ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

Obstacle problems with measure data

CANDIDATE
Paolo Dall'Aglio

SUPERVISOR
Prof. Gianni Dal Maso

Thesis submitted for the degree of "Doctor Philosophiæ"

Academic Year 1997/98

SISSA - SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI
TRIESTE
Via Beirut 2-4

TRIESTE

Il presente lavoro costituisce la tesi presentata da Paolo Dall'Aglio, sotto la direzione del Prof. Gianni Dal Maso, al fine di ottenere il diploma di *"Doctor Philosophiæ"* presso la Scuola Internazionale Superiore di Studi Avanzati, Classe di Matematica, Settore di Analisi Funzionale ed Applicazioni.

Ai sensi del Decreto del Ministero della Pubblica Istruzione n. 419 del 24. 04.1987, tale diploma di ricerca post-universitaria è equipollente al titolo di *"Dottore di Ricerca in Matematica"*

Non chiedermi cos'è il duale,
né cos'è il biduale.
Dualità della dualità,
tutto è dualità.

Nicolas Barbecue

Acknowledgements

First of all, of course, I want to thank my supervisor and teacher Gianni Dal Maso. He has always been ready (and sufficiently patient) to answer questions and to make mathematics together.

Thanks also to my previous teachers: Rina Curti (I still have much to learn from her) and Maria Giovanna Garroni.

I want to thank, just naming them in strictly chaotic order, all the friends who gave sense to my stay in SISSA. Rodica, Fabrizio, Pasquale, Debora, Maria Stella, Roberto, Agata, Daniele, Monica, Alberto and Bianca, Mihai, Paolo, Michela, Carlota, Giovanni, Monica, Andrea...

Chiara Leone deserves to appear outside the list because of the pleasant mathematical collaboration and for the precious help in the last period.

Thank you to the SISSA football team (twice last on two championships).

Special thanks to Andrea Wehrenfennig and all the librarians for their kind assistance and for it was a pleasure to work with them.

Among people outside SISSA I choose to name here Lidia Corelli and Rosetta Stegel, who made easier my stay in Trieste, Miriana Stefano & Co., the friends of "Il Frantoio", the "Miramare" Youth Hostel and particularly il signor Vittorio, Libero Laganis...

Thanks to my friends in Rome: Marco (untiring keyboard-friend), Corrado, Andrea, Giovanni, Fabio, Chiara & Cricca, Bizio, Costanza, Alberto Luisa and Giacomo... for remote support.

Thank you to the Magrini family for their kindness, and above all thank you to Chiara, for she exists. This thesis is dedicated to her.

In conclusion it is impossible to list the reasons to be grateful to my mum, dad, brother and sister. Thank you.

Thank God I made it.

Contents

Introduction

Given a regular bounded open set Ω of $\mathbf{R}^N$, $N \geq 1$, and a linear elliptic operator $\mathcal{A}$ of the form

$$\mathcal{A}u = -\sum_{j,j=1}^{N} D_i(a_{ij}D_j u),$$

with $a_{ij} \in L^\infty(\Omega)$, we study obstacle problems for the operator $\mathcal{A}$ in Ω with homogeneous Dirichlet boundary conditions on $\partial\Omega$, when the datum μ is a bounded Radon measure on Ω and the obstacle ψ is an arbitrary function on Ω.

Obstacle problems have always been studied as part of the well known theory of variational inequalities (see Section 1.2). In this frame the problem consists in finding a function $u \in \mathrm{H}_0^1(\Omega)$ which is above a given function ψ (the obstacle) and is such that

$$\begin{cases} \langle \mathcal{A}u, v-u\rangle \geq \langle f, v-u\rangle \\ \forall v \in \mathrm{H}_0^1(\Omega) \text{ s.t. } v \geq \psi. \end{cases}$$

For such problems (which will be denoted by $VI(f,\psi)$) a wide abstract theory has been developed, and we know that if the datum f belongs to the dual $\mathrm{H}^{-1}(\Omega)$ of the Sobolev space $\mathrm{H}_0^1(\Omega)$, and if there exists at least a function $v \in \mathrm{H}_0^1(\Omega)$ above the obstacle ψ, then there exists one and only one solution. Also many results on continuous dependence with respect to data (both the forcing term and the obstacles) are known. Several characterizations of the solution have also been produced: the one we quote here concerns the fact that the solution of the variational inequality touches the obstacle wherever it is not the solution of the corresponding equation $\mathcal{A}u = f$. More precisely it is possible to prove that $\mathcal{A}u = f + \lambda$ and λ is a nonnegative measure concentrated on the set $\{u = \psi\}$.

In order to extend this theory to problems where the forcing term is a measure various difficulties arise.

The main ones are in writing equations with such data: it is not possible to use a variational formulation and the distributional solution of an elliptic equation with discontinuous coefficients may not be unique. So G. Stampacchia introduced in [41] a formulation (using duality and regularity arguments) which allowed to obtain existence and uniqueness in $\mathrm{W}_0^{1,q}(\Omega)$, with $q < \dfrac{N}{N-1}$, of the solution of an equation of the form

$$\begin{cases} \mathcal{A}u = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{0.0.1}$$

whenever μ is a measure in $\mathcal{M}_b(\Omega)$, the space of bounded Radon measures (see Section 1.3). These solutions coincide with the variational ones when the datum allows both formulations, i.e. when $\mu \in \mathcal{M}_b(\Omega) \cap \mathrm{H}^{-1}(\Omega)$.

Following these ideas we give (Section 2.1) the following definition of obstacle problems: a function u is a solution of this problem, which will be denoted by $OP(\mu, \psi)$, if u is the smallest function with the following properties: $u \geq \psi$ in Ω and u is a solution in the sense of Stampacchia [41] of a problem of the form

$$\begin{cases} \mathcal{A}u = \mu + \lambda & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases} \tag{0.0.2}$$

for some bounded Radon measure $\lambda \geq 0$. The measure λ which corresponds to the solution u of the obstacle problem $OP(\mu, \psi)$ is called the obstacle reaction associated with u.

From now on, with a little abuse of language, we will call obstacle problems the ones with measure data, according to the previous definition, and variational inequalities those with data in $\mathrm{H}^{-1}(\Omega)$, solved in the variational sense.

We want to study the problem with possibly also thin obstacles. This means that we will consider a function u to be above the obstacle ψ when the inequality $u \geq \psi$ holds up to sets of zero harmonic capacity.

For example if $B_1(0)$ is the unit ball of $\mathbb{R}^2$, the segment $I := \{-\frac{1}{2} < x < \frac{1}{2}, y = 0\}$ has nonzero capacity with respect to $B_1(0)$, but has zero 2-dimensional Lebesgue measure. So the sets $\{v \in \mathrm{H}^1_0(B_1(0)) \; : \; v \geq \chi_I \;\text{ a.e.}\}$ and $\{v \in \mathrm{H}^1_0(B_1(0)) \; : \; v \geq \chi_I \;\text{cap} - \text{q.e.}\}$ are different, since the former coincide with $\mathrm{H}^1_0(B_1(0))$. We shall consider only the latter in our formulation of the problem.

In order to simplify the exposition, throughout the thesis we assume that the obstacle ψ is quasi upper semicontinuous. This technical assumption is not restrictive, since we can prove that every obstacle problem can be replaced by an equivalent problem with a quasi upper semicontinuous obstacle (see Proposition 1.1.3).

The only restriction required on the choice of the obstacle is that there exists a nonnegative measure $\rho \in \mathcal{M}^+_b(\Omega)$ such that the solution of equation (0.0.1) with datum ρ is above the obstacle.

This condition, called OP-admissibility, is similar to the one needed for the variational case, but it is not comparable to that. Both are minimal in their context, in a sense that will be made precise in Section 2.5.

In this work we develop the three subjects outlined in the classic case. Existence and uniqueness of solutions, characterization in terms of "contact set" and continuous dependence with respect to data.

In Chapter 2 we prove existence and uniqueness of solutions of $OP(\mu, \psi)$.

To do this we first consider the case of negative obstacles (Section 2.2) so that we are able to show that, when the datum μ belongs both to $H^{-1}(\Omega)$ (for which variational inequalities make sense) and to $\mathcal{M}_b(\Omega)$ (for obstacle problems) the solution to the latter, if it exists is the same as the solution to the former.

Another preparatory result, which is also interesting on its own, says that the reaction λ of the obstacle can not be stronger than the "downward" part μ^- of the load μ, in the sense that

$$|\lambda|(\Omega) \leq |\mu^-|(\Omega).$$

Now, given a general $\mu \in \mathcal{M}_b(\Omega)$ we apply an approximation argument: we build a particular sequence of measures μ_k which belong to $\mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$ and such that $\mu_k \rightharpoonup \mu$ weakly-$*$ in $\mathcal{M}_b(\Omega)$; for any k the solution to $VI(\mu_k, \psi)$ exists and is unique, then we prove that this solutions converge and the limit is the desired solution of $OP(\mu, \psi)$. The result easily extends to the case of a general obstacle, provided the latter is OP-admissible (Section 2.3).

In Section 2.5 we will show that the variational solution to the Obstacle Problem (equation (1.2.1)) coincides with the new one (Definition 2.1.1) when both make sense. Section 2.6 provides a characterization of the solution in terms of approximating sequences of solutions of variational inequalities.

Chapter 3 is devoted to the study of the interaction between obstacles and data. The aim is to obtain something similar to complementarity conditions, but we will see that this is not possible in general.

An important role in this problem is played by the space $\mathcal{M}_b^0(\Omega)$ of all bounded Radon measures on Ω which are absolutely continuous with respect to the harmonic capacity. If the datum μ belongs to $\mathcal{M}_b^0(\Omega)$, so does the obstacle reaction, provided that there exists at least a measure $\sigma \in \mathcal{M}_b^0(\Omega)$ such that the corresponding solution of (0.0.1) is greater than or equal to ψ (see Theorem 3.1.5). Also in this case the obstacle reaction is concentrated on the contact set $\{u = \psi\}$ (see Theorem 3.1.7 proved by C. Leone in [33]). Example 3.0.1, which is a variant of an example proposed by L. Orsina and A. Prignet, shows that this is not always true when μ is not absolutely continuous with respect to the harmonic capacity.

Using the linearity of the operator $\mathcal{A}$, it is easy to see that the obstacle reaction belongs to $\mathcal{M}_b^0(\Omega)$ and is concentrated on the contact set $\{u = \psi\}$ even if just the negative part μ^- of μ belongs to $\mathcal{M}_b^0(\Omega)$. Therefore we concentrate our attention on the case $\mu^- \notin \mathcal{M}_b^0(\Omega)$. Then μ^- can be decomposed as $\mu^- = \mu_a^- + \mu_s^-$, where $\mu_a^- \in \mathcal{M}_b^0(\Omega)$ and μ_s^- is concentrated on a set of capacity zero. We assume that the obstacle ψ satisfies the estimates $-v - \varphi \le \psi \le v$, where $\varphi \in \mathrm{H}^1(\Omega)$ and v is the solution in the sense of Stampacchia of a problem of the form

$$\begin{cases} \mathcal{A}v = \nu & \text{in } \Omega, \\ v = 0 & \text{in } \partial\Omega, \end{cases} \tag{0.0.3}$$

with $\nu \in \mathcal{M}_b^0(\Omega)$. Then we prove (Theorem 3.3.1) that the obstacle problems $OP(\mu, \psi)$ and $OP(\mu^+ - \mu_a^-, \psi)$ have the same solution u, while the corresponding obstacle reactions λ and λ_1 satisfy $\lambda = \lambda_1 + \mu_s^-$. This shows that, under these assumptions, the solution u of $OP(\mu, \psi)$ does not depend on μ_s^-, while the obstacle reaction has the form $\lambda_1 + \mu_s^-$, where λ_1 is a nonnegative measure in $\mathcal{M}_b^0(\Omega)$ which is concentrated on the contact set $\{u = \psi\}$ (Theorem 3.3.5).

These results rely on Lemma 3.2.5 which is the most general form of the following result, which has an intrinsic interest. Let u_μ and u_ν be the solutions of (0.0.3) corresponding to the measures μ and ν, which are not assumed to belong to $\mathcal{M}_b^0(\Omega)$. Suppose that $\mu^+ \perp \nu$ and that $u_\mu \le u_\nu$. Then $\mu^+ \in \mathcal{M}_b^0(\Omega)$. This result is obtained by investigating the behaviour of the potentials of two mutually singular measures near their singular points (Lemmas 3.2.3 and 3.2.4).

Finally Chapter 4 develops the theme of continuous dependence with respect to data.

As for stability with respect to the right hand side, we obtain that if μ_n, $\mu \in \mathcal{M}_b(\Omega)$ are such that $\mu_n \to \mu$ strongly in $\mathcal{M}_b(\Omega)$ then $u_n \to u$ strongly in $\mathrm{W}^{1,q}(\Omega)$, where u_n and u are the solutions of $OP(\mu_n, \psi)$ and $OP(\mu, \psi)$ respectively (Proposition 4.1.1).

This is too strong a condition, because, for instance, we can not approximate all measures $\mu \in \mathcal{M}_b(\Omega)$ by means of more regular measures (say elements of $\mathcal{M}_b(\Omega) \cap \mathrm{H}^{-1}(\Omega)$) with strong convergence. Indeed the strong closure of $\mathcal{M}_b(\Omega) \cap \mathrm{H}^{-1}(\Omega)$ is $\mathcal{M}_b^0(\Omega)$.

So we would need to turn our attention to weak-$*$ convergence. But Example 4.1.4 shows that in general $\mu_n \rightharpoonup \mu$ weakly-$*$ does not imply that $u_n \to u$, even with the obstacle $\psi \equiv 0$.

So we content ourselves with the fact (see Section 2.6) that for any measure $\mu \in \mathcal{M}_b(\Omega)$, there exists a special sequence $\mu_k \rightharpoonup \mu$ weakly-$*$ in $\mathcal{M}_b(\Omega)$, with $\mu_k \in \mathcal{M}_b(\Omega) \cap \mathrm{H}^{-1}(\Omega)$, such that, for any obstacle ψ, $u_k \to u$ strongly in $\mathrm{W}^{1,q}(\Omega)$.

It is easy to see that also stability with respect to obstacles (a first result is proved, for technical reasons, already in Section 2.4) is not always true. To study this question we introduce in Section 4.2 a kind of convergence of functions, the level set convergence, which yields the convergence of solutions under very mild assumptions.

The convergence of ψ_n to ψ in the sense of level sets, defined precisely in Definition 4.2.1, is verified in particular when

$$\operatorname{cap}(\{\psi > t\} \cap B) = \lim_{n \to +\infty} \operatorname{cap}(\{\psi_n > t\} \cap B)$$

for all $t \in \mathbb{R}$ and for all $B \subset\subset \Omega$ (see also Remark 4.2.5).

We will see that without further hypothesis it can only be proved that, calling u_n and u the solutions of $OP(\mu, \psi_n)$ and of $OP(\mu, \psi)$ respectively, if $\psi_n \xrightarrow{\text{lev}} \psi$ then, up to a subsequence, $u_n \to u^* \geq u$ (Proposition 4.2.9)

Then we will see that in all those situations, described in Chapter 3, in which the reaction of the obstacle is concentrated on the contact set, we obtain, from the level set convergence of the obstacles, that u_n converge to u.

In particular, by means of the Mosco convergence of convex sets, we obtain that

- if $\mu^- \in \mathrm{H}^{-1}(\Omega)$ then $u_n \to u$ strongly in $\mathrm{H}^1(\Omega)$;
- if $\mu^- \in \mathcal{M}_b^0(\Omega)$ then $u_n \to u$ strongly in $\mathrm{W}^{1,q}(\Omega)$;
- if ψ is suitably controlled below then $u_n \to u$ strongly in $\mathrm{W}^{1,q}(\Omega)$.

In Section 4.3 we list a few conditions, generalizing Proposition 2.4.1, that, together with level set convergence, ensure the convergence of the solutions.

We conclude this study considering two cases in which the assumptions that the obstacles converge in a stronger way allows to obtain a stronger convergence also for the solutions. When the difference $\psi_n - \psi$ belongs to $\mathrm{H}_0^1(\Omega)$ and tends to zero strongly in this space, then we obtain the same type of convergence for the solutions.

In Section 4.5, we extend the theory so far developed to the case of nonzero boundary values. For any function $g \in \mathrm{H}^1(\Omega)$, we can define the function u to be the solution of $OP(\mu, g, \psi)$ if and only if $u - u_0^g$ is the solution of $OP(\mu, \psi)$, where u_0^g is the solution of

$$\begin{cases} \mathcal{A}u_0^g = 0 & \text{in } \mathrm{H}^{-1}(\Omega) \\ u_0^g - g \in \mathrm{H}_0^1(\Omega). \end{cases}$$

All the results developed in the case of homogeneous boundary conditions can be extended, thanks to the linearity of $\mathcal{A}$.

Using this extension we prove a new characterization: the solution of $OP(\mu, g, \psi)$ is the minimum element among all the supersolutions of $\mathcal{A}-\mu$ which are above the obstacle and greater than or equal to g on the boundary $\partial\Omega$. From this we easily prove that if the obstacles difference $\psi_n - \psi$ tends to zero uniformly then so do the solutions of the corresponding obstacle problems.

The case of nonlinear operators was studied by L. Boccardo and T. Galloüet in [7] and by L. Boccardo and G.R. Cirmi in [5] and [6] when the right-hand side is in $L^1(\Omega)$. With Remark 4.1.3 we will note that our theory is consistent with that one.

More recently C. Leone studied in [33] nonlinear obstacle problems when the right hand side is in $\mathcal{M}_b^0(\Omega)$, using the so called entropy solutions (defined in [4] and [9]). The obstacle problems are defined similarly to the linear case, existence and uniqueness are proved together with coherence with linear theory. The characterization via complementarity conditions is the one we quote here in Theorem 3.1.7.

The content of the chapters 2, 3, and 4 corresponds approximately to the papers [17] with C. Leone, [16] with G. Dal Maso, and [15].

Chapter 1
Notations and preliminary results

Consider first the objects that won't change throughout the work.

Let Ω be a regular bounded open set in $\mathbb{R}^N$, $N \geq 1$ (for the notion of regularity see Definition 1.1.1).

Let $\mathcal{A}u = -\text{div}(A(x)\nabla u)$ be a linear elliptic operator with coefficients in $L^\infty(\Omega)$, that is $A(x) = (a_{ij}(x))$ is an $N \times N$ matrix such that

$$a_{ij} \in L^\infty(\Omega) \text{ and } \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \gamma|\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \text{ for a.e. } x \in \Omega. \tag{1.0.4}$$

Recall that $\mathrm{H}^1(\Omega)$ is the Sobolev space of functions with distributional derivatives in $\mathrm{L}^2(\Omega)$, and $\mathrm{H}^1_0(\Omega)$ is the closure in $\mathrm{H}^1(\Omega)$ of $C_0^\infty(\Omega)$. We shall denote

$$\|u\|_{\mathrm{H}^1(\Omega)} = \left(\int_\Omega |u|^2\,dx + \int_\Omega |Du|^2\,dx\right)^{\frac{1}{2}},$$

$$\|u\|_{\mathrm{H}^1_0(\Omega)} = \left(\int_\Omega |Du|^2\,dx\right)^{\frac{1}{2}},$$

respectively.

1.1. Capacity

We want to consider the obstacle problem also in the case of thin obstacles, so we will need the techniques of capacity theory. For this theory we refer, for instance, to [28].

We recall very briefly that, given a set $E \subseteq \Omega$, its capacity with respect to Ω is given by

$$\text{cap}(E,\Omega) = \inf\{\|z\|^2_{\mathrm{H}^1_0(\Omega)} : z \in \mathrm{H}^1_0(\Omega), z \geq 1 \text{ a.e. in a neighbourhood of } E\}.$$

When the ambient set Ω is clear from the context we will write $\text{cap}(E)$.

A property holds quasi everywhere (abbreviated as q.e.) when it holds up to sets of capacity zero.

A set A is said to be quasi open (resp. quasi closed) if for any $\varepsilon > 0$ there exists an open set V such that $\mathrm{cap}(V) < \varepsilon$ and $A \cup V$ is open (resp. $A \setminus V$ is closed).

A function $v : \Omega \to \overline{\mathbb{R}}$ is quasi continuous (resp. quasi upper semicontinuous) if, for every $\varepsilon > 0$ there exists a set E such that $\mathrm{cap}(E) < \varepsilon$ and $v_{|\Omega \setminus E}$ is continuous (resp. upper semicontinuous) in $\Omega \setminus E$.

We recall also that if u and v are quasi continuous functions and $u \leq v$ a.e. in Ω then also $u \leq v$ q.e. in Ω.

A function $u \in \mathrm{H}_0^1(\Omega)$ always has a quasi continuous representative, that is there exists a quasi continuous function $\tilde{u}$ which equals u a.e. in Ω. We shall always identify u with its quasi continuous representative. With this convention we have

$$\mathrm{cap}(\{u > t\}) \leq \frac{1}{t^2} \int_\Omega |Du|^2\, dx \qquad (1.1.1),$$

for all $t \in \mathbb{R}^+$. From this it follows that if $u_n \to u$ strongly in $\mathrm{H}_0^1(\Omega)$ then there exists a subsequence which converges quasi everywhere.

Moreover, for every set $E \subseteq \Omega$ we have that

$$\mathrm{cap}(E) = \min\{u \in \mathrm{H}_0^1(\Omega) \,:\, u \geq 1 \text{ q.e. in } E\}.$$

If $\mathrm{cap}(E) < +\infty$ the minimizer w_E is called the capacitary potential of E in Ω

We can now introduce the concept of Wiener point (see for instance [28]).

Definition 1.1.1. Given $\Omega \subseteq \mathbb{R}^N$ we say that a point $x_0 \in \partial\Omega$ satisfies the Wiener condition if

$$\int_0^1 \left[\frac{\mathrm{cap}(B_\rho(x_0) \cap \Omega^c, B_{2\rho}(x_0))}{\mathrm{cap}(B_\rho(x_0), B_{2\rho}(x_0))}\right] \frac{1}{\rho}\, d\rho = +\infty.$$

With this definition we can give the following theorem.

Theorem 1.1.2. *Let $\Omega \subseteq \mathbb{R}^N$. The point $x_0 \in \partial\Omega$ is a Wiener point if and only if for every $g \in \mathrm{L}^\infty(\Omega)$ the solution u of*

$$\begin{cases} \mathcal{A}u = g & \text{in } \mathrm{H}^{-1}(\Omega) \\ u \in \mathrm{H}_0^1(\Omega) \end{cases}$$

which is continuous in Ω *by Proposition 1.3.3, is such that*

$$\lim_{x \to x_0} u(x) = 0.$$

From now on we will assume that the set Ω is such that every point of $\partial\Omega$ is a Wiener point. Such a domain is said to be regular. This is not a very strong condition, especially in low dimensions, in particular notice that this condition is satisfied when $\partial\Omega$ is Lipschitz.

Consider the function $\psi : \Omega \to \overline{\mathbb{R}}$, and let us define the convex set

$$K_\psi(\Omega) := \{z \text{ quasi continuous } : z \geq \psi \text{ q.e. in } \Omega\}.$$

Without loss of generality we may always assume that ψ is quasi upper semicontinuous thanks to the following Proposition (it is a consequence of Proposition 1.5 in [19]).

Proposition 1.1.3. *Let* $\psi : \Omega \to \overline{\mathbb{R}}$. *Then there exists a quasi upper semicontinuous function* $\hat{\psi} : \Omega \to \overline{\mathbb{R}}$ *such that:*

1. $\hat{\psi} \geq \psi$ *q.e. in* Ω*;*

2. if $\varphi : \Omega \to \overline{\mathbb{R}}$ *is quasi upper semicontinuous and* $\varphi \geq \psi$ *q.e. in* Ω *then* $\varphi \geq \hat{\psi}$ *q.e. in* Ω*.*

Thus, in particular, $K_\psi(\Omega) = K_{\hat{\psi}}(\Omega)$.

1.2. Variational inequalities

In their natural setting, obstacle problems are part of the theory of variational inequalities (for which we refer to well known books such as [3], [31] and [43]).

For any datum $f \in \mathrm{H}^{-1}(\Omega)$ the variational inequality with obstacle ψ

$$\begin{cases} \langle \mathcal{A}u, v - u\rangle \geq \langle f, v - u\rangle \quad \forall v \in \mathrm{K}_\psi(\Omega) \cap \mathrm{H}^1_0(\Omega) \\ u \in \mathrm{K}_\psi(\Omega) \cap \mathrm{H}^1_0(\Omega) \end{cases} \tag{1.2.1}$$

is denoted by $VI(f, \psi)$, and makes sense whenever the set $\mathrm{K}_\psi(\Omega) \cap \mathrm{H}^1_0(\Omega)$ is nonempty, that is ensured by the condition

$$\exists z \in \mathrm{H}^1_0(\Omega) \;:\; z \geq \psi \text{ q.e. in } \Omega. \tag{1.2.2}$$

In this case we will say that the obstacle is VI-admissible.

Theorem 1.2.1. *For any $f \in \mathrm{H}^{-1}(\Omega)$ and for any ψ satisfying (1.2.2) there exists a unique solution of $VI(f,\psi)$. Moreover, if f_1, $f_2 \in \mathrm{H}^{-1}(\Omega)$ and u_1 and u_2 are the corresponding solutions, then*

$$\|u_1 - u_2\|_{\mathrm{H}_0^1(\Omega)} \leq c\,\|f_1 - f_2\|_{\mathrm{H}^{-1}(\Omega)}, \tag{1.2.3}$$

where c is a constant depending only on $\mathcal{A}$.

In this frame, among all classical results, we recall that the solution of $VI(f,\psi)$ is also characterized as the smallest function $u \in \mathrm{H}_0^1(\Omega)$ such that

$$\begin{cases} \mathcal{A}u - f \geq 0 \text{ in } \mathcal{D}'(\Omega) \\ u \geq \psi \quad \text{q.e. in } \Omega. \end{cases} \tag{1.2.4}$$

Since $\lambda := \mathcal{A}u - f$ is a nonnegative element of $\mathrm{H}^{-1}(\Omega)$, by the Riesz Representation Theorem, it is a nonnegative (not necessarily bounded) Radon measure, that will be called the obstacle reaction associated with u. The measure λ is concentrated on the set where the solution touches the obstacle. More precisely it can be proved that (see, e.g., Theorem 3.2 in [1]) u is the solution of $VI(f,\psi)$ if and only if

$$\begin{cases} \mathcal{A}u - f = \lambda \geq 0 \text{ in } \mathcal{D}'(\Omega) \\ u \geq \psi \quad \text{q.e. in } \Omega \\ \lambda(\{u > \psi\}) = 0. \end{cases} \tag{1.2.5}$$

The last condition can be read also as $u = \psi$ λ-a.e. These are called complementarity conditions.

Of course variational inequalities can be studied also with nonhomogeneous boundary conditions. Indeed for any $g \in \mathrm{H}^1(\Omega)$ we say that ψ is VI_g-admissible if there exists $z \in \mathrm{H}^1(\Omega)$ such that

$$\psi \leq z \quad \text{q.e. in } \Omega \qquad \text{and} \quad z - g \in \mathrm{H}_0^1(\Omega).$$

For such ψ and g we can define the set

$$K_\psi^g(\Omega) := \{z \in \mathrm{H}^1(\Omega) : z \geq \psi \text{ q.e. in } \Omega,\ z - g \in \mathrm{H}_0^1(\Omega)\},$$

which is nonempty. In this case the variational inequality (which will be indicated by $VI(f,g,\psi)$), is

$$\begin{cases} \langle \mathcal{A}u, v - u\rangle \geq \langle f, v - u\rangle \quad \forall v \in K_\psi^g(\Omega) \\ u \in K_\psi^g(\Omega) \end{cases} \tag{1.2.6}$$

and has all the properties of the homogeneous one. Another useful characterization of the solutions to variational inequalities is the following one.

Proposition 1.2.2. *Let $f \in H^{-1}(\Omega)$, $g \in H^1(\Omega)$ and ψ be VI_g-admissible. Then u is the solution of $VI(f,g,\psi)$ if and only if the two following conditions hold:*

$$\begin{cases} \mathcal{A}u - f \geq 0 & \text{in } \mathcal{D}'(\Omega) \\ u \geq \psi & q.e. \text{ in } \Omega \\ u \geq g & \text{on } \partial\Omega \end{cases}$$

and for all v such that

$$\begin{cases} \mathcal{A}v - f \geq 0 & \text{in } \mathcal{D}'(\Omega) \\ v \geq \psi & q.e. \text{ in } \Omega \\ v \geq g & \text{on } \partial\Omega \end{cases}$$

we have $v \geq u$ q.e. in Ω.

Recall that writing $u \geq g$ on $\partial\Omega$ we mean in the sense of $H^1(\Omega)$, i.e. $(u-g)^- \in H^1_0(\Omega)$.

Problems with nonhomogeneous boundary conditions will be taken into account in Section 4.5. Until that point, for the sake of simplicity, we will instead always consider the case when $g = 0$.

The problem of continuous dependence with respect to obstacles was completely solved by U. Mosco in [36]. He introduced a convergence for the convex sets $K_{\psi_n}(\Omega)$ which is defined as follows:

Definition 1.2.3. Let K_n be a sequence of subsets of a Banach space X. The strong lower limit

$$s-\liminf_{n\to+\infty} K_n$$

of the sequence K_n is the set of all $v \in X$ such that there exists a sequence $v_n \in K_n$, for n large, converging to v strongly in X.

The weak upper limit

$$w-\limsup_{n\to+\infty} K_n$$

of the sequence K_n is the set of all $v \in X$ such that there exists a sequence v_k converging to v weakly in X and a sequence of integers n_k converging to $+\infty$, such that $v_k \in K_{n_k}$.

The sequence K_n converges to the set K in the sense of Mosco, shortly $K_n \xrightarrow{M} K$, if

$$s-\liminf_{n\to+\infty} K_n = w-\limsup_{n\to+\infty} K_n = K.$$

Mosco proved that this type of convergence is the right one for the stability of variational inequalities with respect to obstacles. This is the main theorem of his theory.

Theorem 1.2.4. *Let ψ_n and ψ be VI-admissible. Then*

$$K_{\psi_n}(\Omega) \cap \mathrm{H}^1_0(\Omega) \xrightarrow{\mathrm{M}} K_{\psi}(\Omega) \cap \mathrm{H}^1_0(\Omega),$$

if and only if, for any $f \in \mathrm{H}^{-1}(\Omega)$,

$$u_n \to u \quad \textit{strongly in } \mathrm{H}^1(\Omega),$$

where u_n and u are the solutions of $VI(f, \psi_n)$ and $VI(f, \psi)$, respectively.

Several stability results can be proved as corollaries of this theorem by Mosco. In particular we give here the following two results which are well known, but that we prove here for the sake of completeness

Corollary 1.2.5. *Let $f \in \mathrm{H}^{-1}(\Omega)$ and let ψ_n and ψ be VI-admissible obstacles. Let u_n and u be the solutions of $VI(f, \psi_n)$ and $VI(f, \psi)$ respectively. If*

$$\psi_n \le \psi \quad q.e. \ in\ \Omega \qquad \psi_n \to \psi \quad q.e.\ in\ \Omega; \tag{1.2.7}$$

then

$$u_n \to u \quad \textit{strongly in } \mathrm{H}^1(\Omega).$$

Proof. Let us prove that

$$K_{\psi_n}(\Omega) \cap \mathrm{H}^1_0(\Omega) \xrightarrow{\mathrm{M}} K_{\psi}(\Omega) \cap \mathrm{H}^1_0(\Omega),$$

then the conclusion will follow from Theorem 1.2.4. Consider first the case when $\psi_n \le \psi_{n+1}$. Then clearly $K_{\psi_{n+1}}(\Omega) \cap \mathrm{H}^1_0(\Omega) \subseteq K_{\psi_n}(\Omega) \cap \mathrm{H}^1_0(\Omega)$. By Lemma 1.3 in [36] we have that

$$K_{\psi_n}(\Omega) \cap \mathrm{H}^1_0(\Omega) \xrightarrow{\mathrm{M}} S := \left(\bigcap_{n=1}^{+\infty} K_{\psi_n}(\Omega) \right) \cap \mathrm{H}^1_0(\Omega).$$

Let us prove that $S \subseteq K_{\psi}(\Omega) \cap \mathrm{H}^1_0(\Omega)$, the reverse inclusion being trivial. If $v \in S$, by Definition 1.2.3, there exists a sequence $v_n \in K_{\psi_n}(\Omega) \cap \mathrm{H}^1_0(\Omega)$ such that $v_n \to v$ strongly in $\mathrm{H}^1_0(\Omega)$. Then we can pass to the limit quasi everywhere in the inequality $v_n \ge \psi_n$ q.e. in Ω and obtain $v \ge \psi$ q.e. in Ω.

In the case of general ψ_n using the sequence $\varphi_n := \inf_{k \ge n} \psi_k$, which is such that $\varphi_n \le \psi_n$ q.e. in Ω and $\varphi_n \nearrow \psi$ q.e. in Ω, we can refer to the previous case. □

Corollary 1.2.6. *Let $\psi_n, \psi : \Omega \to \overline{\mathbb{R}}$ be VI-admissible. Suppose that $\psi_n - \psi \in \mathrm{H}^1(\Omega)$ and that $\psi_n - \psi \to 0$ strongly in $\mathrm{H}^1(\Omega)$. Let $f \in \mathrm{H}^{-1}(\Omega)$ and let u_n and u be the solutions of $VI(f, \psi_n)$ and $VI(f, \psi)$, respectively. Then*

$$u_n - u \to 0 \text{ strongly in } \mathrm{H}^1_0(\Omega).$$

Moreover if $\psi_n - \psi \in \mathrm{H}^1_0(\Omega)$ then

$$\|u_n - u\|_{\mathrm{H}^1_0(\Omega)} \leq \frac{C}{\gamma} \|\psi_n - \psi\|_{\mathrm{H}^1_0(\Omega)}, \tag{1.2.8}$$

where γ is the ellipticity constant and C is such that $|\langle \mathcal{A}u, v\rangle| \leq C\|u\|_{\mathrm{H}^1_0(\Omega)}\|v\|_{\mathrm{H}^1_0(\Omega)}$, so they depend only on the operator $\mathcal{A}$.

Proof. It is well known that the strong $\mathrm{H}^1(\Omega)$ convergence of ψ_n to ψ implies

$$K_{\psi_n}(\Omega) \cap \mathrm{H}^1_0(\Omega) \xrightarrow{\mathrm{M}} K_{\psi}(\Omega) \cap \mathrm{H}^1_0(\Omega).$$

This by Theorem 1.2.4 implies that $u_n \to u$ strongly in $\mathrm{H}^1_0(\Omega)$.

For the second part, suppose first that $f = 0$. Then we have

$$\langle \mathcal{A}u, v - u\rangle \geq 0 \quad \forall v \in \mathrm{H}^1_0(\Omega),\ v \geq \psi \text{ q.e. in } \Omega,$$

$$\langle \mathcal{A}u_n, v - u_n\rangle \geq 0 \quad \forall v \in \mathrm{H}^1_0(\Omega),\ v \geq \psi_n \text{ q.e. in } \Omega.$$

Consider $v = u_n + (\psi - \psi_n)$ as test function in the first inequality and $v = u + (\psi_n - \psi)$ as test function in the second one.

Using the linearity and the ellipticity of $\mathcal{A}$ we can obtain

$$\begin{aligned}\gamma\|u - u_n\|^2_{\mathrm{H}^1_0(\Omega)} &\leq \langle \mathcal{A}(u - u_n), u - u_n\rangle \\ &\leq \langle \mathcal{A}(u - u_n), \psi - \psi_n\rangle \leq C\|u - u_n\|_{\mathrm{H}^1_0(\Omega)} \|\psi - \psi_n\|_{\mathrm{H}^1_0(\Omega)}\end{aligned}$$

from which the thesis.

For the case of $f \neq 0$ it is enough to observe that $u - u_f$ and $u_n - u_f$ are the solutions of $VI(0, \psi - u_f)$ and $VI(0, \psi_n - u_f)$, respectively, and the obstacles $\psi - u_f$ and $\psi_n - u_f$ satisfy the hypotheses of the theorem. So, thanks to the previous step we conclude

$$\begin{aligned}\|u - u_n\|_{\mathrm{H}^1_0(\Omega)} &= \|u - u_f - (u_n - u_f)\|_{\mathrm{H}^1_0(\Omega)} \\ &\leq \frac{C}{\gamma}\|\psi - u_f - (\psi_n - u_f)\|_{\mathrm{H}^1_0(\Omega)} = \frac{C}{\gamma}\|\psi - \psi_n\|_{\mathrm{H}^1_0(\Omega)}.\end{aligned}$$

□

As for continuous dependence with respect to the right hand side we know from the Theorem 1.2.1 that if f, $f_n \in \mathrm{H}^{-1}(\Omega)$ and u and u_n are the solutions of $VI(f,\psi)$ and $VI(f_n,\psi)$, respectively, then

$$\|u-u_n\|_{\mathrm{H}^1_0(\Omega)} \le c\|f-f_n\|_{\mathrm{H}^{-1}(\Omega)},$$

and from this we obtain the continuous dependence of solutions when the data converge strongly in $\mathrm{H}^{-1}(\Omega)$.

We will see with Example 4.1.4 that the weak convergence of data in $\mathrm{H}^{-1}(\Omega)$ does not give the converge of solutions.

1.3. Equations with measure data

Let now $\mathcal{M}_b(\Omega)$ be the space of bounded Radon measures viewed as the dual of the Banach space $C_0(\Omega)$, the space of continuous functions on $\bar{\Omega}$ which are zero on $\partial\Omega$. We denote by μ^+ and μ^- the positive, bounded and mutually singular measures called the positive and negative part of μ. The norm on $\mathcal{M}_b(\Omega)$ is given by $\|\mu\|_{\mathcal{M}_b(\Omega)} = |\mu|(\Omega) = \mu^+(\Omega)+\mu^-(\Omega)$. If $\mu \in \mathcal{M}_b(\Omega)$ and $A \subseteq \Omega$ is a Borel set, the restriction of μ to A, $\mu \llcorner A$, is defined by $(\mu \llcorner A)(B) = \mu(A \cap B)$ for every Borel set $B \subseteq \Omega$. The measure $\mu \llcorner A$ still belongs to $\mathcal{M}_b(\Omega)$. By duality we say that a sequence $\mu_n \in \mathcal{M}_b(\Omega)$ weakly-$*$ converges to $\mu \in \mathcal{M}_b(\Omega)$ if

$$\int_\Omega h\,d\mu_n \to \int_\Omega h\,d\mu \qquad \forall h \in C_0(\Omega).$$

It is a classical fact that if a sequence μ_n is bounded in the norm of $\mathcal{M}_b(\Omega)$, then there exists a subsequence μ_{n_k} and a measure $\mu \in \mathcal{M}_b(\Omega)$ such that $\mu_{n_k} \rightharpoonup \mu$ weakly-$*$ in $\mathcal{M}_b(\Omega)$.

$\mathcal{M}^0_b(\Omega)$ is the subspace of measures of $\mathcal{M}_b(\Omega)$ vanishing on sets of zero capacity. $\mathcal{M}^+_b(\Omega)$ and $\mathcal{M}^{0,+}_b(\Omega)$ are the corresponding cones of non negative measures. Recall that $\mathrm{H}^{-1}(\Omega) \not\subseteq \mathcal{M}_b(\Omega)$ but $\mathrm{H}^{-1}(\Omega) \cap \mathcal{M}_b(\Omega) \subseteq \mathcal{M}^0_b(\Omega)$.

Any measure $\mu \in \mathcal{M}_b(\Omega)$ can be decomposed as $\mu = \mu_a + \mu_s$, where $\mu_a \in \mathcal{M}^0_b(\Omega)$ and μ_s is concentrated on a set of capacity zero (see [25]).

If $x \in \Omega$, we denote by δ_x the Dirac's delta centered at x.

In order to study the problem

$$\begin{cases} \mathcal{A}u = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $\mu \in \mathcal{M}_b(\Omega)$, we can not use the variational formulation used in the case of right hand side $f \in \mathrm{H}^{-1}(\Omega)$:

$$\begin{cases} \displaystyle\int_\Omega a_{ij}\, D_j u\, D_i v\, dx = \langle f, v\rangle_{\mathrm{H}^{-1}(\Omega)\mathrm{H}_0^1(\Omega)} & \forall v \in \mathrm{H}_0^1(\Omega) \\ u \in \mathrm{H}_0^1(\Omega). \end{cases}$$

Indeed notice that only for $N = 1$, we have that $\mathcal{M}_b(\Omega) \subseteq \mathrm{H}^{-1}(\Omega)$, and hence the variational theory applies. Unfortunately, in general, the term $\langle \mu, v\rangle$ has not always meaning when μ is a measure and $u \in \mathrm{W}_0^{1,p}(\Omega)$, $p < N$. Moreover it is well known that, when $N \geq 2$, the solution of

$$\begin{cases} -\Delta v = -\delta_0 & \text{in } B_1(0) \\ v = 0 & \text{on } \partial B_1(0). \end{cases}$$

does not belong to $\mathrm{H}_0^1(\Omega)$ but only to $\mathrm{W}_0^{1,q}(\Omega)$, with $q < \dfrac{N}{N-1}$.

Hence we need to use a weaker formulation with more regular test functions

$$\begin{cases} \displaystyle\int_\Omega a_{ij}\ D_j u\ D_i v\ dx = \int_\Omega v\, d\mu & \forall v \in \displaystyle\bigcup_{q'>N} \mathrm{W}_0^{1,q'}(\Omega) \\ u \in \displaystyle\bigcap_{q<\frac{N}{N-1}} \mathrm{W}_0^{1,q}(\Omega) \end{cases} \tag{1.3.1}$$

so that everything makes sense, because $\mathrm{W}_0^{1,q'}(\Omega) \subseteq C(\bar{\Omega})$ for $q' > N$.

Now, when $N = 2$, the solution of equation (1.3.1) exists and is unique (see [26]); but in general the solution in this sense may not be unique. Indeed in the following example (given by A. Prignet in [39] modifying the classical one presented by J. Serrin in [40]) shows that the homogeneous equation

$$\begin{cases} \mathcal{A}u = 0 \text{ in } \mathcal{D}'(\Omega) \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

has a nontrivial solution $w \in \displaystyle\bigcap_{q<\frac{N}{N-1}} \mathrm{W}_0^{1,q}(\Omega)$ which does not belong to $\mathrm{H}_0^1(\Omega)$.

Example 1.3.1. Let $\Omega = B_1(0)$, the unit ball in $\mathbb{R}^N$, with $N > 2$. Let

$$a_{ij} := \begin{cases} \delta_{ij} + \left(\frac{1}{\varepsilon^2} - 1\right) \frac{x_i x_j}{|x|^2} & \text{for } i,j = 1,2 \\ \delta_{ij} & \text{for } i,j = 3...N \end{cases}$$

where $\varepsilon > 0$ is still to be fixed. With this choice the coefficients are in $\mathrm{L}^\infty(\Omega)$ and the operator $\mathcal{A}$ satisfies the ellipticity condition (1.0.4).

For ε sufficiently small, the function

$$u(x_1, ..., x_N) = u(x_1, x_2) = x_1({x_1}^2 + {x_2}^2)^{\frac{-1-\varepsilon}{2}}$$

belongs to $\mathrm{W}^{1,q}(\Omega)$ for all $q < \dfrac{N}{N-1}$, but does not belong to $\mathrm{H}^1(\Omega)$.

It is proved that u is a solution of $\mathcal{A}u = 0$ in the sense of distributions, and its trace on $\partial B_1(0)$ belongs to $\mathrm{H}^{\frac{1}{2}}(\partial B_1(0))$. Thanks to this regularity of u we can consider the following Dirichlet problem

$$\begin{cases} \mathcal{A}v = 0 & \text{in } B_1(0) \\ v = u & \text{on } \partial B_1(0) \end{cases}$$

which has a unique variational solution $v \in \mathrm{H}^1(\Omega)$. At this point, by linearity, the function $w := u - v$ satisfies

$$\begin{cases} \mathcal{A}w = 0 & \text{in } B_1(0) \\ w = 0 & \text{on } \partial B_1(0) \end{cases}$$

or, more precisely,

$$\begin{cases} \displaystyle\int_\Omega a_{ij}\, D_j w\, D_i z\, dx = 0 & \forall z \in \displaystyle\bigcup_{p>N} \mathrm{W}_0^{1,p}(\Omega) \\ w \in \displaystyle\bigcap_{q<\frac{N}{N-1}} \mathrm{W}_0^{1,q}(B_1(0)), & \end{cases}$$

that is, w is a solution in the sense of (1.3.1). Clearly, by Lax-Milgram Lemma there exists a unique variational solution in $\mathrm{H}_0^1(\Omega)$ and this is $u \equiv 0$, which is also a solution in the sense of (1.3.1). So we have obtained two solutions in the sense of formulation (1.3.1) which is then proved to be too weak to ensure uniqueness.

G. Stampacchia overcame this difficulty for linear equations, using a wider class of test functions, and gave in [41] (see also [42] and [34]) the following definition, which uses regularity and duality arguments.

Definition 1.3.2. Let $\mu \in \mathcal{M}_b(\Omega)$. A function $u_\mu \in \mathrm{L}^1(\Omega)$ is a solution in the sense of Stampacchia (also called solution by duality) of the equation

$$\begin{cases} \mathcal{A}u_\mu = \mu & \text{in } \Omega \\ u_\mu = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3.2}$$

if

$$\int_\Omega u_\mu g\, dx = \int_\Omega u_g^*\, d\mu, \quad \forall g \in \mathrm{L}^\infty(\Omega), \tag{1.3.3}$$

where u_g^* is the solution of

$$\begin{cases} \mathcal{A}^* u_g^* = g & \text{in } \mathrm{H}^{-1}(\Omega) \\ u_g^* \in \mathrm{H}_0^1(\Omega) \end{cases}$$

and $\mathcal{A}^*$ is the adjoint of $\mathcal{A}$.

The theory of this type of solution relies on the following proposition due to E. De Giorgi and G. Stampacchia.

Proposition 1.3.3. *Let $f \in \mathrm{W}^{-1,q'}(\Omega)$, with $q' > N$, and let $v \in \mathrm{H}_0^1(\Omega)$ be the variational solution of*

$$\begin{cases} \mathcal{A}v = f & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

then $v \in \mathrm{L}^\infty(\Omega) \cap C^{0,\alpha}(\Omega)$ and

$$\|v\|_{\mathrm{L}^\infty(\Omega)} \leq c\|f\|_{\mathrm{W}^{-1,q'}(\Omega)}.$$

The constants c and α depend only on Ω, $\mathcal{A}$ and q'. By the regularity of Ω we have also $v \in C_0(\Omega)$ thanks to Theorem 1.1.2.

Thanks to this we can first of all notice that Definition 1.3.2 makes sense. In the first term of (1.3.3) $g \in \mathrm{L}^\infty(\Omega)$ and $u_\nu \in \mathrm{L}^1(\Omega)$, in the second one $u_g^* \in C_0(\Omega)$

Throughout this work q will be any exponent satisfying $1 < q < \dfrac{N}{N-1}$. The next theorem shows that a solution u_μ exists, is unique, and belongs to $\mathrm{W}_0^{1,q}(\Omega)$; we give the complete proof because it casts light on the duality technique.

Theorem 1.3.4. *Let $\mu \in \mathcal{M}_b(\Omega)$. Then there exists a unique $u_\mu \in \mathrm{W}_0^{1,q}(\Omega)$ solution of the equation*

$$\begin{cases} \mathcal{A}u_\mu = \mu & \text{in } \Omega \\ u_\mu = 0 & \text{on } \partial\Omega \end{cases}$$

in the sense of Definition 1.3.2. Moreover we have

$$\|u_\mu\|_{\mathrm{W}^{1,q}(\Omega)} \leq c\,\|\mu\|_{\mathcal{M}_b(\Omega)}. \tag{1.3.4}$$

Proof. From Proposition 1.3.3 we get that when $g \in \mathrm{W}^{-1,q'}(\Omega)$, with $q' > N$ then u_g^*, belongs to $C_0(\Omega) \cap \mathrm{H}_0^1(\Omega)$. Hence, for any $q' > N$, we can consider the linear functional

$$L : \mathrm{W}^{-1,q'}(\Omega) \longrightarrow \mathbb{R}$$

defined by

$$L(g) = \int_\Omega u_g^*\, d\mu$$

Using the estimate in Proposition 1.3.3 it is easy to see that this functional is continuous:

$$|L(g)| \leq \|u_g^*\|_{\mathrm{L}^\infty(\Omega)}\, \|\mu\|_{\mathcal{M}_b(\Omega)} \leq c\,\|g\|_{\mathrm{W}^{-1,q'}(\Omega)}\, \|\mu\|_{\mathcal{M}_b(\Omega)}. \tag{1.3.5}$$

Hence, by the reflexivity of $\mathrm{W}_0^{1,q}(\Omega)$, we have that there exists a unique function $u_\mu \in \mathrm{W}_0^{1,q}(\Omega)$ (since $q' > N$ we have $q < \dfrac{N}{N-1}$), with $\|u_\mu\|_{\mathrm{W}^{1,q}(\Omega)} = \|L\|_{\mathrm{W}^{-1,q'}(\Omega)}$, such that

$$L(g) = \langle g, u_\mu \rangle_{\mathrm{W}^{-1,q'}(\Omega)\, \mathrm{W}_0^{1,q}(\Omega)}$$

for all $g \in \mathrm{W}^{-1,q'}(\Omega)$. In particular, since $\mathrm{L}^\infty(\Omega) \subseteq \mathrm{W}^{-1,p}(\Omega)$ for any $p \geq 1$, for all $g \in \mathrm{L}^\infty(\Omega)$ we can write

$$\int_\Omega u_g^*\, d\mu = \int_\Omega u_\mu g\, dx,$$

and hence u_μ is the solution we wanted. The estimate on the solution comes from (1.3.5).

□

First of all it is worth to notice that this theory is consistent with the variational one. More precisely, it is possible to prove that, if the datum μ belongs to $\mathcal{M}_b(\Omega) \cap \mathrm{H}^{-1}(\Omega)$, then the solution coincides with the variational one.

From now on we will use the following notation: u_μ denotes the solution of the equation

$$\begin{cases} \mathcal{A}u_\mu = \mu & \text{in } \Omega \\ u_\mu = 0 & \text{on } \partial\Omega, \end{cases}$$

when μ is either a measure in $\mathcal{M}_b(\Omega)$ or an element of $\mathrm{H}^{-1}(\Omega)$. In the former case we refer to the Definition 1.3.2, in the latter to the usual variational one.

Another important fact is the continuous dependence with respect to the data converging weakly-$*$ in $\mathcal{M}_b(\Omega)$. This is useful mainly because a measure in $\mathcal{M}_b(\Omega)$ can be approximated in this way by means of measures of $\mathcal{M}_b(\Omega) \cap \mathrm{H}^{-1}(\Omega)$.

Proposition 1.3.5. *Let μ_n, $\mu \in \mathcal{M}_b(\Omega)$ be such that $\mu_n \rightharpoonup \mu$ weakly-$*$ in $\mathcal{M}_b(\Omega)$ then $u_{\mu_n} \to u_\mu$ strongly in $\mathrm{W}^{1,q}(\Omega)$.*

Proof. By (1.3.4) we have that the sequence u_{μ_n} is bounded in $\mathrm{W}^{1,q}(\Omega)$ and hence, up to a subsequence,

$$u_{\mu_n} \rightharpoonup z \text{ weakly in } \mathrm{W}_0^{1,q}(\Omega).$$

To see that the limit is u_μ itself consider

$$\int_\Omega u_{\mu_n} g \, dx = \int_\Omega u_g^* \, d\mu_n \qquad \forall g \in \mathrm{L}^\infty(\Omega).$$

Since, by Theorem 1.1.2, $u_g^* \in C_0(\Omega)$. Then by the definition of weak-$*$ convergence we can pass to the limit and get

$$\int_\Omega zg \, dx = \int_\Omega u_g^* \, d\mu.$$

Hence, by uniqueness, $z = u_\mu$. This holds for every subsequence and hence for the whole sequence.

To see that the convergence is in fact strong we have to consider that $Du_{\mu_n} \rightharpoonup Du_\mu$ weakly in $(\mathrm{L}^q(\Omega))^N$, for all $q < \dfrac{N}{N-1}$ (from (1.3.4)) together with $Du_{\mu_n} \to Du_\mu$ a.e. in Ω (from the next lemma) gives that $Du_{\mu_n} \to Du_\mu$ strongly in $(\mathrm{L}^q(\Omega))^N$, for all $q < \dfrac{N}{N-1}$. And this concludes the proof. $\square$

In the previous proof we have used this Lemma, due to L. Boccardo and T. Gallouët (see [8] and [11]).

Lemma 1.3.6. *Let $\mu_n, \mu \in \mathcal{M}_b(\Omega)$ be such that $\mu_n \rightharpoonup \mu$ weakly-$*$ in $\mathcal{M}_b(\Omega)$. Then*

$$Du_{\mu_n} \to Du_\mu \quad a.e. \text{ in } \Omega.$$

We point out that the request for regularity of $\partial\Omega$ we made in Section 1.1 is crucial in the last proposition, which will be fundamental in the theory.

Indeed the following example shows that otherwise we can find a weakly-$*$ converging sequence of measures whose Stampacchia solutions do not converge, to the solution corresponding to the limit measure.

Example 1.3.7. Let $\Omega \subseteq \mathbb{R}^N$, $N > 2$. Suppose that $x_0 \in \partial\Omega$ does not satisfy the Wiener condition (see Definition 1.1.1).

Thanks to Theorem 1.1.2 it is possible to construct a function $g \in L^\infty(\Omega)$ such that u_g^*, the solution of

$$\begin{cases} \mathcal{A}^* u_g^* = g & \text{in } H^{-1}(\Omega) \\ u_g^* \in H_0^1(\Omega) \end{cases}$$

has the following property: there exists a sequence $x_n \in \Omega$, $x_n \to x_0$ and $u_g^*(x_n) \to 1$.

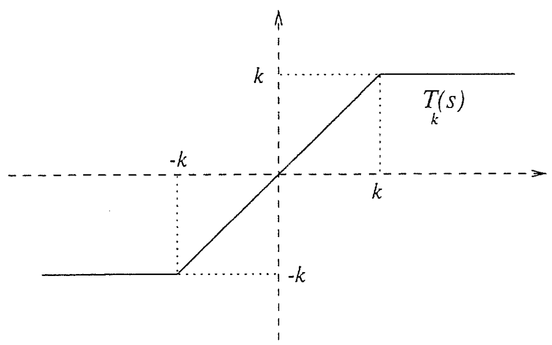
Now consider the sequence of measures $\mu_n := \delta_{x_n}$. Clearly it is such that

$$\mu_n \rightharpoonup 0 \quad \text{weakly-}* \text{ in } \mathcal{M}_b(\Omega),$$

but u_{μ_n} does not converge to 0 because by Definition 1.3.2

$$\int_\Omega u_{\mu_n} g \, dx = \int_\Omega u_g^* \, d\mu_n = u_g^*(x_n) \longrightarrow 1.$$

We notice now that the function u_μ is in fact more regular. Here and in the following $T_k(s) := (-k) \vee (s \wedge k)$ denotes the usual truncation function.



Proposition 1.3.8. *If u_μ is the solution of (1.3.2) then $T_k(u_\mu) \in \mathrm{H}_0^1(\Omega)$, for any $k \in \mathbb{R}^+$. Moreover*

$$\int_\Omega |DT_k(u_\mu)|^2 dx \leq \frac{k}{\gamma} |\mu|(\Omega). \tag{1.3.6}$$

Proof. Let $f_n \in \mathcal{M}_b(\Omega) \cap \mathrm{H}^{-1}(\Omega)$ be such that $f_n \rightharpoonup \mu$ weakly-$*$ in $\mathcal{M}_b(\Omega)$ and $\|f_n\|_{\mathrm{L}^1(\Omega)} \leq \|\mu\|_{\mathcal{M}_b(\Omega)}$. Then

$$\int_\Omega A(x) D_j u_{f_n} D_i v \, dx = \int_\Omega f_n v \, dx \qquad \forall v \in \mathrm{H}_0^1(\Omega);$$

then we can take as test function $v = T_k(u_{f_n})$ and obtain, using ellipticity, that

$$\int_\Omega |DT_k(u_{f_n})|^2 \leq \frac{k}{\gamma} \|f_n\|_{\mathcal{M}_b(\Omega)} \leq \tilde{k}.$$

Hence, up to a subsequence, $T_k(u_{f_n}) \rightharpoonup w$ weakly in $\mathrm{H}_0^1(\Omega)$. The limit w is actually $T_k(u_\mu)$ itself because, by Proposition 1.3.5, we already know that $u_{f_n} \to u_\mu$ a.e. in Ω. Hence $T_k(u_\mu) \in \mathrm{H}_0^1(\Omega)$ for all $k > 0$. □

Proposition 1.3.8 implies that u_μ has a quasi continuous representative. In the rest of the paper we shall always identify u_μ with its quasi continuous representative.

Proposition 1.3.9. *If u_μ is the solution of (1.3.2) then there exists w quasi continuous, with $u_\mu = w$ a.e. in Ω.*

Proof. Since, for all k, $T_k(u_\mu) \in \mathrm{H}_0^1(\Omega)$, then it has a quasi continuous representative. Call it v_k. Notice that, if $h < k$, then $T_h(v_k) = v_h$ a.e. in Ω and hence q.e. in Ω.

Step 1. Suppose that on some $E \subseteq \Omega$ we have that $v_{k|_E}$ is continuous for every k and that $T_h((v_k(x)) = v_h(x)$ for all $x \in E$, for $h < k$. Moreover assume that for every $x \in \Omega$ there exists k such that $|v_k(x)| < k$.

In each $x \in E$ we define $v(x) := v_k(x)$ for such k. This definition is well posed: if there exists another $h < k$ with $|v_h(x)| < h$ then

$$|T_h(v_k(x))| = |v_h(x)| < h,$$

from which $v_h(x) = T_h(v_k(x)) = v_k(x)$.

Let us prove now that v is continuous on E. Fix $x_0 \in E$ and k such that $|v_k(x_0)| < k$, and hence $v(x_0) = v_k(x_0)$. By continuity there exists a neighbourhood of x_0 in E in which $|v_k(x)| < k$, and then $v(x) = v_k(x)$ for all such x. The continuity of v_k says that around the point x_0

$$|v_k(x) - v_k(x_0)| < \varepsilon.$$

In a suitable vicinity of x_0 in E we can replace v_k with v and obtain

$$|v(x) - v(x_0)| < \varepsilon,$$

so that v is continuous on E. And finally $v = u_\mu$ a.e. in E since $T_k(v) = v_k$, $v_k = T_k(u)$ a.e., and we can pass to the limit with respect to k a.e. in E.

Step 2. For the general case, consider the set

$$B := \bigcap_{h=1}^{+\infty} \{|v_h| > h\}.$$

To prove that $\text{cap}(B) = 0$, observe that

$$\text{cap}(B) \leq \text{cap}(\{|v_k| > k\}),$$

for all k, and using (1.1.1) and (1.3.6), we can compute

$$\begin{aligned}\text{cap}(\{v_k| > k\}) &\leq \frac{1}{k^2} \int_\Omega |Dv_k|^2 \, dx \\ &= \frac{1}{k^2} \int_\Omega |DT_k(u)|^2 \, dx \leq \frac{ck}{k^2}.\end{aligned}$$

and obtain the claim as k tends to $+\infty$.

By the definition of quasi continuity, for all k and m there exist $A_m^k \subseteq \Omega$ with

$$\text{cap}(A_m^k) < \frac{1}{2^m} \frac{1}{2^k}$$

such that $v_k|_{\Omega \setminus A_m^k}$ is continuous in $\Omega \setminus A_m^k$. The sequence of sets A_m^k can be taken to be decreasing.

Set

$$A_m := \bigcup_{k=1}^{+\infty} A_m^k \cup B,$$

so that $\text{cap}(A_m) < \frac{1}{2^m}$.

For any m fixed, in the set $\Omega \backslash A_m$, we are in the situation of *Step 1.* Indeed, for all k, $v_{k|_{\Omega\setminus A_m}}$ is continuous and, for all $x \in \Omega \setminus A_m$ there exists k such that $|v_{k|_{\Omega\setminus A_m}}(x)| < k$.

So, by *Step 1*, there exists a continuous function $w_m : \Omega \setminus A_m \to \mathbb{R}$ such that $T_k(w_m) = u$ a.e. in $\Omega \setminus A_m$.

Define now $w : \Omega \to \mathbb{R}$ by

$$w_{|_{\Omega\setminus A_m}} := w_m.$$

The definition in $\bigcap_{m=1}^{+\infty} A_m$ is irrelevant because this set has capacity zero.

The definition is well posed: indeed, for all $x \in \Omega \setminus A_m$ there exists k such that $|v_k(x)| < k$ so that $w_{m+1}(x) = v_k(x) = w_m(x)$.

The function w is quasi continuous, because, for all ε there exists m such that $\text{cap}(A_m) < \varepsilon$ and $w_{|_{\Omega\setminus A_m}} = w_m$ is continuous. Moreover $w = u$ a.e. in Ω since so do the w_m's. □

Remark 1.3.10. From (1.3.6) it follows also that if μ_n, $\mu \in \mathcal{M}_b(\Omega)$ are such that $\mu_n \rightharpoonup \mu$ weakly-$*$ in $\mathcal{M}_b(\Omega)$, then

$$T_k(u_{\mu_n}) \rightharpoonup T_k(u_\mu) \text{ weakly in } \mathrm{H}_0^1(\Omega).$$

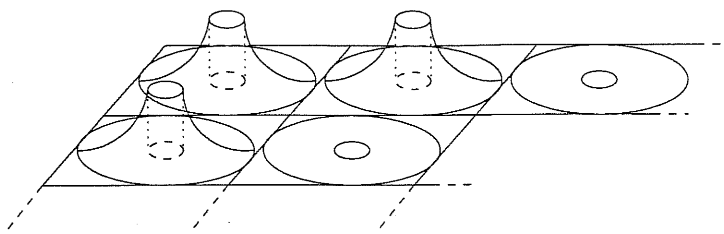
Indeed since the sequence μ_n is bounded, from (1.3.6) it is clear that, up to a subsequence, $T_k(u_{\mu_n}) \rightharpoonup z$ weakly in $\mathrm{H}^1(\Omega)$, where $z \in \mathrm{H}_0^1(\Omega)$. But on the other hand, from Proposition 1.3.5, $T_k(u_{\mu_n}) \to T_k(u_\mu)$ a.e. in Ω, for all $k > 0$. Hence by uniqueness we have the thesis, for the whole sequence.

It is to be noticed anyway that in general it is not possible to obtain the strong convergence of the truncates in $\mathrm{H}_0^1(\Omega)$. This is shown in the following example, which follows a well known construction made by D. Cioranescu and F. Murat in [14].

Example 1.3.11. Let $\Omega = (0,1)^N$ with $N \geq 3$, $\mathcal{A} = -\Delta$.

For each $n \in \mathbb{N}$, divide the whole of Ω into small cubes of side $\frac{1}{n}$. In the centre of each of them take two balls: $B_{\frac{1}{2n}}$, inscribed in the cube, and B_{r_n} of ray $r_n = \left(\frac{1}{2n}\right)^{\frac{N}{N-2}}$.

In each cube define w_n to be the capacitary potential of B_{r_n} with respect to $B_{\frac{1}{2n}}$ extended by zero in the rest of the cube.



Consider the measures

$$\mu_n := -\Delta w_n$$

it is possible to prove (the proof is rather tricky, see [14]) that $\mu_n \in \mathcal{M}_b(\Omega) \cap \mathrm{H}^{-1}(\Omega)$ and that

$$\mu_n \rightharpoonup 0 \text{ both weakly in } \mathrm{H}^{-1}(\Omega) \text{ and weakly-}* \text{ in } \mathcal{M}_b(\Omega)$$

Hence from Proposition 1.3.5 and Remark 1.3.10, and from the fact that $0 \le w_n \le 1$ we get that

$$w_n \to 0 \text{ strongly in } \mathrm{W}^{1,q}(\Omega) \text{ and } w_n \rightharpoonup 0 \text{ weakly in } \mathrm{H}^1_0(\Omega).$$

If we had also that $w_n \to 0$ strongly in $\mathrm{H}^1(\Omega)$ this would imply the convergence in capacity, but it is possible to see that

$$\operatorname{cap}\left(\left\{u_{\mu_n} \ge \frac{1}{2}\right\}\right) \ge c$$

and the constant c does not depend on n. This fact was already observed in [11].

As for the approximation of measures, the following theorem due to L. Boccardo and F. Murat (see [12]) gives a special approximating sequence of measures in $\mathcal{M}_b(\Omega) \cap \mathrm{H}^{-1}(\Omega)$.

Theorem 1.3.12. *Let u_μ be the solution of (1.3.2). Then*

$$\mathcal{A}T_k(u_\mu) \in \mathcal{M}_b(\Omega) \cap \mathrm{H}^{-1}(\Omega).$$

Moreover

$$\mathcal{A}T_k(u_\mu) \rightharpoonup \mu \qquad \textit{weakly-}* \textit{ in } \mathcal{M}_b(\Omega).$$

Remark 1.3.13. Notice that in general it is not possible to approximate any measure $\mu \in \mathcal{M}_b(\Omega)$ by means of measures in $\mathcal{M}_b(\Omega)\cap \mathrm{H}^{-1}(\Omega)$ with respect to the strong topology of $\mathcal{M}_b(\Omega)$. Indeed the strong closure of $\mathcal{M}_b(\Omega)\cap \mathrm{H}^{-1}(\Omega)$ is $\mathcal{M}_b^0(\Omega)$. In particular, for any measure $\mu \in \mathcal{M}_b^0(\Omega)$, we know (see [9]) that we can write it as $\mu = f + F$ where $f \in \mathrm{L}^1(\Omega)$ and $F \in \mathcal{M}_b(\Omega)\cap \mathrm{H}^{-1}(\Omega)$. Hence if we take $\mu_k := T_k(f) + F$ we have that $\mu_k \in \mathcal{M}_b(\Omega)\cap \mathrm{H}^{-1}(\Omega)$ and $\mu_k \to \mu$ strongly in $\mathcal{M}_b(\Omega)$. Conversely, since every measure of $\mathcal{M}_b(\Omega)\cap \mathrm{H}^{-1}(\Omega)$ vanishes on sets of capacity zero, so does every measure in its strong closure.

Also the potential theoretic aspect of this theory is important and will be used in Section 3.2 to study some properties of this kind of solutions.

The Green's function $G_\Omega^{\mathcal{A}}(x,y)$ relative to the operator $\mathcal{A}$ in Ω is defined as the solution, in the sense of Stampacchia (Definition 1.3.2), of the equation

$$\begin{cases} \mathcal{A}^* G_\Omega^{\mathcal{A}}(x,\cdot) = \delta_x & \text{in } \Omega \\ G_\Omega^{\mathcal{A}}(x,\cdot) = 0 & \text{on } \partial\Omega \end{cases}$$

In [41] it is proved that $G_\Omega^{\mathcal{A}} : \Omega \times \Omega \to [0,+\infty]$ is continuous and satisfies the following estimates: for every compact set $J \subseteq \Omega$ there exist four constants $c_1 > 0$, $c_2 > 0$, $d_1 \geq 0$ and $d_2 \geq 0$ such that

$$c_1 G(|x-y|) - d_1 \leq G_\Omega^{\mathcal{A}}(x,y) \leq c_2 G(|x-y|) + d_2, \tag{1.3.7}$$

for every $x, y \in J$, where $G(|x|)$ is the fundamental solution of $-\Delta$ in $\mathbb{R}^N$, i.e.

$$G(|x|) = \begin{cases} \dfrac{1}{(N-2)\sigma_{N-1}} \, |x|^{N-2} & \text{if } N > 2, \\ \dfrac{1}{2\pi} \log\left(\dfrac{1}{|x|}\right) & \text{if } N = 2; \end{cases}$$

here σ_{N-1} is the $(N-1)$-dimensional measure of the boundary of the unit ball in $\mathbb{R}^N$ (see also [27]). Notice that we can take $d_1 = d_2 = 0$ if $N \geq 3$. Thanks to this Stampacchia proves that the solution of (1.3.2) satisfies

$$u_\mu(x) = \int_\Omega G_\Omega^{\mathcal{A}}(x,y)\, d\mu(y) \qquad \text{for q.e. } x \in \Omega. \tag{1.3.8}$$

It can be proved (see [35]) that this potential representation of u_μ is finite q.e. in Ω.

Chapter 2
Obstacle problems with measure data

2.1. Definition

Now we want to arrive to a suitable definition of obstacle problems with measure data. As we have seen, we can not use the variational formulation (1.2.1), because the term of the form $\langle \mu, v-u \rangle$ may not be defined. Also the use of the characterization (1.2.4) is not possible because this, in general, does not determine the solution of the Obstacle Problem. Indeed reconsider Example 1.3.1: with such $\mathcal{A}$, if we choose $\psi \equiv -\infty$, and if u were the minimal supersolution, then we would have $u \leq u + tw$ a.e. in Ω for any t in $\mathbb{R}$, which is a contradiction.

To avoid these problems we give the following definition, in which, roughly speaking, we choose the minimum element among those functions v such that $\mathcal{A}v - \mu$ is not only nonnegative in the sense of distributions but is actually a nonnegative bounded Radon measure.

First of all recall from Section 1.1 that $\psi : \Omega \to \overline{\mathbb{R}}$ is a quasi upper semicontinuous function and $K_\psi(\Omega)$ is the set of all quasi continuous functions z such that $z \geq \psi$ q.e. in Ω.

Definition 2.1.1. We say that the function $u \in \mathrm{K}_\psi(\Omega) \cap W_0^{1,q}(\Omega)$, $1 < q < \dfrac{N}{N-1}$ is a solution of the Obstacle Problem with datum μ and obstacle ψ if

1. there exists a positive bounded measure $\lambda \in \mathcal{M}_b^+(\Omega)$ such that

$$u = u_\mu + u_\lambda;$$

2. for any $\nu \in \mathcal{M}_b^+(\Omega)$, such that $v = u_\mu + u_\nu$ belongs to $\mathrm{K}_\psi(\Omega)$, we have

$$u \leq v \quad \text{q.e. in } \Omega.$$

Also here the positive measure λ, which is uniquely determined, will be called the obstacle reaction relative to u. This problem will be shortly indicated by $OP(\mu, \psi)$.

To show that for any datum μ there exists one and only one solution, we introduce the set

$$\mathcal{F}_\psi(\mu) := \left\{ v \in \mathrm{K}_\psi(\Omega) \cap W_0^{1,q}(\Omega) \ : \ \exists \nu \in \mathcal{M}_b^+(\Omega) \text{ s.t. } v = u_\mu + u_\nu \right\}.$$

We will prove that $\mathcal{F}_\psi(\mu)$ has a minimum element, that is a function $u \in \mathcal{F}_\psi(\mu)$ such that $u \leq v$ q.e. in Ω for any other function $v \in \mathcal{F}_\psi(\mu)$. This is clearly the solution of the Obstacle Problem according to the Definition 2.1.1. If this solution exists it is obviously unique. Hypothesis (1.2.2) does not ensure that $\mathcal{F}_\psi(\mu)$ be nonempty. The minimal hypothesis, instead of (1.2.2), will be

$$\exists \rho \in \mathcal{M}_b(\Omega) \ : \ u_\rho \geq \psi \text{ q.e. in } \Omega; \tag{2.1.1}$$

so the set $\mathcal{F}_\psi(\mu)$ is nonempty for every $\mu \in \mathcal{M}_b(\Omega)$, because it contains the function $u_{\mu^+} + u_\rho$. In this case we will say that the obstacle is OP-admissible.

In Section 3.1 we will need a slightly stronger condition of admissibility. If

$$\exists \sigma \in \mathcal{M}_b^0(\Omega) \ : \ u_\sigma \geq \psi \text{ q.e. in } \Omega; \tag{2.1.2}$$

we will say that the obstacle is OP^0-admissible.

This definition of obstacle problem is given in the case of homogeneous boundary condition. It will be extended to general boundary data only in Section 4.5 to make the exposition simpler.

2.2. Nonpositive obstacles

We begin the proof of existence of solutions to obstacle problems from a case that makes things easier. Throughout this section we assume the obstacle to be nonpositive. In this frame both hypotheses (2.1.1) and (1.2.2) are trivially satisfied, i.e. the obstacle is both VI- and OP-admissible.

We begin with a preparatory result which will be proved in two steps.

Lemma 2.2.1. *Let* $\psi \leq 0$ *q.e. in* Ω *an let* $\mu \in \mathcal{M}_b(\Omega) \cap \mathrm{H}^{-1}(\Omega)$ *such that* μ^+ *and* μ^- *belong to* $\mathrm{H}^{-1}(\Omega)$. *Let* u *be the solution of* $VI(\mu, \psi)$ *and* λ *the obstacle reaction associated with* u. *Then*

$$||\lambda||_{\mathcal{M}_b(\Omega)} \leq ||\mu^-||_{\mathcal{M}_b(\Omega)}.$$

Proof. Observe that the function u_{μ^+} is positive and hence greater than or equal to ψ q.e. in Ω, belongs to $\mathrm{H}_0^1(\Omega)$, and

$$\mathcal{A}u_{\mu^+} - \mu \geq 0 \quad \text{in } \mathcal{D}'(\Omega).$$

By (1.2.4) we have

$$u = u_\mu + u_\lambda \leq u_{\mu^+} \quad \text{a.e. in } \Omega,$$

hence, since both functions are quasi continuous, also q.e., and, by linearity,

$$u_\lambda \leq u_{\mu^-} \quad \text{q.e. in } \Omega. \tag{2.2.1}$$

We will prove that this implies

$$\lambda(\Omega) \leq \mu^-(\Omega) \tag{2.2.2}$$

which is equivalent to the thesis.

To prove (2.2.2) we note that, thanks to (2.2.1)

$$\int_\Omega w \, d\mu^- = \langle \mathcal{A}^* w, u_{\mu^-} \rangle \geq \langle \mathcal{A}^* w, u_\lambda \rangle = \int_\Omega w \, d\lambda, \tag{2.2.3}$$

for every $w \in \mathrm{H}_0^1(\Omega)$, such that $\mathcal{A}^* w \geq 0$ in $\mathcal{D}'(\Omega)$.

It is now easy to find a sequence $\{w_n\}$ in $\mathrm{H}_0^1(\Omega)$ such that $w_n \nearrow 1$ and $\mathcal{A}^* w_n \geq 0$ in $\mathcal{D}'(\Omega)$. For instance, one can choose as w_n the $\mathcal{A}^*$-capacitary potential of J_n, where J_n is an invading family of compact subsets of Ω.

Passing to the limit in (2.2.3), as $n \to \infty$, we obtain (2.2.2). □

Theorem 2.2.2. *Let* $\psi \leq 0$ *and* $\mu \in \mathcal{M}_b(\Omega) \cap \mathrm{H}^{-1}(\Omega)$. *Let* u *be the solution of* $VI(\mu, \psi)$ *and let* λ *be the obstacle reaction relative to* u. *Then*

$$||\lambda||_{\mathcal{M}_b(\Omega)} \leq ||\mu^-||_{\mathcal{M}_b(\Omega)}. \tag{2.2.4}$$

Proof. Thanks to Lemma 3.3 in [21] there exists a sequence of smooth functions f_n such that

$$||f_n - \mu||_{\mathrm{H}^{-1}(\Omega)} \leq \frac{1}{n} \quad \text{and} \quad ||f_n||_{\mathrm{L}^1(\Omega)} \leq ||\mu||_{\mathcal{M}_b(\Omega)}.$$

Thanks to the next lemma, the sequence f_n satisfies

$$f_n^{\pm} \rightharpoonup \mu^{\pm} \text{ weakly-}* \text{ in } \mathcal{M}_b(\Omega) \text{ and } ||f_n^{\pm}||_{\mathrm{L}^1(\Omega)} \to ||\mu^{\pm}||_{\mathcal{M}_b(\Omega)}.$$

Let u_n and u be the solutions of $VI(f_n, \psi)$ and $VI(\mu, \psi)$, respectively. We know from the general theory that $u_n \to u$ in $\mathrm{H}_0^1(\Omega)$. So the measures λ_n and λ associated with u_n and u, respectively, satisfy

$$\lambda_n \to \lambda \text{ in } \mathrm{H}^{-1}(\Omega),$$

$$||\lambda_n||_{\mathcal{M}_b(\Omega)} \leq ||f_n^-||_{\mathrm{L}^1(\Omega)}.$$

So $\lambda_n \rightharpoonup \lambda$ in weakly-$*$ in $\mathcal{M}_b(\Omega)$, and we get the inequality (2.2.4). □

The following lemma is quite simple, but is proved here for the sake of completeness.

Lemma 2.2.3. *Let μ_n and μ be measures in $\mathcal{M}_b(\Omega)$ such that*

$$\mu_n \rightharpoonup \mu \text{ weakly-}* \text{ in } \mathcal{M}_b(\Omega) \text{ and } ||\mu_n||_{\mathcal{M}_b(\Omega)} \to ||\mu||_{\mathcal{M}_b(\Omega)}$$

then

$$\mu_n^+ \rightharpoonup \mu^+ \text{ and } \mu_n^- \rightharpoonup \mu^- \text{ weakly-}* \text{ in } \mathcal{M}_b(\Omega),$$

and

$$||\mu_n^+||_{\mathcal{M}_b(\Omega)} \to ||\mu^+||_{\mathcal{M}_b(\Omega)} \text{ and } ||\mu_n^-||_{\mathcal{M}_b(\Omega)} \to ||\mu^-||_{\mathcal{M}_b(\Omega)}. \tag{2.2.5}$$

Proof. Observe that

$$||\mu_n^{\pm}||_{\mathcal{M}_b(\Omega)} \leq ||\mu||_{\mathcal{M}_b(\Omega)}.$$

so, up to a subsequence,

$$\mu_n^+ \rightharpoonup \alpha \text{ and } \mu_n^- \rightharpoonup \beta \text{ weakly-}* \text{ in } \mathcal{M}_b(\Omega);$$

where $\alpha - \beta = \mu$. Hence, we can compute

$$\begin{aligned} ||\alpha||_{\mathcal{M}_b(\Omega)} + ||\beta||_{\mathcal{M}_b(\Omega)} &\leq \liminf ||\mu_n^+||_{\mathcal{M}_b(\Omega)} + \liminf ||\mu_n^-||_{\mathcal{M}_b(\Omega)} \\ &\leq \liminf ||\mu_n||_{\mathcal{M}_b(\Omega)} = ||\mu||_{\mathcal{M}_b(\Omega)}; \end{aligned}$$

from which we easily deduce that $\alpha = \mu^+$, $\beta = \mu^-$. Therefore the whole sequences μ_n^+ and μ_n^- converge to μ^+ and μ^- respectively. Moreover, as

$$\limsup_{n\to+\infty} \|\mu_n^+\|_{\mathcal{M}_b(\Omega)} + \liminf_{n\to+\infty} \|\mu_n^-\|_{\mathcal{M}_b(\Omega)}$$
$$\leq \lim_{n\to+\infty} \|\mu_n\|_{\mathcal{M}_b(\Omega)} = \|\mu\|_{\mathcal{M}_b(\Omega)} = \|\mu^+\|_{\mathcal{M}_b(\Omega)} + \|\mu^-\|_{\mathcal{M}_b(\Omega)}$$

we obtain easily the first relation in (2.2.5). The second one is obtained in a similar way. □

In order to proceed we need to prove that when both the classical formulation for the obstacle problem and the new one, given in Definition 2.1.1, make sense then the solutions are the same. At present we prove it for a nonpositive obstacle, and we will prove it in the general case in Section 2.5.

Lemma 2.2.4. *Let μ be an element of $\mathcal{M}_b(\Omega)\cap \mathrm{H}^{-1}(\Omega)$ and ψ a nonpositive function; then the solution of $VI(\mu,\psi)$ coincides with the solution of $OP(\mu,\psi)$.*

Proof. Let u be the solution of $VI(\mu,\psi)$ and λ be the corresponding obstacle reaction. Thanks to Theorem 2.2.2 it is an element of $\mathcal{M}_b(\Omega)$; so $u \in \mathcal{F}_\psi(\mu)$. Take v an element in $\mathcal{F}_\psi(\mu)$, then $v = u_\mu + u_\nu$, with $\nu \in \mathcal{M}_b^+(\Omega)$, and $v \geq \psi$ q.e. in Ω.

Consider the approximation of ν, given by $\mathcal{A}T_k(u_\nu) =: \nu_k$. This is such that $\nu_k \rightharpoonup \nu$ weakly-$*$ in $\mathcal{M}_b(\Omega)$ and $\nu_k \in \mathcal{M}_b^+(\Omega)\cap \mathrm{H}^{-1}(\Omega)$ (see Theorem 1.3.12). Set $v_k = u_\mu + u_{\nu_k} = u_\mu + T_k(u_\nu)$. Since trivially $T_k(u_\nu) \nearrow u_\nu$ q.e. in Ω, we have

$$v_k \nearrow v \quad \text{q.e. in } \Omega.$$

Denote now the solutions of $VI(\mu,\psi_k)$ by u_k, where ψ_k are the functions defined by

$$\psi_k := \psi \wedge v_k.$$

From $\psi_k \leq \psi_{k+1}$ q.e. in Ω it easily follows that $u_k \leq u_{k+1}$ q.e. in Ω. Then there exists a function u^* such that $u_k \nearrow u^*$ q.e. in Ω.

So, passing to the limit in $u_k \geq \psi_k$ q.e. in Ω we obtain $u^* \geq \psi$ q.e. in Ω.

Moreover it is easy to see that $\|u_k\|_{\mathrm{H}_0^1(\Omega)} \leq C$. So, thanks to Lemma 1.2 in [20] we get that u^* is a quasi continuous function of $\mathrm{H}_0^1(\Omega)$ such that

$$u_k \rightharpoonup u^* \quad \text{weakly in } \mathrm{H}_0^1(\Omega).$$

Thanks to this, in the inequality

$$\langle \mathcal{A}u_k - \mu, v - u_k \rangle \geq 0$$

using Minty Lemma, we can pass to the limit and obtain

$$\langle \mathcal{A}u^* - \mu, v - u^* \rangle \geq 0,$$

for all $v \in \mathrm{H}^1_0(\Omega)$, with $v \geq \psi$ q.e. in Ω (notice that such v can be used as test function also in the first inequality, since $\psi \geq \psi_k$ q.e. in Ω). So u^* is the solution of $VI(\mu, \psi)$ and by uniqueness $u^* = u$ q.e. in Ω.

Naturally, from the minimality of u_k, we deduce

$$u_k \leq v_k \quad \text{q.e. in } \Omega.$$

so, passing to the limit as $k \to +\infty$ we conclude that $u \leq v$ q.e. in Ω. Since this is true for every $v \in \mathcal{F}_\psi(\mu)$, the function u is the minimum in $\mathcal{F}_\psi(\mu)$, i.e. the solution of $OP(\mu, \psi)$. □

We are now in a position to prove that, for every $\mu \in \mathcal{M}_b(\Omega)$ and for every $\psi \leq 0$, there exists a solution to the obstacle problem according to Definition 2.1.1.

Theorem 2.2.5. *Let $\psi \leq 0$ and $\mu \in \mathcal{M}_b(\Omega)$. Then there exists a (unique) solution of $OP(\mu, \psi)$.*

Proof. Consider the function u_μ and define

$$\mathcal{A}(T_k(u_\mu)) =: \mu_k.$$

We know from Theorem 1.3.12 that

$$\mu_k \rightharpoonup \mu \quad \text{weakly-}* \text{ in } \mathcal{M}_b(\Omega)$$

and $\mu_k \in \mathrm{H}^{-1}(\Omega)$.

Let u_k be the solution of $VI(\mu_k, \psi)$ and denote

$$\mathcal{A}u_k - \mu_k =: \lambda_k,$$

which we know from Theorem 2.2.2 to be a measure in $\mathcal{M}_b^+(\Omega)$ such that

$$||\lambda_k||_{\mathcal{M}_b(\Omega)} \leq ||\mu_k^-||_{\mathcal{M}_b(\Omega)}. \tag{2.2.6}$$

Up to a subsequence $\lambda_k \rightharpoonup \lambda$ weakly-$*$ in $\mathcal{M}_b(\Omega)$. From this, thanks to Proposition 1.3.5, it follows that $u_k \to u$ strongly in $\mathrm{W}^{1,q}(\Omega)$, with $u = u_\mu + u_\lambda$, and also that $T_h(u_k) \rightharpoonup T_h(u)$ weakly in $\mathrm{H}_0^1(\Omega)$, for all $h > 0$.

Now the set

$$E := \{v \in \mathrm{H}_0^1(\Omega) \ : \ v \geq T_h(\psi) \ \text{ q.e. in } \Omega\}$$

is closed and convex in $\mathrm{H}_0^1(\Omega)$, so it is also weakly closed. Since, clearly, $T_h(u_k) \geq T_h(\psi)$ q.e. in Ω, passing to the limit as $k \to +\infty$ we get that also $T_h(u) \in E$, hence $T_h(u) \geq T_h(\psi)$ q.e. in Ω for all $h > 0$. Passing to the limit as $h \to +\infty$ we get $u \geq \psi$ q.e. in Ω In conclusion we deduce $u \in \mathcal{F}_\psi(\mu)$.

To show that u is minimal, take $v \in \mathcal{F}_\psi(\mu)$ so that $v \geq \psi$ q.e. in Ω and $v = u_\mu + u_\nu$.

Let $v_k = u_{\mu_k} + u_\nu$ so that $v_k = T_k(u_\mu) + u_\nu$ and $v_k \rightharpoonup v$ weakly in $\mathrm{W}_0^{1,q}(\Omega)$.

Since $\psi \leq 0$, we have that $v_k \geq \psi$ q.e. in Ω. As u_k is the minimum of $\mathcal{F}_\psi(\mu_k)$, by Lemma 2.2.4, we obtain $u_k \leq v_k$ q.e. in Ω, in the limit $u \leq v$ a.e. and then also q.e. in Ω. Hence u solves $OP(\mu, \psi)$. □

From formula (2.2.6) we see that to extend (2.2.4) to the case of $\mu \in \mathcal{M}_b(\Omega)$ we just need to show that

$$||\mu_k^-||_{\mathcal{M}_b(\Omega)} \to ||\mu^-||_{\mathcal{M}_b(\Omega)};$$

this is proved in the following proposition.

Proposition 2.2.6. *Let $\psi \leq 0$ and $\mu \in \mathcal{M}_b(\Omega)$. Let u be the solution of $OP(\mu, \psi)$ and λ the corresponding obstacle reaction. Then*

$$||\lambda||_{\mathcal{M}_b(\Omega)} \leq ||\mu^-||_{\mathcal{M}_b(\Omega)}.$$

Proof. This proof repeats the one in [12] with minor modifications.

Let f_n be a smooth approximation of μ in the $*$-weak topology of $\mathcal{M}_b(\Omega)$, such that $||f_n||_{\mathrm{L}^1(\Omega)} \leq ||\mu||_{\mathcal{M}_b(\Omega)}$, and let u_n be the solutions of

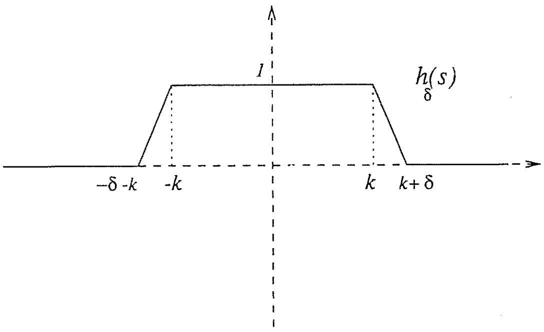
$$\begin{cases} \mathcal{A}u_n = f_n & \text{in } \mathrm{H}^{-1}(\Omega) \\ u_n \in \mathrm{H}_0^1(\Omega). \end{cases}$$

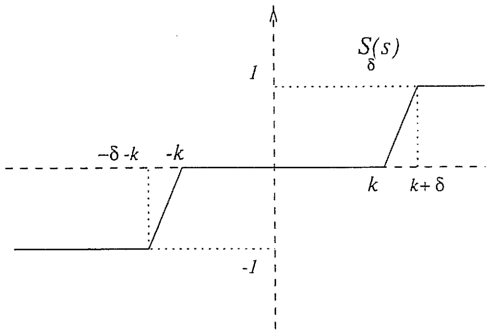
Consider, for $\delta > 0$, the Lipschitz continuous functions h_δ defined by

$$\begin{cases} h_\delta(s) = 1 & \text{if } |s| \leq k \\ h_\delta(s) = 0 & \text{if } |s| \geq k+\delta \\ |h'_\delta(s)| = \dfrac{1}{\delta} & \text{if } k \leq |s| \leq k+\delta, \end{cases}$$

and S_δ defined by

$$\begin{cases} S_\delta(s) = 0 & \text{if } |s| \leq k \\ S_\delta(s) = \operatorname{sign}(s) & \text{if } |s| \geq k+\delta \\ S'_\delta(s) = \dfrac{1}{\delta} & \text{if } k \leq |s| \leq k+\delta. \end{cases}$$





We can compute

$$-\operatorname{div}(h_\delta(u_n)A(x)Du_n) = h_\delta(u_n)f_n + h'_\delta(u_n)A(x)Du_nDu_n$$

$$= h_\delta(u_n)f_n + \frac{1}{\delta}\chi_{\{-k-\delta<u_n<-k\}}(x)A(x)Du_nDu_n - \frac{1}{\delta}\chi_{\{k<u_n<k+\delta\}}(x)A(x)Du_nDu_n.$$

The second term gives

$$\int_{\{-k-\delta<u_n<-k\}} \frac{1}{\delta}A(x)Du_nDu_n\,dx = -\int_\Omega f_nS_\delta^-(u_n)\,dx.$$

Similarly for the third term we obtain

$$\int_{\{k<u_n<k+\delta\}} \frac{1}{\delta}A(x)Du_nDu_n\,dx = \int_\Omega f_nS_\delta^+(u_n)\,dx.$$

So we get that

$$\begin{aligned} &\int_\Omega |-\operatorname{div}(h_\delta(u_n)A(x)Du_n)|\,dx \\ &\leq \int_\Omega |f_n|\left(h_\delta(u_n) + S_\delta^+(u_n) + S_\delta^-(u_n)\right)\,dx \\ &= \int_\Omega |f_n|\,dx \leq ||\mu||_{\mathcal{M}_b(\Omega)}. \end{aligned}$$

Hence

$$\| -\operatorname{div}(h_\delta(u_n)A(x)Du_n)\|_{\mathcal{M}_b(\Omega)} = \| -\operatorname{div}(h_\delta(u_n)A(x)Du_n)\|_{L^1(\Omega)} \leq \|\mu\|_{\mathcal{M}_b(\Omega)}.$$

Now let δ tend to zero, with n and k fixed, and obtain

$$-\operatorname{div}(h_\delta(u_n)A(x)Du_n) \to \mathcal{A}T_k(u_n) \text{ strongly in } H^{-1}(\Omega)$$

which implies

$$\|\mathcal{A}T_k(u_n)\|_{\mathcal{M}_b(\Omega)} \leq \|\mu\|_{\mathcal{M}_b(\Omega)}.$$

As also $n \to +\infty$ we get

$$\|\mu_k\|_{\mathcal{M}_b(\Omega)} \leq \|\mu\|_{\mathcal{M}_b(\Omega)},$$

and thanks to Lemma 2.2.3 we obtain

$$\|\mu_k^-\|_{\mathcal{M}_b(\Omega)} \leq \|\mu^-\|_{\mathcal{M}_b(\Omega)},$$

and, since $\lambda_k \rightharpoonup \lambda$ weakly-$*$ in $\mathcal{M}_b(\Omega)$ and by the lower semicontinuity of the norm

$$\|\lambda\|_{\mathcal{M}_b(\Omega)} \leq \liminf_{n\to+\infty} \|\lambda_k\|_{\mathcal{M}_b(\Omega)} \leq \|\mu^-\|_{\mathcal{M}_b(\Omega)}.$$

□

2.3. The general existence theorem

We come now to prove the existence and uniqueness of the solution to the obstacle problem, without the technical assumption that the obstacle be negative. From now on the only hypothesis will be (2.1.1), i.e. that the obstacle is OP-admissible.

Theorem 2.3.1. *Let ψ be OP-admissible and let $\mu \in \mathcal{M}_b(\Omega)$. Then there exists a (unique) solution of $OP(\mu, \psi)$.*

Proof. It is enough to show that we can reduce the problem to the case $\psi \leq 0$. Indeed define

$$\varphi := \psi - u_\rho,$$

which is, obviously, q.e. negative.

By Theorem 2.2.5 there exists v minimum in $\mathcal{F}_{\varphi}(\mu-\rho)$, and we prove that the function $u := v + u_{\rho}$ is the minimum of $\mathcal{F}_{\psi}(\mu)$.

Trivially $u \geq \psi$ q.e. in Ω and, denoted the positive obstacle reaction associated to v by λ, we have $u = v + u_{\rho} = u_{\mu} + u_{\lambda}$, which shows that u is an element of $\mathcal{F}_{\psi}(\mu)$.

Consider now a function $w \in \mathcal{F}_{\psi}(\mu)$. By similar computations we deduce that $w - u_{\rho}$ belongs to $\mathcal{F}_{\varphi}(\mu-\rho)$ and, by the minimality of v, $v \leq w - u_{\rho}$, so that we conclude $u \leq w$ q.e. in Ω, and λ is the obstacle reaction associated to u. □

Remark 2.3.2. From the previous proof we deduce that in the general case we have the inequality

$$||\lambda||_{\mathcal{M}_b(\Omega)} \leq ||(\mu-\rho)^{-}||_{\mathcal{M}_b(\Omega)}. \tag{2.3.1}$$

We mention here a very simple and very useful result whose proof is immediate, but that is worth stating on its own.

Lemma 2.3.3. *Let $\mu, \nu \in \mathcal{M}_b(\Omega)$ and ψ is OP-admissible. Then u is the solution of $OP(\mu,\psi)$ if and only if $u - u_{\nu}$ is the solution of $OP(\mu-\nu, \psi - u_{\nu})$. The obstacle reaction is the same.*

2.4. A stability result

In this section we want to show a result of continuous dependence of the solutions on the obstacles.

The following proposition is proved here because it will be needed in the following, but the problem will be studied in details in Chapter 4.

Proposition 2.4.1. *Let $\psi_n : \Omega \to \overline{\mathbb{R}}$ be obstacles such that*

$$\psi_n \leq \psi \quad \textit{and} \quad \psi_n \to \psi \quad \textit{q.e. in } \Omega,$$

ψ OP-admissible, and let u_n and u be the solutions of $OP(\mu,\psi_n)$ and $OP(\mu,\psi)$, respectively. Then

$$u_n \to u \quad \textit{strongly in } \mathrm{W}^{1,q}(\Omega).$$

We also obtain that $u_n \to u$ q.e. in Ω and that $T_k(u_n) \to T_k(u)$ weakly in $\mathrm{H}^1_0(\Omega)$, for all $k > 0$.

Proof. Since u is trivially in $\mathcal{F}_{\psi_n}(\mu)$ for any n we have

$$u_n \le u \qquad \text{q.e. in } \Omega. \tag{2.4.1}$$

To every minimum u_n there corresponds a positive obstacle reaction λ_n, satisfying inequality (2.3.1), so we obtain that, up to a subsequence,

$$\lambda_n \rightharpoonup \hat{\lambda} \qquad \text{weakly-}* \text{ in } \mathcal{M}_b(\Omega)$$
$$u_n \to \hat{u} \qquad \text{strongly in } \mathrm{W}_0^{1,q}(\Omega)$$

and

$$\hat{u} = u_\mu + u_{\hat{\lambda}}.$$

Hence, from (2.4.1), $\hat{u} \le u$ a.e. in Ω, and also q.e. On the other side, we have to prove that $\hat{u} \ge \psi$ q.e. in Ω, in order to obtain $\hat{u} \in \mathcal{F}_\psi(\mu)$, and so $u \le \hat{u}$ q.e. in Ω.

First consider the case when $\psi_n \le \psi_{n+1}$ q.e. in Ω.

From this fact it follows that $u_n \le u_{n+1}$ q.e. in Ω, and then $T_k(u_n) \le T_k(u_{n+1})$ q.e. in Ω, for all $k > 0$. Hence this sequence has a quasi everywhere limit. On the other hand, the fact that $\mu + \lambda_n \rightharpoonup \mu + \hat{\lambda}$ weakly-$*$ in $\mathcal{M}_b(\Omega)$ implies that $T_k(u_n) \rightharpoonup T_k(\hat{u})$ weakly in $\mathrm{H}_0^1(\Omega)$ and then, by Lemma 1.2 of [20], $T_k(u_n) \to T_k(\hat{u})$ q.e. in Ω. Since this holds for all $k > 0$ we get also

$$u_n \to \hat{u} \quad \text{q.e. in } \Omega.$$

Then, passing to the limit in $u_n \ge \psi_n$ q.e. in Ω we get $\hat{u} \ge \psi$ q.e. in Ω.

If the sequence ψ_n is not increasing, consider

$$\varphi_n := \inf_{k \ge n} \psi_k, \tag{2.4.2}$$

so that $\varphi_n \nearrow \psi$ q.e. in Ω and $\varphi_n \le \psi_n$ q.e. in Ω. If $\overline{u}_n$ is the solution of $OP(\mu, \varphi_n)$ it is easy to see, using Definition 2.1.1, that $\overline{u}_n \le u_n \le u$ q.e. in Ω. Applying the first case to $\overline{u}_n$ and passing to the limit we get $u_n \to u$ q.e. in Ω. □

2.5. Comparison with the classical solutions

In this section, we want to show that the new formulation of the obstacle problem is consistent with the classical one.

To talk about the equivalence of the two formulations it is necessary that both make sense. So we will work under the hypothesis that $\mu \in \mathcal{M}_b(\Omega) \cap \mathrm{H}^{-1}(\Omega)$ and that the obstacle ψ satisfies

$$\exists z \in \mathrm{H}_0^1(\Omega) \text{ s.t. } z \geq \psi \text{ q.e. in } \Omega; \tag{2.5.1}$$

$$\exists \rho \in \mathcal{M}_b^+(\Omega) \text{ s.t. } u_\rho \geq \psi \text{ q.e. in } \Omega. \tag{2.5.2}$$

or, in other words that ψ is both VI- and OP-admissible.

Later on we will discuss these conditions in deeper details.

Lemma 2.5.1. *If there exists a measure $\sigma \in \mathcal{M}_b(\Omega) \cap \mathrm{H}^{-1}(\Omega)$ such that $u_\sigma \geq \psi$ q.e. in Ω, then the solutions of $VI(\mu,\psi)$ and of $OP(\mu,\psi)$ coincide.*

Proof. Let u be the solution of $VI(\mu,\psi)$. Subtracting u_σ to it, and with the same technique as in the proof of Theorem 2.3.1, we return to the case of negative obstacle and we can use Lemma 2.2.4. □

Theorem 2.5.2. *Under the hypotheses (2.5.1) and (2.5.2), the solutions of $VI(\mu,\psi)$ and of $OP(\mu,\psi)$ coincide.*

Proof. As a first step consider the case of an obstacle bounded from above by a constant M. By Theorem 1.3.12, the measure $\rho_M := \mathcal{A}(T_M(u_\rho))$ is in $\mathcal{M}_b(\Omega) \cap \mathrm{H}^{-1}(\Omega)$ and $T_M(u_\rho) \geq \psi$ q.e. in Ω so that we are in the hypotheses of the previous lemma.

If, instead, ψ is not bounded, we consider the truncates $\psi \wedge k$, and, with respect to this new obstacle, conditions (2.5.1) and (2.5.2) are satisfied by the function $T_k(u_\rho)$, and in addition $\psi \wedge k \nearrow \psi$ q.e. in Ω.

Hence we can apply the first step and say that u_k, solution of $VI(\mu, \psi \wedge k)$, is also the solution of $OP(\mu, \psi \wedge k)$.

From Corollary 1.2.5 we know that the sequence u_k tends in $\mathrm{H}_0^1(\Omega)$, and hence also q.e., to the solution of $VI(\mu,\psi)$, while, from Proposition 2.4.1, u_k converges to the solution of $OP(\mu,\psi)$ q.e. in Ω. □

Remark 2.5.3. At this point we can notice that

$$\exists \sigma \in \mathcal{M}_b^+(\Omega) \cap \mathrm{H}^{-1}(\Omega) \text{ s.t. } u_\sigma \geq \psi \text{ q.e. in } \Omega$$

is not only a sufficient condition for both (2.5.1) and (2.5.2) to hold, but is also a necessary one.

Indeed if ψ satisfies both (2.5.1) and (2.5.2), then it is enough to take u_σ the solution of $OP(0,\psi)$, which, by Theorem 2.5.2, is the same as $VI(0,\psi)$, so that $\sigma \in \mathcal{M}_b^+(\Omega)\cap \mathrm{H}^{-1}(\Omega)$.

A little attention is required in treating conditions (2.5.1) and (2.5.2). Each one is necessary for the corresponding problem to be nonempty, but together they can be somewhat weakened.

First of all we underline that no one of the two conditions is implied by the other. This is seen with the following examples.

Example 2.5.4. Let $\Omega = (-1,1) \subseteq \mathbb{R}$ and let $\mathcal{A} = -\Delta = -u''$. Take $\psi \in \mathrm{H}_0^1(-1,1)$ such that $-\psi''$ is an unbounded positive Radon measure. For instance we may take $\psi = (1-|x|)(1-\log(1-|x|))$.

Now (2.5.1) is trivially true, and the solution of $VI(0,\psi)$ is ψ itself. If also (2.5.2) were true, then ψ would be also the solution of $OP(0,\psi)$. But this is not possible, because, being $-\psi''$ an unbounded measure, we can not write it as u_λ for some $\lambda \in \mathcal{M}_b^+(\Omega)$.

Example 2.5.5. Let $N \geq 3$, $\mathcal{A} = -\Delta$ and $\rho = \delta_{x_0}$, the Dirac's delta in a fixed point $x_0 \in \Omega$.

Take $\psi = u_{\delta_{x_0}}$, the Green's function with pole at x_0. Then (2.5.2) holds, but if also (2.5.1) held we would have $\psi \in \mathrm{L}^{2^*}(\Omega)$ which is not true.

On the other side we already saw in the proof of Theorem 2.5.2 that if an OP-admissible obstacle is also bounded, then it is VI-admissible.

Moreover, if a VI-admissible obstacle is "controlled near the boundary" then it is also OP-admissible: assume that (2.5.1) holds and there exists a compact $J \subset \Omega$, such that $\psi \leq 0$ in $\Omega \setminus J$. Then also (2.5.2) holds. Indeed just take as ρ the obstacle reaction corresponding to u, the solution of $VI(0,\psi)$. Then

$$\mathrm{supp}\rho \subseteq J,$$

and hence $\rho \in \mathcal{M}_b^+(\Omega)$.

A finer condition expressing the "control near the boundary" is that there exist J compact $\subset \Omega$ and $\tau \in \mathcal{M}_b^+(\Omega)\cap \mathrm{H}^{-1}(\Omega)$ sucht that $u_\tau \geq \psi$ q.e. in $\Omega \setminus J$.

To complete this discussion we present here a simple result which will be useful in the future.

Proposition 2.5.6. *Let $\psi \in \mathrm{H}^1(\Omega)$ be OP-admissible. Then it is also VI-admissible.*

Proof. Let $\rho \in \mathcal{M}_b^+(\Omega)$ be such that $\psi \le u_\rho$ q.e. in Ω.

For any $k \in \mathbb{R}^+$, we have $0 \le \psi^+ \wedge k \le u_\rho \wedge k$. Since $u_\rho \wedge k$ belongs to $\mathrm{H}_0^1(\Omega)$, so does $\psi^+ \wedge k$. As

$$\int_\Omega |D(\psi^+ \wedge k)|^2 dx \le \int_\Omega |D\psi^+|^2 dx < +\infty.$$

the function ψ^+ is the limit of the increasing sequence $\psi^+ \wedge k$, which is bounded in $\mathrm{H}_0^1(\Omega)$. This implies that $\psi^+ \in \mathrm{H}_0^1(\Omega)$, hence ψ is VI-admissible. □

In conclusion we want to remark that, in general, in classical variational inequalities, the obstacle reaction associated to the solution is indeed a Radon measure, but it is not always bounded, as Example 2.5.4 shows.

On the other side, in the new setting, the minimum of $\mathcal{F}_\psi(\mu)$ is not, in general, an element of $\mathrm{H}_0^1(\Omega)$.

Hence the two formulations do not overlap completely and no one is included in the other.

2.6. Approximation properties

As we will see in Section 4.1, if we have a sequence μ_n $*$-weakly convergent to μ, we can not, in general deduce convergence of solutions u_n of $OP(\mu_n, \psi)$ to the solution u of $OP(\mu, \psi)$, but, from (2.3.1) we have

$$||\lambda_n||_{\mathcal{M}_b(\Omega)} \le ||(\mu_n - \rho)^-||_{\mathcal{M}_b(\Omega)},$$

where the λ_n are the obstacle reactions relative to the solutions u_n. So, up to a subsequence,

$$\lambda_n \rightharpoonup \hat{\lambda} \text{ weakly-}* \text{ in } \mathcal{M}_b(\Omega)$$

and

$$u_n \to \hat{u} = u_\mu + u_{\hat{\lambda}} \text{ strongly in } \mathrm{W}^{1,q}(\Omega).$$

With the same argument used in the proof of the Theorem 2.2.5 we can show that $\hat{u} \geq \psi$ q.e. in Ω. Hence $\hat{u} \geq u$, the minimum of $\mathcal{F}_\psi(\mu)$.

On the other hand, in Theorem 2.3.1 we have obtained the solution of $OP(\mu, \psi)$ as a limit of the solutions to $OP(\mathcal{A}T_k(u_{\mu-\rho}) + \rho, \psi)$. We remark that if ρ belongs to the ordered dual of $\mathrm{H}^1_0(\Omega)$ that is $V := \{\mu \in \mathcal{M}_b(\Omega) \cap \mathrm{H}^{-1}(\Omega) \ : \ |\mu| \in \mathrm{H}^{-1}(\Omega)\}$ (i.e. ψ is both VI- and OP-admissible), then the approximating problems are actually variational inequalities.

Thanks to these two facts, when the obstacle is VI- and OP-admissible, we can characterize the solution u of $OP(\mu, \psi)$ by approximation with solutions of variational inequalities with data in V as follows.

1. For every sequence μ_k in $\mathcal{M}_b(\Omega)$, with $\mu_k \rightharpoonup \mu$ weakly-$*$ in $\mathcal{M}_b(\Omega)$, we have

$$s\text{-}\mathrm{W}^{1,q}_0(\Omega)\text{-}\lim_{k\to\infty} u_k \geq u.$$

2. There exists a sequence $\mu_k \in V$, with $\mu_k \rightharpoonup \mu$ weakly-$*$ in $\mathcal{M}_b(\Omega)$ such that

$$s\text{-}\mathrm{W}^{1,q}_0(\Omega)\text{-}\lim_{k\to\infty} u_k = u$$

In other words:

$$u = \min\left\{ s\text{-}\lim_{k\to+\infty} u_k \ : \ u_k \text{ sol. } VI(\mu_k, \psi),\ \mu_k \in V,\ \mu_k \rightharpoonup \mu \text{ weakly-}* \text{ in } \mathcal{M}_b(\Omega) \right\}.$$

Chapter 3
Interaction between obstacles and data

As mentioned in Section 1.2, for variational inequalities we have the characterization of solutions via complementarity conditions (1.2.5). These say, roughly speaking, that the solution of a variational inequality $VI(f,\psi)$ always touches the obstacle wherever it is not the solution of the equation itself $\mathcal{A}u = f$.

This fact is no longer true when we pass to consider data in $\mathcal{M}_b(\Omega)$: we show now an example (suggested by L. Orsina and A. Prignet) in which the solution of the obstacle problem with right-hand side measure does not touch the obstacle, though it is not the solution of the equation.

Example 3.0.1. Let $N \geq 2$, Ω be the ball $B_1(0)$, and $\mathcal{A} = -\Delta$. Take the datum μ a negative measure concentrated on a set of zero 2-Capacity and the obstacle ψ negative and bounded below by a constant $-h$. Let u be the solution of $OP(\mu,\psi)$, then $u = u_\mu + u_\lambda$. We want to show that $\lambda = -\mu$.

First observe that, for minimality, $u \leq 0$; on the other hand $u \geq -h$, so that $u = T_h(u)$ and hence $u \in \mathrm{H}_0^1(\Omega)$. This implies that the measure $\mu + \lambda$ is in $\mathcal{M}_b(\Omega) \cap \mathrm{H}^{-1}(\Omega)$, which is contained in $\mathcal{M}_b^0(\Omega)$. In other words $\lambda = -\mu + \hat{\lambda}$, with $\hat{\lambda}$ a measure in $\mathcal{M}_b^0(\Omega)$, and so positive, since λ is positive. Then $u \geq 0$, and finally $u = 0$. Thus the solution can be far above the obstacle, but the obstacle reaction is nonzero, and is exactly $-\mu$.

3.1. Measures vanishing on sets of zero capacity

We want to consider here a class of data for which the above phenomenon is avoided.

Consider, as datum, a measure in $\mathcal{M}_b^0(\Omega)$. In this case we can use the fact (see Remark 1.3.13) that for any such measure μ there exists a function f in $\mathrm{L}^1(\Omega)$ and a functional F in $\mathcal{M}_b(\Omega) \cap \mathrm{H}^{-1}(\Omega)$, such that $\mu = f + F$. If, in addition $\mu \geq 0$, then also f can be taken to be positive.

Assume that the obstacle satisfies also

$$\exists \sigma \in \mathcal{M}_b^0(\Omega) \ : \ u_\sigma \geq \psi \text{ q.e. in } \Omega; \tag{3.1.1}$$

this will be shortened by saying that ψ is OP^0-admissible.

Remark 3.1.1. Notice that, thanks to Remark 2.5.3, if an obstacle is both VI- and OP-admissible then it is also OP^0-admissible

We want to show that also the obstacle reaction λ belongs to $\mathcal{M}_b^0(\Omega)$ and that in this particular case we can write our obstacle problem in a variational way, that is with an "entropy formulation".

Notice that if the datum μ is in $\mathcal{M}_b^0(\Omega)$, but the obstacle is only OP-admissible, then the reaction λ in general does not belong to $\mathcal{M}_b^0(\Omega)$.

Indeed if for instance we take $\mathcal{A} = -\Delta$, $\mu = 0$ and $\psi = u_{\delta_{x_0}}$, the Green's function relative to the Dirac's delta centered at $x_0 \in \Omega$, then the solution of $OP(0, \psi)$ is $u_{\delta_{x_0}}$ itself and hence $\lambda = \delta_{x_0} \notin \mathcal{M}_b^0(\Omega)$.

We begin by considering the case of a negative obstacle.

Lemma 3.1.2. *Let $\psi \leq 0$ and let $\mu_1, \mu_2 \in \mathcal{M}_b(\Omega) \cap \mathrm{H}^{-1}(\Omega)$. Let λ_1 and λ_2 be the reactions of the obstacle corresponding to the solutions u_1 and u_2 of $VI(\mu_1, \psi)$ and $VI(\mu_2, \psi)$, respectively.*

If $\mu_1 \leq \mu_2$ then $\lambda_1 \geq \lambda_2$.

Proof. This proof is inspired by Lemma 2.5 in [22]. We easily have that $u_1 \leq u_2$.

Take now a function $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geq 0$, and set

$$\varphi_\varepsilon := \varepsilon\varphi \wedge (u_2 - u_1) \in \mathrm{H}_0^1(\Omega).$$

Now, using the hypothesis that $\mu_1 \leq \mu_2$ and monotonicity of $\mathcal{A}$, compute

$$\begin{aligned}\langle \lambda_1, \varepsilon\varphi - \varphi_\varepsilon \rangle &\geq \langle \mathcal{A}u_1, \varepsilon\varphi - \varphi_\varepsilon \rangle - \langle \mu_2, \varepsilon\varphi - \varphi_\varepsilon \rangle \\ &= \langle \mathcal{A}u_1 - \mathcal{A}u_2, \varepsilon\varphi - \varphi_\varepsilon \rangle + \langle \lambda_2, \varepsilon\varphi - \varphi_\varepsilon \rangle \\ &\geq \varepsilon \int_{\{u_2 - u_1 \leq \varepsilon\varphi\}} A(x)\nabla(u_1 - u_2)\nabla\varphi + \varepsilon\langle \lambda_2, \varphi \rangle - \langle \lambda_2, \varphi_\varepsilon \rangle.\end{aligned}$$

Now, using u_1 as a test function in $VI(\mu_2, \psi)$ and the fact that $u_2 - u_1 \geq \varphi_\varepsilon \geq 0$ we easily get $\langle \lambda_2, \varphi_\varepsilon \rangle = 0$.

Since, also, $-\langle \lambda_1, \varphi_\varepsilon \rangle \leq 0$ we obtain

$$\langle \lambda_1, \varphi \rangle \geq \int_{\{u_2 - u_1 \leq \varepsilon\varphi\}} A(x)\nabla(u_1 - u_2)\nabla\varphi + \langle \lambda_2, \varphi \rangle.$$

Passing to the limit as $\varepsilon \to 0$ and observing that

$$\int_{\{u_2-u_1\le\varepsilon\varphi\}} A(x)\nabla(u_1-u_2)\nabla\varphi \longrightarrow \int_{\{u_2=u_1\}} A(x)\nabla(u_1-u_2)\nabla\varphi = 0,$$

we get the thesis. □

Let us see now what can we say more if $\mu \in \mathcal{M}_b^0(\Omega)$, still in the case of negative obstacle.

Lemma 3.1.3. *Let $\psi \le 0$ and let $\mu \in \mathcal{M}_b^0(\Omega)$. Then the obstacle reaction relative to the solution of $OP(\mu,\psi)$ is also in $\mathcal{M}_b^0(\Omega)$.*

Proof. It is not restrictive to assume μ to be negative. Indeed, if $\mu = \mu^+ - \mu^-$, then also μ^+ and μ^- are in $\mathcal{M}_b^0(\Omega)$. Hence, by Lemma 2.3.3, the minimum of $\mathcal{F}_\psi(\mu)$ can be written as $u_{\mu^+} + v$ with v minimum in $\mathcal{F}_{\psi-u_{\mu^+}}(-\mu^-)$, and the same obstacle reaction λ; and so we are in the case of a negative measure.

Consider now the decomposition $\mu = f + F$ with $f \le 0$. And let $\mu_k := T_k(f) + F$ so that $\mu_k \to \mu$ strongly in $\mathcal{M}_b(\Omega)$.

Let u_k be the solution of $OP(\mu_k,\psi)$. It is also the solution of $VI(\mu_k,\psi)$ so that $\lambda_k \in \mathcal{M}_b^0(\Omega)$.

Thanks to Proposition 4.1.1 we have that $u_k \to u = u_\mu + u_\lambda$ strongly in $W_0^{1,q}(\Omega)$ and that $\lambda_k \rightharpoonup \lambda$ weakly-$*$ in $\mathcal{M}_b(\Omega)$.

From the fact that $\mu_k \ge \mu_{k+1}$ and from Lemma 3.1.2 we obtain that $\lambda_k \le \lambda_{k+1}$. Hence if we define

$$\hat{\lambda}(B) := \lim_{k\to\infty} \lambda_k(B) \qquad \forall B \text{ Borel set in } \Omega,$$

we know from classical measure theory that it is a bounded Radon measure, it is in $\mathcal{M}_b^0(\Omega)$, since all λ_k are, and necessarily coincides with λ. So $\lambda \in \mathcal{M}_b^0(\Omega)$. □

Remark 3.1.4. It is clear from the proof that the lemma holds as well if we only suppose that $\mu^- \in \mathcal{M}_b^0(\Omega)$.

In order to pass to a signed obstacle we need to require now that the obstacle is OP^0-admissible, i.e. satisfies (3.1.1).

Once we have noticed this, it is easy to use the result for a negative obstacle, as we did in the proof of Theorem 2.3.1 and obtain the following result.

Theorem 3.1.5. *Let ψ satisfy hypothesis (3.1.1), and let $\mu \in \mathcal{M}_b(\Omega)$ with μ^- in $\mathcal{M}_b^0(\Omega)$. Then the obstacle reaction relative to the solution of $OP(\mu,\psi)$ belongs to $\mathcal{M}_b^0(\Omega)$ as well.*

Remark 3.1.6. Notice that thanks to the pointwise convergence we have, in this case, that $\lambda_k \to \lambda$ strongly in $\mathcal{M}_b(\Omega)$.

As pointed out with Example 3.0.1 the fact that the reaction of the obstacle is concentrated on the coincidence set (roughly speaking, the set where the solution u is equal to the obstacle) which is true for variational inequalities (see complementarity conditions (1.2.5)), fails when the datum is a measure.

In the following theorem, proved by C. Leone in [33], it is shown that this holds anyway in the case of data in $\mathcal{M}_b^0(\Omega)$ (in [33] it is proved for obstacles that are both OP- and VI-admissible, but thanks to Lemma 2.3.3 it holds for any OP^0-admissible obstacle).

Theorem 3.1.7. *Let $\mu^- \in \mathcal{M}_b^0(\Omega)$ and let ψ be an OP^0-admissible obstacle. Then the following facts are equivalent*

1. *u is the solution of $OP(\mu,\psi)$ and λ is the corresponding obstacle reaction;*
2. *$\lambda \in \mathcal{M}_b^{0,+}(\Omega)$, $u = u_\mu + u_\lambda$ q.e. in Ω, $u \geq \psi$ q.e. in Ω, and $u = \psi$ λ-a.e. in Ω*

The problem of the interaction between data and obstacles will be deeply investigated in the forthcoming sections. In particular, Theorem 3.3.5 will extend Theorem 3.1.7 to general data, but of course, some restrictive hypothesis on the obstacle will be made.

Remark 3.1.8. These properties of the case of $\mathcal{M}_b^0(\Omega)$ measures, allow us to write the obstacle problem in a "more variational" way, when the obstacle is bounded from above. Namely, if $\mu \in \mathcal{M}_b^0(\Omega)$ and its decomposition is $\mu = f + F$ then the function u solution of $OP(\mu,\psi)$ satisfies also

$$\begin{cases} \langle \mathcal{A}u, T_j(v-u)\rangle \geq \displaystyle\int_\Omega f\, T_j(v-u) + \langle F, T_j(v-u)\rangle \\ \forall v \in \mathrm{H}_0^1(\Omega)\cap \mathrm{L}^\infty(\Omega),\ v \geq \psi \text{ q.e. in } \Omega \end{cases}$$

for any $j \in \mathbb{R}^+$. This is similar to the entropy formulation given by Boccardo and Cirmi in [6] in the case of datum in $\mathrm{L}^1(\Omega)$. Also the proof that such a formulation holds is similar to the one in [6] and is made by approximation. To this aim we choose the particular sequence of measures (see Remark 1.3.13) $\mu_k := T_k(f) + F$, so that $\mu_k \to \mu$ strongly in $\mathcal{M}_b(\Omega)$ and $\mu_k \in \mathcal{M}_b(\Omega) \cap \mathrm{H}^{-1}(\Omega)$. Hence also the solutions of $OP(\mu_k, \psi)$ (and also of $VI(\mu_k, \psi)$) converge strongly in $\mathrm{W}_0^{1,q}(\Omega)$ to u solution of $OP(\mu, \psi)$. Then u_k solves

$$\begin{cases} \langle \mathcal{A}u_k, v - u_k \rangle \geq \langle \mu_k, v - u_k \rangle \\ \forall v \in \mathrm{H}_0^1(\Omega),\ v \geq \psi \ \text{ q.e. in } \Omega \end{cases}$$

In this inequality we can use as test functions $v = T_j(w - u_k) + u_k$, with $w \in \mathrm{H}_0^1(\Omega) \cap \mathrm{L}^\infty(\Omega)$, $w \geq \psi$ q.e. in Ω, so that we obtain

$$\begin{aligned} &\langle \mathcal{A}u_k - \mathcal{A}w, T_j(w - u_k) \rangle + \langle \mathcal{A}w, T_j(w - u_k) \rangle \\ &\qquad \geq \int_\Omega T_k(f) T_j(w - u_k)\, dx + \langle F, T_j(w - u_k) \rangle. \end{aligned} \tag{3.1.2}$$

Consider one by one the four terms of this inequality. From $T_j(w - u_k) \rightharpoonup T_j(w - u)$ weakly in $\mathrm{H}_0^1(\Omega)$ we get

$$\langle \mathcal{A}w, T_j(w - u_k) \rangle \to \langle \mathcal{A}w, T_j(w - u) \rangle$$

and

$$\langle F, T_j(w - u_k) \rangle \to \langle F, T_j(w - u) \rangle;$$

by the dominated convergence theorem, we get

$$\int_\Omega T_k(f) T_j(w - u_k)\, dx \to \int_\Omega f T_j(w - u)\, dx;$$

and thanks to Lemma 1.3.6 and by Fatou's lemma we get

$$\begin{aligned} -\liminf_{k \to +\infty} \langle \mathcal{A}(w - u_k), T_j(w - u_k) \rangle &= -\liminf_{k \to +\infty} \int_\Omega A(x) DT_j(w - u_k) DT_j(w - u_k)\, dx \\ &\leq -\int_\Omega A(x) DT_j(w - u) DT_j(w - u)\, dx = -\langle \mathcal{A}(w - u), T_j(w - u) \rangle. \end{aligned}$$

Hence, letting k tend to infinity in (3.1.2) we get the thesis.

In the following sections it will be discussed how, when the obstacle is controlled from below in an appropriate way (see Theorem 3.3.1), it is possible, roughly speaking, to "isolate" the effect of the singular negative part of the data and refer to the case of datum in $\mathcal{M}_b^0(\Omega)$.

3.2. Some results of potential theory

In this section we prove some results concerning the potential of a measure. The first two lemmas characterize the measures of $\mathcal{M}_b^{0,+}(\Omega)$ in terms of their potentials.

The main result of this section is Lemma 3.2.3 about the behaviour of the potentials of two mutually singular measures near the points where both potentials diverge.

It allows us to study the solutions of two equations of the form (1.3.2) corresponding to mutually singular data. In particular (Lemma 3.2.4) we will compare these solutions near their singular points.

For every $\mu \in \mathcal{M}_b^+(\Omega)$ we consider the potentials $G\mu$ and $G_\Omega^{\mathcal{A}}\mu$ defined for every $x \in \Omega$ by

$$G\mu(x) = \int_\Omega G(|x-y|)\,d\mu(y)$$

and by

$$G_\Omega^{\mathcal{A}}\mu(x) = \int_\Omega G_\Omega^{\mathcal{A}}(x,y)\,d\mu(y).$$

The functions $G(|x|)$ and $G_\Omega^{\mathcal{A}}(x,y)$ were defined in page 25. Note that $-\Delta G\mu = \mu$ in the sense of distributions in Ω. By (1.3.8) $G_\Omega^{\mathcal{A}}\mu$ is a precise representative of the solution u_μ of (1.3.2).

Lemma 3.2.1. *Let $\mu \in \mathcal{M}_b^+(\Omega)$. Then,*

$$\mu \in \mathcal{M}_b^{0,+}(\Omega) \iff G\mu < +\infty \quad \mu\text{-a.e. in } \Omega,$$

Proof. One implication is easy: by a classical result (see, e.g., Theorem 7.33 in [29]) $G\mu$ is finite q.e. in Ω and hence μ-a.e. in Ω if $\mu \in \mathcal{M}_b^0(\Omega)$.

We will prove the converse first in the case $N > 2$, so that $G \geq 0$. We start proving that $\mu^s(\{G\mu < +\infty\}) = 0$.

For every $t > 0$, let $E_t := \{x \in \Omega : G\mu(x) \leq t\}$ and $\mu_t := \mu \llcorner E_t$. Note that E_t is a closed set since $G\mu$ is lower semicontinuous. As $\mu_t \leq \mu$, we have $G\mu_t \leq G\mu$ (recall that $G \geq 0$). In particular $G\mu_t \leq t$ in E_t. By the maximum principle (see, e.g., Theorem 1.10 in [32]) we obtain $G_{\mu_t} \leq t$ in Ω.

Since $G\mu_t$ is superharmonic and bounded it belongs to $\mathrm{H}^1_{loc}(\Omega)$ (see, e.g. Corollary 7.20 in [28]). As $\mu_t = -\Delta G\mu_t$ in the sense of distributions in Ω, we have $\mu_t \in \mathrm{H}^{-1}_{loc}(\Omega)$ and hence $\mu_t \in \mathcal{M}_b^{0,+}(\Omega)$.

Let now $B \subseteq \{G\mu < +\infty\}$ such that $\text{cap}(B) = 0$. Then B is the union of the sets $E_t \cap B$, for $t > 0$ and hence

$$\mu(B) = \sup_{t \in \mathbb{R}^+} \mu(E_t \cap B) = \sup_{t \in \mathbb{R}^+} \mu_t(B) = 0$$

and so $\mu^s(\{G\mu < +\infty\}) = 0$. In conclusion, if, by contradiction, μ^s were not identically zero it would be $\mu^s(\{G\mu = +\infty\}) > 0$, which implies $\mu(\{G\mu = +\infty\}) > 0$ and this would contradict the hypothesis.

The case $N = 2$ can be dealt with by adding a suitable constant c to G so that $G + c \geq 0$ in Ω. The proof is the same with minor modifications; in particular we use the version of maximum principle given in Theorem 1.6 in [32]. □

Now using (1.3.7) we can extend Lemma 3.2.1 to general elliptic operators.

Lemma 3.2.2. *If $\mu \in \mathcal{M}_b^+(\Omega)$ then*

$$\mu \in \mathcal{M}_b^0(\Omega) \iff G_\Omega^{\mathcal{A}}\mu < +\infty \ \mu\text{-a.e. in } \Omega,$$

for all elliptic operators $\mathcal{A}$ satisfying (1.0.4).

Proof. Thanks to (1.3.7) it is easy to prove that for every $x \in \Omega$

$$G\mu(x) < +\infty \iff G_\Omega^{\mathcal{A}}\mu(x) < +\infty \tag{3.2.1}$$

and so the thesis follows from Lemma 3.2.1. □

Also the next result is proved first for the case of the Laplacian on $\mathbb{R}^N$, then it will be extended.

Lemma 3.2.3. *Let $\mu, \nu \in \mathcal{M}_b^+(\Omega)$, with $\mu \perp \nu$ and let*

$$E := \{x \in \mathbb{R}^N : G\mu(x) = G\nu(x) = +\infty\}.$$

Then

$$\lim_{r \to 0^+} \frac{\displaystyle\fint_{B_r(x)} G\nu(y)\, dy}{\displaystyle\fint_{B_r(x)} G\mu(y)\, dy} = 0 \quad \text{for } \mu\text{-a.e. } x \in E, \tag{3.2.2}$$

where $\displaystyle\fint_A$ *denotes the mean value on* A.

Proof. Let $R > 0$ be such that $\Omega \subseteq B_R(0)$. Observing that $\Omega \subseteq B_{2R}(x)$ for every $x \in \Omega$, we have

$$\fint_{B_r(x)} G\nu(y)\,dy = \int_{B_{2R}(x)} G_r(|x-z|)\,d\nu(z),$$

where $G_r(|x-z|) := \fint_{B_r(x)} G(|y-z|)dy$ and ν is defined for every Borel se $B \subseteq \mathbb{R}^N$ by $\nu(B) = \nu(B \cap \Omega)$. As $G(|x|)$ is superharmonic in $\mathbb{R}^N$ and harmonic for $x \neq 0$, we obtain

$$G_r(s) \begin{cases} = G(s) & \text{for } s \geq r \\ \leq G(s) & \text{for } s < r \end{cases}$$

and $G_r(s) \nearrow G(s)$ as $r \searrow 0$.

For any measure $\nu \in \mathcal{M}_b^+(\Omega)$ the following equality holds

$$\int_{B_{2R}(x)} G_r(|x-z|)\,d\nu(z) = G_r(2R)\nu(\Omega) - \int_0^{2R} G_r'(s)\nu(B_s(x))\,ds. \tag{3.2.3}$$

Note that $\nu(\Omega) < +\infty$ and that $G_r(2R) = G(2R)$ for r small enough. Since the left hand side of (3.2.3) tends to $+\infty$, so does the last term.

The same argument can be developed for the denominator so the limit in (3.2.2) is equal to

$$\lim_{r\to 0^+} \frac{\displaystyle\int_0^{2R} G_r'(s)\nu(B_s(x))\,ds}{\displaystyle\int_0^{2R} G_r'(s)\mu(B_s(x))\,ds}. \tag{3.2.4}$$

Given $\delta \in (0, 2R)$, the integrals between δ and $2R$ remain bounded as $r \to 0$, so that (3.2.4) is equal to

$$\lim_{r\to 0^+} \frac{\displaystyle\int_0^{\delta} G_r'(s)\nu(B_s(x))\,ds}{\displaystyle\int_0^{\delta} G_r'(s)\mu(B_s(x))\,ds}. \tag{3.2.5}$$

Since $\mu \perp \nu$, by the Besicovitch differentiation theorem (see, e.g. Chapter 1.6 in [23]), for μ-a.e. $x \in \Omega$ we have

$$\lim_{r\to 0^+} \frac{\nu(B_r(x))}{\mu(B_r(x))} = 0. \tag{3.2.6}$$

Fix $x \in E$ such that (3.2.6) holds. For each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\nu(B_r(x)) < \varepsilon\mu(B_r(x)), \qquad \text{for all } r \in (0,\delta)$$

and since G_r is decreasing in s, we have

$$-\int_0^\delta G_r'(s)\nu(B_s(x))\,ds \le -\varepsilon \int_0^\delta G_r'(s)\mu(B_s(x))\,ds.$$

This shows that the limit in (3.2.5), and hence the one in (3.2.4), is less than or equal to ε. Since ε is arbitrary, the limit (3.2.4) is zero and we get the thesis. $\square$

Using (1.3.7) we can now prove this result in the general case of $\mathcal{A}$ elliptic operator with coefficients in $\mathrm{L}^\infty(\Omega)$.

Lemma 3.2.4. *Let $\mu,\nu \in \mathcal{M}_b^+(\Omega)$, with $\mu \perp \nu$, and let F be the set of points $x \in \Omega$ such that*

$$\lim_{r\to 0^+} \fint_{B_r(x)} u_\mu(z)\,dz = \lim_{r\to 0^+} \fint_{B_r(x)} u_\nu(z)\,dz = +\infty.$$

Then

$$\lim_{r\to 0^+} \frac{\displaystyle\fint_{B_r(x)} u_\nu(z)dz}{\displaystyle\fint_{B_r(x)} u_\mu(z)\,dz} = 0 \qquad \textit{for } \mu\textit{-a.e. } x \in F.$$

Proof. As for the numerator, let us fix $x \in F$ and choose $R > 0$ such that $B_R(x) \subset\subset \Omega$. Then

$$\begin{aligned}\fint_{B_r(x)} u_\nu(z)\,dz &= \fint_{B_r(x)} \int_\Omega G_\Omega^{\mathcal{A}}(z,y)\,d\nu(y)\,dz \\ &= \int_{\Omega\setminus B_R(x)} \fint_{B_r(x)} G_\Omega^{\mathcal{A}}(z,y)\,dz\,d\nu(y) + \int_{B_R(x)} \fint_{B_r(x)} G_\Omega^{\mathcal{A}}(z,y)\,dz\,d\nu(y).\end{aligned}$$

The first term, when $r < \frac{R}{2}$, is bounded, so only the second one is relevant in the limit. The same can be said of the denominator, so that we can study the ratio

$$\frac{\displaystyle\int_{B_R(x)} \fint_{B_r(x)} G_\Omega^{\mathcal{A}}(z,y)\, dz\, d\nu(y)}{\displaystyle\int_{B_R(x)} \fint_{B_r(x)} G_\Omega^{\mathcal{A}}(z,y)\, dz\, d\mu(y)}$$

Thanks to (1.3.7) this is smaller than or equal to

$$\frac{c_2 \displaystyle\int_{B_R(x)} \fint_{B_r(x)} G(|z-y|)\, dz\, d\nu(y) + d_2\nu(B_R(x))}{c_1 \displaystyle\int_{B_R(x)} \fint_{B_r(x)} G(|z-y|)\, dz\, d\mu(y) - d_1\mu(B_R(x))}. \tag{3.2.7}$$

Using again (1.3.7), for every $x \in F$ we have

$$\lim_{r\to 0^+} \fint_{B_r(x)} G\mu(z)\, dz = \lim_{r\to 0^+} \fint_{B_r(x)} G\nu(z)\, dz = +\infty.$$

Since $G\mu$ and $G\nu$ are superharmonic, this implies $G\mu(x) = G\nu(x) = +\infty$ for every $x \in F$.

Considering once again the fact that the integrals over $\Omega \setminus B_R(x)$ remain bounded as $r \to 0^+$, we obtain that the ratio in (3.2.7) tends to zero as $r \to 0^+$ for μ-a.e. $x \in F$. $\square$

Lemma 3.2.5. *Let $\mu, \nu \in \mathcal{M}_b(\Omega)$, let $\lambda \in \mathcal{M}_b^0(\Omega)$ and let $v \in \mathrm{H}^1(\Omega)$. Assume that $\nu \perp \mu^+$ and that*

$$u_\mu \le u_\nu + u_\lambda + v \quad q.e. \text{ in } \Omega.$$

Then $\mu^+ \in \mathcal{M}_b^0(\Omega)$

Proof. First of all the measures ν and λ can be assumed to be positive, replacing them with their positive parts. The function v can be replaced by $w + g$, where g is the solution of

$$\begin{cases} \mathcal{A}g = 0 & \text{in } \mathrm{H}^{-1}(\Omega), \\ g - v^+ \in \mathrm{H}_0^1(\Omega), \end{cases}$$

and $w = (v - g)^+$. Note that g is a nonnegative $\mathcal{A}$-harmonic function and w is a nonnegative function of $\mathrm{H}_0^1(\Omega)$ and we still have $u_\mu \le u_\nu + u_\lambda + w + g$ q.e. in Ω.

Step 1. Consider first the case $u_\mu \le u_\nu$. Then $u_{\mu^+} \le u_\nu + u_{\mu^-}$ q.e. in Ω. Clearly $\mu^+ \perp (\nu + \mu^-)$. Let E be the set of points $x \in \Omega$ such that

$$\lim_{r\to 0^+} \fint_{B_r(x)} u_{\mu^+}(y)\, dy = +\infty$$

Note that, by our hypothesis, E is contained in the set F of Lemma 3.2.4, relative to the measures μ^+ and $\nu + \mu^-$. Therefore

$$\lim_{r\to 0^+} \frac{\fint_{B_r(x)} u_{(\nu+\mu^-)}(z)\, dz}{\fint_{B_r(x)} u_{\mu^+}(z)\, dz} = 0 \quad \text{for } \mu^+\text{-a.e. } x \in E. \tag{3.2.8}$$

But by hypothesis

$$\frac{\fint_{B_r(x)} u_{(\nu+\mu^-)}(z)\, dz}{\fint_{B_r(x)} u_{\mu^+}(z)\, dz} \ge 1 \quad \text{for } B_r(x) \subseteq \Omega. \tag{3.2.9}$$

Hence we have $\mu^+(E) = 0$. As $G_\Omega^{\mathcal{A}}\mu^+$ is lower semicontinuous, we have $G_\Omega^{\mathcal{A}}\mu^+(x)$ for $x \in \Omega \setminus E$, and this implies $\mu^+ \in \mathcal{M}_b^{0,+}(\Omega)$, by Lemma 3.2.2.

Step 2. Let now $u_\mu \le u_\nu + g$ q.e. in Ω. Since g is $\mathcal{A}$-harmonic, it is continuous, hence

$$\lim_{r\to 0^+} \fint_{B_r(x)} g(y)dy = g(x) < +\infty, \quad \forall x \in \Omega.$$

Therefore, if we add this integral in the numerators of (3.2.8) and (3.2.9), we can repeat the same argument and we obtain $\mu^+ \in \mathcal{M}_b^{0,+}(\Omega)$ also in this case.

Step 3. Consider now also the term in λ. Assume $u_\mu \le u_\nu + g + u_\lambda$ q.e. in Ω.

As before $u_{\mu^+} \le u_{(\nu+\mu^-)} + g + u_\lambda$, with $\mu^+ \perp (\nu + \mu^-)$. Let now decompose μ^+ into $\mu_1 + \mu_2$, $\mu_i \in \mathcal{M}_b^+(\Omega)$, with $\mu_1 \ll \lambda$ and $\mu_2 \perp \lambda$. Then $u_{\mu_2} \le u_{(\lambda+\nu+\mu^-)} + g$, and by *Step 2* $\mu_2^+ = \mu_2 \in \mathcal{M}_b^{0,+}(\Omega)$. But $\mu_1 \in \mathcal{M}_b^{0,+}(\Omega)$ since $\lambda \in \mathcal{M}_b^{0,+}(\Omega)$, so we conclude that $\mu^+ \in \mathcal{M}_b^{0,+}(\Omega)$.

Step 4. Consider now the general case $u_\mu \le u_\nu + g + u_\lambda + w$ q.e. in Ω.

Take as obstacle $u_\mu - u_\nu - g - u_\lambda$ which is controlled above both by w and by u_μ so that it is both VI- and OP-admissible. Then the solution u_τ of $OP(0, u_\mu - u_\nu - g - u_\lambda)$ belongs to $\mathrm{H}_0^1(\Omega)$, hence $\tau \in \mathcal{M}_b^+(\Omega) \cap \mathrm{H}^{-1}(\Omega) \subseteq \mathcal{M}_b^{0,+}(\Omega)$.

So $u_\mu \le u_\nu + g + u_{(\lambda+\tau)}$ and we conclude by means of *Step 3*. □

Corollary 3.2.6. *Let* $\mu, \nu \in \mathcal{M}_b(\Omega)$, *let* $\lambda \in \mathcal{M}_b^0(\Omega)$, *and let* $v \in \mathrm{H}^1(\Omega)$. *Assume that* $\nu \perp \mu$ *and that* $|u_\mu| \le u_\nu + v + u_\lambda$ *q.e. in* Ω. *Then* $\mu \in \mathcal{M}_b^0(\Omega)$.

3.3. Interaction between obstacles and singular data

We arrive now to the main theorem in which it is shown that when the obstacle is regular enough, the component of μ^- which is singular with respect to the capacity is completely absorbed by the reaction λ.

Theorem 3.3.1. *Let* $\mu \in \mathcal{M}_b(\Omega)$ *and let* μ_s^- *be the part of* μ^- *which is concentrated in a set of capacity zero. Let* $\psi : \Omega \to \overline{\mathbb{R}}$ *be an obstacle such that*

$$-u_\tau - u_\sigma - \varphi \le \psi \le u_\sigma, \tag{3.3.1}$$

where $\varphi \in \mathrm{H}^1(\Omega)$, $\sigma \in \mathcal{M}_b^0(\Omega)$, *and* $\tau \in \mathcal{M}_b(\Omega)$, *with* $\tau \perp \mu_s^-$. *Let* $u = u_\mu + u_\lambda$ *be the solution of* $OP(\mu, \psi)$. *Then*

$$\lambda = \lambda_1 + \mu_s^-,$$

with $\lambda_1 \in \mathcal{M}_b^{0,+}(\Omega)$.

Proof. Write u as $u_{\mu^+} - u_{\mu_a^-} - u_{\mu_s^-} + u_\lambda$. As

$$u_\mu + u_{\mu^-} + u_\sigma = u_{\mu^+} + u_\sigma \ge \psi \text{ q.e. in } \Omega,$$

by Definition 2.1.1 we have $u_{\mu^+} + u_\sigma \ge u$ q.e. in Ω, which implies $u_\lambda - u_{\mu_s^-} \le u_\sigma + u_{\mu_a^-}$ q.e. in Ω. Hence by Lemma 3.2.5, $(\lambda - \mu_s^-)^+ \in \mathcal{M}_b^0(\Omega)$.

On the other hand $-u_{\mu_s^-} + u_\lambda \ge \psi - u_{\mu^+} + u_{\mu_a^-}$ q.e. in Ω and hence

$$u_{(\mu_s^- - \lambda)} \le u_{\mu^+} - u_\tau + u_\sigma + \varphi \text{ q.e. in } \Omega. \qquad (3.3.2).$$

Now $(\mu^+ + \tau) \perp (\mu_s^- - \lambda)^+$ since $\mu^+ \perp \mu^-$, $\tau \perp \mu_s^-$ and λ is positive. So, by Lemma 3.2.5, $(\mu_s^- - \lambda)^+ \in \mathcal{M}_b^{0,+}(\Omega)$.

As $(\mu_s^- - \lambda)^- = (\lambda - \mu_s^-)^+ \in \mathcal{M}_b^{0,+}(\Omega)$ we conclude that $(\mu_s^- - \lambda) \in \mathcal{M}_b^0(\Omega)$. Therefore $\lambda = \lambda_1 + \mu_s^-$ with $\lambda_1 \in \mathcal{M}_b^0(\Omega)$, hence $\lambda_1 \perp \mu_s^-$. Since $\lambda \geq 0$, we deduce that $\lambda_1 \geq 0$. □

Remark 3.3.2. Hypothesis (3.3.1) is satisfied, for instance, when ψ belongs to $H^1(\Omega)$ and is OP-admissible. Indeed, in this case, we know from Proposition 2.5.6 and Remark 3.1.1 that ψ is OP^0-admissible, then we can take $\varphi = -\psi$ and $\tau = 0$ in (3.3.1).

Remark 3.3.3. In the previous theorem the hypotheses on τ depend also on the datum μ, while the hypotheses on σ and φ depend only on the obstacle ψ.

The presence of τ in (3.3.1) allows anyway to treat situations like the following one. If $\mathcal{A} = -\Delta$, $\Omega = B_1(0)$, $N \geq 2$, the obstacle is $-u_{\delta_0}$ and the datum is $-\delta_{x_0}$ for any $x_0 \neq 0$, then the solution of the obstacle problem is zero, because the theorem applies and, on the other hand, the solution must be less than or equal to zero.

The next result follows easily From Theorem 3.3.1.

Theorem 3.3.4. *Let $\mu \in \mathcal{M}_b(\Omega)$ and let $\psi \in H^1(\Omega)$, OP-admissible. Then the solutions of $OP(\mu, \psi)$ and of $OP(\mu^+ - \mu_a^-, \psi)$ are the same.*

Proof. If u and u' are the respective solutions, then

$$u = u_{\mu^+} - u_{\mu_a^-} + u_{(-\mu_s^- + \lambda)} \geq \psi$$

with $(-\mu_s^- + \lambda)$ positive. Then $u \geq u'$ q.e. in Ω by Definition 2.1.1. Similarly, $u' \geq u$ q.e. in Ω. □

Notice that the solutions coincide but the reactions of the obstacle do not. In fact, if the first one is λ the other one is the λ_1 of Theorem 3.3.1.

Notice also that the case $\psi \in H^1(\Omega)$ is a particular case of this one, thanks to Remark 3.3.2.

We are in a position now to extend Theorem 3.1.7, valid for data in $\mathcal{M}_b^0(\Omega)$, to the case of data in $\mathcal{M}_b(\Omega)$, using the fact that for the above obstacles the negative singular part of the datum disappears.

Theorem 3.3.5. *Let $\mu \in \mathcal{M}_b(\Omega)$ let ψ satisfy the hypothesis (3.3.1). The following facts are equivalent*

1. *u is the solution of $OP(\mu,\psi)$ and λ is the corresponding obstacle reaction;*
2. *$\lambda = \lambda_1 + \mu_s^-$, with $\lambda_1 \in \mathcal{M}_b^{0,+}(\Omega)$ and $u = u_\mu + u_\lambda$ q.e. in Ω, $u \geq \psi$ q.e. in Ω $u = \psi$ λ_1-a.e. in Ω.*

Proof. By Theorem 3.3.4 $u = u_\mu + u_{\lambda_1} + u_{\mu_s^-}$ is the solution of $OP(\mu,\psi)$ if and only if $u = u_\mu + u_{\lambda_1}$ is the solution of $OP(\mu^+ - \mu_a^-, \psi)$. To this case Theorem 3.1.7 can be applied, and the conclusion follows. □

Chapter 4
Stability with respect to data

In this chapter we will be concerned with the stability of obstacle problems with respect to data. Both the dependence on the right-hand side and on the obstacle will be discussed.

The first one is much like the variational case: there is stability with respect to data converging strongly, but not, in general, with respect to data converging weakly-$*$.

As for the second one, we will see how it is influenced by the strange behaviour of solutions due to the singular components of the datum, that we have studied in the previous chapter.

4.1. Stability with respect to right-hand side

As for stability with respect to the right-hand side, we will show later that in general

$$\mu_n \rightharpoonup \mu \;\text{ weakly-}* \text{ in } \mathcal{M}_b(\Omega)$$

does not imply

$$u_n \rightharpoonup u \;\text{ weakly in } \mathrm{W}_0^{1,q}(\Omega),$$

where u_n and u are the solutions relative to μ_n and μ with the fixed obstacle ψ.

However we can give now the following stability result.

Proposition 4.1.1. *Let ψ be OP-admissible and let μ_n and μ be measures in $\mathcal{M}_b(\Omega)$ such that*

$$\mu_n \to \mu \textit{ strongly in } \mathcal{M}_b(\Omega),$$

then

$$u_n \to u \textit{ strongly in } \mathrm{W}_0^{1,q}(\Omega)$$

where u_n and u are the solutions of $OP(\mu_n, \psi)$ and of $OP(\mu, \psi)$, respectively.

Proof. Let λ_n be the obstacle reactions associated to u_n, then

$$\|\lambda_n\|_{\mathcal{M}_b(\Omega)} \leq \|(\mu_n - \rho)^-\|_{\mathcal{M}_b(\Omega)},$$

so, up to a subsequence,

$$\lambda_n \rightharpoonup \hat{\lambda} \text{ weakly-}* \text{ in } \mathcal{M}_b(\Omega)$$

and

$$u_n \to \hat{u} \text{ strongly in } \mathrm{W}^{1,q}(\Omega)$$

$$T_k(u_n) \rightharpoonup T_h(\hat{u}) \text{ weakly in } \mathrm{H}_0^1(\Omega)\ \forall k > 0$$

where $\hat{u} = u_\mu + u_{\hat{\lambda}}$.

As $T_k(u_n) \geq T_k(\psi)$ q.e. in Ω for every $k \geq 0$, and for every n, we have $T_k(\hat{u}) \geq T_k(\psi)$ q.e. in Ω for every $k > 0$.

Passing to the limit as $k \to +\infty$ we obtain that $\hat{u}$ belongs to $\mathcal{F}_\psi(\mu)$.

Let $v \in \mathcal{F}_\psi(\mu)$, with ν the associated measure. Consider now v_n the Stampacchia solution relative to $\zeta_n := \mu_n + (\mu_n - \mu)^- + \nu$. Since $\zeta_n \to \mu + \nu$ strongly in $\mathcal{M}_b(\Omega)$, the sequence v_n converges strongly in $\mathrm{W}_0^{1,q}(\Omega)$ to v.

Moreover $v_n \geq v \geq \psi$ q.e. in Ω; hence $v_n \in \mathcal{F}_\psi(\mu_n)$, then $u_n \leq v_n$ q.e. in Ω, and, in the limit,

$$\hat{u} \leq v \text{ a.e. in } \Omega,$$

and hence also q.e. in Ω. □

Remark 4.1.2. This result was used in the proof of Lemma 3.1.3. We prove it here, but its proof is completely independent of that result.

Remark 4.1.3. Thanks to this result we can say that the solutions obtained in this paper coincide with those given by Boccardo and Cirmi in [5] and [6] when the data are $\mathrm{L}^1(\Omega)$-functions, because those are obtained by strong approximation.

As said above we give now the counterexample showing that in general there is not stability with respect to $*$-weakly convergent data.

Example 4.1.4 Let $\Omega = (0,1)^N$ with $N \geq 3$, $\mathcal{A} = -\Delta$ and $\psi \equiv 0$.

Consider w_n and μ_n as defined in Example 1.3.11 so that

$$\mu_n \rightharpoonup 0 \text{ both weakly in } \mathrm{H}^{-1}(\Omega) \text{ and weakly-}* \text{ in } \mathcal{M}_b(\Omega).$$

Thus $w_n \rightharpoonup 0$ weakly in $\mathrm{H}_0^1(\Omega)$.

Let $\psi \equiv 0$ and let u_n be the solution of $VI(\mu_n, 0)$. Using w_n as test function in the variational inequality we get $||u_n||_{H_0^1(\Omega)} \leq C$. By contradiction assume that its $H_0^1(\Omega)$-weak limit is zero.

Consider the function $z_n := u_n + w_n$ which must then converge to zero weakly in $H_0^1(\Omega)$. Obviously $z_n \geq w_n$ q.e. in Ω and then $z_n \geq 1$ q.e. in Ω on $\bigcup B_{r_n}$. Hence if we define the obstacles

$$\psi_n := \begin{cases} 1 & \text{in } \bigcup B_{r_n} \\ 0 & \text{elsewhere} \end{cases}$$

$z_n \geq \psi_n$. Call v_n the function realizing

$$\min_{\substack{v \geq \psi_n \\ v \in H_0^1(\Omega)}} \int_\Omega |\nabla v|^2 \, dx.$$

A simple computation yields

$$-\Delta z_n = -\Delta u_n - \Delta w_n \geq 0.$$

Then $z_n \geq v_n \geq 0$, so that

$$v_n \rightharpoonup 0 \text{ weakly in } H_0^1(\Omega).$$

But this is not possible because a Γ-convergence result contained in [13] says that there exists a constant $c > 0$ such that v_n tends to the minimum point of

$$\min_{\substack{v \geq 0 \\ v \in H_0^1(\Omega)}} \int_\Omega |\nabla v|^2 \, dx + c \int_\Omega |(v-1)^-|^2 \, dx$$

which is not zero.

4.2. Stability with respect to obstacles: the level set convergence

The question regarding stability with respect to obstacle is much more variegated. In Proposition 2.4.1 we gave a first result, because it was needed in the subsequent discussion. Now we want to treat the subject as generally as possible.

It is possible to see that in general there is no continuous dependence on the obstacles unless we make additional assumptions.

In this section we will define a kind of convergence of functions which will prove to be a good one for the obstacles in obstacle problems with measure data: it is rather general and allows to obtain the convergence of the solutions under very mild assumptions.

Definition 4.2.1. Let ψ_n and ψ be quasi upper semicontinuous function from Ω to $\overline{\mathbb{R}}$. We say that ψ_n tends to ψ in the sense of level sets and write

$$\psi_n \xrightarrow{\text{lev}} \psi$$

if

$$\operatorname{cap}(\{\psi > t\} \cap B) \leq \liminf_{n\to\infty} \operatorname{cap}(\{\psi_n > s\} \cap B') \tag{4.2.1}$$

$$\limsup_{n\to\infty} \operatorname{cap}(\{\psi_n > t\} \cap B) \leq \operatorname{cap}(\{\psi > s\} \cap B') \tag{4.2.2}$$

for all $s, t \in \mathbb{R}$, $s < t$, and for all $B \subset\subset B' \subset\subset \Omega$.

To make sure that this is a good definition we prove now that if the limit exists then it is unique.

To this purpose we give the following lemma from capacity theory which can be found in [24].

Lemma 4.2.2. *Let E and F be quasi closed subsets of Ω such that*

$$\operatorname{cap}(E \cap A) \leq \operatorname{cap}(F \cap A), \quad \forall A \subseteq \Omega \textit{ open}, \tag{4.2.3}$$

then $\operatorname{cap}(E \setminus F) = 0$ *(we say also that E is quasi contained in F).*

Proof. First show that (4.2.3) holds also for all A quasi open. Indeed then there exists an open set V, with $\operatorname{cap}(V) < \varepsilon$ such that $A \cup V$ is open. Then

$$\begin{aligned}\operatorname{cap}(E \cap A) &\leq \operatorname{cap}(E \cap (A \cup V) \\ &\leq \operatorname{cap}(F \cap (A \cup V)) \\ &\leq \operatorname{cap}(F \cap A) + \operatorname{cap}(V) < \operatorname{cap}(F \cap A) + \varepsilon\end{aligned}$$

and conclude by arbitrariness of ε.

Now, since E is quasi closed we can take $A = \Omega \setminus E$ and obtain that $\operatorname{cap}(E \setminus F) = 0$. □

Proposition 4.2.3. *Let ψ_n, ψ and φ be quasi upper semicontinuous functions. If*

$$\psi_n \xrightarrow{\text{lev}} \psi \qquad \text{and} \qquad \psi_n \xrightarrow{\text{lev}} \varphi,$$

then $\psi = \varphi$.

Proof. Let us fix an open set $A \subset\subset \Omega$ and two real numbers $s < t$. Take now two subsets A' and A'' such that $A'' \subset\subset A' \subset\subset A$ and real numbers t' and t'' such that $s < t' < t'' < t$. Then

$$\begin{aligned}\operatorname{cap}(\{\psi > t''\} \cap A'') &\le \liminf_{n\to+\infty} \operatorname{cap}(\{\psi_n > t'\} \cap A') \\ &\le \limsup_{n\to+\infty} \operatorname{cap}(\{\psi_n > t'\} \cap A') \le \operatorname{cap}(\{\varphi \ge s\} \cap A).\end{aligned}$$

Then, since $\{\psi \ge t\} \subseteq \{\psi > t''\}$, we have

$$\operatorname{cap}(\{\psi \ge t\} \cap A'') \le \operatorname{cap}(\{\varphi \ge s\} \cap A),$$

from which, invading A by means of $A'' \subset\subset A$,

$$\operatorname{cap}(\{\psi \ge t\} \cap A) \le \operatorname{cap}(\{\varphi \ge s\} \cap A).$$

Using the fact that ψ and φ are quasi upper semicontinuous and thanks to Lemma 4.2.2 we deduce that $\{\psi \ge t\}$ is quasi contained in $\{\varphi \ge s\}$. Now, fixed t, consider two sequences $t_k \searrow t$ and $s_k \searrow t$, with $t_k > s_k$, so that

$$\{\psi \ge t_k\} \nearrow \{\psi > t\} \quad \text{and} \quad \{\varphi \ge s_k\} \nearrow \{\varphi > t\},$$

and we get that

$$\{\psi > t\} \text{ is quasi contained in } \{\varphi > t\},$$

for all $t \in \mathbb{R}$.

Exchanging the roles of ψ and φ we get the reverse inclusion so that $\{\psi > s\}$ and $\{\varphi > s\}$ coincide up to sets of capacity zero.

Now we recover the values of ψ and φ at quasi every point $x \in \Omega$ thanks to the well known formula

$$\varphi(x) = \sup_{s \in Q} s\, \chi_{\{\varphi > s\}}(x).$$

Since the level sets are the same, the two functions coincide quasi everywhere. □

The main result on level sets convergence is the following theorem which shows the connection with the Mosco convergence introduced in Definition 1.2.3 (for the proof see Theorem 5.9 in [19]).

Theorem 4.2.4. *Let ψ_n and ψ be functions $\Omega \to \overline{\mathbb{R}}$ If*

$$K_{\psi_n}(\Omega) \cap \mathrm{H}_0^1(\Omega) \xrightarrow{\mathrm{M}} K_{\psi}(\Omega) \cap \mathrm{H}_0^1(\Omega).$$

then

$$\psi_n \xrightarrow{\text{lev}} \psi$$

If moreover the obstacles are equicontrolled above, namely

$$\psi_n,\ \psi \le u_\rho \quad \text{with } \rho \in \mathcal{M}_b(\Omega) \cap \mathrm{H}^{-1}(\Omega),$$

then also the reverse implication holds.

Notice that, though very similar to Mosco convergence, the level set convergence concerns also the case of obstacles that are not VI-admissible.

Remark 4.2.5. From the definition it is clear that the level set convergence is implied by

$$\operatorname{cap}(\{\psi > t\} \cap B) = \lim_{n \to +\infty} \operatorname{cap}(\{\psi_n > t\} \cap B). \tag{4.2.4}$$

for all $t \in \mathbb{R}$ and for all $B \subset\subset \Omega$

Remark 4.2.6. From Theorem 4.2.4 it follows that, if ψ_n converge to ψ in capacity, i.e.

$$\operatorname{cap}(\{|\psi_n - \psi| > t\}) \to 0, \quad \forall t \in \mathbb{R}^+,$$

then $\psi_n \xrightarrow{\text{lev}} \psi$.

Another simple observation, which requires no proof, but which is useful to state separately, is the following.

Lemma 4.2.7. *Let $\psi_n \xrightarrow{\text{lev}} \psi$ and let $\Phi : \mathbb{R} \to \mathbb{R}$ be a continuous non decreasing function. Then*

$$\Phi(\psi_n) \xrightarrow{\text{lev}} \Phi(\psi).$$

In the next lemmas we will denote the solution of $OP(\mu, \psi_n)$ by u_n and of $OP(\mu, \psi)$ by u.

Let us show that in general the Mosco convergence (and so also the level set convergence) of the obstacles does not imply the convergence of the solutions.

Example 4.2.8. Let $\Omega = B_1(0) \subseteq \mathbb{R}^N$, with $N > 2$, $\mathcal{A} = -\Delta$ and $\mu = -\delta_0$, the Dirac delta in the origin.

Let the obstacles $\psi_n = -n$, so that clearly

$$K_{\psi_n}(\Omega) \cap \mathrm{H}_0^1(\Omega) \xrightarrow{\mathrm{M}} \mathrm{H}_0^1(\Omega)$$

and, by Theorem 4.2.4, also $\psi_n \xrightarrow{\text{lev}} -\infty$

It is immediate to see that the solutions $u_n = u_{-\delta_0} + u_{\lambda_n}$ of $OP(-\delta_0, -n)$ are less than or equal to zero since the latter is in $\mathcal{F}_{-n}(-\delta_0)$. So $u_n = T_n(u_n)$ and hence is in $\mathrm{H}_0^1(\Omega)$. But then $-\delta_0 + \lambda_n \in \mathrm{H}^{-1}(\Omega) \cap \mathcal{M}_b(\Omega) \subset \mathcal{M}_b^0(\Omega)$, and it must be a positive measure and hence $u_n = 0$ for each n. On the other hand $u = u_{-\delta_0}$ and cannot be the limit of the u_n.

What can be proved without further assumptions is the following result.

Proposition 4.2.9. *Let $\psi_n, \psi \le u_\rho$ q.e. in Ω with $\rho \in \mathcal{M}_b(\Omega)$. Assume that*

$$\psi_n \xrightarrow{\text{lev}} \psi$$

Then there exists a subsequence $u_{n'}$ and a quasi continuous function $u^* \in \mathrm{W}_0^{1,q}(\Omega)$, such that

$$u_{n'} \to u^* \quad \textit{strongly in } \mathrm{W}^{1,q}(\Omega),$$

and

$$u^* \ge u \quad \textit{q.e. in } \Omega.$$

Proof. By Theorem 2.3.1

$$\|\lambda_n\|_{\mathcal{M}_b(\Omega)} \le \left\|(\mu - \rho)^-\right\|_{\mathcal{M}_b(\Omega)} \tag{4.2.5}$$

so that there exists a subsequence $\{\lambda_{n'}\}$ and a measure $\lambda^* \in \mathcal{M}_b^+(\Omega)$ such that $\lambda_n \rightharpoonup \lambda^*$ weakly-$*$ in $\mathcal{M}_b(\Omega)$ and hence $u_{n'} = u_\mu + u_{\lambda_{n'}} \to u^* = u_\mu + u_{\lambda^*}$ strongly in $\mathrm{W}^{1,q}(\Omega)$.

If we show that $u^* \ge \psi$ q.e. in Ω we will have that u^* is in $\mathcal{F}_\psi(\mu)$ and get the thesis, by Definition 2.1.1.

Given $k > 0$, observe that, thanks to (1.3.6) and to (4.2.5)

$$\int_\Omega |DT_k(u_{n'})|^2 dx \le kc$$

and hence $T_k(u_{n'}) \rightharpoonup T_k(u^*)$ weakly in $\mathrm{H}_0^1(\Omega)$.

From Lemma 4.2.7 it follows that

$$T_k(\psi_n) \xrightarrow{\text{lev}} T_k(\psi), \quad \forall k \in \mathbb{R}^+;$$

Since $T_k(u_n) \geq T_k(\psi_n)$ q.e. in Ω for each n and k, and using Theorem 4.2.4 and the definition of Mosco convergence, we get $T_k(u^*) \geq T_k(\psi)$ q.e. in Ω.

Now we can pass to the limit as $k \to +\infty$ and obtain $u^* \geq \psi$ q.e. in Ω. □

We prove now the central lemma of this section.

Lemma 4.2.10. *Let ψ_n and ψ be quasi upper semicontinuous functions controlled above by u_ρ with $\rho \in \mathcal{M}_b(\Omega) \cap \mathrm{H}^{-1}(\Omega)$, and let w be a quasi continuous function. If*

$$\psi_n \xrightarrow{\text{lev}} \psi$$

then

$$\psi_n - w \xrightarrow{\text{lev}} \psi - w.$$

To prove this Lemma we need the following result.

Lemma 4.2.11. *Given a quasi continuous function w, for each $A \subset\subset \Omega$ and for each $\varepsilon > 0$, there exists $u \in C_0^\infty(\Omega)$ such that*

$$\operatorname{cap}(\{|u - w| > \varepsilon\} \cap A) < \varepsilon. \tag{4.2.6}$$

Proof. By definition of quasi continuity there exists a relatively closed subset C such that $\operatorname{cap}(\Omega \backslash C) < \varepsilon$ and $w_{|_C}$ is continuous. By Tietze's theorem there exists a continuous function g which extends $w_{|_{C\cap\overline{A}}}$ to $\mathbb{R}^N$.

Obviously, for any $A \subset\subset \Omega$, we have $\{|w - g| > 0\} \cap A \subseteq \Omega \setminus C$ so that

$$\operatorname{cap}(\{|w - g| > 0\} \cap A) < \varepsilon.$$

On its turn g can be approximated in A with a function $u \in C_0^\infty(\Omega)$ so that $\sup_A |u - g| < \varepsilon$, and again from the fact that $\{|w - u| > \varepsilon\} \cap A \subseteq \{|w - g| > 0\} \cap A$ we get $\operatorname{cap}(\{|w - u| > \varepsilon\} \cap A) < \varepsilon$. □

Proof of 4.2.10. It is immediate to observe that the thesis is true when, instead of w, we have a function u in $\mathrm{H}_0^1(\Omega)$. This is because under our hypotheses level set convergence is equivalent to Mosco convergence (see Theorem 4.2.4) and translating $K_{\psi_n}(\Omega)$ and $K_\psi(\Omega)$ by $u \in \mathrm{H}_0^1(\Omega)$ we get

$$\mathrm{K}_{(\psi_n - u)} \cap \mathrm{H}_0^1(\Omega) \xrightarrow{\mathrm{M}} \mathrm{K}_{(\psi - u)} \cap \mathrm{H}_0^1(\Omega).$$

or equivalently

$$\psi_n - u \xrightarrow{\mathrm{lev}} \psi - u$$

We want to show the inequalities (4.2.1) and (4.2.2) for $\psi_n - w$ and $\psi - w$.

Let us fix $B \subset\subset \Omega$, $\varepsilon > 0$ and a function $u \in C_0^\infty(\Omega)$ such that (4.2.6) holds with respect to B.

Observe now that for any $t \in \mathbb{R}$ we have

$$\{\psi_n - w > t\} \cap B \subseteq (\{\psi_n - u > t - \varepsilon\} \cap B) \cup (\{u - w > \varepsilon\} \cap B),$$

hence, by subadditivity,

$$\mathrm{cap}\,(\{\psi_n - w > t\} \cap B) \leq \mathrm{cap}\,(\{\psi_n - u > t - \varepsilon\} \cap B) + \mathrm{cap}\,(\{|u - w| > \varepsilon\} \cap B).$$

Passing to the limsup and using (4.2.6), we obtain

$$\limsup_{n\to\infty} \mathrm{cap}\,(\{\psi_n - w > t\} \cap B) \leq \limsup_{n\to\infty} \mathrm{cap}\,(\{\psi_n - u > t - \varepsilon\} \cap B) + \varepsilon.$$

We know that, for $\psi - u$, (4.2.2) holds true, so we can use it with $t - \varepsilon$ and $t - 2\varepsilon$ instead of t and s, so that, for $B \subset\subset B' \subset\subset \Omega$, we get

$$\limsup_{n\to\infty} \mathrm{cap}\,(\{\psi_n - w > t\} \cap B) \leq \mathrm{cap}\,(\{\psi - u > t - 2\varepsilon\} \cap B') + \varepsilon. \tag{4.2.7}$$

With an argument similar to before from

$$\{\psi - u > t - 2\varepsilon\} \cap B' \subseteq (\{\psi - w > t - 3\varepsilon\} \cap B') \cup (\{w - u > \varepsilon\} \cap B'),$$

we obtain,

$$\mathrm{cap}\,(\{\psi - u > t - 2\varepsilon\} \cap B') \leq \mathrm{cap}\,(\{\psi - w > t - 3\varepsilon\} \cap B') + \varepsilon,$$

and substituting in (4.2.7) we get

$$\limsup_{n\to\infty} \operatorname{cap}(\{\psi_n - w > t\} \cap B) \leq \operatorname{cap}(\{\psi - w > t - 3\varepsilon\} \cap B') + 2\varepsilon.$$

For any choice of s and t, ε can be taken sufficiently small so that $s < t - 3\varepsilon$. Then we can let $\varepsilon \to 0$ and conclude

$$\limsup_{n\to\infty} \operatorname{cap}(\{\psi_n - w > t\} \cap B) \leq \operatorname{cap}(\{\psi - w > s\} \cap B').$$

Here nothing depends on u so this holds for all $s, t \in \mathbb{R}$, $s < t$ and for all $B \subset\subset B' \subset\subset \Omega$.

Inequality (4.2.1) is proved in a similar way and this concludes the proof. □

Theorem 4.2.12. *Let $\psi_n, \psi \leq u_\rho$ with $\rho \in \mathcal{M}_b(\Omega) \cap \mathrm{H}^{-1}(\Omega)$. Let $\mu \in \mathcal{M}_b(\Omega)$ with $\mu^- \in \mathrm{H}^{-1}(\Omega)$ and let u_n and u be the solutions of $OP(\mu, \psi_n)$ and of $OP(\mu, \psi)$ respectively. If*

$$\psi_n \xrightarrow{\text{lev}} \psi$$

then

$$u_n - u \in \mathrm{H}_0^1(\Omega) \text{ and } u_n - u \to 0 \text{ strongly in } \mathrm{H}_0^1(\Omega).$$

Proof. Thanks to the previous lemma we have that

$$\psi_n - u_\mu \xrightarrow{\text{lev}} \psi - u_\mu,$$

and then, since $\psi_n - u_\mu$, $\psi - u_\mu \leq u_\rho + u_{\mu^-}$ q.e. in Ω, by Theorem 4.2.4,

$$\mathrm{K}_{(\psi_n - u_\mu)} \cap \mathrm{H}_0^1(\Omega) \xrightarrow{\mathrm{M}} \mathrm{K}_{(\psi - u_\mu)} \cap \mathrm{H}_0^1(\Omega).$$

Hence all solutions of variational inequalities converge. In particular if v_n and v are the solutions of $VI(0, \psi_n - u_\mu)$ and $VI(0, \psi - u_\mu)$, respectively, then $v_n \to v$ strongly $\mathrm{H}_0^1(\Omega)$.

By Lemma 2.3.3 we have $u_n = v_n + u_\mu$ and $u = v + u_\mu$. This implies that $u_n - u = v_n - v$ and the conclusion follows. □

The minimal hypothesis on the obstacles ψ_n and ψ in order to have the solutions of $OP(\mu, \psi_n)$ and $OP(\mu, \psi)$ is that they be OP-admissible. Nevertheless if in this theorem we drop the request that they be controlled by a function which is also in $\mathrm{H}^1_0(\Omega)$, the conclusion fails. Indeed there is the following example which derives from Example 4.2.8.

Example 4.2.13. Let us consider the operator $\mathcal{A} = -\Delta$, the domain $\Omega = B_1(0) \subseteq \mathbb{R}^N$, with $N > 2$, and the datum $\mu = 0$.

Consider now as obstacles $\psi_n = u_{\delta_0} - n$, where δ_0 is the Dirac delta centred at zero. They are clearly OP-admissible, and also bounded by the same function u_ρ, but in this case $\rho = \delta_0 \notin \mathrm{H}^{-1}(\Omega)$.

Now for each n the solution u_n of $OP(0, \psi_n)$ is u_{δ_0} itself. Indeed, according to Lemma 2.3.3, $u_n - u_{\delta_0}$ is the solution of $OP(-\delta_0, -n)$, that, as seen in Example 4.2.8, is zero.

But then we have that although $\psi_n \xrightarrow{\text{lev}} -\infty$ and $u_n \to u_{\delta_0}$ while the solution of $OP(0, -\infty)$ is $u = 0$.

When the negative part of the measure μ is only in $\mathcal{M}^0_b(\Omega)$, we can not use the same trick because the the sets $K_{\psi_n - u_\mu} \cap \mathrm{H}^1_0(\Omega)$ might be empty, but anyway we do not fall in the pathology of the Example 4.2.8, and in fact we can prove the following theorem which gives the convergence of the solutions as well, though in a weaker sense.

Theorem 4.2.14. *Let $\psi_n, \psi \leq u_\rho$ with $\rho \in \mathcal{M}_b(\Omega) \cap \mathrm{H}^{-1}(\Omega)$, $\mu \in \mathcal{M}_b(\Omega)$ such that $\mu^- \in \mathcal{M}^0_b(\Omega)$, and let u_n and u be the solutions of $OP(\mu, \psi_n)$ and of $OP(\mu, \psi)$ respectively. If*

$$\psi_n \xrightarrow{\text{lev}} \psi$$

then $u_n \to u$ strongly in $\mathrm{W}^{1,q}(\Omega)$.

Proof. From [18] we know that μ^- can be written as $g\nu$ where $\nu \in \mathcal{M}^+_b(\Omega) \cap \mathrm{H}^{-1}(\Omega)$ and $g \in L^1(\Omega, \nu)$, $g \geq 0$. Hence the measure $\mu^-_k := (g \wedge k)\nu$ is in $\mathrm{H}^{-1}(\Omega)$, so that $\mu_k := \mu^+ - \mu^-_k$ satisfies the hypothesis of the previous theorem.

Call u^k_n and u^k the solutions of $OP(\mu^k, \psi_n)$ and $OP(\mu^k, \psi)$, respectively. By Theorem 4.2.12,

$$u^k_n \to u^k, \quad \text{strongly in } \mathrm{H}^1(\Omega) \qquad \forall k > 0.$$

Now, observing that $\mu^- - \mu^-_k$ is a positive measure, we easily obtain by comparison that $u^k_n \geq u_n$ and $u^k \geq u$, q.e. in Ω.

On the other hand

$$u_n + u_{(\mu^- - \mu_k^-)} = u_\mu + u_{(\lambda + \mu^- - \mu_k^-)},$$

where $\lambda_n \geq 0$ is the obstacle reaction of $OP(\mu, \psi_n)$. Since also $u_n + u_{(\mu^- - \mu_k^-)} \geq u_n \geq \psi_n$, by Definition 2.1.1 we have $u_n + u_{(\mu^- - \mu_k^-)} \geq u_n^k$ q.e. in Ω. In the same way we prove that $u + u_{(\mu^- - \mu_k^-)} \geq u^k$ q.e. in Ω.

Since $\mu^- - \mu_k^- \to 0$ strongly in $\mathcal{M}_b(\Omega)$, we have $u_{(\mu^- - \mu_k^-)} \to 0$ strongly in $\mathrm{W}^{1,q}(\Omega)$, so, from

$$u + u_{(\mu^- - \mu_k^-)} \geq u^k \geq u \quad \text{q.e. in } \Omega$$

letting $k \to \infty$ we get that $u^k \to u$ a.e. in Ω.

Recalling Proposition 4.2.9, let us fix a subsequence $\{u_{n'}\}$ which converges to a function u^* strongly in $\mathrm{W}^{1,q}(\Omega)$, so that from

$$u_{n'} + u_{(\mu^- - \mu_k^-)} \geq u_{n'}^k \geq u_{n'} \quad \text{q.e. in } \Omega$$

letting first $n' \to \infty$ and then $k \to \infty$ we obtain $u^* \geq u \geq u^*$ q.e. in Ω. Therefore $u_n^k \to u$, since the limit does not depend on the subsequence. □

As seen in Example 4.2.13 the request that the obstacles be well controlled can not be dropped, even if the datum is regular. On the other hand Example 4.2.8 showed that the control from above can be not enough to have convergence for all data $\mu \in \mathcal{M}_b(\Omega)$.

In the following theorem we show how, provided we strengthen the assumptions on the obstacles in the way given by Theorem 3.3.4, we can give up any assumption on the datum μ.

Notice that in the examples is always the limit obstacle the one that gives troubles. Indeed we see here that it is enough to require the control from below only for the limit.

Theorem 4.2.15. *Let $\psi_n, \psi \leq u_\rho$ with $\rho \in \mathcal{M}_b(\Omega) \cap \mathrm{H}^{-1}(\Omega)$ and let ψ satisfy (3.3.1). Let $\mu \in \mathcal{M}_b(\Omega)$ and let u_n and u be the solutions of $OP(\mu, \psi_n)$ and of $OP(\mu, \psi)$, respectively. If*

$$\psi_n \xrightarrow{\text{lev}} \psi$$

then $u_n \to u$ strongly in $\mathrm{W}^{1,q}(\Omega)$.

Proof. From Proposition 4.2.9 we know that, up to a subsequence, $u_n \to u^*$ strongly in $\mathrm{W}^{1,q}(\Omega)$, and $u^* \geq u$ q.e. in Ω.

Now consider v_n solution of $OP(\mu^+ - \mu_a^-, \psi_n)$. These, by Theorem 4.2.14, converge to v the solution of $OP(\mu^+ - \mu_a^-, \psi)$, but, according to Theorem 3.3.4, $v = u$.

On the other side $v_n = u_\mu + u_{\lambda_n} + u_{\mu_s^-}$, with $\lambda_n \in \mathcal{M}_b^+(\Omega)$, and $v_n \geq \psi_n$ q.e. in Ω and so, by Definition 2.1.1, we have

$$v_n \geq u_n \quad \text{q.e. in } \Omega.$$

Letting n go to $+\infty$ we obtain $u \geq u^*$ q.e. in Ω. Therefore $u_n \to u$ strongly in $W^{1,q}(\Omega)$. Since the limit does not depend on the subsequence, the whole sequence u_n converges to u.

□

Let us show now a further example, which clarifies more deeply in which cases there is not convergence of the solutions.

In particular we see that the limit obstacle need not be $-\infty$ everywhere, nor in a large portion of Ω; it suffices it is singular in the "right" point.

Example 4.2.16. Let us choose $\mathcal{A} = -\Delta$, $\Omega = B_1(0) \subseteq \mathbb{R}^N$, with $N > 2$, and $\mu = -\delta_0$. Let us consider the obstacles $\psi = -u_{\delta_0}$ and

$$\psi_n := -\left(u_{\delta_0} \wedge n\right) - \left(\frac{1}{2}u_{\delta_0} - n\right)^+$$

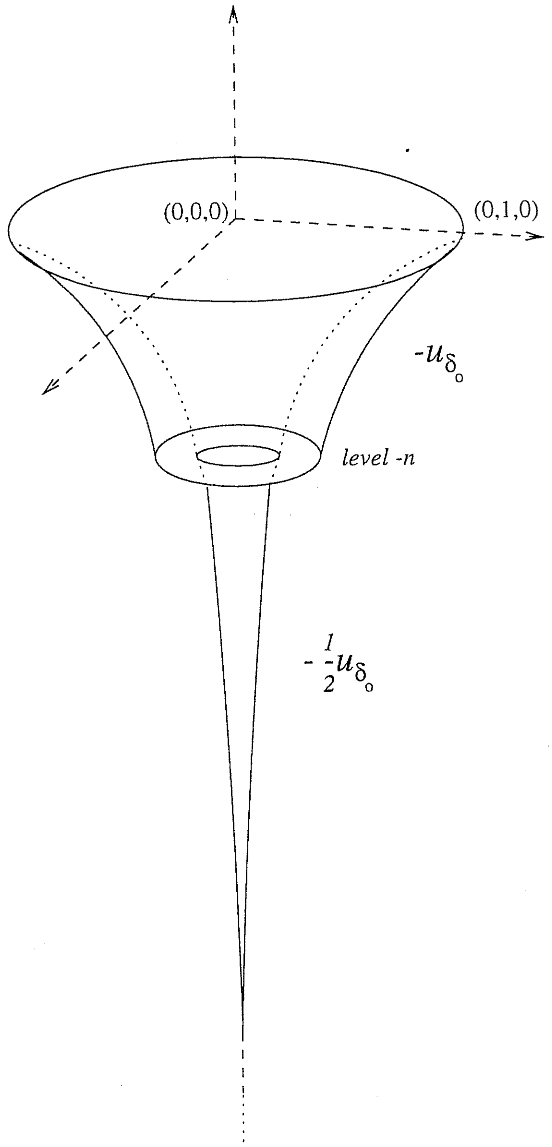
which can also be written as

$$\psi_n(x) = \begin{cases} -\dfrac{1}{2}u_{\delta_0}(x) & \text{if} \quad |x| < a_n \\ -n & \text{if} \quad a_n < |x| < b_n \\ -u_{\delta_0} & \text{if} \quad b_n < |x| \end{cases}$$

where a_n and b_n are appropriate constants, which tend to zero as $n \to +\infty$ (see picture in the next page).

It is easy to verify that $\psi_n \xrightarrow{\text{lev}} \psi$ and that the solution of the limit problem $OP(-\delta_0, \psi)$ is clearly $-u_{\delta_0}$ itself.

Let us prove that the solution of $OP(-\delta_0, \psi_n)$ is $-\dfrac{1}{2}u_{\delta_0}$.



This is indeed in $\mathcal{F}_\psi(\mu)$ because it is of the form $u_\mu + u_{\frac{1}{2}\delta_0}$ and it is above the obstacle for each n.

Fix n and suppose $\nu_n \in \mathcal{M}_b^+(\Omega)$ such that $u_\mu + u_{\nu_n}$ is the solution. Then it is smaller than or equal to $u_\mu + u_{\frac{1}{2}\delta_0}$, or also

$$u_{\nu_n} \leq u_{\frac{1}{2}\delta_0} \quad \text{q.e. in } \Omega.$$

In the small circle $B_{a_n}(0)$ they must be equal. In $B_1(0) \setminus B_{a_n}(0)$, u_{ν_n} is superharmonic and $u_{\frac{1}{2}\delta_0}$ is harmonic and they have the same boundary data. So $u_{\nu_n} \geq u_{\frac{1}{2}\delta_0}$ and they must coincide.

This proves that the solution u_n of $OP(-\delta_0, \psi_n)$ is $-\dfrac{1}{2}u_{\delta_0}$ independently of n, and that u_n does not converge to the solution $u = u_{\delta_0}$ of $OP(-\delta_0, \psi)$.

As of Remark 3.3.3 we point out that in the example it is crucial that the deltas involved are centered in the same point. If for instance, with the same obstacles, we had as datum $\mu = -\delta_{x_0}$ for any $x_0 \neq 0$, we would obtain, thanks to Theorem 3.3.4 that the solutions of $OP(-\delta_{x_0}, \psi_n)$ and of $OP(-\delta_{x_0}, \psi)$ are all identically zero.

The last consideration of this section concerns the fact that passing from Theorem 4.2.12 to Theorem 4.2.14 we lose something on the convergence of the solutions. To see that this loss is not due to the technique of the proof we can consider the following example.

Example 4.2.17. Let $\Omega = B_{\frac{1}{2}}(0) \subseteq \mathbb{R}^N$, with $N > 2$ and let $f \in \mathrm{L}^1(\Omega)$ be the function defined by

$$f(x) = \begin{cases} \dfrac{1}{|x|^N(-\log|x|)^\vartheta} & \text{if } x \in \Omega,\ x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

with $\vartheta > 1$. L. Orsina in [37] noticed that the solution u_f of the equation

$$\begin{cases} -\Delta u_f = f & \text{in } \Omega \\ u_f = 0 & \text{on } \partial\Omega \end{cases}$$

belongs to $\mathrm{W}^{1,q}(\Omega)$ for any $q < \dfrac{N}{N-1}$ but does not belong to $\mathrm{W}^{1,\frac{N}{N-1}}(\Omega)$.

With this choice of $\mathcal{A}$ and Ω, take as datum $\mu = -f$, which clearly belongs to $\mathcal{M}_b^{0,+}(\Omega)$, and as limit obstacle $\psi = -u_f$, which satisfies condition (3.3.1) with $\sigma = f$, so that the solution of $OP(-f, -u_f)$ is $-u_f$ itself.

If we set $\psi_n = -(u_f \wedge n)$ then the solution u_n of $OP(-f, -(u_f \wedge n))$ is between 0 (because f is positive and $0 = u_{-f} + u_f$ is a supersolution) and $-n$. Hence $u_n = T_n(u_n)$ and this implies that $u_n \in \mathrm{H}_0^1(\Omega)$.

Now it is easy to see (use for instance Remark 4.2.5) that $\psi_n \xrightarrow{\text{lev}} \psi$ so that, by Proposition 4.2.15,

$$u_n \to u \ \text{ strongly in } \mathrm{W}^{1,q}(\Omega).$$

It is not possible to have this convergence in the norm of $\mathrm{W}^{1,p}(\Omega)$ with $p \geq \dfrac{N}{N-1}$, because the fact that $u_n \in \mathrm{H}^1(\Omega)$ would imply also that $u \in \mathrm{W}^{1,p}(\Omega)$, which is false since $u = -u_f$.

4.3. Some more conditions for convergence

In the previous section we have seen that the level sets convergence of the obstacles in general it is not enough to give the convergence of the solutions.

In this section we want to generalize Proposition 2.4.1. Pointwise convergence is replaced by level sets convergence, while condition $\psi_n \leq \psi$ q.e. in Ω is weakened. No control from below is required on the limit obstacle.

Proposition 4.3.1. *Let $\psi_n, \psi \leq u_\rho$ with $\rho \in \mathcal{M}_b(\Omega)$ be such that*

$$\psi_n \xrightarrow{\text{lev}} \psi.$$

Suppose in addition that

$$\psi_n \leq \psi_n^* \quad q.e. \ in\ \Omega$$

where ψ_n^* is a sequence of OP-admissible obstacles such that, if v_n are the solutions of $OP(\mu, \psi_n^*)$, then $v_n \to u$ a.e. in Ω. Then

$$u_n \to u \quad strongly\ in\ \mathrm{W}^{1,q}(\Omega).$$

Proof. First apply Proposition 4.2.9 from which we know that, up to a subsequence, $u_n \to u^*$ strongly in $\mathrm{W}^{1,q}(\Omega)$ an that $u^* \geq u$. We need to prove the reverse inequality.

We see easily that

$$v_n \geq u_n \quad \text{q.e. in } \Omega$$

and letting n go to $+\infty$ we get $u \geq u^*$ a.e. in Ω and hence also q.e. in Ω. □

A special case of the previous result is given by the following proposition.

Proposition 4.3.2. *Let ψ_n, $\psi \leq u_\rho$ with $\rho \in \mathcal{M}_b(\Omega)$ be such that*

$$\psi_n \xrightarrow{\text{lev}} \psi.$$

Let u_n and u be the solutions of $OP(\mu, \psi_n)$ and $OP(\mu, \psi)$, respectively.

Suppose in addition that

$$\psi_n \leq \psi + u_{\sigma_n} \quad q.e. \ in\ \Omega$$

where $\sigma_n \in \mathcal{M}_b^+(\Omega)$ is such that $u_{\sigma_n} \to 0$ a.e. in Ω. Then

$$u_n \to u \quad strongly\ in\ \mathrm{W}^{1,q}(\Omega).$$

Proof. We can set $\psi_n^* := \psi + \dot{u}_{\sigma_n}$. If v_n is the solution of $OP(\mu, \psi_n^*)$ it is easy to see, according to Definition 2.1.1, that

$$u \leq v_n \leq u + u_{\sigma_n} \quad \text{a.e. in } \Omega.$$

And letting $n \to +\infty$ we get that $v_n \to u$ a.e. in Ω. So we can apply Proposition 4.3.1 and we obtain the thesis. □

Corollary 4.3.3. *Let ψ_n and ψ be OP-admissible and such that*

$$\psi_n \xrightarrow{\text{lev}} \psi.$$

If in addition $\psi_n \leq \psi$ q.e. in Ω then

$$u_n \to u \quad \text{q.e. in } \Omega.$$

Let us remark that this is a generalized version of Proposition 2.4.1. Indeed under the assumption that $\psi_n \leq \psi$ we have that the quasi everywhere convergence implies the level sets convergence. This fact is proved as follows.

First consider the case of ψ_n monotone increasing. Then it is clear that for all $t \in \mathbb{R}$ and $B \subset\subset \Omega$,

$$\{\psi_n > t\} \cap B \nearrow \left[\bigcup_{n=1}^{+\infty} \{\psi_n > t\} \right] \cap B,$$

Since $\{\psi > t\}$ differs from $\bigcup_{n=1}^{+\infty} \{\psi_n > t\}$ by a set of capacity zero, by the continuity of capacities on increasing sequences of sets, we have

$$\lim_{n \to +\infty} \operatorname{cap}(\{\psi_n > t\} \cap B) = \operatorname{cap}\left(\{\psi > t\} \cap B\right),$$

and we obtain (4.2.4).

In the general case $\psi_n \leq \psi$ but ψ_n not necessarily increasing, there always exists a sequence $\varphi_n \leq \psi_n$ q.e. in Ω (see (2.4.2)) such that $\varphi_n \nearrow \psi$ q.e. in Ω. From the fact that $\varphi_n \leq \psi_n \leq \psi$ q.e. in Ω it easily follows that

$$\operatorname{cap}(\{\varphi_n > t\} \cap B) \leq \operatorname{cap}(\{\psi_n > t\} \cap B) \leq \operatorname{cap}(\{\psi > t\} \cap B)$$

for all $t \in \mathbb{R}$, and $B \subset\subset \Omega$. Passing to the limit, thanks to the previous step, we conclude the proof.

We want to recall here also another condition which comes from a classical result.

Proposition 4.3.4. *Let $\mu \in \mathcal{M}_b(\Omega)$ and let ψ_n and ψ be functions in $\mathrm{W}_0^{1,p}(\Omega)$ with $p > 2$, such that ψ_n, $\psi \le u_\rho$ with $\rho \in \mathcal{M}_b(\Omega)$ and*

$$\psi_n \rightharpoonup \psi \text{ weakly in } \mathrm{W}_0^{1,p}(\Omega) \tag{4.3.1}$$

then

$$u_n \to u \quad \text{strongly in } \mathrm{W}^{1,q}(\Omega).$$

Proof. In [10] L. Boccardo and F. Murat obtained that, under this hypothesis of convergence of the obstacles, for every $f \in \mathrm{H}^{-1}(\Omega)$ the solutions of $VI(f, \psi_n)$ converge strongly in $\mathrm{H}_0^1(\Omega)$ to the solution of $VI(f, \psi)$. In order to obtain the analogue in our case we recall a result of H. Attouch and C. Picard in [2]: they proved that condition (4.3.1) implies the Mosco convergence of the convex sets $K_{\psi_n}(\Omega) \cap \mathrm{H}_0^1(\Omega)$ to $K_\psi(\Omega) \cap \mathrm{H}_0^1(\Omega)$.

By Theorem 4.2.4 we have also that $\psi_n \xrightarrow{\text{lev}} \psi$. Moreover the obstacle ψ satisfies condition (3.3.1), and then we can apply Theorem 4.2.15. These considerations give the proof of our theorem. □

4.4. Obstacles converging in the energy space

Recall that in the case of variational inequalities the convergence of obstacles in the norm of $\mathrm{H}^1(\Omega)$ implies the convergence of the corresponding solutions in the same norm (see Corollary 1.2.6).

We want to prove a similar result for the solutions of Obstacle Problems. In this frame, as we have seen with Example 4.2.17, this can not follow directely from Mosco convergence, as it was in the variational case. The next theorem concerns the case in which the obstacles "have the same boundary value".

Theorem 4.4.1. *Let $\psi_n, \psi : \Omega \to \overline{\mathbb{R}}$ be such that $\psi_n, \psi \le u_\rho$ q.e. in Ω, with $\rho \in \mathcal{M}_b(\Omega)$. Assume $\psi_n - \psi \in \mathrm{H}_0^1(\Omega)$, and let $\psi_n - \psi \to 0$ strongly in $\mathrm{H}^1(\Omega)$. Let $\mu \in \mathcal{M}_b(\Omega)$ and let u_n and u be the solutions of $OP(\mu, \psi_n)$ and $OP(\mu, \psi)$, respectively. Then*

$$u_n - u \in \mathrm{H}_0^1(\Omega) \quad \text{and} \quad u_n - u \to 0 \text{ strongly in } \mathrm{H}_0^1(\Omega),$$

Proof. *Step 1.* As a first step assume that ρ belongs also to $\mathrm{H}^{-1}(\Omega)$, and consider the special sequence μ_k of measures in $\mathcal{M}_b(\Omega) \cap \mathrm{H}^{-1}(\Omega)$ $\mu_k \rightharpoonup \mu$, weakly-$*$ in $\mathcal{M}_b(\Omega)$, such that the solutions of the corresponding obstacle problems converge (see Section 2.6). In particular $u_n^k \to u_n$, strongly in $\mathrm{W}^{1,q}(\Omega)$, for all n, and $u^k \to u$, strongly in $\mathrm{W}^{1,q}(\Omega)$.

Thanks to (1.2.8), for all k we also have

$$\|u_n^k - u^k\|_{\mathrm{H}_0^1(\Omega)} \leq c\|\psi_n - \psi\|_{\mathrm{H}_0^1(\Omega)},$$

so that the sequence $\{u_n^k - u^k\}_k$ is bounded in $\mathrm{H}_0^1(\Omega)$, for each n fixed. Thus, up to a subsequence, there is a limit function z. But we already know that the sequence converges, strongly in $\mathrm{W}^{1,q}(\Omega)$, to $u_n - u$, so this must be also the weak limit in $\mathrm{H}^1(\Omega)$.

By lower semicontinuity of the norm we have

$$\|u_n - u\|_{\mathrm{H}_0^1(\Omega)} \leq \liminf_{k\to\infty} \|u_n^k - u^k\|_{\mathrm{H}_0^1(\Omega)} \leq c\|\psi_n - \psi\|_{\mathrm{H}_0^1(\Omega)}.$$

This says that $u_n - u$ belongs to $\mathrm{H}_0^1(\Omega)$, while u_n and u, in general, do not, and gives the thesis in the first case.

Step 2. Let now ρ be only in $\mathcal{M}_b(\Omega)$. Set $\psi^h := \psi \wedge h$ and $\psi_n^h := \psi_n - \psi + \psi^h$.

These obstacles are equi OP-admissible, because $\psi_n^h \leq \psi_n$ and $\psi^h \leq \psi$ q.e. in Ω. They are also equi VI-admissible since, if $\psi \leq u_\rho$, then $\psi^h \leq T_h(u_\rho) \in \mathrm{H}_0^1(\Omega)$ and $\psi_n^h \leq \psi_n - \psi + T_h(u_\rho) \in \mathrm{H}_0^1(\Omega)$, and we can find a function $\varphi \in \mathrm{H}_0^1(\Omega)$ such that $\varphi \geq \psi_n - \psi$ for all n.

Thanks to Remark 2.5.3 there exists $\rho_h \in \mathcal{M}_b(\Omega) \cap \mathrm{H}^{-1}(\Omega)$ such that $\psi_n^h, \psi^h \leq u_{\rho_h}$ q.e. in Ω. Hence we are in the hypothesis of *Step 1* Moreover $\|\psi_n^h - \psi^h\|_{\mathrm{H}_0^1(\Omega)} = \|\psi_n - \psi\|_{\mathrm{H}_0^1(\Omega)}$.

So by *Step 1*, for each h

$$\|u_n^h - u^h\|_{\mathrm{H}_0^1(\Omega)} \leq c\|\psi_n - \psi\|_{\mathrm{H}_0^1(\Omega)}.$$

On the other side we know that, since $\psi_n^h \nearrow \psi_n$ and $\psi^h \nearrow \psi$, by Proposition 2.4.1 $u_n^h - u^h \to u_n - u$ strongly in $\mathrm{W}^{1,q}(\Omega)$, so that we can conclude as in the first step. □

We want to remark that more generally if the obstacles are such that $\psi_n - \psi \to 0$ in $\mathrm{H}^1(\Omega)$ then they also converge in the sense of level sets, so we can deduce the convergence of the solutions in all those situations given by Theorems 4.2.12, 4.2.14 and 4.2.15, but here we obtain a stronger convergence with no further assumptions on the obstacles and on the data.

We may now wonder what happens when the obstacles converge in the space $W^{1,q}(\Omega)$, with $1 < q < \dfrac{N}{N-1}$. In general this is not enough to obtain the convergence of the solutions. Indeed reconsider Example 4.2.16. Let us prove that $\psi_n \to \psi$ strongly in $W_0^{1,q}(\Omega)$. We have

$$\psi_n - \psi = \begin{cases} \frac{1}{2} u_{\delta_0} & \text{in} \quad |x| < a_n \\ u_{\delta_0} - n & \text{in } a_n < |x| < b_n \\ 0 & \text{otherwise} \end{cases}$$

so that

$$\|\psi_n - \psi\|^q_{W^{1,q}(\Omega)} = \frac{1}{2} \int\limits_{B_{a_n}(0)} |Du_{\delta_0}|^q dx + \int\limits_{B_{b_n}(0)\setminus B_{a_n}(0)} |Du_{\delta_0}|^q dx,$$

which tends to zero, since a_n and b_n tend to zero and by the absolute continuity of the integral. But, as already seen in Example 4.2.16, the solutions of the obstacle problems do not converge.

Anyway it is possible to prove the following result.

Proposition 4.4.2. *Let μ be in $\mathcal{M}_b(\Omega)$ and let ψ_n and ψ be OP-admissible and such that $\psi_n - \psi = u_{\rho_n}$ with $\rho_n \in \mathcal{M}_b(\Omega)$, $\|\rho_n\|_{\mathcal{M}_b(\Omega)} \to 0$. Then*

$$u_n \to u \quad \textit{strongly in } W^{1,q}(\Omega),$$

where u_n and u are the solutions of $OP(\mu, \psi_n)$ and $OP(\mu, \psi)$, respectively.

Proof. Since $\psi_n = \psi - u_{\rho_n}$, we have (using Lemma 2.3.3) that $u_n - u_{\rho_n}$ is the solution of $OP(\mu + \rho_n, \psi)$. So from Theorem 4.1.1 we get that

$$u_n - u_{\rho_n} \to u \text{ strongly in } W^{1,q}(\Omega). \tag{4.4.1}$$

Then

$$\|u_n - u\|_{W^{1,q}(\Omega)} \le \|u_n - u_{\rho_n} - u\|_{W^{1,q}(\Omega)} + \|u_{\rho_n}\|_{W^{1,q}(\Omega)};$$

the first term goes to zero because of (4.4.1), the second one by hypothesis, and we get the thesis. □

4.5. Problems with nonzero boundary data and uniform convergence

In this section we extend the theory of obstacle problems with measure data developed in Chapter 2, to problems with nonzero boundary data. This is standard for variational inequalities as recalled in Section 1.2. Also in this case this generalization is very simple; we will only point out what are the points to be settled.

Let $g \in \mathrm{H}^1(\Omega)$ we will denote by u_0^g the solution of

$$\begin{cases} \mathcal{A}u_0^g = 0 & \text{in } \mathrm{H}^{-1}(\Omega) \\ u_0^g - g \in \mathrm{H}_0^1(\Omega). \end{cases}$$

We will look for solutions of obstacle problems which take the value g on the boundary $\partial\Omega$. So we have to change accordingly the notion of admissibility for the obstacles.

An obstacle $\psi : \Omega \to \overline{\mathbb{R}}$ is said to be OP_g-admissible if

$$\exists \rho \in \mathcal{M}_b^+(\Omega) \text{ s.t. } \psi \leq u_\rho + u_0^g \quad \text{q.e. in } \Omega.$$

Given a measure $\mu \in \mathcal{M}_b(\Omega)$, a boundary datum $g \in \mathrm{H}^1(\Omega)$ and an obstacle ψ OP_g-admissible, the solution of the obstacle problem $OP(\mu, g, \psi)$, if it exists, is the minimum element of the set

$$\mathcal{F}_\psi^g(\mu) := \left\{ v \in \mathrm{W}^{1,q}(\Omega) \,:\, \exists \nu \in \mathcal{M}_b^+(\Omega),\, v = u_\mu + u_0^g + u_\nu;\, v \geq \psi \quad \text{q.e. in } \Omega \right\}.$$

It is immediate to prove the following

Theorem 4.5.1. *Let $\mu \in \mathcal{M}_b(\Omega)$ and let ψ be OP_g-admissible. Then there exists a unique solution of $OP(\mu, g, \psi)$.*

Proof. Consider the obstacle $\psi - u_0^g$. It is OP-admissible. So there exists a unique solution v of $OP(\mu, \psi - u_0^g)$. Then $v + u_0^g$ is our solution: indeed it belongs to $\mathcal{F}_\psi^g(\mu)$, and it is less than or equal to any $z \in \mathcal{F}_\psi^g(\mu)$. □

Remark 4.5.2. From Theorem 4.5.1 and (2.3.1) it follows also that, if $u_\mu + u_0^g + u_\lambda$ is the solution of $OP(\mu, g, \psi)$, then

$$\|\lambda\|_{\mathcal{M}_b(\Omega)} \leq \left\|(\mu - \rho)^-\right\|_{\mathcal{M}_b(\Omega)},$$

independently of g.

Remark 4.5.3. Since ψ is OP-admissible if and only if $\psi - u_0^g$ is OP_g-admissible and $u_n \to u$ strongly in $W^{1,q}(\Omega)$ if and only if $u_n - u_0^g \to u - u_0^g$ strongly in $W^{1,q}(\Omega)$, all the theorems on continuous dependence on the data hold without modifications, in particular Propositions 2.4.1 which will be useful in the following.

Remark 4.5.4. When $\mu \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$ and $\psi \leq u_\rho + u_0^g$ q.e. in Ω with $\rho \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$ then the solution of $OP(\mu, g, \psi)$ coincides with the solution of $VI(\mu, g, \psi)$ as defined in (1.2.6).

We come now to discuss the continuous dependence of the solutions on the obstacles when these converge uniformly.

To do this we will use a characterization via supersolutions similar to Theorem 1.2.2 that holds in the variational case.

To this aim let us introduce the set $\mathcal{G}_\psi^g(\mu)$ of all the functions $v \in W^{1,q}(\Omega)$ with $v \geq \psi$ q.e. in Ω, such that $v = u_\mu + u_0^h + u_\nu$, where $\nu \in \mathcal{M}_b^+(\Omega)$ and $h \in H^1(\Omega)$ such that $h \geq g$ on $\partial\Omega$, i.e. $(h-g)^- \in H_0^1(\Omega)$.

We see now that the solution of $OP(\mu, g, \psi)$ can be compared not only with the functions of $\mathcal{F}_\psi^g(\mu)$, but also with all those that have boundary datum greater than or equal to g.

Proposition 4.5.5. *Let $\mu \in \mathcal{M}_b(\Omega)$ and ψ be OP_g-admissible. If u is the solution of $OP(\mu, g, \psi)$ then it is the minimum element of $\mathcal{G}_\psi^g(\mu)$.*

Proof. Step 1. First consider $\mu \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$ and ψ both VI- and OP-admissible.

Let $v = u_\mu + u_0^h + u_\nu \in \mathcal{G}_\psi^g(\mu)$. We approximate ν by means of the sequence $\nu_k := \mathcal{A}T_k(u_\nu)$. By Theorem 1.3.12 we have that $\nu_k \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$ and that $\nu_k \rightharpoonup \nu$ weakly-$*$ in $\mathcal{M}_b(\Omega)$. Moreover observe that $u_{\nu_k} = T_k(u_\nu)$ tends to u_ν q.e. in Ω and, since u_ν is nonnegative it is an increasing sequence.

Hence if we define $v_k := u_\mu + u_0^h + u_{\nu_k}$, then $v_k \nearrow v$ q.e. in Ω, and setting $\psi_k := \psi \wedge v_k$ also $\psi_k \nearrow \psi$ q.e. in Ω

Let now u_k be the solutions of $VI(\mu, g, \psi_k)$. So, by Proposition 1.2.2, $v_k \geq u_k$. Using Proposition 2.4.1 and Remark 4.5.3 we know that $u_k \to u$ a.e. in Ω. Then $v \geq u$ a.e. in Ω and then also q.e. in Ω.

Step 2. Consider now $\mu \in \mathcal{M}_b(\Omega)$ and ψ still both VI- and OP-admissible. Take again $v \in \mathcal{G}_\psi^g(\mu)$.

Let $\mu_k = \mathcal{A}T_k(u_\mu - u_\rho) + \rho$ be the sequence of measures given in Theorem 2.3.1, so that we know that if u_k are the solutions of $VI(\mu, g, \psi)$ then $u_k \to u$ strongly in $\mathrm{W}^{1,q}(\Omega)$.

Taking now $v_k = u_{\mu_k} + u_0^h + u_\nu$ it is easy to verify that $v_k \geq \psi$ q.e. in Ω for all $k > 0$, and then, by Definition 2.1.1, $v_k \geq u_k$ q.e. in Ω. Also $v_k \to v$ strongly in $\mathrm{W}^{1,q}(\Omega)$ so, passing to the limit, we obtain $v \geq u$ a.e. in Ω and then also q.e in Ω.

Step 3. Finally consider the general case $\mu \in \mathcal{M}_b(\Omega)$ and ψ OP-admissible. The obstacles $\psi_k := \psi \wedge k$ are also VI-admissible and such that $\psi_k \nearrow \psi$ q.e. in Ω. So, if u_k is the solution of $OP(\mu, g, \psi_k)$, by Proposition 2.4.1 and Remark 4.5.3, we have that $u_k \to u$ strongly in $\mathrm{W}^{1,q}(\Omega)$.

Taken any $v \in \mathcal{G}_\psi^g(\mu)$, $v \geq \psi_k$, for all k. Hence, by definition, $v \geq u_k$ q.e. in Ω. Passing to the limit, we get $v \geq u$ a.e. in Ω an also q.e. in Ω. □

From this we point out that the sets $\mathcal{F}_\psi^g(\mu)$ and $\mathcal{G}_\psi^g(\mu)$ have the following lattice property.

Proposition 4.5.6. *Let $\mu \in \mathcal{M}_b(\Omega)$, $g \in \mathrm{H}^1(\Omega)$ and ψ OP_g-admissible. Then*
(i) If $u, v \in \mathcal{F}_\psi^g(\mu)$ then $u \wedge v \in \mathcal{F}_\psi^g(\mu)$;
(ii) If $u, v \in \mathcal{G}_\psi^g(\mu)$ then $u \wedge v \in \mathcal{G}_\psi^g(\mu)$.

Proof. Let us prove only the first statement, the proof of the second being alike.

Set $w := u \wedge v$ and let z be the solution of $OP(\mu, g, w)$. Then $u, v \in \mathcal{F}_w^g(\mu)$ and hence also $w \geq z$.

On the other hand $z \geq w$ and hence they are equal. So $u \wedge v$ is of the form $u_\mu + u_0^g + u_\nu$ and is above ψ, and hence belongs to $\mathcal{F}_\psi^g(\mu)$. □

We can prove now the following continuity result

Theorem 4.5.7. *Let $\mu \in \mathcal{M}_b(\Omega)$, $g \in \mathrm{H}^1(\Omega)$, and ψ_n and ψ be OP-admissible and let u_n and u be the solutions of $OP(\mu, \psi_n)$ and $OP(\mu, \psi)$, respectively. Assume that $\psi_n - \psi \in \mathrm{L}^\infty(\Omega)$ and $\psi_n - \psi \to 0$ in $\mathrm{L}^\infty(\Omega)$. Then*

$$u_n - u \in \mathrm{L}^\infty(\Omega) \quad \textit{and} \quad u_n - u \to 0 \textit{ in } \mathrm{L}^\infty(\Omega).$$

Proof. Set $c_n := \|\psi_n - \psi\|_{\mathrm{L}^\infty(\Omega)}$. Obviously $c_n = u_0^{c_n}$, so that

$$u + c_n = u_\mu + u_0^{c_n+g} + u_\lambda \text{ and } u + c_n \geq \psi_n \text{ q.e. in } \Omega$$

hence $u + c_n \in \mathcal{G}^g_{\psi_n}(\mu)$ and hence $u + c_n \geq u_n$.

The same can be done the other way round to obtain that $u_n + c_n \geq u$. In the end $|u_n - u| \leq c_n$, and, taking the sup over $x \in \Omega$, we obtain the thesis. □

Remark 4.5.8. Also in this case we have to remark that the uniform convergence of the obstacles implies their level set convergence (via Remark 4.2.6). But the result we have obtained in this section does not require that the obstacles be equicontrolled, and the convergence of the solutions is in a different norm.

References

[1] ATTOUCH H., PICARD C.: Problèmes variationnels et théorie du potentiel non linéaire, *Ann. Faculté Sci. Toùlouse*, Vol. **I** (1979), 89-136.

[2] ATTOUCH H., PICARD C.: Inéquations variationnelles avec obstacles et espaces fonctionnels en théorie du potentiel. *Appl. Anal.* **12** (1981), 287-306

[3] BAIOCCHI C., CAPELO A.: *Variational and quasivariational inequalities* Wiley, New York, 1984.

[4] BÉNILAIN Ph., BOCCARDO L., GALLOUËT T., GARIEPY R., PIERRE M., VAZQUEZ J.L.: An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. *J. Funct. Anal.* **50** (1983), 329-386.

[5] BOCCARDO L., CIRMI G.R.: Nonsmooth unilateral problems. In *Nonsmooth optimization: methods and applications* (Proceedings Erice 1991), 1-10. F. Giannessi editor. Gordon and Breach, Amsterdam, 1992.

[6] BOCCARDO L., CIRMI G.R.: Existence and uniqueness of solution of unilateral problems with L^1-data. Manuscript.

[7] BOCCARDO L., GALLOUËT T.: Problèmes unilatéraux avec données dans L^1. *C. R. Acad. Sci. Paris, Série I* **311** (1990), 617-619.

[8] BOCCARDO L., GALLOUËT T.: Nonlinear elliptic equations with right-hand side measures. *Comm. Partial Differential Equations* **17** (1992), 641-655.

[9] BOCCARDO L., GALLOUËT T., ORSINA L.: Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data. *Ann. Inst. Henri Poincaré* **13** (1996), 539-551.

[10] BOCCARDO L., MURAT F.: Nouveaux résultats de convergence dans des problèmes unilateraux. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Volume II,* 64-85. H. Brézis and J.L. Lions editors. Pitman research notes in mathematics n. 60, Pitman, London, 1982

[11] BOCCARDO L., MURAT F.: Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations. *Nonlinear Anal.* **19** (1992), 581-597.

[12] BOCCARDO L., MURAT F.: A property of nonlinear elliptic equations when the right-hand side is a measure *Potential Analysis* **3** (1994), 257-263.

[13] CARBONE L., COLOMBINI F.: On convergence of functionals with unilateral constraints. *J. Math. Pures et Appl.* **59** (1980), 465-500.

[14] CIORANESCU D., MURAT F.: Un terme étrange venu d'ailleurs I and II. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Volume II,* 98-138 and Volume III, 154-178. H. Brézis and J.L. Lions editors. Pitman research notes in mathematics nn. 60 and 70, Pitman, London, 1982

[15] DALL'AGLIO P.: Stability results for solutions of obstacles problems with measure data In preparation, 1998

[16] DALL'AGLIO P., DAL MASO G.: Some properties of solutions of obstacle problems with measure data. In preparation, 1998

[17] DALL'AGLIO P., LEONE C.: Obstacle problems with measure data. Preprint S.I.S.S.A., Trieste, 1997

[18] DAL MASO G.: On the integral representation of certain local functionals. *Ricerche Mat.* **22** (1983), 85-113.

[19] DAL MASO G.: Some necessary and sufficient conditions for the convergence of sequences of unilateral convex sets *J. Funct. Anal.* **62**, 2 (1985), 119-159.

[20] DAL MASO G., GARRONI A.: The capacity method for asymptotic Dirichlet problems. *Asymptotic Anal.* **15** (1997), 299-324.

[21] DAL MASO G., MALUSA A.: Some properties of reachable solutions of nonlinear elliptic equations with measure data. *Ann. Scuola Norm. Sup. Pisa*, to appear.

[22] DAL MASO G., SKRYPNIK I.V.: Capacity theory for monotone operators. *Pot. Anal.* **7** (1997), 765-803.

[23] EVANS L.C., GARIEPY R.F.: *Measure theory and fine properties of functions.* CRC Press, Boca Raton, 1992

[24] FUGLEDE B.: The quasi topology associated with a countably subadditive set function. *Ann. Inst. Fourier* **21,1** (1971), 123-169

[25] FUKUSHIMA M., SATO K., TANIGUCHI S.: On the closable part of pre-Dirichlet forms and the fine supports of underlying measures. *Osaka J. Math.* **28** (1991), 517-535.

[26] GALLOUËT T., HERBIN R.: Existence of a solution to a coupled elliptic system. *Appl. Math. letters* **7** (1994), 49-55

[27] GRÜTER M., WIDMAN K.O.: The Green function for uniformly elliptic equations. *Manuscripta Math.* **37** (1982), 303-342.

[28] HEINONEN J., KILPELÄINEN T., MARTIO O.: *Nonlinear potential theory of degenerate elliptic equations.* Clarendon Press, Oxford, 1993.

[29] HELMS L.L.: *Introduction to potential theory.* Wiley Interscience, New York, 1969.

[30] KILPELÄINEN T., MALÝ J.: Degenerate elliptic equations with measure data and nonlinear potentials. *Ann. Scuola Norm. Sup. Pisa* **19** (1992), 591-613.

[31] KINDERLEHRER D., STAMPACCHIA G.: *An introduction to Variational Inequalities and their applications.* Academic, New York, 1980.

[32] LANDKOF N.S.: *Foundations of potential theory.* Springer Verlag, Berlin, 1972

[33] LEONE C.: Existence and uniqueness of solutions for nonlinear obstacle problems with measure data. Preprint S.I.S.S.A., Trieste, 1998

[34] LITTMAN W., STAMPACCHIA G., WEINBERGER H.F.: Regular points for elliptic equations with discontinuous coefficients. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **17** (1963), 45-79

[35] MALUSA A., ORSINA.: Existence and regularity results for relaxed Dirichlet problems with measure data. *Ann. Mat. Pura Appl.* IV **170** (1996), 57-87.

[36] MOSCO U.: Convergence of convex sets and of solutions of variational inequalities *Adv. in Math.* **3** (1969), 510-585

[37] ORSINA L.: Solvability of linear and semilinear eigenvalue problems with $L^1(\Omega)$ data. *Rend. Sem. Mat. Padova* **90** (1993), 207-238

[38] ORSINA L., PRIGNET A.: Nonexistence of solutions for some nonlinear elliptic equations involving measures. Manuscript.

[39] PRIGNET A.: Remarks on existence and uniqueness of solutions of elliptic problems with right-hand side measures. *Rend. Mat.* **15** (1995), 97-107.

[40] SERRIN J.: Pathological solutions of elliptic differential equations. *Ann. Scuola Norm. Sup. Pisa* **18** (1964), 385-387.

[41] STAMPACCHIA G.: Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. *Ann. Inst. Fourier Grenoble* **15** (1965), 189-258.

[42] STAMPACCHIA G.: *Equations elliptiques du second ordre à coefficients discontinus.* Les presses de l'Université de Montréal, Montréal, 1966.

[43] TROIANIELLO G.M.: *Elliptic differential equations and obstacle problems.* Plenum Press, New York, 1987.