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The role of graphs of groups in 3-dimensional geometry

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Special notations

$\Omega(F)$	Domain of discontinuity of a Kleinian group F
(Γ, G)	graph of groups
$V(\Gamma)$	vertex set of a graph Γ
$E(\Gamma)$	edge set of a graph Γ
$\delta_0: E(\Gamma) \rightarrow V(\Gamma)$	maps edges to initial vertices
$\delta_1: E(\Gamma) \rightarrow V(\Gamma)$	maps edges to final vertices
G_v	vertex group of a vertex v
G_e	edge group of an edge e
$i_e: G_e \rightarrow G_{\delta_0 e}$	edge-to-vertex monomorphism
Γ_e	G -component of $e \in E(\Gamma)$
\mathbb{Z}_n	cyclic group of order n
D_n	dihedral group of order n
A_n	alternating group of order n
S_n	Symmetric group of order n
$(g=0, h_1, h_2, h_3)$ or (h_1, h_2, h_3)	signature of a group ([Mas] X.D)
$(2, 2, n) \xrightarrow{n} (2, 3, n)$	graph of groups with two vertices v_1 and v_2 , $G_{v_1} \cong (2, 2, n)$, $G_{v_2} \cong (2, 3, n)$ and one edge e with $G_e \cong \mathbb{Z}_n$.

Introduction

The starting point of this work are some crucial developments in the theory of Kleinian groups and their interplay with 3-dimensional geometry.

The theory of *Kleinian groups* goes back to Poincare's work in the 19th century.

The original Poincare's idea was to extend to Kleinian groups his geometric approach for *Fuchsian groups* and to develop a correspondent abstract theory.

As a matter of fact there is some analogy between the two cases, because *fundamental polyhedra* for Kleinian groups play the same role as *fundamental polygons* for Fuchsian groups; but Poincare's analysis didn't go very far.

The reason why this attempt had to fail is clear today: on one side the theory of Fuchsian groups is an algebraic counterpart of two-dimensional topology, but on the other one Kleinian groups involve three-dimensional manifolds which are much more complicated than surfaces.

In some way, the algebraic theory of groups must also reflect the complexity of the topological analysis: this is the case and after one century Poincare's approach has not produced so many results, for the technical difficulties involved in a combinatorial description of fundamental polyhedra.

More recently some new ideas have provided powerful tools for a renewed interest in Kleinian groups.

One of the most important ways of constructing new Kleinian groups is to obtain these groups by *combining* simpler groups which are already at hand.

Such a procedure was introduced by Klein, but only in recent years Maskit [Mas], has developed this method and has shown its far-reaching consequences.

For example his *first Combination Theorem* states conditions under which, given two Kleinian groups E_1 and E_2 , the group $F = \langle E_1, E_2 \rangle$

generated by E_1 and E_2 is still Kleinian; moreover the theorem describes the Kleinian structure of F .

A striking result achieved by this way is the classification, up to *similarity*, of *function groups*, that is analytically finite Kleinian groups with one invariant component [Mas].

So large classes of new interesting Kleinian groups can be constructed by starting from simple groups and applying iteratively combination theorems.

The leading idea of this work is that the geometrical intuition in three dimensions explains and renews these results and that in this setting some generalizations which are not so evident otherwise look very natural. The reader can easily understand how much I am indebted to Marden's paper [Mar], even if technical tools and results are completely different.

To this purpose, we borrow an idea which was introduced in [MMZ1] to provide a unified theory for finite actions on handlebodies and surfaces: in this paper the authors show how the fundamental group of a 3-dimensional orbifold admits a geometric presentation as a *graph of groups* and that this presentation can be used to describe a certain class of Kleinian groups (virtually free Kleinian groups). Here this approach is revisited and generalized: graphs of groups are seen as the right formal framework where to develop the theory of iterative constructions of Kleinian groups by combination theorems and where to understand the deep interplay between these theorems and some three-dimensional geometry.

So the ideas of [MMZ1], [Mas] and [Mar] must be considered as the source of this work.

I want now to explain a little bit more the role of graphs of groups in 3-dimensional geometry, by an example: Maskit's classification of geometrically finite function groups [Mas] X.G.2: *Let G and F be geometrically finite function groups. G is a deformation of F if and only if G and F have the same signature.*

The proof of this result is based on the study of limit sets of F and G , in three crucial steps:

- 1) The *planarity theorem* ([Mas] X.A.6) about regular planar coverings of topologically finite Riemann surfaces, allows us

to associate a *signature* (X.E) to each function group.

- 2) The *Jordan theorem*, stating that a surface S is planar if and only if every simple closed curve on S is dividing, implies that *quasiconformally conjugate* function groups have the same signature (X.D.5)
- 3) We can find out the set of *admissible* signatures, by constructing, through the combination theorems, function groups with a given signature.

On one hand the three-dimensional point of view gives an intuitive geometric support to this theorem; on the other one, it suggests various generalizations, which are not so evident by only looking at the limit set of function groups: *generalized function groups*, for example, are a natural geometric extension of the notion of function groups, while their characterization as Kleinian groups (1.4) is a little tricky.

The 3-dimensional counterpart of planarity theorem is *Dehn lemma and Cylinder theorem*, in the following sense. For a geometrically finite function group F uniformizing an orbifold product-with-handles $O = \mathbb{H}^3 \cup \Omega(F)/F$, Dehn lemma and Cylinder theorem give us a *geometric decomposition* of it (see 1.5) along a collection of discal 2-orbifolds. The signature ([Mas] X.5) of F corresponds to the graph of groups (Γ, G) that we associate to the geometric decomposition of O (2.1).

The topological meaning of the signature is so disclosed by the three-dimensional picture. As an example, it becomes easy to understand why quasiconformally conjugate function groups have the same signature. We prove (see 2.1) that the graph of groups associated to an orbifold product-with-handles is essentially unique. In fact the singular part of the orbifold product-with-handles determines, up to some triviality, the geometric decomposition. In some sense it represents a rigid core of the decomposition, reproduced by the graph of groups.

In conclusion the interplay with 3-dimensional geometry explains and renews these previous results on Kleinian groups. Moreover graphs of groups reveal to be very convenient to state classification results. The classification is given in fact in terms of equivalence classes of graphs of groups.

We have also said at the beginning that graphs of groups are a natural framework to develop a theory of iterative constructions of Kleinian groups and to study finite actions on some classes of three-dimensional manifolds. As it will appear evident in a minute, there is a close connection among these different features.

The first chapter addresses the question of *realizing a finite graph of groups as a Kleinian group*, that is to find conditions under which the fundamental group $\pi_1(\Gamma, G)$ of a graph of groups (Γ, G) is a Kleinian group (1.3.4 and 1.4.2).

The technique we use is simply to iterate the application of Maskit's combination theorems, that is to study the conditions under which Maskit's combination theorems can be applied iteratively according to the given geometric structure of a graph (sections 1.1 and 1.2).

Such realizability theorems have a much deeper geometrical meaning than it looks like at a first sight.

In fact, in stating conditions $N_0 - N_4$ or similar ones for graphs of groups, one heavily relies on his geometrical intuition.

So sections 1.3 and 1.4 give realizability conditions for two classes of graphs of groups: in fact the second class includes the first one and, in this sense, this is a more general result.

Notice however that, the larger is the class of groups for which we state realizability, the less are the information we get about the Kleinian structure of the group we realize.

A similar result, but obtained by different methods, can be found in [MMZ1], where the authors analyze graphs of groups (Γ, G) such that $\pi_1(\Gamma, G)$ is *virtually free*, that is admits a free subgroup of finite index. This class of graphs of groups is included in our classes and so their realizability is implied by our realizability theorems: but, as we have already said, the information about the Kleinian structure realized is less.

(1.5.1) and (1.6.1) tell us something more: we prove not only that the classes of graphs of groups satisfying $N_0 - N_4$ and $N_0' - N_4'$ are realizable as Kleinian groups, but we are also able to characterize the set of Kleinian groups realized.

Function groups are a wide set of Kleinian groups including elementary groups, Fuchsian groups, basic groups. From the

topological point of view, they naturally arise as fundamental groups of neighborhoods of compressible surfaces embedded in a three-dimensional manifold.

The second class of groups we find (section 1.6) include function groups and as there does not seem to be a specific name for them in the literature, we have called them *generalized function groups*.

Actually, from the three-dimensional point of view, they are a natural geometric extension of the notion of a function group, even if this connection is not immediately clear from two dimensional topology.

So in the first chapter graphs of groups appear as the natural tool to construct iteratively Kleinian groups and this has a relevant geometrical meaning.

Notice in fact that, instead of applying combinations theorems for Kleinian groups, one can prove these theorems by geometrical arguments as in [MMZ1].

More precisely, given a graph of groups (Γ, G) satisfying $N0-N4$ or $N0'-N4'$, one can regard a regular neighborhood of Γ as an *orbifold* whose singular set reproduces the geometric structure of Γ and it turns out that this orbifold is uniformized by a function group.

Finally I will tell some words about the use of graphs of groups in an algebrogeometrical theory of finite actions on 3-manifolds. This idea has been introduced in [MMZ1] for handlebodies. The reason why it is so efficient is that it includes in a nice way the *Riemann-Hurewitz formula*, providing a powerful calculus for the construction of actions which has been successfully applied to various questions in [KM], [Mc], [MMZ2].

A basic question for finite actions is the so called *Nielsen realization problem*, previously posed for closed surfaces and solved by Kerchoff [Ke] in 1983. Let's state it for a 3-manifold.

Let M be a 3-manifold and H be a finite group acting effectively, smoothly and orientation-preservingly on M . The H -action on M induces a homomorphism $\eta: H \rightarrow \text{Out}(\pi_1(M))$, the group of outer automorphisms of $\pi_1(M)$. The question is: for a given homomorphism $\eta: H \rightarrow \text{Out}(\pi_1(M))$, is there a H -action on M as above which induces it?

For 3-manifolds this is not generally true (for example this fact is shown in [MMZ1] for handlebodies).

For *almost compact* (see 1.5) product-with-handles the solution of the realization problem is a consequence of 1.5.1.

As a matter of fact, if H is a finite group acting on a product-with-handles M , there is a conformal structure on M , uniformized by a function group, for which H acts conformally. By this remark, 1.5.1 gives necessary and sufficient conditions for a group of outer automorphisms of $\pi_1 M$ to be induced by a group of orientation-preserving diffeomorphisms on M (section 3.1).

So 3.1.4 solves the realization problem for almost compact product-with-handles, by showing that it may have a negative answer. I repeat that this technique has been introduced in [MMZ1] for handlebodies and, from this point of view, this work is a natural generalization of [MMZ1] and [MMZ2]. In fact almost compact product-with-handles include handlebodies and products $S \times [0,1]$, where S is any finite surface with boundary.

More generally the theorems above turn questions about finite actions into combinatorial problems involving graphs of groups.

There is plenty of applications and the method has been already successfully used to study, for example, periodic diffeomorphisms on handlebodies [MMZ1], equivalence of actions on handlebodies [KM], actions on nonclosed two-manifolds [MMZ2].

In principle such results could be stated in analogous way for almost compact product-with-handles. There is really nothing new in applying those methods to this more general case except that calculations are a little more messy.

So I have not repeated a parallel analysis and I have confined myself to a few features in which the study of finite actions on almost compact product-with-handles turns out to be really different.

My proposals are:

- 1) An intensive use of *Euler characteristics* in finding admissible graphs of groups (see 3.1.4)
- 2) A method for studying finite actions which is based on the *symmetry group* of the graphs (3.4)

More details about these applications can be found at the beginning of chapter 3.

The work is organized as follows.

In the first chapter we study sufficient conditions under which Maskit's theorems can be applied iteratively to graphs of groups and characterize the set of Kleinian groups realized.

In the second chapter we classify function groups and generalized function groups in terms of equivalence classes of graphs of groups.

In the third chapter we show how one can use graphs of groups to study finite actions on almost compact product-with-handles, with some concrete applications.

1 A graph-theoretical approach to Kleinian groups

In this chapter we address the question of *realizing a finite graph of groups as a Kleinian group*, that is to find conditions under which the fundamental group $\pi_1(\Gamma, G)$ of a graph of groups (Γ, G) is a Kleinian group (1.3.4 and 1.4.2).

The technique we use is simply to iterate the application of Maskit's combination theorems, that is to study the conditions under which Maskit's combination theorems can be applied iteratively according to the given geometric structure of a graph (sections 1.1 and 1.2).

However it should be noticed that such realizability theorems have a much deeper geometrical meaning than it looks like at a first sight.

As we shall see in a minute this geometrical counterpart is due to the interplay between the theory of Kleinian groups on one side and the topology of three-dimensional manifolds on the other one.

In fact, in stating conditions $N_0 - N_4$ or similar ones for graphs of groups, one heavily relies on his geometrical intuition.

So sections 1.3 and 1.4 give realizability conditions for two classes of graphs of groups: in fact the second class includes the first one and, in this sense, this is a more general result.

Notice however that, the larger is the class of groups for which we state realizability, the less are the information we get about the Kleinian structure of the group we realize.

A similar result, but obtained by different methods, can be found in [MMZ1], where the authors analyze graphs of groups (Γ, G) such that $\pi_1(\Gamma, G)$ is *virtually free*, that is admits a free subgroup of finite index. This class of graphs of groups is included in our classes and so its realizability is implied by our realizability theorems: but, as we have already said, the information about the Kleinian structure realized is less.

(1.5.1) and (1.6.1) tell us something more: we prove not only that the classes of graphs of groups satisfying $N0-N4$ and $N0'-N4'$ are realizable as Kleinian groups, but we are also able to characterize the set of Kleinian groups realized.

Function groups are a wide set of Kleinian groups including elementary groups, Fuchsian groups, basic groups. From the topological point of view, they naturally arise as fundamental groups of neighborhoods of compressible surfaces embedded in a three-dimensional manifold.

The second class of groups we find (section 1.6) include function groups and as there does not seem to be a specific name for them in the literature, we have called them *generalized function groups*.

Actually, from the three-dimensional point of view, they are a natural geometric extension of the notion of a function group, even if this connection is not immediately clear from two dimensional topology.

It is interesting to look at the way we prove that function groups and generalized function groups are isomorphic to graphs of groups satisfying $N0 - N4$ and $N0'-N4'$. In fact it is possible to give an algebraic proof of this result by using the same ideas as in [MMZ1]. But if one consider the three-dimensional manifolds corresponding to these groups, then the topological tools give the same answer at once.

For example, function groups uniformize *product-with-handles* (def.1.5.2) and by applying *Dehn lemma*, one gets a nice decomposition of these manifolds in pieces and so a nice presentation of their fundamental group as a graph of groups. This graph of groups is in a simple normalized form and conditions $N0-N4$ correspond to the normalized conditions of [MMZ1]. For an analogous use of topological theorems see [Mar2].

It should be said that, instead of applying combinations theorems for Kleinian groups, one can prove these results by geometrical arguments as in [MMZ1].

More precisely, given a graph of groups (Γ, G) satisfying $N0-N4$ or $N0'-N4'$, one can regard a regular neighborhood of Γ as an *orbifold* whose singular set reproduces the geometric structure of Γ and it turns out that this orbifold is uniformized by a function group.

The chapter is organized as follows.

In the first four sections Maskit's combination theorems are applied to the case of graph of groups with cyclic edge groups to give sufficient conditions for a graph of groups to be Kleinian.

We first analyze the two basic cases, the *free product with amalgamation* along a cyclic group (section 1.1) and the *HNN-extension* (section 1.2). In these first two sections we state the starting theorems (1.1.1 and 1.2.1), which are particular cases of Maskit's combination theorems, and then we study the technical conditions under which they can be applied iteratively according to the given geometric structure of a graph.

In section 1.3 we select a set of sufficient conditions (N0-N4) for realizing a graph of groups as a function group.

In section 1.4 we select analogous sufficient conditions for a wider class of graphs of groups containing the previous one.

In sections 1.5 and 1.6 we characterize the sets of Kleinian groups realized in sections 1.3 and 1.4.

1.1 Free products with amalgamation

We first analyze the case of the *free product with amalgamation* along a cyclic group. We state the starting theorem (1.1.1a and 1.1.1b), which is a particular case of *Maskit's first Combination Theorem* ([Mas] VII.C.2), and then we study the technical conditions under which it can be applied iteratively according to the given geometric structure of a graph.

Notation Let G be a Kleinian group and J a cyclic parabolic subgroup. If J is maximal among all torsion-free parabolic subgroups of G , then we say that J is a *maximal cyclic parabolic subgroup* of G .

1.1.1a Proposition Let E_1 and E_2 be two Kleinian groups with a common maximal finite cyclic subgroup J . Suppose that

- 1) $J \neq E_1, J \neq E_2$
- 2) The two fixed points A, B of the group J on S^2 are not limit points of E_1 and E_2 .

Then it is possible to realize $E_1 *_J E_2$ as a Kleinian group F .

proof:

As A is not a limit point for E_1 there exists a neighborhood U of A in the domain of discontinuity $\Omega(E_1)$ in S^2 i.e. $g(U)$ is either equal to U or disjoint from $U \forall g \in E_1$.

The stabilizer of U in E_1 is a finite group and so it is a finite elliptic group with axis L : so it coincides with J . As a consequence, we can choose a sufficiently small disk neighborhood U of A in S^2 , s.t. $g(U) \cap U = \emptyset \forall g \in E_1 - J$ or $g \in E_2 - J$.

Let now r be the reflection of S^2 at ∂U and define $E_2' = rE_2r^{-1}$. Notice that $rJr^{-1} = J$. So:

E_1 and E_2' are two Kleinian groups with a common maximal cyclic subgroup J . $k = \partial U$ is a simple closed curve in S^2 that divides S^2 into two closed topological disks $B_1 = \text{clos } U$ and $B_2 = S^2 - U$. Moreover B_1 is precisely invariant

under J in E_1 and B_2 is precisely invariant under J in E_2 .

To apply Maskit's first Combination Theorem we have still to prove that the interactive pair $(\text{int}B_1, \text{int}B_2)$ is proper. As a matter of fact the projection of $\text{int}B_1$ to $\Omega(E_1)/E_1$ has non empty exterior and so any fundamental set D for E_1 has non-empty intersection $D \cap B_2$.

By the first combination theorem, we conclude that the group generated by E_1 and E_2' (let's say F) is Kleinian and that $F \cong E_1 *_J E_2$.

q.e.d.

Remark The hypothesis 2) above can be replaced by the following:

2)' One of the two fixed points A, B of the group J on S^2 is not a limit point of E_1 and of E_2 .

This will be useful in the case that E_1 is a Euclidean group.

1.1.1b Proposition Let E_1 and E_2 be two Kleinian groups with a common maximal cyclic parabolic subgroup J . Suppose that

1) $J \neq E_1, J \neq E_2$

2) E_1 and E_2 are geometrically finite ([Mas] VI)

Then it is possible to realize $E_1 *_J E_2$ as a Kleinian group F .

proof:

As E_1 is geometrically finite and J maximal parabolic it is always possible to find an open circular disk $B_1 \subset \mathbb{C} \cup \{\infty\}$ (resp. $B_2 \subset \mathbb{C} \cup \{\infty\}$) which is precisely invariant under J in E_1 (resp. E_2).

Choose B_1 and B_2 with the same diameter. Let r_1 be a rotation carrying B_2 onto B_1 and r_2 the reflection of S^2 at ∂B_1 . Define

$$E_2' = r_2 r_1 E_2 r_1^{-1} r_2^{-1}$$

Notice that $(r_1 r_2) J (r_1 r_2)^{-1} = J$ and that J is still parabolic and maximal among parabolic subgroups of E_2' . We can apply the first Combination Theorem as for the finite case, for the proper interactive pair $(\text{int}B_1, (S^2 - B_1))$.

q.e.d.

Remark

Let E_i $i=1,2$ be a Kleinian group. Let J_i be a finite cyclic subgroup of order n of E_i . Let L_i be the oriented axis of J_i with endpoints A_i and B_i . Let h be a hyperbolic isometry carrying L_1 onto L_2 .

Then hE_1h^{-1} and E_2 have a common cyclic subgroup J . So E_1 and E_2 are also representable as Kleinian groups and that J_1 and J_2 correspond to the same group J in $PSL_2(\mathbb{C})$.

Analogously we have: let E_i $i=1,2$ be a group, J_i an infinite cyclic subgroup. Suppose that E_i is a Kleinian group in such a way that J_i is a parabolic group, maximal among parabolic subgroups of E_i . Then E_1 and E_2 are also representable as Kleinian groups in such a way that J_1 and J_2 correspond to the same group J in $PSL_2(\mathbb{C})$, and J is parabolic and maximal among parabolic subgroups of E_1 and E_2 .

For iterating this construction to the case of a graph of groups we need to prove the following :

1.1.2a Proposition Let E_1 and E_2 be two Kleinian groups satisfying the hypotheses of 1.1.1a or 1.1.1b.

Let K be any finite cyclic subgroup of E_1 not conjugate to a subgroup of J with axis M . Then for some realization F of $E_1 *_J E_2$ as in 1.1.1a (resp. 1.1.1b), the following properties hold:

- 1) If K is maximal in E_1 , then it is a maximal cyclic subgroup of F .
- 2) If an endpoint C of M is not a limit point of E_1 , then it is not a limit point for F .
- 3) If there are no elements in E_1 acting as reflections on M , then there are no such elements in F .

Remark The hypothesis 2) of 1.1.1a can be replaced with 2)': see the remark after 1.1.1a.

1.1.2b Proposition Let E_1 and E_2 be two Kleinian groups satisfying the hypotheses of 1.1.1a or 1.1.1b. Let K be a subgroup of E_1 conjugate to J , with axis M . Then for some realization F of $E_1 *_J E_2$ as in 1.1.1a (resp. 1.1.1b) the following properties hold:

- 1) K is a maximal cyclic subgroup of F .
- 2) If there is no element in E_1 or in E_2 acting as a reflection on the axis of J , then the endpoints of M are not limit points for F .
- 3) If there are no elements in E_1 and in E_2 acting as reflections on the axis of J , then there are no elements in F acting as reflections on M .

Remark We have seen that the hypothesis 2) in 1.1.1a can be replaced with 2'): see the remark after 1.1.1a. This proposition still goes through except for property 2, where you have to require that A, B (see the notation of 1.1.1a) are not limit points of E_1 and one of the two is not a limit point of E_2 .

proof: 1.1.2a - 1); 1.1.2b -1)

Suppose there exists a maximal cyclic subgroup $K' \subset E_1 * J E_2$ s.t. $K \subset K'$. As K' is cyclic, it is indecomposable and, by the Subgroup Theorem, it must be conjugate to a subgroup of E_1 . It follows $K' = K$.

1.1.2a - 2)

Choose a disk U for A like in 1.1.1a. The endpoints of the axis M of K are not contained in $\text{cl } U$ and not even in any E_1 -translate of $\text{cl } U$ as K is not conjugate to a subgroup of J and $\text{cl } U$ has been chosen in such a way that A is an isolated singularity. May be we have amalgamated along a maximal parabolic subgroup J and one endpoint C of K coincides with the fixed point of J on S^2 . But, in this case, C is a limit point and we are not interested in it for the proof of the theorem. By the first Combination Theorem $\Omega(E_1) \cap R = \Omega(F) \cap R$ where R is the complement in S^2 of all the E_1 -translates of $\text{cl } U$.

1.1.2b - 2)

We refer again to the construction in 1.1.1a.

If there is no element, say in E_1 , acting as a reflection on L , then we can choose U so small that the endpoint B is not contained in the closure of $E_1(U)$. We apply then the construction of 1.1.1a to this choice of U .

If B is a limit point for F , $\forall x \in \Omega(F)$ there exists a sequence $\{g_n\}$ $g_n \in F$ s.t. $g_n x \rightarrow B$. As $\Omega(F) \cap \partial U \neq \emptyset$, we can choose $x \in \partial U \cap \Omega(F)$. By a normal form argument and our choice of U , we deduce that this does not happen. So B is not a limit point for this realization F . If there is no element also in E_2 acting as a reflection on L we deduce analogously (for an eventually different realization of F) that it is not a limit point.

If there is one such element $g \in E_2$, A is not a limit point for our realization F as it is F -equivalent to B and $B \in \Omega(F)$.

1.1.2a - 3); 1.1.2b - 3)

Suppose $g \in F$ acts as a reflection on M . Then g^2 leaves M fixed and so it is elliptic of finite order. The group generated by g is cyclic and so is indecomposable. By the Subgroup Theorem it is conjugate to a subgroup K of E_1 or E_2 , let's say E_1 . But K should be a cyclic subgroup of E_1 , leaving invariant $g(M)$, which is absurdum.

q.e.d.

An analogous technical lemma must be proved for parabolic subgroups

1.1.3 Proposition Let E_1, E_2 be Kleinian groups satisfying the hypotheses of 1.1.1a or 1.1.1b. Let K be any cyclic parabolic subgroup of E_1 .

Then for some realizations F of $E_1 *_J E_2$ as in 1.1.1a (resp. 1.1.1b) the following properties hold:

- 1) If K is a maximal cyclic parabolic subgroup of E_1 , then it is a maximal cyclic parabolic subgroup of F .
- 2) F is geometrically finite.

Finally we collect here a statement which will be useful in the third section to ensure that the amalgamated group is a *function group* (that is an analytically finite Kleinian group with one invariant component).

1.1.4 Proposition Let E_1, E_2 be two function groups with a common finite (resp. parabolic) subgroup J like in 1.1.1a or 1.1.1b. If one of the two endpoints of the axis of J (resp. if the fixed point of J on S^2) is in the (resp. the boundary of the) invariant component of E_1 and E_2 , then any realization of $E_1 *_J E_2$ as a Kleinian group F like in 1.1.1a (resp. 1.1.1b) is a function group.

proof: [Mas] page 296/297.

1.2 HNN-extensions

In this section we analyze the case of HNN-extension. Statements and results are exactly analogous to those of the previous section, except that one uses the second Maskit's Combination Theorem ([Mas] VII.E.5).

1.2.1a Proposition Let F be a Kleinian group and J_1, J_2 two maximal finite cyclic subgroups of the same order. Suppose:

- 1) If J_1 and J_2 are conjugate there is no element in F acting as a reflection on the axis L_1 of J_1 .
- 2) The fixed points A_1, B_1, A_2, B_2 of J_1 and J_2 on S^2 are not limit points for F

Then it is always possible to realize $\langle F, t / tat^{-1} = b \rangle$ as a Kleinian group, where a and b are any minimal rotations for J_1 and J_2 . In the case J_1 conjugate to J_2 , a, b have to satisfy the property that there

does not exist $g \in F$ s.t. $gag^{-1} = b^m$ for some $m > 0$ (such a pair of generators always exist in view of 1))

Remark As in the amalgamated case, we can replace hypotheses 1) and 2) above with the following:

1') J_1 and J_2 are not conjugate

2') One of the two endpoints of L_1 and one of L_2 is not a limit point of F .

This will be useful if F is a Euclidean group.

1.2.1b Proposition Let F be a Kleinian group and J_1, J_2 two maximal parabolic subgroups of F .

Suppose:

1) If J_1 and J_2 are conjugate they have singularities of infinite cyclic type (that is we exclude the case of a dihedral singularity) (for this notion see [Mas] VI.A) ..

2) F is geometrically finite.

Then it is always possible to realize $\langle F, t \mid tat^{-1} = b \rangle$ as a Kleinian group, where a and b are any minimal rotations for J_1 and J_2 .

1.2.2a Proposition Let F be a Kleinian group satisfying the hypotheses of 1.2.1a or 1.2.1b. Let K be any finite cyclic subgroup of F not conjugate to a subgroup of J_1 and J_2 , with axis M . Then for some realizations of $\langle F, t \mid tat^{-1} = b \rangle$ as a Kleinian group F' like in 1.2.1a (resp. 1.2.1b)

1) If K is maximal in F then it is maximal in F'

2) If an endpoint C of M is not a limit point of F , then it is not a limit point of F' .

3) If there are no elements in F acting as reflections on the axis M of K , there are no such elements in F' .

Remark Properties 1 and 3 above go through if one replaces the hypotheses 1 and 2 of 1.2.1a with 1' and 2': see the remark after 1.2.1a.

1.2.2b Proposition Let F be a Kleinian group satisfying the hypotheses of 1.2.1a or 1.2.1b. Let K be any finite cyclic subgroup of F conjugate to a subgroup of J_1 or J_2 . Then for some realizations of $\langle F, t \mid tat^{-1}=b \rangle$ as a Kleinian group F' like in 1.2.1a (resp. 1.2.1b)

- 1) If K is maximal in F then it is maximal in F'
- 2) If the following additional hypotheses hold:
 - 1) J_1 and J_2 are finite
 - 2) J_1 and J_2 are not conjugate
 - 3) there are no elements in F acting as reflections on the axes of J_1 and J_2 ;
then the endpoints of the axes of J_1 and J_2 are not limit points for F'
- 3) If the following additional hypotheses hold:
 - 1) J_1 and J_2 are infinite
 - 2) J_1 or J_2 has singularities of infinite cyclic type;
then the endpoints of the axes of J_1 and J_2 are not limit points for F'
- 4) If there are no elements in F acting as reflections on the axis M of K , there are no such elements in F' .

1.2.3 Proposition Let F be a Kleinian group satisfying the hypotheses of 1.2.1a or 1.2.1b. Let K be any cyclic parabolic subgroup of F . Then for some realizations of $\langle F, t \mid tat^{-1}=b \rangle$ as a Kleinian group F' like in 1.2.1a (resp. 1.2.1b), we have

- 1) If K is maximal among parabolic subgroups of F then K is also maximal among parabolic subgroups of F' . If K has a singularity of infinite cyclic type in F , it still has such a singularity in F'
- 2) F' is geometrically finite.

1.2.4 Proposition Let F be a function group with subgroups J_1, J_2 satisfying the hypotheses of 1.2.1a (resp. 1.2.1b). If one of the two fixed points of J_1 on S^2 and one of J_2 (resp. if the fixed points of J_1 and of J_2 on S^2) are in the (resp. the boundary of the) invariant component of F , then any realization of $\langle F, t \mid tat^{-1}=b \rangle$ like in 1.2.1a (resp. 1.2.1b) is a function group (in the construction 1.2.1a identify the fixed points with A_1 and B_2).

1.3 Realization of a graph of groups as a function group

In this section we select a set of sufficient conditions (N0-N4) for realizing a graph of groups as a *function group*. In section 1.5 we will see that these conditions are also necessary, that is they characterize the set of graphs of groups (Γ, G) s.t. $\pi_1(\Gamma, G)$ is a function group.

We recall the usual notations for *graphs of groups*.

A graph Γ consists of a vertex set $V(\Gamma)$ and an edge set $E(\Gamma)$, together with a free involution $e \rightarrow e'$ taking each edge to its reverse edge and a map $\delta_0: E(\Gamma) \rightarrow V(\Gamma)$ taking each edge to its initial vertex. The terminal vertex map is $\delta_1: E(\Gamma) \rightarrow V(\Gamma)$ and it is given by $\delta_1 e = \delta_0 e'$. To each $v \in V(\Gamma)$ and $e \in E(\Gamma)$, G associates groups G_v and G_e respectively, subject to the constraint that $G_e = G_{e'}$. For each $e \in E(\Gamma)$, G assigns an *edge-to-vertex monomorphism* $i_e: G_e \rightarrow G_{\delta_0 e}$. In the graph of groups (Γ, G) a G path is a path $e_1 \dots e_n$ such that $\text{im}(i_{e_i}) = \text{im}(i_{e_{i+1}})$ for $1 \leq i \leq n-1$. A G -loop is a G -path which is a loop (that is, $\delta_1 e_n = \delta_0 e_1$), and a *closed G -loop* is a G -loop which also satisfies $\text{im}(i_{e_n}) = \text{im}(i_{e_1})$. We will say that two edges $e, e' \in E(\Gamma)$ are G -equivalent if there is a G -path $e_1 \dots e_n$ with $e_1 = e$ and $e_n = e'$. Since each edge is G -equivalent to its reverse edge, it follows that G -equivalence defines an equivalence relation on $E(\Gamma)$. For $e \in E(\Gamma)$ we define the G -component of e to be the subgraph Γ_e of Γ whose edge set $E(\Gamma_e)$ is the G -equivalence class of e in $E(\Gamma)$ and whose vertex set is $V(\Gamma_e) = \{\delta_0 e' \mid e' \in E(\Gamma_e)\}$. A vertex $v \in V(\Gamma_e)$ is called a N -vertex in Γ_e if there is an edge $e' \in E(\Gamma_e)$ with $\delta_0 e' = v$ and $N_{G_v}(\text{im}(i_{e'})) \neq \text{im}(i_{e'})$ (N_{G_v} denoting the normalizer in G_v). Finally we will say that a graph of groups (Γ, G) is in *standard form* provided that it satisfies the following two conditions:

- S1 If E_1 and E_2 are edges in Γ such that $\delta_0 e_1 = \delta_0 e_2$, and $\text{im}(f_{e_1})$ is conjugate into $\text{im}(f_{e_2})$, then $\text{im}(f_{e_1}) \subseteq \text{im}(f_{e_2})$
- S2 Γ has no trivial edges, where an edge e is *trivial* if $\text{im}(f_e) = G_{\delta_0 e}$ and $\delta_0 e \neq \delta_1 e$.

Before stating conditions N0-N4, we need also some preliminaries about *canonical presentations* of a group acting discontinuously on the plane.

A geometrically finite *Fuchsian group of the first kind*, a *Euclidean group* or a *finite Kleinian group* (that is the groups we will be concerned about in N1) are isomorphic to a finitely generated, orientation preserving group acting discontinuously on the plane. So ([Zie] page 35) they have the following canonical presentation

$$\langle s_1, \dots, s_k; d_1, \dots, d_n; t_1 u_1, \dots, t_g u_g \mid d_i^{h_i} = 1 (1 \leq i \leq n, h_i \geq 2); \\ \prod_1^k s_i \prod_1^n d_i \prod_1^g t_p u_p t_p^{-1} u_p^{-1} = 1 \rangle$$

From now on we fix a canonical presentation for each non spherical vertex. Then the elements d_i ($1 \leq i \leq n$) and s_j ($1 \leq j \leq k$) are geometrically minimal rotations in the same direction. We call these elements *canonical generators*.

If the vertex group is Fuchsian as above, there are two invariant circular components of the domain of discontinuity on the sphere at infinity. We define as the *positive component* the component S such that if l is the axis of any rotation, the canonical generator corresponds to a clockwise rotation of S with respect to l .

When we construct Kleinian groups according to the basic constructions in the proofs of sections 1.1 and 1.2 we use the positive component.

If the vertex group G is a Euclidean group as above, we choose a canonical presentation such that the canonical generator corresponds to a clockwise rotation of the domain of discontinuity with respect to the axis of any rotation.

These choices fix the correct way of performing the constructions of sections 1.1 and 1.2 to obtain a *function group*.

We can apply iteratively constructions of sections 1.1 and 1.2 to realize as a function group a graph of groups (Γ, G) satisfying the following conditions:

- N0 The graph is in standard form
- N1 Every vertex group of G is a finite Kleinian group (a subgroup of $SO(3)$), a Euclidean group (that is a Kleinian group with one limit point) or a finitely generated Fuchsian group of the first kind (that is a geometrically finite Kleinian group that keeps invariant some circular disk U and such that every point in the boundary of U is a limit point)
- N2 Every edge group of G is cyclic; if it is non-trivial it is maximal cyclic in the adjacent vertex groups. Moreover if it is infinite we suppose it is a parabolic subgroup in the adjacent vertex groups and maximal among parabolic subgroups
- N3 i) Going around a closed edge-path in G maps a generator of an edge-group to the same generator
 ii) Going along an edge from a non-spherical vertex to a non-spherical vertex maps a canonical generator to a canonical generator
- N4 i) For each $v \in V(\Gamma)$ s.t. G_v is spherical and for each nontrivial subgroup H of G_v , there is at most one edge $e \in E(\Gamma)$ with $\text{im}(i_e) = H$ except that there may be two such edges if the pair (G_v, H) is isomorphic to $(\mathbb{Z}_n, \mathbb{Z}_n)$, (D_{2n+1}, \mathbb{Z}_2) or (A_4, \mathbb{Z}_3)
 ii) For each G_v not spherical and for each nontrivial finite subgroup H or rank 1 parabolic subgroup of G_v there is at most one edge $e \in E(\Gamma)$ with $\text{im}(i_e) = H$.

Abuse of language: We will often say 'spherical (Fuchsian, Euclidean) vertex' instead of 'vertex with a spherical (Fuchsian, Euclidean) vertex group'.

1.3.1 Remark A graph of groups (Γ', G') satisfying N0-N4 satisfies also

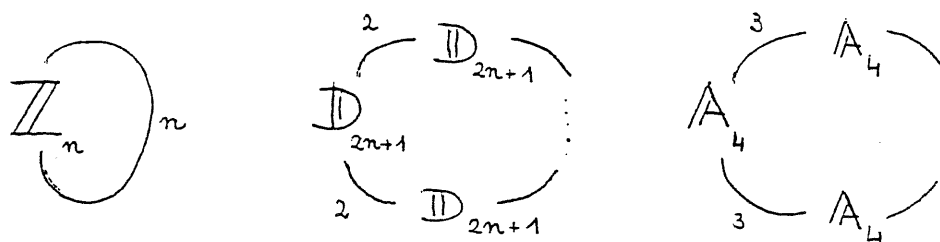
- U For each edge $e \in E(\Gamma)$ with nontrivial edge-group G_e ,
 $\chi(\Gamma_e) \geq 0$ and Γ_e contains at most two finite N -vertices
 In addition if $\chi(\Gamma_e) = 0$ then it has no more than one finite N -vertex, whereas if Γ_e contains a non-trivial closed G -loop

then it has no finite N-vertex. .

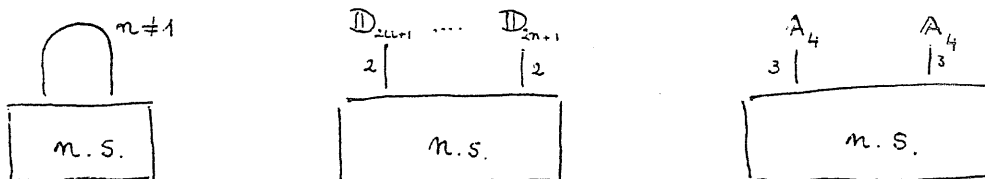
Analogously for each edge $e \in E(\Gamma)$ with nontrivial edge-group G_e , Γ_e contains at most two non-spherical vertices. If Γ_e contains a non-trivial closed G-loop then it has no non-spherical vertex). For a non-closed edge path, initial and endpoints are counted as two different points.

U will be used only in the proof of 1.3.4.

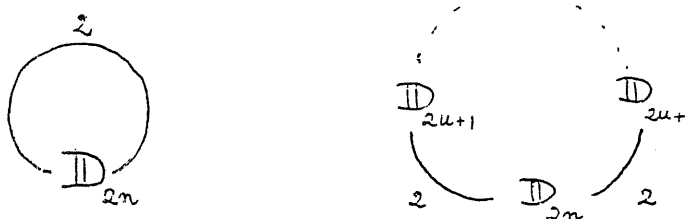
In fact by N0-N1-N2-N4, given an edge $e \in E(\Gamma)$ with non-trivial edge-group G_e , Γ_e contains a non-trivial closed G-loop if and only if it is one of the following types:



So there are neither N-vertices nor nonspherical vertices. Moreover $X(\Gamma_e)=0$ (if there are no G-loops) if and only if it is one of these types:



n.s. stands for nonspherical vertices



and so Γ_e has at most one N-vertex. If Γ_e is a tree, N-vertices or non spherical vertices may be only endpoints in Γ_e .

The algebraic properties of spherical groups enable us to put a graph of groups (Γ, G) satisfying N0-N4 in a simplified form (see condition N5 later), to which constructions of sections 1.1 and 1.2 may be directly applied. To this purpose we need to prove the following technical result

1.3.2 Lemma Let (Γ, G) and (Γ, \underline{G}) be graphs of groups satisfying N0 - N4. Suppose that there exists a system of isomorphisms $\alpha = \{\alpha(v), \alpha(e)\}$ $\alpha(v): G_v \rightarrow \underline{G}_v \forall v \in V(\Gamma)$; $\alpha(e): G_e \rightarrow \underline{G}_e \forall e \in E(\Gamma)$ s.t.

- i) $\alpha(e') = \alpha(e) \forall e \in E(\Gamma)$
- ii) $\alpha(\delta_0 e)(\text{im}(f_e)) = \text{im}(\underline{f}_e) \forall e \in E(\Gamma)$ such that $\delta_0 e$ is not spherical
- iii) if v is not spherical $\alpha(v)$ maps a canonical generator to a canonical generator.

Then $\pi_1(\Gamma, G) \cong \pi_1(\Gamma, \underline{G})$

proof: The proof is essentially as in [MMZ1] page 26.

One shows that there is a system $\alpha = \{\alpha(v), \alpha(e)\}$ of isomorphisms where $\alpha(v): G_v \rightarrow \underline{G}_v \forall v \in V(\Gamma)$; $\alpha(e): G_e \rightarrow \underline{G}_e \forall e \in E(\Gamma)$ s.t. i) holds and which is compatible on Γ , that is $\forall e \in E(\Gamma)$ the following diagram commutes

$$\begin{array}{ccc}
 G_{\delta_0 e} & \xrightarrow{\alpha(\delta_0 e)} & \underline{G}_{\delta_0 e} \\
 \uparrow f_e & & \uparrow \underline{f}_e \\
 G_e & \xrightarrow{\alpha(e)} & \underline{G}_e
 \end{array}$$

Consider the system α of isomorphisms of the hypothesis. By some modifications we can construct a new system of isomorphisms α' such that the property ii) holds for spherical vertices.

If $v \in V(\Gamma)$ is a spherical vertex we perform the two operations of [MMZ1] page 27 which consist of:

- a) choice of different isomorphisms between $G_{\delta_0 e}$ and $\underline{G}_{\delta_0 e}$ for spherical vertices
- b) modification of all edge-to-vertex monomorphisms $G_{\delta_0 e} \rightarrow \underline{G}_{\delta_0 e}$ of some fixed $v = \delta_0 e$ spherical, by inner automorphisms of $\underline{G}_{\delta_0 e}$.

Observe that operation b) changes (Γ, \underline{G}) but the resulting graph of groups still satisfies N0-N4 and has the same fundamental group.

We obtain then two graphs of groups (Γ, G) (Γ, \underline{G}) satisfying N0-N4 and a system of isomorphisms α with the properties i), iii) and s.t. ii) holds for every vertex of Γ .

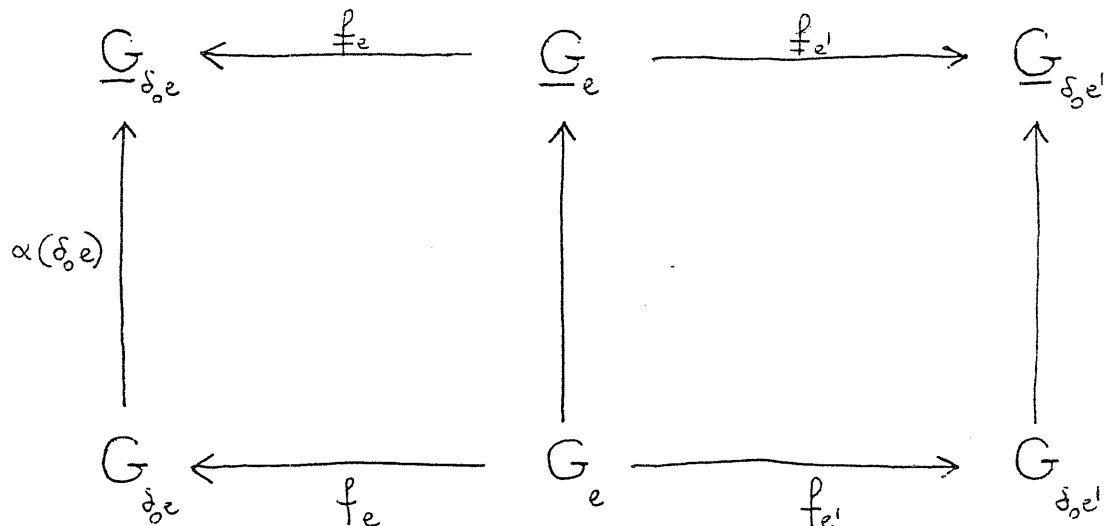
To obtain a system of isomorphisms which is compatible on Γ we proceed as in [MMZ1] page 27/29 for all spherical vertices.

The operations I - II - III of [MMZ1] change the edge-to-vertex monomorphisms in (Γ, \underline{G}) $\underline{f}_e: G_e \rightarrow \underline{G}_{\delta_0 e}$ with $\delta_0 e$ spherical by inner automorphisms of $\underline{G}_{\delta_0 e}$ leading to a graph of groups with the same fundamental group, satisfying N0-N4.

So we have two graphs of groups (Γ, G) , (Γ, \underline{G}) and a system of isomorphisms α with the properties i), ii), iii) and compatible on each spherical vertex.

Finally we arrange for compatibility of non spherical vertices.

Consider the case of an edge $e \in E(\Gamma)$ s.t. $\underline{G}_{\delta_0 e}$ is not spherical and $G_{\delta_0 e}$ is spherical (the case of adjacent non spherical vertices is easier)



By ii) $\phi = (\underline{f}_e, \underline{f}_e^{-1}) \alpha(\delta_0 e) (f_e f_e^{-1}) \alpha(\delta_0 e)^{-1}$ is an automorphism of $\text{im}(\underline{f}_e)$. So there exists $x \in \text{im}(\underline{f}_e)$ s.t. $\mu(x)\phi = 1$. We now define the system α' of isomorphisms on Γ to agree with α except that

$$\alpha'(e) = \underline{f}_e^{-1} \alpha(\delta_0 e) f_e \quad \alpha'(e') = \alpha'(e)$$

and we define a new edge-to-vertex monomorphism $\underline{f}'_e = \mu(x)\underline{f}_e$. The new graph of groups, as there are no other edges $e' \neq e$ s.t.

$\delta_0 e' = \delta_0 e$ and $\text{im} \underline{f}'_e (G_e)$ conjugate to $\text{im} \underline{f}_e (G_e)$, still satisfies N0-N4 and the system α' of isomorphisms still satisfies i), ii), iii), is compatible on each spherical vertex and on e' .

We repeat this operation $\forall e \in E(\Gamma)$ s.t. $\delta_0 e$ is not spherical and we obtain the thesis (notice that by property iii) of the system α , the modifications of the graph of groups (Γ, G') still satisfy N3ii))

q.e.d.

We have already fixed a canonical presentation for the non spherical vertices (see the remark at the beginning of the section). This is important to orient coherently the axes of the rotations of the vertex groups and to be sure to finally obtain, by a correct application of the constructions of sections 1.1 and 1.2 a function group.

Let us fix now a canonical presentation for each spherical vertex group in such a way that canonical generators correspond to clockwise rotations of the boundary component with respect to the axis of the rotation.

In view of these choices the constructions of sections 1.1 and 1.2 are not ambiguous.

The next corollary to 1.3.2 states that, whatever is our choice of a presentation for spherical vertex groups, we can always suppose that our graph of groups has the important property that canonical generators are mapped to canonical generators (N5).

1.3.3 Corollary *Let (Γ, G) be a graph of groups satisfying N0-N4. Fix a presentation of each spherical vertex group as a Kleinian group. Then (Γ, G) is isomorphic to a graph of groups (Γ, G') satisfying N0-N4 and*

N5 For every edge $A \rightarrow B$ of Γ the map i_e, i_e^{-1} maps a canonical generator to a canonical generator.

proof: We want to apply the previous lemma. Suppose v is a non spherical vertex and $e \in E(\Gamma)$ with $\delta_0 e = v$ s.t. $i_e(G_e)$ is not spanned by a canonical generator (notice that by N3ii) this implies that $\delta_1 e$ is a spherical vertex). This means that there exists $g \in G_{\delta_0 e}$ s.t. $g i_e(G_e) g^{-1}$ is a cyclic group spanned by a uniquely determined canonical generator. Define a new graph of groups (Γ, G) , with the same underlying graph and the same edge-to-vertex-monomorphisms except for

$$\underline{i}_e = \mu(g) i_e \quad \quad \quad \underline{i}_{e'} = i_e$$

($\mu(g)$ denotes conjugation by g). The fundamental group of (Γ, G) is unchanged and it still satisfies N0-N4.

By repeating this operation $\forall e \in E(\Gamma)$ as above, we can assume that we are given with a graph of groups (Γ, G) satisfying N0-N4 and s.t. if v is a non spherical vertex and $e \in E(\Gamma)$ is an edge with $\delta_0 e = v$, then $i_e(G_e)$ is a cyclic subgroup of $G_{\delta_0 e}$ spanned by a canonical generator.

Now define a new graph of groups (Γ, G') with the same underlying graph and the same edge and vertex groups. Given $e \in E(\Gamma)$ we define new edge-to-vertex-monomorphisms i_e' in this way:

- if $\delta_0 e$ is not spherical

$$i_e' = i_e \quad \quad i_{e'}' = i_e$$

- if $\delta_0 e$ is a spherical vertex and $\delta_1 e$ is a spherical vertex, define as edge-to-vertex-monomorphisms any pair of maps i_e, i_e' s.t.
1) they satisfy N2 2) i_e, i_e^{-1} maps a canonical generator of $G_{\delta_0 e}$ to a canonical generator of $G_{\delta_1 e}$ according to the fixed presentation.

- if $\delta_0 e$ is a spherical vertex and $\delta_1 e$ is not, $i_e(G_e)$ is, by our construction, a cyclic subgroup of $G_{\delta_1 e}$ generated by a canonical generator d .

Define new edge-to-vertex-monomorphisms

$$i_{e'}' = i_e, \\ i_{e'}', i_e'^{-1}(d) = f$$

where f is a canonical generator of $G_{\delta_0 e}$ s.t. i_e satisfies N2.

If necessary, we can always put (Γ, G') in standard form.

By construction (Γ, G') satisfies N0-N4 and N5 . Moreover (Γ, G) and

(Γ, G') satisfy the hypotheses of 3.2 for the system of isomorphisms $\alpha = \{\alpha(v), \alpha(e)\}$

$$\alpha(v): G_v \rightarrow G'_v = 1_{G_v} \quad \forall v \in V(\Gamma)$$

$$\alpha(e): G_e \rightarrow G'_e = 1_{G_e} \quad \forall e \in E(\Gamma)$$

So (Γ, G) and (Γ, G') are isomorphic.

q.e.d

The main theorem of the section is the following

1.3.4 Theorem Let (Γ, G) be a finite graph of groups satisfying N0-N5. Then there exists a function group F s.t. $\pi_1(\Gamma, G) \cong F$

We break the proof of the theorem into two parts by first realizing as a function group simpler graphs of groups.

Let us construct these simpler graphs of groups from a given (Γ, G) satisfying N0-N5. For any G -equivalence class Γ_e $e \in E(\Gamma)$ we perform some operations on the graph of groups.

- a) If G_e is not trivial and Γ_e has two finite N -vertices there is only one geometric path (in Γ_e) between the two vertices (in fact, by condition U, we have $\chi(\Gamma_e) > 0$). Choose a pair of edges $\{f, f'\}$ in this path and delete it: that is, consider the graph of groups (Γ', G) where $V(\Gamma') = V(\Gamma)$, $E(\Gamma') = E(\Gamma) - \{f, f'\}$ and edge and vertex groups are induced by (Γ, G) on the subgraph Γ' (may be disconnected).

If G_e is not trivial and Γ_e has two non spherical vertices) there is only one geometric path (in Γ_e) between the two vertices (in fact, by condition U, we have $\chi(\Gamma_e) > 0$). Choose a pair of edges $\{f, f'\}$ in this path and delete it: that is, consider the graph of groups (Γ', G) where $V(\Gamma') = V(\Gamma)$, $E(\Gamma') = E(\Gamma) - \{f, f'\}$ and edge and vertex groups are induced by (Γ, G) on the subgraph Γ' (may be disconnected).

- b) If G_e is not trivial and Γ_e contains a non-trivial closed G -loop (there may be at most one non-trivial closed G -loop by U) choose a pair of edges $\{f, f'\}$ in the loop and delete it.

After performing these operations we obtain a subgraph Γ' of Γ . Let (Γ', G) be the graph of groups induced by restriction of (Γ, G) to Γ' . (Γ', G) still satisfies N0 - N5.

Notations I Let $f \in (E(\Gamma) - E(\Gamma'))$ be an edge s.t. $\delta_0 f = v \in \Gamma'$. G_f injects into $\pi_1(\Gamma', G)$ through the map $i_{\delta_0 f} i_f$ where $i_f: G_f \rightarrow G_{\delta_0 f}$ is the injective edge monomorphism of (Γ, G) and $i_{\delta_0 f}: G_{\delta_0 f} \rightarrow \pi_1(\Gamma', G)$ is the natural inclusion. Let $J = i_{\delta_0 f} i_f(G_f) \subset \pi_1(\Gamma', G)$. We will call J exterior subgroup of (Γ', G) with respect to (Γ, G) .

Notations II May be $\delta_1 f$ and $\delta_0 f$ belongs to different connected components Γ_A, Γ_B . So we have two Kleinian groups $\pi_1(\Gamma_A, G), \pi_1(\Gamma_B, G)$ with maximal cyclic subgroups or q maximal parabolic subgroups (by ii)) $i_f(G_f), i_{\bar{f}}(G_{\bar{f}})$ of the same order. Consider for example the finite case. Set $J_1 = i_{\bar{f}}(G_{\bar{f}}), J_2 = i_f(G_f)$. $\phi = i_{\bar{f}} i_f^{-1}$ maps J_2 isomorphically onto J_1 . Consider the oriented axis L_1 (resp. L_2) of J_1 (resp. J_2) with endpoints A_1, B_1 (A_2, B_2). Let h be a hyperbolic isometry carrying L_1 onto L_2 , preserving the orientation of the axis. Then $h\pi_1(\Gamma_A, G)h^{-1}$ and $\pi_1(\Gamma_B, G)$ have a common cyclic subgroup $J = i_f(G_f)$. We call $h\pi_1(\Gamma_A, G)h^{-1}$ the group corresponding to $\pi_1(\Gamma_A, G)$ for J .

Notice that N5 ensures that $\mu(h)\phi$ (where $\mu(h)$ denotes conjugation by h) is the identity on J . Then $\Gamma_A \rightarrow \Gamma_B$ is isomorphic to $(h\pi_1(\Gamma_A, G)h^{-1}) *_J \pi_1(\Gamma_B, G)$.

1.3.7 Lemma Each connected component of (Γ', G) can be realized as a function group H s.t. for any subgroup J exterior with respect to (Γ, G) , the following properties hold:

- i) J is trivial, maximal cyclic or maximal parabolic in H
- ii) $J \neq H$
- iii) a) If J is not trivial and finite the fixed points of J on S^2 are not limit points for H (if the G -equivalence class of

- G_f in (Γ, G) contains Euclidean vertices we only require that one of the two fixed points is not a limit point for H).
- b) H is geometrically finite.
- iv) If J is not trivial and finite and the G -equivalence class of G_f in (Γ, G) contains no N -vertices, then there are no elements in H acting as reflections on the axis of J .
- v) a) If J is not trivial and finite and $\delta_1 f$ belongs to the same connected component of $\delta_0 f$, then one of the two fixed points of J on S^2 is in the invariant component of H
- b) If J is infinite, the fixed point of J on S^2 is in the boundary of the invariant component of H .
- vi) a) If J is not trivial and finite and $\delta_1 f$ belongs to a different connected component from $\delta_0 f$, let's say Γ^* , then one of the two fixed points of J on S^2 is in the invariant component of H and the group corresponding to H^* for J , where H^* is the realization as a function group of $\pi_1(\Gamma^*, G)$.
- b) If J is infinite and $\delta_1 f$ belongs to a different connected component from $\delta_0 f$, let's say Γ^* , then the fixed point of J on S^2 is in the boundary of the invariant component of H and the group corresponding to H^* for J , where H^* is the realization as a function group of $\pi_1(\Gamma^*, G)$.

proof of 1.3.7

By induction on the number of edges and the propositions of sections 1.1 and 1.2.

proof of 1.3.4:

Let $\{e, e'\}$ be a pair of edges of $E(\Gamma)$ deleted in operation a). Suppose, for example, that $\delta_0 e$ and $\delta_0 e'$ belong to different connected components Γ_1' and Γ_2' of Γ' (the other case is analogous).

Consider the graph $\Gamma'', V(\Gamma'') = V(\Gamma) \cup \{e, e'\}$; let (Γ'', G) the graph of groups induced on the subgraph Γ'' by (Γ, G) . By 1.3.7 we can realize (Γ_1', G) and (Γ_2', G) as function groups H_1 and H_2 s.t. for any subgroup J exterior with respect to (Γ, G) i)-vi) hold. As a consequence we claim that we can realize (Γ'', G) as a function group H s.t. for any subgroup J' exterior with respect to

(Γ, G) i)-vi) hold.

First of all H_1 and the group corresponding to H_2 for $i_e(G_e)=K$, satisfy the hypotheses of 1.1.1, 1.1.4 for the subgroup K . In fact they follow from i)-vi). So we can use its conclusions.

(Γ'', G) is realizable as a function group H s.t. for any subgroup J exterior with respect to (Γ, G) the following properties hold:

1.1.2-1 implies i)

1.1.2-2 implies iii): notice that J cannot be conjugate to K)

1.1.2-3 implies iv)

N3 implies v) -vi)

By iterating the procedure we realize a graph of groups (Γ^n, G) obtained by (Γ, G) with operations of type b), as a function group s.t. i)-vi) hold. For b-operations we behave in the same way, by using 1.2.1, 1.2.4 and N3i) (which guarantees the correct application of 1.2.1, 1.2.4).

q.e.d.

1.4 Realization of a graph of groups as a generalized function group

In the previous section we have realized as function groups graphs of groups satisfying conditions N0-N4. As a matter of fact conditions N3ii) and part of N4 are needed only to ensure that the realization of the graph of groups as a Kleinian group has an invariant component. We define now a notion of *generalized function group* and state conditions (N0'-N4') under which a graph of groups is realizable as a generalized function group.

Before stating the definition of generalized function group, we need some preliminaries.

Let F be a geometrically finite Kleinian group with no rank1 maximal parabolic subgroups. Let Ω_1 be a component of the domain of discontinuity $\Omega(F)$ and $\text{Stab}\Omega_1 \subset F$ its stabilizer. Then $\text{Stab}\Omega_1$ is a geometrically finite function group with invariant component Ω_1 . In fact, by [Maskit II F.8] the invariant component of $\text{Stab}\Omega_1$ is obtained from Ω_1 by adding a discrete set of points; as there are no rank1 maximal parabolic subgroups and F is geometrically finite, the invariant component coincides with Ω_1 . So $\text{Stab}\Omega_1$ is a finitely generated Kleinian group with one invariant component, that is a function group. In particular, for each component Ω' of the domain of discontinuity $\Omega'(F)$ we can define the structure subgroups S_1, \dots, S_n of F relative to Ω' as the structure subgroups of $\text{Stab}\Omega'$.

Now we can state the following:

1.4.1 Definition A *generalized function group (g.f.g)* is a geometrically finite Kleinian group with no rank1 maximal parabolic subgroups, such that, for each structure subgroup S_1 which is quasiconformally conjugate to a Fuchsian group and is relative to a boundary component Ω_1 of $\Omega(F)$, there exists a component Ω_2 which is not F -equivalent to Ω_1 and a structure subgroup S_2 relative to Ω_2 which is also a structure subgroup of Ω_1 and is conjugate to S_1 in $\text{Stab}\Omega_1$. Moreover for each Euclidean

structure subgroup S_1 relative to Ω_1 there exists a parabolic fixed point in $\Omega(F)$ which has a stabilizer conjugate to S_1 in $\text{Stab}\Omega_1$.

Remark The definition above includes geometrically finite function groups with no rank 1 maximal parabolic subgroups.

As a generalization of 1.3.4 we have the following statement:

1.4.2 Theorem Let (Γ, G) be a finite graph of groups satisfying:

$N0' = N0$

$N1'$ Every vertex group of G is a finite Kleinian group, a Euclidean group of rank two or a cocompact Fuchsian group of the first kind.

$N2'$ Every edge group of G is cyclic and if it is not trivial it is maximal cyclic in the adjacent vertex groups.

$N3' i) = N3i)$

ii) Going along an edge from a non spherical vertex to a non spherical vertex maps a canonical generator to a canonical generator or the inverse of a canonical generator

$N4' i) = N4i)$

ii) For each $v \in V(\Gamma)$ s.t. G_v is Euclidean and for each nontrivial finite subgroup H of G_v , there is at most one edge $e \in E(\Gamma)$ with $\text{im}(f_e) = H$

iii) For each $v \in V(\Gamma)$ s.t. G_v is Fuchsian and for each nontrivial finite subgroup H of G_v , there are at most two edges $e \in E(\Gamma)$ with $\text{im}(f_e) = H$.

Then (Γ, G) is realizable as a generalized function group

1.5 Geometrically finite function groups

We have seen in section 1.3 that any graph of groups (Γ, G) satisfying the normalized conditions N0-N4 is the fundamental group of a geometrically finite function group. The aim of this section is to prove the converse, that is:

1.5.1 Theorem *An abstract group F is isomorphic to a geometrically finite function group, if and only if there exists a graph of groups (Γ, G) satisfying the conditions N0-N4 s.t. $\pi_1(\Gamma, G) \cong F$*

It is possible to give an algebraic proof of this result by using geometric group theory as in [MMZ1].

But, if one consider the three-dimensional objects corresponding to these groups, the topological tools give the same answer at once.

In fact, by applying *Equivariant Dehn Lemma* and *Cylinder Theorem*, one proves that function groups uniformize *orbifold-with-handles* (see definition below). Orbifold-with-handles admit a natural geometric decomposition in simpler pieces and through this decomposition one gets a nice presentation of function groups as graphs of groups. For an analogous use of topological theorems see [Mar2].

There are many ways in which this decomposition can be done and so there are many graphs of groups satisfying N0-N4 which can be associated to a given function group.

The most direct way is to observe that, if F is a geometrically finite function group, $O = \mathbb{H}^3/F$ is *essentially compact*, that is there is a finite set of disjoint two-orbifolds $S_1 \dots S_m$ s.t., if we cut O along US_i , the resulting orbifold O' is compact. Moreover the possible ends of O are known and one sees that $\pi_1(O') = \pi_1(O)$.

O' admits a decomposition in compact simpler pieces and through this decomposition one gets a presentation of F as a graph of groups.

Moreover the graph of groups so obtained satisfies the following conditions which are more restricted than $N0 - N4$:

$$C0 = N0$$

C1 Every vertex group of G is a finite Kleinian group (a subgroup of $SO(3)$), a Euclidean group of rank 2 (that is a Kleinian group with one limit point with an abelian subgroup of rank 2) or a cocompact Fuchsian group of the first kind (that is a Kleinian group that keeps invariant some circular disk U and such that every point in the boundary of U is a limit point: moreover U/F is a compact surface)

C2 Every edge group of G is finite cyclic; if it is not trivial it is maximal cyclic in the adjacent vertex groups

$$C3 = N3$$

$$C4 = N4$$

We could have stated 1.5.1 also by using conditions $C0 - C4$, instead of $N0 - N4$. As a matter of fact the first part of the proof in which, by using combinations theorems, we give a graph the structure of a Kleinian group, would have been easier.

But the point is that, if we describe F through a simpler graph of groups (Γ, G) satisfying $C0 - C4$, a lot of geometric information about F is lost. As a matter of fact, we know that $\pi_1(\Gamma, G) \cong F$ as an abstract group, but we don't know nothing about the deformation class of F as a function group.

The class of graphs of groups satisfying $C0 - C4$ is suitable for describing function groups F uniformizing compact product-with-handles but they don't tell anything about the possible punctures of $M = \mathbb{H}^3 \cup \Omega(F)/F$ in the general case.

The application of Dehn lemma and Cylinder Theorem we will use is, accordingly, more sophisticated and closer to the approach of

[Mar2] (but he deals with the torsion-free case).

1.5.2 Definition A *product-with-handles* is a connected, orientable 3-manifold M s.t. there exists a finite collection of disks or punctured disks D_1, \dots, D_r mutually disjoint, with all boundaries in the same connected component of ∂M and the property that, by cutting M along D_1, \dots, D_r we obtain a collection of 3-manifolds M_1, \dots, M_n as follows.

M_i $1 \leq i \leq n$ may be

- 1) a 3-ball
- 2) a 3-ball without one diameter
- 3) a product $S \times [0,1]$ where S is a compact, orientable surface without boundary with genus $g > 1$
- 4) $T^2 \times [0,1]$
- 5) a product $\Sigma \times [0,1]$ where Σ is an orientable surface with punctures and free rank $r \geq 2$.

This collection of 3-manifolds M_1, \dots, M_n and of disks D_1, \dots, D_r will be called a *geometric decomposition* for M .

We will sometimes use the notion (see chapter 3) of *almost compact product-with-handles*. The definition of almost compact product-with-handles is the same as above, except that 2) and 5) are not allowed. So the only noncompact elements of a geometric decomposition are homeomorphic to $T^2 \times [0,1]$.

1.5.3 Definition An *orbifold product-with-handles* is a connected, orientable 3-orbifold O s.t. there exists a finite collection of discal 2-orbifolds (that is orbifolds which are isomorphic to D^2/G where D^2 is the 2-disk and G is a finite subgroup of $SO(2)$) or punctured disks D_1, \dots, D_r mutually disjoint with all boundaries in the same connected component of ∂O and the property that, by cutting O along D_1, \dots, D_r we obtain a collection of 3-orbifolds O_1, \dots, O_n as follows.

O_i $1 \leq i \leq n$ may be

- 1) a quotient of B^3 by a finite linear action (a ballic orbifold)
- 2) a quotient of $\mathbb{H}^3 \cup \Omega(E)$ by a Euclidean group E (that is a

- Kleinian group with one limit point)
- 3) a product $S \times [0,1]$, where S is a compact hyperbolic 2-orbifold or a 2-orbifold with some punctures (the case S^2 with one puncture is excluded)

This collection of 3-orbifolds $O_1 \dots O_n$ and of discal 2-orbifolds $D_1 \dots D_r$ will be called a *geometric decomposition* for O . We will show later (2.1) that this decomposition is essentially unique.

1.5.1 is an immediate corollary of the following

1.5.4 Lemma *If F is a geometrically finite function group $M = \mathbb{H}^3 \cup \Omega(F)/F$ is an orbifold product-with-handles.*

Notice the following particular case of 1.5.4

1.5.5 Corollary *If F is a geometrically finite torsion-free function group, then $M = \mathbb{H}^3 \cup \Omega(F)/F$ is a product-with-handles.*

By 1.5.4 we know that, if F is a geometrically finite function group, then $M = \mathbb{H}^3 \cup \Omega(F)/F$ is an orbifold product-with-handles, that is, there exists a collection of mutually disjoint discal 2-orbifolds like in 1.5.3. The geometric decomposition of M gives a presentation of F as a graph of groups (Γ, G) satisfying N1-N4, but not necessarily N0 (they may have trivial edges). Of course it is always possible to collapse trivial edges and obtain, algebraically, a graph of groups in standard form. Something more is true, that is there exists a geometric decomposition of M such that the associated graph of groups is in standard form. Suppose that $e \in E(\Gamma)$ is a trivial edge with $\delta_0 e = v_1$ and $\delta_1 e = v_2$; moreover let $v_1 \neq v_2$ and $i_e: G_e \rightarrow G_{\delta_0 e}$ an isomorphism. Then there exists a discal 2-orbifold D_e (or a punctured disk) in the collection above corresponding to the edge e which can be ignored. In fact one sees

that $D_1 \dots D_{r-1}$ give still a decomposition of M as in 1.5.3. This is due to the fact that we have only two possibilities: the 3-orbifold associated to v_1 in the first decomposition can be a 3-ball with a singular elliptic axis or a 3-ball without one axis. Whatever is the 3-orbifold associated to v_2 in the first decomposition, by property N1, we obtain a homeomorphic 3-orbifold by glueing to it one of these 3-balls. So the collection of discal 2-orbifolds $D_1 \dots D_{r-1}$ gives the same a decomposition of M as an orbifold product-with-handles, with associated graph of groups the graph obtained from (Γ, G) by collapsing the edge e .

proof of 1.5.4: By Selberg's lemma F has a normal torsion-free subgroup of finite index H .

Moreover $\Omega(H) = \Omega(F)$ and so H is still a function group.

If $F/H \cong G$ where G is a finite group, O is isomorphic to M/G where $M = \mathbb{H}^3 \cup \Omega(H)/H$ and M/G is the quotient of M by the action of a finite group of orientation-preserving diffeomorphisms isomorphic to G .

Let Ω be the invariant component of H and $\Omega/H = S$ the correspondent boundary component of $M = \mathbb{H}^3 \cup \Omega(H)/H$. Let $I: \pi_1(S) \rightarrow \pi_1(M)$ the homomorphism induced by the natural inclusion: as Ω is invariant, I is an epimorphism.

We have two cases:

- i) I is an isomorphism
- ii) $\ker I = \emptyset$.

Let's first analyze the case ii).

By applying equivariant Dehn lemma, there exists a collection of r disjoint G -equivariant disks $D_1 \dots D_r$ s.t. $\partial D_i \subset S \ \forall i \ 1 \leq i \leq r$ and $\ker I$ is generated by the boundaries of the disks.

Cut the manifold along these disks: we obtain n 3-dimensional orientable manifolds M_i with one boundary component S_i $1 \leq i \leq m$ given by a submanifold of S collapsed along some boundary circles. The manifolds M_i are stabilized by some finite subgroups of G , $\text{Stab}_1 \dots \text{Stab}_m$. So M can be obtained by glueing the n 3-manifolds M_i along disjoint disks; S can be obtained from S_i by removing the interior of the same disks and by glueing the m pieces S_i obtained along the boundary circles.

From the point of view of fundamental groups this corresponds to a

presentation of $\pi_1 M$ as a graph of groups (Γ, G) with vertex groups isomorphic to $\pi_1 M_i$ $1 \leq i \leq m$ and trivial edge groups; and to a presentation of $\pi_1 S$ as a graph of groups (Γ_0, G_0) with vertex groups isomorphic to $\pi_1 S_i$ $1 \leq i \leq m$ and infinite cyclic edge groups.

The map I defined above maps the vertex groups G_v of (Γ_0, G_0) into the vertex groups of (Γ, G) with kernel generated in G_v by the images of the edge groups G_e of (Γ_0, G_0) with $\delta_0 e = v$ and maps edge groups of (Γ_0, G_0) to edge groups of (Γ, G) .

As I is an epimorphism, the vertex groups of (Γ_0, G_0) are mapped onto the vertex groups of (Γ, G) .

Then the map $J: \pi_1(S_i) \rightarrow \pi_1(M_i)$ $1 \leq i \leq n$ induced by inclusion is an isomorphism. We are so reduced to case i).

If M_i is compact, by [H] th. 10.2, M_i is B^3 or a product $S \times [0, 1]$ where S is a compact orientable surface different from B^3 and with no boundary.

As a matter of fact $T^2 \times [0, 1]$ is excluded as any rank 2 abelian subgroup of a kleinian group is parabolic.

If M_i is not compact and has some non compact ends homeomorphic to $T^2 \times [0, 1]$, we can always choose the embedded end in M_i in such a way that G acts equivariantly on it.

By cutting M_i along $T^2 \times \{0\}$ we obtain two pieces which are precisely invariant under the action of G .

So we can reduce to the case in which noncompact ends are not of this type.

Then by [Mar2] prop. 5.4 \forall fixed i $1 \leq i \leq n$ there exists a certain number n_i of disjoint tubes T_1, \dots, T_{n_i} pairing some punctures on $\partial M_i - S_i$ s.t., by removing T_i we obtain a collection of products (or 3-balls) N_1, \dots, N_{n_i} where $N_j = S_j \times [0, 1]$ and S_j is a finitely generated surface with possible punctures. As F is geometrically finite, we can choose these tubes to be also G -equivariant.

So we conclude that \forall fixed i $1 \leq i \leq n$ there exists a certain number n_i of disjoint G -equivariant punctured disks D_1, \dots, D_{n_i} with $\partial D_j \subset S_i$ $1 \leq j \leq n_i$ and s.t., by cutting M_i along D_i we obtain a collection of products (or 3-balls) N_1, \dots, N_{n_i} where $N_j = S_j \times [0, 1]$ and S_j is a finitely generated surface with possible punctures. Notice that S_j can't be S^2 or a sphere S^2 with one puncture as M is irreducible.

Otherwise stated: we can find in M a finite collection of mutually disjoint disks and punctured disks with all the boundaries in the

same connected component of ∂M s.t., by cutting M along them we obtain a set of 3-manifolds $N_1 \dots N_k$ which are precisely invariant under the action of finite subgroups of G .

N_i may be one of the manifolds 1)-5) listed in 1.5.2.

Now we can conclude the proof of the theorem by studying the action of finite groups of orientation-preserving diffeomorphisms on 3-manifolds of type 1) - 5).

If N is a 3-ball and G a finite group acting on it, the action of G is linear. If $N \cong S \times [0,1]$, then $N/G \cong S' \times [0,1]$, with S' a compact surface with possible elliptic singularities ([H] theorem 8.1). For the non compact cases, one reduces to the previous case, with S a compact surface with nonempty boundary, by cutting N , as F is geometrically finite, along G -equivariant tubes pairing the punctures of $S \times \{0\}$ with $S \times \{1\}$.

q.e.d.

We will also use (see chapter 3) the following analogous but more restricted result:

1.5.6 Theorem *An abstract group F is isomorphic to a geometrically finite function group with no maximal rank1 parabolic subgroups, if and only if there exists a graph of groups (Γ, G) satisfying the conditions C0-C4 s.t. $\pi_1(\Gamma, G) \cong F$.*

From the geometrical point of view, we have a statement corresponding to 1.5.5:

1.5.7 Theorem *If F is a geometrically finite function group with no maximal rank1 parabolic subgroups, then $M = \mathbb{H}^3 \cup \Omega(F)/F$ is an almost compact product-with-handles.*

1.6 Generalized function groups

We have seen that any graph of groups (Γ, G) satisfying the normalized conditions is the fundamental group of a generalized function group. The aim of this section is to prove the converse, that is

1.6.1 Theorem *An abstract group F is isomorphic to a generalized function group if and only if there exists a graph of groups (Γ, G) satisfying the conditions $N0'$ - $N4'$ such that $\pi_1(\Gamma, G) \cong F$.*

As in the previous section we state the following

1.6.2 Definition *A generalized orbifold product-with-handles (g.o.p.) is a connected orientable 3-orbifold M such that there exists a finite collection of discal 2-orbifolds D_1, \dots, D_r mutually disjoint with the property that, by cutting M along D_1, \dots, D_r we obtain a collection of 3-orbifolds M_1, \dots, M_n as follows.*

M_i $1 \leq i \leq n$ may be:

- 1) A quotient of B^3 by a finite linear action
- 2) A quotient of $\mathbb{H}^3 \cup \Omega(E)$ by a Euclidean group E with rank two.
- 3) A product $S \times [0, 1]$, where S is a compact hyperbolic two-orbifold.

1.6.1 is an immediate corollary of the following

1.6.3 Lemma *If F is a g.f.g. then $M = \mathbb{H}^3 \cup \Omega(F)/F$ is a g.o.p.*

As for orbifold product-with-handles, the geometric decomposition

of M gives a presentation of $\pi_1 M$ as a graph of groups (Γ, G) satisfying $N1'-N4'$ and without loss of generality we can suppose it satisfies also $N0'$.

proof of 1.6.3 Instead of applying equivariant Dehn lemma like in 1.5.1, we will use, in an equivalent way, Dehn lemma for orbifolds [Ta]. Let $M = \mathbb{H}^3 \cup \Omega(F)/F$ and fix a connected component Σ of ∂M . If $\ker(i^*: \pi_1 \Sigma \rightarrow \pi_1 M) \neq 1$ (where i denotes an inclusion) we can apply Dehn lemma for orbifolds [Ta] 3.4 and find a collection of disjoint discal 2-orbifolds D_1, \dots, D_r such that $\partial D_1, \dots, \partial D_r \subset \Sigma$ and they generate $\ker i^*$. The boundaries $\partial D_1, \dots, \partial D_r$ are a system of dividers for Σ in the sense of [Mas] X.C. We repeat this operation for any component of ∂M and cut M along them. We obtain n 3-dimensional orientable manifolds M_i which are almost compact: they may admit noncompact ends homeomorphic to H/E where E is a Euclidean subgroup of F of rank two and $H \subset \mathbb{H}^3$ is a horoball precisely invariant under E . Let M_1 be one of these manifolds. It admits one boundary component Σ_1 which is uniformized by a structure subgroup S_1 of F (relative to some component Ω_1 of $\Omega(F)$) acting on its invariant component. As there are no maximal rank 1 parabolic subgroups in F , S_1 cannot be degenerate and is elementary or quasifuchsian. If S_1 is elementary and finite, as M is irreducible, it has only the boundary component Σ_1 and is a ballic orbifold. If S_1 is a Euclidean orbifold, by 1.4.1, $M_1 \cong \mathbb{H}^3 \cup \Omega(S_1)/S_1$. If S_1 is quasiconformally conjugate to a Fuchsian group, by 1.4.1, there exists a different boundary component Σ_2 of ∂M such that $i_1^*(\pi_1 \Sigma_1) \cong i_2^*(\pi_1 \Sigma_2)$ for some inclusions i_1 and i_2 . As i_1^* and i_2^* are injective maps, we conclude, by [MS] and [Wa] lemma 5.1, that $M_1 \cong \Sigma_1 \times [0, 1]$.

q.e.d.

2 Some classification results

This part is the core of the thesis. It shows how the use of 3-dimensional topology greatly simplifies the study of Kleinian groups and gives easily strong classification results.

The crucial example to bear in mind is Maskit's classification of geometrically finite function groups [Mas] X.G.2:

Let G and F be geometrically finite function groups. G is a deformation of F if and only if G and F have the same signature.

The proof of this result is based on three crucial steps:

- 1) The *planarity theorem* ([Mas] X.A.6) about regular planar coverings of topologically finite Riemann surfaces, allows us to associate a *signature* (X.E) to each function group.
- 2) The *Jordan theorem*, stating that a surface S is planar if and only if every simple closed curve on S is dividing, implies that quasiconformally conjugate function groups have the same signature (X.D.5)
- 3) We can find out the set of *admissible* signatures, by constructing, through the combination theorems, function groups with a given signature.

The three-dimensional point of view, on one side sheds a different light on this theorem and, on the other one, suggests various generalizations, which are not so evident by only looking at the limit set of function groups: generalized function groups, for example, are a natural geometric extension of the notion of function groups, while their characterization as Kleinian groups (1.4) is a little tricky.

The first step has been accomplished in chapter 1.

The 3-dimensional counterpart of planarity theorem is *Dehn lemma and Cylinder theorem*, in the following sense. For a geometrically finite function group F uniformizing an orbifold

product-with-handles $O = \mathbb{H}^3 \cup \Omega(F)/F$, Dehn lemma and Cylinder theorem give us a collection of discal 2-orbifolds and punctured disks $D_1 \dots D_n$, such that, by cutting O along them, we obtain a geometric decomposition of it (see 1.5). The boundaries $\partial D_1 \dots \partial D_n$ are a system of dividers for F . By this way the signature ([Mas] X.5) (K, t) of F corresponds to the graph of groups (Γ, G) that we associate to the geometric decomposition of O (2.1). The topological meaning of the system of dividers is so disclosed by the three-dimensional picture.

To obtain a classification of geometrically finite function groups, one has first to prove that quasiconformally conjugate function groups have the same signature. The nice Maskit's proof (X.D.5) relies heavily on the characterizing property of planar surfaces: every simple closed curve is dividing. On the other hand, in this section we prove (see 2.1) that a geometric decomposition of an orbifold product-with-handles is essentially unique. From the 3-dimensional point of view it is easy to understand why: the singular part of the orbifold product-with-handles determines, up to some triviality, the arrangement of the geometric decomposition. In some sense it represents a rigid core of the decomposition.

The final step is constructive: is there any geometrically finite function group admitting a given signature? The determination of the class of admissible signatures and the reconstruction of the corresponding function groups has been done by Maskit. Notice however that it is much easier to prove that a function group admits a given signature (X.F) than a given geometric decomposition (2.1).

In conclusion the interplay with 3-dimensional geometry explains and renews these previous results on Kleinian groups. Moreover in this setting, some generalizations which are not so evident by only analyzing the behaviour of a Kleinian group at infinity, look very natural (see the section 2.2).

A technical consideration. Graphs of groups reveal to be very convenient to state classification results. The classification is given in terms of equivalence classes of graphs of groups.

This chapter is organized as follows.

In 2.1 we classify geometrically finite function groups in terms of equivalence classes of graphs of groups satisfying N0-N4 and prove that this classification is equivalent to Maskit's one.

In 2.2 we prove an analogous result for generalized function groups.

2.1 Classification of function groups

In this section we classify geometrically finite function groups in terms of equivalence classes of graphs of groups. As a first step, we state this equivalence relation.

2.1.1 Definition Two graphs of groups (Γ, G) and (Γ', G') satisfying N0-N4 are *equivalent*, if and only if they have isomorphic fundamental groups and, once removed all edges with trivial edge group in $E(\Gamma)$ and $E(\Gamma')$, they are isomorphic as graphs of groups (possibly disconnected).

Let A be the set of deformations ([Mas] II.J.3) of geometrically finite function groups, B be the set of equivalence classes of graphs of groups satisfying N0-N4. We define the map $f:A \rightarrow B$ which associates to each $[F] \in A$ the class $[(\Gamma, G)]$ of any graph of groups $[(\Gamma, G)]$ satisfying N0-N4 associated to a geometric decomposition of $M = \mathbb{H}^3 \cup \Omega(F)/F$ as in 1.5.

As we are going to prove, this map is a bijection and classifies geometrically finite function groups.

2.1.2 Theorem *The map f is well defined.*

proof: Let F and H be two quasiconformally-conjugate function groups. Then $O = \mathbb{H}^3 \cup \Omega(F)/F \cong \mathbb{H}^3 \cup \Omega(H)/H$. Let (Γ, G) and (Γ', G') be two graphs of groups satisfying N0-N4, associated to two geometric decompositions of O . If $v \in V(\Gamma)$ and G_v is a nonspherical group, then, by the properties of the geometric decomposition (1.5.3), v is associated to a product $S \times [0, 1]$, where S is a compact hyperbolic 2-orbifold or a 2-orbifold with some punctures, or a quotient of $\mathbb{H}^3 \cup \Omega(E)$ by a Euclidean group E . In any case there are no open nontrivial ballic suborbifolds in $S \times [0, 1]$ or $\mathbb{H}^3 \cup \Omega(E)/E$. On the other side, if $v \in V(\Gamma)$ is a spherical vertex, it is associated to a ballic orbifold and the boundary of this orbifold has no common points with any incompressible boundary component (i.b.c.) of O . In the first case, if v is nonspherical, there exists in O a boundary component $S \times \{1\}$ or a noncompact end which has no common points with the compressible boundary component (c.b.c.) of O . In the second case, if v is spherical, there exists a spherical singularity in the interior of O . On the other hand, if Σ is an incompressible boundary component (i.b.c.) of O or a noncompact end, there exists at least one element O_i in the geometric decomposition (see 1.5.3), which contains some points of Σ . By the reasonment above O_i cannot be associated to a spherical vertex (as the boundary of the ballic orbifolds of the decomposition have no common parts with any i.b.c. of O) and so it is associated to a vertex v which is Fuchsian or Euclidean. In the first case we know that $O_i \cong S \times [0, 1]$ for some 2-orbifold S and that $S \times \{1\}$ is an i.b.c. of O . So $S \times \{1\} \cong \Sigma$. In the second case we know that $O_i \cong \mathbb{H}^3 \cup \Omega(E)/E$ for some Euclidean group E and that the noncompact end of O_i is a noncompact end of ∂O . So it

coincides with the given end. Finally for a given nontrivial open ballic suborbifold of O , there exists an element O_i in the geometric decomposition of O which contains the singularity in its interior. O_i is associated to a spherical vertex as we have seen above that only in this case we can find nontrivial open ballic suborbifolds.

Till now we have proved that:

for any graph of groups (Γ, G) in standard form associated to a geometric decomposition of O , the number of vertices is fixed and is determined by the number of i.b.c., the number of noncompact ends which are distinct from the c.b.c., the number of nontrivial open ballic suborbifolds. Moreover if (Γ, H) and (Γ', H') are different geometric decompositions of O , then there exists a natural bijection $\phi: V(\Gamma) \rightarrow V(\Gamma')$ such that corresponding vertex groups are isomorphic.

To analyze the situation for edges, it is useful to consider G -components of (Γ, G) . Each unoriented edge in $E(\Gamma)$ is contained in one and only one G -component. Let Γ_e be a subgraph of Γ which is the G -component of e and (Γ_e, G) the induced graph of groups. Suppose G_e finite (and not trivial). Then Γ_e corresponds to a singular axis in O , which can be closed or have one or two endpoints. If the axis is closed, there are no nonspherical vertices in Γ_e . For example a Fuchsian vertex has an associated element O_i in the geometric decomposition which is a product $S \times [0, 1]$ and any singular axis in this product has an endpoint on an i.b.c. of O (so it can not be closed). Analogous considerations hold for Euclidean vertices. As spherical vertices are in bijective correspondence with nontrivial open ballic suborbifolds of O , we deduce that the type and number of vertices in the G -component is uniquely determined by the geometry of the singular structure of O (precisely by the type and number of nontrivial open ballic suborbifolds of a tubular neighbourhood of the singular axis). Let's now consider the case of an axis with two endpoints. If an endpoint lies in an i.b.c. of O , then (Γ_e, G) has a corresponding Fuchsian vertex at one end, as we have seen that this is the only case in which the corresponding element $O_i \cong S \times [0, 1]$ in the geometric decomposition of O contains points of i.b.c. of O . The type of the Fuchsian vertex is determined by this component. Vice versa, a Fuchsian vertex in (Γ_e, G) implies that the corresponding

singular axis has an endpoint in an i.b.c. of O . Analogous considerations show that the singular axis is open at one end if and only if the G -component (Γ_e, G) admits one Euclidean vertex at one end (and its type is determined by the noncompact end). Then, if the singular axis has an endpoint on the c.b.c., it must correspond to a spherical vertex at the end of (Γ_e, G) . As nonspherical vertices in (Γ_e, G) can appear only at the ends, all other vertices in it are spherical and their type and number is determined by the nontrivial open ball-like suborbifolds of a tubular neighborhood of the singular axis.

Analogous considerations for G -components with infinite edge groups lead to conclude that, given different geometric decompositions of O with associated graphs (Γ, G) and (Γ', G') , there exists a bijection α between edges such that α is compatible with ϕ and corresponding G -components are isomorphic graphs of groups.

These facts are enough to conclude that (Γ, G) and (Γ', G') are equivalent.

q.e.d.

2.1.3 Remark Notice that, in general, there are many inequivalent graphs of groups satisfying N0-N4 which have isomorphic fundamental groups. This is still true even if we restrict to the subset of graphs of groups satisfying C0-C4 and the same equivalence relation. As a matter of fact the techniques of [SW] 7.6 or [Her] or even a direct analysis, give us the following result:

Let (Γ, G) and (Γ', G') be two finite graphs of groups with the following properties:

- 1) *They satisfy C0-C1-C2-C3*
- 2) $\pi_1(\Gamma, G) \cong \pi_1(\Gamma', G')$.

Then

- i) *There exists a bijection $\alpha: V(\Gamma) \rightarrow V(\Gamma')$ such that corresponding vertex groups are conjugate*
- ii) *The number of edges of Γ and Γ' is the same*
- iii) *Let \mathbb{A} be the set of G -components of (Γ, G) and \mathbb{B} the set of G' -components of (Γ', G') . Then there exists a bijection $\beta: \mathbb{A} \rightarrow \mathbb{B}$ such that*
 - *a and $\beta(a)$ are isomorphic graphs of groups $\forall a \in \mathbb{A}$*
 - *If a has vertices v_1, \dots, v_n , $\beta(a)$ has vertices $\alpha(v_1), \dots, \alpha(v_n)$.*

Let B' be the set B minus the equivalence classes of graphs of groups with a subgraph $(2,2,\infty) \xrightarrow{\infty} (2,2,\infty)$.

2.1.4 Theorem f is bijective.

proof: If F and G are two function groups with $f([F]) = f([G]) = [(\Gamma, G)]$, then there exist geometric decompositions of $M = \mathbb{H}^3 \cup \Omega(F)/F$ and $N = \mathbb{H}^3 \cup \Omega(G)/G$ such that the associated graphs of groups (Γ, G) and (Γ', G') are isomorphic graphs of groups up to edges with trivial edge groups. This implies that $M \cong N$ and so f is injective. On the other side f is surjective, because any graph of groups (Γ, G) satisfying $N0-N4$ is realizable, in view of (1.5), [Mas] VII.C.2ix) and VII.E.5ix) as a function group F such that $f([F]) = [(\Gamma, G)]$ (Notice that the geometrically finite function group obtained from $(2,2,\infty) \xrightarrow{\infty} (2,2,\infty)$ by the usual procedure is $(2,2,2,2)$ and that there exist no orbifold-with-handles and geometric decompositions with associated graph of groups $(2,2,\infty) \xrightarrow{\infty} (2,2,\infty)$).

q.e.d.

By 2.1.4 geometrically finite function groups are classified in terms of equivalence classes of graphs of groups. One sees at once that this classification is equivalent to Maskit's classification in terms of signatures in the following sense.

K is a marked two-complex consisting of some Riemann surfaces of finite type $R_1 \dots R_s$ and some 1-cells $v_1 \dots v_k$ (the connectors). t is the Schottky number. There are two bijections $\phi: (R_1 \dots R_s) \rightarrow V(\Gamma)$ and $\chi: (v_1 \dots v_k, 1 \dots t + \tau_0 - 1) \rightarrow \{\text{unoriented edges of } \Gamma\}$ (τ_0 = number of connected components of K) with the properties:

- 1) $G_{\phi(R_i)} \cong \pi_1(R_i)$
- 2) $|G_{\chi(v_i)}| = \text{order of } v_i \text{ and } |G_{\chi(j)}| = 1$
- 3) The images through ϕ of the endpoints of the connectors v are the endpoints of $\chi(v)$.

2.2 Classification of generalized function groups


In this section we classify generalized function groups (g.f.g.) in terms of equivalence classes of graphs of groups satisfying N0'-N4'. To state the equivalence relation we need an additional structure on the graph (a marking).

Let (Γ, G) be a graph of groups satisfying N0'-N4'.

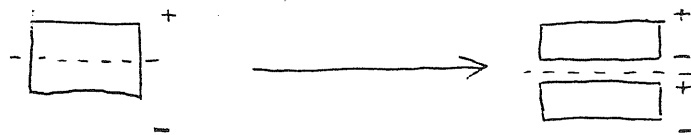
Notation $E_F(\Gamma) = \{e \in E(\Gamma) : \delta_0 e \text{ is Fuchsian}\}$

2.2.1 Definition A marking for (Γ, G) is a map $m: E_F(\Gamma) \rightarrow \{-1, +1\}$ with the following properties:

- i) Let $e \in E(\Gamma)$ such that $\delta_0 e = v$ and $\delta_1 e = w$ are Fuchsian vertices. Then $m(e) = \pm m(e')$ according if i_e, i_e^{-1} maps a canonical generator to a canonical generator or the inverse of a canonical generator.
- ii) Let $e, f \in E(\Gamma)$ be two edges such that $\delta_0 e = \delta_0 f$ and $i_e(G_e) = i_f(G_f)$. If $G_e \cong G_f \neq \{1\}$, then $m(e) = m(f)$.

Consider now the marked graph (Γ, m) . We can associate to (Γ, m) the following picture: $\forall v \in V(\Gamma)$ which is Fuchsian, draw a rectangle with an upper and a lower side () and $\forall e \in E(\Gamma)$ such that $\delta_0 e = v$, join e to the upper or lower side according if $m(e) = \pm 1$. This picture allows us to reconstruct both the graph Γ and the marking m .

Now cut each rectangle into two rectangles with upper and lower sides



We will obtain a finite number of disconnected pictures corresponding to marked graphs $(\Gamma_1, m_1) \dots (\Gamma_s, m_s)$. $\Gamma_1, \dots, \Gamma_s$ are naturally subgraphs of Γ .

2.2.2 Definition Let (Γ, G, m) be a marked graph of groups satisfying $N0'-N4'$. Then we associate to it its *boundary graphs of groups* $(\Gamma_1, G_1) \dots (\Gamma_s, G_s)$ where Γ_i $1 \leq i \leq s$ has been defined above and vertex, edge groups and edge-to-vertex monomorphisms are induced on Γ_i by (Γ, G) .

At last we are ready for defining equivalence between graphs of groups satisfying $N0'-N4'$.

2.2.3 Definition Two marked graphs of groups (Γ, G, m) and (Γ', G', m') satisfying $N0'-N4'$ are *equivalent* if and only if:

- 1) (Γ, G) and (Γ', G') are isomorphic graphs of groups once removed edge with trivial edge group.
- 2) There exists a bijection $b: \{(\Gamma_1, G_1) \dots (\Gamma_s, G_s)\} \rightarrow \{(\Gamma'_1, G'_1) \dots (\Gamma'_r, G'_r)\}$ between boundary groups such that $\pi_1(\Gamma_1, G_1) \cong \pi_1[b(\Gamma_1, G_1)]$.

Remark The condition 2) is needed for fixing the number of edges with trivial edge group in each boundary graph.

Let \mathbb{A} be the set of quasiconformal conjugacy classes of g.f.g., and \mathbb{B} the set of equivalence classes of marked graphs of groups in the sense 2.2.3.

We define a map $f: \mathbb{A} \rightarrow \mathbb{B}$ in the following way.

By 1.5.4 a g.f.g. uniformizes a g.o.p. O . Choose any geometric decomposition of O and fix a positive boundary component for each product $S \times [0, 1]$, where S is a compact 2-orbifold. Pose $m(e) = \pm 1$ according if the handle corresponding to e is glued to the positive or negative component. Property ii) of 2.2.1 is verified and, for any choice of the canonical presentation of Fuchsian vertex groups in (Γ, G) compatible with this choice of positive boundary

components (see 1.3), also property i) holds.

Let f be the map which associates to $[F] \in \mathbb{A}$ the equivalence class $[(\Gamma, G)]$ of a marked graph of groups (Γ, G) constructed as above.

2.2.4 Theorem *The map f is well defined.*

proof: Let F, G be two quasiconformally conjugate g.f.g.. Then $M = \mathbb{H}^3 \cup \Omega(F)/F \cong \mathbb{H}^3 \cup \Omega(G)/G$ and in particular, corresponding boundary components are homeomorphic. Let $D_1 \dots D_r$ be a collection of discal orbifolds for M like in the definition and let $D_1 \dots D_n$ be the subset of discal orbifolds with boundary in one fixed component S of ∂M . The boundaries $\partial D_1 \dots \partial D_n$ are a system of dividers for S in the sense of [Mas] X.C. The signature (K, t) for this system of dividers corresponds to the boundary graph (Γ_1, G_1) of S in the following sense.

K is a marked two-complex consisting of some Riemann surfaces of finite type $R_1 \dots R_s$ and some 1-cells $v_1 \dots v_x$ (the connectors). t is the Schottky number. There are two bijections $\phi: (R_1 \dots R_s) \rightarrow V(\Gamma)$ and $\chi: (v_1 \dots v_x, 1 \dots t + \tau_0 - 1) \rightarrow \{\text{unoriented edges of } \Gamma\}$ (τ_0 = number of connected components of K) with the properties:

- 1) $G_\phi(R_i) \cong \pi_1(R_i)$
- 2) $|G_\chi(v_i)| = \text{order of } v_i \text{ and } |G_\chi(j)| = 1$
- 3) The images through ϕ of the endpoints of the connectors v are the endpoints of $\chi(v)$.

The function group corresponding to this signature is the subgroup $\text{Stab} \Omega$ of F stabilizing a component Ω of the domain of discontinuity of F which covers S (in view of [Mas] II.F.8). Let Ω' be the corresponding component of the domain of discontinuity of G and $\text{Stab} \Omega'$ its stabilizer in G . As F and G are quasiconformally conjugate, $\text{Stab} \Omega$ and $\text{Stab} \Omega'$ are type-preservingly similar. So, by [Mas] X.G.2. the corresponding signatures are the same. By the correspondence above between signatures and graph we conclude that corresponding boundary graphs of groups in two different geometric decompositions of M have isomorphic underlying graphs, once removed edges with trivial edge group. Let denote by (Γ, H, m) and (Γ', H', m') the graphs of groups associated to F and G . By corresponding boundary graphs of groups in (Γ, H, m) and (Γ', H', m') are isomorphic graphs of groups once removed edges with trivial edge group and by

[Mas] X.D.5i) the isomorphism between F and G maps vertex groups of (Γ, G) into vertex groups of (Γ', G') . So (Γ, G) and (Γ', G') are isomorphic graphs of groups once removed edges with trivial edge group.

q.e.d.

Let \mathbb{B}' be the set \mathbb{B} minus the equivalence classes of graphs of groups with a subgraph $(2, 2, \infty) \xrightarrow{\infty} (2, 2, \infty)$.

2.2.5 Theorem f is a bijection onto \mathbb{B}' .

proof: If F and G are two g.f.g. with $f([F]) = f([G]) = [(\Gamma, H)]$, then there exist geometric decompositions of $M = \mathbb{H}^3 \cup \Omega(F)/F$ and $N = \mathbb{H}^3 \cup \Omega(G)/G$ such that the associated graphs of groups (Γ, H) and (Γ', H') are isomorphic graphs of groups up to edges with trivial edge group and corresponding boundary groups are isomorphic as groups. This implies that $M \cong N$ and so f is injective.

On the other hand f is surjective, because any graph of groups (Γ, H) satisfying $N0'-N4'$ is realizable (1.4) as a generalized g.f.g. F such that $f([F]) = [(\Gamma, H)]$ (For the graphs $(2, 2, \infty) \xrightarrow{\infty} (2, 2, \infty)$ there are considerations analogous to the function groups case).

q.e.d.

3 Finite actions on product-with-handles

The last part of the thesis includes some geometrical applications of the previous theorems in the study of 3-dimensional manifolds. The use of graphs of groups as an algebrogeometrical theory of actions on handlebodies has been introduced in [MMZ1]. The theory includes a *Riemann-Hurewitz formula*, providing a powerful calculus for the construction of actions which has been successfully applied to various questions in [KM], [Mc], [MMZ2].

A basic question for finite actions is the so called *Nielsen realization problem*, previously stated for closed surfaces and solved by Kerchhoff [Ke] in 1983. Let's state it for a 3-manifold.

Let M be a 3-manifold and H be a finite group acting effectively, smoothly and orientation-preservingly on M . The H -action on M induces a homomorphism $\eta: H \rightarrow \text{Out}(\pi_1(M))$, the group of outer automorphisms of $\pi_1(M)$. The question is: for a given homomorphism $\eta: H \rightarrow \text{Out}(\pi_1(M))$, is there a H -action on M as above which induces it?

For 3-manifolds this is not generally true (for example this fact is shown in [MMZ1] for handlebodies).

For almost compact product-with-handles the solution of the realization problem is a consequence of 1.5.1.

As a matter of fact, if H is a finite group acting on a product-with-handles M , there is a conformal structure on M , uniformized by a function group, for which H acts conformally. By this remark, 1.5.1 gives necessary and sufficient conditions for a group of outer automorphisms of $\pi_1 M$ to be induced by a group of orientation-preserving diffeomorphisms on M (section 3.1).

So 3.1.4 solves the realization problem for almost compact product-with-handles, by showing that it may have a negative answer.

I repeat that this technique has been introduced in [MMZ1] for

handlebodies and, from this point of view, this work is a natural generalization of [MMZ1] and [MMZ2]. In fact almost compact product-with-handles include handlebodies and products $S \times [0,1]$, where S is any finite surface with boundary.

More generally the theorems above turn questions about finite actions into combinatorial problems involving graphs of groups.

There is plenty of applications and the method has been already successfully used to study, for example, periodic diffeomorphisms on handlebodies [MMZ1], equivalence of actions on handlebodies [KM], actions on nonclosed two-manifolds [MMZ2].

In principle such results could be stated in analogous way for almost compact product-with-handles. There is really nothing new in applying those methods to this more general case except that calculations are a little more involved.

So I have not repeated a parallel analysis and I have confined myself to a few features in which the study of finite actions on almost compact product-with-handles turns out to be really different.

My proposals are:

- 1) A more elaborated use of *Euler characteristics* in studying admissible graphs of groups (see 3.1.4) satisfying C0-C4, which lies upon the coexistence of spherical and Fuchsian vertex groups (3.2)
- 2) A method for studying finite actions which is based on the *symmetry group* of the graphs (3.4).

Finite actions on a fixed almost compact product-with-handles M , correspond to find all graphs of groups (Γ, G) satisfying C0-C4 (see 3.1.4) which are *admissible* for some finite group H , that is such that there exists an epimorphism $\phi: \pi_1(\Gamma, G) \rightarrow H$ with kernel $K \cong \pi_1(M)$.

In general, to list all H -admissible graphs is a boring task and can be practically carried out only for easy cases like finite abelian groups or for product-with-handles with not too complicated topology.

However the question of selecting H -admissible graphs for a finite group H can be addressed by using *Euler characteristics* (see 3.1.5). We study in wide generality the set of Euler characteristics one can define on function groups (3.2).

They are the counterpart for function groups of the *Riemann-Hurewitz theorem* for Fuchsian groups.

It turns out that the existence of many Euler characteristics make considerably easier the study of finite actions.

As an almost randomly chosen illustration of the method we address the following standard question (3.3):

given a compact product-with-handles M , which is the maximal order $|H|$ for a finite group H acting smoothly, effectively and orientation-preservingly on it?

The method of Euler characteristics is practically easy and, at the same time, powerful enough to estimate, for example, the maximal order of finite actions on a given a.c.p.. However our geometric intuition can help us a lot, if we carefully reanalyze the way by which we have obtained 1.5.4. The purpose of section 3.4 is to exhibit another method to study finite actions on a.c.p., which is based on elementary graph theory.

Let's consider again the framework.

Let M be an a.c.p. and H a finite group acting on M effectively and orientation-preservingly. Then there exists a collection of mutually disjoint H -equivariant disks D_1, \dots, D_r with the properties of 1.5.3. The geometric decomposition of M gives a presentation of $\pi_1 M$ as a graph of groups (Γ, G) satisfying C1-C4, not necessarily in standard form.

As the collection of disks is H -equivariant, the action of H on M induces an action of H on Γ and $\forall v \in V(\Gamma)$ an action of $\text{Stab } v$ on the element of the decomposition corresponding to v . The idea of section 3.4 is to decompose the study of actions on M and consider separately finite actions on the graph Γ and actions on the manifolds of the geometric decomposition. By this way we fully exploit the ambivalent geometro-algebraic meaning of the notion of graph of groups, by considering both the graph underlying (Γ, G) and its vertex groups.

As a matter of fact, an action on a graph Γ is simply a representation of H into the symmetry group of the graph and actions on the manifolds of the geometric decompositions are much easier to be studied than actions on M , as they are essentially actions on surfaces..

Let's fix an a.c.p. M and a finite group H . The set of graphs of

groups (Γ, G) satisfying C1-C4 has infinite elements. However, by the remark that the index of a maximal cyclic subgroup in a spherical group can assume only certain values, we conclude (see 3.4.1) that for groups H which have sufficiently high order, we can restrict, without loss of generality, to graphs of groups with no spherical vertices, with a few exceptions.

In view of this result, it is easy to list all realizable quotient orbifolds M/H . As an application of the method, we illustrate the case of periodic actions.

This chapter is organized as follows:

In section 1 we show how 1.5.1 solves the realization problem for compact product-with-handles.

In section 2 we construct a two-dimensional vector space of Euler characteristics on the set of function groups with no rank 1 maximal parabolic subgroups.

In section 3 we address the problem of estimating the maximal order of a finite action on a product-with-handles.

In section 4 we exhibit a method to study finite actions which is based on elementary graph theory.

3.1 The Nielsen realization problem for product-with-handles

In this section we show how 1.5.1 solves the realization problem for almost compact product-with-handles (a.c.p.).

More precisely, we will use the more restricted results 1.5.6 and 1.5.7.

If H is a finite group acting on an a.c.p. M , there exists a conformal structure on M , uniformized by a function group, on which H acts conformally. By this remark, 1.5.6 gives necessary and sufficient conditions for a group of outer automorphisms of $\pi_1(M)$ to be induced by a group of orientation-preserving diffeomorphisms on M . We state this fact as a theorem:

3.1.1 Theorem *Let M be an a.c.p.. Given a finite group H and an abstract kernel $\eta: G \rightarrow \text{Out}(\pi_1(M))$, the following are equivalent:*

a) η is realizable (as an orientation-preserving effective H -action) on M

b) for some extension ε_η with abstract kernel η

$$1 \rightarrow \pi_1(M) \rightarrow E \rightarrow H \rightarrow 1$$

there is a finite graph of groups (Γ, G) with $\pi_1(\Gamma, G) \cong E$ which satisfies conditions C0 - C4.

proof: a) implies b)

Let $\phi: H \rightarrow \text{Diff}^+(M)$ be a realization of η . Then $M/G = O$ is an orbifold-with-handles (see 1.5.4) and $E = \pi_1 O$ gives an extension

$$1 \rightarrow \pi_1 M \rightarrow \pi_1 O \rightarrow H \rightarrow 1$$

with abstract kernel η . By 1.5.6 we know that there is a finite graph of groups (Γ, G) with $\pi_1(\Gamma, G) \cong E$, satisfying C0-C4.

b) implies a)

By 1.5.6 E is isomorphic to a geometrically finite function group F with no rank 1 maximal parabolic subgroups. As $\pi_1 M$ is a subgroup of finite index of F it is isomorphic to a function group F' with no rank 1 maximal parabolic subgroups. By 1.5.7 $\mathbb{H}^3 \cup \Omega(F')/F'$ is an a.c.p. M' and so H is realizable as a group of conformal maps on M' for the conformal structure given by the uniformization through F' . By the subgroup theorem $F' \cong \pi_1 M \cong \pi_1 M'$ admits a unique (up to order) decomposition $F' \cong F_1 * \dots * F_n * \mathbb{Z}^r$ and so M and M' have the same number and type of products and the same number of handles. As they are almost compact, they are homeomorphic.

q.e.d.

Remark Notice that 3.1 does not admit a naive extension to noncompact product-with-handles M and graphs of groups satisfying N0-N4. In fact, given an extension $1 \rightarrow \pi_1(M) \rightarrow E \rightarrow H \rightarrow 1$ with M noncompact and $E \cong \pi_1(\Gamma, G)$ for (Γ, G) a finite graph of groups satisfying N0-N4, we can only conclude that the kernel is realizable (as an orientation-preserving effective G -action) on M' , where M' is a three-manifold with $\pi_1(M') \cong \pi_1(M)$ (compare also with [MMZ2]).

3.1.1 turns questions about finite actions into combinatorial problems involving graphs of groups.

There is plenty of applications. One has to find all graphs of groups (Γ, G) satisfying C0-C4 which are admissible for some finite group H , that is such that there exists an epimorphism $\phi: \pi_1(\Gamma, G) \rightarrow H$ with kernel $K \cong \pi_1(M)$.

In general, to list all H -admissible graphs is a boring task and can be practically carried out only for easy cases like finite abelian groups or for product-with-handles with not too complicated topology.

However the question of selecting H -admissible graphs for a finite group H can be addressed by using Euler characteristics.

We adopt the following definition:

3.1.2 Definition Given a collection of groups A , a map $\chi:A \rightarrow \mathbb{R}$ is called an Euler characteristic for A if it satisfies the following two properties

- 1) If $G, H \in A$ and $G \cong H$, then $\chi(G) = \chi(H)$
- 2) If $G, H \in A$ and G is isomorphic to a subgroup of finite index j of H , then $\chi(G) = j\chi(H)$

3.1.3 Remark The definition implies $\chi(\mathbb{Z}) = 0$ if $\mathbb{Z} \in A$

Studying Euler characteristics is much easier than checking directly which normal subgroups of finite index a group admits. In fact, for a fixed M and a finite group H , an H -admissible graph of groups (Γ, G) will have to satisfy the following necessary condition

$$\chi(\pi_1(M)) = |H| \chi(\Gamma, G)$$

for any Euler characteristic as above. As we will see, this considerably restricts the set of graphs of groups candidates for a finite action.

In the next section we study in wide generality the set of Euler characteristics one can define on the set of groups which are fundamental groups of a.c.p..

3.2 Euler characteristics on function groups

In this section we construct a two-dimensional vector space of Euler characteristics on the set C of function groups with no rank 1 maximal parabolic subgroups.

A standard way of defining an Euler characteristic for the set of groups \mathbb{A} which have finite homological type and are virtually torsion-free (see e.g. [B] page 247) is

$$\chi(A) = \sum_i (-1)^i \text{rank}_{\mathbb{Z}}(H_i A) \quad \text{if } A \in \mathbb{A} \text{ is torsion-free}$$

$$\chi(A) = \chi(B) / (A:B) \quad \text{for any torsion-free subgroup of finite index } B \text{ of } A, \text{ where } A, B \in \mathbb{A}$$

Notice that, if (Γ, G) is a finite graph of groups such that each vertex group and edge group is in \mathbb{A} , then $\pi_1(\Gamma, G) \in \mathbb{A}$ and

$$\chi(\pi_1(\Gamma, G)) = \sum_{v \in V(\Gamma)} \chi(G_v) - \sum_{e \in E(\Gamma)} \chi(G_e)$$

As we are dealing with a restricted subset of \mathbb{A} , namely the set C of groups above, we have a wide variety of definitions of χ . We are interested in the following set:

3.2.1 Proposition *Let $\chi_{\mathbb{A}}$ be an Euler characteristic for the set \mathbb{A} of spherical groups, cocompact Fuchsian groups of the first kind and the Euclidean groups of rank two. Then*

$$\chi: C \rightarrow \mathbb{R}$$

$$\chi(A) = \chi(\Gamma, G) = \sum_{v \in V(\Gamma)} \chi_{\mathbb{A}}(G_v) - \sum_{e \in E(\Gamma)} \chi_{\mathbb{A}}(G_e)$$

where (Γ, G) is any graph of groups satisfying C0-C4 such that

$\pi_1(\Gamma, G) \cong A$, is an Euler characteristic for the set C

3.2.2 Lemma Let χ_A be an Euler characteristic for the set A of trivial groups, cocompact torsion-free Fuchsian groups of the first kind and the abelian group of rank two.

Let B be the set of groups which are fundamental groups of finite graphs of groups (Γ, G) satisfying

- 1) there are no trivial edges
- 2) every vertex group is in A
- 3) every edge group is trivial. Then

$$\chi: B \rightarrow \mathbb{R}$$

$$\chi_B(A) = \chi_B(\Gamma, G) = \sum_{v \in V(\Gamma)} \chi_A(G_v) - \sum_{e \in E(\Gamma)} \chi_A(G_e)$$

where (Γ, G) is any graph of groups satisfying 1-2-3 such that $\pi_1(\Gamma, G) \cong A$, is an Euler characteristic for the set B .

proof: Let's first prove the property 1 of the definition.

If (Γ, G) and (Γ', G') satisfy 1-2-3 and have isomorphic fundamental groups, by [S-W] lemma 7.6

- i) there is a bijection between the vertices of Γ and Γ' such that corresponding groups are isomorphic
- ii) Γ and Γ' have the same number of edges
- i) and ii), together with 3) imply that $\chi(\Gamma, G) = \chi(\Gamma', G')$.

Let us now come to property 2) of the definition of Euler characteristic. Let (Γ, G) be a graph of groups satisfying 1-2-3 and H a subgroup of finite index j of $\pi_1(\Gamma, G)$. By the subgroup theorem, H can be represented as the fundamental group of a graph of groups (Γ_0, G_0) s.t.

- i) the vertices of Γ_0 correspond to the double cosets $HgA_1 \dots HgA_n$ where $A_1 \dots A_n$ are the vertex groups of (Γ, G) and the corresponding groups are $H \cap gA_i g^{-1}$ $1 \leq i \leq n$.
- ii) the edges of Γ_0 correspond to the double cosets $HgB_1 \dots HgB_r$, where $B_1 \dots B_r$ are the edge groups of (Γ, G) and the corresponding groups are trivial.

The number of left cosets of H in one double coset HgA_1 is equal to the index j_1 of $g^{-1}Hg \cap A_1$ in A_1 . If there are n different double cosets $Hg_1A_1 \dots Hg_nA_n$, we have that $j_1 + \dots + j_n = j$. Moreover $\forall 1 \leq i \leq n$ $\chi_A(H \cap g_iA_1g_i^{-1}) = j_i \chi_A(A_1)$ as χ_A is an Euler characteristic and

$$H \cap g_i A_1 g_i^{-1} \in A.$$

So the Euler characteristic of the n vertex groups corresponding to A_1 is $\sum_{1 \leq i \leq n} \chi_A(H \cap g_i A_1 g_i^{-1}) = j \chi_A(A_1)$.

An analogous reasonment holds for every vertex and edge in Γ . This concludes the proof.

q.e.d.

As a consequence of the lemma we can prove the proposition

proof of 3.2.1 . Let's first prove property 1) of χ_C . If (Γ, G) satisfies C1 and has finite edge groups, then $\pi_1(\Gamma, G)$ is a Kleinian group. By Selberg's lemma it admits a subgroup H of finite index j which is torsion-free. By applying again the subgroup theorem as above, we obtain that H is representable as the fundamental group of a graph of groups (Γ_0, G_0) satisfying 2) and 3). Moreover we have the equality

$$(*) \quad \sum_{v \in V(\Gamma_0)} \chi_A(G_v^0) - \sum_{e \in E(\Gamma_0)} \chi_A(G_e^0) = j \chi_C(\Gamma, G)$$

In general (Γ_0, G_0) does not satisfy 1), but we can obtain from it a new graph of groups (Γ_1, G_1) satisfying 1), 2) and 3) such that $\pi_1(\Gamma, G) \cong H$. By the lemma we obtain the property 1) of χ_C .

The proof of property 2) is the same as in the lemma.

q.e.d.

We can exactly say which is the set of Euler characteristics χ_A , in view of the following

3.2.3 Proposition Let \mathbb{B} be a set of groups. Let $\mathbb{B}_0 \subset \mathbb{B}$ be the subset of torsion-free groups. If every $A \in \mathbb{B}$ is virtually torsion-free, then there exists a bijective correspondence between the set E of Euler characteristics of \mathbb{B} and the set E_0 of Euler characteristics of \mathbb{B}_0 .

proof: If $\chi: \mathbb{B} \rightarrow \mathbb{R}$ is an element of E , then the restriction of χ to \mathbb{B}_0 is an element of E_0 . Let $r: E \rightarrow E_0$ be the restriction map. If $r(\chi_1) = r(\chi_2)$, then $\forall A \in \mathbb{B}$ take a torsion-free subgroup A_0 of finite index j . As $A_0 \in \mathbb{B}_0$ $\chi_1(A_0) = \chi_2(A_0)$ and, by the definition of the Euler

characteristic we have $\chi_1(A) = \chi_2(A)$. So r is injective. On the other side, if $\chi_0 \in E_0$, then define $\chi: \mathbb{B} \rightarrow \mathbb{R}$ in the following way. $\forall A \in \mathbb{B}$ take a torsion-free subgroup A_0 of finite index j . As A_0 lies in \mathbb{B}_0 then $\chi_0(A_0)$ is defined. Put $\chi(A) = j\chi_0(A_0)$. One readily checks that χ is well defined and is an element of E . So r is surjective

q.e.d.

In the particular case of the set of groups \mathbb{A} , then the set of groups \mathbb{A}_0 is given by:

- 1) the trivial group
- 2) the rank 2 abelian group
- 3) the groups $F_g = \langle a_1, \dots, b_g \mid [a_1, b_1] \dots [a_g, b_g] = 1 \rangle$ $g \geq 2$

So, by the proposition above, it is equivalent to consider the set of Euler characteristics on \mathbb{A}_0 . Notice that F_g is a subgroup of finite index of F_2 and $\mathbb{Z} \oplus \mathbb{Z}$ is a subgroup of finite index of itself. Then the set of Euler characteristics on \mathbb{A}_0 is completely determined by the following two values

$$\chi(\{e\}) = c$$

$$\chi(\{F_2\}) = -d$$

On the other side any pair of values $(c, d) \in \mathbb{R} \times \mathbb{R}$ corresponds to an Euler characteristic for \mathbb{A}_0 (and so for \mathbb{A}) defined by

$$1) \chi(\{e\}) = c$$

$$2) \chi(\mathbb{Z} \oplus \mathbb{Z}) = 0$$

$$3) \chi(\{F_g\}) = \frac{d}{2}(2-2g) = d(1-g)$$

This definition works as 1) is not a subgroup of finite index of 2) or 3) and we know from the topology that the values $\chi(F_g) = 2-2g$ $g \geq 2$ satisfy the property ii) of the definition of Euler characteristic.

So we have proved

3.2.4 Proposition *The set of Euler characteristics of C contains a two-dimensional vector space*

Notation An Euler characteristic of C in this space will be denoted by the pair (c,d) .

There are two Euler characteristics for C which are worth to be noticed: $(1,2); (1,1)$. In fact they have an immediate topological meaning.

$(1,2)$ is the standard virtual Euler characteristic χ_v ([B] page 247). By the reasonment above it is sufficient to verify this assertion on $\{e\}$ and F_2 : $\chi_v(\{e\})=1$, as an Eilenberg-Mac Lane space for $\{e\}$ is a point and $\chi_v(F_2) = 2-2g$, as an Eilenberg-Mac Lane space for F_2 is any compact orientable surface of genus two.

In fact, if we take a graph of groups (Γ, G) satisfying C0-C4 and with torsion-free fundamental group, then we know that it is the fundamental group of an a.c.p. M and we have $\chi_{(1,2)}(\Gamma, G) = \chi(M)$ where $\chi(M)$ is meant in the topological sense.

Analogously one checks that, in this case, $\chi_{(1,1)}(M) = 1/2 \chi(\Sigma)$ where Σ is the compressible boundary component of M . This topological meaning for $(1,1)$ is occasional: in fact $\chi(\Sigma) = 2\chi(M) - \chi(\Sigma_1) - \dots - \chi(\Sigma_n)$ and $\chi(\Sigma_i) = \chi_{(1,2)}(F_i)$, where F_i is the indecomposable group occurring in any decomposition of $\pi_1(\Gamma, G)$ as fundamental group of Σ_i .

3.3 Upper bounds for finite actions on product-with-handles

In this section we will address the following question:

given an a.c.p. M , which is the maximal order $|H|$ for a finite group H acting smoothly, effectively and orientation-preservingly on it?

In view of 3.1.1 the question of finding the maximal order of a finite group H acting effectively and orientation-preservingly on M can be addressed by answering the following easier question: given an Euler characteristic $\chi: C \rightarrow \mathbb{R}$, which is, if any, the maximal negative value χ_{\max}^- and the maximal positive value χ_{\max}^+ of χ on the set of graphs of groups satisfying conditions C0 - C4?

In fact, if we know χ_{\max}^- and χ_{\max}^+ , we can estimate the maximal order $|H|_{\max}$ of a finite group H acting as above on a fixed M in the following way:

$$|H|_{\max} = \chi(M) / \chi(\Gamma, G) \leq \chi_C(M) / \chi_{\max}$$

where for χ_{\max} we choose χ_{\max}^- or χ_{\max}^+ according if $\chi(M) > 0$ or $\chi(M) < 0$ (we can't obtain any estimate for $\chi(M) = 0$).

According to the chosen Euler characteristic we can estimate $|H|_{\max}$ in terms of the topological Euler characteristic of M , the topological Euler characteristic of the compressible boundary component, and so on...

If M is a handlebody, we can evaluate χ_{\max}^- or χ_{\max}^+ on the subset of C such that the graphs of groups have only spherical vertex groups. In this case

$$\chi(M) = \chi_{\max} = k$$

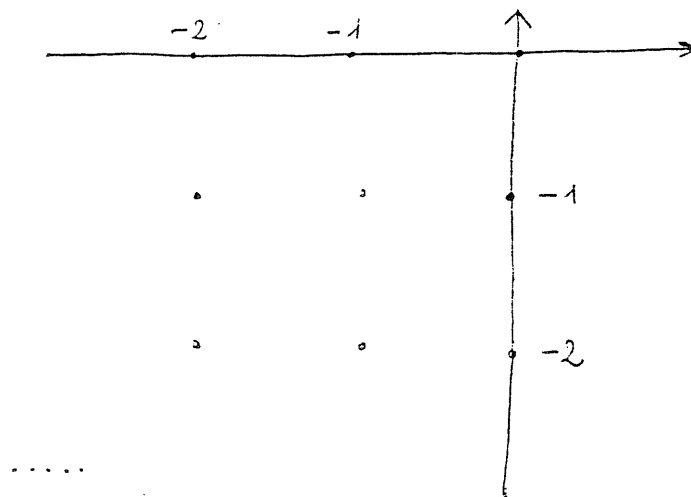
where for χ_{\max} we choose χ_{\max}^- or χ_{\max}^+ and k is a constant value which does not depend on χ .

The existence of many different Euler characteristics for C can be used in various ways for selecting admissible graphs

For a fixed M consider the set of values $\chi(M) = \chi_{\max}$ obtained by considering all possible Euler characteristics on C . If k is the minimal value in the set, realized for a fixed χ , this will be an estimate for all χ . This means that if, for $\chi': C \rightarrow \mathbb{R}$ there exists a graph of groups (Γ, G) satisfying N0-N4 such that $\chi'(M) / \chi'(\Gamma, G) > k$, then (Γ, G) can not be an admissible graph for a finite action on M .

If, for a fixed M , χ is the Euler characteristic for $|H|_{\max}$ and this estimate is realized on the graph of groups (Γ, G) , we can also improve this estimate if we find that (Γ, G) is not admissible, by the following criterion. For all χ , $\chi(M) / \chi(\Gamma, G)$ must be a constant. Luckily enough we do not need to consider Euler characteristics all together, but only a basis of the vector space. Moreover it will turn out that, by considering simultaneously the two elements of a basis, we obtain the best possible estimate in the sense of the first remark.

Denote $(1, 0)$ and $(0, 1)$ by χ_1 and χ_2 . We will consider the set of values $(\chi_1(\Gamma, G), \chi_2(\Gamma, G))$ in \mathbb{R}^2 .



For a given a.c.p. M , $\chi(M)$ will be a point of the lattice drawn in the picture. We can estimate $|H|_{\max}$ as

$$\begin{aligned}
|H|_{\max} &\leq \{x_1(M) + x_2(M)\} / \{x_{\max}^1(M) + x_{\max}^2(M)\} = \\
&= x_1(M) / x_{\max}^1 \\
&= x_2(M) / x_{\max}^2
\end{aligned}$$

where (x_{\max}^1, x_{\max}^2) is the value $(x_1(\Gamma, G), x_2(\Gamma, G))$ next to the origin on the line passing through $(0, 0)$ and $(x_1(M), x_2(M))$.

Remarks

- i) Notice that if M is not a handlebody and, more precisely, $x_2(M)$ is not zero, then $-x_1(M)$ is the number of handles of M .
- ii) As (x_1, x_2) is a basis for the vector space of the Euler characteristics, the estimate above is the best that one can obtain by this way. In fact (c, d) can be expressed as $(c, d) = cx_1 + dx_2$. So, given a graph of groups (Γ, G) , let (Γ_0, G_0) be the graph of groups such that $x_1(\Gamma_0, G_0) = x_{\max}^1$ and $x_2(\Gamma_0, G_0) = x_{\max}^2$. Then
$$\begin{aligned}
x(\Gamma, G) / x(\Gamma_0, G_0) &= \{cx_1(\Gamma, G) + dx_2(\Gamma, G)\} / \{cx_1(\Gamma_0, G_0) + dx_2(\Gamma_0, G_0)\} \\
&= x_1(\Gamma, G) / x_1(\Gamma_0, G_0) = x_2(\Gamma, G) / x_2(\Gamma_0, G_0) = k \in \mathbb{R}.
\end{aligned}$$
This means that, given (Γ, G) , there exists (Γ_0, G_0) such that $x(\Gamma, G) / x(\Gamma_0, G_0)$ is k : so the best estimate obtainable from (c, d) is at least k .
- iii) Among all bases available, (x_1, x_2) is, in some sense, natural. As spherical groups and cocompact Fuchsian groups of the first kind are discrete groups acting discontinuously on the plane, they have a canonical presentation
$$\langle a_1 \dots b_g, h_1 \dots h_r \mid [a_1, b_1] \dots [a_g, b_g] h_1 \dots h_r = 1; h_i^{n_i} = \dots h_r^{n_r} = 1 \rangle$$
which we denote by their signature $\langle g, r \mid h_1 \dots h_r \rangle$. Now
$$\begin{aligned}
x_2(\langle g, r \mid h_1 \dots h_r \rangle) &= 1 - g - \frac{1}{2} \sum_{1 \leq i \leq r} (1 - \frac{1}{h_i}) \\
&\text{for Fuchsian groups, by the Riemann-Hurewitz formula} \\
x_1(\langle 0, r \mid h_1 \dots h_r \rangle) &= 1 - 0 - \frac{1}{2} \sum_{1 \leq i \leq r} (1 - \frac{1}{h_i}) \\
&\text{for spherical groups.}
\end{aligned}$$
So they have the same formal expression.

By the last remark, it is not necessary to do any computation for evaluating (x_{\max}^1, x_{\max}^2) . In fact we have the following proposition:

3.3.1 Proposition Let (Γ, G) be in C . Suppose

i) $\chi_1(\Gamma, G) < 0$

ii) $\chi_1(\Gamma, G) + \chi_2(\Gamma, G) > -1/4$.

Then there exists (Γ_0, G_0) in C with only spherical vertices, such that $\chi_1(\Gamma, G) + \chi_2(\Gamma, G) = \chi_1(\Gamma_0, G_0)$.

proof: If (Γ, G) has no Fuchsian or Euclidean vertices, choose $(\Gamma_0, G_0) = (\Gamma, G)$.

Suppose that $v \in V(\Gamma)$ is a Fuchsian vertex. As a first remark, notice that, in view of ii), Fuchsian vertices of (Γ, G) have signature $\langle g=0, 3|h_1, h_2, h_3 \rangle$ or $\langle g=0, 4|h_1, h_2, h_3, h_4 \rangle$ and at least two values $h_1, h_2=2$.

As $\chi_1(\Gamma, G) < 0$ there is at least one edge $e \in E(\Gamma)$ with $\delta_e v = v$. Let $\delta_e v = v'$ (possibly $v=v'$). As (Γ, G) satisfies N0, then G_v is not isomorphic to \mathbb{Z} . So G_v and $G_{v'}$ will have a signature $\langle g=0, n|h_1, \dots, h_n \rangle$ where n is 3 or 4 (the case $G_e \cong \{1\}$ is excluded because $\chi_1(\Gamma, G) \geq -1/4$).

Let's first consider the case $n=3$ and v different from v' .

Denote by (Γ_1, G_1) the graph $\langle g=0, 3|h_1, h_2, h_3 \rangle \xrightarrow{h_4} \langle g=0, 3|h_1, h_4, h_5 \rangle$ (Γ_1, G_1) , by remark iii), has the same value $\chi_1 + \chi_2$ as

$$(\Gamma_2, G_2) = \langle g=0, 3|k_1, h_2, h_3 \rangle \xrightarrow{k_4} \langle g=0, 3|k_1, h_4, h_5 \rangle \quad k_i \geq 2$$

Moreover for $k_1=2$ $\chi_1(\langle g=0, 3|2, h_2, h_3 \rangle) > 0$ and

$\chi_1(\langle g=0, 3|k_1, h_4, h_5 \rangle) > 0$, that is the two vertices are spherical. In fact $\chi_2(\langle g=0, 3|2, h_2, h_3 \rangle) > -1/4 - \chi_2(\langle g=0, 3|2, h_4, h_5 \rangle) + 1/2 = 1/4 - \chi_2(\langle g=0, 3|2, h_4, h_5 \rangle) \geq 0$.

So, for this case, we can always construct, by using N1 and N2, a graph of groups (Γ_0, G_0) satisfying N0 - N4 obtained by modifying the subgraph (Γ_1, G_1) in (Γ_2, G_2) . We are sure that the value $\chi_1 + \chi_2$ does not change and we reduce the number of vertices which are not spherical by one unit (at least).

Analogous considerations hold for the case $n=3$ and $v = v'$.

Suppose finally that v has signature $\langle g=0, 4|h_1, h_2, h_3, h_4 \rangle$; then the graph

$$(\Gamma_1, G_1) = \langle g=0, 3|h_1, h_2, k \rangle \xrightarrow{k} \langle g=0, 3|h_3, h_4, k \rangle$$

has the property that $(\chi_1 + \chi_2)(\Gamma_1, G_1) = (\chi_1 + \chi_2)(G_v)$ and so we can reduce to the previous case, by modifying the subgraph consisting of the single vertex v in (Γ_1, G_1) .

Notice that the analysis above can be as well applied to Euclidean vertices.

As we are dealing with finite graph of groups, we can obtain a graph of groups (Γ_0, G_0) with the required properties, by iteratively applying the procedure.

q.e.d.

By considering [MMZ1] prop.7.1 we have the following general estimate

3.3.2 Proposition *Let H be a finite group of orientation-preserving diffeomorphisms of an a.c.p. M which has at least one handle (that is $\chi_1(M) < 0$). Then the order of H is less or equal than $-12(\chi_1(M) + \chi_2(M))$.*

proof: $|H|_{\max} \leq \{\chi_1(M) + \chi_2(M)\} / \{\chi_{\max}^1(M) + \chi_{\max}^2(M)\}$. But $\chi_{\max}^1(M) + \chi_{\max}^2(M) \leq \chi_1(\Gamma, G) + \chi_2(\Gamma, G)$ for some (Γ, G) with $\chi_1(\Gamma, G) < 0$. By the proposition above $\chi_1(\Gamma, G) + \chi_2(\Gamma, G) = \chi_2(\Gamma_0, G_0)$ for some (Γ_0, G_0) in C with only spherical vertices. By [MMZ] prop.7.1. $\chi_2(\Gamma_0, G_0) \leq -1/12$ and so the thesis.

q.e.d.

In view of the remarks after 3.2.4, we know that, for an a.c.p. $\chi_1(M) + \chi_2(M) = 1/2 \chi(\Sigma)$, where Σ is the compressible boundary component of M and χ is the topological Euler characteristic. So $\chi_1(M) + \chi_2(M) = 1 - g$, where g is the genus of Σ . Another appealing version of the proposition is then:

Proposition *Let H be a finite group of orientation-preserving diffeomorphisms of an a.c.p. M which has at least one handle (that is $\chi_1(M) < 0$). Then the order of H is less or equal than $12(g-1)$.*

So stated, the proposition is a direct generalization of the analogous estimate for handlebodies, which can also be proved by the same methods [MMZ1] theorem 7.2.

Notice that the corresponding upper bound for finite groups H

acting on compact hyperbolic surfaces of genus g is $|H| \leq 84(g-1)$ (e.g. [Zie] 15.21).

One can summarize the situation in these terms: the maximal possible order of a finite group of orientation-preserving homeomorphisms of an orientable compact bounded surface of algebraic genus g is $12(g-1)$. A finite action on a compact bounded surface of genus g extends, by taking the product with the interval $[0,1]$, to an action on the 3-dimensional handlebody of genus g . However the maximal possible order of a finite group of orientation-preserving homeomorphisms of a handlebody of genus g is still $12(g-1)$. Finally, a.c.p. include handlebodies and the upper bound for finite actions is still the same.

One might wonder if there actually exist finite groups H of orientation-preserving homeomorphisms of an a.c.p. M attaining the upper bound, that is such that $|H| = 12(g-1)$ (where g is the genus of the compressible boundary component of M).

The question has been analyzed for compact bounded surfaces [May] and [MMZ2] and for handlebodies [MMZ1]. In these cases the answer is yes. More precisely, the finite groups occurring for compact bounded surfaces are the surjective images of $D_2 *_{\mathbb{Z}_2} D_3$ such that the surjection is finite-injective ([MMZ1] page 40). On the other hand, the finite groups occurring for handlebodies are the surjective images of $D_2 *_{\mathbb{Z}_2} D_3$, $D_3 *_{\mathbb{Z}_3} A_4$, $D_4 *_{\mathbb{Z}_4} S_4$, $D_5 *_{\mathbb{Z}_5} A_5$ for a finite-injective surjection. The groups occurring in this second class do not coincide with the first ones ([Zim]).

What about a.c.p.?

By (3.1.1) and the table below, the finite groups attaining the upper bound for a.c.p. are the surjective images of $(2,2,n) \xrightarrow{n} (2,3,n)$ for a finite-injective epimorphism.

Various questions arise: are there any finite groups in this set which are not contained in the previous ones?

Some appealing geometric features about these problems can be found in [Zim].

We will not study in detail this problem, but 3.3.1 suggest us some more considerations.

By carefully reading the proof, one deduces all values $(\chi_1(\Gamma, G), \chi_2(\Gamma, G)) \in \mathbb{R}^2$ with $\chi_1 + \chi_2 > -1/4$ and $\chi_1 < 0$. For example,

consider the graph

$(\Gamma_0, G_0) = \mathbb{D}_2 \xrightarrow{2} \mathbb{D}_3$. We have that $\chi_1(\Gamma_0, G_0) = -1/12$ and $\chi_2(\Gamma_0, G_0) = 0$. The set of graphs (Γ, G) with $\chi_1(\Gamma, G) + \chi_2(\Gamma, G) = -1/12$ associated to (Γ_0, G_0) by the proposition are:

- $(2, 2, n) \xrightarrow{n} (2, 3, n) \quad n \geq 2$
- $(2, 2, 2, 3)$

We are not interested in $(2, 2, 2, 3)$ as $\chi_1(2, 2, 2, 3) = 0$. The values of the other graphs can be expressed as

$$(\chi_1, \chi_2) = \begin{cases} (-1/12, 0) & 2 \leq n \leq 6 \\ (-1/2n, 1/12 - 1/2n) & n > 7 \end{cases}$$

In particular, lines in \mathbb{R}^2 going through these points have angular coefficients $n/6 \quad n \geq 1$.

So the maximal order for a finite action on an a.c.p. M can be obtained only if $(\chi_1(M), \chi_2(M))$ lies on those lines.

From [MMZ] Chart B page 48, we obtain the following complete list of graphs of groups (Γ, G) such that $\chi_1(\Gamma, G) + \chi_2(\Gamma, G) > -1/4$; $\chi_1(\Gamma, G) < 0$ and $\chi_2(\Gamma, G) < 0$.

In the last column we have computed the angular coefficient of lines going through them.

$\chi_1(\Gamma, G) + \chi_2(\Gamma, G)$	graph	angular coefficient
$-1/12$	$(2, 2, n) \xrightarrow{n} (2, 3, n) \quad n \geq 6$	$n/6 \quad n \geq 1$
$-1/8$	$(2, 2, n) \xrightarrow{n} (2, 4, n) \quad n \geq 4$	$n/4 \quad n \geq 1$
$-3/20$	$(2, 2, n) \xrightarrow{n} (2, 5, n) \quad n \geq 4$	$3n/10 - 1 \quad n \geq 4$
$-1/6$	$(2, 2, n) \xrightarrow{n} (3, 3, n) \quad n \geq 3$	$n/3 \quad n \geq 1$
$-1/6$	$(2, 3, n) \xrightarrow{n} (2, 3, n) \quad n \geq 6$	$n/6 \quad n \geq 1$
$-5/24$	$(2, 2, n) \xrightarrow{n} (3, 4, n) \quad n \geq 3$	$5n/12 - 1 \quad n \geq 3$
$-5/24$	$(2, 3, n) \xrightarrow{n} (2, 4, n) \quad n \geq 6$	$5n/24 - 1 \quad n \geq 6$
$-5/24$	$(2, 3, 5) \xrightarrow{5} (2, 4, 5)$	$-11/60$
$-7/30$	$(2, 3, n) \xrightarrow{n} (3, 5, n) \quad n \geq 3$	$7n/15 - 1 \quad n \geq 3$
$-7/30$	$(2, 3, n) \xrightarrow{n} (2, 5, n) \quad n \geq 6$	$7n/30 - 1 \quad n \geq 6$
$-7/30$	$(2, 3, 4) \xrightarrow{4} (2, 4, 5)$	$-5/24$
$-7/30$	$(2, 3, 5) \xrightarrow{5} (2, 5, 5)$	$-11/60$

This table should be compared and integrated with the corresponding table in [MMZ1] 7.2, which lists graphs of groups satisfying C0-C4 and with $\chi_1 + \chi_2 > -1/4$, but with only spherical vertices.

By using this table one can estimate for a given a.c.p. M , the value $|H|_{\max}$ by checking the lines passing through $(\chi_1(M), \chi_2(M))$ for any graph of groups (Γ, G) satisfying C0-C4 and with $\pi_1(\Gamma, G) \cong \pi_1 M$.

We give a randomly chosen illustration of the method:

3.3.3 Proposition *Let M be a product-with-handles with 5 handles and $\chi_2(M) \geq -4$. Then $|H|_{\max} \leq 30/7(g-1)$.*

proof: By looking at the tables above one finds that there is only one line passing through a point $(-5, -\chi_2)$ with $\chi_2 \in \{-1, -2, -3, -4\}$. Precisely the line with angular coefficient $2/5$ associated to the graph $(2, 3, 6) \xrightarrow{6} (2, 5, 6)$. But $(\chi_1 + \chi_2)((2, 3, 6) \xrightarrow{6} (2, 5, 6)) = -7/30$.

q.e.d.

3.4 Symmetries of graphs

In this section we exhibit a method to study finite actions which is based on elementary graph theory.

Let M be an a.c.p. and H a finite group acting on M effectively and orientation-preservingly. Then there exists a collection of mutually disjoint H -equivariant disks D_1, \dots, D_r with the properties of 1.5.3. The geometric decomposition of M gives a presentation of $\pi_1 M$ as a graph of groups (Γ, G) satisfying C1-C4 not necessarily in standard form.

Moreover M/H is an orbifold-with-handles with a given geometric decomposition and an associated graph of groups (Γ', G') satisfying C1-C4. As in 1.5 we can always select the collection of disks D_1, \dots, D_r in such a way that (Γ', G') satisfies C0. As the collection of disks is H -equivariant, the action of H on M induces an action of H on Γ and $\Gamma/H = \Gamma'$.

The idea of this section is to decompose the study of actions on M and consider separately finite actions on the graph Γ and actions on the manifolds of the geometric decomposition. By this way we fully exploit the ambivalent geometro-algebraic meaning of the notion of graph of groups, by considering both the graph underlying (Γ, G) and its vertex groups.

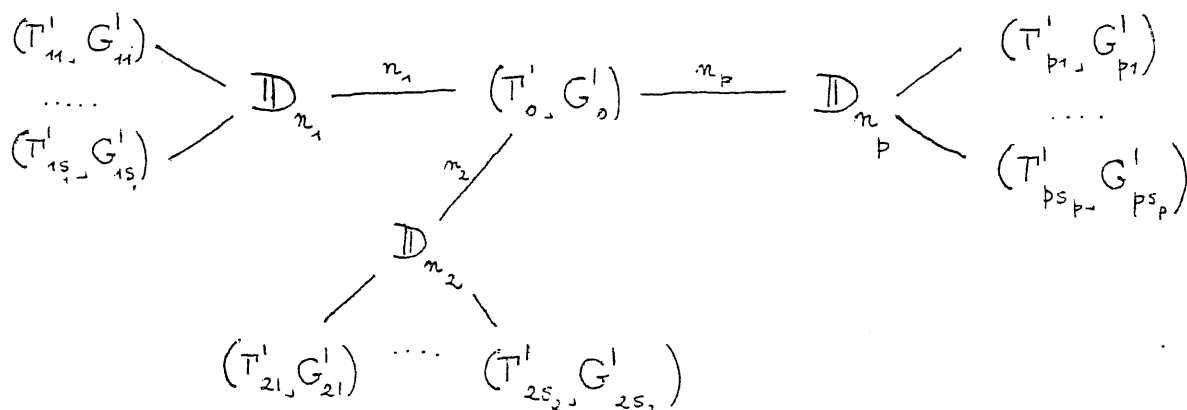
As a matter of fact, an action on a graph Γ is simply a representation of H into the symmetry group of the graph and actions on the manifolds of the geometric decompositions are much easier to be studied than actions on M , as they are essentially actions on surfaces.

Let's fix an a.c.p. M and a finite group H . The set of graphs of groups (Γ, G) satisfying C1-C4 has infinite elements. However, by the remark that the index of a maximal cyclic subgroup in a spherical group can assume only certain values, we conclude (see 3.4.1) that for groups H which have sufficiently high order, we can restrict, without loss of generality, to graphs of groups with no spherical vertices, with a few exceptions.

In view of this result, it is easy to list all realizable quotient orbifolds M/H . As an application of the method, we illustrate the case of periodic actions.

3.4.1 Proposition Let M be an a.c.p. and H a finite group acting on it. Let $\pi_1 M \cong F_1 * \dots * F_p * \mathbb{Z}^r$, where for $1 \leq i \leq p$ F_i is the fundamental group of a compact orientable surface of genus $g > 0$. If $r \leq |H|/6$ then the graph (Γ', G') associated to the quotient orbifold M/H has the following properties

- I) If there exists $v \in V(\Gamma')$ such that H_v is a finite group and H_v is not \mathbb{D}_n for any $n \neq 2$, then there are no other spherical vertices.
- II) If there exists $v \in V(\Gamma')$ such that $H_v \cong \mathbb{D}_n$ for some $n \neq 2$, then (Γ', G') is



where there are no other spherical vertices except in the picture.

It will be useful the following analogous proposition, if the symmetry group of the graph is not too large:

3.4.2 Proposition Let M, H , $\pi_1 M \cong F_1 * \dots * F_p * \mathbb{Z}^r$ as above. Let S be the maximal number of pairwise isomorphic groups in the set $\{F_1, \dots, F_p\}$. Suppose $S \geq 1$. If $r + S - 1 \leq |H|/6$, then the graph of groups

(Γ', G') associated to the quotient orbifold M/H has only (if any) spherical vertices v with $G_v \cong \mathbb{D}_n$ $n \neq 2$.

proof of 3.4.1: Suppose that in (Γ', G') there exists one spherical vertex $v \in V(\Gamma')$ and let $e \in E(\Gamma')$ an edge with $\delta_0 e = v$. Suppose that $a = \delta_1 e$ is not v . Let $A_1, \dots, A_\alpha \in V(\Gamma)$ and $V_1, \dots, V_n \in V(\Gamma)$ the vertices covering a and v ; let $E_1, \dots, E_\varepsilon$ be the edges in $E(\Gamma)$ covering e . Consider the (possibly disconnected) subgraph Λ of Γ with $V(\Lambda) = \{A_1, \dots, A_\alpha; V_1, \dots, V_n\}$ and $E(\Lambda) = \{E_1, \dots, E_\varepsilon\}$.

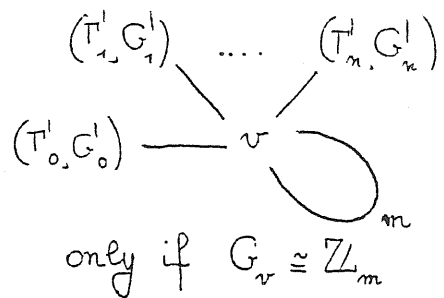
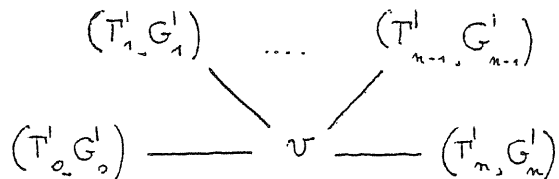
Suppose that there exists a path $p = \{p_1, \dots, p_n\}$ in Γ such that

- 1) $\delta_0 p_1 \in \{A_1, \dots, A_\alpha\}$
- 2) $\delta_1 p_n \in V(\Lambda)$
- 3) $p_i \notin E(\Lambda) \quad \forall i \quad 1 \leq i \leq n$.

The image through H of p still satisfies 1) 2) and 3) and there are at least α different images. Let P be the set of these paths and their inverses. Let us consider the (possibly disconnected) subgraph Λ' of Γ with $E(\Lambda') = E(\Lambda) \cup \{e \in E(\Gamma) \text{ such that there exists } p \in P \text{ with } e \in p\}$; $V(\Lambda') = \{v \in V(\Gamma) \text{ such that there exists } e \in E(\Lambda') \text{ with } \delta_0 e = v\}$. The number of nonseparating edges of Λ' is more than $|H|/|H_\varepsilon| - |H|/|H_v| > |H|/6$, contradicting $r \leq |H|/6$ except for $G_v = \mathbb{D}_s$ and $s \neq 2$.

Then, if G_v is not \mathbb{D}_s with $s \neq 2$, as Γ is connected, there exists edges joining V_1, \dots, V_n unless Λ' is already connected. But, if there exists such edges, then the number of nonseparating edges in Γ should be again more than $|H|/|H_\varepsilon| - |H|/|H_v|$ and this is excluded.

The only possible solutions are the following



On the other hand the solutions for the case $G_v = \mathbb{D}_s$, $s \neq 2$, are those listed in II.

q.e.d.

The proof of 3.4.2 is analogous.

In view of this result, it is easy to list all realizable quotient orbifolds M/H . As an application of the method, we illustrate the case of periodic actions.

Let $M, H, (\Gamma, G), (\Gamma', G')$ as above, with H a cyclic group. Let a be the induced action of H on Γ ($\Gamma/H = \Gamma'$).

We mark the graph Γ' in the following way.

$\forall e \in E(\Gamma')$ we define $n_e = |H_e|$. $\forall v \in V(\Gamma')$ let $e_1, \dots, e_p \in E(\Gamma')$ be the totality of edges such that $\delta_0 e_i = v$ $1 \leq i \leq p$. We define the set $(n_v^0, n_v^1, \dots, n_v^p) = (|Stab_w|, n_{e_1}, \dots, n_{e_p})$ where $w \in V(\Gamma)$ is any vertex covering v and $Stab$ means the stabilizer for a . Finally let g be the genus of the compact orientable surface Σ with $\pi_1 \Sigma \cong G_w$. A marking for Γ is the assignment $m(v) = (g_v, n_v^0, \dots, n_v^p)$ $v \in V(\Gamma)$.

The action of H on M induces $\forall v \in V(\Gamma')$ an action of $\mathbb{Z}_{n_v^0}$ on the compact orientable surface of genus g with branch points of order n_1, \dots, n_p .

So, to a given action of H on M , we have associated a marked graph (Γ', m) and a set of cyclic actions.

On the other side, for the same marked graph (Γ', m) , any set of $\mathbb{Z}_{n_v^0}$ -actions on compact orientable surfaces Σ_v with genus g_v and at least some branch points of order n_v^1, \dots, n_v^p , is induced by a H -action on M , because the action of $\mathbb{Z}_{n_v^0} / \mathbb{Z}_{n_v^1} \dots \mathbb{Z}_{n_v^p}$ on the set of branch points is necessarily cyclic.

Periodic actions on compact orientable surfaces are well-known and classified ([Yo]) and so the problem of finding all possible quotient orbifolds which are obtained by a H -action on M is reduced to the problem of listing all marked graphs (Γ', m) which can be constructed by the procedure above.

This easier task is readily accomplished (in view of the propositions above) by studying the symmetries of the graphs Γ associated to the geometric decompositions of M .

Here are some examples

Examples

Let M be an a.c.p.. Let $\pi_1 M \cong F_1 * \dots * F_p * \mathbb{Z}^r$. Consider the set of all orientation-preserving effective differentiable \mathbb{Z}_n -actions on M . We will answer the following question: which are the 3-orbifolds which can be obtained as a quotient M/\mathbb{Z}_n ?

If we suppose $r \leq n/6$ we can apply 3.4.1. However this bound can be improved in this case, as we know that the fundamental group of the quotient orbifold admits no other spherical vertex groups except cyclic ones. In fact it is enough to suppose $r \leq n/2$ and the conclusions of the proposition hold as well. Analogously, if $r+s-1 \leq n/2$, we can apply 3.4.2.

First example

$$\pi_1 M \cong F * F * F * \mathbb{Z}^r$$

where F is Fuchsian and $r \leq n/2$.

By 3.4.2 (Γ', G') has no spherical vertices (because the action of \mathbb{Z}_n on a 3-ball is equivalent to a cyclic action).

Necessarily we have

$$(*) \quad r+2 = \frac{1}{2} \sum_i n_i / m(e_i)$$

where e_i varies over $E(\Gamma')$. (Γ', G') may have one, two or three Fuchsian vertices.

Three vertices

Necessarily we have

$$(**) \quad m(v) = (g, n) \quad \forall v \in V(\Gamma')$$

where g is determined by F .

So any marked connected graph (Γ', m) satisfying $(*)$, $(**)$ is realizable in the sense above (maybe the set of periodic actions corresponding to a given marked vertex is empty).

Two vertices

$$(**) \quad m(v_1) = (g, n) \quad m(v_2) = (g, n/2) \quad (n \text{ divisible by } 2)$$

Any marked connected graph (Γ', m) satisfying $(*)$, $(**)$ is realizable in the sense above (maybe the set of periodic actions corresponding to a given marked vertex is empty).

One vertex

$$(***) \quad m(v) = (g, n/3) \quad (n \text{ divisible by } 3)$$

Any marked connected graph (Γ', m) satisfying $(*)$, $(**)$ is realizable in the sense above (maybe the set of periodic actions corresponding to a given marked vertex is empty).

Second example

$$\pi_1 M \cong F_1 * \dots * F_p * \mathbb{Z}^r$$

where $r+2 \leq n/2$

Suppose that F_i is not isomorphic to F_j $1 \leq i \neq j \leq p$ and is Fuchsian.

By 3.4.2 (Γ', m) has no spherical vertices and so it has p vertices v_1, \dots, v_p such that

$$(**) \quad m(v_i) = (g_i, n)$$

where g_i is determined by F_i .

Any marked connected graph (Γ', m) satisfying $(*)$, $(**)$ is realizable in the sense above (maybe the set of periodic actions corresponding to a given marked vertex is empty).

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