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# STUDIES ON GRAVITATIONAL WAVES

*Thesis submitted for the degree of*

*" Doctor Philosophiae "*


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Supervisor:

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Academic Year 1990/91

TRIESTE

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# Contents

# INTRODUCTION

Although this work at first might appear to be a collection of writings on disconnected subjects, in fact a single theme underlies the work presented. The idea underlying the logical development of the various parts is the possibility that gravitational waves can gravitationally lense distant light sources. This idea is due to Bertotti ([63]; cited in [25]) and mentioned by Wheeler [25]. Thinking on this topic, new ideas and problems appeared, though here they were sometimes treated in general, and not only regarding their specific application to lensing by gravitational waves. These various topics are located in different chapters, which will appear unrelated at a first sight.

Chapter 1 deals with the propagation of light through exact plane wave solutions of Einstein equations (in particular with the frequency shift effect). This study provides some physical insight into the phenomenon, even if the results can hardly be generalized to realistic (linearized) gravitational waves. However such exact waves are of some relevance to high energy physics, and the propagation of light through such waves is interesting from the mathematical point of view, and has not received much attention in the literature.

Chapter 2 reviews the effects induced by linearized gravitational waves (both of primordial origin, or generated by astrophysical sources) on the light coming from distant sources and propagating through them. Though no original result is contained in this chapter, some critical considerations are pointed out there.

Chapter 3 is devoted to treat the gauge-dependence problems which arise when considering experiments on gravitational waves, and linked to the use of particular gauges (like the transverse-traceless one). A general view of the problem is given, and then an application to gravitational wave detectors is pointed out. The starting point of this work was the difficulty of describing a detailed model of lensing gravitational wave in a coordinate system different from the one achieving the transverse-traceless gauge.

Chapter 4 contains the starting idea of this thesis, namely the possibility of lensing by gravitational waves. Following Ref. [63], the vector formalism for ordinary gravitational lenses is applied to gravitational waves acting as lenses. Though this is not a priori correct, it is proved there that the results obtained in this way are correct to first order in the gravitational wave amplitudes (metric perturbations). This is accomplished by using a suitable version of Fermat principle for arbitrary (nonstationary) spacetimes. This application to fully nonstationary lenses in this thesis generalizes the work of Refs. [83] and [84]. Finally, some order of magnitude estimates rule out certain sources of gravitational waves as realistic lenses, while some other appear to be possible lenses, though the probability of observing their effect on light is probably low.

Chapter 5 arises from the question whether the cosmological gravitational waves taken in consideration in Chapters 2 and 4 are really distributed in a homogeneous and isotropic way in the universe. If this is not the case, the picture given there should be changed. A possible source of inhomogeneity is the following phenomenon: backscattering off the curvature of the background spacetime may cause radiation to propagate in the *interior* of the light cone, in vacuo. The question whether this effect is important for long wavelength radiation, from a cosmological point of view, is the so-called *tail problem*. The relevance of this effect in Friedmann cosmologies is analyzed, reducing the problem to that of one-dimensional scattering of a quantum particle in a suitable potential, and solving the corresponding Schrödinger equation. The standard treatment of the scattering process in terms of reflection and transmission coefficients cannot be applied to this case, as pointed out. The tail problem requires then a radically different approach, and it is not solved here.

In order to understand more deeply the problem, one realizes that the presence of tails for radiation in curved spacetimes has sometimes been understood as a failure of Huygens principle. Chapter 5 clarifies then the relationship between the tail-free property, Huygens principle and the characteristic propagation property discussed in the literature. Huygens principle is reformulated using the Green function representation for the solutions of the wave equations, and the relationship between it and the characteristic propagation property is discussed.

The content of Chapter 1 can be found in Ref. [18]; that of Chapter 2 in Refs. [87, 88]. Chapter 4 is summarized in Ref. [114], and Chapter 5 in Refs. [160] and [153].

**Notation:**

We adopt the notations of Ref. [1]. We use units in which $G = c = 1$ (but sometimes we will restore $G$ and $c$). The metric signature is $+2$. $\nabla_a$ denotes the covariant derivative operator, and $\Box \equiv g^{ab}\nabla_a\nabla_b$.

The abstract index notation is adopted; Latin indices $a$, $b$, $c$, ... on a tensor are part of its name; Greek indices (running from 0 to 3) denote tensor components. Latin indices $i$, $j$, $k$, ... are however needed sometimes to denote spatial components, and run from 1 to 3. The convention on the definition of the Riemann tensor is such that

$$R_{\mu\nu\rho}{}^{\sigma} = \Gamma^{\sigma}_{\mu\rho,\nu} - \Gamma^{\sigma}_{\nu\rho,\mu} + \Gamma^{\alpha}_{\mu\rho}\Gamma^{\sigma}_{\alpha\nu} - \Gamma^{\alpha}_{\nu\rho}\Gamma^{\sigma}_{\alpha\mu} .$$

# Chapter 1

# EXACT PLANE GRAVITATIONAL WAVES

## 1.1 Introduction

Exact, analytic solutions of the Einstein field equations that describe plane gravitational waves have been considered in the literature for a long time (e.g. [2]-[6] and references therein), and it has been shown that plane waves exhibit focusing properties on timelike and null geodesics propagating in them [7, 2, 8, 9]. The focusing property makes the study of propagation of light in exact plane waves nontrivial.

More recent work has considered collisions of plane waves and the consequent formation of singularities ([10]-[13] and references therein); it appears that focusing plays a major role in the generation of such singularities. Further, it has recently been realized that plane fronted waves with parallel rays (*pp-waves*), of which plane waves are a subclass, are exact classical solutions to string theory [14, 15], renewing previous interest [16, 17] of high energy physicists in such waves. Here we consider photons propagating in plane gravitational waves in the geometric optics approximation. We take into account the frequency shift of light. The effects induced on light by an exact pulse of gravitational waves can be hardly generalized to realistic gravitational waves, because they are enormously exaggerated, or are due to non generic symmetries of spacetime. However by examining these effects we can get some insight into the properties of lensing by gravitational waves. Moreover, the argument is interesting from a mathematical point of view. In addition, since plane waves can be interpreted as the gravitational fields generated by highly energetic massless particles, they could be applied to describe situations occurring in a hot, primordial universe, or near black holes [16]. Finally, since they describe fully non-linear gravitational wave pulses, if strong fields with properties similar to those of our examples exist, they could be useful to study phenomena near astrophysical sources of gravitational waves. As far as this point is concerned, it has been suggested [10] that mutual focusing of gravitational waves could produce gravitational waves with amplitudes larger than we would expect on the basis of the linearized theory.

## 1.2 The frequency shift effect

Apparently, the frequency shift for light propagating through *exact* gravitational waves has never received much attention in the literature, even if its astrophysical consequences for *linearized* waves (see later) have been taken into account in various works. The following is based on Ref. [18].

### 1.2.1 The model

Let us consider the metric (Ref. [8], p. 166)

$$ds^2 = -dt^2 + p^2(u)\, dx^2 + q^2(u)\, dy^2 + dz^2 \tag{2.1}$$

where $u = t - z$ and the functions $p(u)$ and $q(u)$ are chosen so that spacetime is curved only between the null hyperplanes $u = 0$ and $u = a^2$ (*wavezone*). The metric Eq. (3.1) represents a particular case of plane fronted wave with parallel rays (*pp-wave*). Emptiness of spacetime is equivalent to

$$\frac{p''}{p} + \frac{q''}{q} = 0$$

(where a prime means derivation with respect to $u$), while flatness is equivalent to

$$p'' = q'' = 0 \; .$$

The simplest nontrivial way given in the literature (Ref. [8] p. 166; Ref. [9] p. 229) to satisfy emptiness is

$$p(u) = \begin{cases} 1 & u < 0 \\ \cos(u/a) & 0 < u < a^2 \\ \alpha + \beta u & u > a^2 \end{cases}$$

and

$$q(u) = \begin{cases} 1 & u < 0 \\ \cosh(u/a) & 0 < u < a^2 \\ \gamma + \delta u & u > a^2 \ , \end{cases}$$

where $\alpha$, $\beta$, $\gamma$ and $\delta$ are constants determined by requiring that $p$ and $q$ are continuous with their first derivatives. This gives

$$\begin{aligned} \alpha &= \cos a + a \sin a \ , \\ \beta &= -\sin a/a \ , \\ \gamma &= \cosh a - a \sinh a \ , \\ \delta &= \sinh a/a \ . \end{aligned}$$

The metric Eq. (2.1) represents then a plane sandwich gravitational wave propagating along the positive $z$-axis. It takes the Minkowskian form in the *beforezone* ($u < 0$) in coordinates $(t, x, y, z)$ and in the *afterzone* ($u > a^2$) in coordinates $(T, X, Y, Z)$ given by the transformation

$$\begin{aligned} T &= t + \frac{\beta}{2} p(u)\, x^2 + \frac{\delta}{2} q(u)\, y^2 \ , \\ X &= p(u)\, x \ , \\ Y &= q(u)\, y \ , \\ Z &= z + \frac{\beta}{2} p(u)\, x^2 + \frac{\delta}{2} q(u)\, y^2 \ . \end{aligned} \tag{2.2}$$

Since we have

$$\text{curvature in the wavezone} \sim |p''/p| = |q''/q| = 1/a^2 \ ,$$

a small $a$ gives a *strong* wave; we are actually interested in this case, even though we allow any $a > 0$ for the moment.

### 1.2.2 Null and timelike geodesics

The only nonvanishing Christoffel symbols are

$$
\begin{aligned}
\Gamma^0_{11} = \Gamma^3_{11} &= pp' \,, \\
\Gamma^0_{22} = \Gamma^3_{22} &= qq' \,, \\
\Gamma^1_{01} = \Gamma^1_{10} = -\Gamma^1_{13} = -\Gamma^1_{31} &= \frac{p'}{p} \,, \\
\Gamma^2_{02} = \Gamma^2_{20} = -\Gamma^2_{23} = -\Gamma^2_{32} &= \frac{q'}{q} \,.
\end{aligned} \tag{2.3}
$$

The equation of null geodesics gives

$$
\begin{aligned}
&\frac{dp^t}{d\lambda} + pp'(p^x)^2 + qq'(p^y)^2 = 0 \,, \\
&\frac{dp^x}{d\lambda} + 2\frac{p'}{p}p^x(p^t - p^z) = 0 \,, \\
&\frac{dp^y}{d\lambda} + 2\frac{q'}{q}p^y(p^t - p^z) = 0 \,, \\
&\frac{dp^z}{d\lambda} + pp'(p^x)^2 + qq'(p^y)^2 = 0 \,,
\end{aligned}
$$

where $\lambda$ is an affine parameter along null geodesics and $p^\mu$ are the components of the tangent to the null geodesic (photon's 4-momentum). $p_x$, $p_y$ and $p_v = -p^u/2$ (which is the momentum conjugate to the null coordinate $v \equiv t+z$) are constants of motion, since they are conjugated to cyclic coordinates. Moreover, we get $p^u \equiv du/d\lambda = \pm 1$, with the positive or negative sign according to the possibility that the photon is propagating with a component of its momentum in the positive or negative $z$-direction. The solution of the null geodesic equation is given by

$$p^t = \pm\left(\frac{c_1^2}{2p^2} + \frac{c_2^2}{2q^2}\right) \pm 1/2 \,, \tag{2.4}$$

$$p^x = \frac{c_1}{p^2} \,, \tag{2.5}$$

$$p^y = \frac{c_2}{q^2} \,, \tag{2.6}$$

$$p^z = \pm\left(\frac{c_1^2}{2p^2} + \frac{c_2^2}{2q^2}\right) \mp 1/2 \,, \tag{2.7}$$

where $p^u = \pm 1$ respectively, and $c_1 = p_x$, $c_2 = p_y$, and the normalization $p_\mu p^\mu = 0$ has been used to eliminate a third integration constant. Moreover,

$$\begin{aligned} t(u) &= \pm\left(\frac{c_1^2}{2}\,A + \frac{c_2^2}{2}\,B\right) \pm u/2\ , \\ x(u) &= c_1 A + x_S\ , \\ y(u) &= c_2 B + y_S\ , \\ z(u) &= \pm\left(\frac{c_1^2}{2}\,A + \frac{c_2^2}{2}\,B\right) \mp u/2\ , \end{aligned}$$

where

$$A(u) \equiv \int_{u_S}^{u} \frac{du}{p^2(u)}\ , \tag{2.8}$$

$$B(u) \equiv \int_{u_S}^{u} \frac{du}{q^2(u)}\ , \tag{2.9}$$

and $x_S = X_S$, $y_S = Y_S$ and $u_S$ are the transverse and null coordinates of the light source, respectively.

The equation of timelike geodesics gives

$$\begin{aligned} &\frac{du^t}{ds} + pp'(u^x)^2 + qq'(u^y)^2 = 0\ , \\ &\frac{du^x}{ds} + 2\,\frac{p'}{p}\,u^x(u^t - u^z) = 0\ , \\ &\frac{du^y}{ds} + 2\,\frac{q'}{q}\,u^y(u^t - u^z) = 0\ , \\ &\frac{du^z}{ds} + pp'(u^x)^2 + qq'(u^y)^2 = 0\ , \end{aligned} \tag{2.10}$$

where $s$ is proper time and $u^\mu$ is the tangent to the timelike geodesics. One gets

$$\begin{aligned} u^t &= \frac{\alpha_1^2}{2\,p^2} + \frac{\alpha_2^2}{2\,q^2} + 1\ , \\ u^x &= \frac{\alpha_1}{p^2}\ , \\ u^y &= \frac{\alpha_2}{q^2}\ , \\ u^z &= \frac{\alpha_1^2}{2\,p^2} + \frac{\alpha_2^2}{2\,q^2}\ , \end{aligned}$$

where $\alpha_1$ and $\alpha_2$ are integration constants (corresponding to $p_x/m$ and $p_y/m$, where $p^\mu$ is the 4-momentum and $m$ the mass of a test particle; again $p_x$ and $p_y$ are conserved quantities), and where the normalization $u_\mu u^\mu = -1$ has been used[1]. We are interested in considering test particles with $u^\mu = \delta^{0\mu}$ in the beforezone; this implies $\alpha_1 = \alpha_2 = 0$ and $u^\mu = \delta^{0\mu}$ everywhere, in coordinates $(t, x, y, z)$. This coordinate system has the property that observers initially at rest relative to each other in flat space remain such forever (this is not the case however if $u^\mu \neq \delta^{0\mu}$ initially, i.e. $\alpha_1$,$\alpha_2 \neq 0$).

### 1.2.3 The frequency shift effect

Let us consider a test particle with 4-velocity $v^\mu = \delta^{0\mu}$ in coordinates $(t, x, y, z)$, emitting light in the beforezone. The light signal propagates through the sandwich wave and is received by an observer (with 4-velocity $u^\mu = \delta^{0\mu}$) in the afterzone; the light will suffer a frequency shift. We limit our study to the geometric optics approximation, that holds if the wavelength of the electromagnetic signal is much smaller than the scale of variation of the gravitational wave: $\lambda << a$. The (angular) frequency emitted by the light source in the beforezone is

$$\omega_S = -g_{\mu\nu} p^\mu v^\nu|_S = \frac{1}{2}(c_1^2 + c_2^2 + 1) \tag{2.11}$$

(where we consider null geodesics with $p^u = -1$). The (angular) frequency received by the observer in the afterzone is

$$\omega_O = -g_{\mu\nu}\, p^\mu u^\nu|_O = \frac{1}{2}\left[\frac{c_1^2}{p^2(u_O)} + \frac{c_2^2}{q^2(u_O)} + 1\right] . \tag{2.12}$$

The frequency shift is given by

$$\frac{\omega_O}{\omega_S} = \left[\frac{c_1^2}{p^2(u_0)} + \frac{c_2^2}{q^2(u_O)} + 1\right] (c_1^2 + c_2^2 + 1)^{-1} . \tag{2.13}$$

The integration constants $c_1$ and $c_2$ parameterize the spatial direction of the light rays. We can write the condition for red-/blue- shift in observer's coordinates $(T, X, Y, Z)$ by writing the solution of the null geodesic equation in

[1]The equations of null and timelike geodesics and their solutions are valid in all the three zones in which spacetime is divided by the wave. In the afterzone they must however be written in coordinates $(T, X, Y, Z)$, which are the most convenient there.

coordinates $(T, X, Y, Z)$ using Eq. (2.2), and obtaining from these

$$c_1 = \frac{X - p(u)X_S}{p(u)\,A} \,, \tag{2.14}$$

$$c_2 = \frac{Y - q(u)Y_S}{q(u)\,B} \,, \tag{2.15}$$

where

$$\begin{aligned}
p(u)A &= -u_S\,(\cos a - a \sin a) + a\,(\sin a + a\xi \cos a) \,, \\
q(u)B &= -u_S\,(\cosh a + a\xi \sinh a) + a\,(\sinh a + a\xi \cosh a) \,, \\
\xi &\equiv \frac{u}{a^2} - 1 \,,
\end{aligned}$$

so that

$$\begin{aligned}
\frac{\omega_0}{\omega_S} = \left[\frac{a_1}{p^2(u_0)}\,(X - b_1 X_S)^2 + \frac{a_2}{q^2(u_0)}\,(Y - b_2 Y_S)^2 + 1\right] \cdot \\
\cdot \left[a_3\,(X - b_3 X_S)^2 + a_4\,(Y - b_4 Y_S)^2 + 1\right]^{-1} \,,
\end{aligned} \tag{2.16}$$

where

$$\begin{aligned}
a_1 &= \left[a\,(\sin a + a\,\xi_O \cos a) - u_S\,(\cos a - a\,\xi_O \sin a)\right]^{-2} \,, \\
a_2 &= \left[a\,(\sinh a + a\,\xi_O \cosh a) - u_S\,(\cosh a + a\,\xi_O \sinh a)\right]^{-2} \,, \\
a_3 &= \left[a\,(\sin a + a\,\xi_S \cos a) - u_S\,(\cos a - a\,\xi_S \sin a)\right]^{-2} \,, \\
a_4 &= \left[a\,(\sinh a + a\,\xi_S \cosh a) - u_S\,(\cosh a + a\,\xi_S \sinh a)\right]^{-2} \,, \\
b_1 &= \cos a - a\,\xi_O \sin a \,, \\
b_2 &= \cosh a + a\,\xi_O \sinh a \,, \\
b_3 &= \cos a - a\,\xi_S \sin a \,, \\
b_4 &= \cosh a + a\,\xi_S \sinh a \,.
\end{aligned}$$

We limit our considerations to *strong* waves and Taylor-expand for $a \ll 1$. Moreover, we consider an observer at $z = 0$: the sandwich wave is incident on him at $t = 0$ and leaves him at $t = a^2$. We then set $\xi_O = 0$, getting

$$a_1 \simeq u_S^{-2}\left[1 + \frac{2a^2}{u_S}\left(1 + \frac{u_S}{2}\right)\right] \,,$$

$$
\begin{aligned}
a_2 &\simeq u_S^{-2}\left[1+\frac{2a^2}{u_S}\left(1-\frac{u_S}{2}\right)\right] , \\
a_3 &\simeq u_S^{-4}\left[1+\frac{2a^2}{u_S}\left(1+\frac{u_S}{6}\right)\right] , \\
a_4 &\simeq u_S^{-4}\left[1+\frac{2a^2}{u_S}\left(1-\frac{u_S}{6}\right)\right] , \\
b_1 &\simeq 1-\frac{a^2}{2} , \\
b_2 &\simeq 1+\frac{a^2}{2} , \\
b_3 &\simeq 1-u_S+\frac{a^2}{2}\left(1+\frac{u_S}{3}\right) , \\
b_4 &\simeq 1+u_S-\frac{a^2}{2}\left(1-\frac{u_S}{3}\right) , \\
p^2(a^2) &\simeq 1-a^2 , \\
q^2(a^2) &\simeq 1+a^2 .
\end{aligned}
$$

The condition for redshift ($\omega_O/\omega_S < 1$) for $\xi_O = 0$ and $X_S = Y_S = 0$ is

$$
\left[\frac{a_1}{p^2(a^2)} - a_3\right] X^2 < \left[a_4 - \frac{a_2}{q^2(a^2)}\right] Y^2 .
$$

Since

$$
\begin{aligned}
\frac{a_1}{p^2(a^2)} - a_3 &\simeq u_S^{-4}\left[u_S^2 - 1 + a^2\left(2u_S^2 + 2u_S - \frac{2}{u_S} - \frac{1}{3}\right)\right] , \\
a_4 - \frac{a_2}{q^2(a^2)} &\simeq u_S^{-4}\left[1 - u_S^2 + a^2\left(2u_S^2 - 2u_S + \frac{2}{u_S} - \frac{1}{3}\right)\right] ,
\end{aligned}
$$

we get that if $-1 < u_S < 0$ the coefficient of $X$ is negative, while that of $Y$ is positive, and there is redshift in the whole $(X, Y)$ plane (except at $(0,0)$). If $u_S < -1$ the signs of the coefficients of $X$ and $Y$ are reversed, and there is blueshift in the whole $(X, Y)$ plane (except at $(0,0)$). If $u_S = -1$ the condition for redshift is $|Y| > |X|$, and the regions of red- and blue- shift are separated by the lines $Y = \pm X$, in the $(X, Y)$ system that has the polarization axes of the wave as coordinate axes and the observer at the origin (see Fig. 1.1). By continuity of the function $(\omega_O/\omega_S) - 1$, these results hold also for small, positive $\xi_O$. We have a frequency shift even if $g_{00} = -1$ (a possibility already pointed out by Burke [19]) and $g_{0i} = 0$ ( $i = 1, 2, 3$ ), and a spatial dependence

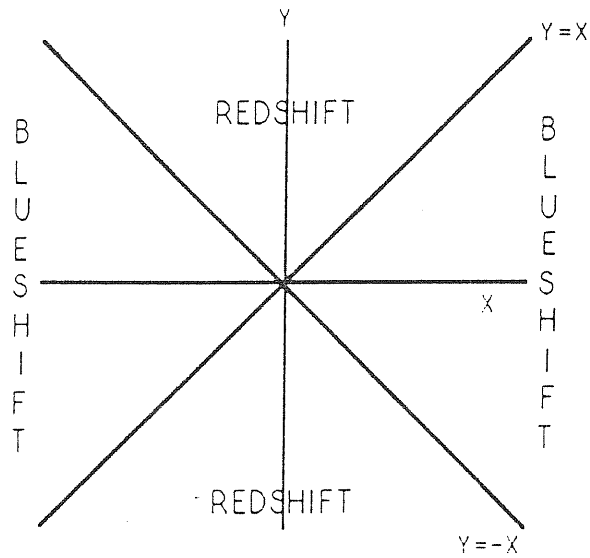


Figure 1.1: The pattern of frequency shifts for light propagating in a strong plane sandwich wave, in the case $u_S = -1$, $(X_S, Y_S) = (0,0)$.

of the frequency shift (a differential effect) even if the wavefront is perfectly uniform [$\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are Killing vectors of the metric Eq. (2.1)].

The previous formulae do not hold if $u_S = 0$. Actually it can be interesting to examine this case, corresponding to emission of light taking place when the plane wave collides with the light source. In this case we have (for $X_S = Y_S = 0$)

$$\frac{\omega_O}{\omega_S} = \left[\frac{\alpha_1}{p^2(u_O)} X^2 + \frac{\alpha_2}{q^2(u_O)} Y^2 + 1\right] \left(\alpha_3 X^2 + \alpha_4 Y^2 + 1\right)^{-1} ,$$

where

$$\begin{aligned} \alpha_1 &= a^{-2}(\sin a + a\,\xi_O \cos a)^{-2} , \\ \alpha_2 &= a^{-2}(\sinh a + a\,\xi_O \cosh a)^{-2} , \\ \alpha_3 &= a^{-2}(\sin a - a \cos a)^{-2} , \\ \alpha_4 &= a^{-2}(\sinh a - a \cosh a)^{-2} . \end{aligned}$$

The condition for redshift if $\xi_0 = 0$ is

$$\left[\frac{\alpha_1}{p^2(a^2)} - \alpha_3\right] X^2 < \left[\alpha_4 - \frac{\alpha_2}{q^2(a^2)}\right] Y^2 .$$

Expanding for small $a$, we get

$$\begin{aligned} \frac{\alpha_1}{p^2(a^2)} - \alpha_3 &\simeq -\left(1 + a^{-2}\right) < 0 , \\ \alpha_4 - \frac{\alpha_2}{q^2(a^2)} &\simeq a^{-2} - 1 > 0 . \end{aligned}$$

Then there is redshift in the whole $(X, Y)$ plane (except at $(0,0)$).

For small $\xi_O$ we have

$$\begin{aligned} \alpha_1 &\simeq u_O^{-2}(1 + a^2) , \\ \alpha_2 &\simeq u_O^{-2}(1 - a^2) , \end{aligned}$$

and $q(u_0)$ increases linearly with $u_0$, while $p(u_0)$ decreases linearly, vanishing at $u_* \equiv a(a + \cot a)$ ($\simeq 1 + 2a^2/3$ for strong waves): the factor $\omega_O/\omega_S$ diverges at $u_*$ (regardless of the position in the $(X, Y)$ plane). This reflects the fact that all null and timelike geodesics are focused on the $(Z, Y)$ plane (see Refs. [7, 2, 8] for the focusing property). The divergence of $\omega_O/\omega_S$ is however a spurious effect, since it is not physically meaningful to consider the frequency shift when $p = 0$. To see this, consider a circular cloud of test particles (possibly emitting light) in the plane $z = 0$ before the wave arrives, and suppose that light is emitted by the test particles when they are in the afterzone (when the gravitational wave has already passed); any frequency shift can be explained by Doppler effect due to relative motions induced by the gravitational wave, since we are now in flat space (while if we consider light emitted in the beforezone there is an additional "gravitational"contribution due to the wave field). We describe the situation in coordinates $(t, x, y, z)$: when $p$ approaches zero we have, retaining only the leading terms

$$\begin{aligned} p^x &\sim \alpha_1/p^2 , \\ p^z &\sim -\alpha_1^2/2p^2 , \\ p^y &< \infty , \\ dx/dz &= p^x/p^z \sim -2/\alpha_1 , \end{aligned}$$

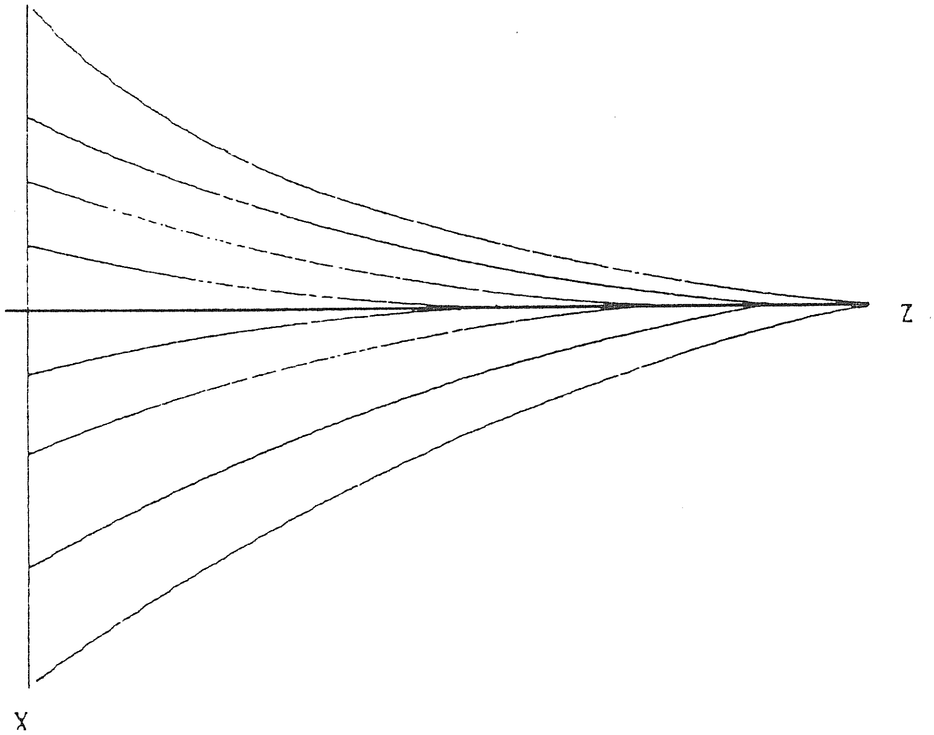


Figure 1.2: The "funnelling" of light rays near the line $p = 0$ for a strong plane sandwich wave.

$$
\begin{aligned}
dy/dz &\sim 0\,, \\
dy/dx &\sim 0\,,
\end{aligned}
\tag{2.17}
$$

so light rays tend to move in the $(x, z)$ plane but not in the $(y, z)$ plane and they are "funneled" near the line $p = 0$ (see Fig. 1.2).

The picture in a plane transversal to the $z$-axis is the following: the projection of the originally circular cloud of test particles on such a plane is now an ellipse (due to the wave's shear) with size shrinking to zero in the $x$-direction. For a test particle on the ellipse, $dx/dt \sim -2/\alpha_1$, $dy/dt \sim 0$ and $dz/dt \sim 1$. In the limit $p \to 0$ such an observer A will receive light only from a test particle B with $y_B = y_A$ $(dy/dx \sim 0)$, that is from a test particle on the $z$-axis, approaching him at the velocity of light $(dz/dt \sim 1)$. The formula for the Doppler effect gives then

$$
\frac{\omega_O}{\omega_S} = \frac{\sqrt{1-v^2}}{1-v} = \frac{\sqrt{1+v}}{\sqrt{1-v}} \to \infty \qquad as \quad v \to 1
$$

(the motion of test particles in coordinates $(t, x, y, z)$ seems to violate causality; see Refs. [8] and [9] for examples of motion of test particles in plane sandwich waves). We conclude that the singularity in the frequency shift effect at $p = 0$ is not physically meaningful; it comes from drawing conclusions with a formalism failing on the line $p = 0$, that is a singularity of the geodesic congruence. Any congruence of timelike geodesics in the sandwich wave spacetime has expansion

$$\theta = \frac{p'}{p} + \frac{q'}{q} \longrightarrow \infty \quad as \quad p \longrightarrow 0 \; . \tag{2.18}$$

This scalar quantity is purely kinematical and its divergence does not at all imply the existence of a singularity (the vanishing of $g_{11}$ at $p = 0$ is a purely coordinate singularity, since the metric can be put in the Minkowskian form in the afterzone), but it shows the relative motion of the timelike geodesic observers we considered in our heuristic explanation. The situation is very much like in Milne's spacetime, the Minkowski space with a cloud of particles expanding uniformly into it, that has scale function $a(t) = t$ (where $t$ is the cosmic time) and where the expansion of a congruence of fundamental observers is

$$\theta(t) = 3\,\frac{\dot{a}(t)}{a(t)}$$

(a dot means derivation with respect to $t$); $\theta$ diverges at the singularity $a = 0$.

## 1.3 Another example

Let us consider the metric [20]-[22], [7, 5, 6]

$$ds^2 = -dudv + dx^2 + dy^2 - H(u, x, y)\, du^2 \tag{3.1}$$

where $u = t - z$ and $v = t + z$. The metric given by Eq. (2.1) corresponds to a particular choice of $H$ and of the $(x, y)$ coordinates [23]. Equating to zero the Ricci tensor gives

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = 0 \; .$$

We choose $H$ to decrease (at least) as $1/r$ as $r \equiv (x^2 + y^2)^{1/2}$ tends to infinity; the metric Eq. (3.1) describes an exact, sourceless gravitational wave propagating along the positive $z$-axis and spatially localized.

Light rays propagating in the weak field region (large $r$'s) suffer only small deflections $\delta x^\mu$. The 4-momentum of such a photon is

$$p^\mu = p^\mu_{(0)} + \delta p^\mu \ ,$$

where $dp^\mu_{(0)}/d\lambda = 0$ ($\lambda$ is an affine parameter along null geodesics) and $\delta p^\mu$ are small quantities, so that the photon path is the curve

$$x^\mu(\lambda) = x^\mu_{(0)} + \delta x^\mu = a^\mu \lambda + b^\mu + \delta x^\mu(\lambda)$$

(where $a^\mu$ and $b^\mu$ are constants). Since the $\delta p^\mu$, $H$ and its derivatives are small in the far region, we neglect quantities of order $H \cdot \delta$, $\delta^2$, $\delta \cdot \partial_i H$ in the equation of null geodesics, getting

$$\frac{d(\delta p^\mu)}{d\lambda} + \Gamma^\mu_{\nu\sigma} p^\nu_{(0)} p^\sigma_{(0)} \simeq 0 \ ,$$

or

$$\delta p^\mu = -\int_S^O d\lambda \ \Gamma^\mu_{\nu\sigma} \ p^\nu_{(0)} \ p^\sigma_{(0)} \ ,$$

where the integral is computed along the photon's path from the source to the observer. Since the only non-vanishing Christoffel symbols are

$$\Gamma^1_{00} = H_u \quad \Gamma^1_{02} = \Gamma^1_{20} = H_x \quad \Gamma^1_{03} = \Gamma^1_{30} = H_y$$
$$\Gamma^2_{00} = H_x/2 \quad \Gamma^3_{00} = H_y/2$$

(where $H_\alpha \equiv \partial H / \partial x^\alpha$) we get, to first order,

$$\begin{aligned}
\delta p^u &\simeq 0 \ , \\
\delta p^v &\simeq -\int_{\lambda_S}^{\lambda_O} d\lambda \ \left[ H_u (a^u)^2 + 2 H_x a^u a^x + 2 H_y a^u a^y \right] \ , \\
\delta p^x &= -\frac{1}{2} \int_{\lambda_S}^{\lambda_O} d\lambda \ H_x (a^u)^2 \ , \\
\delta p^y &= -\frac{1}{2} \int_{\lambda_S}^{\lambda_O} d\lambda \ H_y (a^u)^2 \ .
\end{aligned}$$

The exact null geodesic equation gives

$$p^u = a^u + \delta p^u = 1 \ ;$$

to our order of approximation we may substitute $a^u$ with 1, getting

$$\begin{aligned}
\delta p^u(u) &\simeq 0 , \\
\delta p^v(u) &\simeq -\int_{u_S}^{u} du' \left(H_u + 2H_x a^x + 2H_y a^y\right) , \\
\delta p^x(u) &\simeq -\frac{1}{2}\int_{u_S}^{u} du' \, H_x , \\
\delta p^y(u) &\simeq -\frac{1}{2}\int_{u_S}^{u} du' \, H_y .
\end{aligned}$$

Moreover,

$$\frac{d(\delta p^t)}{d\lambda} = \frac{d(\delta p^z)}{d\lambda} = \frac{1}{2}\frac{dp^v}{d\lambda} ,$$

$$\delta p^t(O) = -\int_{u_S}^{u_O} du' \left(H_x a^x + H_y a^y + H_u/2\right) . \tag{3.2}$$

Consider an observer and a source at rest relatively to each other in the weak field region, with 4-velocities $u^\mu = (1 + \delta u^t, \delta u^x, \delta u^y, \delta u^z)$ and $v^\mu = (1 + \delta v^t, \delta v^x, \delta v^y, \delta v^z)$ in coordinates $(t, x, y, z)$ respectively, where $\delta u^\mu$, $\delta v^\mu$ are small velocity perturbations induced by the gravitational wave. The (angular) frequency received by the observer is

$$\omega_O = -g_{\mu\nu}\, p^\mu u^\nu = \left(1 + \delta p^t + \delta u^t - \delta u^x + H\right) |_O + O(2) .$$

The normalization $u_\nu u^\nu = -1$ gives

$$\delta u^t = -H/2 + O(2) ,$$

so that

$$\omega_O = 1 + \delta p^t(O) - \delta u^x + H(O)/2 + O(2) . \tag{3.3}$$

Analogously we find the (angular) frequency emitted by the source, $\omega_S = -g_{\mu\nu} p^\mu v^\nu|_S$. The frequency shift is given by the redshift parameter

$$z \equiv \frac{\omega_O}{\omega_S} - 1 =$$

$$= (\delta v^x - \delta u^x) + \left\{\delta p^t(O) + \left[H(O) - H(S)\right]/2\right\} + O(2) ,$$

where $\delta p^t(O)$ is given by Eq. (3.2). The first term can be interpreted as a Doppler effect due to the relative motion between source and observer induced by the gravitational wave; the remaining term can be interpreted as a "gravitational" contribution due to the wave's field. (Such a separation of the two contributions is however only a matter of interpretation, since only the total effect is in principle observable). It is worth noting that the frequency shift effect is of the first order in the metric and velocity perturbations; this is in agreement with the works on linearized waves (see Chapter 2).

In conclusion, the study of propagation of light in exact wave-like solutions of Einstein equations shows that the temporal and spatial dependence of the frequency perturbation is a fundamental feature and is present even when one does not expect it, due to the spacetime symmetries. There is a characteristic pattern of frequency shifts caused by a simple plane wave; it changes at different instants, for fixed positions of the source and the observer. The typical situation is such that one has, at a fixed instant and for a fixed source position, red– (or blue–) shift at all the observer's spatial positions (with the value of the redshift parameter depending on the observer's position). There is a particular source position for which one has, at a fixed instant, redshift or blueshift, according to the observer's spatial position.

# Chapter 2

# LINEARIZED GRAVITATIONAL WAVES

## 2.1 Introduction

Gravitational waves in real world are described by *linearized* General Relativity [1, 24] in most (if not all) astrophysical situations. The possibility that propagation of light through gravitational waves gives rise to observable effects was firstly considered by Bertotti [63], and by Wheeler [25] in a qualitatively way. This possibility was then considered by various authors, who most commonly took into account propagation of light through the stochastic gravitational wave background analogous to the 2.7 K electromagnetic background expected on the basis of the standard big bang theory of the universe (see Refs. [26, 27, 28] and references therein), or in unconventional cosmologies [29, 30, 31]. Sometimes however, light propagation through single bursts of gravitational radiation of noncosmological origin has been considered. If one considers isolated bursts of gravitational radiation from astrophysical sources, certain conditions must be satisfied in order that the photons coming from a luminous object cross the gravitational waves giving rise to a net effect. This can reflect in a small probability that the effect takes place (unless one considers very particular geometries in which the light source and the source of gravitational radiation are continuously emitting, and light is forced to cross a beam of gravitational waves). On the other hand the gravitational wave background is present always and everywhere, and the light reaching us from distant sources travels through it, so any effect induced by gravitational waves is surely present (but it is not necessarily so strong to be observable). In this case, under some assumptions on the frequency range and the cosmological density of lensing gravitational waves, one describes the various effects they induce on photons propagating through them. Comparison with observations allows one to put constraints on the gravitational wave amplitudes. In other situations, the effects on light due to cosmological gravitational waves have been proposed as an explanation of some puzzling observations.

The effects arising from the interaction of light with gravitational waves, which could possibly be relevant in astrophysics, are

- amplification and intensity fluctuations of a light beam;
- frequency shift;
- spatial deflection of light rays;
- multiple imaging and associated high amplification events;
- phase shift of electromagnetic waves;
- shear of a light beam.

Suppose that gravitational waves can be characterized by a single period $P$. Then, if the observation times are longer than, or comparable to $P$, an observer will notice the time-dependence of the effect induced by gravitational waves on the light propagating through them. If instead the observation times are shorter than $P$ (this occurs when ultralow frequency gravitational waves of cosmological origin are involved), the effect will appear to be "frozen" in time (wavelengths $\lambda \sim 1$ Mpc have been considered; they generate effects varying on timescales $\sim 10^6$ years).

## 2.2 Amplification and intensity fluctuations of a light beam

Let us consider, in the geometric optics approximation, a light beam propagating through gravitational waves, which induce a time-dependence in the beam intensity. **Intensity fluctuations** (together with the time-dependent deflection of light rays) cause scintillation of a distant light source. Various authors [32]-[35], [37, 38] have considered intensity fluctuations for a beam of light coming from a distant source and travelling through the gravitational wave background.

Zipoy [32] derives a formula for the intensity fluctuations of a light beam propagating through an arbitrarily varying gravitational field. He applies it to the gravitational wave background, and finds for a small size source at a distance $\sim 10^9$ light years, an average intensity fluctuation

$$\langle \left( \frac{\delta I}{I} \right)^2 \rangle^{1/2} \sim 10^{-5}$$

for the extreme case of a homogeneous and isotropic universe dominated by gravitational radiation. Zipoy considers also the case of a plane linearized wave with amplitude $h_{\mu\nu}$ (the spacetime metric being $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ in an asymptotically Cartesian coordinate system). The wave has arbitrary profile, a well defined direction of propagation, and is localized in a region of size $l$. The intensity fluctuations it causes in a light beam are found to be

$$\frac{\delta I}{I} \simeq h^2 \frac{D}{l} (1 - \cos\theta) \ , \qquad (2.1)$$

where $\theta$ is the angle between the directions of propagation of the gravitational wave and the light beam, $D \equiv D_L D_{LS}/D_S$ and $D_S$, $D_L$ and $D_{LS}$ are the observer-source, observer-lens and lens-source distances respectively (since the background spacetime is flat these are Euclidean distances). Note that according to this formula:

- the intensity fluctuations are of second order in the gravitational wave amplitudes;
- if the light beam and the gravitational wave propagate along the same axis ($\theta = 0$), the intensity fluctuations vanish;
- the effect results from a balance between (very small values of) $h^2$ and the ratio $D/l$ (which is very large for a light source at cosmological distance and a well localized pulse of gravitational radiation).

An optimistic estimate of the amplitude $h$ comes from considering a pulse of gravitational waves generated by a violent explosion involving a mass of order $10^8 M_\odot$ at a distance $\sim 10^9$ light years, which gives $\delta I/I \sim 10^{-15}$. The effect appears to be exceedingly small.

Winterberg [33] considered this effect basing his calculations on a formula derived in studies of scintillation produced by atmospheric turbulence, treating gravitational waves as a randomly fluctuating medium with refractive index $n = n_0 + \delta n$ (where $\delta n$ describe stochastic fluctuations). The result is

$$\langle \left(\frac{\delta I}{I_0}\right)^2 \rangle = \frac{32\sqrt{\pi}}{3} \langle \left(\frac{\delta n}{n_0}\right)^2 \rangle \left(\frac{D}{l}\right)^3 , \tag{2.2}$$

where $I_0$ is the time average of the beam intensity, $D$ is the distance between the light source and the observer, and $l$ is the characteristic length scale of the density inhomogeneities in the medium (which is connected with the auto-correlation function $\langle \delta n(\underline{x}_1), \delta n(\underline{x}_2) \rangle$). Winterberg used Eq. (2.2) to describe the scintillation of a distant light source, taking into account waves emitted by close binaries in our galaxies, by distant quasars, and the gravitational wave background. He found large intensity fluctuations $\delta I/I \sim 1$ in the last two cases, contrary to Zipoy's [32] conclusions. As shown explicitly by Zipoy and Bertotti [34], this result is wrong, and the large effect found by Winterberg is an untrue coordinate effect. Eq. (2.2) leads to two mistakes: firstly, there is no first order effect in the gravitational wave amplitudes. The amplification of a light beam is a second order effect, and consequently it is very small [35, 34]. This can be understood by considering a congruence of null geodesics in the field of linearized gravitational waves around the Minkowski metric [36, 35], and studying the propagation equations for the optical scalars. If $k^a$ is the (null) tangent field to the null geodesics, the optical scalars expansion, shear and vorticity are defined by

$$\begin{aligned}
\theta &\equiv \frac{1}{2} k^a{}_{;a} , \\
|\sigma|^2 &\equiv \frac{1}{2} k_{(a;b)} k^{a;b} - \theta^2 , \\
\omega^2 &\equiv \frac{1}{2} k_{[a;b]} k^{a;b} ,
\end{aligned}$$

respectively, and the complex shear is (apart from a phase factor)

$$\sigma = k_{a;b}\,\bar{t}^a\,t^b = \frac{1}{\sqrt{2}}\left[k_{(a;b)}\,k^{a;b} - \frac{1}{2}\left(k^a{}_{;a}\right)^2\right]^{1/2} ,$$

where $t^a$ is a complex null vector orthogonal to $k^a$ and normalized according to $t_a\bar{t}^a = -1$. The optical scalars satisfy the equations

$$\begin{aligned}
\frac{d\theta}{d\lambda} &= -\theta^2 - |\sigma|^2 + \omega^2 - \frac{1}{2}\,R_{ab}\,k^a k^b , \\
\frac{d\sigma}{d\lambda} &= -2\,\theta\,\sigma - C_{abcd}\,\bar{t}^a\,k^b\,\bar{t}^c\,k^d \equiv -2\,\theta\,\sigma - C(\lambda) ; \\
\frac{d\omega}{d\lambda} &= -2\,\omega\,\theta ,
\end{aligned}$$

where $\lambda$ is an affine parameter along null geodesics. An approximate solution to these equations for weak gravitational fields is [36]

$$\theta = \frac{1}{\lambda} + \delta\theta \ , \quad \sigma = \delta\sigma \ , \quad \omega = \delta\omega = 0$$

(where we have chosen $\lambda = 0$ at the position of the light source). In this approximation, in empty space, $\delta\theta$ and $\delta\sigma$ obey

$$\begin{aligned}
\frac{1}{\lambda^2}\frac{d}{d\lambda}\left(\lambda^2\delta\theta\right) &= -|\sigma|^2 , \\
\frac{1}{\lambda^2}\frac{d}{d\lambda}\left(\lambda^2\delta\sigma\right) &= -C(\lambda) .
\end{aligned}$$

Clearly, if $h$ is the amplitude of perturbations of the Minkowski metric, $\delta\sigma \sim h$ and $\delta\theta \sim |\delta\sigma|^2 \sim h^2$. Thus the change in the expansion (and then in the amplification of a pencil of light rays) vanishes to first order, reducing to a second order effect [1].

The second mistake to which one is led by using Eq. (2.2) is that the relative fluctuations $\delta I/I$ are proportional to $D^{3/2}$, where $D$ is the length of

[1] This suggests an interesting observation concerning ordinary gravitational lenses: the shear they induce on a beam of light propagating for a long time in a inhomogeneous universe can generate an appreciable amount of expansion (a typical value for the dimensionless Newtonian potential of galaxies is $\Phi/c^2 \sim 10^{-6}$, which is much larger than the amplitude $h$ of gravitational waves in most situations). Dyer and Oattes [39] have found that this nonlinear effect influences the light beam more than the common linear treatments suggest.

the path travelled by the photons. Such a dependence is wrong, since Eq. (2.2) does not apply to a situation in which the inhomogeneities of the refracting medium propagate with the speed of light (we will comment in the following the dependence of the wave-induced effects on the path length). We have quoted these works because it is important to realize that the amplification induced by gravitational waves on a light beam is a second order effect in the wave amplitudes in ordinary situations (a special treatment is however necessary for situations involving multiple images - see later), and that, though treating the gravitational waves as a medium with fluctuating index of refraction is in principle correct [40, 41], one must be very careful in doing this, in order to avoid mistakes like the dependence of $\delta I/I$ on the travelled path length, that completely vitiates the final results when numbers are put in.

Bertotti and Catenacci [38] studied the influence of a dispersive medium on the dependence on the travelled distance for light propagating in the gravitational wave background. They examined fluctuations in the intensity and position of a light source, and found a different dependence of these effects on the source distance $D$, according to the fact that the refractive index of the dispersive medium is greater or less than 1. If $n > 1$ a Cerenkov type resonance produces scintillation proportional to $D^{3/2}$. If instead $n < 1$, the effect is proportional to $D^{1/2}$. They use this result to put an upper limit to the spectrum of gravitational waves, but these limits are not very interesting (as themselves remark), due to the fact that the interstellar and intergalactic plasmas are too rarified.

## 2.3 Frequency shifts

Light propagating through gravitational waves suffers a time- and position-dependent **frequency shift** (see Chapter 3 for detailed calculations). Various authors dealt with this effect from different points of view. As one easily finds by solving the null geodesic equation, the redshift parameter $z \equiv (\lambda_{observed}/\lambda_{emitted}) - 1$ is of the order of the gravitational wave amplitudes (metric perturbations): $z \sim h$. Then small effects are expected for astrophysically generated gravitational waves, while waves of cosmological origin can be more interesting.

Zipoy [32] considers the frequency shift due to the gravitational wave background and finds $z \sim 10^{-8}$ for a universe dominated by gravitational radiation.

Kaufmann [42] considers explosions near the centre of our Galaxy involving masses of order $10^8 M_\odot$ moving at velocities $v \sim 10^{-2}$, at a distance $r \sim 10^4$ light years, and finds

$$z \sim h \sim \frac{GM}{c^2 r} \sim 10^{-13}$$

(corresponding to velocities $\sim 3 \cdot 10^{-3}$ $cm/s$). He points out that this effect is within the sensitivity limit of devices based on the Mössbauer effect (which may reach sensitivities up to $\sim 10^{-15}$), but it is not at all clear how to separate the effect due to gravitational waves from other contributions. Events outside our galaxy (in the local supercluster) are not detectable, even in principle, by these devices.

Rees [29, 43, 30] considers primordial gravitational waves in the wavelength range 1–10 Mpc (a size intermediate between the separations of neighbouring galaxies and the diameter of large clusters), and finds that the frequency shift effect is of the same order of the Doppler effect due to the relative velocities between galaxies induced by gravitational waves. Galaxies in a cluster are treated as particles separated by a distance $l$ and it is found that

$$z \sim \begin{cases} \sqrt{\Omega_{g.w.}}\,(\lambda/\lambda_H) & \text{if } l \geq \lambda \\ \sqrt{\Omega_{g.w.}}\,(\lambda/\lambda_H)\cdot(l/\lambda) & \text{if } l \ll \lambda\ , \end{cases}$$

where $\Omega_{g.w.}$ is the energy density of gravitational waves in units of the critical density $\rho_c \simeq 2 \cdot 10^{-29}$ $g \cdot cm^{-3}$ (defined according to the prescription by Isaacson [44]) in the frequency band considered, and $\lambda_H \sim H^{-1}$ is the Hubble radius ($\lambda_{H_0} \sim 3000$ Mpc). Note that $z \sim h$ for $l \geq \lambda$, as follows from the equation

$$h \sim \sqrt{\Omega_{g.w.}}\,\frac{\lambda}{\lambda_H} \qquad (3.1)$$

[26]. In the canonical big bang theory of the universe one expects [26] a gravitational wave background with $\Omega_{g.w.}$ not exceeding the energy density of electromagnetic radiation $\Omega_{e.m.} \simeq 2.5 \cdot 10^{-5}$. This gives amplitudes by far too small to generate appreciable effects. Also astrophysically generated waves cannot give rise to any effects in galaxy groups and clusters because their wavelengths

are too short ($\lambda \leq 1$ pc). The waves considered by Rees are instead remnants of primordial chaos in the early universe, in an unconventional cosmological model [29, 30, 31] in which the universe is chaotic at $z > 1000$, and the spectrum of chaos is truncated on scales larger than clusters of galaxies, to account for the observed homogeneity of the universe on large scales. These waves are expected to be broad-band, with a predominant wavelength in the range 1–10 Mpc, and can have $\Omega_{g.w.} \sim 1$ without contraddicting present observations [29]. They can give rise to substantial effects: if $l \geq \lambda$ apparent velocity dispersions $\sim 300$ Km/s would be generated, being capable of accounting for the redshift anomalies observed in small groups and clusters of galaxies. The apparent motions in the Local Group (and in particular the negative velocity of M31) are also considered [29, 43]. Though the purpose of Refs. [29, 43] (see also Ref. [45]) is mainly to treat the effect of waves on the dynamics of clusters and groups, these works are relevant here because the redshift variations we are interested in are of the same order of the Doppler effect due to wave-induced velocities of galaxies [29]. Eqs. (2.3) later show that waves with $\Omega_{g.w.} \sim 1$ and any $\lambda$ are capable of completely cancelling the Hubble recession on scales $\leq \lambda$. These considerations do not depend on the precise spectrum of gravitational waves.

Burke [19] proposed to explain part of the **apparent velocity dispersion in cluster galaxies** with redshift fluctuations induced by long wavelength random gravitational waves of cosmological origin. The wavelength should be tuned such that the gravitational wave field is correlated across a single galaxy, but is uncorrelated at different galaxies. Since such wave periods ($\lambda \sim 1$ Mpc), are greater than any reasonable observation time, the effect appears to be "frozen" in time. Photons emitted by galaxies in the same cluster but located at different positions, where the metric perturbations have different values, will undergo different frequency shifts. Photons from some galaxies will be redshifted, photons from others will be blueshifted, so these galaxies will appear to have a relative velocity dispersion. The redshift parameter is found to satisfy

$$\begin{aligned} \langle z \rangle &= 0 \, , \\ \sqrt{\langle z^2 \rangle} &\simeq \sqrt{\Omega_{g.w.}} \, \frac{\lambda}{\lambda_H} \sim h \, . \end{aligned}$$

Dautcourt [46] proposed to explain the redshift anomalies observed in galaxy clusters, groups and chains by the the frequency shifts induced by waves with wavelength greater than the Megaparsec scale. These waves should form a non-thermal background originated in the early stages of the Universe by turbulence in primordial matter motions. Ultralow frequency waves (wavelengths in the range 1–3000 Mpc) can have amplitudes $h$ in the range $3 \cdot 10^{-4}$–1, i.e. extremely large when compared to the small amplitudes expected for radiation impinging the Earth ($h \sim 10^{-18}$ for supernovae in our galaxy, $h \sim 10^{-21}$ for events in the Virgo cluster). Such large amplitudes are possible without supercritical cosmological densities of gravitational waves, due to the very long wavelengths involved. In fact $h$, $\Omega_{g.w.}$ and $\lambda$ are related by Eq. (3.1), where $h$ is an increasing function of the wavelength and can become appreciable if $\lambda$ is some fraction of $\lambda_H$, $\Omega_{g.w.}$ being not too far from 1.

Dautcourt considers perturbations $h_{\mu\nu}$ to the metric of a $K = 0$ Friedmann-Lemaitre-Robertson-Walker (hereafter FLRW) spacetime (in the synchronous gauge), treated as stochastic Fourier integrals, and derives equations for the spectral densities. These are solved in the high frequency approximation $\lambda/\lambda_H \ll 1$ (that allows also the introduction of the energy density of gravitational waves according to the prescription by Isaacson [44]). Due to gravitational waves, the observed redshift of a light source will be

$$z_{total} = z + \delta z \ ,$$

where $z$ is the non-random mean of the redshift related to the distance by $D = cz/H_0$ for small $z$, while the fluctuating contribution $\delta z$ is a random variable changing irregularly with the source position. The explicit dependence of $\delta z$ on the spectral densities is derived. If the gravitational radiation is quasimonochromatic $\delta z$ exhibits **periodicities in the source distance** $z$ **and in the angular distance** $\theta$. The predicted periods are

$$\begin{aligned} \Delta z &\simeq (1+z)\frac{\lambda}{\lambda_H} \ , \\ \Delta\theta &\simeq \frac{1+z}{z}\frac{\lambda}{\lambda_H} \ , \end{aligned} \tag{3.2}$$

respectively. For quasimonochromatic radiation he finds

$$\sqrt{\langle\delta z^2\rangle} \simeq \frac{\lambda}{\lambda_H}\sqrt{\frac{4}{5}\,\Omega_{g.w.}\left(1+z+\frac{z^2}{2}\right)}\,.$$

As a consequence, distances of extragalactic objects determined by redshift measurements are subject to errors of order $H_0^{-1}\sqrt{\langle\delta z^2\rangle} \sim \lambda\sqrt{\Omega_{g.w.}}$ The redshift component $\delta z$ induced by gravitational waves will appear to be frozen in time. The values of $\delta z$ for single galaxies vary with the depth and the apparent sky position of the galaxy. One effect is an increase of the **velocity dispersions of galaxies in clusters and groups** (the same effects considered by Burke for galaxies, but on a larger scale).

If the gravitational waves have wavelengths greatly exceeding the cluster diameter, the effect is not to increase the apparent velocity dispersions, but rather an anomalous redshift with nearly the same value for all galaxies. Waves with angular correlation length $\Delta\theta$ given by Eq. (3.2) will produce a **systematic change of galaxy redshifts across a cluster or a group of galaxies.** This could explain, e.g. the systematic redshift variations in the chain of galaxies observed by Gregory and Connolly [47]: a redshift difference of $\sim$ 750 Km/s is observed between the ends of the chain and apparently cannot be explained in terms of peculiar motions. Waves with $\lambda < 1$ Mpc are required in order to explain it. The same effect could also explain large discrepancies in other groups of galaxies that almost surely are physically related.

It has been proposed that gravitational waves with $\lambda$ greater or of the order 100 Mpc with a not too small cosmological density could explain the anisotropy and nonlinearity of the **redshift distribution of galaxies in the local supercluster**. Dautcourt finds that the redshift perturbation consists of a change $\delta z$ along the supergalactic equator, mainly of quadrupole type, due to the local wave field in the neighborhood of the observer. As remarked in Ref. [19], this quadrupole type anisotropy affects the redshifts of nearby galaxies, cluster of galaxies, quasars and the microwave background. Limits on $\Omega_{g.w.}$ coming from observations of the microwave background constrain simple models explaining the local redshift anomaly with a wave-induced systematic redshift perturbation. The question whether refined models can account for the observed redshift anomalies without contraddicting observational limits on the isotropy of the microwave background apparently has not yet been answered.

A **periodic redshift clustering** for all extragalactic objects is predicted if the waves generating the previous effects exist. The number density of sources of a given type with redshifts between $z$ and $z + \delta z$ is

$$n_{total}(z) = n(z) - \frac{\partial}{\partial z}(n\delta z) \ ,$$

where $n(z)$ is the corresponding density were the waves absent. If the gravitational waves have a broad spectrum, periodicities in $n_{total}$ are smeared out. If, on the other hand, gravitational radiation is quasimonochromatic with wavelength $\bar{\lambda}$, one has redshift clustering on a scale $\Delta z \simeq (1+z)\bar{\lambda}/\lambda_H$. A systematic study of this conjecture considering all the present observations of periodicities in the redshift of extragalactic objects apparently has not been given.

Dautcourt's approach treats the waves as coherent in phase, while a phase-amplitude relation should exist. Moreover, it fails when $\lambda$ is some fraction $\alpha$ of $\lambda_H$, i.e. in the region where $h$ takes its larger values, unless $\Omega_{g.w.}$ peaks at wavelengths much smaller than $\alpha\lambda_H$ (however one expects that primordial gravitational waves are broad-band [29, 43, 26]). In addition, there exists a redshift $z^*$ such that for $z > z^*$ the high frequency approximation breaks down. In fact, if $a(t) = (t/t_0)^n$ is the scale factor (normalized to 1 at the present epoch) in the $K = 0$ Friedmann model used, one has [2]

$$\begin{aligned} \frac{\lambda}{\lambda_0} &= (1+z)^{-1} \ , \\ \frac{\lambda_H}{\lambda_{H_0}} &= \frac{H_0}{H} = (1+z)^{-1/n} \ , \end{aligned}$$

and then $\lambda/\lambda_H = (1+z)^{1/n-1}\left(\lambda/\lambda_H\right)_0 \sim 1$ at $z^*$. If $\Omega_{g.w.} = 1$ for waves around $\lambda_0 = 3$ Mpc, $n = 1/2$ and $z^* \sim 1000$.

The possibility that cosmological gravitational waves affect also microwave background photons has been considered by various authors ([48, 46, 41, 49]; [50]-[52] and references therein). The temperature of the microwave background is

$$T = T_0 + \delta T \ ,$$

[2]using $H = (t/t_0)^{-1} n/t_0$ and $1+z = (t/t_0)^{-n}$

where $\delta T$ describes the frequency shift induced by gravitational waves. Dautcourt [46] considers a $K = 0$ Friedmann model with metric perturbations $h_{\mu\nu}$ (in the synchronous gauge) and with scale factor $a(t) = (t/t_0)^n$, and Fourier-decomposes the temperature fluctuations:

$$\delta T = \int \tau(\underline{k})\, e^{i\underline{k}\cdot\underline{x}-ikt}\, d^3\underline{k} \quad + \text{ complex conjugate ,}$$

and integrates the equation of radiative transfer for $\tau$, taking into account Thomson scattering and the interaction with gravitational waves. He finds

$$\tau(\underline{k}) = \frac{kT_0 h_{ij} n^i n^j}{2\,(\underline{k}\cdot\underline{n}+k)} \left\{1-(1+z_1)\exp\left\{-\frac{\lambda_H}{\lambda_c}\,\overline{q} + \frac{in\lambda_H\,(\underline{k}\cdot\underline{n}+k)}{1-n}\left[1-(1+z_1)^{1-1/n}\right]\right\}\right\}, \tag{3.3}$$

where it is assumed that $\tau = 0$ at some initial epoch $z_1$, $\lambda_c = (n_0\sigma_T)^{-1}$ (where $n_0$ is the present density of matter, $\sigma_T$ is the Thomson cross section) and

$$\overline{q} = \int_0^{z_1} dz'\, q(z')\,(1+z')^{2-1/n} \;;$$

$q(z')$ is the degree of ionization of intergalactic matter at the epoch $z'$.

$\tau$ consists of two terms: the first is slowly varying across the sky, and the second (containing the exponential function) is a rapidly changing function of the direction of observation $\underline{n}$. The slowly varying contribution is due to the local wave field in the neighborhood of the observer, while the rapidly changing term is due to photons' interaction with gravitational waves at some early instant prior to the recombination [53]. A similar decomposition is found by Burke [19].

The temperature variation on a large angular scale is decomposed in spherical harmonics and it is found [53] that the induced anisotropy is of quadrupole type, a dipole-type contribution being negligible. From observational limits on the microwave background quadrupole anisotropy one sets the upper limit on the cosmological density of gravitational waves with $\lambda \sim \lambda_H$ [28]

$$\Omega_{g.w.} \leq 2.5 \cdot 10^{-9}\, \Omega_{e.m.} \,.$$

The anisotropy induced by waves produced at decoupling ($z \sim 1000$) with wavelengths of order of the horizon is on scales minutes to degree. Observational limits on these scales give [28]

$$\Omega_{g.w.} \leq 4 \cdot 10^{-10}\, \Omega_{e.m.}$$

for $\lambda \sim 100$ Mpc.

The frequency shift induced by gravitational waves affects radio signals propagating from pulsars to the Earth, and has been taken into account extensively in the literature on pulsar timing (see Ref. [54] for a review). Pulsars are extraordinarily accurate clocks and the frequency shift can be observable as the time derivative of the timing residuals in the observational data. The observational limits on the residual noise in pulsar timing can then be used to set limits on the cosmological density of the gravitational wave background [55]. The spectral response was studied theoretically by Mashoon [56] and by Bertotti *et al.* [57], while Romani and Taylor [58] and Hellings and Downs [59] performed actual data analysis. The result is the upper limit

$$\Omega_{g.w.} < 1.4 \cdot 10^{-4} \text{ for } P \geq 10^8 \text{ s} \quad .$$

Data from the millisecond pulsar PSR 1937+21 give [60]

$$\Omega_{g.w.} < 5 \cdot 10^{-4} \text{ for } P \sim 10^7 \text{ s} \quad .$$

Bertotti *et al.* considered data from the binary pulsar PSR 1913+16 (in which the knowledge of the orbital dynamics permits one to predict the time derivative of the orbital period $P_0$, contrary to what happens for the spin period $P_s$ of pulsars, for which $\dot{P}_s$ is determined by a not well known physics) in order to set an analogous limit in the wavelength range 1–$10^4$ light years. They find $\Omega_{g.w.} < 2$. This is not an exciting result, but it is expected to improve as data on PSR 1913+16 continue to accumulate.

## 2.4 Spatial deflection of light rays

The time-varying field of gravitational waves deflects the null rays propagating in it. Zipoy [32] finds for the angular deflection $\delta\theta$ of a light ray in a stochastic gravitational wave background

$$\langle \delta\theta^2 \rangle = \frac{5}{3} \int_0^\infty d\omega \, \frac{P(\omega)}{\omega^4} \; ,$$

where $P(\omega)$ is the spectrum of gravitational waves (Zipoy's definition of $P(\omega)$ however is somewhat different from the usual one – compare Refs. [32] and [26]). He estimates angular deflections of order $10^{-5}$ arcseconds.

Linder [61] considers light propagating through a medium with refractive index

$$n(x,t) = n_0 \left[1 + \epsilon(x,t)\right] \ ,$$

where the inhomogeneities are described by $\epsilon$ ($|\epsilon| \ll 1$) and propagate with velocity $v$. For the case of the gravitational wave background and of a photon propagating along the $x$-axis, $v = 1$ and $\epsilon = h_{xx}/2$ in the TT gauge [3].

The angular deviation for a signal whose unperturbed path is along the $x$-axis is, to first order,

$$\theta_A = \int_0^D \partial_A \epsilon \, dx$$

($A = y, z$), where it is assumed that the path length $D$ the signal traverses is much greater than the scale of inhomogeneities $\lambda$. When the medium with fluctuating refractive index is the gravitational wave background, we have

$$\epsilon(x,t) = Re\left(\epsilon_0 \, e^{ik_\mu x^\mu}\right) \ .$$

In non-relativistic treatments of wave propagation through inhomogeneous, turbulent media, the mean square angular deflections $\langle \theta^2 \rangle$ grow as the path length ("$D$–effect"). This can be understood by thinking that if $N$ is the number of scatterings the propagating wave has undergone, the $N$ deflections add stochastically, so that $\sqrt{\langle \theta^2 \rangle} \propto \sqrt{N}$, according to the well-known law for the standard deviation. However, when the inhomogeneities in the medium propagate with relativistic velocities (as for the gravitational wave background), the situation is different and the $D$–effect vanishes, a striking and counterintuitive result.

Considering the "+" polarization, Linder finds

$$\langle \theta^2 \rangle = \frac{1}{2} \, kD \langle Re^2 \epsilon_+ \rangle \, I \ ,$$

where $k = |\underline{k}|$, $I = \sum_{n=-2}^{2s} a_n I_n$, $s$ is the spin of the field responsible for inhomogeneities, $a_n$ are constants independent of $kD$, and

$$I_n = (kD)^{-n-2} \int_{-kD(1+v)/2}^{kD(1-v)/2} dy \, y^n \sin^2 y$$

[3] Note that for a monochromatic plane gravitational wave propagating along the $x$-axis, one has $h_{xx} = 0$ in the TT gauge, and the effect vanishes (in any gauge, of course). This is due to the transversality of the gravitational wave.

($y \equiv kD(\cos\theta - v)/2$). It is found that the integrals $I_n$ for $n \geq 0$ do not vary as any positive power of $D$. $I_{-1}$ and $I_{-2}$ contain no $D$–effect in the limit $v \to 1$. The absence of the $D$–effect for the gravitational wave background is understood as due to the transversality of gravitational waves, and to the equality between the velocities of propagation of the light signal and of the inhomogeneities, so the vanishing of the effect is not a matter of relativistic, but rather of relative velocities. (The absence of secular effects had also been noted in Refs. [32, 38]). Linder actually treats the more general situation of wave signals propagating with velocity $V$ through a medium where the inhomogeneities have velocity $v$ and are due to a field of spin $s$ ($v$ and $V$ are not necessarily 1, and $s$ can be 0, 1 or 2) [4].

The spatial deflection of null rays by random gravitational waves (together with the already considered intensity fluctuations induced in a light beam) causes scintillation of a light source. Dautcourt [46, 62] considers ultralow frequency gravitational waves which give a frozen effect, i.e. a positional shift of all sources in a region of the sky which varies on unobservably long time scales. It is suggested that, if the displacements are correlated over the sky, the position shift could lead to an apparent clustering. This would concern general clustering tendencies, not distinct, well recognizable, clusters of galaxies. The mean square root angular displacement is found to be

$$\sqrt{\langle\theta^2\rangle} \sim 0.1\sqrt{\Omega_{g.w.}}\,\frac{\lambda}{\lambda_H}$$

for quasimonochromatic radiation of wavelength $\sim \lambda$, and for light sources at redshifts $z \ll 1$. The angular displacement can be decomposed into two contributions, analogously to what happens for the redshift effect. The first contribution is due to the local field of gravitational waves in the neighborhood of the observer and shows correlations over a large part of the sky. It gives a rather large (for long wavelengths $\lambda$ and appreciable densities $\Omega_{g.w.}$) effect. Since it is a systematic position shift on a large scale it is in practice unobservable.

[4]It is interesting to note that if one considers lensing by ordinary mass concentrations in an inhomogeneous universe as lenses, the deflections suffered by a pencil of light rays cumulate with the travelled distance (at least for light sources at moderate redshifts). This results in a "fuzzy" structure of the past null cone perceived by an observer, imposing limits on the reliability of standard observations in determining the structure of the universe [39].

The second contribution to $\sqrt{\langle\theta^2\rangle}$ is connected to the wave field at the source, and is smaller than the first by a factor $\lambda/(2\pi\lambda_H)$. This contribution is responsible for the apparent clustering, and it is found to be

$$\sqrt{\langle\theta^2\rangle} \simeq \frac{(1+z)^2}{z}\left(\frac{\lambda}{2\pi\lambda_H}\right)^2\sqrt{\Omega_{g.w.}} \; .$$

This gives a *differential* effect on a smaller scale than the first contribution. The deflections can reach minutes of arc for waves with $\lambda \geq 100$ Mpc. However, it is difficult to separate apparent effects induced by gravitational waves from real clustering tendencies, which are expected to produce stronger deviations of the index of clumpiness from 1. As a consequence, Dautcourt concludes that existing galaxy counts put no significant upper limits on the gravitational wave amplitudes [62].

## 2.5 Multiple imaging and high amplification events

The probability distribution of amplitudes for the waves considered by Dautcourt is not known, so one cannot rule out the possibility that high amplitudes occur. If nonlinear waves really occur, they may have given rise to a network of caustics due to the focusing property mentioned in Chapter 1 [62].

The possibility that gravitational waves multiply image light sources was treated heuristically by Wheeler [25] and applied by Bertotti [63]. High amplification events are associated to the existence of caustics and critical lines which separate regions corresponding to different numbers of images (see later Chapter 4). The formalisms used in the literature to compute intensity fluctuations of a light beam due to gravitational waves [32, 35, 34] give very small effects. However they are known to break down when caustics describing multiple imaging are concerned, so it is possible that the known results do not tell the whole story. These formalisms look very much like the *propagation formalism* ([64] and references therein) used in the theory of ordinary gravitational lenses. This is not able to describe multiple images and high amplification events associated to the crossing of caustics by a light source. As an example, we quote the amplification found by Zipoy [32] for a light beam crossing the time-dependent gravitational field of a binary system. Studies of the two point-mass gravitational lens performed with the standard theory [65]-[68] give a different amplification. The formula found by Zipoy shows only a caustic point ($b = 0$), while the caustic manifold found in [65]-[67] is more complicated[5]. Thus it appears that the statement that gravitational waves *always* give a very small amplification of a pencil of light rays can be untrue in circumstances in which multiple imaging is involved. We know that for ordinary gravitational lenses multiple imaging is not always possible, since a certain condition on the projected mass density of the lens must be satisfied [69]. Thus a lens capable of multiple imaging is more an exception than a rule, so we can ask whether a lensing gravitational wave capable of multiple imaging is realistic. If the answer is positive, we should expect high amplification events from such a wave. The problem whether realistic gravitational waves can be capable of multiple imaging is considered in Chapter 4, and it is found that the answer is positive in some situations in which astrophysically generated waves are involved. They do not appear to be very frequent, but there is room for the possibility of multiple imaging and the associated high amplification events.

## 2.6 Phase shift of electromagnetic waves

A light beam propagating through gravitational waves experiences a phase shift. This effect has been considered in order to build interferometric detectors of gravitational waves ([70, 71, 72] – see [73] for gauge-dependence problems connected with these detectors).

Braginsky *et al.* [74] have given a detailed treatment of this effect. They solve Maxwell equations in the field of a monochromatic plane gravitational wave, in the high frequency approximation (with $\lambda/\lambda_H \ll 1$) and to first order in the wave amplitude $h$. Then they apply the result to a random superposition of linearized waves to find the phase shift of a plane electromagnetic

[5] One could raise doubts on the validity of the standard vector formalism (see later) used in the above references for the case of a rapidly rotating binary lens, since the hypothesis of lens stationarity is explicitly required to apply it. One can however be confident that at least the order of magnitude of the amplification it gives is correct, since it is not expected that the time variability of the gravitational field will cancel the effect.

wave. They find that the phase shift does not increase with the travelled distance (the usual absence of secular effects), due to the transverse character of gravitational waves and to the equality between the velocity of light and of gravitational waves. An explicit comparison with a nonrelativistic randomly fluctuating medium is made. They consider also the phase shift effect due to random gravitational waves in presence of a plasma with refractive index $n > 1$. They find that the phase shift in this case cumulates with the travelled distance, but even assuming that the density of matter in the Universe is critical, and that the travelled distance is the Hubble radius, the effect turns out to be extremely weak. This result is similar to that found by Bertotti and Catenacci [38] for the intensity fluctuations of a light beam. To the second order approximation, an effect quadratic in the wave amplitude $h$ is found to be relevant only for inadmissibly high densities $\Omega_{g.w.}$. Finally, they analize the capability of space radio interferometry to detect, by means of the phase shift effect, gravitational radiation of both astrophysical and cosmological origin. Space radio interferometry is found to be competitive with other detection methods for wavelengths $\lambda \leq 1$ light year.

## 2.7 Shear of a light beam

Using the equations of propagation for the optical scalars of a congruence of null geodesics, one finds that the shear of a light beam due to gravitational waves is of first order in the gravitational wave amplitudes: $\sigma \sim h/\lambda$. For cosmological gravitational waves one has

$$\sigma \sim \frac{\sqrt{\Omega_{g.w.}}}{\lambda_H} ,$$

i.e. the shear depends on the wavelength of gravitational radiation only through its energy spectrum.

Let us consider a single plane gravitational wave; the shear it induces on a bundle of null rays is zero when averaged over many wave periods. For a random superposition of plane waves one expects that the shear vanishes even prior to the time averaging (due to the ensemble averaging). To find some effect one must consider a single plane wave with period longer than the observation times. Gravitational waves generated by astrophysical sources have roughly $P \sim 10\, R_S$ and $h \sim \epsilon\, R_S/D$ (where $\epsilon$ is the efficiency of the process generating waves, $R_S$ and $D$ are the Schwarzschild radius and the distance of the source). A catastrophic event involving a mass $\sim 10^8\, M_\odot$ gives $P \sim 10^4$ s, $h \sim 10^{-14}$ (assuming $D \sim 100$ Mpc and a generous efficiency $\sim 0.1$), and $\sigma \sim 10^{-18}\ s^{-1}$. These numbers suggest that waves generated by astrophysical sources are too weak to produce appreciable effects. However (as we have noted before) when multiple imaging is involved, considerations based on the optical scalar equations no longer hold, and the images of a light source suffer substantial changes in shape (the giant arcs observed in galaxy clusters, the small elongated images accompanying them, and the radio rings being dramatic examples). Multiple imaging by astrophysically generated gravitational waves cannot be ruled out (see Chapter 4). Unless $P$ is of the order of a few hours (the time scale required for observing faint, distant objects) or larger, the time averaging of the shear makes the effect vanish. So we are left with a situation that probably is very unlikely to occur: multiple imaging must take place, due to a gravitational wave generated by an astrophysical source, in a narrow range of amplitudes and periods. In addition, the observations should take place when the lensing event (which is usually rare and does not last forever) occurs. If, despite the difficulties suggested by these rough considerations, such an event should ever be observed, a region of the sky would appear anisotropic, in the sense that luminous objects would be elongated in one direction and flattened in the perpendicular direction. It would be possible to observe variations in the shape of the images during a single night of observations if the period of the lensing wave is $P \sim 10^4$ s (longer periods are not expected by astrophysical sources).

Another (perhaps more realistic) possibility is that a fluctuation arises in the gravitational wave background, with a well defined direction of propagation and polarization [63]. In such a case, for wavelengths greater than $\sim 10$ pc one would see the instantaneous, and not the time averaged, shear in the shape of distant sources. The effect is inversely proportional to the wavelength of the fluctuation. The simultaneous presence of a certain number of such lenses in the sky would affect the distribution of ellipticities of high redshift galaxies. A similar effect has been considered for ordinary gravitational lenses [75]-[77].

Photon scattering in an anisotropic medium is accompanied by polarization of the scattered photon. The effect of shear due to a long wavelength plane wave is to create an anisotropy in the plane orthogonal to the wave propagation. This phenomenon has been considered for the microwave background [28]. The expected polarization is

$$p \sim \sigma\lambda \sim h \sim \frac{\delta T}{T}$$

(model calculations give $p \sim 0.3\, \delta T/T$). While temperature fluctuations tend to be erased by scattering, polarization requires it. The scale on which the polarization effect is relevant is greater, or of the order of the horizon size at decoupling, since for smaller scales the superposition of many regions with uncorrelated polarizations in the sky will destroy the effect. Observational limits on the polarization anisotropy are comparable to limits on temperature anisotropies, and the upper limits one obtains on $\Omega_{g.w.}$ are comparable with those obtained for the redshift fluctuations $z \sim \delta T/T$ [28].

# Chapter 3

# GAUGE-DEPENDENCE PROBLEMS

## 3.1 Introduction

Among the various effects induced by gravitational waves on the light propagating in them, there is the possibility that they create multiple images of distant light sources, in the same way as mass concentrations perturbing the background curvature of the universe do. This possibility has been suggested by Bertotti ([63]; mentioned in [25]). In order to describe it, one would like to use, as far as possible, the standard formalisms adopted for ordinary gravitational lenses. Here we describe in detail only the vector formalism, which we will actually employ.

The **vector formalism** ([78] and references therein) is the most commonly adopted in the literature on gravitational lenses. It is the most well-suited to treat point-like lenses (e.g. in microlensing), and is sometimes called "Newtonian approach", because it describes deflection of rays with linearized General Relativity and the lens by Newtonian theory. It assumes the following:

- linearized General Relativity;
- geometric optics;
- small-angle scattering;
- transparent, bounded, and stationary lenses;

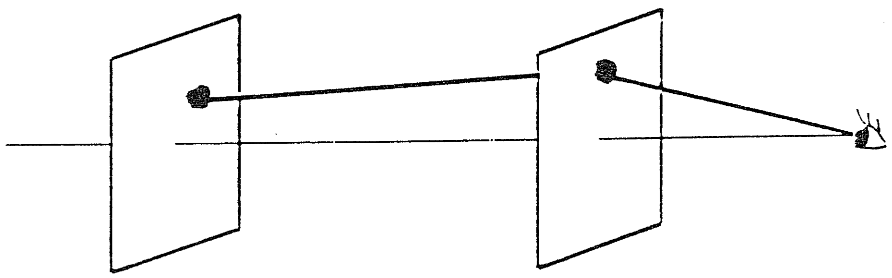


Figure 3.1: The geometry adopted in the vector formalism for ordinary gravitational lenses.

- thin screen approximation.

The propagation of a photon through the universe and the lens field is described by a mapping from the *image plane* (the sky as we see) to the *source plane* (the sky as it would look were the lens absent). The image plane is taken orthogonal to the source-observer line of sight, and passing through the lens; the source plane is analogously defined, passing through the source. Cartesian coordinates $\underline{r} = (x, y)$ and $\underline{s} = (s_x, s_y)$ are used for the image and source positions, respectively. The mapping is defined *from the observer* to the source, because one wants to have a single-valued function even in case of multiple imaging. The origin of coordinates in the sky is determined by a conveniently chosen optic axis. $\underline{s}$ can be defined as the image coordinate in the image plane, were the lens absent (see Fig. 3.1).

Let us consider the linear positions of the source and the image *in the source plane*; they are obtained from the linear positions in the image plane and the angles they subtend, as seen by the observer – see Fig. 3.2, where

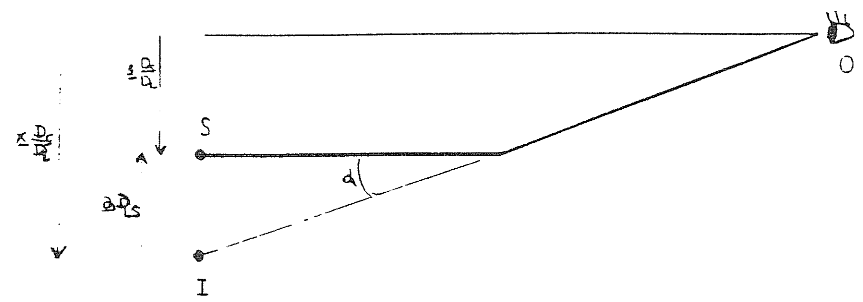


Figure 3.2: A simplified geometry for an heuristic derivation of the lens equation.

$\underline{x}\, D_S/D_L$ image linear position in the source plane;

$\underline{s}\, D_S/D_L$ source linear position in the source plane;

$\underline{\alpha} D_{LS}$ linear deflection in the source plane;

$D_L$ observer-lens distance;

$D_S$ observer-source distance;

$D_{LS}$ lens-source distance.

We get

$$\underline{x}\,\frac{D_S}{D_L} = \underline{s}\,\frac{D_S}{D_L} + (-\underline{\alpha}) D_{LS} \; .$$

Introducing

$$D \equiv \frac{D_L D_{LS}}{D_S} \; , \tag{1.1}$$

we get the **lens equation**

$$\underline{s} = \underline{x} + D\underline{\alpha} \; , \tag{1.2}$$

where the deflection angle $\underline{\alpha}$ can be obtained from linearized General Relativity considering the path of a photon with 4-momentum

$$p^\mu = \frac{dx^\mu}{d\lambda} = (1+\delta p^0, \delta p^1, \delta p^2, 1+\delta p^3) \ ,$$

suffering a net deflection

$$\Delta p^\mu = -\int d\lambda \left( h^\mu{}_{\lambda,\lambda} - \frac{1}{2} h_{\lambda\lambda}{}^{,\mu} \right) = -\frac{2}{c^2} \int d\lambda \, \Phi^{,\mu} + O\left[ (\delta p)^2 \right] \ . \tag{1.3}$$

This gives the bending angle

$$\underline{\alpha} = -\frac{2}{c^2} \int dl \, \underline{\nabla} \Phi \ , \tag{1.4}$$

where the integral is taken along the unperturbed photon's path between the source and the observer. In the approximation employed, $\underline{\alpha}$ is a 2-dimensional vector with no component parallel to the photon's path, and depending on the projected (2-dimensional) structure of the lens. The angular diameter distances in the FLRW universe are used for the $D$'s to describe cosmology. The angular diameter distance between two objects at redshifts $z_A$ and $z_B$ (with $z_A < z_B$) is given by

$$D_{AB} = \frac{2}{H_0} \frac{(1-\Omega_0)(G_A - G_B) + (G_A G_B^2 - G_A^2 G_B)}{\Omega_0^2 (1+z_A)(1+z_B)^2} \tag{1.5}$$

[79], where

$$G_i \equiv (1+\Omega_0 z_i)^{1/2} \quad i = A, B \ ,$$

and $H_0$, $\Omega_0$, and $q_0$ are the present Hubble parameter, density parameter, and deceleration parameter, respectively. Results in gravitational lensing theory are more sensitive to the choice of the lens model than to the choice of $\Omega_0$. A $\Omega_0 = 1$ ($K = 0$) Einstein-De Sitter model is most commonly used in literature. The transformation described by the lens equation has the Jacobian matrix

$$J(\underline{x}) = \frac{\partial \underline{s}}{\partial \underline{x}} \ .$$

Its inverse represents the *amplification tensor*

$$A(\underline{x}) \equiv \frac{\partial \underline{x}}{\partial \underline{s}} \ ,$$

whose determinant is the *scalar amplification*

$$\mathcal{A} \equiv \mathrm{Det}\left(\frac{\partial \underline{x}}{\partial \underline{s}}\right) .$$

The loci of points in the image plane in which $\mathrm{Det}(J) = 0$ are called *critical lines* and the corresponding lines in the source plane are called *caustics*. They separate regions in the source plane corresponding to different numbers of images. As a point source is moved in the source plane, images appear, or disappear, in pairs when the source crosses a caustic. Amplification is virtually infinite on critical lines, but in practice it is limited by wave optics to the values

$$\mathcal{A} \sim \left(\frac{GM}{\lambda c^2}\right)^{1/3} \tag{1.6}$$

on a fold caustic, and

$$\mathcal{A} \sim \left(\frac{GM}{\lambda c^2}\right)^{1/2} \tag{1.7}$$

on a cusp caustic, where $M$ is the lens mass, the intensification becoming chromatic [80]; $\mathcal{A}$ can however be very large. In case of multiple imaging, different images will generally have different amplifications.

### Complex notation

The 2-dimensional nature of the bending angle $\underline{\alpha}$ allows the use of complex notation; $\underline{\alpha}$ is replaced by the complex number $\alpha = \alpha_x + i\alpha_y$ and is completely represented by the *scattering function*

$$I(\underline{x}) = \frac{1}{2G}\left[\int_{-\infty}^{+\infty} \frac{\partial \Phi(\underline{x}, l)}{\partial x}\, dl - i \int_{-\infty}^{+\infty} \frac{\partial \Phi(\underline{x}, l)}{\partial y}\, dl\right] \tag{1.8}$$

via the expression

$$\alpha = -\frac{4G}{c^2} I^* . \tag{1.9}$$

Eqs. (1.9) and (1.4) are equivalent. One makes also the substitutions

$$\underline{x} \equiv (x, y) \mapsto z = x + iy ,$$
$$\underline{s} \equiv (s_x, s_y) \mapsto z_s = s_x + is_y .$$

The scattering function $I(z)$ is analytic outside the matter distribution (as can be proved using the Cauchy-Riemann conditions) and its knowledge is equivalent to that of the projected lensing potential. The lens equation is

$$z_s = z - \frac{4GD}{c^2} I^* \,, \tag{1.10}$$

and can have more than one solution $z$ for a given source position $z_s$ (multiple imaging). The amplification tensor is

$$A = \frac{\partial z}{\partial z_s} = \frac{1}{\mathcal{G}^2 - |\mathcal{F}|^2} \begin{pmatrix} \mathcal{G} + \mathcal{R}e(\mathcal{F}) & -\mathcal{I}m(\mathcal{F}) \\ -\mathcal{I}m(\mathcal{F}) & \mathcal{G} - \mathcal{R}e(\mathcal{F}) \end{pmatrix} \,, \tag{1.11}$$

where

$$\begin{aligned} \mathcal{G} &\equiv 1 - \frac{2GD}{c^2} (\partial_x I + i\partial_y I) \,, \\ \mathcal{F} &\equiv \frac{2GD}{c^2} (\partial_x I - i\partial_y I) \,. \end{aligned}$$

The function $\mathcal{G}$ is real and can be written

$$\mathcal{G} = 1 - \frac{4\pi GD}{c^2} \Sigma(x,y) \equiv 1 - \chi \,, \tag{1.12}$$

where

$$\Sigma(x,y) \equiv \int_{-\infty}^{+\infty} dl \; \rho(x,y,l) \tag{1.13}$$

is the projected surface density of the lens, and $\chi$ (called *convergence*) is the same quantity measured in units of

$$\Sigma_c \equiv \frac{c^2}{4\pi GD} \,. \tag{1.14}$$

We have $\chi \geq 0$ and $\mathcal{G} \leq 1$ . $\mathcal{F}$ is a complex function. Clearly, $\mathcal{G} \to 1$ and $|\mathcal{F}| \to 0$ as $r \to \infty$ . The eigenvalues of the amplification tensor $A$ are

$$\lambda_\pm = \frac{\mathcal{G} \pm |\mathcal{F}|}{\mathcal{G}^2 - |\mathcal{F}|^2} \tag{1.15}$$

and are real. Their product is the amplification of the image located at $z$ resulting from a small source located at $z_s$:

$$\mathcal{A} = \frac{1}{\mathcal{G}^2 - |\mathcal{F}|^2} \,. \tag{1.16}$$

In fact, since surface brightness is conserved by lensing [81], the amplification (ratio of intensities with and without the lens) is simply the ratio

$$\frac{\text{area of an infinitesimal region in the image plane}}{\text{area of the corresponding region in the source plane}} \,.$$

A small disk source will be imaged into an ellipse whose eccentricity $\epsilon$ is given by the ratio of the eigenvalues of $A$

$$(1-\epsilon^2)^{1/2} = \left|\frac{\lambda_+}{\lambda_-}\right| = \left|\frac{\mathcal{G}-|\mathcal{F}|}{\mathcal{G}+|\mathcal{F}|}\right| , \qquad (1.17)$$

while the eigenvectors of $A$ give the orientation of the ellipse.

### Transverse motions of the source or lens

Relative motion of the source and the lens will cause displacements $\delta\underline{x}$ of the image which can be computed using the lens equation, giving

$$\delta\underline{x} = A\,\delta\underline{s} , \qquad (1.18)$$

where $\delta\underline{s}$ is the displacement of the source in the image plane. For source motions it is given by

$$\delta\underline{s} = \frac{D_L}{D_S}\,\underline{v}_S\,\delta t_S ,$$

and for lens motions it is given by

$$\delta\underline{s} = \underline{v}_L\,\delta t_L ,$$

where

$\underline{v}_S$ is the transverse velocity of the source (measured at the emission time $t_S$);

$\underline{v}_L$ is the transverse velocity of the lens measured at the time $t_L$ the light passes the lens;

$\delta t_S$ is the time interval during which the source moves (measured at emission times $t_S$);

$\delta t_L$ is the time interval during which the lens moves (measured at times $t_L$).

An observer measures a time interval $\delta t_O$ during which the image moves. We have

$$\delta t_S = \delta t_O(1+z_S)^{-1}$$

for source motions, and

$$\delta t_L = \delta t_O(1+z_L)^{-1}$$

for lens motions, where $z_S$ and $z_L$ are the source and lens redshifts respectively. For source motions we get

$$\delta \underline{x} = A \frac{D_L}{D_S} \underline{v}_S \, \delta t_O (1 + z_S)^{-1} , \tag{1.19}$$

and for lens motions

$$\delta \underline{x} = A \, \underline{v}_L \, \delta t_O (1 + z_L)^{-1} . \tag{1.20}$$

Realistic sources and lenses are at cosmological distances; however the way in which cosmology is included in the model by using the relativistic angular diameter distance as a distance measure is not satisfactory. It could ultimately reveal to be correct since angular quantities are measured in observations, but it mixes exact relativistic cosmology and linearized theory leaving the more mathematically-minded workers unsatisfied. One would like to have more rigorous bases for gravitational lensing theory. We will not try to tackle this problem because it should be treated in a forthcoming book [82] [1].

Two other formalisms are currently used in gravitational lens theory; the *scalar formalism* is based on Fermat's principle, that holds in stationary spacetimes and can easily be used to treat ordinary gravitational lenses (which are assumed to be time-independent). It allows a topological classification of multiple images using catastrophe optics, and uses angular diameter distances to describe cosmology. This formalism cannot be applied directly to our situation, but a generalization of Fermat principle to nonstationary spacetimes [83, 84] will allow us to describe lensing by gravitational waves (see Chapter 4).

The *propagation formalism* actually collects several approaches which describe the propagation of light in an inhomogeneous spacetime without a well defined lens plane and use the relativistic optical scalar equations (OSE) by Sachs [85] and Penrose [86]. The thin screen and weak field approximations, and the hypothesis of bounded and stationary lens are not needed. This formalism unfortunately does not describe multiple images, but only the amplification

[1] When one is under pressure, it is very depressing and time-consuming to obtain results that one believes to be original but instead are already known, a thing that happened to the author three times while writing this thesis.

of a bundle of null rays, and so is not useful for studying multiple imaging by gravitational waves.

When trying to apply the vector formalism to the problem of lensing by gravitational waves, one immediately encounters a problem: the gravitational wave profiles available in the literature, which one would like to adopt as lens models, are given in the transverse-traceless (TT) gauge. A reference frame can be naturally associated to the TT gauge, but it cannot be realized by means of material bodies. On the other hand the vector formalism singles out a special asymptotically Cartesian coordinate system which represents the sky as seen from the Earth's position (neglecting the Earth's motions). The further introduction of the notion of cosmological distance in some sense transforms to a comoving reference frame in a Friedmann universe (this transformation however suffers from the problems we have pointed out above). This special coordinate system has nothing to do with the frame associated with the TT gauge, and describing gravitational waves in this system appears to be difficult. This problem arises very often when dealing with gravitational wave detection [87]. We will then approach it from a more general point of view than our situation suggests. The following is based on Refs. [87, 88].

## 3.2 The gauge-dependence problem

It is well known that General Relativity has a gauge freedom, that can ultimately be seen as freedom of coordinate transformations. This freedom is usefully employed to simplify calculations in a variety of problems concerning the effects of weak gravitational radiation, by achieving suitable gauges. A situation in which we are interested will be used as an example, namely the frequency perturbation induced by linearized gravitational waves on light in the geometric optics approximation (that we have already considered for exact plane waves in Chapter 1). The fact that light travelling through time fluctuating media undergoes Doppler-like shifts is well known ([89, 90, 91] and references therein) and has been tested with recent laboratory experiments [92, 93]. This effect could be present in photons coming from extragalactic sources or in the microwave background, and could possibly explain redshift anomalies and periodicities [94] in galaxy clusters, galaxy chains and groups, due to the light propagation in a background of cosmological gravitational waves, or in single gravitational pulses, as seen in Chapter 2. Calculations of this effect in the literature consider linearized gravitational waves around a flat, or a $K = 0$ FLRW background, and are commonly performed in particular gauges (the radiation or TT gauge in the former, the synchronous gauge in the latter situation). More generally, the calculation of effects due to small metric perturbations will be more easily performed in a particular gauge; the literature deals extensively with gauge conditions suitable for numerical calculations of emission of gravitational radiation by compact sources (Ref. [95] and references therein), and with non-covariant formalisms used in the analysis of tensor, vector and scalar cosmological perturbations of Friedmann universes [96]. However, adopting a particular gauge means choosing a particular coordinate system "adapted" to the problem (in our case one with coordinates "following" the gravitational wave), so that the results will be valid in that particular system only, unless they refer to scalar quantities. Alternatively one can perform calculations in a coordinate system that seems physically more appropriate than others, like a freely-falling (geodesic) frame, or an asymptotically Cartesian frame. In fact physically meaningful observers are naturally associated with such systems. Usually, if $h_{\mu\nu}$ are the (small) metric perturbations, one performs calculations at order $O(h^n)$ for some $n$, and the gauge-dependent part of the results is of order $m$, completely negligible if $m > n$; however this is not always the case, as we are going to show.

### 3.2.1 Linearized G. R. and the radiation gauge

Let us consider small perturbations of Minkowski spacetime; the metric is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \; , \tag{2.1}$$

where we assume the existence of an asymptotically Cartesian coordinate system $\{x^\mu\}$, $\eta_{\mu\nu}$ are the components the Minkowski metric, and $|h_{\mu\nu}| \ll 1$ . We raise and lower tensor indices with $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$. The linearized vacuum Einstein equations turn out to be

$$\partial^\rho \partial_\rho \bar{h}_{\mu\nu} = 0 \; , \tag{2.2}$$

where

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\, h^{\rho}{}_{\rho} \,, \tag{2.3}$$

and with the gauge choice

$$\partial^{\nu}\bar{h}_{\mu\nu} = 0 \,, \tag{2.4}$$

which is obtained by performing the gauge transformation

$$h_{\mu\nu} \longrightarrow h_{\mu\nu} + \partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu} \,, \tag{2.5}$$

where $\xi^{\mu}$ is a vector field solution of the equation

$$\partial^{\nu}\partial_{\nu}\xi_{\mu} = -\partial^{\nu}\bar{h}_{\mu\nu} \,. \tag{2.6}$$

One can perform a further gauge transformation Eq. (2.5), provided that

$$\partial^{\nu}\partial_{\nu}\xi^{\mu} = 0 \,.$$

This restricted gauge freedom can be employed to achieve the *radiation gauge*

$$h^{\mu}{}_{\mu} = 0 \,,$$

$$h_{0i} = 0$$

$(i = 1,2,3)$, from which we also obtain $h_{00} = 0$. How do we achieve this gauge? We solve on an initial time surface $t = t_0$ the equations

$$-\frac{\partial\xi_0}{\partial t} + \underline{\nabla}\cdot\underline{\xi} = -h^{\mu}{}_{\mu}/2 \,, \tag{2.7}$$

$$-\nabla^2\xi_0 + \underline{\nabla}\cdot\left(\frac{\partial\underline{\xi}}{\partial t}\right) = -\frac{1}{2}\frac{\partial h^{\mu}{}_{\mu}}{\partial t} \,, \tag{2.8}$$

$$\frac{\partial\xi_i}{\partial t} + \frac{\partial\xi_0}{\partial x^i} = -h_{0i} \,, \tag{2.9}$$

$$\nabla^2\xi_i + \frac{\partial}{\partial x^i}\left(\frac{\partial\xi_0}{\partial t}\right) = -\frac{\partial h_{0i}}{\partial t} \,, \tag{2.10}$$

where $i = 1,2,3$ and $\underline{\xi} = (\xi_1, \xi_2, \xi_3)$, to obtain the initial values $\xi_{\mu}^{(0)}$, $\partial\xi_{\mu}^{(0)}/\partial t$ (on the $t = t_0$ surface). Then we define the vector field $\xi^{\mu}$ to be the solution of Eqs. (2.7)–(2.10) with these initial data. In the coordinate system that achieves the radiation gauge ("R. G. system") one has

$$h^{\mu}{}_{\mu} = \frac{\partial h^{\mu}{}_{\mu}}{\partial t} = h_{0i} = \frac{\partial h_{0i}}{\partial t} = 0$$

($i = 1, 2, 3$) on the initial surface. The theory of the initial value problem for hyperbolic partial differential equations guarantees that $h^\mu{}_\mu$ and $h_{0i}$ vanish everywhere in the source-free region. Moreover, Eq. (2.4) gives

$$\partial_t h_{00} = 0 \ ,$$

and

$$\partial^\mu \partial_\mu h_{00} = 0$$

gives

$$\nabla^2 h_{00} = 0 \ .$$

Together these imply $h_{00} = 0$. So far this is standard theory (see e.g. Ref. [14]). The gauge transformation Eq. (2.5) can be obtained by

$$x^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu \ ,$$

where the order of magnitude of $\xi^\mu$ is clearly related to the smallness of $h_{\mu\nu}$; Eqs. (2.7)-(2.10) give $\partial_\mu \xi_\nu = O(h)$. The transformation equations

$$x'^\mu = x^\mu + \xi^\mu(x^\alpha) \ ,$$

$$x^\mu = x'^\mu - \xi^\mu(x^\alpha) \simeq x'^\mu - \xi^\mu(x'^\alpha)$$

give

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu + \frac{\partial \xi^\mu}{\partial x^\nu} \ ,$$
$$\frac{\partial x^\mu}{\partial x'^\nu} = \delta^\mu_\nu - \frac{\partial \xi^\mu}{\partial x'^\nu} \ .$$

Let us consider now a tensor quantity with components $A^{\mu_1 \mu_2 \cdots}{}_{\nu_1 \nu_2 \ldots} = O(h^n)$ in the asymptotically Cartesian ("A. C. ") system. Its components in the R. G. system will be

$$\begin{aligned} & A'^{\mu_1 \mu_2 \cdots}{}_{\nu_1 \nu_2 \ldots} = \\ & = \left(\delta^{\mu_1}_{\alpha_1} + \frac{\partial \xi^{\mu_1}}{\partial x^{\alpha_1}}\right) \left(\delta^{\mu_2}_{\alpha_2} + \frac{\partial \xi^{\mu_2}}{\partial x^{\alpha_2}}\right) \cdots \left(\delta^{\beta_1}_{\nu_1} - \frac{\partial \xi^{\beta_1}}{\partial x^{\nu_1}}\right) \left(\delta^{\beta_2}_{\nu_2} - \frac{\partial \xi^{\beta_2}}{\partial x^{\nu_2}}\right) \cdots A^{\alpha_1 \alpha_2 \cdots}{}_{\beta_1 \beta_2 \ldots} = \\ & = A^{\mu_1 \mu_2 \cdots}{}_{\nu_1 \nu_2 \ldots} + O(h^{n+1}) \ . \end{aligned}$$

The metric perturbations $h_{\mu\nu}$ clearly are not components of a tensor, since they do not satisfy the previous equation; this is evident from Eq. (2.5). In fact the metric perturbations in the R. G. system are not obtained from those in the A. C. system via the tensor transformation law for $h_{\mu\nu}$, but are *defined* according to the transformation of the *complete* metric tensor (that is a real tensor) as deviations from the Minkowski metric in the R. G. system. In fact we have

$$g'_{\mu\nu} = g_{\mu\nu} - \frac{\partial \xi_\mu}{\partial x^\nu} - \frac{\partial \xi_\nu}{\partial x^\mu} + O(h^2) \ ,$$

which gives

$$h'_{\mu\nu} \equiv h_{\mu\nu} - \left( \frac{\partial \xi_\mu}{\partial x^\nu} + \frac{\partial \xi_\nu}{\partial x^\mu} \right) \ .$$

Here the term in brackets is of order $h$ (the difference in the sign of $\xi^\mu$ with respect to Eq. (2.5) is not relevant, since it can be absorbed in the definition of the vector field $\xi^\mu$ via Eqs. (2.7)–(2.10)). The point here is that $\eta_{\mu\nu}$ and $h_{\mu\nu}$ are not tensors, except with respect to Lorentz transformations, and the decomposition Eq. (1.3) is meaningful only in a given coordinate system (the A. C. system).

Note that the definition of the metric perturbations in the R. G. system does not change their smallness:

$$O(h'_{\mu\nu}) = O(h_{\mu\nu}) \ .$$

In conclusion, if we are performing calculations to order $h^n$, we can forget the problem of the gauge-dependence of the results only for real tensor quantities of order $n$, not for the metric perturbations, or quantities constructed from them.

The relation between the derivatives of the metric perturbations in the R. G. system and in the A. C. system will be useful in the following:

$$\begin{aligned} \frac{\partial h'_{\mu\nu}}{\partial x'^\alpha} &= \frac{\partial h'_{\mu\nu}}{\partial x^\beta}\frac{\partial x^\beta}{\partial x'^\alpha} = \\ &= (\delta^\beta_\alpha - \frac{\partial \xi^\beta}{\partial x'^\alpha})\frac{\partial}{\partial x^\beta}(h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu) = \\ &= \frac{\partial h_{\mu\nu}}{\partial x^\alpha} + \underbrace{\partial_\alpha \partial_\mu \xi_\nu + \partial_\alpha \partial_\nu \xi_\mu}_{O(h)} + O(h^2) \ , \end{aligned} \tag{2.11}$$

where Eq. (2.5) has been used: note that the difference between the two derivatives is of order $h$.

### 3.2.2 The frequency shift effect induced by linearized gravitational waves around a flat background

As already remarked, the basic fact that light travelling through a fluctuating medium undergoes Doppler-like shifts is well known [89, 90, 91], and is astrophysically important when the fluctuating medium is composed of gravitational waves, since it could explain redshift anomalies and periodicities [94] in galaxy clusters, galaxy chains and groups (see Chapter 2). It seems reasonable to believe that, if the present programs to detect gravitational waves are successful, the very existence of a gravitational wave background implies the observation of a non-cosmological frequency shift, at least on certain scales, in a not too far future. Moreover, the phenomenon of frequency shift can be tested with laboratory experiments [92], and the interesting possibility of connecting them with extragalactic observations has been suggested [93].

Let us consider linearized gravitational waves around a flat background, an asymptotically Cartesian system $\{x^\mu\}$, and a light ray whose unperturbed path is parallel to the $x$-axis. The photon describing this ray has 4-momentum

$$p^\mu = (1+\delta p^0, 1+\delta p^1, \delta p^2, \delta p^3) ,$$

where $\delta p^\mu$ are small deflections of order $h_{\mu\nu}$, and the unperturbed photon's 4-momentum is $p^\mu_{(0)} = (1,1,0,0)$. We work in the geometric optics approximation and compute the frequency shift effect caused by gravitational waves to first order. The null geodesic equation gives

$$\delta p^0 = -\int_S^O dx \left[\frac{1}{2}(h_{11}-h_{00})_{,0} - (h_{00}+h_{01})_{,1}\right] + O(h^2) , \qquad (2.12)$$

where the integral is computed along the *unperturbed* photon's path from the source to the observer (computing it along the perturbed path simply adds a correction of order $h^2$). Let us consider now an observer and a light source initially at rest in the A. C. system and freely-falling, with 4-velocities

$$\begin{aligned} u^\mu &= u^\mu_{(0)} + \delta u^\mu = (1+\delta u^0, \delta u^1, \delta u^2, \delta u^3) , \\ v^\mu &= v^\mu_{(0)} + \delta v^\mu = (1+\delta v^0, \delta v^1, \delta v^2, \delta v^3) , \end{aligned}$$

respectively. We get (using also the normalizations $u_\nu u^\nu = v_\nu v^\nu = -1$) the angular frequency received by the observer

$$\omega_O = -p_\mu u^\mu = 1 + \delta p^0(O) - \delta u^1 - [h_{00}/2 + h_{01}]_O + O(h^2) , \qquad (2.13)$$

and the angular frequency emitted by the source

$$\omega_S = -p_\mu v^\mu = 1 + \delta p^0(S) - \delta v^1 - [h_{00}/2 + h_{01}]_S + O(h^2) \ . \tag{2.14}$$

The frequency perturbation induced by the gravitational waves is described by the redshift parameter

$$z \equiv \frac{\omega_O}{\omega_S} - 1 = (\delta v^1 - \delta u^1) - \int_S^O dx \left[ \frac{1}{2} (h_{11} - h_{00})_{,0} - (h_{00} + h_{01})_{,1} \right] + [h_{00}/2 + h_{01}]_S^O + O(h^2) \ . \tag{2.15}$$

Moreover,

$$\delta u^\mu = \int d\tau \left( h^\mu{}_{0,0} - \frac{1}{2} h_{00}{}^{,\mu} \right) + O(h^2) \ , \tag{2.16}$$

where $\tau$ is the proper time of the observer, and the integral is along the observer's worldline. An analogous formula holds for $\delta v^\mu$. In the R. G. system $\delta u^1 = \delta v^1 = 0$. This tells us that the coordinate system achieving the radiation gauge "follows" the oscillations between the source and the observer, so that they see each other at a fixed distance, and consequently there is no Doppler effect, i.e. no first term in Eq. (2.15). Since $z$ is a scalar, we can compute it in the R. G. system:

$$z = -\frac{1}{2} \int_S^O dx \, h_{11,0} \, |_{RG} + O(h^2) \ . \tag{2.17}$$

We cannot exchange the values of $\delta u^\mu$, $\delta v^\mu$ and $h_{\mu\nu}$ in the R. G. system with those in the A. C. system, since $\delta u^\mu$ and $\delta v^\mu$ are not real vectors (considerations analogous to those for $h_{\mu\nu}$ apply) and $h_{\mu\nu}$ is not a real tensor.

We regard the A. C. system as physically more meaningful than the R. G. system (which is in some sense "adapted" to the gravitational waves), and consequently we keep Eq. (2.15), with quantities computed in the A. C. system as our final result. This is because we choose as fundamental observers the inertial observers of the flat background. Consider for example the situation in which a gravitational wave comes from a distant source. Before the wave arrives, we refer to the inertial observers that see the light source at rest in their own coordinate system ($v^\mu = \delta^{0\mu}$); the subsequent evolution of these observers can be computed from Eq. (2.16).

Moreover, the notions of the unperturbed photon path and its deflection, and the integrals on this path in Eqs. (2.15), (2.17) are well defined in the A. C.

system, while they are not intuitive in the R. G. system. However the literature on the subject uses the expression in the radiation gauge [61, 97], being rather ambiguous on such concepts. The order of magnitude of the results is the same in the two coordinate systems, but their expressions and values are different.

### 3.2.3 The frequency shift effect induced by linearized gravitational waves around a FLRW background

Linearized gravitational waves around a FLRW background with curvature index $K = 0$, and the frequency perturbation effect have been considered by Dautcourt [46] and Linder [49], using the synchronous gauge [98]

$$\begin{aligned} g_{00} &= -1 , \\ g_{0i} &= 0 , \\ g_{ij} &= a^2(t)\,(\delta_{ij} + h_{ij}) , \end{aligned} \tag{2.18}$$

in $(t, x, y, z)$ coordinates, where $a(t)$ is the scale factor. They derive expressions for the angular deflection and frequency perturbation suffered by a light ray travelling near the $x$-axis and propagating through gravitational waves. The redshift parameter is [49]

$$z = \frac{a(t_O)}{a(t_E)} - 1 - \frac{1}{2}\,\frac{a(t_O)}{a(t_E)} \int_{t_E}^{t_O} dt\; h_{11,0} , \tag{2.19}$$

where $t_O$ and $t_E$ are the observation and emission times, and the integral must be computed in the synchronous gauge. When the universe does not expand ($a = 1$), we recover the result already found in the radiation gauge for weak perturbations of Minkowski spacetime. Again, the meaning of the integral in Eq. (2.19) and its evaluation in the synchronous gauge are rather ambiguous for a physical observer whose unperturbed worldline coincides with the average motion of matter (plus eventually a known peculiar velocity). Criticism of the use of the synchronous gauge when considering tensor (vector and scalar) perturbations in a cosmological context can be found in Ref. [96].

### 3.2.4 Remarks

We pointed out the difficulty of comparing the eventual outcoming of observations with the results of calculations available in the literature. This is due to the fact that these are always performed in the radiation gauge, when considering perturbations of a flat background (eventually relevant for laboratory experiments), or in the synchronous gauge when dealing with perturbations of a FLRW universe (relevant for astrophysical applications). The problem is substantially that realistic measurements are not performed in the coordinate systems achieving the radiation or synchronous gauge, while the literature gives results in terms of quantities computed in such coordinate systems, where their evaluation (and sometimes even their physical meaning) is unclear, since these gauge choices greatly simplify the calculations. In other words, the observers considered in the literature are not clearly specified, while physically meaningful observers are naturally associated, for example, with an asymptotically Cartesian system. The frequency shift effect is important in itself because of its astrophysical implications (redshift anomalies and periodicities), and for the possible connection with laboratory experiments. Our problem however is not limited to the frequency shift effect on light travelling through gravitational waves, but is more general: one expects to face it whenever a situation of theoretical or experimental interest, in which weak (that is realistic, in our neighborhood) gravitational radiation is involved, and the final results are expressed in terms of quantities evaluated in such gauges. As we pointed out, however, exchanging the values of the relevant quantities in the particular gauges we considered with those in the physical observers' coordinate systems does not change the order of magnitude of the results.

From the previous considerations it appears that the gauge dependence problem is somewhat obscure. We present now a more concrete application of the previous considerations to the problem of gravitational wave detection.

## 3.3 Theoretical problems on gravitational wave detectors

As seen before, in General Relativity there are situations in which one cannot, in practice, avoid the choice of a particular coordinate system in which to perform calculations. This situation occurs when dealing with experimental gravitation. Here we consider the detection of gravitational waves, for which a number of coordinate systems and gauge choices can be found in the literature; among them, the TT gauge is widely used. Theoretical works related to electromagnetic (e.m.) detectors claimed that Fermi normal coordinates (FNC), instead of the usual TT-coordinates, must be used when describing the interaction of gravitational waves with the e.m. field in the detector [99, 100]. Actually, in general, a coordinate system has no physical meaning in itself, being only a chart on the spacetime manifold; one needs to specify a *family of observers*, rather than a coordinate system. For the particular case of FNC however, the coordinate system is associated with a well-defined observer, that turns out to be the physical observer making experiments with the detector, provided that this has size much smaller than the wavelength of the gravitational waves to be detected (we will limit our considerations to this case for the moment). On the contrary, it seems that the physical meaning of the observers associated with the TT system ("TT observers") has never been investigated. Here we will try to remedy this lack, by describing (to first order in the gravitational wave amplitudes) how they are accelerated, or rotated with respect to a freely falling (geodesic) observer in the field of the gravitational wave. It turns out that the TT observers undergo very complicated motions with respect to freely falling ones, as one could expect (and infer from Ref. [100]), since TT coordinates are in some sense "adapted" to the gravitational wave. We show that the metric perturbations describing the incoming waves (that, by definition, are not gauge-invariant) differ in the TT and in the FNC systems not only in their functional dependence on the coordinates, but also by a factor $(l/\lambda)^2$, where $l$ is the size of the detector, and $\lambda$ the wavelength of gravitational radiation. This fact is well known in the theory of detectors, but is particularly clear when seen using the geometric construction of FNC. As a consequence, the Maxwell equations in the field of the gravitational wave, when specified to a FNC system, show clearly the way in which e.m. detectors operate, and their difference with respect to mechanical detectors. Finally, we consider detectors with $l/\lambda \geq 1$, and point out the difficulty of giving a precise definition of the physical family of observers carrying out measurements in this case.

## 3.3.1 Detectors with $l/\lambda \ll 1$

### The TT and FNC systems

Until explicitly specified, we will always limit our considerations to a detector with size $l$ much smaller than the wavelength $\lambda$ of the gravitational radiation to be detected (we assume that it can be characterized by a single wavelength; in most cases real detectors are sensitive only to waves whose wavelengths are close to the resonance values. Wavelengths much smaller or larger than these will not influence the detector). This is described, in any coordinate system, by the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{3.1}$$

and by linearized theory.

As remarked in Ref. [100], the interaction between gravitational waves and the parts of a *mechanical* detector are described by the curvature tensor $R_{\mu\nu\rho\sigma} = O(h)$ (since such a device essentially measures tidal accelerations due to the waves) that, being a real tensor of order 1, is gauge independent to first order, according to the previous considerations.

The interaction of a gravitational wave with the e.m. field of an e.m. detector is described by the metric perturbations $h_{\mu\nu}$. According to Ref. [100], this fact obliges us to consider the problem of the coordinate system for e.m. detectors, while it is unimportant for mechanical detectors. In our opinion the problem is conceptually important for both kinds of devices, since one needs to specify unambiguously the observers performing measurements.

Let us consider now the construction of a FNC system ([102]; [155] pp. 327-332): a sufficiently small ($l/\lambda \ll 1$) detector can be described by the worldline of its centre of mass. We assume that no force other than the field of the gravitational wave acts on the detector (any other known force can, in principle, be taken into account separately), so that its worldline is a timelike geodesic $\Gamma$, the worldline of the "FNC-observer". We choose his proper time $\tau$ as an affine parameter along $\Gamma$. From any point $P_0$ on $\Gamma$ depart spacelike geodesics orthogonal to $\Gamma$ (i.e. their tangent vectors are orthogonal to the tangent vector $e^a_{(0)}$ to $\Gamma$, the 4-velocity of the FNC-observer, in $P_0$). We choose three such geodesics (that can be fixed using gyroscopes) with unit tangent vectors $e^a_{(1)}, e^a_{(2)}, e^a_{(3)}$. Then the tetrad $\left\{e^a_{(0)}, e^a_{(1)}, e^a_{(2)}, e^a_{(3)}\right\} \equiv \left\{e^a_{(\mu)}\right\}$ satisfies

$$\begin{aligned} g_{ab}e^a_{(0)}e^b_{(0)} &= -1 \ , \\ g_{ab}e^a_{(0)}e^b_{(i)} &= 0 \ , \\ g_{ab}e^a_{(i)}e^b_{(j)} &= \delta_{ij} \ , \end{aligned}$$

i.e. it is a orthonormal tetrad in $P_0$. We can define it at any other point $P$ on $\Gamma$ by parallel transport:

$$e^a_{(0)}\nabla_a e^b_{(\mu)} = 0 \ .$$

Adopting this tetrad as a vector basis, the Fermi normal coordinates of a point $Q$ near $\Gamma$ are given by $\{x^\mu\} = \{\tau, s^i\}$, where $s^i$ is the proper length of the i-th spacelike geodesic from $P$ to $Q$. Far away from $P$, FNC are not defined, since the spacelike geodesics can cross, due to the curvature of spacetime. The transformation equations from FNC to another, general, coordinate system, to first order, are derived in Ref. [100] (see also Refs. [99, 101]). Other versions of Fermi coordinates for an observer accelerating and/or rotating with respect to a freely falling one have been considered in the literature [102, 155]. The metric to first order for these other versions of Fermi systems is given in Ref. [104] (see also Refs. [24, 155]). We will use the term FNC only for the system constructed above.

### The physical meaning of the TT system

In order to explore the physical meaning of the TT observers, we consider a particularly simple example: a plane monochromatic gravitational wave propagating in the direction of the $y^3$-axis in the TT coordinate system $\{y^\mu\}$. The TT components of the wave 4-vector are $k^\mu = (k, 0, 0, k)$. In the TT gauge the only nonzero wave amplitudes are

$$\begin{aligned} h_{11} &= -h_{22} = A_+ \sin\left[k\left(y^3 - y^0\right)\right] , \\ h_{12} &= h_{21} = A_\times \sin\left[k\left(y^3 - y^0\right)\right] . \end{aligned}$$

For calculation economics with this wave, the TT gauge is clearly the most convenient, since the metric components are simplified to the maximum possible extent. This happens because TT coordinates are in some sense "adapted" to the wave; note that they satisfy [105] the wave equation

$$\nabla^a \nabla_a y^\mu = 0 .$$

We consider a family of TT observers, whose 4-velocity field $u^a$ (a geometrical object, defined in a coordinate-independent way) has TT components $u^{(TT)\mu} = \delta^{0\mu}$ (from which $u^{(TT)}_\mu = -\delta_{0\mu}$). Its components in the FNC system $\{x^\mu\}$ are given by

$$u^{(FNC)\mu} = g^{(FNC)\mu\nu} u^{(FNC)}_\nu , \tag{3.2}$$

where

$$u^{(FNC)}_\mu = \frac{\partial y^\nu}{\partial x^\mu} u^{(TT)}_\nu = -\frac{\partial y^0}{\partial x^\mu} . \tag{3.3}$$

The transformation equations from the FNC system $\{x^\mu\}$ to the TT system $\{y^\mu\}$ are derived in Refs. [100, 99, 101]:

$$\begin{aligned} y^0(x^\alpha) &= x^0 - \frac{A_+\left[(x^1)^2 - (x^2)^2\right] + 2A_\times x^1 x^2}{2x^3} \cdot \left[\frac{\cos(ku) - \cos(kx^0)}{kx^3} - \sin(kx^0)\right] , \\ y^1(x^\alpha) &= x^1 + \left(A_+ x^1 + A_\times x^2\right)\left[\frac{\cos(ku) - \cos(kx^0)}{kx^3} - \frac{\sin(kx^0)}{2}\right] , \\ y^2(x^\alpha) &= x^2 + \left(A_\times x^1 - A_+ x^2\right)\left[\frac{\cos(ku) - \cos(kx^0)}{kx^3} - \frac{\sin(kx^0)}{2}\right] , \\ y^3(x^\alpha) &= x^3 - \frac{A_+\left[(x^1)^2 - (x^2)^2\right] + 2A_\times x^1 x^2}{2x^3} \cdot \left[\frac{\cos(ku) - \cos(kx^0)}{kx^3} - \sin(kx^0)\right] \end{aligned}$$

(where $u \equiv y^3 - y^0 = x^3 - x^0$) for $x^3 \neq 0$, and

$$
\begin{aligned}
y^0 &= x^0 + \frac{k}{4}\cos(kx^0)\left\{A_+\left[(x^1)^2 - (x^2)^2\right] + 2A_\times x^1 x^2\right\} , \\
y^1 &= x^1 + \frac{1}{2}\sin(kx^0)\left(A_+ x^1 + A_\times x^2\right) , \\
y^2 &= x^2 + \frac{1}{2}\sin(kx^0)\left(A_\times x^1 - A_+ x^2\right) , \\
y^3 &= \frac{k}{4}\cos(kx^0)\left\{A_+\left[(x^1)^2 - (x^2)^2\right] + 2A_\times x^1 x^2\right\}
\end{aligned}
$$

for $x^3 = 0$. From these one obtains the derivatives $\partial y^\alpha / \partial x^\beta$, and

$$
\begin{aligned}
u_0^{(FNC)} &= -1 + \frac{A_+\left[(x^1)^2 - (x^2)^2\right] + 2A_\times x^1 x^2}{2x^3} \cdot \left[\frac{\sin(ku) + \sin(kx^0)}{x^3} - k\cos(kx^0)\right] , \\
u_1^{(FNC)} &= \frac{A_+ x^1 + A_\times x^2}{x^3} \cdot \left[\frac{\cos(ku) - \cos(kx^0)}{kx^3} - \sin(kx^0)\right] , \\
u_2^{(FNC)} &= \frac{A_\times x^1 - A_+ x^2}{x^3} \cdot \left[\frac{\cos(ku) - \cos(kx^0)}{kx^3} - \sin(kx^0)\right] , \\
u_3^{(FNC)} &= \frac{A_+\left[(x^1)^2 - (x^2)^2\right] + 2A_\times x^1 x^2}{2(x^3)^2} \cdot \left[\frac{2\cos(kx^0) - 2\cos(ku)}{2(x^3)^2} + \sin(kx^0) - \sin(ku)\right]
\end{aligned}
$$

The metric tensor in the FNC system is given in Refs. [99, 101]:

$$
\begin{aligned}
h_{00}^{(FNC)} &= K_{0l0m} x^l x^m \left[\cos(kx^0)\,\frac{kx^3 - \sin(kx^3)}{(kx^3)^2} + \sin(kx^0)\,\frac{\cos(kx^3) - 1}{(kx^3)^2}\right] , \\
h_{0i}^{(FNC)} &= K_{il0m} x^l x^m \left\{\cos(kx^0)\left[\frac{kx^3 - \sin(kx^3)}{(kx^3)^2} - \frac{\cos(kx^3) - 1 + (kx^3)^2/2}{(kx^3)^3}\right] + \right. \\
&\quad \left. + \sin(kx^0)\left[\frac{\cos(kx^3) - 1}{(kx^3)^2} - \frac{\sin(kx^3) - kx^3}{(kx^3)^3}\right]\right\} , \\
h_{ij}^{(FNC)} &= K_{iljm} x^l x^m \left\{\cos(kx^0)\left[\frac{2}{(kx^3)^2} - \frac{\sin(kx^3)}{(kx^3)^2} - 2\,\frac{\cos(kx^3)}{(kx^3)^3}\right] + \right. \\
&\quad \left. + \sin(kx^0)\,\frac{kx^3\cos(kx^3) - 2\sin(kx^3) + kx^3}{(kx^3)^3}\right\} ,
\end{aligned}
$$

where

$$
K_{0l0m} x^l x^m = k^2\left\{A_+\left[(x^2)^2 - (x^1)^2\right] - 2A_\times x^1 x^2\right\} ,
$$

$$\begin{aligned}
K_{il0m}x^l x^m &= k^2\left\{A_+\left[-x^1x^3\delta_i^1+(x^1)^2\delta_i^3+x^2x^3\delta_i^2-(x^2)^2\delta_i^3\right]+\right.\\
&\quad\left.+\,A_\times\left(-x^2x^3\delta_i^1-x^1x^3\delta_i^2+2x^1x^2\delta_i^3\right)\right\}\ ,\\
K_{iljm}x^l x^m &= k^2\left\{A_+\left[(\delta_i^2\delta_j^2-\delta_i^1\delta_j^1)(x^3)^2-x^2x^3(\delta_i^2\delta_j^3+\delta_j^2\delta_i^3)+\right.\right.\\
&\quad+x^1x^3(\delta_i^1\delta_j^3+\delta_j^1\delta_i^3)+\delta_i^3\delta_j^3\left[(x^2)^2-(x^1)^2\right]+\\
&\quad+A_\times\left[-(\delta_i^1\delta_j^2+\delta_i^2\delta_j^1)(x^3)^2+x^2x^3(\delta_i^3\delta_j^1+\delta_i^1\delta_j^3)+\right.\\
&\quad\left.\left.+x^1x^3(\delta_i^3\delta_j^2+\delta_i^2\delta_j^3)-2x^1x^2\delta_i^3\delta_j^3\right]\right\}\ .
\end{aligned}$$

Now, the inverse metric tensor is given, to first order, by

$$g^{(FNC)\mu\nu}=\eta^{\mu\nu}-h^{(FNC)\mu\nu}$$

(where $h^{(FNC)\mu\nu}\equiv\eta^{\mu\alpha}\eta^{\nu\beta}h^{(FNC)}_{\alpha\beta}$), that gives

$$\begin{aligned}
g^{00} &= -1-h^{(FNC)}_{00}\ ,\\
g^{11} &= 1-h^{(FNC)}_{11}\ ,\\
g^{22} &= 1-h^{(FNC)}_{22}\ ,\\
g^{33} &= 1-h^{(FNC)}_{33}\ ,\\
g^{01} &= g^{10}=h^{(FNC)}_{01}\ ,\\
g^{02} &= g^{20}=h^{(FNC)}_{02}\ ,\\
g^{03} &= g^{30}=h^{(FNC)}_{03}\ ,\\
g^{12} &= g^{21}=-h^{(FNC)}_{12}\ ,\\
g^{13} &= g^{31}=-h^{(FNC)}_{13}\ ,\\
g^{23} &= g^{32}=-h^{(FNC)}_{23}
\end{aligned}$$

in the FNC system. From Eq. (3.2) we get

$$\begin{aligned}
u^{(FNC)0} &= g^{00}u_0^{(FNC)}+O(2)=\\
&= 1+\left\{A_+\left[(x^2)^2-(x^1)^2\right]-2A_\times x^1x^2\right\}\cdot\\
&\quad\cdot\left\{\left[\frac{\sin(ku)+\sin(kx^0)}{x^3}-k\cos(kx^0)\right]+\right.\\
&\quad\left.+k^2\left[\cos(kx^0)\,\frac{kx^3-\sin(kx^3)}{(kx^3)^2}+\sin(kx^0)\,\frac{\cos(kx^3)-1}{(kx^3)^2}\right]\right\}\ ,
\end{aligned}$$

$$
\begin{aligned}
u^{(FNC)1} &= g^{10}u_0^{(FNC)} + g^{11}u_1^{(FNC)} + O(2) = \left(A_+ x^1 + A_\times x^2\right)\cdot \\
&\cdot\left\{k^2 x^3 \cos(kx^0)\left[\frac{kx^3 - \sin(kx^3)}{(kx^3)^2} - \frac{\cos(kx^3) - 1 + (kx^3)^2/2}{(kx^3)^3}\right] + \right.\\
&+k^2 x^3 \sin(kx^0)\left[\frac{\cos(kx^3) - 1}{(kx^3)^2} - \frac{\sin(kx^3) - kx^3}{(kx^3)^2}\right] + \\
&\left. +\frac{1}{x^3}\left[\frac{\cos(ku) - \cos(kx^0)}{kx^3} - \sin(kx^0)\right]\right\}\ , \\
u^{(FNC)2} &= g^{02}u_0^{(FNC)} + g^{22}u_2^{(FNC)} + O(2) = \left(A_\times x^1 - A_+ x^2\right)\cdot \\
&\cdot\left\{k^2 x^3 \cos(kx^0)\left[\frac{kx^3 - \sin(kx^3)}{(kx^3)^2} - \frac{\cos(kx^3) - 1 + (kx^3)^2/2}{(kx^3)^3}\right] + \right.\\
&+ k^2 x^3 \sin(kx^0)\left[\frac{\cos(kx^3) - 1}{(kx^3)^2} - \frac{\sin(kx^3) - kx^3}{(kx^3)^3}\right] + \\
&\left. +\frac{1}{x^3}\left[\frac{\cos(ku) - \cos(kx^0)}{kx^3} - \sin(kx^0)\right]\right\}\ , \\
u^{(FNC)3} &= g^{30}u_0^{(FNC)} + g^{33}u_3^{(FNC)} + O(2) = \left\{A_+\left[(x^1)^2 - (x^2)^2\right] + 2A_\times x^1 x^2\right\}\cdot \\
&\cdot\left\{-k^2 \cos(kx^0)\left[\frac{kx^3 - \sin(kx^3)}{(kx^3)^2} - \frac{\cos(kx^3) - 1 + (kx^3)^2/2}{(kx^3)^3}\right] - \right.\\
&- k^2 \sin(kx^0)\left[\frac{\cos(kx^3) - 1}{(kx^3)^2} - \frac{\sin(kx^3) - kx^3}{(kx^3)^3}\right] + \\
&\left. +\frac{1}{2(x^3)^2}\left[\frac{2\cos(kx^0) - 2\cos(ku)}{kx^3} + \sin(kx^0)\right]\right\}\ .
\end{aligned}
$$

The 4-acceleration field of the TT-observers in the FNC system is given by

$$
a^\mu = e^a_{(0)}\,\nabla_a\, u^{(FNC)\mu} = \partial_0 u^{(FNC)\mu}\ ,
$$

that is

$$
\begin{aligned}
a^0 &= \left\{A_+\left[(x^2)^2 - (x^1)^2\right] - 2A_\times x^1 x^2\right\}\cdot\left\{\frac{k\cos(kx^0) - k\cos(ku)}{x^3} + k^2\sin(kx^0) - \right.\\
&\left. -k\sin(kx^0)\,\frac{kx^3 - \sin(kx^3)}{(x^3)^2} + k\cos(kx^0)\,\frac{\cos(kx^3) - 1}{(x^3)^2}\right\}\ , \\
a^1 &= \left(A_+ x^1 + A_\times x^2\right)\cdot\left\{-k^3 x^3 \sin(kx^0)\left[\frac{kx^3 - \sin(kx^3)}{(kx^3)^2} - \right.\right.\\
&\left. -\frac{\cos(kx^3 - 1 + (kx^3)^2/2}{(kx^3)^2}\right] + k^2 x^3 \cos(kx^0)\left[\frac{\cos(kx^3) - 1}{(kx^3)^2} - \frac{\sin(kx^3) - kx^3}{(kx^3)^3}\right] +
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1}{x^3}\left[\frac{\sin(ku)+\sin(kx^0)}{x^3}-k\cos(kx^0)\right]\Bigg\} \ , \\
a^2 &= \left(A_\times x^1 - A_+ x^2\right)\cdot\Bigg\{-k^3x^3\sin(kx^0)\left[\frac{kx^3-\sin(kx^3)}{(kx^3)^2}-\right. \\
& \left. -\frac{\cos(kx^3)-1+(kx^3)^2/2}{(kx^3)^2}\right]+k^3x^3\cos(kx^0)\left[\frac{\cos(kx^3)-1}{(kx^3)^2}-\frac{\sin(kx^3)-kx^3}{(kx^3)^3}\right]+ \\
& +\frac{1}{x^3}\left[\frac{\sin(ku)+\sin(kx^0)}{x^3}-k\cos(kx^0)\right]\Bigg\} \ , \\
a^3 &= \left\{A_+\left[(x^1)^2-(x^2)^2\right]+2A_\times x^1x^2\right\}\cdot\Bigg\{k^3\sin(kx^0)\left[\frac{kx^3-\sin(kx^3)}{(kx^3)^2}-\right. \\
& \left. -\frac{\cos(kx^3)-1+(kx^3)^2/2}{(kx^3)^3}\right]-k^3\cos(kx^0)\left[\frac{\cos(kx^3)-1}{(kx^3)^2}-\frac{\sin(kx^3)-kx^3}{(kx^3)^3}\right]+ \\
& +\frac{1}{2x^3}\left[\frac{-2\sin(kx^0)-2\sin(ku)}{x^3}+k\cos(kx^0)+k\cos(ku)\right]\Bigg\} \ .
\end{aligned}
$$

From the expressions of $u^{(FNC)\mu}$ and $a^\mu$ one can see that the 3-dimensional velocity and acceleration of a TT observer with respect to the freely falling FNC observer are of order $h^{(TT)}_{\mu\nu}$. Moreover, a TT observer on the propagation axis of the gravitational wave (that coincides in the TT and in the FNC system), has zero (3-dimensional) acceleration and velocity with respect to the FNC observer. On a fixed plane $x^3$ = constant, and at a fixed instant $x^0$, the level curves of $u^1$, $u^2$, $a^1$, $a^2$ in the $(x^1, x^2)$ plane are straight lines, while the level curves of $u^3$ and $a^3$ are hyperbolas. TT observers aligned along a straight line $x^2 = px^1$ in a plane $x^3$ = constant experience, at a fixed instant $x^0$, an acceleration with components $a^1$, $a^2$ linear in $x^1$, and component $a^3$ quadratic in $x^1$, in the FNC system:

$$
\begin{aligned}
a^1 &= \text{const.}\ (A_+ + pA_\times)\,x^1 \ , \\
a^2 &= \text{const.}\ (A_\times - pA_+)\,x^1 \ , \\
a^3 &= \text{const.}\ \left[A_+(1-p^2)+2pA_\times\right](x^1)^2 \ .
\end{aligned}
$$

As a conclusion, TT observers undergo "strange" motions, when seen from a freely falling frame, with velocities and accelerations that are very small, but of the same order of magnitude of the quantities to be detected.

### The effects of gravitational waves in the FNC and in the TT system

In FNC one can Taylor-expand the metric in powers of the normal (spacelike) distance from $P \in \Gamma$, with vanishing first order terms (i.e. with vanishing Christoffel symbols), getting [102, 155]

$$\begin{aligned} g_{00} &= -1 - R_{0l0m}(P)\, x^l x^m + \dots \,, \\ g_{0i} &= 0 - \frac{2}{3}\, R_{0lim}(P)\, x^l x^m + \dots \,, \qquad (3.4) \\ g_{ij} &= \delta_{ij} - \frac{1}{3}\, R_{iljm}(P)\, x^l x^m + \dots \,, \qquad (3.5) \end{aligned}$$

where the Riemann components are evaluated at the spatial coordinates of $P$, and contain the dependence on the timelike coordinate $x^0 = \tau$. This expansion actually motivates the introduction of FNC [102], and shows why they are adequate only for detectors with size smaller than the curvature radius of the spacetime.

We can now relate the gravitational wave amplitudes in the TT and in the FNC systems. One has in the TT system

$$g_{\mu\nu} = \eta_{\mu\nu} + h^{(TT)}_{\mu\nu} \,,$$

$$R_{\mu\nu\rho\sigma} \sim h^{(TT)}/\lambda^2 \,,$$

while in FNC one has from the expansion Eq. (3.4), using $x^i \sim l$

$$g^{(FNC)}_{\mu\nu} = \eta_{\mu\nu} + h^{(TT)}_{\mu\nu}\,(l/\lambda)^2 \,,$$

so that

$$h^{(FNC)} \approx h^{(TT)}\,(l/\lambda)^2 \ll h^{(TT)} \,. \qquad (3.6)$$

Since the interaction laws between the e.m. field in a e.m. detector are described by the metric deviations from the Minkowski metric, it seems from these results that small size e.m. detectors are completely useless; however this is not the case, as we are going to show.

### The e.m. field in a e.m. detector

The interaction between the e.m. field in a detector and the inpinging gravitational waves are described by Maxwell equations that, in the absence of charges and currents, are

$$\nabla_\nu F^{\mu\nu} = \frac{1}{\sqrt{-g}}\,\partial_\nu\left(\sqrt{-g}\,F^{\mu\nu}\right) = 0 \ , \tag{3.7}$$

$$\partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} + \partial_\mu F_{\nu\rho} = 0 \ . \tag{3.8}$$

The e.m. tensor $F^{ab}$ can be decomposed, in any coordinate system, into the sum of an unperturbed term and a small perturbation [106, 107, 101]

$$F_{\mu\nu} = F^{(0)}_{\mu\nu} + F^{(1)}_{\mu\nu} \tag{3.9}$$

with $|F^{(1)}_{\mu\nu}| \ll |F^{(0)}_{\mu\nu}|$. The equations of motion for $F^{(1)}_{\mu\nu}$ are derived, in the TT system, in Ref. [106]:

$$\partial_\beta F^{(1)\alpha\beta} = \eta^{\alpha\mu} h^{\beta\gamma} \partial_\beta F^{(0)}_{\mu\gamma} + \eta^{\beta\nu}\partial_\beta h^{\alpha\mu} F^{(0)}_{\mu\nu} \ . \tag{3.10}$$

Baroni *et al.* [107, 101] were able to show that other expressions of $F^{(1)}_{\mu\nu}$ in the literature are wrong (because they do not take into account the Lorentz condition). Moreover, they derive the equations of motion for $F^{(1)}_{\mu\nu}$ in a generic, non-harmonic coordinate system:

$$\partial_\beta {F^{(1)}}^{\alpha\beta} = \eta^{\alpha\mu}\partial_\beta \bar{h}^{\beta\nu} F^{(0)}_{\mu\nu} + \eta^{\alpha\mu} h^{\beta\gamma} F^{(0)}_{\mu\gamma,\beta} + \eta^{\beta\nu}(\partial_\beta h^{\alpha\mu}) F^{(0)}_{\mu\nu} \tag{3.11}$$

where $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\,h^\rho{}_\rho$. These equations reduce to Eq. (3.10) in the TT gauge. Due to the argument of the previous section, we get

$$ {F^{(1)}}^{FNC} \approx {F^{(1)}}^{TT}\,(l/\lambda)^2 \ , $$

since the perturbation in the e.m. field is given by the unperturbed field "weighted" by the metric perturbations (the same reasoning may, of course, be done using Maxwell potentials). Thus the direct coupling between gravitational waves and the e.m. field is completely negligible in any detector with $l/\lambda \ll 1$ [24, 108, 109]. This shows clearly the deep difference between a mechanical and a e.m. detector; the former detects tidal accelerations by direct coupling between the gravitational wave and the normal modes of the oscillator, while the latter operates through the *parametric* influence exerted by gravitational waves on the e.m. field ([110, 109] and references therein).

### 3.3.2 Breakdown of FNC and detectors with $l/\lambda \geq 1$

Situations can occur in which FNC are not useful for practical purposes, even for detectors with $l/\lambda \ll 1$. This can happen for microwave resonant cavities in which the indirect coupling of gravitational waves to the e.m. field (i.e. gravitational waves interact directly with the cavity walls, and the walls' motion couples directly to the modes of the e.m. field) is not negligible: the e.m. field is determined from Maxwell's equations with boundary conditions on the moving cavity walls. A tricky possibility [108] is to transform to new coordinates in which the walls are at rest, getting usual boundary conditions: this coordinate transformation is performed only in a small region near the walls, retaining FNC in the other parts of the resonator. Then one regards the curved space Maxwell equations as flat space Maxwell equations for a moving anisotropic medium [111]. This approach treats the detector as a test particle with structure, whose centre of mass moves (to first order) along a timelike geodesic, but whose walls do not; this corresponds to choosing non-FNC fundamental observers at the cavity walls. The physical meaning of these "boundary observers" however is rather obscure; since they are "adapted" to the walls' motion, they look much like TT observers.

When a detector with $l/\lambda \geq 1$ is considered, it is no longer useful, in general, to specify the non rotating and non accelerating observers with respect to the centre of mass of the detector. Gravitational waves will act at different instants, and with different strength on the different parts of the detector; then it is not trivial to define the family of physical observers associated with the detector. FNC are no longer useful; they are adequate only for small distances from a timelike geodesics. Different parts of the detectors would define different worldlines that, in general, will not be geodesics. The detector should then be seen as an extended body, satisfying Papapetrou's equations of motion [105]. A lucky situation occurs when the walls of the resonator move with a frequency that is much smaller than the gravitational wave frequency: in this case one neglects completely the walls' motion [109, 112], and adopts a synchronous gauge [98, 113], in which

$$g_{00} = -1 \,,$$

$$\begin{aligned} g_{0i} &= 0 \,, \\ g_{ij} &= \delta_{ij} + h_{ij} \,. \end{aligned}$$

A geometric construction of this coordinate system [113] shows that, when matter is treated as a dust (pressure $P = 0$, i.e. as a collection of test particles), and vorticity vanishes, then the time lines[2] are timelike geodesics, that one can conceivably identify with the worldlines of the different parts of the detector ("synchronous observers"). The resonator walls are described by $x^i = constant$ by the synchronous observers.

In general, however, there is not a unique description, and one must specialize to particular detectors; this is beyond the purpose of this work. The guiding line in this direction should however be the following: the family of fundamental observers has always to be specified (a not trivial task when handling complicated devices described in *ad hoc* coordinate systems), and the "right" coordinate system is associated to the physical family of observers performing measurements with the detector.

### 3.3.3 Remarks

The physical meaning of the observers associated with the particular coordinate systems adopted to describe gravitational wave detectors is not always clear. For detectors with size much smaller than the wavelength of the radiation to be detected, FNC are associated with the detector's centre of mass and its timelike geodesic. The observers associated with the commonly used TT gauge cannot be regarded as the physical observers performing measurements; they undergo very strange motions when seen by a freely falling observer.

However, when the detector must be regarded as a test particle with structure, FNC are no longer appropriate, even for $l/\lambda \ll 1$. For detectors with $l/\lambda \geq 1$ the situation is less clear: in some lucky situations the coordinate system used to perform calculations is associated to a well defined family of observers, which can be regarded as the family of the physical observers making measurements with the detector. In general however, one is faced with tricky ways of performing computations in *ad hoc* coordinate systems, depending on

[2]The coordinate $x^0$ represents the proper time at each point of space.

the particular kind of device considered. The family of fundamental observers usually is not specified, or its physical meaning is not clear. The problem of the physical significance of the family of fundamental observers is relevant when one tries to compare the outcome of experiments with theoretical predictions, in view of the day in which gravitational waves will be detected and the data of a new astronomy will need to be understood.

# Chapter 4

# MULTIPLE IMAGING BY GRAVITATIONAL WAVES

## 4.1 Introduction

As we have seen in Chapter 3, the application of the vector formalism for ordinary gravitational lensing to gravitational waves leads to problems in describing gravitational waves in the appropriate coordinate system (a rather general problem). Moreover, the vector formalism breaks down when nonstationary lenses are considered. We have shown that the gauge-dependence problem does not affect the order of magnitude of the relevant quantities. Then, if we do not pretend to go in details, and limit ourselves to rough considerations concerning mostly order of magnitude estimates, we can forget about this problem. It seems reasonable to expect that the use of the vector formalism still can tell us something on lensing by gravitational waves, and in particular on multiple imaging. The results we will obtain are in fact correct to first order, and not only heuristic, as we will prove using the Fermat principle in a suitable formulation for nonstationary spacetimes.

Gravitational waves can act as lenses for the light propagating in them [25, 63], in the same way as mass concentrations perturbing the background curvature of the universe do, giving rise to the spectacular objects observed by the astronomers: multiple quasars [115], giant arcs [116] and radio rings [117]. Here we study how linearized, realistic, gravitational waves deflect the light coming from a distant source, and their capability of creating multiple images, approaching the problem with the usual vector formalism. We find that a rough condition for multiple imaging analogous to that for ordinary lenses holds. Some order of magnitude estimates show that this condition can be satisfied by astrophysical sources of gravitational waves considered in the literature [118]-[125]. This is due to the fact that the lensing waves can have perturbation amplitudes greater than those expected in the Solar System, and to the balance between the (large) distances of the lens and source and the (small) wave amplitudes. On the other hand, the gravitational wave background considered in the literature [26]-[28] is not likely to be capable of multiple imaging, on average. We conclude that multiple imaging by gravitational waves has to be taken as a serious possibility, but the probability of observing such a phenomenon is not high.

## 4.2 The vector formalism for lensing gravitational waves

Let us consider gravitational waves localized in a region of space between a light source and an observer. The spacetime metric is given, in an asymptotically Cartesian coordinate system, by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ ,$$

where $\eta_{\mu\nu}$ are the components of the Minkowski metric and $|h_{\mu\nu}| \ll 1$. Let us consider a light ray whose unperturbed path is parallel to the $z$-axis. The photon describing this ray has 4-momentum

$$p^{\mu} = p^{\mu}_{(0)} + \delta p^{\mu} = (1 + \delta p^0, \delta p^1, \delta p^2, 1 + \delta p^3) \ ,$$

where $\delta p^{\mu}$ are small deflections of order $h_{\mu\nu}$, and the unperturbed photon 4-momentum is $p^{\mu}_{(0)} = (1, 0, 0, 1)$. We work again in the geometric optics approximation, that holds if the wavelength $\lambda_{g.w.}$ of the gravitational wave is much larger than the photon wavelength $\lambda$, and if

$$\lambda > \lambda_{g.w.} \cdot (\lambda_{g.w.}/D_L) \tag{2.1}$$

(where $D_L$ is the observer-lens distance). Eq. (2.1) ensures us that the size of the interference fringes which eventually form at the observer's position is not comparable with the "geometrical shadow" of the lens [35]. In order to make computations to second order in $h$, we will lower and raise tensor indices with $g_{\mu\nu}$ and with

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + O(h^2) \equiv \eta^{\mu\nu} - \eta^{\mu\rho}\eta^{\nu\sigma} h_{\rho\sigma} + O(h^2) \ . \tag{2.2}$$

The equation of null geodesics gives

$$\frac{d(\delta p^\mu)}{d\lambda} + \Gamma^\mu_{\rho\sigma} \left(p^\rho_{(0)} + \delta p^\rho\right) \left(p^\sigma_{(0)} + \delta p^\sigma\right) = 0 \ , \tag{2.3}$$

where $\lambda$ is an affine parameter along the null geodesics and

$$\Gamma^\mu_{\rho\sigma} = \frac{1}{2}\, g^{\mu\nu} \left(h_{\nu\rho,\sigma} + h_{\nu\sigma,\rho} - h_{\rho\sigma,\nu}\right) \ . \tag{2.4}$$

We have then

$$\begin{aligned}
\frac{d(\delta p^\mu)}{d\lambda} &+ \eta^{\mu\nu} \left(h_{\nu 0,0} + h_{\nu 0,3} + h_{\nu 3,0} - h_{03,\nu} - \frac{1}{2}\, h_{00,\nu} - \frac{1}{2}\, h_{33,\nu}\right) \\
&+ \eta^{\mu\nu} \left(h_{\nu 0,\sigma} + h_{\nu 3,\sigma} + h_{\nu\sigma,0} + h_{\nu\sigma,3} - h_{0\sigma,\nu} - h_{3\sigma,\nu}\right) \delta p^\sigma \\
- \eta^{\mu\alpha}\eta^{\nu\beta} h_{\alpha\beta} &\left(h_{\nu 0,0} + h_{\nu 0,3} + h_{\nu 3,0} + h_{\nu 3,3} - h_{03,\nu} - \frac{1}{2}\, h_{00,\nu} - \frac{1}{2}\, h_{33,\nu}\right) \\
&+ \frac{1}{2}\, \eta^{\mu\nu} \left(h_{\nu\rho,\sigma} + h_{\nu\sigma,\rho} - h_{\rho\sigma,\nu}\right) \delta p^\rho \delta p^\sigma \\
- \eta^{\mu\alpha}\eta^{\nu\beta} h_{\alpha\beta} &\left(h_{\nu 0,\sigma} + h_{\nu 3,\sigma} + h_{\nu\sigma,0} + h_{\nu\sigma,3} - h_{0\sigma,\nu} - h_{3\sigma,\nu}\right) \delta p^\sigma = 0 \ ,
\end{aligned} \tag{2.5}$$

where the term in the first bracket is of order $h$, those in the second and the third brackets are of order $h^2$, and those in the fourth and the fifth brackets are of order $h^3$.

We now set the geometry of the problem as customary in gravitational lens theory (see Fig. 3.1). We consider only waves which are localized near the plane $z = z_w$. In the thin lens approximation, the deflection takes place essentially in this plane (*lens* or *image plane* in the usual language). Let $\underline{x} = (x, y)$ be the apparent source position in the $z = z_w$ plane, and $\underline{s} = (s_x, s_y)$ the true source position (i.e. its position were the lensing wave absent). The deflection is described by a 2-dimensional vector field $\delta p^A(\underline{x})$ $(A = 1, 2)$ in this plane. The action of the lens can be described by a plane-to-plane mapping $x^A \longmapsto s^A$, where $\underline{s}$ is given by the lens equation

$$s^A = x^A - \frac{D_L D_{LS}}{D_S}\, \delta p^A(\underline{x}) \ , \tag{2.6}$$

where $D_L$ is the observer-lens distance, $D_{LS}$ is the lens-source distance, and $D_S$ is the observer-source distance. As customary in gravitational lens theory, we could fit cosmology into the model by taking the $D$'s as angular diameter distances in a FLRW universe. However we assume, for the sake of simplicity, that the background is flat, so that the $D$'s denote euclidean distances, and $D_S = D_L + D_{LS}$.

The map described by the lens equation has the Jacobian matrix

$$J\left(\frac{\underline{s}}{\underline{x}}\right) = \left(\frac{\partial s^A}{\partial x^B}\right) = \begin{pmatrix} 1 - D\,\partial_x(\delta p^x) & -D\,\partial_y(\delta p^y) \\ -D\,\partial_x(\delta p^y) & 1 - D\,\partial_y(\delta p^y) \end{pmatrix} , \qquad (2.7)$$

where $D \equiv D_L D_{LS}/D_S$. The inverse matrix $A = J^{-1}$ represents the *amplification tensor*, while its determinant $\mathcal{A} = \mathrm{Det}(J)^{-1}$ is the (scalar) *amplification.* Let us consider now the Jacobian determinant

$$\mathrm{Det}(J) = 1 - D\,\frac{\partial(\delta p^A)}{\partial x^A} - D^2\left[\partial_y(\delta p^x)\cdot\partial_x(\delta p^y) - \partial_x(\delta p^x)\cdot\partial_y(\delta p^y)\right] \;; \quad (2.8)$$

the divergence $\partial(\delta p^A)/\partial x^A$ vanishes to first order [35] (see Appendix A). This can be understood if one considers the Raychaudhuri's equation without the matter term for a congruence of null rays

$$-\frac{d\theta}{d\tau} = \frac{\theta^2}{3} + \sigma^2 - \omega^2 \;,$$

where $\theta$, $\sigma^2$ and $\omega^2$ are the expansion, shear and vorticity scalars of the congruence. The right hand side is of order $h^2$; any variation $\delta\theta$ in the expansion (and then in the amplification of the source) induced by gravitational waves is then of order $h^2$, as seen in the previous chapter. We can write

$$\mathrm{Det}(J) = 1 + \sqrt{f(\alpha)}\,D_S\,\frac{\partial(\delta p^A)}{\partial x^A} + f(\alpha)D_S^2\left[\partial_x(\delta p^x)\cdot\partial_y(\delta p^y) - \partial_y(\delta p^x)\cdot\partial_x(\delta p^y)\right] \equiv$$
$$\equiv 1 + J_1 + J_2 \;,$$

where $\alpha \equiv D_{LS}/D_S$ and $f(\alpha) = \alpha^2(1-\alpha)^2$. The polynomial $f(\alpha)$ is symmetric about $\alpha = 1/2$ (corresponding to $D_L = D_{LS}$), where it assumes its maximum value $1/16 \simeq 0.0625$. In order for $\mathrm{Det}(J)$ to vanish (i.e. to have multiple images), we must have $J_1 + J_2 < 0$. Since $J_1$ and $J_2$ are terms of order $h^2$ times $D/P$ or $(D/P)^2$ respectively (see Eq. (2.12) below and the following considerations), in order to have $\mathrm{Det}(J) = 0$, a large value of the distance $D$

must balance small values of the wave amplitudes. Moreover, for large values of $D/P$, $J_1$ is negligible in comparison with $J_2$. In order to have multiple imaging, we need

$$f(\alpha)\left[D_S\,\partial_A(\delta p^B)\right]^2 \sim 1 \; .$$

For standard gravitational lenses the probability of lensing of a distant source is (roughly) maximum when the lens is halfway between the source and the observer [126]. Something similar should presumably happen for lensing gravitational waves, and values of the numerical factor $f(\alpha)$ very far from the maximum would not be statistically significant. We take $f(\alpha)$ in the range $\frac{1}{100}$ - $\frac{1}{16}$. Then, in order to have multiple imaging, we need

$$D\,\frac{\delta p}{P} \sim \frac{Dh}{P} \geq 4 - 10 \; ,$$

where $P$ is the period of the gravitational wave (computed at the redshift of the lens plane, if the $D's$ are chosen to represent angular diameter distances in a FLRW universe), and where we used $\delta p \sim h$ (see Eq. (2.12) and the following considerations). We write this inequality in the form

$$\frac{h}{P} \geq \mathcal{S}_c \; , \tag{2.9}$$

where $\mathcal{S}_c \equiv (4 - 10)\,c/D$. The rough condition for multiple imaging (2.9) involves the "strength" $h$ and the "size" $P$ of gravitational waves, the geometry of the problem (through $D$), and the fundamental constant $c$. (2.9) is strictly analogous to the well-known condition for multiple imaging by ordinary gravitational lenses [69]

$$\Sigma \geq \Sigma_c \equiv \frac{c^2}{4\pi G D} \; , \tag{2.10}$$

where $\Sigma$ is the 2-dimensional (projected) density of the lens, and the *critical density* $\Sigma_c$ is determined only by the geometric factor $D$ and by fundamental constants. This is a condition on the "strength" (the mass) and the size of the lens. Note that, since for gravitational waves the energy is not localized, a condition on the amplitudes and the period is probably the best one can expect when searching for the analog of (2.10).

The odd image number theorem [127] holds, since its proof requires only boundedness, smoothness and transparency of the lens (conditions satisfied by localized gravitational waves).

Let us compute now the deflection angles appearing in Eq. (2.6). We introduce the notation [63]

$$A_{,\lambda} \equiv p^{\alpha} \frac{\partial}{\partial x^{\alpha}} .$$

We have

$$\Gamma^{\mu}_{\lambda\lambda} = \eta^{\mu\nu} \left( h_{\nu\lambda,\lambda} - \frac{1}{2} \, h_{\lambda\lambda,\nu} \right) + O(h^2)$$

so that, from Eq. (2.5), we get

$$\delta p^{\mu} = - \int_S^O d\lambda \, \left( h^{\mu}{}_{\lambda,\lambda} - \frac{1}{2} \, h_{\lambda\lambda}{}^{,\mu} \right) + O(h^2) ,$$

where the indices are now raised with $\eta^{\mu\nu}$ and the integral is computed along the photon path from the source to the observer. Since we consider a *localized* pulse of gravitational waves, the first term in the integrand gives a vanishing boundary term, and we get [63]

$$\begin{aligned} \delta p^{A}(\lambda) &= -\frac{1}{2} \int_S^O d\lambda \, h_{\lambda\lambda}{}^{,A} + O(h^2) = \\ &= -\frac{1}{2} \int_S^O d\lambda \, \left( h_{00}{}^{,A} + 2 h_{03}{}^{,A} + h_{33}{}^{,A} \right) + O(h^2) . \qquad (2.11) \end{aligned}$$

If we substitute the integral along the actual photon path with the integral along the *unperturbed* path, we make an error of order $h^2$, so we can write, to first order,

$$\delta p^{A}_{(1)} = -\frac{1}{2} \int_S^O dz \, \left( h_{00}{}^{,A} + 2 h_{03}{}^{,A} + h_{33}{}^{,A} \right) . \qquad (2.12)$$

For pulses of gravitational waves the integral is extended to a region of space of order $P$ (the characteristic period of the waves) where the pulse is localized, so that we have, in order of magnitude, $\delta p \sim P \cdot h/P = h$. As far as the gravitational wave background is concerned, it could seem that computing the integral along the whole photon path from the source to the observer gives $\delta p \sim Dh/P$. Such a secular effect however is not present, i.e. the deflection does not cumulate with the travelled distance, as discussed in Chapter 2.

In order to compute $\delta p^{\mu}$ to second order (and the divergence $\partial_A \delta p^A$), we must insert the first order expression of $\delta p^{\mu}_{(1)}$ given by Eq. (2.12) into the equation of null geodesics

$$\frac{d(\delta p^{\mu})}{d\lambda} + \Gamma^{\mu}_{\lambda\lambda} + 2\Gamma^{\mu}_{\lambda\sigma} \delta p^{\sigma}_{(1)} + O(h^3) = 0 ,$$

which gives

$$\delta p^\mu = \delta p^\mu_{(1)} + \delta p^\mu_{(2)} \,,$$

where

$$\begin{aligned}
\delta p^\mu_{(2)} &= -2\int_S^O d\lambda\ \Gamma^\mu_{\lambda\sigma}\delta p^\sigma_{(1)} = \\
&= -\int_S^O dz\ \left(\partial_\sigma h^\mu{}_0 + \partial_\sigma h^\mu{}_3 + \partial_0 h^\mu{}_\sigma + \partial_3 h^\mu{}_\sigma - \partial^\mu h_{0\sigma} - \partial^\mu h_{3\sigma}\right)\delta p^\sigma_{(1)} + \\
&+\ O(h^3)\,,
\end{aligned} \tag{2.13}$$

with $\delta p^\sigma_{(1)}$ given by Eq. (2.12), and where we have again performed the integration along the unperturbed photon's path instead of the actual path, the difference contributing only by a third order term. Eq. (2.13) permits us to compute the second order divergence

$$\begin{aligned}
\partial_A(\delta p^A) = -\int_S^O dz\ &\left\{\delta p^\sigma_{(1)}\left(h^A{}_{0,\sigma A} + h^A{}_{3,\sigma A} + h^A{}_{\sigma,0A} + h^A{}_{\sigma,3A}\right)\right. \\
&\left.+\partial_A\left(\delta p^\sigma_{(1)}\right)\left(h^A{}_{0,\sigma} + h^A{}_0 + h^A{}_{\sigma,0} + h^A{}_{\sigma,3} - h_{0\sigma}{}^{,A} - h_{3\sigma}{}^{,A}\right)\right\}
\end{aligned} \tag{2.14}$$

from which we get, in order of magnitude

$$J_1 \sim D\,\partial_A(\delta p^A) \sim \frac{D}{P}\,h^2 \,.$$

Since $J_2 \sim (D/P)^2 h^2$, we can neglect $J_1$ in comparison to $J_2$ in the Jacobian determinant in Eq. (2.8) when large values of $D/P$ are considered.

## 4.3 Lensing by gravitational waves and Fermat principle

As already remarked, the vector formalism commonly used to describe ordinary gravitational lenses requires explicitly the stationarity of the lens potential ($\partial\Phi/\partial t = 0$), so one expects the previous results for lensing gravitational waves (a highly nonstationary case) to be correct only in order of magnitude. However, in the literature on gravitational lenses, the hypotesis of lens stationarity is needed only to compute the deflection angle in the lens plane [1].

[1]The transverse motion of the lens in such a plane (to which we can reduce every relative motions of the lens and the observer), or along the line of sight, is almost unimportant,

Since we have computed the deflection angle caused by gravitational waves independently, starting from the geodesic equation, we may hope that the vector formalism nevertheless gives correct results (to our order of approximation). This section approaches the problem of lensing by gravitational waves using Fermat principle in a way parallel to the so-called scalar formalism [132, 133] for ordinary lenses. We will prove that the results of the previous section are correct to first order.

Fermat principle was derived by Weyl [128] for static spacetimes and by Pham Mau Quan [129] and by Brill [130] for stationary spacetimes. These formulations admit a straightforward generalization to the conformally stationary case [131], and have proven to be very useful in the application to gravitational lens theory [132, 133, 134, 135]. Kovner [83], in a rather obscure article, generalized Fermat principle to arbitrary spacetimes, in order to be able to consider essentially nonstationary lenses like cosmic strings, or nonstationary lens components like gravitational waves superimposed to an ordinary gravitational lens (an idea introduced by Mc Breen and Metcalfe [136], who suggested that such a composite lens could be responsible for $\gamma$-ray bursts, resulting from the crossing of small hot cores of BLLac objects by microcaustics. In their model the burst-like nature of the source is explained by the caustics motion caused by gravitational waves). The validity of Fermat principle in arbitrary spacetimes has been clearly explained and satisfactorily proved in a later paper by Perlick [84].

For a photon deflected by a gravitational wave, the crossing time of the lens is $\tau_c \sim \lambda_{g.w.}/c = P$, while the variation scale of the lensing field is $\tau \sim P$. For ordinary lenses $\tau >> \tau_c$, while in our case $\tau \sim \tau_c$. Under the assumption that the lens is geometrically thin and the background is Euclidean, the actual light rays differ from straight lines only within the gravitational field of the wave, i.e. in a region much smaller than the distance travelled by the photons. This permits us to substitute the true photon path in the 3-dimensional space with

even if it takes place with the velocity of light, since the displacement of the lens on a timescale of, say, years, is negligible in comparison to the huge distances between lens, source and observer. Such motions will possibly cause the merging/appearing of pairs of images on timescales shorter than those characteristic of ordinary gravitational lenses, due to the crossing of caustics.

a zig-zag path composed of two straight lines from the source to the lens, and from the lens to the observer. This zig-zag construction is accurate to first order [83].

The zig-zag path is given by

$$\dot{x}^\mu(\lambda) = \begin{cases} \left(\dot{x}^\mu(\lambda), \dot{\xi}(\lambda)\, \vec{n}_{ol}\right) & \text{if } \lambda < \lambda_l \\ \left(\dot{x}^\mu(\lambda), \dot{\xi}(\lambda)\, \vec{n}_{ls}\right) & \text{if } \lambda > \lambda_l, \end{cases}$$

where $\lambda$ is the parameter along the photon path (with value $\lambda_l$ at the lens position), and $\vec{n}_{ol}$ and $\vec{n}_{ls}$ are three-dimensional vectors satisfying

$$\vec{n}_{ol}^2 = \vec{n}_{ls}^2 = 1\ ,$$

pointing in the lens-observer and in the source-lens directions respectively. The zig-zag path is assumed to be a null curve satisfying the on shell condition

$$g_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu = 0\ , \tag{3.1}$$

which gives

$$(-1 + h_{00})(\dot{x}^0)^2 + 2\, h_{0i} n^i\, \dot{x}^0\, \dot{\xi} + \dot{\xi}^2\, (\delta_{ij} + h_{ij})\, n^i n^j = 0\ .$$

This can be written as a second degree equation for the variable $\dot{\xi}/\dot{x}^0$

$$\left(\frac{\dot{\xi}}{\dot{x}^0}\right)^2 \left(1 + h_{ij} n^i n^j\right) + 2\left(h_{0i} n^i\right)\left(\frac{\dot{\xi}}{\dot{x}^0}\right) + (h_{00} - 1) = 0$$

which has the solutions

$$\frac{\dot{\xi}}{\dot{x}^0} = \frac{-h_{0i} n^i \pm \sqrt{(h_{0i} n^i)^2 + (1 - h_{00})(1 + h_{ij} n^i n^j)}}{1 + h_{ij} n^i n^j}\ .$$

Choosing the "+" sign (corresponding to $d\xi/dx^0 > 0$, i.e. to photons travelling in the direction of increasing $\xi$) we get

$$\frac{\dot{\xi}}{\dot{x}^0} = 1 - \frac{1}{2}\left(h_{00} + 2h_{0i} n^i + h_{ij} n^i n^j\right) + O(h^2)\ . \tag{3.2}$$

We adopt now the geometry used for ordinary gravitational lenses, and in the previous chapter. Perlick's [84] first coordinate version of Fermat principle is

**Theorem:** let $(M, g_{ab})$ be a spacetime; let $U \subseteq M$ be an open set on which a local ($\mathcal{C}^\infty$) coordinate system $(x^1, x^2, x^3, x^4)$ is defined, such that $\partial_0$ is timelike;

a future pointing null $\mathcal{C}^\infty$ curve $\lambda : [a, b] \to U$ with coordinate representation $s \mapsto x^i(s)$ is a geodesic iff it makes the functional

$$\omega(\lambda) \equiv \int_S^O \left[ \left( \hat{g}_{ij} \dot{x}^i \dot{x}^j \right)^{1/2} - \hat{\phi}_i \dot{x}^i \right] (s)\, ds \tag{3.3}$$

stationary with respect to ($\mathcal{C}^\infty$) variations satisfying

$$\delta x^\mu(S) = 0$$

$$\delta x^i(O) = 0$$

$$\delta \left( \dot{x}^0 - \left( \hat{g}_{ij} \dot{x}^i \dot{x}^j \right)^{1/2} + \hat{\phi}_i \dot{x}^i \right) = 0 \ ,$$

where

$$\begin{aligned} f &\equiv \frac{1}{2} \log |g_{00}| \ , && (3.4) \\ \hat{\phi}_i &\equiv -e^{-2f} g_{0i} \ , && (3.5) \\ \hat{g}_{ij} &\equiv e^{-2f} g_{ij} + \hat{\phi}_i \hat{\phi}_j \ . && (3.6) \end{aligned}$$

Computing the functional Eq. (3.3), we get

$$\begin{aligned} \omega(\lambda) &= \int_S^O \dot{\xi} \left[ \sqrt{\left( \frac{h_{0i} n^i}{1 - h_{00}} \right)^2 + \frac{1 + h_{ij} n^i n^j}{1 - h_{00}}} + \frac{h_{0i} n^i}{1 - h_{00}} \right] (s)\, ds = \\ &= \int_S^O \dot{\xi} \left[ 1 + \frac{1}{2} \left( h_{00} + 2\, h_{0i} n^i + h_{ij} n^i n^j \right) \right] (s)\ ds + O(h^2) = \\ &= \int_S^O \dot{x}^0\, ds + O(h^2) \ , \end{aligned} \tag{3.7}$$

where Eq. (3.1) has been used. Note that the zig-zag path satisfies, to first order, the conditions required to the varied curves in the theorem:

$$\delta x^\mu(S) = 0 \ ,$$

$$\delta x^i(O) = 0 \ ,$$

and

$$\delta\left(\dot{x}^0 - \left(\hat{g}_{ij}\dot{x}^i\dot{x}^j\right)^{1/2} + \hat{\phi}_i\dot{x}^i\right) = \delta\left(\dot{x}^0 - \dot{\xi}\sqrt{\left(\frac{h_{0i}n^i}{1-h_{00}}\right)^2 + \frac{1+h_{ij}n^in^j}{1-h_{00}}} - \dot{\xi}\frac{h_{0i}n^i}{1-h_{00}}\right) =$$
$$= \delta\left(\dot{x}^0 - \dot{x}^0\left[1 - \frac{1}{2}(h_{00} + 2h_{0i}n^i + h_{ij}n^in^j)\right]\cdot\left[1 + \frac{1}{2}(h_{00} + 2h_{0i}n^i + h_{ij}n^in^j)\right]\right) =$$
$$= 0 + O(h^2)\,.$$

The zig-zag paths approximating the real null geodesics are extrema of the travel time

$$\begin{aligned}\tilde{t} &= \int_S^O dx^0 = \int_S^O \frac{\dot{x}^0}{\dot{\xi}}\, d\xi = \\ &= \int_S^O \left[1 + \frac{1}{2}(h_{00} + 2h_{0i}n^i + h_{ij}n^in^j)\right] d\xi + O(h^2)\,.\end{aligned}$$

By subtracting from this integral the travel time calculated along a straight line were the lens absent (compare the case of ordinary gravitational lenses in Ref. [83]), we obtain

$$\tilde{t} = \text{constant} + \frac{1}{2c}\frac{D_S}{D_L D_{LS}}\left(\underline{x} - \underline{s}\right)^2 - \mathcal{H} + O(h^2)\,, \tag{3.8}$$

where

$$\mathcal{H} \equiv -\frac{1}{2c}\int_S^O \left(h_{00} + 2h_{0i}n^i + h_{ij}n^in^j\right) d\xi\,. \tag{3.9}$$

The first term in square brackets in Eq. (3.9) is called the *geometrical time delay*, and results from the extra length travelled by the photon due to the deflection. The second term in square brackets is the *gravitational time delay* resulting from the wave's field. Its analog for the case of an ordinary gravitational lens is familiar from experiments in the Solar System.

If we adopt a FLRW universe as a background, the expression of the time delay given by Eq. (3.8) is valid as long as the time $\tau_c$ of crossing the lens is small compared to the Hubble time.

The lens equation is now given by requiring stationarity of the arrival time

$$\underline{\nabla}_x \tilde{t} = 0\,, \tag{3.10}$$

or

$$\underline{s} = \underline{\nabla}_x T(\underline{x}, t)\,, \tag{3.11}$$

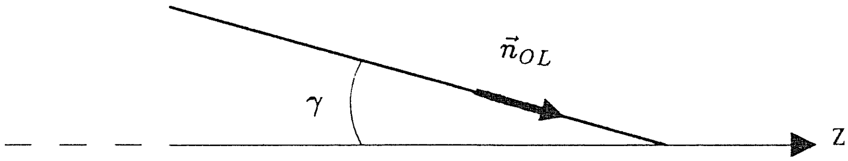


Figure 4.1: The geometry used in the calculation of $\mathcal{H}$.

where

$$T(\underline{x}, t) \equiv \frac{\underline{x}^2}{2} - c\,\frac{D_L D_{LS}}{D_S}\,\mathcal{H}(\underline{x}, t)\ . \tag{3.12}$$

If we assume that the unperturbed photon path is the $z$-axis, we can substitute the integral along the zig-zag path in Eq. (3.8) with the integral along the $z$-axis (up to second order terms), getting

$$\mathcal{H} = -\frac{1}{2c}\int_S^O dz\,(h_{00} + 2h_{03} + h_{33})$$

(see Fig. 4.1). The lens equation (3.10) can now be written

$$\underline{s} = \underline{x} + \frac{1}{2c}\frac{D_L D_{LS}}{D_S}\int_S^O dz\,\underline{\nabla}_x(h_{00} + 2h_{03} + h_{33})\ . \tag{3.13}$$

Notice that the deflection angle and the lens equation coincide with those given by Eqs. (2.11) and (2.6). This confirms our observation that the hypotesis of lens stationarity in the vector formalism for ordinary gravitational lenses is needed only in the computation of the deflection angle, and proves that our results in the previous section are valid not only heuristically, but are correct to first order.

## 4.4 Comparison between a gravitational wave and an ordinary gravitational lens

It is instructive to compare the action of a gravitational wave with that of an ordinary gravitational lens. The latter is a mass distribution described by a Newtonian potential $\Phi$ (satisfying the Poisson equation $\nabla^2\Phi = 4\pi\rho$, where $\rho$ is the lens mass density), and is smooth, bounded and stationary, i.e.

- $\Phi$ is continuous with its first and second derivatives;
- $\Phi \to 0$ and $\nabla\Phi \to 0$ as $r \equiv (x^2+y^2+z^2)^{1/2} \to +\infty$;
- $\partial\Phi/\partial t \simeq 0$.

The plane-to-plane mapping describing the lens action is given by the lens equation, and the Jacobian matrix can be written [64]

$$J = \begin{pmatrix} 1-\chi-\lambda & -\mu \\ -\mu & 1-\chi+\lambda \end{pmatrix} ,$$

where

$$\begin{aligned} \Sigma &\equiv \int_{-\infty}^{+\infty} dl\, \rho \, , \\ \chi &\equiv \frac{\Sigma}{\Sigma_c} \, , \\ \lambda &\equiv \frac{D}{c^2}\int_{-\infty}^{+\infty} dl \left(\frac{\partial^2\Phi}{\partial x^2} - \frac{\partial^2\Phi}{\partial y^2}\right) \, , \\ \mu &\equiv \frac{D}{c^2}\int_{-\infty}^{+\infty} dl\, \frac{\partial^2\Phi}{\partial x\partial y} \end{aligned}$$

(where $D$ has the same meaning as before). The Jacobian determinant is

$$\mathrm{Det}(J) = (1-\chi)^2 - (\lambda^2+\mu^2) \, .$$

The *convergence* $\chi$ describes the action of matter in the light beam, while $\lambda$ and $\mu$ describe the action of shear. For a lensing gravitational wave we can write (taking first order quantities)

$$J_{g.w.} = \begin{pmatrix} 1-\lambda_1 & -\mu_1 \\ -\mu_2 & 1-\lambda_2 \end{pmatrix} , \tag{4.1}$$

where

$$\lambda_1 \equiv \frac{D}{2}\int_S^O d\lambda\ h_{\lambda\lambda}{}^{,x}{}_{,x} + O(h^2)\,, \tag{4.2}$$

$$\lambda_2 \equiv \frac{D}{2}\int_S^O d\lambda\ h_{\lambda\lambda}{}^{,y}{}_{,y} + O(h^2)\,, \tag{4.3}$$

$$\mu_1 \equiv \frac{D}{2}\int_S^O d\lambda\ h_{\lambda\lambda}{}^{,x}{}_{,y} + O(h^2)\,, \tag{4.4}$$

$$\mu_2 \equiv \frac{D}{2}\int_S^O d\lambda\ h_{\lambda\lambda}{}^{,y}{}_{,x} + O(h^2)\,, \tag{4.5}$$

and where the integrals are computed along the photon's path, from the source to the observer. To first order, $\lambda_2 = -\lambda_1 \equiv -\lambda$ and $\mu_1 = \mu_2 \equiv \mu$ (see Appendix B), so we can write (to this order)

$$J_{g.w.} = \begin{pmatrix} 1-\lambda & -\mu \\ -\mu & 1+\lambda \end{pmatrix}$$

and

$$\mathrm{Det}(J_{g.w.}) = 1 - (\lambda^2 + \mu^2)$$

(where the term in brackets is of second order in $hD/P$). The convergence term is clearly absent, the lens action being due only to the shear.

We expect that the images of a distant source created by a gravitational wave show a variability on timescales of the order of the wave period. This could possibly be used to explain the short-scale variability of some AGNs or active galaxies. Moreover, the details of the images configuration will depend on the detailed shape and parameters of the lensing wave, such as its spatial and time profile, its duration, direction of propagation and polarization.

## 4.5 Order of magnitude estimates

In order to apply the previous theory, and the multiple images to be detectable, the following conditions must be satified:

1. geometric optics holds;
2. the scale of separation between different images must not be smaller than $10^{-3}$ arcseconds. In fact structures on scales $\sim 10^{-3}$ arcseconds can be resolved with VLBI, while VLA and optical techniques apply on larger scales ;

3. the lens must not be exceptionally rare, i.e. the rate of occurrence of the event generating the lensing wave must not be too low;

4. in order to appreciate variability in the images induced by a lensing gravitational wave, its period must not be too short, let us say $P < 10^8$ s.

In order to satisfy 1), we limit our considerations to electromagnetic radiation with wavelength $\lambda$ satisfying $\lambda_{g.w.} \cdot (\lambda_{g.w.}/D) < \lambda < \lambda_{g.w.}$.

To satisfy 2), note that, if $\delta \sim h$ is the deflection angle, the separation scale $\delta$ between the images must not be smaller than $\sim 10^{-3}$ arcseconds, which gives

$$h \geq 5 \cdot 10^{-9} \ .$$

3) depends on the particular processes generating gravitational radiation; since these are almost all purely speculative, their rates of occurrence are largely or completely unknown, and we can only try to guess their values. Note that a continuous source of gravitational radiation will give rise to a permanent lens, while a gravitational wave burst will constitute a temporarily lens.

The sources of gravitational waves that are most often considered in the literature are

- stellar collapse with non-spherical symmetry;
- formation of massive black holes in active galactic nuclei;
- neutron star collision;
- black hole collision;
- close binary systems;
- black hole accretion.

The last two phenomena are *continuous* sources of gravitational radiation, while the others give *bursts*. In addition, we will consider the stochastic gravitational wave background, both primordial or generated ([26]-[28] and references therein). Gravitational waves generated by a process involving a body of mass $M$ and size $R$ have dimensionless amplitudes (near the source) of order

$$h_S \sim \epsilon \, \frac{M}{R} \, ,$$

where the *efficiency* $\epsilon$ is defined as the fraction of energy radiated away. For processes involving neutron stars or black holes one can assume, respectively, $M/R \sim 1/20$ and $M/R \sim 1$.

Multiple imaging by gravitational waves is realistic only if they satisfy the rough condition (2.9). We examine the astrophysical sources of gravitational radiation considered in the literature, and check if this is possible. When the event generating gravitational waves involves neutron stars or black holes, the ordinary lensing associated to these objects ("microlensing" [137]-[140]) should, in principle, be taken into account: however the separation scale between microimages of a distant source created by a compact object is of order $10^{-6}$ arcseconds, not detectable with present techniques. Moreover, the flux of a microimage decreases with the distance $r$ from the microlens approximately as $r^{-4}$ [121, 64], while the wave amplitudes $h$ decrease as $R_S/r$ (where $R_S$ is the Schwarzschild radius of the compact object). Thus we will neglect ordinary microlensing, restricting our considerations to events which take place at some distance from the source of gravitational waves (they are much more likely than lensing very near the compact object(s)). Finally, the condition $r > \lambda_{g.w.}$ which ensures us that we are considering fields in the wave zone (i.e. far enough from the source of gravitational waves) must be satisfied.

## Stellar core collapse

The gravitational collapse of stellar cores from the point of view of gravitational waves generation has been reviewed by Eardley [122]. A research programme on collapsing homogeneous ellipsoids has been taken over by Saenz and Shapiro [118]-[120] ; they have found that the expected maximal efficiences for a "cold" and "hot" equation of state are $\epsilon \sim 10^{-2}$ and $\epsilon \sim 10^{-4}$ respectively, the spectrum of emitted gravitational radiation being broadly peaked between 100 Hz and 1 KHz. Taking the lower value $\epsilon \sim 10^{-4}$ we get

$h_S \sim 5 \cdot 10^{-6}$ for the wave amplitudes near the collapsing core, and the condition $h \sim h_S R_S/r > 5 \cdot 10^{-9}$ implies $r < 10^3 R_S \sim 10^9$ cm. On the other hand we must have $r > \lambda_{g.w.} \simeq 3 \cdot 10^7 - 10^8$ cm. There is a rather narrow permitted range for the impact parameter ($r \sim 10^8$ cm), that gives $Dh/P \sim 10$ if $D \sim 6 \cdot 10^{15}$ cm.

If the late phase when the ellipsoid has settled down as a rapidly rotating neutron star (evolving with timescale $\sim 1$ sec) is taken into account, it is found [141] that the emitted spectrum is very narrowly peaked ($\Delta\nu/\nu \sim 10^{-3}$), and $\epsilon \sim 10^{-6}$ at $\nu \sim 1$ KHz, that gives $h_S \sim 5 \cdot 10^{-8}$; $h > 5 \cdot 10^{-9}$ requires $r < 10 R_S \sim 10^7$ cm. On the other hand, the condition $r > \lambda_{g.w.} \simeq 3 \cdot 10^7$ cm does not allow for lensing on a relevant scale to take place in the wave zone.

If the core keeps bouncing, its eccentricity becomes large after enough bounces; this asymmetry [120] makes the efficiency almost uniformly near its maximum value for any initial period above 1 sec to several hundred seconds. The growth of asymmetry can be understood as a parametric resonance, with the core radial mode acting as the "pump" oscillator, and where a nonspherical mode of the core is the driven oscillator [122]. One gets $h_S \sim 10^{-2}$ at $\nu \sim 1$ KHz; $h > 5 \cdot 10^{-9}$ gives $r < 2 \cdot 10^6 R_S \sim 2 \cdot 10^{12}$ cm, while $r > \lambda_{g.w.} \simeq 3 \cdot 10^7$ cm. A rather large range of values of the impact parameter is permitted; we get $Dh/P \sim 10$ if $D \sim 10^{12}$ cm, $r \sim 3 \cdot 10^7$ cm, and if $D \sim 6 \cdot 10^{16}$ cm, $r \sim 2 \cdot 10^{12}$ cm.

Studies on the perturbations of pressureless spherical collapse leading to the formation of a black hole [125, 143, 144] give results that could possibly be extrapolated to larger deviations from spherical symmetry [122], getting $\epsilon \sim 2 \cdot 10^{-2} \left(\frac{J}{M^2}\right)^4$ at $\nu \sim 1KHz \cdot (M/10M_\odot)^{-1}$ for $J/M^2 \ll 1$ (where $J$ is the angular momentum, and $J = M^2$ corresponds to a maximally rotating Kerr black hole). Taking $J \sim 0.1\,M^2$ and $M = 10M_\odot$, we get $h_S \sim 2 \cdot 10^{-6}$, and $h > 5 \cdot 10^{-9}$ implies $r < 400 R_S \sim 4 \cdot 10^8$ cm, while $r > \lambda_{g.w.} \simeq 3 \cdot 10^7$ cm. A rather narrow range of values of $r \sim 10^8$ is permitted, that gives $Dh/P \sim 10$ if $D \sim 3 \cdot 10^{16}$ cm.

## Final decay of a neutron star/neutron star binary

Rough estimates for the final decay of a binary system composed of two neutron stars [124] give $\epsilon \sim 5 \cdot 10^{-3}$ at $\nu \leq 2 - 3$ KHz. Taking $\nu \sim 500$ Hz we get $h_S \sim 2.5 \cdot 10^{-4}$; $h > 5 \cdot 10^{-9}$ gives $r < 5 \cdot 10^4 R_S \sim 5 \cdot 10^{10}$ cm, while $r > 6 \cdot 10^7$ cm. A large range of values of $r$ is permitted; we get $Dh/P \sim 10$ if $D \sim 3 \cdot 10^{16}$ cm, $r \sim 5 \cdot 10^{10}$ cm, and if $D \sim 6 \cdot 10^7$ cm, $r \sim 6 \cdot 10^7$ cm.

## Black hole collisions

The head-on collision of two equally massive, non-rotating black holes has been studied numerically [123], leading to a single, larger, black hole, with efficiency $\epsilon \sim 7 \cdot 10^{-4}$. If the two initial black holes have nearly enough angular momentum to go into orbit before coalescing, a formula derived from extrapolation of perturbation theory [122, 123] gives $\epsilon \sim 3 \cdot 10^{-2}$. This efficiency is expected to hold for $P \sim 1$ s; $h > 5 \cdot 10^{-9}$ gives $r < 6 \cdot 10^6 R_S \sim 6 \cdot 10^{12}$ cm, while $r > \lambda_{g.w.} \simeq 3 \cdot 10^{10}$ cm. In the permitted range of values of $r$ we have $Dh/P \sim 10$ if $D \sim 3 \cdot 10^{17}$ cm, $r \sim 3 \cdot 10^{10}$ cm, and if $D \sim 6 \cdot 10^{19}$ cm, $r \sim 10^{12}$ cm.

## The binary pulsar

The binary pulsar PSR 1913+16 [145, 146] is believed to radiate gravitational waves in a continuous way, according to the predictions of General Relativity ([147] and references therein ). The estimated distance of the binary system (believed to be a neutron star/neutron star system) is $D \sim 5$ Kpc [145], and the frequency of the radiation is twice the orbital frequency (due to the quadrupole nature of the radiation). From these values we get[2] $D/P \sim 3.5 \cdot 10^7$, and $Dh/P \sim 10$ if $h \sim 3 \cdot 10^{-7}$. An estimate of the amplitude of the waves emitted by the binary pulsar gives

$$h \sim \frac{\ddot{Q}}{r} \sim \frac{Ma^2\omega^2}{r}$$

where $Q$ is the quadrupole moment, $M$ is the mass, and $a \sim 7 \cdot 10^{10}$ cm is the semimajor axis of the binary system [147], so that $h_S \sim (a\omega/c)^2 \sim 10^{-6}$. $h > 5 \cdot 10^{-9}$ implies then $r < 2 \cdot 10^2 R_S \sim 2 \cdot 10^8$ cm, while the wave zone is

[2]The orbital period is $\sim 2.8 \cdot 10^4$ s [147].

approached for $r > \lambda_{g.w.} \simeq 3 \cdot 10^{14}$ cm. Multiple imaging by the gravitational waves emitted by the binary pulsar is not possible.

## The gravitational wave background

A primordial gravitational wave background analogous to the 2.7 K electromagnetic background has been considered in the literature ([26] and references therein). In addition, there can be a generated background coming from the overlapping of gravitational waves generated by certain astrophysical processes. For both of them one has [26]

$$h \sim \sqrt{\Omega_{g.w.}} \, \frac{P_0}{R} \, ,$$

where $P_0$ is the present gravitational wave period, $R$ is the radius of the universe and $\Omega_{g.w.}$ is the cosmological density of gravitational waves (in units of the critical density). We have

$$\frac{Dh}{P} \sim \sqrt{\Omega_{g.w.}} \, \frac{D}{R} \, .$$

Upper bounds on $\Omega_{g.w.}$ have been set in various bands of frequencies (see Refs. [26]-[28] and references therein). One has $\Omega_{g.w.} < 1$ for all frequencies, and $\Omega_{g.w.} \ll 1$ in many bands. Moreover, $D/R < 1$, so it is likely that

$$\frac{Dh}{P} \ll 1$$

(the exact value depending on the frequency band), and then one concludes that multiple imaging by the gravitational wave background is impossible, or only marginally possible. There can however be exceptionally large fluctuations in the gravitational wave background, with a definite frequency, direction of propagation and polarization, which could conceivably be capable of multiple imaging [63]. Such exceptionally rare events are not unrealistic, but have to be expected in any large enough sample; they have been considered for the microwave background radiation [148]. Effects that are not due to localized exceptionally large fluctuations cannot be approached with the thin lens approximation; the gravitational wave background requires then a particular treatment that will be the subject of further work.

## 4.6 Remarks

Gravitational waves affect the propagation of light like other gravitational fields do. Approaching the problem in the same way as for "ordinary" gravitational lenses [35], one gets that a rough condition strictly analogous to that holding for ordinary lenses must be satisfied in order to have multiple images of a distant light source. Certain astrophysical sources considered in the literature [118]-[125] are shown to satisfy it, due to the balance between the large values of the distances involved, and the small values of the gravitational wave amplitudes. Moreover, lensing takes place in regions not too far from the sources of gravitational waves, where their amplitudes are larger than those expected in the Solar System. The gravitational wave background [26]-[28], on the other hand, is not likely to produce multiple images, apart possibly from exceptionally strong fluctuations. As a conclusion, multiple imaging of a distant source by gravitational waves should be taken as a serious possibility in certain favourable situations, including the collapse of stellar cores, the final decay of neutron star/neutron star binaries, and black hole collisions. As far as stellar core collapses are concerned, if the correlations found between the records of gravitational wave antennae and neutrino detectors during supernova SN1987A ([149, 150] and references therein) really represent gravitational waves [3], then the signals would be $10^4 - 10^6$ times larger than expected, and the gravitational collapse activity would last for a few hours, instead of a fraction of a second. This would certainly increase the probability of observing multiple imaging by gravitational waves.

Unfortunately, these events are not very frequent (black hole collisions in particular are believed to be very rare), so that the probability of observing the phenomenon certainly is not high, mainly because the duration of the multiple images would be limited to the period of intense emission of gravitational radiation. Continuous sources of gravitational waves like the binary pulsar PSR 1913+16 would give a much higher probability, but unfortunately they emit gravitational waves too weakly.

[3] It seems hard this can really be the case.

# Chapter 5

# THE TAIL PROBLEM IN COSMOLOGY

## 5.1 Introduction

In the second and fourth chapters we considered cosmological gravitational waves, assuming that they are distributed in a homogeneous and isotropic way through the universe, according to the cosmological principle. This assumption is motivated by the analogy with the electromagnetic background, which is observed to be homogeneous and isotropic with a high degree of accuracy. However scattering off the background curvature of the universe can confine gravitational radiation of very large wavelength to distant regions, even in a Friedmann universe. This phenomenon is not present for electromagnetic radiation in a FLRW spacetime. This could possibly change the considerations of Chapters 2 and 4, but the problem is interesting in itself, apart from the applications that Chapters 2 and 4 suggest, so we will consider it in general. The following is based on Ref. [160].

The propagation of waves in a curved spacetime is sometimes accompanied by diffusive effects: The usual picture of propagation along the light cone is in these cases altered by the fact [151]-[153] that a nonvanishing part of the radiation lies in the interior of the null cone. This phenomenon has been studied for radiation around a compact object, in which case it appears not to be particularly relevant from the astrophysical viewpoint, because the curvature responsible for scattering dies off rapidly with the distance from the object itself [154]-[158]. Within a cosmological context, the corresponding problem has been outlined by Ellis and Sciama [159]. Since the cosmological curvature is present throughout all of spacetime, rather than being localized like happens for a compact object, one cannot in principle exclude physically relevant effects. The problem of finding solutions of the scalar wave equation which are free of "diffusive" components in symmetric spacetimes is treated in Refs. [161]-[166] (and references therein).

The presence of diffusive components of radiation can be understood by examining the form of the Green function for the corresponding wave equation [153]. In the simple case of a scalar field $\Phi$ satisfying

$$\Box\Phi - \xi\, R\,\Phi = 0 \tag{1.1}$$

where $\xi$ is a numerical constant, the Green function $G(x', x)$ is defined as a solution of

$$[\Box' - \xi\, R(x')]\, G(x', x) = -\delta(x', x)\ , \tag{1.2}$$

where $\delta(x', x)$ is the delta function on spacetime. The retarded Green function can be written [152] in a normal domain as

$$G^{(-)}\,(x', x) = \Sigma\,(x', x)\ \delta^{(-)}\,(\Gamma(x', x)) + V\,(x', x)\,\Theta^{(-)}\,(-\Gamma(x', x))\ , \tag{1.3}$$

where $\Gamma(x', x)$ is the square of the proper distance calculated along the unique geodesic connecting $x'$ and $x$, whereas $\delta^{(-)}$ and $\Theta^{(-)}$ are, respectively, the Dirac delta distribution and the Heaviside step function with support in the past of $x'$. It can be shown [151, 167, 153] that a necessary and sufficient condition for the absence of tails is

$$V(x', x) = 0\ . \tag{1.4}$$

In Minkowski spacetime Eq. (1.4) is satisfied; therefore the responsible for tails is the spacetime curvature. Actually, there are tails also in three and two-dimensional Minkowski spacetime [151, 153], in which

$$V = \frac{1}{2\pi\sqrt{-\Gamma}} \tag{1.5}$$

and

$$V = 1/2 \tag{1.6}$$

respectively. This can be seen immediately by integrating the Green function in the four-dimensional Minkowski spacetime

$$G^{(4)}(x_0', x_1', x_2', x_3'; x_0, x_1, x_2, x_3) = \frac{1}{2\pi}\,\delta(\Gamma(x_0', x_1', x_2', x_3'; x_0, x_1, x_2, x_3)) \quad (1.7)$$

with respect to one of the spatial coordinates of the point $x$, to get

$$G^{(3)}(x_0', x_1', x_2'; x_0, x_1, x_2) = \frac{\Theta(-\Gamma(x_0', x_1', x_2'; x_0, x_1, x_2))}{2\pi\sqrt{-\Gamma(x_0', x_1', x_2'; x_0, x_1, x_2)}} \; . \quad (1.8)$$

Repeating the same procedure for $G^{(3)}$, one obtains

$$G^{(2)}(x_0', x_1'; x_0, x_1) = \frac{1}{2}\,\Theta(-\Gamma(x_0', x_1'; x_0, x_1)) \; . \quad (1.9)$$

The presence of tails in these cases can be intuitively understood regarding the Green functions $G^{(3)}$ and $G^{(2)}$ as the solutions of the inhomogeneous wave equation in four dimensions, with impulsive sources concentrated on the line $x_3 = 0$, or on the plane $x_2 = x_3 = 0$, respectively. At a fixed point of space and at arbitrarily late times, the field $\Phi$ is nonvanishing, since there are always contributions coming from points located far away on the line or on the plane. In this paper, however, we are interested in tails due to the curvature, rather than to the specific topology.

Even if the responsible for tails is the spacetime curvature, $R_{abc}{}^{d} \neq 0$ is not a sufficient condition for the development of tails; in fact in plane wave spacetimes one has $V = 0$ as well, although some components of the Riemann tensor are nonvanishing [167]. It might be tempting to attribute to the term $\xi R\,\Phi$ the responsability for the diffusive propagation. Actually this non minimal coupling influences the occurrence of tails, but the latter arise even when $\xi = 0$ (e.g., in de Sitter spacetime $V(x', x)$ does not vanish [152] even if $\xi = 0$). The occurrence of tails can therefore be studied excluding nonessential effects by considering simply the scalar wave equation

$$\Box\Phi = 0 \; , \quad (1.10)$$

in which the curvature-field coupling occurs only through the d'Alembertian $\Box$.

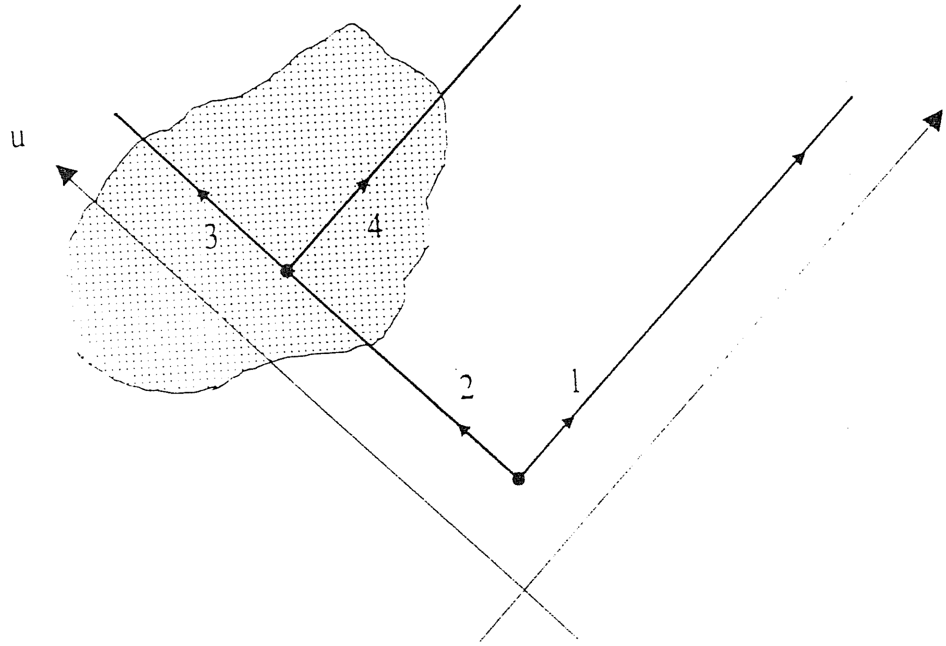


Figure 5.1: A very schematic representation of tail generation: Curvature is nonzero only in the shaded region. The ray 1 not entering the region propagates freely, whereas ray 2 is decomposed into a transmitted (3) and a reflected (4) part (null coordinates are employed).

The formation of tails can be understood as an effect of the backscattering of curvature on the waves; this is schematically represented in Fig. 5.1, which represents a portion of spacetime in null coordinates $u$, $v$. The curvature is nonzero only in the shaded region, and scatters waves analogously to what happens under the action of a potential. The ray 1, which does not enter the curved region, remains unperturbed, whereas the ray 2 can be decomposed into a transmitted (3) and a reflected (4) part; nevertheless we emphasize that such a representation is purely pictorial and does not properly describe many features of the real phenomenon, which has a continuous, rather than localized character: A better, although still schematic, picture is given in Fig. 5.2.

As in the usual scattering processes reflection is due to the variations of the potential, so here it is caused by the curvature. In fact, a region of spacetime containing a homogeneous gravitational field is flat: Consequently Eq. (1.4) is satisfied and there is no diffusion. Even in presence of curvature, however, reflection will not affect equally all the frequency components of the field $\Phi$; it is well known from the study of scattering in more common situations that only wavelengths greater than a typical length defined by the shape and size of the potential are significantly scattered. We expect therefore that reflection of waves by the background spacetime is important only for wavelengths greater than the typical scale of variation of the gravitational field (i.e., than the radius of curvature).

The decomposition (1.3) is formal; it does not give information neither about the importance of the diffusive term as compared to the "sharp" one, nor about the relative effect of backscattering on components of different frequencies. The previous remarks suggest that an alternative treatment in terms of reflection and transmission coefficients should be more helpful in clarifying these points. This method can be successfully applied to describe scattering by compact objects [154]-[158], but fails in the case of diffusion by the cosmological curvature, as will be discussed later.

Here we study the problem in a FLRW universe, deserving attention in particular to the $K = -1$ case, in which the situation is not complicated by the possibility that radiation could travel more than once through space sections, giving rise to spurious effects. In the case $K = 0$, on the other hand, the phenomenon turns out to be absent.

In the next section we reduce the problem to the study of a stationary Schrödinger equation, which is solved exactly in the following section. Then it is explained why the standard treatment in terms of reflection and transmission coefficients is not adequate to investigate this topic.

## 5.2 Tails in FLRW spacetimes

The scattering off the background curvature has been considered for waves emitted by a compact source [154]-[158]. Scattering appears to be relevant only in the induction zone and for long wavelengths, becoming negligible at large distances from the source and at late times, so that it is irrelevant for the problem of wave propagation in presence of astrophysical objects [24]. These general results can be qualitatively understood remembering that, as pointed

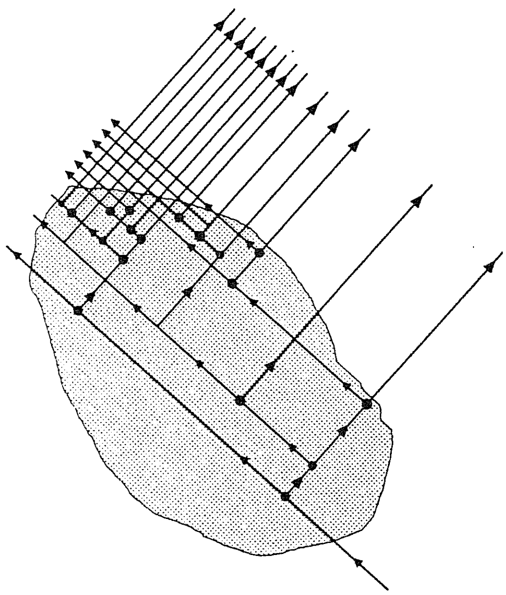

Figure 5.2: Another schematic representation of tail generation in a curved (shaded) region of spacetime.

out before, scattering occurs only in the regions of appreciable curvature. Since around a compact object the curvature drops off quickly as $M/r^3$, where $M$ is the mass of the object and $r$ is a Schwarzschild-like radial coordinate, the region of scattering is localized very near the source, as in Fig. 5.3. Moreover, $r > 2M$ and the "potential barrier" turns out to be not very high; consequently, most of the radiation escapes in the region where the curvature is negligible, and where it is no more backscattered. Reflection is appreciable only for wavelengths $\lambda \gtrsim M$.

The situation can be quite different when considering wave propagation in a cosmological model. In this case, the curvature never drops off, and scattering can in principle last forever; hence the fraction of radiation present at late times in the form of tails might be non negligible (see Fig. 5.4). Whether this effect is physically important constitutes the so-called *tail problem*; apparently such a possibility has been considered in the literature only in Ref. [159].

Electromagnetic radiation in conformally flat spacetimes (and in particu-

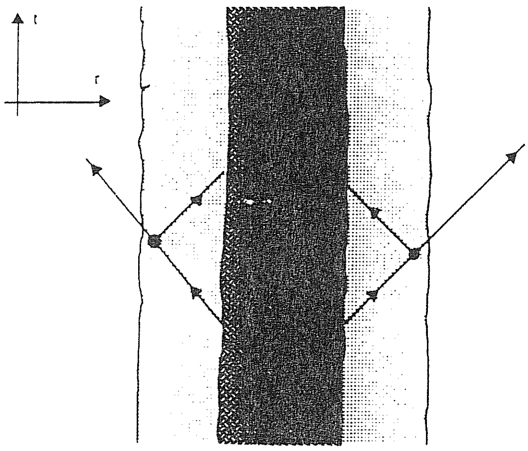

Figure 5.3: The darker region represents the world-tube of a compact object emitting radiation. In the grey region around the compact object curvature is not negligible and scatters the waves, like a "potential barrier". Outside this region the curvature drops off quickly, and scattering does not take place appreciably.

lar in FLRW spacetimes – see e.g. Ref. [23]) has no tails [153]. However the problem still holds for scalar and gravitational radiation. For the sake of simplicity, we shall treat explicitly only the case of a scalar field; the problem will be reduced to the solution of a quantum scattering problem, described by a Schrödinger equation with a suitable potential, in analogy to Refs. [155, 158]. Since the scalar and tensor wave equations admit the same kind of integral representation with Green functions, our conclusions are intended to hold broadly for gravitational radiation as well, with the FLRW metric as a background.

The simple wave equation (1.10) can be rewritten in the form

$$\partial_a(\sqrt{-g}\, g^{ab}\, \partial_b \Phi) = 0\,. \tag{2.1}$$

In a FLRW background,

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + f^2(\chi)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right] \tag{2.2}$$

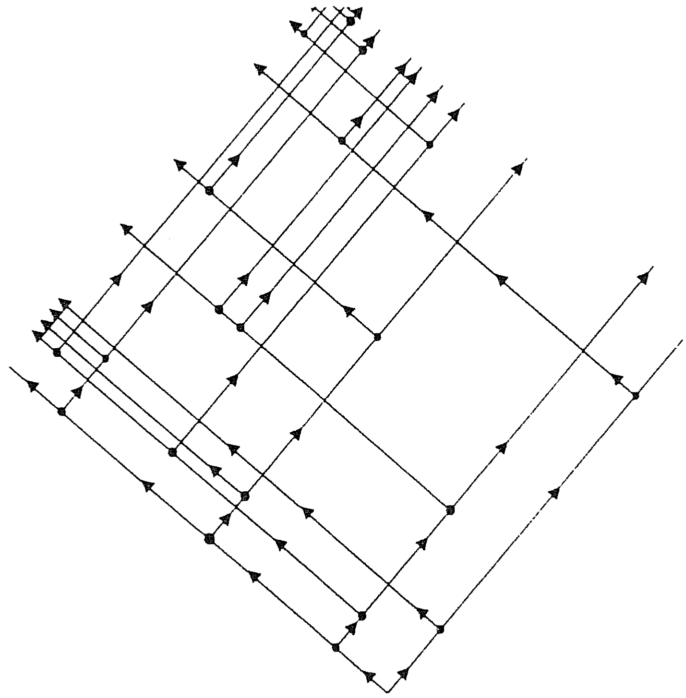

Figure 5.4: Tail generation in a cosmological situation; since the curvature never drops off, scattering can last forever, and the fraction of radiation in tails can, in principle, be relevant.

where

$$f(\chi) = \begin{cases} \sinh\chi & \text{if} \quad K = -1 \\ \chi & \text{if} \quad K = 0 \\ \sin\chi & \text{if} \quad K = +1\,, \end{cases} \tag{2.3}$$

Eq. (2.1) becomes

$$-\frac{1}{a}\,\partial_t\left(a^3\partial_t\Phi\right) + \frac{1}{f^2}\left[\partial_\chi\left(f^2\partial_\chi\Phi\right) + \frac{1}{\sin\theta}\,\partial_\theta\left(\sin\theta\,\partial_\theta\Phi\right) + \frac{1}{\sin^2\theta}\,\partial_\varphi\partial_\varphi\Phi\right] = 0\,. \tag{2.4}$$

Separation of time and space variables

$$\Phi(t,\chi,\theta,\varphi) = T(t)S(\chi,\theta,\varphi) \tag{2.5}$$

leads to

$$\frac{1}{a}\frac{d}{dt}\left(a^3\frac{dT}{dt}\right) + kT = 0 \tag{2.6}$$

and

$$\partial_\chi\left(f^2\partial_\chi S\right) + \frac{1}{\sin\theta}\,\partial_\theta\left(\sin\theta\,\partial_\theta S\right) + \frac{1}{\sin^2\theta}\,\partial_\varphi\partial_\varphi S + kSf^2 = 0\,, \tag{2.7}$$

where $k$ is a separation constant. Further separation of radial and angular coordinates

$$S(\chi,\theta,\varphi) = X(\chi)\,Y(\theta,\varphi)$$

leads to the usual equation for spherical harmonics $Y_{lm}(\theta,\varphi)$

$$\frac{1}{\sin\theta}\,\partial_\theta\left(\sin\theta\,\partial_\theta Y_{lm}\right) + \frac{1}{\sin^2\theta}\,\partial_\varphi\partial_\varphi Y_{lm} = -l(l+1)Y_{lm}\ , \tag{2.8}$$

and to

$$\frac{d}{d\chi}\left(f^2\,\frac{dX_l}{d\chi}\right) + kf^2X_l = l(l+1)X_l\ . \tag{2.9}$$

Setting $\Psi_l(\chi) \equiv f(\chi)X_l(\chi)$ and using Eq. (2.3), the radial equation becomes

$$\frac{d^2\Psi_l}{d\chi^2} + \left[2E - \frac{l(l+1)}{f^2(\chi)}\right]\Psi_l = 0\ , \tag{2.10}$$

where

$$2E \equiv k + K\ . \tag{2.11}$$

Eq. (2.10) describes also the stationary Schrödinger problem for a particle subject to the central potential

$$V_l(\chi) \equiv \frac{l(l+1)}{2}\left(\frac{1}{f^2(\chi)} - \frac{1}{\chi^2}\right) \tag{2.12}$$

($\chi$ plays now the role of an ordinary radial coordinate in the three-dimensional Euclidean space), simply writing the radial part of the Schrödinger wave function as $\Psi_l(\chi)/\chi$. The problem of wave propagation in a FLRW model is thus reduced to the study of the solutions of Eq. (2.10), i.e. of the behaviour of stationary states with given orbital quantum number $l$ for a particle moving in the potential (2.12). In particular, the existence and relevance of tails can be investigated by considering the process of emission of particles from the centre $\chi = 0$ within this analogy.

For $K = +1$ the spacetime has closed spatial sections; the transmitted radiation might therefore be allowed to pass more than once through a given point of space, superposing to the fraction of radiation which is possibly reflected. Hence, the analysis of diffusion produced by the background curvature would be, in this model, complicated by such a spurious effect. Being interested only in the essential features of the phenomenon of tails, we shall not deal any more with this case.

For $K = 0$, on the other hand, Eq. (2.10) becomes

$$\frac{d^2\Psi_l}{d\chi^2} + \left[2E - \frac{l(l+1)}{\chi^2}\right]\Psi_l = 0 \ , \tag{2.13}$$

which is familiar from the quantum mechanical description of a *free* particle in spherical coordinates (see e.g. Ref. [168]). Its general solution is a superposition of spherical waves, and corresponds to the absence of any scattering off the background curvature, i.e. to the absence of tails.

The case $K = -1$ is more interesting, and we shall concentrate on it throughout all the rest of the paper. Eq. (2.10) takes the form

$$\frac{d^2\Psi_l}{d\chi^2} + \left[2E - \frac{l(l+1)}{\sinh^2\chi}\right]\Psi_l = 0 \ , \tag{2.14}$$

which allows the eigenvalue $E$ to assume any positive value: It is convenient to define

$$2E \equiv p^2 \ , \tag{2.15}$$

with $p > 0$. The shape of $V_l(\chi)$ for this case is reported in Fig. 5.5 for various values of $l$, and corresponds to a potential well whose depth and size increase with $l$. Insight in scattering problems suggests that reflection will be important only for values of $E$ smaller than $l(l+1)$, the depth of the well, i.e. for

$$p \lesssim l \ . \tag{2.16}$$

The quantity $p$ plays the role of a wave number; therefore the conclusion that only waves with $p \lesssim l$ undergo reflection can be interpreted saying that the backscattering off curvature is efficient only for wavelengths greater than $a(t)/l$. The phenomenon admits an interesting alternative representation observing that, for travelling waves, $k > 0$ can be written as $k = \omega^2$; Eqs. (2.11) and (2.15) give then

$$\omega^2 = p^2 + 1 \ , \tag{2.17}$$

which can be regarded as the dispersion relation for waves with an effective "mass" $a(t)^{-1}$: Only components with frequency and wave numbers much greater than $a(t)^{-1}$ propagate without diffusion.

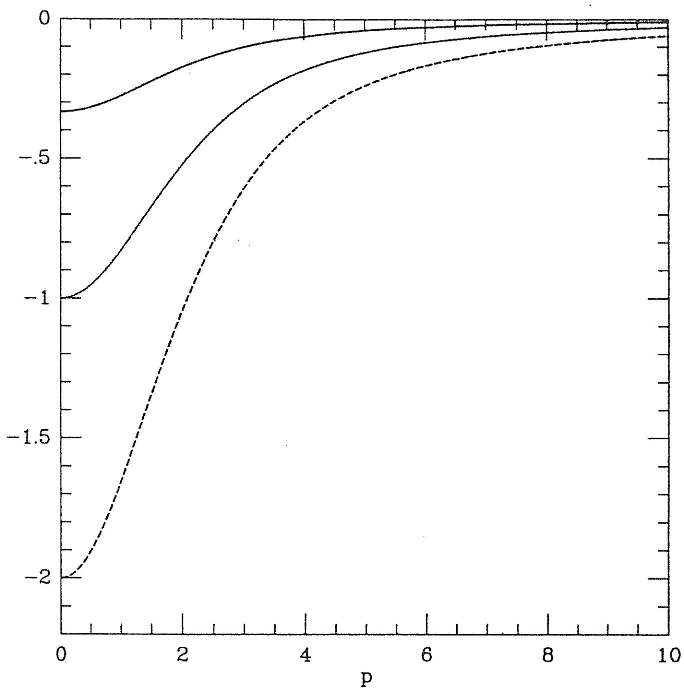


Figure 5.5: The potential $V_l(\chi)$ for $l = 1$ (solid line), $l = 2$ (dotted line), and $l = 3$ (dashed line). The value of the minimum, attained for $\chi = 0$, is $-l(l+1)/6$.

## 5.3 Exact solution

Let us rewrite Eq. (2.14) using the definition (2.15):

$$\frac{d^2\Psi_l}{d\chi^2} + \left[p^2 - \frac{l(l+1)}{\sinh^2\chi}\right]\Psi_l = 0 \ . \tag{3.1}$$

Writing

$$\Psi_l(\chi) \equiv e^{ip\chi}\,\phi_l(\chi) \ , \tag{3.2}$$

Eq. (3.1) transforms in the following equation for $\phi_l$:

$$\frac{d^2\phi_l}{d\chi^2} + 2\,ip\,\frac{d\phi_l}{d\chi} - \frac{l(l+1)}{\sinh^2\chi}\,\phi_l = 0 \ . \tag{3.3}$$

It is now convenient to define the new variable $z \in (-\infty, 0)$ as

$$z \equiv \frac{1}{2}\,(1 - \coth\chi) = \frac{1}{1 - e^{2\chi}} \ ; \tag{3.4}$$

in terms of $z$, Eq. (3.3) takes the form

$$z\,(1-z)\,\frac{d^2\phi_l}{dz^2} + (1 - ip - 2z)\,\frac{d\phi_l}{dz} + l(l+1)\phi_l = 0 \ , \tag{3.5}$$

which is immediately recognized as an hypergeometric equation [168] with coefficients

$$\begin{aligned}\alpha &= -l \\ \beta &= l+1 \\ \gamma &= 1-ip\ .\end{aligned} \tag{3.6}$$

This equation admits two independent solutions [168],

$$\phi_l^{(1)} = F(\alpha,\beta,\gamma;z) = F(-l,l+1,1-ip;z)\ , \tag{3.7}$$

and

$$\begin{aligned}\phi_l^{(2)} &= z^{1-\gamma}(1-z)^{\gamma-\alpha-\beta}F(1-\beta,\,1-\alpha,\,2-\gamma;z) = \\ &= (-1)^{ip}e^{-2ip\chi}F(-l,l+1,1+ip;z)\ ,\end{aligned} \tag{3.8}$$

where $F$ denotes the hypergeometric function. Correspondingly, Eq. (3.2) gives the two independent solutions of Eq. (3.1):

$$\Psi_l^{(1)}(\chi) = e^{ip\chi}F(-l,l+1,1-ip;z(\chi))\ ; \tag{3.9}$$

$$\Psi_l^{(2)}(\chi) = (-1)^{ip}e^{-ip\chi}F(-l,l+1,1+ip;z(\chi))\ . \tag{3.10}$$

The general solution of Eq. (3.1) is thus

$$\Psi_l(\chi) = A_l\,\Psi_l^{(1)}(\chi) + B_l\,\Psi_l^{(2)}(\chi)\ , \tag{3.11}$$

with $A_l, B_l \in \mathbb{C}$ arbitrary.

In our specific physical problem, the presence of tails is characterized by the fact that a pulse of radiation emitted at $\chi = 0$ is partially backscattered. A stationary emission must therefore correspond to a solution $\Psi_l(\chi)$ of Eq. (3.1) which for $\chi \to 0$ contains incoming (reflected) as well as outgoing (emitted) radiation, whereas for $\chi \to +\infty$ only outgoing (transmitted) waves are present. Therefore, we must impose to the general solution (3.11) a boundary condition that corresponds to the absence of incoming waves at $\chi \to +\infty$. This is easily accomplished by noticing that for $\chi \to +\infty$ one has $z \to 0$, and $F \to 1$. Hence, the asymptotic form of $\Psi_l^{(1)}$ and $\Psi_l^{(2)}$ is

$$\Psi_l^{(1)}(\chi \to +\infty) \approx e^{ip\chi}\ , \tag{3.12}$$

$$\Psi_l^{(2)}(\chi \to +\infty) \approx (-1)^{ip}\,e^{-ip\chi}\ , \tag{3.13}$$

which correspond, respectively, to outgoing and ingoing waves. The required boundary condition is therefore $B_l = 0$, leading to

$$\Psi_l(\chi) = A_l\, e^{ip\chi} F(-l, l+1, 1-ip; z(\chi)) \tag{3.14}$$

as the specific solution of our problem.

It is important to remark that, since $\alpha = -l$, the series defining $F$ terminates [168], and $F$ is therefore a polynomial of degree $l$ in $z$:

$$F(-l, l+1, 1-ip; z) = \sum_{n=0}^{l} \begin{pmatrix} l+n \\ l \end{pmatrix} \frac{(-l)_n}{(1-ip)_n}\, z^n \,, \tag{3.15}$$

where $\forall \alpha \in \mathbb{C}$,

$$(\alpha)_n \equiv \alpha(\alpha+1)...(\alpha+n-1) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \,, \tag{3.16}$$

and $(\alpha)_0 \equiv 1$. This result is useful in comparing the exact solution (3.14) with the asymptotic one for $\chi \to 0$.

For $l = 0$, the hypergeometric function is identically equal to 1, and Eq. (3.14) reduces to

$$\Psi_0(\chi) = A_0\, e^{ip\chi} \,, \tag{3.17}$$

which contains no incoming waves for $\chi \to 0$. Therefore, in this case reflection is absent.

## 5.4 Asymptotic analysis for $\chi \to 0$

Looking for solutions of Eq. (3.1) of the form

$$\Psi_l(\chi) \sim \chi^m + O(\chi^{m+1}) \,, \tag{4.1}$$

one finds immediately that either $m = -l$ or $m = l + 1$. The general solution of Eq. (3.1) can therefore be written as

$$\Psi_l(\chi) = C_l\, f_l(\chi) + D_l\, g_l(\chi) \,, \tag{4.2}$$

with $C_l$, $D_l \in \mathbb{C}$ arbitrary and $f_l$, $g_l$ two particolar solutions which behave, for $\chi \to 0$, as

$$f_l(\chi) = \frac{1}{\chi^l} + O(\chi^{1-l}) \,, \tag{4.3}$$

and

$$g_l(\chi) = \chi^{l+1} + O(\chi^{l+2}) \,. \tag{4.4}$$

The coefficients $C_l$ and $D_l$ for any particular solution $\Psi_l$ can be obtained by comparing Eqs. (4.2)–(4.4) with the expansion of $\Psi_l(\chi)$ for $\chi \to 0$. In the particular case of Eq. (3.14) one gets, by Eq. (C.8),

$$C_l = A_l \, \frac{(2l-1)!!}{(1-ip)_l} \,, \tag{4.5}$$

and

$$D_l = A_l \, \frac{ip\,(1+ip)_l}{(2l+1)!!} \,. \tag{4.6}$$

However, as far as a calculation of the reflection and transmission coefficients is concerned, one does not need $C_l$ and $D_l$, but rather the coefficients $A_l^{(\pm)}$ characterized by writing $\Psi_l(\chi)$ as

$$\Psi_l(\chi) = A_l^{(+)}\,\Psi_l^{(+)}(\chi) + A_l^{(-)}\,\Psi_l^{(-)}(\chi) \,, \tag{4.7}$$

where $\Psi_l^{(\pm)}$ are independent solutions of Eq. (3.1) corresponding to waves which are purely outgoing (+) and ingoing (−) for $\chi \to 0$. Since, as done above, a comparison between Eqs. (4.2)–(4.4) and the exact solution (3.14) allows one to determine $C_l$ and $D_l$, but not directly $A_l^{(\pm)}$, one needs to express the former coefficients in terms of the latter ones.

In order to do this, notice that, since $\Psi_l^{(\pm)}$ are particular solutions of Eq. (3.1), they can be written, according to Eqs. (4.2)–(4.4), as

$$\Psi_l^{(\pm)}(\chi) = C_l^{(\pm)}\,f_l(\chi) + D_l^{(\pm)}\,g_l(\chi) \,. \tag{4.8}$$

The coefficients $C_l^{(\pm)}$ and $D_l^{(\pm)}$ in Eq. (4.8) should be chosen in such a way as to guarantee the prescribed character (purely outgoing or ingoing) of the solutions $\Psi_l^{(\pm)}$ for $\chi \to 0$. Unfortunately, such solutions do not exist for $l \neq 0$. This can be qualitatively understood by observing that in the neighbourhood of $\chi = 0$ the centrifugal potential $l(l+1)/\chi^2$ varies extremely rapidly; therefore, waves in that region are continuously backscattered[1]. As a consequence, one cannot select purely outgoing or ingoing solutions of Eq. (3.1) as $\chi \to 0$.

---

[1]In the analogy with the quantum particle, based on the Schrödinger equation (3.1), one can remark that particles with nonzero angular momentum cannot "enter" or "exit" from the centre $\chi = 0$.

Formally, this can be realized by characterizing such solutions as asymptotic eigenstates of the radial momentum operator $\hat{P}_\chi \equiv -id/d\chi$, i.e.

$$-i\,\frac{d\Psi_l^{(\pm)}}{d\chi} = \pm\lambda\,\Psi_l^{(\pm)} + (\text{higher powers of } \chi)\ , \qquad (4.9)$$

with $\lambda > 0$. Since for $l \neq 0$

$$\left[\hat{H}_l, \hat{P}_\chi\right] = O(\chi^{-3}) \neq \hat{0}\ , \qquad (4.10)$$

where

$$\hat{H}_l \equiv -\frac{1}{2}\frac{d^2}{d\chi^2} + \frac{l(l+1)}{2\sinh^2\chi}\ , \qquad (4.11)$$

it follows that there are no common eigenstates of the operators $\hat{H}_l$ and $\hat{P}_\chi$, i.e. there are no solutions of Eq. (3.1) which satisfy the asymptotic condition (4.9). An explicit check can be performed by substituting Eq. (4.8) into (4.9) and requiring the coefficients of the leading powers of $\chi$ to vanish; one obtains the trivial result $C_l^{(\pm)} = D_l^{(\pm)} = 0$.

The non-existence of solutions of Eq. (3.1) which are purely ingoing or outgoing in the region $\chi \to 0$, prohibits one to define reflection and transmission coefficients as in the usual treatments of scattering problems. To the authors' knowledge, no method has been developed in the literature to deal with similar situations. The case considered here is quite different from the simpler ones arising when studying diffusion of waves by Schwarzschild black holes, where a coordinate transformation can be found for which the corresponding one dimensional Schrödinger problem involves a localized potential barrier [155].

## 5.5 Remarks

The most appropriate treatment of the physical problem pointed out in this paper would be the explicit determination of the reflection coefficient describing backscattering by the cosmological curvature, and characterizing quantitatively the fraction of radiation which does not propagate along the light cone. The impossibility of carrying on this approach leads one to look for alternative ways of studying the phenomenon. A straightforward idea could be to pursue the formal analogy between the tail problem and quantum scattering, computing the cross section for the central potential $V_l(\chi)$. Although such a calculation is in principle possible, it cannot however be regarded as a satisfactory solution to our specific physical problem, which concerns waves emitted from $\chi = 0$ rather than incoming from infinity. Consequently, the expression of the cross section will give nothing but the qualitative bounds already discussed at the end of Sec. 5.2. A satisfactory study of the subject seems thus to require a radically different approach.

In spite of these formal deficiencies of the treatment, some features of the phenomenon emerge clearly: The tail problem for radiation in FLRW spacetimes regards only waves whose wavelengths are at least of order $a(t)$; waves with wavelengths much smaller than the scale factor are not affected by scattering off the background curvature. Radiation of such a long wavelength is then "confined" at high redshifts. Since this effect regards only a very small, extreme part of the spectrum of the gravitational radiation content of the universe, and since the variability of these waves on scales of order $a(t)$ is too slow to be observed, it appears that the effect is hardly observable. However, no conclusion can be drawn without a careful investigation; in fact radiation of very long wavelengths can possibly lead to physically relevant effects (see e.g. Ref. [169]). These considerations about the way the background curvature affects scalar waves can be extended to gravitational waves, due to the common structure of the integral representation for the scalar and tensor wave equation [153], and to the fact that the effect is present even without direct curvature coupling. Moreover, the tail problem still holds in cosmological models others than the FLRW ones, and, what is more important, it concerns also electromagnetic radiation in non conformally flat spacetimes.

Scalar fields satisfying the wave equation

$$\Box\Phi + \frac{\partial V}{\partial \Phi} = 0 \ , \tag{5.1}$$

(which reduces to Eq. (1.10) in the slowly rolling regime), where $V(\Phi)$ is a suitable function, are considered in the inflationary models of the universe (see e.g. [170] and references therein). The tail problem for the inflaton regards length scales relevant for cosmology, so it could possibly have some importance in problems connected to the physics of the early universe. The cosmological model most commonly used in inflationary theories is de Sitter spacetime, and the considerations of Ref. [162] apply. This is however beyond the scope of the present work.

## 5.6 Huygens' principle and characteristic propagation property for waves in curved spacetimes

The occurrence of tails for scalar, electromagnetic, and gravitational radiation is particularly evident from the form of the Green function for the corresponding wave equation, which can be decomposed, at any point $x$ of spacetime, into a "sharp" and a "diffusive" term, with support along and inside the light cone through $x$, respectively [152, 171].

This circumstance has been associated in the literature both to the failure of Huygens' principle [152, 23, 151, 172] and to violations of the "characteristic propagation property" (CPP) [173]. However, the relationship between these two characterizations of the effect has never received much attention, remaining somewhat unclear.

Our purpose is to give a unitary formulation of the properties mentioned above, which allows one to compare them. Sec. 5.7 is devoted to the introduction and discussion of the "tail-free property", that characterizes formally the absence of diffusive effects. In Sec. 5.8 the relationship between the tail-free and the characteristic propagation properties is investigated. It is found that the CPP holds independently of the tail-free property only in the case of minimal coupling in two spacetime dimensions. The peculiarity of gauge fields, for which tails in the potentials do not necessarily correspond to physical effects, is considered in Sec. 5.9 for the case of electromagnetism. Sec. 5.10 contains the conclusions and outlines possible generalizations. The following is based on Ref. [153].

## 5.7 Formal characterization of tails

The propagation of weak gravitational radiation on a curved spacetime can be studied writing the metric as

$$\tilde{g}_{ab} = g_{ab} + h_{ab} \ ,$$

where $h_{ab}$ are "small" perturbations around the background metric $g_{ab}$. In the transverse-traceless gauge [2]

$$\nabla^a h_{ab} = 0\,, \tag{7.1}$$

$$g^{ab} h_{ab} = 0\,, \tag{7.2}$$

the vacuum Einstein equations lead to the wave-like equation

$$\Box h_{ab} - 2R^c{}_{ab}{}^d h_{cd} = 0\,. \tag{7.3}$$

Analogously, the electromagnetic potential on a curved spacetime in the absence of sources and in the Lorentz gauge

$$\nabla^a A_a = 0 \tag{7.4}$$

satisfies the wave equation

$$\Box A_a - R_{ab} A^b = 0\,. \tag{7.5}$$

Together with Eqs. (7.3) and (7.5) one can consider also the wave equation for a scalar field

$$\Box\Phi - \xi R\,\Phi = 0\,, \tag{7.6}$$

where $\xi$ is a numerical constant.

The behaviour of the fields $\Phi$; $A_a$ and $h_{ab}$ satisfying Eqs. (7.6), (7.5) and (7.3) can be conveniently studied by introducing the concept of Green function [152, 171]. Denoting as $G(x',x)$, $G^{a'}{}_a(x',x)$ and $G^{(S)\,a'b'}{}_{ab}(x',x)$ the Green functions for a scalar, vector and symmetric tensor field respectively, they satisfy the following equations:

$$\left[\Box' - \xi R(x')\right] G(x',x) = -\delta(x',x)\,, \tag{7.7}$$

$$\left[\delta^{a'}{}_{b'}\Box' - R^{a'}{}_{b'}\right] G^{b'}{}_a(x',x) = -\delta^{a'}{}_a(x',x)\,, \tag{7.8}$$

$$\left[\delta^{a'}{}_{c'}\,\delta^{b'}{}_{d'}\,\Box' - 2\,R^{a'}{}_{c'd'}{}^{b'}\right] G^{(S)\,c'd'}{}_{ab}(x',x) = -\delta^{(S)\,a'b'}{}_{ab}(x',x)\,, \tag{7.9}$$

where $\delta(x',x)$ is the delta function on spacetime such that for each test function [174] $f$,

$$\int d^4x'\,\sqrt{-g(x')}\,f(x')\,\delta(x',x) = f(x)\,, \tag{7.10}$$

[2] All quantities except $h_{ab}$ are associated with the unperturbed spacetime.

and $\delta^{a'}{}_a\,(x',x)$, $\delta^{(S)\,a'b'}{}_{ab}\,(x',x)$ are the "elementary sources", which can be constructed out of $\delta\,(x',x)$ and the two-point vector of geodesic parallel transport [171, 175].

If $\mathcal{N}$ is a normal domain of spacetime not containing sources, the use of the Green functions allows one to write the fields at each point $x \in \mathcal{N}$ in terms of their values on the boundary $\partial\mathcal{N}$ as

$$\Phi(x) = \int_{\partial\mathcal{N}} dS^{a'}(x')\, G\,(x',x)\overleftrightarrow{\nabla}_{a'}\,\Phi(x')\ , \tag{7.11}$$

$$A_a(x) = \int_{\partial\mathcal{N}} dS^{b'}(x')\, G^{a'}{}_a\,(x',x)\overleftrightarrow{\nabla}_{b'}\,A_{a'}(x')\ , \tag{7.12}$$

$$h_{ab}(x) = \int_{\partial\mathcal{N}} dS^{c'}(x')\, G^{(S)\,a'b'}{}_{ab}\,(x',x)\overleftrightarrow{\nabla}_{c'}\,h_{a'b'}(x')\ , \tag{7.13}$$

where $dS^{a'}(x')$ is the oriented volume element on the hypersurface $\partial\mathcal{N}$ at $x'$, and

$$f_1\overleftrightarrow{\nabla} f_2 \equiv f_1\nabla f_2 - f_2\nabla f_1\ , \tag{7.14}$$

for any differentiable functions $f_1$, $f_2$. Since the integral representation is essentially the same for $\Phi$, $A_a$ and $h_{ab}$ we shall limit ourselves, in the following, to consider a scalar field, keeping in mind that the results can be generalized to the other cases without introducing substantial conceptual differences.

The *advanced* $(+)$ and *retarded* $(-)$ Green functions can be written [152] as

$$G^{(\pm)}\,(x',x) = \Sigma\,(x',x)\ \delta^{(\pm)}(\Gamma(x',x)) + V\,(x',x)\ \Theta^{(\pm)}(-\Gamma(x',x))\ , \tag{7.15}$$

where $\Gamma\,(x',x)$ is the square of the proper distance calculated along the unique geodesic connecting $x'$ and $x$ in the normal domain $\mathcal{N}$; $\delta^{(\pm)}$ and $\Theta^{(\pm)}$ are, respectively, the Dirac delta distribution and the Heaviside step function with support in the future $(+)$ and past $(-)$ of $x'$. Hereafter we shall restrict ourselves to consider the retarded Green function, dropping the $(-)$. The functions $\Sigma$ and $V$ can be determined uniquely once the spacetime is given [152, 171]. The equations for $V$ are hard to solve, and no analytic expression for $V\,(x',x)$ is known in a general spacetime. The only known solutions correspond to Minkowski, plane wave, and de Sitter spacetimes. In the first two cases, $V$ turns out to be identically zero [167] and the diffusive term is thus absent;

moreover, it has been proved [176] that these are the only empty spacetimes which enjoy this property. In the case of de Sitter spacetime the explicit expression for $V$ is given in Ref. [152] (p. 163); in particular, if $\xi = 0$, $V$ is constant [3].

In general $V$ is different from zero, and both the terms in the right hand side of Eq. (7.15) contribute to the propagation of waves. The first term has support entirely along the null cone, and represents therefore a lightlike propagation of the initial data, according to Eq. (7.11). The second one accounts for the possibility of a diffusive contribution from the interior of the light cone, that is for a *timelike* propagation of the initial data. Hence, a nonvanishing function $V(x', x)$ might convey part of the radiation in the interior of the light cone. This effect has sometimes been regarded as a deviation from Huygens' principle [23, 151, 172], because it leads one to conclude that radiation is not, in general, concentrated on the wavefront [4] (for example, using the retarded function and restricting, for sake of simplicity, to a spacetime that is spherically symmetric around some point $r = 0$, one sees that the radiation emitted as a pulse at $r = 0$, $t = 0$ is distributed, at $t > 0$, within an entire region $r \leq R(t)$, and is not simply concentrated on the sphere $r = R(t)$).

From now on, we shall refer to the part of radiation present in the interior of the null cone as the *tail*. This name is justified by the fact that this fraction of the radiation stays back with respect to that propagating on the wavefront.

These heuristic considerations can be formalized as follows. Let $\mathcal{M}$ be a globally hyperbolic region of spacetime, and let $\mathcal{S}$ be a Cauchy hypersurface with normal $n^a$, over which a set of smooth data $\mathcal{D}$ for Eq. (7.6) is specified [5]. Moreover, let $\mathcal{D}$ have compact support $\mathcal{C} \subset \mathcal{S}$. We say that the data $\mathcal{D}$ *produce*

[3] Remarkably enough, when $\xi = 1/6$ one has $V(x', x) = 0$. This fact can be explained by noticing that for such value of $\xi$, Eq. (7.6) is conformally invariant [1]: The absence of diffusion in Minkowski spacetime is therefore transferred to conformally flat spacetimes (de Sitter being an example).

[4] Huygens' principle (see [151] for a precise formulation) is violated by any linear, normal hyperbolic partial differential equation in which the solution depends on an *odd* number of variables [151, 172]. When the number of variables is even, as in a four-dimensional spacetime, Huygens' principle may or may not hold. The tail-free property defined below turns out to be equivalent to the Huygens' principle defined in Ref. [151].

[5] Where $\mathcal{S}$ is spacelike ($n_a n^a = -1$), $\mathcal{D} = \{(\Phi, n^a \nabla_a \Phi)\}$; where $\mathcal{S}$ is null ($n_a n^a = 0$), $\mathcal{D} = \{\Phi\}$ (cf. Ref. [151], p. 179).

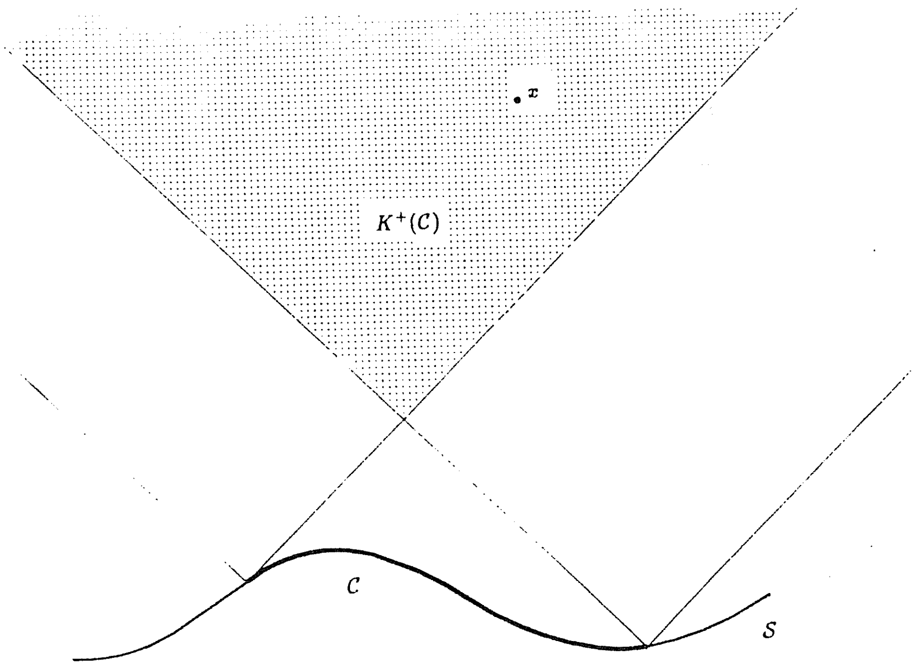


Figure 5.6: The spacetime region $K^+(\mathcal{C})$.

*no tails* in $\mathcal{M}$ iff $\forall x \in K^+(\mathcal{C}) \cap \mathcal{M}$, $\Phi(x) = 0$, where $\Phi$ is the unique solution of the Cauchy problem for Eq. (7.6) with initial data $\mathcal{D}$, and $K^+(\mathcal{C})$ is the set of all points in $J^+(\mathcal{C})$ (the causal future of $\mathcal{C}$) which cannot be reached from $\mathcal{C}$ by a null geodesic (see Fig. 5.6). Eq. (7.6) is said to be *tail-free* in the region $\mathcal{M}$ iff each set of initial data $\mathcal{D}$ in $\mathcal{M}$ with compact support produces no tails in $\mathcal{M}$.

When $\mathcal{M}$ is also a normal domain, the previous definitions can be reformulated more explicitly. In fact, given the Cauchy hypersurface $\mathcal{S}$, the region $J^+(\mathcal{S}) \cap \mathcal{M}$ is normal as well, and we can use the integral representation, Eq. (7.11), to specify the value of $\Phi$ at each $x \in J^+(\mathcal{S}) \cap \mathcal{M}$ in terms of the data $\mathcal{D}$, as

$$\Phi(x) = \int_{\mathcal{S}} dS^{a'}(x')\, G(x', x) \overleftrightarrow{\nabla}_{a'} \Phi(x') \,. \tag{7.16}$$

In particular, if $\mathcal{D}$ have compact support $\mathcal{C}$ on $\mathcal{S}$, and $x \in K^+(\mathcal{C}) \cap \mathcal{M}$, the $\delta$-term in Eq. (7.15) gives a vanishing contribution to the right hand side of Eq. (7.16), which becomes

$$\Phi(x) = \int_{\mathcal{C}} dS^{a'}(x')\, \left[\Theta\left(-\Gamma\left(x', x\right)\right) V\left(x', x\right)\right] \overleftrightarrow{\nabla}_{a'} \Phi(x') \,. \tag{7.17}$$

Since

$$\nabla_{a'}\left[\Theta\left(-\Gamma\left(x',x\right)\right)V\left(x',x\right)\right] = -\delta\left(\Gamma\left(x',x\right)\right)\nabla_{a'}\Gamma\left(x',x\right)V\left(x',x\right) + \\ +\Theta\left(-\Gamma\left(x',x\right)\right)\nabla_{a'}V\left(x',x\right)\ , \quad (7.18)$$

we have that $\forall x \in K^{+}(\mathcal{C}) \cap \mathcal{M}$

$$\Phi(x) = \int_{\mathcal{C}} dS^{a'}(x')\, V\left(x',x\right) \overleftrightarrow{\nabla}_{a'} \Phi(x')\ . \quad (7.19)$$

The no-tail condition can therefore be expressed in terms of vanishing of the integral on the right hand side of Eq. (7.19). It is obvious from this equation that

$$V\left(x',x\right) = 0 \quad (7.20)$$

is a sufficient condition for Eq. (7.6) to be tail-free in $\mathcal{M}$; in particular we recover the result that the propagation of radiation in Minkowski and plane wave spacetimes is not accompanied by tails. Actually, Eq. (7.20) is also a necessary condition; this can be proved taking advantage of the arbitrariness of the initial data $\mathcal{D}$. Choosing in fact $\mathcal{C}$ spacelike and, $\forall x' \in \mathcal{C}$,

$$\Phi(x') = 0 \quad (7.21)$$

and

$$n^{a'}(x')\nabla_{a'}\Phi(x') = \Phi_0\, \delta_{\mathcal{C}}\left(x', \bar{x}'\right)\ , \quad (7.22)$$

where $\bar{x}'$ is an arbitrary point in $\mathcal{C}$, $\Phi_0$ is an arbitrary nonzero constant, and $\delta_{\mathcal{C}}$ is the three-dimensional delta function in the region $\mathcal{C}$, we trivially get Eq. (7.20).

## 5.8 Relationship between tail-free and characteristic propagation property

Even if Eq. (7.6) is not tail-free in a region $\mathcal{M}$, one might nevertheless be interested in data that produce no tail in $\mathcal{M}$. In the literature particular attention has been devoted to the characteristic Cauchy problem, in which the support $\mathcal{C}$ is null; in this context, the characteristic propagation property for

Eq. (7.6) has been defined [173]. With our terminology, Eq. (7.6) *satisfies the CPP in a region* $\mathcal{M}$ iff each set of data with null compact support $\mathcal{C}$ produces no tail in $\mathcal{M}$. It is obvious that in a tail-free region, Eq. (7.6) satisfies the CPP. As far as the converse is concerned, apparently the CPP is a weaker condition, but the question whether the CPP implies the tail-free property is not a trivial one. The present section is devoted to this problem.

As the tail-free property, so the CPP becomes particularly treatable if $\mathcal{M}$ is assumed to be a normal domain. In this case the field at each point $x \in K^+(\mathcal{C}) \cap \mathcal{M}$ is given by Eq. (7.19) which becomes, using the Leibnitz rule,

$$\Phi(x) = \int_{\mathcal{C}} dS^{a'}(x')\nabla_{a'}\left[V(x',x)\,\Phi(x')\right] - 2\int_{\mathcal{C}} dS^{a'}(x')\,\Phi(x')\nabla_{a'}V(x',x)\,. \quad (8.1)$$

Let now $\mathcal{U} \subseteq \mathcal{M}$ be a region containing $\mathcal{C}$, and let $f : \mathcal{U} \to \mathbb{R}$ be a suitable function defining $\mathcal{C}$ through the condition $f(x) =$ constant. If $\mathcal{C}$ is a null hypersurface, we can introduce in $\mathcal{U}$ the coordinates [177]

$$\left(x^1 \equiv f, x^2 \equiv \rho, x^3, x^4\right)\ , \quad (8.2)$$

where $\rho$ is the affine parameter along the geodesics that are integral curves of the vector field

$$n^a = g^{ab}\nabla_b f\ , \quad (8.3)$$

which is tangent to $\mathcal{C}$; $x^3$ and $x^4$ label these geodesics on $\mathcal{C}$. In such coordinates, the components of the metric $g^{ab}$ can be written [177] as

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & & & \\ 0 & & \left(g^{\alpha\beta}\right) & \\ 0 & & & \end{pmatrix}\ , \quad (8.4)$$

where now $\alpha, \beta \in \{2, 3, 4\}$; the components of the normal vector are

$$n^a = \delta^a_2 \quad (8.5)$$

and

$$n_a = \delta^1_a\ . \quad (8.6)$$

The first term on the right hand side of Eq. (8.1) can now be transformed as

$$\int_C dS^{a'}(x')\nabla_{a'}\left[V(x',x)\,\Phi(x')\right] =$$

$$= \int_C d^3x'\sqrt{-g(x')}\,n^{\alpha'}(x')\,\partial_{\alpha'}\left[V(x',x)\,\Phi(x')\right] =$$

$$= -\int_C dS^{a'}(x')\,\Phi(x')V(x',x)\frac{1}{\sqrt{-g(x')}}\,\partial_{\alpha'}\left[\sqrt{-g(x')}\,n^{\alpha'}(x')\right] \; , \quad (8.7)$$

where Gauss' theorem in $\mathbb{R}^3$ has been applied, and a term has been dropped using the compactness of $\mathcal{C}$. Eq. (8.5) allows us to write

$$\frac{1}{\sqrt{-g(x')}}\,\partial_{\alpha'}\left[\sqrt{-g(x')}\,n^{\alpha'}(x')\right] = \frac{1}{\sqrt{-g(x')}}\,\partial_{a'}\left[\sqrt{-g(x')}\,n^{a'}(x')\right] = \nabla_{a'}n^{a'}(x') \; ; \quad (8.8)$$

consequently, Eq. (8.1) becomes, using Eq. (8.3),

$$\Phi(x) = -\int_C dS(x')\,\Phi(x')\left[V(x',x)\Box' f(x') + 2\nabla^{a'}f(x')\nabla_{a'}V(x',x)\right] \; , \quad (8.9)$$

which expresses the value of the field at $x \in K^+(\mathcal{C})\cap\mathcal{M}$ in terms of its values on the compact null region $\mathcal{C}$, defined by the condition $f(x) =$ constant. Therefore, Eq. (7.6) satisfies the CPP iff the right hand side of Eq. (8.9) vanishes for each choice of data on an arbitrary $\mathcal{C}$ with the properties above. We shall now show that necessary and sufficient conditions for this to happen are

$$V(x',x) = \text{constant} \quad (8.10)$$

and

$$\Box f = 0 \quad (8.11)$$

for each $f$ such that

$$g^{ab}\,\nabla_a f\,\nabla_b f = 0 \; . \quad (8.12)$$

Obviously, conditions (8.10), (8.11), and (8.12) are sufficient. To prove that they are also necessary, let us choose the initial data on $\mathcal{C}$ as

$$\Phi(x') = \Phi_0\,\delta_{\mathcal{C}}(x',\overline{x}') \; , \quad (8.13)$$

with $\overline{x}' \in \mathcal{C}$ arbitrary and $\Phi_0$ an arbitrary nonzero constant; this leads one to

$$V(x',x)\Box' f(x') + 2\nabla^{a'}f(x')\nabla_{a'}V(x',x) = 0 \; , \quad (8.14)$$

which must be satisfied by every $f$ for which Eq. (8.12) holds. The case $V = 0$ corresponds to the tail-free property, which evidently contains the CPP; let us therefore suppose $V \neq 0$. Eq. (8.14) becomes

$$\Box' f(x') = -2\nabla^{a'} f(x') \nabla_{a'} \ln |V(x', x)| \,, \tag{8.15}$$

which can be satisfied only if conditions (8.10) and (8.11) hold separately.

As already remarked, if $V = 0$ the CPP holds trivially, since Eq. (7.6) is tail-free in that case. The interesting situation corresponds to

$$V(x', x) = \text{constant} \neq 0 \,, \tag{8.16}$$

supplemented by the condition that Eq. (8.11) holds for each $f$ which satisfies Eq. (8.12). We shall now comment separately about these two requirements. Let us consider (8.16) first. For generic initial data with support (spacelike, null or both) $\mathcal{C}$, Eq. (7.19) leads to conclude that $\Phi$ is constant throughout $K^+(\mathcal{C}) \cap \mathcal{M}$, with value

$$V \int_{\mathcal{C}} dS^{a'}(x') \, \nabla_{a'} \Phi(x') \,. \tag{8.17}$$

Since the field expressed by (8.17) must satisfy Eq. (7.6), one gets immediately, for nontrivial choices of the data, that $\forall x \in K^+(\mathcal{C}) \cap \mathcal{M}$ it must be

$$\xi \, R(x) \, V = 0 \,. \tag{8.18}$$

Considering all the possible specifications of $\mathcal{C}$, it is obvious that Eq. (8.18) must hold everywhere in $\mathcal{M}$. We arrive therefore to the conclusion that $V$ can be a nonvanishing constant only for a field satisfying the equation

$$\Box \Phi = 0 \,. \tag{8.19}$$

De Sitter and anti-de Sitter spacetimes are examples of this circumstance [152]; it is interesting to notice that Eq. (8.16) weakens the no-tail property only by allowing the presence of uniform tails – a feature involving only zero multipole moments. If one requires, as usual, that $\Phi$ vanishes approaching the spatial infinity, such constant tails are of course ruled out.

Regarding the other condition for the CPP, i.e. the requirement that Eq. (8.11) be satisfied by every $f$ for which Eq. (8.12) holds, it is easy to see that it is extremely restrictive. Consider in fact an $n$-dimensional ($n \geq 2$) Minkowski spacetime, with usual coordinates $(t, \vec{x})$. The functions

$$f^{(\pm)} \equiv t \pm |\vec{x}| \tag{8.20}$$

trivially satisfy Eq. (8.12); nevertheless

$$\Box f^{(\pm)} = \pm \vec{\nabla}^2 |\vec{x}| = \pm \frac{n-2}{|\vec{x}|} \tag{8.21}$$

does not vanish unless $n = 2$. Since in any spacetime, coordinates can always be chosen in which the metric takes locally the Minkowskian form, this simple example shows that, for $n \geq 2$, null hypersurfaces exist for which Eq. (8.11) does not hold [6]. Consequently, the CPP cannot be satisfied independently of the tail-free property whenever $n > 2$. In other words, the CPP is equivalent to the tail-free property in every spacetime with more than two dimensions. For $n = 2$, the condition (8.11) is automatically fulfilled by any function $f$ for which Eq. (8.12) holds. This can be proved remembering that any two-dimensional spacetime is conformally flat [1], and that the metric can therefore be written as

$$g_{ab} = \Omega^2 \, \eta_{ab} \tag{8.22}$$

where $\Omega$ is a nonvanishing function and $\eta_{ab}$ is the metric of Minkowski spacetime. Introducing the null coordinates

$$u \equiv \frac{1}{\sqrt{2}} (t - x) \ , \tag{8.23}$$

$$v \equiv \frac{1}{\sqrt{2}} (t + x) \ , \tag{8.24}$$

Eqs. (8.11) and (8.12) become, respectively:

$$\Box f = -\frac{2}{\Omega^2} \frac{\partial^2 f}{\partial u \partial v} = 0 \ ; \tag{8.25}$$

[6] For $n = 4$ this can be physically understood as follows. To require that Eq. (8.11) holds whenever $f$ satisfies Eq. (8.12), would amount to say that, if $f$ satisfies Eq. (8.12), then an arbitrary function $F(f)$ solves the wave equation (8.19). In the particular case of $f^{(\pm)}$ given by Eq. (8.20), it would follow that $F(t \pm |\vec{x}|)$ are local solutions of Eq. (8.19) near $t = |\vec{x}| = 0$. However, this would be an erroneous conclusion, since we know that the generic advanced and retarded solutions of Eq. (8.19) with spherical symmetry are, locally, $F(t \pm |\vec{x}|) / |\vec{x}|$.

$$\frac{\partial f}{\partial u}\frac{\partial f}{\partial v} = 0 \ . \tag{8.26}$$

Eq. (8.26) can be satisfied only if $f$ depends on only one of the two coordinates $u$, $v$, i.e., if either

$$f = f_1(u) \tag{8.27}$$

or

$$f = f_2(v) \ . \tag{8.28}$$

But both (8.27) and (8.28) satisfy identically Eq. (8.25), thus proving our assert.

This result leaves open a last chance for the CPP to be satisfied independently of the tail-free property, namely, that condition (8.16) be fulfilled in a two-dimensional spacetime. As previously seen, (8.16) necessarily requires [7] that $\Phi$ satisfies Eq. (8.19), which is conformally invariant [1] for $n = 2$: The corresponding Green function is therefore the same as in Minkowski spacetime. It is straightforward to find, performing the calculations in $u$, $v$ coordinates, that this is

$$G(x', x) = \frac{1}{2}\,\Theta\left(-\Gamma(x', x)\right) \ , \tag{8.29}$$

where $\Gamma$ has now the simple expression

$$\Gamma(x', x) = \eta^{ab}\,(x' - x)_a (x' - x)_b \ . \tag{8.30}$$

Since Eq. (8.29) satisfies indeed the condition (8.16), we are led to the remarkable conclusion that *Eq. (7.6) satisfies the CPP independently of the tail-free property only in the case of minimal coupling in two spacetime dimensions.*

The independence of the CPP from the tail-free property in the two dimensional case, can be checked explicitly calculating the value of $\Phi$ at a generic point $x \in K^+(\mathcal{C}) \cap \mathcal{M}$ according to Eq. (7.19). The result is

$$\Phi(x) = \frac{1}{2}\int_{\mathcal{C}} dS^{a'}(x')\nabla_{a'}\Phi(x') \ , \tag{8.31}$$

where now the integral is performed over the one-dimensional support $\mathcal{C}$ of data, with normal $n^a$. It is trivial to verify that, in the case of $\mathcal{C}$ spacelike, it

[7] The proof holds obviously for any number of dimensions.

is always possible to choose data on $\mathcal{C}$ such that the constant right hand side of Eq. (8.31) is nonzero (for example, choosing on $\mathcal{C}$

$$\nabla_a\Phi(x) = \Phi_0\, n_a(x)\ , \tag{8.32}$$

with $\Phi_0 \neq 0$ a nonzero constant): Eq. (8.19) is therefore not tail-free. Nevertheless, if $\mathcal{C}$ is null, then we can always choose it either in the form

$$\{(u,v)\,|u = u_0,\quad v \in [v_1, v_2]\}\ , \tag{8.33}$$

or

$$\{(u,v)\,|u \in [u_1, u_2],\quad v = v_0\}\ , \tag{8.34}$$

where $u_0, u_1, u_2, v_0, v_1$ and $v_2$ are specified values of the coordinates. Eq. (8.31) becomes respectively

$$\Phi(x) = \frac{1}{2}\int_{v_1}^{v_2} dv\, \Omega^2(u_0,v)\, \frac{\delta_v^a}{\Omega^2(u_0,v)}\, \frac{\partial\Phi}{\partial x^a}(u_0,v) = \frac{1}{2}\int_{v_1}^{v_2} dv\, \frac{\partial\Phi}{\partial v}(u_0,v) = 0 \tag{8.35}$$

and

$$\Phi(x) = \frac{1}{2}\int_{u_1}^{u_2} du\, \Omega^2(u,v_0)\, \frac{\delta_u^a}{\Omega^2(u,v_0)}\, \frac{\partial\Phi}{\partial x^a}(u,v_0) = \frac{1}{2}\int_{u_1}^{u_2} du\, \frac{\partial\Phi}{\partial u}(u,v_0) = 0\ . \tag{8.36}$$

The validity of the CPP for Eq. (8.19) in a two-dimensional spacetime is thus explicitly established.

We can get insight into the possibility to have CPP independently of the tail-free property in two dimensions with the following argument. Let $\mathcal{D}$ be some data with compact spacelike support $\mathcal{C}$; the value of $\Phi$ at $x \in K^+(\mathcal{C}) \cap \mathcal{M}$ is given by Eq. (8.31), and is a constant that we can always make different from zero by a suitable choice of the initial data, as already remarked. We can, however, give another form to the initial value problem. Let us propagate the data to $H^+(\mathcal{C})$, the future Cauchy horizon of $\mathcal{C}$; in two dimensions $H^+(\mathcal{C})$ is composed of null lines. For sake of simplicity we restrict ourselves to the simplest case in which there are only two components $\mathcal{C}_L$ and $\mathcal{C}_R$ with one point $p$ in common [8] – see Fig. 5.7. The values of the field and its gradient on $H^+(\mathcal{C})$

[8] Examples of more complicated situations can be easily constructed by removing regions in $\mathcal{M}$.

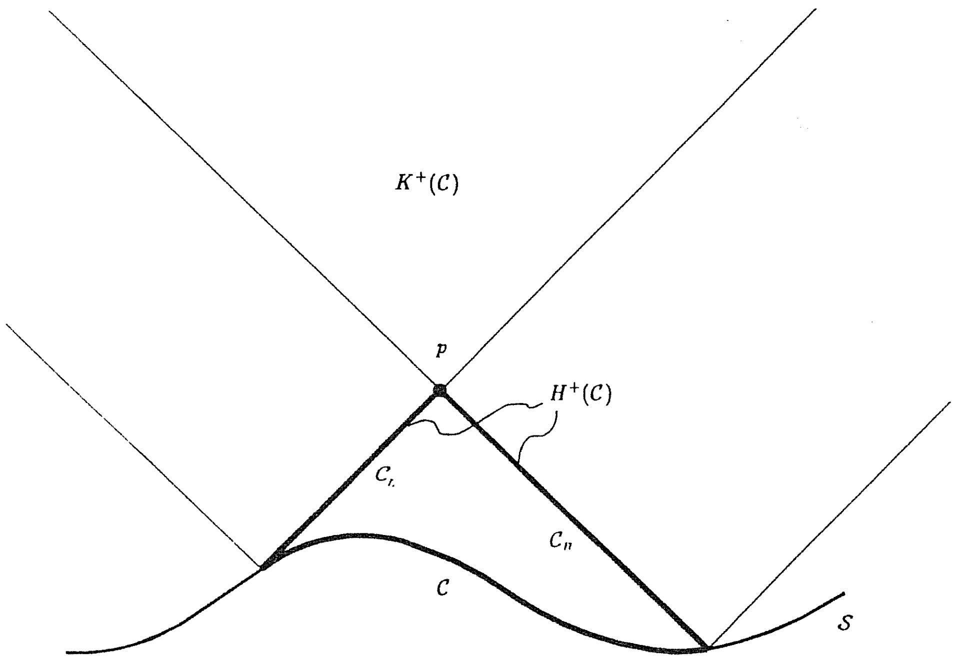


Figure 5.7: The new Cauchy problem obtained propagating initial data from $\mathcal{C}$ to $H^+(\mathcal{C})$.

can be chosen as data for a new Cauchy problem, which must of course be compatible with the previous one. The integral

$$\Phi(x) = \frac{1}{2} \int_{H^+(\mathcal{C})} dS^{a'}(x')\nabla_{a'}\Phi(x') \ , \tag{8.37}$$

analog to the one in Eq. (8.31), results from three contributions, two of which – coming from integration over $\mathcal{C}_L$ and $\mathcal{C}_R$ – vanish due to the CPP, while the third one – deriving from integration in the neighbourhood of $p$ – requires a particular care, since the integrand exhibits a delta-like singularity at $p$. The problem is essentially that $H^+(\mathcal{C})$ fails to be a differentiable manifold at $p$; we can however regard the integral in Eq. (8.37) as the limit of a sequence of integrals over smooth approximations to $H^+(\mathcal{C})$. These can be constructed by substituting a small region $O(p) \subset H^+(\mathcal{C})$ around $p$ with a spacelike curve $\mathcal{O}$ smoothly joined to $H^+(\mathcal{C}) - O(p)$ (see Figs. 5.8 and 5.9). The sequence of integrals is constant since each of them differs from the others only by integrations over null regions which give vanishing contributions thanks to the CPP. The constant value is therefore due to the integration over $\mathcal{O}$, which in

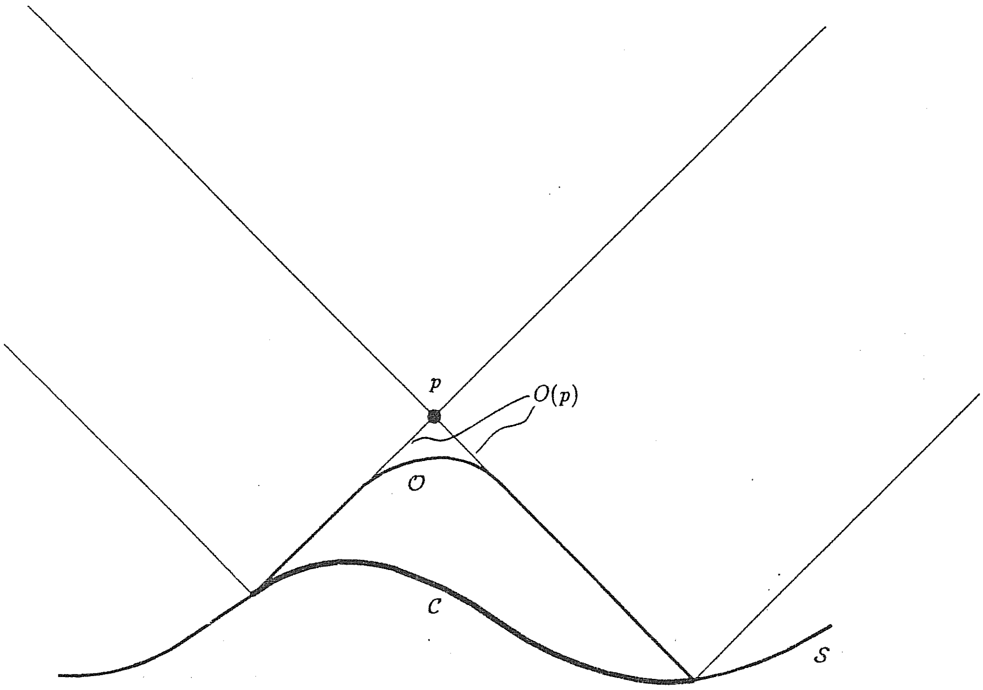


Figure 5.8: A smooth submanifold approximating $H^+(C)$.

the limit amounts to a contribution from the point $p$ alone.

## 5.9 Electromagnetic radiation

As far as the electromagnetic field is concerned, it is well known [1] that Maxwell equations in vacuum,

$$\nabla^b F_{ab} = 0 \,, \tag{9.1}$$

$$\nabla_{[a} F_{bc]} = 0 \,, \tag{9.2}$$

are conformally invariant in four dimensions. Consequently, electromagnetic radiation in a conformally flat spacetime has no tails. However, this point deserves a few comments. In fact, although the equation for the electromagnetic potential

$$\Box A_a - R_a{}^b A_b - \nabla_a \nabla^b A_b = 0 \tag{9.3}$$

is conformally invariant, the wave equation (7.5) is not. A straightforward way to realize this is to notice that, under a conformal transformation with

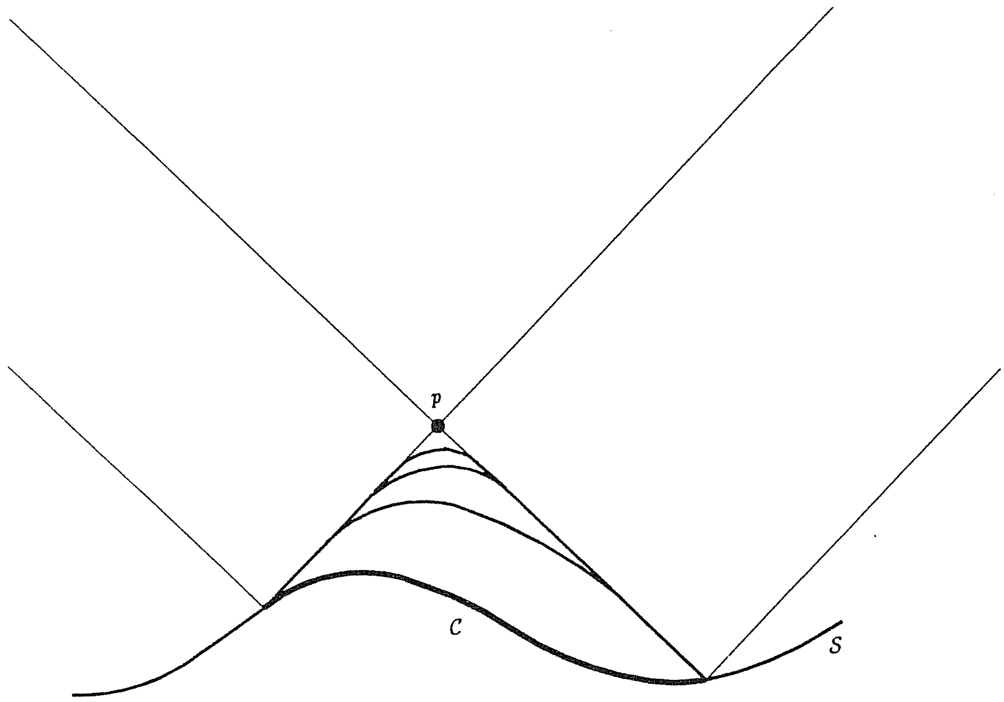


Figure 5.9: The sequence of smooth submanifolds approximating $H^+(\mathcal{C})$.

conformal factor $\Omega$, the gauge condition (7.4) gives

$$\tilde{\nabla}^a \tilde{A}_a = 2\Omega^{-3}\tilde{\nabla}^a\Omega\, \tilde{A}_a \neq 0 \ , \tag{9.4}$$

where a tilde denotes transformed quantities. The conformal invariance of Maxwell equations does not therefore guarantee that the solutions of Eq. (7.5) have no tails in a conformally flat spacetime. The possibility of having tails in the potentials, but not in the fields [167] is not, however, a problem, provided the tails in $A_a$ are purely gauge terms, corresponding to $F_{ab} = 0$. A necessary and sufficient condition on $V^{a'}{}_a(x', x)$ for this to occur in a normal region $\mathcal{M}$ is [167]

$$\nabla_{[b} V^{a'}{}_{a]}(x', x) = 0 \ . \tag{9.5}$$

To prove this, we use Eq. (7.12) and

$$F_{ab} = \nabla_a A_b - \nabla_b A_a$$

to express the field at a point $x \in K^+(\mathcal{C}) \cap \mathcal{M}$ as

$$F_{ab}(x) = -2 \int_{\mathcal{C}} dS^{b'}(x')\, \nabla_{[b} V^{a'}{}_{a]}(x', x) \overleftrightarrow{\nabla}_{b'} A_{a'}(x') \ , \tag{9.6}$$

where a decomposition analogous to (7.15) has been adopted for $G^{a'}{}_{a}(x', x)$. $F_{ab}$ is tail-free in $\mathcal{M}$ iff the integral on the right hand side of Eq. (9.6) vanishes identically for each choice of the initial data subject to the constraints deriving from the gauge condition (7.4). Choosing $\mathcal{C}$ spacelike with unit normal vector $n^a$, we write such a constraint as

$$n^b \left(n_a \nabla^a A_b\right) = D^a A_a \tag{9.7}$$

where $D_a$ is the covariant derivative operator on $\mathcal{C}$ induced by $\nabla_a$ (see e.g. Ref. [1]). With the specific choice of data on $\mathcal{C}$

$$A_{a'}(x') = 0 \ ,$$

$$n^{a'}(x')\nabla_{a'}A_{b'}(x') = B_{b'}(x')\,\delta_{\mathcal{C}}(x', \bar{x}') \ ,$$

where $\bar{x}' \in \mathcal{C}$ is arbitrary and $B^a$ is orthogonal to $n^a$ (as follows from Eq. (9.7)) one gets immediately Eq. (9.5).

Due to the gauge freedom of General Relativity, we expect an analogous situation for gravitational radiation. Possible tails for $h_{ab}$ are not necessarily associated to physical effects in $K^+$, since they could be removed by a coordinate transformation. This subject will be investigated further in future work.

## 5.10 Outlooks

The formulation of the tail-free property discussed in this paper is based essentially on the requirement that the field in the region $K^+$ vanish. This definition allows one to give a unitary treatment both of the Huygens' principle and of the CPP, and is therefore particularly useful in order to clarify the relationship between them. In a normal region of spacetime, the validity of these properties is expressed in terms of simple conditions on the coefficient $V(x', x)$ in the diffusive part of the Green function.

Considering the scalar wave equation (7.6), we have been led to conclude that the CPP holds independently of the tail-free property only for the case of minimal coupling in a two-dimensional spacetime[9].

Wave equations in two spacetime dimensions, of a more general form than Eq. (7.6), have been considered in Refs. [173] as describing wave propagation in four dimensional spherically symmetric spacetimes. It has been shown that in this case Eq. (7.6) reduces, expanding the solution in spherical harmonics, to a family of equations of the form

$$\frac{\partial^2 \phi_l}{\partial u \partial v} + \alpha_l \frac{\partial \phi_l}{\partial u} + \beta_l \frac{\partial \phi_l}{\partial v} + \gamma_l \phi_l = 0 , \tag{10.1}$$

where $l \in \{0, 1, 2, \cdots\}$ labels the various multipole moments, and $\alpha_l$, $\beta_l$, $\gamma_l$ are given functions of $u$, $v$. Within this context, it may be interesting to investigate, with the methods employed here, the validity of the CPP and of the tail-free property for Eq. (10.1), since the results apply to the wave equation (7.6) in four dimensional spacetime as well. This could extend and clarify the existing work on the subject [173].

[9] Regarding the dimensionality of spacetime, it is interesting to note that the anthropic principle requires Huygens' principle, and thus that the spacetime in which we live has even dimension [178].

# Chapter 6

# APPENDICES

## 6.1 Appendix A: the divergence $\partial_A(\delta p^A)$

From Eq. (2.11) we get

$$\frac{\partial(\delta p^A)}{\partial x^A} = -\frac{1}{2}\int_S^O d\lambda\ \left(h_{\lambda\lambda}{}^{,x}{}_{,x} + h_{\lambda\lambda}{}^{,y}{}_{,y}\right) + O(h^2)\,. \tag{A.1}$$

Introducing the null coordinates $u \equiv t - z$ and $v \equiv t + z$ one easily gets

$$\begin{aligned}\Box \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2} &= \\ = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 8\,\frac{\partial^2}{\partial u \partial v}\,.\end{aligned}$$

Since we are considering gravitational waves propagating in the radiation zone along a definite direction, we have $\Box h_{\mu\nu} = 0$ and $\partial^2 h_{\mu\nu}/\partial u \partial v = 0$, so that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) h_{\mu\nu} = 0 \tag{A.2}$$

and

$$\partial_A(\delta p^A) + O(h^2) = 0\,. \tag{A.3}$$

## 6.2 Appendix B: $\lambda_2 = -\lambda_1$ and $\mu_1 = \mu_2$ to first order

From Eqs. (4.2)-(4.5) and (A.2), we get

$$\lambda_1 + \lambda_2 = \frac{D}{2}\int_S^O d\lambda\ \left(h_{\lambda\lambda}{}^{,x}{}_{,x} + h_{\lambda\lambda}{}^{,y}{}_{,y}\right) + O(h^2) =$$

$$= \frac{D}{2} \int_S^O d\lambda\ p^\alpha p^\beta \left( h_{\alpha\beta}{}^{,x}{}_{,x} + h_{\alpha\beta}{}^{,y}{}_{,y} \right) + O(h^2) \qquad = 0 + O(h^2)$$

and

$$\lambda_2 = -\lambda_1 + O(h^2)\ . \tag{B.1}$$

Since, to first order, we raise and lower tensor indices with $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$, and $\eta^{AB} = \eta_{AB} = \delta_{AB}$ for $A, B = 1, 2$, we have (to this order)

$$h_{\mu\nu}{}^{,A}{}_{,B} = h_{\mu\nu}{}^{,B}{}_{,A}\ ,$$

and

$$\mu_1 = \mu_2 + O(h^2)\ . \tag{B.2}$$

## 6.3 Appendix C: Asymptotic expansion of $\Psi_l(\chi)$ for $\chi \to 0$

The function $z(\chi)$ given by Eq. (3.4) can be expanded for small $\chi$ as (see Ref. [179], p. 1076)

$$z(\chi) = -\frac{1}{2\chi} \sum_{k=0}^{+\infty} \frac{2^k B_k}{k!} \chi^k\ , \tag{C.1}$$

where $B_k$ are the Bernoulli numbers. From Eq. (3.15) it follows that the hypergeometric function can be expanded in powers of $\chi$ if an expression is available for $z^n$, with $n \geq 1$. The latter can be obtained using the following generalization of the well known Cauchy formula for the square of a series,

$$\left( \sum_{k=0}^{+\infty} a_k \right)^n = \sum_{k_n=0}^{+\infty} \sum_{k_{n-1}=0}^{k_n} \cdots \sum_{k_1=0}^{k_2} a_{k_n - k_{n-1}} \cdots a_{k_2 - k_1} a_{k_1}\ , \tag{C.2}$$

whose proof is straightforward (it suffices to apply $n-1$ times the Cauchy relation). One obtains, for $n \geq 1$,

$$\left( \sum_{k=0}^{+\infty} \frac{2^k B_k}{k!} \chi^k \right)^n = \sum_{k=0}^{+\infty} \alpha_{nk} \chi^k\ , \tag{C.3}$$

where

$$\alpha_{nk} \equiv 2^k \sum_{k_{n-1}=0}^{k} \cdots \sum_{k_1=0}^{k_2} \frac{B_{k-k_{n-1}} \cdots B_{k_1}}{(k-k_{n-1})! \cdots k_1!}\ . \tag{C.4}$$

Defining

$$\alpha_{0k} \equiv \delta_{0k} \; , \tag{C.5}$$

where $\delta_{0k}$ is the Kronecker symbol, the following expression holds $\forall n \geq 0$:

$$z^n(\chi) = \frac{1}{(-2)^n \chi^n} \sum_{k=0}^{+\infty} \alpha_{nk} \, \chi^k \; . \tag{C.6}$$

Expanding the exponential and applying Eq. (C.2) with $n = 2$, we get

$$e^{ip\chi} \, z^n(\chi) = \frac{1}{(-2)^n} \sum_{h=0}^{n+l+1} \sum_{k=0}^{h} \alpha_{nk} \, \frac{(ip)^{h-k}}{(h-k)!} \, \chi^{h-n} + O(\chi^{l+2}) \; , \tag{C.7}$$

and finally, by Eqs. (3.14) and (3.15),

$$\Psi_l(\chi) = A_l \sum_{n=0}^{l} \frac{1}{(-2)^n} \begin{pmatrix} n+l \\ l \end{pmatrix} \frac{(-l)_n}{(1-ip)_n} \sum_{h=0}^{n+l+1} \sum_{k=0}^{h} \alpha_{nk} \, \frac{(ip)^{h-k}}{(h-k)!} \, \chi^{h-n} + O(\chi^{l+2}) \; . \tag{C.8}$$

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