subseteq \mathbb{C}^n$ and $D_2 \subseteq \mathbb{C}^m$ be domains and $f \in Hol(D_1, D_2)$. Then*

$$c_{D_2}(f(z), f(w)) \leq c_{D_1}(z, w)$$

for all z, $w \in D_1$ and all $f \in Hol(D_1, D_2)$. Moreover,

$$c_\Delta \equiv \omega.$$

Therefore, every holomorphic mapping is a contraction for the Carathéodory pseudodistance. In some sense, we can say that the Carathéodory pseudodistance has a built-in Schwarz Lemma. In particular, a biholomorphic mapping is an isometry with respect to the Carathéodory pseudodistance, just as we wished.
It turns out that the Carathéodory pseudodistance is a continuous logarithmically plurisubharmonic function ([Kobayashi 1976], a proof is supplied in [Vesentini 1982 a]). Thus, in particular, the standard topology of a domain is finer than the topology induced by the Carathéodory pseudodistance. If $D \subseteq \mathbb{C}^n$ is (biholomorphic to) a bounded domain, then c_D induces the standard topology ([Jarnicki-Pflug 1993]).
It is clear that c-hyperbolicity is a necessary condition for c_D to induce the standard topology on D. As far as sufficiency of this condition is concerned, it turns out that if D is a planar domain, then c-hyperbolicity implies that c_D induce on D the standard

topology; but, for any $n \geq 3$, a c-hyperbolic domain $D \subseteq \mathbb{C}^n$ such that the standard topology of D is strictly finer than the c_D- topology has been constructed ([Jarnicki-Pflug-Vigué 1991], [Jarnicki-Pflug 1993] pg. 30, see also [Vigué 1984 a], where an example of a complex analytic space bearing this property is given). On the other hand Sibony, in [Sibony 1975], proved that if the closed Carathéodory balls (see below for the definition) are compact in a c-hyperbolic domain $D \subseteq \mathbb{C}^n$, then c_D induces the standard topology on D.

Let

$$B_{c_D}(z_o, r) = \{ \ z \in D \mid c_D(z, z_o) < r \ \}$$

be the open Carathéodory ball with centre at $z_o \in D$ and radius $r > 0$, and

$$\overline{B}_{c_D}(z_o, r) = \{ \ z \in D \mid c_D(z, z_o) \leq r \ \}$$

be the closed Carathéodory ball. If $\overline{B_{c_D}(z_o, r)}$ denotes the closure of $B_{c_D}(z_o, r)$ with respect to the standard topology of D, then one may ask whether $\overline{B}_{c_D}(z_o, r)$ and $\overline{B_{c_D}(z_o, r)}$ coincide. The answer is that, in general,

$$\overline{B_{c_D}(z_o, r)} \subseteq \overline{B}_{c_D}(z_o, r).$$

Note that it may well happen that

$$\overline{B_{c_D}(z_o, r)} \neq \overline{B}_{c_D}(z_o, r). \tag{1.2.1}$$

In fact, for any $n \geq 2$, a bounded strongly pseudoconvex domain $D \subseteq \mathbb{C}^n$, with real analytic boundary, has been worked out in such a way that, for some $z_o \in D$ and some $r > 0$, (1.2.1) holds ([Jarnicki-Pflug-Vigué 1992], [Jarnicki-Pflug 1993], pg. 41).

Let d_D be a continuous distance on a domain $D \subseteq \mathbb{C}^n$. If any sequence $\{ z_n \}_{n \in \mathbb{N}} \subseteq D$, which is a Cauchy sequence with respect to the distance d_D, converges to a point $\tilde{z} \in D$ with respect to the standard topology of D, then the domain D is said to be d_D-complete.

If all d_D balls are relatively compact with respect to the standard topology of D, then the domain D is said to be d_D-finitely compact.

Remark that d_D-finitely compactness implies d_D-completeness.

It turns out that a bounded strongly pseudoconvex domain $D \subseteq \mathbb{C}^n$ is c_D-finitely compact. Also, any convex bounded domain $D \subseteq \mathbb{C}^n$ is c_D-finitely compact.

In [Pflug 1984], it has been proved that any bounded pseudoconvex Reinhardt domain containing the origin is c-finitely compact.

We refer the reader to [Jarnicki-Pflug 1993] for a detailed discussion about the above questions.

Since the Carathéodory pseudodistance on a domain D is defined by means of holomorphic mappings from D into Δ, it is natural to look at mappings from Δ into D to see whether they can be used to define another invariant pseudodistance on D. This was done at the end of the Sixties by Kobayashi ([Kobayashi 1967 a, b]).

Given a domain $D \subseteq \mathbb{C}^n$, a "natural" definition should be

$$\widetilde{k}_D(z, w) = \inf\{\ \omega(\xi, \eta) \mid f \in Hol(\Delta, D)\ f(\xi) = z\ f(\eta) = w\}$$

for all z, $w \in D$. Such a definition is meaningful, since the set $\{\ f \in Hol(\Delta, D) \mid z, w \in f(\Delta)\ \}$ is not empty and therefore $\widetilde{k}_D(z, w)$ is a non-negative real number.

The function $\widetilde{k}_D$ is symmetric, i.e. $\widetilde{k}_D(z, w) = \widetilde{k}_D(w, z)$, but, in general, it does not satisfy the triangle inequality, as the next example shows.

Example 1.2.2 Let, for $\varepsilon > 0$,

$$D_\varepsilon = \{\ (z, w) \in \mathbb{C}^2 \mid |z| < 1,\ |w| < 1,\ |zw| < \varepsilon\ \}.$$

Let $P = (1/2, 0)$, $Q = (0, 1/2)$ and $O = (0, 0)$. It has been proved ([Lempert 1981]) that if ε is sufficiently small, then

$$\widetilde{k}_{D_\varepsilon}(P, Q) > \widetilde{k}_{D_\varepsilon}(P, O) + \widetilde{k}_{D_\varepsilon}(O, Q).$$

The Kobayashi pseudodistance on a domain $D \subseteq \mathbb{C}^n$ is defined to be the largest pseudodistance which is smaller than $\widetilde{k}_D$. It turns out that (see, for example, [Jarnicki-Pflug 1993], [Dineen 1989], [Abate 1989 a])

$$k_D(z, w) = \inf\left\{ \sum_{j=1}^{n} \widetilde{k}_D(z_j, z_{j+1}) \mid n \in \mathbb{N} \quad \{z = z_1, \ldots, w = z_{n+1}\} \subseteq D \right\}.$$

The Kobayashi pseudodistance can be defined for domains in complex normed spaces of infinite dimension ([Franzoni-Vesentini 1980]) and for complex manifolds ([Kobayashi 1970, 1976]) as well.

There is a remarkable case in which k_D coincides with $\widetilde{k}_D$, namely ([Lempert 1981])

Theorem 1.2.3 *Let $D \subseteq \mathbb{C}^n$ be a convex domain. Then*

$$\widetilde{k}_D = k_D.$$

Since the above result was proved by Lempert, then, for any domain $D \subseteq \mathbb{C}^n$, we will call $\widetilde{k}_D$ the Lempert function of D.

The main properties of the Kobayashi pseudodistance are analogous to the properties of the Carathéodory pseudodistance

Proposition 1.2.4 *Let* $D_1 \subseteq \mathbb{C}^n$ *and* $D_2 \subseteq \mathbb{C}^m$ *be arbitrary domains. Then*

$$k_{D_2}(f(z), f(w)) \leq k_{D_1}(z, w)$$

for all $z, w \in D_1$ *and all* $f \in Hol(D_1, D_2)$, *and*

$$k_\Delta \equiv \omega.$$

A domain $D \subseteq \mathbb{C}^n$ is said to be taut if for any sequence $\{ f_n \}_n \subseteq Hol(\Delta, D)$ there exists a subsequence $\{ f_{n_j} \}_j$ such that either $\{ f_{n_j} \}$ converges uniformly on compact subsets to an element of $Hol(\Delta, D)$ or, for each compact subset $K_1 \subseteq \Delta$ and each compact subset $K_2 \subseteq D$, one has that $f_{n_j}(K_1) \cap K_2 = \emptyset$ for all sufficiently large j ([Wu 1967]).
It turns out that a bounded taut domain is necessarily pseudoconvex. Viceversa, a strongly pseudoconvex domain is taut. Every bounded convex domain is taut ([Abate 1989 a], [Jarnicki-Pflug 1993]).

The Kobayashi pseudodistance is a continuous function, while the Lempert function is only upper semicontinuous in general. However, the Lempert function of a taut domain is continuous ([Jarnicki-Pflug 1993]).
A domain D is called k-hyperbolic if k_D is a distance. Contrary to what happens in the Carathéodory case, k-hyperbolicity implies that the Kobayashi distance induces the standard topology ([Barth 1972], [Royden 1971]). Taut domains are k-hyperbolic ([Kiernan 1970]).
Harris, in [Harris 1979], proved that a convex domain in $\mathbb{C}^n$ is k-hyperbolic if, and only if, it is biholomorphic to a bounded domain.
Every c-complete domain is k-complete too. It has been proved that a k-complete domain is necessarily taut. It turns out that the converse of this statement fails to hold ([Jarnicki-Pflug 1991 a]). For a convex domain, c-finitely compactness is proved to be equivalent to k-hyperbolicity. See [Jarnicki-Pflug 1993] to have an exhaustive discussion on Kobayashi completeness and related questions.

The Carathéodory and the Kobayashi pseudodistances are generalizations of the Poincaré distance which both contract holomorphic mappings. In general, we call Schwarz-Pick system of pseudodistances a system which assigns a pseudodistance to each domain in each normed complex linear space in such a way that (see [Harris 1979])

- the pseudodistance assigned to Δ is ω;

- if d_1 and d_2 are the pseudodistances assigned to the domains D_1 and D_2, respectively, and if $f \in Hol(D_1, D_2)$, then, for all $z, w \in D_1$,

$$d_2(f(z), f(w)) \leq d_1(z, w).$$

This definition provides a convenient setting for comparing different invariant pseudodistances. It has been proved that the Carathéodory and the Kobayashi pseudodistances are the "smallest" and the "largest" Schwarz-Pick systems, respectively, i.e. it has been proved that ([Harris 1979])

$$c_D \leq d_D \leq k_D$$

whenever $\{d_D\}$ is a Schwarz-Pick system.

3. The Carathéodory and the Kobayashi pseudometrics

Beside Schwarz-Pick systems of pseudodistances, it is likewise important to study holomorphically contractible pseudometrics which are generalizations of the Poincaré metric. In what follows we define the Carathéodory and the Kobayashi pseudometrics, which are the infinitesimal versions of the Carathéodory and the Kobayashi pseudodistances, respectively.
The Carathéodory pseudometric has been introduced by Carathéodory himself in [Carathéodory 1928], but it has not been studied extensively until the Sixties, when the works by Reiffen [Reiffen 1963, 1965] appeared.
Let $D \subseteq \mathbb{C}^n$ be a domain. The Carathéodory pseudometric is defined as follows (see, for example, [Jarnicki-Pflug 1993], [Dineen 1989], [Abate 1989 a])

$$\gamma_D(z; v) = \sup\{ \langle df(z)v \rangle_{f(z)} \mid f \in Hol(D, \Delta) \}$$

for all $z \in D$ and all $v \in \mathbb{C}^n$. We refer to [Franzoni-Vesentini 1980] for the study of the Carathéodory pseudometric on domains of complex normed spaces of infinite dimension. One checks that γ_D is a pseudometric. Moreover, for all $z \in D$, the function $\gamma_D(z; .)$ is a seminorm on $\mathbb{C}^n$, i.e.

$$\gamma_D(z; \lambda v) = |\lambda| \gamma_D(z; v)$$

$$\gamma_D(z; v + w) \leq \gamma_D(z; v) + \gamma_D(z; w)$$

for all $v, w \in \mathbb{C}^n$ and all $\lambda \in \mathbb{C}$. If $\gamma_D(z; .)$ is a norm for all $z \in D$, then we say that D is γ-hyperbolic. It turns out that bounded domains are γ-hyperbolic ([Jarnicki-Pflug 1993]).

The Carathéodory pseudometric is a locally Lipschitz logarithmically plurisubharmonic function ([Franzoni-Vesentini 1980], [Vesentini 1982 b]).

Proposition 1.3.1 *Let* $D_1 \subseteq \mathbb{C}^n$ *and* $D_2 \subseteq \mathbb{C}^m$ *be domains and* $f \in Hol(D_1, D_2)$, *then*

$$\gamma_{D_2}(f(z); df(z)v) \leq \gamma_{D_1}(z; v)$$

for all $z \in D_1$, $v \in \mathbb{C}^n$. *Moreover, on* Δ *the Carathéodory pseudometric coincides with the Poincaré metric, i.e.*

$$\gamma_D(z; v) = \langle v \rangle_z$$

for all $z \in \Delta$ *and all* $v \in \mathbb{C}$.

The inner Carathéodory pseudodistance for a domain $D \subseteq \mathbb{C}^n$ is the integrated form of the Carathéodory pseudometric, i.e. ([Harris 1979])

$$c^i_D(z, w) = \left\{ \int_0^1 \gamma_D(f(t); f'(t))dt \mid f \in B(z, w) \right\}$$

for all $z, w \in D$, where $B(z, w) = \{ f : [0, 1] \to D$ piecewise C^1 curve $: f(0) = z,\ f(1) = w \}$.
The inner Carathéodory pseudodistances form a Schwarz-Pick system, therefore

$$c_D \leq c^i_D \leq k_D.$$

Let $A = \{ \xi \in \mathbb{C} \mid 1/r < \xi < r \}$ for some $r > 0$. It has been proved that ([Jarnicki-Pflug 1990])

$$c_A(1, -1) < c^i_A(1, -1).$$

Notice that the annulus A is among the domains for which the Carathéodory metric and the inner Carathéodory distances are explicitly computed ([Simha 1975], [Jarnicki-Pflug 1993]).
Another example of this phenomenon can be found in [Vigué 1983], where it has been proved that

$$c_D((0, 0), (x, x)) < c^i_D((0, 0), (x, x))$$

for the domain $D = \{ (z, w) \in \mathbb{C}^2 \mid |z|+|w| < 1,\ |z\,w| < 1/16 \}$ and where $1/8 < |x| < 1/4$.

It turns out that c^i_D is a continuous function. If D is c^i-hyperbolic (i.e. if c^i_D is a distance), then c^i_D induces on D the standard topology ([Jarnicki-Pflug 1993]).

It has been proved that, given a domain $D \subseteq \mathbb{C}^n$ ([Harris 1979])

$$\lim_{\substack{z_1, z_2 \to z_o \\ z_1 \neq z_2 \\ \frac{z_1 - z_2}{\|z_1 - z_2\|} \to v}} \frac{c_D(z_1, z_2)}{\|z_1 - z_2\|} = \gamma_D(z_o; v) \tag{1.3.1}$$

for all $z_o \in D$ and all $v \in \mathbb{C}^n$ such that $\|v\| = 1$.
Whenever a pseudodistance d_D on D and a pseudometric δ_D on D are related as in (1.3.1), we say that δ_D is the derivative of d_D.

The infinitesimal version of the Kobayashi pseudodistance was defined by Royden in [Royden 1971], on any domain $D \subseteq \mathbb{C}^n$, as follows (see, for example, [Jarnicki-Pflug 1993], [Dineen 1989], [Abate 1989 a])

$$\kappa_D(z; v) = \inf\{ \langle\lambda\rangle_\xi \mid f \in Hol(\Delta, D) : \; f(\xi) = z \; df(\xi)\lambda = v \}$$

for all $z \in D$ and all $v \in \mathbb{C}^n$. Besides the usual reference to [Franzoni-Vesentini 1980] for the infinite dimensional case, we mention [Venturini 1996] where the Kobayashi pseudometric is defined on complex spaces too.
The fact that κ_D is a pseudometric on the domain D results directly from the definition, but, in general, $\kappa_D(z; .)$ is not a seminorm: for example, in [Kaup 1982] it has been noticed that the Kobayashi metric at the origin of the domain $D = \{ (z, w) \in \mathbb{C}^2 \mid |z| < 1, \; |w| < 1, \; 2|zw| < 1 \}$ is not a seminorm on $\mathbb{C}^2$.
The following properties hold

Proposition 1.3.2 *Let $D_1 \subseteq \mathbb{C}^n$ and $D_2 \subseteq \mathbb{C}^m$ be domains. Let $f \in Hol(D_1, D_2)$. Then*

$$\kappa_{D_2}(f(z); df(z)v) \leq \kappa_{D_1}(z; v)$$

for all $z \in D_1$, $v \in \mathbb{C}^n$. Moreover, the Kobayashi pseudometric on Δ coincides with the Poincaré metric, i.e.

$$\kappa_D(z; v) = \langle v\rangle_z.$$

In general, the Kobayashi pseudometric is an upper semicontinuous function ([Royden 1971]); it is continuous on taut domains ([Jarnicki-Pflug 1993]).
It can be proved that the Kobayashi pseudodistance is the integrated form of the Kobayashi pseudometric ([Royden 1971], [Royden 1974], [Vesentini 1982 b]), i.e.

$$k_D(z, w) = \left\{ \int_0^1 \kappa_D(f(t); f'(t))dt \mid f \in B(z, w) \right\}$$

for all $z, w \in D$, where $B(z,w) = \{ f : [0,1] \to D$ piecewise C^1 curve $: f(0) = z,\ f(1) = w \}$. This result is fundamental to prove the following property ([Jarnicki-Pflug 1993]), which does not hold in the Carathéodory case (cf. (1.2.1))

Proposition 1.3.3 *Let $D \subseteq \mathbb{C}^n$ be a k-hyperbolic domain. Then*

$$\overline{B_{k_D}(z_o, r)} = \overline{B}_{k_D}(z_o, r)$$

for all $z_o \in D$ and all $r > 0$ i.e. the closed Kobayashi balls coincide with the closure of the open Kobayashi balls.

The Kobayashi pseudometric need not to be the derivative of the Kobayashi pseudodistance. For example, in [Venturini 1989 a] it is proved that the Kobayashi metric at the origin of the domain D_ε defined in Example 1.2.2 is not the derivative of the Kobayashi distance k_{D_ε} provided that $\varepsilon > 0$ is sufficiently small.

Analogously to Schwarz-Pick systems of pseudodistances, one can define a Schwarz-Pick system of pseudometrics as a system which assigns a pseudometric to each domain in each complex normed linear space in such a way that ([Harris 1979])

- the pseudometric assigned to Δ is the Poincaré metric;
- if δ_1 and δ_2 are the pseudometrics assigned to the domain D_1 of the space N_1 and to the domain D_2 of the space N_2, respectively, and if $f \in Hol(D_1, D_2)$, then

$$\delta_2(f(z); df(z)v) \leq \delta_1(z; v)$$

for all $z \in D_1$ and all $v \in N_1$.

The Carathéodory and the Kobayashi pseudometrics form the "smallest" and the "largest" Schwarz-Pick systems of pseudometrics ([Harris 1979]), i.e.

$$\gamma_D \leq \delta_D \leq \kappa_D$$

whenever $\{ \delta_D \}$ is a Schwarz-Pick system of pseudometrics.

Second Chapter

COMPLEX GEODESICS

A holomorphic map from the unit disc $\Delta \subseteq \mathbb{C}$ into a domain $D \subseteq \mathbb{C}^n$ is a complex geodesic for the Carathéodory (or for the Kobayashi) pseudodistance if it is an isometry with respect to the Poincaré distance on Δ and the Carathéodory (or the Kobayashi) pseudodistance on D. The notion of complex geodesic was introduced by Vesentini in 1979 to study the automorphism group of the unit ball of $L^1(M, \mu)$, where (M, μ) is a measure space ([Vesentini 1979]). Actually, we should mention that already in the Twenties Carathéodory considered the image of a complex geodesic (for the Carathéodory pseudodistance) under the name of "metrische Ebene" ([Carathéodory 1927]) and that Kritikos exploited them to study the automorphism group of the complex ellipsoid $\mathcal{E} = \{ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1| + |z_2| < 1 \}$ ([Kritikos 1927]).

Complex geodesics turned out to be a useful tool for the investigation of several questions concerning complex analysis and complex geometry.

The characterization of complex geodesics in convex domains as one-dimensional holomorphic retracts entails equality of Carathéodory and Kobayashi distances on such domains ([Lempert 1982]).

It has been proved that a subset of a convex bounded domain $D \subseteq \mathbb{C}^n$ is the fixed point set of a holomorphic endomorphism of D if, and only if, it is a holomorphic retract of D ([Vigué 1985 a]). As a consequence, a relation between fixed point sets and images of complex geodesics in convex bounded domains is obtained (see also [Vigué 1984 b]). In particular, these results allow to characterize completely those subsets of a convex bounded domain $D \subseteq \mathbb{C}^2$ which are the fixed point set of a holomorphic endomorphism of D: the empty set, a single point, the image of a complex geodesic, D itself ([Vesentini 1982 a]).

Knowledge of explicit formulas for complex geodesics on a complex domain D allows to

compute the Carathéodory (or the Kobayashi) metric on D ([Blank et alia 1992], [Jarnicki-Pflug 1993], example 8.4.8).
By means of the existence and uniqueness of complex geodesics in strictly convex bounded and balanced domains $D \subseteq \mathbb{C}^n$, Vigué gave conditions under which a holomorphic endomorphism $\varphi : D \to D$ is necessarily a linear automorphism of D ([Vigué 1991]).
Also, some results in the same vein of the Schwarz Lemma had been obtained: if $D_1 \subseteq \mathbb{C}^n$ is a convex taut domain, $D_2 \subseteq \mathbb{C}^n$ is a domain and $\varphi \in Hol(D_1, D_2)$ is such that $\gamma_{D_2}(\varphi(z); d\varphi(z)v) = \gamma_{D_1}(z; v)$ for one point $z \in D_1$ and all $v \in \mathbb{C}^n$, then φ is a biholomorphic map ([Vigué 1985 b]). The same result holds if D_1 is taut, D_2 is strictly convex and bounded and φ is a holomorphic isometry for the Kobayashi metric at one point $z \in D_1$ ([Graham 1989], see also [Venturini 1989 b]). Inspired by the techniques used in [Vigué 1985 b] and in [Graham 1989], Abate and Patrizio proved analogue results for holomorphic mappings between Teichmüller spaces ([Abate-Patrizio 1997]).
Complex geodesics can be used to prove some characterizations of the unit ball of $\mathbb{C}^n$ by its automorphism group. For example, a theorem due to Rosay states that if $D \subseteq \mathbb{C}^n$ is a domain with C^2 boundary and $Aut(D)$ is transitive, then D is biholomorphic to the unit ball $B_n \subseteq \mathbb{C}^n$ ([Rosay 1979], see [Wong 1977] and [Lin-Wong 1990] for more general results).
Complex geodesics are involved in some characterizations of circular domains in $\mathbb{C}^n$ ([Abate-Patrizio 1992]) and of the polydisc $\Delta^n \subseteq \mathbb{C}^n$ ([Stanton 1980]) in terms of properties of Carathéodory and Kobayashi metrics (see [Stanton 1983] for a similar result concerning the unit ball of $\mathbb{C}^n$).
Abate used complex geodesics to generalize Shields' Theorem (by proving the existence of a common fixed point of a family of commuting holomorphic endomorphisms of a strongly convex domain $D \subseteq \mathbb{C}^n$) and Julia-Wolff-Carathéodory Theorem (by giving sufficient conditions for a holomorphic endomorphism of a strongly convex domain $D \subseteq \mathbb{C}^n$ with C^3 boundary to admit angular derivative at a point $z \in \partial D$) to strongly convex domains ([Abate 1989 b, 1989 a, 1990] and [Abate 1991] for a further generalization of the Julia-Wolff-Carathéodory Theorem to strongly pseudoconvex domains).
Complex geodesics played a rôle in a result by Krantz and Burns on boundary rigidity for holomorphic endomorphisms of bounded strongly pseudoconvex domains in $\mathbb{C}^n$ with C^3 boundary ([Burns-Krantz 1994], see also [Huang 1994, 1995]).
In his fundamental work [Lempert 1981], Lempert found the relation between complex geodesics and the complex Monge-Ampére equation on a convex bounded domain and exploited his results on existence and uniqueness of complex geodesics to improve Fefferman Theorem on smooth extension of biholomorphic mappings between smoothly strongly

pseudoconvex domains: Lempert proved that a biholomorphic map between two strongly pseudoconvex domains with C^k boundary, $k \geq 6$, extends to a diffeomorphism of class C^{k-4} between the closures of the domains (see also [Lempert 1986]).
The above account would only let feel the flavour of the various fields in which complex geodesics have been used. It is influenced by author's own taste and it is not meant to be complete.

1. Complex geodesics

Let $D \subseteq \mathbb{C}^n$ be a domain. Let d_D be the pseudodistance assigned to D by a Schwarz-Pick system. For every holomorphic map $\varphi : \Delta \to D$, one has that

$$d_D(\varphi(\xi), \varphi(\eta)) \leq \omega(\xi, \eta)$$

for all $\xi, \eta \in \Delta$. If there exist $\xi_1, \xi_2 \in \Delta$ such that

$$d_D(\varphi(\xi_1), \varphi(\xi_2)) = \omega(\xi_1, \xi_2),$$

then φ is called a complex geodesic for d_D at $\varphi(\xi_1)$ and $\varphi(\xi_2)$. When φ is a complex geodesic for d_D at $\varphi(\xi_1)$ and $\varphi(\xi_2)$ for all $\xi_1, \xi_2 \in \Delta$, we say that φ is a complex geodesic for d_D.
A complex geodesic for c_D is necessarily a complex geodesic for the pseudodistance assigned to D by any Schwarz-Pick system; in particular, it is a complex geodesics for k_D.
A complex geodesic $\varphi : \Delta \to D$ for d_D is necessarily injective and, if d_D is a distance, $\varphi(\Delta)$ is closed with respect to the d_D-topology ([Vesentini 1981]).
This fact implies, in particular, that a bounded non simply connected domain $D \subseteq \mathbb{C}$ does not admit any complex geodesic for k_D.
Let $\varphi : \Delta \to D \subseteq \mathbb{C}^n$ be a complex geodesic for d_D. Since any automorphism g of Δ is an isometry for the Poincaré distance, then the holomorphic map $\varphi \circ g : \Delta \to D$ is a complex geodesic too and $\varphi \circ g(\Delta) = \varphi(\Delta)$. It can be proved that if $\varphi : \Delta \to D$ and $\psi : \Delta \to D$ are complex geodesics such that $\varphi(\Delta) = \psi(\Delta)$, then there exists an automorphism f of Δ such that $\varphi = \psi \circ f$ ([Vesentini 1981]). Therefore, every time we will discuss about uniqueness of complex geodesics we will mean uniqueness up to a composition with an automorphism of Δ, i.e. complex geodesics with the same image will be identified.
If δ_D is the pseudometric assigned to D by a Schwarz-Pick system, then a holomorphic map $\varphi : \Delta \to D$ is called an infinitesimal complex geodesic for δ_D at $\varphi(\xi_o)$ if

$$\delta_D(\varphi(\xi_o); d\varphi(\xi_o)v) = \langle v \rangle_{\xi_o}$$

for all $v \in \mathbb{C}$. An infinitesimal complex geodesic $\varphi : \Delta \to D$ for δ_D is, by definition, an infinitesimal complex geodesic for δ_D at $\varphi(\xi)$ for all $\xi \in \Delta$.

Notice that an infinitesimal complex geodesic for the Carathéodory pseudometric is necessarily an infinitesimal complex geodesic for the Kobayashi pseudometric.

The fact that the Carathéodory pseudometric is the derivative of the Carathéodory pseudodistance together with Montel Theorem and Schwarz-Pick Lemma allow to prove the following result ([Vesentini 1982 a])

Proposition 2.1.1 *Let $D \subseteq \mathbb{C}^n$ be a domain. Let $\varphi \in Hol(\Delta, D)$. Then the following facts are equivalent*

(a) there exist $\xi_1, \xi_2 \in \Delta$ such that φ is a complex geodesic for c_D at $\varphi(\xi_1)$ and $\varphi(\xi_2)$;

(b) φ is a complex geodesic for c_D;

(c) there exists $\xi_o \in \Delta$ such that φ is an infinitesimal complex geodesic for γ_D at $\varphi(\xi_o)$;

(d) φ is an infinitesimal complex geodesic for γ_D.

Therefore, in the Carathéodory case, the notions of complex geodesic and infinitesimal complex geodesic coincide and, moreover, in order to know whether a given holomorphic map is a complex geodesic for c_D it suffices to check equality at a single couple of points only.

As far as complex geodesics for the Kobayashi distance are concerned, the situation goes as follows ([Venturini 1989 b])

Proposition 2.1.1′ *Let $D \subseteq \mathbb{C}^n$ be a domain. Let $\varphi : \Delta \to D$ be a complex geodesic for k_D at $\varphi(\xi_1)$ and $\varphi(\xi_2)$, for some $\xi_1, \xi_2 \in \Delta$. Let S be the arc of the Riemannian geodesic for the Poincaré metric joining ξ_1 and ξ_2 in Δ. Then*

(a) φ is a complex geodesic for k_D at $\varphi(\eta_1)$ and $\varphi(\eta_2)$ for all $\eta_1, \eta_2 \in S$;

(b) φ is an infinitesimal complex geodesic for κ_D at $\varphi(\xi)$ for all $\xi \in S$.

Proposition 2.1.1 and the Hahn-Banach Theorem yield the following ([Vesentini 1981])

Proposition 2.1.2 *Let $D \subseteq \mathbb{C}^n$ be a convex and balanced domain. Let μ_D be its Minkowski functional. Let $z \in D$ be such that $\mu_D(z) > 0$. Then the holomorphic map*

$$\begin{aligned} \Delta &\to D \\ \lambda &\mapsto \lambda \frac{1}{\mu_D(z)} z \end{aligned}$$

is a complex geodesic for c_D.

Therefore, in particular, for each point z in a convex balanced domain D, there exists at least one complex geodesic for c_D whose image contains the origin and z.

If $\psi : D \to D$ is an automorphism and $\varphi : \Delta \to D$ is a complex geodesic, then $\psi \circ \varphi$ is a complex geodesic too. Consequently, if a convex balanced domain D has "enough" automorphisms, then for any two points of D we can find a complex geodesic whose image contains them.
Since for most domains we do not have a deep knowledge of the automorphism groups and, above all, complex geodesics are intended as a tool to study also the automorphisms of a domain, then we should look for another way to study the question of existence of complex geodesics.
By the Schwarz-Pick Lemma, we get the following characterization of complex geodesics for the Carathéodory pseudodistance ([Jarnicki-Pflug 1993], [Reiffen 1963]).

Proposition 2.1.3 *Let $D \subseteq \mathbb{C}^n$ be a domain. Then a holomorphic map $\varphi : \Delta \to D$ is a complex geodesic for c_D if, and only if, there exists a holomprphic map $\psi : D \to \Delta$ such that $\psi \circ \varphi = Id_\Delta$.*

If $\varphi : \Delta \to D$ is a complex geodesic for c_D, then it is a complex geodesic also for k_D and

$$c_D(\varphi(\xi), \varphi(\eta)) = \widetilde{k}_D(\varphi(\xi), \varphi(\eta)) = k_D(\varphi(\xi), \varphi(\eta)) = \omega(\xi, \eta)$$

for all $\xi, \eta \in \Delta$. Therefore on the image of a Carathéodory complex geodesic the Carathéodory pseudodistance coincides with the Kobayashi pseudodistance and with the Lempert function.
For any two points z and w in a taut domain $D \subseteq \mathbb{C}^n$ there exists a holomorphic map $\psi : \Delta \to D$ such that $\psi(0) = z$, $\psi(\xi) = w$ and $\widetilde{k}_D(z, w) = \omega(0, \xi)$. If, moreover, $c_D = \widetilde{k}_D$, then by Proposition 2.1.1 for any $z, w \in D$ there exists a complex geodesic (for c_D) $\varphi : \Delta \to D$ such that $z, w \in \varphi(\Delta)$. This fact and Proposition 2.1.3 yield the following ([Vigué 1985 a])

Proposition 2.1.4 *Let $D \subseteq \mathbb{C}^n$ be a taut domain. Then the following conditions are equivalent*
(a) $c_D = \widetilde{k}_D$;
(b) for any $z, w \in D$ there exist holomorphic maps $\varphi : \Delta \to D$ and $f : D \to \Delta$ such that $z, w \in \varphi(\Delta)$ and $f \circ \varphi = Id_\Delta$.

These results, however, are not satisfactory because it is not at all an easy question to decide whether the Carathéodory pseudodistance coincide with the Lempert function.

2. Complex geodesics in convex domains.

Up to now, the most comprehensive results on this matter are due to Lempert and concern the existence (and uniqueness) of complex geodesics for the Kobayashi distance on (striclty) convex domains.
In the fundamental work [Lempert 1981] it is proved that for any two distinct points of a convex bounded domain D there exists a complex geodesic for $k_D(=\widetilde{k}_D)$ whose image contains them. Moreover, Lempert gives a characterization of complex geodesics (in strongly convex bounded domains with C^3 boundary) which can be used to find explicit formulas for complex geodesics.
Recall that on a convex domain the Kobayashi distance and the Lempert function coincide (Theorem 1.2.3).
Let $D \subseteq \mathbb{C}^n$ be a strongly convex domain with C^3 boundary. The starting point of Lempert work is the observation that an infinitesimal complex geodesic $\varphi : \Delta \to D$ for κ_D at $(z, v) \in D \times \mathbb{C}^n$ is a solution of the extremal problem

$$\sup\{\ |g'(0)|\ |\ g \in Hol(\Delta, D) :\ g(0) = z \quad g'(0) = \lambda v \quad \lambda > 0\ \}.$$

Considerations from calculus of variation led Lempert to believe that a holomorphic map $\varphi : \Delta \to D$ is an infinitesimal complex geodesic for κ_D if, and only if,

(a) $\varphi \in Hol(\Delta, D) \cap C^{1/2}(\overline{\Delta}, \overline{D})$;

(b) $\varphi(\partial\Delta) \subseteq \partial D$;

(c) there exists a function $p \in C^{1/2}(\partial\Delta, \mathbb{R}^+)$ such that the map

$$\begin{aligned} \partial\Delta &\to \mathbb{C}^n \\ \xi \quad &\mapsto \xi\, p(\xi)\, \overline{\nu(\varphi(\xi))} \end{aligned}$$

extends to a map $\widetilde{\varphi} \in Hol(\Delta, \mathbb{C}^n) \cap C^{1/2}(\overline{\Delta}, \mathbb{C}^n)$, where $\nu(z)$ is the unit outer normal vector to ∂D at $z \in \partial D$ and $C^{1/2}(B_1, B_2)$ denotes the set of 1/2-Hölder maps from the metric space B_1 to the metric space B_2.

For the moment, we call stationary a holomorphic map $\varphi : \Delta \to D$ satisfying conditions (a), (b) and (c).
Strict convexity of the domain and condition (c) allow to prove that a stationary map $\varphi : \Delta \to D$ is the unique infinitesimal complex geodesic for κ_D at $(\varphi(0), \varphi'(0))$.
Once it has been noticed that the composition $\varphi \circ f : \Delta \to D$ of a stationary map $\varphi : \Delta \to D$ with an automorphism f of Δ is again a stationary map, one gets the following

Proposition 2.2.1 *Let $D \subseteq \mathbb{C}^n$ be a strongly convex domain with C^3 boundary. A stationary map $\varphi : \Delta \to D$ is the unique infinitesimal complex geodesic for κ_D at $(\varphi(\xi), \varphi'(\xi))$ for all $\xi \in \Delta$.*

Analogously, one can prove that

Proposition 2.2.2 *Let $D \subseteq \mathbb{C}^n$ be a strongly convex domain with C^3 boundary. A stationary map $\varphi : \Delta \to D$ is the unique complex geodesic for k_D at $\varphi(\xi)$ and $\varphi(\eta)$ for all $\xi, \eta \in \Delta$.*

One of the deepest results of Lempert concerns the relation between the regularity of the boundary of the domain D and the regularity up to the boundary of Δ of a stationary map, namely

Proposition 2.2.3 *Let $D \subseteq \mathbb{C}^n$ be a strongly convex domain with C^k boundary, $k \geq 3$. Let $\varphi : \overline{\Delta} \to \overline{D}$ be a stationary map. Let $p \in C^{1/2}(\partial\Delta, \mathbb{R}^+)$ and $\widetilde{\varphi} \in Hol(\Delta, \mathbb{C}^n) \cap C^{1/2}(\overline{\Delta}, \mathbb{C}^n)$ be the mappings related by the equation $\widetilde{\varphi}(\xi) = \xi\, p(\xi)\, \overline{\nu(\varphi(\xi))}$ for all $\xi \in \partial\Delta$, which exist by the definition of a stationary map. Then φ, $\widetilde{\varphi} \in C^{k-2}(\overline{\Delta}, \mathbb{C}^n)$.*

Let us define $w \bullet z = \langle w, \overline{z} \rangle = \sum_{j=1}^{n} w_j z_j$ for $z, w \in \mathbb{C}^n$, where $\langle\ ,\ \rangle$ is the usual Hermitian product on $\mathbb{C}^n$. It turns out that, if $\varphi : \Delta \to D \subseteq \mathbb{C}^n$ is a stationary map, then $\varphi'(\xi) \bullet \widetilde{\varphi}(\xi)$, $\xi \in \Delta$, is a positive constant function on $\overline{\Delta}$. This fact helps to prove that a stationary map imbeds $\overline{\Delta}$ into $\mathbb{C}^n$ and implies that, given a stationary map φ, the corresponding mappings p and $\widetilde{\varphi}$ are determined up to multiplication by a positive constant. Therefore, with the above notations, one can choose $p(\xi) = \left(\xi\varphi'(\xi) \bullet \overline{\nu(\varphi(\xi))}\right)^{-1}$, for all $\xi \in \partial\Delta$.
The uniqueness result in Proposition 2.2.1 implies that the images of two stationary maps either coincide or consist of a single point, but a stronger statement has been proved, that is

Proposition 2.2.4 *Let $D \subseteq \mathbb{C}^n$ be a strongly convex domain with C^4 boundary. Let $\varphi \not\equiv \psi : \Delta \to D$ be two stationary maps. Let $z = \varphi(0) = \psi(0)$. Then $\varphi(\overline{\Delta}) \cap \psi(\overline{\Delta}) = \{\ z\ \}$.*

The main result, namely the existence of stationary maps, is proved by using a method which involves the perturbation of domains.

Theorem 2.2.5 *Let $D \subseteq \mathbb{C}^n$ be a strongly convex bounded domain with C^6 boundary. Then for any two point $z, w \in D$ there exists a unique stationary map $\varphi : \Delta \to D$ for z and w.*

Propositions 2.2.1 and 2.2.2 and Theorem 2.2.5 yield existence and uniqueness of complex geodesics for k_D and infinitesimal complex geodesics for κ_D in any strongly convex domain $D \subseteq \mathbb{C}^n$ with C^6 boundary. By an approximation process, one gets we the following

Theorem 2.2.6 *Let $D \subseteq \mathbb{C}^n$ be a convex bounded domain. Then*
(a) for any two points $z, w \in D$ there exists a complex geodesic for k_D $\varphi : \Delta \to D$ such that $z, w \in \varphi(\Delta)$;
(b) for any point $z \in D$ and any vector $v \in \mathbb{C}^n$ there exists an infinitesimal complex geodesic for κ_D $\varphi : \Delta \to D$ such that $z = \varphi(0)$ and $v = \lambda\,\varphi'(0)$ for some $\lambda > 0$.

It turns out that complex geodesics for the Carathéodory distance on a strongly convex domain with C^2 boundary are continuous up to the boundary of Δ (in [Jarnicki-Pflug 1993] it has been proved that the same result also holds for strongly pseudoconvex domains with C^2 boundary)

Proposition 2.2.7 *Let $D \subseteq \mathbb{C}^n$ be a strongly convex domain with C^2 boundary. Let $\varphi : \Delta \to D$ be a complex geodesic for c_D. Then $\varphi \in C^{1/2}(\overline{\Delta}, \overline{D})$.*

As far as complex geodesics with prescribed boundary data are concerned, in [Chang-Hu-Lee 1988] and in [Abate 1989 b] the following result is proved

Theorem 2.2.8 *Let $D \subseteq \mathbb{C}^n$ be a strongly convex domain with C^3 boundary. Then for any two points $z, w \in \overline{D}$ there exists a unique (up to automorphisms of Δ) complex geodesic $\varphi : \Delta \to D$ for k_D such that $z, w \in \varphi(\overline{\Delta})$.*

The infinitesimal version of this result is more involved, namely ([Chang-Hu-Lee 1988])

Theorem 2.2.9 *Let $D \subseteq \mathbb{C}^n$ be a strongly convex domain with C^{14} boundary. Let $z \in \partial D$, $i\,v \in T_z^{\mathbb{R}}(\partial D)$ with $v \bullet \overline{\nu(z)} > 0$. Then there exists a unique (up to automorphisms of Δ) infinitesimal complex geodesic $\varphi : \Delta \to D$ for κ_D such that $\varphi(1) = z$ and $\varphi'(1) = v$.*

If the boundary of the domain is sufficiently regular, then the following characterization holds

Theorem 2.2.10 *Let $D \subseteq \mathbb{C}^n$ be a strongly convex domain with C^3 boundary. Then a holomorphic map $\varphi : \Delta \to D$ is a complex geodesic for k_D if, and only if,*
(a) $\varphi \in Hol(\Delta, D) \cap C^{1,1/2}(\overline{\Delta}, \overline{D})$;
(b) $\varphi(\partial\Delta) \subseteq \partial D$;
(c) there exists a continuous function $p \in C^{1/2}(\partial\Delta, \mathbb{R}^+)$ such that the map

$$\begin{aligned} \partial\Delta &\to \mathbb{C}^n \\ \xi \ &\mapsto \xi\, p(\xi)\, \overline{\nu(\varphi(\xi))} \end{aligned}$$

extends to a map $\widetilde{\varphi} \in Hol(\Delta, \mathbb{C}^n) \cap C^{1,1/2}(\overline{\Delta}, \mathbb{C}^n)$.

This result allows to prove a characterization of complex geodesics for the Kobayashi distance which was more or less immediate in the Carathéodory case, namely ([Lempert 1982])

Theorem 2.2.11 *Let* $D \subseteq \mathbb{C}^n$ *be a strongly convex domain with* C^3 *boundary. Then a holomorphic map* $\varphi : \Delta \to D$ *is a complex geodesic for* k_D *if, and only if, there exists a holomorphic map* $\psi : D \to \Delta$ *such that* $\psi \circ \varphi = Id_\Delta$.

Given a complex geodesic $\varphi : \Delta \to D$, the mapping $\psi : D \to \Delta$ appearing in the above Theorem is determined by the equation $(z - \varphi(\psi(z))) \bullet \widetilde{\varphi}(\psi(z)) = 0$ for all $z \in D$, where $\widetilde{\varphi}$ is the holomorphic map defined in Theorem 2.2.10.
A subset S of a domain $D \subseteq \mathbb{C}^n$ is called a holomorphic retract if, and only if, there exists a holomorphic map $r : D \to D$ such that $r(D) \subseteq S$ and $r(z) = z$ for all $z \in S$. Theorem 2.2.11 can be rephrased by saying that the images of complex geodesics on strongly convex domains with C^3 boundary are exactly the one-dimensional holomorphic retracts.
By an approximation process, Theorem 2.2.11 yields the following

Theorem 2.2.12 *Let* $D \subseteq \mathbb{C}^n$ *be a convex domain. Then*

$$c_D = k_D \qquad \text{and} \qquad \gamma_D = \kappa_D.$$

Therefore, on a convex domain, we can discuss about complex geodesics *tout court*.
Recall that if, on a given taut domain $D \subseteq \mathbb{C}^n$, the Carathéodory and the Kobayashi distances concide, then one gets the existence of complex geodesics (Proposition 2.1.4).
In [Dineen-Timoney-Vigué 1985] it has been proved that the Carathéodory and the Kobayashi distances on any convex domain of a locally convex space coincide.
As a consequence of Theorem 2.2.5, Lempert obtained the following result about the regularity of the Kobayashi distance

Theorem 2.2.13 *Let* $D \subseteq \mathbb{C}^n$ *be a strongly convex domain with* C^k *boundary,* $k \geq 6$. *Then the Kobayashi distance*

$$k_D : D \times D - \{ (z, z) \mid z \in D \} \to \mathbb{R}^+$$

is a C^{k-4} *function.*

By using a different approach Royden and Wong obtained a characterization of complex geodesics in a convex bounded domain D without any regularity assumption on the boundary of D ([Royden-Wong 1983]).

The dual Minkowski functional of μ_D is defined as follows

$$\hat{\mu}_D(w) \stackrel{\text{def}}{=} \sup\left\{ \frac{\operatorname{Re}(w \bullet z)}{\mu_D(z)} \mid z \in \mathbb{C}^n - \{0\} \right\} = \max\{ \operatorname{Re}(w \bullet z) \mid z \in \partial D \}$$

for all $w \in \mathbb{C}^n$.
The result stated by Royden and Wong is the following

Theorem 2.2.14 *Let $D \subseteq \mathbb{C}^n$ be a bounded convex domain with $0 \in D$. Then a holomorphic mapping $\varphi : \Delta \to D$ is a complex geodesic if, and only if,*

$$(a) \qquad \varphi^*(\xi) \in \partial D \qquad \text{for a.a. } \xi \in \partial\Delta$$

and there exists $h \in H^1(\Delta, \mathbb{C}^n)$, $h \not\equiv 0$, such that

$$(b) \qquad \operatorname{Re}\left(\varphi^*(\xi) \bullet \frac{1}{\xi} h^*(\xi)\right) = \hat{\mu}_D\left(\frac{1}{\xi} h^*(\xi)\right) \qquad \text{for a.a. } \xi \in \partial\Delta$$

It has been proved that for any $w_o \in \mathbb{C}^n - \{0\}$ the following equation holds (cf. [Jarnicki-Pflug 1993, Remark 8.2.3])

$$\operatorname{Re}(w_o \bullet z_o) = \hat{\mu}_D(w_o)$$

if, and only if, $\overline{w}_o$ is an outer normal vector to ∂D at z_o.
The Minkowski functional μ_D of a convex bounded domain $D \subseteq \mathbb{C}^n$ such that $0 \in D$ turns out to be differentiable almost everywhere (with respect to the Lebesgue measure of ∂D) on ∂D, and therefore such a domain has a unique outer normal vector to ∂D at almost all $w \in \partial D$ (see Appendix). Thus the following theorem gives a satisfactory description of complex geodesics in a convex bounded domain

Theorem 2.2.15 *Let $D \in \mathbb{C}^n$ be a bounded convex domain with $0 \in D$. Let $\varphi : \Delta \to D$ be a holomorphic map such that $\varphi^*(\xi) \in \partial D$ for a.a. $\xi \in \partial\Delta$ and that the unit outer normal vector to ∂D is defined at $\varphi^*(\xi) \in \partial D$ for a.a. $\xi \in \partial\Delta$. Then φ is a complex geodesic if, and only if, there exist $h \in H^1(\Delta, \mathbb{C}^n)$, $h \not\equiv 0$, and $p : \partial\Delta \to \mathbb{R}^+$ such that*

$$(a) \qquad \frac{1}{\xi} h_j^*(\xi) = p(\xi) \frac{\partial \mu_D}{\partial z_j}(\varphi^*(\xi)) \qquad \text{for a.a. } \xi \in \partial\Delta$$

$j = 1, \ldots, n$.

3. A class of functions in $H^1(\Delta)$

In [Gentili 1986 b] it is proved the following

Lemma 2.3.1 *Let $f \in H^1(\Delta)$ be such that*

$$\frac{f^*(\xi)}{\xi} \in \mathbb{R} \qquad \textit{for a.a. } \xi \in \partial\Delta,$$

then there exist $\alpha \in \mathbb{C}$ and $r \in \mathbb{R}$ such that

$$f(\lambda) = \overline{\alpha} + r\,\lambda + \alpha\,\lambda^2 \qquad \lambda \in \Delta.$$

If, in particular, $f^*(\xi)/\xi \in \mathbb{R}^+$ for a.a. $\xi \in \partial\Delta$, then $r \geq 2\,|\alpha|$ and there exist $r_o > 0$ and $\alpha_o \in \overline{\Delta}$ such that

$$f(\lambda) = r_o(\lambda - \alpha_o)(1 - \overline{\alpha}_o\,\lambda)$$

for all $\lambda \in \Delta$.
In the sequel we will use the following generalization of this result

Lemma 2.3.2 *Let $f \in H^1(\Delta)$ be such that*

$$\mathrm{Re}\left(\frac{f^*(\xi)}{\xi}\right) > 0 \qquad \textit{for a.a. } \xi \in \partial\Delta,$$

then there exist $\alpha \in \mathbb{C}$ and $k \in H^1(\Delta)$ such that Re $(k(\lambda)) > 0$ *for all $\lambda \in \Delta$ and that*

$$f(\lambda) = \overline{\alpha} + \lambda\,k(\lambda) - \alpha\,\lambda^2 \qquad \lambda \in \Delta.$$

Proof: Let $f(\lambda) = \overline{\alpha} + g(\lambda)$, where g is such that $g(0) = 0$. Therefore $\psi(\lambda) = g(\lambda)/\lambda \in H^1(\Delta)$. By using this notation we have

$$\frac{f(\lambda)}{\lambda} = \frac{\overline{\alpha}}{\lambda} + \psi(\lambda) \qquad \lambda \in \Delta.$$

Since, by hypothesis, Re $(\overline{\alpha}/\xi + \psi^*(\xi)) > 0$ for a.a. $\xi \in \partial\Delta$, then

$$\mathrm{Re}\left(\frac{\overline{\alpha}}{e^{i\theta}} + \psi^*(e^{i\theta})\right) = \mathrm{Re}\left(\overline{\alpha\, e^{i\theta}} + \psi^*(e^{i\theta})\right) = \mathrm{Re}\left(\alpha\, e^{i\theta} + \psi^*(e^{i\theta})\right) > 0$$

for a.a. $\theta \in \mathbb{R}$. The function $k(\lambda) = \alpha\,\lambda + \psi(\lambda)$, $\lambda \in \Delta$, belongs to $H^1(\Delta)$ and $\mathrm{Re}(k^*(\xi)) > 0$ for a.a. $\xi \in \partial\Delta$. Therefore, by the Poisson integral representation formula, $\mathrm{Re}(k(\lambda)) > 0$ for all $\lambda \in \Delta$. Finally, since $k(\lambda) = \alpha\,\lambda + f(\lambda)/\lambda - \overline{\alpha}/\lambda$, then

$$f(\lambda) = \overline{\alpha} + \lambda\, k(\lambda) - \alpha\,\lambda^2 \qquad\qquad \lambda \in \Delta.$$

QED

We remark that the proof of Lemma 2.3.2 closely follows that of Lemma 2.3.1.
Notice that, if $f(\xi)/\xi \in \mathbb{R}^+$ for a.a. $\xi \in \partial\Delta$, then

$$\frac{\overline{\alpha}}{\xi} + k^*(\xi) - \alpha\,\xi = \overline{\alpha\,\xi} - \alpha\,\xi + k^*(\xi) = -2\,i\,\mathrm{Im}(\alpha\,\xi) + k^*(\xi) \in \mathbb{R}$$

for a.a. $\xi \in \partial\Delta$ and this happens if, and only if,

$$\mathrm{Im}(k^*(\xi)) - 2\,\mathrm{Im}(\alpha\,\xi) = 0$$

for a.a. $\xi \in \partial\Delta$. Therefore $k(\lambda) - 2\,\alpha\,\lambda = c \in \mathbb{R}$ for all $\lambda \in \Delta$ and so $f(\lambda) = \overline{\alpha} + c\,\lambda + \alpha\,\lambda^2$.

4. Complex geodesics in convex balanced domains

The aim of this Section is to find out as much information as possible on the function $p : \partial\Delta \to \mathbb{R}^+$ appearing in Theorem 2.2.15.
Let $\varphi : \Delta \to D$ be a complex geodesic; then Theorem 2.2.15 implies that there exist $h \in H^1(\Delta, \mathbb{C}^n)$, $h \not\equiv 0$, and $p : \partial\Delta \to \mathbb{R}^+$ such that

$$\text{(2.4.1)} \qquad \frac{1}{\xi} h_j^*(\xi) \varphi_j^*(\xi) = p(\xi)\, \frac{\partial \mu_D}{\partial z_j}(\varphi^*(\xi)) \varphi_j^*(\xi) \qquad\qquad j = 1, \ldots, n$$

for a.a. $\xi \in \partial\Delta$. Summing up these equations we get

$$\text{(2.4.2)} \qquad \frac{1}{\xi} \sum_{j=1}^{n} h_j^*(\xi) \varphi_j^*(\xi) = p(\xi) \sum_{j=1}^{n} \frac{\partial \mu_D}{\partial z_j}(\varphi^*(\xi))\, \varphi_j^*(\xi)$$

for a.a. $\xi \in \partial\Delta$. Since, by convexity of D and the fact that $0 \in D$,

$$\mathrm{Re}\left(\sum_{j=1}^{n} \frac{\partial \mu_D}{\partial z_j}(\varphi^*(\xi))\, \varphi_j^*(\xi) \right) = \mathrm{Re}\left(\langle \varphi^*(\xi) - 0, \nu(\varphi^*(\xi)) \rangle \right) > 0$$

then

$$\operatorname{Re}\left(\frac{1}{\xi}\sum_{j=1}^{n} h_j^*(\xi)\varphi_j^*(\xi)\right) > 0$$

for a.a. $\xi \in \partial\Delta$. By Lemma 2.3.2 we have that there exist $\alpha \in \mathbb{C}$ and $k \in H^1(\Delta)$, $k : \Delta \to \{\, \xi \in \mathbb{C} \mid \operatorname{Re}\xi > 0 \,\}$, such that

$$\sum_{j=1}^{n} h_j(\lambda)\varphi_j(\lambda) = \alpha + \lambda\, k(\lambda) - \overline{\alpha}\,\lambda^2$$

for all $\lambda \in \Delta$. Hence we have the equation

$$\alpha\,\overline{\xi} + k^*(\xi) - \overline{\alpha}\,\xi = -2\,i\operatorname{Im}(\overline{\alpha}\,\xi) + k^*(\xi) = p(\xi)\sum_{j=1}^{n}\frac{\partial\mu_D}{\partial z_j}(\varphi^*(\xi))\,\varphi_j^*(\xi) \tag{2.4.3}$$

for a.a. $\xi \in \partial\Delta$, which is equivalent to the following real equations

$$\operatorname{Re}(k^*(\xi)) = p(\xi)\operatorname{Re}\left(\sum_{j=1}^{n}\frac{\partial\mu_D}{\partial z_j}(\varphi^*(\xi))\,\varphi_j^*(\xi)\right)$$

$$-2\operatorname{Im}(\overline{\alpha}\,\xi) + \operatorname{Im}(k^*(\xi)) = p(\xi)\operatorname{Im}\left(\sum_{j=1}^{n}\frac{\partial\mu_D}{\partial z_j}(\varphi^*(\xi))\,\varphi_j^*(\xi)\right)$$

for a.a. $\xi \in \partial\Delta$. Therefore

$$p(\xi) = \frac{\operatorname{Re}(k^*(\xi))}{\operatorname{Re}\left(\sum_{j=1}^{n}\frac{\partial\mu_D}{\partial z_j}(\varphi^*(\xi))\,\varphi_j^*(\xi)\right)} \tag{2.4.4}$$

for a.a. $\xi \in \partial\Delta$, where the function k satisfies the condition

$$\operatorname{Im}(k^*(\xi)) = \frac{\operatorname{Im}\left(\sum_{j=1}^{n}\frac{\partial\mu_D}{\partial z_j}(\varphi^*(\xi))\,\varphi_j^*(\xi)\right)}{\operatorname{Re}\left(\sum_{j=1}^{n}\frac{\partial\mu_D}{\partial z_j}(\varphi^*(\xi))\,\varphi_j^*(\xi)\right)}\operatorname{Re}(k^*(\xi)) + 2\operatorname{Im}(\overline{\alpha}\,\xi)$$

for a.a. $\xi \in \partial\Delta$.
In particular,

$$\frac{1}{\xi}\sum_{j=1}^{n} h_j^*(\xi)\varphi_j^*(\xi) \in \mathbb{R}^+$$

for a.a. $\xi \in \partial\Delta$ if, and only if,

$$\sum_{j=1}^{n}\frac{\partial\mu_D}{\partial z_j}(\varphi^*(\xi))\,\varphi_j^*(\xi) \in \mathbb{R}^+$$

for a.a. $\xi \in \partial\Delta$.

Let us notice that if $D \subseteq \mathbb{C}^n$ is a (convex bounded) domain symmetric with respect to the origin $0 \in \mathbb{C}^n$ (i.e. such that if $z \in D$ then $-z \in D$), then $\mu_D(t\,z) = |t|\mu_D(z)$ for all $z \in \mathbb{C}^n$ and all $t \in \mathbb{R}$. In this case

$$\frac{d}{dt}(\mu_D(t\,z)) = 2\,\mathrm{Re}\sum_{j=1}^{n}\frac{\partial\mu_D}{\partial z_j}(t\,z)z_j$$

$$\frac{d}{dt}(|t|\mu_D(z)) = \frac{t}{|t|}\mu_D(z)$$

for a.a. $z \in \mathbb{C}^n$ and all $t \in \mathbb{R} - \{0\}$. Therefore, for $t = 1$ we get

$$\mathrm{Re}\left(\sum_{j=1}^{n}\frac{\partial\mu_D}{\partial z_j}(z)z_j\right) = \frac{1}{2}\mu_D(z)$$

for a.a. $z \in \mathbb{C}^n$ and the function $p : \partial\Delta \to \mathbb{R}^+$ appearing in equation (2.4.4) becomes

$$p(\xi) = 2\,\mathrm{Re}\,(k^*(\xi))$$

for a.a. $\xi \in \partial\Delta$, where $k \in H^1(\Delta)$ is such that $\mathrm{Re}\,(k(\lambda)) > 0$ for all $\lambda \in \Delta$ and that there exists $\alpha \in \mathbb{C}$ with

$$\mathrm{Im}\,(k^*(\xi)) = 2\left(\mathrm{Im}\left(\sum_{j=1}^{n}\frac{\partial\mu_D}{\partial z_j}(\varphi^*(\xi))\,\varphi_j^*(\xi)\right)\mathrm{Re}\,(k^*(\xi)) + \mathrm{Im}(\overline{\alpha}\,\xi)\right)$$

for a.a. $\xi \in \partial\Delta$.
We collect these results in the following

Theorem 2.4.1 *Let $D \in \mathbb{C}^n$ be a bounded convex domain with $0 \in D$. Let $\varphi : \Delta \to D$ be a holomorphic map such that $\varphi^*(\xi) \in \partial D$ for a.a. $\xi \in \partial\Delta$ and that the unit outer normal vector to ∂D is defined at $\varphi^*(\xi) \in \partial D$ for a.a. $\xi \in \partial\Delta$. Then $\varphi : \Delta \to D$ is a complex geodesic if, and only if, there exist $h \in H^1(\Delta, \mathbb{C}^n)$, $h \not\equiv 0$, $k : \Delta \to \{\ \xi \in \mathbb{C} \mid \mathrm{Re}\,\xi > 0\ \}$, $k \in H^1(\Delta)$ and $\alpha \in \mathbb{C}$ such that*

(a) $$\frac{1}{\xi}h_j^*(\xi) = \frac{\mathrm{Re}\,(k^*(\xi))}{\mathrm{Re}\,(\sum_{j=1}^{n}\frac{\partial\mu_D}{\partial z_j}(\varphi^*(\xi))\,\varphi_j^*(\xi))}\frac{\partial\mu_D}{\partial z_j}(\varphi^*(\xi)) \qquad \text{for a.a. } \xi \in \partial\Delta$$

$j = 1, \ldots, n$, and
(b)

$$\mathrm{Im}(k^*(\xi)) = \frac{\mathrm{Im}(\sum_{j=1}^{n}\frac{\partial\mu_D}{\partial z_j}(\varphi^*(\xi))\,\varphi_j^*(\xi))}{\mathrm{Re}(\sum_{j=1}^{n}\frac{\partial\mu_D}{\partial z_j}(\varphi^*(\xi))\,\varphi_j^*(\xi))}\mathrm{Re}\,(k^*(\xi)) + 2\,\mathrm{Im}(\overline{\alpha}\,\xi) \qquad \text{for a.a. } \xi \in \partial\Delta.$$

If, moreover, D is symmetric with respect to the origin $0 \in \mathbb{C}^n$, then

$$\mathrm{Re}\left(\sum_{j=1}^{n} \frac{\partial \mu_D}{\partial z_j}(\varphi^*(\xi))\,\varphi_j^*(\xi)\right) = \frac{1}{2} \qquad \textit{for a.a. } \xi \in \partial\Delta.$$

Finally, let us suppose $D \subseteq \mathbb{C}^n$ be a convex bounded and balanced domain, i.e. the domain D is such that $\mu_D(\lambda z) = |\lambda|\mu_D(z)$ for all $z \in \mathbb{C}^n$ and all $\lambda \in \mathbb{C}$.
We have

$$\frac{d}{d\lambda}(\mu_D(\lambda z)) = \sum_{j=1}^{n} \frac{\partial \mu_D}{\partial z_j}(\lambda z) z_j$$

$$\frac{d}{d\lambda}(|\lambda|\mu_D(z)) = \frac{\overline{\lambda}}{2|\lambda|}\mu_D(z)$$

for a.a. $z \in \mathbb{C}^n$ and all $\lambda \in \mathbb{C}$. Therefore, for $\lambda = 1$, we get

$$\sum_{j=1}^{n} \frac{\partial \mu_D}{\partial z_j}(z) z_j = \frac{1}{2}\mu_D(z)$$

for a.a. $z \in \mathbb{C}^n$.
Let $\varphi : \Delta \to D$ be a complex geodesic. Since $\varphi^*(\xi) \in \partial D$ for a.a. $\xi \in \partial\Delta$, then equation (2.4.2) becomes in this case

$$\frac{1}{\xi}\sum_{j=1}^{n} h_j^*(\xi)\varphi_j^*(\xi) = \frac{1}{2}\,p(\xi)\,\mu_D(\varphi^*(\xi)) = \frac{1}{2}\,p(\xi) > 0$$

for a.a. $\xi \in \partial\Delta$. Therefore, by Lemma 2.3.1

$$\sum_{j=1}^{n} h_j(\lambda)\varphi_j(\lambda) = r_o(\lambda - \alpha_o)(1 - \overline{\alpha}_o\lambda) \qquad \lambda \in \Delta \tag{2.4.5}$$

for some $r_o > 0$ and $\alpha_o \in \overline{\Delta}$, and consequently

$$p(\xi) = 2\,r_o|1 - \overline{\alpha}_o\xi|^2 \qquad \text{for a.a. } \xi \in \partial\Delta.$$

So, we can state the following

Theorem 2.4.2 *Let $D \Subset \mathbb{C}^n$ be a convex bounded and balanced domain. Let $\varphi : \Delta \to D$ be a holomorphic map such that $\varphi^*(\xi) \in \partial D$ for a.a. $\xi \in \partial\Delta$ and that the unit outer normal* *vector to ∂D is defined at $\varphi^*(\xi) \in \partial D$ for a.a. $\xi \in \partial\Delta$. Then $\varphi : \Delta \to D$ is a complex geodesic if, and only if, there exist $h \in H^1(\Delta, \mathbb{C}^n)$, $h \not\equiv 0$, $r_o > 0$ and $\alpha_o \in \overline{\Delta}$, such that*

$$(a) \qquad \frac{1}{\xi} h_j^*(\xi) = r_o |1 - \overline{\alpha}_o \xi|^2 \frac{\partial \mu_D}{\partial z_j}(\varphi^*(\xi)) \qquad \textit{for a.a. } \xi \in \partial\Delta$$

$j = 1, \ldots, n$.

Example 2.4.3 As a simple consequence of this Theorem we can find a large family of complex geodesics through the origin of a convex bounded balanced domain $D \subseteq \mathbb{C}^n$. Let $z \in \partial D$ be such that there exists $\nu(z)$. Then the holomorphic map

$$\varphi : \Delta \to D$$
$$\lambda \mapsto \lambda z$$

is a complex geodesic in D.
First of all, since, $\mu_D(\lambda z) = |\lambda| \mu_D(z)$, then $\varphi(\Delta) \subseteq D$ and $\varphi^*(\partial\Delta) \subseteq \partial D$. One has that

$$(2.4.6) \qquad \frac{\partial}{\partial z_j} \mu_D(\lambda z) = |\lambda| \frac{\partial \mu_D}{\partial z_j}(z)$$

for all $\lambda \in \mathbb{C}$. Therefore the unit outer normal vector $\nu(\varphi^*(\xi))$ is defined at $\varphi^*(\xi)$ for all $\xi \in \partial\Delta$. Thus the hypotheses of Theorem 2.4.2 are verified and we can look for $h \in H^1(\Delta, \mathbb{C}^n)$, $h \not\equiv 0$ and $\alpha_o \in \overline{\Delta}$ satisfying condition (a).
Since $\varphi(0) = 0$, then, by equation (2.4.5), we have $\alpha_o = 0$. It turns out that

$$\frac{\partial}{\partial z_j} \mu_D(\lambda z) = \frac{\partial \mu_D}{\partial z_j}(\lambda z)\lambda$$

for all $\lambda \in \mathbb{C}$. By equation (2.4.6) it follows that

$$\frac{\partial \mu_D}{\partial z_j}(\lambda z) = \frac{|\lambda|}{\lambda} \frac{\partial \mu_D}{\partial z_j}(z)$$

for all $\lambda \in \mathbb{C} - \{0\}$. Therefore φ satisfies condition (a) in Theorem 2.4.2 as soon as we define $r_o = 1$ and

$$h_j(\lambda) = \frac{\partial \mu_D}{\partial z_j}(z) \qquad \lambda \in \Delta$$

for all $j = 1, \ldots, n$.
We remark that the same result follows from the Hahn-Banach Theorem without assuming the existence of the unit outer normal vector $\nu(z)$ to ∂D at z (cf. [Vesentini 1982 a]). However, we believe that this approach will be useful also in the study of those complex geodesics which do not pass through the origin.

5. Uniqueness of complex geodesics

Since uniqueness of complex geodesics is important in many applications, in what follows we give a brief account on this question

Let $D \subseteq \mathbb{C}^n$ be a strictly convex bounded domain. Let $z, w \in D$. Suppose that there exist two different complex geodesics $\varphi_\circ, \varphi_1 : \Delta \to D$ such that

$$\varphi_\circ(0) = \varphi_1(0) = z$$

$$\varphi_\circ(r) = \varphi_1(r) = w$$

for some $r \in (0,1)$. For all $t \in [0,1]$, define the holomorphic map $\varphi_t = t\varphi_1 + (1-t)\varphi_\circ$. By convexity of D, it turns out that $\varphi_t : \Delta \to D$. Moreover, since

$$c_D(\varphi_t(0), \varphi_t(r)) = c_D(z, w) = \omega(0, r),$$

then φ_t is a complex geodesic. The boundary values of a complex geodesic cannot belong to the interior of D, therefore we can improve a little Theorem 2.2.6 by stating the following ([Dineen 1989])

Theorem 2.5.1 *Let $D \subseteq \mathbb{C}^n$ be a strictly convex bounded domain. Then for any $z, w \in D$ there exists a unique (up to automorphisms of Δ) complex geodesic through z and w.*

As far as uniqueness of complex geodesics through the origin of a convex bounded balanced domain $D \subseteq \mathbb{C}^n$ is concerned, the notion of complex extreme point comes on stage.

Let $D \subseteq \mathbb{C}^n$ be an arbitrary domain. A point $z \in \overline{D}$ is called a complex extreme point if $w = 0$ is the only vector such that $z + \xi w \in \overline{D}$ for all $\xi \in \Delta$.

Notice that if D is a strictly convex domain, then every boundary point of D is a complex extreme point of $\overline{D}$. However, the domain $\mathcal{E} = \{\ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1| + |z_2| < 1\ \}$ is not strictly convex, nevertheless every boundary point of $\mathcal{E}$ is a complex extreme point of $\overline{\mathcal{E}}$.

A boundary point $w = (w_1, \ldots, w_n)$ of the polydisc $\Delta^n = \{\ z \in \mathbb{C}^n \mid \max_{j=1,\ldots n} |z_j| < 1\ \}$ is not a complex extreme point whenever $|w_j| < 1$ for some $j = 1, \ldots, n$.

In [Vesentini 1982 a] the following result has been proved

Proposition 2.5.2 *Let $D \subseteq \mathbb{C}^n$ be a convex bounded balanced domain. Let μ_D be the Minkowski functional of D. Let $z \in D$. Then the holomorphic mapp*

$$\varphi : \Delta \to D$$
$$\lambda \mapsto \lambda \frac{z}{\mu_D(z)}$$

is the unique (up to automorphisms of Δ) complex geodesic for c_D whose image contains $0 \in D$ and z if, and only if, $\varphi(\overline{\Delta}) \cap \partial D$ contains a complex extreme point of $\overline{D}$.

Further investigation about the relation between non-uniqueness of complex geodesics and complex extreme points has been carried out in [Gentili 1986 a]. Let $D \subseteq \mathbb{C}^n$ be a convex bounded balanced domain. Let μ_D be the Minkowski functional of D. Let $z \in D$. Define $\mathcal{M}(z)$ to be the family of all convex balanced subsets P of $\mathbb{C}^n$ such that $z + P \subseteq \overline{D}$. The Zorn Lemma implies that there exists a maximal element (with respect to inclusion) $M(z)$ of the family $\mathcal{M}(z)$. The "amount" of complex geodesics through the origin and z depends on the "size" $M(z)$ as the following result explains ([Gentili 1986 a])

Proposition 2.5.3 *Let $D \subseteq \mathbb{C}^n$ be a convex bounded balanced domain. Let μ_D be the Minkowski functional of D. Let $z \in D$. If, either $h : \Delta \to M\left(\frac{z}{\mu_D(z)}\right)$ is a holomorphic map having at least two zeroes in Δ, or $h : \Delta \to M\left(\frac{z}{\mu_D(z)}\right)$ is a holomorphic map such that $h(0) = 0$, or $h : \Delta \to t\,M\left(\frac{z}{\mu_D(z)}\right)$ is a holomorphic map and $t \in (0,1)$; then the holomorphic map*

$$\begin{aligned} \varphi &: \Delta \to D \\ \lambda &\mapsto \lambda \frac{z}{\mu_D(z)} + h(\lambda) \end{aligned}$$

is a complex geodesic.

A consequence of the above result is the following: if there exists a point $z_\circ \in \partial D$ on the boundary of a convex bounded balanced domain $D \subseteq \mathbb{C}^n$ such that $M(z_\circ) \neq \{ 0 \}$, then there are complex geodesics of D which are not continuous on $\overline{\Delta}$.
In [Gentili 1985] it has been proved that, in some sense, non-uniqueness of complex geodesics is a "global" property, namely

Proposition 2.5.4 *Let $D \subseteq \mathbb{C}^n$ be a convex bounded domain. Let $\varphi : \Delta \to D$ be a complex geodesic. Then the following facts are equivalent:*
(a) there exist $\lambda_1 \neq \lambda_2 \in \Delta$ such that φ is not the unique complex geodesic through $\varphi(\lambda_1)$ and $\varphi(\lambda_2)$;
(b) there exists $\lambda_\circ \in \Delta$ such that φ is not the unique complex geodesic tangent to $\varphi'(\lambda_\circ)$ at the point $\varphi(\lambda_\circ)$;
(c) for all $\xi \neq \eta \in \Delta$ φ is not the unique complex geodesic of D through $\varphi(\xi)$ and $\varphi(\eta)$;
(d) for all $\lambda \in \Delta$ φ is not the unique complex geodesic of D tangent to $\varphi'(\lambda)$ at $\varphi(\lambda)$.

Third Chapter

HOW TO COMPUTE COMPLEX GEODESICS IN REINHARDT DOMAINS

We have seen that, for any two points in a convex bounded domain, there exists a complex geodesic whose image contains them. As far as explicit examples are concerned, a satisfactory description of the complex geodesics through the origin of a convex balanced domain has been obtained ([Vesentini 1982 a], [Gentili 1986 a]) and effective formulas for the whole family of complex geodesics have been worked out in some cases, e.g. convex complex ellipsoids and some of their possible generalizations ([Blank et alia], [Gentili 1986 b], [Jarnicki-Pflug 1995], [Jarnicki-Pfulg-Zeinstra 1993], [Zwonek 1995 a], [Visintin 1995]). However, a general method to determine explicitly the family of all complex geodesics of a given convex bounded domain is not available at the moment.

In this chapter, we generalize and extend to an arbitrary convex bounded Reinhardt domain the particular procedures used to solve the question of describing complex geodesics in the cases quoted above.

The main statement proved is Theorem 3.2.1, which characterizes complex geodesics by means of a system of equations involving the module at the boundary of the single components and the unit outer normal vector to the boundary of the domain.

This result gives a method (simpler than that furnished by Theorem 2.4.2) to check whether a holomorphic map from the unit disc Δ into a convex bounded Reinhardt domain is or not a complex geodesic, and can be used to find explicit formulas for complex geodesics in such domains. Moreover, some of its consequences give a better insight of the behaviour of complex geodesics. For example, we have proved that each component of a complex geodesic in a strictly convex bounded Reinhardt domain may have at most one zero in Δ and that, for some particular families of convex bounded Reinhardt domains, the knowledge of the complex geodesics of one domain of the family suffices to determine the complex geodesics of all other domains belonging to that family.

1. Complex geodesics in Reinhardt domains

Let D be a convex and bounded Reinhardt domain. Let us recall that D is a Reinhardt domain if, and only if, the following condition is satisfied:

$$\text{if}\quad (z_1,\ldots,z_n)\in D\quad\text{then}\quad (e^{i\theta_1}z_1,\ldots,e^{i\theta_n}z_n)\in D\quad\text{for all}\quad \theta_1,\ldots,\theta_n\in\mathbb{R}.$$

A convex Reinhardt domain is obviously a balanced domain, therefore Theorem 2.4.2 applies to this case too.

By the definition of Minkowski functional it immediately follows that

(*) D is a Reinhardt domain if, and only if,

$$\mu_D((z_1,\ldots,z_n))=\mu_D((|z_1|,\ldots,|z_n|))$$

for all $z\in C^n$;

(**) if D is a convex Reinhardt domain and $(z_1,\ldots,z_n)\in D$ then $(\lambda_1 z_1,\ldots,\lambda_n z_n)\in D$ for all $\lambda_1,\ldots,\lambda_n\in\overline{\Delta}$; therefore $\mu_D((\lambda_1 z_1,\ldots,\lambda_n z_n))\le\mu_D((z_1,\ldots,z_n))$ for all $z\in\mathbb{C}^n$ and for all $\lambda_1,\ldots,\lambda_n\in\overline{\Delta}$.

By (*) we can compute, for a.a. $z\in\mathbb{C}^n$,

$$\frac{\partial}{\partial z_j}\mu_D((z_1,\ldots,z_n))=\frac{\partial}{\partial z_j}\mu_D((|z_1|,\ldots,|z_n|))=\frac{\partial\mu_D}{\partial|z_j|}(|z_1|,\ldots,|z_n|)\frac{\overline{z}_j}{2|z_j|}$$

and

$$\frac{1}{2}\mu_D(z)=\sum_{j=1}^n\frac{\partial\mu_D}{\partial z_j}(z)z_j=\sum_{j=1}^n\frac{\partial\mu_D}{\partial|z_j|}(|z_1|,\ldots,|z_n|)\frac{\overline{z}_j}{2|z_j|}z_j=\frac{1}{2}\sum_{j=1}^n\frac{\partial\mu_D}{\partial|z_j|}(|z_1|,\ldots,|z_n|)|z_j|.$$

Therefore

$$\sum_{j=1}^n\frac{\partial\mu_D}{\partial|z_j|}(|z_1|,\ldots,|z_n|)|z_j|=\mu_D(z)\tag{3.1.1}$$

for a.a. $z\in(\mathbb{C}-\{0\})^n$. By (**) it follows that

$$\frac{\partial\mu_D}{\partial|z_j|}(|z_1|,\ldots,|z_n|)\ge 0$$

for a.a. $z \in \mathbb{C}^n$. The following result clarifies the behaviour of the partial derivatives $\partial\mu_D/\partial|z_j|$.

Lemma 3.1.1 *Let $D \subseteq \mathbb{C}^n$ be a convex bounded Reinhardt domain. Let $\widetilde{z} \in \partial D$ be such that $\nu(\widetilde{z})$ is defined, $\frac{\partial\mu_D}{\partial|z_j|}(\widetilde{z}) = 0$ and $\widetilde{z}_j \neq 0$.*
Then D is not strictly convex at $\widetilde{z} \in \partial D$.
Precisely, $(\widetilde{z}_1, \ldots, \widetilde{z}_{j-1}, t\, z_j + (1-t)\, \widetilde{z}_j, \widetilde{z}_{j+1}, \ldots, \widetilde{z}_n) \in \partial D$ for all $t \in [0,1]$ and all $|z_j| < |\widetilde{z}_j|$.

Proof: Convexity of μ_D implies

$$\mu_D(z) \geq \mu_D(\widetilde{z}) + \mathrm{Re}(\langle z - \widetilde{z}, \nu(\widetilde{z})\rangle)$$

for all $z \in \mathbb{C}^n$. Let $z = (z_1, \ldots, z_n)$ be such that $z_k = \widetilde{z}_k$ for all $k \neq j$ and $|z_j| < |\widetilde{z}_j|$. Therefore $\mu_D(z) \leq \mu_D(\widetilde{z}) = 1$, i.e. $z \in \overline{D}$. Let $w(t) = t\, z + (1-t)\widetilde{z} = (\widetilde{z}_1, \ldots, \widetilde{z}_{j-1}, t\, z_j + (1-t)\, \widetilde{z}_j, \widetilde{z}_{j+1}, \ldots, \widetilde{z}_n)$ for all $t \in [0,1]$. Obviously $w(t) \in \overline{D}$ and

$$\mu_D(w(t)) \geq \mu_D(\widetilde{z}) + \mathrm{Re}\left(\sum_{k=1}^{n} \frac{\partial\mu_D}{\partial z_k}(\widetilde{z})(w_k(t) - \widetilde{z}_k)\right) = \mu_D(\widetilde{z}) = 1$$

for all $t \in [0,1]$. Therefore

$$1 = \mu_D(\widetilde{z}) \leq \mu_D(w(t)) = t\, \mu_D(z) + (1-t)\mu_D(\widetilde{z}) \leq 1$$

and so $w(t)$ and z are in ∂D.

QED

Let us define the holomorphic maps

$$P_k : \mathbb{C}^n \to \mathbb{C}^{n-1}$$
$$(z_1, \ldots, z_n) \mapsto (z_1, \ldots, \hat{z}_k, \ldots, z_n)$$

and

$$I_k : \mathbb{C}^{n-1} \to \mathbb{C}^n$$
$$(z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_n) \mapsto (z_1, \ldots, z_{k-1}, 0, z_{k+1}, \ldots, z_n)$$

Lemma 3.1.2 *Let $D \subseteq \mathbb{C}^n$ be a convex bounded Reinhardt domain. Let $\varphi : \Delta \to D$ be a holomorphic map such that $\varphi_k \equiv 0$. Then φ is a complex geodesic in D if, and only if, $P_k \circ \varphi$ is a complex geodesic in $P_k(D)$.*

Proof: Let $\varphi : \Delta \to D$ be a complex geodesic in D such that $\varphi_k \equiv 0$. Since $I_k : P_k(D) \to D$, then, for all $\lambda, \xi \in \Delta$,

$$\omega(\lambda, \xi) = c_D(\varphi(\lambda), \varphi(\xi)) = c_D((I_k \circ P_k \circ \varphi)(\lambda), (I_k \circ P_k \circ \varphi)(\xi)) \leq$$

$$\leq c_{P_k(D)}((P_k \circ \varphi)(\lambda), (P_k \circ \varphi)(\xi)) \leq \omega(\lambda, \xi).$$

Therefore $P_k \circ \varphi$ is a complex geodesic in $P_k(D)$.
Viceversa, let $P_k \circ \varphi$ be a complex geodesic in $P_k(D)$. Since $P_k : D \to P_k(D)$, then, for all $\lambda, \xi \in \Delta$,

$$\omega(\lambda, \xi) = c_{P_k(D)}((P_k \circ \varphi)(\lambda), (P_k \circ \varphi)(\xi)) \leq c_D(\varphi(\lambda), \varphi(\xi)) \leq \omega(\lambda, \xi),$$

i.e. φ is a complex geodesic in D.

QED

By this Lemma, it suffices to study the complex geodesics $\varphi : \Delta \to D$ of D such that $\varphi_j \not\equiv 0$ for all $j = 1, \ldots, n$.
The general situation goes as follows.
Define the holomorphic map

$$I_{c_k} : \mathbb{C}^{n-1} \to \mathbb{C}^n$$
$$(z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_n) \mapsto (z_1, \ldots, z_{k-1}, c_k, z_{k+1}, \ldots, z_n)$$

where $c_k \in \mathbb{C}$ is a complex constant.
Let $D \subseteq \mathbb{C}^n$ be a domain and $D_{c_k} \overset{\text{def}}{=} D \cap \{ z \in \mathbb{C}^n \mid z_k = c_k \}$.

Lemma 3.1.2′ *Let $D \subseteq \mathbb{C}^n$ be a domain. Let $\varphi : \Delta \to D$ be a holomorphic map in D such that $\varphi_k \equiv c_k$, i.e. $\varphi = I_{c_k} \circ P_k \circ \varphi$. Then*
(a) if φ is a complex geodesic in D, then $P_k \circ \varphi$ is a complex geodesic in $P_k(D_{c_k})$;
(b) if $P_k \circ \varphi$ is a complex geodesic in $P_k(D_{c_k})$ (or in $P_k(D)$), then φ is a complex geodesic in D_{c_k} (or in D).

Proof: (a) We have

$$\omega(\lambda, \xi) = c_D(\varphi(\lambda), \varphi(\xi)) = c_D((I_{c_k} \circ P_k \circ \varphi)(\lambda), (I_{c_k} \circ P_k \circ \varphi(\xi)) \leq$$

$$\leq c_{P_k(D_{c_k})}(P_k(\varphi(\lambda)), P_k(\varphi(\xi))) \leq \omega(\lambda, \xi)$$

for all $\lambda, \xi \in \Delta$.

(b) A computation leads to

$$\omega(\lambda,\xi) = c_{P_k(D_{c_k})}(P_k(\varphi(\lambda)), P_k(\varphi(\xi))) \leq$$
$$c_{D_{c_k}}(\varphi(\lambda),\varphi(\xi)) \leq \omega(\lambda,\xi)$$

QED

If D is a convex Reinhardt domain, then $P_k(D_k) = P_k(D)$, where $D_k = D_{0_k}$, and Lemma 3.1.2 can be seen as a consequence of Lemma 3.1.2$'$.

2. Characterization of complex geodesics in convex bounded Reinhardt domains

The following Theorem characterizes the complex geodesics $\varphi = (\varphi_1,\ldots,\varphi_n) : \Delta \to D$ of a convex bounded Reinhardt domain D such that $\varphi_j \not\equiv 0$, for all $j = 1,\ldots,n$, and that the unit outer normal vector $\nu(\varphi^*(\xi)) = (\nu_1(\varphi^*(\xi)),\ldots,\nu_n(\varphi^*(\xi)))$ to ∂D is defined at $\varphi^*(\xi)$ for a.a. $\xi \in \partial\Delta$. Remark that, by Lemma 3.1.2, and since the unit outer normal vector $\nu(z)$ is defined at almost all points z belonging to the boundary of a convex domain, then neither the former assumption nor the latter is very restrictive.
In the proof of the Theorem we will use the Factorization Theorem for holomorphic functions belonging to $H^p(\Delta)$ (see Appendix).
It turns out that the moduli of the outer factors of the φ_j's solve a system of equations involving $\nu(\varphi^*(\xi))$, $\varphi(\xi)$ and functions such as $r_j|1-\overline{\alpha}_j\xi|^2/|1-\overline{\alpha}_o\xi|^2$, where $r_j \geq 0$, $\alpha_j, \alpha_o \in \overline{\Delta}$.
As far as the inner factors M_j's of the φ_j's are concerned, one has that, if $\nu_j(\varphi^*(\xi)) \neq 0$ for a.a. $\xi \in \partial\Delta$, then M_j is either constant or a Möbius transformation. If, on the contrary, $\nu_j(\varphi^*(\xi)) = 0$ for a.a. $\xi \in \partial\Delta$, then we have no information on the inner function M_j.
Finally, condition (b) ensures that $\varphi^*(\xi) \in \partial D$ for a.a. $\xi \in \partial\Delta$.

Theorem 3.2.1 *Let $D \subseteq \mathbb{C}^n$ be a convex bounded Reinhardt domain. Let $\varphi = (\varphi_1,\ldots,\varphi_n)$: $\Delta \to D$ be a non-constant holomorphic map such that $\varphi_j \not\equiv 0$ for all $j = 1,\ldots,n$ and that the boundary values $\varphi^*(\xi)$ belong to ∂D for a.a. $\xi \in \partial\Delta$. Let M_j be the inner factor of φ_j. Moreover, suppose that the unit outer normal vector to ∂D is defined at $\varphi^*(\xi)$ for a.a. $\xi \in \partial\Delta$.*
Then φ is a complex geodesic if, and only if, there exist $\alpha_o,\alpha_1,\ldots,\alpha_n \in \overline{\Delta}$, $r_1,\ldots,r_n \geq 0$ such that

$$(a) \qquad r_j\frac{|1-\overline{\alpha}_j\xi|^2}{|1-\overline{\alpha}_o\xi|^2} = \frac{\partial\mu_D}{\partial|z_j|}(|\varphi_1^*(\xi)|,\ldots,|\varphi_n^*(\xi)|)|\varphi_j^*(\xi)| \qquad \text{for a.a. } \xi \in \partial\Delta$$

for all $j = 1, \ldots, n$*;*

$$(b) \qquad \alpha_o = \sum_{j=1}^{n} r_j \alpha_j \qquad\qquad 1 + |\alpha_o|^2 = \sum_{j=1}^{n} r_j (1 + |\alpha_j|^2).$$

If $r_j > 0$*, then there exists* $\theta_j \in \mathbb{R}$ *such that*

$$(c) \qquad M_j(\lambda) = e^{i\,\theta_j} \left(\frac{\lambda - \alpha_j}{1 - \overline{\alpha}_j \lambda} \right)^{s_j} \qquad\qquad \lambda \in \Delta,$$

where $s_j \in \{ 0, 1 \}$ *and* $s_j = 1$ *implies* $\alpha_j \in \Delta$*.*

Proof: Let $\varphi : \Delta \to D$ be a complex geodesic such that $\varphi_j \not\equiv 0$ for all $j = 1, \ldots, n$. Then, by Theorem 2.4.2, there exist $h \in H^1(\Delta, \mathbb{C}^n)$, $h \not\equiv 0$, $r_o > 0$ and $\alpha_o \in \overline{\Delta}$, such that

$$\frac{1}{\xi} h_j^*(\xi) = r_o |1 - \overline{\alpha}_o \xi|^2 \frac{\partial \mu_D}{\partial |z_j|} (|\varphi_1^*(\xi)|, \ldots, |\varphi_n^*(\xi)|) \frac{\overline{\varphi_j^*}(\xi)}{2|\varphi_j^*(\xi)|} \qquad\qquad j = 1, \ldots, n$$

for a.a. $\xi \in \partial\Delta$. Since

$$\frac{\partial \mu_D}{\partial |z_j|} (|z_1|, \ldots, |z_n|) \geq 0$$

for a.a. $z \in \mathbb{C}^n$, then we have that

(3.1.2)

$$\frac{1}{\xi} h_j^*(\xi) \varphi_j^*(\xi) = \frac{r_o}{2} |1 - \overline{\alpha}_o \xi|^2 \frac{\partial \mu_D}{\partial |z_j|} (|\varphi_1^*(\xi)|, \ldots, |\varphi_n^*(\xi)|) |\varphi_j^*(\xi)| \geq 0 \qquad\qquad j = 1, \ldots, n$$

for a.a. $\xi \in \partial\Delta$. It is not restrictive to suppose $r_o = 2$.
Let H_j be the outer factor of $h_j \varphi_j$. If

$$\frac{\partial \mu_D}{\partial |z_j|} (|\varphi_1^*(\xi)|, \ldots, |\varphi_n^*(\xi)|) = 0$$

for all ξ in a subset Λ of positive measure in $\partial\Delta$, then $|H_j(\xi)| = 0$ for all $\xi \in \Lambda$. Therefore ([Rudin 1974], Thm. 17.18) it must be $H_j(\lambda) \equiv 0$, for all $\lambda \in \Delta$. Since $\varphi_j \not\equiv 0$, then $h_j(\lambda) \equiv 0$, for all $\lambda \in \Delta$, and we have no information at all on the inner factor of φ_j. Notice that in this case we have that necessarily

$$\frac{\partial \mu_D}{\partial |z_j|} (|\varphi_1^*(\xi)|, \ldots, |\varphi_n^*(\xi)|) = 0$$

for a.a. $\xi \in \partial\Delta$.
On the other hand, if

$$\frac{\partial \mu_D}{\partial |z_j|} (|\varphi_1^*(\xi)|, \ldots, |\varphi_n^*(\xi)|) > 0$$

for a.a. $\xi \in \partial\Delta$, then, by Lemma 2.3.1,

$$h_j(\lambda)\,\varphi_j(\lambda) = r_j\,(\lambda - \alpha_j)(1 - \overline{\alpha}_j\,\lambda) \qquad\qquad \lambda \in \Delta$$

for some $r_j > 0$ and some $\alpha_j \in \overline{\Delta}$. Therefore in this latter case both φ_j and h_j has at most one zero at $\alpha_j \in \overline{\Delta}$ and their singular factors are identically equal to 1.

Let us set $r_j = 0$ for those $j \in \{1, \ldots, n\}$ such that $\frac{\partial \mu_D}{\partial |z_j|}(|\varphi_1^*(\xi)|, \ldots, |\varphi_n^*(\xi)|) = 0$ for a.a. $\xi \in \partial\Delta$. By using this convention, we have the equation

$$\sum_{j=1}^{n} r_j\,(\lambda - \alpha_j)(1 - \overline{\alpha}_j\,\lambda) = (\lambda - \alpha_o)(1 - \overline{\alpha}_o\,\lambda) \qquad \text{for all } \lambda \in \Delta.$$

It follows that either $\alpha_o \in \Delta$ or $\alpha_j = \alpha_o \in \partial\Delta$ for all $j = 1, \ldots, n$. Moreover, we get the conditions

$$\sum_{j=1}^{n} r_j\,\alpha_j = \alpha_o \qquad \sum_{j=1}^{n} r_j(1 + |\alpha_j|^2) = (1 + |\alpha_o|^2).$$

Finally, system (3.1.2) becomes

$$r_j|1 - \overline{\alpha}_j\xi|^2 = |1 - \overline{\alpha}_o\xi|^2\,\frac{\partial \mu_D}{\partial |z_j|}(|\varphi_1^*(\xi)|, \ldots, |\varphi_n^*(\xi)|)|\varphi_j^*(\xi)| \geq 0 \qquad\qquad j = 1, \ldots, n$$

for a.a. $\xi \in \partial\Delta$.

Viceversa, let $\varphi : \Delta \to D$ be a non-constant holomorphic map satisfying conditions (a), (b) and (c).

By (a), (b) and equation (3.1.1) it follows that $\varphi^*(\xi) \in \partial D$ for a.a. $\xi \in \partial\Delta$.

We will prove that φ satisfies condition (a) in Theorem 2.4.2.

If $r_j = 0$ we set $h_j(\lambda) \equiv 0$, $\lambda \in \Delta$.

Now, suppose $r_j > 0$.

By Theorem 2.1 (e) (Appendix), it follows that there exists $0 < \beta_j < \infty$ such that

$$\frac{\partial \mu_D}{\partial |z_j|}(|\varphi_1^*(\xi)|, \ldots, |\varphi_n^*(\xi)|) \leq \beta_j$$

for a.a. $\xi \in \partial\Delta$. Therefore the function

$$g(\xi) = \frac{1}{2}|1 - \overline{\alpha}_o\xi|^2\,\frac{\partial \mu_D}{\partial |z_j|}(|\varphi_1^*(\xi)|, \ldots, |\varphi_n^*(\xi)|)$$

defined for a.a. $\xi \in \partial\Delta$ belongs to $L^\infty(\partial\Delta)$ and we can set ([Rudin 1974], Thm. 17.16) H_j as the outer function determined by

$$|H_j^*(\xi)| = g(\xi) = \frac{1}{2}|1 - \overline{\alpha}_o\xi|^2\,\frac{\partial \mu_D}{\partial |z_j|}(|\varphi_1^*(\xi)|, \ldots, |\varphi_n^*(\xi)|)$$

for a.a. $\xi \in \partial\Delta$. It turns out that $H_j \in H^\infty(\Delta)$.
If $s_j = 1$ let

$$h_j(\lambda) = e^{-i\theta_j} H_j(\lambda) \qquad\qquad \lambda \in \Delta.$$

If $s_j = 0$ and $\alpha_j \in \Delta$ let

$$h_j(\lambda) = e^{-i\theta_j} \frac{\lambda - \alpha_j}{1 - \overline{\alpha}_j\,\lambda} H_j(\lambda) \qquad\qquad \lambda \in \Delta.$$

Finally, if $s_j = 0$ and $\alpha_j \in \partial\Delta$ let

$$h_j(\lambda) = -e^{-i\theta_j} \alpha_j\, H_j(\lambda) \qquad\qquad \lambda \in \Delta.$$

By (i) and the definition of h_j it follows that

$$|h_j^*(\xi)\, \varphi_j^*(\xi)| = \frac{1}{2} r_j\, |1 - \overline{\alpha}_j\, \xi|^2$$

for a.a. $\xi \in \partial\Delta$. Therefore the outer factor of $h_j\varphi_j$ is $\frac{1}{2} r_j\, (1 - \overline{\alpha}_j\, \lambda)^2$, $\lambda \in \Delta$. Moreover, by the definitions, one has that the inner factor of $h_j\varphi_j$ is either the Möbius transformation $B(\lambda) = \frac{\lambda - \alpha_j}{1 - \overline{\alpha}_j\,\lambda}$, $\lambda \in \Delta$, if $\alpha_j \in \Delta$, or the constant $-\alpha_j$ if $\alpha_j \in \partial\Delta$. Therefore, in any case, we have that

$$h_j(\lambda)\, \varphi_j(\lambda) = \frac{1}{2} r_j (\lambda - \alpha_j)(1 - \overline{\alpha}_j\, \lambda)$$

for all $\lambda \in \Delta$. Finally, by (a) it follows that

$$\frac{1}{\xi} h_j^*(\xi)\varphi_j^*(\xi) = \frac{1}{2} r_j |1 - \overline{\alpha}_j \xi|^2 = \frac{1}{2} r_o |1 - \overline{\alpha}_o \xi|^2 \frac{\partial \mu_D}{\partial |z_j|}(|\varphi_1^*(\xi)|, \ldots, |\varphi_n^*(\xi)|)|\varphi_j^*(\xi)|$$

for a.a. $\xi \in \partial\Delta$ and all $j = 1, \ldots, n$. This equation is equivalent to condition (a) in Theorem 2.4.2.

QED

Lemma 3.1.1 and Theorem 3.2.1 immediately yield the following

Corollary 3.2.2 *Let $D \subseteq \mathbb{C}^n$ be a strictly convex bounded Reinhardt domain. Let $\varphi = (\varphi_1, \ldots, \varphi_n) : \Delta \to D$ be a non-constant holomorphic map such that $\varphi_j \not\equiv 0$ for all $j = 1, \ldots, n$ and that the boundary values $\varphi^*(\xi)$ belong to ∂D for a.a. $\xi \in \partial\Delta$. Let Q_j be the outer factor of φ_j. Moreover, suppose that the unit outer normal vector to ∂D is defined at $\varphi^*(\xi)$ for a.a. $\xi \in \partial\Delta$.*
Then φ is a complex geodesic if, and only if, there exist $\alpha_o, \alpha_1, \ldots, \alpha_n \in \overline{\Delta}$, $r_1, \ldots, r_n > 0$, $\theta_1, \ldots, \theta_n \in \mathbb{R}$ such that, for all $j = 1, \ldots, n$,

$$\varphi_j(\lambda) = e^{i\,\theta_j} \left(\frac{\lambda - \alpha_j}{1 - \overline{\alpha}_j \lambda} \right)^{s_j} \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\exp(it) + \lambda}{\exp(it) - \lambda} \log |Q_j^*(\exp(it))|\, dt \right\} \qquad \lambda \in \Delta,$$

where:

(a) $s_j \in \{ 0, 1 \}$;

(b) $s_j = 1$ *implies* $\alpha_j \in \Delta$;

(c) *the following equations are satisfied*

$$(c.1) \qquad r_j \frac{|1-\overline{\alpha}_j \xi|^2}{|1-\overline{\alpha}_o \xi|^2} = \frac{\partial \mu_D}{\partial |z_j|}(|Q_1^*(\xi)|, \ldots, |Q_n^*(\xi)|)|Q_j^*(\xi)| \qquad \text{for a.a. } \xi \in \partial\Delta,$$

for all $j = 1, \ldots, n$;

$$(c.2) \qquad \alpha_o = \sum_{j=1}^{n} r_j \alpha_j \qquad 1 + |\alpha_o|^2 = \sum_{j=1}^{n} r_j (1 + |\alpha_j|^2).$$

Therefore the inner factor M_j of a component $\varphi_j = M_j Q_j$ of a complex geodesic of a strictly convex bounded Reinhardt domain is deeply related to the corresponding outer factor Q_j : if M_j is not a constant, then it is a Möbius transformation whose zero is the α_j appearing in equation (c.1) of the above Corollary.
On the other hand, if D is a convex, but not strictly convex, bounded Reinhardt domain, then the inner factor M_j of a component $\varphi_j = M_j Q_j$ of a complex geodesic of D can be whatever inner function, provided that $\nu_j(\varphi^*(\xi)) = 0$ for a.a. $\xi \in \partial\Delta$. In particular, we can state the following (cf. [Pflug-Zwonek 1996])

Corollary 3.2.3 *Let* $D \subseteq \mathbb{C}^n$ *be a convex bounded Reinhardt domain. Let* $\varphi = (\varphi_1, \ldots, \varphi_n)$: $\Delta \to D$ *be a complex geodesic such that* $\varphi_j \not\equiv 0$ *for all* $j = 1, \ldots, n$ *and that the unit outer normal vector to* ∂D *is defined at* $\varphi^*(\xi)$ *for a.a.* $\xi \in \partial\Delta$. *Let*

$$\varphi_j = S_j M_j Q_j$$

be the factorization of φ_j, *where* S_j *is a singular inner function,* B_j *is a Blaschke product and* Q_j *is an outer function. Let* Z_j *be the zero set of* B_j. *Let* $\widetilde{Z}_j \subseteq Z_j$ *and define* $\widetilde{B}_j$ *as the Blaschke product whose zero set is* $\widetilde{Z}_j$. *Then the holomorphic map* $\widetilde{\varphi} = (\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n)$: $\Delta \to D$ *whose components are*

$$\widetilde{\varphi}_j = S_j^{s_j} \widetilde{B}_j Q_j,$$

where $s_j \in \{0, 1\}$, *for all* $j = 1, \ldots, n$, *is a complex geodesic.*

Now, let $f : (\mathbb{R}^+ \cup \{0\})^n \to \mathbb{R}^+ \cup \{0\}$ be a separately non-decreasing function, i.e. such that

$$f_j(x) \stackrel{\text{def}}{=} f(x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_n)$$

is a non-decreasing function for all $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \in (\mathbb{R}^+ \cup \{0\})^{n-1}$. Let $\mathbf{p} = (p_1, \ldots, p_n) \in (\mathbb{R}^+)^n$ and define

$$f_{\mathbf{p}}(z) = f(|z_1|^{p_1}, \ldots, |z_n|^{p_n}) \qquad \text{for all} \quad z \in \mathbb{C}^n.$$

Suppose that the Reinhardt domain

$$D_{f,\mathbf{p}} = \{\, z \in \mathbb{C}^n \mid f_{\mathbf{p}}(z) < 1 \,\}$$

is convex and bounded. The Minkowski functional $\mu_{D_{f,\mathbf{p}}}$ of $D_{f,\mathbf{p}}$ is determined by the equation

$$f_{\mathbf{p}}\left(\frac{1}{\mu_{D_{f,\mathbf{p}}}(z)} z\right) = 1.$$

The convexity of $D_{f,\mathbf{p}}$ implies that, for a.a. $z \in \mathbb{C}^n$, we can compute

$$0 = \frac{\partial}{\partial |z_j|} f_{\mathbf{p}}\left(\frac{1}{\mu_{D_{f,\mathbf{p}}}(z)}(|z_1|, \ldots, |z_n|)\right) =$$

$$= \sum_{k=1}^{n} \frac{\partial f_{\mathbf{p}}}{\partial |z_k|}\left(\frac{1}{\mu_{D_{f,\mathbf{p}}}(z)}(|z_1|, \ldots, |z_n|)\right) \frac{\partial}{\partial |z_j|}\left(\frac{1}{\mu_{D_{f,\mathbf{p}}}(z)}|z_k|\right) =$$

$$= \sum_{k=1}^{n} \frac{\partial f_{\mathbf{p}}}{\partial |z_k|}\left(\frac{1}{\mu_{D_{f,\mathbf{p}}}(z)}(|z_1|, \ldots, |z_n|)\right)\left(-\frac{1}{\mu_{D_{f,\mathbf{p}}}(z)^2}\frac{\partial \mu_{D_{f,\mathbf{p}}}}{\partial |z_j|}(z)|z_k| + \frac{1}{\mu_{D_{f,\mathbf{p}}}(z)}\frac{\partial |z_k|}{\partial |z_j|}\right).$$

Therefore, for a.a. $z \in \partial D_{f,\mathbf{p}}$, we have that

$$\frac{\partial \mu_{D_{f,\mathbf{p}}}}{\partial |z_j|}(z) = \frac{\frac{\partial f_{\mathbf{p}}}{\partial |z_j|}(z)}{\sum_{k=1}^{n} \frac{\partial f_{\mathbf{p}}}{\partial |z_k|}(z)|z_k|}. \tag{3.1.3}$$

Let $\mathbf{q} \in (\mathbb{R}^+)^n$ be such that $D_{f,\mathbf{q}}$ is convex and bounded. Since, for all $z \in (\mathbb{C} - \{0\})^n$, the following equation holds

$$f_{\mathbf{q}}(z_1^{p_1/q_1}, \ldots, z_n^{p_n/q_n}) = f(|z_1|^{p_1}, \ldots |z_n|^{p_n}) = f_{\mathbf{p}}(z),$$

then we have that $z \in \partial D_{f,\mathbf{p}} \cap (\mathbb{C} - \{0\})^n$ if, and only if, $(z_1^{p_1/q_1}, \ldots, z_n^{p_n/q_n}) \in \partial D_{f,\mathbf{q}} \cap (\mathbb{C} - \{0\})^n$.

Let us compute

$$\frac{\partial f_{\mathbf{q}}}{\partial |z_j|}(z) = \frac{\partial f}{\partial x_j}(|z_1|^{q_1}, \ldots, |z_n|^{q_n}) q_j |z_j|^{q_j - 1}$$

and

$$\frac{\partial f_{\mathbf{q}}}{\partial |z_j|}(z_1^{p_1/q_1}, \ldots, z_n^{p_n/q_n}) = \frac{q_j}{p_j}\frac{\partial f_{\mathbf{p}}}{\partial |z_j|}(z)|z_j|^{1-p_j/q_j}.$$

Therefore, for a.a. $z \in \partial D_{f,\mathbf{p}} \cap (\mathbb{C} - \{0\})^n$, we have that

$$\frac{\partial \mu_{D_{f,\mathbf{q}}}}{\partial |z_j|}(z_1^{p_1/q_1}, \ldots, z_n^{p_n/q_n})|z_j|^{p_j/q_j} =$$

$$= \frac{\frac{\partial f_{\mathbf{q}}}{\partial |z_j|}(z_1^{p_1/q_1}, \ldots, z_n^{p_n/q_n})|z_j|^{p_j/q_j}}{\sum_{k=1}^n \frac{\partial f_{\mathbf{q}}}{\partial |z_k|}(z_1^{p_1/q_1}, \ldots, z_n^{p_n/q_n})|z_k|^{p_k/q_k}} =$$

$$= \frac{\frac{q_j}{p_j}\frac{\partial f_{\mathbf{p}}}{\partial |z_j|}(z)|z_j|}{\sum_{k=1}^n \frac{q_k}{p_k}\frac{\partial f_{\mathbf{p}}}{\partial |z_k|}(z)|z_k|} =$$

$$= \frac{\frac{q_j}{p_j}\frac{\partial f_{\mathbf{p}}}{\partial |z_j|}(z)|z_j|}{\sum_{k=1}^n \frac{\partial f_{\mathbf{p}}}{\partial |z_k|}(z)|z_k|} \frac{1}{\sum_{k=1}^n \frac{q_k}{p_k}\left(\frac{\frac{\partial f_{\mathbf{p}}}{\partial |z_k|}(z)|z_k|}{\sum_{\ell=1}^n \frac{\partial f_{\mathbf{p}}}{\partial |z_\ell|}(z)|z_\ell|}\right)} =$$

$$= \frac{q_j}{p_j}\frac{\partial \mu_{D_{f,\mathbf{p}}}}{\partial |z_j|}(z)|z_j| \frac{1}{\sum_{k=1}^n \frac{q_k}{p_k}\frac{\partial \mu_{D_{f,\mathbf{p}}}}{\partial |z_k|}(z)|z_k|}.$$

Now, let Let $\varphi = (\varphi_1, \ldots, \varphi_n) : \Delta \to D_{f,\mathbf{p}}$ be a complex geodesic such that $\varphi_j \not\equiv 0$ for all $j = 1, \ldots, n$ and that the unit outer normal vector to $\partial D_{f,\mathbf{p}}$ is defined at $\varphi^*(\xi)$ for a.a. $\xi \in \partial\Delta$. Let $\varphi_j = M_j Q_j$, $j = 1, \ldots n$, where M_j is the inner factor of φ_j and Q_j is the outer factor of φ_j. Define the holomorphic map $\psi(\lambda) = (\psi_1(\lambda), \ldots, \psi_n(\lambda))$ for all $\lambda \in \Delta$, where $\psi_j(\lambda) = M_j(\lambda)Q_j(\lambda)^{p_j/q_j}$. It turns out that $\psi : \Delta \to D_{f,\mathbf{q}}$. By Theorem 3.2.1, there exist $\alpha_o, \alpha_1, \ldots, \alpha_n \in \overline{\Delta}$, $r_1, \ldots, r_n \geq 0$ such that

$$r_j \frac{|1 - \overline{\alpha}_j \xi|^2}{|1 - \overline{\alpha}_o \xi|^2} = \frac{\partial \mu_{D_{f,\mathbf{p}}}}{\partial |z_j|}(|\varphi_1^*(\xi)|, \ldots, |\varphi_n^*(\xi)|)|\varphi_j^*(\xi)| \qquad \text{for a.a. } \xi \in \partial\Delta$$

for all $j = 1, \ldots, n$; and that

$$\alpha_o = \sum_{j=1}^n r_j \alpha_j \qquad 1 + |\alpha_o|^2 = \sum_{j=1}^n r_j (1 + |\alpha_j|^2).$$

Let us compute

$$\frac{\partial \mu_{D_{f,\mathbf{q}}}}{\partial |z_j|}(|\psi_1^*(\xi)|, \ldots, |\psi_n^*(\xi)|)|\psi_j^*(\xi)| =$$

$$= \frac{\partial \mu_{D_{f,\mathbf{q}}}}{\partial |z_j|}(|Q_1^*(\xi)|^{p_1/q_1}, \ldots |Q_n^*(\xi)|^{p_n/q_n})|Q_j^*(\xi)|^{p_j/q_j} =$$

$$= \frac{\frac{q_j}{p_j}\frac{\partial \mu_{D_{f,\mathbf{p}}}}{\partial |z_j|}(|Q_1^*(\xi)|, \ldots, |Q_n^*(\xi)|)|Q_j(\xi)|}{\sum_{k=1}^n \frac{q_k}{p_k}\frac{\partial \mu_{D_{f,\mathbf{p}}}}{\partial |z_k|}(|Q_1^*(\xi)|, \ldots, |Q_n^*(\xi)|)|Q_k(\xi)|} =$$

$$= \frac{q_j}{p_j} r_j \frac{|1-\overline{\alpha}_j\xi|^2}{|1-\overline{\alpha}_\circ\xi|^2} \frac{1}{\sum_{k=1}^n \frac{q_k}{p_k} r_k \frac{|1-\overline{\alpha}_k\xi|^2}{|1-\overline{\alpha}_\circ\xi|^2}} =$$

$$= \frac{q_j}{p_j} r_j |1-\overline{\alpha}_j\xi|^2 \frac{1}{\sum_{k=1}^n \frac{q_k}{p_k} r_k |1-\overline{\alpha}_k\xi|^2}$$

By Lemma 2.3.1, it follows that there exist $r_\circ > 0$ and $\beta_\circ \in \overline{\Delta}$ such that

$$\sum_{k=1}^n \frac{q_k}{p_k} r_k |1-\overline{\alpha}_k\xi|^2 = r_\circ |1-\overline{\beta}_\circ\xi|^2. \tag{3.1.4}$$

By setting $a_j = \frac{q_j}{p_j}\frac{r_j}{r_\circ}$ we have that

$$\frac{\partial \mu_{D_{f,\mathbf{q}}}}{\partial |z_j|}(|\psi_1^*(\xi)|, \ldots, |\psi_n^*(\xi)|)|\psi_j^*(\xi)| = a_j \frac{|1-\overline{\alpha}_j\xi|^2}{|1-\overline{\beta}_\circ\xi|^2}$$

Moreover, equation (3.1.4) implies that

$$\beta_\circ = \sum_{j=1}^n a_j \alpha_j \qquad\qquad 1+|\beta_\circ|^2 = \sum_{j=1}^n a_j(1+|\alpha_j|^2)$$

Therefore, by Theorem 3.2.1, $\psi : \Delta \to D_{f,\mathbf{q}}$ is a complex geodesic and we can state the following (cf. [Jarnicki-Pflug 1995])

Proposition 3.2.4 *Let $f : (\mathbb{R}^+ \cup \{0\})^n \to \mathbb{R}^+ \cup \{0\}$ be a separately non-decreasing function. Let $\mathbf{p}, \mathbf{q} \in (\mathbb{R}^+)^n$. Let the Reinhardt domains $D_{f,\mathbf{p}}$ and $D_{f,\mathbf{q}}$ be convex and bounded. Let $\varphi = (\varphi_1, \ldots, \varphi_n) : \Delta \to D_{f,\mathbf{p}}$ be a complex geodesic such that $\varphi_j \not\equiv 0$ for all $j = 1, \ldots, n$ and that the unit outer normal vector to $\partial D_{f,\mathbf{p}}$ is defined at $\varphi^*(\xi)$ for a.a. $\xi \in \partial\Delta$. Let $\varphi_j = M_j Q_j$, $j = 1, \ldots n$, where M_j is the inner factor of φ_j and Q_j is the outer factor of φ_j. Define the holomorphic map $\psi = (\psi_1, \ldots, \psi_n) : \Delta \to D_{f,\mathbf{q}}$ where $\psi_j(\lambda) = M_j(\lambda)Q_j(\lambda)^{p_j/q_j}$. Then $\psi : \Delta \to D_{f,\mathbf{q}}$ is a complex geodesic.*

Thus, if $f : (\mathbb{R}^+ \cup \{0\})^n \to \mathbb{R}^+ \cup \{0\}$ is a separately non-decreasing function and if $A = \{\ \mathbf{p} \in (\mathbb{R}^+)^n \mid D_{f,\mathbf{p}}$ is convex $\}$, then in order to determine the complex geodesics of the domains $D_{f,\mathbf{p}}$, $\mathbf{p} \in A$, it suffices to choose one particular $\widetilde{\mathbf{p}} \in A$ and find the complex geodesics of $D_{f,\widetilde{\mathbf{p}}}$: by Proposition 3.1.6, given the complex geodesics of $D_{f,\widetilde{\mathbf{p}}}$ one can build up the complex geodesics of $D_{f,\mathbf{p}}$ for any $\mathbf{p} \in A$ and viceversa.
This method can be useful when f is a homogeneous polynomial, because in this case $\sum_{j=1}^n \frac{\partial f}{\partial x_j}(x)x_j = deg(f)\, f(x)$ (cf. equation (3.1.3) and the proof of Theorem 4.1.1).

Another simple consequence of Theorem 3.2.1 is the following result

Proposition 3.2.5 *Let $D \subseteq \mathbb{C}^n$ be a convex bounded Reinhardt domain. Suppose that the unit outer normal vector exists at $z = (z_1, \ldots, z_n) \in \partial D$. Define the holomorphic map*

$$\varphi : \Delta \to D$$
$$\lambda \mapsto (\lambda^{p_1} z_1, \ldots, \lambda^{p_n} z_n)$$

where $p_j \in \{0, 1\}$ for all $j = 1, \ldots, n$ and at least one of the p_j's is 1. Then φ is a complex geodesic in D.

Proof: We will apply Theorem 3.2.1 to prove that φ is a complex geodesic in D.
By $(**)$ it follows that $\varphi(\Delta) \subseteq D$ and by $(*)$ we have that $\varphi(\partial\Delta) \subseteq \partial D$. Since

$$\frac{\partial \mu_D}{\partial |z_j|}(|\varphi_1^*(\xi)|, \ldots, |\varphi_n^*(\xi)|)|\varphi_j^*(\xi)| = \frac{\partial \mu_D}{\partial |z_j|}(|z_1|, \ldots, |z_n|)|z_j|$$

for all $\xi \in \partial\Delta$, all $j = 1, \ldots, n$ and

$$\sum_{j=1}^{n} \frac{\partial \mu_D}{\partial |z_j|}(|z_1|, \ldots, |z_n|)|z_j| = \mu_D(z) = 1,$$

then the map φ satisfies conditions (a), (b) and (c) of Theorem 3.2.1. In fact, it suffices to define

$$r_j = \frac{\partial \mu_D}{\partial |z_j|}(|z_1|, \ldots, |z_n|)|z_j|,$$

$\alpha_o = \alpha_j = 0$, $s_j = p_j$ and $\theta_j = 0$ for all $j = 1, \ldots, n$.

QED

It is interesting to point out that, already in 1927, Kritikos was able to prove that, for any $\zeta_o \in \Delta$, the map

$$\varphi : \Delta \to \{(z_1, z_2) \mid |z_1| + |z_2| < 1\}$$
$$\lambda \mapsto (z_o, (1 - |z_o|)\lambda)$$

is a complex geodesic.

Fourth Chapter

EXPLICIT COMPUTATION OF COMPLEX GEODESICS

This chapter is devoted to the explicit computation of complex geodesics in three classes of examples in order to show how Theorem 3.2.1 can be used.
The first class of examples to be considered is that of complex ellipsoids.

1. Complex geodesics in convex complex ellipsoids

We recall that the complex ellipsoids are in the class of domains whose complex geodesics have been explicitly found ([Gentili 1986 b], [Jarnicki-Pflug 1995], [Jarnicki-Pflug-Zeinstra 1993]).
Let $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$, $p_j > 0$ for all $j = 1, \ldots, n$. By definition, the bounded Reinhardt domain

$$\mathcal{E}(p) = \left\{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n \;\middle|\; E(|z_1|, \ldots, |z_n|) \stackrel{\text{def}}{=} \sum_{j=1}^{n} |z_j|^{2p_j} < 1 \right\}$$

is said to be a complex ellipsoid.
One can easily check that $\mathcal{E}(p)$ is convex if, and only if, $p_j \geq 1/2$ for all $j = 1, \ldots, n$. Since

$$\frac{\partial E}{\partial z_j}(|z_1|, \ldots, |z_n|) = \frac{\partial E}{\partial |z_j|}(|z_1|, \ldots, |z_n|)\frac{\partial |z_j|}{\partial z_j} = p_j\, |z_j|^{2(p_j-1)}\overline{z}_j,$$

then the unit outer normal vector to $\partial\mathcal{E}(p)$ is not defined at $(z_1, \ldots, z_n) \in \partial\mathcal{E}(p)$ if, and only if, $z_j = 0$ and $1/2 \leq p_j < 1$ for some $j \in \{1, \ldots, n\}$. Therefore Theorem 3.2.1 characterizes all complex geodesics $\varphi = (\varphi_1, \ldots, \varphi_n) : \Delta \to \mathcal{E}(p)$ such that $\varphi_j \not\equiv 0$ for all

$j = 1, \ldots, n$. Recall that by Lemma 3.1.2 it suffices to study complex geodesics of the last type only.

We can state the following

Proposition 4.1.1 *Let $p_j \geq 1/2$ for $j = 1, \ldots, n$. Let $\varphi = (\varphi_1, \ldots, \varphi_n) : \Delta \to \mathcal{E}(p)$ be such that $\varphi_j \not\equiv 0$ for all $j = 1, \ldots, n$.*
Then φ is a complex geodesic if, and only if, there exist $\alpha_o \in \Delta$, $\alpha_1, \ldots, \alpha_n \in \overline{\Delta}$, $r_1, \ldots, r_n > 0$, $\theta_1, \ldots, \theta_n \in \mathbb{R}$ such that

$$\varphi_j(\lambda) = \exp(i\theta_j) \left(\frac{\lambda - \alpha_j}{1 - \overline{\alpha}_j \lambda} \right)^{s_j} r_j^{1/(2p_j)} \left(\frac{1 - \overline{\alpha}_j \lambda}{1 - \overline{\alpha}_o \lambda} \right)^{1/p_j} \qquad \lambda \in \Delta$$

$j = 1, \ldots, n$, where $s_j \in \{0, 1\}$, $s_j = 1$ implies $\alpha_j \in \Delta$ and

$$\alpha_o = \sum_{j=1}^{n} r_j \alpha_j \qquad 1 + |\alpha_o|^2 = \sum_{j=1}^{n} r_j (1 + |\alpha_j|^2).$$

Moreover, the case $s_j = 0$ for all $j \in \{1, \ldots, n\}$ and $\alpha_o = \alpha_1 = \ldots = \alpha_n \in \Delta$ is excluded.

Proof: By Proposition 3.2.4, it suffices to prove the statement for one particular complex ellipsoid: we choose the unit ball $B_n = \mathcal{E}(1, \ldots, 1)$. By Corollary 3.2.2, we have that a non-constant holomorphic map $\varphi = (\varphi_1, \ldots, \varphi_n) : \Delta \to B_n$, such that $\varphi^*(\xi) \in \partial B_n$ for a.a. $\xi \in \partial\Delta$ and that $\varphi_j \not\equiv 0$ for all $j = 1, \ldots, n$, is a complex geodesic if, and only if, there exist $\alpha_o, \alpha_1, \ldots, \alpha_n \in \overline{\Delta}$, $r_1, \ldots, r_n \geq 0$ such that

$$r_j \frac{|1 - \overline{\alpha}_j \xi|^2}{|1 - \overline{\alpha}_o \xi|^2} = \frac{\partial \mu_{B_n}}{\partial |z_j|} (|Q_1^*(\xi), \ldots Q_n^*(\xi)) |Q_j^*(\xi)| = |Q_j^*(\xi)|^2 \qquad \text{for a.a. } \xi \in \partial\Delta$$

$j = 1, \ldots, n$, where Q_j is the outer factor of φ_j and the following equations are satisfied

$$\alpha_o = \sum_{j=1}^{n} r_j \alpha_j \qquad 1 + |\alpha_o|^2 = \sum_{j=1}^{n} r_j (1 + |\alpha_j|^2).$$

Then we have that

$$Q_j(\lambda) = r_j^{1/2} \frac{1 - \overline{\alpha}_j \lambda}{1 - \overline{\alpha}_o \lambda}$$

As far as the inner factor M_j of φ_j is concerned, since $|Q_j^*(\xi)| > 0$ for a.a. $\xi \in \partial\Delta$ and for all $j = 1, \ldots, n$, then M_j is either a constant $\exp(i\theta_j)$ or, only if $\alpha_j \in \Delta$, the Möbius transformation

$$M_j(\lambda) = \exp(i\theta_j) \frac{\lambda - \alpha_j}{1 - \overline{\alpha}_j \lambda} \qquad \lambda \in \Delta$$

for some $\theta_j \in \mathbb{R}$.
If $\alpha_o \in \partial\Delta$, then $\alpha_1 = \ldots = \alpha_n = \alpha_o \in \partial\Delta$ and φ should be constant. This is a contradiction, since a complex geodesic cannot be a constant map.
Also if $s_j = 0$ for all $j \in \{1, \ldots, n\}$ and $\alpha_o = \alpha_1 = \ldots = \alpha_n \in \Delta$, then φ is a constant map and therefore this case must be excluded.

QED

Notice that there are complex geodesics on $\mathcal{E}(p)$ which are merely Hölder continuous up to the boundary if $\mathcal{E}(p) \neq B_n$ and that the only strongly convex complex ellipsoid is the unit ball B_n. Thus, in order to have that all complex geodesics are C^1 up to the boundary of Δ, strongly convexity is a necessary condition in Proposition 2.2.3 even if the boundary of the domain is smooth.
By using this explicit characterization of complex geodesics in $\mathcal{E}(p)$, in [Chang-Lee 1993] it has been proved the following

Proposition 4.1.2 *Let $p_j \geq 1/2$ for all $j = 1, \ldots, n$. Let $\tilde{z} = (1, 0, \ldots, 0) \in \partial\mathcal{E}(p)$. Then*
(a) for $z \in \partial\mathcal{E}(p) - \{\tilde{z}\}$ there is a complex geodesic passing through z_o and z except when $p_1 > 1$ and $0 < |z_1| < 1$;
(b) for $z \in \mathcal{E}(p)$ there exists a positive number α depending on $|z_1|$ such that, if $\sum_{j=2}^{n} |z_j|^{2p_j} < \alpha$, then there exists a complex geodesic through $\tilde{z}$ and z;
(c) for any $\mathcal{E}(p)$ with $p_1 > 1$ there exists $z \in \partial\mathcal{E}(p)$ such that there are at least two distinct complex geodesics through $\tilde{z}$ and z;
(d) for any $\mathcal{E}(p)$ with $p_1 > 1$ there exists $z \in \mathcal{E}(p)$ such that there are at least two distinct complex geodesics through $\tilde{z}$ and z.

This result shows that strongly convexity is a necessary hypothesis in order to have existence and uniqueness of complex geodesics with prescribed boundary data (cf. Proposition 2.2.8).

2. Complex geodesics in other classes of domains

For $n \geq 2$ and $a \geq 0$, let us consider the homogeneous polynomial

$$f_a(x_1, \ldots, x_n) = x_1^2 + 2\,a\,x_1x_2 + x_2^2 + \sum_{j=3}^{n} x_j^2$$

and define

$$D_{a,\mathbf{p}} = D_{f_a,\mathbf{p}} = \left\{ z \in \mathbb{C}^n \;\middle|\; |z_1|^{2p_1} + 2\,a\,|z_1|^{p_1}|z_2|^{p_2} + |z_2|^{2p_2} + \sum_{j=3}^{n} |z_j|^{2p_j} < 1 \right\}.$$

The case $a = 0$ is known, since $D_{0,\mathbf{p}} = \mathcal{E}(\mathbf{p}) \stackrel{\text{def}}{=} \{ z \in \mathbb{C}^n \mid \sum_{j=1}^n |z_j|^{2p_j} < 1 \}$ is a complex ellipsoid.

The domains $D_{a,\mathbf{p}}$ are always bounded and they are convex if, for example, $p_1 \geq 1$, $p_2 \geq 1$, $p_j \geq \frac{1}{2}$ for $j = 3, \ldots, n$ and

$$0 \leq a \leq a(p_1, p_2) = \sqrt{\frac{(2p_1 - 1)(2p_2 - 1)}{p_1 + p_2 - 1}}.$$

It can be easily seen that $a(p_1, p_2) \geq 1$ for every $(p_1, p_2) \in [1, \infty) \times [1, \infty)$, whence $D_{a,\mathbf{p}}$ is convex and bounded whenever $0 \leq a \leq 1$, $p_1 \geq 1$, $p_2 \geq 1$ and $p_j \geq \frac{1}{2}$ for $j = 3, \ldots, n$. From now on we will consider the convex case only.

Proposition 4.2.1 *Let $0 \leq a \leq 1$, $p_1 \geq 1$, $p_2 \geq 1$ and $p_j \geq \frac{1}{2}$ for $j = 3, \ldots, n$. Let $\varphi : \Delta \to \mathbb{C}^n$ be a holomorphic map such that $\varphi_j \not\equiv 0$ for all $j = 1, \ldots, n$. Then φ is a complex geodesic in $D_{a,\mathbf{p}}$ if, and only if, there exist $\alpha_o \in \Delta$, $\alpha_1, \ldots, \alpha_n \in \overline{\Delta}$, $r_1, \ldots, r_n \in \mathbb{R}^+$, $\theta_1, \ldots, \theta_n \in [0, 2\pi]$ such that, for all $\lambda \in \Delta$,*

$$\varphi_1(\lambda) = \exp(i\theta_1) \left(\frac{\lambda - \alpha_1}{1 - \overline{\alpha}_1 \lambda} \right)^{s_1} 2^{1/(2p_1)}\, r_1^{1/p_1} \frac{(1 - \overline{\alpha}_1 \lambda)^{2/p_1}}{(1 - \overline{\alpha}_o \lambda)^{1/p_1}} \widetilde{Q}_1(\lambda)$$

$$\varphi_2(\lambda) = \exp(i\theta_2) \left(\frac{\lambda - \alpha_2}{1 - \overline{\alpha}_2 \lambda} \right)^{s_2} 2^{1/(2p_2)}\, r_2^{1/p_2} \frac{(1 - \overline{\alpha}_2 \lambda)^{2/p_2}}{(1 - \overline{\alpha}_o \lambda)^{1/p_2}} \widetilde{Q}_2(\lambda)$$

$$\varphi_j(\lambda) = \exp(i\theta_j) \left(\frac{\lambda - \alpha_j}{1 - \overline{\alpha}_j \lambda} \right)^{s_j} r_j^{1/(2p_j)} \left(\frac{1 - \overline{\alpha}_j \lambda}{1 - \overline{\alpha}_o \lambda} \right)^{1/p_j} \qquad j = 3, \ldots, n$$

where:

(a) $s_j \in \{0, 1\}$, for all $j = 1, \ldots, n$;

(b) $s_j = 1$ implies $\alpha_j \in \Delta$, for all $j = 1, \ldots, n$;

(c) and the following equations are satisfied

$$\text{(c.1)} \qquad \widetilde{Q}_\ell(\lambda) = \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\exp(it) + \lambda}{\exp(it) - \lambda} \frac{1}{2p_\ell} \log\left(g_\ell(\exp(it))\right) dt \right\} \qquad \ell = 1, 2,$$

for all $\lambda \in \Delta$;

$$\text{(c.2)} \qquad g_1(\xi) = \frac{1}{a^2(r_2|1 - \overline{\alpha}_2\xi|^2 - r_1|1 - \overline{\alpha}_1\xi|^2) + 2\, r_1|1 - \overline{\alpha}_1\xi|^2 + a\, g(\xi)}$$

for all $\xi \in \partial\Delta$;

$$\text{(c.3)} \qquad g_2(\xi) = \frac{1}{a^2(r_1|1 - \overline{\alpha}_1\xi|^2 - r_2|1 - \overline{\alpha}_2\xi|^2) + 2\, r_2|1 - \overline{\alpha}_2\xi|^2 + a\, g(\xi)}$$

for all $\xi \in \partial\Delta$;

(c.4) $$g(\xi) = \sqrt{a^2(r_2|1-\overline{\alpha}_2\xi|^2 - r_1|1-\overline{\alpha}_1\xi|^2)^2 + 4\,r_1|1-\overline{\alpha}_1\xi|^2\,r_2|1-\overline{\alpha}_2\xi|^2}$$

for all $\xi \in \partial\Delta$;

(c.5) $$\alpha_o = \sum_{j=1}^{n} r_j\alpha_j \qquad 1+|\alpha_o|^2 = \sum_{j=1}^{n} r_j(1+|\alpha_j|^2).$$

Moreover the case $s_j = 0$, $\alpha_o = \alpha_j$ *for all* $j = 1,\ldots,n$ *is excluded.*

Proof: By Proposition 3.2.4 it suffices to prove the assertion in the case $\mathbf{p} = \mathbf{2} = (2,2,\ldots,2)$.
Let $\varphi : \Delta \to \mathbb{C}^n$ be a non-constant holomorphic mapping such that $\varphi_j \not\equiv 0$ for all $j = 1,\ldots,n$. By Theorem 3.2.1, φ is a complex geodesic in $D_{a,\mathbf{2}}$ if, and only if, there exist $r_1,\ldots r_n \geq 0$ and $\alpha_o,\alpha_1,\ldots,\alpha_n \in \overline{\Delta}$ such that, for $j = 1,\ldots,n$

$$r_j\frac{|1-\overline{\alpha}_j\xi|^2}{|1-\overline{\alpha}_o\xi|^2} = \frac{\partial\mu_{D_{a,2}}}{\partial|z_j|}(|\varphi_1^*(\xi)|,\ldots,|\varphi_n^*(\xi)|)|\varphi_j^*(\xi)| \qquad \text{for a.a. } \xi \in \partial\Delta.$$

Since $\frac{\partial\mu_{D_{a,2}}}{\partial|z_j|}(|\varphi_1^*(\xi)|,\ldots,|\varphi_n^*(\xi)|) > 0$ for a.a. $\xi \in \partial\Delta$ and $|\varphi_j| \not\equiv 0$ for all $j = 1,\ldots,n$, then $r_j > 0$ for all $j = 1,\ldots,n$. After some computations, we get the system

(4.2.1) $$\begin{cases} r_1|1-\overline{\alpha}_1\xi|^2 = |1-\overline{\alpha}_o\xi|^2\left(|\varphi_1^*(\xi)|^2 + a\,|\varphi_2^*(\xi)|^2\right)|\varphi_1^*(\xi)|^2 \\ r_2|1-\overline{\alpha}_2\xi|^2 = |1-\overline{\alpha}_o\xi|^2\left(|\varphi_2^*(\xi)|^2 + a\,|\varphi_1^*(\xi)|^2\right)|\varphi_2^*(\xi)|^2 \\ r_j|1-\overline{\alpha}_j\xi|^2 = |1-\overline{\alpha}_o\xi|^2\,|\varphi_j^*(\xi)|^4 \qquad j = 3,\ldots,n. \end{cases}$$

where the following relations hold

$$\alpha_o = \sum_{j=1}^{n} r_j\alpha_j \qquad 1+|\alpha_o|^2 = \sum_{j=1}^{n} r_j(1+|\alpha_j|^2)$$

If $\alpha_o \in \partial\Delta$, then $\alpha_1 = \alpha_2 = \ldots = \alpha_n = \alpha_o$ and φ should be constant, which contradicts the hypothesis. Hence $\alpha_o \in \Delta$. System (4.2.1) enables us to find $|\varphi_j^*|$ for all $j = 1,\ldots,n$ and to know the outer factors of all the components of φ. After some computations, one gets

$$|\varphi_1^*(\xi)|^4 = |Q_1^*(\xi)|^4 = \frac{2\,r_1^2\,|1-\overline{\alpha}_1\xi|^4}{|1-\overline{\alpha}_o\xi|^2}\,g_1(\xi) \qquad \text{for a.a. } \xi \in \partial\Delta$$

$$|\varphi_2^*(\xi)|^4 = |Q_2^*(\xi)|^4 = \frac{2\,r_2^2\,|1-\overline{\alpha}_2\xi|^4}{|1-\overline{\alpha}_o\xi|^2}\,g_2(\xi) \qquad \text{for a.a. } \xi \in \partial\Delta$$

$$|\varphi_j^*(\xi)|^4 = |Q_j^*(\xi)|^4 = r_j \left|\frac{1-\overline{\alpha}_1\xi}{1-\overline{\alpha}_o\xi}\right|^2 \qquad \text{for a.a. } \xi \in \partial\Delta$$

(where Q_j denotes the outer factor of φ_j) and

$$g_1(\xi) = \frac{1}{a^2(r_2|1-\overline{\alpha}_2\xi|^2 - r_1|1-\overline{\alpha}_1\xi|^2) + 2\,r_1|1-\overline{\alpha}_1\xi|^2 + a\,g(\xi)} \qquad \xi \in \partial\Delta$$

$$g_2(\xi) = \frac{1}{a^2(r_1|1-\overline{\alpha}_1\xi|^2 - r_2|1-\overline{\alpha}_2\xi|^2) + 2\,r_2|1-\overline{\alpha}_2\xi|^2 + a\,g(\xi)} \qquad \xi \in \partial\Delta$$

$$g(\xi) = \sqrt{a^2(r_2|1-\overline{\alpha}_2\xi|^2 - r_1|1-\overline{\alpha}_1\xi|^2)^2 + 4\,r_1|1-\overline{\alpha}_1\xi|^2\,r_2|1-\overline{\alpha}_2\xi|^2} \qquad \xi \in \partial\Delta.$$

As far as the inner factor M_j of φ_j is concerned, since $r_j > 0$, then it follows that M_j is either a constant $\exp(i\theta_j)$ or, only if $\alpha_j \in \Delta$, the Möbius transformation

$$M_j(\lambda) = \exp(i\gamma_j)\frac{\lambda-\alpha_j}{1-\overline{\alpha}_j\lambda} \qquad \lambda \in \Delta$$

for some $\theta_j \in \mathbb{R}$.

QED

This result holds whenever $D_{a,\mathbf{p}}$ is convex. In fact, the proof works for any $\mathbf{p}$: performing similar computations, one can find that each component of a complex geodesic has no singular factor and at most one zero in Δ, moreover a system analogous to (4.2.1) allows to find the outer factors of the components of a complex geodesic in $D_{a,\mathbf{p}}$.

Now, let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and let

$$f_{\mathbf{a}}(x_1, \ldots, x_{2n}) = \sum_{j=1}^{n}(x_j^2 + 2a_jx_jx_{j+n} + x_{j+n}^2)$$

$$g_{\mathbf{a}}(x_1, \ldots, x_{2n+1}) = \sum_{j=1}^{n}(x_j^2 + 2a_jx_jx_{j+n} + x_{j+n}^2) + x_{2n+1}^2.$$

It is easily seen that $D_{f_{\mathbf{a}},\mathbf{p}}$ and $D_{g_{\mathbf{a}},\mathbf{p}}$ are convex (and bounded) if $0 \le a_j \le 1$, $(j = 1, \ldots, n)$, and $p_j \ge 1$, $(j = 1, \ldots, 2n)$, $p_{2n+1} \ge \frac{1}{2}$. By using the same techniques of Proposition 4.2.1, one can prove the following results

Proposition 4.2.2 *Let* $\mathbf{a} \in ([0,1])^n$. *Let* $\mathbf{p} \in ([1,\infty))^{2n}$, *let* $\varphi : \Delta \to \mathbb{C}^{2n}$ *be a holomorphic map such that* $\varphi_j \not\equiv 0$ *for all* $j = 1, \ldots, 2n$. *Then* φ *is a complex geodesic in* $D_{f_{\mathbf{a}},\mathbf{p}}$ *if, and only if, there exist* $\alpha_o \in \Delta$, $\alpha_1, \ldots, \alpha_{2n} \in \overline{\Delta}$, $r_1, \ldots, r_{2n} > 0$, $\theta_1, \ldots, \theta_{2n} \in [0, 2\pi]$ *such that, for* $j = 1, \ldots, n$,

$$(a) \qquad \varphi_j(\lambda) = \exp(i\theta_j)\left(\frac{\lambda-\alpha_j}{1-\overline{\alpha}_j\lambda}\right)^{s_j} 2^{1/(2p_j)}\, r_j^{1/p_j}\, \frac{(1-\overline{\alpha}_j\lambda)^{2/p_j}}{(1-\overline{\alpha}_o\lambda)^{1/p_j}}\, \widetilde{Q}_j(\lambda)$$

(b)

$$\varphi_{j+n}(\lambda) = \exp(i\theta_{j+n}) \left(\frac{\lambda - \alpha_{j+n}}{1 - \overline{\alpha}_{j+n}\lambda}\right)^{s_{j+n}} 2^{1/(2p_{j+n})} r_{j+n}^{1/p_{j+n}} \frac{(1 - \overline{\alpha}_{j+n}\lambda)^{2/p_{j+n}}}{(1 - \overline{\alpha}_o\lambda)^{1/p_{j+n}}} \widetilde{Q}_{j+n}(\lambda)$$

for all $\lambda \in \Delta$, *where:*

(c) $s_j \in \{0, 1\}$, *for all* $j = 1, \ldots, 2n$;

(d) $s_j = 1$ *implies* $\alpha_j \in \Delta$, *for all* $j = 1, \ldots, 2n$;

(e) *and the following equations are satisfied*

$$\text{(e.1)} \qquad \widetilde{Q}_\ell(\lambda) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\exp(it) + \lambda}{\exp(it) - \lambda} \frac{1}{2p_\ell} \log\left(g_\ell(\exp(it))\right) dt\right\} \qquad \ell = 1, \ldots, 2n$$

for all $\lambda \in \Delta$;

$$\text{(e.2)} \qquad g_j(\xi) = \frac{1}{a_j^2(r_{j+n}|1 - \overline{\alpha}_{j+n}\xi|^2 - r_j|1 - \overline{\alpha}_j\xi|^2) + 2\,r_j|1 - \overline{\alpha}_j\xi|^2 + a_j\,\tilde{g}_j(\xi)};$$

$$\text{(e.3)} \qquad g_{j+n}(\xi) = \frac{1}{a_j^2(r_j|1 - \overline{\alpha}_j\xi|^2 - r_{j+n}|1 - \overline{\alpha}_{j+n}\xi|^2) + 2\,r_{j+n}|1 - \overline{\alpha}_{j+n}\xi|^2 + a_j\,\tilde{g}_j(\xi)}$$

$$\text{(e.4)} \qquad \tilde{g}_j(\xi) = \sqrt{a_j^2(r_{j+n}|1 - \overline{\alpha}_{j+n}\xi|^2 - r_j|1 - \overline{\alpha}_j\xi|^2)^2 + 4\,r_j|1 - \overline{\alpha}_j\xi|^2\,r_{j+n}|1 - \overline{\alpha}_{j+n}\xi|^2}$$

for all $\xi \in \partial\Delta$, *for* $j = 1, \ldots, n$;

$$\text{(e.5)} \qquad \alpha_o = \sum_{j=1}^{2n} r_j\alpha_j \qquad 1 + |\alpha_o|^2 = \sum_{j=1}^{2n} r_j(1 + |\alpha_j|^2).$$

Moreover the case $s_j = 0$, $\alpha_o = \alpha_j$ *for all* $j = 1, \ldots, 2n$ *is excluded.*

Proposition 4.2.3 *Let* $\mathbf{a} \in ([0,1])^n$. *Let* $\mathbf{p} \in ([1,\infty))^{2n} \times [\frac{1}{2}, \infty)$, *let* $\varphi : \Delta \to \mathbb{C}^{2n+1}$ *be a holomorphic map such that* $\varphi_j \not\equiv 0$ *for all* $j = 1, \ldots, 2n+1$. *Then* φ *is a complex geodesic in* $D_{g_{\mathbf{a}},\mathbf{p}}$ *if, and only if, there exist* $\alpha_o \in \Delta$, $\alpha_1, \ldots, \alpha_{2n+1} \in \overline{\Delta}$, $r_1, \ldots, r_{2n+1} \in \mathbb{R}^+$, $\theta_1, \ldots, \theta_{2n+1} \in [0, 2\pi]$ *such that, the first* $2n$ *components of* φ *are given by formulas* (a) *and* (b) *in Proposition 4.2.2 such that equations* (e.1), (e.2), (e.3), (e.4) *hold, and*

$$\varphi_{2n+1}(\lambda) = \exp(i\theta_{2n+1}) \left(\frac{\lambda - \alpha_{2n+1}}{1 - \overline{\alpha}_{2n+1}\lambda}\right)^{s_{2n+1}} r_{2n+1}^{1/(2p_{2n+1})} \left(\frac{1 - \overline{\alpha}_{2n+1}\lambda}{1 - \overline{\alpha}_o\lambda}\right)^{1/p_{2n+1}}$$

where

(a) $s_j \in \{0, 1\}$, *for all* $j = 1, \ldots, 2n+1$;

(b) $s_j = 1$ *implies* $\alpha_j \in \Delta$, *for all* $j = 1, \dots, 2n+1$;
and the following equations are satisfied

$$\alpha_o = \sum_{j=1}^{2n+1} r_j \alpha_j \qquad\qquad 1 + |\alpha_o|^2 = \sum_{j=1}^{2n+1} r_j (1 + |\alpha_j|^2).$$

Moreover the case $s_j = 0$, $\alpha_o = \alpha_j$ *for all* $j = 1, \dots, 2n+1$ *is excluded.*

Finally, we will use Theorem 3.2.1 to determine the complex geodesics of the following class of convex bounded Reinhardt domain of $\mathbb{C}^2$

$$D = \left\{ \; z = (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; F(|z_1|, |z_2|) \stackrel{\text{def}}{=} |z_1|^{p_1} + |z_2|^{p_2} + |z_2|^{2p_2} < 1 \; \right\}$$

where $p_1 \geq 1$ and $p_2 \geq 1$. Since

$$\frac{\partial F}{\partial z_1}(z) = \frac{p_1}{2} |z_1|^{p_1 - 2} \overline{z}_1 \qquad \text{and} \qquad \frac{\partial F}{\partial z_2}(z) = \frac{p_2}{2} |z_2|^{p_2 - 2} (1 + 2\,|z_2|^{p_2}) \overline{z}_2,$$

then the unit outer normal vector to ∂D is not defined at $(z_1, z_2) \in \partial D$ if, and only if, either $z_1 = 0$ and $p_1 < 2$ or $z_2 = 0$ and $p_2 < 2$. Therefore Theorem 3.2.1 characterizes all complex geodesics $\varphi = (\varphi_1, \varphi_2) : \Delta \to D$ such that $\varphi_1 \not\equiv 0 \not\equiv \varphi_2$. We are going to prove the following

Proposition 4.2.4 *Let* $p_1 \geq 1$, $p_2 \geq 2$ *and* $D = \{ \; (z_1, z_2) \in \mathbb{C}^2 \mid F(|z_1|, |z_2|) = |z_1|^{p_1} + |z_2|^{p_2} + |z_2|^{2p_2} < 1 \; \}$. *Let* $\varphi = (\varphi_1, \varphi_2) : \Delta \to D$ *be such that* $\varphi_1 \not\equiv 0 \not\equiv \varphi_2$.
Then φ *is a complex geodesic if, and only if, there exist* $\alpha_o \in \Delta$, $\alpha_1, \alpha_2 \in \overline{\Delta}$, $r_1, r_2 > 0$, $\theta_1, \theta_2 \in \mathbb{R}$ *such that*

$$\varphi_j(\lambda) = \exp(i\theta_j) \left(\frac{\lambda - \alpha_j}{1 - \overline{\alpha}_j \lambda} \right)^{s_j} \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\exp(it) + \lambda}{\exp(it) - \lambda} \log |Q_j^*(\exp(it))| \, dt \right\} \qquad \lambda \in \Delta$$

$j = 1, 2$, *where:*
(a) $s_j \in \{0, 1\}$, *for* $j = 1, 2$;
(b) $s_j = 1$ *implies* $\alpha_j \in \Delta$, *for* $j = 1, 2$;
(c) *and the following equations hold*

$$(c.1) \qquad \alpha_o = r_1 \alpha_1 + r_2 \alpha_2 \qquad\qquad 1 + |\alpha_o|^2 = r_1 (1 + |\alpha_1|^2) + r_2 (1 + |\alpha_2|^2),$$

$$(c.2) \qquad |Q_1^*(\xi)|^{p_1} = \frac{r_1 |1 - \overline{\alpha}_1 \xi|^2 (5\,|1 - \overline{\alpha}_o \xi|^2 + 4\, r_1 |1 - \overline{\alpha}_1 \xi|^2 - Q(\xi))}{2(|1 - \overline{\alpha}_o \xi|^2 + r_1 |1 - \overline{\alpha}_2 \xi|^2)^2};$$

$$(c.3) \qquad |Q_2^*(\xi)|^{p_2} = \frac{-|1-\overline{\alpha}_o\xi|^2 + Q(\xi)}{2(|1-\overline{\alpha}_o\xi|^2 + r_1|1-\overline{\alpha}_2\xi|^2)};$$

$$(c.4) \qquad Q(\xi) = \sqrt{|1-\overline{\alpha}_o\xi|^4 + 4\,r_2\,|1-\overline{\alpha}_2\xi|^2(|1-\overline{\alpha}_o\xi|^2 + r_1|1-\overline{\alpha}_1\xi|^2)};$$

for a.a. $\xi \in \partial\Delta$*. Moreover, the case* $s_1 = s_2 = 0$ *and* $\alpha_o = \alpha_1 = \alpha_2 \in \Delta$ *is excluded.*

Proof: For all $z \in \mathbb{C}^2$, the Minkowski functional $\mu_D(z)$ is defined by the equation

$$F\left(\frac{1}{\mu_D(z)}(|z_1|, |z_2|)\right) = 1.$$

One computes

$$\frac{\partial \mu_D}{\partial |z_1|}(z) = \mu_D(z)\frac{p_1(|z_1|/\mu_D(z))^{p_1}}{p_1|z_1|^{p_1}/\mu_D(z)^{p_1-1} + (1+2\,(|z_2|/\mu_D(z))^{p_2})\,p_2|z_2|^{p_2}/\mu_D(z)^{p_2-1}}$$

$$\frac{\partial \mu_D}{\partial |z_2|}(z) = \frac{(1+2\,(|z_2|/\mu_D(z))^{p_2})p_2(|z_2|/\mu_D(z))^{p_2-1}}{p_1|z_1|^{p_1}/\mu_D(z)^{p_1-1} + (1+2\,(|z_2|/\mu_D(z))^{p_2})\,p_2|z_2|^{p_2}/\mu_D(z)^{p_2-1}}.$$

By Theorem 3.2.1, a nonconstant holomorphic map $\varphi = (\varphi_1, \varphi_2) : \Delta \to D$, such that $\varphi^*(\xi) \in \partial D$ for a.a. $\xi \in \partial\Delta$ and that $\varphi_1 \not\equiv 0 \not\equiv \varphi_2$, is a complex geodesic if, and only if, there exist $\beta_o, \alpha_1, \alpha_2 \in \overline{\Delta}$, $a_1, a_2 \geq 0$ such that

$$(4.2.2) \qquad a_1\frac{|1-\overline{\alpha}_1\xi|^2}{|1-\overline{\beta}_o\xi|^2} = \frac{p_1|Q_1^*(\xi)|^{p_1}}{p_1|Q_1^*(\xi)|^{p_1} + (1+2\,|Q_2^*(\xi)|^{p_2})p_2|Q_2^*(\xi)|^{p_2}} \qquad \text{for a.a. } \xi \in \partial\Delta$$

$$(4.2.3) \qquad a_2\frac{|1-\overline{\alpha}_2\xi|^2}{|1-\overline{\beta}_o\xi|^2} = \frac{(1+2\,|Q_2^*(\xi)|^{p_2})p_2|Q_2^*(\xi)|^{p_2}}{p_1|Q_1^*(\xi)|^{p_1} + (1+2\,|Q_2^*(\xi)|^{p_2})p_2|Q_2^*(\xi)|^{p_2}} \qquad \text{for a.a. } \xi \in \partial\Delta$$

where Q_j is the outer factor of φ_j and if the following equations are satisfied

$$\beta_o = a_1\,\alpha_1 + a_2\,\alpha_2 \qquad\qquad 1+|\beta_o|^2 = a_1(1+|\alpha_1|^2) + a_2(1+|\alpha_2|^2).$$

As far as the inner factor M_j of φ_j is concerned, since $|Q_j^*(\xi)| > 0$ for a.a. $\xi \in \partial\Delta$, for $j = 1, 2$, then M_j is either a constant $\exp(i\theta_j)$ or, only if $\alpha_j \in \Delta$, the Möbius transformation

$$M_j(\lambda) = \exp(i\theta_j)\frac{\lambda - \alpha_j}{1-\overline{\alpha}_j\lambda} \qquad\qquad \lambda \in \Delta$$

for some $\theta_j \in \mathbb{R}$. Remark that $\varphi^*(\xi) \in \partial D$ for a.a. $\xi \in \partial\Delta$ if, and only if,

$$(4.2.4) \qquad |Q_1^*(\xi)|^{p_1} + |Q_2^*(\xi)|^{p_2} + |Q_2^*(\xi)|^{2p_2} = 1 \qquad \text{for a.a. } \xi \in \partial\Delta.$$

Now, we suppose that $\varphi : \Delta \to D$ is a complex geodesic. We will exploit equations (4.2.2), (4.2.3) and (4.2.4) to determine Q_1 and Q_2. Set $r_j = a_j/p_j$ for $j = 1, 2$.
Dividing equation (4.2.2) by equation (4.2.3) we get

$$\frac{r_1|1-\overline{\alpha}_1\xi|^2}{r_2|1-\overline{\alpha}_2\xi|^2} = \frac{|Q_1^*(\xi)|^{p_1}}{(1+2\,|Q_2^*(\xi)|^{p_2})|Q_2^*(\xi)|^{p_2}} \qquad \text{for a.a. } \xi \in \partial\Delta.$$

Therefore

$$|Q_1^*(\xi)|^{p_1} = \frac{r_1|1-\overline{\alpha}_1\xi|^2}{r_2|1-\overline{\alpha}_2\xi|^2}(1+2\,|Q_2^*(\xi)|^{p_2})|Q_2^*(\xi)|^{p_2} \qquad \text{for a.a. } \xi \in \partial\Delta.$$

Now, we can solve equation (4.2.4) for $|Q_2^*(\xi)|$ obtaining

$$|Q_2^*(\xi)|^{p_2} = \frac{-|1-\overline{\alpha}_o\xi|^2 + Q(\xi)}{2(|1-\overline{\alpha}_o\xi|^2 + r_1|1-\overline{\alpha}_2\xi|^2)} \qquad \text{for a.a } \xi \in \partial\Delta,$$

where

$$Q(\xi) = \sqrt{|1-\overline{\alpha}_o\xi|^4 + 4\,r_2\,|1-\overline{\alpha}_2\xi|^2(|1-\overline{\alpha}_o\xi|^2 + r_1|1-\overline{\alpha}_1\xi|^2)} \qquad \text{for a.a. } \xi \in \partial\Delta$$

and $\alpha_o \in \overline{\Delta}$ is such that

$$\alpha_o = r_1\,\alpha_1 + r_2\,\alpha_2 \qquad\qquad 1 + |\alpha_o|^2 = r_1(1+|\alpha_1|^2) + r_2(1+|\alpha_2|^2);$$

consequently

$$|Q_1^*(\xi)|^{p_1} = \frac{r_1|1-\overline{\alpha}_1\xi|^2(5\,|1-\overline{\alpha}_o\xi|^2 + 4\,r_1|1-\overline{\alpha}_1\xi|^2 - Q(\xi))}{2(|1-\overline{\alpha}_o\xi|^2 + r_1|1-\overline{\alpha}_2\xi|^2)^2} \qquad \text{for a.a } \xi \in \partial\Delta.$$

Now, it suffices to recall that an outer function is determined by the modulus of its boundary values to get that

$$\varphi_j(\lambda) = \exp(i\theta_j)\left(\frac{\lambda-\alpha_j}{1-\overline{\alpha}_j\lambda}\right)^{s_j} \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\exp(it)+\lambda}{\exp(it)-\lambda}\log|Q_j^*(\exp(it))|\,dt\right\} \qquad \lambda \in \Delta.$$

Remark that if $\alpha_o \in \partial\Delta$, then $\alpha_o = \alpha_1 = \alpha_2 \in \partial\Delta$. Therefore Q_1 and Q_2 should be constant and $s_1 = s_2 = 0$, but this is a contradiction, since φ is nonconstant. For the same reason we must exclude also the case $s_1 = s_2 = 0$ and $\alpha_o = \alpha_1 = \alpha_2 \in \Delta$.
Viceversa, if $\varphi : \Delta \to D$ is a nonconstant map defined as

$$\varphi_j(\lambda) = \exp(i\theta_j)\left(\frac{\lambda-\alpha_j}{1-\overline{\alpha}_j\lambda}\right)^{s_j} \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\exp(it)+\lambda}{\exp(it)-\lambda}\log|Q_j^*(\exp(it))|\,dt\right\} \qquad \lambda \in \Delta$$

$j = 1, 2$, where $\alpha_o \in \Delta$, $\alpha_1, \alpha_2 \in \overline{\Delta}$, $r_1, r_2 > 0$, $\theta_1, \theta_2 \in \mathbb{R}$, $s_j \in \{0, 1\}$, where $s_j = 1$ implies $\alpha_j \in \Delta$ and where the following equations hold

$$\alpha_o = r_1\,\alpha_1 + r_2\,\alpha_2 \qquad\qquad 1 + |\alpha_o|^2 = r_1(1 + |\alpha_1|^2) + r_2(1 + |\alpha_2|^2),$$

$$|Q_1^*(\xi)|^{p_1} = \frac{r_1|1 - \overline{\alpha}_1\xi|^2(5\,|1 - \overline{\alpha}_o\xi|^2 + 4\,r_1|1 - \overline{\alpha}_1\xi|^2 - Q(\xi))}{2(|1 - \overline{\alpha}_o\xi|^2 + r_1|1 - \overline{\alpha}_2\xi|^2)^2} \qquad \text{for a.a } \xi \in \partial\Delta$$

$$|Q_2^*(\xi)|^{p_2} = \frac{-|1 - \overline{\alpha}_o\xi|^2 + Q(\xi)}{2(|1 - \overline{\alpha}_o\xi|^2 + r_1|1 - \overline{\alpha}_2\xi|^2)} \qquad \text{for a.a } \xi \in \partial\Delta$$

$$Q(\xi) = \sqrt{|1 - \overline{\alpha}_o\xi|^4 + 4\,r_2\,|1 - \overline{\alpha}_2\xi|^2(|1 - \overline{\alpha}_o\xi|^2 + r_1|1 - \overline{\alpha}_1\xi|^2)} \qquad \text{for a.a. } \xi \in \partial\Delta,$$

then it is easy to check that φ satisfies equations (4.2.2), (4.2.3) and (4.2.4). Therefore, by Theorem 3.2.1, it is a complex geodesic.

QED

As for other possible applications of Theorem 3.2.1, the only bounds stay in the imagination of the reader and in the capability to solve systems of partial differential equations.

Fifth Chapter

CARATHÉODORY BALLS AND NORM BALLS IN CONVEX BOUNDED REINHARDT DOMAINS

Let $D \subseteq \mathbb{C}^n$ be a domain. Define

$$B^*_{c_D}(\widetilde{z}, r) = B_{c_D}(\widetilde{z}, \operatorname{arctanh} r) = \{ \; z \in D \mid c_D(z, \widetilde{z}) < \operatorname{arctanh} r \; \}$$

as the Carathéodory ball with centre at $\widetilde{z} \in D$ and radius $\operatorname{arctanh} r$, for $r \in (0,1)$. The boundedness of D implies that $\overline{B^*_{c_D}(\widetilde{z}, r)} \subseteq D$ (cf. Proposition 1.3.3).
Now, if $D \subseteq \mathbb{C}^n$ is a bounded, convex, balanced domain, the μ_D-norm ball with centre at $\widetilde{w} \in \mathbb{C}^n$ and radius $s > 0$ is defined as follows

$$B_D(\widetilde{w}, s) = \{ \; z \in \mathbb{C}^n \mid \mu_D(z - \widetilde{w}) < s \; \}.$$

We will simply say "norm ball" instead of "μ_D-norm ball", when this does not give rise to misunderstanding.
Since, by Proposition 2.1.2,

$$c_D(0, z) = \omega\,(0, \mu_D(z)) = \operatorname{arctanh}\,(\mu_D(z))$$

for all $z \in D$, then one has that

$$B^*_{c_D}(0, r) = B_D(0, r)$$

for all $r \in (0,1)$, i.e. Carathéodory balls centred at the origin and μ_D-norm balls centred at the origin are the same.
The question arises, for which norms the balls not centred at the origin are, as well, of both kinds.

In the case of the unit disc of $\mathbb{C}$, by Lemma 1.1.2, we have that every Carathéodory ball (Poincaré disc) is a norm ball (Euclidean disc).
In dimension greater than one the situation is quite different: in the case of the Euclidean ball $B_n = \{ z \in \mathbb{C}^n \mid \| z \|_2 = \sum_{j=1}^n |z_j|^2 < 1 \}$ every Carathéodory ball not centred at the origin is an ellipsoid and not an Euclidean ball ([Rudin 1980]); while a Carathéodory ball with centre at $\widetilde{z} = (\widetilde{z}_1, \ldots, \widetilde{z}_n) \in \Delta^n$ in the polydisc $\Delta^n = \{ z \in \mathbb{C}^n \mid \| z \|_\infty = \max_{j=1,\ldots,n}\{ |z_j| \} < 1 \}$ is a $\| \, . \, \|_\infty$-norm ball if, and only if, $|\widetilde{z}_1| = \ldots = |\widetilde{z}_n|$. Also in the case of the set M of all $n \times n$ complex matrices with spectral norm < 1 considered as a subset of $\mathbb{C}^{n^2}$, it has been proved that there is a set of Carathéodory balls (depending on $n^2 + 2$ real parameters) which are spectral norm balls ([Schwarz 1990]).
Remark that B_n, Δ^n and M are homogeneous domains, and this fact suggests a way to handle the question: it suffices to study the image of a Carathéodory ball centred at the origin by an automorphism of the domain and check in which cases this image is a norm ball.
As for non-homogeneous domains, the case of convex complex ellipsoids $\mathcal{E}(p_1, \ldots, p_n) \stackrel{\text{def}}{=} \{ z \in \mathbb{C}^n \mid \sum_{j=1}^n |z_j|^{2p_j} < 1 \}$, where $n \geq 2$ and $p_j \geq 1/2$ for all $j = 1, \ldots, n$, has been discussed.
Let us consider the following assertion:

$(*)$ - the only Carathéodory balls which are also $\mu_{\mathcal{E}(p_1,\ldots,p_n)}$-norm balls are those centred at the origin.

At first, it has been proved that statement $(*)$ holds in $\mathcal{E}(1/2, 1/2) \subseteq \mathbb{C}^2$ ([Schwarz 1993]) and in $\mathcal{E}(1/2, \ldots, 1/2) \subseteq \mathbb{C}^n$, then it has been generalized to the case $\mathcal{E}(p/2, \ldots, p/2) \subseteq \mathbb{C}^n$ where $p > 1$ is not an even integer ([Srebro 1995], [Zwonek 1995 b], [Schwarz-Srebro 1996]). Afterwards this question has been treated for general convex complex ellipsoids and it has been shown that if $p_j \neq 1$ for all $j = 1, \ldots, n$, then the only Carathéodory balls which are also $\mu_{\mathcal{E}(p_1,\ldots,p_n)}$-norm balls are those centred at the origin; besides, an example has been found, to point out that this is not the general rule: if $n = 2$, $p_1 = 1/2$ and $p_2 = 1$, then $B^*_{c_{\mathcal{E}(1/2,1)}}((0, w_2), r)$ is a norm ball for any $(0, w_2) \in \mathcal{E}(1/2, 1)$ and for any $r \in (0, 1)$ ([Zwonek 1996]).
This last result is drawn exploiting a particular family of complex geodesics through $(0, w_2)$ and the fact that for every $(0, w_2) \in \mathcal{E}(1/2, 1)$ there exists an automorphism Ψ of $\mathcal{E}(1/2, 1)$ such that $\Psi((0, 0)) = (0, w_2)$.
The proofs of statement $(*)$ are obtained by contradiction and rely essentially upon:
(a) - the existence of some particular families of complex geodesics, namely the linear ones through the origin and those which are "parallel" to some axis;

(b) - the following result: let $f(z) = \sum_{j=1}^{n} b_j |z_j|^{2p_j}$ for $z \in \mathbb{C}^n$ and $b \in (\mathbb{R}^+)^n$; if z and w are such that $f(z + e^{it}w)$ is constant for all $t \in \mathbb{R}$, then $w_j \neq 0$ implies $z_j = 0$ (this statement fails to hold if $p_j = 1$ for all $j = 1, \ldots, n$ and this is the reason why this kind of proofs does not include the case of the Euclidean ball).
Here, we will consider the question for a class $\mathcal{D}$ of convex bounded Reinhardt domains of $\mathbb{C}^n$, which are a generalization of complex ellipsoids. We prove that in this case a Carathéodory ball in $D \in \mathcal{D}$ is a norm ball if, and only if, D is a complex ellipsoid $\mathcal{E}(p_1, \ldots, p_n)$ such that $p_k = 1$ for exactly one $k \in \{1, \ldots, n\}$, $p_j = 1/2$ for all $j \neq k$ and the centre lies on the z_k-axis.
In particular, we completely solve the question for convex complex ellipsoids too.
At first, we prove that, in any strictly convex bounded Reinhardt domain with C^1-boundary, a Carathéodory ball not centred on some axis cannot be a norm ball. Since the assertion in (b) is not so easy to generalize, we tried a different way: we considered a "large" subset of the boundary of the domains we are to discuss about as a real $(2n-1)$-manifold $\widetilde{S}$ with its real tangent spaces. If a Carathéodory ball is also a norm ball, then we can define a smooth non constant curve $\alpha(t)$ in $\widetilde{S}$, whose tangent vector $\alpha'(t)$ must be orthogonal to $T_{\alpha(t)}\widetilde{S}$. Computations show that last condition leads to a contradiction. This method does not work if one tries to prove that a Carathéodory ball centred on one axis cannot be a norm ball, nevertheless it allows to show that a Carathéodory ball not cetred on some axis cannot be a norm ball in the case in which the domain is the Euclidean ball.
As far as Carathéodory balls centred on an axis are concerned, the families of complex geodesics quoted in (a) do not suffice to get a contradiction.
In this case we exploit the explicit formulas of other families of complex geodesics passing through the centre of the Carathéodory ball $B^*_{c_D}(\widetilde{z}, r)$ in order to prove that $B^*_{c_D}(\widetilde{z}, r)$ can be a norm ball only if $D = \mathcal{E}(p_1, \ldots, p_n)$ where $p_k = 1$ for exactly one $k \in \{1, \ldots, n\}$, $p_j = 1/2$ for all $j \neq k$ and the centre lies on the z_k-axis.
Moreover if, say, $D = \mathcal{E}(1, 1/2, \ldots, 1/2)$ and $\widetilde{z} = (\widetilde{z}_1, 0, \ldots, 0)$, then the formulas of all complex geodesics through $\widetilde{z}$ allow us to check that $B^*_{c_D}(\widetilde{z}, r) = B_D\left(\frac{1-r^2}{1-r^2|\widetilde{z}_1|^2}\widetilde{z}, \frac{1-|\widetilde{z}_1|^2}{1-r^2|\widetilde{z}_1|^2} r\right)$.
We remark that to obtain the results of this chapter we do not use the existence of automorphisms that shift the origin along the z_1-axis in the complex ellipsoid $\mathcal{E}(1, 1/2, \ldots, 1/2)$.

1. Carathéodory balls and norm balls

Let $D \subseteq \mathbb{C}^n$ be a convex bounded Reinhardt domain. Let $z = (z_1, \ldots, z_n) \in D$, i.e. let z be such that $\mu_D(z) < 1$. There exists one, and only one, positive real number $\mu_j(z) > 0$ such that

$$\mu_D(z_1, \ldots, z_{j-1}, \mu_j(z), z_{j+1}, \ldots, z_n) = 1.$$

The fact that μ_D is a non-decreasing function of $|z_j|$ implies that $|z_j| < \mu_j(z)$.
The following lemma gives a sufficient condition for the centre and the radius of a Carathéodory ball to determine the centre of a norm ball; the same condition turns out to be equivalent for the Carathéodory ball and the norm ball at issue to coincide, as Lemma 5.1.2 states (cf. [Zwonek 1996]).

Lemma 5.1.1 *Let $D \subseteq \mathbb{C}^n$ be a strictly convex bounded Reinhardt domain with C^1 boundary. Let $\partial B^*_{c_D}(\widetilde{z}, r) \subseteq \partial B_D(\widetilde{w}, s)$ for some $\widetilde{z}$, $\widetilde{w} \in D$, $r \in (0,1)$, $s \in \mathbb{R}^+$. Then*

$$\widetilde{w}_j = \nu_j\, \widetilde{z}_j \quad \textit{where} \quad \nu_j = \frac{1 - r^2}{1 - r^2 \frac{|\widetilde{z}_j|^2}{\mu_j(\widetilde{z})^2}} \tag{5.1.1}$$

for $j = 1, \ldots, n$.

Proof: Let

$$\begin{aligned} A_j =& \{\ \xi \in \mathbb{C} \mid (\widetilde{z}_1, \ldots, \widetilde{z}_{j-1}, \xi, \widetilde{z}_{j+1}, \ldots, \widetilde{z}_n) \in \partial B^*_{c_D}(\widetilde{z}, r)\ \} = \\ =& \{\ \xi \in \mathbb{C} \mid c_D((\widetilde{z}_1, \ldots, \widetilde{z}_{j-1}, \xi, \widetilde{z}_{j+1}, \ldots, \widetilde{z}_n), \widetilde{z}) = \operatorname{arctanh} r\ \}. \end{aligned}$$

Since, by Proposition 3.2.5, the map

$$\begin{aligned} \Delta \to& D \\ \xi \mapsto& (\widetilde{z}_1, \ldots, \widetilde{z}_{j-1}, \xi\mu_j(\widetilde{z}), \widetilde{z}_{j+1}, \ldots, \widetilde{z}_n) \end{aligned}$$

is a complex geodesic, then

$$A_j = \left\{\ \xi \in \mathbb{C} \;\middle|\; \omega\left(\frac{\xi}{\mu_j(\widetilde{z})}, \frac{\widetilde{z}_j}{\mu_j(\widetilde{z})}\right) = \operatorname{arctanh} r\ \right\}$$

and by Lemma 1.1.2,

$$A_j = \{\ \xi = \xi_j + \eta_j e^{i\theta} \mid \theta \in \mathbb{R}\ \}$$

where

$$\xi_j = \frac{1 - r^2}{1 - r^2 \frac{|\widetilde{z}_j|^2}{\mu_j(\widetilde{z})^2}} \widetilde{z}_j \qquad \text{and} \qquad \eta_j = \frac{1 - \frac{|\widetilde{z}_j|^2}{\mu_j(\widetilde{z})^2}}{1 - r^2 \frac{|\widetilde{z}_j|^2}{\mu_j(\widetilde{z})^2}}\, r\, \mu_j(\widetilde{z}) > 0.$$

By hypothesis, if $\xi \in A_j$ then $(\widetilde{z}_1, \ldots, \widetilde{z}_{j-1}, \xi, \widetilde{z}_{j+1}, \ldots, \widetilde{z}_n) \in \partial B_D(\widetilde{w}, s)$, i.e.

$$\mu_D\big((\widetilde{z}_1, \ldots, \widetilde{z}_{j-1}, \xi, \widetilde{z}_{j+1}, \ldots, \widetilde{z}_n) - \widetilde{w}\big) = s. \tag{5.1.2}$$

where $\xi = \xi_j + \eta_j e^{i\theta}$, $\eta_j \neq 0$ and θ is arbitrary in $\mathbb{R}$. Therefore,

$$\frac{d}{d\theta}\mu_D\big((\widetilde{z}_1, \ldots, \widetilde{z}_{j-1}, \xi_j + \eta_j e^{i\theta}, \widetilde{z}_{j+1}, \ldots, \widetilde{z}_n) - \widetilde{w}\big) =$$

$$= -\frac{\partial \mu_D}{\partial |z_j|}\big((\widetilde{z}_1, \ldots, \widetilde{z}_{j-1}, \xi_j + \eta_j e^{i\theta}, \widetilde{z}_{j+1}, \ldots, \widetilde{z}_n) - \widetilde{w}\big)\frac{\operatorname{Im}(\eta_j(\overline{\xi}_j - \overline{\widetilde{w}}_j)e^{i\theta})}{|\xi_j - \widetilde{w}_j + \eta_j e^{i\theta}|} = 0$$

for all $\theta \in \mathbb{R}$. Now, strict convexity of D implies that

$$\widetilde{w}_j = \xi_j = \frac{1 - r^2}{1 - r^2\frac{|\widetilde{z}_j|^2}{\mu_j(\widetilde{z})^2}}\widetilde{z}_j.$$

QED

Remark that this result also holds for complex ellipsoids $\mathcal{E}(\mathbf{p})$ with $1/2 \leq p_j < 1$, because $\frac{\partial \mu_{\mathcal{E}_{\mathbf{p}}}}{\partial |z_j|}(z_1, \ldots, z_n)$ fails to exist only if $z_j = 0$.

Lemma 5.1.2 *Let $D \subseteq \mathbb{C}^n$ be a strictly convex bounded Reinhardt domain with C^1-boundary. Let $\widetilde{z}, \widetilde{w} \in D$, $r \in (0,1)$, $s \in \mathbb{R}^+$. Then $\partial B^*_{c_D}(\widetilde{z}, r) \subseteq \partial B_D(\widetilde{w}, s)$ if, and only if, $B^*_{c_D}(\widetilde{z}, r) = \widetilde{B}_D(\widetilde{w}, s)$.*

Proof: It is clear that $B_{c_D}(\widetilde{z}, r) = B_D(\widetilde{w}, s)$ implies that $\partial B_{c_D}(\widetilde{z}, r) \subseteq \partial \widetilde{B}_D(\widetilde{w}, s)$, therefore it suffices to prove the other implication.
First of all, we prove that $\widetilde{z} \in B_D(\widetilde{w}, s)$ and that $\widetilde{w} \in B^*_{c_D}(\widetilde{z}, r)$.
With the same notations as in the above lemma, we have that, for any $j = 1, \ldots, n$,

$$\mu_D(\widetilde{z} - \widetilde{w}) < \mu_D\big((\widetilde{z}_1, \ldots, \widetilde{z}_{j-1}, \xi, \widetilde{z}_{j+1}, \ldots, \widetilde{z}_n) - \widetilde{w}\big) = s,$$

since

$$\xi - \widetilde{w}_j = \eta_j e^{i\theta} = \frac{1 - \frac{|\widetilde{z}_j|^2}{\mu_j(\widetilde{z})^2}}{1 - r^2\frac{|\widetilde{z}_j|^2}{\mu_j(\widetilde{z})^2}}\, r\, \mu_j(\widetilde{z})e^{i\theta}, \qquad \widetilde{z}_j - \widetilde{w}_j = (1 - \nu_j)\widetilde{z}_j = \frac{1 - \frac{|\widetilde{z}_j|^2}{\mu_j(\widetilde{z})^2}}{1 - r^2\frac{|\widetilde{z}_j|^2}{\mu_j(\widetilde{z})^2}}\, r^2\widetilde{z}_j$$

and $r|\widetilde{z}_j| \leq |\widetilde{z}_j| < \mu_j(\widetilde{z})$. Therefore $\widetilde{z} \in B_D(\widetilde{w}, s)$.
Now, by contradiction, let us suppose that $\widetilde{w} \notin B^*_{c_D}(\widetilde{z}, r)$. This means that $c_D(\widetilde{w}, \widetilde{z}) >$ arctanh r and hence that there exists $t \in (0,1)$ such that $t\,\widetilde{z} + (1-t)\widetilde{w} \in \partial B^*_{c_D}(\widetilde{z}, r) \subseteq \partial B_D(\widetilde{w}, s)$. It follows that

$$\mu_D\big(t\,(\widetilde{z} - \widetilde{w})\big) = \mu_D\big(t\,\widetilde{z} + (1-t)\widetilde{w} - \widetilde{w}\big) = s,$$

but

$$\mu_D\big(t\,(\widetilde{z}-\widetilde{w})\big) < \mu_D\big(\widetilde{z}-\widetilde{w}\big) < s$$

because $\widetilde{z} \in B_D(\widetilde{w}, s)$. Contradiction.
Let us prove that $B^*_{c_D}(\widetilde{z}, r) = B_D(\widetilde{w}, s)$.
Let $z \in B^*_{c_D}(\widetilde{z}, r)$ and consider a complex geodesic $\varphi : \Delta \to D$ such that $\varphi(0) = \widetilde{z}$, $\varphi(\widetilde{r}) = z$ and $\widetilde{r} \in (0, 1)$. The hypothesis implies that $\widetilde{r} < r$. Let us define the function

$$m : \Delta \to \mathbb{R}$$
$$\lambda \mapsto \mu_D\big(\varphi(\lambda) - \widetilde{w}\big)$$

which is subharmonic. Since

$$\varphi(e^{i\theta} r) \in \partial B^*_{c_D}(\widetilde{z}, r) \subseteq \partial B_D(\widetilde{w}, s),$$

then $m(e^{i\theta} r) = \mu_D\big(\varphi((e^{i\theta} r)) - \widetilde{w}\big) = s$. The subharmonicity of m and the inequality $m(0) = \mu_D(\widetilde{z} - \widetilde{w}) < s$ imply that $m(\lambda) < s$ for all $|\lambda| < r$ and, in particular, we get that $m(\widetilde{r}) = \mu_D(z - \widetilde{w}) < s$, i.e. $z \in B_D(\widetilde{w}, s)$.
Finally, let $w \in B_D(\widetilde{w}, s)$. By contradiction, let us suppose that $w \notin B^*_{c_D}(\widetilde{z}, r)$. Since $\widetilde{w} \in B^*_{c_D}(\widetilde{z}, r)$, there exists $t \in (0, 1)$ such that $t\,w + (1-t)\widetilde{w} \in \partial B^*_{c_D}(\widetilde{z}, r) \subseteq \partial B_D(\widetilde{w}, s)$. Hence we have that

$$\mu_D\big(t\,(w - \widetilde{w})\big) = \mu_D\big(t\,w + (1-t)\widetilde{w} - \widetilde{w}\big) = s,$$

but this contradicts the hypothesis, since

$$\mu_D\big(t\,(w - \widetilde{w})\big) < \mu_D\big(w - \widetilde{w}\big) < s.$$

Therefore $w \in B^*_{c_D}(\widetilde{z}, r)$ and the proof is complete.

QED

Now, let $\widetilde{z} = (\widetilde{z}_1, \ldots, \widetilde{z}_n) \in D \subseteq \mathbb{C}^n$ be such that $\widetilde{z}_k = 0$ for some $k \in \{1, \ldots, n\}$ and consider the projection

$$P_k \; : \; \mathbb{C}^n \to \mathbb{C}^{n-1}$$
$$(z_1, \ldots, z_n) \mapsto (z_1, \ldots, \hat{z}_k, \ldots, z_n)$$

where $\hat{z}_k$ means that z_k is missing. Let $\psi = (\psi_1, \ldots, \psi_{k-1}, \psi_{k+1} \ldots, \psi_n) : \Delta \to P_k(D) \subseteq \mathbb{C}^{n-1}$ be a complex geodesic such that $\psi(0) = P_k(\widetilde{z})$.
Then the map $\widetilde{\psi} = (\psi_1, \ldots, \psi_{k-1}, 0, \psi_{k+1} \ldots, \psi_n) : \Delta \to D$ is a complex geodesic of D such that $\widetilde{\psi}(0) = \widetilde{z}$ and therefore

$$c_D(z, \widetilde{z}) = c_{P_k(D)}\big(P_k(z), P_k(\widetilde{z})\big)$$

for all $z = (z_1, \dots, z_{k-1}, 0, z_{k+1}, \dots, z_n) \in D \cap \{ z \in \mathbb{C}^n \mid z_k = 0 \}$. Thus

$$B^*_{c_{P_k(D)}}(P_k(\widetilde{z}), r) = P_k\big(B^*_{c_D}(\widetilde{z}, r)\big).$$

If $z = (z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n) \in P_k(D)$ then $(z_1, \dots, z_{k-1}, 0, z_{k+1}, \dots, z_n) \in D$, therefore $\mu_D\big((z_1, \dots, z_{k-1}, 0, z_{k+1}, \dots, z_n)\big) \leq \mu_{P_k(D)}\big(P_k(z)\big)$. This inequality implies that

$$B_{P_k(D)}(P_k(\widetilde{z}), s) \subseteq P_k(B_D(\widetilde{z}, s)).$$

If $z \in D$ then $P_k(z) \in P_k(D)$, therefore

$$\mu_{P_k(D)}(P_k(z)) \leq \mu_D(z).$$

In particular, $\mu_D\big((z_1, \dots, z_{k-1}, 0, z_{k+1}, \dots, z_n)\big) = \mu_{P_k(D)}\big(P_k(z)\big)$.
Now, if

$$w = (w_1, \dots, w_{k-1}, w_{k+1}, \dots, w_n) \in P_k(B_D(\widetilde{z}, s)),$$

then

$$(w_1, \dots, w_{k-1}, 0, w_{k+1}, \dots, w_n) \in B_D(\widetilde{z}, s)$$

and therefore $w \in B_{P_k(D)}(P_k(\widetilde{z}), s)$. Hence

$$P_k(B_D(\widetilde{z}, s)) \subseteq B_{P_k(D)}(P_k(\widetilde{z}), s),$$

and finally,

$$P_k(B_D(\widetilde{z}, s)) = B_{P_k(D)}(P_k(\widetilde{z}), s).$$

Thus we can state the following

Lemma 5.1.3 *Let $D \subseteq \mathbb{C}^n$ be a convex bounded Reinhardt domain. Let $B^*_{c_D}(\widetilde{z}, r) = B_D(\widetilde{w}, s) \subseteq D$ for some $\widetilde{z} \in D$, $r \in (0,1)$, $s \in \mathbb{R}^+$. If $\widetilde{z}_k = 0$, then $B^*_{c_{P_k(D)}}(P_k(\widetilde{z}), r) = B_{P_k(D)}(P_k(\widetilde{z}), s)$.*

2. On the centre of coinciding balls

We can now prove that there are no Carathéodory balls which are also norm balls if the centre does not lie on some axis

Theorem 5.2.1 *Let $D \subseteq \mathbb{C}^n$ be a strictly convex bounded Reinhardt domain with C^1 boundary. Let $\widetilde{z} \in D$ be such that at least two of its components are non zero. Then*

$$B^*_{c_D}(\widetilde{z}, r) \neq B_D(\widetilde{w}, s)$$

for all $\widetilde{w} \in D$, $r \in (0,1)$, $s \in \mathbb{R}^+$.

Proof: By contradiction, let us suppose that $B^*_{c_D}(\widetilde{z}, r) = B_D(\widetilde{w}, s)$ for some $\widetilde{w} \in D$, $r \in (0,1)$, $s \in \mathbb{R}^+$. By Lemma 5.1.1 we know $\widetilde{w}$ as a function of $\widetilde{z}$, and by Lemma 5.1.3 it is not restrictive to suppose that $\prod_{j=1}^n \widetilde{z}_j \neq 0$.
Let

$$A = \{\ \xi \in \mathbb{C} \mid \xi\,\widetilde{z} \in \partial B^*_{c_D}(\widetilde{z}, r)\ \} = \{\ \xi \in \mathbb{C} \mid c_D(\xi\,\widetilde{z}, \widetilde{z}) = \operatorname{arctanh} r\ \}.$$

The map

$$\Delta \to D$$

$$\xi \mapsto \xi \frac{1}{\mu_D(\widetilde{z})} \widetilde{z}$$

is a complex geodesic , therefore

$$A = \{\ \xi \in \mathbb{C} \mid \omega(\xi\,\mu_D(\widetilde{z}), \mu_D(\widetilde{z})) = \operatorname{arctanh} r\ \} =$$

$$= \{\ \xi = \tau + \widetilde{\tau} e^{i\theta} \mid \theta \in \mathbb{R}\ \}$$

where

$$\tau = \frac{1 - r^2}{1 - r^2 \mu_D(\widetilde{z})^2} \qquad \text{and} \qquad \widetilde{\tau} = r\,\frac{1 - \mu_D(\widetilde{z})^2}{1 - r^2 \mu_D(\widetilde{z})^2} \frac{1}{\mu_D(\widetilde{z})} > 0.$$

By hypothesis, $\xi \in A$ if, and only if, $\xi\,\widetilde{z} \in \partial B_D(\widetilde{w}, s)$, therefore we have a curve

$$\beta : \mathbb{R} \to \partial B_D(\widetilde{w}, s)$$

$$\theta \mapsto (\tau + \widetilde{\tau} e^{i\theta})\,\widetilde{z}$$

which is non constant since $\widetilde{\tau} \neq 0$.
Thus $\mu_D(\beta(\theta) - \widetilde{w}) = s$ for all $\theta \in \mathbb{R}^+$. Let us compute

$$\frac{d}{d\theta} \mu_D((\tau + \widetilde{\tau} e^{i\theta})\widetilde{z} - \widetilde{w}) = -\sum_{j=1}^n \frac{\partial \mu_D}{\partial |z_j|}((\tau + \widetilde{\tau} e^{i\theta})\widetilde{z} - \widetilde{w}) \frac{\operatorname{Im}(\widetilde{\tau}\widetilde{z}_j(\tau \overline{\widetilde{z}}_j - \overline{\widetilde{w}}_j) e^{i\theta})}{|\tau \widetilde{z}_j - \widetilde{w}_j + \widetilde{\tau} \widetilde{z}_j e^{i\theta}|} =$$

$$= -\sum_{j=1}^n \frac{\partial \mu_D}{\partial |z_j|}((\tau + \widetilde{\tau} e^{i\theta})\widetilde{z} - \widetilde{w}) \frac{\widetilde{\tau} |\widetilde{z}_j|^2 (\tau - \nu_j) \operatorname{Im} e^{i\theta}}{|\tau \widetilde{z}_j - \widetilde{w}_j + \widetilde{\tau} \widetilde{z}_j e^{i\theta}|} = 0$$

for all $\theta \in \mathbb{R}$. Since $\mu_D(\widetilde{z}) < 1$ and $|\widetilde{z}_j| < \mu_j(\widetilde{z})$, then $\tau - \nu_j > 0$ and so we get a contradiction.

QED

This Theorem also holds for complex ellipsoids $\mathcal{E}(\mathbf{p})$ with some $1/2 \leq p_j < 1$ because $(\tau + \widetilde{\tau} e^{i\theta})\widetilde{z}_j - \widetilde{w}_j = 0$ if, and only if,

$$\tau + \widetilde{\tau} e^{i\theta} = \frac{1 - r^2}{1 - r^2 \frac{|\widetilde{z}_j|^2}{\mu_j(\widetilde{z})^2}}$$

and it is easily seen that this equation can hold only if $\theta = \pm\pi$. Analogously, we have the following

Corollary 5.2.2 *Let $\widetilde{z} \in D_{a,\mathbf{p}}$ be such that at least two of its components are non zero. Then*

$$B^*_{c_{D_{a,\mathbf{p}}}}(\widetilde{z}, r) \neq B_{D_{a,\mathbf{p}}}(\widetilde{w}, s)$$

for all $\widetilde{w} \in D_{a,\mathbf{p}}$, $r \in (0,1)$, $s \in \mathbb{R}^+$.

3. Carathéodory balls and norm balls in the domains $D_{a,\mathbf{p}} \subseteq \mathbb{C}^n$ when $n > 2$

From now on we restrict our attention to the domains $D_{a,\mathbf{p}}$.
By Corollary 5.2.2, it suffices to study the Carathéodory balls centred on some axis.
In this section by exploiting the formulas for the complex geodesics of the domains $D_{a,\mathbf{p}}$, we can completely understand the situation when our domains are in a space of dimension greater than 2.

Theorem 5.3.1 *Let $n > 2$, let $D_{a,\mathbf{p}} \subseteq \mathbb{C}^n$ be convex. Let $\widetilde{z} \in D_{a,\mathbf{p}}$ and $k \in \{1, \ldots, n\}$, be such that $\widetilde{z}_j = 0$ for all $j \neq k$ and $\widetilde{z}_k \neq 0$. Let us suppose that*

$$B^*_{c_{D_{a,\mathbf{p}}}}(\widetilde{z}, r) = B_{D_{a,\mathbf{p}}}(\widetilde{w}, s)$$

for some $\widetilde{w} \in \mathbb{C}^n$, $r \in (0,1)$, $s \in \mathbb{R}^+$. Then $a = 0$, $p_k = 1$ and $p_j = 1/2$ for all $j \neq k$, i.e. $D_{a,\mathbf{p}} = \mathcal{E}(p_1, \ldots, p_n)$ is a convex complex ellipsoid.

Proof: Let $\widetilde{z} = (0, \ldots, \widetilde{z}_k, \ldots, 0) \in D_{a,\mathbf{p}}$ and suppose $B^*_{c_{D_{a,\mathbf{p}}}}(\widetilde{z}, r) = B_{D_{a,\mathbf{p}}}(\widetilde{w}, s)$ for some $\widetilde{w} \in \mathbb{C}^n$, $r \in (0,1)$, $s \in \mathbb{R}^+$. By Lemma 5.1.1 and recalling that $\mu_k(\widetilde{z}) = 1$ we know that

$$\widetilde{w} = \frac{1 - r^2}{1 - r^2|\widetilde{z}_k|^2}\widetilde{z}.$$

Moreover, since $f_{a,\mathbf{p}}\left(\frac{1}{s}z\right) = 1$ if and only if $\mu_{D_{a,\mathbf{p}}}\left(\frac{1}{s}z\right) = 1$, then from equation (5.1.2) we get

$$\frac{r}{s} = \frac{1 - r^2|\widetilde{z}_k|^2}{1 - |\widetilde{z}_k|^2}. \tag{5.3.1}$$

The hypothesis $B^*_{c_{D_{a,\mathbf{p}}}}(\widetilde{z}, r) = B_{D_{a,\mathbf{p}}}(\widetilde{w}, s)$ implies that

$$f_{a,\mathbf{p}}\left(\frac{1}{s}(\varphi(r) - \widetilde{w})\right) = 1 \tag{5.3.2}$$

for all complex geodesics $\varphi : \Delta \to D_{a,\mathbf{p}}$ such that $\varphi(0) = \widetilde{z}$.
By Proposition 4.2.1 we know explicitly all complex geodesics $\varphi : \Delta \to D_{a,\mathbf{p}}$ such that $\varphi_j \not\equiv 0$ for all $j = 1, \ldots, n$ and $\varphi(0) = \widetilde{z}$.
In order to obtain all complex geodesics $\varphi : \Delta \to D_{a,\mathbf{p}}$ such that $\varphi(0) = \widetilde{z}$ it suffices to consider $r_j \geq 0$ for all $j \neq k$ (where the notations are those used in Proposition 4.2.1). In fact, if $r_j = 0$, then $\varphi_j \equiv 0$ and the projection $\widetilde{\varphi} \stackrel{\text{def}}{=} (\varphi_1, \ldots, \hat{\varphi}_j, \ldots, \varphi_n) : \Delta \to D_{a,\mathbf{p}'} \subseteq \mathbb{C}^{n-1}$ is a complex geodesic in $D_{a,\mathbf{p}'}$, because it satisfies Proposition 4.2.1. This implies that φ is a complex geodesic in $D_{a,\mathbf{p}}$.
Now, let $k > 2$.
Let us consider the complex geodesics $\varphi : \Delta \to D_{a,\mathbf{p}}$ such that $\varphi(0) = \widetilde{z}$, $s_k = 0$ and $r_j > 0$ for one and only one $j \neq k$ (where the notations are those used in Proposition 4.2.1). In this case

$$\alpha_o = r_k\,\alpha_k \qquad 1 + r_k^2|\alpha_k|^2 = r_j + r_k(1 + |\alpha_k|^2)$$
$$\varphi_k(0) = \exp{(i\theta_k)} r_k^{1/(2p_k)} = \widetilde{z}_k$$

and equation (5.3.2) becomes

$$\left(\frac{r}{s}\right)^{2p_j} r_j \frac{1}{|1 - r_k\,\overline{\alpha}_k\,r|^2} + \frac{1}{s^{2p_k}}|\widetilde{z}_k|^{2p_k} \left|\left(\frac{1 - \overline{\alpha}_k r}{1 - r_k\overline{\alpha}_k r}\right)^{1/p_k} - \frac{1 - r^2}{1 - r^2|\widetilde{z}_k|^2}\right|^{2p_k} = 1. \tag{5.3.3}$$

If $\alpha_k = 0$, we get $r_j = 1 - r_k = 1 - |\widetilde{z}_k|^{2\,p_k}$ and equation (5.3.3) becomes

$$\left(\frac{r}{s}\right)^{2p_j} (1 - |\widetilde{z}_k|^{2p_k}) + \left(\frac{r}{s}\right)^{2p_k} \left(\frac{1 - |\widetilde{z}_k|^2}{1 - r^2|\widetilde{z}_k|^2}\right)^{2p_k} |\widetilde{z}_k|^{2\,p_k} r^{2\,p_k} = 1.$$

Hence, for all $j \neq k$,

$$\left(\frac{r}{s}\right)^{2p_j} = \left(\frac{1 - r^2|\widetilde{z}_k|^2}{1 - |\widetilde{z}_k|^2}\right)^{2\,p_j} = \frac{1 - r^{2p_k}|\widetilde{z}_k|^{2p_k}}{1 - |\widetilde{z}_k|^{2p_k}}. \tag{5.3.4}$$

If $\alpha_k \in \overline{\Delta}$ one gets

$$r_j = (1 - |\widetilde{z}_k|^{2p_k})(1 - |\widetilde{z}_k|^{2p_k}|\alpha_k|^2)$$

and by using equation (5.3.4), one can write equation (5.3.3) as follows

$$\frac{1 - r^{2p_k}|\widetilde{z}_k|^{2p_k}}{1 - |\widetilde{z}_k|^{2p_k}} \frac{(1 - |\widetilde{z}_k|^{2p_k})(1 - |\widetilde{z}_k|^{2p_k}|\alpha_k|^2)}{|1 - |\widetilde{z}_k|^{2p_k}\,\overline{\alpha}_k\,r|^2} +$$

$$+\frac{|\widetilde{z}_k|^{2p_k}}{s^{2p_k}}\left|\left(\frac{1-\overline{\alpha}_k r}{1-|\widetilde{z}_k|^{2p_k}\overline{\alpha}_k r}\right)^{1/p_k}-\frac{1-r^2}{1-r^2|\widetilde{z}_k|^2}\right|^{2p_k}=1$$

(5.3.5)

$$\frac{|\widetilde{z}_k|^{2p_k}}{s^{2p_k}}\left|\left(\frac{1-\overline{\alpha}_k r}{1-|\widetilde{z}_k|^{2p_k}\overline{\alpha}_k r}\right)^{1/p_k}-\frac{1-r^2}{1-r^2|\widetilde{z}_k|^2}\right|^{2p_k}=1-\frac{(1-r^{2p_k}|\widetilde{z}_k|^{2p_k})(1-|\widetilde{z}_k|^{2p_k}|\alpha_k|^2)}{|1-|\widetilde{z}_k|^{2p_k}\,\overline{\alpha}_k\,r|^2}.$$

In particular, if $\alpha_k = r \in (0,1)$ one computes

$$1-\frac{(1-r^{2p_k}|\widetilde{z}_k|^{2p_k})(1-|\widetilde{z}_k|^{2p_k}r^2)}{|1-|\widetilde{z}_k|^{2p_k}\,r^2|^2}=\frac{|\widetilde{z}_k|^{2p_k}(r^{2p_k}-r^2)}{1-|\widetilde{z}_k|^{2p_k}\,r^2}.$$

Now, since the left side of equation (5.3.5) is always non-negative, then the same must hold true for the right side of it, hence it must be $r^{2p_k} - r^2 \geq 0$. This means that $p_k \leq 1$; and by recalling the convexity of our domains, we get $1/2 \leq p_k \leq 1$. One can check that for all $p_k \in [1/2, 1]$ the left side of equation (5.3.5) has a zero $\alpha_k \in (0,1)$ and this implies that, as function of $\alpha_k \in (-1,1)$, the left side of equation (5.3.5) is not smooth, unless $p_k = 1$, while the right side of equation (5.3.5) is smooth for all $p_k \in [1/2, 1]$.
So far, we have proved that, in our hypothesis, $p_k = 1$.
Therefore equation (5.3.4) becomes

$$\left(\frac{1-r^2|\widetilde{z}_k|^2}{1-|\widetilde{z}_k|^2}\right)^{2p_j}=\frac{1-r^2|\widetilde{z}_k|^2}{1-|\widetilde{z}_k|^2}$$

for all $j \neq k$ and hence $p_j = 1/2$ for all $j \neq k$.
If $p_1 = p_2 = 1/2$, then the convexity of $D_{a,\mathbf{p}}$ implies $a = 0$ and so $D_{a,\mathbf{p}} = \mathcal{E}(\mathbf{p})$.
As for the cases $k = 1$ and $k = 2$, it suffices to consider just one of them.
So, let us suppose that $\widetilde{z} = (\widetilde{z}_1, 0, \ldots, 0)$ with $\widetilde{z}_1 \neq 0$.
The discussion of the case $k > 2$ works to prove that, for $k = 1$, it must be $p_1 = 1$ and $p_j = 1/2$ for all $j \geq 3$.
In order to determine p_2, let us consider the complex geodesic φ whose only non-zero components are

$$\varphi_1(\lambda) \equiv \widetilde{z}_1$$
$$\varphi_2(\lambda) = \lambda\, \mu_2(\widetilde{z}).$$

Equation (5.3.2) in this case yields

$$1 = r^{2p_1}|\widetilde{z}_1|^{2p_1} + 2\,a\,r^{p_1}|\widetilde{z}_1|^{p_1}\left(\frac{r}{s}\right)^{p_2}(\mu_2(\widetilde{z}))^{p_2} + \left(\frac{r}{s}\right)^{2p_2}(\mu_2(\widetilde{z}))^{2p_2}.$$

Therefore

(5.3.6)

$$\left(\frac{1-r^2|\widetilde{z}_1|^2}{1-|\widetilde{z}_1|^2}\right)^{p_2} = \left(\frac{r}{s}\right)^{p_2} = \left(\frac{\mu_2(r\widetilde{z})}{\mu_2(\widetilde{z})}\right)^{p_2} = \frac{-a\,r^{p_1}|\widetilde{z}_1|^{p_1} + \sqrt{1-(1-a^2)r^{2p_1}|\widetilde{z}_1|^{2p_1}}}{-a|\widetilde{z}_1|^{p_1} + \sqrt{1-(1-a^2)|\widetilde{z}_1|^{2p_1}}}.$$

Now, we want to check that it is possible to choose one complex geodesic such that $s_1 = 0$, $\alpha_1 = 0$, $r_2 \geq 0$ and $r_j > 0$ for one index $j \geq 3$ only. This choice gives the following complex geodesics

$$\varphi_1(\lambda) = \exp(i\theta_1)\sqrt{2}\,r_1 \frac{1}{\sqrt{a^2(r_2-r_1)+2\,r_1+a\sqrt{a^2(r_2-r_1)^2+4\,r_1\,r_2}}} = \widetilde{z}_1$$

$$\varphi_2(\lambda) = \exp(i\theta_2)\,\lambda\,2^{\frac{1}{2p_2}}\,r_2^{\frac{1}{p_2}} \frac{1}{\left(a^2(r_1-r_2)+2\,r_2+a\sqrt{a^2(r_1-r_2)^2+4\,r_1\,r_2}\right)^{\frac{1}{2p_2}}}$$

$$\varphi_j(\lambda) = \exp(i\theta_j)\,\lambda\,r_j$$

$$\varphi_k(\lambda) = 0 \qquad k \geq 3, \quad k \neq j$$

$$r_1 + r_2 + r_j = 1.$$

The condition $\varphi_1(0) = \widetilde{z}_1$ is equivalent to the following equation

$$|\widetilde{z}_1|^2 = 2\,r_1^2 \frac{1}{\left(a^2(r_2-r_1)+2\,r_1+a\sqrt{a^2(r_2-r_1)^2+4\,r_1\,r_2}\right)}. \tag{5.3.7}$$

One checks that this equation is satisfied if $r_1 = |\widetilde{z}_1|^2$ and $r_2 = 0$. If $a \neq 0$, equation (5.3.7) can be solved for r_2 and we obtain

$$a^2\,|\widetilde{z}_1|^2\,r_2 = (r_1 - |\widetilde{z}_1|^2)(r_1 - (1-a^2)|\widetilde{z}_1|^2).$$

Therefore, when $a \neq 0$,

$$r_2 = f(r_1) = \frac{1}{a^2\,|\widetilde{z}_1|^2}(r_1 - |\widetilde{z}_1|^2)(r_1 - (1-a^2)|\widetilde{z}_1|^2).$$

Since $f(|\widetilde{z}_1|^2) = 0$ and $f'(|\widetilde{z}_1|^2) = \frac{1}{a^2\,|\widetilde{z}_1|^2}(2|\widetilde{z}_1|^2 - (2-a^2)|\widetilde{z}_1|^2) = 1 > 0$, then there exists a right neighborhood of $|\widetilde{z}_1|^2$, say U, such that for all $r_1 \in U$ we have $r_2 = f(r_1) > 0$ and $0 < r_1 + r_2 = r_1 + f(r_1) < 1$. In this situation we can define $r_j = g(r_1) = 1 - r_1 - r_2 = 1 - r_1 - f(r_1)$ so that the function $\varphi_{r_1} : \Delta \to D_{a,\mathbf{p}}$ defined as follows

$$(\varphi_{r_1})_1(\lambda) = \widetilde{z}_1$$

$$(\varphi_{r_1})_2(\lambda) = \exp(i\theta_2)\,\lambda\, 2^{\frac{1}{2p_2}}\, r_2^{\frac{1}{p_2}} \frac{1}{\left(a^2(r_1-r_2)+2\,r_2+a\,\sqrt{a^2(r_2-r_1)^2+4\,r_1\,r_2}\right)^{\frac{1}{2p_2}}}$$

$$(\varphi_{r_1})_j(\lambda) = \exp(i\theta_j)\,\lambda\, r_j$$

$$(\varphi_{r_1})_k(\lambda) = 0 \qquad \text{for all} \quad k \neq j, 1, 2$$

(where $r_1 \in U$, $r_2 = f(r_1)$ and $r_j = g(r_1)$) is a complex geodesic of $D_{a,\mathbf{p}}$ such that $\varphi_{r_1}(0) = \widetilde{z}$. Equation (5.3.2) applies to all complex geodesics such that $\varphi(0) = \widetilde{z}$, hence also to φ_{r_1}. Set $y = \left(\frac{1}{r}|(\varphi_{r_1})_2(r)|\right)^{p_2}$ and compute

$$f_{a,\mathbf{p}}\left(\frac{1}{s}((\varphi_{r_1})(r) - \nu_1 \widetilde{z})\right) =$$

$$= r^2|\widetilde{z}_1|^2 + 2\,a\,r|\widetilde{z}_1|\left(\frac{r}{s}\right)^{p_2} y + \left(\frac{r}{s}\right)^{2p_2} y^2 + \left(\frac{r}{s}\right)^{2p_j} r_j = 1. \tag{5.3.8}$$

We recall equations (5.3.6) and (5.3.1) for $(\frac{r}{s})^{p_2}$ and $(\frac{r}{s})$ respectively. Recall also that, since $\varphi_{r_1}(\exp(i\theta) \in \partial D_{a,\mathbf{p}}$ for almost all $\theta \in \mathbb{R}$, i.e.

$$|\widetilde{z}_1|^2 + 2a|\widetilde{z}_1|\,y + y^2 + r_j = 1,$$

one gets

$$y = -a|\widetilde{z}_1| + \sqrt{1-(1-a^2)|\widetilde{z}_1|^2 - r_j}.$$

Therefore (5.3.8) is an equation in r_j, which does not depend on r_1, and so $r_j = g(r_1) = 1 - r_1 - f(r_1)$ should give the same value for all $r_1 \in U$. But this is a contradiction. Therefore a must be 0. Equation (5.3.6), then, implies $p_2 = 1/2$.

QED

Let us consider now a complex ellipsoid $\mathcal{E}(\mathbf{p})$ such that $p_k = 1$ for one index $k \in \{1, \ldots, n\}$ and $p_j = 1/2$ for all $j \neq k$. Let $\widetilde{z} \in \mathcal{E}(\mathbf{p})$ be such that $\widetilde{z}_k \neq 0$ and $\widetilde{z}_j = 0$ for all $j \neq k$. One can check that, in this situation, equation (5.3.2) is fulfilled for all complex geodesics $\varphi : \Delta \to \mathcal{E}(\mathbf{p})$ such that $\varphi(0) = \widetilde{z}$. This fact implies that $\partial B^*_{c_{\mathcal{E}(\mathbf{p})}}(\widetilde{z}, r) \subseteq \partial B_{\mathcal{E}(\mathbf{p})}\left(\frac{1-r^2}{1-r^2|\widetilde{z}_k|^2}\widetilde{z}, \frac{1-|\widetilde{z}_k|^2}{1-r^2|\widetilde{z}_k|^2}r\right)$, which, by Lemma 5.1.2, is equivalent to the condition $B^*_{c_{\mathcal{E}(\mathbf{p})}}(\widetilde{z}, r) = B_{\mathcal{E}(\mathbf{p})}\left(\frac{1-r^2}{1-r^2|\widetilde{z}_k|^2}\widetilde{z}, \frac{1-|\widetilde{z}_k|^2}{1-r^2|\widetilde{z}_k|^2}r\right)$. Therefore we can state the following

Theorem 5.3.2 *Let $\mathcal{E}(p_1, \ldots, p_n) \subseteq \mathbb{C}^n$ with $p_j \geq 1/2$ for all $j = 1, \ldots, n$. Let $\widetilde{z} \in \mathcal{E}(p_1, \ldots, p_n)$ and $k \in \{1, \ldots, n\}$ be such that $\widetilde{z}_k \neq 0$ and $\widetilde{z}_j = 0$ for all $j \neq k$. Let $p_k = 1$ and $p_j = 1/2$ for all $j \neq k$. Then*

$$B^*_{c_{\mathcal{E}(p_1,\ldots,p_n)}}(\widetilde{z}, r) = B_{\mathcal{E}(p_1,\ldots,p_n)}\left(\frac{1-r^2}{1-r^2|\widetilde{z}_k|^2}\widetilde{z}, \frac{1-|\widetilde{z}_k|^2}{1-r^2|\widetilde{z}_k|^2}r\right)$$

for all $r \in (0,1)$.

As a particular case of our results we can state the following

Corollary 5.3.3 *Let* $\mathcal{E}(p_1,\ldots,p_n) \subseteq \mathbb{C}^n$ *with* $p_j \geq 1/2$ *for all* $j = 1,\ldots,n$. *Let* $\widetilde{z} \in \mathcal{E}(p_1,\ldots,p_n)$ *be such that* $\widetilde{z}_k \neq 0$ *for exactly one* $k \in \{1,\ldots,n\}$. *Let us suppose that*

$$B^*_{c_{\mathcal{E}(p_1,\ldots,p_n)}}(\widetilde{z},r) = B_{\mathcal{E}(p_1,\ldots,p_n)}(\widetilde{w},s)$$

for some $\widetilde{w} \in \mathbb{C}^n$, $r \in (0,1)$, $s \in \mathbb{R}^+$.
Then $p_k = 1$ *and* $p_j = 1/2$ *for all* $j \neq k$.
Moreover, if $p_k = 1$ *and* $p_j = 1/2$ *for all* $j \neq k$, *then*

$$B^*_{c_{\mathcal{E}(p_1,\ldots,p_n)}}(\widetilde{z},r) = B_{\mathcal{E}(p_1,\ldots,p_n)}\left(\frac{1-r^2}{1-r^2|\widetilde{z}_k|^2}\widetilde{z}, \frac{1-|\widetilde{z}_k|^2}{1-r^2|\widetilde{z}_k|^2}r\right)$$

for all $r \in (0,1)$.

We remark that this result holds for $n = 2$ too.

4. The case in which $D_{a,\mathbf{p}} \subseteq \mathbb{C}^2$

It remains to consider the case

(5.4.1) $$B^*_{c_{D_{a,\mathbf{p}}}}(\widetilde{z},r) = B_{D_{a,\mathbf{p}}}(\widetilde{w},s)$$

for convex domains $D_{a,\mathbf{p}} \subseteq \mathbb{C}^2$ for $a \geq 0$ and $\widetilde{z} = (\widetilde{z}_1, 0)$ or $\widetilde{z} = (0, \widetilde{z}_2)$.
First of all, let us remark that it suffices to discuss the case $B^*_{c_{D_{a,\mathbf{p}}}}(\widetilde{z},r) = B_{D_{a,\mathbf{p}}}(\widetilde{w},s)$ for $\widetilde{z} = (\widetilde{z}_1, 0)$.
The aim of this section is the proof of the following

Theorem 5.4.1 *Let* $D_{a,\mathbf{p}} \subseteq \mathbb{C}^2$ *be convex. Let* $\widetilde{z} = (\widetilde{z}_1, 0) \in D_{a,\mathbf{p}}$. *Let*

$$B^*_{c_{D_{a,\mathbf{p}}}}(\widetilde{z},r) = B_{D_{a,\mathbf{p}}}(\widetilde{w},s)$$

for some $\widetilde{w} \in D_{a,\mathbf{p}}$, $r \in (0,1)$, $s \in \mathbb{R}^+$. *Then*

$$D_{a,\mathbf{p}} = \mathcal{E}(1, 1/2).$$

By Corollary 5.3.3 it suffices to consider the case $a > 0$.
So, let us suppose that (5.4.1) holds for some $a > 0$ and some $\widetilde{z} = (\widetilde{z}_1, 0)$.

By Lemma 5.1.1 and recalling that $\mu_1(\widetilde{z}) = 1$ we know that $\widetilde{w} = \frac{1-r^2}{1-r^2|\widetilde{z}_1|^2}\widetilde{z} = \nu_1\widetilde{z}$. Moreover from equation (5.1.2) we get $\frac{r}{s} = \frac{1-r^2|\widetilde{z}_1|^2}{1-|\widetilde{z}_1|^2}$. Recall that equation (5.3.6) holds in this case too:

(5.4.2)

$$\left(\frac{r}{s}\right)^{p_2} = \left(\frac{1-r^2|\widetilde{z}_1|^2}{1-|\widetilde{z}_1|^2}\right)^{p_2} = \frac{-ar^{p_1}|\widetilde{z}_1|^{p_1} + \sqrt{1-(1-a^2)r^{2p_1}|\widetilde{z}_1|^{2p_1}}}{-a|\widetilde{z}_1|^{p_1} + \sqrt{1-(1-a^2)|\widetilde{z}_1|^{2p_1}}} = \left(\frac{\mu_2(r\widetilde{z})}{\mu_2(\widetilde{z})}\right)^{p_2}.$$

Hypothesis

$$B^*_{c_{D_{a,\mathbf{p}}}}(\widetilde{z}, r) = B_{D_{a,\mathbf{p}}}\left(\frac{1-r^2}{1-r^2|\widetilde{z}_1|^2}\widetilde{z}, \frac{1-|\widetilde{z}_1|^2}{1-r^2|\widetilde{z}_1|^2}r\right) \tag{5.4.3}$$

for some a (such that $D_{a,\mathbf{p}}$ is convex), some r and some $|\widetilde{z}_1|$ implies that

$$f_{a,\mathbf{p}}\left(\frac{1}{s}(\varphi(r) - \nu_1\widetilde{z})\right) = \tag{5.4.4}$$

$$= \frac{1}{s^{2p_1}}|\varphi_1(r) - \nu_1\widetilde{z}_1|^{2p_1} + 2a\frac{1}{s^{p_1}}|\varphi_1(r) - \nu_1\widetilde{z}_1|^{p_1}\frac{1}{s^{p_2}}|\varphi_2(r)|^{p_2} + \frac{1}{s^{2p_2}}|\varphi_2(r)|^{2p_2} = 1$$

for all complex geodesics $\varphi = (\varphi_1, \varphi_2) : \Delta \to D_{a,\mathbf{p}}$ such that $\varphi(0) = \widetilde{z}$, and in particular for those ones such that φ_1 has no zeroes in Δ, namely

$$\varphi_1(\lambda) = \frac{\widetilde{z}_1}{|\widetilde{z}_1|}2^{1/(2p_1)}\, r_1^{1/p_1}\frac{(1-\overline{\alpha}_1\lambda)^{2/p_1}}{(1-r_1\overline{\alpha}_1\lambda)^{1/p_1}}\,\widetilde{Q}_1(\lambda) \tag{5.4.5}$$

$$\varphi_2(\lambda) = \lambda\, 2^{1/(2p_2)}\,((1-r_1)\,(1-r_1\,|\alpha_1|^2))^{1/p_2}\frac{1}{(1-r_1\overline{\alpha}_1\lambda)^{1/p_2}}\,\widetilde{Q}_2(\lambda) \tag{5.4.6}$$

where, $\alpha_1 \in \overline{\Delta}$, $r_1 \in (0,1)$ and

$$\widetilde{Q}_\ell(\lambda) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\exp(it)+\lambda}{\exp(it)-\lambda}\,\frac{1}{2p_\ell}\log\left(g_\ell(\exp(it))\right)dt\right\} \qquad \ell = 1,2$$

$$g_1(\xi) = \frac{1}{a^2((1-r_1)\,(1-r_1\,|\alpha_1|^2) - r_1|1-\overline{\alpha}_1\xi|^2) + 2\,r_1|1-\overline{\alpha}_1\xi|^2 + a\,g(\xi)} \qquad \xi \in \partial\Delta$$

$$g_2(\xi) = \frac{1}{a^2(r_1|1-\overline{\alpha}_1\xi|^2 - (1-r_1)\,(1-r_1\,|\alpha_1|^2)) + 2\,(1-r_1)\,(1-r_1\,|\alpha_1|^2) + a\,g(\xi)}$$

$$g(\xi) = \sqrt{a^2((1-r_1)\,(1-r_1\,|\alpha_1|^2) - r_1|1-\overline{\alpha}_1\xi|^2)^2 + 4\,r_1|1-\overline{\alpha}_1\xi|^2\,(1-r_1)\,(1-r_1\,|\alpha_1|^2)}.$$

Moreover we require

$$|\varphi_1(0)| = 2^{1/(2p_1)}\,r_1^{1/p_1}\,|\widetilde{Q}_1(0)| = |\widetilde{z}_1|. \tag{5.4.7}$$

Notice that the left side of equations (5.4.4) and (5.4.7) can be viewed as C^1 functions of the variables $\alpha_1 \in (-1,1)$ and $r_1 \in (0,1)$. Therefore we define

$$F(r_1,\alpha_1) = |\varphi_1(0)|^{2p_1} - |\tilde{z}_1|^{2p_1}$$

and

$$G(r_1,\alpha_1) = \frac{1}{s^{2p_1}}|\varphi_1(r) - \nu_1\tilde{z}_1|^{2p_1} + 2a\frac{1}{s^{p_1}}|\varphi_1(r) - \nu_1\tilde{z}_1|^{p_1}\frac{1}{s^{p_2}}|\varphi_2(r)|^{p_2} + \frac{1}{s^{2p_2}}|\varphi_2(r)|^{2p_2} - 1.$$

where $\varphi_1(r)$ and $\varphi_2(r)$ are the values attained in $\lambda = r$ by the holomorphic functions defined by equations (5.4.5) and (5.4.6).
Using this notation, we can say that the hypothesis (5.4.1) implies that the function G should be zero for all $(r_1,\alpha_1) \in (0,1)\times(-1,1)$ such that $F(r_1,\alpha_1) = 0$.
If $\alpha_o = 0$ and $r_1 = \tilde{r}_1 \stackrel{\text{def}}{=} |\tilde{z}_1|^{p_1}(|\tilde{z}_1|^{p_1} + a(\mu_2(\tilde{z}))^{p_2})$, where we remind that $(\mu_2(\tilde{z}))^{p_2} = -a|\tilde{z}_1|^{p_1} + \sqrt{1-(1-a^2)|\tilde{z}_1|^{2p_1}}$, we obtain the complex geodesic $\tilde{\varphi}(\lambda) = (\tilde{z}_1, \lambda\,\mu_2(\tilde{z}))$.
Moreover, one computes

$$\left.\frac{\partial}{\partial r_1}F(r_1,\alpha_1)\right|_{(\tilde{r}_1,0)} = 2\,|\tilde{z}_1|^{2p_1}\left(\frac{1}{\tilde{r}_1} - g_1\left(1-a^2\right)\left(1+\frac{a}{g}(1-2\,\tilde{r}_1)\right)\right) > 0$$

We also have $\frac{\partial}{\partial \alpha_1}F(r_1,\alpha_1)|_{(\tilde{r}_1,0)} = 0$. Therefore we can apply the Implicit Function Theorem to the C^1 function F and find a neighborhood U of $\alpha_1 = 0$, a neighborhood V of $\tilde{r}_1$ and a function $f: U \to V$ such that

$$F\big(f(\alpha_1),\alpha_1\big) = 0 \qquad \forall\,\alpha_1 \in U$$

Moreover

$$\left.\frac{\partial f}{\partial \alpha_1}\right|_{\alpha_1} = -\frac{\left.\frac{\partial F}{\partial \alpha_1}\right|_{(f(\alpha_1),\alpha_1)}}{\left.\frac{\partial F}{\partial r_1}\right|_{(f(\alpha_1),\alpha_1)}} \quad \text{and} \quad \left.\frac{\partial f}{\partial \alpha_1}\right|_{\alpha_1=0} = 0$$

Since $F(f(\alpha_1),\alpha_1) = 0$ for all $\alpha_1 \in U$, then also $G(f(\alpha_1),\alpha_1) = 0$ for all $\alpha_1 \in U$.
Therefore

$$\frac{d}{d\alpha_1}G(f(\alpha_1),\alpha_1) = 0 \qquad \forall\,\alpha_1 \in U$$

In particular

$$\left.\frac{d}{d\alpha_1}G(f(\alpha_1),\alpha_1)\right|_{\alpha_1=0} = \left.\frac{\partial G}{\partial r_1}\right|_{(\tilde{r}_1,0)}\left.\frac{\partial f}{\partial \alpha_1}\right|_{\alpha_1=0} + \left.\frac{\partial G}{\partial \alpha_1}\right|_{(\tilde{r}_1,0)} = \left.\frac{\partial G}{\partial \alpha_1}\right|_{(\tilde{r}_1,0)} = 0$$

After a long computation one gets that $\frac{\partial G}{\partial \alpha_1}|_{(\tilde{r}_1,0)} = 0$ if, and only if,

$$\frac{1-r^2|\tilde{z}_1|^2}{r^2(1-|\tilde{z}_1|^2)}(r|\tilde{z}_1|)^{p_1}\left((r|\tilde{z}_1|)^{p_1} + a\mu_2(r\tilde{z})^{p_2}\right)\mu_2(\tilde{z})^{p_2}\left(a|\tilde{z}_1|^{p_1} + \mu_2(\tilde{z})^{p_2}\right) =$$

$$= \mu_2(r\widetilde{z})^{p_2}\big(a(r|\widetilde{z}_1|)^{p_1} + \mu_2(r\widetilde{z})^{p_2}\big)|\widetilde{z}_1|^{p_1}\big(|\widetilde{z}_1|^{p_1} + a\mu_2(\widetilde{z})^{p_2}\big).$$

Therefore we can state the following

Lemma 5.4.2 *Let $D_{a,\mathbf{p}} \subseteq \mathbb{C}^2$ be convex, let $a > 0$, $\widetilde{z} = (\widetilde{z}_1, 0)$ and $r \in (0,1)$. Let*

$$B^*_{c_{D_{a,\mathbf{p}}}}(\widetilde{z}, r) = B_{D_{a,\mathbf{p}}}\left(\frac{1-r^2}{1-r^2|\widetilde{z}_1|^2}\widetilde{z}, \frac{1-|\widetilde{z}_1|^2}{1-r^2|\widetilde{z}_1|^2}r\right).$$

Then

$$\frac{1-(r|\widetilde{z}_1|)^2}{(r|\widetilde{z}_1|)^2}\,\frac{(r|\widetilde{z}_1|)^{p_1}(r|\widetilde{z}_1|)^{p_1} + a\mu_2(r\widetilde{z})^{p_2})}{\mu_2(r\widetilde{z})^{p_2}\left(a(r|\widetilde{z}_1|)^{p_1} + \mu_2(r\widetilde{z})^{p_2}\right)} =$$

(5.4.8)
$$= \frac{(1-|\widetilde{z}_1|^2)}{|\widetilde{z}_1|^2}\,\frac{|\widetilde{z}_1|^{p_1}(|\widetilde{z}_1|^{p_1} + a\mu_2(\widetilde{z})^{p_2})}{\mu_2(\widetilde{z})^{p_2}(a|\widetilde{z}_1|^{p_1} + \mu_2(\widetilde{z})^{p_2})}$$

hold.

The above result will be used in the sequel together with the following

Lemma 5.4.3 *Let $D \subseteq \mathbb{C}^n$ be a convex bounded domain. Let $\widetilde{z} \in D$ and $r \in (0,1)$. Let $\varphi : \Delta \to D$ be a complex geodesic such that $\varphi(0) = \widetilde{z}$. Then the holomorphic map*

$$\psi_\varphi : \Delta \to B_{c_D}(\widetilde{z}, r)$$
$$\lambda \mapsto \varphi(r\,\lambda)$$

is a complex geodesic in $B_{c_D}(\widetilde{z}, r)$ such that $\psi_\varphi(0) = \widetilde{z}$.

Proof: By Proposition 2.1.3 a holomorphic map $\varphi : \Delta \to D$ is a complex geodesic if, and only if, there exists a holomorphic map $\widetilde{\varphi} : D \to \Delta$ such that $\widetilde{\varphi} \circ \varphi = Id_\Delta$. Let $w \in B_{c_D}(\widetilde{z}, r)$ and compute

$$\omega(0, \widetilde{\varphi}(w)) = \omega(\widetilde{\varphi}(\widetilde{z}), \widetilde{\varphi}(w)) \le c_D(\widetilde{z}, w) < \omega(0, r).$$

Therefore $|\widetilde{\varphi}(w)| < r$. Thus, we can define the holomorphic map

$$\widetilde{\psi} : B_{c_D}(\widetilde{z}, r) \to \Delta$$
$$w \mapsto \frac{1}{r}\widetilde{\varphi}(w).$$

On the other hand, the holomorphic map

$$\psi_\varphi : \Delta \to D$$
$$\lambda \mapsto \varphi(r\,\lambda)$$

is such that $\psi_\varphi(\Delta) \subseteq B_{c_D}(\widetilde{z}, r)$; in fact, for all $\lambda \in \Delta$, we have

$$c_D(\psi_\varphi(0), \psi_\varphi(\lambda)) = c_D(\widetilde{z}, \varphi(r\,\lambda)) = \omega(0, r\,\lambda) < \omega(0, r).$$

Finally, let us compute $\widetilde{\psi} \circ \psi_\varphi(\lambda) = \widetilde{\psi}(\varphi(r\,\lambda)) = \frac{1}{r}\widetilde{\varphi}(\varphi(r\,\lambda) = \frac{1}{r} r\,\lambda = \lambda$. Therefore $\psi_\varphi : \Delta \to B_{c_D}(\widetilde{z}, r)$ is a complex geodesic in $B_{c_D}(\widetilde{z}, r)$ such that $\psi_\varphi(0) = \widetilde{z}$.

QED

The next result will help us to simplify the dicsussion of equation (5.4.8)

Lemma 5.4.4 *Let $D_{a,\mathbf{p}} \subseteq \mathbb{C}^2$ be convex, let $a > 0$ and $\widetilde{z} = (\widetilde{z}_1, 0)$. Let*

$$B^*_{c_{D_{a,\mathbf{p}}}}(\widetilde{z}, r) = B_{D_{a,\mathbf{p}}}\left(\frac{1-r^2}{1-r^2|\widetilde{z}_1|^2}\widetilde{z}, \frac{1-|\widetilde{z}_1|^2}{1-r^2|\widetilde{z}_1|^2}r\right). \tag{5.4.9}$$

Then $p_1 = 2$.

Proof: The hypothesis (5.4.9) allows us to define the biholomorphism

$$\begin{aligned} \iota : D_{a,\mathbf{p}} &\to B^*_{c_{D_{a,\mathbf{p}}}}(\widetilde{z}, r) \\ (z_1, z_2) &\mapsto (sz_1 + \nu_1\widetilde{z}_1, sz_2) \end{aligned}$$

whose inverse is

$$\begin{aligned} \kappa = \iota^{-1} : B^*_{c_{D_{a,\mathbf{p}}}}(\widetilde{z}, r) &\to D_{a,\mathbf{p}} \\ (w_1, w_2) &\mapsto \left(\frac{1}{s}(w_1 - \nu_1\widetilde{z}_1), \frac{1}{s}w_2\right). \end{aligned}$$

Now, by Lemma 5.4.3 and using the biholomorphic map κ, starting from a complex geodesic φ for $D_{a,\mathbf{p}}$ such that $\varphi(0) = \tilde{z}$ we can define a complex geodesic ψ_φ (for $D_{a,\mathbf{p}}$) such that $\psi_\varphi(0) = r\tilde{z}$, namely

$$\psi_\varphi(\lambda) = \left(\frac{1}{s}(\varphi_1(r\lambda) - \nu_1\tilde{z}_1), \frac{1}{s}\varphi_2(r\lambda)\right).$$

Notice that ψ_φ is defined on a disc of radius $1/r > 1$.
Now, since $(\nu_1\widetilde{z}_1, s) = \left(\frac{1-r^2}{1-r^2|\tilde{z}_1|^2}\widetilde{z}, \frac{1-|\tilde{z}_1|^2}{1-r^2|\tilde{z}_1|^2}r\right) \in \partial B_{\mu_{c_{D_{a,\mathbf{p}}}}}\left(\frac{1-r^2}{1-r^2|\tilde{z}_1|^2}\widetilde{z}, \frac{1-|\tilde{z}_1|^2}{1-r^2|\tilde{z}_1|^2}r\right)$, the hypothesis (5.4.9) implies the existence of a complex geodesic $\varphi : \Delta \to D_{a,\mathbf{p}}$ such that $\varphi(0) = \tilde{z} = (\tilde{z}_1, 0)$ and $\varphi(r) = (\nu_1\widetilde{z}_1, s)$. The corresponding complex geodesic $\psi = (\psi_1, \psi_2) = \psi_\varphi : \Delta \to D_{a,\mathbf{p}}$ defined as above is such that

$$\psi(0) = r\tilde{z} = (r\tilde{z}_1, 0) \quad \text{and} \quad \psi(1) = (0, 1). \tag{5.4.10}$$

According to Proposition 4.2.1, there exist $\alpha_o \in \Delta$, $\alpha_1, \alpha_2 \in \overline{\Delta}$, $r_1, r_2 \in \mathbb{R}^+$, $\theta_1, \theta_2 \in [0, 2\pi]$ such that

$$\psi_1(\lambda) = \exp(i\theta_1)\left(\frac{\lambda - \alpha_1}{1 - \overline{\alpha}_1\lambda}\right)^{s_1} 2^{1/(2p_1)}\, r_1^{1/p_1}\frac{(1-\overline{\alpha}_1\lambda)^{2/p_1}}{(1-\overline{\alpha}_o\lambda)^{1/p_1}}\widetilde{Q}_1(\lambda)$$

$$\psi_2(\lambda) = \exp(i\theta_2)\left(\frac{\lambda-\alpha_2}{1-\overline{\alpha}_2\lambda}\right)^{s_2} 2^{1/(2p_2)}\, r_2^{1/p_2} \frac{(1-\overline{\alpha}_2\lambda)^{2/p_2}}{(1-\overline{\alpha}_o\lambda)^{1/p_2}} \widetilde{Q}_2(\lambda)$$

where

$$\widetilde{Q}_\ell(\lambda) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\exp(it)+\lambda}{\exp(it)-\lambda}\frac{1}{2p_\ell}\log\left(g_\ell(\exp(it))\right)dt\right\} \qquad \ell = 1,2$$

$$g_1(\xi) = \frac{1}{a^2(r_2|1-\overline{\alpha}_2\xi|^2 - r_1|1-\overline{\alpha}_1\xi|^2) + 2\,r_1|1-\overline{\alpha}_1\xi|^2 + a\,g(\xi)} \qquad \xi \in \partial\Delta$$

$$g_2(\xi) = \frac{1}{a^2(r_1|1-\overline{\alpha}_1\xi|^2 - r_2|1-\overline{\alpha}_2\xi|^2) + 2\,r_2|1-\overline{\alpha}_2\xi|^2 + a\,g(\xi)} \qquad \xi \in \partial\Delta$$

$$g(\xi) = \sqrt{a^2(r_2|1-\overline{\alpha}_2\xi|^2 - r_1|1-\overline{\alpha}_1\xi|^2)^2 + 4\,r_1|1-\overline{\alpha}_1\xi|^2\, r_2|1-\overline{\alpha}_2\xi|^2} \qquad \xi \in \partial\Delta$$

$$\alpha_o = r_1\alpha_1 + r_2\alpha_2 \qquad\qquad 1+|\alpha_o|^2 = r_1(1+|\alpha_1|^2) + r_2(1+|\alpha_2|^2).$$

For $j = 1,2$, $s_j \in \{0,1\}$ and $s_j = 1$ implies $\alpha_j \in \Delta$. Moreover conditions (5.4.10) must be satisfied.

First of all, $\psi_2(0) = 0$ implies $s_2 = 1$ and $\alpha_2 = 0$. Since ψ_1 is holomorphic on $\Delta_{1/r}$, condition $\psi_1(1) = 0$ implies that $\psi_1(\lambda) = (1-\lambda)\phi(\lambda)$, where ϕ is a holomorphic function defined on $\Delta_{1/r}$; then

$$\exp(i\theta_1)\left(\frac{\lambda-\alpha_1}{1-\overline{\alpha}_1\lambda}\right)^{s_1} 2^{1/(2p_1)}\, r_1^{1/p_1} \frac{(1-\overline{\alpha}_1\lambda)^{2/p_1}}{(1-\overline{\alpha}_o\lambda)^{1/p_1}} \widetilde{Q}_1(\lambda) = (1-\lambda)\phi(\lambda) \qquad \forall\,\lambda \in \Delta.$$

We know that ([Rudin 1974])

$$\lim_{\lambda\to\xi\in\partial\Delta}\left|\exp(i\theta_1)\left(\frac{\lambda-\alpha_1}{1-\overline{\alpha}_1\lambda}\right)^{s_1} 2^{1/(2p_1)}\, r_1^{1/p_1} \frac{(1-\overline{\alpha}_1\lambda)^{2/p_1}}{(1-\overline{\alpha}_o\lambda)^{1/p_1}} \widetilde{Q}_1(\lambda)\right| =$$

$$= 2^{1/(2p_1)}\, r_1^{1/p_1} \frac{|1-\overline{\alpha}_1\xi|^{2/p_1}}{|1-\overline{\alpha}_o\xi|^{1/p_1}} g_1(\xi)^{1/(2p_1)}$$

for almost all $\xi \in \partial\Delta$. Therefore

$$2^{1/(2p_1)}\, r_1^{1/p_1} \frac{|1-\overline{\alpha}_1\xi|^{2/p_1}}{|1-\overline{\alpha}_o\xi|^{1/p_1}} g_1(\xi)^{1/(2p_1)} = |1-\xi|\,|\phi(\xi)| \tag{5.4.11}$$

for almost all $\xi \in \partial\Delta$. Now, both the left and the right side of equation (5.4.11) are continuous function on $\partial\Delta$, therefore equation (5.4.11) holds for all $\xi \in \partial\Delta$. Since $g_1(\xi) \neq 0$ for all $\xi \in \partial\Delta$, then it must be $\alpha_1 = 1$ (and $s_1 = 0$).

Thus

$$\exp(i\theta_1) 2^{1/(2p_1)}\, r_1^{1/p_1} \frac{(1-\lambda)^{2/p_1}}{(1-r_1\lambda)^{1/p_1}} \widetilde{Q}_1(\lambda) = (1-\lambda)\phi(\lambda) \qquad \forall\,\lambda \in \Delta$$

and

$$2^{1/(2p_1)}r_1^{1/p_1}\frac{|1-\xi|^{2/p_1}}{|1-r_1\xi|^{1/p_1}}g_1(\xi)^{1/(2p_1)} = |1-\xi|\,|\phi(\xi)| \qquad \forall \xi \in \partial\Delta.$$

Last equation can be written as follows

$$2^{1/(2p_1)}r_1^{1/p_1}\frac{1}{|1-r_1\xi|^{1/p_1}}g_1(\xi)^{1/(2p_1)} = |1-\xi|^{1-2/p_1}\,|\phi(\xi)|.$$

Since $g_1(1)$ is well-defined and positive, being $r_1 < 1$, then it must be $2/p_1 = 1$.

QED

By combining Lemmas 5.4.2 and 5.4.4 we can now prove Theorem 5.4.1.

Proof of Theorem 5.4.1: By Lemma 5.4.4, $p_1 = 2$, and equation (5.4.8) (of Lemma 5.4.2) in this case becomes

$$1-(r|\tilde{z}_1|)^2\frac{(1-a^2)(r|\tilde{z}_1|)^2 + a\sqrt{1-(1-a^2)(r|\tilde{z}_1|)^4}}{(-a(r|\tilde{z}_1|)^2+\sqrt{1-(1-a^2)(r|\tilde{z}_1|)^4})\sqrt{1-(1-a^2)(r|\tilde{z}_1|)^4}} =$$

(5.4.12)
$$= (1-|\tilde{z}_1|^2)\frac{(1-a^2)|\tilde{z}_1|^2 + a\sqrt{1-(1-a^2)|\tilde{z}_1|^4}}{(-a|\tilde{z}_1|^2+\sqrt{1-(1-a^2)|\tilde{z}_1|^4})\sqrt{1-(1-a^2)|\tilde{z}_1|^4}}.$$

Let us consider the function

$$\Omega(x) = (1-x^2)\frac{(1-a^2)x^2 + a\sqrt{1-(1-a^2)x^4}}{(-ax^2+\sqrt{1-(1-a^2)x^4})\sqrt{1-(1-a^2)x^4}} =$$

$$= \frac{a + x^2\sqrt{1-(1-a^2)x^4}}{\sqrt{1-(1-a^2)x^4}(1+x^2)}$$

defined for $x \in (0,1)$. Let us compute

$$\Omega'(x) = \frac{2x((1-(1-a^2)x^4)(\sqrt{1-(1-a^2)x^4}-a) + a(1-a^2)x^2(1+x^2))}{(1+x^2)^2(1-(1-a^2)x^4)\sqrt{1-(1-a^2)x^4}}.$$

It is easy to check that $\Omega'(x) < 0$ if $a > 1$ and that $\Omega'(x) > 0$ if $a < 1$. Therefore the function Ω is injective if $a \neq 1$, while if $a = 1$ the equation (5.4.12) is always satisfied. Then equation (5.4.12) is satisfied (for all $r \in (0,1)$ and $\tilde{z}_1 \in \Delta$) if and only if $a = 1$. Now, if $p_1 = 2$ and $a = 1$, equation (5.4.2) becomes

$$\left(\frac{1-r^2|\tilde{z}_1|^2}{1-|\tilde{z}_1|^2}\right)^{p_2} = \frac{1-r^2|\tilde{z}_1|^2}{1-|\tilde{z}_1|^2}$$

which is satisfied if and only if $p_2 = 1$.
It is staightforward to notice that $D_{1,(2,1)} = \mathcal{E}(1, 1/2)$.

QED

Finally, we summarize the results of this chapter in the following

Theorem 5.4.5 *Let $D_{a,\mathbf{p}} \subseteq \mathbb{C}^n$ be convex. Let $\widetilde{z}, \widetilde{w} \in D_{a,\mathbf{p}}$, $\widetilde{z} \neq 0$, $r \in (0,1)$ and $s \in \mathbb{R}^+$. Then*

$$B^*_{c_{D_{a,\mathbf{p}}}}(\widetilde{z}, r) = B_{D_{a,\mathbf{p}}}(\widetilde{w}, s)$$

if and only if

$$D_{a,\mathbf{p}} = \mathcal{E}(p_1, \ldots, p_n)$$

where $p_k = 1$ for exactly one $k \in \{1, \ldots, n\}$, $p_j = 1/2$ for all $j \neq k$, $\widetilde{z}_j = 0$ for all $j \neq k$,

$$\widetilde{w} = \frac{1 - r^2}{1 - r^2 |\widetilde{z}_k|^2} \widetilde{z} \quad \text{and} \quad s = \frac{1 - |\widetilde{z}_k|^2}{1 - r^2 |\widetilde{z}_k|^2} r.$$

Sixth Chapter

COMPLEX GEODESICS IN NON-CONVEX DOMAINS

1. Complex geodesics in balanced pseudoconvex domains

The question of existence and uniqueness of complex geodesics for the Carathéodory (or for the Kobayashi) pseudodistance on non-convex domains is far from being understood, nevertheless there are some interesting partial results.

Example 6.1.1 Let ([Lempert 1982])

$$D = \{ (z_1, z_2) \in \mathbb{C}^2 \mid (1+|z_1|^2)(1+|z_2|^2) < 25 \}.$$

The point $\widetilde{z} = (2,2) \in \partial D$ is a "non-convex" point, i.e. D does not admit a supporting hyperplane at $\widetilde{z}$. Let $\widetilde{w} = (1,1)$. As a consequence of the Schwarz Lemma, it follows that the map

$$\begin{aligned} \varphi : \Delta &\to D \\ \lambda &\mapsto \lambda \widetilde{z} \end{aligned}$$

is the unique extremal map for $\widetilde{k}_D$ at 0 and $\widetilde{w}$, i.e. $\widetilde{k}_D(0,\widetilde{w}) = \omega(0,1/2)$. Lempert has proved that if one assumes that there exists a holomorphic retract S containing 0 and $\widetilde{w}$, then it should be $S = \varphi(\Delta)$. Moreover, it has been proved that if there exist a holomorphic retraction $r : D \to S$, then D should admit a supporting hyperplane at $\widetilde{z}$ and this is a contradiction.

Therefore, on an arbitrary domain one cannot try to characterize complex geodesics for the Kobayashi distance as holomorphic one-dimensional retracts.

Since complex geodesics for the Carathéodory distance are characterized as one-dimensional holomorphic retracts, then the above example shows also that on an arbitrary non-

convex domain it may happen that there does not exists any complex geodesic for the Carathéodory distance through two given points.
On the other hand, let

$$\begin{aligned} \varphi : \Delta &\to D \\ \lambda &\mapsto (\sqrt{24}\lambda, 0) \end{aligned}$$

and let

$$\begin{aligned} r : D &\to \Delta \\ (z_1, z_2) &\mapsto \left(\frac{1}{\sqrt{24}} z_1, 0\right). \end{aligned}$$

One checks that $r \circ \varphi = Id_\Delta$, therefore φ is a complex geodesic for c_D. Notice that the points $(e^{i\theta}\sqrt{24}, 0)$ are "convex" points of ∂D.

The situation above is typical of balanced pseudoconvex domains, in fact we have the following results ([Venturini 1990])

Proposition 6.1.1 *Let $D \subseteq \mathbb{C}^n$ be a balanced pseudoconvex domain. Let $\hat{D}$ be the convex hull of D. Let μ_D and $\mu_{\hat{D}}$ be the Minkowski functionals of D and $\hat{D}$ respectively. Then*
(a) a holomorphic map $\varphi : \Delta \to D \subseteq \hat{D}$ such that $\varphi(0) = 0$ is a complex geodesic for c_D in D if, and only if, φ is a complex geodesic for $c_{\hat{D}}$ and in this case we have that

$$\mu_{\hat{D}}(\varphi(\lambda)) = \mu_D(\varphi(\lambda))$$

for all $\lambda \in \Delta$ and that

$$\mu_{\hat{D}}(\varphi'(0)) = \mu_D(\varphi'(0);$$

(b) the holomorphic map

$$\begin{aligned} \varphi : \Delta &\to D \\ \lambda &\mapsto \lambda \frac{z}{\mu_D(z)}, \end{aligned}$$

where $z \in D$ is such that $\mu_D(z) > 0$, is a complex geodesic for c_D in D if, and only if, $\mu_D(z) = \mu_{\hat{D}}(z)$.

Proposition 6.1.2 *Let $D \subseteq \mathbb{C}^n$ be a balanced pseudoconvex domain. Let $\hat{D}$ be the convex hull of D. Let μ_D and $\mu_{\hat{D}}$ be the Minkowski functionals of D and $\hat{D}$ respectively. Then*
(a) there exists a complex geodesic for c_D passing through $0 \in D$ and $z \in D$ if, and only if, $\mu_D(z) = \mu_{\hat{D}}(z)$;
(b) there exists an infinitesimal complex geodesic for γ_D $\varphi : \Delta \to D$ such that $\varphi(0) = 0$ and $\varphi'(0) = v \in \mathbb{C}^n$ if, and only if, $\mu_D(z) = \mu_{\hat{D}}(z) = 1$.

Moreover, we have the following characterization of balanced convex domains among balanced pseudoconvex domains

Proposition 6.1.3 *Let D be a balanced pseudoconvex domain. Let μ_D be its Minkowski functional. Then the following facts are equivalent*
(a) D is convex;
(b) the Kobayashi pseudometric at the origin $\kappa_D(0;.)$ is the derivative of the Kobayashi pseudodistance;
(c) for any $v \in \mathbb{C}^n$ such that $\mu_D(v) = 1$ there exists a complex geodesic for c_D $\varphi : \Delta \to D$ such that $\varphi(0) = 0$ and $\varphi'(0) = v$;
(d) for any $v \in \mathbb{C}^n$ one has that $\kappa_D(0; v) = \gamma_D(0; v)$.

In particular, a balanced pseudoconvex domain biholomorphic to a convex domain is necessarily convex.
The above Propositions are obtained as consequences of the following ([Venturini 1990])

Theorem 6.1.4 *Let $D \subseteq \mathbb{C}^n$ be a balanced pseudoconvex domain. Let $\hat{D}$ be the convex hull of D. Let μ_D and $\mu_{\hat{D}}$ be the Minkowski functionals of D and $\hat{D}$ respectively. Let $z \in D$. Then one has that*

$$\widetilde{k}_D(0, z) = k_D(0, z)$$

if, and only if,

$$\mu_D(z) = \mu_{\hat{D}}(z);$$

and in this case one has that

$$c_D(0, z) = k_D(0, z).$$

In general, one can compute the Lempert function on a balanced domain as follows ([Jarnicki-Pflug 1993])

Proposition 6.1.5 *Let $D \subseteq \mathbb{C}^n$ be a balanced pseudoconvex domain. Let μ_D be the Minkowski functional of D. Let $z \in D$. Then one has that*

$$\widetilde{k}_D(0, z) = \omega(0, \mu_D(z)).$$

Now, let $D \subseteq \mathbb{C}^n$ be an arbitrary domain. Let $z \in D$. The <u>Carathéodory indicatrix</u> at z is defined as follows

$$C_z(D) = \{ v \in \mathbb{C}^n \mid \gamma_D(z; v) < 1 \}.$$

Analogously, the Kobayashi indicatrix at z is defined as follows

$$K_z(D) = \{ \ v \in \mathbb{C}^n \mid \kappa_D(z;v) < 1 \ \}.$$

In [Barth 1983] a remarkable result on the indicatrices of the Carathéodory and the Kobayashi at the origin of a balanced domain has been proved, namely

Theorem 6.1.6 *Let $D \subseteq \mathbb{C}^n$ be a balanced domain. Then*
(a) $D \subseteq K_0(D) \subseteq C_0(D)$;
(b) if D is convex, then $D = C_0(D)$;
(c) if D is pseudoconvex, then $D = K_0(D)$.

A simple consequence of this theorem is the following

Corollary 6.1.7 *Let $D \subseteq \mathbb{C}^n$ be a balanced domain. Then*
(a) if D is convex, then $\gamma_D(0;v) = \mu_D(v)$ for all $v \in \mathbb{C}^n$;
(b) if D is pseudoconvex, then $\kappa_D(0;v) = \mu_D(v)$ for all $v \in \mathbb{C}^n$.

Now, we can produce an example of a domain D (balanced and pseudoconvex) such that there exist an infinitesimal complex geodesic $\varphi : \Delta \to D$ for κ_D at $0 \in D$ which is not a complex geodesic for k_D at any couple of points in $\varphi(\Delta)$.

Example 6.1.8 Let

$$D = \{ \ (z_1, z_2) \in \mathbb{C}^2 \mid \ |z_1| < 1, \ |z_2| < 1, |z_1 z_2| < \varepsilon^2 \ \},$$

where $0 < \varepsilon < 1/2$. Let us define the holomorphic map

$$\begin{aligned} \varphi : \Delta &\to D \\ \lambda &\mapsto \lambda\,(\varepsilon, \varepsilon). \end{aligned}$$

One has that $\varphi(0) = 0$ and that $\varphi'(0) = (\varepsilon, \varepsilon)$. Since D is a balanced pseudoconvex domain, then Corollary 6.1.7 implies that

$$\kappa_D(\varphi(0); \varphi'(0)) = \kappa_D(0; (\varepsilon, \varepsilon)) = \mu_D((\varepsilon, \varepsilon)) = 1$$

and therefore φ is an infinitesimal complex geodesic for κ_D at the origin $0 \in D$. On the other hand, (since $(\varepsilon, \varepsilon)$ is a "non-convex" point of ∂D, then) one has that $\mu_{\hat{D}}((\varepsilon, \varepsilon)) < \mu_D((\varepsilon, \varepsilon))$ and therefore, by Theorem 6.1.4 and Proposition 6.1.5,

$$k_D(0, \lambda(\varepsilon, \varepsilon)) < \widetilde{k}_D(0, \lambda(\varepsilon, \varepsilon)) = \omega(0, \lambda\mu_D((\varepsilon, \varepsilon)) = \omega(0, \lambda)$$

for all $\lambda \in \Delta$. Thus φ is not a complex geodesic for k_D at any couple of points in $\varphi(\Delta)$.

2. Complex geodesics on pseudoconvex domains

We have seen that on an arbitrary balanced pseudoconvex domain complex geodesics may not exist. On the other hand, the following example shows that, sometimes, many different complex geodesics passing through a given point with the same direction can exist as well.

Example 6.2.1 Let, for any $t > 0$ and any $m \in \mathbb{N}$,

$$D_{t,m} = \{ \ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 + t\,|z_1^{2m} - z_2^m|^2 < 1 \ \}.$$

In [Sibony 1979] (see [Jarnicki-Pflug 1993]) it has been proved that for any $m \in \mathbb{N}$ there exists $t(m) > 0$ such that, for each $t \geq t(m)$, there are at least m different infinitesimal complex geodesics $\varphi : \Delta \to D_{t,m}$ for $\gamma_{D_{t,m}}$ such that $\varphi(0) = (0,0)$ and $\varphi'(0) = a\,(1,0)$ for some $a > 0$.

Besides some examples, there are few results concerning the existence of complex geodesics on strongly pseudoconvex domains and, as far as we know, they rely upon the fact that one can embed in a "nice" way a strongly pseudoconvex domain in a strongly convex domain. For example, in [Burns-Krantz 1994] the following result on the existence of complex geodesics for the Carathéodory distance on a strongly pseudoconvex domain whose image lie arbitrarily "near" the boundary is proved (see also [Huang 1995])

Proposition 6.2.2 *Let $D \subseteq \mathbb{C}^n$ be a strongly pseudoconvex domain with C^k boundary, $k \geq 6$. Let $\widetilde{z} \in \partial D$. Then there exist $\widetilde{w} \in D$ and an open neighborhood $W \subseteq D$ of $\widetilde{w}$ such that for every $w \in W$ there exist holomorphic mappings*

$$\varphi_w : \Delta \to D \qquad\qquad \psi_w : D \to \Delta,$$

which are C^{k-4} up to the boundary, such that
(a) $\varphi(0) = w$
(b) $\varphi(1) = \widetilde{z}$
(c) $\psi_w \circ \varphi_w = Id_\Delta$.
Moreover, given any neighborhood $U \subseteq \mathbb{C}^n$ of $\widetilde{z}$, one can assume that both $\widetilde{w}$ and $\varphi_w(\Delta)$ lie in $\overline{D} \cap U$.

We have seen that on strongly convex domains with C^3 boundary complex geodesics are characterized as stationary maps. This is not the case on a pseudoconvex domain, nevertheless in [Pang 1993] the following result is proved

Theorem 6.2.3 *Let $D \subseteq \mathbb{C}^n$ be a strongly pseudoconvex domain with smooth boundary. Let $\varphi : \Delta \to D$ be a holomorphic embedding of Δ into D of class C^2 up to the boundary of Δ. Let φ be an infinitesimal complex geodesic for κ_D at $\varphi(0)$. Then φ is a stationary map.*

A strongly pseudoconvex domain $D \subseteq \mathbb{C}^n$ with smooth boundary is taut and therefore for any $z \in D$ and any $v \in \mathbb{C}^n$ there exists an infinitesimal complex geodesic for κ_D $\varphi : \Delta \to D$ such that $\varphi(0) = z$ and $t\varphi'(0) = v$ for some $t > 0$. The above Theorem gives a system to find such an infinitesimal complex geodesic for κ_D: one should determine the stationary maps $\psi : \Delta \to D$ such that $\psi(0) = z$ and $t_\psi \psi'(0) = v$ for some $t_\psi > 0$, then the desired infinitesimal complex geodesic φ is the stationary map ψ such that the corresponding t_ψ is minimum (cf. [Pflug-Zwonek 1996], where this technique is used to compute the Kobayashi metric for the non-convex complex ellipsoids $\mathcal{E}(1, m) \subseteq \mathbb{C}^2$, with $0 < m < 1/2$).
The example below is due to Sibony and shows that a stationary map need not to be an infinitesimal complex geodesic for the Kobayashi pseudometric.

Example 6.2.4 Let (see [Pang 1993])

$$\Omega_t = \{ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 - t\,\mathrm{Re}(\overline{z}_1^4\, z_2^2) < 1 \}$$

where $t > 0$. One checks that the defining function of Ω_t is strictly plurisubharmonic near $\partial\Delta \times \{0\}$ for any $t > 0$. The holomorphic map

$$\begin{aligned} \varphi : \Delta &\to \Omega_t \\ \lambda &\mapsto (\lambda, 0) \end{aligned}$$

is proved to be a stationary map for all $t > 0$.
Let $t \leq 1$. In this case, the domain Ω_t is contained in the domain $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| < 1\}$ and φ is a holomorphic retract of Ω. Therefore φ is a holomorphic retract of Ω_t and then φ is a complex geodesic for c_{Ω_t}.
On the other hand, if $t > 1$, then φ is no more an infinitesimal complex geodesic at $\varphi(0)$ neither for the Kobayashi pseudometric. In fact, let $\varphi_s(\lambda) = ((1 + c\,s^2)\lambda, s\lambda^2)$, for all $\lambda \in \Delta$, where $c = (t - 1 - \varepsilon)/2$ and $\varepsilon > 0$. If ε is sufficiently small, then it turns out that $\varphi_s(\Delta) \subseteq \Omega_t$. One computes that $\varphi_s'(0) = (1 + c\,s^2, 0)$, which has bigger magnitude than $\varphi'(0) = (1, 0)$ whenever $s \neq 0$.
We can use this example also to see that it is possible to have a smooth family of infinitesimal complex geodesics for the Kobayashi pseudometric tangent to the same direction at a given point.
In fact, for $t = 1$, the holomorphic maps $\varphi_s(\lambda) = (\lambda, s\,\lambda^2)$ map Δ into Ω_1 for small s. Since $\varphi_s(0) = (0, 0) = \varphi(0)$ and $\varphi_s'(0) = (1, 0) = \varphi'(0)$, then, for small s, φ_s are infinitesimal complex geodesics for κ_{Ω_1} tangent to $(1, 0)$ at the origin $0 \in \Omega_1$.

Since Theorem 3.2.1 on complex geodesics on convex bounded Reinhardt domains is proved by exploiting the characterization of complex geodesics on convex bounded domains as stationary maps, then, as a consequence of Theorem 6.2.3, we can state the following

Theorem 6.2.5 *Let $D \subseteq \mathbb{C}^n$ be a bounded Reinhardt strongly pseudoconvex domain with smooth boundary. Let $\varphi : \overline{\Delta} \to \overline{D}$ be a holomrphic embedding of class C^2 up to the boundary of Δ and such that $\varphi_j \not\equiv 0$ for all $j = 1, \ldots n$. If φ is an infinitesimal complex geodesic for κ_D at $\varphi(0)$, then there exist $\alpha_\circ, \alpha_1, \ldots, \alpha_n \in \overline{\Delta}$, $r_1, \ldots, r_n \geq 0$ such that*

$$(a) \qquad r_j \frac{|1-\overline{\alpha}_j\xi|^2}{|1-\overline{\alpha}_\circ\xi|^2} = \frac{\partial \mu_D}{\partial |z_j|}(|\varphi_1(\xi)|, \ldots, |\varphi_n(\xi)|)|\varphi_j(\xi)| \qquad \textit{for all} \quad \xi \in \partial\Delta$$

for all $j = 1, \ldots, n$;

$$(b) \qquad \alpha_\circ = \sum_{j=1}^{n} r_j \alpha_j \qquad 1 + |\alpha_\circ|^2 = \sum_{j=1}^{n} r_j(1 + |\alpha_j|^2).$$

If $r_j > 0$, then there exists $\theta_j \in \mathbb{R}$ such that

$$M_j(\lambda) = e^{i\,\theta_j} \left(\frac{\lambda - \alpha_j}{1 - \overline{\alpha}_j \lambda} \right)^{s_j} \qquad \textit{for all} \quad \lambda \in \Delta,$$

where $s_j \in \{0, 1\}$ and $s_j = 1$ implies $\alpha_j \in \Delta$.

APPENDIX

1. Factorization Theorem

Let $\Delta = \{ \xi \in \mathbb{C} \mid |\xi| < 1 \}$ be the open unit disc of $\mathbb{C}$. For any $0 < p \leq \infty$, the symbol $H^p(\Delta)$ denotes, as usual, the Hardy space consisting of all holomorphic functions $h \in Hol(\Delta, \mathbb{C})$ from Δ into $\mathbb{C}$ such that $\| h \|_p < \infty$, while the symbol $H^p(\Delta, \mathbb{C}^n)$ denotes the set of all holomorphic mappings $h = (h_1, h_2, \ldots, h_n) : \Delta \to \mathbb{C}^n$ such that $h_j \in H^p(\Delta)$ for all $j = 1, \ldots, n$. Given $h \in H^1(\Delta)$ (or $h \in H^1(\Delta, \mathbb{C}^n)$), by h^* we mean the boundary value of h, which is defined almost everywhere on $\partial\Delta$ ([Rudin 1974]). Finally, $L^p(\partial\Delta)$, $0 < p \leq \infty$, stands for the set of all complex Lebesgue measurable functions h defined on $\partial\Delta$ for which $\| h \|_p < \infty$.

Theorem 1.1 *Let $0 < p \leq \infty$. Let $\varphi \in H^p(\Delta)$ be non identically 0. Then* $\log | \varphi^* | \in L^1(\partial\Delta)$*, the function*

$$Q_\varphi(\lambda) = \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\exp(it) + \lambda}{\exp(it) - \lambda} \log | \varphi^*(\exp(it)) | \, dt \right\} \qquad \lambda \in \Delta$$

is in $H^p(\Delta)$, there exists $M_\varphi \in H^\infty(\Delta)$ such that $|M_\varphi^*(\lambda)| = 1$ a.e. on $\partial\Delta$ and

$$\varphi(\lambda) = M_\varphi(\lambda) \, Q_\varphi(\lambda) \qquad \forall \lambda \in \Delta.$$

The function Q_φ is called the *outer factor* of φ and it depends only on the modulus of the boundary values of φ. A function $M \in H^\infty(\Delta)$ such that $|M_\varphi^*(\lambda)| = 1$ a.e. on $\partial\Delta$ is called an *inner function.* It can be proved that every inner function is of the following form ([Garnett 1981])

$$M(\lambda) = c \, B(\lambda) \, S(\lambda) \qquad \forall \lambda \in \Delta$$

where $c \in \partial\Delta$, B is a *Blaschke product* and S is a *singular function*, i.e.

$$B(\lambda) = \lambda^m \prod_{\alpha_k \neq 0} \frac{-\overline{\alpha}_k}{|\alpha_k|} \frac{\lambda - \alpha_k}{1 - \overline{\alpha}_k \lambda} \qquad \lambda \in \Delta$$

where $\{\alpha_k\}$ is a sequence of points in Δ such that $\sum_{k=1}^{\infty}(1 - |\alpha_k|) < \infty$ and m is a nonnegative integer, and

$$S(\lambda) = \exp\left\{ -\int_{-\pi}^{\pi} \frac{\exp(it) + \lambda}{\exp(it) - \lambda}\, d\mu(t) \right\} \qquad \lambda \in \Delta$$

for some finite positive Borel measure μ on $\partial\Delta$ which is singular with respect to Lebesgue measure. It can also be proved that $S^*(\lambda) = 1$ for a.a. (with respect to Lebesgue measure) $\lambda \in \partial\Delta$ and that $S^*(\lambda) = 0$ for a.a. (with respect to μ) $\lambda \in \partial\Delta$.

2. Minkowski functionals

Let $D \subseteq \mathbb{C}^n$ be a convex domain containing the origin $0 \in \mathbb{C}^n$ and let μ_D be its Minkowski functional (cf. [Rudin 1973], for example)

$$\mu_D(z) \stackrel{\text{def}}{=} \inf\left\{ t > 0 \;\middle|\; \frac{1}{t} z \in D \right\} < \infty \qquad z \in \mathbb{C}^n.$$

It turns out that $D = \{ z \in \mathbb{C}^n \mid \mu_D(z) < 1 \}$ and that $\partial D = \{ z \in \mathbb{C}^n \mid \mu_D(z) = 1 \}$. Moreover, one has that μ_D is a norm if, and only if, D is bounded convex and balanced. One easily checks that the Minkowski functional of a convex domain containing the origin is a convex function. We quote some results from the theory of convex functions which are be of interest ([Rockafellar 1970])

Theorem 2.1 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then*
(a) f is continuous and the following limit

$$f'(x; y) \stackrel{\text{def}}{=} \lim_{t \to 0+} \frac{f(x + t\,y) - f(x)}{t}$$

exists for all $x, y \in \mathbb{R}^n$; moreover, if f is differentiable at x, then $f'(x; y) = \langle\langle \nabla f(x), y \rangle\rangle$ for all $y \in \mathbb{R}^n$, where $\langle\langle .\,, . \rangle\rangle$ is the usual scalar product in $\mathbb{R}^n$;
(b) f is differentiable at $x \in \mathbb{R}^n$ if, and only if, the n two-sided partial derivatives $\frac{\partial f}{\partial x_j}(x)$ exist at x and are finite;
(c) f is differentiable almost everywhere on $\mathbb{R}^n$ and the gradient mapping $x \mapsto \nabla f(x)$ is continuous where it is defined;
(d) if f is differentiable at x_o, then the direction of $\nabla f(x_o)$ is the unique outer normal vector to $\{ x \in \mathbb{R}^n \mid f(x) \le f(x_o) \}$ at x_o; moreover, in this case,

$$f(x) \ge f(x_o) + \langle\langle \nabla f(x_o), x - x_o \rangle\rangle$$

for all $x \in \mathbb{R}^n$; *conversely, if there is a unique vector* $x^* \in \mathbb{R}^n$ *such that*

$$f(x) \geq f(x_o) + \langle\!\langle x^*, x - x_o \rangle\!\rangle$$

for all $x \in \mathbb{R}^n$, *then* f *is differentiable at* x_o;
(e) if $\emptyset \neq S \subseteq \mathbb{R}^n$ *is closed and bounded, then there exists* $0 < \beta < \infty$ *such that*

$$\langle\!\langle f'(x_o; y), x \rangle\!\rangle \leq \beta \parallel x \parallel$$

for all $x_o \in S$ *and all* $x \in \mathbb{R}^n$.

Since

$$\lim_{t \to 0} \frac{\mu_D(s\,z + t\,v) - \mu_D(s\,z)}{t} = \lim_{t \to 0} \frac{\mu_D\left(z + \frac{t}{s} v\right) - \mu_D(z)}{\frac{t}{s}}$$

for all $s > 0$, then, by Theorem 2.1 (*b*), we have that μ_D is differentiable at z_o if, and only if, it is differentiable at $s\,z_o$. Therefore we can state the following

Lemma 2.2 *Let* $D \subseteq \mathbb{C}^n$ *be a convex domain such that* $0 \in D$. *Then* μ_D *is continuously differentiable almost everywhere on* ∂D.

If $z_o \in \partial D$ is such that there exists a uniquely determined unit outer normal vector $\nu(z_o)$ to ∂D at z_o, then

$$\nu(z_o) = n(z_o) \overline{\left(\frac{\partial \mu_D}{\partial z_1}(z_o), \ldots, \frac{\partial \mu_D}{\partial z_n}(z_o) \right)}$$

where $n(z_o) = \left\| \left(\frac{\partial \mu_D}{\partial z_1}(z_o), \ldots, \frac{\partial \mu_D}{\partial z_n}(z_o) \right) \right\|^{-1}$ and $\mathrm{Re}(\langle z - z_o, \nu(z_o) \rangle) < 0$ for all $z \in D$, where $\langle .\,,. \rangle$ is the usual Hermitian product in $\mathbb{C}^n$.

3. Convexity and pseudoconvexity

A domain $D \subseteq \mathbb{C}^n$ is said to be <u>convex</u> if $z, w \in \overline{D}$ implies that $t\,z + (1-t)\,w \in \overline{D}$ for all $t \in [0, 1]$.
A domain $D \subseteq \mathbb{C}^n$ is said to be <u>strictly convex</u> if $z, w, \frac{1}{2}(z + w) \in \partial D$ implies that $z = w$.
A domain $D \subseteq \mathbb{C}^n$ has C^k-boundary, $k = 1, 2, \ldots, \infty, \omega$, if there exists a C^k function $\rho : \mathbb{C}^n \to \mathbb{R}$, which is called a <u>defining function</u> for D, such that:

(*a*) $D = \{ z \in \mathbb{C}^n \mid \rho(z) < 0 \}$;
(*b*) $\partial D = \{ z \in \mathbb{C}^n \mid \rho(z) = 0 \}$;
(*c*) the complex gradient vector $\left(\frac{\partial \rho}{\partial \bar{z}_1}(z), \ldots, \frac{\partial \rho}{\partial \bar{z}_n}(z) \right)$ is not vanishing for all $z \in \partial D$.

Let $D \subseteq \mathbb{C}^n$ be a domain with C^1-boundary. The real tangent space to ∂D at $z \in \partial D$ is given by

$$T_z^{\mathbb{R}}(\partial D) = \left\{ w \in \mathbb{C}^n \mid \operatorname{Re}\left(\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(z) w_j\right) = 0 \right\}$$

while the complex tangent space to ∂D at $z \in \partial D$ is given by

$$T_z^{\mathbb{C}}(\partial D) = \left\{ w \in \mathbb{C}^n \mid \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(z) w_j = 0 \right\}.$$

A domain $D \subseteq \mathbb{C}^n$ with C^2 boundary is said to be weakly convex at $z \in \partial D$ if

$$\operatorname{Re}\left(\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \, \partial z_k}(z) w_j w_k\right) + \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \, \partial \overline{z}_k}(z) w_j \overline{w}_k \geq 0$$

for all $w \in T_z^{\mathbb{R}}(\partial D)$.
A domain $D \subseteq \mathbb{C}^n$ with C^2 boundary is said to be strongly convex at $z \in \partial D$ if the above inequality is strict for all $w \in T_z^{\mathbb{R}}(\partial D) - \{0\}$.
One can prove that a weakly convex domain is convex ans that a strongly convex domain is strictly convex.
A domain $D \subseteq \mathbb{C}^n$ with C^2 boundary is said to be strictly linearly convex at $z \in \partial D$ if

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \, \partial \overline{z}_k}(z) w_j \overline{w}_k \geq \left| \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \, \partial z_k}(z) w_j w_k \right|$$

for all $w \in T_z^{\mathbb{C}}(\partial D)$.
A strongly convex domain is strictly linearly convex, but not viceversa ([Lempert 1984], in [Jarnicki-Pflug 1993] an example of a non-convex strictly linearly convex domain is exhibited).
A domain $D \subseteq \mathbb{C}^n$ with C^2 boundary is said to be (weakly) pseudoconvex at $z \in \partial D$ if

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \, \partial \overline{z}_k}(z) w_j \overline{w}_k \geq 0$$

for all $w \in T_z^{\mathbb{C}}(\partial D)$.
A domain $D \subseteq \mathbb{C}^n$ with C^2 boundary is said to be strongly pseudoconvex at $z \in \partial D$ if the above inequality is strict for all $w \in T_z^{\mathbb{C}}(\partial D)$.
A convex domain with C^2 boundary is pseudoconvex.

The following result is due to Narasimhan (see [Krantz 1992])

Lemma 3.1 *Let $D \subseteq \mathbb{C}^n$ be a relatively compact domain with C^2 boundary. Let $z \in \partial D$ be a point of strong pseudoconvexity. Then there exist a neighborhood $U \subseteq \mathbb{C}^n$ of z and a biholomorphic map Ψ on U such that $\Psi(U \cap \partial D$ is strongly convex.*

In [Fornaess 1976], Narasimhan Lemma has been improved as follows

Theorem 3.2 *Let $D \subseteq \mathbb{C}^n$ be a strongly pseudoconvex domain with C^2 boundary. Then there exist an integer $m > n$, a strongly convex domain $\Omega \subseteq \mathbb{C}^m$, a neighborhood $\hat{D}$ of $\overline{D}$, and a ono-to-one imbedding $\Psi : \hat{D} \to \mathbb{C}^m$ such that*
(a) $\Psi(D) \subseteq \Omega$;
(b) $\Psi(\partial D) \subseteq \partial\Omega$;
(c) $\Psi(\hat{D} - \overline{D}) \subseteq \mathbb{C}^m - \overline{\Omega}$;
(d) $\Psi(\hat{D})$ *is transversal to* $\partial\Omega$.

Sibony has given an example of a weakly pseudoconvex domain that cannot be mapped properly into any weakly convex domain of any dimension ([Sibony 1986]).
We refer to [Krantz 1992] for a detailed discussion on convexity and pseudoconvexity.

4. Subharmonic and plurisubharmonic functions

A function $u : \Omega \subseteq \mathbb{C} \to [-\infty, +\infty)$ defined on an open set Ω is called subharmonic if

(a) u is upper semicontinuous, that is the set $\{ \lambda \in \Omega \mid u(\lambda) < s \}$ is open for every $s \in \mathbb{R}$;

(b) for every compact set $K \subseteq \Omega$ and every continuous function h on K which is harmonic in the interior of K and is such that $h_{\partial K} \geq u_{\partial K}$, then it holds that $h \geq u$ on K.

A function $u : D \subseteq \mathbb{C}^n \to [-\infty, +\infty)$ defined on an open set D is called plurisubharmonic if

(a) u is upper semicontinuous;

(b) for every $z, w \in \mathbb{C}^n$, the function $\lambda \mapsto u(z + \lambda w)$ is subharmonic on the set $\{ \lambda \in \mathbb{C} \mid z + \lambda w \in D \}$.

A function $u : D \subseteq \mathbb{C}^n \to (0, +\infty)$ defined on an open set D is said to be logarithmically plurisubharmonic if $\log u$ is a plurisubharmonic function.
The following result gives an analytic condition on C^2 functions which is equivalent to plurisubharmonicity

Theorem 4.1 *A function $u : D \subseteq \mathbb{C}^n \to [-\infty, +\infty)$ of class C^2 is plurisubharmonic if, and only if,*

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \, \partial \overline{z}_k}(z) w_j \overline{w}_k \geq 0$$

for all $z \in D$ and all $w \in \mathbb{C}^n$.

Theorem 4.2 (Maximum Principle for subharmonic functions)
Let $D \subseteq \mathbb{C}^n$ be a bounded domain. Let $u : D \to [-\infty, +\infty)$ be a non-constant plurisubharmonic function. Then

$$u(z) < \sup_{\xi \in \partial D} \left\{ \limsup_{\substack{w \to \xi \\ w \in D}} u(w) \right\}$$

for all $z \in D$.

Proposition 4.3 *Let $\varphi : D_1 \subseteq \mathbb{C}^{n_1} \to D_2 \subseteq \mathbb{C}^{n_2}$ be a holomorphic map. Let $u : D_2 \to [-\infty, +\infty)$ be a plurisubharmonic function. Then $u \circ \varphi : D_1 \to [-\infty, +\infty)$ is a plurisubharmonic function.*

Proposition 4.4 *Let $\mu : \mathbb{C}^n \to \mathbb{R}^+ \cup 0$ be a seminorm. Then μ is a plurisubharmonic function.*

See [Dineen 1989] or [Hörmander 1979] for further results on plurisubharmonicity.
A domain $D \subseteq \mathbb{C}^n$ is called pseudoconvex if the function $-\log \operatorname{dist}(., \partial D)$ is plurisubharmonic, where

$$\operatorname{dist}(z, \partial D) = \inf_{w \in \partial D} \{ \parallel z - w \parallel \}$$

for all $z \in D$.

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