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INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

TRIESTE

Thesis submitted for the degree of Doctor Philosophiæ

Issues in Matrix String Theory

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Accademic year 1998/99

Contents

Chapter 1

From String Theory to Matrix String Theory

1.1 Introduction

String theory [1] is a beautiful and enjoying construction in mathematical physics. It is a vast and fun toy-room full of everyday new amusements for the playful ones who like such a kind of games.

It pretends, actually, to be also something more than this. String theory by now represents the best candidate framework to try to understand, in an unique scheme and deeply, both gravity and quantum mechanics. This scheme is growing and a lot has been learned about this theory. However, due to the fact that string theory pretends to be the theory of everything, it is reasonable that it will take some more time to be settled in a good shape. String theory is about 30 years old and we do not have a reasonably satisfying formulation of it, yet. Moreover, there remain still a lot of fundamental unsolved questions. Of paramount importance is the problem of the uniqueness of the vacuum state which is to formulate the theory in such a way that its low energy limit is so constrained by symmetry requirements to be uniquely determined. It should eventually reproduce a picture of the low energy world as we know it [2].

[1]There exist in the literature a lot of good review texts about string theory. See for example [1, 2].

[2]Notice that there is a big hole in our experimental knowledge of the low energy world: gravity has been tested – I heard this embarrassing datum by C. Bachas at Strings99 – only for length scales ≥ 1 mm. Therefore, by now, there could be possible deviations at microscopic scales from General Relativity. This is of course a stimulating situation.

Perturbative string theory is formulated in a peculiar mathematical language. Notwithstanding its unquestionable beauty, this language is in fact unsuitable to be extended to a nonperturbative formulation of string theory. It has been discovered during the past few years that the perturbative string spectrum has to be augmented by the inclusion of higher dimensional extended objects [3, 4], which have been called *branes*. The description of these objects has been implemented in string theory only at the semi-classical level, while it has not been found a way to include them as fully dynamical quantum degrees of freedom. This limitation of the language in which string theory is formulated is one of the best reasons not to believe it as a definitive theory in this formulation. In general, there can be several reasons for a problem to be stated in a too bounded language. One of these, which seems to apply to our case, is that there is a lack of full comprehension of what one is talking about and that an effort is needed in order to find a better defined logical basis to organize ideas.

During the last few years a new approach to string theory has been developed, together with a new attitude. This new approach arose to embody the dream of a non perturbative formulation of string theory which should reproduce the five well defined perturbative string theories in an unique scheme. This new formulation of string theory has been called M-theory [4, 5, 6], where however 'M', by now, still stands for Moon. Despite the great activity in this direction, the full structure of M-theory still remains elusive. Anyhow, it seems that this new framework can be thought of as a really higher point of view in string theory, although not definitive.

One of the major novelty in this new attitude is a renewed role which is assigned to supersymmetric Yang-Mills (YM) gauge theories. As it is well known, both General Relativity and Elementary Particle Standard Model are formulated as gauge theories. So it is pleasant to find this same class of theories at the heart of string theory. Supersymmetric YM theories entered the game as it has been understood [7] that their moduli space encodes a way to realize the target space embedding of branes. This turns out to be a consequence of the fact that the dynamics of the SQCD multiplets describes the low energy theory of such extended objects.

Crucial features of supersymmetric gauge theories are their duality properties[8]. This means that it may happen that different supersymmetric gauge theories in different regimes turns out to be equivalent under a suitable mapping of their moduli spaces, coupling constants and degrees of freedom. This kind of maps are known as *duality* maps. Thanks to supersymmetry, a full control on this strong characterization of gauge theories is made feasible by the existence of exact methods to estimate their quantum effective actions.

As the perturbative spectrum of each string theory has been augmented by its particular types of branes, this opens up the possibility to conjecture possible dualities between different string theories at different regimes of their couplings. The outcome of this reasoning is that different regimes of different perturbative string theories can be grouped in duality equivalence classes. As long as this program is expected to close successfully, the existence of a superior theory could be conjectured. The most welcome feature of M-theory should be to generate all known equivalence classes of perturbative string theories as different perturbative expansions about definite points of its moduli space. By now the word M-theory still refers to a collection of facts, links, conjectures, observations and personal tastes.

The major proposal for M-theory definition has been given in [9] where it has been conjectured to be a supersymmetric matrix quantum mechanical system of particles of infinite possible types whose underlying space-time geometry results, in a sense, quantized. Therefore, M-theory has been proposed to be a matrix theory. Let us notice that while from some points of view this formulation seems to be much simpler than the usual perturbative string theory, its logical basis and motivations are still unclear and under scrutiny. In this formulation the dimension of the matrices has to be infinite to generate extended objects (M-branes) in the spectrum of the theory. It has been conjectured that the resulting theory describes M-theory as an eleven dimensional theory in the infinite momentum frame. One evident bound of this proposed definition is that it is not covariant.

A possible way to confirm this proposal is to understand if it does really reproduce, in the appropriate corners of the moduli space, the known features of perturbative string theory itself. A potential answer to this question has been presented under the name of the *matrix string theory* (MST) hypothesis [10, 11, 12] and it is the subject of this thesis. We will try to convince the reader that MST has good chances to be pushed forward as part of a nonperturbative definition of string theory.

In the rest of this introductory chapter we review how to give a well defined statement to the MST hypothesis and outline the strategy we will follow to face it. The discussion about the results we have obtained is postponed to the last chapter.

A great part of the material presented in this thesis has been worked out in the collaborations [13, 14, 15, 16].

1.2 M-theory as a matrix theory

M-theory is the surmised nonperturbative theory that should encode all known string theories as different perturbative limits [4, 5]. Different vacua of different perturbative theories should be recognized as different perturbative expansions of M-theory around particular corners of its moduli space. Around some of these points it appears as an eleven dimensional theory. In particular, the compactification on a circle S^1 of M-theory is conjectured to be equivalent to ten dimensional type IIA string theory.

Type IIA string theory is a closed string theory with a ten dimensional target space and 32 supercharges. The low energy effective theory of its massless states is ten dimensional type IIA supergravity. The type IIA massless spectrum is made of a NS-NS sector and a R-R one. The bosonic part of the NS-NS sector is given by the metric $g_{\mu\nu}$, the dilaton scalar ϕ and the antisymmetric tensor $B_{\mu\nu}$, while the bosonic part of the R-R sector is given by the 1-form potential A_μ and the 3-form potential $A'_{\mu\nu\rho}$, where $\mu, \nu, \rho = 0, \ldots, 9$. The bosonic part of the type IIA supergravity action is given by

$$\begin{aligned} S^{IIA} = \frac{1}{2}\int d^{10}x\sqrt{g}\left[e^{-2\phi}\left(R + 4\left(\nabla\phi\right)^2 - \frac{1}{12}(dB)^2\right) + \right. \\ \left. -\frac{1}{2}(dA)^2 - \frac{1}{24}(dA')^2\right] - \frac{1}{4}\int dA' \wedge dA' \wedge B\,. \end{aligned} \tag{1.1}$$

From the low energy effective theory point of view [4] one can in fact reproduce the type IIA low energy field content as the massless spectrum of the S^1 compactification of eleven dimensional super-gravity. The bosonic part of the eleven dimensional supergravity multiplet is given by the metric G_{AB} and by a 3-form potential with components C_{ABC}, where $A, B, C = 1, \ldots, 10$. The number of physical degrees of freedom of the eleven dimensional super-graviton is given by $\frac{1}{2}(11-1)(11-2) - 1 = 44$ for the metric and $\frac{1}{3!}(11-2)(11-3)(11-4) = 84$ for the 3-form potential, which is a total number of 128 bosonic physical degrees of freedom. The gravitino field, by supersymmetry, gives the same number of fermionic degrees of freedom which implies that the full eleven dimensional super-graviton multiplet is composed by 256 states. The bosonic part of the eleven dimensional supergravity action is given by

$$S^{11SG} = \frac{1}{2}\int d^{11}x\sqrt{G}\left(R + (dC)^2\right) + \int dC \wedge dC \wedge C\,. \tag{1.2}$$

Under S^1 compactification the eleven dimensional supergravity fields generate a ten dimensional massless spectrum given by dimensional reduction as follows

$$G_{10\,10} \quad = e^{(4/3)\phi}$$

$$\begin{aligned} G_{10\mu} &= e^{(4/3)\phi} A_\mu \\ G_{\mu\nu} &= e^{(-2/3)\phi} g_{\mu\nu} \\ C_{10\,\mu\nu} &= B_{\mu\nu} \\ C_{\mu\nu\rho} &= A'_{\mu\nu\rho} \, . \end{aligned} \tag{1.3}$$

The spectrum of fields that we have just obtained coincides with the bosonic part of the spectrum of type IIA supergravity. Actually, the specific map between the field spectra of the theories is fixed by the request that the eleven dimensional supergravity action (1.2) exactly reproduces the ten dimensional type IIA one (1.1) upon dimensional reduction.

In type IIA superstring the vacuum expectation value of e^ϕ is interpreted as the string coupling constant g_s. Therefore, from the first line of (1.3), we get the relation between the eleven dimensional radius R_{11} measured in eleven dimensional Planck units and the string coupling constant

$$\frac{R_{11}}{l_p} = g_s^{2/3} \, , \tag{1.4}$$

where l_p is the eleven dimensional Planck length. This relation means that the decompactification regime $R_{11} \to \infty$ has to be related to the strong coupling regime of type IIA theory where it should appear as a genuine eleven dimensional theory. On the other side, the perturbative regime corresponds to the compactification regime in which the radius of the circle is very small.

In the latter case, all the Kaluza-Klein massive modes decouple. Indeed the ten dimensional metric is given by $g_{\mu\nu} = e^{(2/3)\Phi} G_{\mu\nu}$, while the KK mass spectrum is quantized in units $M_{KK} = \frac{1}{R_{11}}$, where now R_{11} is measured in ten dimensional units. Therefore, we get

$$\frac{R_{11}}{l_s} = \frac{l_p}{l_s} \cdot \frac{R_{11}}{l_p} g_s^{1/3} \cdot g_s^{2/3} = g_s \, , \tag{1.5}$$

where $l_s = \sqrt{\alpha'}$ is the string length, α' being related to the string tension $T = (2\pi\alpha')^{-1}$. M_{KK} is thus quantized in units of $\frac{1}{g_s l_s}$ which means that KK states are infinitely massive in the weak coupling limit and hence decouple.

On the contrary, in the strong coupling regime KK modes are massless and will be interpreted as non perturbative states of type IIA superstring theory. KK states appear as particles minimally coupled to the $G_{10\mu} \sim A_\mu$ field with charge equal to their mass. The KK multiplet is BPS saturated (ultrashort), being its dimension 2^8 equal to that of the graviton.

As we already pointed out, the perturbative spectra of superstring theories have to be augmented by the inclusion of nonperturbative states related to extended objects.

These include *Dp-branes* [3], which are defined in the perturbative theory language as $p+1$ dimensional space-time defects where strings can end. D-branes are the carriers of RR charges and are 1/2 BPS states, that is, being their mass saturated to their charge, they belong to short representations of the supersymmetry algebra. This property implies that D-brane states are protected from quantum corrections and hence are stable with respect to the coupling constant flowing from one regime to another. In particular, type IIA string theory requires D-branes with an even number of space dimensions. These are in correspondence to the RR potentials they minimally couple to. In particular, D0-brane states carry the RR charge relative to the 1-form potential A_μ quantized in units of $\frac{1}{g_s l_s}$. This is obtained by the appropriate disk diagram.

From the supersymmetry algebra point of view, D-branes appear as central charge tensors which are their total RR charges [3]. For example, the ten dimensional supersymmetry algebra relevant to type IIA is

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^\mu)_{\alpha\beta}P_\mu + (C\Gamma^{11})_{\alpha\beta}Z^{(0)} + \ldots \tag{1.6}$$

where Q_α are the ten dimensional supersymmetry charges, C is the charge matrix, Γ^μ are the ten dimensional Γ-matrices, P_μ is the ten dimensional super-graviton momentum, Γ^{11} is the ten dimensional chirality matrix and $Z^{(0)}$ the D0-brane charge[4].

Let us now go back to the problem of finding the interpretation of KK states in the spectrum of type IIA string theory. The eleven dimension super-algebra is similar to (1.6) and reads as follows

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^A)_{\alpha\beta}P_A + \ldots \tag{1.7}$$

where P_A is the eleven dimensional super-graviton momentum. As we have already shown, upon compactification on S^1 the eleven dimensional super-graviton gives the ten dimensional super-graviton plus an infinite tower of KK massive states. It means that the eleven dimensional momenta get split as

$$(P_A) \to \left(\frac{n}{g_s l_s}, P_\mu\right), \tag{1.8}$$

where n is the integer labelling the spectrum of KK masses. Substituting (1.8) in (1.7) we obtain the super-algebra for the compactified theory. The result corresponds

[3]These numbers are infinite if the D-branes are extended along non-compact dimensions and are identified with D-branes charge density × volume. Charge densities enter the supercurrents OPE algebra.

[4]Dots indicate other charges which are unnecessary in this discussion.

to (1.6) if D0-brane charge $Z^{(0)}$ is identified with KK mass. Notice that this makes sense due to the fact that KK masses and D0-charges are quantized in the same units. This indicates that D0-branes are the nonperturbative states in type IIA theory we were looking for. This is possible only if the representations they belong to are the same, that is if they have the same dimension and if D0-brane bound states exist. As we already stated, KK multiplets belong to the ultra short 2^8 representation of the ten dimensional theory. As far as D0-branes are concerned, the question is a bit longer.

The effective theory of open strings is given by ten dimensional supersymmetric $\mathcal{N} = 1$ Yang - Mills whose gauge group is dictated by the Chan - Paton factors attached to their ends. On the other hand, the theory of strings ending on the branes is the theory of the excitations of the brane itself. Since these are open strings, their low energy effective ten dimensional theory is the corresponding supersymmetric gauge theory. In [7] the low energy effective field theory of the strings describing N parallel D-p branes in type II theories has been shown to be the maximally supersymmetric $U(N)$ Yang - Mills theory in $p+1$ dimensions. It can be obtained by dimensional reduction of the 10-dimensional $\mathcal{N} = 1$ supersymmetric $U(N)$ YM theory down to $p+1$ dimensions. The ten dimensional theory is defined by the action functional

$$S^{SYM} = \frac{1}{g_{10}^2} \int d^{10}x \mathrm{Tr} \left(-\frac{1}{4}\mathrm{F}^2 + \frac{\mathrm{i}}{2}\Psi^{\mathrm{T}}\Gamma^0\Gamma^\mu \mathrm{D}_\mu \Psi \right) , \tag{1.9}$$

where F is the gauge curvature and Ψ is a sixteen components Majorana-Weyl spinor of $SO(1,9)$ in the adjoint of the gauge group. The action is invariant under the supersymmetry transformations

$$\delta_\epsilon A_\mu = \frac{i}{2}\bar{\epsilon}\Gamma_\mu \Psi \quad \text{and} \quad \delta_\epsilon \Psi = -\frac{1}{4}\Gamma^{\mu\nu}F_{\mu\nu}\epsilon + \epsilon' \cdot \mathbf{1}_N ,$$

where ϵ and ϵ' are constant Majorana-Weyl spinors. By dimensional reduction from 10 to $p+1$ dimensions one breaks the full Lorentz group as $SO(1,9) \to SO(1,p) \times SO(9-p)$, which is the reduced Lorentz group of the p-brane world-volume times the R-symmetry isotopic group which is interpreted as the transverse rotation group. The gauge connection decomposes along this path in the reduced $p+1$ dimensional gauge field plus $9-p$ adjoint scalars which describe the transverse oscillations of the brane system, while the spinor Ψ decomposes under the relative reduction of the spinorial representation it belongs to.

In particular, the low energy effective theory action of D-particles in type IIA is obtained as the dimensional reduction, along the 9 space directions, from 10 to 1 dimension of S^{SYM} as is given in (1.9). This is a supersymmetric quantum mechanical system of particles with a Galilean-like nine dimensional invariance. Its one particle spectrum at null transverse momentum has the same quantum numbers of the 2^8 states of the KK multiplet. As D0-branes bound states at threshold are concerned, let us notice that one should be able to solve the relevant wave-function equation, but this is a difficult and open problem. The outcome should confirm the existence of bound stated at threshold for any value of the D0-brane charge to be associated to the full KK tower of states. This concludes our arguments in favor of the identification of KK states with D0-branes.

What we said until now is a first indication that the strong coupling regime of type IIA string theory can be described as an eleven dimensional theory. Let us notice that, as the supergravity multiplet contains a 3-form potential, there should exist in the eleven dimensional theory some membrane states which minimally couple to it. The tension of the membrane is completely determined by the eleven dimensional Planck scale to be $1/(2\pi l_p^3)$. Upon S^1 compactification these membranes can wrap around the circle by giving in the ten dimensional theory string states with tension $R_{11}/(2\pi l_p^3) = 1/(2\pi l_s^2)$ which is[5] the tension of the fundamental string. The membrane tension by itself is the tension of the D2-brane which arises as the duality map image in string theory of unwrapped membranes[6]. This fact seems to suggest that M-theory may be formulated as a theory whose perturbative degrees of freedom are membranes. Despite some remarkable attempts [17] to describe a fully satisfactory 11-dimensional quantized membrane theory, it does not still seem to exist a formulation of it which does the right job.

The BFSS proposal [9] gives a formulation[7] of M-theory as the supersymmetric quantum mechanical system of N D-particles in the large N limit. This takes its origin from the correspondence that we just have explained between KK particle states and D0-branes. Since these are massless states in the eleven dimensional theory, one may wander if they can be considered as fundamental degrees of freedom of M-theory in this regime. As we saw before, the resulting supersymmetric quantum mechanical system describes particles within a nine dimensional Euclidean space whose rotation group gets identified with the R-symmetry group $SO(9)$. In order to interpret this theory as an eleven dimensional one, one has to explain where the

[5]We used (1.4) and (1.5).

[6]Along the same line of reasoning, one can complete the correspondence between the full brane contents of the two theories. We are not going to review all its points. We just notice that the eleven dimensional theory spectrum contains by consistence also 5-branes, which are the magnetic duals of membranes. Upon compactification they give rise to D4-branes and NS5-branes.

[7]See also [19, 20, 21] for some related material as well as [22] for some of the basic ideas.

other two dimensions have disappeared. This is done by imagining that the quantum mechanical system describes the theory in the infinite momentum frame. This means that the system has been boosted in a definite direction with a momentum amount much greater than any energy scale of the theory. The choice of this direction and the boost break $SO(1,10) \to SO(1,1) \times SO(9)$ for any kind of particle system and thus leaves the system as effectively Galilean. In formulas, suppose that a particle system is given in eleven dimensions and let us denote by p_a the individual momenta. Let $P = \sum_a p_a$ be the total momentum of the system and let us decompose each individual momentum as

$$p_a = w_a P + p_a^t \,, \tag{1.10}$$

where w_a is the fraction of the total momentum carried by the a-th particle and $P \cdot p_a^t = 0$. For generic configurations, one can boost the system in the P direction until $w_a > 0$. Further boosting, doesn't change the picture. The energy of each particle is given by

$$E_a = \sqrt{p_a^2 + m_a^2} \sim w_a P + \frac{(p_a^t)^2 + m_a^2}{2 w_a P} + O\left(\frac{1}{P^2}\right)$$

which has the structure of a Galilean theory in nine dimensions. Once the theory is described in the infinite momentum frame, one interprets the resulting picture as a partonic one.

Therefore, one is led to conjecture that the supersymmetric quantum mechanical system represents a partonic picture of M-theory in the infinite momentum frame. This has been tested by calculating particle scattering amplitudes, identifying multi-particles asymptotic states as block diagonal matrices, each block representing a particle. The results agree with graviton scattering amplitudes within eleven dimensional supergravity [9, 23]. As we already noticed, M-theory should contain in its spectrum also membrane states, as well as their magnetic duals 5-brane states. These are realized in matrix theory as large N effects. In the large N limit the $U(\infty)$ symmetry group realizes the reparametrization symmetry group of the membrane at fixed time [9, 22, 18]. The central charges of M-branes are given by traces of commutators of infinite dimensional matrices which are not forced to vanish in this limit.

The conjectured identification [9] is in fact between

- M-theory compactified on a circle S^1 of radius R_{11} in the infinite momentum frame
- the $\mathcal{N} = 16$ matrix quantum mechanics with symmetry group $U(N)$ and coupling constant R_{11}, in the large N limit.

The large N limit [24] has the effect of reincluding into the game the 11^{th} direction as effectively uncompactified. Notice also that l_p arises as the typical length scale of the bosonic matrix fields and hence have a dynamical origin.

The most evident limit of the matrix theory definition of M-theory is that it is not eleven dimension Lorentz covariant and that a lot of features are transferred to the large N limit which is still not well understood.

At this point, a natural question arises. If M-theory is a matrix theory, where is string theory? As we already remarked, M-theory compactified on a circle should reproduce type IIA string theory in the compactification regime, while the matrix description seems to be peculiar of the decompactification regime. Therefore, to mix these aspects of M-theory while retaining the full parton spectrum, one is led to further compactify the theory, say in the 9^{th} direction, along another circle S^1 of radius R_9. This subsequent compactification has to be understood carefully. In fact, the natural radii hierarchy is $R_9 >> R_{11}$, while one is looking for a situation in which R_{11} is not considered as a small scale. Therefore, once the theory is further compactified, one needs to flip the roles of the two circles to retain M-theoretic parton spectrum properties. This will be shown to be equivalent to a chain of duality maps.

Before discussing matrix theory compactification, let us first pave the way on the string theory side. Apart from type IIA, there exists another ten dimensional closed superstring theory with 32 supercharges which is called type IIB and whose low energy effective theory is ten dimensional type IIB supergravity. The massless spectrum of this theory again consists of a NS-NS and a R-R sectors. The bosonic part of the NS-NS sector is equal to the type IIA one, being given by the metric $g_{\mu\nu}$, the dilaton scalar ϕ and the antisymmetric tensor $B_{\mu\nu}$. On the contrary, the bosonic part of the R-R sector is given by the scalar a, the 2-form potential $B'_{\mu\nu}$ and the 4-form potential $A''_{\mu\nu\rho\sigma}$ with self-dual curvature $F'' = dA''$, where $\mu, \nu, \rho, \sigma = 0, \ldots, 9$. Due to the nature of the R-R fields, the perturbative spectrum of type IIB theory has to be augmented by means of D-branes with odd space dimensions.

Due to the self-duality of the F'' curvature, there is no action that gives the full low energy field equations, but for solutions with $F'' = 0$. The remaining field equations can be obtained by the action

$$S^{IIB} = \int d^{10}x\sqrt{\bar{g}}\left(\bar{R} + 4\mathrm{Tr}(\partial_\mu \mathrm{S}\partial^\mu \mathrm{S}^{-1}) - \frac{1}{12}\mathrm{Tr}(\mathrm{H}^{\mathrm{t}}_{\mu\nu\rho}\mathrm{S}\mathrm{H}^{\mu\nu\rho})\right) , \tag{1.11}$$

where $\bar{g}_{\mu\nu} = e^{\phi/2}g_{\mu\nu}$ is the metric in the Einstein frame,

$$S = e^{\phi}\begin{pmatrix} |a|^2 + e^{-2\phi} & a \\ a & 1 \end{pmatrix} , \quad \text{and} \quad H = \begin{pmatrix} dB \\ dB' \end{pmatrix} .$$

The supergravity type IIB field equations are invariant under the action of the $SL(2,\mathbb{R})$ group which leaves F'' and $\bar{g}_{\mu\nu}$ invariant and acts on the other fields by

$$S \to \Lambda S \Lambda^t \quad \text{and} \quad H \to \Lambda^{-1} H\,, \quad \text{with} \quad \Lambda \in SL(2,\mathbb{R})\,.$$

In the full string theory, charge quantization leaves only a discrete $SL(2,\mathbb{Z})$ group out of the above continuous invariance. This is called the $SL(2,\mathbb{Z})$ self-duality group of type IIB. The element corresponding to $\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and is called S (self-)duality. Its action interchanges B and B', which is that interchanges fundamental string states with D1-brane states. Moreover it transforms (for the case $a = 0$) the string coupling into its inverse, which is that interchanges the weak and the strong coupling regimes of the theory. In particular, the weakly coupled theory of strings, which is perturbative type IIB, is then equivalent via S duality to the strong coupling limit theory where D1-branes arise as fundamental perturbative degrees of freedom.

Let us consider now type IIA and type IIB theories each one compactified on a circle of radius R_A and R_B respectively. By comparing their field content upon dimensional reduction to nine dimensions, one finds them to be equal. The two theories are really equal if the two compactification radii are related by the relation $R_A R_B = 2\pi l_s^2$. The correspondence between the theories exchanges the relative R-R sectors as a consequence of the chirality reversal which acts on the string oscillation fermionic modes. This also changes the boundary conditions of the bosonic fields in such a way that even branes are exchanged with odd ones and viceversa. This correspondence between type IIA and type IIB is implemented as a discrete transformation called T-duality which fulfills the property $T = T^{-1}$.

Let us now go back to the compactification of matrix theory. The compactification procedure [25] consists of a cyclic matrix identification which represents the quotient $S^1 = \mathbb{R}/\mathbb{Z}$ in the matrix coordinates space. After Fourier transform, the neat effect is of obtaining supersymmetric Yang-Mills theory with $\mathcal{N} = (8,8)$ and $U(N)$ gauge group on the cylinder $\mathbf{R} \times S^{1'}$, where $S^{1'}$ is the T-dual of the circle in the 9^{th} direction.

This resulting theory can be obtained as the dimensional reduction from 10 to 2 dimensions of the ten dimensional $\mathcal{N} = 1$ SYM (1.9). Therefore, it is a theory of weakly coupled D1-branes wrapped around the dual circle $S^{1'}$ and has to be identified with type IIB in the strong coupling regime. As we just saw the latter is S dual to weakly coupled type IIB and under S duality D1-brane states are exchanged with fundamental string states. Therefore, the theory is the same as type IIB in the perturbative regime. This still results to be compactified on the circle $S^{1'}$. By T duality it is then equivalent to weakly coupled type IIA compactified on S^1. See

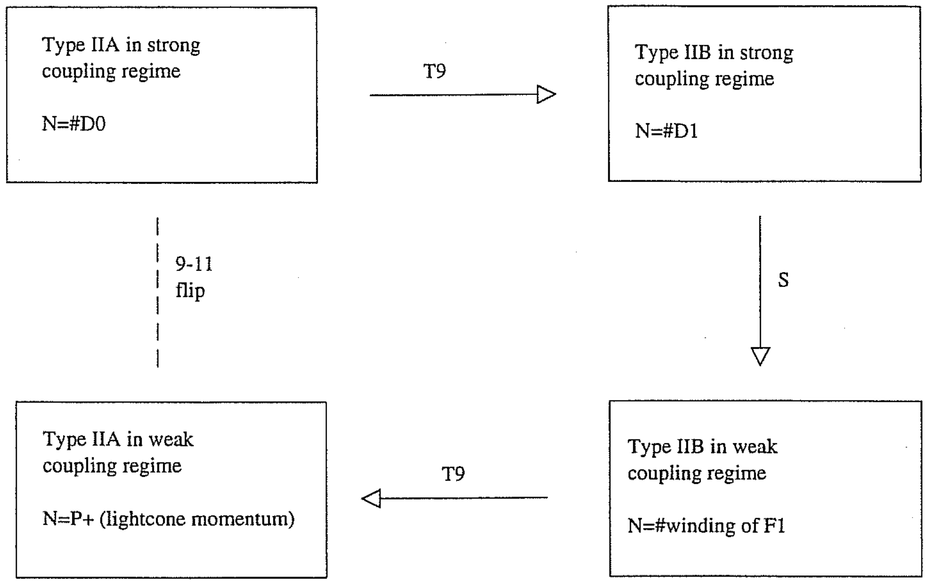

Fig. 1: *The duality chain and the interpretation of N at the various stages. The full duality chain realizes the 9-11 flip.*

Fig. 1. for a summarizing scheme. Therefore, the S^1 compactification of matrix theory is $T \cdot S \cdot T$ dual to type IIA string theory in the perturbative regime (See [26] for further details). Following [7], the gauge coupling, being the mass scale for the transverse bosonic excitations, gets interpreted as the tension parameter of the strings stretched between the D1-branes which is proportional to the inverse of the perturbative string coupling. In fact, under the $T \cdot S$ duality chain the gauge coupling gets identified with the inverse type IIA string coupling (in $l_s = 1$ units). We will find duality independent evidence of this fact.

The outcome of what we said is that supersymmetric Yang-Mills theory with $\mathcal{N} = (8,8)$ and $U(N)$ gauge group on the cylinder $\mathbf{R} \times S^1$ in the strong coupling regime should represent type II A string theory in its perturbative coupling regime.

1.3 A first bit of MST

Let us give an introduction to the string interpretation of the strongly coupled theory. In this section we show that the classical moduli space of MST is a space of free strings in $\mathbb{R}^8$.

The action of the supersymmetric Yang-Mills theory with $\mathcal{N} = (8,8)$ and $U(N)$

gauge group on the cylinder $\mathbf{R} \times S^1$ is [8]

$$S = \frac{1}{\pi} \int_C d^2w \, \mathrm{Tr} \left(\mathrm{D_w X^i D_{\bar{w}} X^i} - \frac{1}{4\mathrm{g}^2} \mathrm{F}^2_{\mathrm{w\bar{w}}} - \frac{\mathrm{g}^2}{2} [\mathrm{X^i}, \mathrm{X^j}]^2 + \quad [\text{fermions}] \quad \right) .$$

In the $g >> 1$ regime all the fields are projected into the Cartan directions

$$\left[A_w, X^i\right] = 0 \quad \text{and} \quad \left[X^i, X^j\right] = 0$$

by the potential and the covariant derivative terms. The neat result is that the surviving Cartan gauge sector decouples from the rest of the projected non-Abelian multiplet and the theory remains the theory of the N Cartan components of the X^i's

$$S_{g \sim \infty} = \frac{1}{\pi} \int_C d^2w \, \mathrm{Tr} \; \partial_{\mathrm{w}} \mathrm{X^i} \partial_{\bar{\mathrm{w}}} \mathrm{X^i} + \quad [\text{fermions}] \quad + \quad [\text{decoupled gauge sector}]$$

where $X^i \sim \hat{x}^i = \mathrm{diag}\left\{x^i_{(1)}, \ldots, x^i_{(N)}\right\}$ so that

$$S_{g \sim \infty} = \frac{1}{\pi} \sum_{\alpha=1}^{N} \int_C d^2w \, \partial_w x^i_{(\alpha)} \partial_{\bar{w}} x^i_{(\alpha)} + \quad [\text{others}] \, .$$

The true X^i fields are given by $X^i = U \hat{x}^i U^\dagger$ where U is a unitary matrix and these fields are well defined on the cylinder, i.e. they remain unchanged under a 2π shift of the σ variable.

The entries of the diagonal $\hat{x}^i$ are instead in some given twisted sector (the Weyl group of $U(N)$ is the permutation group of N objects S_N). This means that the X^i eigenvalues can get exchanged around the cylinder by a permutation

$$x^i_{(\alpha)} \quad \to \quad x^i_{(\alpha')} \quad \alpha' = g(\alpha) \, , \; g \in S_N \, .$$

This is the realization of the residual gauge invariance under the Weyl group. By decomposing g in irreducible classes $[g] = (1)^{n_1} \cdot (2)^{n_2} \cdot \ldots \cdot (N)^{n_N}$ with $\sum_{i=1}^N i \cdot n_i = N$ and (i) being the irreducible $\mathbb{Z}_i$ block, we can reinterpret the above fields as periodic on the union of some multiple cylinders. As a consequence the theory above is a theory of periodic bosons on these big cylinders identified by the tying of the various components in each simple cycle (See *Fig.* 2.). We call this phenomenon world-sheet generation. We associate with each twisted sector a collection of big cylinders which are naturally interpreted as free string world-sheets.

Notice that from the naive point of view we are considering here, we are able to construct the on shell free string propagation but we are unable to recognize directly string interactions.

[8]The fermions will be introduced later on when we will analyze the theory in deeper detail.

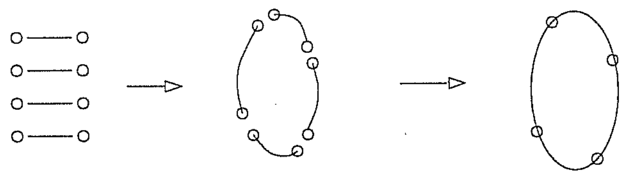

Fig. 2: *World-sheet generation. The ends of the images of the fixed time circle get tied together making up a long string.*

To be more general and clear, let us consider the limit

$$\begin{pmatrix} U(N) \\ SYM \\ \mathcal{N}=(8,8) \end{pmatrix} \quad \stackrel{g\to\infty}{\longrightarrow} \quad \begin{pmatrix} (\mathbf{R}^8)^N/S_N \\ CFT \\ \mathcal{N}=(8,8) \end{pmatrix}$$

The IR CFT has to be the free theory twisted by the $U(N)$-Weyl group: in fact there is no interacting realization of the $D = 2$, $(8,8)$-superconformal algebra [27]. The only freedom we are left with is then the choice of the two-manifold the theory in the IR is well defined on.

The asymptotic states of our theory are naturally labeled by S_N conjugation classes $[g] = (1)^{n_1} \cdot (2)^{n_2} \cdot \ldots \cdot (N)^{n_N}$ where $N = \sum_{a=1}^{N} a n_a$ and string configurations are associated to these twisted sectors: $[g]$ represents $\sum_a n_a$ strings each of length a.

In the CFT Hilbert space these sectors are orthogonal, therefore, in order to let strings interact, one should exit the conformal point by adding to the action a relevant vertex. It can be proved [12] that there exists only one compatible vertex. More precisely, it can be constructed in a unique way by requiring it to be made of spin fields, to be compatible with the $\mathcal{N} = (8,8)$ supersymmetry of the theory and to be of minimal conformal dimension. The structure of this vertex encodes in a specific way the coupling between different twisted sectors by describing an elementary cycle change.

This vertex results to be essentially the Mandelstam [30] string vertex counterpart and properly generates the superstring expansion in the light-cone by explicitly constructing the elementary joining/splitting interaction. This approach is, however, not fully satisfactory since it doesn't trace the string interaction directly from the interacting SYM theory and doesn't give a prescription to fully link the strong coupling limit of the gauge theory with perturbative superstring theory.

The problem we are facing with is then to find superstring interaction from the very structure of the SYM theory.

1.4 Superstrings from super Yang – Mills

The attitude we are going to keep in the following is to try to recognize from the full SYM partition function the light-cone superstring perturbative one. More precisely, what one would like to realize is that the asymptotic behavior of SYM at strong coupling matches the perturbative light-cone superstring theory. We will obtain precisely this.

As already noted there are a lot of constraints coming from supersymmetry which prevents getting a simple limit procedure unless we do find a way to sort out string interactions so that the theory seems to be free in the infrared, while the interaction is introduced directly by some fundamental mechanism which generates the interacting string world-sheets.

As we will show, what happens is in fact that the interacting string world-sheets are naturally encoded in BPS classical fields trajectories interpolating between different asymptotic free string configurations. These classical configurations preserve $1/2$ of the supersymmetry of the theory, which is a $(4,4)$ sub-sector, and we will call such a configuration *stringy instanton*. In this way string diagrams are shown to be at the heart of the gauge theory and, being stringy instantons dominant in the strong coupling, they turn out to be the fundamental degrees of freedom of the theory in the strong coupling phase.

This mechanism looks like as a resumming of the Mandelstam expansion of the light-cone superstring theory. In the Mandelstam [30] approach string interaction is undone by cutting the interacting string world-sheet into cylinders which are then glued along some cuts (see *Fig.* 3.). At the beginning of each cut strings join and at the end they split. The interaction is then kept into account via a vertex insertion which assigns the right weight in the coupling constant to each intermediate process and performs explicitly the cylinders gluing at the interaction points. What we will obtain is an intermediate result where the effect of the vertex insertion is already kept into account by the natural emergence of the right factor in the string coupling constant. The representation of string theories in the light-cone gauge does need anyway Mandelstam diagrams to picture string interaction. In fact, it is not possible to chose flat coordinates all along the full world-sheet and one needs to concentrate curvature somewhere on it in order to keep mathematical consistency. This is done by choosing an almost flat metric representative whose curvature is delta-

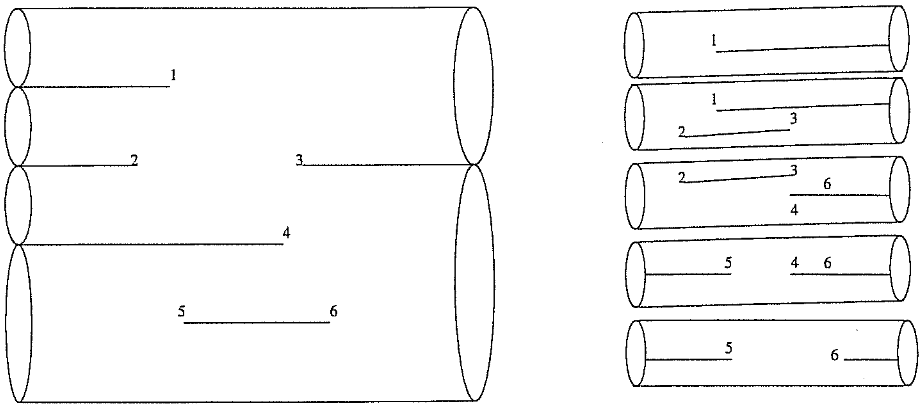

Fig. 3: *An example of branched cover structure.*

like concentrated on one point for each simple string splitting/joining. The choice of this point is arbitrary and, being a modulus, it is integrated over leaving the theory Lorentz invariant. Let us notice that, apart from the obvious breaking of the Lorentz group $SO(1,9) \to SO(1,1) \times SO(8)$ which is due to the light-cone gauge choice [30, 31], each single interacting process further breaks $SO(8) \to SO(2) \times SO(6)$. This breaking doesn't in fact occur in the theory and is just a decomposition of the $SO(8)$ representation in the physical interacting string state space needed to represent the interaction. There is an explicitly analogous mechanism in MST.

What we have realized so far is that SYM on a D=2 cylinder with gauge group $U(N)$ and $\mathcal{N} = (8,8)$ reduces naively in the strong YM-coupling limit to the free Green-Schwarz superstring theory in the light-cone.

As we already anticipated, we will find some $(4,4)$ preserving classical field configurations which encode Mandelstam diagrams in a natural way. A crucial point is the appearance into the game of the $U(N)$ Hitchin [44, 45] system [9] on the cylinder. It is an integrable system whose solutions are parametrized by certain spectral curves. These spectral curves are interpreted as the intrinsic definition of the interacting string worldsheet and are represented as branched covers of the cylinder we started from. The solution space of the above Hitchin system, which will be integrated over in the calculation of path integrals, can be analyzed in the large N perspective and it turns out to be a discretization of the moduli space of Riemann surface which the more accurate the bigger N is.

Then we will perform the strong YM-coupling expansion of the SYM partition

[9]The Hitchin system already entered the string theory game in [28].

function around these stringy instantons. This is done by splitting the fields in their stringy instanton parts and the fluctuations around them. The expansion has a predilection for the Cartan fluctuations whose fate is to embody the interacting string embedding degrees of freedom, while leaving to the non Cartan fluctuations a passive role to play in the evaluation of the partition function.

What we will realize at the end is a strong agreement between the strong coupling limit of $U(N)$ SYM partition function on the cylinder and the light-cone Green-Schwarz superstring partition function. The various points in this correspondence will be explained in full detail in the last chapter and will not be explained here further. Also the question of the MST way to generate physical string amplitudes will be addressed and mostly solved.

All this anyhow doesn't seem to indicate that MST has to be considered as a good operational scheme for perturbative superstring theory. In fact, the forced discretization of the light-cone momentum puts strong restrictions on the external data at low N. Moreover, also the discretization of the moduli space means that, apart the tree level which doesn't undergo any discretization, already for genus one amplitudes one encounters a sum over a discrete set of inequivalent tori which is only a subset of the full genus one moduli space. Actually, we were not looking for a reliable computational approximation for string theory, but for an equivalence between string theory and SYM in order to show that matrix theory in the appropriate corners of its moduli space correctly reproduces the perturbative expansion of string theory. If this formulation might have some useful computational side in string theory, it seems to be in its high energy limits where string fragmentation occurs [33] and the genus expansion badly diverges. In such a situation *finite* N MST proposes a natural cut off to this divergent behavior as well as nonperturbative contributions which may eventually cure this divergence. This is in the spirit of the string-bit models[34], although in such a more advanced framework. Let us recall that, as long as N is finite, we obtain a discretized version of type IIA superstring theory. This outcome is in strong agreement with [50] where a similar conjecture was sketched.

MST's hypothesis is formulated putting our trust in the existence of a deep connection between string and gauge theories. It uses the language of gauge theory to speak about strings in a new and constructive framework. An outcome of this is then a new possible point of view on some sectors of gauge theories in the strongly coupled phase.

Chapter 2

Stringy Instantons

2.1 Introduction

In this chapter we develop the theory of *stringy instantons.* These are $(4,4)$ supersymmetry preserving classical solutions of Matrix String Theory and result to be the solutions of the Hitchin equations for gauge group $U(N)$ on the cylinder $S^1 \times \mathbf{R}$ satisfying certain boundary conditions. The latter implement the structure of the *in* and *out* string state while at intermediate times the instanton solution describes the tunneling between the two string configurations. Each instanton solution is in fact completely fixed by an interpolating Riemann surface, which is a branched covering of the base cylinder. This structure turns out to be equal to that of a Mandelstam light-cone string diagram. We begin the study of the moduli space of stringy instantons whose full structure will be given in Chapter 4.

2.2 (4,4) preserving instanton equation

The MST action is

$$S = \frac{1}{\pi}\int_C d^2w \,\mathrm{Tr}\Big(D_w X^i D_{\bar{w}} X^i - \frac{1}{4g^2}F^2_{w\bar{w}} - \frac{g^2}{2}[X^i, X^j]^2 +$$

$$+ i(\theta_s D_{\bar{w}}\theta_s + \theta_c D_w \theta_c) + 2ig\theta_s\gamma_i[X^i, \theta_c]\Big) \tag{2.1}$$

where $(X^i, \theta_s, \theta_c)$ are in the adjoint representation of the $U(N)$ gauge group and in the $(8_v, 8_s, 8_c)$ of the $SO(8)$ R-symmetry group. The gauge connection is an R-singlet, the covariant derivatives are $D_w = \partial_w + i\mathrm{ad}_{A_w}$ and $D_{\bar{w}} = \partial_{\bar{w}} + i\mathrm{ad}_{A_{\bar{w}}}$ and the gauge curvature arises as $[D_w, D_{\bar{w}}] = i\text{ad}_{F_{w\bar{w}}}$. The cylinder $\mathcal{C}$ coordinates are $w = \tau + i\sigma$ where $(\sigma, \tau) \in S^1 \times \mathbf{R}$ and in our units the radius of the S^1 is set to 1. The γ^i are the 8×8 γ-matrices used in [2].

The theory is $\mathcal{N} = (8,8)$-supersymmetry invariant. The relative fields' variations are the following ones

$$\delta X^i = \frac{i}{g}(\epsilon_s \gamma^i \theta_c + \epsilon_c \tilde{\gamma}^i \theta_s)$$

$$\delta\theta_s = (-\frac{i}{2g^2} F_{w\bar{w}} + \frac{1}{2}[X^i, X^j]\gamma_{ij})\epsilon_s - \frac{1}{g} D_w X^i \gamma_i \epsilon_c$$

$$\delta\theta_c = (\frac{i}{2g^2} F_{w\bar{w}} + \frac{1}{2}[X^i, X^j]\tilde{\gamma}_{ij})\epsilon_c - \frac{1}{g} D_{\bar{w}} X^i \tilde{\gamma}_i \epsilon_s$$

$$\delta A_w = -2\epsilon_s \theta_s, \qquad \delta A_{\bar{w}} = -2\epsilon_c \theta_c$$

where ϵ_c and ϵ_s are 8_c and 8_s constant spinors.

The easier way to show the above point is to notice that $(8,8)$ SYM $D = 2$ can be obtained from $\mathcal{N} = 1$ SYM $D = 10$ by dimensional reduction of 8 space dimensions as we explained in the introduction. From the ten-dimensional theory one easily recovers both the action functional and the supersymmetry transformations.

We look now for classical supersymmetric configurations that preserve $(4,4)$ supersymmetry. These are classical solutions sets on which the $(8,8)$ supersymmetry acts in a reduced way. To characterize them, it is enought to single out a point in the orbit of the supersymmetry action.

The action (2.1) can be written schematically as

$$S = \int_{Cyl} d^2w \left[F^2(b) - f\partial_b F(b) f\right]$$

where b are the bosonic fields, f are the fermionic fields and $F(b)$ is a quadratic local functional of the bosons only. The supersymmetry variations are $\delta b = f\epsilon$ and $\delta f = F(b)\epsilon$ and the action is supersymmetry invariant due to specific spinorial index identities (and the triality property of $SO(8)$). The strategy now is the following: we set $f = 0$, which singles out the point to be susy stationary in the whole b directions, and require $\delta f = 0$, which is $F(b)\epsilon = 0$, along the reduced susy directions. Let $\P$ be the projector along these directions, i.e. we choose $\P\epsilon = 0$. Then the relevant equation is $F(b)(1 - \P) = 0$. Decompose now the bosonic fields as $b = b_1 \oplus b_2$ and fix to a constant $b_2 = \bar{b}_2$ in such a way that $F(b_1 \oplus \bar{b}_2)\P = 0$ identically. Calculating the reduced action at $f = 0$ and $b_2 = \bar{b}_2$ one gets $S_{red} = \int_{Cyl} d^2w \left\{[F(b_1 \oplus \bar{b}_2)(1 - \P)]^2\right\}$ (up to boundary terms due to the projecting procedure) which is minimal at the instanton equation $F(b_1 \oplus \bar{b}_2) = 0$.

Applying to the case at hand, we set $\theta = 0$ and look for solutions of the equations $\delta\theta_{(s,c)} = 0$ along a $(4,4)$ space, i.e.

$$(\frac{i}{2g^2}F_{w\bar{w}} + \frac{1}{2}[X^i, X^j]\tilde{\gamma}_{ij})\epsilon_c = 0, \quad D_w X^i \gamma_i \epsilon_c = 0 \tag{2.2}$$

$$(-\frac{i}{2g^2}F_{w\bar{w}} + \frac{1}{2}[X^i, X^j]\gamma_{ij})\epsilon_s = 0, \quad D_{\bar{w}} X^i \tilde{\gamma}_i \epsilon_s = 0\,. \tag{2.3}$$

Solutions of these equations that preserve one half supersymmetry are the following ones: set $X^i = x^i \mathbf{1}_N$ for $i = 3, \ldots, 8$ with x^i real constants and introduce the complex notation $X = X^1 + iX^2$, $\bar{X} = X^1 - iX^2 = X^\dagger$ for the remaining two X's.

To extract the equations, let us remark that γ_{12} is an antisymmetric 8×8 matrix and fulfills $\gamma_{12}^2 = -1$ and therefore its eigenvalues are $\pm i$ (moreover $\tilde{\gamma}_{12} = \gamma_{12}$). It is easy to show that there exists ϵ_s and ϵ_c, each with four independent components, such that [1]

$$\gamma_{12}\epsilon_s = i\epsilon_s, \quad \tilde{\gamma}_{12}\epsilon_c = i\epsilon_c, \quad \gamma_1\epsilon_s = -i\gamma_2\epsilon_s, \quad \tilde{\gamma}_1\epsilon_c = i\tilde{\gamma}_2\epsilon_c\,. \tag{2.4}$$

Then the conditions to be satisfied in order to preserve one half supersymmetry are

$$F_{w\bar{w}} + ig^2[X, \bar{X}] = 0 \tag{2.5}$$

$$D_w X = 0, \quad D_{\bar{w}} \bar{X} = 0\,. \tag{2.6}$$

It is easy to verify that, if solutions to such equations exist, they satisfy the equations of motion of the action (4.14).

These equations can be derived from the matrix string theory action with $\theta = 0, X^i = x^i \mathbf{1}_N$ for $i = 3, \ldots 8$ which is,

$$S_{red} = \frac{1}{2\pi}\int d^2w \,\mathrm{Tr}\left(D_w X D_{\bar{w}}\bar{X} + D_w\bar{X}D_{\bar{w}}X - \frac{1}{2g^2}F^2_{w\bar{w}} + \frac{g^2}{2}[X,\bar{X}]^2\right). \tag{2.7}$$

It is elementary to prove that S_{red} is minimized in correspondence to solutions of (2.5, 2.6). The value of the action on such a configuration is

$$S^{(4,4)}_{cl} = \frac{1}{2\pi}\int d^2w \,\mathrm{Tr}\left(D_w\bar{X}D_{\bar{w}}X\right). \tag{2.8}$$

[1]There exists also the possibility of choosing the opposite polarization for the ϵ parameters. This corresponds to exchanging the definition $X \to \bar{X}$ and $\bar{X} \to X$ in the following equations. This operation just corresponds to a parity operation on the cylinder, but does not produce any new phenomenon.

From a mathematical point of view, (2.5, 2.6) are easily seen to identify a *Hitchin system* [44, 45] on the cylinder (which is conformal to a sphere with two punctures). In such systems, $F = F_{w\bar{w}} dw \wedge d\bar{w}$ is the gauge curvature two form in reference to a gauge vector bundle V, and $\bar{X}$ is the holomorphic section of the bundle $EndV \otimes K$, where K is the canonical line bundle over the base which is trivial in our case.

The moduli space of $(4,4)$-instanton solutions and the moduli space of the solutions of the Hitchin systems are now identified.

Let us notice for completeness that there exist also a class of $(2,2)$ supersymmetry preserving solutions and can be obtained by following a method similar [2] to the one we used above. In this class of solutions one has $\theta = 0$, $X^I = x^I \mathbf{1}_N$ for $I = 5, \ldots, 8$ and a set of Hitchin-like equations for the two complex fields $X = X^1 + iX^2$ and $X' = X^3 + iX^4$ which are

$$F_{w\bar{w}} + ig^2[X,\bar{X}] + ig^2[X',\bar{X}'] = 0 \quad [X,X'] = 0\,, \tag{2.9}$$

$$D_w X = 0, \qquad D_w X' = 0\,. \tag{2.10}$$

Let us notice that again they minimize the MST action and that the value of the action in correspondence of these configurations is given by

$$S_{cl}^{(2,2)} = \frac{1}{2\pi}\int d^2w\,\mathrm{Tr}\left(D_w\bar{X}D_{\bar{w}}X + D_w\bar{X}'D_{\bar{w}}X' + \frac{g^2}{2}[X,\bar{X}'][X',\bar{X}]\right). \tag{2.11}$$

For a reason that we will shortly explain, we are not going to study this class of solutions. The reason is that we are going to study the strong coupling expansion of the partition function of MST about its classical solutions. If we do so, a pre-factor $e^{-S_{cl}}$ is to be kept into account for the corresponding classical solution. While for the $(4,4)$ class of solutions this factor is non zero, for the $(2,2)$ type the classical action contains a positive factor proportional to g^2 which dumps the classical configuration with a pre-factor as $e^{-g^2 \cdot const}$ and therefore these configurations do contribute to the strong coupling expansion only as highly suppressed effects. Let us notice that it could be that const.$=0$ in the dumping factor and that in those cases a brief inspection to (2.11) reveals that one should add to eq. (2.10) also the condition $[X,\bar{X}'] = 0$. This leads the $(2,2)$ solution to be in fact of the $(4,4)$ preserving type.

Therefore, in what follows, we will consider only the $(4,4)$ preserving class of solutions as the interesting one in faith of the fact that if less supersymmetry is

[2] With respect to the general scheme, we are not fixing the projected bosonic fields b_2 in the same way. Nonetheless the solutions of the following equations are solutions of the classical equations of motion.

preserved then the classical configuration contributes to the strong coupling limit only as a suppressed effect.

2.3 The spectral curve

Let us consider the Hitchin equations

$$F_{w\bar{w}} + ig^2 \left[X, \bar{X}\right] = 0 \quad D_w X = 0 \quad D_{\bar{w}} \bar{X} = 0 \,.$$

We want to study their solutions with boundary conditions which implement the interpolation between $[in]$ and $[out]$ free string configurations at $\tau \sim \mp\infty$.

A free string configuration is given by a cycle $[\pi] = \prod(i)^{(n_i)}$ with $\sum_i i n_i = N$ which characterizes the multivaluedness of the eigenvalues of the X field as $x_\alpha \to x_{\pi\alpha}$ around the basis cylinder. On the other hand, the X field has to be *well defined* on the cylinder, in the sense that the matrix fields are all periodic ($X \to X$ as $\sigma \to \sigma + 2\pi$, etc.) by supersymmetry. Moreover X and $\bar{X}$ have to satisfy the condition that $[X, \bar{X}] = 0$ which implies that X is diagonalized by a unitary matrix U, i.e. $X = U\hat{x}U^\dagger$ with $\hat{x}$ a diagonal matrix of its eigenvalues. Let us notice that the ordering of the eigenvalues in the diagonal matrix $\hat{x}$ is inessential in determining the free string state content and that the unitary matrix U is only constrained to satisfy the multivaluedness condition $U_{i\alpha} \to U_{i\pi\alpha}$ around the cylinder [3]: in fact any two choices U and U' are related by gauge equivalence under a gauge transform with unitary field $U'U^\dagger$ which is *well defined* on the cylinder. Therefore, we are left to the problem of generating at least one unitary matrix U which does the right job. To this end, let us consider [37] the $N \times N$ matrix

$$S_{i\alpha} \equiv x_\alpha^{N-i} \equiv H_{ij} U_{j\alpha}$$

which we split in its hermitian part $H = \sqrt{SS^\dagger}$ and its unitary part $U = H^{-1}S$. By definition, $U \equiv H^{-1}S$ is a good unitary matrix.

Now we are ready to properly implement the free string boundary conditions to the X field in the Hitchin equations. Suppose that free string asymptotic configurations are given in terms of anti-analytic $x_\alpha^{[as]}$ such that

$$x_{(\alpha)}^{[as]} \to x_{(\pi^{[as]}\alpha)}^{[as]}$$

[3] The i-index and the α-index are formally distinguished since the i-index doesn't undergo any permutation around the cylinder, while the α-index does.

with respect to a given $[\pi]$. Then construct

$$S^{[as]}_{i\alpha} \equiv \left(x^{[as]}_{(\alpha)}\right)^{N-i} \quad ; \quad S^{[as]} = H^{[as]}U^{[as]}$$

where $H^{[as]}$ is hermitian and $U^{[as]}$ is unitary and define

$$X^{[as]} \equiv U^{[as]}\hat{x}^{[as]}U^{[as]\dagger} \quad A^{[as]}_{w} \equiv i\partial_w U^{[as]} \cdot U^{[as]\dagger} .$$

The relative boundary conditions to be imposed to the fields are

$$X \sim X^{[as]} \quad \text{and} \quad A_w \sim A^{[as]}_w \quad \text{for} \quad \text{Re}w \sim \pm\infty . \tag{2.12}$$

Notice that the asymptotic free string configurations satisfy the Hitchin system for any value of the coupling since $D^{[as]}_w X^{[as]} = 0$ together with $F^{[as]}_{w\bar{w}} = 0$ and $\left[X^{[as]}, \bar{X}^{[as]}\right] = 0$.

The problem now arises of understanding if and in what sense the solutions to the Hitchin system with the boundary conditions (2.12) do represent Riemann surfaces interpolating between the $[in]$ and $[out]$ free string configurations. To this end, let us consider that we can rewrite our boundary conditions as an assignment to the spectrum of the X field as follows: consider the X-characteristic polynomial

$$P_X(x) = \text{Det}(x - X) = \prod_\alpha \left(x - x_{(\alpha)}\right)$$

and the curve

$$P_X(x) = 0 \tag{2.13}$$

which is called the **spectral curve** of the Hitchin system. From this point of view, the boundary conditions can be read as a prescription for the asymptotic form of the spectral curve (2.13) as Re$w \sim \mp\infty$. More precisely, the boundary condition can be interpreted [4] as an assignment to the index-spectrum of the Puiseax expansion of the curve (2.13) around the above points.

Therefore, the spectral curve defines a Riemann surface interpolating between $x^{[in/out]}$. Let us notice that the spectral curve is a branched cover of the cylinder, i.e. it is composed of the union of the images of the base cylinder given by the spectrum of the X field, and that this structure is the same as the structure of a Mandelstam closed string diagram. The connection between the two subjects will be explained in full details in Chapter 4. Moreover, $D_w X = 0$ implies that this branched cover is

[4] See the last section of this chapter for details.

anti-holomorphic (this will be very important later on when we will discuss in what sense MST generates the Riemann surface moduli space).

For completeness, before going on, we review some few basics about branched coverings of Riemann surfaces.

2.3.1 Holomorphic branched coverings

In this section we review some basics of the mathematics on holomorphic branched coverings.

Let C and Σ be two Riemann surfaces and let there exists a projection map between the two surfaces

$$\Pi : \Sigma \rightarrow C$$

The rank N of Π^{-1} at a generic point of C is called the rank of the covering. If it is maximal up to isolated points, then we say that the curve Σ is a branched cover of C. The isolated points where the rank lowers are called branching points. If the map is holomorphic, then the covering is said to be holomorphic.

Let us review in a constructive framework the holomorphic case, which is the one we are interested in. Let C be a complex domain $\mathcal{A}$ and z be a coordinate on it. Let $a_i(z)$, $i = 0, \ldots, N-1$, be analytic functions on $\mathcal{A}$ and consider the curve Σ in $\mathcal{A} \times \mathbf{C}$ defined by

$$P(x) = x^N + \sum_{i=0}^{N-1} a_i(z) x^i = \prod_{\alpha=1}^{N} \left(x - x_{(\alpha)}(z) \right) = 0$$

Notice that, for generic $a_i(z)$, the root functions $x_{(\alpha)}(z)$ are not one-valued functions on $\mathcal{A}$. In fact they can exchange by continuing along paths encircling points where two (or more) of them may coincide. These points are the branching points of the covering. The explicit covering structure is given in terms of the image copies of the domain $\mathcal{A}$ under the root functions $\mathcal{A}_k = \mathrm{Im}\,(\mathcal{A}, x_k)$ as follows: on each copy coherently give a cuts system connecting the branching points and possibly the boundary $\partial\mathcal{A}_k$, then glue them together along the cuts to get the surface.

In MST, $\mathcal{A}$ is the punctured complex plane $X_2 = \mathbf{C} \setminus \{0\}$ which is conformally equivalent to the cylinder ($z = e^{\bar{w}}$) and the polynomial equation is the spectral curve of the Hitchin system. So, we let $\mathcal{A} = \mathbf{C} \setminus \{0\}$ in the following. Notice that X_2 is naturally equipped with a flat metric $|dw|^2$ which can be lifted to Σ giving a picture of it as flat almost everywhere.

To be concrete let us give an easy example, which is the pants diagram with rank

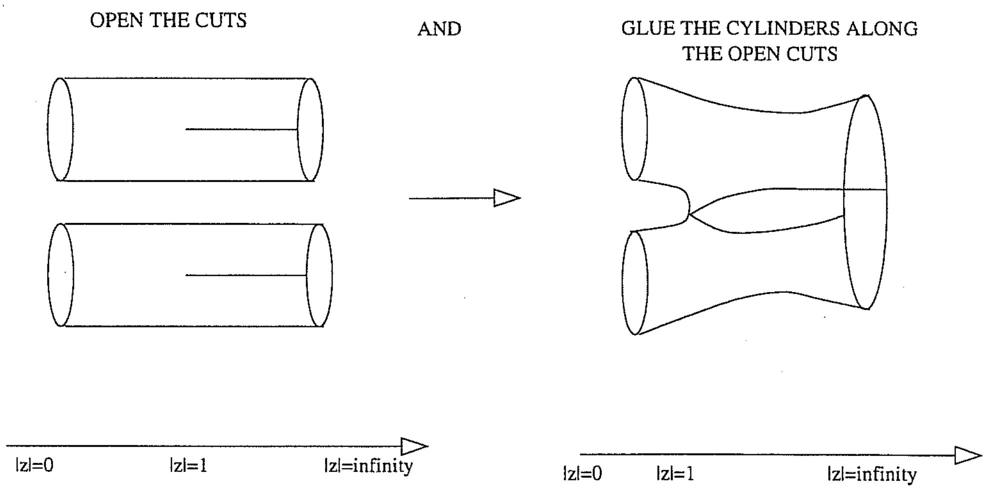

Fig. 4: *The branched cover structure of the elementary pants diagram.*

2. Take the equation to be

$$x^2 = z - 1$$

whose branching locus is $\{1, \infty\}$ and root functions $x_\pm = \pm\sqrt{z-1}$. The surface Σ is made of two identical cylinders (see *Fig.* 4.) cut and glued along the images of the straight line $\{\text{Im}z = 0\,, \quad \text{Re}z \in [1, \infty]\}$. The induced metric is $|d\ln(1 + x_\pm^2)|^2$ whose curvature turns out to be proportional to $\delta(z-1)$, which is then concentrated on the branched locus of the covering.

2.4 The Hitchin system on the Cylinder

Let us now come back to the problem of constructing the solution space of the Hitchin system with free string boundary conditions. What we want to show is that they are fully characterized by the associated spectral curve.

Let therefore the spectral curve be given as the curve

$$P_X(x) = x^N + \sum_i a(z)_i x^i = \text{Det}(x - X) = 0$$

in terms of N analytic functions $a_i(z)$ on X_2.

Notice that if two fields F_i satisfy $P_X(F_i) = 0$, then they are related by a similarity equation $F_1 = Y^{-1}F_2Y$ for some Y. Moreover, $P_X(X) = 0$ by definition and the field

$$M = \begin{pmatrix} -a_{N-1} & -a_{N-2} & \dots & \dots & -a_0 \\ 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \tag{2.14}$$

do satisfy the spectral condition $P_X(M) = 0$ for any choice of the spectral curve functions a_i. Therefore we have

$$X = Y^{-1}MY$$

for some *well defined* Y field on the cylinder. Moreover, by definition, the equation $\partial_w M = 0$ holds.

Notice now that the *ansatz* [5] $A_w = -iY^{-1}\partial_w Y$ solves identically the equation $D_w X = 0$. In the parameterization

$$A_w = -iY^{-1}\partial_w Y \ , \quad X = Y^{-1}MY$$

we get for the curvature

$$F_{w\bar{w}} \equiv \partial_w A_{\bar{w}} - \partial_{\bar{w}} A_w + i\left[A_w, A_{\bar{w}}\right] = iY^{-1}\partial_w \left[\partial_{\bar{w}}\left(YY^\dagger\right)\left(YY^\dagger\right)^{-1}\right] Y$$

and for the relevant commutator term

$$\left[X, \bar{X}\right] = Y^{-1}\left[M, \left(YY^\dagger\right) M^\dagger \left(YY^\dagger\right)^{-1}\right] Y \ .$$

Therefore the only equation left on Y is effective on its gauge independent part $\Omega \equiv YY^\dagger$ and reads

$$\partial_w \left(\partial_{\bar{w}}\Omega\Omega^{-1}\right) + g^2\left[M, \Omega M^\dagger \Omega^{-1}\right] = 0 \ . \tag{2.15}$$

Notice that this equation is the extremality condition for the deformed WZNW action

$$I_g[\Omega] = \frac{1}{16\pi}\int d^2w \mathrm{Tr}\Omega^{-1}\partial_w\Omega\Omega^{-1}\partial_{\bar{w}}\Omega +$$
$$+\frac{1}{24\pi}\int_{B|\partial B = Cyl.} d^3y \varepsilon^{ijk}\mathrm{Tr}\left(\Omega^{-1}\partial_i\Omega\Omega^{-1}\partial_j\Omega\Omega^{-1}\partial_k\Omega\right) - \frac{g^2}{4\pi}\int d^2w \mathrm{Tr}\left(\Omega^{-1}M\Omega M^\dagger\right) \ .$$

[5] At first sight it could seem that we should add to A_w an arbitrary function of M as $A_w = -iY^{-1}\partial_w Y + Y^{-1}f(M)Y$, but this can be reabsorbed by a redefinition of the Y field which is still unconstrained. Therefore our *ansatz* does generate the general solution.

The relevant boundary conditions on Ω can be easily obtained to be [6]

$$\Omega \sim \left[SS^{\dagger}\right]^{[as]} \quad \text{as } \Re w \sim \pm\infty .$$

Notice that one could be tempted by trying to extend the boundary conditions to a full solution of the equation as $\Omega = SS^{\dagger}$. This however can't be done since it implies $\left[M, \Omega M^{\dagger}\Omega^{-1}\right] = 0$ identically and then one should fulfill the equation $\partial_w\left(\partial_{\bar{w}}\Omega\Omega^{-1}\right) = 0$ which is not true. In fact, some simple calculations show that on the right hand side one gets in this case some δ-functions located on the branched points of the spectral curve.

The above observation is in any case teaching us that, away from the branched points, we can well approximate the Ω field by $SS^{\dagger}$. The deviation from this solutions will be bigger and bigger as we approach the branched points.

Notice that the only dimensional parameter in the above equation is g whose dimension is [length^{-1}]. This parameter fixes the size of the neighborhood of branching points where Ω is significantly different from $SS^{\dagger}$ to be of order g^{-1}. Therefore the general structure of the solutions of the Hitchin system on the cylinder can be described as a fixed structure on the spectral curve which is deformed in the vicinity of the branching points with a width of order g^{-1}.

Let us notice that the spectral curve is independent of the coupling g. What depends on the coupling is only the Y field (which is Ω) peaked around the branching points over a fixed background.

2.4.1 The strong coupling limit of stringy instantons

In this subsection we will study the strong coupling limit of stringy instantons.

Let us begin by defining

$$A_w^{\infty} = i\partial_w UU^{+} \text{ and } X^{\infty} = U\hat{x}U^{+}$$

and parameterize the general solution of the Hitchin system as

$$A_w = -iZ(\partial_w + iA_w^{\infty})Z^{-1} \text{ and } X = ZX^{\infty}Z^{-1} .$$

[6]Recall that considering any branched covering of the cylinder

$$P(x) = x^N + \sum_i a(z)_i x^i = \prod_{\alpha}(x - x_{(\alpha)}) = 0$$

we construct out of its root functions the matrix $S_{i\alpha} \equiv x_{(\alpha)}^{N-i}$ and its unitary part U (as before $S = HU$ etc.).

The Hitchin system is then equivalent to the following equation ($\Psi = ZZ^+$)

$$iF^\infty_{w\bar{w}} + D^\infty_w \left(\Psi^{-1} D^\infty_{\bar{w}} \Psi\right) - g^2 \left[X^\infty, \Psi^{-1} \bar{X}^\infty \Psi\right] = 0$$

where $F^\infty_{w\bar{w}}$ is the curvature of A^∞. A close inspection reveals that in so doing we are isolating the delta function singularities we found above in a single term, which is this curvature term.

Following an analysis equal to the one we performed in the previous paragraph, we reach the conclusion that the Ψ-field is peaked on the branching locus with characteristic peak width $\propto \frac{1}{g}$ over a $\Psi = 1$ background.

In the strong coupling limit $g \to \infty$ we have therefore, up to gauge transformations,

$$(X, A_w) \quad \to \quad (X^\infty, A^\infty_w) \quad \text{on } X_2 \setminus \{b.p.\}\,. \tag{2.16}$$

At the branching points the solution seems to be not well defined to compensate the δ-like term effect at the branching locus since the above equation exactly breaks up there. By the discussion we did in the preceding chapter we know anyhow that the total solution is smooth. Therefore we can safely extend the solution (2.16) to the whole cylinder by continuation. We will come back on these issues in next sections where a constructive method to treat the equation for the dressing factor Y will be explained.

Some useful properties of this stringy instantons at strong coupling are the followings: given

$$A^\infty_w = i\partial_w U U^\dagger \text{ and } X^\infty = U\hat{x}U^\dagger$$

along fixed time curves on the cylinder $\mathrm{Re}\, w = T$ we get

$$\hat{x} \ \to P^\dagger_T \cdot \hat{x} \cdot P_T \quad \text{and} \quad U \ \to U \cdot P_T$$

with $P_T \in S_N$ is the permutation whose cycle $[P_T]$ describes the intermediate string state. Moreover, U defines a Cartan sub-algebra in $u(N)$ which is $\mathtt{t} = U\mathtt{t}_d U^\dagger$, where $\mathtt{t}_d$ is the diagonal and therefore $\mathtt{t}$ is by definition the U-rotated diagonal one.

2.5 The $N = 2$ instanton

In this section we treat in detail the instanton construction for the particular case $N = 2$. It is not restrictive to take the surface as the curve of equation

$$X^2 = a \tag{2.17}$$

where $\partial_{\bar{z}} a = 0$ on $\mathbf{C}^*$. The relative reference matrix solution

$$M = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}$$

is such that $M^2 = a\mathbf{1}_2$. Let us parameterize the relevant Y field as

$$Y \equiv \begin{pmatrix} e^{\phi/4} & 0 \\ 0 & e^{-\phi/4} \end{pmatrix} \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix}$$

whose specific form can always be chosen by a gauge rotation. Here ϕ is real and ν complex. The resulting equations are the following ones

$$\frac{1}{2}\partial_w \partial_{\bar{w}} \phi + e^{-\phi}|\partial_w (e^{\phi/2}\nu)|^2 + g^2 e^{\phi}(|ae^{-\phi} - \nu^2|^2 - 1) = 0$$

$$e^{\phi/2}\partial_{\bar{w}}\left[e^{-\phi}\partial_w(e^{\phi/2}\nu)\right] - 2g^2 e^{\phi}\left[\nu - \bar{\nu}(ae^{-\phi} - \nu^2)\right] = 0$$

(and the conjugate of the last one). The relevant boundary conditions for the field-parameters ϕ and ν corresponding to the vanishing of $[X, \bar{X}]$ and of $F_{w\bar{w}}$ at $\tau \sim \mp\infty$ are given by

$$e^{\phi} \sim \frac{|a|}{1+\rho_{\mp}^2} \text{ and } \quad \nu \sim \rho_{\mp}\left(\frac{a}{\bar{a}}\right)^{1/4}$$

with $\rho_{\mp}$ two real numbers. To fix these two constants let us compare with the resulting form of the asymptotic

$$X \sim \frac{\sqrt{|a|}}{\sqrt{1+\rho_{\mp}^2}} \begin{pmatrix} -\rho_{\mp}\,(a/\bar{a})^{1/4} & a/|a| \\ 1 & \rho_{\mp}\,(a/\bar{a})^{1/4} \end{pmatrix}$$

which is well defined on the cylinder only if $\rho_{\mp} = 0$. Therefore the only consistent boundary conditions are with $\rho_{\mp} = 0$.

Note now that $\nu = 0$ solves identically the equations and the boundary conditions. We believe this solution to be unique. So we are left with a single equation for ϕ which is

$$\frac{1}{2}\partial_w \partial_{\bar{w}} \phi + g^2\left(|a|^2 e^{-\phi} - e^{\phi}\right) = 0 \text{ with } \quad e^{\phi} \sim |a| \text{ as } \quad \tau \sim \mp\infty\,. \tag{2.18}$$

As we already pointed out, the solution of this equation is everywhere regular on the cylinder. In the simple case at hand, we can in fact prove it rigorously. Let us study equation (2.18) by mapping it to the punctured plane $z = e^{\bar{w}}$ where it becomes

$$|z|^2 \partial_z \partial_{\bar{z}} \phi + 2g^2\left(e^{\phi} - |a|^2 e^{-\phi}\right) = 0 \text{ with } \quad e^{\phi} \sim |a| \text{ as } \quad |z| \sim 0, \infty\,.$$

We can prove[7] that ϕ is a limited function on the punctured plane for generic values of the coupling g. In order to do this, let us consider a domain D on $\mathbb{C}\setminus\{0\}$ such that $\phi|_{\partial D}$ is a constant and do not contain the point ∞. The prove is done in three steps. Let us first notice that for each point in D the potential is an increasing function in ϕ and that it is majored by $c_1 e^{|\phi|}+c_2$ at any point in D, where c_1 and c_2 are real positive constants. This allows the application of known theorems about monotonic operators whose outcome is that $\phi \in H^1(D)$, where $H^1(D) = \{\phi | \int_D(|\phi|^2 + |\nabla\phi|^2) < +\infty\}$. As a second step one can show that the following Trudinger's inequality takes place

$$\int_D e^{k_1|\phi|^2} \leq k_2 \int_D |\nabla\phi|^2 \quad \text{for any } \phi \in H^1(D)\,.$$

Trudinger's inequality can be used[8] to show that, given the specific form of the potential of the equation, the potential function calculated on the solution ϕ itself is an $L^p(D)$ function for any $p \in \mathbb{N}$, where $L^p(D) = \{\phi | \int_D |\phi|^p < +\infty\}$. As a third step, a regularity result can be now advocated to obtain that ϕ is a continuous function on $\bar{D} = D \cup \partial D$ and hence is limited on it.

This proves that the $N = 2$ dressing factor e^{ϕ} is regular for any finite value of the coupling.

One can further study the behavior of the $N = 2$ dressing factor e^{ϕ} by shifting the field parameter to a field with vanishing boundary conditions at infinity $\phi \equiv \varphi + \ln|a|$. If we do this, we obtain a version of the sh-Gordon equation with a source

$$\frac{1}{2}\partial_w\partial_{\bar{w}}\varphi + g^2|a|\left(e^{-\varphi} - e^{\varphi}\right) = -\frac{1}{2}\partial_w\partial_{\bar{w}}\ln|a| \text{ with} \quad \varphi \sim 0 \text{ as} \quad \tau \sim \mp\infty$$

Notice now that the right-hand-side of this equation is

$$-\frac{1}{2}\partial_w\partial_{\bar{w}}\ln|a| = -\frac{\pi}{2}|\partial_w\bar{a}|^2\delta(a)$$

which shows that getting rid of the boundary conditions at infinity, we are exposing a singular behavior of the field parameter at the zeros of the function a, i. e. at the branch points of the spectral curve $X^2 = a$ of the instanton. Moreover, away from the branch points φ tends to vanish due to sh-like potential which dumps it more and more as long as g increases [9]. The competition of these two effects can be better

[7] We would like to thank G. dal Maso and M. Negri who helped us for this proof.

[8] This is obtained by decomposing the domain D as $D = D_+ \cup D_-$ where $D_+ = \left\{z \in D | \, |\phi| \geq \frac{p}{k_1}\right\}$ and $D_- = \left\{z \in D | \, |\phi| < \frac{p}{k_1}\right\}$. Once this is done, one uses Trudinger's inequality to prove that $c_1 e^{|\phi|} + c_2 \in L^p(D)$.

[9] This has also been obtained in [38, 13] with numerical methods.

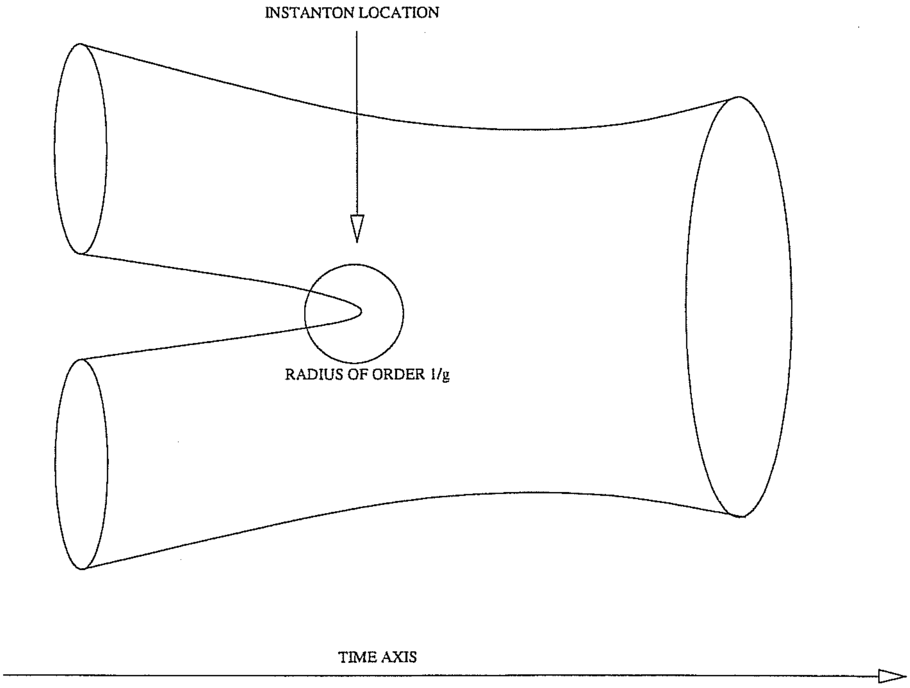

Fig. 5: *The pant diagram describes the joining of two length one strings into a single length two string. The location of the instanton is shown.*

recognized from the functional

$$\int_{cyl} d^2w \left[\frac{1}{4}|\partial_w\varphi|^2 + g^2|a|\left(e^{-\varphi} + e^{\varphi} - 2\right) - \frac{1}{2}\ln|a|\partial_w\partial_{\bar{w}}\varphi\right]$$

which is minimal in correspondence with the $N = 2$ instanton equation on the space of φ's vanishing at infinity.

One can speculate on the fate of the solution in the strong coupling $g \sim \infty$. In this regime one can see that φ vanishes in any point where $a \neq 0$ and is point-like peaked about a finite number of points, that is the zeros of $|a|$, where the value of the field remains unknown. We then identify the limiting $g \to \infty$ solution as $\varphi = 0$. This means that the limiting value of the dressing factor is, as we have already supposed, given by $e^{\phi} = |a|$.

Let us now give pictorial interpretation of the instanton. To simplify the discussion, let us consider the pant diagram (depicted in *Fig.* 5.) of equation $x^2 = z - 1$, i.e. $a = z - 1$ has a single simple zero at $w = 0$ corresponding to the branch point about which the instanton is peaked.

2.6 The moduli space of stringy instantons

In this section we start the study of the moduli space of solutions of the Hitchin system on the cylinder. A further mathematical paragraph is needed.

2.6.1 Local fields on branched coverings

In this subsection we will review some other mathematics about branched coverings of the cylinder which will be useful very soon.

If Σ is a branched covering of the cylinder $\mathcal{C}$, then, as we saw, there exists a projection map $\Pi : \Sigma \to \mathcal{C}$ whose inverse image is N-valued up to finitely many isolated points.

In the case of an holomorphic covering we can rephrase this explicitly in terms of the root function set as

$$\Pi^{-1} : w \to \left(x_{(1)}(w), \ldots, x_{(N)}(w)\right). \tag{2.19}$$

The meaning of this representation is simply that we cut Σ into N disjoint cylinders along the cut system and we are in fact representing it as the union of these cylinders by giving them in terms of the root functions. The problem we want to face now is how to describe local fields on Σ from the point of view of the decomposition into cylinders. Suppose a local complex field $\tilde{\psi}$ is given on Σ. Its counter-image on the cylinder $\Pi^{-1}\tilde{\psi}$ can be represented as an N-tuple $\left(\psi_{(1)}(w), \ldots, \psi_{(N)}(w)\right)$ assigning the value of the field on each copy of the cylinder $\mathcal{C}$ that composes the covering structure of Σ. The elements on the N-tuple $\psi_{(i)}(w)$'s are related by the appropriate monodromy properties along the curves $\Re w = const$, i.e. they get exchanged along $\sigma \to \sigma + 2\pi$ at given $\tau = T$ by the same intermediate monodromy exchanging the roots.

What we just said is complete for scalar fields. Suppose now a field with some differential weight (p, q) (a section of $K^p \times \bar{K}^q$) is given on Σ. With reference to some coordinate system it is given as $\tilde{\psi}^{(p,q)} = \{\psi'_\alpha dz^p_\alpha d\bar{z}^q_\alpha\}$ with the usual patching conditions. With some care we can represent it also in the covering decomposition choosing patches in one to one correspondence with the cylinders in such a way that each patch covers one cylinder completely and partly its neighbor in order to close the atlas structure of Σ as a two dimensional manifold. Therefore on each cylinder we can use these coordinates and write $\tilde{\psi}^{(p,q)}_\alpha = \psi'_\alpha dx^p_\alpha d\bar{x}^q_\alpha = \psi_\alpha(w) dw^p d\bar{w}^q$. Therefore we reach the conclusion that to represent fields with some non zero differential weight we need to keep into account their weight by multiplying their representative elements by the corresponding powers of the differentials [10] dw and $d\bar{w}$.

Let us now give the field theory application of this last point. Suppose a well defined field $\psi^{\mathfrak{t}}$ in the Cartan sub-algebra is given on the cylinder. Exposing the U-rotation we can rewrite $\psi^{\mathfrak{t}} = U\hat{\psi}U^{\dagger}$ and by definition the entries ψ_{α} of the diagonal matrix $\hat{\psi}$ get exchanged as a local field's representation elements. This means that any field $\psi^{\mathfrak{t}}$ can be reinterpreted as a local field on Σ once multiplied by the corresponding powers in dw and $d\bar{w}$ respectively. We will refer to this by saying that $\psi^{\mathfrak{t}}$ gets lifted to the corresponding single field $\tilde{\psi}$ on Σ.

Another point we want to review in this subsection is the problem of counting harmonic 1-differentials on open surfaces as Σ is [11].

If Σ was a closed Riemann surface of genus h, then their number would be h. Their counting in the present case is done following the Ahlfors' doubling trick [49].

The double $\hat{\Sigma}$ of Σ is obtained as follows. Duplicate the surface by considering an exact mirror copy Σ' of the original surface Σ and glue Σ and Σ' together along the corresponding equal boundary components: $\hat{\Sigma} \sim \Sigma \cup \Sigma'$ has genus $\hat{h} = 2h+b-1$, where b is the number of boundaries of Σ, and admits an anti-conformal involution (a $\mathbb{Z}_2$ auto-morphism) whose the set of fixed points corresponds exactly to the boundary of Σ. We can count now the number of analytic differentials on Σ that extend to $\hat{\Sigma}$, that is the so–called analytic Schottky differentials: their number is $\hat{h}$ and they correspond to the harmonic 1-differentials on Σ.

Notice that the number of harmonic scalars remains, by the same argument, equal to 1 on open surfaces as on closed ones.

2.6.2 The dimension of the moduli space of the Hitchin system on a cylinder

In this subsection we count the dimension of the solution space of Hitchin equations on a cylinder up to gauge transformations. The strategy will be the following: first we linearize the system transferring the calculation to the tangent space, then we calculate the dimension of the subspace of the tangent space which is orthogonal to infinitesimal gauge transformations in the strong coupling limit $g \to \infty$. To prove

[10] dw is in fact an abelian differential $\omega = dw$ with imaginary periods which is canonical, i.e. is fixed only by the complex structure of the surface. See later on for an extensive discussion of this point. $d\bar{w}$ is just its complex conjugate.

[11] Punctures have to be considered as boundaries of zero length. Recall that punctures represent in and out strings.

that this dimension is independent on the value of the coupling g, we will make use of the index theorem for the relative complex. The equations in the $g \to \infty$ limit are in a form which can be lifted on the relative spectral curve Σ. This way the calculation reduces to a zero modes counting on Σ. The exact result is then obtained by taking into account the degeneration in the plane curve representation of the spectral curve Σ.

The Hitchin equations are

$$F_{w\bar{w}} + ig^2[X,\bar{X}] = 0\,, \quad D_w X = 0\,, \quad D_{\bar{w}}\bar{X} = 0\,, \tag{2.20}$$

and their linearization is

$$D_w \delta A_{\bar{w}} - D_{\bar{w}}\delta A_w - ig^2\left[\bar{X},\delta X\right] + ig^2\left[X,\delta\bar{X}\right] = 0 \tag{2.21}$$

$$D_w\delta X + i\left[\delta A_w, X\right] = 0\,, \quad D_{\bar{w}}\delta\bar{X} + i\left[\delta A_{\bar{w}},\bar{X}\right] = 0\,. \tag{2.22}$$

The condition of orthogonality to gauge transformations is obtained in the form

$$D_w \delta A_{\bar{w}} + D_{\bar{w}}\delta A_w + ig^2\left[\bar{X},\delta X\right] + ig^2\left[X,\delta\bar{X}\right] = 0 \tag{2.23}$$

making use of the scalar product

$$< (\delta_1 A, \delta_1 X)|(\delta_2 A, \delta_2 X) >= \int_{Cyl} d^2w \mathrm{Tr}\left[\delta_1 A_w \delta_2 A_{\bar{w}} + g^2 \delta_1 X \delta_2 \bar{X} + \quad \text{h.c.}\right]\,. \tag{2.24}$$

Let us now take the $g \to \infty$ limit of the above equations. The base point solution is then a generic strong coupling limit instanton relative to a spectral curve Σ

$$A_w^\infty = -iU\partial_w U^+\,, \quad X^\infty = U\hat{X}U^+ \tag{2.25}$$

and the equations remain the following ones

$$D_{\bar{w}}^\infty \delta A_w = 0\,, \tag{2.26}$$

$$\left[\bar{X}^\infty, \delta X\right] = 0 \tag{2.27}$$

$$D_w^\infty \delta X + i\left[\delta A_w, X^\infty\right] = 0\,, \tag{2.28}$$

and their hermitian conjugates. To solve these equations we are going to use the lifting technique we exploited in the previous paragraph.

Let $\mathtt{t}$ be the Cartan sub-algebra in $u(N)$ obtained as $U\mathtt{t}_d U^\dagger$ where $\mathtt{t}_d$ is the diagonal one. Since $\bar{X} \in \mathtt{t}$, then, by (2.27), also $\delta X \in \mathtt{t}$. Using the properties of (2.25) it is easy to see that also $D_w \delta X \in \mathtt{t}$ and by (2.28) and $X \in \mathtt{t}$ we obtain that $\delta A_w \in \mathtt{t}$. This means that all the variations are in the Cartan $\mathtt{t}$ and the above equations reduce to

$$D^{\infty}_{\bar{w}} \delta A_w = 0\,, \quad D^{\infty}_{w} \delta X = 0 \quad \text{where} \quad \delta A_w\,,\ \delta X \in \mathtt{t} \tag{2.29}$$

and their hermitian conjugates. To count the solutions of Eq.(2.29), we have to lift it to the spectral surface relative to the base-point background field (2.25) and it naturally reduces to

$$\partial_{\bar{z}} \widetilde{\delta A_z} = 0\,, \quad \partial_z \widetilde{\delta X} = 0 \tag{2.30}$$

where $\widetilde{\delta A_z} d\,z$ is therefore an holomorphic differential on Σ and $\widetilde{\delta X}$ an harmonic scalar. Following the doubling trick explained in the last paragraph we get a total number of solutions equal to $\hat{h} + 1 = (2h + n - 1) + 1 = 2h + n$.

Let us now show that the above dimension is independent on the value of the coupling. To do this we have to apply the index theorem to the complex

$$\Omega^0_{s.a.}(\text{ad}_g) \xrightarrow{\delta_g} \Omega^0(\text{ad}_g) \bigoplus \Omega^1(\text{ad}_g) \xrightarrow{T} \Omega^2_{s.a.}(\text{ad}_g) \bigoplus \Omega^1(\text{ad}_g) \tag{2.31}$$

where $\Omega^p(\text{ad}_g)$ is the space of p-forms on the cylinder in the adjoint representation with respect to the gauge group, $s.a.$ indicates that the elements of these spaces are taken to be self adjoint, δ_g denotes the operator of infinitesimal gauge transformations

$$\delta_g\,: u \to (-i[X, u], -D_w u) \tag{2.32}$$

and T is the operator associated to the linearization of the Hitchin equations (2.22)

$$T\,: (\delta X, \delta A) \to \Big((D_w \delta X + i\,[\delta A_w, X])\, dw + \Big(D_{\bar{w}} \delta \bar{X} + i\,\big[\delta A_{\bar{w}}, \bar{X}\big] \Big)\, d\bar{w},$$

$$\Big(D_w \delta A_{\bar{w}} - D_{\bar{w}} \delta A_w - ig^2 \big[\bar{X}, \delta X\big] + ig^2 \big[X, \delta \bar{X}\big] \Big)\, d^2 w \Big) \tag{2.33}$$

The index of the complex (2.31) is by definition given by

$$\text{index} = \dim_{\mathbf{R}} H^0 - \dim_{\mathbf{R}} H^1 + \dim_{\mathbf{R}} H^2 \tag{2.34}$$

where

$$H^0 = \text{Ker}\delta_g\,, \quad H^1 = \frac{\text{Ker}T}{\text{Im}\delta_g} = \text{Ker}T \cap \text{Ker}\delta_g^+\,, \quad H^2 = \text{Ker}T^+ \tag{2.35}$$

where T^+ and δ_g^+ are the relevant adjoint operators, that is

$$\delta_g^+ : (\delta X, \delta A) \to \left(D_w \delta A_{\bar{w}} + D_{\bar{w}} \delta A_w + ig^2 \left[\bar{X}, \delta X\right] + ig^2 \left[X, \delta \bar{X}\right]\right) \tag{2.36}$$

and

$$T^+ : (\delta f, \delta A) \to \Big(D_w \delta A_{\bar{w}} - g^2 [X, \delta f] \,,$$

$$i \left(D_w \delta f + [\delta A_w, X]\right) dw - i \left(D_{\bar{w}} \delta f - \left[\delta A_{\bar{w}}, \bar{X}\right]\right) d\bar{w}\Big) \,. \tag{2.37}$$

Notice that to obtain Eq.(2.37) we used the pairing

$$< (\delta_1 A, \delta_1 f) | (\delta_2 A, \delta_2 F) >= \int_{Cyl} d^2 w \mathrm{Tr} \left[\delta_1 A_w \delta_2 A_{\bar{w}} + i \delta_1 f \delta_2 F_{w\bar{w}} + \quad \text{h.c.}\right] .$$

Using the lifting technique again one can easily calculate the value of the index at $g = \infty$ which turns out to be index $= 0$. This number is independent of the value of the coupling g. To show that $\dim H^1$ is independent of the value of the coupling it is then sufficient to show that $\dim H^0 + \dim H^2$ is.

To do this we first formulate a vanishing formula for the T^+ operator that is: if $(\delta f, \delta A) \in \mathrm{Ker} T^+$, then

$$0 = \int_{Cyl} d^2 w \left(|D_{\bar{w}} \delta A_w|^2 + g^2 |\left[\delta A_w, X\right]|^2 + g^2 |D_w \delta f|^2 + g^4 |\left[\delta f, X\right]|^2\right) \tag{2.38}$$

which can be obtained by integration by parts.

In view of (2.38) and (2.32), the calculation of $\dim H^0 + \dim H^2$ reduces to counting the solutions of

$$D_{\bar{w}} \delta A_w = 0 \,, \quad [\delta A_w, X] = 0 \tag{2.39}$$

$$D_w \delta \phi = 0 \,, \quad [\delta \phi, X] = 0 \tag{2.40}$$

(and their adjoint equations) where $\delta \phi = \delta f + iu$.

As we have pointed out in the previous section, the finite coupling solutions of the Hitchin equation which we consider here can be related to the ones at infinite coupling by a dressing factor Z, that is

$$X = Z X^\infty Z^{-1} \,, \quad A_w = Z \left(-i \partial_w + A_w^\infty\right) Z^{-1} . \tag{2.41}$$

Substituting (2.41) in (2.39) and (2.40) we get

$$D_{\bar{w}}^\infty \delta A_w^Z = 0 \,, \quad \left[\delta A_w^Z, X^\infty\right] = 0 \tag{2.42}$$

$$D^{\infty}_{w}\delta\phi^{Z} = 0\,, \quad \left[\delta\phi^{Z}, X^{\infty}\right] = 0 \tag{2.43}$$

where we defined $\delta\phi^Z = Z^{-1}\delta\phi Z$ and $\delta A^Z_{\bar{w}} = Z^{-1}\delta A_{\bar{w}} Z$. At this point it is evident that the number of solutions of (2.42) and (2.43), that is $\dim H^0 + \dim H^2$, is independent of the value of the coupling.

The result is that for any value of the coupling g

$$\dim_{\mathbf{C}} H^1 = \dim_{\mathbf{C}}\left(\mathrm{Ker}T \cap \mathrm{Ker}\delta^+_g\right) = \dim_{\mathbf{C}} H^1|_{g=\infty} = 2h + n \tag{2.44}$$

To obtain the true moduli space dimension we have to consider that there are some degrees of freedom in the plane curve representation of the spectral surface that have to be further subtracted from this number since they represent equivalent curve representations. As it will be clarified later on, the polynomial representation of the spectral curve

$$P_X(x) = X^N + \sum_{i=0}^{N-1} a_i(z) X^i = 0$$

is of degree N in X and in z and monic in X. Therefore one can consider all the curves obtained by moving

$$z \to \alpha z \quad \text{and} \quad X \to X + \beta z + \gamma\,, \tag{2.45}$$

with α, β and γ being arbitrary complex numbers, as equivalent. This means that this group of moves leaves the instanton as effectively the same one and hence have to be subtracted from the computation of the effective dimension of the moduli space.

Therefore, the moduli space of the Hitchin system on a cylinder $\mathcal{M}^H_n$ is organized in strata $\mathcal{M}^H_{h,n}$ fixed by the genus h and the number of punctures n of the spectral curve and

$$\dim\mathcal{M}^H_{h,n} = 2h + n - 3\,.$$

In Chapter 4 it will be shown that each stratum corresponds to a complex codimension h sub-manifold of the moduli space $\overline{\mathcal{M}_{h,n}}$ of Riemann surfaces of genus h and n punctures.

2.6.3 Fermionic zero modes

As we said, stringy instantons are classical configurations preserving a $(4,4)$ sub-supersymmetry. Its complement, the broken part, acts on the bosonic representative orbit point, that we calculated, generating also the fermionic completition of the Hitchin pair. It can be easily proved that in so doing one generates spinor fields along the preserved directions which are, due to the properties of the (A, X) pair, still solutions of the classical equation of motions (coupled Dirac equations) of the theory.

These fermion zero modes will come in some number that we want to calculate. To do this, we can just make use of supersymmetry. Let us recall that at fixed curve representation the number (2.44) of bosonic moduli is $2h+n$. By supersymmetry, there will be also $2h+n$ fermionic moduli. In fixing the dimension of the space up to equivalent curves, for the bosonic moduli we subtracted the dimension of the group of transformations (2.45). By looking at the supersymmetry transformations of the fermions and to their equations of motion, constant translations of the X variable do not act on the fermion fields, while the other do act. Therefore, we calculate the number of fermionic moduli up to equivalences as

$$\#_f = 2h + n - 2\,. \tag{2.46}$$

2.7 Construction of instanton solutions

In this section we generalize to generic spectral curves the constructive analysis that we already performed for the $N=2$ case. Each solution of (2.5), (2.6) consists of two parts: a branched covering of the cylinder via the relative X characteristic polynomial and a group theoretical factor.

Let us start by recalling our construction of the solutions of eqs. (2.5), (2.6). By this we mean a couple (X, A_w) which are solutions of (2.5), (2.6) and are smooth everywhere on $\mathcal{C}$. We parameterize them as

$$X = Y^{-1}MY, \qquad A_w = -iY^{-1}\partial_w Y\,. \tag{2.47}$$

The group theoretical factor Y takes values in the complex group $SL(N,\mathbf{C})$ while the matrix M determines the branched covering (see below). The dependence on the Yang-Mills coupling constant g is contained in the Y factor, while M does not depend on g. We have shown that $Y = Y^{\infty}Z$, where Z, the dressing factor, tends to 1 in the strong coupling limit outside the string interaction points, while Y^{∞} is a special matrix, independent of g, endowed with the property that$Y^{\infty -1}MY^{\infty}$ and $Y^{\infty\dagger}M^{\dagger}(Y^{\infty\dagger})^{-1}$ are simultaneously diagonalizable. The construction of Y^{∞} and Z in the general case is rather subtle. One first diagonalizes M by means of a matrix S of $SL(N,\mathbf{C})$. Then one introduces a matrix K such that $KS = U$ is unitary. As it turns out, K may have singularities at the points of $\mathcal{C}$ where any two eigenvalues of M coincide: these correspond to the branch points of the spectral covering (K is also allowed to diverge in a prescribed way for $w = \pm\infty$, but we disregard this issue for the time being). Therefore KMK^{-1} is in general singular at these points. We therefore introduce into the game a new matrix L, with the purpose of canceling the singularities of KMK^{-1} in such a way that $LKMK^{-1}L^{-1}$ is smooth and satisfies (2.5), (2.6). In order for this to be true the entries of L must satisfy equations of the WZNW type with delta-function-type sources at the branch points. By construction K is independent of g while L does depend on g. We will show that in fact $L \to 1$ as $g \to \infty$. We therefore see that K^{-1} plays the role of Y^∞ and L^{-1} is to be identified with the dressing factor Z, so that $LK = Y^{-1}$.

We will deal with the general construction of Z and Y^∞ in detail in subsection 2.7.1. Here we would like to stress the double 'miracle' of the above construction: we construct an everywhere smooth solution by means of two non smooth matrices K and L, which are such that on the one hand $L \to 1$ as $g \to \infty$ and on the other hand K form with S a unitary matrix $U = KS$.

2.7.1 The unitary and dressing factors

As explained in the previous section, the MST instantons consist of two pieces: a group theory factor Y and a branched covering (plane curve) parametrized by the matrix M. Let us now concentrate on the former.

The unitary factor The factor Y^∞ can be constructed as follows. As we already saw, the matrix M can be diagonalized

$$M = SDS^{-1}, \qquad D = \mathrm{Diag}(\lambda_1, \ldots, \lambda_N) \tag{2.48}$$

by means of the following matrix $S \in SL(N, \mathbf{C})$:

$$S = \Delta^{-\frac{1}{N}} \begin{pmatrix} l_1^{N-1} & l_2^{N-1} & \ldots & \ldots & l_N^{N-1} \\ l_1^{N-2} & l_2^{N-2} & \ldots & \ldots & l_N^{N-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & \ldots & \ldots & 1 \end{pmatrix}, \tag{2.49}$$

where

$$\Delta = \prod_{1 \le i < j \le N} (l_i - l_j)\,. \tag{2.50}$$

We now introduce a matrix K such that $U = KS$ is unitary. If such a matrix exists than $KMK^{-1} = UDU^{-1}$ and $(K^\dagger)^{-1}M^\dagger K^\dagger = UD^\dagger U^{-1}$ do commute. Then K can play the role of $Y^{\infty-1}$. We refer to $U = KS$ as the *unitary factor* in the construction of the background solution X.

One such K can be constructed with the Gram-Schmidt procedure. The result is the following upper triangular matrix belonging to $SL(N, \mathbf{C})$:

$$K = \begin{pmatrix} k_{11} & k_{12} & \dots & \dots & k_{1N} \\ 0 & k_{22} & \dots & \dots & k_{2N} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & k_{NN} \end{pmatrix}, \tag{2.51}$$

where

$$k_{pp} = \sqrt{\frac{Q^{(N-p)}}{Q^{(N-p+1)}}} |\Delta|^{\frac{1}{N}}, \qquad 1 \leq p \leq N$$

$$k_{pq} = \frac{Q^{(N-p+1)}_{N-q,N-p}}{\sqrt{Q^{(N-p)} Q^{(N-p+1)}}} |\Delta|^{\frac{1}{N}}, \qquad 1 \leq p < q \leq N-1 \, .$$

The symbols in the previous formulae have the following meaning. For $2 \leq k \leq N$ we set

$$Q^{(k)} = \sum_{1 \leq j_1 < j_2 < \ldots < j_k \leq N} \left(\prod_{1 \leq l \leq k-1} |l_{j_l} - l_{j_{l+1}}|^2 |l_{j_l} - l_{j_{l+2}}|^2 \ldots |l_{j_l} - l_{j_k}|^2 \right),$$

$$Q^{(k)}_{n,m} = (-1)^{n+m} \sum_{1 \leq j_1 < j_2 < \ldots < j_{k-1} \leq N} S_{k-1-n}(l_{j_1}, l_{j_2}, \ldots, l_{j_{k-1}}) S_{k-1-m}(\bar{l}_{j_1}, \bar{l}_{j_2}, \ldots, \bar{l}_{j_{k-1}}) \times$$
$$\times \left(\delta_{k,2} + \prod_{1 \leq l \leq k-2} |l_{j_l} - l_{j_{l+1}}|^2 |l_{j_l} - l_{j_{l+2}}|^2 \ldots |l_{j_l} - l_{j_{k-1}}|^2 \right), \quad 0 \leq n \leq m \leq N-1$$

where $S_p(x_1, \ldots, x_s)$ denotes the elementary symmetric polynomial of order p in $x_1, \ldots, x_s$ $(s \geq p)$:

$$S_p(x_1, \ldots, x_s) = \sum_{1 \leq j_1 < j_2 < \ldots < j_p \leq s} x_{j_1} \ldots x_{j_p}, \qquad S_0 = 1 \, .$$

Moreover, by definition, $Q^{(0)} = 1, Q^{(1)} = N$. Notice that

$$Q^{(N)} = |\Delta|^2, \qquad k_{11} k_{22} \ldots k_{NN} = 1 \, .$$

The $Q^{(k)}_{n,m}$ are homogeneous polynomials of order $\frac{k(k-1)}{2} - n$ in the variables l_i and of order $\frac{k(k-1)}{2} - m$ in the complex conjugates $\bar{l}_i$. $Q^{(k)}$ are homogeneous polynomials of order $\frac{k(k-1)}{2}$ in both l_i and $\bar{l}_i$.

For example, in the case $N = 3$, the matrix K is given by

$$K = \begin{pmatrix} \frac{\sqrt{\sum_{i<j}|l_i-l_j|^2}}{|\Delta|^{\frac{2}{3}}} & -\frac{\sum_{i<j}(l_i-l_j)|l_i-l_j|^2}{\sqrt{\sum_{i<j}|l_i-l_j|^2}\,|\Delta|^{\frac{2}{3}}} & \frac{\sum_{i<j}l_il_j|l_i-l_j|^2}{\sqrt{\sum_{i<j}|l_i-l_j|^2}\,|\Delta|^{\frac{2}{3}}} \\ 0 & \sqrt{\frac{3}{\sum_{i<j}|l_i-l_j|^2}}|\Delta|^{\frac{1}{3}} & -\frac{\sum_i l_i}{\sqrt{3\sum_{i<j}|l_i-l_j|^2}}|\Delta|^{\frac{1}{3}} \\ 0 & 0 & \frac{|\Delta|^{\frac{1}{3}}}{\sqrt{3}} \end{pmatrix}, \quad 1 \le i, j \le 3\,.$$

This completes the construction of K. We remark that generally the entries of $U = KS$ contain as a factor some fractional power of $|\Delta|$. Therefore they may vanish or diverge with some fractional power whenever two of the eigenvalues of M coincide. This corresponds to a simple branch point in the spectral covering, as we have seen above. [12] Outside these points the unitary factor U is smooth.

The dressing factor Let us come now to the Z factor. We have just noted that the entries of K are generically singular whenever two eigenvalues of M coincide, that is at the site of a branch point of the covering. Then KMK^{-1} shares the same singularities and it is not a satisfactory ansatz for our solution X of (2.5), (2.6), which we want to be everywhere smooth (except perhaps at $w = \pm\infty$). To this end we introduce a new matrix L with the requirement that $LKMK^{-1}L^{-1}$ is smooth and is the desired solution X of (2.5) ,(2.6). While K is independent of g, L will depend on g. L^{-1} is our candidate for the dressing factor Z.

Since L has to smooth out the singularities of K, it is enough to take for it an upper triangular matrix belonging to $SL(N, \mathbf{C})$. A possible parameterization for L is the following

$$L = \begin{pmatrix} e^{u_1} & e^{u_1}\psi_{12} & e^{u_1}\psi_{13} & \dots & e^{u_1}\psi_{1N} \\ 0 & e^{u_2-u_1} & e^{u_2-u_1}\psi_{23} & \dots & e^{u_2-u_1}\psi_{2N} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e^{u_{N-1}-u_{N-2}} & e^{u_{N-1}-u_{N-2}}\psi_{N-1N} \\ 0 & 0 & \dots & 0 & e^{-u_{N-1}} \end{pmatrix}. \qquad (2.52)$$

The fields u and ψ, which generalize the field parameters φ and ν that we introduced in the treatment of the $N = 2$ case, have to satisfy certain differential equations in order to comply with the requirements. We just plug the ansatz for X (and the connection) into (2.5) and work out the relevant equations. We will not write them

[12]The entries of K contains other factors, beside Δ, that may vanish when more than two eigenvalues coincide. This corresponds to multiple branch points, which we disregard for sake of simplicity.

down here. A few examples were given in [13, 14]. They are all equations of the WZNW type and can be cast in the general form

$$\partial_w \partial_{\bar{w}} \phi + \ldots \sim \partial_w \partial_{\bar{w}} \ln|\Delta| = \pi \frac{\partial \Delta}{\partial w} \frac{\partial \bar{\Delta}}{\partial \bar{w}} \delta(\Delta) \,, \tag{2.53}$$

where ϕ denotes any field u or ψ, while dots represent all the other terms, which are irrelevant in the cancellation of singularities. Let us refer to these equations as the 'dressing equations'. On the right-hand side we see the typical delta-function type source which characterizes them. The sources are point-like and located at the zeroes of Δ, that is at the branch points of the covering.

Naturally the solution X exists with the required properties only if the 'dressing equations' admit solutions that vanish at $w = \pm\infty$. To our best knowledge, not much is known in the literature concerning the existence of such solutions. Some simple cases were discussed also in [14], where we presented a few numerical solutions. We deem these sufficient for us to assume that the 'dressing equations' do admit solutions that vanish at $w = \pm\infty$. Once one assumes this, it is rather easy to argue, on a completely general ground, that in the strong coupling limit, $g \to \infty$, such solutions vanish outside the zeroes of the discriminant.

The argument goes as follows. Consider a candidate solution of (2.5) in which $u = \psi = 0$ outside the zeroes of the discriminant. Then, there, $L = 1$, and $X = KMK^{-1}$. As we have already noticed before, in such a situation $[X, \bar{X}] = 0$, since both X and $\bar{X}$ are simultaneously diagonalized by the matrix $U = KS$. This is most welcome since, if $[X, \bar{X}]$ were not to vanish (outside the zeroes of the discriminant), it would be impossible to have any finite solution of (2.5). Next, we have to show that also $F_{w\bar{w}}$ vanishes in the same region when $L = 1$. In fact when $L = 1$,

$$A_w = -iK\partial_w K^{-1} = -i(KSS^{-1})\partial_w(SS^{-1}K^{-1}) = -iU(\partial_w + \tilde{A}_w)U^{-1} \,, \tag{2.54}$$

where $\tilde{A}_w = S^{-1}\partial_w S$. But $\partial_w S \equiv 0$ due to holomorphicity of the eigenvalues of M. In conclusion (2.5) is identically satisfied by the ansatz $L = 1$ outside the zeroes of the discriminant. Since the solutions are uniquely determined by their boundary conditions, we can conclude that, as $g \to \infty$, the only solution of the dressing equations outside the zeroes of the discriminant, is the identically vanishing solution. We infer from this argument that the solutions of the dressing equations for large g are concentrated around the branch points and become more and more spiky as g grows larger and larger. Therefore the matrix L is just the dressing factor Z.

Let us here warn the reader, however, that the above argument assumes specific boundary conditions for the fields at infinity. One can envisage in fact other types of solutions. For instance, let A_w and X be any g-dependent solution considered so far. Then call $A_w^{(1)}, X^{(1)}$ their value at $g = 1$. The latter satisfy (2.5), (2.6) for $g = 1$. Now set $X_g = X^{(1)}/g$. Then $A_w^{(1)}, X_g$ satisfy the same equation, but, as $g \to \infty$, $X_g \to 0$ while $F_{w\bar{w}}^{(1)} \neq 0$. In such a situation, however, the eigenvalues of X_g for $g \to \infty$ do not describe any asymptotic string configuration and we do not admit such configurations for the simple reason that they do not fulfill the required boundary conditions on the X spectrum.

2.8 Plane curves and branched coverings

In this section we present some mathematical results in order to complete the above discussion and to introduce the reader to the theory of plane curves. This will reveal useful later on when we will compare the MST result with superstring theory.

The main ingredient of a stringy instanton solution is a branched covering of the cylinder. In dealing with branched coverings it is however sometimes more convenient to map, by the standard mapping $z = e^{\bar{w}}$, the infinite cylinder $\mathcal{C}$ to the punctured complex z-plane $\mathbf{C} \setminus \{0\} = \mathbf{C}^*$.

2.8.1 The punctures

Let us consider again the polynomial

$$P_X(x) = \mathrm{Det}(x - X) = x^N + \sum_{i=0}^{N-1} x^i a_i \,,$$

where x is a complex indeterminate and the set of functions $\{a_i\}$ are analytic on $\mathbf{C}^*$, and therefore they are allowed in principle to have poles at $z = 0$ and $z = \infty$. The structure of these functions reveals the allowed realization of the asymptotic free string configurations by the root functions.

To have a better understanding of plane curves, let us introduce the concept of Puiseux expansion. The Puiseux expansion is a technique to study the local structure of the curve by means of a local series expansion of the root functions.

Suppose one wish to locally solve the root functions in a neighborhood of some given value $z = z_0$. There is a well defined way to do this. Let $\{x_{0\alpha}\}$ be the solutions of $P_X(x, z_0) = 0$ and let us consider, for each α,

$$Q_\alpha(\hat{x}, \hat{z}) \equiv P_X(x_{0\alpha} + \hat{x}, z_0 + \hat{z}) = \sum_{a + b\mu_\alpha \geq \nu_\alpha} c^\alpha_{ab} \hat{x}^b \hat{z}^a = 0 \,,$$

where a and b are integers [13], whose solution can be written as $\hat{x} = t_\alpha \hat{z}^{\mu_\alpha}$+(higher order terms), with μ_α rational and positive. We can calculate the coefficients t_α by substituting $\hat{x} = t\hat{z}^{\mu_\alpha}$ in Q_α and considering that $Q_\alpha(t\hat{z}^{\mu_\alpha}, \hat{z}) = \hat{z}^{\nu_\alpha} T_\alpha(t)$+(higher order terms). Therefore the t_α's are the solutions of $T_\alpha(t) = 0$ and we have a well defined expansion of the root functions as $x_\alpha = x_{0\alpha} + t_\alpha (z - z_0)^{\mu_\alpha} (1 + o(z - z_0))$. Using the same method one can in principle calculate also the higher order correction terms and obtain a series (which can also be finite) as

$$x_\alpha = x_{0\alpha} + t_{\alpha 1} ((z - z_0)^{\mu_{\alpha 1}} + t_{\alpha 2} ((z - z_0)^{\mu_{\alpha 2}} + t_{\alpha 3} ((z - z_0)^{\mu_{\alpha 3}} + \dots \tag{2.55}$$

where the $\{\mu_{\alpha i}\}$ are rational and increasing. The series 2.55 can be shown to converge in a sufficiently small neighborhood of z_0.

From the Puiseux series one can reconstruct the local structure of the curve by studying how the root functions permute following their fate about a small loop $z = z_0 + \epsilon e^{i\theta}$, $\theta \in [0, 2\pi]$ for sufficiently small ϵ, about a given point z_0.

If one considers the Puiseux series about $z_0 = 0$, then the above procedure shows how it contains in its structure the free string asymptotics, i.e. the realization of the relevant cycle $[g]$ with $g \in S^N$. There is, however, a subtility in that. We said that the functions a_i are allowed to have poles at $z = 0$, since this point does not lay in the curve. If so the above procedure do not work at these points. Therefore we are led to consider curves which are obtained by means of regular a_i functions at $z = 0$. The story is not still at its end. In fact, there are analogous problems at $z = \infty$ which admit a similar solution by constraining the a_i functions to be polynomial in $\bar{z}$ of order not greater than $N - i$. All in all this means that $P_X(x, z)$ is a polynomial in x and z of total degree N monic in x.

There is, however, another mathematically serious reason for choosing this class of curves. This is connected with the concept of *tameness* of the Hitchin pair [62] on punctured base surfaces. A Hitchin pair is said to be tame at a boundary point if the Puiseux expansion of its spectral curve is regular there, which is the same as what we required above [14]. Tameness guarantees the existence of a well-behaved bundle

[13]The coefficients μ_α and ν_α are uniquely fixed by requiring the above expansion to hold. They can be obtained directly by examining the *Newton Polygon* of the curve. We are not going to review this classical subject in this thesis and we refer to [35] about this and other themes. Here we will give just some material suitably adapted to the case of punctured curves.

[14]Notice that at first sight it might seem to the reader that there is a mismatch with the tameness definition as it has been given by the author of [62]. There it is defined by saying that the root functions at the point have to be not more singular than a simple pole. To connect to our notation, however, there is a multiplicative Jacobian factor $\partial_{\bar{w}} z = z$ which keeps into account the different definitions.

metric in the V gauge bundle relative to the Hitchin system. This property implies the good definition of the volume elements that we use in Chapter 3 to define the path-integral measure.

2.8.2 Singular points and the genus of the curve

Let us restart again from the polynomial equation

$$P(x,z) \equiv \sum_{p,q} a_{p,q} x^q z^p = 0, \qquad (x,z) \in \mathbf{C} \times \mathbf{C}^*, \tag{2.56}$$

where P is a polynomial of degree N monic in x. The independent non-vanishing coefficients $a_{p,q}$ can be varied without changing, at generic values, the topological type (h,n) of the curve. They are the *moduli* of the plane curve.

A *singular point* of the curve (2.56) is a point where

$$P(x,z) = \partial_x P(x,z) = \partial_z P(x,z) = 0$$

. This means that singular points are points of the curve admitting more than one tangent line. When no such points are present the curve is smooth. Let us notice that it may well be that the punctures, i.e. points located at $z = 0, \infty$, are formally singular, i.e. they fulfill the singularity condition just stated. It doesn't however mean that the curve is singular since these points are not points of the surface.

Let us here introduce the genus formula for the curves we are studying. The genus formula for the compactified curve, i.e. for the curve augmented by the addition of the punctures with $z \in \mathbf{CP}$ and $x \in \mathbf{CP}$, is given by [15]

$$h = \frac{1}{2}(N-1)(N-2) - \sum_{s'} \delta_{s'}$$

where the the sum runs over the singular points of the curve and $\delta_{s'} \in \mathbf{N}_+$ is the contribution of each singular point. These integer numbers can be calculated from the specific form of the defining equation $P(x,z) = 0$, but we are not going to review this point in detail, preferring to discuss it in view of its application to MST.

Let us give an idea of how this result can be obtained in the completely smooth case. Define for any point $p_0 = (x_0, z_0)$ of the curve $P(x,z) = 0$ its ramification index ν_{p_0} as the order of x_0 as a zero of $P(x, z_0) = 0$. The branching locus of the curve is then given by $B \equiv \{p \in \Sigma | \nu_p > 1\}$ and can be obtained as the zero set of the discriminant of the curve $\Delta \equiv \sum_{\alpha \neq \beta}(x_\alpha - x_\beta)$, where x_α are the root functions. Applying the standard

[15]See for example [35] for a clear derivation of the genus formula.

Riemann-Hurwitz formula to cylinder branched coverings – the Euler characteristic of the cylinder is zero – one obtains that $\chi_\Sigma = -\sum_{p\in B}(\nu_p - 1) = -(\#\Delta - \#\Delta|_0 - \#\Delta|_\infty)$. If the curve is sufficiently generic we can assume that the degree of the discriminant is given by $N(N-1)$. Moreover, by construction, the number of punctures of the surface is given by $(N - \#\Delta_0)$ for the punctures at $z = 0$ plus $(N - \#\Delta_\infty)$ for the punctures at $z = \infty$. Therefore we obtain that

$$\chi_\Sigma = -N(N-1) + 2N - n = 2 - 2g - n \text{ with } \quad g = \frac{1}{2}(N-1)(N-2)\,.$$

Starting from such a smooth situation, we can move around the plane curves moduli space and eventually lower the genus of the plane curve, while keeping the degree constant, by allowing for singular points.

Let us note here just the following fact. By definition singular points are branch points of the surface. If the surface is smooth, then these points are all playing the same role in the construction of the branched covering being the points where the cuts start/end. Suppose now that two or more of these points come to coincide. At these points the surface then can develop a singularity, which is a point where the surface looks like a multiple cone. The *local structure* of the curve about a singular point is typically dominated by an equation of the type $(x - x_0)^p = Q(z - z_0)$, with p an integer greater than 1 and $Q(z - z_0)$ a polynomial having $z = z_0$ as a multiple zero. This can be desingularized easily by considering a deformed Q_ϵ polynomial whose zero at z_0 looses its multiplicity.

The curves we are considering are punctured Riemann surfaces, while the genus formula we gave above refers to unpunctured surfaces. It still remains valid, of course, but its interpretation changes due to the fact that one can admit a contribution from formal singularities at the punctures which lowers the genus of the curve without making it singular. In other word, let us rewrite the genus formula as

$$h = \left(\frac{1}{2}(N-1)(N-2) - \delta_{z=0} - \delta_{z=\infty}\right) - \sum_s \delta_s$$

where we separated the contribution from the singular points of the surface and contribution coming from the formal singularity at the punctures. This formula for punctured surfaces then means that we can well have any value of the genus of the curve, compatible with the upper bound $\frac{1}{2}(N-1)(N-2)$, while keeping the curve smooth by allowing for formal singularity at the punctures which lowers the genus but does not make the punctured curve singular at all.

All this means that far from discarding singular curves, we have to take them into full account in MST. We will come back to this subject in Chapter 4 where we will see the relevance of singular curves and how they play a necessary role in MST in comparison with string theory.

Chapter 3

The strong coupling expansion

In this chapter we set out to compute the strong YM coupling limit $g \sim \infty$ of Matrix String Theory's partition function.

We will expand the action about the stringy instantons we studied before by splitting the quantum modes in two sets: the Cartan and the non–Cartan modes. It turns out that the non–Cartan modes can be neatly integrated out while it remains a quadratic action for the Cartan modes. However these modes are not individually well defined fields on the cylinder. A field interpretation is possible if we lift them all together to the spectral curve of the individual instanton Σ. We show that, if we do so, we obtain the Green–Schwarz theory plus a decoupled Maxwell theory on the world–sheet Σ. In the calculation of the partition function of this theory in the strong YM coupling limit a fundamental role is played by the Maxwell field zero modes. With a careful computation one can show that this sector contributes with a constant factor proportional to g^{χ_Σ}, where χ_Σ is the Euler characteristic of the world-sheet surface.

Notice that, even if we start expanding the theory about classical configurations which break the R-symmetry group $SO(8)$ to $SO(2) \times SO(6)$, the limiting resulting theory is again fully $SO(8)$ invariant. Let us recall here also the discussion we had in Chapter 2 about other possible saddle points about which one could expand the action. We have shown there that only $(4,4)$ classical solutions dominate the path integral, being the others suppressed by a classical action which goes like $S_{cl} \sim g^2$. Therefore in the sequel we will consider only the $(4,4)$ sector and expand the path integral about stringy instantons.

3.1 The MST partition function

In this section we introduce the MST partition function, that is the partition function of the SYM theory with $\mathcal{N} = (8,8)$ and gauge group $U(N)$ on the cylinder. The partition function that we are going to define will be suitable for a string theory interpretation only if the right boundary conditions for the various fields are taken into account. These boundary conditions will be interpreted as an assignment of the incoming and outgoing string states.

The partition function we want to extimate is the following

$$\mathcal{Z}_{[\pi^{in}][\pi^{out}]}(g) \equiv \int_{[in/out]} D[\Phi]\, \mathrm{e}^{-S[\Phi]} \tag{3.1}$$

where S is the MST action (2.1) depending on all the relative fields which we denoted collectivelly by Φ. All the fields Φ are subject to the following boundary conditions at $\mathrm{Re}w = \tau \sim \mp\infty$ which we noticed by $[in/out]$ in the path integral:

$$X = X^1 + iX^2 \sim U^{as}\hat{x}^{as}U^{as\dagger}$$

$$X^I \sim x^I \mathbf{1}_N \text{ where } I = 3,\ldots,8$$

$$A_w \sim i\partial_w U^{as} U^{as\dagger}$$

$$\theta \sim 0$$

where the x^I's are 6 real constants, $[\pi^{as}]$ are two fixed cycles in $[S_N]$ representing the asymthotic free string configuration, the diagonal matrix $\hat{x}^{as}$ is composed by a set of analitic functions implementing the asymthotic cycle and U^{as} is the related unitary matrix obtained as in Chapter 2.

Our aim is in fact to calculate strong coupling $g \sim \infty$ asympthotic expansion of (3.1). In order to compute it we will first split the fields in background stringy instanton classical configurations and fluctuation about them. The path integral is then splitted in a finite integral over the stringy instanton moduli space and a path integral over the fluctuations. Once this point will be reached, it will be possible to perform the strong coupling analysis in the proper way.

3.2 The strong coupling expansion of the action

To start our study of the strong coupling expansion, we first study the expansion of the action about a generic *stringy instanton* in its strong coupling limit. To this aim it is useful first to rewrite the action functional as

$$S = \frac{1}{\pi}\int d^2w\, \mathrm{Tr}\Big(D_w X^I D_{\bar{w}} X^I - \frac{g^2}{2}[X^I, X^J]^2 - g^2[X^I, X][X^I, \overline{X}] +$$

$$+D_w X D_{\bar{w}} \overline{X} - \frac{1}{4g^2}\left(F_{w\bar{w}} + ig^2[X,\overline{X}]\right)^2 + i(\theta_s D_{\bar{w}}\theta_s + \theta_c D_w \theta_c)$$

$$+2ig\theta_s\gamma_i[X^i,\theta_c]\Big) + \frac{i}{2\pi}\oint_{\partial C} \mathrm{Tr}[\overline{X}\bar{D}X - X\bar{D}\overline{X} + \overline{X}DX - XD\overline{X}]\,,$$

where $I = 3, 4, ..., 8$ and $D = dwD_w$ and $\bar{D} = d\bar{w}D_{\bar{w}}$. To proceed with the expansion, decompose any field Φ as

$$\Phi = \Phi^{(b)} + \phi^{\mathrm{t}} + \phi^{\mathrm{n}} \equiv \Phi^{(b)} + \phi \equiv \Phi^{\circ} + \phi^{\mathrm{n}}\,, \tag{3.2}$$

where $\Phi^{(b)}$ is the background value of the field at generic coupling, ϕ^{t} are the fluctuations along the Cartan directions relative to the background field and ϕ^{n} are the fluctuations along the non-Cartan directions. In Eq.(3.2) we also defined ϕ as the total fluctuation of the field and Φ° as its total Cartan component.

To obtain a finite answer from the evaluation of any gauge theory partition function, it is necessary to keep into account the gauge degeneracy in the representation of the degrees of freedom. In the case of a background expansion, the gauge fixing condition naturally depends on it. The relevant gauge fixing condition is

$$\mathcal{G}_{w\bar{w}} = D^{\circ}_w a_{\bar{w}} + D^{\circ}_{\bar{w}} a_w + ig^2([X^{\circ},\bar{x}] + [\bar{X}^{\circ},x]) + 2ig^2[X^{\circ I},x^I] = 0\,,$$

Let us notice that this choice will reveal the appropriate one to get a meaningful and finite strong coupling limit of the partition funcion. This inspired by the t' Hooft abelian gauges and by the gauge choice in [46]. In order to fix the gauge in the quantum theory, we apply the Faddeev–Popov procedure by adding to the action the following term

$$S_{FP} = S_{gf} + S_{ghost} = \frac{1}{4\pi g^2}\int d^2w\;\mathcal{G}^2_{w\bar{w}} - \frac{1}{2\pi g^2}\int d^2w\;\bar{c}\frac{\delta\mathcal{G}_{w\bar{w}}}{\delta c}c\,,$$

where δ represents the gauge transformation with infinitesimal parameter c. Notice that the ghost fields c and $\bar{c}$ are Grassmann scalars in the adjoint of the gauge group on the base cylinder and it is understood that they are to be integrated over in the partition function. The total action is then given by the sum

$$S_{tot.} = S + S_{FP}\,.$$

In order to get a finite and well defined strong coupling limit, we rescale the fields as follows

$$A_w = A^{(b)}_w + ga^{\mathrm{t}}_w + a^{\mathrm{n}}_w, \quad X = X^{(b)} + x^{\mathrm{t}} + \frac{1}{g}x^{\mathrm{n}}, \quad X^I = X^{I(b)} + x^{I\mathrm{t}} + \frac{1}{g}x^{I\mathrm{n}},$$

$$\theta = \theta^{\mathrm{t}} + \frac{1}{\sqrt{g}}\theta^{\mathrm{n}}, \quad c = gc^{\mathrm{t}} + \sqrt{g}c^{\mathrm{n}}, \quad \bar{c} = g\bar{c}^{\mathrm{t}} + \frac{1}{\sqrt{g}}\bar{c}^{\mathrm{n}}. \tag{3.3}$$

These rescalings induce a unit Jacobian in the path integral measure of the non–zero modes, but they may produce a non-trivial factor due to the presence of non canceling zero modes. We will come back later on to this point to calculate the important exact factor induced by this rescaling.

After the above rescalings the action can be lastly expanded as

$$S = S_{cl}^{(4,4)} + S_{sc} + Q_{\mathrm{n}} + o\left(\frac{1}{\sqrt{g}}\right),$$

where

$$S_{cl}^{(4,4)} = \frac{1}{2\pi}\int d^2w \,\mathrm{Tr}\left(D_w^{(b)}\bar{X}^{(b)}D_{\bar{w}}^{(b)}X^{(b)}\right),$$

comes from the boundary term [1] and is the value of the classical action on the instanton,

$$S_{sc} = \frac{1}{\pi}\int_{C_0} d^2w \,\mathrm{Tr}\left[D_w^{(b)}x^{I\mathrm{t}}D_{\bar{w}}^{(b)}x^{I\mathrm{t}} + D_w^{(b)}x^{\mathrm{t}}D_{\bar{w}}^{(b)}\bar{x}^{\mathrm{t}} + i(\theta_s^{\mathrm{t}}D_{\bar{w}}^{(b)}\theta_s^{\mathrm{t}} + \theta_c^{\mathrm{t}}D_w^{(b)}\theta_c^{\mathrm{t}})\right.$$
$$\left. + D_w^{(b)}a_{\bar{w}}^{\mathrm{t}}D_{\bar{w}}^{(b)}a_w^{\mathrm{t}} + D_w^{(b)}\bar{c}^{\mathrm{t}}D_{\bar{w}}^{(b)}c^{\mathrm{t}}\right]$$

is the part of the action including Cartan fluctuations only and Q_{n} is a quadratic term in the non-Cartan fluctuations ϕ^{n} which will be soon tackled. The background fields are from now on understood to be at their infinite coupling limit values.

The exact evaluation of the strong coupling limit of the partition function is then calculable as a double Gaussian integral.

Let us evaluate first the contribution from the background

$$S_{cl}^{(4,4)} = \frac{1}{2\pi}\int d^2w \,\mathrm{Tr}\left(D_w^{(b)}\bar{X}^{(b)}D_{\bar{w}}^{(b)}X^{(b)}\right).$$

Substituting the value of the strong coupling limit of the instanton string configuration and undoing the U dressing we find $\frac{1}{2\pi}\int_\Sigma \partial\bar{\tilde{x}}_{cl}\bar{\partial}\tilde{x}_{cl}$, where $\tilde{x}_{cl}$ is the lifting to Σ of the spectrum of the $X^{(b)}$ field as a scalar field.

Let us now show that integrating along the non-Cartan direction do not contribute to the effective action. The quadratic term in the non-Cartan fluctuations is

$$Q_{\mathrm{n}} = \frac{1}{\pi}\int d^2w \mathrm{Tr}\left[x^{i\mathrm{n}}\mathcal{Q}x^{i\mathrm{n}} + a_{\bar{w}}^{\mathrm{n}}\mathcal{Q}a_w^{\mathrm{n}} + \bar{c}^{\mathrm{n}}\mathcal{Q}c^{\mathrm{n}} + i(\theta_s^{\mathrm{n}}, \theta_c^{\mathrm{n}})\mathcal{A}\begin{pmatrix}\theta_s^{\mathrm{n}} \\ \theta_c^{\mathrm{n}}\end{pmatrix}\right],$$

[1]Fluctuations have vanishing boundary term due to the fact that they vanish there by definition.

where

$$\mathcal{Q} = \mathrm{ad}_{X^{oi}} \cdot \mathrm{ad}_{X^{oi}} + \mathrm{ad}_{a^t_{\bar{w}}} \cdot \mathrm{ad}_{a^t_w}$$

and

$$\mathcal{A} = \begin{pmatrix} i\mathrm{ad}_{a^t_{\bar{w}}} & \gamma_i \mathrm{ad}_{X^{oi}} \\ \tilde{\gamma}_i \mathrm{ad}_{X^{oi}} & i\mathrm{ad}_{a^t_w} \end{pmatrix}.$$

Since Q_{n} is a purely quadratic term in the ϕ^{n} fluctuations, these can be easily integrated over since the path-integral is Gaussian. The algebraic nature of the operators $\mathcal{Q}$ and $\mathcal{A}$ entering the expression of Q_{n} ensures that the integral doesn't introduce any factor due to non-canceling zero-mode rescaling.

The integration over a^{n} and c^{n} exactly cancels to 1 for simple boson-fermion compensation.

The integration over x^{n} and θ^{n} gives the ratio $\left((\mathrm{Det}\mathcal{A})^{16}/(\mathrm{Det}\mathcal{Q})^{8}\right)^{N^2-N}$, but, using their definition, we have $\mathrm{Det}\mathcal{A} \equiv \sqrt{\mathrm{Det}(-\mathcal{A}\mathcal{A}^\dagger)}$ and $\mathcal{A}\mathcal{A}^\dagger = \mathcal{A}^\dagger\mathcal{A} = -\mathcal{Q}$, and again determinants cancel to 1 by simple boson-fermion cancellation.

Therefore, the total result of integrating over the non–Cartan modes is 1 as it was also expected from supersymmetry.

In conclusion, in the strong coupling limit we are left with the quadratic action S_{sc} over the Cartan modes.

3.3 Lifting to the world - sheet

In this section we are going to recast S_{sc} in its natural form by giving a unified field interpretation to the Cartan modes. The outcome is that S_{sc} equals the Green–Schwarz superstring action plus a free Maxwell action on the world-sheet Σ identified by the spectral curve of the relevant background instanton.

Let us start by rewriting the strong coupling limit of the action

$$S_{sc} = \frac{1}{\pi}\int_{C_0} d^2w \,\mathrm{Tr}\left[D^{(b)}_w x^{I\mathrm{t}} D^{(b)}_{\bar{w}} x^{I\mathrm{t}} + D^{(b)}_w x^{\mathrm{t}} D^{(b)}_{\bar{w}} \bar{x}^{\mathrm{t}} + i(\theta^{\mathrm{t}}_s D^{(b)}_{\bar{w}} \theta^{\mathrm{t}}_s + \theta^{\mathrm{t}}_c D^{(b)}_w \theta^{\mathrm{t}}_c)\right.$$
$$\left. + D^{(b)}_w a^{\mathrm{t}}_{\bar{w}} D^{(b)}_{\bar{w}} a^{\mathrm{t}}_w + D^{(b)}_w \bar{c}^{\mathrm{t}} D^{(b)}_{\bar{w}} c^{\mathrm{t}}\right]$$

and by recalling the expression of the strong coupling limit of stringy instantons. Together with the fact that the Cartan sub-algebra $\mathfrak{t}$ has been defined as the U rotated diagonal one, we can completely get rid of all the U factors in the expression of the action so that it can be rewritten exposing the diagonal core of the Cartan sub-algebra: the result of this algebraic manipulation is that the covariant derivative

$D_w^{(b)}$ becomes the simple partial derivative ∂_w and the Cartan subalgebra gets rotated to the diagonal one

$$S_{sc} = \frac{1}{\pi}\int_C d^2w \,\mathrm{Tr}\left[\partial_w x^{I\mathrm{t}_d}\partial_{\bar{w}} x^{I\mathrm{t}_d} + \partial_w x^{\mathrm{t}_d}\partial_{\bar{w}}\bar{x}^{\mathrm{t}_d} + i(\theta_s^{\mathrm{t}_d}\partial_{\bar{w}}\theta_s^{\mathrm{t}_d} + \theta_c^{\mathrm{t}_d}\partial_w\theta_c^{\mathrm{t}_d}) + \right.$$
$$\left. + \partial_w a_{\bar{w}}^{\mathrm{t}_d}\partial_{\bar{w}} a_{w}^{\mathrm{t}_d} + \partial_w \bar{c}^{\mathrm{t}_d}\partial_{\bar{w}} c^{\mathrm{t}_d}\right].$$

Since all the matrices are diagonal we can rewrite this action in terms of the diagonal modes $\phi^{\mathrm{t}_d} = \mathrm{diag}\left(\phi_{(1)}, \ldots, \phi_{(N)}\right)$

$$S_{sc} = \frac{1}{\pi}\int_C d^2w \sum_{n=1}^{N}\left[\partial_w x^i_{(n)}\partial_{\bar{w}} x^i_{(n)} + i(\theta_{s(n)}\partial_{\bar{w}}\theta_{s(n)} + \theta_{c(n)}\partial_w\theta_{c(n)}) + \right.$$
$$\left. + \partial_w a_{\bar{w}(n)}\partial_{\bar{w}} a_{w(n)} + \partial_w \bar{c}_{(n)}\partial_{\bar{w}} c_{(n)}\right].$$

Notice that the individual components $\phi_{(i)}$ are not well defined fields on the cylinder. In fact, since $\phi^{\mathrm{t}} = U\phi^{\mathrm{t}_d}U^+$ is well-defined on the cylinder and since, along the circle $\mathrm{Re}\, w = T$, $U \to U \cdot P_T$ with $P_T \in S_N$, then along $\mathrm{Re}\, w = T$, we get

$$\phi^{\mathrm{t}_d} \to P_T^+ \cdot \phi^{\mathrm{t}_d} \cdot P_T . \tag{3.4}$$

Therefore, the theory at hand is not a field theory on the base cylinder. To understand where it is defined on, we have to recall what does (3.4) means from the previous chapter. The outcome of the above analisys is that the generic field ψ^d represents a well-defined field on Σ when rescaled with the appropriate $\omega = dw$ factor and can be lifted to a well defined field $\tilde{\psi}$ on the surface.

Let us notice here that for half integer fields there is an ambiguity in the choice of the spin structure, i. e. in the choice of the square root of the canonical line bundle $K^{1/2}$ on Σ. The pleasent outcome is that we have to sum over all its different possible choices in the partition function, i.e. we have to mediate over all the spin structures.

Resumming over the image cylinders we then get a single integral over Σ which is

$$S_{sc}^{\Sigma} = S_{GS}^{\Sigma} + S_{Maxwell}^{\Sigma}$$

where

$$S_{GS}^{\Sigma} = \frac{1}{\pi}\int_\Sigma d^2\xi \left(\partial_\xi \tilde{x}^i \partial_{\bar{\xi}}\tilde{x}^i + i(\tilde{\theta}_s\partial_{\bar{\xi}}\tilde{\theta}_s + \tilde{\theta}_c\partial_\xi\tilde{\theta}_c)\right)$$

is the Green-Schwarz superstring action on Σ and

$$S_{Maxwell}^{\Sigma} = \frac{1}{\pi}\int_\Sigma d^2\xi \left(g^{\xi\bar{\xi}}\partial_\xi\tilde{a}_{\bar{\xi}}\partial_{\bar{\xi}}\tilde{a}_\xi + \partial_\xi\tilde{\bar{c}}\partial_{\bar{\xi}}\tilde{c}\right)$$

is a Maxwell $U(1)$ action on Σ. The metric in the Maxwell term is $g_{\xi\bar{\xi}} = \omega_\xi\omega_{\bar{\xi}}$ and ξ is a system of local coordinates on Σ.

At this point it is evident that, even if we started from an $SO(8)$ breaking background configuration set, the expansion about them is nonetheless $SO(8)$ symmetric. In fact, the eight fields $\tilde{x}^i$, as well as their susy counterparts, are all at equal footing in the Green-Schwarz superstring action and they represent the emdedding of Σ in the 8 transverse directions being well-defined scalar fields (chiral multiplets once considered together with the θ's) on Σ.

3.4 A nice present from the Maxwell sector

In order to obtain the effective action for the superstring fields, let us now integrate over the Maxwell sector. Since the action is quadratic the integration produces a simple ratio of determinants

$$Z^{\Sigma}_{Maxwell}(g) = \int \mathcal{D}\left[\tilde{a},\tilde{c}\right]\; e^{-S^{\Sigma}_{Maxwell}(\tilde{a},\tilde{c})} \propto \frac{\mathrm{Det}'\Delta_c}{\mathrm{Det}'\Delta_a}$$

where Δ denotes the relevant laplacian and $'$ means that the zero modes have been excluded from the computation of the regularized determinants.

The proportionality factor has to be calculated by taking account of the zero modes for the fields that have been rescaled. The relevant rescalings are the following ones

$$\tilde{a}_{\xi} \to g\,\tilde{a}_{\xi}\,, \quad \tilde{a}_{\bar{\xi}} \to g\,\tilde{a}_{\bar{\xi}}\,, \quad \tilde{c} \to g\,\tilde{c}\,, \quad \tilde{\bar{c}} \to g\,\tilde{\bar{c}}$$

and the Maxwell partition function is then [2]

$$Z^{\Sigma}_{Maxwell}(g) = \int \mathcal{D}\left[\tilde{a},\tilde{c}\right]\; e^{-S^{\Sigma}_{Maxwell}(\tilde{a},\tilde{c})} = \frac{\mathrm{Det}'\Delta_c}{\mathrm{Det}'\Delta_a} \times g^{\P_c - \P_a}\,,$$

where $\P_c$ is the nubmer of ghosts (scalar) zero modes and $\P_a$ is the nubmer of vector potential (1-form) zero modes As for the ghost fields, which are scalars, the only zero modes of the laplacian operator Δ on Σ is the constant. Hence, $\P_c = 1$.

The zero modes of the Maxwell field correspond to the harmonic 1-differentials on Σ. In the previous chapter we have seen how to compute their number: the result turned out to be $\P_a = 2h + b - 1$.

Therefore, the zero-mode rescaling factor is $\P_c - \P_a = 1 - \hat{h} = 2 - 2h - b = \chi_\Sigma$ which is the Euler characteristic of the spectral curve Σ and we get

$$Z^{\Sigma}_{Maxwell}(g) \quad = \quad g^{\chi_\Sigma}\,\frac{\mathrm{Det}'\Delta^M_0}{\mathrm{Det}'\Delta^M_1}$$

[2]For a similar calculation see also [48].

where Δ_n^M is the laplacian on n-differentials on Σ with respect to the Mandelstam metric $g^M_{\xi\bar{\xi}} = |\omega_\xi|^2$.

3.5 The strong coupling expansion of the partition function

In this section we first discuss the well definiteness of the background/fluctuation split of the fields and then recollect the various things we calculated along this chapter.

As a first step, let us notice that, due to the fact that the measure is induced by the Φ volume element given by $||\delta\Phi||^2 = \int_{Cyl.} d^2w \mathrm{Tr}[(\delta\Phi)^2]$ and the fact that the Cartan subalgebra is Tr-orthogonal with respect to its complement, the splitting along Cartan and non-Cartan modes doesn't introduce any Jacobian factor.

As the Cartan sector is concerned, let us start by recalling that the path integral of the partition function is performed over a space of field whose fluctuations ϕ are *zero* on the boundary while the background part $\Phi^{(b)}$ fullfils by construction the boundary condidion imposed on the total field Φ.

The background/fluctuation split is, from a general prespective, pathological if there can be an overlap between moduli translations and fluctuations, i.e. if they are not well defined as flat directions and effective fluctuations. In our case, it does not happen. In the strong coupling limit it is manifest. In fact, as we saw in the previous section, the action for the $\phi^{\mathfrak{t}}$ fluctuations is quadratic and positive definite, being positive definite the laplacian operator, on the space of these fields once the null boundary condition is considered. This means that the background configurations are true minima of the total action and that the moduli/fluctuation split has been well defined.

There remains, of course, the problem of giving a good representation to the true measure on the stringy instanton moduli space which is hence induced by the field theory one

$$D[\Phi^{(b)}] = |\det \int_{Cyl} d^2w \mathrm{Tr}[\partial_{m_a}\Phi^{(b)}\partial_{m_b}\Phi^{(b)}]|^{1/2} d\, m = \mu dm \,,$$

where the set $\{m_a\}$ are some good coordinates on the Hitchin moduli space on the cylinder and on the fermionic moduli. We will come back again to this question.

We conclude that the path integral measure in the partition function splits as

$$D[\Phi] \equiv \mu dm \cdot D[\phi^{\mathfrak{t}}] \cdot D[\phi^{\mathfrak{n}}] \,.$$

Let us now recollect the various terms to exhibit the $g \sim \infty$ asymptotic of the partition function:

$$Z_{[\pi^{in}][\pi^{out}]}(g) \quad \sim \quad \sum_h (1/g)^{-\chi} \int_{\mathcal{M}^H_{h,n}} \mu dm \mathcal{N} \sum_\nu \int D\left[\tilde{x}, \tilde{\theta}\right] e^{-S_{GS}\left[\tilde{x},\tilde{\theta}\right]}, \tag{3.5}$$

where $\chi = 2 - 2h - n$ is the Euler characteristic of the typical spectral curve Σ, $\mathcal{M}^H_{h,n}$ is the space of instantons at strong coupling whose spectral curve has genus h and reproduces along its punctures the relevant asymptotic cycles and on the fermionic moduli, $\mathcal{N} = \frac{\mathrm{Det}'\Delta^M_0}{\mathrm{Det}'\Delta^M_1} e^{-S^{(4,4)}_{cl}}$ is the remnant of the decoupled Maxwell sector and of the classical action, μdm is the field theory induced measure on $\mathcal{M}^H_{h,n}$, $\sum_\nu$ is the sum over the spin structures relative to the fermion fields and S_{GS} is the Green-Schwarz superstring action on Σ.

The reader will find natural to recognize in (3.5) a striking similarity with the perturbative superstring partition function in the light-cone gauge once the string coupling is identified as $g_s = g^{-1}$.

Chapter 4

The string interpretation

4.1 Light-cone String Theory vs MST

In this section we compare the strong coupling expansion of MST with light-cone Green-Schwarz superstring theory.

Superstring theory can be formulated in two different ways. The NS-R theory [29] is a formulation admitting a covariant quantization [40] and where one has a good control of the structure of the multi-loop amplitudes [1]. Its quantization in the light-cone gauge was developed earlier [30] and shown to lead to a unitary well defined set of amplitudes. The two quantization schemes propose different looking amplitudes which have been proved to be equal [39], thus establishing their equivalence. The covariant NS-R superstring realizes ten dimensional supersymmetry as an uneasy outcome and from this point of view is unnatural. The G-S theory [31] is intrinsically ten dimensional supersymmetric but, being highly non linear in its covariant form [32], it doesn't admit a straightforward covariant formulation. However, it greatly simplifies in the light-cone gauge, where it becomes a free theory on the worldsheet and can be proved to be on shell equivalent to the NS-R theory in the light-cone.

Covariant quantization of superstring theory is well established in the super-ghost formalism [40] which, being based on the BRST formulation of the theory, at the end guarantees the unitarity of the theory. However, this formulation being covariant is unsuitable for a comparison with the MST results, while it is evident, from all that we have shown in the previous chapters, that MST should be compared with G-S light-cone string theory. If an eleven dimensional covariant formulation of M-theory would exist, then its compactification on S^1 could probably be compared with the covariant formulation of type IIA superstring theory.

In this section we are going to review some basics of this subject and at the same time perform a comparison with the results of the MST outcome.

4.1.1 The partition functions

The picture we want to compare with is the light-cone formulation of the G-S superstring theory. Let us start from the covariant NS-R formulation and single out the light-cone as a reference choice. Here it has been chosen from the very beginning the reference metric on the world-sheet to be the Mandelstam one and the two-dimensional gravitino field has been also fixed up to a finite number of super moduli. The choice of the metric can be traced to be $\hat{g}_{z\bar{z}} = \partial_z X \partial_{\bar{z}} \bar{X}$ with $X = X^9 + iX^0$ – after the Wick rotation – which manifestly breaks the $SO(10)$ covariance to $SO(8) \times SO(2)$.

In so doing we are effectively singling out the light-cone condition by leaving the theory globally supersymmetric with respect to two types of supersymmetries which are a dynamical one and a kinematical one. These are the remnants of the supersymmetry which completes the above choice of the reference metric and whose effect is to break the local worldsheet supersymmetry down to a global one.

In the path integral light-cone formulation, the light-cone part of the spinor field is integrated over together with the light-cone scalar X, while making use of the usual $SO(8)$ triality together with bosonization techniques, we switch the transverse fermion fields to the Green-Schwarz representation as $\psi_L^i \to \theta_s$ and $\psi_R^i \to \theta_c$,

The worldsheet is taken to be a punctured Riemann surface Σ where punctures represent the external legs of the string diagram hosting incoming/outgoing external states. Physical amplitudes are then obtained by suitable vertex-wave function insertions that we will review later on.

Therefore, one has the partition function, or better a pre-amplitude without wave function insertions, as

$$Z = \sum_h g_s^{2h+n-2} \int_{\overline{\mathcal{M}_{h,n}}} d(M) \mathcal{N} \sum_\nu \int D[x,\theta] e^{-S_{GS}[x,\theta]} \tag{4.1}$$

where $\overline{\mathcal{M}_{h,n}}$ is the moduli space of n-punctured Riemann surfaces with genus h and $d(M)$ the relevant Mandelstam measure on it and on gravitino zero modes. The sum $\sum_h$ is over the genus of the world sheet, weighted by the usual $g_s^{-\chi}$ perturbative factor where g_s is the string coupling constant. $\mathcal{N}$ is a normalization and kinematical factor and $\sum_\nu$ is the sum over the 2^{2h} spin structures.

Let us now recall the results in MST obtained in the previous chapter. We found that the $g \sim \infty$ asymptotic of the partition function is

$$Z_{[\pi^{in}][\pi^{out}]}(g) \quad \sim \quad \sum_h (1/g)^{-\chi} \int_{\mathcal{M}^H_{h,n}} (dm\mu') e^{-S^{(4,4)}_{cl}} \sum_\nu \int D[x,\theta]\, e^{-S_{GS}[x,\theta]}, \quad (4.2)$$

where $\mathcal{M}^H_{h,n}$ is the space of instantons at strong coupling whose spectral curve has genus h and reproduces along its punctures the relevant asymptotic cycles, $\mu' = \mu \cdot \frac{\mathrm{Det}'\Delta^M_0}{\mathrm{Det}'\Delta^M_1}$ is the remnant of the decoupled Maxwell sector and the field theory induced measure on $\mathcal{M}^H_{h,n}$ and on fermionic moduli, $\chi = 2 - 2h - n$ is the Euler characteristic of Σ, $\sum_\nu$ is the sum over the spin structures induced by the ambiguity of the lifting procedure of the worldsheet fermions and S_{GS} is the Green-Schwarz superstring action. With respect to the previous chapter we dropped the tildes from the fields just to simplify the notation.

Let us compare now the partition functions (4.1) and (4.2) and trying to identify them. This can be done if we identify the inverse YM coupling with the string coupling (in $l_s = 1$ units) as $g_s = g^{-1}$. Moreover, once the instanton spectral curve has been identified with the string worldsheet, the large N limit moduli space of $U(N)$ Hitchin pairs on the cylinder whose spectral curve has genus h and n punctures has to be identified with the moduli space of stable curves of genus h and n punctures. The measures on the two moduli spaces have to be compared together with the relevant normalization factors. Moreover, the fermionic moduli have to be compared with gravitino zero modes.

One should also be interested in understanding if the wave function operators which are inserted in the superstring partition function to construct physical amplitudes do have a MST counterpart.

4.1.2 The interpretation of the spectral curve and of the punctures

We interpret the Riemann surface defined by the relevant branched covering of the cylinder as the classical carrier of a string process, i.e. with the string world-sheet. The polynomial equation $P(x,z) = 0$ which describes it as embedded in the (x,z) space $\mathbf{C} \times \mathbf{C}^*$ is interpreted as an intrinsic representation of the curve rather than a target embedding. In fact, once the lifting procedure of the Cartan fields has been performed, the only remnant of the branched covering representation of the spectral curve concerns its moduli and the eight lifted x^i fields represent its actual physical target space embedding. We will see along this section how this moduli representation exactly fits the Mandelstam diagram parameters. The branch points represent joining and splitting processes of strings of different lengths. Generically we have the joining of two strings to form a unique string or the splitting of one string into two. We may also have multiple branch points in which more then two incoming or outgoing strings are involved at the same light-cone time.

The inverse images under Π of $z = 0$ and $z = \infty$ are punctures in Σ with a definite string interpretation: they represent incoming and outgoing strings, respectively. More precisely they represent the points where incoming strings enter (outgoing strings leave) the process represented by the Riemann surface Σ.

Let us discuss further properties of the punctures corresponding to $z = 0$ (an analogous discussion holds for $z = \infty$). The counter images of $z = 0$ by Π may be N distinct points, i.e. the solutions of the algebraic equation (2.13) at $z = 0$ may be all distinct. In such a case we say we have N small incoming strings (of length 1 each). However, in general, the inverse image of $z = 0$ may contain several branch points $P_1^{in}, \ldots, P_s^{in}$, with multiplicity $n_1, \ldots, n_s$, respectively. In this case the process represented by Σ involves s incoming strings of length $n_1, \ldots, n_s$. The physical interpretation of the string length has been given in Chapter 1. In the framework of the light-cone quantization of type IIA superstring, the string length is identified with the momentum component in the light-cone direction of the string in suitably normalized units.

4.1.3 Mandelstam diagrams

As we already remarked, in the light-cone picture one starts by directly fixing light-cone gauge direction from the very beginning by equating the time-like string world-sheet direction and the light-cone time. This is done by choosing an appropriate metric on the world-sheet which gives a particular representation of its moduli. Let us review this point in view of a comparison with MST.

Let $\tilde{\Sigma}$ be a compact Riemann surface of genus h and let ω_I, $I = 1, \ldots, h$ be a set of holomorphic differentials on $\tilde{\Sigma}$ normalized by $\oint_{\alpha_J} \omega_I = \delta_{IJ}$, while $\oint_{\beta_J} \omega_I = \Omega_{IJ}$ is the period matrix. We fix n punctures $\{Q_1, \ldots, Q_n\}$ on $\tilde{\Sigma}$ and define the divisor $D = Q_1 \cdot \ldots \cdot Q_n$. We also introduce a set of n real numbers $R = \{r_1, \ldots, r_n\}$ such that $\sum_i r_i = 0$.

Now, let ω be the differential which is holomorphic on the punctured surface $\Sigma = \tilde{\Sigma} \setminus D$ with simple poles at D with $\text{res}_{Q_i}\omega = r_i$ and $\text{Re} \oint_{\alpha_I} \omega = 0 = \text{Re} \oint_{\beta_I} \omega$.

The above differential defines a nice procedure which allows us to look at Σ as a *topological* covering of a cylinder: one can easily decompose Σ into pants along the

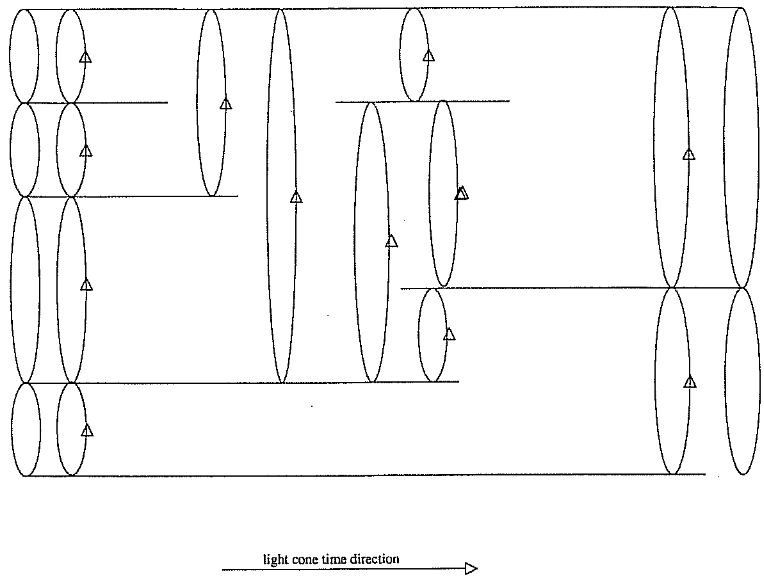

Fig. 6: *A typical Mandelstam diagram. The pants decomposition is shown together with the relevant cycles defining the Mandelstam moduli parameters.*

level lines of the function $\tau(P) \equiv \mathrm{Re} \int^P \omega$ which is the light-cone time. In this sense, ω induces on Σ the structure of a Mandelstam diagram (see *Fig.* 6.).

The Mandelstam parameters are the twist-angles θ_b, $b = 1, \ldots, 3h + n - 3$, along the junctures of the pants decomposition and the relative light-cone time coordinates $\tau_a - \tau_0$, $a = 1, \ldots, 2h + n - 3$, of the $2h + n - 2$ interaction points. h additional real parameters are the internal light-cone momenta $p_I^+ = \oint_{\alpha_I} \omega$. Altogether they form a set of $6h - 6 + 2n$ real parameters. In [47] it was shown that these parameters represent good coordinates on the moduli space of genus h Riemann surfaces with n punctures $\mathcal{M}_{h,n}$.

To complete the picture the elements in the set R are identified with the relevant components of the external light-cone momenta of the diagram, i.e. the periods of ω around the punctures. We also have the relations $\oint_{\beta_I} \omega = \frac{i}{2\pi} p_K^+ \mathcal{W}^I_{Kb} \theta_b$, where $\mathcal{W}^I$ are integer-valued matrices which depends on the pants decomposition of the Riemann surface and its intersections with the α and β cycles.

In superstring theory there is also a fermionic counterpart of the moduli space which is parametrized by the space of zero modes of the gravitino field. This is the space of harmonic 3/2 differentials whose dimension is equal to minus the Euler characteristic of the world-sheet surface $-\chi_\Sigma = 2h - 2 + n$ by Riemann-Roch theorem.

4.1.4 Mandelstam diagrams from MST

What we are now going to do is to examine the consequence of the main new input from MST, that is the holomorphicity of the covering map which defines the would be Mandelstam diagram. The result will be a set of constraints on the kinematical data of the diagram which turn out to be a quantization condition for some of the Mandelstam parameters. In the large N limit these constraints loose their effectiveness and allow us to recover the full moduli space of the string diagrams.

Our strategy now is the following. We first construct an explicit form for ω, in terms of the prime-form, the ω_I's and the period matrix of Σ. Then we compare this ω with the one that comes from MST. The relevant new input consists in the fact that MST induces on Σ the structure of a *holomorphic* covering of the cylinder. By this we mean that, if $z : \tilde{\Sigma} \to \mathbf{CP}^1$ is the covering map in the MST scheme, the coordinate z is a meromorphic function on Σ and hence holomorphic on the cylinder. The role of ω in MST is played by $d\ln z$, therefore we have to identify them. This condition becomes a constraint on the data of the Mandelstam diagram. In fact, it means that $D^R \equiv Q_1^{r_1} \cdot \ldots \cdot Q_n^{r_n}$, being the divisor of the meromorphic function z, is a principal divisor on Σ, so that in particular $r_i \in \mathbb{Z}$. As a consequence, some constraints appear in the data of the Mandelstam diagram and these conditions induce a complex co-dimension h slicing of the moduli space. This can be seen as follows.

Let $\omega_{P_+P_-}$ be the holomorphic differential on $\tilde{\Sigma} \setminus \{P_+, P_-\}$ with simple poles at $P_\pm$ with residues ± 1 and imaginary periods. It can be written as [1]

$$\omega_{P_+P_-}(P) = d_{(P)}\ln\left[\frac{E(P,P_+)}{E(P,P_-)} \cdot \mathrm{e}^{2\pi i \mathrm{Im} \int_{P_-}^{P_+} \omega_I \Omega^{(2)}{}^{-1}_{IJ} \int^P \omega_J}\right] = d_{(P)}\ln H(P, P_+, P_-)\,, \quad (4.3)$$

where $E(P,Q)$ is the prime form on $\tilde{\Sigma}$, $\Omega^{(2)}$ is the imaginary part of the period matrix and $d_{(P)} = dP \cdot \frac{\partial}{\partial P}$.

In terms of the above differentials we can write

$$\omega = \sum_{l=1}^{n-1} k_l \omega_{Q_l Q_{l+1}}\,, \quad (4.4)$$

where $k_i - k_{i-1} = r_i$ and $k_0 = 0 = k_n$; substituting (4.3) into (4.4) we obtain

$$\omega(P) = d_{(P)}\ln\tilde{z}(P)$$

where

$$\tilde{z}(P) = \prod_{l=1}^{n-1} \left[H\left(P, Q_l, Q_{l-1}\right)\right]^{k_l}. \quad (4.5)$$

[1] $H(P, P_+, P_-)$ depends also on a base point which is irrelevant in this context and, for the sake of simplicity, is not specified.

Now, as anticipated above, we make the identification $\omega = d\ln z$. This requires that $\tilde{z} = z$ up to a multiplicative constant, which implies that $\tilde{z}$ is a well defined meromorphic function on $\tilde{\Sigma}$. On one hand this imposes that the residues r_l be quantized in integer values while, on the other hand, it requires that the differential $d\tilde{z}$ has vanishing periods along α and β cycles. The latter condition is fulfilled iff

$$\sum_{l=1}^{n} r_l \int^{Q_l} \omega_I = m_I + n_J \Omega_{JI} \tag{4.6}$$

for some m_I, $n_I \in \mathbf{Z}$. At this point the situation is clear: (4.6) is the vanishing condition for the Abel map and says that the divisor D^R is principal.

Conversely, let z be a meromorphic function on $\tilde{\Sigma}$ and D^R its divisor. By definition (4.6) holds and $\mathrm{res}_{Q_l} d_{(P)} \ln z = r_l$.

Notice that the periods of ω are quantized in integral values as

$$\oint_{\alpha_I} \omega = 2\pi i n_I \quad \text{and} \quad \oint_{\beta_I} \omega = -2\pi i m_I\,, \tag{4.7}$$

and this condition is equivalent to (4.6).

Eq. (4.7) means that the internal light-cone momenta of the diagram are quantized and that, in addition, there are h discretizing constraints on the twist-angles of the Mandelstam diagram. Since these variables, together with the relative interaction times which have been left untouched, are the coordinates of the moduli space, we are left with a discrete slicing of the moduli space $\mathcal{M}_{h,n}$, each slice being of complex dimension $2h-3+n$. This discretized moduli space is the fixed genus stratum $\mathcal{M}^H_{h,n}$ of the Hitchin moduli space on the cylinder.

In the large N limit, however, the quantization condition disappears in a continuum of values [2]

$$\lim_{N\to\infty} \frac{1}{N} \left[\mathbf{Z}^h \bigoplus \Omega \mathbf{Z}^h\right] = \mathbf{C}^h\,. \tag{4.8}$$

Simultaneously, for large N also the upper bound $\frac{1}{2}(N-1)(N-2)$ on the genus of the plane curves in MST, becomes ineffective, and we recover the full moduli space of string theory.

It is interesting to review the genus 0 and 1 cases in detail.

On the sphere the prime form is simply $E(P,Q) = P - Q$ and then, up to a multiplicative constant,

$$z = \prod_{i=1}^{n} (P - Q_i)^{r_i}\,. \tag{4.9}$$

[2] In some sense, the topological reconstruction of the Mandelstam diagram can be seen as an infinitely-sheeted holomorphic covering of the cylinder.

Splitting the divisor $D^R = D_0 \cdot D_\infty^{-1}$ into its zero and polar parts — where $D_0 = \prod_{r_i>0} Q_i^{r_i}$ and $D_\infty = \prod_{r_i<0} Q_i^{-r_i}$ are its positive and negative parts respectively — we get

$$z = \frac{\prod_{i|r_i>0}(P-Q_i)^{r_i}}{\prod_{i|r_i<0}(P-Q_i)^{-r_i}} \tag{4.10}$$

and the independent complex parameters are $n-3$. Therefore in this case there is no moduli quantization at all and the moduli of plane curves cover the full moduli space $\mathcal{M}_{0,n}$. These curves correspond to tree level Mandelstam diagrams with n external strings each of light-cone momentum r_i. Therefore, in this case, MST gives exact results at finite N (except for the fact that the + components of the external momenta are discrete). To relate (4.10) to the plane curve form, it is sufficient to fix one of the punctures at $z = \infty$ and $X = \infty$ to reach an equation of the form $P_{in}(X) - zP_{out}(X) = 0$ where P_{in} is monic in X and of degree N while P_{out} is of degree lower than N.

The first case in which moduli quantization becomes effective is at $h = 1$. So, let us specialize to the torus the above construction. In this case, the prime form is proportional to the odd theta function $E(P,Q) \propto \Theta_{odd}(P-Q|\tau)$ and there is a unique holomorphic differential $\omega_1 = dP$. Therefore

$$\omega_{P_+P_-}(P) = d_{(P)}\ln\left[\frac{\Theta_{odd}(P-P_+|\tau)}{\Theta_{odd}(P-P_-|\tau)} \cdot \exp\left(2\pi i\mathrm{Im}\,(P_- - P_+)\,(\mathrm{Im}\tau)^{-1}(P-P_0)\right)\right],$$

and we get

$$\omega(P) = \sum_{l=1}^{n-1} k_l\omega_{Q_lQ_{l+1}}(P) = d_{(P)}\ln\prod_{l=1}^{n}\left[\Theta_{odd}(P-Q_l|\tau)\right]^{r_l} \cdot e^{r_l 2\pi i\mathrm{Im}(Q_l)(\mathrm{Im}\tau)^{-1}P} = d_{(P)}\ln z\,.$$

Using the standard modular properties of Θ-functions one obtains that z is a well defined meromorphic function on the torus $T_\tau \equiv \frac{\mathbf{C}}{<1,\tau>}$ iff $r_l \in \mathbf{Z}$ and

$$\sum_{l=1}^{n} r_l Q_l = m + n\tau \tag{4.11}$$

for some integers n and m. This is the discretizing condition for the genus one case. Due to the genus upper bound, the first case in which one can encounter such a type of situation in MST is $N = 3$. Consider, for example, the following curve

$$X^3 + 3\alpha zX + 2z(z-1) = 0 \text{ with } \quad \alpha^3 \neq 0, 4$$

whose (extended) branching locus is $B = \{0^2, z_+, z_-, \infty^2\}$, where $z_\pm$ are the solutions of $(1-z)^2 + \alpha^3 z = 0$. This corresponds to a genus one curve with two punctures at

$(z, x) = (0, 0)$ and $(z, x) = (\infty, \infty)$ associated to incoming/outgoing strings of length 3. Along the loop one unit of light-cone momentum is circulating.

In Chapter 2 we encountered also the fermionic counterpart of the moduli space of Hitchin system on the cylinder which is the space of fermionic zero modes. As we have shown, its dimension equals $2h + n - 2$ which coincides with the number of two dimensional gravitino zero modes. This strongly suggests to pair them while comparing MST with G-S superstring theory.

Let us here touch another aspect of the approximation of the full moduli space of curves by means of the moduli space of the Hitchin system on the cylinder. What we want to show here is that the approximation is finer and finer as long as N increases. What one can show is, in fact, that the Hitchin moduli space of $U(N)$ pairs (A, X) contains properly the moduli space of $U(N')$ pairs (A', X') if $N > N'$. This is a consequence of the simple following observation: along the possible $U(N)$ pairs, one can choose the subset whose spectral curve is reducible as the characteristic polynomial factorizes $P_X = P_{X'}P_{X''}$. These correspond to $U(N)$ gauge bundles obtained as direct sums of $U(N')$ and $U(N'')$ bundles, where $N'' = N - N'$. Therefore one can decompose, in this particular case, the Hitchin pair as $(A, X) = (A', X') \oplus (A'', X'')$ and the $U(N)$ Hitchin equations factor in the two $U(N')$ and $U(N'')$ sub-cases. This shows the existence of a proper embedding. Notice, moreover, that as long as N is increased one generates a finer and finer discretization of the moduli space of punctured curves since there are more and more possible solutions to eq. (4.6). Therefore, as long as one allows a greater variety for the values of the length of the incoming/outgoing strings, together with the increasing of the number of sheets which compose the covering, one has more and more possible values of the internal loops light-cone momenta and of the twisting angles.

4.1.5 The compactification of the moduli space

A classical problem in string theory is that of the compactification of the moduli space of curves. As it has been analyzed by several authors [63, 55, 1], the true moduli space one has to integrate over is the Deligne-Mumford compactification of it. This means that one has to integrate over a compact completion of the moduli space which includes all possible singular curves obtained by shrinking to zero the length of non contractible cycles. These curves are in fact singular and they represent the boundary of this compactified moduli space. One can represent the compactification as the union of the moduli space of smooth curves plus a boundary component as

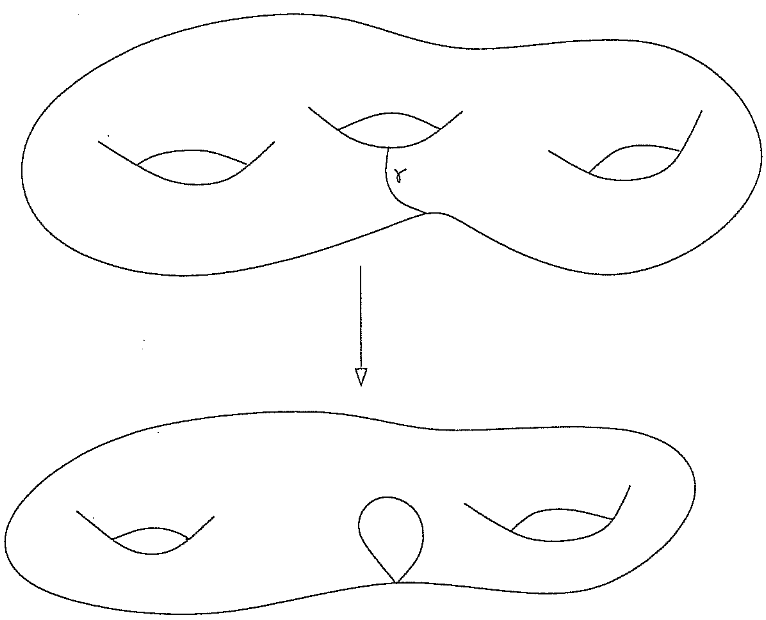

Fig. 7: *The shrinking of the γ cycle generates the node.*

follows.

Consider a smooth curve point in $\mathcal{M}_{h,n}$ represented by the curve Σ and the complete metric with constant scalar curvature on it, that is the Poincare metric, as a reference metric. We can reach the boundary of the space of smooth curves by shrinking to zero geodesic length any non contractible loop $\gamma \in H_1(\Sigma, \mathbb{Z})$. This is called a shrinking cycle. In the vicinity of the shrunk cycle the curve develops a node, i.e. locally looks like a multiple cone. The geodesic length of the cycle l_γ is then a parameter which leads the curve to become singular as long as $l_\gamma \to 0$ and therefore is transverse to the boundary of the moduli space.

The boundary component can be differently composed depending on the connectivity of $\Sigma \backslash \gamma$. If γ divides Σ in L disjoint components, then the boundary component which is reached is of type $\mathcal{M}_{h',n'} \times \mathcal{M}_{h'',n''} \times \ldots \times \mathcal{M}_{h^{L'},n^{L'}}$. One can disentangle this shrinking procedure into elementary steps by taking γ a connected not self intersecting loop. In this case there are two possibilities: if $\Sigma \setminus \gamma$ is connected, then the boundary component is of type $\mathcal{M}_{h-1,n+2}$, while if $\Sigma \setminus \gamma$ has two connected components – and they cannot be more than two – the boundary component is of type $\mathcal{M}_{h-h',n-n'+1} \times \mathcal{M}_{h',n'+1}$ for some h' and n'. Here the new punctures have to be considered as identified point on the surface corresponding to singular points (see *Fig.* 7.).

Explicitly, consider a neighborhood collar along γ. We can represent it by choosing local coordinates in which it can be given as $\{s_1 s_2 = l_\gamma\}$ with s_i being in some not too big complex disks. As long as we shrink the cycle γ we approach the node $\{s_1 s_2 = 0\}$ which is given by the union of two complex disks where we identified their centers.

The Deligne-Mumford compactification of the moduli space of curves $\overline{\mathcal{M}_{h,n}}$ is the closure of $\mathcal{M}_{h,n}$ under the above boundary operation and one has that the boundary $\partial\overline{\mathcal{M}_{h,n}} = \overline{\mathcal{M}_{h,n}} \setminus \mathcal{M}_{h,n}$ is composed by a factor of type $\overline{\mathcal{M}_{h-1,n+2}}$ plus factors of type $\overline{\mathcal{M}_{h-h',n-n'+1}} \times \overline{\mathcal{M}_{h',n'+1}}$.

The curves in $\overline{\mathcal{M}_{h,n}}$ are called *stable* and can be described by means of a suitable extension of the concept of two dimensional manifold. Stable curves are equipped with an atlas of open sets which are given by a finite number of disks with their centers identified, i. e. a multi-cone. It is clear that about a smooth point the number of disks is one and one recovers the usual two dimensional manifold local structure.

4.1.6 Singular curves from MST

The spectral curve Σ of stringy instantons are plane curves, i. e. they are represented as polynomial curves. As we saw at the end of Chapter 2, plane curves can be singular. Let us discuss a realistic point of view to link the two perspectives.

What we want to study is how to read the shrinking procedure that we reviewed above in the language of plane curves. The problem at hand concerns the following: the shrinking procedure has been defined with respect to the length of cycles measured with the constant curvature metric on the surface while singular branching covering structures, i.e. the Mandelstam diagram, are defined by means of the Mandelstam metric, i.e. the Mandelstam moduli parameters. The shrinking procedure is well described in the Fenchel-Nielsen coordinates, or better in their generalization to punctured surfaces, in which one uses as moduli parameters the geodesic lengths of the simple boundary cycles relative to a pant decomposition of the relevant Riemann surface together with the twist-angle around them. Singularities in the Mandelstam diagrams arise if some cut is shortened to zero length (and the resulting branch points then moved to coincide) and/or if some internal light-cone momenta are zero. In both cases we reach the boundary of the moduli space. The difference is just that we are describing the same phenomenon in different coordinates of the same moduli space [47].

Of course, as long as N is finite some directions in the moduli space could be not viable due to the discretizing conditions which constrain the possible deformations.

However in the large N limit these constraints are no more effective.

As we said in Chapter 2, singularities of the plane curve lower the genus on the surface. This is described by the genus formula

$$h = \left(\frac{1}{2}(N-1)(N-2) - \delta_{z=0} - \delta_{z=\infty}\right) - \sum_s \delta_s$$

where the the sum runs over the singular points of the curve with $\delta_s \in \mathrm{N}_+$ is the contribution of each singular point and $\delta_{z=0}$ and $\delta_{z=\infty}$ are the contributions from the formal singularity at the punctures. Let us recall that, due to the fact that punctures are not points of the curve, even if they are formally singular from the point of view of the compact plane curve, they contribute by lowering the genus of the curve by leaving it smooth. In general, δ_s depends on the specific way in which the tangent lines at s proliferate and superpose and can be calculated from the Puiseux expansion of the curve about the point. The analogous of the simple shrinking procedure that we meet above is the ordinary double point contribution. A singular point is said to be an ordinary double point if it has two distinct tangent lines. The local prototype of such a point is the nodal cubic curve $(x-x_0)^3 + (x-x_0)^2 - (z-z_0)^2 = 0$. In such a case the singular point contribution is calculated to be $\delta_s = 1$ and corresponds to the shrinking of a connected not self-intersecting cycle.

4.1.7 Physical String Amplitudes

Physical superstring amplitudes are usually calculated as vertex insertions in the path-integral. However we have to compare MST with the superstring formulation with punctures in the light-cone. Schematically one has

$$A_{ss}(1,2,\ldots,L) = \\ = \sum_h g_s^{2h+n-2} \int_{\mathcal{M}_{h,n}} d(M)\mathcal{N} \sum_\nu \int D[x,\theta] e^{-S_{GS}[x,\theta]} \prod_{p=1}^{L} W_p(Q_p, E_p) \qquad (4.12)$$

where $W_p(Q_p, E_p)$ are wave-functions of the world sheet fields depending on their specific structure in the external state that one wants to single out, on the values of its kinematical data E_p (i. e. momentum and polarization tensor) and on the puncture position on the worldsheet which is integrated over as a coordinate parameter of the moduli space of curves. There is a simple wave function/vertex operator correspondence which reads as follows. The generic vertex is usually taken to be the light-cone reduction of $V(E) = \int_\Sigma d^2\xi u_{\xi\bar{\xi}} e^{ik_\mu x^\mu}$ where k_μ is the momentum and $u_{\xi\bar{\xi}}$ is a local polynomial in the derivatives of x^μ – with equal number of ∂_ξ and $\partial_{\bar{\xi}}$ –

uniquely completed by the addition of the relevant terms to fulfill the supersymmetry invariance of the V vertex. In particular, for massless states $u_{\xi\bar{\xi}}$ is quadratic in the first derivatives. The relative wave function is, up to the exponential of a boundary term about the puncture location, the integrand of the vertex operator $V(E)$ calculated on the relative puncture location (see [41, 1] for more details). In the puncture formalism the integration over the position of the insertion is provided by the integral over the moduli space.

We can try to mimic this structure by defining an analogous one in MST as

$$A_{MST}(1,2,\ldots,L) = \int_{[in/out]} D[A, X^i, \Theta]\, \mathrm{e}^{-S_{MST}} \prod_{p=1}^{L} W_p^{MST}[X,\Theta,A](Q_p, E_p)\,, \quad (4.13)$$

where W_p^{MST} are some analogous of the above wave functions. These have to be supersymmetry and gauge invariant.

Keeping the strong coupling procedure and the lifting mechanism in mind, we are immediately led to guess their form to be something like $W^{MST}[X,\Theta,A](E) = \mathrm{Tr} U_{w\bar{w}} e^{ik_i X^i}$ where $U_{w\bar{w}}$ is a local symmetric polynomial in the covariant derivatives of X^i and in the ω_w's again uniquely completed by the addition of the relevant terms to fulfill the supersymmetry invariance of the W^{MST} wave function.

Let us keep this ansatz and see what happens in the strong coupling limit. Under the relevant rescalings (3.3) we have $X^i = X^{oi} + O(1/g)$ and

$$D_w X^i = D^{(b)} X^{oi} - i\mathrm{ad}_{X^{oi}} a_w^{\mathrm{n}} + i\mathrm{ad}_{a_w^{t}} x^{\mathrm{n}i} + O(1/g)\,.$$

Note that, if it were not for the additional commutators in the previous formula, everything would be nice at this point with $U_{w\bar{w}}$ being the symmetrized reduced $u_{\xi\bar{\xi}}$ polynomial. Their effect is to potentially generate contact terms in the vertices as a consequence of the integration along the non Cartan sector. What we need to do is to show if these potential contact terms spoil the local structure of the wave function insertions or if their effect can be kept into account by some simple subtraction.

To see this, recall that the action term in the non-Cartan fluctuation is

$$Q_{\mathrm{n}} = \frac{1}{\pi}\int d^2w \mathrm{Tr}\left[x^{i\mathrm{n}} \mathcal{Q} x^{i\mathrm{n}} + a_{\bar{w}}^{\mathrm{n}} \mathcal{Q} a_w^{\mathrm{n}} + \ldots\right]\,,$$

where

$$\mathcal{Q} = \mathrm{ad}_{X^{oi}} \cdot \mathrm{ad}_{X^{oi}} + \mathrm{ad}_{a_{\bar{w}}^{t}} \cdot \mathrm{ad}_{a_w^{t}}$$

and the dots refer to terms which are not relevant now. Since Q_{n} is purely quadratic in the ϕ^{n} fluctuations, they can be easily integrated over since the path-integral is Gaussian. Moreover, since $\mathcal{Q}$ is purely algebraic in the fields, the relative contact terms are absolutely localized. Therefore their effect can be of two types: a self contraction inside the single wave function or a contraction between two wave functions located at colliding punctures. (At this point, keeping into account also the supersymmetry of the model, one could suspect that these contact terms do completely cancel. We didn't checked this point.) All that means, anyhow, that (4.13) is correct up to some potential local counter-terms in the structure of the wave function insertions that we have potentially to keep into account to get rid of these contact terms.

What we said until now is strictly true for massless states only. In fact, multiple covariant derivatives $D^p_w D^q_{\bar{w}} X^i$ insertions do not have a straightforwardly well defined strong coupling limit as the single covariant derivatives. This means that for massive states the above ansatz may be too naive and needs to be better refined or that some unseen cancellations of diverging terms due to supersymmetry take place.

We are not going to further develop this subject by contenting ourselves in stating the existence of the MST analog of wave function insertions. Let us just notice that a similar problem with massive states vertices arises also in [41] while proving the equivalence of the light-cone and the covariant string scattering amplitudes. In the latter case, the problem is solved by using the DDF construction [54], which provides a translation dictionary between covariant and light-cone vertices, applied to the factorized amplitudes. It may be that some analogous construction would help also in this case. This question wouldn't arise if we had a covariant M-theory formulation and hence a covariant MST.

4.1.8 Odd points

In the above comparative section we tried and convince the reader of the fact that MST does indeed represent a discretized version of type IIA superstring theory.

The most evident odd points at this phase of this program are, from our point of view, the following ones.

The measure on the moduli space We didn't even try and exploit the actual form of the measure on the moduli space of $U(N)$ Hitchin pairs on the cylinder [3]. It is field theoretically induced by the path integral definition of MST and one should be able to show that it really reduces to a discretized form of the Mandelstam measure on the moduli space of curves. In any case, let us observe that it is out of question

[3]Actually one can use some two dimensional field theory techniques to work out the measure, but the resulting form does not still take a form suitable to be lifted to the world-sheet

at this point that the large N limit of MST do represent a closed string theory very close to type IIA. If another measure would arise it would then mean that there does not exist a unique way in resumming moduli in string theory and that MST defines a new and unknown superstring theory. We prefer to think that such kind of other string theories do not exist. From a technical point of view, let us notice that a direct calculation of the actual measure may be singled out by means of localization methods applied to hyperKahler quotients [4]. One can, anyhow, guess the result to be given by the string measure on the full moduli space with δ insertions implementing the h discretizing conditions on the Mandelstam parameters. This kind of problems are, in any case, of a strong mathematical difficulty: the (same) problem for the measure on the ADHM moduli space is in fact a long standing one and the most recent account about it is still based on an ansatz [61].

Supermoduli In Chapter 2 we counted the number of fermionic zero modes in MST and during this section we identified them with the gravitino moduli in superstring theory. One should study more precisely this correspondence and if the dependence on these parameters generates the G-S analog of the fermion vertex insertions.

Physical string amplitudes In the last subsection we failed in giving a complete account of the MST way of calculating physical string amplitudes for massive states.

Let us here add the following observation which is due to a sake of completeness. In Chapter 3 we found for the classical action the following expression

$$S_{cl}^{(4,4)} = \frac{1}{2\pi}\int_\Sigma \partial\tilde{\bar{x}}_{cl}\bar{\partial}\tilde{x}_{cl}\,,$$

where $\tilde{x}_{cl}$ is the lifting to Σ of the spectrum of the $X^{(b)}$ field as a scalar field. This enters the path integral as a multiplicative factor $\mathrm{e}^{-S_{cl}^{(4,4)}}$. By definition $\tilde{x}_{cl}$ is the harmonic scalar on Σ with boundary conditions around the punctures such that $\oint_{Q_i} d\tilde{x}_{cl} = r_i$. It follows at this point that $d\tilde{x}_{cl} = \omega$ and then

$$S_{cl}^{(4,4)} = \frac{1}{2\pi}\int_\Sigma \omega\bar{\omega} = \frac{1}{2\pi}\mathcal{A}_\Sigma^M$$

equals the area of the surface measured by means of the Mandelstam metric. Recalling that $\omega = \sum_i r_i \omega_{Q_i,P_0}$, where P_0 is an arbitrary point in Σ, we get that

$$S_{cl}^{(4,4)} = \frac{1}{2\pi}\sum_{ij} r_i r_j G_{P_0}(Q_i, Q_j)\,,$$

[4]The moduli space of Hitchin pairs is an hyperKahler quotient defined as the space of (X, A) pairs modulo the Hitchin equations which are interpreted as a triple composing the hyperKahler momentum map [44].

where $G_P(Q,T) = \int_\Sigma \omega_{Q,P}\bar{\omega}_{T,P}$. The Riemann bilinear identity [36] for this particular case gives $G_P(Q,T) = \mathrm{Re}\int_P^Q \omega_{P,T}$. Actually, the function $G_P(Q,T)$ is the Green function of the scalar Laplacian on the Riemann surface, as can be seen by considering

$$\Delta_Q G_P(Q,T) = -2\frac{\partial}{\partial \bar{Q}}\omega_{PT}(Q) = 2\pi(\delta(Q,T) - \delta(Q,P))\,.$$

Therefore, the factor $e^{-S_{cl}^{(4,4)}}$ which appears in the MST path integral has again a direct counterpart in the kinematical factors one encounters in calculating amplitudes in superstring theory. Notice the striking fact that such a term could never arise from the path integral along the transverse Cartan fluctuation.

Apart what we said, a detailed account of all the kinematical and normalization factors is still lacking.

We hope we will be able in solving the above problems with some further efforts. In any case we consider the arguments we gave above to be sufficient to believe that MST does work fine in reconstructing perturbative superstring theory.

4.2 MST as a nonperturbative string theory

Once the path integral formulation of perturbative string theory had been settled, it posed the question of the nature of the perturbative genus expansion of string theory. It is soon clear that the genus sum

$$\mathcal{Z}_n(g_s) = \sum_h g_s^{n+2h-2} \int_{\overline{\mathcal{M}_{h,n}}} \mathcal{Z}_{h,n}$$

has to be interpreted, as most perturbative expansions, as an asymptotic series representation of the actual function of the string coupling about $g_s \sim 0$. In this situation one may be interested to know if string theory admits a nonperturbative formulation, i.e. a formulation in which the genus expansion does not take place from the very beginning, but is a subsequent effect of a perturbative evaluation of the functions of the string coupling.

If such a theory exists, then it has been conjectured [55] that the genus expansion is the effect of a perturbative evaluation of integrals over a universal moduli space on

n-punctured Riemann surfaces of any genus $\overline{\mathcal{M}_n}$ as[5]

$$\mathcal{Z}_n(g_s) = \int_{\overline{\mathcal{M}_n}} \mathrm{e}^{-\ln g_s \chi_\Sigma} \mathcal{Z}'_h(m)$$

where χ_Σ is the Euler characteristic of the worldsheet and $\mathcal{Z}'_h(m)$ is the integrand over the putative universal moduli space. The actual form of the universal moduli space has been variously conjectured to be the space of inequivalent uniformization maps of the possible world-sheets [56] or an appropriate telescopic construction over the family of the moduli spaces at fixed genus obtained in an analogous way as the infinite dimensional sphere is generated recursively from finite dimensional ones [57].

MST gives a definite solution to this question. In fact we can compare the above point of view with the results coming from MST where a precise recipe is given in order to construct the universal moduli space. It is simply the moduli space of $U(N)$ Hitchin system on the cylinder in the large N limit with fixed asymptotic.

Another observation is in order. The interpretation of the genus expansion as a perturbative expansion of string theory regards the intrinsic two dimensional gravity sector of string theory. Due to the fact that the Einstein action equals the Euler characteristic χ_Σ of the worldsheet, two dimensional gravity is a topological theory. This was the reason why the above manipulations resulted well defined.

One of the formulations of 2D gravity has been given in terms of a matrix model [58] of the "old" type. Let us observe that in this theory one organizes perturbative expansions of matrix integrals as a ribbon graph expansion. Each ribbon graph Γ is associated with a triangularizion of a Riemann surface Σ via the identification of the ribbon graph with integral curves of a quadratic holomorphic Strebel differential $\mathcal{Q}$ on Σ with specific poles at the punctures and can be expressed as $\left(\frac{\partial_\xi f}{f}\right)(d\xi)^2$ for a meromorphic function of Σ with zeros and poles at the punctures. Notice that in MST something similar happens while expanding over stringy instantons. In fact, for each such an instanton, the spectral curve is associated to a meromorphic function z which is fully characterized by the branching cover prescription. The Riemann surface pants decomposition is then obtained as a slicing up along the integral curves of the holomorphic one-differential $\omega = \frac{\partial_\xi z}{z} d\xi$. Once we identify the two meromorphic functions, then an interesting link between "old" and "new" matrix models appear as one identifies $\mathcal{Q} \equiv \omega^2$ and one can read the two ways of generating the worldsheet as a first and second order respectively procedures. One can try to speculate about a

[5]The function may depend also on other external variable sources which are generating the relevant string observables. We do not write them down in order not to make formulas too dull reading.

more deep connection between the two. If this connection would be established, then it could represent a solution also to the problem of finding the actual measure on the moduli space of Hitchin systems on the cylinder. Another signal in that direction comes from the well known link of "old" matrix models with integrable systems. In [59] it was conjectured that the string partition function is to be considered as a τ-function of some integrable hierarchy. Moreover, the pre-potential theory for topological σ-models is directly communicating with integrable systems theory [58]. Far from being shortly linked to these facts, we only notice that in MST a very important role is played by the Hitchin system which is an integrable one. It remains an appetizing open question to understand if MST can represent a way to better characterize topological sectors of string theory and its links with rational curves enumeration.

If MST is a non perturbative formulation of superstring theory, we then expect that higher order corrections to the leading order expansion that we have calculated here, should represent non perturbative corrections to superstring theory. In particular one expects that (see also [38]) emission of $D0 - \bar{D}0$ virtual pairs takes place to correct the perturbative series [6].

From the MST point of view this phenomenon should be taken into account by sub-leading non-BSP classical solutions. In fact $D0 - \bar{D}0$ pair configurations break all supersymmetries. This is in accordance with the picture we found in which the less the supersymmetry is preserved and the less the configuration contributes to the MST path integral.

Notice that in such a class of configurations one expects that the electric field do play a special role [12, 38]. If this is true, it raises another problem about the meaning of the higher order terms in the expansion about stringy instantons. The answer seems to be that simply they vanish as is confirmed by a first analysis [64]. One can expect, in fact, that using localization arguments and supersymmetry the exactness of the semi-classical expansion we did can be proved.

In [7] it has been proposed that D1/F1 perpendicular bound state systems in type IIB should be described by topological vacua of the relative two dimensional gauge theory. This system is T dual to D0/F1 bound states in type IIA. Topological vacua on the cylinder are homotopy classes of field configurations whose gauge connection part is charged also under some background charges placed on the boundary of the two dimensional space-time, which in our case is the cylinder. As we saw at the beginning of chapter one, at the IR conformal fixed point MST reduces to a theory of

[6]Remember that only extended branes arise as a pure large N limit effect.

free superstrings on the big cylinders defined by the relative cycles in the equivalence classes of the Weyl group S_N and a decoupled Cartan gauge sector. As we saw later on, for BPS field trajectories this Cartan gauge sector is completely bound to be flat in the strong coupling limit. Such configurations correspond to pure strings. There could exist anyhow other configurations with non vanishing electric field which we interpret as D0/F1 bound states where the quantized value of the electric field $E = g^2 m$, where $m \in \mathbb{Z}$, represents D0 charge. We can then conclude that $D0 - \bar{D}0$ pair production have to be represented by appropriate electric field configurations. In [38] an attempt of describing this process has been performed. The outcome is in agreement with supergravity/superstring under the assumption that D0 charge production is a high energy effect which is strongly favored in the fragmented string phase. We assume the validity of this result. From this point of view D-particles should be described by electrically charged short strings. In the large N limit one expects that the stringy modes, while becoming infinitely massive, completely decouple from the center of mass degrees of freedom by leaving an effective particle like state. Therefore we obtain that electrically charged short strings should represent new degrees of freedom in string theory whose effect should be evident at high energies, while actual D0/F1 bound states should arise as electrically charged long strings.

Suppose that a theory is given in terms of an expansion of its scattering amplitudes. In order to know if this theory is complete, one has to investigate all the possible scattering configurations in the scattering data space. The range of validity of the given theory is defined to be the subspace where one is able to give a meaning to the series expansion (at least Borel summability). If the range of validity is reached then one usually has to improve the theory by adding new scattering channels related to new degrees of freedom.

High energy string scattering of massless states at fixed angle has been analyzed in [33] where it is shown to be dominated by the saddle point worldsheet configurations of the exponential factor

$$\mathrm{e}^{-E(m,p_i)} = \mathrm{e}^{-\sum_{i,j} p_i \cdot p_j G_m(\xi_i,\xi_j)}$$

varied along the moduli m. The results are not conclusive, but a class of saddle points has been identified. Given the in/out string configuration the genus h leading contribution should be given by an $(h+1)$-sheet cover of the relative tree level diagram. In the branched cover language this corresponds to consider the curves of the form

$$z^{h+1} = \frac{P_{in}(x)}{P_{out}(x)}$$

where the zeros of $P_{in/out}$ represent the asymptotic free strings. The value of the saddle point energy is given by

$$E(\bar{m}, p_i) = \frac{1}{h+1} E(p_i)|_{tree}$$

where $E(p_i)|_{tree}$ is the tree level saddle point contribution and becomes more and more important as the genus of the worldsheet increases. This is a signal of a tendency of strings to break in little fragments as long as the total energy of the scattering channel is increased. By keeping into account the full amplitude one obtains that in such a situation the perturbative genus expansion badly diverges, due to a $(h+1)^{9h+3}$ factor, and looses its sense opening to discussions about various possible nonperturbative contributions to the scattering amplitudes in such a phase. A similar bad behavior has also been found in string diffraction scattering at large s and fixed t [60].

In this regime *finite* N MST provides a natural upper bound to the genus expansion as well as to string fragmentation cutting off this divergence at the minimal string length 1. At this point $D0 - \bar{D}0$ pair production should be effective thus adding new nonperturbative channels to the string scattering amplitude which should no more diverge as we remove the cut off by taking N to infinity. It is an open question if this picture does really work. Let us only notice that the appearance of a new role of the electric field may change, for this kind of background solutions, the computation of the zero-modes rescalings in the Maxwell sector and thus may give a new structure to the genus expansion. This is exactly what we would call a correction to the perturbative expansion.

4.3 Heterotic Matrix String Theory

The purpose of this section is to extend to Heterotic Matrix String Theory (HMST),[66, 67, 68, 69, 70, 71, 72, 73, 74, 75] the results obtained before for Matrix String Theory.

HMST is an $O(2N)$ SYM theory with chiral (anomaly free) matter and $\mathcal{N} = (8,0)$ supersymmetry. The treatment parallels very much what has been already developed for MST, but some details are different and somewhat subtler due to the chiral nature of the theory. However the final result is similar to MST. For this reason we will only sketch the derivation of the results and refer the reader to [16] for a full exposition.

The HMST arises from the compactification of eleven-dimensional Matrix Theory on $S^1/\mathbb{Z}_2 \times S^1$. The orbifold structure of this manifold is implemented by a doubling in the rank of the gauge group $U(N) \to U(2N)$ followed by the relevant projection $U(2N) \to O(2N)$ which also chirally reduces the supersymmetries $(8,8) \to (8,0)$.

4.3.1 The model

The HMST action is

$$S = \frac{1}{\pi}\int_{Cyl} d^2w \left\{\mathrm{Tr}\Big(D_w X^i D_{\bar{w}} X^i + \frac{1}{4g^2}F^2_{w\bar{w}} - \frac{g^2}{2}[X^i, X^j]^2 + i(\theta_s D_{\bar{w}}\theta_s - \theta_c D_w \theta_c) - 2g\theta_s\Gamma_i[X^i, \theta_c]\Big) + i\chi D_w \chi\right\}, \tag{4.14}$$

where we used the same complex notation as in Chapter 2, X^i (with $i = 1, \ldots, 8$) are real symmetric $2N \times 2N$ matrix fields and A_0, A_1 are real antisymmetric $2N \times 2N$ matrices, i.e. in the Lie algebra $o(2N)$. The covariant derivative is therefore defined as $D_w X^i = \partial_w X^i - [A_w, X^i]$. $\theta_s(\theta_c)$ represents 8 $2N \times 2N$ symmetric (antisymmetric) matrices, whose entries are 2D real spinors, in the $\mathbf{8_s}$ ($\mathbf{8_c}$) representation of $SO(8)$. χ are 2D fermions transforming according to the fundamental representation of $O(2N)$ and the fundamental representation of $SO(32)$. They arise as the twisted sector of the above orbifold projection. All the fields are again *well defined*, i.e. periodic, on the cylinder.

The action (4.14) is chiral, however the fermion representations are such that the gauge anomaly vanishes. This is the condition which fixes the content of the twisted sector of the theory.

The HMST, in the strong coupling limit, is expected to represent heterotic string theories. We can enrich the content of (4.14) by introducing Wilson lines. The corresponding term in the integrand of (4.14) is $\chi B \chi = \sum_{a=1}^{32} \chi^a B_{ab} \chi^b$, where B is a real 32×32 antisymmetric constant matrix.

A specific choice of B leads to a theory in which, for example, $SO(32)$ is broken to $SO(16) \times SO(16)$. Via a T-duality transformation this theory can be related to the *ten-dimensional* $E_8 \times E_8$ heterotic theory broken down to $SO(16) \times SO(16)$ by another suitable Wilson line, see [72, 73, 74, 75]. In the absence of Wilson lines it is expected to represent $SO(32)$ heterotic string theory compactified to nine dimension on a very small circle (see *Fig.* 8.). All this, as well as the S-duality connection with type IA and type IB theories [76, 77], is well-known. However the introduction of Wilson lines does not affect our subsequent derivation. Therefore we will drop them throughout and comment about them at the end of the section.

4.3.2 The strong coupling limit

In this subsection we review the strong coupling limit of HMST. The discussion starts with a naive strong coupling limit which will lead us directly to the core of the problem.

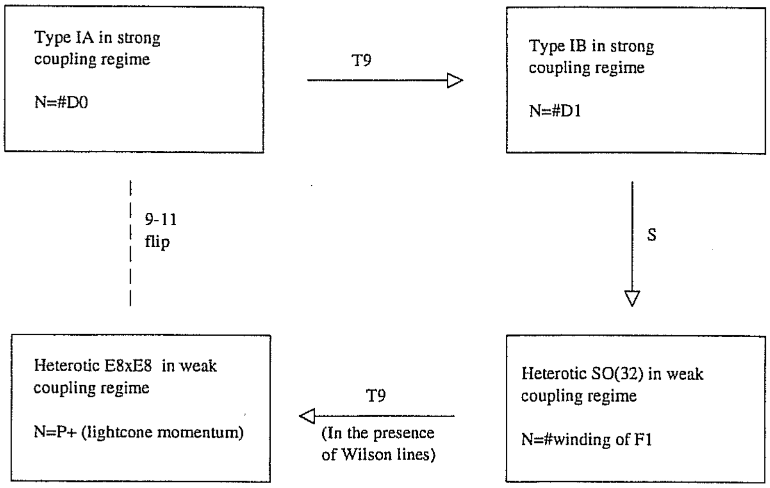

Fig. 8: *The relevant duality chain for HMST.*

The naive strong coupling limit ($g \to \infty$) in the action, after rescaling $A \to gA$, tells us that all the X^i, $A_w, A_{\bar{w}}$ and θ commute, and $\chi A_w \chi = 0$.

Denoting with a bar the fields in the strong coupling limit, one sees that there are two types of solutions. Let us denote by $\mathbf{1}$ the 2×2 identity matrix and $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The first type of solution is: $\bar{A}_w = \bar{a} \otimes \epsilon, \bar{\theta}_c = \bar{\vartheta}_c \otimes \epsilon$, $\bar{X}^i = \bar{x}^i \otimes \mathbf{1}$ and $\bar{\theta}_s = \bar{\vartheta}_s \otimes \mathbf{1}$, where $\bar{a}, \bar{x}^i, \bar{\vartheta}_c$ and $\bar{\vartheta}_s$ are diagonal matrices. Moreover $\bar{\chi}$ is such that $\bar{\chi}\bar{A}_w\bar{\chi} = 0$, which implies that half of the degrees of freedom of $\bar{\chi}$ must vanish.

A second group of solution is characterized by $\bar{A}_w = 0$. Consequently $\bar{X}^i$ and $\bar{\theta}_s$ are generic diagonal matrices (without the two by two identification of eigenvalues as before). Supersymmetry then requires that $\bar{\theta}_c = 0$. The $\bar{\chi}$ are left generic.

In the following we refer to these two group of solutions as first and second branch, respectively. There exists also mixed branches which factor in sub-blocks of the above types.

It is well-known that in the strong coupling theory there is a residual gauge freedom, which contains the Weyl group of $O(2N)$ and allows a variety of boundary conditions. Each one of them, in the first branch, defines a string configurations. Let us consider, for example, a solution from the first branch. We have in particular $\bar{X}^i = Diag(x^i_1, \ldots, x^i_N) \otimes \mathbf{1}$. The distinct eigenvalues of the latter can be interpreted as free strings of various lengths. This is possible by the fact that the $O(2N)$ Weyl group contains the permutation group S_N.

For an analogous treatment of the second branch solutions, one would need a much bigger Weyl group than the actual $O(2N)$ one which is not possible. In fact S_{2N} is not generically contained in the $O(2N)$ Weyl group. Moreover, in the second branch there is no decoupled Cartan gauge sector. This is a difficulty since as we saw the Cartan gauge field turn out to be essential for the generation of the perturbative string weight g_s^{2h-2+n} in the partition function. Therefore the solutions in the second and mixed branches do not allow a string interpretation. In any case, since it can't be used to interpolate between asymptotic free string configuration, this sector gets automatically excluded from the strong coupling limit.

One may notice now that also the first branch solutions do have problems, of different nature however. In fact, the first branch spectrum does seem to allow for two different string interpretations. The root of this phenomenon is to be found in the fact that the spectrum of this branch does not define a consistent, i.e. anomaly free, theory and therefore the true strong coupling limit should rescale out some other fermion in order to keep the theory anomaly free.

Suppose now that the effective strong coupling limit scales out the full set of $\bar{\chi}$ degrees of freedom. We would obtain in this way, apart from the decoupled gauge degrees of freedom, the spectrum of the free type IIA superstring theory. This is however impossible and the reason for this is simply that HMST is a chiral theory, while the strong coupling limit would not be chiral. To see it more in detail, one can imagine to gauge the chiral $SO(32)$ symmetry of HMST and consider its anomaly which is not vanishing and coupling independent. It follows that also the strong coupling limit should have a chiral asymmetry and an analogous anomaly which is however incompatible with the supposed type IIA-like strong coupling limit. Therefore we exclude the possibility of such a strong coupling limit.

This means that the fermionic degrees of freedom which are scaled out in the strong coupling limit are the $\bar{\theta}_c$'s.

2D instantons and Hitchin systems The action (4.14)) is invariant under the $\mathcal{N} = (8,0)$ supersymmetry transformations

$$\begin{aligned}
\delta X^i &= \frac{i}{g}\epsilon_c \tilde{\gamma}^i \theta_s \\
\delta \theta_s &= -\frac{1}{g} D_w X^i \gamma_i \epsilon_c \\
\delta \theta_c &= (\frac{i}{2g^2} F_{w\bar{w}} - \frac{i}{2}[X^i, X^j]\tilde{\gamma}_{ij})\epsilon_c \\
\delta A_{\bar{w}} &= -2\epsilon_c \theta_c
\end{aligned}$$

$$\delta A_w = 0, \qquad \delta\chi = 0, \tag{4.15}$$

We look now for classical Euclidean supersymmetric configurations that preserve half supersymmetry. We can follow what we did in Chapter 2 applied to (4.15) and find again $X^I = x^I \mathbf{1}_{2N}$ for $I = 3, \ldots, 8$ with x^I real constants and the Hitchin equations for the complex symmetric field $X = X^1 + iX^2$

$$F_{w\bar{w}} + g^2[X, \bar{X}] = 0, \qquad D_w X = 0, \tag{4.16}$$

and we want to construct solutions of eqs. (4.16) that interpolate between any two asymptotic string states ($w = \pm\infty$) as the ones considered in the previous subsection.

There is a very clear way to do this by mapping the above system to the relative $U(N)$ one. In fact, as is well known, there exists a standard representation R of $U(N)$ in $O(2N)$ and we can use it to solve our problem in terms of the already solved $U(N)$ Hitchin system. Fix the representation as $R_U \equiv U_1 \otimes \mathbf{1}_2 + U_2 \otimes \epsilon$, where $U \in U(N)$ and $U = U_1 + iU_2$ and let us indicate here with boldface the $U(N)$ pair $(\mathbf{X}, \mathbf{A})$. The induced $O(2N)$ pair is then given by $X^i = \Re\mathbf{X^i} \otimes \mathbf{1_2} + \Im\mathbf{X^i} \otimes \epsilon$ with $i = 1, 2$ and by $A = -\Im\mathbf{A} \otimes \mathbf{1_2} + \Re\mathbf{A} \otimes \epsilon$. Moreover, R maps the $U(N)$ Hitchin system into the corresponding $O(2N)$ one.

Notice that the so defined X field do have the right doubled asymptotics, schematically $X \sim \hat{x} \otimes \mathbf{1}_2$, and therefore fulfills our asymptotic conditions.

What we have shown is that the moduli space of heterotic stringy instantons is exactly the same as the moduli space of plane curves in the same sense as in the $U(N)$ case.

As is clear from the MST treatment, what we have to do now is to expand the action about a given instanton solution and to extract the strong coupling limit. Moreover one would get again as an effective theory on Cartan sector the heterotic superstring after the lifting to the spectral curve. This can be done and one obtains, following essentially [7] the same steps as in Chapter 3, that the partition function of HMST in the strong coupling limit reconstructs the perturbative heterotic superstring one.

This requires that the corresponding amplitude be proportional to $g_s^{-\chi}$, where χ is the Euler characteristic of the Riemann surface, as explained in the introduction.

[7] As we said at the beginning of the section there are some intriguing differences due to the chiral nature of the theory. The calculation is made feasible by a careful balancing of the left/right fermionic sectors (see [16]).

This factor can only come from a Maxwell field theory on the covering of the cylinder, which, in turn, can only be a consequence of a surviving part of the original gauge symmetry of the theory. We recall that in MST the $U(N)$ gauge symmetry of the theory breaks down, in the strong coupling configurations on the basis cylinder, to $(U(1))^N$, and that this is the basis for the persistence of a $U(1)$ gauge symmetry on the covering in the strong coupling limit. In the instantons constructed in the previous subsection, the strong coupling configurations break down the gauge symmetry $O(2N)$ in such a way that a $(O(2))^N$ symmetry survives. This guarantees that a Maxwell theory will survive on the corresponding covering in the strong coupling limit.

4.3.3 The addition of the Wilson lines

As pointed out in the previous subsection, in the absence of Wilson lines, HMST in its strong coupling regime gives the $SO(32)$ theory compactified to nine dimensions on a circle of very small radius. To obtain other heterotic string theories, one must introduce Wilson lines $\chi B \chi$, which at strong coupling become $\chi B \chi \to \bar{\chi} B \bar{\chi}$. The latter term is lifted to the covering as $\tilde{\chi} B \tilde{\chi}$ and accounts for the breaking of $SO(32)$ to some suitable subgroup. As remarked at the beginning, with a standard choice of Wilson lines, one can break $SO(32)$ to $SO(16) \times SO(16)$, and relate the model to the *ten-dimensional* $E_8 \times E_8$ string also broken to $SO(16) \times SO(16)$.

4.4 Gauge Theory and (Matrix) Strings

The first aim of the pioneers in string theory was to use it as a possible new language to describe the confining phase of QCD by recognizing in the string spectrum the sequences of known hadronic resonances. Some important ideas by Polyakov [43] are still oriented in these directions and the development of this subject is still to be completed. New ideas in this framework arose in the last two years following the AdS/CFT conjecture [51], but their limits seem to be reached by now[8]. In fact, as things are presently understood, there are a lot of difficulties in overcoming supersymmetry as a crucial ingredient and in pushing the analysis to the strong YM coupling regime at finite rank of the gauge group.

Apart all that, MST seems to suggest another possible point of view which we are going to explain in the following. Classical solutions of 4D Yang - Mills theory are

[8] We refer the reader to the most comprehensive review on the subject [52] both for an account of the present status of this theme and for a long list of references.

obtained by solving the (anti-)self-duality equation

$$F = \pm^* F \tag{4.17}$$

where F is the 4-dimensional gauge curvature two-form and * is the Hodge $*$ operator. Instantons are finite action classical solutions of YM theory [42]. These configurations are localized in 4 dimensions about some space-time points whose number gives a natural topological classification to the strata of the moduli space of 4-dimensional instantons. However, there are other interesting classical solutions.

If the Yang - Mills theory is defined on a 4-manifold $\mathcal{B}^{\mathcal{F}}$ which is a 2 dimensional fibration $\mathcal{F}$ over a base curve $\mathcal{B}$ equipped with a metric of the type $g = g^{\mathcal{B}}_{x\bar{x}}(x)[dx \otimes d\bar{x} + d\bar{x} \otimes dx] + g^{\mathcal{F}}_{y\bar{y}}(y,x)[dy \otimes d\bar{y} + d\bar{y} \otimes dy]$, where $g^{\mathcal{B}}$ is a metric on the base $\mathcal{B}$ and $g^{\mathcal{F}}(x)$ is a metric on the local fiber $\mathcal{F}$ at the point x of the base curve $\mathcal{B}$, the self duality equations read [9]

$$g^{\mathcal{B}}_{x\bar{x}} F_{y\bar{y}} = g^{\mathcal{F}}_{y\bar{y}} F_{x\bar{x}} \ , \ F_{xy} = 0 \ , \ F_{\bar{x}\bar{y}} = 0 \, . \tag{4.18}$$

Suppose now that in some strong coupling regime one is calculating vevs of an observable $\mathcal{O}$ such that, in the path-integral formalism, the dependence on the fiber curve – whose coordinates have been called y – can be neglected (for example it is the case for the theory on flat $\mathbb{R}^4$ when one is evaluating the vev of a large planar Wilson loop or in general a large cylindrically symmetric distribution). In such a case[10], the relevant classical equations (4.18) reduce to the Hitchin system on the base $\mathcal{B}$ where the component along the fiber direction of the original connection becomes the adjoint X field. In formulas, one has the connection decomposition $A = A_{\mathcal{B}} + gA_{\mathcal{F}}$ with $A_{\mathcal{F}} = \Phi_y dy + \Phi_{\bar{y}} d\bar{y}$ which implies that the curvature along the $\mathcal{F}\mathcal{F}$ direction is $F_{y\bar{y}} = i[\Phi_y, \Phi_{\bar{y}}]$ and that the curvature along the mixed directions is $F_{xy} = D_x \Phi_y$ so that one is left with the system

$$g^{\mathcal{F}}_{y\bar{y}} F_{x\bar{x}} = g^2 g^{\mathcal{B}}_{x\bar{x}} i[\Phi_y, \Phi_{\bar{y}}] \ , \ D_x \Phi_y = 0 \, ,$$

which becomes the (twisted) Hitchin system on the base $\mathcal{B}$ if the fiber $\mathcal{F}$ is flat. In fact, if $\mathcal{F}$ is a Riemann surface of zero Euler characteristic, we can choose $g^{\mathcal{F}}_{y\bar{y}} = 1$ and freely twist $\Phi_y \to X$ a scalar since $K_{\mathcal{F}}$ admits a global trivialization. In so doing we obtain

$$F_{x\bar{x}} = ig^2 g^{\mathcal{B}}_{x\bar{x}}[X, X^\dagger] \ , \ D_{\bar{x}} X = 0 \, .$$

[9] The analogous reduction of the anti-self-duality equations can be obtained by exchanging the role of x and $\bar{x}$ in the following.

[10] See also [78, 79] for similar treatments of the dimensionally reduced YM theory.

Extending now what we saw in Chapter 2 [11], the space of solutions is represented by the space of spectral curves, which is the space B_R of branched cover of $\mathcal{B}$ of degree equal to the rank R of the gauge group. Therefore, if one performs a background/fluctuation split to the gauge connection and calculates $< \mathcal{O} >$ as a sum over the solution of the resulting Hitchin system, one ends up with a sum over the moduli space of B_R curves and the path integral over the quantum fluctuations about it.

From this point of view one can speculate on a string interpretation of some sectors of YM_4 theory in its strong coupling regime. The outcome is the appearance of an integral over the B_R moduli space which resembles a string path integral. As long as the rank R of the gauge group is finite, which is the physically interesting case, the moduli space structure gives a restriction on the possible string diagrams that we have to integrate over.

4.5 Other open questions

In this final section we collect some conclusive or not conclusive observations.

In a sense MST gives a new perspective to link gauge and string theories. It uses gauge theory as a language to describe string theory, thus reversing the previous attitude in which string theory was expected to describe gauge theories. As we just observed in the last section, once this new experience has been done, one can try to reverse again the perspective. The resulting point of view seems to suggest a possible new way to describe strongly coupled gauge theories in a discretized stringy way.

An open problem in MST is if it admits a generalization to open string theories at weak coupling. In such an hypothetical generalization one would like to be able to form long open strings. Notice that to this end it doesn't seem to be appropriate to formulate SYM on a strip instead of on the cylinder. Anyhow, one can speculate in the following direction. MST encodes the mapping method often used to calculate string amplitudes as a crucial point. It could be that some new insights can be obtained by some matrix translation of that representation of string world-sheets. Let us notice that the existence of such an extension would be crucial to test that matrix theory conjecture works correctly. Anyhow, this is not guaranteed since it could well be that for some reasons the M-theory moduli space corners corresponding to perturbative open string theories at weak coupling are not visible once M-theory has been boosted along the infinite momentum frame. This enhances again the urgency of a covariant

[11] See also [45] for the untwisted case.

M-theory formulation. Anyhow, nobody can tell that M-theory appears as eleven dimensional in every corner of its moduli space.

We obtained string theory on flat $\mathbb{R}^{10}$ as a target space. It would be interesting to have similar theories for other 10-dimensional non flat manifolds. To this end it has been proposed to use the language of gauged linear σ-models [53] to generate a curved target MST and some results have been obtained. The methods we developed here could be generalized also to these situations. Another interesting generalization of this kind of instanton analysis of the two dimensional gauge theory could be done also for the D1/D5 system.

Acknowledgments

I would like to thank

F. Nesti and A. Tomasiello for the scientific collaboration,

E. Aldrovandi, D. Amati, L. Andrianopoli, U. Bruzzo, C.S. Chu, F. Cordaro, P. Goatin, T. Grava, G. dal Maso, G. Falqui, G. Ferretti, R. Iengo, D. Martelli, M. Negri, C. Reina, R. Russo, A. Santambrogio, S. Terna and A. Zampa for useful discussions,

all the SISSA staff for the stimulating atmosphere they created at the School,

C.A. Scrucca and M. Serone for useful and deep criticisms,

M. Trigiante for useful discussions about the need of a careful handling of the Montonen-Olive duality,

A.B. Hammou and J.F. Morales for their constant wish of comparing our points of views during very stimulating discussions.

Particular thanks are due to G. Pinato Space Red.

I would like to thank especially M. Bertolini for the most sincere wish of sharing knowledge, for long and deep discussions about the ultimate fate of the universe – which includes also superstring theory – and for constant encouragement

and L. Bonora for the highest quality training he offered me along the past years of scientific collaboration and for his ability in sharing his passionate attitude in doing theoretical physics.

Most of all, I thank Cristiana and Matilde for their love.

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