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JACOBI GROUPS, JACOBI FORMS AND THEIR APPLICATIONS

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Introduction

In the theory of discrete groups a particular role is played by reflection groups: in order of complexity the first to be studied are *finite reflection groups* or *Coxeter groups*. They include the Weyl groups of simple Lie algebras and some exceptional groups which do not correspond to any Lie algebra. All of them can be realized as a group of *linear* reflections of a finite dimensional metric affine space $V_{\mathbb{R}}$ over the field of real numbers: their fundamental chamber is a dihedral cone.

Next there are *affine* Weyl groups, which are obtained by addition of an extra reflection w.r.t. an affine hyperplane: their fundamental chamber is a simplex.

The action of both these groups can be extended on $V_{\mathbb{C}} = \mathbb{C} \otimes \mathbb{R}$, where the fundamental domain of affine Weyl groups is not anymore compact.

If we require compactness we go to the next level, represented by *Complex Crystallographic Coxeter groups* (CCC): by definition they are reflection groups acting on $V_{\mathbb{C}}$ such that the fundamental chamber is compact and such that the linear part is real (namely there exist a basis in which the matrices representing them are real). Such groups were considered extensively and classified in [BS86], where the authors show that they are labelled by a (irreducible) reduced root system (hence a Dynkin diagram) and a complex parameter τ in the Poincaré upper half plane.¹

All these classes of groups preserve a nondegenerate (positive definite in the real case) second rank symmetric tensor, namely they are group of (possibly complex) isometries: we will refer to this tensor as the *intersection form*.

If we give up the requirement of nondegenerateness of the tensor then the problem of classification gets rapidly more and more involved: Saito has studied the class of reflection groups generated by *marked extended affine root systems* [Sa85, Sa90]. They are generalized root systems of affine type which preserve a semidefinite positive bilinear pairing with two-dimensional kernel: the “marked” refers to a choice of a sublattice of rank one in the kernel, in order to extend the metric to a preserved hyperbolic metric (nondegenerate and indefinite). They generalize CCC groups and hence have a generalized Dynkin diagrammatics.

Already Looijenga noticed that these reflection groups carry a natural $SL(2, \mathbb{Z})$ -action (in the case of CCC groups) and hence one can define a semidirect product which we denote by $J(\mathfrak{g})$ and call **Jacobi group** of type \mathfrak{g} (the Lie algebra associated to the Dynkin diagram).

They can be made to act in space preserving a nondegenerate symmetric tensor *up to conformal invariance*² (or they could be made to preserve a corank 2 symmetric tensor in the sense of Saito).

¹More generally one may study the discrete reflection groups acting on spaces of constant curvature.

²Again, this tensor is referred to as the “intersection form”.

Since they extend CCC groups, they are classified as well by Dynkin diagrams.

The basic problem which is related to the study of discrete groups is to formulate a convenient invariant theory, namely the theory of functions which are invariant in some sense under the action of the group (the class of function depending on the particular setting).

Finite reflection groups have a well established invariant theory: it was proved by Chevalley [Ch55] that the algebra of invariant polynomials is a graded polynomial algebra freely generated by l invariant polynomials y_1, \dots, y_l (l being the rank of the group or equivalently the real dimension of $V_{\mathbb{R}}$).

The degrees of such generators are uniquely fixed and can be read off the spectrum of the so-called ‘‘Coxeter element’’ c of the group: by definition c is (any element of the conjugacy class of) the product of all fundamental reflections which generate the group.

The corresponding theory for affine Weyl group proves similarly that the algebra of invariant functions is freely generated by –again– l generators Y_1, \dots, Y_l : in view of the invariance under the lattice of translations of the group, they are *Fourier* polynomials in the coordinates of $V_{\mathbb{R}}$. The algebra of such Fourier polynomials is not naturally graded, contrarily to the previous case.

For CCC groups the invariant theory can be traced in the works by Looijenga [Lo80] and Bernstein–Schwarzman [BS86]: now, however, we cannot look anymore for invariant functions because the quotient space is a multidimensional complex torus and hence any holomorphic function on it is a constant. Instead one has to lift the action of the CCC group on $V_{\mathbb{C}}$ to a suitable line bundle (corresponding to the choice of a 1-cocycle with values in $\mathcal{O}^*(V_{\mathbb{C}})$) and seek invariant sections in the (graded) tensor algebra of this bundle.

The result is that the algebra is freely generated by $l + 1$ ‘‘theta functions’’ $\Theta_0, \dots, \Theta_l$ and has a natural \mathbb{N} -grading called *index* (in the context of Affine Lie Algebras (Kac–Moody) it is called the *level*): the degrees of the generators are then related to the labels of the Dynkin diagram (they correspond to the labels in the simply laced cases). This algebra is often called Theta algebra.

As for Jacobi groups the invariant theory has been introduced by Eichler and Zagier [EZ] (who firstly introduced the name) for the case A_1 and generalized to other cases by Wirthmüller [Wi92]. Now we are looking for invariant section in the tensor algebra of the line bundle obtained by tensoring the ‘‘theta’’ line bundle with the canonical bundle of \mathcal{H} , the Poincaré upper half plane. This amounts to searching for invariant functions of the Theta algebra tensored with the graded algebra of modular forms³. The result is a bigraded algebra $J_{\bullet, \bullet}^{(\mathfrak{g})}$ (\mathfrak{g} denotes the Lie algebra corresponding to the Dynkin diagram), where the two gradations are the \mathbb{N} -gradation of the index and the \mathbb{Z} -gradation of the weight, defined as customary for modular forms.

This algebra⁴ is a freely generated polynomial algebra over the graded ring M_{\bullet} of modular functions: there are $l + 1$ such generators $\varphi_0, \dots, \varphi_l$ whose bigrades can be specified and are the same indices of the previous case and weights which are zero and minus the degrees of the invariant polynomials y_1, \dots, y_l of the corresponding Weyl group. The algebra $J_{\bullet, \bullet}^{(\mathfrak{g})}$ is called the algebra of **Jacobi forms** (the name being given firstly by Eichler and Zagier in loc.cit.).

In Chapter 2 we explicitly construct the generators of the algebra of Jacobi forms in some cases.

³Certain conditions of boundedness when the modular parameter tends to a cusp must be added.

⁴This has been proven in [Wi92] for Dynkin diagrams except E_8

The first case is the series A_l : we can define a generating (elliptic) function in one auxiliary variable v , where the coefficients of the Laurent tail at the pole $v = 0$ provide the desired generators.

This generating function is essential to compute the “intersection elements” $\mathfrak{M}_{i,k}$: they are the quantities equivalent to the matrix elements of the Killing form \langle, \rangle on the basis of invariant polynomials. The result is a formula giving an elliptic deformation of a formula computed by Saito, Yano and Sekiguchi in [SYS80] for the case of polynomial invariants.

Moreover we can compute the expressions for the components of the (contravariant) connection in the basis of Jacobi invariants with the aid of the generating function: this computation is useful in the application of Frobenius manifolds to the dispersionless limit of integrable hierarchies. Indeed, although conceptually straightforward, such a computation would involve nontrivial Fay’s identities for theta functions and in any case it would not be viable in closed form for any rank of A_l .

The series B_l can be treated with similar devices and also the corresponding intersection elements can be expressed in terms of a generating function: again this gives an elliptic deformation of the corresponding polynomial case in [SYS80].

Unfortunately we did not find any generating function for the remaining two classical series, the main obstacle being that the generators have different indices while in the cases A_l and B_l they are all of index one. However some specific cases can be solved, namely the C_3 , C_4 , D_4 ones.

The case C_3 is especially easy because the algebra $J_{\bullet,\bullet}^{(C_3)}$ is a subalgebra of $J_{\bullet,\bullet}^{(A_3)}$ due to the fact that the finite Weyl group $W^{(C_3)}$ is a \mathbb{Z}_2 extension of $W^{(A_3)}$.

The case D_4 (from which the case C_4 follows easily) requires heavy (but straightforward) computations in order to analyze the modular properties of the classical Theta functions (the construction is a “brute force” computation).

Among the exceptional Lie algebras the case G_2 is the easiest because (similarly to C_3) $J_{\bullet,\bullet}^{(G_2)} \subset J_{\bullet,\bullet}^{(A_2)}$ due to the fact that $W^{(G_2)}$ is a \mathbb{Z}_2 -extension of $W^{(A_2)}$.

We remark that C_3, D_4, G_2 are all *codimension-one* cases in the theory of Saito, namely among the generators of the algebra of theta functions there exist only one generator of maximal index. This is relevant in the application of Jacobi forms to Frobenius manifolds (see later).

Beside the theory of (generalized) root systems, there exists a second context where Jacobi groups arise analogously to Weyl groups, namely singularity theory: this is historically the first occurrence.

We recall that Weyl groups of the $A - D - E$ type arise in the study of miniversal deformations of simple hypersurface singularities. For example, in the case of the A_l series, such hypersurface is defined by $F : \mathbb{C}^2 \rightarrow \mathbb{C}$, $F(x, y) = x^{l+1} + y^2 = 0$, and the Weyl group of type A_l appears as follows: consider the miniversal deformation $\tilde{F}_a(x, y) = x^{l+1} + a_1 x^{l-1} + \dots + a_{l-1} x + a_l + y^2 =: P_a(x) + y^2$ and a basis in the homology H_1 of the regular level surface $V_a := \tilde{F}_a^{-1}(0)$ (in a neighborhood of the origin). Now, for generic a ’s, V_a is regular and it becomes singular on the *discriminant* Δ . In this case the discriminant is the set of a ’s for which the polynomial $P_a(x)$ has non-simple zeroes. As the parameters a ’s undergo a loop which avoids the discriminant Δ , the basis gets reshuffled. The fundamental group of $\mathbb{C}^l \setminus \Delta$ is isomorphically mapped into $GL(H_1, \mathbb{Z})$ and one can check that this is exactly the action of the Weyl group of type A_l .

In a similar way, the Jacobi group of type E_6 arises in the study of the miniversal deformation of

the simple *elliptic* singularity specified by the function $F : \mathbb{C}^3 \rightarrow \mathbb{C}$, $F(x, y, z) = x^3 + y^3 + z^3 + \lambda xyz$ [Sa74].

There exists still another parallel between ordinary Weyl groups and Jacobi groups: indeed in Chapter 1 we prove that one can define an infinite dimensional Lie algebra $\widehat{T(\mathfrak{g})}$ in such a way that Jacobi groups naturally appear as “Weyl” groups of a suitable Cartan subalgebra \mathfrak{h} in $\widehat{T(\mathfrak{g})}$. This is obtained by considering a (nontrivial) central extension of the algebra of smooth \mathfrak{g} -valued functions over a complex torus of modular parameter τ and adding a derivation δ w.r.t. the antiholomorphic vector field of the torus, in the same spirit of Kac–Moody algebras (the aforementioned central extension is a modification of the one considered by Etingof and Frenkel in [EF94]). In $\widehat{T(\mathfrak{g})}$ there exists an obvious CSA \mathfrak{h} which is spanned by the derivation δ , the central element and the CSA \mathfrak{h} of \mathfrak{g} : the result is that the group of automorphisms of $\widehat{T(\mathfrak{g})}$ which leave \mathfrak{h} (modulo the centralizer of \mathfrak{h}) invariant is proved to be isomorphic with the Jacobi group $J(\mathfrak{g})$.

This realization of $J(\mathfrak{g})$ is deeply related with the moduli theory of flat unitary vector bundles over the elliptic curve E_τ and Jacobi forms are hereby interpreted as section of a line bundle over such moduli space (Chapt. 2) (this is also related to geometric quantization of Hitchin systems [GaTra98]).

Along these lines we provide the first effective application of the theory of Jacobi forms to the Wess–Zumino–Novikov–Witten model on the torus (Chapt. 4). Indeed, following the works of Falceto and Gawedzki [FaGa96, FaGa95, FaGa92] we can express Chern–Simons states with no insertions (the chiral part of the partition function of such models) as Jacobi forms: this allows us to decompose the space of such states into subspaces of definite weight under the action of the modular group. The spectrum of these weights can be expressed at any level by means of a generating function. With the same function we can compute the dimensions of the vector spaces of such states for any level \varkappa (corresponding to the index) and any group (except E_8) in the case of “zero insertions”: this approach is of different origin w.r.t. the celebrated Verlinde’s formula for the dimension of Chern–Simons states [Ve88].

The case with insertions is more difficult because involves “vector-valued Jacobi forms” carrying a representation of the (finite) Weyl group. Nonetheless we are able to reduce it to the study of ordinary Jacobi forms which are *not* invariant under the finite Weyl subgroup of the Jacobi group.

The theory of polynomial invariants for finite reflection groups has found a relevant application to a recent topic of mathematical physics: indeed it is known from [Du92] that the spectrum of these algebras (i.e. the underlying manifold structure of the quotient space) can be endowed with a very rich geometric structure called “Frobenius manifold” [Du93]; Frobenius manifolds arise in a different, –physical–, context as intrinsic formulation of the *Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations of associativity* for a two dimensional topological field theory.

Since there are many points of contact between the polynomial case (i.e. Coxeter groups) and the elliptic one (i.e. Jacobi groups), it is natural to work out an analogous structure in the context of Jacobi forms.

A generalization to the case of “extended” affine Weyl groups was already made in [DZ98]; the authors considered a particular extension of the affine Weyl groups of simple Lie algebras which

endows the algebra of invariants with a grading operator analogous to the usual grading of Coxeter-invariant polynomials.

In Chapter 5 we go in the same direction by constructing a Frobenius structure on (a suitable covering of) the orbit space of the Jacobi group of type $A_l, B_l, \forall l = 1, \dots, G_2$ and C_3 .

In the case of the two series, this provides a series of solution of WDVV which are polynomials in the variables $t_1, \dots, t_l, t_{l+1}^{-1}, t_{l+1}$ and depend on t_0 via modular functions (here t_j are flat coordinates of the invariant metric of the Frobenius structure). Having a singularity at $\mathbf{t} = 0$ (due to the presence of the modular functions of t_0 and a pole in t_{l+1}), these solutions to the associativity equations do not satisfy the “good analyticity properties” in the sense of [Du93], Appendix A, but do provide interesting examples of twisted structure.

Notice moreover that the flat coordinates t_0, t_1, \dots, t_{l+1} are flat theta invariants on a suitable covering of the orbit space of the complex crystallographic affine Weyl group (though not precisely in the sense of [Sa90]). The explicit computations enable us to carry out the corresponding formulae for the generators and the intersection form in the case C_3 and G_2 . As we have remarked these two cases lie in the class of “codimension-one cases” studied by Saito in [Sa90]: this implies that there exists a natural pencil of flat structures (this implies a Frobenius structure) on the spectrum of the invariant algebra as described in [Sa90]. On the other hand, however, the study of the A_2, A_3 cases show that there exist different and independent flat structures, which are described in example 5.3.2 and Sections 5.5, 5.6: these structures come from the fact that the algebras $J_{\bullet, \bullet}^{(G_2)}$ and $J_{\bullet, \bullet}^{(C_3)}$ are embedded into $J_{\bullet, \bullet}^{(A_{2,3})}$ as a consequence of the fact that the respective Jacobi groups are in the reversed inclusion.

We did not work out explicitly the case D_4 but a simple reasoning shows that again there are two independent flat structures: the Saito’s one and the corresponding prepotential has been computed by Satake in [Sat98], the other comes from the relation with the Jacobi algebra of type B_4 (of which the algebra of D_4 is an extension).

The occurrence of multiple pencils of flat structures sharing a base point is not peculiar of these examples. In fact we have found a very simple example of this phenomenon in the polynomial case of type B_l (in Appendix B): in this particular instance there are l pencils of flat metrics having as common basepoint the intersection form.

The layout of the thesis is the following.

In Chapter 1, Section 1.1 we recall the construction of Jacobi groups and their usual representation.

In Section 1.2 we provide the details of the realization of Jacobi groups as groups of automorphisms of a current algebra which preserve a Cartan subalgebra.

In Chapter 2, Section 2.1 we recall the invariant theory for Jacobi groups and give their interpretation (Sec. 2.2.1) as sections of a line bundle over moduli space of flat G -bundles over an elliptic curve.

In Chapter 3 we provide the explicit expressions of the generators of Jacobi forms of type A_l, B_l, G_2, C_3 and D_4 (C_4). In the cases G_2 and C_3 we also give expressions of the particular set of theta functions which are flat coordinates for Saito’s metric (since these two cases are of

“codimension one”).

In Chapter 4 we briefly recall the setting of the Wess–Zumino–Witten model over a curve and then specialize to the case of interest to us, namely the elliptic one. We describe Chern–Simons states reducing the general problem to the study of non–Weyl–invariant Jacobi forms. In Section 4.1.2 we give the formulae for the dimension of the space of Chern–Simons states with zero insertions and arbitrary level.

In Chapter 5 we relate Jacobi forms to Frobenius manifolds as told before.

Finally in the Appendices we provide the formulae for elliptic functions and classical theta functions which have been used in the body of the work. Moreover (App. B) as we said, we give the example of multiple pencils of flat metrics in the polynomial B_l case.

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Chapter 1

Jacobi groups

In this chapter we introduce one of the main objects of the thesis: Jacobi groups. Jacobi groups are extensions of complex crystallographic groups [BS86] namely discrete groups acting on a vector space in such a way that the quotient is compact. The extension is with the full modular group $SL(2, \mathbb{Z})$ in a rather natural way. This extension was pointed out by Looijenga in [Lo76, Lo80] and then studied in the context of invariant theory by Wirthmüller in [Wi92].

Historically they are related to the study of the period mapping for the miniversal unfolding of the simple elliptic singularity of the function [Sa74]

$$F(X, Y, Z) = X^3 + Y^3 + Z^3 + \lambda XYZ ,$$

which corresponds to the Jacobi group $J(E_6)$ in the following notation. The generalization to other Lie algebras is straightforward, however it does not correspond to a period mapping of a hypersurface singularity but, possibly to singularity of subvarieties of higher codimension.

Deeply related to these groups is the study of *marked* extended affine root systems, carried out by K. Saito [Sa85, Sa90, Sa98].

For our purposes these groups are rather easily described and each of them is associated to one of the finite dimensional simple Lie algebras.

In the first section we give the definition of Jacobi groups $J(\mathfrak{g})$ for any simple, finite dimensional complex Lie algebra \mathfrak{g} and also provide the standard faithful representation as an action over a suitable cone.

In the second section we introduce a different description of these groups as “Weyl” groups of a suitable current algebra on the torus. To be more precise, we build an infinite dimensional current algebra which is very close to Kac–Moody’s algebras: then we pick up a natural Cartan subalgebra \mathfrak{h} and show that the automorphisms of the algebra which preserve \mathfrak{h} (modulo the action of the centralizer) form exactly the Jacobi group $J(\mathfrak{g})$. This enforces the parallel between the invariant theory of these groups (to be recalled in the next chapter) and the invariant theory of Weyl groups, which leads to the study of invariant polynomials. Moreover this introduces the application to be developed in Ch. 4 to the elliptic Chern–Simons states and Wess–Zumino–Novikov–Witten models on the torus.

1.1 The group $J(\mathfrak{g})$

In this section all the objects we will refer to (Weyl groups, root lattices, etc.) are the objects corresponding to a complex finite dimensional simple Lie algebra \mathfrak{g} of rank l .

Let W be the Weyl group, Q the root lattice (as an abelian group).

Let H_R be the Heisenberg group obtained as a central extension of $Q \times Q$ by \mathbb{Z} using the cocycle defined by the invariant Killing form \langle, \rangle normalized to 2 for the short roots, (this implies that $\forall \lambda \in Q, \langle \lambda, \lambda \rangle \in 2\mathbb{Z}$).

This group is obtained by definition of the product in $Q \times Q \times \mathbb{Z}$ as

$$\begin{aligned} \forall (\lambda, \mu, k), (\lambda', \mu', k') \in Q \times Q \times \mathbb{Z} \\ (\lambda, \mu, k) \cdot (\lambda', \mu', k') := (\lambda + \lambda', \mu + \mu', k + k' + \langle \mu, \lambda' \rangle). \end{aligned}$$

Since the Killing form \langle, \rangle is Weyl invariant and the Weyl group W of \mathfrak{g} acts on Q , we can take the semidirect product $W \rtimes H_R$: this way we obtain an (infinite) discrete group which we denote by \mathcal{W} where the semidirect product is specified by the multiplication rule

$$\begin{aligned} \mathcal{W} := W \rtimes H_R, \quad s.t. \quad \forall w, w' \in W, \mathbf{t} = (\lambda, \mu, k), \mathbf{t}' = (\lambda', \mu', k') \in H_R \\ (w, \mathbf{t}) \cdot (w', \mathbf{t}') := (ww', w \cdot \lambda' + \lambda, w \cdot \mu' + \mu, k + k' + \langle \mu, \lambda' \rangle). \end{aligned}$$

This group is rather well known and its invariants are theta functions, as we will explain in due course.

We now can give the definition of Jacobi group

Definition 1.1.1 *The Jacobi group $J (= J(\mathfrak{g}))$ is the semidirect product $\mathcal{W} \rtimes SL(2, \mathbb{Z})$. The action of $SL(2, \mathbb{Z})$ on the group $W \rtimes H_R$ is defined by*

$$\begin{aligned} Ad_\gamma(w) &:= w \\ Ad_\gamma(\mathbf{t}) &:= \left(a\lambda - b\mu, -c\lambda + d\mu, k + \frac{ac}{2} \langle \lambda, \lambda \rangle - bc \langle \lambda, \mu \rangle + \frac{bd}{2} \langle \mu, \mu \rangle \right), \end{aligned} \quad (1.1)$$

for $w \in W, \mathbf{t} = (\lambda, \mu, k) \in H_R, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$. Then the multiplication rule is defined by $(\forall (w, \mathbf{t}, \gamma), (w', \mathbf{t}', \gamma') \in W \times H_R \times SL(2, \mathbb{Z}))$

$$(w, \mathbf{t}, \gamma) \cdot (w', \mathbf{t}', \gamma') := (w \cdot w', \mathbf{t} \cdot Ad_\gamma(\mathbf{t}'), \gamma \cdot \gamma').$$

1.1.1 Faithful representation of J

Let us consider the cone $\Omega := \mathbb{C} \oplus \mathfrak{h} \oplus \mathcal{H} \ni (u, \mathbf{x}, \tau)$, where \mathfrak{h} is the complex Cartan subalgebra of \mathfrak{g} and \mathcal{H} denotes the Poincaré upper half plane.

In the literature it is often called the *Tits cone* [Lo80] and it is the union of all the images under \mathcal{W} of the closure of a fundamental chamber \mathcal{C} ; therefore it is an invariant cone for the action of the group \mathcal{W} (and of J as well).

Let us consider $\tau \in \mathcal{H}$; we have an embedding of $Q \times Q$ in \mathfrak{h} as the complex crystallographic lattice $Q + \tau Q$ and we can define the action as in the following

Proposition 1.1.1 The Jacobi group J is represented on Ω by definition of the action of $w \in W$, $t \in H_R$ and $\gamma \in SL(2, \mathbb{Z})$ as

$$\begin{aligned} w(u, \mathbf{x}, \tau) &:= (u, w\mathbf{x}, \tau) \\ t(u, \mathbf{x}, \tau) &:= \left(u + k - \langle \mathbf{x}, \mu \rangle - \frac{\tau}{2} \langle \mu, \mu \rangle, \mathbf{x} + \lambda + \tau\mu, \tau \right) \\ \gamma(u, \mathbf{x}, \tau) &:= \left(u + \frac{c \langle \mathbf{x}, \mathbf{x} \rangle}{2(c\tau + d)}, \frac{\mathbf{x}}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right). \end{aligned}$$

The proof is straightforward although rather long, and it is left to the reader.

Remark 1.1.1 The action of the Jacobi group is non linear, but it could be made such by realizing it as a discrete subgroup of the conformal group: in fact, as we will see, its action is a conformal action for the metric $du \otimes d\tau + d\tau \otimes du + \langle dx \otimes dx \rangle$.

1.2 Jacobi group as Weyl group of a current algebra

In this part of the chapter we provide an interpretation of Jacobi groups which is completely unrelated to singularity theory. This will realize Jacobi groups as the group of automorphisms preserving the Cartan subalgebra of a suitable algebra.

The algebra which we are going to define is an extension of a variation of the well-known current algebra considered in [EF94] (but see also [PrSe]), hence it is an infinite dimensional algebra. More precisely it is the result of adjoining a derivation to a specific central extension of a current algebra, much in the same spirit of Kac-Moody's algebras, except for the fact that S^1 is here substituted by a complex torus.

Inside this algebra we find a natural Cartan subalgebra and study the action of the group of internal and external automorphisms leaving it invariant (modulo the centralizer of the subalgebra): this is the analog in the present context of the Weyl group for semisimple Lie algebras. We will see that this group is exactly the Jacobi group $J(\mathfrak{g})$.

In order to fix the notation and recall the necessary objects and definitions we preliminarily report some well-known facts.

Let Σ be a smooth compact surface of genus g , G a simple, simply connected Lie group, $\mathfrak{g} = Lie(G)$, \langle, \rangle an invariant bilinear form on \mathfrak{g} : for us this will be the Killing form with the same normalization as before, namely $\langle \alpha, \alpha \rangle = 2$ for *short* roots.

Let $T(G)$ be the Lie group of all smooth maps from Σ to G , $T(\mathfrak{g})$ be the corresponding Lie algebra of smooth maps $\Sigma \mapsto \mathfrak{g}$.

When fixing a complex structure on Σ , we will denote the surface by \mathcal{C} ; let $H_{\mathcal{C}}$ be the complex vector space of abelian differentials of the first kind (i.e. holomorphic).

We will first shortly account for the construction of a particular central extension of the current group $T(G)$ given in [EF94] of which we will need a rather straightforward modification.

Let Φ denote the identity element in $H_{\mathcal{C}} \otimes H_{\mathcal{C}}^*$ and define a central extension of $T(\mathfrak{g})$ with values in $H_{\mathcal{C}}^*$ by means of the 2-cocycle Ω (see [EF94])

$$\Omega(\xi, \eta) := \int_{\Sigma} \Phi \wedge \langle \xi, d\eta \rangle, \quad \forall \xi, \eta \in T(\mathfrak{g}).$$

This defines a g -dimensional central extension $\widehat{T(\mathfrak{g})}$ which can be obtained from the so-called *universal central extension* by means of a quotient; indeed we have that

Proposition 1.2.1 [[PrSe] Section 4.2] The **universal central extension** $U\mathfrak{g}^\Sigma$ is an extension of $T(\mathfrak{g})$ by means of the infinite-dimensional space $\mathfrak{a} := \Omega^1(\Sigma)/d\Omega^0(\Sigma)$ of complex-valued one-forms on Σ modulo exact forms. This extension is defined by the \mathfrak{a} -valued cocycle

$$u(\xi, \eta) := \langle \xi, d\eta \rangle \text{ mod } d\Omega^0(\Sigma), \quad \forall \xi, \eta \in T(\mathfrak{g}).$$

Notice that the cocycle is skew-symmetric only up to an exact form: the homomorphism which maps $UT(\mathfrak{g})$ to $\widehat{T(\mathfrak{g})}$ is the factorization by the abelian subgroup of the universal center \mathfrak{a} of all $\omega \in \mathfrak{a}$ such that

$$\int_{\mathcal{C}} \omega \wedge \Phi = 0 \in H_{\mathcal{C}}^*.$$

Now, we have the following results.

Proposition 1.2.2 [[EF94] Proposition 1.2] (i) Let $\mathcal{C}_1, \mathcal{C}_2$ be two Riemann surfaces with complex structure: the Lie algebras $\widehat{T_1(\mathfrak{g})}, \widehat{T_2(\mathfrak{g})}$ are isomorphic iff \mathcal{C}_1 and \mathcal{C}_2 are conformally equivalent (i.e. biholomorphically isomorphic).

(ii) Any automorphism f of $T(\mathfrak{g})$ can be uniquely represented as a composition $f = h \circ \phi_*$ where h is a conjugation by an element of $T(Aut(\mathfrak{g}))$ and ϕ_* is the direct image map induced by a conformal diffeomorphism $\phi : \Sigma \rightarrow \Sigma$.

This means that the group of the outer automorphisms of $T(\mathfrak{g})$ is induced by the holomorphic automorphisms of \mathcal{C} .

This construction at the level of Lie algebras can be integrated at the level of Lie groups, as the following theorems show us.

Theorem 1.2.1 [[EF94], Thm. 2.2] There exists a central extension $\widehat{T(G)}$ of the current group $T(G)$ by means of the Jacobian variety J of \mathcal{C} ; the Lie algebra of this central extension is exactly $\widehat{T(\mathfrak{g})}$.

A similar result holds for the universal central extension.

Theorem 1.2.2 [[PrSe] Section 4.10] (i) The universal central extension of $UT(G)$ of the group $T(G)$ is an extension of the universal covering group $\widetilde{T(G)}$ of $T(G)$ by means of the infinite-dimensional abelian group \mathcal{A} of complex-valued one-forms on Σ modulo closed one-forms with integer periods.

(ii) $\pi_n(UT(G)) \otimes \mathbb{R} = 0$ for $n < 3$;

(iii) the group \mathcal{A} is homotopy equivalent to a $2g$ -dimensional real torus. The natural map $\pi_2(T(G)) \rightarrow \pi_1(\mathcal{A})$ associated to the fiber bundle $UT(G) \rightarrow T(G)$ is an isomorphism up to torsion.

Proposition 1.2.3 [[EF94] Prop. 2.4] The universal central extension $UT(G)$ is homotopy equivalent to $\widehat{T(G)}$.

In the following we slightly modify the cocycle Ω ; the result is homotopy equivalent to the previous in view of Thm. 1.2.3.

Let \overline{H}_C be the complex vector space of *purely antiholomorphic* differentials; it is obtained by taking the complex conjugate of the abelian differentials.

The two spaces H_C and \overline{H}_C are naturally paired by

$$B(\omega, \overline{\eta}) := \int_{\Sigma} \omega \wedge \overline{\eta} ,$$

and this is a nondegenerate pairing. The new central extension will be with values in $\overline{H}_C^* \simeq H_C$; indeed, let Φ be the identity element in $\overline{H}_C \otimes \overline{H}_C^*$ and redefine the cocycle by

$$\Omega(\xi_1, \xi_2) := \int_{\Sigma} \langle \partial \xi_1, \xi_2 \rangle \wedge \Phi .$$

Again it is a straightforward computation to check the cocycle properties.

1.2.1 Extension of the algebra and its automorphisms

We are going to define an extension $\widehat{\mathbb{T}(\mathfrak{g})}$ of $\mathbb{T}(\mathfrak{g})$.

This algebra will *not* be the Lie algebra of any Lie group (exactly in the same way as Virasoro's algebra) but has the property that it supports in a natural way (an extension of) the coadjoint representation of $\widehat{\mathbb{T}(G)}$. In addition any automorphism of $\widehat{\mathbb{T}(\mathfrak{g})}$ extends to an automorphism of $\widehat{\mathbb{T}(G)}$; we will realize explicitly this action and see that the subgroup of automorphisms preserving a Cartan subalgebra is isomorphic to the represented Jacobi group $J(\mathfrak{g})$.

For the sake of completeness we recall the basics of the coadjoint representation for current groups.

This representation acts over the infinite dimensional complex vector space \mathcal{A} of all connections over the trivial \mathfrak{g} -bundle on \mathcal{C} . Such connections can be realized as operators of the form $D = w \frac{\partial}{\partial \bar{z}} + \xi$, where $w \in \mathbb{C}$ and $\xi \in \mathbb{T}(\mathfrak{g})$. Any such D defines thus a complex structure for a trivial G -vector bundle V over \mathcal{C} in the usual manner by decreeing that holomorphic sections are those annihilated by D . As a holomorphic vector bundle we will denote it by $B(D)$ and in general it will not be a trivial holomorphic vector bundle.

1.2.1.1 Extension of the natural coadjoint representation of $\mathbb{T}(G)$ for elliptic curves

We now perform the extension which was anticipated at the beginning of the previous section: the action of $\mathbb{T}(G)$ on the space of connections $\mathcal{A} := \{D = w\overline{\partial} + \xi d\bar{z}\} \simeq \mathbb{C}\delta \oplus \mathbb{T}(\mathfrak{g})$, will be extended in a natural way to a larger space which is naturally identified with an extension of the Lie algebra $\widehat{\mathbb{T}(\mathfrak{g})}$: here we denote with δ the derivation $\frac{\partial}{\partial \bar{z}}$.

In [EF94] the vector space \mathcal{A} is naturally realized as the dual of the algebra $\widehat{\mathbb{T}(\mathfrak{g})}$ under the invariant pairing

$$B(D, \widehat{\eta}) := wy + \int_{\Sigma} dz \wedge d\bar{z} \langle \xi, \eta \rangle , \text{ with } D := w\overline{\partial} + \xi d\bar{z} , \widehat{\eta} = y\overline{\omega}^{\vee} + \eta ,$$

\bar{w}^\vee being the unique (in genus $g = 1$) central element belonging to $\overline{H}_C^* \simeq H_C^1$.

Consider now the extended vector space $\widehat{\mathbb{T}(\mathfrak{g})} := \mathbb{C}\delta \oplus \widehat{\mathbb{T}(\mathfrak{g})}$. We introduce coordinates on it of the form $(w, \xi, z) \in \mathbb{C} \times \mathbb{T}(\mathfrak{g}) \times \mathbb{C}$ and identify naturally with the pairs $(D, u) \equiv (w \frac{\partial}{\partial \bar{z}} + \xi, u) \in \mathcal{A} \times \mathbb{C}$. We define the commutation relations (extending the previous ones) [PrSe]

$$\left[(D_1, v_1), (D_2, v_2) \right] := \left([\xi_1, \xi_2] + w_1 \frac{\partial}{\partial \bar{z}} \xi_2 - w_2 \frac{\partial}{\partial \bar{z}} \xi_1, \Omega(\xi_1, \xi_2) \right) = ([D_1, D_2], \Omega(\xi_1, \xi_2)) ,$$

where we recall the form of the cocycle

$$\Omega(\xi_1, \xi_2) := \int_{\Sigma} \left\langle \xi_1, \frac{\partial}{\partial z} \xi_2 \right\rangle dz \wedge d\bar{z} .$$

The adjoint action can be computed by straightforward calculations adapting those for ordinary Kac–Moody’s algebras.

Proposition 1.2.4 [See Prop. 4.9.4 in [PrSe]] If $\gamma := \exp(\xi)$ belongs to the identity component of $\mathbb{T}(G)$, then the adjoint action of γ on the extended algebra $\widehat{\mathbb{T}(\mathfrak{g})}$ is given by

$$\begin{aligned} \widehat{Ad}_\gamma(D, v) &= \left(w \frac{\partial}{\partial \bar{z}} + Ad_\gamma \eta + w \gamma^{-1} \frac{\partial}{\partial \bar{z}} \gamma , \right. \\ &\left. v + \int_{\Sigma} dz \wedge d\bar{z} \left\langle \gamma^{-1} \frac{\partial \gamma}{\partial z}, \eta \right\rangle + \frac{w}{2} \int_{\Sigma} dz \wedge d\bar{z} \left\langle \gamma^{-1} \frac{\partial \gamma}{\partial z}, \gamma^{-1} \frac{\partial \gamma}{\partial \bar{z}} \right\rangle \right) . \end{aligned}$$

Remark 1.2.1 The extended algebra $\widehat{\mathbb{T}(\mathfrak{g})}$ fails to be integrated to a Lie group because of the presence of the derivation $\frac{\partial}{\partial \bar{z}}$.

The extended algebra $\widehat{\mathbb{T}(\mathfrak{g})}$ contains the obvious Cartan subalgebra

$$\mathfrak{h} := \delta \oplus \mathfrak{h} \oplus \bar{w}^\vee = \left\{ (D, v) = \left(w \frac{\partial}{\partial \bar{z}} + 2i\pi \frac{\mathbf{x}}{\tau - \bar{\tau}}, v \right) \right\} ,$$

where \mathfrak{h} is the abelian subalgebra of constant maps with values in a fixed Cartan subalgebra of \mathfrak{g} . In this definition we consider w as coordinate along the derivation $\delta = \frac{\partial}{\partial \bar{z}}$ and v as coordinate along the central element \bar{w}^\vee . The normalization $\frac{2i\pi}{\tau - \bar{\tau}}$ is for later convenience.

We now consider the group of automorphisms in $Aut(\widehat{\mathbb{T}(\mathfrak{g})})$ which leave invariant \mathfrak{h} (modulo the centralizer).

First of all we prove

Lemma 1.2.1 *Any automorphism of $\widehat{\mathbb{T}(\mathfrak{g})}$ is the lift of an automorphism of $\widehat{\mathbb{T}(\mathfrak{g})}$.*

¹This explains why the dual space of the central extended Lie algebra $\mathbb{T}(\mathfrak{g})$ is taken as the space of connections of the form $D = w \frac{\partial}{\partial \bar{z}} + \xi$; this is the space that naturally carries the co-adjoint action of $\mathbb{T}(G)$ induced by $B(D, \bar{\eta})$

Proof.

We must check that any automorphism $F \in \text{Aut}(\widehat{\mathbb{T}(\mathfrak{g})})$ leaves invariant $\widehat{\mathbb{T}(\mathfrak{g})} \leftrightarrow \widehat{\mathbb{T}(\mathfrak{g})}$. Indeed, if p_1 is the projection onto the abelian subalgebra spanned by the derivation δ we have, for any two elements $a, b \in \widehat{\mathbb{T}(\mathfrak{g})}$

$$0 = p_1([Fa, Fb]) = p_1 \circ F[a, b].$$

But $\widehat{\mathbb{T}(\mathfrak{g})} = \left[\widehat{\mathbb{T}(\mathfrak{g})}, \widehat{\mathbb{T}(\mathfrak{g})} \right]$, hence $p_1 \circ F \equiv 0$; this implies that $F\widehat{\mathbb{T}(\mathfrak{g})} \subseteq \widehat{\mathbb{T}(\mathfrak{g})}$.

Therefore the automorphisms of the extended algebra are a subgroup of the automorphisms of $\widehat{\mathbb{T}(\mathfrak{g})}$; but again, any automorphism of $\widehat{\mathbb{T}(\mathfrak{g})}$ can be lifted to $\widehat{\mathbb{T}(\mathfrak{g})}$ acting as the identity on the coordinate w . Q.E.D

Therefore we pass to consider all automorphisms of $\widehat{\mathbb{T}(\mathfrak{g})}$ which stabilize $\hat{\mathfrak{h}}$; we know that any automorphism of the extended algebra must be an extension of an automorphism of the algebra $\widehat{\mathbb{T}(\mathfrak{g})}$. In view of Prop. (1.2.2), the latter are the composition of an inner automorphism and an automorphism of the underlying complex curve.

To understand the action of the group of automorphisms of the complex curve on the extended algebra it is convenient to introduce a fibration; the base space will be the Poincaré upper half plane $\mathcal{H} := \{\tau, \Im(\tau) > 0\}$ while the fibers will be the extended algebras just defined

$$\mathcal{G} := \begin{array}{c} \widehat{\mathbb{T}(\mathfrak{g})} \\ \downarrow \\ \mathcal{H} \end{array}.$$

This fibration is acted upon by the group $SL(2, \mathbb{Z})$, namely the group of complex isomorphisms of elliptic curves; here it appears the full group $SL(2, \mathbb{Z})$ and not only $PSL(2, \mathbb{Z})$ because the action of the isomorphism of complex curves on the derivation can distinguish between plus or minus the identity in $SL(2, \mathbb{Z})$.

We introduce coordinates on $\mathcal{G} (D, v, \tau)$ by adjoining the modulus of the elliptic curve τ .

As far as the action of $\widehat{\mathbb{T}(\mathfrak{g})}$ on $\widehat{\mathbb{T}(\mathfrak{g})}$ is concerned we can regard τ as a parameter.

1.2.1.2 Analysis of the Weyl group

We are to analyze the automorphisms fixing our Cartan subalgebra, namely the Weyl group of $\widehat{\mathbb{T}(\mathfrak{g})}$.

The obvious ones are the element of the Weyl group of \mathfrak{h} considered as constant elements of the gauge group $\widehat{\mathbb{T}(\mathfrak{g})}$. The action of such an element w is given by

$$w \left(w \frac{\partial}{\partial \bar{z}} + 2i\pi \frac{\mathbf{x}}{\tau - \bar{\tau}}, v, \tau \right) = \left(w \frac{\partial}{\partial \bar{z}} + 2i\pi \frac{w \cdot \mathbf{x}}{\tau - \bar{\tau}}, v, \tau \right).$$

Along with these we should also consider the action of the outer automorphisms of \mathfrak{g} , namely those corresponding to the symmetries of the Dynkin diagram.

Another class of automorphisms are those deriving by nonconstant gauge transformations. We have

Lemma 1.2.2 *Any non-constant gauge transformation $F(z, \bar{z}) \in T(G)$ leaving invariant \mathfrak{H} is of the form*

$$F(z, \bar{z}) := w \circ \exp \left[2i\pi \frac{\rho \bar{z} - \bar{\rho} z}{\tau - \bar{\tau}} \right],$$

with w an element of the Weyl group of \mathfrak{h} and $\rho = \lambda + \tau\mu \in Q \oplus \tau Q \hookrightarrow \mathfrak{h}$.

Remark 1.2.2 The embedding of the root lattice Q in \mathfrak{h} via our bilinear form \langle, \rangle , coincides with the usual *coroot* lattice in \mathfrak{h} : this is due to the unconventional normalization of the Killing form $\langle \alpha, \alpha \rangle = 2$ for short roots. This has the effect that $\exp(2i\pi\lambda)$ equals the identity of the group G iff λ belongs to our root lattice.

Proof. Any such transformation $F(z, \bar{z})$ must satisfy $Ad_F H + F^{-1} \partial_{\bar{z}} F = K(H) = \text{const} \in \mathfrak{h}$ for any H belonging to \mathfrak{h} . The operator $K(H)$ is clearly affine w.r.t. H and its linear part must be an inner automorphism of G preserving \mathfrak{h} because $K(H) \in \mathfrak{h}$, $\forall H \in \mathfrak{h}$. Therefore there exists some $w \in W$ and a $\rho \in CSA$ such that

$$K(H) = w \cdot H + \rho.$$

There follows that F is of the form $wF_0(z, \bar{z})$ with F_0 a $\exp(\mathfrak{h})$ -valued function. Plugging into the equation we have

$$F_0^{-1} \partial_{\bar{z}} F_0 = \rho \in \mathfrak{h}.$$

The conditions of periodicity force $\rho = 2i\pi(\lambda + \tau\mu)$ with λ and μ belonging to the (co)root lattice. Q.E.D

The action of an element of the above form (with $w = \text{id}_{\mathfrak{h}}$) on \mathfrak{H} is the following

$$\begin{aligned} & \left(w \frac{\partial}{\partial \bar{z}} + 2i\pi \frac{\mathbf{x}}{\tau - \bar{\tau}}, v \right) \mapsto \\ & \mapsto \left(w \frac{\partial}{\partial \bar{z}} + 2i\pi \frac{\mathbf{x} + w\lambda + w\tau\mu}{\tau - \bar{\tau}}, v + \frac{(2i\pi)^2}{\tau - \bar{\tau}} \left(\frac{\bar{\tau}\tau}{2} w \|\mu\|^2 + \bar{\tau} \langle \mu, \mathbf{x} \rangle + \frac{w}{2} \|\lambda\|^2 + \langle \lambda, \mathbf{x} \rangle \right) \right), \end{aligned}$$

so that we obtain that the action on our Cartan subalgebra is

$$\rho(w, \mathbf{x}, v, \tau) = \left(w, \mathbf{x} + w\lambda + w\tau\mu, v + \frac{(2i\pi)^2}{\tau - \bar{\tau}} \left(\frac{\bar{\tau}\tau}{2} w \|\mu\|^2 + \bar{\tau} \langle \mu, \mathbf{x} \rangle + \frac{w}{2} \|\lambda\|^2 + \langle \lambda, \mathbf{x} \rangle \right) \right).$$

Finally we compute the action of the change of complex structure of the torus to an isomorphic one; this amounts to study the action of the transformations $\tau \mapsto \tau' = \frac{a\tau + b}{c\tau + d}$, with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$.

We focus on the action of the two generators $\tau \rightarrow \tau + 1$ and $\tau \rightarrow -\frac{1}{\tau}$. It is easy to see that their action is

$$\begin{aligned} \tau \mapsto \tau + 1 : & \quad \left(\frac{\partial}{\partial \bar{z}} + 2i\pi \frac{\mathbf{x}}{\tau - \bar{\tau}}, v, \tau \right) \mapsto \left(\frac{\partial}{\partial \bar{z}} + 2i\pi \frac{\mathbf{x}}{\tau - \bar{\tau}}, v, \tau + 1 \right) \\ \tau \mapsto -\frac{1}{\tau} : & \quad \left(\frac{\partial}{\partial \bar{z}} + 2i\pi \frac{\mathbf{x}}{\tau - \bar{\tau}}, v, \tau \right) \mapsto \left(\frac{\partial}{\partial \bar{z}'} + 2i\pi \frac{\frac{\mathbf{x}}{\tau}}{\tau' - \bar{\tau}'}, v, -\frac{1}{\tau} \right), \end{aligned}$$

where, in the second line z' is the holomorphic coordinate on the complex torus of parameter $\tau' = -\frac{1}{\tau}$. We summarize these formulae in the following

Proposition 1.2.5 The action of the group of automorphisms of $\widehat{\mathbb{T}(\mathfrak{g})}$ preserving the Cartan subalgebra $\mathfrak{h} = \left\{ (D, v) = \left(w \frac{\partial}{\partial z} + 2i\pi \frac{\mathbf{x}}{\tau - \bar{\tau}}, v \right) \right\} \simeq \mathbb{C}\delta \oplus \mathfrak{h} \oplus \mathbb{C}\omega^\vee$ is generated by the following transformations

$$\begin{aligned} (w, \mathbf{x}, v, \tau) &\mapsto (w, w \cdot \mathbf{x}, v, \tau) \\ (w, \mathbf{x}, v, \tau) &\mapsto \left(w, \mathbf{x} + \lambda + \tau\mu, v + \frac{(2i\pi)^2}{\tau - \bar{\tau}} \left(\frac{\bar{\tau}\tau}{2} w \|\mu\|^2 + \bar{\tau} \langle \mu, \mathbf{x} \rangle + \frac{w}{2} \|\lambda\|^2 + \langle \lambda, \mathbf{x} \rangle \right), \tau \right) \\ (w, \mathbf{x}, v, \tau) &\mapsto \left(w, \frac{\mathbf{x}}{c\tau + d}, v, \frac{a\tau + b}{c\tau + d} \right). \end{aligned}$$

The abstract group underlying this action is exactly the Jacobi group $J(\mathfrak{g})$: the action is clearly different from the one considered in Section 1.1.1 but can be made equal by a (nonlinear) change of coordinates. Indeed we introduce the coordinate

$$u := \frac{v}{2i\pi} - 2i\pi \frac{\|\mathbf{x}\|^2}{2(\tau - \bar{\tau})}.$$

One can check that the same action in with this coordinate is exactly the same as in Section 1.1.1 if we set the *level* w equal to 1 (this is not a problem, since the underlying group is independent of w which is left invariant). We have thus proved the

Theorem 1.2.3 The Jacobi group $J(\mathfrak{g})$ is isomorphic to the Weyl group of $\widehat{\mathbb{T}(\mathfrak{g})}$, namely the group of automorphisms of $\widehat{\mathbb{T}(\mathfrak{g})}$ which preserve the Cartan subalgebra \mathfrak{h} modulo its centralizer.

Chapter 2

Invariant theory for Jacobi groups

In this chapter we review the results on the invariant theory of Jacobi groups. This is the analogue of the invariant theory of Coxeter (or, more specifically, Weyl) groups. They are usually realized as groups of reflections of a linear space preserving a positive definite bilinear form (the “intersection form”). The study of invariant theory amounts to the study of the algebra of invariant functions over this linear space.

In the case of Coxeter groups one studies the algebra of invariant *polynomials* and the main result, due to Chevalley [Ch55] is that this algebra is a free polynomial algebra in l generators y_1, \dots, y_l , l being the rank of the linear space. These generators are not uniquely determined but their degrees d_1, \dots, d_l are.

More precisely the degrees are obtained from the exponents of the so-called Coxeter element [Bo]. In the present context of Jacobi groups a similar theorem holds (Thm. 2.1.2, [Wi92]) for all groups except the one associated to the exceptional Lie algebra E_8 .

Here we recall such theory and definition of the proper functional space (this is necessary since the group is infinite) where to look for the invariant functions. Moreover in the spirit of Chapter 4 and following the realization of Jacobi groups as “Weyl” groups of the algebra $\widehat{T(\mathfrak{g})}$ as in Chapt. 1, we give an interpretation of Jacobi forms as sections of a line-bundle over the moduli space of flat G -bundles over an elliptic curve E_τ (Section 2.2.1).

2.1 Jacobi forms

Since we want to consider the orbit space $\Omega/J(\mathfrak{g})$, it is necessary to study the algebra of invariant functions; this is the analog of the study of polynomials over a vector space which are invariant under the action of a Coxeter group (which gives the orbit space a structure of weighted projective space¹). In this case one studies better the $SL(2, \mathbb{Z})$ -equivariant functions for the Jacobi group; it means that they are invariant under the action of the normal subgroup \mathcal{W} , and transform under some representation of the metaplectic group (as in Definition 2.1.3); in more geometrical terms

¹Recall that a weighted projective space $\mathbb{P}_{n_0, n_1, \dots, n_l}$ with weights $n_0, \dots, n_l \in \mathbb{N}$ is the quotient space of $\mathbb{C}^{l+1}/\{0\}$ with respect to the \mathbb{C}^* -action determined by the formula $\rho(t)(z_0, \dots, z_l) = (t^{n_0} z_0, \dots, t^{n_l} z_l)$.

one studies the invariant sections of a suitable line–bundle over the orbit space.

We will come back to this picture later, but for now we give the definitions and the results of the theory.

Following [Wi92] we will consider holomorphic functions on Ω with the further property that they are *locally bounded* (in u, \mathbf{x}) as $\Im(\tau) \rightarrow +\infty$; this is natural since τ is interpreted as modular parameter of an elliptic curve and the Jacobi forms will realize the holomorphic sections of a certain line bundle over it, which one wants to extend at the cusps of $SL(2, \mathbb{Z})$.

To be more definite, we will study the invariant modular forms, *Jacobi forms*, of weight k , $\mathcal{F}(u, \mathbf{x}, \tau) (d\tau)^{k/2}$, which have the following definition

Definition 2.1.1 ([Wi92]) *The Jacobi forms of weight k and index m are holomorphic functions on the Tits cone $\Omega = \mathbb{C} \oplus \mathfrak{h} \oplus \mathcal{H} \ni (u, \mathbf{x}, \tau)$ which satisfy*

$$\begin{aligned} \mathbf{E}\varphi(u, \mathbf{x}, \tau) &:= \frac{1}{2i\pi} \partial_u \varphi(u, \mathbf{x}, \tau) = m \varphi(u, \mathbf{x}, \tau) \\ \varphi(u, \mathbf{x}, \tau) &= \varphi(u, w \cdot \mathbf{x}, \tau) ; \\ \varphi(u, \mathbf{x}, \tau) &= \varphi\left(u - \langle \mathbf{t}, \mathbf{x} \rangle - \frac{\tau}{2} \langle \mathbf{t}, \mathbf{t} \rangle, \mathbf{x} + \tau \mathbf{t} + \lambda, \tau\right) ; \\ \varphi(u, \mathbf{x}, \tau) &= (c\tau + d)^{-k} \varphi\left(u + \frac{c \langle \mathbf{x}, \mathbf{x} \rangle}{2(c\tau + d)}, \frac{\mathbf{x}}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right), \end{aligned} \quad (2.1)$$

and are locally bounded functions of \mathbf{x} as $\Im(\tau) \rightarrow +\infty$.

The space of Jacobi forms of weight k and index m is denoted by $J_{k,m}$.

This means that the Jacobi forms are the invariants of the group $\mathcal{W} = W \rtimes H_R$ and transform as $d\tau^{k/2}$ under the modular group $SL(2, \mathbb{Z})$, being also eigenfunctions of the **Euler vector** $\mathbf{E} = \frac{1}{2i\pi} \partial_u$ with eigenvalue m .

Remark 2.1.1 Instead of thinking in terms of line–bundles, we could define equivalently a truly invariant algebra by adjoining to Ω a coordinate of the line–bundle as follows; consider the trivial line bundle $Y := \mathbb{C} \times \Omega$ with coordinates (v, u, \mathbf{x}, τ) . Then define the action of the Jacobi group on Y as

$$\gamma(v, u, \mathbf{x}, \tau) := \left(\frac{v}{c\tau + d}, u + \frac{c \langle \mathbf{x}, \mathbf{x} \rangle}{2(c\tau + d)}, \frac{\mathbf{x}}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right),$$

and in the obvious way (leaving v unchanged) for the elements of \mathcal{W} .

It follows that we can associate to any Jacobi form $\varphi \in J_{k,m}$ an invariant function on Y as

$$\Phi(v, u, \mathbf{x}, \tau) := v^k \varphi(u, \mathbf{x}, \tau)$$

and hence define a second grading vector $\mathcal{K} := v \frac{\partial}{\partial v}$ which could be called the *weighting operator*.

Digression 1 *The action of the group \mathcal{W} is the well known action in the theory of theta functions, namely, the invariance under $\mathcal{W} = W \rtimes H_R$ can be rephrased by saying that the Jacobi forms of index m are in particular theta functions of “level m ” for the affine Weyl group in consideration. We recall briefly some basic facts (but for a complete reference, see [KP84]). Adapting the notion to our setting and notations, if P is the lattice of weights (i.e. $\langle P, Q \rangle = \mathbb{Z}$),*

Definition 2.1.2 (Section III in [KP84]) Let $m \in \mathbb{N}$ be fixed. The space of theta functions \widetilde{Th}_m is defined as the set of holomorphic functions of (u, \mathbf{x}, τ) which are invariant under H_R and of degree m for the Euler field \mathbb{E} .

The theta functions of characteristic $\mathbf{p} \in P$ and degree $m \in \mathbb{N}$ for \mathbf{p} in $P \bmod mQ$ are defined by

$$\Theta_{\mathbf{p},m}(u, \mathbf{x}, \tau) := e^{-2i\pi m u} \sum_{\lambda \in Q + \frac{1}{m}\mathbf{p}} e^{im\pi\tau\|\lambda\|^2 - 2im\pi\langle \lambda, \mathbf{x} \rangle}.$$

The \mathbb{C} linear span of them inside \widetilde{Th}_m is denoted by Th_m .

It can be shown that

Proposition 2.1.1 [Lemma 3.12 and Prop. 3.13 in [KP84]] For $m \in \mathbb{Z}$ we have

- (a) $\widetilde{Th}_0 = \mathcal{O}(\mathcal{H})$ (where $\mathcal{O}(\mathcal{H})$ denotes the holomorphic functions of τ);
- (b) $\widetilde{Th}_m = \{0\}$ for $m < 0$;
- (c) the space \widetilde{Th}_m is a $\mathcal{O}(\mathcal{H})$ module over Th_m and hence the (graded) ring $\widetilde{Th}_\bullet := \bigoplus_{m \in \mathbb{N}} \widetilde{Th}_m$ is a free module over $\mathcal{O}(\mathcal{H})$ with basis $\{\Theta_{\mathbf{p},m}, m \in \mathbb{N}, m > 0, \mathbf{p} \in P \bmod mQ\} \cup \{1\}$.

Let us denote by \widetilde{Th}_\bullet^W the Weyl-invariant part of this ring (which is clearly also an algebra); regarding its structure we have

Theorem 2.1.1 [Thm 2.7 in [Lo76]] The algebra \widetilde{Th}_\bullet^W is a (graded) polynomial $\mathcal{O}(\mathcal{H})$ -algebra freely generated by $l+1$ W -invariant theta functions $\theta_0, \dots, \theta_l$. These generators are given by

$$\theta_j := W \cdot \Theta_{\mathbf{p}_j, m_j};$$

here $W \cdot \Theta$ denotes the symmetrization w.r.t. the action of the (finite) Weyl group, \mathbf{p}_j are the fundamental weights (we have set $\mathbf{p}_0 := 0$) and m_j are the integers appearing in the decomposition of the dual of the highest coroot $\widetilde{\alpha}^{\vee}$ in the basis of roots, namely $\widetilde{\alpha}^{\vee} = \sum_{j=1}^l m_j \alpha_j$, (setting for brevity $m_0 := 1$).

We now go back to the Jacobi forms.

The set of Jacobi forms of any weight and index has a natural structure of bi-graded algebra where the two gradings are the *weight* and the *index*; the following theorem is the the analog of Chevalley's theorem for invariant polynomials of a Coxeter group

Theorem 2.1.2 [Thm. (3.6) in [Wi92]] Given the Jacobi group associated to any finite dimensional simple lie algebra \mathfrak{g} (possibly except E_8)

1. the bigraded algebra of Jacobi forms $J_{\bullet, \bullet} := \bigoplus_{k,m} J_{km}$ is freely generated by $l+1$ fundamental Jacobi forms $\{\varphi_0, \dots, \varphi_l\}$ over the graded ring of modular forms $\bigoplus_k \mathbf{M}_k$

$$J_{\bullet, \bullet} = \mathbf{M}_\bullet[\varphi_0, \dots, \varphi_l]; \quad (2.2)$$

2. The generator φ_j has weight $-k_j \leq 0$ and index $m_j > 0$ (for $j = 0 \dots l$); the indices m_j are the integers appearing in

$$\widetilde{\alpha}^{\vee} = \sum_{j=1}^l m_j \alpha_j^{\vee},$$

$\widetilde{\alpha}^{\vee}$ being the dual of the highest coroot $\widetilde{\alpha}^{\vee}$.

3. the weights are $k_0 = 0$, and k_j are the Coxeter exponents plus one namely, if c is the product of the fundamental reflections of the Weyl group, then $\left\{ \frac{-2i\pi(k_1-1)}{h}, \dots, \frac{-2i\pi(k_l-1)}{h} \right\}$ are its eigenvalues: notice that k_j , $j = 1..l$, equal the degrees of the invariant polynomials that generate the invariant algebra $\mathbb{C}[V_0]^W$.

Remark 2.1.2 The second statement of Thm. 2.1.2 about the indices of the generators is not surprising if we recall that the bigraded algebra $J_{\bullet,\bullet}$ is a subalgebra of the Weyl-invariant part of the (graded) $\mathcal{O}(\mathcal{H})$ algebra of theta functions \widetilde{Th}_{\bullet} .

This means that the generators are graded-linear (w.r.t. the grading induced by the index) combinations of the generators $\theta_0, \dots, \theta_l$, with coefficients which depend holomorphically on τ only. This dependence is forced because the generators θ_j transform as a multiplet of modular forms under $SL(2, \mathbb{Z})$. There is only one case in which a theta function is also a Jacobi form: it happens for the theta function of characteristic zero in the simply-laced cases for which the lattice of (co)roots is self dual, namely $Q = Q^\vee = P$. This occurs only for E_8 , for which the theorem has not been proved.

The index zero subspace $J_{\bullet,0} \subset J_{\bullet,\bullet}$ is the graded algebra of *modular forms* M_{\bullet} . To show this we observe that $J_{\bullet,0}$ is spanned –by definition– by Jacobi forms independent of u and hence they cannot depend on \mathbf{x} either; indeed they are (for τ fixed) *bounded* functions on $\mathfrak{h}/(Q + \tau Q)$, which is compact, and hence they are constant w.r.t. \mathbf{x} .

Before going on, it is useful to remind some definitions and facts on the modular group and forms.

Definition 2.1.3 (see [KP84]) *The metaplectic group $Mp(2, \mathbb{R})$ is defined as the set*

$$Mp(2, \mathbb{R}) := \{(A, j_A) \in SL(2, \mathbb{R}) \times \{f : \mathcal{H} \rightarrow \mathbb{C}\} \text{ s.t. } j_A(\tau)^2 = c\tau + d\}$$

endowed with the multiplication rule

$$(A, j_A) \cdot (B, j_B) := (A \cdot B, j_A(B\tau)j_B(\tau))$$

Clearly we can consider the discrete subgroup $Mp(2, \mathbb{Z})$ and give the following

Definition 2.1.4 *Let Γ be a finite index subgroup of $Mp(2, \mathbb{Z})$ and a group homomorphism $\chi : \Gamma \rightarrow S^1$; a function $f : \mathcal{H} \rightarrow \mathbb{C}$ is called a **modular form of weight k and multiplier system χ** for Γ if it is holomorphic and for any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ we have*

$$f(A\tau) = \chi(A, j_A)(c\tau + d)^k f(\tau) .$$

An important example of modular form with multiplier system is the **Dedekind's η function** (which we will use in the following)

$$\eta(\tau) := e^{\frac{i\pi\tau}{12}} \prod_{n=1}^{\infty} (1 - e^{2i\pi\tau n}) . \quad (2.3)$$

It has the following transformations properties under the metaplectic group;

$$\begin{aligned}\eta(\tau + 1) &= e^{\frac{i\pi}{12}}\eta(\tau) ; \\ \eta\left(-\frac{1}{\tau}\right) &= \sqrt{(-i\tau)}\eta(\tau) ,\end{aligned}$$

where the determination of the square-root is in the right half plane [Ba]: notice that the η function is a modular form of weight $1/2$ with multiplier system such that $\chi\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = e^{i\pi/12}$ and $\chi\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = e^{-i\pi/4}$.

We finally recall a fundamental theorem for modular forms which can be found in [Se].

Theorem 2.1.3 [Corollary 2, Ch. VII in [Se]] The algebra of holomorphic modular forms \mathbf{M}_\bullet is a free graded algebra over \mathbb{C} generated by the two *Eisenstein series*

$$G_2(\tau) := \sum_{m^2+n^2 \neq 0} \frac{1}{(m+n\tau)^4} \in \mathbf{M}_4 ; \quad G_3(\tau) := \sum_{m^2+n^2 \neq 0} \frac{1}{(m+n\tau)^6} \in \mathbf{M}_6 ,$$

of weights, respectively, 4 and 6.

Consequently the subspace of modular forms of weight k is spanned by the monomials $G_2^a G_3^b$ with $4a + 6b = k$, namely

$$\mathbf{M}_k := \mathbb{C} \left[G_2^a G_3^b, \forall a, b, \in \mathbb{N} \text{ s.t. } 4a + 6b = k \right] .$$

It will be useful to introduce the following special notations

$$\begin{aligned}g_1(\tau) &:= \frac{\eta'(\tau)}{\eta(\tau)}; & g_2(\tau) &:= 60G_2(\tau); & g_3(\tau) &:= 140G_3(\tau) \\ P(\tau) &:= \frac{12\eta'(\tau)}{i\pi\eta(\tau)}; & Q(\tau) &:= \frac{3}{4\pi^4}g_2(\tau); & R(\tau) &:= \frac{27}{8\pi^6}g_3(\tau) .\end{aligned} \quad (2.4)$$

The g_1 function (and so $P(\tau)$) is not a modular function as it has the transformation rules

$$g_1(\tau + 1) = g_1(\tau) , \quad g_1\left(-\frac{1}{\tau}\right) = \tau^2 g_1(\tau) + \frac{1}{2} .$$

There is a natural connection ∇_τ on modular forms given in the following theorem–definition.

Theorem 2.1.4 Given a modular function F of weight k , ($F \in \mathbf{M}_k$), the **modular connection** ∇_τ is defined by the formula

$$(\nabla_\tau F)(\tau) := \eta^{2k}(\tau) \frac{d}{d\tau} \left(\eta^{-2k}(\tau) F(\tau) \right) \in \mathbf{M}_{k+2} ,$$

and maps \mathbf{M}_k in \mathbf{M}_{k+2} .

Example 2.1.1 For example, one can compute the following covariant derivatives of the fundamental Eisenstein series

$$\begin{aligned}\frac{\partial P(\tau)}{\partial \tau} &= \frac{i\pi}{6}(P^2(\tau) - Q(\tau)) \\ \frac{\partial Q(\tau)}{\partial \tau} &= \frac{2i\pi}{3}(P(\tau)Q(\tau) - R(\tau)) \\ \frac{\partial R(\tau)}{\partial \tau} &= i\pi(P(\tau)R(\tau) - Q^2(\tau)).\end{aligned}$$

Remark 2.1.3 While the holomorphic modular forms are of *positive weight* and are generated by modular forms of degrees 4, 6, the Jacobi forms can be of *negative weight* and are generated by Jacobi forms of *negative weight*.

As a corollary to the previous Thm. 2.1.2 and Thm. 2.1.3 we immediately find

Corollary 2.1.1 The dimensions $j_{k,m} := \dim(J_{k,m})$ are obtained from the generating function

$$f(X, Y) := \frac{1}{(1 - X^6)(1 - X^4)} \prod_{j=0}^l \frac{X^{k_j}}{(X^{k_j} - Y^{m_j})} = \sum_{m=0}^{\infty} \sum_{k \in \mathbb{Z}} j_{k,m} X^k Y^m.$$

2.2 Geometric interpretation

Jacobi forms are holomorphic sections of a certain line bundle over the quotient space Ω/J ; care should be taken in considering the orbifold points and cusps for the action of $SL(2, \mathbb{Z})$, but here we shall be slightly cavalier over these (important) details.

We now describe the line-bundle by displaying the transition functions.

Let \mathfrak{h} be the Cartan subalgebra of the lie algebra \mathfrak{g} and consider the fibration (over \mathcal{H}) of complex crystallographic lattices $Q + \tau Q \subset \mathfrak{h}$.

Consider the quotient $E_\tau^l := \mathfrak{h}/(Q + \tau Q)$: it is isomorphic to a product of l copies of elliptic curves of modular parameter τ (consider the coordinates of \mathfrak{h} induced by the fundamental roots). We regard E_τ^l naturally as a fibration of elliptic curves over \mathcal{H} .

Now a linear fractional transformation of τ with integer coefficients induces an isomorphism on the fibers of this fibration: namely if $\tau' = \frac{a\tau+b}{c\tau+d}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ then the fibers E_τ^l and $E_{\tau'}^l$ are isomorphic. The explicit isomorphism is given by

$$\begin{aligned}\Phi_{\tau, \tau'} : \mathfrak{h}/(Q + \tau Q) &\rightarrow \mathfrak{h}/(Q + \tau' Q) \\ \mathbf{x} &\mapsto \Phi_{\tau, \tau'}(\mathbf{x}) = \frac{\mathbf{x}}{c\tau + d}.\end{aligned}$$

The group $SL(2, \mathbb{Z})$ acts as a discrete group on the fiber bundle $\mathcal{E} := \begin{matrix} E_\tau^l \\ \downarrow \\ \mathcal{H} \end{matrix}$. Over this space we

consider the family of line bundles $\mathcal{L}_{k,m}$ indexed by $(k, m) \in \mathbb{Z} \times \mathbb{N}$, whose transition functions are

described hereafter.

This line bundle is defined over the fibration \mathcal{E} previously introduced. The open cover on which the transition functions are defined is constructed as follows; let C_e be the usual fundamental domain of the action of $PSL(2, \mathbb{Z})$ on \mathcal{H} , namely $C_e := \{\tau \in \mathcal{H} : |\tau| > 1, -\frac{1}{2} < \Re(\tau) < \frac{1}{2}\}$, and let $i_e := i, \rho_e := e^{i\pi/3}$ and $\rho'_e := e^{2i\pi/3}$ the orbifold points at the boundary of C_e : let us denote, for any $\gamma \in SL(2, \mathbb{Z})$ the transformed fundamental region by C_γ (where the point $i\infty$ is mapped either in itself or on some rational), and accordingly the boundary orbifold points by $i_\gamma, \rho_\gamma, \rho'_\gamma$.

Let $\mathcal{A}_{0,0}(\tau)$ be the fundamental poly-mesh of $\mathfrak{h}/(Q + \tau Q)$ namely

$$\mathcal{A}_{0,0}(\tau) := \{\mathbf{x} \in \mathfrak{h} : \forall j = 1..l, \langle \mathbf{x}, \mathbf{p}_j \rangle \in \text{fundamental mesh of } \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})\},$$

where the fundamental mesh is defined as containing the segments $[0, 1)$ and $[0, \tau)$. We consequently define the translated fundamental poly-meshes as $\mathcal{A}_{\lambda,\mu}(\tau) := \lambda + \tau\mu + \mathcal{A}_{0,0}(\tau)$. The trivializing charts of the line bundle $\mathcal{L}_{k,m}$ are the sets

$$U_{\gamma,\lambda,\mu} := \pi \left\{ (\mathbf{x}, \tau) \in \bigcup_{\tau \in C_\gamma} \mathcal{A}_{\lambda,\mu}(\tau) \right\},$$

where $\pi : \mathfrak{h} \oplus \mathcal{H} \rightarrow (\mathfrak{h} \oplus \mathcal{H})/J(\mathfrak{g})$ is the canonical projection. We define the transition functions for the line bundle $\mathcal{L}_{k,m}$ between the two charts of the orbit space corresponding to (1) := $(e, 0, 0)$ and (2) := (γ, λ, μ) by

$$g_{(1),(2)}^{(k,m)}(\mathbf{x}, \tau) := (c\tau + d)^{-k} \exp \left(m \frac{c\|\mathbf{x}\|^2}{2(c\tau + d)} - m \langle \mu, x \rangle - m \frac{\tau}{2} \|\mu\|^2 \right).$$

Notice that they are tensor products of an appropriate power of the pull-back of the canonical bundle of \mathcal{H} and of the line-bundle of classical theta functions over the fiber bundle \mathcal{E} previously introduced.

Now one realizes that there is a one-to-one correspondence between the W -invariant sections of this bi-graded family of line bundles and the Jacobi forms of the corresponding index and weight: the correspondence is given simply taking a Jacobi form $\varphi \in J_{k,m}$ and setting $u \equiv 0$. Indeed it follows from the definitions of Jacobi forms and of the above line bundles that if $\varphi \in J_{k,m}$, then $\varphi(0, \mathbf{x}, \tau) \in \Gamma(\mathcal{L}_{k,m})$.

2.2.1 Interpretation as sections over moduli of flat G -bundles

As we saw in Section 1.2, Jacobi groups are ‘‘Weyl’’ groups of Kac–Moody type algebra $\widehat{T(\mathfrak{g})}$: this will enable us to interpret the line bundle in the previous section as a line bundle over the moduli space of flat, unitary holomorphic G -bundles over the elliptic curve E_τ .

Indeed the vector space \mathcal{A} of connections $D = w \frac{\partial}{\partial \bar{z}} + \xi(z, \bar{z})$ is in one-to-one correspondence with the complex structures on the trivial \mathfrak{g} bundle on E_τ . Orbits of a generic connection D under the gauge group $T(G)$ corresponds to isomorphism classes of holomorphic principal G -bundles (we should restrict to the semistable ones which, however, constitute a Zariski open set). In the present case it is known (here we swap the role of Q and Q^\vee in view of Remark 1.2.2) (see [EF94]), that

Proposition 2.2.1 [[EF94] Prop. 4.1] The space of equivalence classes of flat and unitary holomorphic G -bundles over the complex torus is isomorphic to $\mathfrak{h}/(W \rtimes (Q \oplus \tau Q))$.

We recall that a G -bundle is called flat if it is associated to a G -representation of the fundamental group, and it is unitary if this representation lands on a fixed maximal compact subgroup. We now describe the above isomorphism.

Let, as before $D = w \frac{\partial}{\partial \bar{z}} + \xi$. If the bundle $B(D)$ is flat (which happens for generic ξ) then the equation $D\psi = 0$ on a G -valued smooth function ψ will have a solution $\psi(z, \bar{z})$ with the properties

$$\widehat{\psi}(z+1, \bar{z}+1) = \widehat{\psi}(z, \bar{z})K_1, \quad \widehat{\psi}(z+\tau, \bar{z}+\bar{\tau}) = \widehat{\psi}(z, \bar{z})K_\tau,$$

with K_1, K_τ in the maximal compact subgroup of G . Since $\pi_1(\Sigma) = \mathbb{Z}^2$ is abelian, then $K_1 K_\tau K_1^{-1} K_\tau^{-1} = id_G$. If $\mathfrak{k} \in \mathfrak{g}$ is any element such that $\exp(\mathfrak{k}) = K_1$, then the function $\psi(z, \bar{z}) := \widehat{\psi}(z, \bar{z}) \exp(-\mathfrak{k}z)$ is another solution and satisfies

$$\psi(z+1, \bar{z}+1) = \psi(z, \bar{z}), \quad \psi(z+\tau, \bar{z}+\bar{\tau}) = \psi(z, \bar{z})A$$

where A is given by $\exp(-\tau\mathfrak{k})K_\tau$ (and does not belong anymore to the compact subgroup but to its complexification). Under genericity assumptions A will be semisimple, hence we can assume that it belongs to a fixed complex maximal torus $T_{\mathbb{C}} \hookrightarrow G$.

Let \mathfrak{h} be the corresponding abelian Lie algebra (which is a Cartan subalgebra).

The element A completely determines $B(D)$; indeed, if $\mathfrak{x} \in \mathfrak{h}$ satisfies $A = \exp(2i\pi\mathfrak{x})$, setting

$$F(z) := \psi(z) \exp\left(-2i\pi\mathfrak{x} \frac{z - \bar{z}}{\tau - \bar{\tau}}\right)$$

we obtain a smooth doubly periodic G -valued function, i.e. $F \in T(G)$. Its action on the connection D then gives

$$F \circ D = \lambda \left(\frac{\partial}{\partial \bar{z}} - 2i\pi \frac{\mathfrak{x}}{\tau - \bar{\tau}} \right).$$

However, different \mathfrak{x} may correspond to the same bundle $B(D)$. Indeed, if μ belongs to the root lattice Q , then $\exp(2i\pi\mu) = id_G$; now, the previous solution can be changed to $\widehat{\psi} := \psi \exp(2i\pi z\mu)$ so that $\mathfrak{x} \in \mathfrak{h}$ is equivalent to $\mathfrak{x} + \tau Q$; conjugation by an element of the Weyl group W of $T_{\mathbb{C}}$ does lead to isomorphic bundles as well.

Since now $T_{\mathbb{C}} \simeq \mathfrak{h}/Q$, then the equivalence classes of bundles are indeed labelled by elements in $\mathfrak{h}/W \rtimes (Q \oplus \tau Q)$.

Remark 2.2.1 The situation is here very much the same as in the case of finite dimensional simple Lie algebras: indeed in this latter context the above theorem says that *the generic element of \mathfrak{g} is semisimple*, namely is Ad_G -equivalent to an element in an arbitrarily fixed Cartan subalgebra \mathfrak{h} . Here it is the same, except for the fact that D is in the *dual space* to the Lie algebra $\widehat{T(G)}$.

Recall that the linear space of connections \mathcal{A} was extended to the Lie algebra $\widehat{T(\mathfrak{g})}$ by adjoining a central element. We now consider the connection D as the projection of the element $D \oplus v\bar{w}^\vee$

(which we denote also (D, v)). Then, using the formulae in Section 1.2 we see that in the same orbit under the (co)adjoint action of the gauge group $\mathbb{T}(G)$ for generic element we have

$$w\delta + \xi + v\bar{w}^\vee \stackrel{\mathbb{T}(G)}{\simeq} w\delta + \left(2i\pi w \frac{\mathbf{x}}{\tau - \bar{\tau}}\right) + \tilde{v}\bar{w}^\vee,$$

where \tilde{v} is given by

$$\tilde{v} = v + \int_{\Sigma} dz \wedge d\bar{z} \left\langle F^{-1} \frac{\partial F}{\partial z}, \xi \right\rangle + \frac{w}{2} \int_{\Sigma} dz \wedge d\bar{z} \left\langle F^{-1} \frac{\partial F}{\partial z}, F^{-1} \frac{\partial F}{\partial \bar{z}} \right\rangle,$$

and $F \in \mathbb{T}(G)$ is the inner automorphism realizing the gauge equivalence.

Now \mathbf{x} is not uniquely determined, for we can consider the further gauge equivalence

$$\widehat{F}(z, \bar{z}) := F(z, \bar{z}) \exp \left[2i\pi \frac{\rho\bar{z} - \bar{\rho}z}{\tau - \bar{\tau}} \right],$$

where $\rho = \lambda + \tau\mu \in Q \oplus \tau Q \hookrightarrow \mathfrak{h}$. This new gauge brings the element into

$$\begin{aligned} w\delta + \xi + v\bar{w}^\vee &\xrightarrow{\widehat{F}} w\delta + 2i\pi w \left(\frac{\mathbf{x} + \lambda + \tau\mu}{\tau - \bar{\tau}} \right) + \\ &+ \left[\tilde{v} + (2i\pi)^2 w \frac{1}{\tau - \bar{\tau}} \left(\frac{\bar{\tau}\tau}{2} \|\mu\|^2 + \bar{\tau} \langle \mu, \mathbf{x} \rangle + \frac{1}{2} \|\lambda\|^2 + \langle \lambda, \mathbf{x} \rangle \right) \right] \bar{w}^\vee. \end{aligned}$$

Taking the quotient of these transformations we obtain a \mathbb{C}^* -bundle over $\mathfrak{h}/(Q + \tau Q)$: this line bundle is holomorphically equivalent to a theta bundle. To see that it is sufficient to change the holomorphic transition functions (w.r.t. the coordinates in the Cartan subalgebra \mathfrak{h}); indeed, holomorphic sections of this line bundle over $\mathfrak{h}/(Q + \tau Q)$ may be realized as holomorphic functions $\varphi(\mathbf{x})$ over \mathfrak{h} with the property (we rescale for commodity, $u := \frac{v}{2i\pi w}$)

$$\psi(\mathbf{x} + \lambda + \tau\mu|\tau) = \exp 2i\pi \left[\frac{1}{\tau - \bar{\tau}} \left(\frac{\bar{\tau}\tau}{2} \|\mu\|^2 + \bar{\tau} \langle \mu, \mathbf{x} \rangle + \frac{1}{2} \|\lambda\|^2 + \langle \lambda, \mathbf{x} \rangle \right) \right] \psi(\mathbf{x}|\tau).$$

If we change the transition functions by $\exp \left(\frac{2i\pi}{2(\tau - \bar{\tau})} \|\mathbf{x}\|^2 \right)$ (which is a holomorphic function of \mathbf{x}) we obtain a line bundle \mathcal{L} whose transition functions force section to behave like

$$\begin{aligned} \varphi(\mathbf{x}|\tau) &:= e^{-\frac{2i\pi}{2(\tau - \bar{\tau})} \|\mathbf{x}\|^2} \psi(\mathbf{x}|\tau) \\ \varphi(\mathbf{x} + \lambda + \tau\mu|\tau) &= \exp \left(\frac{\tau}{2} \|\mu\|^2 + \langle \mu, \mathbf{x} \rangle \right) \varphi(\mathbf{x}|\tau). \end{aligned}$$

Taking quotient by the Weyl group W of G imposes the further constraint

$$\varphi(w \cdot \mathbf{x}|\tau) = \varphi(\mathbf{x}|\tau), \quad \forall w \in W.$$

If we consider also the $SL(2, \mathbb{Z})$ action, then we must take further quotient; then the line bundle imposes the following relation

$$\varphi \left(\frac{\mathbf{x}}{c\tau + d} \middle| \frac{a\tau + b}{c\tau + d} \right) = e^{\frac{2i\pi c}{2(c\tau + d)} \|\mathbf{x}\|^2} \varphi(\mathbf{x}|\tau).$$

By tensoring with appropriate power of the canonical bundle of \mathcal{H} , we thus obtain the same line bundle considered in Section 2.2, where now sections of this line bundle are really to be considered as gauge invariant functions on the space of connections over the elliptic curve E_τ . We will return on this interpretation in Chapter 4, where this very line bundle will be interpreted as the geometric quantization of an Hitchin system [GaTra98] and sections as gauge invariant states for the Wess–Zumino–Novikov–Witten model on the torus.

Chapter 3

Explicit form of the generators

In this chapter we provide explicit formulae expressing the generators of the algebra of Jacobi forms in the case of the two series A_l , and B_l and for the cases C_3 , C_4 , D_4 , G_2 . As for the series we provide a closed formula for a generating function of the fundamental forms. In these cases the generators $\varphi_0, \dots, \varphi_l$ belong to the index 1 subspace, hence the generating functions is a (meromorphic) function of an auxiliary variable v with values in this subspace.

The construction of the Jacobi forms for the series A_l allows us to find the Jacobi forms for G_2 and C_3 as will be explained in the corresponding sections. The remaining two cases C_4 and D_4 actually amount to the same computation since the Jacobi forms of C_4 form a subalgebra of the Jacobi forms of D_4 (this holds true for any rank l). We are not able to produce a generating function for the remaining two series (one would suffice), hence the construction of the Jacobi forms for D_4 is accomplished directly by studying the ring of theta functions.

3.1 The generators of the algebra of Jacobi forms for $J(A_l)$ and $J(B_l)$

We are going to build some explicit analytic form of the generators of the algebra of Jacobi forms for the series A_l and B_l , but some preliminary formulae do apply for general Lie algebras \mathfrak{g} .

Contrarily to what happens in the study of polynomial invariants for Coxeter groups, the generators we are to build will be essentially unique up to weighted *linear* transformations; this is a feature only of the Jacobi forms of type A_l and B_l due to the fact that the generators are all of index 1.

During the construction in the A_l case it will appear a natural extension of the representation cone to a larger one, $\Omega' \supset \Omega$, and a natural vector field \mathcal{Z} on $T\Omega'$ which will realize a recursive construction of the remaining generators starting from the lightest: this has to do with the weighting operator of Rem. 2.1.1. In both cases A_l, B_l , the Jacobi forms will be given by a generating function which has some notably resemblances with the generating function of the invariant polynomials.

First of all, in the algebra of Jacobi forms there can be defined two natural operators, one

linear $\mathfrak{D} : J_{\bullet,\bullet} \rightarrow J_{\bullet,\bullet}$ and the other bilinear $\mathfrak{M} : J_{\bullet,\bullet} \otimes J_{\bullet,\bullet} \rightarrow J_{\bullet,\bullet}$. We need some preliminary definitions

Definition 3.1.1 *The intersection form is the covariant second-rank tensor \mathfrak{I} obtained by inversion of the contravariant tensor \mathfrak{I}^* given by*

$$\mathfrak{I}^* := -\partial_\tau \otimes \partial_u - \partial_u \otimes \partial_\tau + \langle \partial_x \otimes \partial_x \rangle^* \Rightarrow \mathfrak{I} := -du \otimes d\tau - d\tau \otimes du + \langle dx \otimes dx \rangle ;$$

its associated Laplace-Beltrami operator is denoted by

$$\Delta := -2\partial_\tau \partial_u + \Delta_x ,$$

where Δ_x is the Laplacian associated to the positive definite metric \langle, \rangle (proportional to the Killing form of \mathfrak{h}).

The intersection form \mathfrak{I} is conformally invariant under the Jacobi group J , namely

$$\gamma_* \mathfrak{I}^* = (c\tau + d)^2 \mathfrak{I}^* .$$

The Laplacian Δ is not conformally invariant (contrary to what stated in [KP84] on page 190) but enjoys the property shown in next lemma.

Lemma 3.1.1 *Let $\varphi(u, \mathbf{x}, \tau)$ be any (holomorphic) function: we have*

$$\Delta \left[\tau^{-\frac{l}{2}} \varphi \left(u - \frac{\|\mathbf{x}\|^2}{2\tau}, \frac{\mathbf{x}}{\tau}, -\frac{1}{\tau} \right) \right] = \tau^{-2-\frac{l}{2}} (\Delta \varphi) \left(u - \frac{\|\mathbf{x}\|^2}{2\tau}, \frac{\mathbf{x}}{\tau}, -\frac{1}{\tau} \right) .$$

Proof. It is a straightforward computation. Q.E.D

This lemma can be used to define an operator on sections of the line bundles $\mathcal{L}_{k,m}$; we do this hereafter.

Definition 3.1.2 *Let $\varphi \in J_{-k,m}$ and $\psi \in J_{-h,n}$; then we define*

$$\begin{aligned} \mathfrak{D}(\varphi) &:= \eta^{-2k-l} \Delta(\eta^{2k+l} \varphi) \in J_{-k+2,m} ; \\ \mathfrak{M}(\varphi, \psi) &:= \eta^{-2k-2h} \mathfrak{I}^* \left(d(\eta^{2k} \varphi), d(\eta^{2h} \psi) \right) \in J_{-k-h+2,m+n} . \end{aligned}$$

We will call the matrix elements $\mathfrak{M}(\varphi_i, \varphi_j)$, for two arbitrary generators (whose existence and properties are stated in Thm. 2.1.2), the **intersection elements**.

In this way we have defined two \mathbb{C} -linear maps $\mathfrak{D}, \mathfrak{M}$ of bi-graded modules which are both of bi-grading $(2, 0)$, i.e.

$$\begin{aligned} \mathfrak{D} : J_{k,m} &\mapsto J_{k+2,m} \\ \mathfrak{M} : J_{k,m} \otimes J_{h,n} &\mapsto J_{k+h+2,m+n} . \end{aligned}$$

We have still to check that the definition is well posed, namely that we have not added any singularity;

Proposition 3.1.1 For any $\varphi \in J_{k,m}$, $\psi \in J_{h,n}$, we have $\mathfrak{D}(\varphi) \in J_{k+2,m}$ and $\mathfrak{M}(\varphi, \psi) \in J_{h+k+2,m+n}$.

Proof. We must check that the results of the application of the two operators are still invariant Jacobi forms; invariance is obvious and follows from invariance of the intersection form and of the Laplacian. Also the bigradings of the resulting functions are obvious and follow from Lemma 3.1.1. We must check that $\mathfrak{D}(\varphi)$ and $\mathfrak{M}(\varphi, \psi)$ are still locally bounded functions of \mathbf{x} as $\Im(\tau) \rightarrow +\infty$; now the Dedekind's η has neither zeroes nor poles in \mathcal{H} and vanishes as $q^{1/24}$ at $q = 0$ (where $q = e^{2i\pi\tau}$). Hence, $\Delta(\eta^{2k}\varphi)$ is holomorphic and goes like $q^{\frac{k}{12}}$ as $\tau \rightarrow i\infty$, so that finally $\eta^{-2k}\Delta(\eta^{2k}\varphi)$ is well defined and holomorphic also as $\tau \rightarrow i\infty$: this proves that $\mathfrak{D}(\varphi)$ is still locally bounded. A similar reasoning also holds for \mathfrak{M} .

In passing we notice that $\mathfrak{D}(\varphi) \notin \mathbf{M}_\bullet[\varphi]$, namely it is never proportional to itself for the reason that $\mathbf{M}_2 = \{0\}$; for the same reason $\mathfrak{M}(\varphi, \psi) \notin \mathbf{M}_\bullet[\varphi\psi]$. Q.E.D

The function τ and the generators of the algebra $J_{\bullet,\bullet}$, being algebraically independent, form a set of coordinates $\{\varphi_{-1} := \tau, \varphi_1, \dots, \varphi_{l+1}\}$ in a neighborhood of the generic point of the Tits cone Ω ; they play a similar role to that of invariant polynomials of Coxeter groups. In the case of Coxeter invariants, the contravariant intersection form can be expressed in the local coordinates given by the invariant polynomials and we want to perform a similar computation in the context of Jacobi groups. To this end, the operator \mathfrak{M} and the intersection elements will be very important; we stress that the intersection elements are not the entries of the contravariant intersection form in the coordinates $\{\varphi_\mu\}_{\mu=-1,1,\dots,l+1}$ nonetheless they are useful to compute them as we show now. It follows from the definition that

$$\mathfrak{I}^* \left(d(\eta^{2k_i}\varphi_i), d(\eta^{2k_j}\varphi_j) \right) = \eta^{2k_j+2k_i} \mathfrak{M}(\varphi_i, \varphi_j) .$$

This formula shows that the intersection elements can be used to compute the intersection form.

It appears convenient to introduce a special notation which will shorten formulae; indeed, from the above formula we see that it is of advantage to consider the functions $\eta^{2k_i}\varphi_i$, $i = 1..l+1$ as coordinates (indeed, they are still algebraically independent). Therefore we introduce a **hat operator** which transforms a Jacobi form of weight k into a Jacobi form of weight zero but with some multiplier system under the metaplectic group; it will be defined by

Definition 3.1.3 For any Jacobi form $\varphi \in J_{k,m}$, its **hat-form** is defined by $\widehat{\varphi} := \eta^{-2k}\varphi$.

It is clear that the hatted-forms still are section of a line-bundle (since η never vanishes on \mathcal{H}) which now has *constant* transition functions.

Moreover the hat-operator preserves the index, and it satisfies the identity, $\widehat{\varphi\psi} = \widehat{\varphi}\widehat{\psi}$, for any $\psi, \varphi \in J_{\bullet,\bullet}$.

With this notation we can write

$$\eta^4 \widehat{\mathfrak{M}(\varphi, \psi)} = \mathfrak{I}^* \left(d(\widehat{\varphi}), d(\widehat{\psi}) \right) .$$

This means that when we have computed the intersection elements for a set of generators, by putting hats over every Jacobi forms (including the modular forms), we obtain the *tensor* \mathfrak{I}^*

expressed in the coordinates $\widehat{\varphi}_j$, which are locally well-defined on the orbit space; this will shorten much of the computations we are dealing with.

Remark 3.1.1 The explicit form of the operator \mathfrak{D} is rather simple: for any $\varphi \in J_{-k,m}$ we find

$$\mathfrak{D}\varphi = -4i\pi m(2k+l)g_1\varphi - 4i\pi m\partial_\tau\varphi + \Delta_{\mathbf{x}}\varphi .$$

We observe that if the lowest-degree homogeneous polynomial in the Taylor expansion (w.r.t. \mathbf{x}) of φ is $P(\mathbf{x})$ (possibly depending on τ), then the Taylor expansion of $\mathfrak{D}(\varphi)$ begins with $\Delta_{\mathbf{x}}(P)$.

Analogously we find for $\varphi \in J_{-k,m}$ and $\psi \in J_{-f,n}$ that

$$\mathfrak{M}(\varphi, \psi) = -2i\pi m\varphi\partial_\tau\psi - 2i\pi f\psi\partial_\tau\varphi - 4i\pi(km + fn)g_1\varphi\psi + \langle d_{\mathbf{x}}\varphi, d_{\mathbf{x}}\psi \rangle^* .$$

Again, the leading term in the Taylor expansion w.r.t. \mathbf{x} is simply $\langle d_{\mathbf{x}}P, d_{\mathbf{x}}Q \rangle$, if P, Q are the leading terms for φ, ψ .

Now suppose that we have the generator of $J_{\bullet,\bullet}$ of the minimal weight (the *lightest*), $\varphi_l \in J_{-k_l,m}$ whose existence is stated in Thm. 2.1.2: it follows from the above that $\mathfrak{D}\varphi_l \in J_{-k_l+2,m}$ if not zero; in this latter case one can prove

Proposition 3.1.2 [Prop. 13.3 in [Ka]] The kernel of the Laplacian Δ on the space \widetilde{Th}_m , for $m \geq 0$ is spanned over \mathbb{C} by the theta functions $\Theta_{\mathfrak{p},m}$, where $\mathfrak{p} \in \Lambda \bmod mP$.

Therefore if $\mathfrak{D}\varphi_j = 0$ then $\eta^{2k_j+l}\varphi_j$ is a linear combination with constant coefficients of the theta functions; this could occur because the function $\eta^{2k_j+l}\varphi_l$ transforms under the inversion $\tau \mapsto -\frac{1}{\tau}$ as a weight $l/2$ form (with some multiplier system χ which doesn't matter in this context), and all also the \mathbb{C} -linear combinations of theta functions transform as a (multiplet) of *weight* $l/2$ Jacobi forms (for details see [KP84]).

Since the leading order in the Taylor expansion w.r.t. \mathbf{x} of a generator (actually, of any Jacobi form) is an invariant polynomial, then a sufficient condition for $\mathfrak{D}\varphi_l$ not to vanish is that its leading term $P_{k_l}(\mathbf{x})$ is not a harmonic polynomial. In the analogous context of Coxeter groups, as one can check directly, for the classical root systems (namely the series A_l, B_l, C_l, D_l), all polynomials obtained by recursive application of the invariant Laplacian $\Delta_{\mathbf{x}}$ are algebraically independent as these examples show (we will use this information later).

Example 3.1.1 For A_l (see [Bo] planche I), we realize the Cartan subalgebra in \mathbb{C}^{l+1} with coordinates z_1, \dots, z_l such that $\sum_{j=1}^{l+1} z_j = 0$. The coordinates \mathbf{x} are chosen as $z_1 = x_1, z_2 = x_2 - x_1 \dots z_{l+1} = -x_l$, or the coefficients of the vector in the root basis. The W -invariant polynomial of maximal degree $l+1$ can be chosen as

$$P_{l+1}(\mathbf{x}) := \left[\prod_{j=1}^{l+1} z_j \right]_{|\Sigma z_j=0} .$$

Applying $\Delta_{\mathbf{x}}$ iteratively we obtain all the polynomials whose degree has the same parity

$$P_{l+1-2k}(\mathbf{x}) := (\Delta_{\mathbf{x}})^k P_{l+1}(\mathbf{x}) .$$

To obtain the remaining we notice that, applying the operator $\sum_{j=1}^{l+1} \frac{\partial}{\partial z_j}$ we get the invariant polynomial

$$P_l(\mathbf{x}) := \left[\sum_{j=1}^{l+1} \prod_{k \neq j} z_k \right]_{|\Sigma z_j=0} ,$$

from which we recover all the remaining applying $\Delta_{\mathbf{x}}$. We remark that an alternative definition of these polynomials which is completely equivalent (up to multiplicative constants) is the following, using a generating polynomial in an auxiliary indeterminate λ , as in

$$\mathcal{P}_{\mathbf{x}}(\lambda) := \prod_{j=1}^{l+1} (\lambda + z_j) = \lambda^{l+1} + p_2(\mathbf{x})\lambda^{l-1} + \dots + p_l(\mathbf{x})\lambda + p_{l+1}(\mathbf{x}).$$

It is useful for what is coming to rewrite the same polynomial in an equivalent form which better generalizes to the case of Jacobi forms.

To do this let us remark that the Weyl group of A_l is represented as the permutation group of $l+1$ elements acting on \mathbb{C}^{l+1} with coordinates z_1, \dots, z_{l+1} , restricted to $\Sigma := \{\sum z_j = 0\}$. Now there exist a unique (up to scalar) holomorphic vector field in \mathbb{C}^{l+1} which is orthogonal to Σ and Weyl invariant, namely

$$\mathbf{Z} := \sum_{j=1}^{l+1} \frac{\partial}{\partial z_j}.$$

This means that it is a derivation on the algebra of W invariant extended polynomials (namely all symmetric polynomials in the indeterminates z_1, \dots, z_{l+1}). We can recover the invariant polynomials for A_l as

$$\mathcal{P}_{\mathbf{x}}(\lambda) := \left[e^{\lambda \mathbf{Z}} \cdot \left(\prod_{j=1}^{l+1} z_k \right) \right]_{\Sigma, z_j=0}$$

Notice that

$$\frac{\partial^2}{\partial \lambda^2} \mathcal{P}_{\mathbf{x}}(\lambda) = -(l+1) \Delta_{\mathbf{x}} \mathcal{P}_{\mathbf{x}}(\lambda), \quad (3.1)$$

which follows from the fact that $\Delta_{\mathbf{z}} = \Delta_{\mathbf{x}} + (l+1) \partial_{\lambda}^2$: this will hold true, mutatis mutandis, also for the Jacobi forms.

As for B_l, C_l ([Bo], planches II, III), they are realized in \mathbb{C}^l and the highest degree invariant polynomial can be chosen as

$$P_{2l}(\mathbf{x}) := \prod_{j=1}^l (x_j)^2,$$

and the Laplacian is just the usual one in \mathbb{C}^l . Then it is an easy computation to show that the invariant polynomials

$$P_{2l-2k}(\mathbf{x}) := 2^{-k} (\Delta_{\mathbf{x}})^k P_{2l}(\mathbf{x})$$

are non-zero and algebraically independent. Notice that B_l, C_l have the same invariant polynomials because their Weyl groups are isomorphic and their action actually coincides in this realization; in fact [Bo] both Weyl groups act by permutation of the x_i and independent change of signs and are of order $2^l l!$.

Again we introduce the generating polynomial in two forms as we did for A_l

$$\begin{aligned} \mathcal{P}_{\mathbf{x}}(\lambda) &:= \prod_{k=1}^l (\lambda + x_k^2) = \lambda^l + P_2(\mathbf{x})\lambda^{l-1} + \dots + P_{2l-2}(\mathbf{x})\lambda + P_{2l}(\mathbf{x}) \\ \mathcal{P}_{\mathbf{x}}(\lambda) &\equiv e^{\frac{\lambda}{2} \Delta_{\mathbf{x}}} \cdot \left(\prod_{j=1}^l (x_j)^2 \right) \equiv \left(\lambda - \sum_{j=1}^l \frac{\partial}{\partial (x_j^2)} \right)^l \cdot \left(\prod_{j=1}^l (x_j)^2 \right). \end{aligned} \quad (3.2)$$

As for D_l we have exactly the same construction as before but starting with the highest polynomial

$$P_{2l-2}(\mathbf{x}) := \sum_{j=1}^l \prod_{k \neq j} (x_k)^2$$

and adding the middle degree $\tilde{P}_l = \prod_{j=1}^l x_j$, which is invariant because the Weyl group of D_l acts by permutations of the x_j and by change of an even number of signs (and hence has order $2^{l-1}l!$). Notice that $\Delta_{\mathbf{x}} \tilde{P}_l = 0$, namely it is a harmonic polynomial.

In a completely similar manner we can build the Jacobi forms starting from the lightest generator (\leftrightarrow highest degree polynomial) as we see hereafter.

3.2 The root system of type A_l : fundamental Jacobi forms.

In the case of A_l , it follows from Thm. 2.1.2 that the fundamental Jacobi forms $\varphi_0, \varphi_2, \dots, \varphi_{l+1}$ belong to the spaces (we labelled the forms according to minus their weight) $J_{-l-1+k,1}$ for $k = l+1, l-1, \dots, 0$.

If we realize the Cartan subalgebra \mathfrak{h} of sl_{l+1} as in [Bo], planche I, then Wirthmüller [Wi92] found that the lightest generator (which corresponds to a maximal degree generating polynomial in the setting of the corresponding Coxeter group) is given

$$\varphi_{l+1}(u, \mathbf{x}, \tau) := e^{2i\pi u} \prod_{j=1}^{l+1} \alpha(z_j, \tau) \Big|_{\sum z_i=0} \in J_{-l-1,1},$$

where the function α is defined by

$$\alpha(v, \tau) := \frac{\theta_1(v, \tau)}{\theta_1'(0, \tau)} = -\frac{1}{2\pi} \frac{\theta_1(v, \tau)}{\eta(\tau)^3} = v + O(v^2),$$

θ_1 being the Jacobi theta function. It enjoys the following property (from the properties of θ_1 , see e.g. [Ba])

$$\begin{aligned} \partial_\tau \alpha(x, \tau) &= \frac{1}{4i\pi} \partial_x^2 \alpha(x, \tau) - 3g_1 \alpha(x, \tau) \\ \alpha(-v, \tau) &= -\alpha(v, \tau); \quad \alpha(v+1, \tau) = -\alpha(v, \tau); \quad \alpha(v, \tau+1) = \alpha(v, \tau) \\ \alpha(v+\tau, \tau) &= -e^{-2i\pi v - i\pi\tau} \alpha(v, \tau) \Rightarrow \alpha(v+n\tau, \tau) = (-1)^n e^{-2i\pi n v - i\pi n^2 \tau} \alpha(v, \tau); \\ \alpha\left(\frac{v}{\tau}, -\frac{1}{\tau}\right) &= \frac{1}{\tau} e^{i\pi \frac{v^2}{\tau}} \alpha(v, \tau), \end{aligned} \tag{3.3}$$

We can obtain the Jacobi forms of weights $l+1-2k$ by recursive application of \mathfrak{D} . To obtain the remaining we define the function

$$\varphi_l(u, \mathbf{x}, \tau) := \sum_{k=1}^{l+1} \alpha'(z_k, \tau) \prod_{j \neq k} \alpha(z_j, \tau).$$

It is a straightforward exercise to show that this is a Jacobi form of weight $-l$.

Therefore we can now build the remaining Jacobi forms for A_l of weight $-l + 2k$ by application of \mathfrak{D}^k as in

Proposition 3.2.1 A system of generators for the algebra $J_{\bullet,\bullet}$ of the Jacobi group $J(A_l)$ is given by

$$\begin{aligned}\varphi_{l+1-2k} &:= \mathfrak{D}^k(\varphi_{l+1}) \\ \varphi_{l-2k} &:= \mathfrak{D}^k(\varphi_l),\end{aligned}$$

where φ_{l+1} and φ_l have been defined above.

Any other set of generators is a weighted *linear* combination of these with coefficients in \mathbf{M}_{\bullet} .

Proof. Since we already saw that these are Jacobi forms belonging to the spaces $J_{-l-1+2k,1}$ and $J_{-l+2k,1}$ we only have to show that they are algebraically independent: but this is promptly seen by looking at the lowest term in the Taylor expansion w.r.t. \mathbf{x} for

$$\begin{aligned}\varphi_{l+1-2k} &= e^{2i\pi u} \left(\Delta^k (P_{l+1}(\mathbf{x})) + O(\|\mathbf{x}\|^{l-2k}) \right) \\ \varphi_{l-2k} &= e^{2i\pi u} \left(\Delta^k (P_l(\mathbf{x})) + O(\|\mathbf{x}\|^{l-2k-1}) \right)\end{aligned}$$

and this suffices to show their algebraic independence.

As for the second assertion, since any other set of generators must have index $m = 1$ then it must be at most linear in these generators, and if we want that they have definite weights, the combination must be actually a weighted one. Q.E.D

Remark 3.2.1 In the setting of Coxeter groups, the set of generators of the algebra of invariant polynomials is uniquely specified up to weighted polynomial transformations; on the other hand, here we can only perform linear transformations with coefficients in the modular forms.

We can describe these forms much more concisely with the aid of a generating function. In order to show how to build it, first of all we consider the enlarged space $(u, \mathbf{z}, \tau) := (u, z_1, \dots, z_{l+1}, \tau) \in \Omega' := \mathbb{C} \oplus \mathbb{C}^{l+1} \oplus \mathcal{H}$ and the following vector field

$$\mathcal{Z} := \mathbf{Z} - 4i\pi \left(\sum_{j=1}^{l+1} z_j \right) g_1(\tau) \mathbf{E},$$

where, as before, $\mathbf{Z} := \sum_{j=1}^{l+1} \frac{\partial}{\partial z_j}$.

Lemma 3.2.1 The generators of the algebra $J_{\bullet,\bullet}$ are given by

$$\varphi_{l+1-k}(u, \mathbf{x}, \tau) := \left[\mathcal{Z}^k \left(e^{2i\pi u} \prod_{j=1}^{l+1} \alpha(z_j) \right) \right]_{|\Sigma z_j=0}.$$

Followingly, they are the coefficients of the generating function

$$\Phi_{u, \mathbf{x}, \tau}(v) := \left[e^v \mathcal{Z} \left(e^{2i\pi u} \prod_{j=1}^{l+1} \alpha(z_j, \tau) \right) \right]_{|\Sigma z_j=0}.$$

Proof. We extend the action of the Jacobi group J to the enlarged space in such a way that the complex crystallographic Weyl group acts –exactly as before– by permutation of the coordinates and translation by the root lattice, while the metaplectic group acts by

$$(u, \mathbf{z}, \tau) \mapsto \left(u + \frac{c\|\mathbf{z}\|^2}{2(c\tau + d)}, \frac{\mathbf{z}}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right),$$

where the Killing form has been extended to the obvious $ds^2 = \sum_{j=1}^{l+1} dz_j^2$. We realize that the vector \mathcal{Z} is conformally invariant of weight +1 (namely it increases by one the weight of an extended Jacobi form); this means that if $\varphi(u, \mathbf{z}, \tau)$ is an invariant function of the extended Jacobi group, then (setting $p := (l+1) \sum_1^{l+1} z_i = p(\mathbf{z})$)

$$\begin{aligned} & \left(\mathbf{Z} - 4i\pi \frac{p}{l+1} g_1(\tau) \mathbf{E} \right) \left[\varphi \left(u - \frac{\|\mathbf{z}\|^2}{2\tau}, \frac{\mathbf{z}}{\tau}, -\frac{1}{\tau} \right) \right] = \\ &= -\frac{2i\pi}{l+1} \frac{p}{\tau} (\mathbf{E}\varphi)(*) + \frac{1}{\tau} (\mathbf{Z}\varphi)(*) - 4i\pi \frac{p}{l+1} \left[\frac{\eta'(-1/\tau)}{\eta(-1/\tau)} \frac{1}{\tau^2} - \frac{1}{2\tau} \right] (\mathbf{E}\varphi)(*) = \\ &= \frac{1}{\tau} \left[\left(\mathbf{Z} - \frac{4i\pi}{l+1} p \left(\frac{\mathbf{z}}{\tau} \right) \frac{\eta'(-1/\tau)}{\eta(-1/\tau)} \mathbf{E} \right) \varphi \right] (*). \end{aligned}$$

One may ask why we put a term proportional to $p = (l+1) \sum_1^{l+1} z_i$ since in the end we want to restrict to $p = 0$: the reason is that when we apply more than once the operator \mathcal{Z} and do not include the term we get a non J -covariant result as we see in this

Counterexample. Consider the push-forward under the map \mathfrak{S} of the second iterate of \mathbf{Z} :

$$\begin{aligned} & \mathbf{Z} \left(\mathbf{Z} \left(\varphi \left(u - \frac{\|\mathbf{z}\|^2}{\tau}, \frac{\mathbf{z}}{\tau}, -\frac{1}{\tau} \right) \right) \right) = \mathbf{Z} \left(-\frac{1}{\tau} v(l+1) (\partial_u \varphi)(*) + \frac{1}{\tau} (\mathbf{Z}\varphi)(*) \right) = \\ &= -\frac{l+1}{\tau} \partial_u \varphi(*) + \frac{v^2(l+1)^2}{\tau^2} \partial_u^2 \varphi(*) - 2 \frac{p(l+1)}{\tau} (\mathbf{Z}\varphi)(*) + \frac{1}{\tau^2} (\mathbf{Z}^2 \varphi)(*), \end{aligned}$$

where $*$ stands for the point $\left(u - \frac{\|\mathbf{z}\|^2}{\tau}, \frac{\mathbf{z}}{\tau}, -\frac{1}{\tau} \right)$, and we see that even restricting on $p = 0$ we get

$$\mathbf{Z}(\mathbf{Z}(\varphi(*))) = +\frac{1}{\tau^2} (\mathbf{Z}^2 \varphi)(*) - 2i\pi \frac{l+1}{\tau} \mathbf{E}\varphi(*).$$

We see that the vector \mathcal{Z} commutes with \mathbf{E} , hence preserves the index; we have a natural Jacobi form on the enlarged space which is simply $e^{2i\pi u} \prod_{j=1}^{l+1} \alpha(z_j)$. Since the conformal weight of \mathcal{Z} is +1 we can write alternatively the fundamental Jacobi forms by means of the simpler formula

$$\varphi_{l+1-k}(u, \mathbf{x}, \tau) := \left[\mathcal{Z}^k \left(e^{2i\pi u} \prod_{j=1}^{l+1} \alpha(z_j) \right) \right]_{|\Sigma z_j=0}.$$

We should check that with this definition they are not algebraically dependent, but again this is obtained by looking at the lowest order in the Taylor expansion w.r.t. \mathbf{x} . This proves that the functions defined in the Lemma are a set of generators.

Next, in analogy with the case of the Weyl-invariant polynomials, it is useful to introduce the generating function, which obviously has the form

$$\Phi_{u,\mathbf{x},\tau}(v) := \left[e^{v\mathcal{Z}} \prod_{j=1}^{l+1} \alpha(z_j, \tau) \right]_{|\Sigma z_i=0} = \varphi_{l+1} + v\varphi_l + v^2\varphi_{l-1} + \dots + v^{l+1}\varphi_0 + \dots ;$$

by $e^{v\mathcal{Z}}$ we mean the flow generated by \mathcal{Z} on the extended cone Ω' . Q.E.D

We remark that the series does not stop, but, by virtue of the structure of the algebra of Jacobi forms, the higher terms are polynomial combinations of these with coefficients in \mathbf{M}_\bullet . We will re-sum the series afterwards, while now it is useful to write down explicitly the generating function; to do this we must integrate the flow of \mathcal{Z} on Ω' . We could perform this straightforward computation in a direct way, but it is interesting to point out that \mathcal{Z} is a covariantly flat vector for a suitable *flat* metric on $T(\Omega')$ (the tangent bundle). This metric extends the intersection form of $T(\Omega)$ under the natural embedding $\Omega \hookrightarrow \Omega'$ and it is worked out in the following section. After finding the flat coordinates of this extended intersection form, we will have also the integration of the flow of \mathcal{Z} by shifting the coordinate along \mathcal{Z} .

The extended intersection form will be used later to compute the intersection elements.

3.2.1 Extension of the intersection form

We are to build a metric on $T(\Omega')$ which extends the intersection form on $T\Omega$; we ask the following conditions on the extended intersection form:

- i) the extended metric must be *flat*;
- ii) it must coincide with the previous one when restricted on the hypersurface $\sum_{j=1}^{l+1} z_j = 0$.

To this end we introduce the coordinates on \mathbb{C}^{l+1} as

$$\begin{cases} z_1 = x_1 + p \\ z_2 = x_2 - x_1 + p \\ \dots \\ z_{l+1} = -x_l + p \end{cases}$$

In a concise form we have $\mathbf{z} = \sum_{j=1}^l x_j \alpha_j^\vee + p[1, 1, \dots, 1]$: in the following we will often write $z_i(\mathbf{x}, p)$ or $z_i(\mathbf{x}) := z_i(\mathbf{x}, 0)$.

In these coordinates the flat metric $\sum^{l+1} dz_i^2$ becomes

$$dl^2 = \langle d\mathbf{x}, d\mathbf{x} \rangle + (l+1)dp^2 .$$

The most easy form for the extended metric suitable for our purposes is

$$\tilde{\mathcal{J}} = -du \otimes d\tau - d\tau \otimes du + dl^2 + pB(\tau)(dp \otimes d\tau + d\tau \otimes dp) + p^2 B'(\tau) d\tau^2$$

One can check directly that the curvature vanishes for any choice of the function $B(\tau)$ (whose explicit form will be fixed to our convenience later), but it is sufficient to introduce the new coordinate

$$s := u - \frac{1}{2}p^2 B(\tau)$$

and the metric becomes

$$\tilde{\mathcal{J}} = -ds \otimes d\tau - d\tau \otimes ds + dl^2 .$$

If we choose

$$B(\tau) := -\frac{2}{l+1}g_1(\tau) ,$$

we obtain that the previously introduced vector field \mathcal{Z} in the coordinates (s, \mathbf{x}, p, τ) now reads

$$\mathcal{Z} = \sum_{j=1}^{l+1} \frac{\partial}{\partial z_j} - \frac{2}{l+1}g_1(\tau)p\partial_u = \partial_p .$$

The vector field $\mathcal{T} := \partial_\tau$ of these coordinates reads, in the old ones u, \mathbf{z}, τ as

$$\mathcal{T} = -\frac{p^2}{l+1}(g_1(\tau))'\partial_u + \partial_\tau , \quad (3.4)$$

and the extended intersection form $\tilde{\mathcal{J}}^*$ reads

$$\tilde{\mathcal{J}}^* = -\mathcal{T} \otimes \mathcal{Z} - \mathcal{Z} \otimes \mathcal{T} + \sum_{j=1}^{l+1} \frac{\partial}{\partial z_j} \otimes \frac{\partial}{\partial z_j} . \quad (3.5)$$

Now we can integrate easily the flow generated by \mathcal{Z} simply shifting p with constant s ; after these computation we can rewrite the generating function of the Jacobi forms as in

Proposition 3.2.2 The generating function can be written as

$$\Phi_{u, \mathbf{x}, \tau}(v) := \left[e^{v\mathcal{Z}} \prod_{j=1}^{l+1} \alpha(z_j, \tau) \right]_{|\Sigma z_i=0} = e^{2i\pi(u-(l+1)v^2g_1(\tau))} \prod_{j=1}^{l+1} \alpha(z_j(\mathbf{x}, v), \tau) .$$

It follows from the transformation rules of α that the generating function enjoys the following properties under the Jacobi group:

$$\Phi \left(u - \frac{\|\mathbf{x}\|^2}{2\tau}, \frac{\mathbf{x}}{\tau}, v, -\frac{1}{\tau} \right) = \tau^{-l-1} \Phi(u, \mathbf{x}, \tau v, \tau)$$

$$\Phi(u, \mathbf{x}, v, \tau + 1) = \Phi(u, \mathbf{x}, v, \tau)$$

$$\Phi \left(u + \langle \mu, \mathbf{x} \rangle + \frac{\tau}{2}\|\mu\|^2, \mathbf{x} + \tau\mu + \mathbf{t}, v, \tau \right) = \Phi(u, \mathbf{x}, v, \tau)$$

$$\Phi(u, \mathbf{x}, v + 1, \tau) = (-1)^{l+1} \Phi(u, \mathbf{x}, v, \tau)$$

$$\Phi(u, \mathbf{x}, v + \tau, \tau) = (-e^{-2i\pi v - i\pi\tau})^{l+1} \Phi(u, \mathbf{x}, v, \tau) .$$

Remark 3.2.2 The function $v^{-l-1}\Phi_{u,\mathbf{x},\tau}(v)$ is an invariant function of index 1 on the total space of the line-bundle as in remark (2.1.1); notice that it has a pole of order $l+1$ in v as a consequence of the fact that the spectrum of the weighting operator \mathcal{K} on the subspace $J_{\bullet,1}$ is $-l-1, -l, \dots, -2, 0, 1, \dots$.

It will be useful later to consider the function $\lambda(v) := \sigma^{-l-1}(v)\Phi_{u,\mathbf{x},\tau}(v) = \alpha^{-l-1}(v)e^{2i\pi u} \prod_{j=1}^{l+1} \alpha(z_j(\mathbf{x}, v))$, which has the same singular tail (in v) but is also invariant under the complex lattice $\mathbb{Z} + \tau\mathbb{Z}$.

3.2.2 The generating function for A_l : re-summation of the series.

As we saw, the first $l+1$ coefficients in the Taylor expansion w.r.t. v of the extended Jacobi form

$$\Phi(u, \mathbf{x}, v, \tau) := \Phi_{u,\mathbf{x},\tau}(v) = \left[e^{v\mathcal{Z}} \left(e^{2i\pi u} \prod_{j=1}^{l+1} \alpha(z_j, \tau) \right) \right]_{\sum z_j=0} = e^{2i\pi u - 2i\pi v^2(l+1)\frac{\eta'}{\eta}} \prod_{k=1}^{l+1} \alpha(z_k(\mathbf{x}) + v, \tau)$$

provide us with the desired fundamental Jacobi forms. Unlikely to the case of Coxeter invariants, this generating function is not just a polynomial in the auxiliary variable (in this case v) but a full series. It is therefore useful to re-sum this series in order also to analyze its coefficients.

We realize that the generating function is a classical theta function belonging to Th_1^W for the lattice of A_l depending on the parameters v, τ . Since we know from Thm. 2.1.2 and 2.1.1, that the Jacobi forms span Th_1^W we can write the following equality

$$\Phi(s, \mathbf{x}, v, \tau) = e^{-2i\pi(l+1)v^2 g_1(\tau)} \sum_{k=0}^{l+1} C_k(v, \tau) \varphi_{l+1-k}(s, \mathbf{x}, \tau)$$

and we must find the coefficients $C_k(v, \tau)$. If we analyze the modular properties of Φ , we promptly find that the coefficients $C_k(v, \tau)$ are Jacobi forms of weight $-k$ for the lattice $\mathbb{C} \bmod 2(\mathbb{Z} + \tau\mathbb{Z})$, namely

$$\begin{aligned} C_k(v+1, \tau) &= (-1)^{l+1} C_k(v, \tau); \\ C_k(v+\tau, \tau) &= (-1)^{l+1} e^{-2i\pi v - i\pi(l+1)\tau} C_k(v, \tau); \\ C_k\left(\frac{v}{\tau}, -\frac{1}{\tau}\right) &= \tau^{-k} e^{i\pi(l+1)\frac{v^2}{\tau}} C_k(v, \tau). \end{aligned}$$

The formula which describes concisely the generating function is contained in the

Theorem 3.2.1 Up to normalization, the generators of the algebra of Jacobi forms of type A_l already defined recursively in Prop. 3.2.1, are given by the generating function

$$\Phi_{u,\mathbf{x},\tau}(v) := e^{2i\pi u - 2i\pi(l+1)v^2 g_1} \prod_{j=1}^{l+1} \alpha(z_j(\mathbf{x}) - v) = \sigma^{l+1}(v) \sum_{j=0}^{l+1} \frac{(-1)^{l-1-k}}{(l-k)!} \wp^{(k-2)}(v) \varphi_k(u, \mathbf{x}, \tau) =$$

$$\begin{aligned}
& \det \begin{pmatrix} 1 & \wp(v) & \wp'(v) & \dots & \wp^{(l-1)}(v) \\ 1 & \wp(z_1) & \wp'(z_1) & \dots & \wp^{(l-1)}(z_1) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \wp(z_l) & \wp'(z_l) & \dots & \wp^{(l-1)}(z_l) \end{pmatrix} \\
= & \frac{-1}{l!} \frac{\det \begin{pmatrix} 1 & \wp(z_1) & \wp'(z_1) & \dots & \wp^{(l-2)}(z_1) \\ 1 & \wp(z_2) & \wp'(z_2) & \dots & \wp^{(l-2)}(z_2) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \wp(z_l) & \wp'(z_l) & \dots & \wp^{(l-2)}(z_l) \end{pmatrix}}{\det \begin{pmatrix} 1 & \wp(z_1) & \wp'(z_1) & \dots & \wp^{(l-1)}(z_1) \\ 1 & \wp(z_2) & \wp'(z_2) & \dots & \wp^{(l-1)}(z_2) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \wp(z_l) & \wp'(z_l) & \dots & \wp^{(l-1)}(z_l) \end{pmatrix}} \sigma^{l+1}(v) \prod_{j=0}^{l+1} \alpha(z_j(\mathbf{x})) . \quad (3.6)
\end{aligned}$$

Proof. We recall a classical formula which can be found in [Du93] pag. 199. If $\sum_1^{l+1} z_i = 0$ then

$$\frac{\prod_1^{l+1} \alpha(z_i - v)}{\alpha^{l+1}(v) \prod_1^{l+1} \alpha(z_j)} = \frac{\prod_1^{l+1} \sigma(z_i - v)}{\sigma^{l+1}(v) \prod_1^{l+1} \sigma(z_j)} = \frac{-1}{l!} \frac{\det \begin{pmatrix} 1 & \wp(v) & \wp'(v) & \dots & \wp^{(l-1)}(v) \\ 1 & \wp(z_1) & \wp'(z_1) & \dots & \wp^{(l-1)}(z_1) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \wp(z_l) & \wp'(z_l) & \dots & \wp^{(l-1)}(z_l) \end{pmatrix}}{\det \begin{pmatrix} 1 & \wp(z_1) & \wp'(z_1) & \dots & \wp^{(l-2)}(z_1) \\ 1 & \wp(z_2) & \wp'(z_2) & \dots & \wp^{(l-2)}(z_2) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \wp(z_l) & \wp'(z_l) & \dots & \wp^{(l-2)}(z_l) \end{pmatrix}} .$$

Expanding the determinant w.r.t. the first row of the determinant in the numerator and rearranging terms we promptly get

$$\begin{aligned}
e^{2i\pi u} \prod_1^{l+1} \alpha(z_i(\mathbf{x}) - v) &= \left[\frac{(-1)^{l-1}}{l!} \alpha^{l+1}(v) \wp^{(l-1)}(v) \right] \varphi_{l+1}(\mathbf{x}) + \left[\frac{(-1)^{l-2}}{(l-1)!} \alpha^{l+1}(v) \wp^{(l-2)}(v) \right] \varphi_l(\mathbf{x}) + \\
&+ \dots + \left[\alpha^{l+1}(v) \wp(v) \right] \varphi_2(\mathbf{x}) + \alpha^{l+1}(v) \varphi_0(\mathbf{x}) = \\
&:= C_0(v) \varphi_{l+1}(\mathbf{x}) + C_1(v) \varphi_l(\mathbf{x}) + \dots + C_{l-1}(v) \varphi_2(\mathbf{x}) + C_{l+1}(v) \varphi_0(\mathbf{x}). \\
C_k(v) &:= \frac{(-1)^{l-1-k}}{(l-k)!} \alpha^{l+1}(v) \wp^{(l-1-k)}(v) \quad (3.7)
\end{aligned}$$

In this formula the coefficients of the fundamental Jacobi forms are chosen in such a way that the leading term in the v expansion in front of the Jacobi form φ_{l+1-k} is v^k .

In order to be sure that these φ_k are really the Jacobi forms we are looking for, we first notice that they have no singularities because the LHS in (3.7) is regular and the coefficients C_k are linearly independent.

Moreover since the coefficients C_k do satisfy the transformation rule

$$C_k \left(\frac{v}{\tau}, -\frac{1}{\tau} \right) = \tau^{-k} C_k(v, \tau)$$

and the LHS is of weight $-l-1$ under the map $\mathbf{x} \mapsto \frac{\mathbf{x}}{\tau}$, $v \mapsto \frac{v}{\tau}$, $\tau \mapsto -\frac{1}{\tau}$, it follows by counting the weight that φ_k is of weight $-k$ as well. We now have to check that we exactly re-summed the series

in Prop. 3.2.2: first of all notice that $\wp(v) = \frac{1}{v^2} + \text{regular}$ and hence for $k = 0, 1, 2, \dots, l-1, l+1$ we find

$$\begin{aligned} \wp^{(k)}(v) &= (-1)^k k! v^{-k-2} + \text{regular} \downarrow \\ e^{-2i\pi(l+1)v^2 g_1} C_k(v, \tau) &= \left\{ v^{-l-1+k} + O(1) \right\} (v^{l+1} + O(v^{l+3})) = v^k + O(v^{l+1}) \end{aligned}$$

and therefore

$$\left[\partial_v^j e^{-2i\pi(l+1)v^2 g_1} C_k(v, \tau) \right]_{|v=0} = \delta_{jk}, \quad k, j = 1, 2, \dots, l-1, l+1.$$

This proves that we actually gave the re-summation of the previous series and that the above functions φ_k are indeed the searched Jacobi forms. Q.E.D

Remark 3.2.3 As in the case of the generating function for the invariant polynomials (eq. 3.1) we have for the “gauge transformed” function $\mathcal{P}(v) := e^{2i\pi(l+1)v^2 g_1} \Phi(v)$ (see next section)

$$\frac{d^2}{dv^2} \mathcal{P}_{u, \mathbf{x}, \tau}(v) = -(l+1) \mathfrak{D}_{u, \mathbf{x}, \tau} \mathcal{P}_{u, \mathbf{x}, \tau}(v) + 4i\pi(l+1)g_1 \mathcal{P}_{u, \mathbf{x}, \tau}(v) \quad (3.8)$$

where we have put a subscript to the operator \mathfrak{D} to stress on which variables it acts. This formula follows from the fact that (recall $\mathbf{z} = \mathbf{z}(\mathbf{x}, v)$)

$$\tilde{\mathfrak{D}}(\mathcal{P}(u, \mathbf{z}, \tau)) := \eta^{-3l-3} \tilde{\Delta}(\eta^{3l+3} \mathcal{P}(v)) = -4i\pi \left(\partial_\tau + (3l+3)g_1 + \sum_{j=1}^{l+1} \frac{\partial^2}{\partial z_j^2} \right) e^{2i\pi u} \prod_{k=1}^{l+1} \alpha(z_k) = 0,$$

which is a consequence of the properties of $\alpha(z, \tau)$. But now, expressing the Laplacian in the coordinates \mathbf{x}, v we get

$$0 = \tilde{\mathfrak{D}}(\mathcal{P}(u, \mathbf{z}, \tau)) = \left(-4i\pi g_1 + \frac{1}{l+1} \frac{\partial^2}{\partial v^2} \right) \mathcal{P}(u, \mathbf{z}(\mathbf{x}, v), \tau) + \mathfrak{D}_{u, \mathbf{x}, \tau}(\mathcal{P}(u, \mathbf{z}(\mathbf{x}, v), \tau)),$$

which proves eq. 3.8. The comparison with the formula 3.1 has to be made considering the limit (genus 0 limit)

$$\lim_{\epsilon \rightarrow 0} \epsilon^{-l+1} (\text{Eq. 3.8})_{\epsilon u, \epsilon \mathbf{x}, \tau}(\epsilon v) = (\text{Eq. 3.1}).$$

The formula 3.8 can be written in a very concise form; recalling that $\mathcal{P}(v) = \sum_{j=0}^{l+1} C_{l+1-k}(v, \tau) \varphi_k(u, \mathbf{x}, \tau)$, we have

$$\tilde{\mathfrak{D}}\mathcal{P} = \sum_{j=0}^{l+1} [\mathfrak{D}(C_{l+1-k}) \varphi_k + C_{l+1-k} \mathfrak{D}(\varphi_k)],$$

where we have defined

$$\mathfrak{D}(C_k) := -4i\pi \eta^{-2k-1} \partial_\tau (\eta^{2k+1} C_k) + \frac{1}{l+1} \partial_v^2 C_k.$$

3.2.3 Computation of the intersection form with the generating function

In this paragraph we compute the generating function of the matrix elements of the intersection form in terms of our fundamental Jacobi forms. This computation is the translation in the present

setting of the analog formula by Saito, Yano and Sekiguchi for the polynomial invariants in [SYS80]. To simplify the computational steps we will consider the functions

$$\begin{aligned} \mathcal{P}(u, z_1 \dots z_{l+1}) &:= e^{2i\pi u} \prod_1^{l+1} \alpha(z_i) = e^{2i\pi(l+1)v^2 g_1(\tau)} \Phi(v) \\ \lambda(v) &:= \varphi_0 + \wp(v)\varphi_2 - \frac{1}{2}\wp'(v)\varphi_3 + \dots + \frac{(-1)^{l-1}}{l!}\wp^{(l-1)}(v)\varphi_{l+1} = \alpha^{-l-1}(v)\mathcal{P}(v), \end{aligned} \quad (3.9)$$

with no particular relation between the z_i 's. The first one is clearly related to the previous generating function Φ by a *gauge* transformation

$$\mathcal{P}(u, \mathbf{z}(\mathbf{x}, v), \tau) := e^{2i\pi(l+1)v^2 \frac{\eta'}{\eta}} \Phi(u, \mathbf{x}, v, \tau) = e^{2i\pi u} \prod_{i=1}^{l+1} \alpha(z_i(\mathbf{x}, v)).$$

We now prove the

Theorem 3.2.2 The intersection elements $\mathfrak{M}(\varphi_k, \varphi_j)$ are recovered from the generating function

$$\begin{aligned} &\sum_{k,j=0}^{l+1} \frac{(-1)^{k+j}}{(k-1)!(j-1)!} \wp^{(k-2)}(v) \wp^{(j-2)}(v') \mathfrak{M}(\varphi_k, \varphi_j) = \\ &= 2i\pi (\mathbb{D}^\bullet \lambda(v) \lambda(v') + \lambda(v) \mathbb{D}^\bullet \lambda(v')) - \frac{1}{l+1} \lambda'(v) \lambda'(v') + \\ &+ \frac{1}{2} \frac{\wp'(v) + \wp'(v')}{\wp(v) - \wp(v')} \left(\lambda(v) \frac{d}{dv'} \lambda(v') - \lambda(v') \frac{d}{dv} \lambda(v) \right), \end{aligned} \quad (3.10)$$

where $\mathbb{D}_\tau^\bullet \lambda(v) := \sum_{j=0}^{l+1} \frac{(-1)^{j-1}}{(j-1)!} \mathbb{D}(\wp^{(j-2)}(v)) \varphi_j$ (see appendix for the definition of \mathbb{D}).

In practice, to find the coefficients $\mathfrak{M}(\varphi_k, \varphi_j)$ we have to multiply both sides by $\wp(v) - \wp(v')$ and compare the Laurent expansions w.r.t. v, v' (see example for A_2 later).

The proof of this short formula involves many steps and lemmata.

Lemma 3.2.2 For the extended intersection operator $\widetilde{\mathfrak{M}}$ we have

$$\begin{aligned} &e^{2i\pi(l+1)(v^2+v'^2)g_1(\tau)} \widetilde{\mathfrak{M}}(\Phi(v), \Phi(v')) = \\ &= 2i\pi(l+1) \frac{\nabla_\tau \alpha(v-v')}{\alpha(v-v')} \mathcal{P}(v) \mathcal{P}(v') + \frac{\alpha'(v-v')}{\alpha(v-v')} \left\{ \mathcal{P}(v) \frac{d}{dv'} \mathcal{P}(v') - \mathcal{P}(v') \frac{d}{dv} \mathcal{P}(v) \right\}. \end{aligned} \quad (3.11)$$

Proof. The extended intersection form is given by (see formula (3.5))

$$\widetilde{\mathcal{J}}^* = -\mathcal{T} \otimes \mathcal{Z} - \mathcal{Z} \otimes \mathcal{T} + \sum_{j=1}^{l+1} \frac{\partial}{\partial z_j} \otimes \frac{\partial}{\partial z_j},$$

and the extended intersection operator $\widetilde{\mathfrak{M}}$ is defined accordingly as

$$\widetilde{\mathfrak{M}}(\Phi(v), \Phi(v')) = \eta^{-4(l+1)} \widetilde{\mathcal{J}}^* \left(d \left(\eta^{2l+2} \Phi(v) \right), d \left(\eta^{2l+2} \Phi(v') \right) \right).$$

Since the “gauge-transformed” vectors \mathcal{Z} and \mathcal{T} (the latter being defined in (3.4)) are

$$e^{+2i\pi(l+1)v^2 g_1} \mathcal{Z} e^{-2i\pi(l+1)v^2 g_1} = \sum_1^{l+1} \frac{\partial}{\partial z_i} = \partial_v$$

$$e^{+2i\pi(l+1)v^2 g_1} \mathcal{T} e^{-2i\pi(l+1)v^2 \frac{g_1'}{\eta}} = \partial_\tau$$

we have that for the function $\mathcal{P}(v) := \mathcal{P}(u, \mathbf{z}(\mathbf{x}, p + v), \tau) := \mathcal{P}_{u, \mathbf{z}, \tau}(v)$ (where we consider the z_j unconstrained, and v a parameter of deformation) the *gauge transformed* intersection element reads

$$e^{2i\pi(l+1)(v^2 + v'^2)g_1(\tau)} \widetilde{\mathfrak{M}}(\Phi(v), \Phi(v')) = -\partial_u \mathcal{P}(v) \nabla_\tau \mathcal{P}(v') - \partial_u \mathcal{P}(v') \nabla_\tau \mathcal{P}(v) + \sum_{j=1}^{l+1} \frac{\partial \mathcal{P}(v)}{\partial z_j} \frac{\partial \mathcal{P}(v')}{\partial z_j}.$$

We now compute

$$\begin{aligned} \text{lhs of (3.11)} &= \\ &= -2i\pi \eta^{-2l-2} \left[\mathcal{P}(v) \partial_\tau \left(\eta^{2l+2} \mathcal{P}(v') \right) + \mathcal{P}(v') \partial_\tau \left(\eta^{2l+2} \mathcal{P}(v) \right) \right] + \sum_1^{l+1} \frac{\partial}{\partial z_i} \mathcal{P}(v) \frac{\partial}{\partial z_i} \mathcal{P}(v') = \\ &= -\frac{1}{2} \sum_1^{l+1} \left(\frac{\alpha''(z_i)}{\alpha(z_i)} + \frac{\alpha''(w_i)}{\alpha(w_i)} - 2 \frac{\alpha'(z_i) \alpha'(w_i)}{\alpha(z_i) \alpha(w_i)} \right) \mathcal{P}(v) \mathcal{P}(v') + 4i\pi(l+1)g_1 \mathcal{P}(v) \mathcal{P}(v'), \end{aligned}$$

where $z_i := z_i(\mathbf{x}, p + v)$ and $w_i := z_i(\mathbf{x}, p + v')$. We now must analyze the terms in round brackets: using the formulae in Prop. A.3.1, the intersection form becomes (recall that $z_i - w_i = v - v'$, $\forall i = 1..l+1$)

$$\begin{aligned} \text{lhs of (3.11)} &= \\ &= 2i\pi(l+1) \left[\frac{\partial_\tau \alpha(v-v')}{\alpha(v-v')} + 2g_1 \right] \mathcal{P}(v) \mathcal{P}(v') - \frac{\alpha'(v-v')}{\alpha(v-v')} \left\{ \mathcal{P}(v') \frac{d}{dv} \mathcal{P}(v) - \mathcal{P}(v) \frac{d}{dv'} \mathcal{P}(v') \right\} = \\ &= 2i\pi(l+1) \frac{\nabla_\tau \alpha(v-v')}{\alpha(v-v')} \mathcal{P}(v) \mathcal{P}(v') + \frac{\alpha'(v-v')}{\alpha(v-v')} \left\{ \mathcal{P}(v) \frac{d}{dv'} \mathcal{P}(v') - \mathcal{P}(v') \frac{d}{dv} \mathcal{P}(v) \right\}. \end{aligned}$$

This ends the proof of lemma. Q.E.D

Although very concise, this formula gives the intersection elements of the extended intersection operator \mathfrak{M} : this is insufficient to our purposes since we want to analyze the intersection operator \mathfrak{M} .

In order to find the intersection elements $\mathfrak{M}(\varphi_j, \varphi_k)$, we have the further lemma.

Lemma 3.2.3 *For the intersection elements of \mathfrak{M} we have the formula*

$$\begin{aligned} &\sum_{k,j} C_k(v) C_j(v') \mathfrak{M}(\varphi_{l+1-k}, \varphi_{l+1-j}) = \\ &= 2i\pi(l+1) \frac{\nabla_\tau \alpha(v-v')}{\alpha(v-v')} \mathcal{P}(v) \mathcal{P}(v') + \frac{\alpha'(v-v')}{\alpha(v-v')} \left\{ \mathcal{P}(v) \frac{d}{dv'} \mathcal{P}(v') - \mathcal{P}(v') \frac{d}{dv} \mathcal{P}(v) \right\} + \\ &- \sum_{k,j} \mathfrak{M}(C_k(v), C_j(v')) \varphi_{l+1-k} \varphi_{l+1-j}, \end{aligned} \tag{3.12}$$

where $C_l(v)$ are defined in (3.7) and we have set for short

$$\mathfrak{M}(C_k(v), C_j(v')) := -2i\pi (C_k(v)\eta^{-2j}\partial_\tau(\eta^{2j}C_j(v')) + C_j(v')\eta^{-2k}\partial_\tau(\eta^{2k}C_k(v))) + \frac{1}{l+1}C'_k(v)C'_j(v').$$

Proof. From the definitions, by a straightforward rearrangement of terms, we have

$$\begin{aligned} e^{2i\pi(l+1)(v^2+v'^2)g_1}\widetilde{\mathfrak{M}}(\Phi(v), \Phi(v')) &= \sum_{k,j} C_k(v)C_j(v')\mathfrak{M}(\varphi_{l+1-k}, \varphi_{l+1-j}) + \\ &+ \sum_{k,j} \mathfrak{M}(C_k(v), C_j(v'))\varphi_{l+1-k}\varphi_{l+1-j}, \end{aligned} \quad (3.13)$$

where we have set for short (as in the statement of lemma)

$$\begin{aligned} \mathfrak{M}(C_k(v), C_j(v')) &:= -2i\pi \left(C_k(v)\eta^{-2j}\partial_\tau(\eta^{2j}C_j(v')) + C_j(v')\eta^{-2k}\partial_\tau(\eta^{2k}C_k(v)) \right) + \\ &+ \frac{1}{l+1}C'_k(v)C'_j(v'). \end{aligned}$$

We can now recast eq. (3.13) using eq. (3.11) into the formula in the statement of the lemma, which is now proved. Q.E.D

Proof of Theorem 3.2.2 We use the function $\lambda(v)$ defined here below

$$\begin{aligned} \lambda(v) &:= \varphi_0 + \wp(v)\varphi_2 - \frac{1}{2}\wp'(v)\varphi_3 + \dots + \frac{(-1)^{l-1}}{l!}\wp^{(l-1)}(v)\varphi_{l+1} = \alpha^{-l-1}(v)\mathcal{P}(v), \\ \mathcal{P}'(v) &= \alpha^{l+1}(v) \left[\lambda'(v) + (l+1)\frac{\alpha'(v)}{\alpha(v)}\lambda(v) \right]. \end{aligned}$$

Rewriting the formula in Lemma 3.2.2 for this $\lambda(v)$, we have

$$\begin{aligned} &\frac{e^{2i\pi(l+1)(v^2+v'^2)g_1}\widetilde{\mathfrak{M}}(\Phi(v), \Phi(v'))}{\alpha^{l+1}(v)\alpha^{l+1}(v')} = \\ &= (l+1) \underbrace{\left\{ 2i\pi \frac{\nabla_\tau \alpha(v-v')}{\alpha(v-v')} - \frac{\alpha'(v-v')}{\alpha(v-v')} \left[\frac{\alpha'(v)}{\alpha(v)} - \frac{\alpha'(v')}{\alpha(v')} \right] \right\}}_{:=Q} \lambda(v)\lambda(v') + \\ &+ \frac{\alpha'(v-v')}{\alpha(v-v')} \left(\lambda(v) \frac{d}{dv'} \lambda(v') - \lambda(v') \frac{d}{dv} \lambda(v) \right). \end{aligned} \quad (3.14)$$

On the other hand we can compute¹, recalling the definition of the modular connection ∇_τ^\bullet and the elliptic connection \mathbb{D}^\bullet defined in Appendix, Prop. A.1,

$$\begin{aligned}
& \sum_{k,j} \frac{\mathfrak{M}(C_k(v), C_j(v'))}{\alpha^{l+1}(v)\alpha^{l+1}(v')} \varphi_{l+1-k}\varphi_{l+1-j} = \\
& = -2i\pi (\nabla_\tau^\bullet \lambda(v)\lambda(v') + \lambda(v)\nabla_\tau^\bullet \lambda(v')) - 2i\pi(l+1) \left[4g_1(\tau) + \frac{\partial_\tau \alpha(v)}{\alpha(v)} + \frac{\partial_\tau \alpha(v')}{\alpha(v')} \right] + \\
& + \frac{1}{l+1} \left[\lambda'(v) + (l+1)\frac{\alpha(v)}{\alpha(v)}\lambda(v) \right] \left[\lambda'(v') + (l+1)\frac{\alpha(v')}{\alpha(v')}\lambda(v') \right] = \\
& = -2i\pi (\nabla_\tau^\bullet \lambda(v)\lambda(v') + \lambda(v)\nabla_\tau^\bullet \lambda(v')) + \\
& \quad \quad \quad = -Q \text{ by means of eq. (A.2)} \\
& - (l+1) \overbrace{\left[8i\pi g_1(\tau) + 2i\pi \left(\frac{\partial_\tau \alpha(v)}{\alpha(v)} + \frac{\partial_\tau \alpha(v')}{\alpha(v')} \right) - \frac{\alpha'(v')\alpha'(v)}{\alpha(v')\alpha(v)} \right]} \lambda(v)\lambda(v') + \\
& + \frac{1}{l+1} \lambda'(v)\lambda'(v') + \frac{\alpha'(v)}{\alpha(v)}\lambda(v)\lambda'(v') + \frac{\alpha'(v')}{\alpha(v')}\lambda(v')\lambda'(v) = \\
& = -2i\pi (\nabla_\tau^\bullet \lambda(v)\lambda(v') + \lambda(v)\nabla_\tau^\bullet \lambda(v')) + (l+1)Q\lambda(v)\lambda(v') + \frac{1}{l+1}\lambda'(v)\lambda'(v') + \\
& + \frac{\alpha'(v)}{\alpha(v)}\lambda(v)\lambda'(v') + \frac{\alpha'(v')}{\alpha(v')}\lambda(v')\lambda'(v) = \\
& = -2i\pi [\lambda(v)\mathbb{D}^\bullet \lambda(v') + \lambda(v')\mathbb{D}^\bullet \lambda(v)] + (l+1)Q\lambda(v)\lambda(v') + \frac{1}{l+1}\lambda'(v)\lambda'(v') + \\
& + \left[\frac{\alpha'(v)}{\alpha(v)} - \frac{\alpha'(v')}{\alpha(v')} \right] \left(\lambda(v)\frac{d}{dv'}\lambda(v') - \lambda(v')\frac{d}{dv}\lambda(v) \right). \tag{3.15}
\end{aligned}$$

Therefore we finally find (for compactness we define $\wp^{(-2)}(v) := 1$ and $\wp^{(-1)}(v) := 0$)

$$\begin{aligned}
& \sum_{k,j=0}^{l+1} \frac{(-1)^{k+j}}{(k-1)!(j-1)!} \wp^{(k-2)}(v)\wp^{(j-2)}(v')\mathfrak{M}(\varphi_k, \varphi_j) = (3.14) - (3.15) = \\
& = 2i\pi (\mathbb{D}^\bullet \lambda(v)\lambda(v') + \lambda(v)\mathbb{D}^\bullet \lambda(v')) +
\end{aligned}$$

¹We shall use the notation $\nabla_\tau^\bullet \lambda(v)$ and $\mathbb{D}^\bullet \lambda(v)$ understanding them as

$$\begin{aligned}
\nabla_\tau^\bullet \lambda(v) & := \sum_{j=0}^{l+1} \frac{(-1)^{j-1}}{(j-1)!} \eta^{2j} \partial_\tau \left(\eta^{-2j} \wp^{(j-2)}(v) \right) \varphi_j ; \\
\mathbb{D}_\tau^\bullet \lambda(v) & := \sum_{j=0}^{l+1} \frac{(-1)^{j-1}}{(j-1)!} \mathbb{D}(\wp^{(j-2)})(v) \varphi_j ,
\end{aligned}$$

namely the modular and elliptic connections are supposed to operate only on the functions depending on v .

$$+ \left[\underbrace{\frac{\alpha'(v-v')}{\alpha(v-v')} - \frac{\alpha'(v)}{\alpha(v)} + \frac{\alpha'(v')}{\alpha(v')}}_{=\zeta(v-v')+\zeta(v')-\zeta(v)} \right] \left(\lambda(v) \frac{d}{dv'} \lambda(v') - \lambda(v') \frac{d}{dv} \lambda(v) \right) - \frac{1}{l+1} \lambda'(v) \lambda'(v') \quad (3.16)$$

From the classical pseudo-addition formula [Ba]

$$\zeta(v-v') + \zeta(v') - \zeta(v) = \frac{1}{2} \frac{\wp'(v) + \wp'(v')}{\wp(v) - \wp(v')},$$

we can finally prove the theorem. Q.E.D

Notice the resemblance of eq. 3.11 with the formula worked out in [SYS80] by Saito, Yano and Sekiguchi in the case of the finite Weyl group A_l : in that case they had $\mathcal{P}(v) = \prod_1^{l+1} (z_i - v)$ and the formula was

$$\sum_{i=1}^{l+1} \frac{\partial}{\partial z_i} \mathcal{P}(v) \frac{\partial}{\partial z_i} \mathcal{P}(v') = \frac{1}{v' - v} \left\{ \mathcal{P}(v') \frac{d}{dv} \mathcal{P}(v) - \mathcal{P}(v) \frac{d}{dv'} \mathcal{P}(v') \right\}. \quad (3.17)$$

We can realize that eq. 3.11 is an elliptic deformation of eq. 3.17.

Indeed, under a suitable limit the former formula goes into the latter

$$\lim_{\epsilon \rightarrow 0} \epsilon^{-2l} \left[e^{2i\pi(l+1)(v^2+v'^2)g_1} \widetilde{\mathfrak{M}}(\Phi(v), \Phi(v')) \right]_{(u, z, v, v') \rightarrow (\epsilon u, \epsilon z, \epsilon v, \epsilon v')} = \text{RHS of eq. 3.17}.$$

3.2.4 Connection components

With similar computations we can spell out a formula for the components of the connection in the coordinates provided by the hatted Jacobi forms. As this is the result of a change of coordinates from a flat system the connection is a flat one; nonetheless it is useful to know its components when studying Poisson structures on suitable loop spaces. For further reference and motivation see [Du93], Lecture 6.

In the case of polynomial invariants, easy computations give the result which we report only for the sake of comparison:

Proposition 3.2.3 The contravariant connection in the basis of polynomial invariants $\{y_j\}$ appearing in the generating function

$$P(u) := \prod_{j=1}^{l+1} (u - z_j) \Big|_{\sum z_j = 0} = u^{l+1} + y_2(\mathbf{z})u^{l-1} + \dots + y_{l+1}(\mathbf{z}),$$

is given by

$$\sum_{i,j=1}^{l+1} u^{l+1-i} v^{l+1-j} \nabla_{dy_i^\#} dy_j \equiv \nabla_{dP(u)^\#} dP(v) =$$

$$= \frac{1}{(v-u)^2} \left[P(v)dP(u) - P(u)dP(v) \right] - \frac{1}{u-v} \left[P(u)d(P'(v)) - P'(u)dP(v) \right] - \frac{P'(u)dP'(v)}{l+1},$$

where dy_j^\sharp is the dual vector to dy_j under the natural isomorphism established by the intersection form (which here reads simply $ds^2 = \sum_j dz_j^2$ restricted to $\sum_j z_j = 0$).

Now, we will not spell out all the necessary steps in the case of Jacobi invariants: they are much more involved than those we followed in the computation of the intersection form. We give just the final result in Thm. 3.2.3.

Theorem 3.2.3 The component of the contravariant connection are given by

$$\begin{aligned} & \sum_{\mu, \nu=0}^{l+1} \frac{(-1)^{\mu+\nu}}{(\mu-1)!(\nu-1)!} \wp^{(\mu-2)}(u) \wp^{(\nu-2)}(v) \nabla_{d\widehat{\varphi}^\mu}^\sharp d\widehat{\varphi}^\nu = \\ & = \frac{1}{2} \frac{\wp'(u) + \wp(v)}{\wp(u) - \wp(v)} \left[\lambda(u)d\lambda'(v) - \lambda'(u)d\lambda(v) \right] - \left(\wp(u-v) - 4i\pi g_1(\tau) \right) \left[\lambda(v)d\lambda(u) - \lambda(u)d\lambda(v) \right] + \\ & + 2i\pi \left(\mathbb{D}^\bullet \lambda(u)d\lambda(u) + \lambda(u)d(\mathbb{D}^\bullet \lambda(v)) \right) - \frac{1}{l+1} \lambda'(u)d\lambda'(v) + \\ & - \widehat{C}_\mu(u) \partial_\tau \widehat{C}_\nu(v) \mathcal{I}^*(d\widehat{\varphi}^\mu, d\widehat{\varphi}^\nu) d\tau + \\ & + \left[\partial_\tau \left(\frac{\alpha'(v)}{\alpha(v)} \right) (\lambda(u)\lambda'(v) - \lambda'(u)\lambda(v)) - \partial_\tau \left(\frac{\alpha'(u-v)}{\alpha(u-v)} \right) \lambda'(u)\lambda(v) \right] d\tau, \end{aligned}$$

where $\widehat{C}_\mu(u) := \eta^{-2\mu} C_\mu(u)$, and $d\widehat{\varphi}_\mu^\sharp$ stands for the vector dual to the one-form $d\widehat{\varphi}_\mu$ by means of the intersection form.

It can be seen by means of a limit process as before, that the formula in Prop. 3.2.3 is a limit case of the one in Thm. 3.2.3: notice that for the comparison we are also to neglect the part multiplying the differential $d\tau$.

Example 3.2.1 The case of A_2 In this example we compute explicitly the intersection form and the corresponding connection in the coordinates $\varphi_{-1} = \tau$, $\widehat{\varphi}_0 = \varphi_0$, $\widehat{\varphi}_2 = \eta^4 \varphi_2$, $\widehat{\varphi}_3 = \eta^6 \varphi_3$; this result will be useful in the computations for G_2 .

Using the generating function in Thm. 3.2.1 one can compute explicitly the Jacobi forms to be (recall that we have $z_1 = x_1$; $z_2 = x_2 - x_1$; $z_3 = -x_2$)

$$\begin{aligned} \varphi_3(u, \mathbf{x}) &= -2e^{2i\pi u} \alpha(z_1) \alpha(z_2) \alpha(z_3); \\ \varphi_2(u, \mathbf{x}) &= e^{2i\pi u} \alpha(z_1) \alpha(z_2) \alpha(z_3) \frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)} = -2e^{2i\pi u} \alpha(z_1) \alpha(z_2) \alpha(z_3) [\zeta(z_1) + \zeta(z_2) + \zeta(z_3)] \\ \varphi_0(u, \mathbf{x}) &= e^{2i\pi u} \alpha(z_1) \alpha(z_2) \alpha(z_3) \frac{\wp(z_1)\wp'(z_2) - \wp'(z_1)\wp(z_2)}{\wp(z_2) - \wp(z_1)} = e^{2i\pi u} \alpha(z_1) \alpha(z_2) \alpha(z_3) \cdot \\ & \cdot \left\{ \frac{1}{3} [\wp'(z_1) + \wp'(z_2) + \wp'(z_3)] + \frac{2}{3} [\zeta(z_1) + \zeta(z_2) + \zeta(z_3)] (\wp(z_1) + \wp(z_2) + \wp(z_3)) \right\}, \end{aligned}$$

where we have used some classical formulae in dealing with the \wp functions [WW].

We now compute the elements $\mathfrak{M}_{i,j} := \mathfrak{M}(\varphi_i, \varphi_j)$ since we are going to use them for G_2 later. Using formula

(3.10) and setting $\varphi_{-1} := \tau$ we find

$$\mathfrak{M}(\varphi_i, \varphi_j) = \begin{pmatrix} 0 & -2i\pi\varphi_0 & -2i\pi\varphi_2 & -2i\pi\varphi_3 \\ -2i\pi\varphi_0 & \frac{1}{24}g_2^2\varphi_3^2 - \frac{1}{6}g_2\varphi_0\varphi_2 - \frac{1}{2}g_3\varphi_2^2 & \frac{3}{4}g_3\varphi_3^2 - \frac{1}{3}g_2\varphi_2^2 & -\frac{5}{12}g_2\varphi_2\varphi_3 \\ -2i\pi\varphi_2 & \frac{3}{4}g_3\varphi_3^2 - \frac{1}{3}g_2\varphi_2^2 & \frac{1}{2}g_2(\varphi_3)^2 - 2\varphi_0\varphi_2 & -3\varphi_3\varphi_0 \\ -2i\pi\varphi_3 & -\frac{5}{12}g_2\varphi_3\varphi_2 & -3\varphi_3\varphi_0 & \frac{2}{3}(\varphi_2)^2 \end{pmatrix} \quad (3.18)$$

The connection is given by

$$\begin{aligned} \nabla_{\circ} d\tau &= 0; \quad \nabla_{d\tau} d\widehat{\varphi}_\nu = -2i\pi d\widehat{\varphi}_\nu, \nu = 0, 2, 3 \\ \nabla_{d\widehat{\varphi}_0} d\widehat{\varphi}_0 &= \left[\left(\frac{1}{40} \frac{i\widehat{\varphi}_2^2 \widehat{g}_2(\tau)^2}{\pi} + \frac{1}{4} \frac{I\widehat{\varphi}_0 \widehat{\varphi}_2 \widehat{g}_3(\tau)}{\pi} - \frac{1}{8} \frac{i\widehat{\varphi}_3^2 \widehat{g}_2(\tau) \widehat{g}_3(\tau)}{\pi} \right) \eta(\tau)^8 + \right. \\ &+ \left. \left(-\frac{1}{3} \widehat{\varphi}_0 \widehat{g}_2(\tau) \widehat{\varphi}_2 - \widehat{\varphi}_2^2 \widehat{g}_3(\tau) + \frac{1}{12} \widehat{\varphi}_3^2 \widehat{g}_2(\tau)^2 \right) g_1(\tau) \eta(\tau)^4 - \frac{24}{5} i\pi \widehat{\varphi}_2^2 \widehat{g}_2(\tau) g_1(\tau)^2 \right] d\tau + \\ &- \left[\frac{1}{12} \eta(\tau)^4 \widehat{g}_2(\tau) \widehat{\varphi}_2 \right] d\widehat{\varphi}_0 - \left[\left(\frac{1}{2} \widehat{g}_3(\tau) \widehat{\varphi}_2 + \frac{1}{12} \widehat{\varphi}_0 \widehat{g}_2(\tau) \right) \eta(\tau)^4 \right] d\widehat{\varphi}_2 + \left[\frac{1}{24} \widehat{\varphi}_3 \widehat{g}_2(\tau)^2 \eta(\tau)^4 \right] d\widehat{\varphi}_3, \\ \nabla_{d\widehat{\varphi}_2} d\widehat{\varphi}_0 &= \left[\left(\frac{3}{4} \frac{i\widehat{\varphi}_2^2 \widehat{g}_3(\tau)}{\pi} - \frac{1}{12} \frac{i\widehat{\varphi}_3^2 \widehat{g}_2(\tau)^2}{\pi} \right) \eta(\tau)^8 + \right. \\ &+ \left. \left(-\frac{2}{3} \widehat{\varphi}_2^2 \widehat{g}_2(\tau) + \frac{3}{2} \widehat{\varphi}_3^2 \widehat{g}_3(\tau) \right) g_1(\tau) \eta(\tau)^4 - 24i\pi g_1(\tau)^2 \widehat{\varphi}_0 \widehat{\varphi}_2 \right] d\tau + \\ &- \left[4i\pi g_1(\tau) \widehat{\varphi}_2 \right] d\widehat{\varphi}_0 + \left[4i\pi g_1(\tau) \widehat{\varphi}_0 - \frac{1}{3} \eta(\tau)^4 \widehat{g}_2(\tau) \widehat{\varphi}_2 \right] d\widehat{\varphi}_2 + \left[\frac{3}{4} \eta(\tau)^4 \widehat{g}_3(\tau) \widehat{\varphi}_3 \right] d\widehat{\varphi}_3, \\ \nabla_{d\widehat{\varphi}_3} d\widehat{\varphi}_0 &= \left[-\frac{5}{6} \eta(\tau)^4 \widehat{g}_2(\tau) \widehat{\varphi}_2 \widehat{\varphi}_3 g_1(\tau) + \frac{i\eta(\tau)^8 \widehat{\varphi}_3 \widehat{\varphi}_2 \widehat{g}_3(\tau)}{\pi} - 36i\pi g_1(\tau)^2 \widehat{\varphi}_0 \widehat{\varphi}_3 \right] d\tau + \\ &- \left[4i\pi g_1(\tau) \widehat{\varphi}_3 \right] d\widehat{\varphi}_0 - \left[\frac{1}{3} \eta(\tau)^4 \widehat{g}_2(\tau) \widehat{\varphi}_3 \right] d\widehat{\varphi}_2 + \left[4i\pi g_1(\tau) \widehat{\varphi}_0 - \frac{1}{12} \eta(\tau)^4 \widehat{g}_2(\tau) \widehat{\varphi}_2 \right] d\widehat{\varphi}_3; \\ \nabla_{d\widehat{\varphi}_0} d\widehat{\varphi}_2 &= \left[\left(-\frac{1}{24} \frac{i\widehat{\varphi}_3^2 \widehat{g}_2(\tau)^2}{\pi} + \frac{1}{4} \frac{i\widehat{\varphi}_2^2 \widehat{g}_3(\tau)}{\pi} \right) \eta(\tau)^8 + \left(-\frac{2}{3} \widehat{\varphi}_2^2 \widehat{g}_2(\tau) + \frac{3}{2} \widehat{\varphi}_3^2 \widehat{g}_3(\tau) \right) g_1(\tau) \eta(\tau)^4 + \right. \\ &+ 24i\pi g_1(\tau)^2 \widehat{\varphi}_0 \widehat{\varphi}_2 \left. \right] d\tau + \left[4i\pi g_1(\tau) \widehat{\varphi}_2 \right] d\widehat{\varphi}_0 - \left[4i\pi g_1(\tau) \widehat{\varphi}_0 + \frac{1}{3} \eta(\tau)^4 \widehat{g}_2(\tau) \widehat{\varphi}_2 \right] d\widehat{\varphi}_2 + \\ &+ \left[\frac{3}{4} \eta(\tau)^4 \widehat{g}_3(\tau) \widehat{\varphi}_3 \right] d\widehat{\varphi}_3, \\ \nabla_{d\widehat{\varphi}_2} d\widehat{\varphi}_2 &= \left[-\frac{3}{4} \frac{i\eta(\tau)^8 \widehat{\varphi}_3^2 \widehat{g}_3(\tau)}{\pi} + \left(\widehat{\varphi}_3^2 \widehat{g}_2(\tau) - 4\widehat{\varphi}_2 \widehat{\varphi}_0 \right) g_1(\tau) \eta(\tau)^4 \right] d\tau - \left[\eta(\tau)^4 \widehat{\varphi}_2 \right] d\widehat{\varphi}_0 + \\ &- \left[\eta(\tau)^4 \widehat{\varphi}_0 \right] d\widehat{\varphi}_2 + \left[\frac{1}{2} \widehat{\varphi}_3 \widehat{g}_2(\tau) \eta(\tau)^4 \right] d\widehat{\varphi}_3, \\ \nabla_{d\widehat{\varphi}_3} d\widehat{\varphi}_2 &= \left[-12i\pi g_1(\tau)^2 \widehat{\varphi}_2 \widehat{\varphi}_3 - 6\eta(\tau)^4 \widehat{\varphi}_0 \widehat{\varphi}_3 g_1(\tau) \right] d\tau - \left[2\widehat{\varphi}_3 \eta(\tau)^4 \right] d\widehat{\varphi}_0 + \\ &- \left[4i\pi g_1(\tau) \widehat{\varphi}_3 \right] d\widehat{\varphi}_2 + \left[4i\pi g_1(\tau) \widehat{\varphi}_2 - \widehat{\varphi}_0 \eta(\tau)^4 \right] d\widehat{\varphi}_3; \\ \nabla_{d\widehat{\varphi}_0} d\widehat{\varphi}_3 &= \left[\frac{1}{4} \frac{i\eta(\tau)^8 \widehat{\varphi}_3 \widehat{\varphi}_2 \widehat{g}_3(\tau)}{\pi} - \frac{5}{6} \eta(\tau)^4 \widehat{g}_2(\tau) \widehat{\varphi}_2 \widehat{\varphi}_3 g_1(\tau) + 36i\pi g_1(\tau)^2 \widehat{\varphi}_0 \widehat{\varphi}_3 \right] d\tau + \end{aligned}$$

$$\begin{aligned}
& + \left[4i\pi g_1(\tau) \widehat{\varphi}_3 \right] d\widehat{\varphi}_0 - \left[\frac{1}{12} \eta(\tau)^4 \widehat{g}_2(\tau) \widehat{\varphi}_3 \right] d\widehat{\varphi}_2 - \left[4i\pi g_1(\tau) \widehat{\varphi}_0 + \frac{1}{3} \eta(\tau)^4 \widehat{g}_2(\tau) \widehat{\varphi}_2 \right] d\widehat{\varphi}_3 , \\
\nabla_{d\widehat{\varphi}_2} d\widehat{\varphi}_3 & = \left[12i\pi g_1(\tau)^2 \widehat{\varphi}_2 \widehat{\varphi}_3 - 6\eta(\tau)^4 \widehat{\varphi}_0 \widehat{\varphi}_3 g_1(\tau) \right] d\tau - \left[\widehat{\varphi}_3 \eta(\tau)^4 \right] d\widehat{\varphi}_0 + \left[4i\pi g_1(\tau) \widehat{\varphi}_3 \right] d\widehat{\varphi}_2 + \\
& + \left[-2\widehat{\varphi}_0 \eta(\tau)^4 - 4i\pi g_1(\tau) \widehat{\varphi}_2 \right] d\widehat{\varphi}_3 , \\
\nabla_{d\widehat{\varphi}_3} d\widehat{\varphi}_3 & = \left[\frac{2}{3} \widehat{\varphi}_2 \eta(\tau)^4 \right] d\widehat{\varphi}_2 + \left[\frac{4}{3} \widehat{\varphi}_2^2 g_1(\tau) \eta(\tau)^4 \right] d\tau .
\end{aligned}$$

3.3 Jacobi forms of type G_2 . Saito's flat invariants.

The Cartan subalgebra for G_2 is realized in [Bo], planche IX as the subspace of \mathbb{C}^3 such that $z_1 + z_2 + z_3 = 0$; the root lattice is the same as the one of A_2 and the Weyl group is the dihedral group of order 12. It can be seen that $W(G_2)/W(A_2) \simeq \mathbb{Z}_2$ and it is generated by the involution

$$\mathfrak{S} : (z_1, z_2, z_3) \mapsto (-z_3, -z_2, -z_1) .$$

The Jacobi forms we have to build are

$$\varphi_0 \in J_{0,1}; \quad \varphi_2 \in J_{-2,1}; \quad \varphi_6 \in J_{-6,2} .$$

They can be built directly as follows:

1. As for φ_6 we can take

$$\varphi_6(u, \mathbf{x}, \tau) := 2e^{4i\pi u} \prod_{j=1}^3 \alpha^2(z_j, \tau) = \frac{1}{2} \left(\varphi_3^{(A_2)} \right)^2 \in J_{-6,2}$$

namely the square of the lightest Jacobi form for A_2 , which is clearly invariant under the involution \mathfrak{S} defined above.

2. As for φ_2, φ_0 one can check that the same Jacobi forms which work for A_2 do work for this case (we only have to check invariance under the extra involution \mathfrak{S}) in fact we have for $z_1 + z_2 + z_3 = 0$,

$$\begin{aligned}
\varphi_2(u, \mathbf{x}) & = e^{2i\pi u} \alpha(z_1) \alpha(z_2) \alpha(z_3) \frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)} = \\
& = -2e^{2i\pi u} \alpha(z_1) \alpha(z_2) \alpha(z_3) [\zeta(z_1) + \zeta(z_2) + \zeta(z_3)] \\
\varphi_0(u, \mathbf{x}) & = e^{2i\pi u} \alpha(z_1) \alpha(z_2) \alpha(z_3) \frac{\wp(z_1) \wp'(z_2) - \wp'(z_1) \wp(z_2)}{\wp(z_2) - \wp(z_1)} = \\
& = e^{2i\pi u} \alpha(z_1) \alpha(z_2) \alpha(z_3) \left\{ \frac{1}{3} [\wp'(z_1) + \wp'(z_2) + \wp'(z_3)] + \right. \\
& \quad \left. + \frac{2}{3} [\zeta(z_1) + \zeta(z_2) + \zeta(z_3)] (\wp(z_1) + \wp(z_2) + \wp(z_3)) \right\} ,
\end{aligned}$$

and since both are product of two anti-invariant functions, they are invariant under \mathfrak{S} .

This identification allows us to compute the Jacobi invariant intersection form \mathfrak{M} since it is the same as for A_2 being the Weyl invariant inner product the same $[(dz_1)^2 + (dz_2)^2 + (dz_3)^2]_{|z_1+z_2+z_3=0}$ in both cases. Therefore we can easily compute the intersection elements $\mathfrak{M}(\varphi_j, \varphi_k)$ out of those computed for A_2 in eq. (3.18) and find the following expression for the intersection form in the coordinates $\tau =: \widehat{\varphi}_{-1}, \widehat{\varphi}_0, \widehat{\varphi}_2, \widehat{\varphi}_6$ (where the hat on the Jacobi forms are given in Def. 3.1.3)

$$\begin{aligned} & \mathcal{J}^*(d(\widehat{\varphi}_i), d(\widehat{\varphi}_j)) = \\ & = \eta^4 \begin{pmatrix} 0 & -2i\pi\widehat{\varphi}_0\eta^{-4} & -2i\pi\widehat{\varphi}_2\eta^{-4} & -4i\pi\widehat{\varphi}_6\eta^{-4} \\ -2i\pi\widehat{\varphi}_0\eta^{-4} & \frac{1}{12}\widehat{g}_2^2\widehat{\varphi}_6 - \frac{1}{6}\widehat{g}_2\widehat{\varphi}_0\widehat{\varphi}_2 - \frac{1}{2}\widehat{g}_3\widehat{\varphi}_2^2 & \frac{3}{2}\widehat{g}_3\widehat{\varphi}_6 - \frac{1}{3}\widehat{g}_2\widehat{\varphi}_2^2 & -\frac{5}{6}\widehat{g}_2\widehat{\varphi}_6\widehat{\varphi}_2 \\ -2i\pi\widehat{\varphi}_2\eta^{-4} & \frac{3}{2}\widehat{g}_3\widehat{\varphi}_6 - \frac{1}{3}\widehat{g}_2\widehat{\varphi}_2^2 & \widehat{g}_2\widehat{\varphi}_6 - 2\widehat{\varphi}_0\widehat{\varphi}_2 & -6\widehat{\varphi}_6\widehat{\varphi}_0 \\ -4i\pi\widehat{\varphi}_6\eta^{-4} & -\frac{5}{6}\widehat{g}_2\widehat{\varphi}_6\widehat{\varphi}_2 & -6\widehat{\varphi}_6\widehat{\varphi}_0 & \frac{4}{3}\widehat{\varphi}_6\widehat{\varphi}_2^2 \end{pmatrix} \end{aligned} \quad (3.19)$$

Notice that the matrix is linear in the generator φ_6 : this is obvious when one counts the indices of the matrix elements and it is connected with the fact that G_2 is a ‘‘codimension one’’ case in the sense of [Sa90].

The matrix $\frac{\partial}{\partial \widehat{\varphi}_6} \mathcal{J}^*$ (which is the tensor J^* in Saito’s notation) reads

$$\frac{\partial}{\partial \widehat{\varphi}_6} \mathcal{J}^* = \eta^4 \begin{pmatrix} 0 & 0 & 0 & -4i\pi\eta^{-4} \\ 0 & \frac{1}{12}\widehat{g}_2^2 & \frac{3}{2}\widehat{g}_3 & -\frac{5}{6}\widehat{g}_2\widehat{\varphi}_2 \\ 0 & \frac{3}{2}\widehat{g}_3 & \widehat{g}_2 & -6\widehat{\varphi}_0 \\ -4i\pi\eta^{-4} & -\frac{5}{6}\widehat{g}_2\widehat{\varphi}_2 & -6\widehat{\varphi}_0 & \frac{4}{3}\widehat{\varphi}_2^2 \end{pmatrix}.$$

It follows from Saito’s paper [Sa90] that this tensor defines a contravariant metric whose covariant form is flat; one could now try and look for the flat coordinates of this metric (the ‘‘flat theta invariants’’).

We are in a position to give the flat coordinates associated to this second flat metric.

In order to find them we must integrate the geodesic equations; it is a rather muscular though non completely straightforward exercise for which it is very helpful the paper [Sat93]. Hereafter we report only the result.

Proposition 3.3.1 The flat coordinates of Saito’s metric are τ, t_1, t_2, t_6 , given in the following system

$$\begin{aligned} \widehat{\varphi}_0 &= (2\pi)^2 2^{-2/3} \frac{i\pi}{\eta^2} \left(F_1'(\tau)t_1 + F_2'(\tau)t_2 \right) \\ \widehat{\varphi}_2 &= (2\pi)^2 2^{-2/3} \eta^2 \left(F_1(\tau)t_1 + F_2(\tau)t_2 \right) \\ \widehat{\varphi}_6 &= t_6 - 4i\pi g_1 t_1 t_2 + \frac{3^2}{2^{11}\pi^{12}} \widehat{g}_3 \widehat{\varphi}_0^2 - \frac{1}{2^{11}\pi^{12}} \widehat{g}_2^2 \widehat{\varphi}_0 \widehat{\varphi}_2 + \frac{3}{2^{13}\pi^{12}} \widehat{g}_2 \widehat{g}_3 \widehat{\varphi}_2^2, \end{aligned}$$

where the two functions F_1, F_2 are given, in terms of the modular invariant

$$z(\tau) := \frac{1}{2} \left[\frac{\sqrt{-27}}{(2\pi)^6} \widehat{g}_3 + 1 \right]; \quad \frac{dz}{d\tau} = \frac{\sqrt{-27}}{2(2\pi)^6} \frac{\eta^4}{6i\pi} \widehat{g}_2^2,$$

by,

$$F_1(\tau) = F_1(z(\tau)) = z(\tau)^{\frac{1}{3}}; \quad F_2(\tau) = F_2(z(\tau)) = (z(\tau) - 1)^{\frac{1}{3}}.$$

In these coordinates Saito's metric becomes,

$$J^*(dt_i, dt_j) = \begin{pmatrix} 0 & 0 & 0 & -4i\pi \\ 0 & 0 & -2 & 0 \\ 0 & -2 & 0 & 0 \\ -4i\pi & 0 & 0 & 0 \end{pmatrix}. \quad (3.20)$$

The flat coordinates show up a nontrivial monodromy around $z(\tau) = 0, 1$; these points both correspond to the elliptic curves where $g_2 = 0$, namely the symmetric tori.

3.4 The root system of type B_l : Jacobi forms

The Jacobi forms for $J(B_l)$ can be constructed in a similar but easier way as those for $J(A_l)$.

Similar computations leading to the generating function for A_l , took us to the following

Theorem 3.4.1 The generating function

$$\begin{aligned} \mathcal{P}(v) &:= e^{2i\pi v} \prod_{i=1}^l \alpha(v - x_i) \alpha(v + x_i) := \\ &= \frac{\wp^{(2l-2)}(v)}{(2l-1)!} \alpha^{2l}(v) \varphi_{2l}(\mathbf{x}) + \frac{\wp^{(2l-4)}(v)}{(2l-3)!} \alpha^{2l}(v) \varphi_{2l-2}(\mathbf{x}) + \dots + \alpha^{2l}(v) \varphi_0(\mathbf{x}) = \\ &= \frac{\det \begin{pmatrix} 1 & \dots & \wp^{(2l-2)}(v) \\ \vdots & \vdots & \vdots \\ 1 & \dots & \wp^{(2l-2)}(x_l) \end{pmatrix}}{\det \begin{pmatrix} 1 & \dots & \wp^{(2l-4)}(x_1) \\ \vdots & \vdots & \vdots \\ 1 & \dots & \wp^{(2l-4)}(x_l) \end{pmatrix}} \alpha^{2l}(v) \prod_{j=1}^l \alpha^2(x_j) = \\ &= \varphi_{2l}(\mathbf{x}) + v^2 \varphi_{2l-2}(\mathbf{x}) + \dots + v^{2l} \varphi_0(\mathbf{x}) + O(v^{2l+2}), \end{aligned} \quad (3.21)$$

gives a basis of generators for the algebra of Jacobi forms of type B_l .

Any other set of generators is a weighted *linear* combination of these with coefficients in \mathbf{M}_\bullet .

Proof. We must show first of all, that the functions defined by the formula have all the properties of smoothness. This follows from the fact that the ratio of determinants is a holomorphic function of all its variables without poles.

In order to show the properties of invariance, consider now the elliptic function

$$F(v) := \det \begin{pmatrix} 1 & \wp(v) & \wp''(v) & \dots & \wp^{(2l-2)}(v) \\ 1 & \wp(x_1) & \wp''(x_1) & \dots & \wp^{(2l-2)}(x_1) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & \wp(x_l) & \wp''(x_l) & \dots & \wp^{(2l-2)}(x_l) \end{pmatrix}.$$

Since only even derivatives of the \wp functions are involved, this is clearly invariant under the change of sign of any between v, x_1, \dots, x_l . As a function of v it has a pole at $v = 0$ of order $2l$ and hence it has as many zeroes; it is clear that these are situated at $\pm x_1, \dots, \pm x_l$. The same holds clearly for the variables x_i therefore if we express it by means of the function σ [WW], considering the antisymmetry we must have

$$F(v) \propto \frac{\prod_{i=1}^l \sigma(v - x_i) \sigma(v + x_i) \prod_{i < j} \sigma(x_i - x_j) \sigma(x_i + x_j)}{\sigma^{2l}(v) \prod_{i=1}^l \sigma^{2l}(x_i)}.$$

Therefore we can compute

$$\frac{\det \begin{pmatrix} 1 & \wp(v) & \wp''(v) & \dots & \wp^{(2l-2)}(v) \\ 1 & \wp(x_1) & \wp''(x_1) & \dots & \wp^{(2l-2)}(x_1) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \wp(x_l) & \wp''(x_l) & \dots & \wp^{(2l-2)}(x_l) \end{pmatrix}}{\det \begin{pmatrix} 1 & \wp(x_1) & \wp''(x_1) & \dots & \wp^{(2l-4)}(x_1) \\ 1 & \wp(x_2) & \wp''(x_2) & \dots & \wp^{(2l-4)}(x_2) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \wp(x_l) & \wp''(x_l) & \dots & \wp^{(2l-4)}(x_l) \end{pmatrix}} = (2l-1)! \frac{\prod_{i=1}^l \sigma(v - x_i) \sigma(v + x_i)}{\sigma^{2l}(v) \prod_{i=1}^l \sigma^2(x_i)}$$

Expanding the LHS w.r.t. the first row of the determinant in the numerator and multiplying both sides by

$e^{2i\pi u + 2i\pi \frac{\tau'}{\tau} (\|\mathbf{x}\|^2 + 2lv^2)} \sigma^{2l}(v) \prod_{i=1}^l \sigma^2(x_i)$ we obtain exactly the formula (3.21).

As for the second statement, the proof is exactly as in the case A_l . Q.E.D

Notice that, up to a normalization, all the Jacobi forms can be recovered by applying the operator \mathfrak{D} to the lightest one

$$\varphi_{2l}(u, \mathbf{x}, \tau) = \prod_{j=1}^l (x_j)^2 + O(\|\mathbf{x}\|^{2l+1})$$

and hence for any $b = 1..l$ the Jacobi forms $\varphi_{l-b} := \mathfrak{D}^b(\varphi_l)$ do not vanish identically because their leading term is $\Delta_{\mathbf{x}}^b \left(\prod_{j=1}^l (x_j)^2 \right) \neq 0$. This gives the whole basis of fundamental invariant Jacobi forms; moreover (as in the A_l case) any other basis is obtained from this one by a weighted linear transformation with coefficients in \mathbf{M}_\bullet .

Proposition 3.4.1 The formula

$$\mathcal{P}(v) = e^{2i\pi u} \alpha^{2l}(v) \left[\prod_{j=1}^l \left(\wp(v) - \wp(x_j) \right) \right] \prod_{k=1}^l \alpha^2(x_k) = \alpha^{2l}(v) \sum_{j=0}^l (\wp(v))^j \psi_{2j}(\mathbf{x}),$$

defines a basis of Jacobi forms which is equivalent to that in formula (3.21).

Proof. It is clear that, as a function of v , this is the same generating function because it has the same poles and zeroes, but the fundamental Jacobi forms ψ_{2j} are weighted linear combination of those defined previously; indeed, expanding the product we obtain the Jacobi forms as the coefficients in front of powers of $\wp(v)$ while in the former formula they were the coefficients in front of even derivatives of \wp (recall that even derivatives of \wp can be expressed as polynomials in \wp). Q.E.D

Although this is a more compact formula, the previous one is more effective in actual computations of the intersection elements; indeed in the next paragraph the intersection elements will be computed only for the first basis of Jacobi forms. However this formula for the generating function is the closest to the classical one (3.2) as one can compute that

$$\mathcal{P}(v) = \exp(v^2 \mathcal{D}) \prod_{j=1}^l [e^{2i\pi u} \alpha^2(x_j)] ,$$

which follows from the formula (coming from the case A_1)

$$\left(-4i\pi \nabla_\tau + \frac{1}{2} \frac{d^2}{dx^2} \right) \alpha^2(x) = \wp(x) \alpha^2(x) .$$

Remark 3.4.1 Although the Weyl group of C_l is the same as the one of B_l , the *affine Weyl groups* (namely the complex crystallographic lattices) are different: indeed the root lattice of B_l is simply \mathbb{Z}^l while the one of C_l is $(z_1, \dots, z_l) \in \mathbb{Z}^l$ such that $\sum z_j \in 2\mathbb{Z}$. There follows in particular that the normalization of the invariant Killing form to 2 for the shortest roots, is different: in this realization of the Cartan subalgebra (which follows the corresponding planches in [Bo]) the Killing metrics are $ds^2 := 2 \sum dx_j^2$ for B_l and $\frac{1}{2} ds^2 = \sum_j dx_j^2$ for C_l .

A consequence of this fact is that if $\varphi(u, \mathbf{x}, \tau) \in J_{k,m}^{(B_l)}$ then $\varphi(2u, \mathbf{x}, \tau)$ belongs to $J_{k,2m}^{(C_l)}$. This amounts to saying that the algebra $J_{\bullet,\bullet}^{(B_l)}$ of Jacobi form for B_l is isomorphic to a subalgebra of $J_{\bullet,\bullet}^{(C_l)}$ and the isomorphism doubles the index.

Now, from [Wi92], we know that the generators belong to the spaces

$$\varphi_0 \in J_{0,1}; \quad \varphi_2 \in J_{-2,1}; \quad \varphi_4 \in J_{-4,1}; \quad \varphi_{2k} \in J_{-2k,2} , \quad k = 3 \dots l.$$

The isomorphism which injects $J_{\bullet,\bullet}^{(B_l)} \hookrightarrow J_{\bullet,\bullet}^{(C_l)}$ allows us to identify the $l-2$ generators of index 2 for the Jacobi algebra of type C_l as the images of the corresponding generators of B_l .

3.4.1 Computation of the intersection form with the generating function: B_l

As we did in the A_l case, we can now exploit the generating function to compute the intersection elements $\mathfrak{M}_{2j,2k} := \mathfrak{M}(\varphi_{2j}, \varphi_{2k})$; again the computational details are quite cumbersome but the result is a remarkably simple formula which –again– is in deep analogy with the corresponding formula for the invariant polynomials in [SYS80]. The result will be a generating function in two variables for the elements $\mathfrak{M}_{2j,2k}$ and is contained in Thm. 3.4.2.

The intersection form for the Jacobi group of type B_l reads

$$\mathcal{I} := -du \otimes d\tau - d\tau \otimes du + 2 \sum_{j=1}^l dx_j^2 ,$$

where (following [Bo] planche II), we have realized the CSA of B_l as the vectors $\mathbf{x} := (x_1, \dots, x_l) \in \mathbb{C}^l$; notice the normalization of the Weyl invariant inner product.

Theorem 3.4.2 The intersection elements $\mathfrak{M}(\varphi_{2i}, \varphi_{2j})$ are recovered from the generating function

$$\begin{aligned} \sum_{j,k=0}^l \frac{\wp^{(2k)}(v') \wp^{(2j)}(v)}{(2k-1)! (2j-1)!} \mathfrak{M}(\varphi_{2j}, \varphi_{2k}) &= 2i\pi [\lambda(v)\mathbb{D}^*\lambda(v') + \lambda(v')\mathbb{D}^*\lambda(v)] + \\ &+ \frac{1}{2(\wp(v) - \wp(v'))} \left\{ \wp'(v')\lambda(v) \frac{d}{dv'} \lambda(v') - \wp'(v)\lambda(v') \frac{d}{dv} \lambda(v) \right\}. \end{aligned} \quad (3.22)$$

Proof. We compute

$$\begin{aligned} \mathfrak{M}(\mathcal{P}(v), \mathcal{P}(v')) &:= \eta^{-8l} \mathcal{J}^* \left(d \left(\eta^{4l} \mathcal{P}(v) \right), d \left(\eta^{4l} \mathcal{P}(v') \right) \right) = \\ &= -2i\pi \eta^{4l} \left[\mathcal{P}(v) \partial_\tau \left(\eta^{4l} \mathcal{P}(v') \right) + \mathcal{P}(v') \partial_\tau \left(\eta^{4l} \mathcal{P}(v) \right) \right] + \frac{1}{2} \sum_1^l \frac{\partial}{\partial x_i} \mathcal{P}(v) \frac{\partial}{\partial x_i} \mathcal{P}(v') = \\ &= -\frac{1}{2} \sum_1^l \left(\frac{\alpha''(v-x_i)}{\alpha(v-x_i)} + \frac{\alpha''(v+x_i)}{\alpha(v+x_i)} + \frac{\alpha''(v'-x_i)}{\alpha(v'-x_i)} + \frac{\alpha''(v'+x_i)}{\alpha(v'+x_i)} - \frac{\alpha'(v-x_i)\alpha'(v'-x_i)}{\alpha(v-x_i)\alpha(v'-x_i)} \right. \\ &\quad \left. - \frac{\alpha'(v+x_i)\alpha'(v'+x_i)}{\alpha(v+x_i)\alpha(v'+x_i)} - \frac{\alpha'(v-x_i)\alpha'(v'+x_i)}{\alpha(v-x_i)\alpha(v'+x_i)} - \frac{\alpha'(v+x_i)\alpha'(v'-x_i)}{\alpha(v+x_i)\alpha(v'-x_i)} \right) \mathcal{P}(v)\mathcal{P}(v') + \\ &\quad + 8i\pi l g_1 \mathcal{P}(v)\mathcal{P}(v') = \\ &= \sum_{i=1}^l \left[2i\pi \frac{\partial_\tau \alpha(v-v')}{\alpha(v-v')} + 2i\pi \frac{\partial_\tau \alpha(v+v')}{\alpha(v+v')} + \right. \\ &\quad \left. - \frac{1}{2} \left(\frac{\alpha'(v-v')}{\alpha(v-v')} + \frac{\alpha'(v+v')}{\alpha(v+v')} \right) \left(\frac{\alpha'(v-x_i)}{\alpha(v-x_i)} + \frac{\alpha'(v+x_i)}{\alpha(v+x_i)} \right) + \right. \\ &\quad \left. + \frac{1}{2} \left(\frac{\alpha'(v-v')}{\alpha(v-v')} - \frac{\alpha'(v+v')}{\alpha(v+v')} \right) \left(\frac{\alpha'(v'-x_i)}{\alpha(v'-x_i)} + \frac{\alpha'(v'+x_i)}{\alpha(v'+x_i)} \right) \right] \mathcal{P}(v)\mathcal{P}(v') + 8i\pi l g_1 \mathcal{P}(v)\mathcal{P}(v') = \\ &= 2i\pi l \left[\frac{\nabla_\tau \alpha(v-v')}{\alpha(v-v')} + \frac{\nabla_\tau \alpha(v+v')}{\alpha(v+v')} \right] \mathcal{P}(v)\mathcal{P}(v') + \\ &\quad + \frac{1}{2} \left(\frac{\alpha'(v-v')}{\alpha(v-v')} - \frac{\alpha'(v+v')}{\alpha(v+v')} \right) \mathcal{P}(v) \frac{d}{dv'} \mathcal{P}(v') - \frac{1}{2} \left(\frac{\alpha'(v-v')}{\alpha(v-v')} + \frac{\alpha'(v+v')}{\alpha(v+v')} \right) \mathcal{P}(v') \frac{d}{dv} \mathcal{P}(v). \end{aligned} \quad (3.23)$$

Remark 3.4.2 Notice again the resemblance of eq. 3.23 with Saito's formula [SYS80] in the case of the finite Weyl group B_l (here $\mathcal{P}(v) = \prod_1^l ((x_i)^2 - v^2)$)

$$\sum_{i=1}^{l+1} \frac{\partial}{\partial x_i} \mathcal{P}(v) \frac{\partial}{\partial x_i} \mathcal{P}(v') = \frac{2}{(v')^2 - (v)^2} \left\{ \mathcal{P}(v') v^2 \frac{d}{dv} \mathcal{P}(v) - \mathcal{P}(v) (v')^2 \frac{d}{dv'} \mathcal{P}(v') \right\}. \quad (3.24)$$

We can actually obtain Saito's formula with the limit

$$\lim_{\epsilon \rightarrow 0} \epsilon^{-4l+2} [\mathfrak{M}(\mathcal{P}_{\mathbf{x}}(v), \mathcal{P}_{\mathbf{x}}(v'))]_{u, \mathbf{x}, v, v' \rightarrow \epsilon u, \epsilon \mathbf{x}, \epsilon v, \epsilon v'} = \text{RHS of eq. 3.24}.$$

Again, we can recast equation (3.23) into a more useful form using the function

$$\lambda(v) := \alpha^{-2l}(v)\mathcal{P}(v) = \sum_{j=0}^l \frac{\wp^{(2j-2)}(v)}{(2j-1)!} \varphi_{2j} =: \alpha^{-2l}(v) \sum_{j=0}^l C_j(v) \varphi_{2(l-j)}.$$

First of all, since $\alpha^{-2l}(v)\nabla_\tau^\bullet \alpha^{2l}(v) = \nabla_\tau^\bullet + 2l \frac{\nabla_\tau \alpha(v)}{\alpha(v)}$, then a straightforward computation gives

$$\begin{aligned} & \mathfrak{M}(\mathcal{P}(v), \mathcal{P}(v')) - \sum_{j,k=0}^l C_k(v') C_j(v) \mathfrak{M}(\varphi_{2l-2j}, \varphi_{2l-2k}) = \\ & = -2i\pi \sum_{k,j} (\nabla_\tau C_k(v') C_j(v) + \nabla_\tau C_j(v) C_k(v')) \varphi_{2l-2k} \varphi_{2l-2j} = \\ & = -2i\pi \alpha^{2l}(v') \alpha^{2l}(v) \left\{ [\lambda(v') \nabla_\tau^\bullet \lambda(v) + \lambda(v) \nabla_\tau^\bullet \lambda(v')] + 2l \left(\frac{\nabla_\tau \alpha(v')}{\alpha(v')} + \frac{\nabla_\tau \alpha(v)}{\alpha(v)} \right) \lambda(v') \lambda(v) \right\}. \end{aligned}$$

Using now formula (3.23) for $\mathfrak{M}(\mathcal{P}(v), \mathcal{P}(v'))$, we can finally compute

$$\begin{aligned} & \sum_{j,k} \frac{\wp^{(2j)}(v)}{(2j-1)!} \frac{\wp^{(2k)}(v')}{(2k-1)!} \mathfrak{M}(\varphi_{2k}, \varphi_{2j}) = \\ & = 2i\pi [\lambda(v) \nabla_\tau^\bullet \lambda(v') + \lambda(v') \nabla_\tau^\bullet \lambda(v)] + \\ & + 2i\pi l \left[2 \frac{\nabla_\tau \alpha(v')}{\alpha(v')} + 2 \frac{\nabla_\tau \alpha(v)}{\alpha(v)} + \frac{\nabla_\tau \alpha(v-v')}{\alpha(v-v')} + \frac{\nabla_\tau \alpha(v+v')}{\alpha(v+v')} \right] \lambda(v) \lambda(v') + \\ & + l \left\{ \frac{\alpha'(v-v')}{\alpha(v-v')} \left[\frac{\alpha'(v')}{\alpha(v')} - \frac{\alpha'(v)}{\alpha(v)} \right] - \frac{\alpha'(v+v')}{\alpha(v+v')} \left[\frac{\alpha'(v')}{\alpha(v')} + \frac{\alpha'(v)}{\alpha(v)} \right] \right\} \lambda(v) \lambda(v') + \\ & + \frac{1}{2} \left[\frac{\alpha'(v-v')}{\alpha(v-v')} - \frac{\alpha'(v+v')}{\alpha(v+v')} \right] \lambda(v) \frac{d}{dv'} \lambda(v') - \frac{1}{2} \left[\frac{\alpha'(v-v')}{\alpha(v-v')} + \frac{\alpha'(v+v')}{\alpha(v+v')} \right] \lambda(v') \frac{d}{dv} \lambda(v) = \\ & = 2i\pi [\lambda(v) \nabla_\tau^\bullet \lambda(v') + \lambda(v') \nabla_\tau^\bullet \lambda(v)] + \\ & + \frac{1}{2} \left[\frac{\alpha'(v-v')}{\alpha(v-v')} - \frac{\alpha'(v+v')}{\alpha(v+v')} \right] \lambda(v) \frac{d}{dv'} \lambda(v') - \frac{1}{2} \left[\frac{\alpha'(v-v')}{\alpha(v-v')} + \frac{\alpha'(v+v')}{\alpha(v+v')} \right] \lambda(v') \frac{d}{dv} \lambda(v) = \\ & = 2i\pi [\lambda(v) \mathbb{D}^\bullet \lambda(v') + \lambda(v') \mathbb{D}^\bullet \lambda(v)] + \frac{1}{2} \left[\frac{\alpha'(v-v')}{\alpha(v-v')} - \frac{\alpha'(v+v')}{\alpha(v+v')} + 2 \frac{\alpha'(v')}{\alpha(v')} \right] \lambda(v) \frac{d}{dv'} \lambda(v') + \\ & - \frac{1}{2} \left[\frac{\alpha'(v-v')}{\alpha(v-v')} + \frac{\alpha'(v+v')}{\alpha(v+v')} - 2 \frac{\alpha'(v)}{\alpha(v)} \right] \lambda(v') \frac{d}{dv} \lambda(v) = \\ & = 2i\pi [\lambda(v) \mathbb{D}^\bullet \lambda(v') + \lambda(v') \mathbb{D}^\bullet \lambda(v)] + \frac{1}{2} [\zeta(v-v') - \zeta(v+v') + 2\zeta(v')] \lambda(v) \frac{d}{dv'} \lambda(v') + \\ & - \frac{1}{2} [\zeta(v-v') + \zeta(v+v') - 2\zeta(v)] \lambda(v') \frac{d}{dv} \lambda(v). \end{aligned}$$

To write this in the final form we use the classical formula (see [WW] pag.458, example 18)

$$-\zeta(w-u) + \zeta(w-v) + \zeta(v) - \zeta(u) = \frac{1}{2} \left\{ \frac{\wp'(u) + \wp'(w)}{\wp(u) - \wp(w)} - \frac{\wp'(v) + \wp'(w)}{\wp(v) - \wp(w)} \right\},$$

which allows us to complete the proof by substitution. Q.E.D

3.5 Jacobi forms of type C_3 . Saito's flat invariants

In a similar way as we did for the case G_2 , we construct the Jacobi forms of type C_3 by suitably embedding the Cartan subalgebra of C_3 into that of A_3 . Let x_1, x_2, x_3 be orthogonal coordinates for the CSA of C_3 , $\mathfrak{h}^{(C_3)}$ and z_1, z_2, z_3, z_4 be the usual realization of the CSA of type A_3 , $\mathfrak{h}^{(A_3)}$. Consider the map $\Phi : \mathfrak{h}^{(C_3)} \rightarrow \mathfrak{h}^{(A_3)}$

$$\Phi(x_1, x_2, x_3) = \left(\frac{x_1 + x_2 + x_3}{2}, \frac{x_1 - x_2 - x_3}{2}, \frac{-x_1 + x_2 - x_3}{2} + \frac{-x_1 - x_2 + x_3}{2} \right).$$

Then this map realizes an isometry also of the corresponding Killing forms

$$ds_{(C_3)}^2 = dx_1^2 + dx_2^2 + dx_3^2 = \Phi^*(ds_{(A_3)}^2).$$

Moreover, the root lattice of type C_3 , namely the set

$$Q := \{\mathbf{x} \in \mathbb{Z}^3, x_1 + x_2 + x_3 \in 2\mathbb{Z}\}$$

is bijectively mapped onto the (co)root lattice of type A_3 .

The Weyl group of type C_3 becomes an extension of that of type A_3 by \mathbb{Z}_2 (notice that $3! \times 2^3 = 2 \times 4!$), where the extra generator is the change of sign $\mathbf{z} \mapsto -\mathbf{z}$.

It is straightforward to check that the already constructed Jacobi forms for A_3 are invariant up to sign: in particular, the only one that changes sign is φ_3 , so we will take the following generators for the Jacobi algebra of type C_3 :

$$\varphi_0 = \varphi_0^{(A_3)} \in J_{0,1}^{(C_3)}; \quad \varphi_2 = \varphi_2^{(A_3)} \in J_{-2,1}^{(C_3)}; \quad \varphi_4 = \varphi_4^{(A_3)} \in J_{-4,1}^{(C_3)}; \quad \varphi_6 = \left(\varphi_3^{(A_3)}\right)^2 \in J_{-6,2}^{(C_3)}.$$

where we recall that

$$\varphi_0^{(A_3)} + \varphi_2^{(A_3)} \wp(v) + \varphi_3^{(A_3)} \wp'(v) + \varphi_4^{(A_3)} \wp''(v) = \prod_{i=1}^4 \alpha(z_i) \frac{\det \begin{pmatrix} 1 & \wp(v) & \wp'(v) & \wp''(v) \\ 1 & \wp(z_1) & \wp'(z_1) & \wp''(z_1) \\ 1 & \wp(z_2) & \wp'(z_2) & \wp''(z_2) \\ 1 & \wp(z_3) & \wp'(z_3) & \wp''(z_3) \end{pmatrix}}{\det \begin{pmatrix} 1 & \wp(z_1) & \wp'(z_1) \\ 1 & \wp(z_2) & \wp'(z_2) \\ 1 & \wp(z_3) & \wp'(z_3) \end{pmatrix}}$$

We can easily compute the intersection form in these generators and the (Saito's) contravariant metric obtained by derivation along φ_6 (we use the P, Q, R notation defined in formula 2.4)

$$J^* := \begin{bmatrix} 0 & 0 & 0 & 0 & -4i\pi \\ 0 & \frac{2}{27} \frac{\pi^8 Q^2}{\eta^{12}} & \frac{2}{9} \frac{\pi^6 R}{\eta^8} & 0 & -\frac{10}{9} \frac{\pi^4 \psi_2 Q}{\eta^4} + \frac{8}{27} \frac{\pi^6 \psi_4 R}{\eta^8} \\ 0 & \frac{2}{9} \frac{\pi^6 R}{\eta^8} & \frac{2}{3} \frac{Q \pi^4}{\eta^4} & 0 & \frac{8}{9} \frac{\psi_4 Q \pi^4}{\eta^4} - 6\eta^4 \psi_0 \\ 0 & 0 & 0 & \frac{3}{4} \eta^4 & \eta^4 \psi_2 \\ -4i\pi & -\frac{10}{9} \frac{\pi^4 \psi_2 Q}{\eta^4} + \frac{8}{27} \frac{\pi^6 \psi_4 R}{\eta^8} & \frac{8}{9} \frac{\psi_4 Q \pi^4}{\eta^4} - 6\eta^4 \psi_0 & \eta^4 \psi_2 & 4\eta^4 \psi_2^2 + \frac{16}{9} \frac{Q \pi^4 \psi_4^2}{\eta^4} - 16\eta^4 \psi_4 \psi_0 \end{bmatrix}$$

Again it is straightforward although rather cumbersome to find the flat coordinates of this metric, therefore we omit the computations and just give the result: we find (we use the variables ψ_j instead of the hatted forms)

Proposition 3.5.1 The flat coordinates of Saito's flat metric J^* are given by τ and, setting $Z = \frac{1}{144} \frac{i\sqrt{3}R}{(\eta)^{12}} + \frac{1}{2}$

$$\psi_0 = \sqrt{3} \pi^3 \frac{2}{3} i \eta^2 2^{(1/6)} (t_1 Z^{(2/3)} + t_2 (Z-1)^{(2/3)})$$

$$\psi_2 = \pi \frac{1}{2} \eta^2 2^{(5/6)} ((Z-1)^{(1/3)} t_1 + Z^{(1/3)} t_2),$$

$$\psi_4 = -\frac{1}{4} \frac{i \sqrt{3} \sqrt{2} t_4 \eta^2}{\pi}$$

$$\psi_6 = t_6 + P \left(\frac{1}{12} t_2 t_1 + \frac{1}{6} t_4^2 \right) + \frac{4}{3} \psi_4 \psi_2 - \frac{1}{576} \frac{Q^2 \psi_2 \psi_0}{\pi^4 \eta^{16}} + \frac{1}{3456} \frac{Q \psi_2^2 R}{\pi^2 \eta^{20}} + \frac{1}{384} \frac{\psi_0^2 R}{\pi^6 \eta^{12}}$$

and the contravariant tensor J^* in the flat coordinates $t_0 = \tau, t_1, t_2, t_4, t_6$ reads

$$J^*(dt_i, dt_j) = \begin{bmatrix} 0 & 0 & 0 & 0 & -4i\pi \\ 0 & 0 & -8\pi^2 & 0 & 0 \\ 0 & -8\pi^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\pi^2 & 0 \\ -4i\pi & 0 & 0 & 0 & 0 \end{bmatrix}$$

As in the G_2 case, the flat coordinates show a nontrivial monodromy around the orbifold points $Z = 0, 1$ which corresponds to tori with extra symmetries.

3.6 Root system of type D_4 : fundamental Jacobi forms

The construction of Jacobi forms for the classical series A_l and B_l does not generalize unfortunately to the remaining series D_l and C_l . As we remarked, the Jacobi forms of type B_l are also Jacobi forms of type C_l and D_l under the suitable identifications: in particular their index is doubled. Therefore we should look for the generators of index 1 in the rational extension of the algebra. The most straightforward example is the Jacobi form for D_l of index 1 and weight $-l$ of the form $\psi_l = \exp(2i\pi u) \prod_{j=1}^l \alpha(x_j | \tau)$, which is exactly the square root of the lightest Jacobi form of type B_l .

We already know the only fundamental Jacobi form of index 2 and weight -6 which is inherited from B_4 and is given by

$$\varphi_6 := \alpha^2(x_1) \alpha^2(x_2) \alpha^2(x_3) \alpha^2(x_4) \sum_{j=1}^4 \wp(x_j).$$

We need to construct the remaining four fundamental Jacobi forms of index 1 and weights, 0, -2 , -4 , -4 , respectively, one of which is of the aforementioned form. They are certain linear combi-

nations of the theta functions of level 1 which are expressed by

$$\Theta_j := \sum_{\lambda \in Q + \omega_j} e^{i\pi\tau\|\lambda\|^2 - 2i\pi\langle \mathbf{x}, \lambda \rangle},$$

where $\omega_0 := 0$ and

$$\omega_1 := (1, 0, 0, 0); \quad \omega_3 := \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right); \quad \omega_4 := \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).$$

(Recall that the theta function corresponding to ω_2 has level 2.)

Now it is also easy to check that, $\forall w \in W$, $w\omega_j = \omega_j \bmod Q$, hence these theta functions are already Weyl invariant.

Under the action of the modular group they transform as

$$\begin{aligned} \Theta_0(\mathbf{x}, \tau + 1) &= \Theta_0(\mathbf{x}, \tau), & \Theta_1(\mathbf{x}, \tau + 1) &= -\Theta_1(\mathbf{x}, \tau) \\ \Theta_3(\mathbf{x}, \tau + 1) &= -\Theta_3(\mathbf{x}, \tau), & \Theta_4(\mathbf{x}, \tau + 1) &= -\Theta_4(\mathbf{x}, \tau), \end{aligned}$$

since we have $\|\omega_j\|^2 = 1$ for $j = 1, 3, 4$. Under the other generator of the modular group we find the transformation

$$\begin{aligned} \Theta_0\left(\frac{\mathbf{x}}{\tau}, -\frac{1}{\tau}\right) &= -\tau^2 e^{i\pi\frac{\|\mathbf{x}\|^2}{\tau}} \frac{1}{2} \{\Theta_0(\mathbf{x}, \tau) + \Theta_1(\mathbf{x}, \tau) + \Theta_3(\mathbf{x}, \tau) + \Theta_4(\mathbf{x}, \tau)\} \\ \Theta_1\left(\frac{\mathbf{x}}{\tau}, -\frac{1}{\tau}\right) &= -\tau^2 e^{i\pi\frac{\|\mathbf{x}\|^2}{\tau}} \frac{1}{2} \{\Theta_0(\mathbf{x}, \tau) + \Theta_1(\mathbf{x}, \tau) - \Theta_3(\mathbf{x}, \tau) - \Theta_4(\mathbf{x}, \tau)\} \\ \Theta_3\left(\frac{\mathbf{x}}{\tau}, -\frac{1}{\tau}\right) &= -\tau^2 e^{i\pi\frac{\|\mathbf{x}\|^2}{\tau}} \frac{1}{2} \{\Theta_0(\mathbf{x}, \tau) - \Theta_1(\mathbf{x}, \tau) + \Theta_3(\mathbf{x}, \tau) - \Theta_4(\mathbf{x}, \tau)\} \\ \Theta_4\left(\frac{\mathbf{x}}{\tau}, -\frac{1}{\tau}\right) &= -\tau^2 e^{i\pi\frac{\|\mathbf{x}\|^2}{\tau}} \frac{1}{2} \{\Theta_0(\mathbf{x}, \tau) - \Theta_1(\mathbf{x}, \tau) - \Theta_3(\mathbf{x}, \tau) + \Theta_4(\mathbf{x}, \tau)\} \end{aligned}$$

We immediately notice that the linear combinations

$$\psi := \Theta_3 - \Theta_4, \quad \varphi := \Theta_1 - \Theta_4$$

transform as

$$\begin{aligned} \varphi(\mathbf{x}, \tau + 1) &= -\varphi(\mathbf{x}, \tau), & \varphi\left(\frac{\mathbf{x}}{\tau}, -\frac{1}{\tau}\right) &= -\tau^2 e^{i\pi\frac{\|\mathbf{x}\|^2}{\tau}} \varphi(\mathbf{x}, \tau) \\ \psi(\mathbf{x}, \tau + 1) &= -\psi(\mathbf{x}, \tau), & \psi\left(\frac{\mathbf{x}}{\tau}, -\frac{1}{\tau}\right) &= -\tau^2 e^{i\pi\frac{\|\mathbf{x}\|^2}{\tau}} \psi(\mathbf{x}, \tau) \end{aligned}$$

Moreover, as $\tau \rightarrow i\infty$ we have,

$$\varphi = O(q) = \psi, \quad q := e^{i\pi\tau}$$

therefore $\varphi_4 := \eta^{-12}(\tau)\varphi, \psi_4 := \eta^{-12}(\tau)\psi$ are Jacobi forms of index 1 and weight -4

Lemma 3.6.1 *Setting $\theta_j(\tau) := \Theta_j(0, \tau)$ we have*

$$\theta_1(\tau) = \theta_3(\tau) = \theta_4(\tau).$$

Proof. This is not completely trivial; The equivalence $\theta_3 = \theta_4$ follows immediately

$$\begin{aligned}\theta_3 &= \sum_{n \in \mathbb{Z}^2, \sum n_i \in 2\mathbb{Z}} \exp \left\{ i\pi\tau \left[(n_1 + 1/2)^2 + (n_2 + 1/2)^2 + (n_3 + 1/2)^2 + (n_4 - 1/2)^2 \right] \right\} = \\ &= \sum_{n \in \mathbb{Z}^2, \sum n_i \in 2\mathbb{Z}} \exp \left\{ i\pi\tau \left[(n_1 + 1/2)^2 + (n_2 + 1/2)^2 + (n_3 + 1/2)^2 + (-n_4 - 1/2)^2 \right] \right\} = \theta_4 .\end{aligned}$$

The second comes from the change of basis in the lattice according to the unitary transformation

$$U := \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

which maps ω_4 into ω_1 . Q.E.D

Remark 3.6.1 In the case of D_l , we have, for the four theta function of level 1,

$$\begin{aligned}\Theta_0(\mathbf{x}, \tau + 1) &= \Theta_0(\mathbf{x}, \tau) \\ \Theta_1(\mathbf{x}, \tau + 1) &= -\Theta_1(\mathbf{x}, \tau) \\ \Theta_{l-1}(\mathbf{x}, \tau + 1) &= e^{i\pi\frac{l}{4}} \Theta_{l-1}(\mathbf{x}, \tau) \\ \Theta_l(\mathbf{x}, \tau + 1) &= e^{i\pi\frac{l}{4}} \Theta_l(\mathbf{x}, \tau) \\ \Theta_0\left(\frac{\mathbf{x}}{\tau}, -\frac{1}{\tau}\right) &= e^{i\pi\frac{\|\mathbf{x}\|^2}{\tau}} (-i\tau)^{\frac{l}{2}} \frac{1}{2} \{\Theta_0 + \Theta_1 + \Theta_{l-1} + \Theta_l\} \\ \Theta_1\left(\frac{\mathbf{x}}{\tau}, -\frac{1}{\tau}\right) &= e^{i\pi\frac{\|\mathbf{x}\|^2}{\tau}} (-i\tau)^{\frac{l}{2}} \frac{1}{2} \{\Theta_0 + \Theta_1 - \Theta_{l-1} - \Theta_l\} \\ \Theta_{l-1}\left(\frac{\mathbf{x}}{\tau}, -\frac{1}{\tau}\right) &= e^{i\pi\frac{\|\mathbf{x}\|^2}{\tau}} (-i\tau)^{\frac{l}{2}} \frac{1}{2} \{\Theta_0 - \Theta_1 + e^{i\pi\frac{l}{2}} \Theta_{l-1} + e^{i\pi\frac{l-2}{2}} \Theta_l\} \\ \Theta_l\left(\frac{\mathbf{x}}{\tau}, -\frac{1}{\tau}\right) &= e^{i\pi\frac{\|\mathbf{x}\|^2}{\tau}} (-i\tau)^{\frac{l}{2}} \frac{1}{2} \{\Theta_0 - \Theta_1 + e^{i\pi\frac{l-2}{2}} \Theta_{l-1} + e^{i\pi\frac{l}{2}} \Theta_l\} .\end{aligned}$$

Hence we have that $\psi := \Theta_{l-1} - \Theta_l$ transforms as

$$\begin{aligned}\psi(\mathbf{x}, \tau + 1) &= e^{i\pi\frac{l}{4}} \psi(\mathbf{x}, \tau) \\ \psi\left(\frac{\mathbf{x}}{\tau}, -\frac{1}{\tau}\right) &= e^{i\pi\frac{\|\mathbf{x}\|^2}{\tau}} e^{i\pi\frac{l}{2}} \psi(\mathbf{x}, \tau)\end{aligned}$$

and therefore $\psi_l := \eta^{-3l}(\tau)\psi$ is (the) a Jacobi form of weight $-l$ (notice the regularity at $\tau \rightarrow i\infty$).

Definition 3.6.1 The theta constants are the functions of τ only obtained from

$$\theta_j(\tau) := \Theta_j(0, \tau) .$$

Studying the k -linear combinations of these four theta constants of “level” 1 we can find automorphic forms. Namely we consider

$$\mathfrak{A}_k(\tau) := \sum_{i_1, \dots, i_k=0,1} C_{i_1, \dots, i_k} \prod_{j=1}^k \theta_{i_j}$$

where we have used the fact that $\theta_1 = \theta_3 = \theta_4$ and enforce the identities

$$\begin{aligned}\mathfrak{A}(\tau + 1) &= \lambda \mathfrak{A}(\tau) \\ \mathfrak{A}\left(-\frac{1}{\tau}\right) &= \tau^{2k} \mu \mathfrak{A}(\tau),\end{aligned}$$

thus obtaining a linear system for the coefficients of the polynomial. We have to take into account the behavior at $\tau \rightarrow i\infty$ given by

$$\theta_0 \rightarrow 1, \quad \theta_1 = \mathcal{O}(q).$$

Proposition 3.6.1 The following identities hold true

$$\begin{aligned}\theta_0^2 + 3\theta_1^2 &= E_4 \\ \theta_1\theta_0^2 - \theta_1^3 &= \eta^{12} \\ \theta_0^3 - 9\theta_0\theta_1^2 &= E_6\end{aligned}$$

Proof. The proof is a straightforward (and very long) computation. Q.E.D

We now look for the remaining two generators of weights -2 and 0 respectively. First of all, a similar computation as before shows

Proposition 3.6.2 The following identity hold true

$$\begin{aligned}\theta_0 \{\Theta_1(\mathbf{x}, \tau) + \Theta_3(\mathbf{x}, \tau) + \Theta_4(\mathbf{x}, \tau)\} - 3\theta_1 \Theta_0(\mathbf{x}, \tau) &= \eta^{12} \varphi_2(\mathbf{x}, \tau), \text{ with } \varphi_2 \in J_{1,-2} \\ (2\theta_1\theta_0\Theta_0(\mathbf{x}, \tau) + (\theta_1^2 + \theta_0^2)\Theta_1(\mathbf{x}, \tau) - 2\theta_1^2(\Theta_3(\mathbf{x}, \tau) + \Theta_4(\mathbf{x}, \tau))) &=: \eta^{12} \varphi_0 \text{ with } \varphi_0 \in J_{1,0}\end{aligned}$$

Notice that the defined Jacobi forms have the properties, as $\mathbf{x} \rightarrow 0$

$$\begin{aligned}\varphi_4(\mathbf{x}, \tau) &= \mathcal{O}(\|\mathbf{x}\|^4) \\ \psi_4(\mathbf{x}, \tau) &= \mathcal{O}(\|\mathbf{x}\|^4) \\ \varphi_2(\mathbf{x}, \tau) &= \mathcal{O}(\|\mathbf{x}\|^2) \\ \varphi_0(\mathbf{x}, \tau) &= \mathcal{O}(1).\end{aligned}$$

In particular they are algebraically independent because the theta functions are. We thus have

Theorem 3.6.1 The Jacobi forms of type D_4 are given by

$$\begin{aligned}\varphi_0 &:= \eta^{-12} (2\theta_1\theta_0\Theta_0(\mathbf{x}, \tau) + (\theta_1^2 + \theta_0^2)\Theta_1(\mathbf{x}, \tau) - 2\theta_1^2(\Theta_3(\mathbf{x}, \tau) + \Theta_4(\mathbf{x}, \tau))) \\ \varphi_2 &:= \eta^{-12} (\theta_0 \{\Theta_1(\mathbf{x}, \tau) + \Theta_3(\mathbf{x}, \tau) + \Theta_4(\mathbf{x}, \tau)\} - 3\theta_1\Theta_0(\mathbf{x}, \tau)) \\ \varphi_4 &:= \eta^{-12} \left(\Theta_1 - \frac{1}{2}\Theta_3 - \frac{1}{2}\Theta_4 \right) \\ \psi_4 &:= \eta^{-12} (\Theta_3 - \Theta_4)\end{aligned}$$

These Jacobi forms could be used to compute the intersection elements: the easiest way to do so is to embed them into the algebra of Jacobi forms of type B_4 . Indeed we could extend the crystallographic group of type D_4 to that of type B_4 : the extra generators clearly do not leave invariant the Jacobi forms of type D_4 and index 1 but act with a nontrivial representation. Knowing that, we can look for invariants in the bilinear products of the form $J_{\bullet,1} J_{\bullet,1}$. One can check that, indeed, there are exactly four invariants corresponding to the forms $\varphi_0^{(B_4)}$, $\varphi_2^{(B_4)}$, $\varphi_4^{(B_4)}$, $\varphi_8^{(B_4)}$. With the knowledge of this relation one could finally compute the intersection elements by change of coordinates from the intersection elements of type B_4 .

Since this case too falls in the codimension-one cases studied by Saito, one could also look for Saito's flat invariants and express them in terms of Jacobi forms.

A partial result in this direction is the work [Sat98], where Saito's flat invariants are expressed in terms of Theta functions and the prepotential ("free energy" in the jargon of chapter 5) is computed.

Moreover, in the study of the prepotential there appears a system of ODEs which is recast in Halphen's system [Ha1881]: this is a property that could show up in all these systems in a way to generalize said system, possibly in the direction of [Har99].

Chapter 4

Jacobi forms, WZW models and elliptic Chern–Simons states

In this chapter we provide an application of the theory of Jacobi forms to the Wess–Zumino–Novikov–Witten model on the torus. For an thorough introduction to the subject for novices we recommend [Tra98, Ga89, Ga99].

The basics have already been established in Section 2.2.1 and will be used here in a slightly different context. We first briefly recall the definition and properties of WZNW’s model and then follow the works [FaGa96, FaGa95, GaTra98] where this model is related to elliptic Chern–Simons’s theory. In particular at any level \varkappa (which corresponds to the “index” in the context of Jacobi forms) we can rather easily find the dimension of the space of Chern–Simons states without insertions, namely give a generating function for such dimensions.

This, jointly with the results in Ch. 3, enables to write explicitly all these states in terms of Jacobi forms for the cases A_l, B_l, C_3, D_4, G_2 (namely the cases where we have explicit formulae for the Jacobi forms). In perspective, thus, the study of Jacobi invariants looks quite promising as for this application: unfortunately we manage to reduce the study of Chern–Simons’s states with insertions to the study of Jacobi forms which are not (generically) Weyl invariant.

4.1 WZW model on the torus

We begin with a cursory general description of the model in general and later on we specialize to the case of the torus.

Let Σ be a compact smooth Riemann surface of genus g and G a compact, simple, simply connected Lie group with Lie algebra \mathfrak{g} ; the WZW model is the quantum theory over the functional space G^Σ of G -valued functions over Σ ; we will assume that we are given a complex structure on Σ , by means of which the Riemann surface becomes thus a complex *curve* \mathcal{C} ; the (chiral part of the) theory is defined by the action

$$S(g, \xi) := -\frac{i}{4\pi} \int_{\Sigma} \langle g^{-1} \partial g, g^{-1} \bar{\partial} g \rangle + \frac{i}{2\pi} \int_{\Sigma} \langle g \partial (g^{-1}), \xi \rangle - \frac{i}{12\pi} \int_{\Sigma} d^{-1} \langle [g^{-1} dg, g^{-1} dg], g^{-1} dg \rangle .$$

Here ξ stands for an antiholomorphic connection on the trivial G -bundle over Σ and has to be thought of as a source for the action.

The action defines the partition function which is formally written as

$$\mathcal{Z}(\xi) := \int e^{-\kappa S(g, \xi)} \prod_{z \in \mathcal{C}} dg(z) ,$$

where $\prod_{z \in \mathcal{C}} dg(z)$ is a formal expression to mean the product of Haar invariant measures¹ over the functional space G^Σ .

One may also consider the (unnormalized) correlation functions for fields Φ_1, \dots, Φ_n carrying a representation $\rho_n : G \rightarrow GL(V_n)$, of the gauge group G : this correlation function takes value in $\otimes_{n=1}^N GL(V_n)$

$$\Psi(\xi|\mathbf{z}) := \langle \Phi_1(z_1) \dots \Phi_N(z_N) \rangle_\xi = \int_{G^\Sigma} \left(\bigotimes_{n=1}^N \rho_n(g(z_n)) \right) e^{-\kappa S(g, \xi)} \prod_{z \in \mathcal{C}} dg(z) .$$

For the sake of brevity we will use the notation $g_{(n)} := \rho_n(g)$, for any $g \in G$.

Following [FaGa96, Ga99], the resulting Chern–Simons states Ψ are holomorphic functionals with values in $V := \otimes_{n=1}^N \text{End}(V_n)$ over the space \mathcal{A} of antiholomorphic connections A , depending also on the position of the “insertions”. These states satisfy the chiral Ward identity

$$\Psi(\xi^g|\mathbf{z}) = e^{-\kappa S(g, \xi)} \left(\bigotimes_{n=1}^N g(z_n)_{(n)} \right) \Psi(\xi, \mathbf{z}) , \quad \xi^g := \text{Ad}_g \xi + g \bar{\partial} g^{-1} .$$

In the case Σ an elliptic curve E_τ we saw in Section 2.2.1 that the generic connection ξ is gauge equivalent to a constant connection with values in \mathfrak{h}

$$\mathfrak{g}^\Sigma \ni \xi \simeq \frac{2i\pi \mathbf{x}}{\tau - \bar{\tau}} d\bar{z} , \quad \mathbf{x} \in \mathfrak{h}_\mathbb{C} .$$

By means of the gauge covariance of the Chern–Simons states we can reconstruct them once we know their value at any point of the orbit of ξ under G^Σ ; clearly, since they are quantum states, the invariance has to be understood in the quantum sense, namely projectively. Let us denote by $\Psi(\mathbf{x}, \mathbf{z})$ the value of the state at the constant connection $\frac{2i\pi \mathbf{x}}{\tau - \bar{\tau}} d\bar{z}$. Next define the function $\varphi(\mathbf{x}, \mathbf{z})$ by means of the equation

$$\Psi(\mathbf{x}|\mathbf{z}) =: \exp \left(2i\pi \kappa \frac{\|\mathbf{x}\|^2}{2(\tau - \bar{\tau})} \right) \left(\bigotimes_{n=1}^N \exp \left(2i\pi \frac{z_n - \bar{z}_n}{\tau - \bar{\tau}} \mathbf{x} \right)_{(n)} \right) \varphi(\mathbf{x}|\mathbf{z}) ,$$

where $\|\mathbf{x}\|^2 := \langle \mathbf{x}, \mathbf{x} \rangle$. Recall that there is a remnant ambiguity in the definition of $\mathbf{x} \in \mathfrak{h}$, which should really be thought as belonging to $\mathfrak{h}/(W \rtimes Q \times Q)$. The element realizing the gauge equivalence between \mathbf{x} and $\mathbf{x} + \lambda + \tau\mu$ is

$$h_{\lambda, \mu}(z, \bar{z}) := \exp \left(2i\pi \frac{z - \bar{z}}{\tau - \bar{\tau}} \lambda + 2i\pi \frac{\bar{\tau}z - \tau\bar{z}}{\tau - \bar{\tau}} \mu \right) . \quad (4.1)$$

¹Rigorously speaking such a measure does not exist, nevertheless this formal expression is used in the physical literature to evince mathematical features of the correlation functions.

Let us now compute the gauged WZW action for the transformations of h of the form in eq. (4.1). This is a map from the complex torus Σ to a S^1 inside a maximal torus T into G , therefore the Wess–Zumino term $\Gamma(h)$ is zero. The remaining gives (recall that the area of the torus is $Area = \frac{\tau - \bar{\tau}}{2i}$)

$$S\left(h_{\lambda, \mu}, \frac{2i\pi \mathbf{x}}{\tau - \bar{\tau}}\right) = 2i\pi \frac{\langle \lambda + \tau\mu, \lambda + \bar{\tau}\mu \rangle}{2(\tau - \bar{\tau})} + 2i\pi \frac{\langle \lambda + \bar{\tau}\mu, \mathbf{x} \rangle}{\tau - \bar{\tau}}.$$

Therefore

$$\begin{aligned} \Psi(\mathbf{x} + \lambda + \tau\mu | \mathbf{z}, \tau) &= e^{\varkappa S(h, \mathbf{x})} \bigotimes_{n=1}^N \exp\left(2i\pi \frac{\bar{\rho}z_n - \rho\bar{z}_n}{\tau - \bar{\tau}}\right)_{(n)} \Psi(\mathbf{x} | \mathbf{z}, \tau) = \\ &= e^{\varkappa S(h, \mathbf{x}) + 2i\pi \varkappa \frac{\langle \mathbf{x}, \mathbf{x} \rangle}{\tau - \bar{\tau}}} \bigotimes_{n=1}^N \exp\left(2i\pi \frac{z_n - \bar{z}_n}{\tau - \bar{\tau}} \mathbf{x} + 2i\pi \frac{\bar{\rho}z_n - \rho\bar{z}_n}{\tau - \bar{\tau}}\right)_{(n)} \varphi(\mathbf{x} | \mathbf{z}, \tau). \end{aligned}$$

On the other hand, by definition

$$\Psi(\mathbf{x} + \lambda + \tau\mu | \mathbf{z}, \tau) = e^{2i\pi \varkappa \frac{\langle \mathbf{x} + \lambda + \tau\mu, \mathbf{x} + \lambda + \tau\mu \rangle}{2(\tau - \bar{\tau})}} \bigotimes_{n=1}^N \left(e^{2i\pi \frac{z_n - \bar{z}_n}{\tau - \bar{\tau}} (\mathbf{x} + \lambda + \tau\mu)} \right)_{(n)} \varphi(\mathbf{x} + \lambda + \tau\mu | \mathbf{z}, \tau).$$

By comparison we compute the scalar exponential

$$\begin{aligned} &\varkappa S(h, \mathbf{x}) + 2i\pi \varkappa \frac{\langle \mathbf{x}, \mathbf{x} \rangle}{\tau - \bar{\tau}} - 2i\pi \varkappa \frac{\langle \mathbf{x} + \lambda + \tau\mu, \mathbf{x} + \lambda + \tau\mu \rangle}{2(\tau - \bar{\tau})} = \\ &= 2i\pi \varkappa \frac{\langle \lambda + \tau\mu, \lambda + \bar{\tau}\mu \rangle}{2(\tau - \bar{\tau})} + 2i\pi \varkappa \frac{\langle \lambda + \bar{\tau}\mu, \mathbf{x} \rangle}{\tau - \bar{\tau}} + 2i\pi \varkappa \frac{\langle \mathbf{x}, \mathbf{x} \rangle}{\tau - \bar{\tau}} + \\ &- 2i\pi \varkappa \frac{\langle \mathbf{x} + \lambda + \tau\mu, \mathbf{x} + \lambda + \tau\mu \rangle}{2(\tau - \bar{\tau})} = 2i\pi \varkappa \left[\frac{\tau}{2} \langle \mu, \mu \rangle + \langle \mu, \mathbf{x} \rangle \right]. \end{aligned}$$

We now turn the attention to the outer automorphisms of the holomorphic vector bundle associated to the connection $D := \bar{\partial} + \xi d\bar{z}$, namely those coming from the automorphisms of the underlying surface.

Let $\mathcal{C}_\tau := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ and $\mathcal{C}_{\tau'} := \mathbb{C}/(\mathbb{Z} + \tau'\mathbb{Z})$ be two isomorphic tori; the two moduli must be related by

$$\tau' := \frac{a\tau + b}{c\tau + d}, \quad \text{with } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$

We will consider the case $\tau' = -\frac{1}{\tau}$; the explicit isomorphism between the two curves is given by

$$\begin{aligned} \mathcal{C}_\tau &\xrightarrow{f} \mathcal{C}_{\tau'} \\ z &\longrightarrow z' = f(z) = \frac{z}{\tau}. \end{aligned}$$

Now, under the (co-)adjoint action of the gauge group, the generic connection ξ is equivalent to $2i\pi \frac{\mathbf{x}}{\tau - \bar{\tau}}$. Under the isomorphism φ corresponding to the inversion $\tau \mapsto \tau' = -\frac{1}{\tau}$ it is easy to show

that $\mathbf{x} \mapsto \frac{\mathbf{x}}{\tau}$; indeed, the if $D' := w\bar{\partial}' + \xi d\bar{z}'$ is the connection on $\mathcal{C}_{\tau'}$, then the pull-back connection is

$$D := f^*(D') = w\bar{\partial} + \frac{1}{\tau} \xi d\bar{z} .$$

If $\gamma(z', \bar{z}')$ is the gauge which puts D' into canonical form, then it acts on $D = f^*(D')$ recasting it into

$$\gamma^{-1} \circ D \circ \gamma = w\bar{\partial} + 2i\pi \frac{1}{\tau} \frac{1}{\tau' - \bar{\tau}'} \mathbf{x} = w\bar{\partial} + 2i\pi \frac{1}{\tau - \bar{\tau}} \frac{\mathbf{x}}{\tau} .$$

Since this is not a gauge transformation and the WZW action is conformally invariant, the Chern-Simons states must be projectively² invariant under the automorphism group of \mathcal{C} , hence

$$\Psi(f^*\xi) = j_\gamma(\tau) \Psi(\xi) ,$$

where the proportionality factor may depend on the moduli of the curve.

Since the dimension of the space of states does not depend on the moduli τ of the curve, such a function must be nowhere vanishing for $\tau \in \mathcal{H}$; moreover the consistency implies that it is a cocycle for $SL(2, \mathbb{Z})$ with values in $\mathcal{O}(\mathcal{H})$, hence of the form $j_\gamma(\tau) = j(c\tau + d)$.

It can be proved then (e.g. [Gu]) that $j_\gamma(\tau) = (c\tau + d)^k$ for some constant k .

Passing to the moduli space of connections, the above reads

$$\Psi(\mathbf{x}|\mathbf{z}, \tau) = (\tau)^{-k} \Psi\left(\frac{\mathbf{x}}{\tau} \middle| \frac{\mathbf{z}}{\tau}, -\frac{1}{\tau}\right) .$$

Again, comparing the two sides we find

$$\begin{aligned} LHS &= \exp\left(2i\pi\kappa \frac{\|\mathbf{x}\|^2}{2(\tau - \bar{\tau})}\right) \bigotimes_{n=1}^N \exp\left(2i\pi \frac{z_n - \bar{z}_n}{\tau - \bar{\tau}} \mathbf{x}\right)_{(n)} \varphi(\mathbf{x}|\mathbf{z}, \tau) \\ RHS &= (\tau)^{-k} \exp\left(2i\pi\kappa \frac{\|\mathbf{x}\|^2}{2\tau^2 \left(-\frac{1}{\tau} + \frac{1}{\bar{\tau}}\right)}\right) \bigotimes_{n=1}^N \exp\left(2i\pi \frac{\bar{\tau}z_n - \tau\bar{z}_n}{\tau - \bar{\tau}} \frac{\mathbf{x}}{\tau}\right)_{(n)} \varphi\left(\frac{\mathbf{x}}{\tau} \middle| \frac{\mathbf{z}}{\tau}, -\frac{1}{\tau}\right) = \\ &= (\tau)^{-k} \exp\left[2i\pi\kappa \left(\frac{\|\mathbf{x}\|^2}{2(\tau - \bar{\tau})} - \frac{\|\mathbf{x}\|^2}{2\tau}\right)\right] \bigotimes_{n=1}^N \exp\left(2i\pi \frac{\bar{\tau}z_n - \tau\bar{z}_n}{\tau - \bar{\tau}} \frac{\mathbf{x}}{\tau}\right)_{(n)} \varphi\left(\frac{\mathbf{x}}{\tau} \middle| \frac{\mathbf{z}}{\tau}, -\frac{1}{\tau}\right) . \end{aligned}$$

By comparison we find

$$(\tau)^{-k} \varphi\left(\frac{\mathbf{x}}{\tau} \middle| \frac{\mathbf{z}}{\tau}, -\frac{1}{\tau}\right) = e^{2i\pi\kappa \frac{\|\mathbf{x}\|^2}{2\tau}} \bigotimes_{n=1}^N \exp\left(2i\pi z_n \frac{\mathbf{x}}{\tau}\right)_{(n)} \varphi(\mathbf{x}|\mathbf{z}, \tau) .$$

We summarize the transformation properties (cf. [FaGa96, FaGa95, GaTra98]): to the previous ones one must add a regularity requirement in order that the states do extend also on the singular orbits of the gauge group. We do not enter the details which can be found in loc. cit., but only report the relevant formulae in the following

²This is a very common feature of symmetries in QFT, namely a classical invariance -upon quantization- goes over to a projective invariance.

Proposition 4.1.1 The normalized Chern–Simons states $\varphi(\mathbf{x}|\mathbf{z}, \tau)$ satisfy the following properties of invariance:

$$\begin{aligned}\varphi(\mathbf{x} + \lambda + \tau\mu|\mathbf{z}, \tau) &= e^{2i\pi\kappa\langle\mu, \mathbf{x}\rangle + i\pi\kappa\tau\|\mu\|^2} \bigotimes_{n=1}^N \exp(2i\pi z_n \mu)_{(n)} \varphi(\mathbf{x}|\mathbf{z}, \tau), \quad \forall \mu, \lambda \in Q; \\ \varphi(w \cdot \mathbf{x}|\mathbf{z}, \tau) &= \bigotimes_{n=1}^N w_{(n)} \varphi(\mathbf{x}|\mathbf{z}, \tau), \quad \forall w \in W; \\ \varphi\left(\frac{\mathbf{x}}{c\tau + d} \middle| \frac{\mathbf{z}}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) &= (c\tau + d)^k e^{i\pi\kappa \frac{c\|\mathbf{x}\|^2}{(c\tau + d)}} \bigotimes_{n=1}^N \exp\left(2i\pi z_n \frac{\mathbf{x}}{c\tau + d}\right)_{(n)} \varphi(\mathbf{x}|\mathbf{z}, \tau), \\ &\quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).\end{aligned}$$

To the above we must add the *regularity* condition

$$\left(\sum_{n=1}^N e^{-2i s \pi z_n} (e_\alpha)_{(n)}\right)^p \varphi(\mathbf{x} + \epsilon \mathbf{p}^\vee|\mathbf{z}, \tau) \stackrel{\epsilon \rightarrow 0}{\equiv} \mathcal{O}(\epsilon^p),$$

for any simple root α and weight \mathbf{p} with $\langle \mathbf{p}, \alpha \rangle = 1$ and for $\langle \mathbf{x}, \alpha \rangle = r + s\tau$, $r, s \in \mathbb{Z}$.

Moreover, as the theory is well defined also on degenerate tori, the states must be bounded as $\tau \rightarrow 0$ or, which is the same by virtue of conformal invariance, $\tau \rightarrow +i\infty$ ³.

In order to match with the previous notation we introduce the coordinate u , and multiply the functions φ by $\exp(2i\pi\kappa u)$: therefore we will re-denote $\varphi^z(u, \mathbf{x}, \tau) := e^{2i\pi\kappa u} \varphi(\mathbf{x}|\mathbf{z}, \tau)$. Accordingly, the previous formulae become

$$\begin{aligned}\varphi^z\left(u - \langle \mu, \mathbf{x} \rangle - \frac{\tau}{2}\|\mu\|^2, \mathbf{x} + \lambda + \tau\mu, \tau\right) &= \bigotimes_{n=1}^N \exp(2i\pi z_n \mu)_{(n)} \varphi^z(u, \mathbf{x}, \tau), \quad \forall \mu, \lambda \in Q; \\ \varphi^z(u, w \cdot \mathbf{x}, \tau) &= \bigotimes_{n=1}^N w_{(n)} \varphi^z(u, \mathbf{x}, \tau), \quad \forall w \in W; \\ \varphi^z\left(u - \frac{c\|\mathbf{x}\|^2}{2(c\tau + d)}, \frac{\mathbf{x}}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) &= (c\tau + d)^k \bigotimes_{n=1}^N \exp\left(2i\pi z_n \frac{\mathbf{x}}{c\tau + d}\right)_{(n)} \varphi^z(u, \mathbf{x}, \tau).\end{aligned}$$

Let $\Omega := \{(u, \mathbf{x}, \tau) \in \mathbb{C} \times \mathfrak{h} \times \mathcal{H}\}$ and let $V := \bigotimes_{n=1}^N V_n$; consider the trivial vector bundle $\Omega \times V$ and take its quotient by the action above. What we get is a vector bundle over $\Omega/J(\mathfrak{g})$, where $J(\mathfrak{g})$ denotes the Jacobi group associated to the lie algebra \mathfrak{g} .

The spectrum of weights (namely the possible exponents k appearing in the modular transformations) are a priori unrestricted so far.

However, with the theory of Jacobi forms, we can show that in the case of no insertions, such exponents are related in a simple way to the spectrum of weights of the fundamental Jacobi forms.

³In other terms, as functions of τ they must extend over the cusps of $SL(2, \mathbb{Z})$

4.1.1 Space of states

We now explicitly describe the space of states with N insertions.

The states which have physical interpretation [FaGa96, FaGa95] take value in the zero weight subspace $V[0] \hookrightarrow V := \bigotimes_{n=1}^N V_n$. Here V_n is a highest weight, finite dimensional irreducible representation with highest weight $\Lambda_n \in P_+$ (P_+ is the semigroup generated by positive weights) and highest-weight-vectors Ω_n . We set

$$\begin{aligned}\Lambda &:= \sum_{n=1}^N \Lambda_n = \sum_{j=1}^l m_j \alpha_j \\ \Omega &:= \bigotimes_{n=1}^N \Omega_n ,\end{aligned}$$

where l is the rank of \mathfrak{g} and $\{\alpha_j\}_{j=1..l}$ is a basis of simple positive roots.

Let $\{f^j, e^j\}$ be step generators corresponding to the simple roots α_j .

We can associate to the sequence of integers $\{m_1, \dots, m_l\}$ a *color* function [FV95] c on $\{1, 2, \dots, m := \sum_{j=1}^l m_j\}$ which is the only non-decreasing function

$$c : \{1, \dots, m\} \mapsto \{1, \dots, l\}$$

such that $|c^{-1}(\{j\})| = m_j$ for all $j = 1 \dots l$.

Let $P(c, N)$ be the set of sequences

$$P(c, N) := \{I := (i_1^1, \dots, i_{s_1}^1; \dots; i_1^N, \dots, i_{s_N}^N) : s_k^n \geq 0, i_k^n \in \{1 \dots l\}, |\{i_k^n = j\}| = m_j = |c^{-1}(\{j\})|\} ,$$

and then set

$$f_I \Omega := f_{i_1^1} \cdots f_{i_{s_1}^1} \Omega_1 \otimes \cdots \otimes f_{i_1^N} \cdots f_{i_{s_N}^N} \Omega .$$

Notice that they span $V[0]$ and are eigenvectors of $H_{(n)}$ for all $H \in \mathfrak{h}$ with eigenvalue $\Lambda_n(H) - \sum_{k=1}^{s_n} \alpha_{i_k^n}(H)$.

By the very definition, we find that

$$\bigotimes_{n=1}^N \exp(2i\pi z_n \mu)_{(n)} f_I \Omega = \exp\left(2i\pi \left\langle \mu, \sum_{n=1}^N z_n \Lambda_n^I \right\rangle\right) f_I \Omega ,$$

where we have set

$$\Lambda_n^I := \Lambda_n - \sum_{k=1}^{s_n} \alpha_{i_k^n} .$$

Therefore we have that the states are expressible as

$$\varphi(\mathbf{x}|\mathbf{z}, \tau) = \sum_{I \in P(c, N)} \Theta_I \left(\mathbf{x} - \frac{1}{\mathcal{X}} \sum_{n=1}^N z_n \Lambda_n^I, \tau \right) f_I \Omega .$$

Let us see which properties for the functions Θ_I are implied by Prop. 4.1.1: the easiest is the transformation law under the action of $W \rtimes (Q \times Q)$.

The behavior under the translations by $Q \times Q$ tells us that they are theta functions of level \varkappa . We now analyze the action of the Weyl group. First of all let us define the natural action of W over $P(c, N)$ as follows

$$f_{w \cdot I} \Omega := w \cdot f_I \Omega .$$

The covariance under the Weyl group is then translated into

$$\Theta_I(\mathbf{x}) = \Theta_{w \cdot I}(w \cdot \mathbf{x}) .$$

As for the modular properties, let us study the behavior under the map $(\mathbf{x}|\mathbf{z}, \tau) \mapsto \left(\frac{\mathbf{x}}{\tau} \middle| \frac{\mathbf{z}}{\tau}, -\frac{1}{\tau}\right)$.

After a straightforward computation we find

$$\begin{aligned} & \Theta_I \left(\frac{\mathbf{x}}{\tau} - \frac{1}{\varkappa} \sum_{n=1}^N \frac{z_n}{\tau} \Lambda_n^I, -\frac{1}{\tau} \right) = \\ & = \exp \left(i\pi \varkappa \frac{\|\mathbf{x}\|^2}{\tau} - 2i\pi \left\langle \mathbf{x}, \sum_{n=1}^N z_n \Lambda_n^I \right\rangle \right) \Theta_I \left(\mathbf{x} - \frac{1}{\varkappa} \sum_{n=1}^N z_n \Lambda_n^I, \tau \right) . \end{aligned}$$

A small manipulation will give a better transformation; namely, consider the new function

$$F_I := \exp \left(\frac{2i\pi}{\varkappa} \frac{\eta'(\tau)}{\eta(\tau)} \left\| \sum_{n=1}^N z_n \Lambda_n^I \right\|^2 \right) \Theta_I \left(\mathbf{x} - \frac{1}{\varkappa} \sum_{n=1}^N z_n \Lambda_n^I, \tau \right) ,$$

where $\eta(\tau)$ is the Dedekind's eta function; setting $g_1(\tau) := \frac{\eta'(\tau)}{\eta(\tau)}$, we find that

$$F_I \left(\frac{\mathbf{x}}{\tau} \middle| \frac{\mathbf{z}}{\tau}, -\frac{1}{\tau} \right) = (c\tau + d)^k \exp \left(\frac{i\pi \varkappa}{\tau} \left\| \mathbf{x} - \frac{1}{\varkappa} \sum_{n=1}^N z_n \Lambda_n^I \right\|^2 \right) F_I(\mathbf{x}|\mathbf{z}, \tau) .$$

Therefore we find that F_I are **Jacobi forms** of index \varkappa^4 and some (unspecified) weight

$$F_I(\mathbf{x}|\mathbf{z}, \tau) = \varphi_I \left(\mathbf{x} - \frac{1}{\varkappa} \sum_{n=1}^N z_n \Lambda_n^I, \tau \right) .$$

We therefore have

Proposition 4.1.2 The Chern–Simons states with N insertions are given by

$$\varphi(\mathbf{x}|\mathbf{z}, \tau) = \sum_{I \in P(c, N)} \exp \left(-\frac{2i\pi}{\varkappa} g_1(\tau) \left\| \sum_{n=1}^N z_n \Lambda_n^I \right\|^2 \right) \varphi_I \left(\mathbf{x} - \frac{1}{\varkappa} \sum_{n=1}^N z_n \Lambda_n^I, \tau \right) f_I \Omega ,$$

where $\varphi_I(\mathbf{x}, \tau)$ are generically non Weyl-invariant Jacobi forms of index \varkappa .

We are thus lead to the study of the algebra of non Weyl invariant Jacobi forms of weight k (in general) and index \varkappa ; unfortunately this algebra has not been studied, except in the case of *Weyl-invariant* Jacobi forms.

Note that this mathematical description must be supplemented by other requirements to match the regularity condition. We are not going any further in this direction in this thesis.

⁴In the context of Jacobi forms, the level is called index

4.1.2 States with zero insertions

In the case of zero insertions, Chern–Simons states are in correspondence with Weyl–invariant Theta functions of index \varkappa .

We know that the linear span of Weyl–invariant Theta functions coincides with the $\mathcal{O}(\mathcal{H})$ –linear span of Jacobi forms. Moreover we have already seen (Thm. 2.1.1) that the algebra of Weyl–invariant Theta functions \widetilde{Th}_\bullet^W is freely generated by $l + 1$ Theta functions.

It appears that the basis of fundamental Jacobi forms is simply a more convenient although equivalent basis of generators for this algebra, inasmuch they have well-definite properties of modular invariance.

It is known [Ve88] that the Chern–Simons states on the elliptic curve E_τ belong to the linear span of the characters $\chi_{\mathbf{p},\varkappa}$ of the irreducible extremal (highest-weight) unitary representations of Kac–Moody’s algebra. Such characters are labelled by (dominant) weights $\mathbf{p} \in P \bmod \varkappa Q$ and are invariant under the finite Weyl group, therefore they belong to the $\mathcal{O}(\mathcal{H})$ –linear span of Weyl invariant Theta function of index \varkappa .

Such characters are not invariant under the modular group, but transform according to a unitary representation: in particular, under the inversion \mathfrak{S} (i.e. $\tau \mapsto -\frac{1}{\tau}$) they transform according to

$$\chi_{\mathbf{p},\varkappa} \mapsto \sum_{\mathbf{p}'} S_{\mathbf{p},\mathbf{p}'} \chi_{\mathbf{p}',\varkappa} .$$

The unitary matrix $S_{\mathbf{p},\mathbf{p}'}$ can be computed explicitly [KP84].

The dimension of the space of Chern–Simons states is independent of the complex structure, hence, as a function of \mathbf{x} , they never identically vanish as τ varies in \mathcal{H} . The same is true for the fundamental Jacobi forms $\{\varphi_0, \dots, \varphi_l\}$ and for any their polynomial with constant coefficients. Therefore, in particular, the basis of Th_\varkappa provided by monomials in the fundamental Jacobi forms of total level (\leftrightarrow index) \varkappa is a basis of projectively invariant Chern–Simons states.

This basis gives the space \widetilde{Th}_\varkappa a natural (non-positive) grading provided by the weight: $\widetilde{Th}_\varkappa = \mathcal{O}(\mathcal{H}) \bigoplus_{k \leq 0} Th_{\varkappa,k}$. We argue that any other projectively invariant basis gives the same gradation. Indeed, any other such basis is obtained by a linear (quasi-homogeneous) transformation with coefficients in \mathbf{M}_\bullet : since this is a *positively* graded ring, such a transformation is “upper triangular”. If the determinant of the whole change of basis is a modular form of strictly positive weight, then it vanishes for some τ in \mathcal{H} , which we cannot allow⁵ because the dimension of the space of Chern–Simons states is independent of the conformal class of the torus.

Therefore the change of basis in $Th_{\varkappa,k}$ is given by an invertible matrix with constant coefficients plus a linear (quasi-homogeneous) map with coefficients in \mathbf{M}_\bullet from the subspaces of weight less

⁵In fact for any modular function G_K one has the following formula for the number of zeroes (or poles) in a fundamental domain [Se]

$$v_\infty + \frac{1}{2}v_i + \frac{1}{3}v_\rho + \sum_{p \in \mathcal{H}/\Gamma}^* v_p = \frac{K}{6} ,$$

where v_p denotes the order of the function at the point p , $\rho := e^{\frac{2\pi i}{3}}$, and the star on the symbol of summation means a summation over the points of \mathcal{H}/Γ distinct from the classes of i and ρ . This formula implies that there are always zeroes in a fundamental domain.

than k into $Th_{\varkappa,k}$.

Therefore the dimensions of these subspaces are preserved. We have proved

Proposition 4.1.3 The space of modular projectively invariant Chern-Simons states of level \varkappa with zero insertions can be identified with the subspace of polynomials with *constant* coefficients of index \varkappa in the fundamental generators $\{\varphi_0, \dots, \varphi_l\}$ of the algebra of Jacobi forms. This basis is not canonical but the grading induced by the weight is.

We pass to the computation of the dimensions. We can build a generating function

$$F(y) := \prod_{j=0}^l \frac{1}{(1 - y^{m_j})} = \sum_{\varkappa=0}^{\infty} d_{\varkappa} y^{\varkappa} ,$$

for the dimensions d_{\varkappa} of the space of Chern Simons states with zero insertions⁶.

Moreover we are able to compute explicitly the spectrum of the weights of the corresponding projectively invariant Chern-Simons states. Indeed, since we are considering \mathbb{C} polynomials in the fundamental Jacobi forms, we can decompose as before the space of Chern-Simons states at level \varkappa into subspaces of weight $-k$ (with k nonnegative). The spectrum and dimensions of such subspaces are then read out of the generating function

$$F(x, y) := \prod_{j=0}^l \frac{1}{1 - y^{m_j} x^{-k_j}} = \sum_{\varkappa=0}^{\infty} \sum_{k=0}^{\infty} y^{\varkappa} x^{-k} j_{-k,m} .$$

It is an interesting question whether we can associate to this spectrum of weights a *physically* relevant operator, and which is its interpretation.

This subject will not be presently pursued any further but left to subsequent publications.

⁶In fact this formula is nothing but the character formula for the dimensions of the space of level \varkappa Weyl-invariant Theta functions as follows from [Lo80].

Chapter 5

Relation between Jacobi groups and Frobenius structures on Hurwitz spaces

In this chapter we identify the Jacobi group of type A_l , B_l and G_2 with the monodromy of a suitable Frobenius structure; the identification will be based on the explicit formula for the generating function of the Jacobi forms.

In order to be self-contained as far as it is possible, we recall the due definitions of the objects we are going to use, namely Frobenius manifolds on one hand, Hurwitz spaces on the other and explain how it is possible to give Hurwitz space a structure of Frobenius manifold, following [Du93]. The main result of this part can be expressed as follows:

the quotient space $\Omega/J(\mathcal{A}_l)$ is naturally isomorphic to the moduli space of elliptic functions of degree $l + 1$ with only one pole.

This identification will appear explicitly and allows us to build a structure of Frobenius manifold over a suitable covering of this space; this covering branches around a divisor in Ω/J defined by the zero locus of the lightest Jacobi form.

Before entering the detail we give an account of the necessary mathematical objects.

5.1 Frobenius manifolds

We recall the basic definitions and properties of a Frobenius manifold.

Definition 5.1.1 *A Frobenius algebra \mathcal{A} is a unital, commutative, associative (\mathbb{C}) algebra endowed with a invariant nondegenerate bilinear pairing $\eta(\circ, \circ) : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathbb{C}$, in the sense that*

$$\eta(A \cdot B, C) = \eta(A, B \cdot C) \quad \forall A, B, C \in \mathcal{A} .$$

It follows that $\eta(\cdot, \cdot)$ is symmetric, for $\eta(A, B) = \eta(\mathbf{1}, A \cdot B) = \eta(\mathbf{1}, B \cdot A) = \eta(B, A)$. The notion of Frobenius manifold is now the following

Definition 5.1.2 A Frobenius manifold M is a smooth manifold which is endowed with a structure of Frobenius algebra in the tangent space at each point (and henceforth a nondegenerate symmetric tensor $\eta(\cdot, \cdot)$ of type $(0, 2)$ and a $(0, 3)$ (symmetric) tensor of the structure constants $g(X, Y, Z) := \eta(X, Y \cdot Z)$) and the following properties hold:

1. the Levi-Civita connection defined by the metric $\eta(\circ, \circ)$ is flat;
2. the unit vector field $\mathbf{1}$ is parallel, namely $\nabla_X \mathbf{1} = 0$;
3. the $(0, 4)$ tensor of its (covariant) derivatives $(\nabla_X g)(Y, Z, W)$ is completely symmetric;
4. there exist a vector field \mathbf{E} (the Euler vector) which is covariantly linear $(\nabla \nabla)_{X, Y} \mathbf{E} = 0$, $\forall X, Y \in \Gamma(TM)$, and

$$(a) \quad [\mathbf{E}, \mathbf{1}] = -\mathbf{1};$$

$$(b) \quad (\mathcal{L}_{\mathbf{E}} \eta)(X, Y) := \mathbf{E} \eta(X, Y) - \eta([\mathbf{E}, X], Y) - \eta(X, [\mathbf{E}, Y]) = (2 - d) \eta(X, Y) ;$$

$$(c) \quad (\mathcal{L}_{\mathbf{E}} g)(X, Y, Z) = \mathbf{E}(g(X, Y, Z)) - g([\mathbf{E}, X], Y, Z) - g(X, [\mathbf{E}, Y], Z) - g(X, Y, [\mathbf{E}, Z]) = (3 - d)g(X, Y, Z);$$

namely the \mathbf{E} generates conformal rescalings of the metric and of the Frobenius structure.

5. The $(4, 0)$ -tensor (∇g) is totally symmetric, or equivalently

$$(\nabla g)(X, Y, Z, W) := (\nabla_W g)(X, Y, Z) = (\nabla_X g)(W, Y, Z) .$$

Observe that since \mathbf{E} is a conformal Killing vector field it must satisfy also $div(\mathbf{E}) = -\frac{(2-d)}{2n}$; moreover it follows from the above axioms that

$$[\mathbf{E}, X \cdot Y] - [\mathbf{E}, X] \cdot Y - X \cdot [\mathbf{E}, Y] = X \cdot Y . \tag{5.1}$$

On the spectrum of \mathbf{E} we have

Lemma 5.1.1 If the grading operator $Q := \nabla \mathbf{E}$ is diagonalizable, then the Euler vector can be represented by

$$\mathbf{E} = \sum_1^n ((1 - q_i)t_i + r_i) \partial_i \tag{5.2}$$

for suitable constants q_i, r_i and suitable flat coordinates t_i .

Notice that, up to a translation in the flat coordinates, we can then recast the Euler vector in the form

$$\mathbf{E} = \sum_1^n (1 - q_i)t_i \partial_i + \sum_{i|q_i=1} r_i \partial_i .$$

Remark 5.1.1 The flat coordinates t_i 's which diagonalize the grading operator $\nabla \mathbf{E}$ are unique up to linear transformations which do not mix coordinates with different scaling dimension.

Moreover notice that $\mathbf{1}$ is an isotropic vector for η except in the case $d = 0$; indeed $(2 - d)\eta(\mathbf{1}, \mathbf{1}) = (\mathcal{L}_{\mathbf{E}} \eta)(\mathbf{1}, \mathbf{1}) = \mathbf{E} \eta(\mathbf{1}, \mathbf{1}) - 2\eta(\mathcal{L}_{\mathbf{E}} \mathbf{1}, \mathbf{1}) = 2\eta(\mathbf{1}, \mathbf{1})$.

We now define the scaling exponents as

Definition 5.1.3 A function $\varphi : M \rightarrow \mathbb{C}$ is said to be quasi-homogeneous of scaling exponent d_φ if it is an eigenfunction of the Euler vector,

$$\mathbf{E}(\varphi) = d_\varphi \varphi .$$

This means that the coordinate functions t_i defined before are quasihomogeneous with scaling dimensions $d_i = (1 - q_i)$.

We now give the

Proposition 5.1.1 The structure constants tensor $g(X, Y, Z)$ is the third covariant derivative of a locally well defined function F (called **Free energy**),

$$g(X, Y, Z) = (\nabla \nabla \nabla F)_{XYZ} . \tag{5.3}$$

Moreover this function is almost-quasihomogeneous of degree $d_F = 3 - d$, namely quasihomogeneous up to a function in the kernel of $\nabla \nabla \nabla$ (i.e. a function which is at most quadratic in local flat coordinates)

Proof. Since ∇g is completely symmetric, the proof of existence of the local function F follows easily in flat coordinates.

From $(\mathcal{L}_{\mathbf{E}}g) = (3 - d)g$ it follows (using $\mathcal{L}_{\mathbf{E}} \circ \nabla = \nabla \circ \mathcal{L}_{\mathbf{E}}$) that

$$\nabla \nabla \nabla ((3 - d)F - \mathbf{E}(F)) = 0 .$$

The proof is thus complete. Q.E.D

5.1.1 Intersection form

Since the invariant metric η gives a isomorphism between the tangent and cotangent bundle, we can define a Frobenius structure on the cotangent bundle as well, which we will indicate again as $\omega \cdot \alpha$, for $\omega, \alpha \in \Gamma(T^*M)$.

Definition 5.1.4 The intersection form is the bilinear pairing in T^*M defined by

$$(\omega, \alpha)^* := \underbrace{(\omega \cdot \alpha)}_{\in \Gamma(T^*M)} (\mathbf{E}) .$$

One can prove that it is almost everywhere nondegenerate, hence it defines a new “metric” (denoted by $\mathcal{I}(\cdot)$ as its associated (2,0) tensor) on the tangent bundle: one can show that [Du93]

Proposition 5.1.2 The metric $\mathcal{I}(\cdot)$ is flat and $\forall \lambda \in \mathbb{C}$, the contravariant metric $G^*(\lambda) := \eta^*(\cdot) + \lambda \mathcal{I}^*(\cdot)$ is flat as well and the contravariant connection $\tilde{\nabla}_{(\lambda)}$ on forms is given by $\tilde{\nabla}_{(\lambda)} := \tilde{\nabla}^{(\eta)} + \lambda \tilde{\nabla}^{(\mathcal{I})}$, where $\tilde{\nabla}^{(\eta)}$ and $\tilde{\nabla}^{(\mathcal{I})}$ denote the contravariant connections (acting on one-forms) of the metrics η and \mathcal{I} respectively¹ The family of metrics $G^*(\lambda)$ as λ varies, is called a *flat pencil* of metrics.

¹By contravariant connection on one-forms of a metric g over a manifold M we mean the map

$$\nabla : T^*M \otimes \Gamma(T^*M) \rightarrow \Gamma(T^*M)$$

Properties and relations. There are some differential relation between the two metrics, as it is shown hereafter; these relation allow to reconstruct the invariant metric η and the free energy from the knowledge of the intersection form and the unit vector field as we will see.

Lemma 5.1.2 *We have $\mathcal{L}_1 g = \nabla_1 g = 0$.*

Proof. For any $X, Y, Z \in \Gamma(TM)$ and from the definitions (recalling that $\nabla \mathbf{1} = 0$) we find

$$\begin{aligned} (\mathcal{L}_1 g)(X, Y, Z) &= \mathbf{1} (g(X, Y, Z)) - g([\mathbf{1}, X], Y, Z) - g(X, [\mathbf{1}, Y], Z) - g(X, Y, [\mathbf{1}, Z]) = \\ &= (\nabla_1 g)(X, Y, Z) = (\nabla_Z g)(X, Y, \mathbf{1}) = (\nabla_Z \eta)(X, Y) = 0 . \end{aligned}$$

Q.E.D

Hence we find

Lemma 5.1.3 *We have, for any $x, y \in \Gamma(T^*M)$,*

$$\eta^*(x, y) = (\mathcal{L}_1 \mathcal{J}^*)(x, y) .$$

Proof. Let X, Y denote the dual vectors to x, y (which are co-vectors) (explicitly $x(\bullet) =: \eta(X, \bullet)$), we find (recall $\mathcal{J}^*(x, y) := g(X, Y, \mathbf{E})$)

$$\begin{aligned} (\mathcal{L}_1 \mathcal{J}^*)(x, y) &= \mathbf{1} (g(X, Y, \mathbf{E})) - g(\overbrace{[\mathbf{1}, X]}^{=\nabla_1 X}, Y, \mathbf{E}) - g(X, \overbrace{[\mathbf{1}, Y]}^{\nabla_1 Y}, \mathbf{E}) = \\ &= \underbrace{(\nabla_1 g)}_{=0}(X, Y, \mathbf{E}) + g(X, Y, \underbrace{\nabla_1 \mathbf{E}}_{=1}) = \eta(X, Y) =: \eta^*(x, y) . \end{aligned}$$

Q.E.D

Lemma 5.1.4 *The functions $G^{ij} := \mathcal{J}^*(dt_i, dt_j)$ are homogeneous of degree $d_{ij} = (1 + d) - q_i - q_j$.*

Proof. Recalling that $[\mathbf{E}, \frac{\partial}{\partial t_i}] = (q_i - 1) \frac{\partial}{\partial t_i}$ we find

$$\begin{aligned} \mathbf{E}((dt_i, dt_j)^*) &= \mathbf{E} \left(g \left((dt_i)^\#, (dt_j)^\#, \mathbf{E} \right) \right) = (\mathcal{L}_{\mathbf{E}} g) \left((dt_i)^\#, (dt_j)^\#, \mathbf{E} \right) + \\ &+ g \left([\mathbf{E}, (dt_i)^\#], (dt_j)^\#, \mathbf{E} \right) + g \left((dt_i)^\#, [\mathbf{E}, (dt_j)^\#], \mathbf{E} \right) = (**) . \end{aligned}$$

We now recall that (from $\mathcal{L}_{\mathbf{E}}(\eta^{-1}) = (d - 2)\eta^{-1}$) we have, for an arbitrary $\omega \in \Gamma(T^*M)$,

$$[\mathbf{E}, \omega^\#] = (d - 2)\omega^\# + (\mathcal{L}_{\mathbf{E}}(\omega))^\# ,$$

and hence $[\mathbf{E}, (dt_i)^\#] = (d - 2 + 1 - q_i)(dt_i)^\#$. Therefore we can complete the chain of equation (**), as follows

$$(**) = [3 - d + (d - 1 - q_i) + (d - 1 - q_j)] \mathcal{J}^*(dt_i, dt_j) = (1 + d - q_i - q_j) \mathcal{J}^*(dt_i, dt_j) .$$

Q.E.D

$$\xi \otimes \alpha \rightarrow \nabla_{\xi^\sharp} \alpha ,$$

where ∇ is the Levi-Civita connection and ξ^\sharp is the vector associated to ξ by means of the isomorphism given by the metric.

Lemma 5.1.5 *We have*

$$\mathcal{J}^*(dt_i, dt_j) = (1 + d - q_i - q_j) \nabla_{(dt_i)^\#} \nabla_{(dt_j)^\#} F .$$

Proof. Recalling that

$$\mathcal{J}^*(dt_i, dt_j) = (\nabla \nabla \nabla F)_{(dt_i)^\# (dt_j)^\# \mathbf{E}}$$

and that $[\mathcal{L}_{\mathbf{E}}, \nabla] = 0$ we find (for brevity we denote $(dt_i)^\#, (dt_j)^\#$ by I, J respectively)

$$\begin{aligned} (\nabla \nabla \nabla F)_{\mathbf{E} I J} &= \nabla_{\mathbf{E}} \nabla_I \nabla_J F = \mathbf{E}(\nabla_I \nabla_J F) = \mathcal{L}_{\mathbf{E}}(\nabla_I \nabla_J F) + \nabla_{[\mathbf{E}, I]} \nabla_J F + \nabla_I \nabla_{[\mathbf{E}, J]} F = \\ &= \underbrace{\mathcal{L}_{\mathbf{E}}(\nabla_I \nabla_J F)}_{=(3-d)\nabla_I \nabla_J F} + (d-1-q_i + d-1-q_j) \nabla_I \nabla_J F = (1+d-q_i-q_j) \nabla_{(dt_i)^\#} \nabla_{(dt_j)^\#} F . \end{aligned}$$

Q.E.D

5.1.2 Reconstruction

Let us suppose we are given a Frobenius manifold M and we know only the scaling dimensions d, q_1, \dots, q_n , the Euler vector field \mathbf{E} , the unit vector field $\mathbf{1}$ and the intersection form $\mathcal{J}(\cdot, \cdot)$; then, from the previous lemmas we have as a corollary that we can uniquely reconstruct the full Frobenius structure by setting

$$\eta^{-1} := \mathcal{L}_{\mathbf{1}}(\mathcal{J}^*)$$

and finding the flat coordinates of η as homogeneous functions and then find the structure constants by imposing that

$$(1 + d - q_i - q_j) H^F \left((dt_i)^\#, (dt_j)^\# \right) = \mathcal{J}^*(dt_i, dt_j) . \quad (5.4)$$

Of course this procedure goes through if $q_i + q_j \neq d + 1, \forall i, j = 1..n$, otherwise there may be some obstruction or ambiguity in the construction of the free energy F ; this is the only effective way to find the free energy in many actual examples as we shall do for G_2 later.

5.1.3 Monodromy group of a Frobenius manifold

Both metrics $\eta(X, Y)$ and $\mathcal{J}(X, Y)$ are flat and it is natural to study their mutual relations.

Since \mathcal{J} comes from the inversion of *almost everywhere* nondegenerate bilinear pairing of the cotangent bundle, it is defined almost everywhere, namely outside the locus Δ where the determinant of $\mathcal{J}^* : T^*M \rightarrow TM$ vanishes.

Now the flat complex manifold, $(M/\Delta, \mathcal{J})$ is not simply connected and hence we have a nontrivial holonomy group at any point, which is a discrete subgroup of $O(n, \mathbb{C})$.

To be more specific let y_1, \dots, y_n be the flat normal coordinates at $p_0 \in M$ (fixed and outside Δ) of the intersection form \mathcal{J} , and express them as functions of the flat coordinates t_1, \dots, t_n of the invariant metric η ; since \mathcal{J} degenerates on Δ , its Christoffel symbols (which enter in the equation defining the flat coordinates y_i) have singularities on Δ . As a result, the germs of functions $y_i(t_1, \dots, t_n)$ will be in general multivalued for loops around the discriminant. This implies that the result of a non-contractible loop γ around the discriminant will be a linear affine transformation

of the y_i 's whose linear part is clearly an \mathcal{J} -orthogonal transformation: it is also clear that the correspondence which associates to each non-contractible loop $\gamma \in \pi_1(p_0; M/\Delta)$ this linear affine transformation of the coordinates y_j is a group homomorphism. This map

$$\begin{aligned} \mathbf{M} : \pi_1(p_0; M/\Delta) &\longrightarrow \text{Aff}(\mathcal{J}, n) \\ \gamma &\mapsto \mathbf{M}_\gamma, \end{aligned}$$

is a group homomorphism from the fundamental group of M/Δ onto a (discrete) subgroup of the affine transformations of the functions y_i 's: the image under \mathbf{M} of the fundamental group $\pi_1(p_0; M/\Delta)$ is called the **monodromy group of the Frobenius manifold**.

Vice-versa one could ask to solve the inverse problem, namely that of finding a Frobenius manifold whose monodromy group is a given discrete group of affine transformations preserving a (nondegenerate) bilinear form \mathcal{J} : this is actually the problem that motivated this study.

5.2 Hurwitz spaces and Frobenius structures

In this section we rephrase the contents of [Du93], Lecture 5, and adapt the notation to the present purposes.

Hurwitz spaces are moduli spaces of certain meromorphic functions on algebraic smooth curves of fixed genus. Hereafter we give a short account of their structure and definitions and later we explain how there can be defined Frobenius structures.

Let C be a compact Riemann surface of genus g and let $\lambda : C \rightarrow \mathbb{C}P^1$ of degree N ; we moreover fix the type of ramification over the point at infinity $\infty \in \mathbb{C}P^1$ assuming that $\lambda^{-1}(\infty) = \{\infty_0, \infty_1, \dots, \infty_m \in C\}$ and that the respective degrees at these points be $n_0 + 1, n_1 + 1, \dots, n_m + 1$. These data (namely the genus g , the number of sheets and the ramification at infinity) fix uniquely the total number of ramification points in C , say $\{P_1, \dots, P_n, \infty_0, \dots, \infty_m\}$; by means of the Riemann-Roch theorem we find (notice that we must have $N = m + 1 + n_0 + \dots + n_m$)

$$\overbrace{\text{deg}(K_C)}^{=2g-2} = N \overbrace{\text{deg}(K_{\mathbb{C}P^1})}^{=-2} + \overbrace{\text{deg}(B)}^{=n+n_0+\dots+n_m} \Rightarrow n = 2g + n_0 + \dots + n_m + 2m .$$

Therefore the smooth (i.e. non-orbifold) part of the moduli space $M_{g,m;\{n_j\}}$ of these data has dimension n .

As parameters for the point in this space we can take $(u_1, \dots, u_n) := (\lambda(P_1), \dots, \lambda(P_n)) \in (\mathbb{C}P^1)^{\times n}$; Summarizing

Definition 5.2.1 *The Hurwitz space $M_{g,m;n_0,\dots,n_m}$ is the moduli space of curves C of genus g endowed with a N branched covering λ of $\mathbb{C}P^1$, $\lambda : C \rightarrow \mathbb{C}P^1$ with $m + 1$ branching points over $\infty \in \mathbb{C}P^1$ of branching degree $n_\nu + 1$, $\nu = 0..m$.*

In this kind of spaces the usual notion of equivalence involves also the quotient by the automorphism group of the target space (here $\mathbb{C}P^1$) [Na84], but for the present purposes we will consider the (trivial) principal bundle with structure group the automorphisms of $\mathbb{C}P^1$ which fix one point (the

infinity); this simply means that in the kind of spaces we are considering the notion of equivalence involves only the automorphism group of C and moreover the affine group $\mathbb{C}^* \times \mathbb{C}$ acts on the functions λ as $a\lambda + b$, for $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$. For the sake of clarity we specify the relevant notion of equivalence:

Definition 5.2.2 *Two pairs (C, λ) and $(\tilde{C}, \tilde{\lambda})$ are Hurwitz-equivalent if there exists an analytic isomorphism $\vartheta : C \rightarrow \tilde{C}$ such that $\lambda \circ \vartheta = \tilde{\lambda}$.*

In the following we will use a covering of the Hurwitz spaces which we now describe. First of all notice that, since we are actually considering curves of genus g with $m + 1$ marked points, for $2g + m + 1 \geq 3$ the curve C is stable, that is the group of automorphisms is discrete; for $g \geq 1$ the covering of their moduli space is accomplished considering a symplectic basis of cycles in the homology of the curve².

We have also to consider the points ∞_ν , $\nu = 0..m$: therefore, for each of them, ∞_ν we fix a local uniformizing function, namely a function w_ν such that $w_\nu^{n_\nu+1} = \lambda$ in a neighborhood of ∞_ν . Summarizing

Definition 5.2.3 *The covering space $\widehat{M}_{g,m;n_0\dots n_m}$ is defined as the sets of genus g curves with $m + 1$ marked points endowed with a symplectic basis in the homology and a N -sheeted covering λ of $\mathbb{C}P^1$ with a fixation of local uniformizing coordinates $\{w_\nu\}$,*

$$\widehat{M}_{g,m;n_0\dots n_m} := \{(C, \lambda; w_0, \dots, w_m; \{a_1..a_g, b_1\dots, b_g\})\} .$$

5.2.1 Frobenius structures on $\widehat{M}_{g,m;n_0\dots n_m}$

Over the space $M_{g,m;n_0\dots n_m}$ we consider the coordinates u_1, \dots, u_n as spanning a semisimple commutative, associative, graded and unital algebra in the tangent space, with $\{u_i = \lambda(P_i) \mid d\lambda(P_i)\}$ as local coordinates. Explicitly the algebra structure is as follows (setting $\partial_i := \frac{\partial}{\partial u_i}$)

1. Multiplication: $\partial_i \cdot \partial_j = \delta_{ij} \partial_i$;
2. Unity: $\mathbf{1} := \sum_1^n \partial_i$;
3. Euler field $\mathbf{E} := \sum_1^n u_i \partial_i$.

In order to obtain a Frobenius structure we must define an invariant inner product such that the resulting metric is flat and satisfies all the axioms of Frobenius manifold.

For any one form $\Omega \in \Gamma(T^*M)$ we can define an invariant inner product as

$$\langle X, Y \rangle_\Omega := \Omega(X \cdot Y)$$

which is nondegenerate provided that Ω nowhere vanishes. There is a natural way to build one forms on the Hurwitz space starting from a representative as follows. Let Q be a quadratic

²We recall that this amounts to choosing $2g$ cycles $\{a_1, \dots, a_g, b_1, \dots, b_g\}$ which have intersection number $a_i \cdot a_j = b_i \cdot b_j = 0$ and $a_i \cdot b_j = \delta_{ij}$.

differential on C (i.e. $Q \in \Gamma(T^*C \otimes T^*C)$) and set

$$\Omega_Q := \sum_1^n du_i \operatorname{res}_{P_i} \frac{Q}{d\lambda}$$

Notice that this definition of the one form Ω_Q is independent of the representative and also invariant under the structure group (here simply the affine transformations of the plane): in fact, if $\vartheta : \tilde{C} \rightarrow C$ and $\tilde{\lambda} = a\lambda + b$, then $\tilde{u}_i := a\lambda \circ \vartheta + b = au_i + b$, $\tilde{Q} = \vartheta^*Q$, therefore

$$\sum_1^n d\tilde{u}_i \operatorname{res}_{P_i} \frac{\tilde{Q}}{d\tilde{\lambda}} = \sum_1^n du_i \operatorname{res}_{P_i} \frac{Q}{d\lambda},$$

and hence the definition is well posed.

It is clear that if the differential Q is **$d\lambda$ -divisible**, namely if it has the form $\phi \otimes d\lambda$ and ϕ is holomorphic on the zeroes of $d\lambda$, then the corresponding differential on the Hurwitz space is zero; this allows for an enlargement of the class of quadratic differential that we can use. In fact we can consider the larger class of quadratic differentials on the universal covering of the curve C which have the property that their continuation along a closed curve $\gamma \subset C$ is changed by a $d\lambda$ -divisible differential, namely

$$Q \mapsto Q + q_\gamma \otimes d\lambda.$$

One can consider quadratic forms Q which are squares of a differential ϕ of certain type, namely $Q = \phi \otimes \phi$: these differentials ϕ are called **primary differentials**. The types that lead to Frobenius structures are listed below

1. An Abelian differential of the second kind³ with poles only at the poles of λ , $\infty_0, \dots, \infty_m$ with orders less than n_ν (in such a way that $\frac{\phi \otimes \phi}{d\lambda}$ has no residues there) and such that

$$\oint_{a_j} \phi = 0, \quad i = 1..g.$$

(Such differentials are said *normalized*)

2. An Abelian differential of the second kind of the form

$$\phi = \sum_{k=1}^m D_k \theta_k$$

where θ_k are normalized Abelian differentials of the second kind with only one pole at ∞_k

$$\theta_k = -d\lambda + \text{regular terms near } \infty_k.$$

³On a curve C the Abelian differentials (namely the analytic one forms) are said to be of the **first kind** if it is holomorphic everywhere, of the **second kind** if it has only poles without residues and of the **third kind** if there are poles with residues.

3. An Abelian differential of the third kind

$$\phi = \sum_{k=1}^m D_k \rho_k$$

where ρ_k are normalized Abelian differentials of the third kind and simple poles at ∞_k and ∞_0 of residues $+1$ and -1 respectively.

4. A multivalued normalized differential (namely a differential on the universal cover of C) with increment along one of the b -cycle, say the j -th, of the form

$$\phi \mapsto \phi + \delta_{ij} d\lambda.$$

5. Any differential of the first kind, which can be written for convenience as

$$\phi = \sum_{j=1}^g C_j \omega_j$$

where the holomorphic differentials ω_j are the Poincare' duals to the b -cycles, namely $\oint_{a_k} \omega_j = \delta_{jk}$.

In [Du93] it was showed that any quadratic form $Q = \phi \otimes \phi$ where ϕ is one of the above primary differentials gives rise to a Frobenius structure, namely to a flat invariant metric η along the lines we drew before, namely

$$\eta \left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} \right) = \sum_{i=1}^n \operatorname{res}_{P_i} \frac{\phi \otimes \phi}{d\lambda} du_i \left(\frac{\partial}{\partial u_i} \cdot \frac{\partial}{\partial u_j} \right) = \sum_{i=1}^n \operatorname{res}_{P_i} \frac{\phi \otimes \phi}{d\lambda}.$$

In order to build the **superpotential** of this Frobenius structure we take λ considered as a function on the universal covering \tilde{C} (namely a multivalued function) by considering it as depending on the multivalued coordinate

$$v(P) := \int_{\infty_0}^P \phi,$$

where the principal value prescription (if necessary) is understood in this integral. The necessity to consider λ as a function of v is that we need to make differentiations of λ along the moduli space and it must be clear which is the (local) coordinate w.r.t. make the differentiation. We cite the Theorem to be found in [Du93] and which we will use later.

Theorem 5.2.1 For any primary differential ϕ the corresponding invariant metric η_ϕ along with the canonical multiplication rules in the coordinates u_i , endows the Hurwitz space $\hat{M}_{g,m;n_0..n_m}$ with a structure of Frobenius manifold.

The flat coordinates of the invariant Frobenius metric η are the $n = 2g + 2m + n_0 + \dots + n_m$ coordinates

$$t_{\nu;a} := \operatorname{res}_{\infty_\nu} (w_\nu)^a v d\lambda; \quad \nu = 0..m, \quad a = 1..n_\nu;$$

$$\begin{aligned}
v_\nu &:= \int_{\infty_0}^{\infty_\nu} dv = \int_{\infty_0}^{\infty_\nu} \phi; & V_\nu &:= -\operatorname{res}_{\infty_\nu} \lambda dv = -\operatorname{res}_{\infty_\nu} \lambda \phi; & \nu &= 1..m; \\
B_j &:= \oint_{b_j} dv; & C_j &:= \oint_{a_j} \lambda dv; & j &= 1..g,
\end{aligned}$$

and the nonzero entries of η in these coordinates read

$$\begin{aligned}
\eta_{t_{\nu;a}, t_{\mu;b}} &= \frac{1}{n_\nu + 1} \delta_{\mu\nu} \delta_{a+b, n_\nu+1} \\
\eta_{v_\nu, V_\mu} &= \frac{1}{n_\nu + 1} \delta_{\mu\nu} \\
\eta_{B_j, C_k} &= \frac{1}{2i\pi} \delta_{jk} .
\end{aligned}$$

In order to identify the remaining objects in the framework of Frobenius manifolds, we notice that

Proposition 5.2.1 [Remark 5.3 in [Du93]] The intersection form associated to the Frobenius structure specified in Theorem 5.2.1 is given by

$$\mathfrak{J} = \sum_{d\lambda(P_i)=0} (du_i)^2 \operatorname{res}_{P_i} \left(\frac{dv \otimes dv}{\lambda d\lambda} \right) .$$

The flat coordinates of this metric are the described by the same formulae as above with the substitution $\lambda \mapsto \log(\lambda)$ and with the $t_{\nu;a}$ changed into the

$$z_i = v(Q_i) , \quad \text{s.t. } \lambda(Q_i) = 0 .$$

Remark 5.2.1 If we have any other set of Hurwitz moduli and we consider derivations w.r.t. these ones, we find

$$\mathfrak{J}(\partial, \partial') = \sum_{d\lambda=0} \operatorname{res}_{d\lambda=0} \frac{\partial \lambda dv \otimes \partial' \lambda dv}{\lambda d\lambda} = \sum_{d\lambda=0} \operatorname{res}_{d\lambda=0} \frac{\partial(\log(\lambda)) dv \otimes \partial'(\log(\lambda)) dv}{d(\log(\lambda))} .$$

Proposition 5.2.2 Scaling dimensions The scaling dimensions of the flat invariants of η are given by

$$\begin{aligned}
\mathbf{E}(t_{\nu;a}) &= \frac{n_\nu + 1 - a}{n_\nu + 1} t_{\nu;a} , & \nu &= 0, \dots, m; & a &= 1, \dots, n_\nu ; \\
\mathbf{E}(v_\nu) &= 0 , & \mathbf{E}(V_\nu) &= V_\nu ; & \nu &= 1, \dots, m ; \\
\mathbf{E}(B_j) &= 0 , & \mathbf{E}(C_j) &= C_j ; & j &= 1, \dots, g .
\end{aligned}$$

Proof. The proof is immediate considering the expressions of the invariant and noticing that the Euler vector field is a rescaling in the variable λ , leaving unchanged the primary differential dv .
Q.E.D

In the following section we apply this general theorems to the case of genus 1.

5.3 The space $M_{1,0;l}$ as orbit space of $J(A_l)$

In this section we build a Frobenius structure over the Hurwitz space $M_{1,0;l}$ (or better its covering $\widehat{M}_{1,0;l}$); now C is a torus and the superpotential λ has only one pole ∞_0 of degree $l + 1$.

Applying the general theory we shall identify the flat coordinates of the intersection form and find the monodromy group of the resulting Frobenius manifold: it will turn out that this is the Jacobi group of type A_l , $J(A_l)$ and that the moduli of the superpotential -when expressed as functions of the flat coordinates of the intersection form-, are the generators of the algebra of Jacobi forms $J_{\bullet,\bullet}$ (see formula 3.7).

We call τ the modular parameter of the torus C and we think of it as $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. It is useful to use the Weierstrass uniformization, realizing the torus $C(\tau)$ as the affine curve⁴

$$\begin{aligned} Y^2 &= 4X^3 - g_2(\tau)X - g_3(\tau) \\ X &= \wp(v) ; \quad Y = \wp'(v) . \end{aligned}$$

The point ∞_0 can be chosen (modulo the automorphisms of the torus) as the point $v = 0$ of $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. Since λ must be a meromorphic elliptic function with a pole of order $l + 1$ at $v = 0 \pmod{\mathbb{Z} + \tau\mathbb{Z}}$, it follows that the most generic form is (using a convenient normalization of the moduli)

$$\begin{aligned} \lambda(v) &= \sum_{j=0}^{l+1} \frac{(-1)^{l+1-j}}{(l-j)!} \wp^{(l-1-j)}(v) \varphi_{l+1-j} = \frac{1}{v^{l+1}} \varphi_{l+1} + \frac{1}{v^l} \varphi_l + \frac{1}{v^{l-1}} \varphi_{l-1} + \dots + \frac{1}{v^2} \varphi_2 + O(1) \\ \partial_k \lambda &:= \frac{\partial \lambda}{\partial \varphi_k} = \frac{(-1)^k}{(k-1)!} \wp^{(k-2)}(v) , \end{aligned}$$

where we numbered the coefficients φ_k according to the order of the pole and we have set, for notational brevity, $\wp^{(-1)}(v) \equiv 0$, $\wp^{(-2)}(v) \equiv 1$.

From the modular properties of \wp it follows that the parameters φ_k transform as modular forms of weight $-k$ under $SL(2, \mathbb{Z})$: indeed by assumption the superpotential has to be independent of the isomorphism class of the torus namely,

$$\lambda(v|\tau) = \lambda \left(\frac{v}{c\tau + d} \middle| \frac{a\tau + b}{c\tau + d} \right) , \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \Rightarrow \varphi_k \xrightarrow{\tau \mapsto \tau'} (c\tau + d)^{-k} \varphi_k .$$

We observe that the moduli φ_j 's are equivariant and we anticipate here that they will be identified with the fundamental Jacobi forms. At this stage they are simply some parameters which play

⁴In this section we use a slightly different definition for the Weierstrass functions, namely

$$\begin{aligned} \sigma(v|\tau) &:= e^{-2i\pi v^2 g_1(\tau)} \frac{\theta_1(v, \tau)}{\theta_1'(0, \tau)} = e^{-2i\pi v^2 g_1(\tau)} \alpha(v|\tau) \\ \zeta(v|\tau) &:= \frac{d}{dv} \log(\sigma(v|\tau)) \\ \wp(v|\tau) &:= -\frac{d}{dv} \zeta(v|\tau) = \frac{1}{v^2} + \sum_{m^2+n^2 \neq 0} \left(\frac{1}{(v+m+n\tau)^2} - \frac{1}{(m+n\tau)^2} \right) \end{aligned}$$

the same role as the Jacobi forms in formula (3.9), but the identification will be complete in Thm. 5.3.1. In view of this identification we shall occasionally call them “Jacobi forms” by an abuse of language.

The primary differential we will use is simply the holomorphic differential $\phi = dv$ and hence λ will be the superpotential of our Frobenius manifold as a function of the multivalued coordinate v on the torus (hence as a function on the Jacobian of the torus).

Let us analyze the structure:

1. The parameters $\varphi_0, \varphi_2, \dots, \varphi_{l+1}$ and the modular parameter $\tau =: \varphi_{-1}$ are local coordinates of the Frobenius manifold $M := M_{1,0,l}$ and the invariant metric $\eta(\cdot, \cdot)$ is given by ⁵

$$\eta\left(\frac{\partial}{\partial\varphi_i}, \frac{\partial}{\partial\varphi_j}\right) = \sum_{\substack{\lambda'(v_n)=0 \\ i, j = -1, 0, 2, \dots, l+1}} \frac{\partial u_n}{\partial\varphi_i} \frac{\partial u_n}{\partial\varphi_j} \underset{v=v_n}{res} \frac{dv}{\lambda'(v)} = \sum_{\lambda'=0} \underset{res}{res} \frac{\partial_i \lambda(v) \partial_j \lambda(v) dv}{\lambda'(v)},$$

For $i, j = 0..l+1$ the functions $\frac{\wp^{(i-2)}(v)\wp^{(j-2)}(v)}{\lambda'(v)}$ are elliptic functions (recall that we have set $\wp^{(-2)} \equiv 1$; $\wp^{(-1)} \equiv 0$) and hence the sum of all residues in a fundamental mesh is zero: therefore we can compute the residues at the points defined by $\lambda' = 0$ by computing the residue at $v = 0$ with opposite sign, (we suppress the v dependence to shorten the formulae)

$$\eta\left(\partial_i, \partial_j\right) = - \underset{v=0}{res} \frac{\partial_i \lambda \partial_j \lambda dv}{\lambda'}.$$

A different problem is to compute the matrix elements where $i = -1$ ($\varphi_{-1} := \tau$) for in this case the function $\partial_\tau \lambda$ is *not* an elliptic function, but we can compute the residues for

$$\begin{aligned} \mathbb{D}^\bullet \lambda(v) &\equiv \nabla_\tau^\bullet \lambda(v) - \frac{1}{2i\pi} \frac{\alpha'(v)}{\alpha(v)} \lambda'(v) = \partial_\tau \lambda(v) - \frac{\eta'}{\eta} \sum_{j=0}^{l+1} 2j \wp^{(j-2)}(v) \varphi_j - \frac{1}{2i\pi} \frac{\alpha'(v)}{\alpha(v)} \lambda'(v) = \\ &=: \partial_\tau \lambda(v) + \Sigma(v) + \gamma(v) \lambda'(v) \end{aligned}$$

(which is *elliptic*: notice that Σ is elliptic). Hence if $F(v)$ is any elliptic function with a pole *only* at the origin, we find

$$\sum_{\lambda'=0} \underset{res}{res} \frac{\partial_\tau \lambda F dv}{\lambda'} = \sum_{\lambda'=0} \underset{res}{res} \left[\frac{(\mathbb{D}^\bullet \lambda - \Sigma) F dv}{\lambda'} \right] =$$

⁵The formula follows from the Jacobian of the parameters u_i in terms of the moduli φ_k as follows: the defining equation of the u 's are

$$\begin{cases} u_i(\varphi) := \lambda(P_i; \varphi) \\ \lambda'(P_i; \varphi) = 0 \end{cases}$$

and upon differentiation w.r.t. φ_k we get

$$\begin{cases} \frac{\partial u_i(\varphi)}{\partial \varphi_k} = \frac{\partial \lambda}{\partial \varphi_k}(P_i; \varphi) + \lambda'(P_i; \varphi) \frac{\partial P_i}{\partial \varphi_k} = \frac{\partial \lambda}{\partial \varphi_k}(P_i; \varphi) \\ d\lambda(P_i; \varphi) = 0 \end{cases}.$$

$$= -\operatorname{res}_{v=0} \frac{(\mathbb{D}^\bullet \lambda - \Sigma) F dv}{\lambda'} = -\operatorname{res}_{v=0} \left[\left(\frac{\partial_\tau \lambda F dv}{\lambda'(v)} \right) + \gamma F dv \right].$$

The computational advantage is that we have converted a sum over residues which we do not know where are situated, into a single evaluation at a fixed pole. Along these lines we can compute

$$\begin{aligned} \eta(\partial_{-1}, \partial_k) &= -\operatorname{res}_{v=0} \left[\left(\frac{\partial_\tau \lambda \partial_k \lambda dv}{\lambda'} \right) + \gamma \partial_k \lambda dv \right] = \\ &= \operatorname{res}_{v=0} \left[\frac{\alpha'(v)}{2i\pi\alpha(v)} \partial_k \lambda(v) dv \right], \end{aligned}$$

where, in the last equality, the term with $\left(\partial_\tau \lambda(v) \partial_k \lambda(v) \right) / \lambda'(v) \propto (\partial_\tau \lambda(v) \wp^{(k-2)}(v)) / \lambda'(v)$ has disappeared because it is regular at $v = 0$ for $k = 2..l + 1$.

In order to compute $\eta(\partial_{-1}, \partial_{-1})$ we use a similar trick

$$\begin{aligned} \sum_{\lambda'=0} \operatorname{res} \frac{(\partial_\tau \lambda)^2}{\lambda'} &= \sum_{\lambda'=0} \operatorname{res} \left\{ \frac{(\mathbb{D}^\bullet \lambda - \Sigma) (\mathbb{D}^\bullet \lambda - \Sigma) dv}{\lambda'} \right\} = \\ &= -\operatorname{res}_{v=0} \left\{ \frac{(\partial_\tau \lambda + \gamma \lambda') (\partial_\tau \lambda + \gamma \lambda') dv}{\lambda'} \right\} = -\operatorname{res}_{v=0} \left\{ \frac{(\partial_\tau \lambda)^2 dv}{\lambda'} + 2\gamma \partial_\tau \lambda dv + \gamma^2 \lambda' dv \right\} = \\ &= -\operatorname{res}_{v=0} \left\{ (2\partial_\tau \lambda + \gamma \lambda') \gamma dv \right\}. \end{aligned}$$

2. The multiplication is defined as

$$\eta(\partial_i, \partial_j \cdot \partial_k) = \sum_{\lambda'=0} \operatorname{res} \frac{\partial_i \lambda \partial_j \lambda \partial_k \lambda dv}{\lambda'}.$$

Again, if all indices are nonnegative we can evaluate the residue at zero changing sign; as for the remaining cases we get, after computations similar to those of before,

$$\begin{aligned} \eta(\partial_i, \partial_j \cdot \partial_k) &= -\operatorname{res}_{v=0} \left(\frac{\partial_i \lambda \partial_j \lambda \partial_k \lambda dv}{\lambda'} \right) \\ \eta(\partial_{-1}, \partial_i \cdot \partial_j) &= -\operatorname{res}_{v=0} \left\{ \frac{\partial_\tau \lambda}{\lambda'} \partial_i \lambda \partial_j \lambda dv + \gamma \partial_i \lambda \partial_j \lambda dv \right\} \\ \eta(\partial_{-1}, \partial_{-1} \cdot \partial_i) &= -\operatorname{res}_{v=0} \left\{ (2\partial_\tau \lambda + \gamma \lambda') \partial_i \lambda dv \right\} \\ \eta(\partial_{-1}, \partial_{-1} \cdot \partial_{-1}) &= -\operatorname{res}_{v=0} \left\{ 3\gamma (\partial_\tau \lambda)^2 dv + 3\gamma^2 \partial_\tau \lambda \lambda' dv + \gamma^3 (\lambda')^2 dv \right\}. \end{aligned}$$

Applying Theorem 5.2.1 we find

Proposition 5.3.1 The flat coordinates of the invariant Frobenius metric η are (the principal values prescriptions are understood in the integrations)

$$\begin{aligned}
 t_0 &:= \oint_b dv = \int_0^\tau dv = \tau \\
 t_1 &:= \oint_a \lambda(v)dv = \int_0^1 \lambda(v)dv = \varphi_0 + \left[\sum_{k=0}^{l-1} \frac{(-1)^{l+1-k}}{(l-k)!} \zeta^{(l-1-k)}(v) \varphi_{l+1-k} \right]_{0+\epsilon}^{1+\epsilon} = \\
 &= \varphi_0 + 4i\pi g_1 \varphi_2; \\
 t_a &:= \operatorname{res}_{v=0} \left(v \lambda^{-\frac{a-1}{l+1}}(v) d\lambda(v) \right); \quad a = 2..l + 1;
 \end{aligned}$$

$$\eta(\partial_t, \partial_t) = \begin{pmatrix} 0 & \frac{1}{2i\pi} & 0 & 0 \dots 0 & 0 \\ \frac{1}{2i\pi} & 0 & 0 & 0 \dots 0 & 0 \\ 0 & 0 & 0 & 0 \dots 0 & \frac{1}{l+1} \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \dots & \frac{1}{l+1} & 0 \dots 0 & 0 \end{pmatrix}$$

While the flat (local) coordinates of the intersection form are the zeroes of $\lambda(v)$, τ and $u := \frac{1}{2i\pi} \oint_a \log(\lambda)dv$.

Notice that the second Jacobi form φ_2 is always a quadratic polynomial in the flat coordinates t_2, \dots, t_{l+1} and the lightest Jacobi form φ_{l+1} is a power of the coordinate t_{l+1} as we now prove.

Corollary 5.3.1 The second Jacobi form φ_2 satisfies

$$\varphi_2 = \frac{1}{2} \sum_{i,j \neq 1,0} \eta_{t_i, t_j} t_i t_j = \frac{1}{2(l+1)} \sum_{j=2}^{l+1} t_j t_{l+3-j},$$

while for the lightest one φ_{l+1} we find

$$\varphi_{l+1} = (-l-1)^{-(l+1)} (t_{l+1})^{l+1}.$$

Proof. The second statement is easily proven by computing the residue

$$\begin{aligned}
 t_{l+1} &= \operatorname{res}_{v=0} \left(v \lambda^{-\frac{1}{l+1}} d\lambda \right) = \operatorname{res}_{v=0} \left(v \left(\frac{\varphi_{l+1}}{v^{l+1}} + \dots \right)^{-\frac{1}{l+1}} \left((-1)^{l+1} (l+1) \frac{\varphi_{l+1}}{v^{l+2}} + \dots \right) \right) = \\
 &= \operatorname{res}_{v=0} \left((-1)^{l+1} (l+1) \varphi_{l+1} \frac{1}{v^{l+1}} \frac{1}{v} + O(1) \right).
 \end{aligned}$$

As for the first statement, we introduce the local coordinate $z = (\lambda)^{-\frac{1}{l+1}}$ (choosing one branch of the root), and hence find $(\lambda = \frac{1}{z^{l+1}})$

$$t_j = \operatorname{res}_{v=0} \left(v \lambda^{\frac{j-1}{l+1}} d\lambda \right) = -(l+1) \operatorname{res}_{z=0} \left(v z^{j-l-3} dz \right), \quad j = 2, \dots, l+1,$$

where we implicitly solve the equation $\lambda(v) = \frac{1}{z^{l+1}}$ for v as a function of z ; in other words the flat coordinates t_2, \dots, t_{l+1} are the coefficients in the expansion of v

$$v(z) = \frac{-1}{l+1} \left(t_{l+1} z + t_l z^2 + \dots + t_2 z^l + O(z^{l+2}) \right),$$

in which the term in z^{l+1} vanishes because $\text{res}_{v=0}(vd\lambda) = 0$. We now compute

$$\frac{1}{l+1} \sum_{j=2}^{l-1} t_j t_{l+3-j} = \text{res}_{z=0} \frac{(l+1)v^2(z)dz}{z^{l+1}} = -\text{res}_{v=0} v^2 d\lambda = 2\varphi_2,$$

and this ends the proof. Q.E.D

From this corollary it follows that

$$t_1 = \varphi_0 + 2i\pi \frac{\eta'}{\eta} \sum_{i,j \neq 0,1} \eta_{t_i, t_j} t_i t_j.$$

In order to complete the description of the Frobenius structure it is important to express the two vector fields $\mathbf{1}$ and \mathbf{E} in these coordinates.

Recall that the unity vector field $\mathbf{1}$ in the coordinates $u_i \equiv \lambda(P_i)$ where P_i are the critical points, read $\mathbf{1} = \sum_{i=1}^{l+2} \frac{\partial}{\partial u_i}$, while the Euler vector field is given by $\sum_{i=1}^{l+2} u_i \frac{\partial}{\partial u_i}$. We now prove that

Proposition 5.3.2 The unity vector field $\mathbf{1}$ and the Euler vector field \mathbf{E} in the coordinates t_0, \dots, t_{l+1} read,

$$\mathbf{1} = \frac{\partial}{\partial t_1}; \quad \mathbf{E} = \sum_{k=1}^{l+1} \frac{l+2-k}{l+1} t_k \frac{\partial}{\partial t_k}.$$

Proof. The effect of these two vectors on λ are of shifting or dilating it, hence they read $\mathbf{1} = \frac{\partial}{\partial \lambda}$ and $\mathbf{E} = \lambda \frac{\partial}{\partial \lambda}$; now, from Corollary 5.3.1 it follows that the vector $\frac{\partial}{\partial t_1}$ exactly shifts λ , while the expression for the Euler vector field comes from the degrees of the flat coordinates as stated in Proposition 5.2.2. Q.E.D

Since the superpotential is invariant under the group $SL(2, \mathbb{Z})$, it follows easily that

Proposition 5.3.3 Under a transformation of the modular group with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$, the flat coordinates transform as follows: they are invariant under the map corresponding to $\tau \mapsto \tau + 1$ and under the inversion $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we find

$$\begin{aligned} t_0 &\mapsto t'_0 = -\frac{1}{t_0} \\ t_1 &\mapsto t'_1 = t_1 + i\pi \frac{1}{t_0} \sum_{j=2}^{l+1} t_j t_{l+3-j} \\ t_j &\mapsto t'_j = \frac{1}{t_0} t_j, \quad j = 2 \dots l+1 \end{aligned}$$

namely, the flat coordinates t_2, \dots, t_{l+1} behave like modular forms of weight -1 .

Proof. The transformation rule for $t_0 = \tau$ is obvious since it is the modular parameter of the torus; the one for t_1 follows immediately once expressing it in terms of the Jacobi form φ_0 and φ_2 and from the modular properties of the Dedekind η -function. As for the transformation rule for the remaining, this is a consequence of their definition as residues w.r.t. a weight -1 form which is $v \lambda^\times d\lambda$ (recall that under the inversion the coordinate v on the universal covering of the torus is mapped to $\frac{v}{\tau}$). Q.E.D

5.3.0.1 Twisted Frobenius structure

The Frobenius structure which has been constructed is well defined on $M_{1,0;l}$ only locally, because the automorphism group of the curve C ($SL(2, \mathbb{Z})$ in this case, full modular group) acts in a non-trivial way by means of a *symmetry* (see [Du93] for details on symmetries of Frobenius structures). This picture can be interpreted in two ways; we can consider that the invariant metric η takes values in a suitable line bundle L over $M = M_{1,0;l}$; namely we consider the Frobenius structure to be well-defined on $L \otimes TM$ rather than on TM itself.

Equivalently we can consider the covering space $\widehat{M} = \widehat{M}_{1,0;l}$ and then we get a *bona fide* equivariant Frobenius structure on $T\widehat{M}$.

Notice that the coordinates t_j 's have to be thought of as coordinates on \widehat{M} in view of the branching around the surface $\varphi_{l+1} = 0$: in fact in the definition of \widehat{M} , the fixation of the branching of the root around the infinities makes t_{l+1} a one valued function.

5.3.1 Free energy

To complete this study we must give the structure constants of the bundle of Frobenius algebras on the tangent bundle of the manifold; in principle this could be done by changing coordinates from the φ_j 's to the t_j 's, since the structure constants form a tensor, but it is more useful and satisfactory to express the free energy in terms of the flat coordinates.

To achieve this goal we compute free energy F by means of the particular bilinear pairing of forms which is described in Appendix A.5; it is proven in [Du93] that the **free energy** for the Frobenius structure we have build is given by

$$F := -\frac{1}{2} \langle v d\lambda, v d\lambda \rangle .$$

where we use, in the notation of the cited appendix, $\Phi = \Psi = v d\lambda$. In order to find all the due constants we first notice that no logarithmic polydromy is present in this differential; hence we are to find the coefficients c_k ($\equiv c_{\Phi,k}$ in the notation of the appendix): we already know the first l , which are (up to a constant) the flat coordinates t_2, \dots, t_l . In order to find the remaining we have to write the expansion of the differential around the infinity point ∞_0 in terms of the local coordinate $z := (\lambda)^{-\frac{1}{l+1}}$ and therefore we must expand v as a function of z by means of the inversion formula

$$\lambda(v) = \frac{1}{z^{l+1}} \Rightarrow v = v(z) = -\frac{1}{l+1} \left(t_{l+1}z + t_l z^2 + \dots + t_2 z^l + c_0 z^{l+2} + \dots + c_{l-1} z^{2l+1} + O\left(z^{2l+2}\right) \right) ,$$

where we have numbered the coefficients in a convenient way for the following application. Plugging into the formula we find

$$\begin{aligned} v d\lambda &= v(z) d\left(z^{-l-1}\right) = -(l+1) \frac{v(z)}{z^{l+2}} dz = \\ &= \left[t_{l+1} z^{-l-1} + t_l z^{-l} + \dots + t_2 z^{-2} + c_0 + z c_1 + \dots + c_{l-1} z^{l-1} + O\left(z^l\right) \right] dz, \end{aligned}$$

where we notice that $c_{-1} = 0$ because the differential $v d\lambda$ has no residue. As for the polydromy we compute

$$v d\lambda \xrightarrow{v \rightarrow v+a} v d\lambda - d\lambda \Rightarrow A(\lambda) \equiv \lambda$$

$$v d\lambda \xrightarrow{v \rightarrow v+b} v d\lambda - \tau d\lambda \Rightarrow B(\lambda) \equiv \tau \lambda.$$

Finally we can write the pairing

$$\langle v d\lambda, v d\lambda \rangle = - \sum_{k=0}^{l-1} c_k \frac{1}{k+1} c_{-k-2} + \frac{1}{2i\pi} \oint_a v d\lambda \oint_b v d\lambda - \frac{\tau}{2i\pi} \oint_a \lambda v d\lambda + \frac{1}{2i\pi} \oint_b \lambda v d\lambda.$$

To compute the periods in the formula we recall that we have to realize the cycles a, b as paths with base-point a zero of λ ; in the specific we can think of them as the segments $a \equiv [x, x+1]$ and $b \equiv [x, x+\tau]$ in the complex v plane (which realizes the universal covering of the torus; here x is a zero of λ in a fundamental mesh) and hence compute

$$\begin{aligned} \oint_{a(x)} v d\lambda &= - \oint_a \lambda dv = -t_1, \quad \oint_a \lambda dv = \varphi_0 + 4i\pi g_1 \varphi_2 =: t_1 \\ \oint_{b(x)} v d\lambda &= - \oint_b \lambda dv = -t_0 t_1 - 2i\pi \varphi_2, \\ \oint_b \lambda dv &= \tau \varphi_0 + 4i\pi \tau g_1 \varphi_2 + 2i\pi \varphi_2 = \tau t_1 + 2i\pi \varphi_2 =: t_0 t_1 + 2i\pi \varphi_2 \\ \left(\oint_{a(x)} v d\lambda \right) \left(\oint_{b(x)} v d\lambda \right) &= t_0 (t_1)^2 + 2i\pi t_1 \varphi_2. \end{aligned}$$

The other periods give

$$\begin{aligned} \oint_{a(c)} B(\lambda) v d\lambda &= \tau \oint_{a(c)} v \lambda d\lambda = -\frac{\tau}{2} \oint_a \lambda^2 dv \\ \oint_{b(c)} A(\lambda) v d\lambda &= \oint_{b(c)} v \lambda d\lambda = -\frac{1}{2} \oint_b \lambda^2 dv \\ \oint_{b(c)} A(\lambda) v d\lambda - \oint_{a(c)} B(\lambda) v d\lambda &= \frac{\tau}{2} \oint_a \lambda^2 dv - \frac{1}{2} \oint_b \lambda^2 dv = \\ &= -\frac{1}{2} \left(\oint_b - \tau \oint_a \right) \lambda^2(v) dv = -i\pi \operatorname{res}_{v=0} (v \lambda^2(v) dv). \end{aligned}$$

where we notice that the differential $\lambda^2 dv$ has only one pole at $v = 0$ and no residue⁶: we have then used formula (A.3) so that finally (recall that $\tau = t_0$)

Proposition 5.3.4 The *Free energy* of the Frobenius structure associated to the primary differential dv is

$$\begin{aligned} F &:= -\frac{1}{2} \langle v d\lambda, v d\lambda \rangle = \\ &= \frac{1}{2} \sum_{k=0}^{l-1} c_{-2-k} \frac{1}{k+1} c_k - \frac{1}{4i\pi} \left[\int_{b(x)} v d\lambda \int_{a(x)} v d\lambda - \tau \int_{a(x)} v \lambda d\lambda + \int_{b(x)} v \lambda d\lambda \right] = \\ &= \frac{1}{2} \sum_{k=0}^{l-1} c_k \frac{1}{k+1} c_{-k-2} - \frac{1}{4i\pi} t_0 (t_1)^2 - \frac{1}{2} t_1 \varphi_2 + \frac{1}{4} \operatorname{res}_{v=0} (v \lambda^2(v) dv) = \\ &= \frac{1}{2} \sum_{k=0}^{l-1} c_k \frac{1}{k+1} t_{k+2} - \frac{1}{4i\pi} t_0 (t_1)^2 - \frac{1}{2} t_1 \varphi_2 + \frac{1}{4} \operatorname{res}_{v=0} (v \lambda^2(v) dv) . \end{aligned}$$

We will give an explicit example in the case of A_2 later.

5.3.2 Flat coordinates of the intersection form

We now analyze the structure of the flat coordinates of the intersection form. This enables to identify this Frobenius manifold as a suitable covering of the orbit space of the Jacobi group of type A_l .

In particular it will become clear in which sense the moduli φ_k are Jacobi forms; the point is that as functions of the flat coordinates of the intersection form, they are exactly the previously studied generators of the algebra $J_{\bullet, \bullet}$.

From Thm. 5.2.1 we know that the flat coordinates of the intersection form are the functions $v(Q_i)$ with Q_i a zero of the superpotential λ , and $\oint_b dv = \tau$ and $u = \frac{1}{2i\pi} \oint_a \log(\lambda) dv$.

To begin with, the number of zeroes of $\lambda(v)$ is $l+1$ because λ is an elliptic function with a pole of order $l+1$ at the origin; they are linearly related since the divisor of zeroes must be congruent to the divisor of poles. Therefore we have $\sum_1^{l+1} z_i = 0 \pmod{\mathbb{Z} + \tau\mathbb{Z}}$.

We already know how to express the parameters $\varphi_0, \varphi_2, \dots, \varphi_{l+1}$ in terms of the zeroes of $\lambda(v)$; this follows from the explicit construction of the Jacobi forms for A_l , which is accomplished in eq. (3.7). This clearly leaves an arbitrariness, since the knowledge of the divisor of zeroes fixes a function modulo multiplication by a nonzero number; this multiplicative coefficient will be denoted by $e^{2i\pi s}$.

The explicit formula which expresses the superpotential as a function of its zeroes $z_1 \dots z_{l+1}$ (s.t. $\sum_1^{l+1} z_i = 0$), of τ and s is thus

$$\lambda(v) = e^{2i\pi s + 2i\pi g_1 \|z\|^2} \frac{\prod_1^{l+1} \sigma(z_i - v)}{\sigma^{l+1}(v)} =$$

⁶This follows from the fact that λ^2 is an elliptic function with only one pole, and from the general fact that the sum of residues in a fundamental mesh of an elliptic function is zero

$$= \frac{-1}{l!} \underbrace{\left(e^{2i\pi s} \prod_{j=1}^{l+1} \alpha(z_j) \right)}_{=\varphi_{l+1}} \frac{\det \begin{pmatrix} 1 & \wp(v) & \wp'(v) & \dots & \wp^{(l-1)}(v) \\ 1 & \wp(z_1) & \wp'(z_1) & \dots & \wp^{(l-1)}(z_1) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \wp(z_l) & \wp'(z_l) & \dots & \wp^{(l-1)}(z_l) \end{pmatrix}}{\det \begin{pmatrix} 1 & \wp(z_1) & \wp'(z_1) & \dots & \wp^{(l-2)}(z_1) \\ 1 & \wp(z_2) & \wp'(z_2) & \dots & \wp^{(l-2)}(z_2) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \wp(z_l) & \wp'(z_l) & \dots & \wp^{(l-2)}(z_l) \end{pmatrix}}$$

In order to identify completely the flat coordinates of the intersection form with the coordinates we used in constructing the Jacobi forms of A_l in formula (3.7) we have to compute explicitly the flat coordinate u ; to this end we give the

Lemma 5.3.1 *The flat coordinate $u = \frac{1}{2i\pi} \oint_a dv \log(\lambda)$ equals exactly s .*

Proof. We have

$$u := \frac{1}{2i\pi} \oint_a dv \log(\lambda) = \frac{1}{2i\pi} \int_0^1 dv \log(\lambda) = s + g_1 \|\mathbf{z}\|^2 + \frac{1}{2i\pi} \int_0^1 dv \log \left(\frac{\prod_{j=1}^{l+1} \sigma(v - z_j)}{\sigma^{l+1}(v)} \right).$$

Considering this function depending on z_i we have

$$\begin{aligned} \frac{\partial}{\partial z_k} \int_0^1 dv \left[\sum_{j=1}^{l+1} \log \left(\frac{\sigma(v - z_j)}{\sigma(v)} \right) \right] &= \int_0^1 dv \zeta(v - z_k) = \\ &= \log \left(\frac{\sigma(1 - z_k)}{\sigma(z_k)} \right) = \left(-4i\pi \frac{\eta'}{\eta} z_k \right) = -2i\pi g_1 \frac{\partial}{\partial z_k} (\|\mathbf{z}\|^2). \end{aligned}$$

Since for $\mathbf{z} = 0$ the integral is obviously zero we have finally

$$\int_0^1 dv \left[\sum_{j=1}^{l+1} \log \left(\frac{\sigma(v - z_j)}{\sigma(v)} \right) \right] = -2i\pi \frac{\eta'}{\eta} \|\mathbf{z}\|^2,$$

and therefore $s \equiv u$. Q.E.D

From this explicit formula we recognize that the Hurwitz moduli $\varphi_{-1}, \varphi_0, \varphi_2, \dots, \varphi_{l+1}$ –as functions of the flat coordinates of the intersection form, τ, u, x_1, \dots, x_l – are exactly the invariant Jacobi forms constructed in 3.7 for the Jacobi group $J(A_l)$; we have thus proven

Theorem 5.3.1 The Hurwitz moduli $\varphi_{-1} = \tau, \varphi_0, \varphi_2, \dots, \varphi_{l+1}$ as functions of the flat coordinates of the intersection form $u, z_1, \dots, z_{l+1}, \tau$ (where $\sum z_i = 0$), are the Jacobi forms for the Jacobi group of type A_l .

Moreover the intersection form in Proposition 5.2.1 coincides with the intersection form in Definition 3.1.1.

5.3.3 Monodromy

By means of the identification in Thm. 5.3.1 between the moduli $(\varphi_0, \varphi_2, \dots, \varphi_{l+1}, \tau)$ on the space $\hat{M}_{1,0;l}$ and the Jacobi forms for the Jacobi group associated to A_l , we have therefore constructed a Frobenius structure on a suitable covering of the orbit space $\frac{\Omega}{J} := \frac{\mathbb{C} \oplus \mathbb{C}^l \oplus \mathcal{H}}{J}$. From the above formulae expressing the flat coordinates in terms of the Jacobi forms it is clear that the multivaluedness of this covering comes uniquely from the $t_l = -(l+1)(\varphi_{l+1})^{\frac{1}{l+1}}$ coordinate; looking at the explicit form (defining $x_{l+1} := 0 =: x_0$)

$$\varphi_{l+1}(x_1, \dots, x_l) = \prod_{j=1}^{l+1} \alpha(x_i - x_{i-1})$$

we realize that the zeroes are situated at the walls (recall that α_k^\vee are the coroots spanning over \mathbb{Z} the lattice Q of A_l in \mathbb{C}^{l+1})

$$\alpha_j^\vee(\mathbf{x}) = 0 \pmod{\mathbb{Z} + \tau\mathbb{Z}} .$$

namely at the walls of the alcove for the complex crystallographic group

$$\mathcal{A} := \{\mathbf{x} \in \mathfrak{h} \mid \alpha_j^\vee(\mathbf{x}) \in \mathcal{A}_0\}$$

where \mathcal{A}_0 is the fundamental mesh of $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$.

In other words for fixed τ the quotient of $\mathfrak{h} \simeq \mathbb{C}^l$ by the complex crystallographic group is a torus \mathbb{T}^l which is simply the product of identical tori of modular parameter τ ; our Frobenius structure lives on a $l+1$ sheeted covering of this torus with branching divisor $Y := \{\varphi_{l+1} = 0\}$.

Example 5.3.1 The case A_1 *This example was worked out explicitly also in [Du93] but it is useful to use the present formalism. The superpotential is*

$$\lambda(v) := \wp(v)\varphi_2 + \varphi_0 ,$$

and the Jacobi forms read

$$\varphi_2(u, x, \tau) = -e^{2i\pi u} \alpha^2(x) ; \quad \varphi_0(u, x, \tau) = -e^{2i\pi u} \alpha^2(x) \wp(x) .$$

The flat coordinates of the invariant metric η are

$$t_0 := \tau ; \quad t_1 := \varphi_0 + 4i\pi g_1 \varphi_2 ; \quad t_2 := -2\sqrt{\varphi_2} = -2\alpha(x)$$

$$\eta := \begin{pmatrix} 0 & \frac{1}{2i\pi} & 0 \\ \frac{1}{2i\pi} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} .$$

The free energy is found to be (the dependence of P on τ is understood)

$$F^{(A_1)} := \frac{1}{4i\pi} t_0(t_1)^2 + \frac{1}{4} t_1(t_2)^2 + \frac{1}{96\pi^2} P t_2^4$$

Notice that these flat coordinates live on a double covering of the quotient space $\mathbb{C} \oplus \mathbb{C} \oplus \mathcal{H}/J$, inasmuch the coordinate t_2 changes sign under the action of the Weyl group (in this case the Weyl group is simply \mathbb{Z}_2 acting as $x \mapsto -x$); this comes from the fact that t_2 is a square-root of the truly invariant lightest Jacobi form φ_2 , and this is what happens in the general case.

Example 5.3.2 The case A_2 We have the superpotential

$$\lambda(v) := -\frac{1}{2}\wp'(v)\varphi_3 + \wp(v)\varphi_2 + \varphi_0$$

where the numbers $\varphi_3, \varphi_2, \varphi_0$ in terms of the zeroes of the superpotential read (we set $z_1 = x_1, z_2 = x_2 - x_1, z_3 = -x_2$)

$$\begin{aligned}\varphi_3(u, \mathbf{x}) &= -2e^{2i\pi u}\alpha(z_1)\alpha(z_2)\alpha(z_3); \\ \varphi_2(u, \mathbf{x}) &= e^{2i\pi u}\alpha(z_1)\alpha(z_2)\alpha(z_3)\frac{\wp'(x_1) - \wp'(x_2 - x_1)}{\wp(x_1) - \wp(x_2 - x_1)}; \\ \varphi_0(u, \mathbf{x}) &= e^{2i\pi u}\alpha(z_1)\alpha(z_2)\alpha(z_3)\frac{\wp(x_1)\wp'(x_2 - x_1) - \wp'(x_1)\wp(x_2 - x_1)}{\wp(x_2 - x_1) - \wp(x_1)}.\end{aligned}$$

The flat coordinates of η and its entries are

$$\begin{aligned}t_0 &= \tau; & t_2 &= -\frac{\varphi_2}{(\varphi_3)^{\frac{1}{3}}}; & t_3 &= -3(\varphi_3)^{\frac{1}{3}}; \\ t_1 &= \oint_a \lambda dv = \varphi_0 + 4i\pi g_1 \varphi_2 = \varphi_0 + \frac{4i\pi}{3} g_1 t_2 t_3 = \varphi_0 + \frac{2i\pi}{3} \frac{\eta'}{\eta} \sum_{j=2}^3 t_j t_{l+1-j}; \\ \varphi_0 &= t_1 - \frac{4}{3} i\pi g_1 t_2 t_3 \\ \varphi_2 &= \frac{1}{3} t_2 t_3; & \varphi_3 &= -\frac{1}{27} (t_3)^3; \\ \eta_{t_i, t_j} &= \begin{pmatrix} 0 & \frac{1}{2i\pi} & 0 & 0 \\ \frac{1}{2i\pi} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & 0 \end{pmatrix}\end{aligned}$$

and the free energy reads

$$F^{(A_2)} := \frac{1}{4i\pi} t_0 t_1^2 + \frac{1}{3} t_2 t_3 t_1 - \frac{1}{12} \frac{t_2^4}{t_3^2} + \frac{1}{54\pi^2} P t_3^2 t_2^2 - \frac{\pi^2}{43740} Q t_3^6.$$

These computations can be handled algorithmically by a computer; we list the next free energies of the series A_l (for the definitions of P, Q, R , see Eqs. (2.4))

$$\begin{aligned}F^{(A_3)} &:= \frac{1}{4i\pi} t_0 t_1^2 + \left(\frac{1}{4} t_2 t_4 + \frac{1}{8} t_3^2\right) t_1 + \frac{1}{6} \frac{t_2^3}{t_4} + \left(-\frac{1}{2} \frac{t_3^2}{t_4^2} + \frac{1}{96} \frac{t_4^2 P}{\pi^2}\right) t_2^2 \\ &+ \left(\frac{1}{4} \frac{t_3^4}{t_4^3} + \frac{1}{96} \frac{t_4 P t_3^2}{\pi^2} + \frac{1}{46080} t_4^5 Q \pi^4\right) t_2 - \frac{1}{24} \frac{t_3^6}{t_4^4} + \frac{1}{384} \frac{P t_3^4}{\pi^2} - \frac{1}{18432} t_3^2 t_4^4 Q \pi^4 \\ &+ \frac{1}{18579456} t_4^8 \pi^6 R\end{aligned}$$

$$\begin{aligned}F^{(A_4)} &:= -\frac{1}{4} \frac{I t_0 t_1^2}{\pi} + \left(\frac{1}{5} t_2 t_5 + \frac{1}{5} t_3 t_4\right) t_1 + \left(\frac{1}{2} \frac{t_3}{t_5} - \frac{1}{2} \frac{t_4^2}{t_5^2} + \frac{1}{150} \frac{t_5^2 P}{\pi^2}\right) t_2^2 \\ &+ \left(-\frac{t_4 t_3^2}{t_5^2} + \left(\frac{t_4^3}{t_5^3} + \frac{1}{75} \frac{t_5 P t_4}{\pi^2}\right) t_3 - \frac{1}{5} \frac{t_4^5}{t_5^4} + \frac{1}{28125} t_5^4 Q \pi^4 t_4\right) t_2 - \frac{1}{8} \frac{t_3^4}{t_5^2} + \frac{2}{3} \frac{t_4^2 t_3^3}{t_5^3} \\ &+ \left(-\frac{3}{4} \frac{t_4^4}{t_5^4} + \frac{1}{150} \frac{P t_4^2}{\pi^2} - \frac{1}{37500} t_5^4 Q \pi^4\right) t_3^2 \\ &+ \left(\frac{3}{10} \frac{t_4^6}{t_5^5} - \frac{1}{56250} t_5^3 Q \pi^4 t_4^2 - \frac{1}{14765625} t_5^7 \pi^6 R\right) t_3 - \frac{1}{24} \frac{t_4^8}{t_5^6} - \frac{1}{37500} Q \pi^4 t_5^2 t_4^4 \\ &+ \frac{1}{6328125} t_4^2 t_5^6 \pi^6 R - \frac{1}{10546875000} t_5^{10} \pi^8 Q^2\end{aligned}$$

5.4 The space $M_{1,0;2l}^{\mathbb{Z}_2}$ as orbit space of $J(B_l)$

In this section we build a Frobenius structure over the Hurwitz space $M_{1,0;l}^{\mathbb{Z}_2}$ (or better its covering $\widehat{M}_{1,0;l}^{\mathbb{Z}_2}$); now the superpotential λ has only one pole ∞_0 of degree $2l$ and the \mathbb{Z}_2 group acts on the torus by means of the involution $v \mapsto -v \in \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$.

Applying the general theory we shall identify the flat coordinates of the intersection form and find the monodromy group of the resulting Frobenius manifold: it will turn out that this is the Jacobi group of type B_l , $J(B_l)$ and that the moduli of the superpotential -when expressed as functions of the flat coordinates of the intersection form-, are the generators of the algebra of Jacobi forms $J_{\bullet,\bullet}$ (see formula 3.21).

The point ∞_0 , being the only marked point, must be left fixed by the involution and hence is the point $v = 0$ of $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. Since λ must be a meromorphic even elliptic function with a pole of order $l + 1$ at $v = 0 \pmod{\mathbb{Z} + \tau\mathbb{Z}}$, it follows that the most generic form is

$$\lambda(v) = \sum_{j=0}^l \frac{1}{(2l-2j)!} \wp^{(2l-2j-2)}(v) \varphi_{l-j} = \frac{1}{v^{2l}} \varphi_l + \frac{1}{v^{2l-2}} \varphi_{l-1} + \dots + \frac{1}{v^2} \varphi_1 + O(1)$$

$$\partial_k \lambda := \frac{\partial \lambda}{\partial \varphi_k} = \frac{1}{(2k-2)!} \wp^{(2k-4)}(v) ,$$

where we numbered the coefficients φ_k according to the order of the pole and we have set, for notational brevity, $\wp^{(-1)}(v) \equiv 0$, $\wp^{(-2)}(v) \equiv 1$.

As for the case of A_l the parameters φ_k transform as modular forms of weight $-2k$ under $SL(2, \mathbb{Z})$: again they are to be identified with the fundamental Jacobi forms. The primary differential we will use is the holomorphic differential $\phi = dv$: hence λ will be the superpotential of our Frobenius manifold as an even function of the multivalued coordinate v on the torus .

The structure is analyzed by similar means: since the problems and computation are essentially the same as in the case of A_l , we report only the results.

1. The parameters $\varphi_0, \varphi_1, \dots, \varphi_l$ and the modular parameter $\tau =: \varphi_{-1}$ are local coordinates of the Frobenius manifold $M := M_{1,0;2l}^{\mathbb{Z}_2}$ and the invariant metric $\eta(\cdot, \cdot)$ is given by

$$\eta \left(\frac{\partial}{\partial \varphi_i}, \frac{\partial}{\partial \varphi_j} \right) = \frac{1}{2} \sum_{\lambda'(v_n)=0} \frac{\partial u_n}{\partial \varphi_i} \frac{\partial u_n}{\partial \varphi_j} \underset{v=v_n}{res} \frac{dv}{\lambda'(v)} = \frac{1}{2} \sum_{\lambda'=0} \underset{res}{res} \frac{\partial_i \lambda(v) \partial_j \lambda(v) dv}{\lambda'(v)} ,$$

$$i, j = -1, 0, 2, \dots, l + 1 .$$

The factor $\frac{1}{2}$ takes into account the double counting of poles due to the symmetry of λ .

In the case $i, j = 0..l$ the functions $\frac{\wp^{(2i-2)}(v) \wp^{(2j-2)}(v)}{\lambda'(v)}$ are elliptic functions (recall that we have set $\wp^{(-2)} \equiv 1$; $\wp^{(-1)} \equiv 0$) and hence the sum of all residues in a fundamental mesh is zero: therefore we can compute the residues at the points defined by $\lambda' = 0$ by computing the residue at $v = 0$ with opposite sign, (we suppress the v dependence to shorten the formulae)

$$\eta \left(\partial_i, \partial_j \right) = -\frac{1}{2} \underset{v=0}{res} \frac{\partial_i \lambda \partial_j \lambda dv}{\lambda'}$$

As for the matrix elements where $i = -1$ ($\varphi_{-1} := \tau$), the same trick used in for A_l applies, so that we find

$$\eta(\partial_{-1}, \partial_k) = \frac{1}{2} \operatorname{res}_{v=0} \left[\frac{\alpha'(v)}{2i\pi\alpha(v)} \partial_k \lambda(v) dv \right].$$

Along the same lines one finds

$$\eta(\partial_{-1}, \partial_{-1}) = -\frac{1}{2} \operatorname{res}_{v=0} \left\{ (2\partial_\tau \lambda + \gamma \lambda') \gamma dv \right\}.$$

2. The multiplication is defined as

$$\eta(\partial_i, \partial_j \cdot \partial_k) = \frac{1}{2} \sum_{\lambda'=0} \operatorname{res} \frac{\partial_i \lambda \partial_j \lambda \partial_k \lambda dv}{\lambda'}.$$

Again, if all indices are nonnegative we can evaluate the residue at zero changing sign; as for the remaining cases we get, after computations similar to those of before,

$$\begin{aligned} \eta(\partial_i, \partial_j \cdot \partial_k) &= -\frac{1}{2} \operatorname{res}_{v=0} \left(\frac{\partial_i \lambda \partial_k \lambda \partial_j \lambda dv}{\lambda'} \right) \\ \eta(\partial_{-1}, \partial_i \cdot \partial_j) &= -\frac{1}{2} \operatorname{res}_{v=0} \left\{ \frac{\partial_\tau \lambda}{\lambda'} \partial_i \lambda \partial_j \lambda dv + \gamma \partial_i \lambda \partial_j \lambda dv \right\} \\ \eta(\partial_{-1}, \partial_{-1} \cdot \partial_i) &= -\frac{1}{2} \operatorname{res}_{v=0} \left\{ (2\partial_\tau \lambda + \gamma \lambda') \partial_i \lambda dv \right\} \\ \eta(\partial_{-1}, \partial_{-1} \cdot \partial_{-1}) &= -\frac{1}{2} \operatorname{res}_{v=0} \left\{ 3\gamma (\partial_\tau \lambda)^2 dv + 3\gamma^2 \partial_\tau \lambda \lambda' dv + \gamma^3 (\lambda')^2 dv \right\}. \end{aligned}$$

Applying Theorem 5.2.1 we find

Proposition 5.4.1 The flat coordinates of the invariant Frobenius metric η are (the principal values prescriptions are understood in the integrations)

$$\begin{aligned} t_0 &:= \oint_b dv = \int_0^\tau dv = \tau \\ t_1 &:= \oint_a \lambda(v) dv = \int_0^1 \lambda(v) dv = \varphi_0 + \left[\sum_{k=0}^{l-1} \frac{1}{(2l-2k)!} \zeta^{(2l-2k-1)}(v) \varphi_{l-k} \right]_{0+\epsilon}^{1+\epsilon} = \\ &= \varphi_0 + 4i\pi g_1 \varphi_1; \\ t_a &:= \operatorname{res}_{v=0} \left(v \lambda^{-\frac{2a-3}{2l}}(v) d\lambda(v) \right); \quad a = 2..l + 1; \end{aligned}$$

$$\eta(\partial_t, \partial_t) = \begin{pmatrix} 0 & \frac{1}{2i\pi} & 0 & 0 \cdots 0 & 0 \\ \frac{1}{2i\pi} & 0 & 0 & 0 \cdots 0 & 0 \\ 0 & 0 & 0 & 0 \cdots 0 & \frac{1}{2l} \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & \frac{1}{2l} & 0 \cdots 0 & 0 \end{pmatrix}$$

While the flat (local) coordinates of the intersection form are the zeroes of $\lambda(v)$ (modulo the involution $v \mapsto -v$), τ and $u := \frac{1}{2i\pi} \oint_a \log(\lambda) dv$.

Similarly to the A_l case we find

Corollary 5.4.1 The second Jacobi form φ_1 satisfies

$$\varphi_1 = \frac{1}{2} \sum_{i,j \neq 1,0} \eta_{t_i, t_j} t_i t_j = \frac{1}{4l} \sum_{j=2}^{l+1} t_j t_{l+3-j} ,$$

while for the lightest one φ_l we find

$$\varphi_l = (2l)^{-2l} t_{l+1}^{2l} .$$

Proof. Essentially the same. Q.E.D

From this corollary it follows that

$$t_1 = \varphi_0 + 2i\pi \frac{\eta'}{\eta} \sum_{i,j \neq 0,1} \eta_{t_i, t_j} t_i t_j .$$

In order to complete the description of the Frobenius structure it is important to express the two vector fields $\mathbf{1}$ and \mathbf{E} in these coordinates.

A straightforward computation gives

Proposition 5.4.2 The unity vector field $\mathbf{1}$ and the Euler vector field \mathbf{E} in the coordinates t_0, \dots, t_{l+1} read,

$$\mathbf{1} = \frac{\partial}{\partial t_1} ; \quad \mathbf{E} = \sum_{k=1}^{l+1} \frac{l+2-k}{l+1} t_k \frac{\partial}{\partial t_k} .$$

5.4.1 Free energy

In order to compute the free energy we have similar techniques as before and hence we only report the results.

Proposition 5.4.3 The *Free energy (prepotential)* of the Frobenius structure associated to the primary differential dv is

$$\begin{aligned} F &:= -\frac{1}{2} \langle v d\lambda, v d\lambda \rangle = \\ &= \frac{1}{2} \sum_{k=0}^{l-1} c_{-2-k} \frac{1}{k+1} c_k - \frac{1}{4i\pi} \left[\int_{b(x)} v d\lambda \int_{a(x)} v d\lambda - \tau \int_{a(x)} v \lambda d\lambda + \int_{b(x)} v \lambda d\lambda \right] = \\ &= \frac{1}{2} \sum_{k=0}^{l-1} c_k \frac{1}{k+1} c_{-k-2} - \frac{1}{4i\pi} t_0 (t_1)^2 - \frac{1}{2} t_1 \varphi_1 + \frac{1}{4} \operatorname{res}_{v=0} (v \lambda^2(v) dv) = \\ &= \frac{1}{2} \sum_{k=0}^{l-1} c_k \frac{1}{k+1} t_{k+2} - \frac{1}{4i\pi} t_0 (t_1)^2 - \frac{1}{2} t_1 \varphi_1 + \frac{1}{4} \operatorname{res}_{v=0} (v \lambda^2(v) dv) . \end{aligned}$$

We will give an explicit example in the case of B_2 and B_3 later.

5.4.2 Flat coordinates of the intersection form

The flat coordinates of the intersection form enable us to interpret the superpotential as the generating function of the Jacobi forms of type B_l ; this allows to identify this Frobenius manifold as a suitable covering of the orbit space of the Jacobi group of type B_l .

The arguments are essentially the same as before except for the obvious changes.

$$\begin{aligned} \lambda(v) &= e^{2i\pi s + 2i\pi g_1 \|z\|^2} \frac{\prod_1^l \sigma(x_i - v) \sigma(x_i + v)}{\sigma^{2l}(v)} = \\ &= \frac{1}{(2l)!} \underbrace{\left(e^{2i\pi s} \prod_{j=1}^l \alpha^2(x_j) \right)}_{=\varphi_{l+1}} \frac{\det \begin{pmatrix} 1 & \wp(v) & \wp''(v) & \dots & \wp^{(2l-2)}(v) \\ 1 & \wp(x_1) & \wp''(x_1) & \dots & \wp^{(2l-2)}(x_1) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \wp(x_l) & \wp''(x_l) & \dots & \wp^{(2l-2)}(x_l) \end{pmatrix}}{\det \begin{pmatrix} 1 & \wp(x_1) & \wp''(x_1) & \dots & \wp^{(2l-4)}(x_1) \\ 1 & \wp(x_2) & \wp''(x_2) & \dots & \wp^{(2l-4)}(x_2) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \wp(x_l) & \wp''(x_l) & \dots & \wp^{(2l-4)}(x_l) \end{pmatrix}} \end{aligned}$$

In particular some changes are needed for the computation of the flat coordinate u but the result is the same. From this explicit formula we recognize that the Hurwitz moduli $\varphi_{-1}, \varphi_0, \varphi_1, \dots, \varphi_l$ – as functions of the flat coordinates of the intersection form, τ, u, x_1, \dots, x_l – are exactly the invariant Jacobi forms constructed in 3.21 for the Jacobi group $J(B_l)$; we have thus proven

Theorem 5.4.1 The Hurwitz moduli $\varphi_{-1} = \tau, \varphi_0, \varphi_1, \dots, \varphi_l$ as functions of the flat coordinates of the intersection form u, x_1, \dots, x_l, τ , are the Jacobi forms for the Jacobi group of type B_l . Moreover the intersection form in Proposition 5.2.1 coincides with the intersection form in Definition 3.1.1.

5.4.3 Monodromy

By means of the identification in Thm. 5.4.1 between the moduli $(\varphi_0, \varphi_1, \dots, \varphi_l, \tau)$ on the space $\hat{M}_{1,0,l}^{\mathbb{Z}_2}$ and the Jacobi forms for the Jacobi group associated to B_l , we have therefore constructed a Frobenius structure on a suitable covering of the orbit space $\frac{\Omega}{J} := \frac{\mathbb{C} \oplus \mathbb{C}^l \oplus \mathbb{H}}{J}$. From the above formulae expressing the flat coordinates in terms of the Jacobi forms it is clear that the multivaluedness of this covering comes uniquely from the $t_l = -(2l)(\varphi_l)^{\frac{1}{2l}}$ coordinate; looking at the explicit form

$$\varphi_l(x_1, \dots, x_l) = \prod_{j=1}^l \alpha^2(x_j)$$

we realize that the zeroes are situated at the walls (recall that α_k^\vee are the coroots spanning over \mathbb{Z} the lattice Q of A_l in \mathbb{C}^{l+1})

$$\alpha_j^\vee(\mathbf{x}) = 0 \pmod{\mathbb{Z} + \tau\mathbb{Z}}.$$

namely at the walls of the alcove for the complex crystallographic group

$$\mathcal{A} := \{\mathbf{x} \in \mathfrak{h} \mid \alpha_j^\vee(\mathbf{x}) \in \mathcal{A}_0\}$$

where \mathcal{A}_0 is the fundamental mesh of $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$.

In other words for fixed τ the quotient of $\mathfrak{h} \simeq \mathbb{C}^l$ by the complex crystallographic group is a torus \mathbb{T}^l which is simply the product of identical tori of modular parameter τ ; our Frobenius structure lives on a l sheeted covering of this torus with branching divisor $Y := \{\varphi_l = 0\}$.

Example 5.4.1 The case B_2

$$\left\{ \phi_0 = t_1 - I\pi g_1(\tau) t_2 t_3, \phi_1 = \frac{1}{4} t_2 t_3, \phi_2 = \frac{1}{256} t_3^4, \tau = t_0 \right\}$$

$$\begin{aligned} F^{(B_2)} := & -\frac{1}{4} \frac{I t_0 t_1^2}{\pi} + \frac{1}{4} t_1 t_2 t_3 + \frac{1}{6} \frac{t_2^3}{t_3} - \frac{1}{8} I t_3^2 \pi g_1(\tau) t_2^2 + \frac{1}{61440} t_3^5 g_2(t_0) t_2 \\ & + \frac{1}{5505024} t_3^8 g_3(t_0) \end{aligned}$$

Example 5.4.2 The case B_3

$$\left\{ \begin{aligned} \tau = t_0, \phi_0 = t_1 - \frac{1}{3} I\pi g_1(\tau) (2t_2 t_4 + t_3^2), \\ \phi_1 = \frac{1}{12} t_3^2 + \frac{1}{6} t_2 t_4, \phi_3 = \frac{1}{46656} t_4^6, \phi_2 = \frac{1}{216} t_3 t_4^3 \end{aligned} \right\}$$

$$\begin{aligned} F^{(B_3)} := & -\frac{1}{4} \frac{I t_0 t_1^2}{\pi} + \frac{1}{6} t_4 t_2 t_1 + \frac{1}{12} t_3^2 t_1 + \frac{1}{2} \frac{t_3 t_2^2}{t_4} - \frac{1}{18} I t_4^2 \pi g_1(\tau) t_2^2 - \frac{1}{3} \frac{t_3^3 t_2}{t_4^2} \\ & - \frac{1}{18} I t_4 \pi g_1(\tau) t_3^2 t_2 + \frac{1}{77760} t_4^4 g_2(t_0) t_3 t_2 + \frac{1}{39191040} t_4^7 g_3(t_0) t_2 + \frac{1}{10} \frac{t_3^5}{t_4^3} \\ & - \frac{1}{72} I \pi g_1(\tau) t_3^4 + \frac{1}{155520} t_4^3 g_2(t_0) t_3^3 + \frac{1}{3732480} t_4^6 g_3(t_0) t_3^2 + \frac{1225}{209952} t_4^9 g_2(t_0)^2 t_3 \\ & + \frac{7}{1209323520} t_4^{12} G_5(t_0) \end{aligned}$$

Example 5.4.3 The case B_4

$$\left\{ \begin{aligned} \phi_0 = t_1 - \frac{1}{2} I\pi g_1(\tau) (t_2 t_5 + t_4 t_3), \phi_1 = \frac{1}{8} t_4 t_3 + \frac{1}{8} t_2 t_5, \\ \phi_2 = \frac{3}{1024} t_4^2 t_5^2 + \frac{1}{512} t_3 t_5^3, \phi_3 = \frac{1}{32768} t_4 t_5^5, \phi_4 = \frac{1}{16777216} t_5^8, \tau = t_0 \end{aligned} \right\}$$

$$\begin{aligned} F^{(B_4)} := & \frac{1}{8} t_4 t_3 t_1 + \frac{1}{8} t_5 t_2 t_1 - \frac{1}{32} I t_5^2 \pi g_1(\tau) t_2^2 + \frac{1}{2} \frac{t_2^2 t_4}{t_5} + \frac{1}{36700160} t_5^6 g_3(t_0) t_2 t_4 \\ & + \frac{1}{163840} t_5^3 g_2(t_0) t_4^2 t_2 + \frac{525}{8388608} t_5^9 g_2(t_0)^2 t_2 + \frac{1}{4} \frac{t_4^4 t_2}{t_5^3} + \frac{1}{245760} t_5^4 g_2(t_0) t_3 t_2 \\ & - \frac{1}{16} I t_5 \pi g_1(\tau) t_4 t_3 t_2 - \frac{t_4^2 t_3 t_2}{t_5^2} + \frac{1}{2} \frac{t_3^2 t_2}{t_5} - \frac{1}{2} \frac{t_4 t_3^3}{t_5^2} + \frac{3}{2147483648} t_5^{11} G_5(t_0) t_3 \\ & + \frac{1}{163840} g_2(t_0) t_5^2 t_4^3 t_3 + \frac{3}{18350080} t_5^5 g_3(t_0) t_4^2 t_3 + \frac{29925}{8388608} t_5^8 g_2(t_0)^2 t_4 t_3 - \frac{3}{4} \frac{t_4^5 t_3}{t_5^4} \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{32} I \pi g_1(\tau) t_4^2 t_3^2 + \frac{1}{22020096} t_5^6 g_3(t_0) t_3^2 + \frac{1}{245760} t_5^3 g_2(t_0) t_3^2 t_4 + \frac{7}{6} \frac{t_4^3 t_3^2}{t_5^3} \\
& + \frac{33}{274877906944} t_5^{13} G_6(t_0) t_4 + \frac{297}{21474836480} t_5^{10} G_5(t_0) t_4^2 \\
& + \frac{429}{985162418487296} t_5^{16} G_7(t_0) + \frac{6}{35} \frac{t_4^7}{t_5^5} + \frac{3}{29360128} g_3(t_0) t_5^4 t_4^4 + \frac{11025}{2097152} t_5^7 g_2(t_0)^2 t_4^3 \\
& - \frac{1}{4} \frac{I t_0 t_1^2}{\pi}
\end{aligned}$$

5.5 The Frobenius structures of the orbit space of $J(G_2)$

The exceptional root system G_2 displays an interesting and unexpected double Frobenius structure in the sense we are going to explain. We have seen in section 3.3 that the Jacobi algebra $J_{\bullet, \bullet}^{(G_2)}$ is naturally embedded as a subalgebra of $J_{\bullet, \bullet}^{(A_2)}$; hence, the flat structure on $\text{Spec}(J_{\bullet, \bullet}^{(G_2)})$ can be interpreted as a flat structure also on the (suitable covering of the) orbit space of the Jacobi group $J(G_2)$. Indeed, recall that the Jacobi forms of G_2 are given by

$$\varphi_0^{(G_2)} = \varphi_0^{(A_2)}; \quad \varphi_2^{(G_2)} = \varphi_2^{(A_2)}; \quad \varphi_6^{(G_2)} = \frac{1}{2} \left(\varphi_3^{(A_2)} \right)^2$$

and therefore the flat coordinates associated to the invariant metric $\eta^*(,) = \mathcal{L}_{\frac{\partial}{\partial \varphi_0}} \mathcal{J}^*(,)$ are exactly the same as those computed in example 5.3.2, namely

$$\varphi_0^{(G_2)} = t_1 - \frac{4}{3} i \pi g_1(\tau) t_2 t_3; \quad \varphi_2^{(G_2)} = \frac{1}{3} t_2 t_3; \quad \varphi_6^{(G_2)} = \frac{1}{2 \cdot 27^2} (t_3)^6;$$

therefore we have a Frobenius structure inherited from the one of A_2 , where the free energy is given by $F^{(G_2)} = F^{(A_2)} := \frac{1}{4i\pi} t_0 t_1^2 + \frac{1}{3} t_2 t_3 t_1 - \frac{1}{12} \frac{t_4^2}{t_3^2} - \frac{2i\pi}{9} t_3^2 t_2^2 g_1(t_0) - \frac{1}{24 \cdot 3^6} t_3^6 g_2(t_0)$.

On the other hand G_2 falls also into the category of “codimension one cases” [Sa90], and hence it has another natural flat structure which we have computed in Prop. 3.3.1. An easy computation of indices shows that Saito’s tensor $J^* := \mathcal{L}_{\frac{\partial}{\partial \varphi_6}} \mathcal{J}^*$ form a linear pencil of flat metric with the intersection form, $\mathcal{J}^* + \lambda J^*$. As a consequence there exists a unique associated Frobenius structure whose free energy can be recovered by application of the formulae described in section 5.1.2. In order to find it, we only have to find the scaling dimensions q_j and the Euler vector. Now, since the unit vector must have scaling dimension -1 , and is given by $\mathbf{1} := \frac{\partial}{\partial \varphi_6}$, then we must take as Euler vector

$$E := \frac{1}{2} \mathbf{E} = \frac{1}{4i\pi} \partial_u = t_6 \frac{\partial}{\partial t_6} + \frac{t_1}{2} \frac{\partial}{\partial t_1} + \frac{t_2}{2} \frac{\partial}{\partial t_2} = \varphi_6 \frac{\partial}{\partial \varphi_6} + \frac{\varphi_0}{2} \frac{\partial}{\partial \varphi_0} + \frac{\varphi_2}{2} \frac{\partial}{\partial \varphi_2};$$

from this expression we read off the the scaling degrees, $q_\tau = 1$, $q_1 = q_2 = \frac{1}{2}$, $q_6 = 0$ (namely, $d_\tau = 0, d_1 = d_2 = \frac{1}{2}, d_6 = 1$) and the scaling of the metric of equation (3.20), namely $\mathcal{L}_1 J = J$. The free energy of this Frobenius–Saito structure is found to be

$$F_{(Saito)}^{(G_2)} := \frac{1}{8i\pi} \tau t_6^2 + \frac{1}{2} t_6 t_1 t_2 + \frac{i\pi^2 \sqrt{3}}{36} (2z)^{1/3} (t_1^4 - 2 t_2^3 t_1) \eta^4 +$$

$$-i\pi g_1(t_0)t_2^2t_1^2 + \frac{i\pi^2\sqrt{3}}{36} (2(z-1))^{1/3} (t_2^4 - 2t_2t_1^3)\eta^4,$$

where we recall the definition $z(\tau) := \frac{1}{2} \left[\frac{\sqrt{-27}}{(2\pi)^6} \widehat{g}_3(\tau) + 1 \right]$. Again this expression follows from a straightforward but quite long computation which we spare. Notice that this free energy is well-behaved in t_1, t_2, t_6 and has singularities only in τ coming from the Dedekind's eta function and from the branching around the divisor $z(\tau) = 0$ and $z(\tau) = 1$, which both correspond to the \mathbb{Z}_3 -symmetric torus with $g_2(\tau) = 0$.

Summarizing, we have two Frobenius structures in which the Euler and unit vectors are

1. $\mathbf{E} = \varphi_0 \partial_{\varphi_0} + \varphi_2 \partial_{\varphi_2} + 2\varphi_6 \partial_{\varphi_6}$ and $\mathbf{1} = \partial_{\varphi_0}$ for the Frobenius structure inherited from the one of A_2 ;
2. $\mathbf{E} = \frac{1}{2}\varphi_0 \partial_{\varphi_0} + \frac{1}{2}\varphi_2 \partial_{\varphi_2} + \varphi_6 \partial_{\varphi_6}$ and $\mathbf{1} = \partial_{\varphi_6}$ for the Frobenius structure associated with Saito's flat structure.

The situation is that we have a two-dimensional linear pencil of flat metrics; by this we mean that the (contravariant) metric

$$G^* := \mathcal{I}^* + a\eta^* + bJ^*,$$

is flat for any choice of the constants $a, b \in \mathbb{C}$ with $ab = 0$ and the (contravariant) Levi-Civita connection is the same linear combination of the corresponding connections.

The fact that \mathcal{I}^* is linear in φ_0 is not predictable from the counting of bi-grades; in fact one could expect *a priori* the occurrence of a term proportional to φ_0^2 in $\mathfrak{M}(\varphi_0, \varphi_2) \in J_{0,2}^{(G_2)}$, which is not the case –as a matter of fact– as we saw in the explicit computation.

In the picture of [Sa90] the flat structure we have constructed in the case of A_l , and –as a by-product – in the case of G_2 is unexpected; it essentially comes from the additional structure provided by the modular properties of the Jacobi forms, which are “unseen” in the framework of the theta invariants studied in loc. cit.

There are hints at the possibility that a similar double Frobenius structure could be found also in the other exceptional cases F_4, E_6, E_7 (the latter two corresponding, in Saito's notation, to the “simple-elliptic cases” $\widetilde{E}_6, \widetilde{E}_7$). In another feasible case we find a similar structure: this happens for C_3 in next section.

We notice that the occurrence of multiple structures is not peculiar of Jacobi groups: indeed it occurs in the polynomial (Coxeter) case as well as we will show in Appendix B.

5.6 The Frobenius structures of the orbit space of $J(C_3)$

As in the G_2 case, we saw in section 3.5 that the Jacobi algebra $J_{\bullet, \bullet}^{(C_3)}$ is naturally embedded as a subalgebra of $J_{\bullet, \bullet}^{(A_3)}$; hence, the flat structure on $\text{Spec}(J_{\bullet, \bullet}^{(A_3)})$ can be interpreted as a flat structure also on the (suitable covering of the) orbit space of the Jacobi group $J(C_3)$. The Jacobi forms of C_3 are given by

$$\varphi_0^{(C_3)} = \varphi_0^{(A_3)}; \quad \varphi_2^{(C_3)} = \varphi_2^{(A_3)}; \quad \varphi_4^{(C_3)} = \varphi_4^{(A_3)}; \quad \varphi_6^{(C_3)} = \left(\varphi_3^{(A_3)} \right)^2.$$

As before we have a Frobenius structure inherited from the one of A_2 , where the free energy is given by the same free energy $F^{(A_3)}$ listed in Example 5.3.2. However C_3 falls also into the category of “codimension one cases”, and hence it has another natural flat structure which we have computed in Prop. 3.5.1. An easy computation of indices shows that Saito’s tensor $J^* := \mathcal{L}_{\frac{\partial}{\partial \varphi_6}} \mathcal{J}^*$ form a linear pencil of flat metric with the intersection form, $\mathcal{J}^* + \lambda J^*$. Again, there exists a unique associated Frobenius structure whose free energy can be recovered by application of the formulae described in section 5.1.2. The scaling dimensions q_j and the Euler vector of this new structure are

$$E := \frac{1}{2} \mathbf{E} = \frac{1}{4i\pi} \partial_u = \varphi_6 \frac{\partial}{\partial \varphi_6} + \frac{\varphi_0}{2} \frac{\partial}{\partial \varphi_0} + \frac{\varphi_2}{2} \frac{\partial}{\partial \varphi_2} + \frac{\varphi_4}{2} \frac{\partial}{\partial \varphi_4};$$

from this expression we read off the the scaling degrees, $q_\tau = 1$, $q_1 = q_2 = q_4 = \frac{1}{2}$, $q_6 = 0$ (namely, $d_\tau = 0, d_1 = d_2 = d_3 = \frac{1}{2}, d_6 = 1$) and the scaling of the metric of equation (3.20), namely $\mathcal{L}_1 J = J$. The free energy for this Frobenius structure is given by

$$\begin{aligned} F_{(Saito)}^{(C_3)} = & \frac{1}{8} \frac{i \tau t_6^2}{\pi} - \frac{(\frac{1}{4} t_4^2 + \frac{1}{8} t_2 t_1) t_6}{\pi^2} - \frac{1}{192} \frac{(2 t_4^2 + t_2 t_1)^2 P}{\pi^2} + \\ & + \frac{1}{576} \frac{i \sqrt{3} 2^{(1/3)} (-12 t_4^2 t_1^2 + 2 t_2 t_1^3 - t_2^4 - 16 t_4^3 t_2) \eta^4 z^{(1/3)}}{\pi^2} + \\ & - \frac{1}{576} \frac{i \sqrt{3} 2^{(1/3)} (t_1^4 - 2 t_1 t_2^3 + 16 t_1 t_4^3 + 12 t_4^2 t_2^2) \eta^4 (z-1)^{(1/3)}}{\pi^2} \end{aligned}$$

where we recall the definition $z(\tau) := \frac{1}{2} \left[\frac{\sqrt{-27}}{(2\pi)^6} \widehat{g}_3(\tau) + 1 \right]$. Again this expression follows from a straightforward but quite long computation which we spare. As in the G_2 case this free energy is well-behaved in t_1, t_2, t_4, t_6 and has singularities only in τ coming from the Dedekind’s eta function and from the branching around the divisor $z(\tau) = 0$ and $z(\tau) = 1$, which both correspond to the \mathbb{Z}_3 -symmetric torus with $g_2(\tau) = 0$.

Appendix A

Formulae

In these appendix we list and derive all formulae involving the elliptic functions that are used in the previous sections, and we explain the pairing of forms used in the context of Hurwitz spaces.

A.1 Normalized elliptic functions

In the text we have always used the normalized elliptic functions.

In order to match with the more common formulae we give here the table of conversion.

Here *normalized* means that the periods are 1 and τ while usually the half periods are ω_1, ω_2 , where $\tau = \frac{\omega_2}{\omega_1}$. We define the Weierstrass eta functions by means of the quasi-periodicity of the unnormalized Weierstrass zeta function

$$\begin{aligned}\zeta(z + 2\omega_1) &= \zeta(z) + 2\eta_1 \\ \zeta(z + 2\omega_2) &= \zeta(z) + 2\eta_2 \\ \eta_1 &:= \zeta(\omega_1) ; \quad \eta_2 := \zeta(\omega_2) \\ \eta_1 \omega_2 - \eta_2 \omega_1 &= \frac{i\pi}{2} .\end{aligned}$$

If $\eta(\tau)$ is the Dedekind's eta function we have the following relation

$$\begin{aligned}2\omega_1\eta_1 &= -2i\pi g_1(\tau) \\ 2\omega_1\eta_2 &= -2i\pi \tau g_1(\tau) - i\pi ,\end{aligned}$$

and hence the quasi-periodicity of the normalized zeta function (here –and only here– we put a subscript N to mean the normalized function)

$$\begin{aligned}\zeta_N(v) &:= 2\omega_1\zeta(z) \\ \zeta_N(v + 1) &= \zeta_N(v) - 4i\pi g_1(\tau) \\ \zeta_N(v + \tau) &= \zeta_N(v) - 4i\pi \tau g_1(\tau) - 2i\pi .\end{aligned}$$

In the previous sections we have always used normalized elliptic functions.

A.2 The elliptic connection \mathbb{D}^\bullet

Proposition A.2.1 Let $F(v|\tau)$ be an elliptic function of weight k , namely

$$F\left(\frac{v}{\tau} \middle| -\frac{1}{\tau}\right) = \tau^k F(v|\tau), \quad (\text{A.1})$$

then the function

$$(\mathbb{D}^{(k)}F)(v|\tau) := \nabla_\tau F(v|\tau) - \frac{\alpha'(v, \tau)}{2i\pi\alpha(v, \tau)} F'(v|\tau) := \eta^{2k} \partial_\tau \left(\eta^{-2k} F(v|\tau) \right) - \frac{\alpha'(v, \tau)}{2i\pi\alpha(v, \tau)} F'(v|\tau)$$

is an elliptic function of weight $k+2$.

Proof. Differentiating both members of eq. (A.1) w.r.t. τ we get

$$\begin{aligned} (\partial_\tau F)\left(\frac{v}{\tau} \middle| -\frac{1}{\tau}\right) &= \tau^{k+2} \partial_\tau F(v|\tau) + k\tau^{k+1} F(v|\tau) + \tau^{k+3} v F'(v|\tau); \\ (\partial_\tau F)(v + \tau|\tau) &= \partial_\tau F(v|\tau) - F'(v|\tau); \quad (\partial_\tau F)(v + 1|\tau) = \partial_\tau F(v|\tau), \end{aligned}$$

and in particular we find

$$\begin{aligned} (\nabla_\tau F)\left(\frac{v}{\tau} \middle| -\frac{1}{\tau}\right) &= \tau^{k+2} \nabla_\tau F(v|\tau) + \tau^{k+3} v F'(v|\tau); \\ (\nabla_\tau F)(v + \tau|\tau) &= \nabla_\tau F(v|\tau) - F'(v|\tau). \end{aligned}$$

Recall now that

$$\begin{aligned} \gamma(v|\tau) &:= \frac{1}{2i\pi} \frac{\alpha'(v|\tau)}{\alpha(v|\tau)} = 2vg_1(\tau) + \frac{\zeta(v|\tau)}{2i\pi} \\ \gamma\left(\frac{v}{\tau} \middle| -\frac{1}{\tau}\right) &= \tau\gamma(v|\tau) + v\tau^2; \quad \gamma(v + \tau) = \gamma(v|\tau) - 1; \quad \gamma(v + 1|\tau) = \gamma(v|\tau). \end{aligned}$$

Therefore we finally have

$$\begin{aligned} (\mathbb{D}^{(k)}F)\left(\frac{v}{\tau} \middle| -\frac{1}{\tau}\right) &= \tau^{k+2} \mathbb{D}^{(k)}F(v|\tau); \quad (\mathbb{D}^{(k)}F)(v|\tau + 1) = \mathbb{D}^{(k)}F(v|\tau) \\ (\mathbb{D}^{(k)}F)(v + \tau|\tau) &= \mathbb{D}^{(k)}F(v|\tau); \quad (\mathbb{D}^{(k)}F)(v + 1|\tau) = \mathbb{D}^{(k)}F(v|\tau). \end{aligned}$$

This ends the proof. Q.E.D

In order to interpret this operator we consider the **universal torus**, namely the fibration over $(\mathbb{C}^2)_+ := \left\{ (\omega, \omega') \text{ s.t. } \frac{\omega'}{\omega} \in \mathcal{H} \right\}$ whose fiber is the unnormalized torus $\mathbb{C}/(2\omega\mathbb{Z} + 2\omega'\mathbb{Z})$, or better

$$\mathcal{M} := \begin{array}{c} E_\Lambda \\ \downarrow \\ \mathcal{L} \end{array},$$

where \mathcal{L} is the set of all lattices $\Lambda := 2\omega\mathbb{Z} + 2\omega'\mathbb{Z}$.

Cursorily speaking we can take $(z; \omega, \omega')$ as local coordinates over \mathcal{M} (which can be seen also as a smooth principal fiber bundle with structure group $S^1 \times S^1$).

The analytic sections of this bundle are the (unnormalized) elliptic functions with periods $2\omega, 2\omega'$. Over this fiber bundle we have three natural vector fields [Du93]

$$\begin{aligned} D_1 &:= \omega \frac{\partial}{\partial \omega} + \omega' \frac{\partial}{\partial \omega'} + z \frac{\partial}{\partial z} \\ D_2 &:= \frac{\partial}{\partial z} \\ D_3 &:= \eta_1 \frac{\partial}{\partial \omega} + \eta_2 \frac{\partial}{\partial \omega'} + \zeta \frac{\partial}{\partial z} \end{aligned}$$

where the ζ function is the unnormalized one.

Consider now the projection over the **universal elliptic curve** \mathcal{E} defined as

$$\begin{aligned} \Pi : \quad \mathcal{M} &\longrightarrow \mathcal{E} \\ (z; \omega, \omega') &\mapsto (v|\tau) := \left(\frac{z}{2\omega} \middle| \frac{\omega'}{\omega} \right) \end{aligned}$$

and the push-forward of the above three vectors. It appears that D_1 is the vertical vector field and Π realizes the universal torus \mathcal{M} as a trivial \mathbb{C}^* -principal fibre-bundle over the universal elliptic curve; the other two vectors are horizontal and hence they induce a natural connection over this fibration $\Pi : \mathcal{M} \rightarrow \mathcal{E}$.

The sections of the associated line bundle is the sheaf \mathcal{S} of *homogeneous elliptic functions*

$$\begin{aligned} f(cz; c\omega, c\omega') &= c^{-k} f(z; \omega, \omega') \in \mathcal{S}^{\otimes k} \\ f(z + 2m\omega + 2n\omega'; \omega, \omega') &= f(z; \omega, \omega') , \end{aligned}$$

and the horizontal connection D induced by the vectors D_2, D_3 is

$$\begin{aligned} D_z &= \frac{\partial}{\partial v} \\ D_\tau &:= D_\tau^{(k)} = \eta^{2k} \frac{\partial}{\partial \tau} \eta^{-2k} - \frac{\alpha'(v|\tau)}{2i\pi\alpha(v|\tau)} \frac{\partial}{\partial v} . \end{aligned}$$

It follows that the elliptic connection $\mathbb{D}^{(k)}$ is just the horizontal vector D_τ acting on section of the sheaf $\mathcal{S}^{\otimes k}$. Since we also have a natural structure of graded algebra, we see that we can define the connection on the whole algebra by

$$\mathbb{D}^\bullet := \bigoplus_{j=0}^{\infty} \mathbb{D}^{(k)} .$$

A.3 Pseudo addition formulae

Proposition A.3.1 The following formula holds

$$\left(\frac{\alpha''(x)}{\alpha(x)} + \frac{\alpha''(y)}{\alpha(y)} - 2 \frac{\alpha'(x)\alpha'(y)}{\alpha(x)\alpha(y)} \right) = -4i\pi \frac{\partial_\tau \alpha(x-y)}{\alpha(x-y)} + 2 \frac{\alpha'(x-y)}{\alpha(x-y)} \left\{ \frac{\alpha'(x)}{\alpha(x)} - \frac{\alpha'(y)}{\alpha(y)} \right\} .$$

Proof. Recall first that $\alpha(x) = \frac{1}{2\pi}\eta^{-3}e^{2i\pi\frac{\eta'}{\eta}x^2}\sigma(x)$ where σ is the Weierstrass sigma function. Recall also the standard notation

$$\zeta := \frac{d}{dx} \log(\sigma); \quad \wp := -\frac{d}{dx} \zeta = \frac{d^2}{dx^2} \log(\sigma).$$

Thus we can compute¹ (putting $A := 2i\pi\frac{\eta'}{\eta}$)

$$\begin{aligned} & \left(\frac{\alpha''(x)}{\alpha(x)} + \frac{\alpha''(y)}{\alpha(y)} - 2 \frac{\alpha'(x)\alpha'(y)}{\alpha(x)\alpha(y)} \right) = \\ & = 4A + 4A^2(x-y)^2 + 4A(x-y)(\zeta(x) - \zeta(y)) + \frac{\sigma''(x)\sigma(y) + \sigma''(y)\sigma(x) - 2\sigma'(x)\sigma'(y)}{\sigma(x)\sigma(y)} \end{aligned}$$

and the long fraction involving the sigma functions can be rewritten as

$$\frac{\sigma(y)\sigma''(x) + \sigma(x)\sigma''(y) - 2\sigma'(x)\sigma'(y)}{\sigma(x)\sigma(y)} = -\wp(x) - \wp(y) + (\zeta(x) - \zeta(y))^2.$$

We now need the following pseudo-addition theorem for the ζ function

Theorem A.3.1 [see [WW], pag 446] If $x + y + z = 0$ then

$$[\zeta(x) + \zeta(y) + \zeta(z)]^2 = \wp(x) + \wp(y) + \wp(z)$$

Applying this formula with the substitution $y \mapsto -y$ we can write

$$[\zeta(x) - \zeta(y)]^2 - \wp(x) - \wp(y) = \wp(z) - \zeta^2(z) - 2\zeta(z)[\zeta(x) - \zeta(y)]$$

and hence, expanding and rearranging terms we obtain

$$\begin{aligned} & \left(\frac{\alpha''(x)}{\alpha(x)} + \frac{\alpha''(y)}{\alpha(y)} - 2 \frac{\alpha'(x)\alpha'(y)}{\alpha(x)\alpha(y)} \right) = \\ & = 12i\pi g_1(\tau) - \frac{\alpha''(x-y)}{\alpha(x-y)} + 2 \frac{\alpha'(x-y)}{\alpha(x-y)} \left\{ \frac{\alpha'(x)}{\alpha(x)} - \frac{\alpha'(y)}{\alpha(y)} \right\} = \\ & = -4i\pi \frac{\partial_\tau \alpha(x-y)}{\alpha(x-y)} + 2 \frac{\alpha'(x-y)}{\alpha(x-y)} \left\{ \frac{\alpha'(x)}{\alpha(x)} - \frac{\alpha'(y)}{\alpha(y)} \right\} \end{aligned} \tag{A.2}$$

This ends the proof. Q.E.D

Corollary A.3.1 From the previous it follows

$$4i\pi \left[\frac{\nabla_\tau \alpha(x)}{\alpha(x)} + \frac{\nabla_\tau \alpha(y)}{\alpha(y)} \right] - 2 \frac{\alpha'(x)\alpha'(y)}{\alpha(x)\alpha(y)} = -4i\pi \frac{\nabla_\tau \alpha(x-y)}{\alpha(x-y)} + 2 \frac{\alpha'(x-y)}{\alpha(x-y)} \left\{ \frac{\alpha'(x)}{\alpha(x)} - \frac{\alpha'(y)}{\alpha(y)} \right\}$$

¹We use the straightforward formulae

$$\alpha'(x) = e^{Ax^2} (2xA\sigma(x) + \sigma'(x)); \quad \alpha''(x) = e^{Ax^2} (4x^2A^2\sigma(x) + 4xA\sigma'(x) + 2A\sigma(x) + \sigma''(x)),$$

with $A := 2i\pi\frac{\eta'}{\eta}$.

Proof. It follows from the proposition recalling the definition $\nabla_\tau \alpha(x) := \partial_\tau \alpha(x) + 2g_1 \alpha(x)$, and the equation

$$\partial_\tau \alpha(x) = \frac{1}{4i\pi} \alpha''(x) - 3g_1(\tau) \alpha(x) .$$

Q.E.D .

A.4 Periods of elliptic functions

Let $\Phi(v)$ be a meromorphic elliptic differential of the second kind (i.e. without residues), namely $\Phi(v) = f(v) dv$, with $f(v)$ meromorphic elliptic function.

For the sake of simplicity assume that ϕ has only one pole at, say, $v = 0$; if the principal part is

$$f(v) = \sum_{-N}^0 C_k v^k + O(v) \quad C_{-1} = 0 ,$$

then the periods of the differential ϕ are computed by expressing $f(v)$ in terms of the ζ function

$$f(v) = C_0 - \sum_{k=2}^N \frac{(-1)^k}{(k-1)!} C_k \zeta^{(k-1)}(v) .$$

Then we promptly find

$$\begin{aligned} \oint_a \Phi &= \int_c^{c+1} f(v) dv = C_0 + 4i\pi g_1 C_{-2} \implies \operatorname{res}_{v=0} \left[\frac{f(v) dv}{v} + 4i\pi g_1 v f(v) dv \right] \\ \oint_b \Phi &= \int_c^{c+\tau} f(v) dv = \tau C_0 + 4i\pi (\tau g_1 + 2i\pi) C_{-2} \implies \operatorname{res}_{v=0} \left[\tau \frac{f(v) dv}{v} + (4i\pi \tau g_1 + 2i\pi) v f(v) dv \right] \\ \oint_b \Phi - \tau \oint_a \Phi &= 2i\pi \operatorname{res}_{v=0} (v f(v) dv) . \end{aligned} \tag{A.3}$$

A.5 The pairing of forms.

Let $\widehat{M} := \widehat{M}_{g,m;n_0,\dots,n_m}$ be the covering of the Hurwitz space $M_{g,m;n_0,\dots,n_m}$ introduced in Def. 5.2.3; this means that its points are the set of data $(C, \lambda, \{a_1, \dots, a_g; b_1, \dots, b_g\}, k_0, k_1, \dots, k_m)$ where

- i) C is a smooth curve of genus g with $m+1$ marked points $\infty_0, \dots, \infty_m$;
- ii) λ is a map $\lambda : C \rightarrow \mathbb{C}P^1$ with poles at the marked points and of branching degree respectively n_0, \dots, n_m ;
- iii) $\{a_1, \dots, a_g; b_1, \dots, b_g\}$ is a symplectic basis in the homology of the curve C ;
- iv) k_0, k_1, \dots, k_m are some fixed branches of the roots of λ at the marked points (infinities), namely

$$k_\nu := (\lambda)^{\frac{1}{n_\nu+1}} ; \quad \nu = 0, \dots, m .$$

Let \mathbf{C} be the universal covering of the punctured curve $C \setminus \{\infty_0, \dots, \infty_m\}$ with canonical projection Π

$$\Pi : \mathbf{C} \longrightarrow C \setminus \{\infty_0, \dots, \infty_m\}$$

and let Φ, Ψ be two holomorphic differentials on this covering (namely with poles possibly only at the poles of λ). We moreover assume that their polydromy is fixed and *independent of the moduli* u_1, \dots, u_n of the Hurwitz space. Explicitly we work in the following hypotheses: let $\{a_1, \dots, a_g; b_1, \dots, b_g\}$ be a symplectic basis of cycles on the curve C realized by paths which avoid the infinities, with basepoint $P_0 \in C$ such that $\lambda(P_0) = 0$; let s_0, \dots, s_m be pairwise nonintersecting paths joining P_0 to the infinities and $\gamma_0, \dots, \gamma_m$ small loops around the infinities in counterclockwise orientation.

Let \tilde{C} be a simply connected domain in the universal covering \mathbf{C} constructed as follows: we lift the cycle a_1 to \mathbf{C} and hence obtain a segment from Q_0 to Q_1 such that $\Pi(Q_0) = \Pi(Q_1) = P_0$. We then lift the cycles $a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, \dots, b_g, a_g^{-1}, b_g^{-1}$ in this order and consequently obtain $4g$ points Q_0, Q_1, \dots, Q_{4g} on the covering. Now we lift the paths $s_0, \gamma_0, s_0^{-1}, s_1, \gamma_1, s_1^{-1}, \dots, s_m, \gamma_m, s_m^{-1}$ and at the end, by definition of universal covering, we have come back to the initial point Q_0 and hence have a boundary $\partial\tilde{C}$.

We now make the following assumptions on the differentials Φ, Ψ ;

1. **Polydromy** we assume that the two differentials are $d\lambda$ -**multivalued**, namely there exist suitable analytic functions $A_{\Phi}^j, B_{\Phi}^j, A_{\Psi}^j, B_{\Psi}^j$ on the complex plane such that for any cycles a_j, b_j we have the following properties

$$\begin{aligned} \Phi(P + a_j) &= \Phi(P) + dA_{\Phi}^j(\lambda) = \Phi(P) + \frac{d}{d\lambda} A_{\Phi}^j(\lambda) d\lambda \\ \Phi(P + b_j) &= \Phi(P) + dB_{\Phi}^j(\lambda) = \Phi(P) + \frac{d}{d\lambda} B_{\Phi}^j(\lambda) d\lambda, \end{aligned}$$

and similar formulae for Ψ ; we stress that the cycles are to be meant such that their lift to the universal covering lies inside the simply connected domain $\tilde{C} \subset \mathbf{C}$ defined above.

We further ask that the branching around the infinities is at most $d\lambda$ -logarithmic, namely there exist suitable analytic functions $F_{\Phi}^{\mu}(\lambda), F_{\Psi}^{\mu}(\lambda)$, $\mu = 0, 1, \dots, m$ such that, near ∞_{μ} the polydromy is the same of the differential $d(F_{\Phi}^{\mu}(\lambda) \log(\lambda))$.

All these polydromy data are assumed to be *independent of the moduli*.

2. **Periods** We assume that the a -periods are independent of the moduli.
3. **Poles** We assume that the singular part (up to the polydromy described above) does not depend on the moduli, namely, near an infinity ∞_{μ} , in the *fixed local coordinate*² $z_{\mu} := \frac{1}{k_{\mu}} = \frac{1}{n_{\mu} + 1\sqrt{\lambda}}$

$$\Phi(P) = \sum_{-N}^{\infty} c_{\Phi, k}^{\mu} z_{\mu}^k dz_{\mu} + d(F_{\Phi}^{\mu}(\lambda) \log(\lambda)),$$

and the singular part, namely the numbers $c_{\Phi, -N}^{\mu}, \dots, c_{\Phi, -1}^{\mu}$ are all independent on the moduli.

²This is to stress that the coefficients are to be defined w.r.t. the Laurent expansion w.r.t. these variables

We remark that we have used the fixed branches k_μ of the roots of λ at the infinities ∞_μ , which is a datum of the Hurwitz space.

Definition A.5.1 *The pairing is defined by means of*

$$\begin{aligned} \langle \Phi, \Psi \rangle := & - \sum_{\mu=0}^m \left\{ \sum_{k \geq 0} c_{\Phi, -k-2}^\mu \frac{c_{\Psi, k}^\mu}{k+1} + c_{\Phi, -1}^\mu v.p. \int_{P_0}^{\infty_\mu} \Psi + v.p. \int_{P_0}^{\infty_\mu} F_\Phi(\lambda) \Psi \right\} + \\ & + \frac{1}{2i\pi} \sum_{j=1}^g \left\{ - \int_{P_0}^{P_0+a_j} B_\Phi(\lambda) \Psi + \int_{P_0}^{P_0+b_j} A_\Phi(\lambda) \Psi + \int_{a_j} \Phi \int_{b_j} \Psi \right\}. \end{aligned}$$

The expression is rather cumbersome, though it is important to notice that the dependence of the moduli enters only through Ψ .

Consider the tautological fiber bundle over the Hurwitz space such that the fiber is the punctured curve C ; onto this fiber bundle we define a lift of the vectors $\frac{\partial}{\partial u_i}$

Definition A.5.2 *We define the connection on the locally trivial fiber-bundle*

$$\begin{array}{c} \downarrow C \\ M \end{array}$$

defined implicitly by the formula

$$\nabla_i \lambda \equiv 0.$$

This means that all derivatives w.r.t. the moduli have to be done *at λ constant*; this implies that if we realize the curves C_u as locally identical curves, the connection defines some vector fields on this curve by means of the formula (in a local coordinate w on the curve)

$$\frac{\partial w}{\partial u_i} = - \frac{1}{\lambda'(w; u)} \frac{\partial \lambda(w; u)}{\partial u_i}.$$

Notice that if w_i is a local coordinate in the neighborhood of the point P_i we have that

$$\frac{\partial w_j}{\partial u_i} = - \frac{1}{\lambda'(w_j; u)} \frac{\partial \lambda(w_j; u)}{\partial u_i}$$

is a function with a simple pole only if $i = j$ and regular otherwise.

We have the following

Lemma A.5.1 [*Lemma 5.1 in [Du93]*] *Let $u_i := \lambda(P_i)$, s.t. $d\lambda(P_i) = 0$ be the moduli of the Hurwitz space, then we have $\forall i = 1..n$*

$$\frac{\partial}{\partial u_i} (\langle \Phi, \Psi \rangle) = - \operatorname{res}_{P_i} \frac{\Phi \Psi}{d\lambda}.$$

Corollary A.5.1 *The pairing is symmetric up to a constant independent of the moduli.*

Appendix B

Multilinear pencils of flat metrics: a simple example

We have encountered an unexpected issue: the occurrence of *two* different, compatible Frobenius structures on the orbit space of the Jacobi group of type G_2 and C_3 . In that case we saw that there exist a plane of flat metrics and not only a pencil. Correspondingly, at the level of the semiclassical dispersionless integrable hierarchy, we obtain a *triple* of Poisson structures.

One might think that this oddity is a sort of accident occurring only in this isolated case. However, after some research we found a very simple example which displays this feature at the extremum. Indeed, we can find a flat metric over \mathbb{C}^l which admits exactly l different Frobenius structures.

The example starts on the orbit space of the Coxeter group of type B_l . We compute here the intersection form (which is common to all structures) and the different invariant metrics, which correspond to the choice of a unit vector. We prove that each choice provides a flat pencil of metrics and compute the free energy associated to each of them.

B.1 Intersection form

Consider the case of the Weyl group of B_l acting on the orthogonal coordinates x_1, \dots, x_l and the generating function for the invariant polynomials

$$P(v) := \prod_{j=1}^l (v^2 - x_j^2) = \sum_{j=0}^l v^{2(l-j)} y_j, \quad y_0 := 1.$$

Then we have

Proposition B.1.1 The coefficients of the intersection form in the y coordinates are to be read of the equation

$$\sum_{j,k} \langle dy_j dy_k \rangle v^{2(l-j)} u^{2(l-k)} = \langle dP(u), dP(v) \rangle = \frac{2}{(u^2 - v^2)} (vP'(v)P(u) - uP'(u)P(v)) .$$

They are at most linear in each of the y coordinates.

Proof. We have

$$\begin{aligned} \langle dP(u), dP(v) \rangle &= P(u)P(v) \sum_k \frac{4x_k^2}{(v^2 - x_k^2)(u^2 - x_k^2)} = \\ &= P(u)P(v) \sum_k \frac{4}{(u^2 - v^2)} \left(\frac{v^2}{v^2 - x_k^2} - \frac{u^2}{u^2 - x_k^2} \right) = \frac{2}{(u^2 - v^2)} (vP'(v)P(u) - uP'(u)P(v)) . \end{aligned}$$

Now to show linearity it suffices to spell out the RHS

$$\frac{2}{(u^2 - v^2)} (vP'(v)P(u) - uP'(u)P(v)) = \frac{4}{(u^2 - v^2)} \sum_{j,k=0}^l (j - k)v^{2(l-j)}u^{2(l-k)}y_j y_k .$$

This ends the proof. Q.E.D

Proposition B.1.2 The components of the contravariant connection are to be read off the formula

$$\begin{aligned} \sum_{i,j} u^{2(l-i)}v^{2(l-j)} \nabla_{dy_i^\#} dy_j &= \nabla_{dP(u)^\#} dP(v) = \\ &= \frac{2v^2}{(u^2 - v^2)^2} \left[P(u)dP(v) - P(v)dP(u) \right] + \frac{2}{u^2 - v^2} \left[P(u)v dP'(v) - uP'(u)dP(v) \right] . \end{aligned}$$

They are linear in all the y coordinates.

Proof. We have

$$\begin{aligned} \nabla_{dP(u)^\#} dP(v) &= \sum_{r,k} \partial_k P(u) \partial_k \partial_r P(v) dx_r = \\ &= P(u)P(v) \sum_{k,r} \frac{-2x_k}{v^2 - x_k^2} \left[\frac{-2\delta_{rk}}{u^2 - x_k^2} + 4 \frac{(1 - \delta_{rk})x_k x_r}{(u^2 - x_k^2)(v^2 - x_r^2)} \right] dx_r = \\ &= \frac{2}{u^2 - v^2} \left(P(v)dP(u) - P(u)dP(v) \right) - 8P(u)P(v) \sum_{k \neq r} \frac{x_k^2 x_r}{(u^2 - x_k^2)(v^2 - x_r^2)(u^2 - x_k^2)} dx_r \end{aligned}$$

Now we have

$$\begin{aligned} \frac{x_k}{(v^2 - x_k^2)(u^2 - x_k^2)} &= \frac{1}{(u^2 - v^2)} \left[\frac{x_k}{v^2 - x_k^2} - \frac{x_k}{u^2 - x_k^2} \right] \\ \frac{x_k^2 x_r}{(u^2 - x_k^2)(v^2 - x_k^2)(v^2 - x_r^2)} &= \frac{1}{u^2 - v^2} \frac{x_r}{v^2 - x_r^2} \left[\frac{v^2}{v^2 - x_k^2} - \frac{u^2}{u^2 - x_k^2} \right] \end{aligned}$$

so that the second addend

$$-8P(u)P(v) \sum_{k \neq r} \frac{x_k^2 x_r}{(u^2 - x_k^2)(v^2 - x_r^2)(u^2 - x_k^2)} dx_r =$$

$$\begin{aligned}
&= -8 \frac{P(u)P(v)}{u^2 - v^2} \sum_{k \neq r} \frac{x_r}{v^2 - x_r^2} \left[\frac{v^2}{v^2 - x_k^2} - \frac{u^2}{u^2 - x_k^2} \right] dx_r = \\
&= \frac{2}{u^2 - v^2} P(u)v dP'(v) - \frac{2}{u^2 - v^2} uP'(u)dP(v) - 8 \frac{P(u)P(v)}{u^2 - v^2} \sum_k \frac{x_k u^2}{(v^2 - x_k^2)(u^2 - x_k^2)} = \\
&= \frac{2}{u^2 - v^2} P(u)v dP'(v) - \frac{2}{u^2 - v^2} uP'(u)dP(v) - \frac{4u^2}{(u^2 - v^2)^2} \left(P(v)dP(u) - P(u)dP(v) \right) = \\
&= \frac{2}{u^2 - v^2} \left(P(u)v dP'(v) - uP'(u)dP(v) \right) - \frac{4u^2}{(u^2 - v^2)^2} \left(P(v)dP(u) - P(u)dP(v) \right)
\end{aligned}$$

Hence

$$\begin{aligned}
\nabla_{dP(u)} dP(v) &= \frac{2}{u^2 - v^2} \left(P(v)dP(u) - P(u)dP(v) \right) + \\
&+ \frac{2}{u^2 - v^2} \left(P(u)v dP'(v) - uP'(u)dP(v) \right) - \frac{4u^2}{(u^2 - v^2)^2} \left(P(v)dP(u) - P(u)dP(v) \right) = \\
&= \frac{2v^2}{(u^2 - v^2)^2} \left(P(u)dP(v) - P(v)dP(u) \right) + \frac{2}{u^2 - v^2} \left(P(u)v dP'(v) - uP'(u)dP(v) \right).
\end{aligned}$$

To show linearity it is sufficient to notice that the formula is linear in $P(u)$, $P(v)$. Q.E.D

B.2 Frobenius structure

The Frobenius structures arising from the different choices of unit vector field are recovered from the superpotential

$$Q_{(k,l)}(v) := \frac{y_l}{v^{2(l-k)}} + \frac{y_{l-1}}{v^{2(l-k)-2}} + \dots + y_k + y_{k+1}v^2 + \dots + y_1v^{2k-2} + v^{2k} = \frac{P(v)}{v^{2(l-k)}}.$$

The flat coordinates are

$$\begin{aligned}
t_j &:= \frac{1}{2j-1} \operatorname{res}_{v=\infty} (Q_{(k,l)}(v))^{\frac{2j-1}{2k}}, \quad j = 1..k \\
t_{j+k} &:= \frac{1}{2j-1} \operatorname{res}_{v=0} (Q_{(k,l)}(v))^{\frac{2j-1}{2(l-k)}}, \quad j = 1..l-k,
\end{aligned}$$

and the corresponding free energies are

$$F_{k,l}(t) := -\frac{(2k)^2}{2} \left[\frac{c_{k+1}}{2k+1} + \sum_{n=0}^{k-1} t_{k-n} \frac{c_{n+1}^{(\infty)}}{(2n+1)} \right] - \frac{(2l-2k)^2}{2} \sum_{m=0}^{l-k-1} t_{l-m} \frac{c_{m+1}^{(0)}}{(2m+1)},$$

where the coefficients $c_n^{(\infty)}$, $c_m^{(0)}$ are defined as

$$\begin{aligned}
v &= z + \sum_{j=1}^k t_j \frac{1}{z^{2j-1}} + \sum_{j=k+1}^{2k} c_{j-k} \frac{1}{z^{2j-1}} + O\left(\frac{1}{z^{2k+1}}\right), \quad z := Q_{(k,l)}(v)^{\frac{1}{2k}} \rightarrow \infty \\
v &= \sum_{j=1}^{l-k} t_{k+j} z^{2j-1} + \sum_{j=l-k+1}^{2(l-k)} c_{j-l+k} z^{2j-1} + O\left(z^{2l-2k+1}\right), \quad z := Q_{(k,l)}(v)^{-\frac{1}{2(l-k)}} \rightarrow 0.
\end{aligned}$$

The flat contravariant invariant metric of this Frobenius structure is block diagonal with diagonal blocks of dimension k and $l - k$ respectively and both blocks with $\frac{1}{2k}$ and $\frac{1}{2(l-k)}$ on the principal antidiagonal and zero elsewhere

$$\eta^{ij} := \begin{pmatrix} 0 & 0 & \cdots & \frac{1}{2k} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots & 0 & 0 & \cdots & 0 \\ \frac{1}{2k} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \frac{1}{2(l-k)} \\ \vdots & 0 & \cdots & \vdots & 0 & 0 & \ddots & 0 \\ \vdots & 0 & \cdots & \vdots & 0 & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 0 & \frac{1}{2(l-k)} & 0 & \cdots & 0 \end{pmatrix}$$

B.3 The case B_3

Notice that in the case B_3 we have three relevant structures: the three corresponding free energies are

$$\begin{aligned} F_{3,3} &:= \frac{6}{35} t_1^7 + 6 t_1^3 t_2^2 + (18 t_3^2 + 6 t_2^3) t_1 + 18 t_2^2 t_3 \\ F_{3,2} &:= \frac{4}{15} t_1^5 - 2 t_1^2 t_3^2 + 8 t_1 t_2^2 + 4 t_2 t_3^2 \\ F_{3,1} &:= \frac{2}{3} t_1^3 + 8 t_1 t_2 t_3 - \frac{1}{3} t_2^4 - \frac{8}{3} \frac{t_3^3}{t_2} . \end{aligned}$$

Notice that the usual case is $F_{3,3}$ and is, of course polynomial. But then also $F_{3,2}$ is polynomial and corresponds to D_3 (which is the same as A_3 : this is clear and holds true in the general case, namely $F_{l,l-1}$ is always the free energy of D_l). The third free energy is not polynomial.

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