ISAS - INTERNATIONAL SCHOOL
FOR ADVANCED STUDIES

On First Order Congruences of Lines

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Thesis submitted for the degree of "Doctor Philosophiæ"

Academic Year 1998/99

SISSA - SCUOLA
INTERNAZIONALE
SUPERIORE
DI STUDI AVANZATI
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Il presente lavoro costituisce la tesi presentata da Pietro De Poi, sotto la direzione della Prof. Emilia Mezzetti, al fine di ottenere il diploma di *"Doctor Philosophiæ"* presso la Scuola Internazionale Superiore di Studi Avanzati, Classe di Matematica, Settore di Analisi Funzionale ed Applicazioni.

Ai sensi del Decreto del Ministero della Pubblica Istruzione n. 419 del 24. 04.1987, tale diploma di ricerca post-universitaria è equipollente al titolo di *"Dottore di Ricerca in Matematica"*

1991 *Mathematics Subject Classification.* Primary 14J40, 14M15, 14N05, Secondary 51N35.

ABSTRACT. We discuss projective families of lines, and in particular congruences of order one. We give examples of congruences in arbitrary dimension and then we concentrate studying the cases of $\mathbb{P}^3$ and $\mathbb{P}^4$, in which we give complete classifications.

Contents

Introduction

By definition a congruence of lines in $\mathbb{P}^n$ is a family of lines of dimension $(n-1)$. The order of a congruence is the number of lines passing through a general point of $\mathbb{P}^n$. Here we are interested in classifying congruences of order one, mainly in $\mathbb{P}^4$.

Our present motivation for studying these congruences is linked to some conjectures made by F. L. Zak about non-degenerate projective m-dimensional varieties X of $\mathbb{P}^n$. In particular we report the following conjectures, concerning their j-normality and the general projections:

CONJECTURE 1. *Let i and j be two integers such that $i \geq 1$, and $j \geq 0$; then*

1. $H^i(\mathbb{P}^n, \mathcal{I}_X(j)) = 0$ *for* $i + j < \frac{m}{n-m-1}$*;*
2. *for* $i + j = \frac{m}{n-m-1}$ *it is possible to classify all the varieties for which* $H^i(\mathbb{P}^n, \mathcal{I}_X(j)) \neq 0$.

CONJECTURE 2. *Let $\pi : X \to \mathbb{P}^r$ be a general projection and k a natural number. If $X' = \pi(X)$ is the image of the projection and $\Sigma_{r,k} \subset X'$ the corresponding k-tuple locus (i.e. for $x' \in \Sigma_{r,k}$ the set $\pi^{-1}(x')$ is formed by at least k points), then*

1. $\Sigma_{r,k}$ *is irreducible for* $r < m + \frac{m}{k}$*;*
2. *for* $r = m + \frac{m}{k}$ *it is possible to classify all the varieties for which* $\Sigma_{r,k}$ *is reducible.*

In the study of the varieties on the boundary of Conjecture 1 (*i.e.* the varieties of case (2)) for $i = 1$, that is the varieties X failing to be $\frac{2m-n+1}{n-m-1}$-normal, it is expected that they are characterized by the property that through the general point $P \notin X$ there is a unique $\frac{n-1}{n-m-1}$-secant line ℓ_P of the variety X, while the variety of $\frac{m}{n-m-1}$-secant lines through P is reducible and consists of $\frac{n-1}{n-m-1}$ components intersecting at ℓ_P. It is expected that these varieties are the same as the varieties on the boundary of Conjecture 2 (*i.e.* the varieties of case (2)) for $r = n-1$ (*i.e.* $k = \frac{m}{n-m-1}$) and, *vice versa*, the varieties on the boundary of Conjecture 2 are the same as the varieties on the boundary of Conjecture 1 for $n = r+1$, $i = 1$ and $j = k-1$.

Therefore, studying the varieties on the boundary of these Conjectures, one is induced to study first order congruences of $\mathbb{P}^n$, since we have seen that all the known examples are varieties whose l–secant lines (with l appropriate integer) give such congruences.

For example, in the first non-trivial case, *i.e.* for $n = 4$ and so $k = 2$, the Franchetta's Theorem says that the only general surface of $\mathbb{P}^4$ whose projection to $\mathbb{P}^3$ has a reducible curve as its singular locus is the Veronese surface (see **[MP97]** for a modern proof of Franchetta's Theorem); and in fact the trisecant lines of the Veronese surface in $\mathbb{P}^4$ generate a first order congruence of $\mathbb{P}^4$: see Theorem 4.26.

Returning to the argument of this Thesis, we see that the first case to be analysed is the case of congruences of $\mathbb{P}^3$; in the spirit of the Zak's Conjectures, the most natural way of getting congruences is to consider the secant lines of a non-planar curve; by the Castelnuovo's Bound it is easy to see that the only smooth irreducible curve of $\mathbb{P}^3$ which generates a first order congruence is the rational normal cubic.

Passing to $\mathbb{P}^4$, the natural generalization is to consider the family of trisecant lines of a non-degenerate surface; or more generally, one can consider the $(n-1)$-secant lines of a non-degenerate projective variety of codimension 2 of $\mathbb{P}^n$.

The systematic study of the congruences of lines of $\mathbb{P}^3$ was introduced by the classical school of Kummer, in [**Kum75**], and successively developed by many classical algebraic geometers, such as Reye, Schumacher, Bordiga, C. Segre, Castelnuovo, Fano, Jessop, Semple e Roth.

More recently, congruences of lines of $\mathbb{P}^3$ were studied in [**Gol85**] by N. Goldstein, who (as Marletta, see below) classified the congruences from the point of view of the focal locus. Successively, Z. Ran in [**Ran86**] studied the surfaces of order one in the general Grassmannian $\mathbb{G}(k,n)$ *i.e.* families of k-planes of $\mathbb{P}^n$ for which the general $(n-k-2)$-space meets only one k-plane of the family. He gives a classification of such surfaces, in particular obtaining a modern and more correct proof of the classification of the first order congruences of lines of $\mathbb{P}^3$ of Kummer [**Kum75**] and the fact that for $n=3$ and for the class greater than three the surface is not smooth, as conjectured by I. Sols. Another proof of the Kummer's classification is given by F. L. Zak and others in [**ZILO**].

The starting point of the case of $\mathbb{P}^4$ is an Ascione's work [**Asc97**], whose aim is to classify the projective surfaces of $\mathbb{P}^4$ with only one apparent triple point, *i.e.* he wants to classify the general surfaces whose family of trisecant lines generates a first order congruence of $\mathbb{P}^4$. Unfortunately, his proof of this classification had a gap, which was filled up by Severi in [**Sev01**], who found a surface which did not appear in Ascione's classification. The congruences of lines in $\mathbb{P}^4$ were considered by the Sicilian mathematician G. Marletta. Marletta's point of view in [**Mar09b**] and [**Mar09a**] is different: he is interested in classifying congruences of lines of $\mathbb{P}^4$ of bidegree $(1,n)$ *i.e.* of order one. He studies in particular the focal locus of a congruence of lines and classifies the congruence according to the dimension and number of the irreducible components of this locus. It has, in general, dimension 2 and the case of an irreducible surface is the one of Ascione and Severi.

As for $\mathbb{P}^n$, with $n>4$, there is a Sgroi's paper, [**Sgr27**], in which he begins the study in $\mathbb{P}^5$ giving a first coarse classification from the point of view of the focal loci and some examples. The Marletta's influence on Sgroi is evident as in other papers (also published on the "Accademia Gioenia") of other Sicilian mathematicians of the period between the two World Wars. It is worth noting that in this period many other important mathematicians worked at Catania's University, or more generally in Sicily, such as Albanese, Bagnera and De Franchis.

This paper studies congruences in the spirit of Marletta's work by a modern point of view. In particular we apply the technique of focal diagrams of a projective flat family and the Schubert calculus to the congruences of lines of order one. We give some general results and examples in $\mathbb{P}^n$ and, from these, we deduce Kummer and Marletta's classifications in $\mathbb{P}^3$ and $\mathbb{P}^4$.

The first Chapter is devoted to introduce mainly the focal diagram of a family of projective schemes and apply it to the case of congruences of k-planes of $\mathbb{P}^n$, *i.e.* subschemes of $\mathbb{G}(k,n)$ of dimension $(n-k)$. The focal locus of a family of schemes was introduced, classically, by C. Segre, in [**Seg88**]. More recently, there has been a renew of interest in this subject, thanks to a C. Ciliberto and E. Sernesi's work, [**CS92**], in which the focal locus is introduced in modern terms, in particular with the focal diagram. The first Chapter contains also a summary on the Schubert cycles, the universal bundles and Plücker embeddings on Grassmannians. The Chapter ends with the proof of the fact that—in sufficiently general hypotheses—the general k-plane is tangent to the focal locus.

The second Chapter treats of first order congruences of lines of $\mathbb{P}^n$. After recalling what is the sequence of degrees, we define the fundamental d-loci. Then, we prove Proposition 2.3, which is the key ingredient for the "Classification Theorem" 2.8 which gives a first general way of classifying the first order congruences from the point of view of their fundamental locus. Another important concept introduced in this Chapter is that of parasitic $(n-2)$-plane, *i.e.* a linear space contained in the fundamental locus which is not a fundamental $(n-2)$-locus. Finally, we give some general examples of first order congruences: the linear congruences, *i.e.* general linear sections of the Grassmannian; congruences which are given by multisecant lines of some degeneracy loci of maps of vector bundles and, finally, we classify all the congruences for which the fundamental locus is—set-theoretically—a $(n-2)$-space only. Of these congruences we give a complete description. We finish this Chapter by giving a degree bound on the $(n-2)$-dimensional varieties whose $(n-1)$-secant lines give a first order congruence.

In the third Chapter we study first order congruences of lines of $\mathbb{P}^3$, applying what we proved in the preceding chapter. The main result of Chapter 3 is the following:

THEOREM 0.1. *The focal locus of a congruence of lines of $\mathbb{P}^3$ of order one can be:*

1. *an irreducible curve, which can be one of the following:*
   (a) *a rational normal curve C^3 of $\mathbb{P}^3$, in which case the congruence is given by the secant lines of C^3; vice versa the secant lines of a rational normal curve generate a first order congruence;*
   (b) *a line ℓ, and the congruence is obtained in this way: fix an isomorphism φ between ℓ and $\mathbb{P}^1_\ell$, the pencil of planes containing ℓ:*

   $$\varphi : \ell \to \mathbb{P}^1_\ell;$$

   let $\mathbb{P}^1_P$ be the pencil of lines passing through P and lying in $\varphi(P)$; then the congruence is formed by the lines of the pencils $\mathbb{P}^1_P$ as P varies in ℓ i.e. it is $\cup_{P\in\ell}\mathbb{P}^1_P$;
2. *a rational curve C_1 of degree m_1 union a line C_2 such that* $\text{length}(C_1 \cap C_2) = m_1 - 1$*; in this case the congruence is given by the lines meeting C_1 and C_2,; vice versa, for every rational curve C_1 of degree m_1 which possesses a (m_1-1)-secant line C_2, the join of C_1 and C_2 gives a first order congruence;*
3. *a point, i.e. the congruence is a star of lines.*

In particular, in Chapter 3 we first consider the case in which the components of the focal locus are all focal lines; then we introduce the pure fundamental curve, *i.e.* the fundamental locus without focal lines and we analyse first the case in which this locus is irreducible, then the case in which it is reducible; finally, we prove the main result, *i.e.* Theorem 0.1.

Chapter four is devoted to first order congruences of lines of $\mathbb{P}^4$; in particular, after classifying all the cases in which the fundamental locus is—set-theoretically—a plane in Theorem 4.7, we study the four cases corresponding to the possible splittings of the pure (*i.e.* without parasitic planes) fundamental locus. In particular, for the first type—*i.e.* if the fundamental surface is irreducible—we get the following result:

THEOREM 0.2. *If the fundamental surface F of a first order congruence of lines of $\mathbb{P}^4$, is irreducible general and non-linear, then it can be:*

1. *a (projected) Veronese surface, which has not parasitic planes;*
2. *a projection of a Del Pezzo surface of $\mathbb{P}^5$, which has 5 1-parasitic planes;*

3. *a projection of a rational normal scroll* $S_{1,4}$ *of* $\mathbb{P}^6$ *from a line contained in the 4-subspace generated by the unisecant quartic* C_4 *of the scroll.*

   The surface F *contains 4 parasitic planes, one is the 4-parasitic plane which contains the quartic curve image of* C_4*, and the other 3 are 1-parasitic. They are the planes containing the (projection of the) unisecant line* C_1 *and the couples of ruling lines passing through the 3 singular points of the quartic plane curve contained in* F*;*
4. *a Bordiga surface, i.e.* F *is a blow up of* $\mathbb{P}^2$ *in 10 points* $x_1, \dots, x_{10}$ *embedded in* $\mathbb{P}^4$ *by the linear system*

$$|D| := |\pi^* 4L - E_1 - E_2 - \cdots - E_{10}|$$

   where $\pi : F \to \mathbb{P}^2$ *is the blow up in the points* $x_1, \dots, x_{10}$*,* L *is a line in* $\mathbb{P}^2$ *and* E_i *is the fibre of* π *over* x_i*. In fact* F *contains 10 distinct lines and 10 distinct plane cubics such that each line meets a single cubic. It has 10 1-parasitic planes, which are the planes of the 10 plane cubics.*

In particular, the congruence is given by the family of the trisecant lines of F*.*

Vice versa the trisecant lines of one of the above surfaces F*, generate a first order congruence.*

Concerning the second case, after studying the case of the proper intersections of two surfaces and classifying them in Proposition 4.49, we prove the following result:

THEOREM 0.3. *If the fundamental surface of a first order congruence of lines of* $\mathbb{P}^4$ *has, as irreducible components two surfaces, then the congruence is given by the secant lines of the first surface* F_1 *that meet the second surface* F_2 *also; we have that:*

1. *either* F_2 *is a plane, in which case, if none of the singular points of* F_1 *is in* F_2*, then* $\deg F_1 \in \{3, \dots, 8\}$*;*
2. *or* F_2 *is not a plane, in which case* F_1 *is the rational cubic scroll* $S_{1,2}$ *or the cone* $S_{0,3}$ *and* F_2 *is rational.*

If F_2 is a plane and F_1 is smooth, we obtain a complete classification:

THEOREM 0.4. *If the fundamental surface of a first order congruence of lines of* $\mathbb{P}^4$ *has, as irreducible components two surfaces, a plane* F_2 *and a smooth surface* F_1*, then the congruence is given by the secant lines of* F_1 *that meet the plane* F_2 *also; besides, we have the following possibilities:*

1. F_1 *is a rational quintic with sectional genus 2 (a Castelnuovo surface);* F_1 *is a blow up of* $\mathbb{P}^2$ *in 8 points* $x_1, \dots, x_8$ *embedded in* $\mathbb{P}^4$ *by the linear system*

$$|D| := |\pi^* 4L - 2E_1 - E_2 - \cdots - E_8|$$

   where $\pi : F_1 \to \mathbb{P}^2$ *is the blow up in the points* $x_1, \dots, x_8$*,* L *is a line in* $\mathbb{P}^2$ *and* E_i *is the fibre of* π *over* x_i*.* F_1 *intersects* F_2 *along an irreducible conic, which is the proper transform of a line through two of the 7 simple points of the blow up, and in fact the secants lines of* F_1 *meeting* F_2 *generate a first order congruence. Besides we have 7 1-parasitic planes;*
2. F_1 *is a rational sextic with sectional genus 3;* F_1 *is a blow up of* $\mathbb{P}^2$ *in 10 points* $x_1, \dots, x_{10}$ *embedded in* $\mathbb{P}^4$ *by the linear system*

$$|D| := |\pi^* 4L - E_1 - E_2 - \cdots - E_{10}|$$

   where $\pi : F_1 \to \mathbb{P}^2$ *is the blow up in the points* $x_1, \dots, x_{10}$*,* L *is a line in* $\mathbb{P}^2$ *and* E_i *is the fibre of* π *over* x_i*.* F_1 *intersects* F_2 *along an irreducible cubic, which is the proper transform of a cubic through 9 of the 10 points of the blow up and in fact the secants lines of* F_1 *meeting* F_2 *generate a first order congruence. Besides we have 9 1-parasitic planes;*

3. *F_1 is a quartic with sectional genus 1 (a Del Pezzo surface); F_1 is a blow up of $\mathbb{P}^2$ in 5 points $x_1, \ldots, x_5$ embedded in $\mathbb{P}^4$ by the linear system*
$$|D| := |\pi^* 3L - E_1 - E_2 - \cdots - E_5|$$
where $\pi : F_1 \to \mathbb{P}^2$ is the blow up in the points $x_1, \ldots, x_5$, L is a line in $\mathbb{P}^2$ and E_i is the fibre of π over x_i. F_1 intersects F_2 along a line, which is the proper transform of a line through 2 of the 5 points of the blow up and in fact the secants lines of F_1 meeting F_2 generate a first order congruence. Besides we have 5 1-parasitic planes.
4. *F_1 is the cubic scroll $S_{1,2}$, and the two surfaces F_1 and F_2 are in general position, i.e. they meet in three points. Besides we have 3 1-parasitic planes, given by the conics of the scroll passing through two by two of the three points of the proper intersection $F_1 \cap F_2$;*
5. *F_1 the cubic scroll $S_{1,2}$; C is the unisecant line of the scroll and F_2 is a general plane containing C.*

Vice versa the secant lines of one of the above surfaces F_1 meeting its correspondent surface F_2, generate a first order congruence.

For the third case, after, as usual, classifying the possible proper intersections and classifying them in Theorem 4.80, we prove the following:

THEOREM 0.5. *If the fundamental surface of a first order congruence of lines of $\mathbb{P}^4$ has, as irreducible components three surfaces, then the congruence is given by the lines meeting all of them; we have that at least one of the three surfaces is a plane and*

1. *if two of the surfaces are planes, then the third is rational;*
2. *if one of the surfaces is a plane, then of the other two, one is a rational scroll and the other is rational.*

Finally, for the last case, after the classification of the proper intersections in Theorem 4.104, we have the following:

THEOREM 0.6. *If the fundamental locus of a first order congruence of lines of $\mathbb{P}^4$ has, as irreducible components, a surface F_1 and a curve C, with $\deg(F_1) := m_1$, $\deg(C) := m_2$ and $c := \text{length}(F_1 \cap C)$, then the congruence is given by the lines meeting both F_1 and C; we have that*

1. *either $C \subset F_1$ and then:*
   (a) *C is a line and F_1 is a rational surface of degree m_1 and, if we suppose that F_1 has only isolated singularities, with sectional genus $m_1 - 2$; or*
   (b) *C is a conic and F_1 is a projection of a rational normal scroll of type $S_{m_1-2k,2k}$, with $m_1 \geq 3$, one of its unisecant curves is C and a general hyperplane through C intersects F_1 in C with multiplicity k and in a line;*
2. *or $C \not\subset F_1$ and then we have:*
   (a) *F_1 is a plane, then C is a rational curve such that $c = m_2 - 1$; or*
   (b) *C is a line and F_1 is a rational surface and, if we suppose that F_1 has only isolated singularities, with sectional genus $m_1 - 2$; besides, $c = m_1 - 1$; or*
   (c) *C is a rational curve with a point P of multiplicity $m_2 - 1$ and F_1 is a cone with vertex in P and basis a rational curve and the intersection of F_1 with the plane of C is given by $m_1 - 1$ lines (and so, $c \geq (m_1 - 1)m_2$).*

If we suppose that C and F_1 are smooth, we get a finite list of possibilities:

THEOREM 0.7. *If the fundamental locus of a first order congruence of lines of $\mathbb{P}^4$ has, as irreducible components, a smooth surface F_1 and a smooth curve C, with* $\deg(F_1) := m_1$, $\deg(C) := m_2$ *and* $c := \text{length}(F_1 \cap C)$, *then the congruence is given by the lines meeting both F_1 and C; we have that*

1. *either $C \subset F_1$ and then:*
   (a) *F_1 is the rational normal scroll $S_{1,2}$ of degree 3 linearly normal in $\mathbb{P}^4$, and C is a unisecant conic;*
   (b) *C is a line and F_1 is a speciality one rational surface, i.e. we have the following possibilities (see* [**Ale92**]*)*
      (i) *F_1 has degree eight and it is linked to a Veronese surface in a complete intersection of a cubic and a quartic. F_1 is a blow up of $\mathbb{P}^2$ in 16 points $x_1, \dots, x_{16}$ embedded in $\mathbb{P}^4$ by the linear system*
      $$|D| := |\pi^* 6L - 2E_1 - \dots - 2E_4 - E_5 - \dots - E_{16}|$$
      where $\pi : F_1 \to \mathbb{P}^2$ is the blow up in the points $x_1, \dots, x_{16}$, L is a line in $\mathbb{P}^2$ and E_i is the fibre of π over x_i;
      (ii) *F_1 has degree nine and it lies on a net of quartics; it is given in the following way: the first and the second adjunctions of F_1 give a canonical sequence of birational morphisms of rational surfaces*
      $$S \xrightarrow{f_1} S_1 \xrightarrow{f_2} S_2$$
      where S_2 is canonically a cubic surface in $\mathbb{P}^3$. The morphism f_2 blows up in three distinct (closed) points x_1, x_2, x_3, while f_1 blows up in six distinct points $x_4, \dots, x_9$. Let K_1 and K_2 be the inverse images of the canonical divisors of S_1 and S_2 respectively, the linear system of the hyperplane sections of F_1 is given by
      $$|H| = |-K - K_1 - K_2|$$
      where K is the canonical divisor on F_1;
      (iii) *F_1 has degree ten and it is a blow up of $\mathbb{P}^2$ in 13 points $x_1, \dots, x_{13}$ embedded in $\mathbb{P}^4$ by the linear system*
      $$|D| := |\pi^* 14L - 6E_1 - 4E_2 - \dots - 4E_{10} - 2F_{11} - E_{12} - E_{13}|$$
      where $\pi : F_1 \to \mathbb{P}^2$ is the blow up in the points $x_1, \dots, x_{13}$, L is a line in $\mathbb{P}^2$ and E_i is the fibre of π over x_i;
      (iv) *F_1 is "possibly[1]" a speciality one rational surface of degree eleven;*
2. *or $C \not\subset F_1$ and then we have that F_1 is a plane and C a conic meeting in one point, i.e. $c = 1$.*

Vice versa the lines meeting one of the above surfaces F_1 and a curve C contained in it, generate a first order congruence.

ACKNOWLEDGMENTS. My warmest thanks to Prof. Emilia Mezzetti for her substantial help in the elaboration of this paper both with stimulating discussions and for reviewing it.

I would also like to thank Prof. Enrique Arrondo for useful discussions on these topics and Prof. Fedor L. Zak for some references, especially [**ZILO**].

I thank also Prof. Fabrizio Catanese and Prof. Rita Pardini for inviting me at the Department of Mathematics "Leonida Tonelli" of the University of Pisa and the Department itself where a great part of this Thesis was written. Finally, I want to thank the people of the Doctoral School of this Department for their friendship, help and the warm atmosphere I found in Pisa.

[1]possibly, since it is not known if speciality one rational surfaces of $\mathbb{P}^4$ of degree 11 do exist

CHAPTER 1

Foundation Material

1. Notations and Definitions

We will work with schemes and varieties over the complex field $\mathbb{C}$. By *variety* we mean a reduced and irreducible algebraic $\mathbb{C}$-scheme.

Let us now recall some definitions about families of schemes (see, for example [**CS89**]).

1.1. DEFINITIONS. A *family of schemes* is a triple $(\mathcal{F}, B, p)$, where $\mathcal{F}$ and B are schemes and $p : \mathcal{F} \to B$ is a surjective morphism of schemes (we will suppose, for simplicity, that the scheme B is irreducible). $\mathcal{F}$ is called *total space* and B *parameter space* of the family. We denote with $\mathcal{F}_b$ ($b \in B$) the fibre over the (closed) point b. The fibres of the family are also called the *elements* of the family; the family can be identified with $\{\mathcal{F}_b\}_{b\in B}$, *i.e.* the set of its elements.

1.2. DEFINITION. A *projective family of schemes* $(\mathcal{F}, B, p)$, is a family where $\mathcal{F}$ is a closed subscheme of $B \times \mathbb{P}^n$ and p is the restriction to $\mathcal{F}$ of the projection to the first factor.

We observe that in this case $\mathcal{F}_b$ is a projective scheme.

More generally we are interested in projective families of schemes for which we have that $\mathcal{F} \subset B \times Y$, where $Y \subset \mathbb{P}^n$.

1.3. DEFINITION. For *dimension of a family* we mean the dimension of the parameter space.

1.4. DEFINITION. A family is said to be *flat* if $\forall t \in \mathcal{F}$ and $p(t) = b$, the local ring $\mathcal{O}_{\mathcal{F},t}$ is flat as an $\mathcal{O}_{B,b}$-module.

We recall (without proof) a well known result about flat families:

THEOREM 1.1. *If $(\mathcal{F}, B, p)$ is a flat projective family, then all its elements have the same Hilbert polynomial, i.e.*

$$\mathrm{hilb}(\mathcal{F}_b) = q(x)$$

for a (fixed) polynomial $q(x) \in \mathbb{Q}[x]$ independent of b.

The converse is true if the parameter space B is integral.

1. EXAMPLE. The first nontrivial example (and one of the most important ones) is the family of the linear k-spaces of a fixed $\mathbb{P}^n$. This is given by the triple $(\mathcal{H}_{k,n}, \mathbb{G}(k,n), p)$, where $\mathbb{G}(k,n)$ is the Grassmann variety,

$$\mathcal{H}_{k,n} := \{(v, x) \in \mathbb{G}(k,n) \times \mathbb{P}^n \mid x \in v\}$$

is the incidence variety and p is the restriction of the projection on $\mathbb{G}(k,n)$. It can be shown that this is the *universal family* of the k-planes of $\mathbb{P}^n$: every projective family $(\mathcal{F}, B, p)$ of k-planes of $\mathbb{P}^n$ is given by a pull-back of $\mathcal{H}_{k,n}$ via a morphism $B \to \mathbb{G}(k,n)$ *i.e.* there is a bijection between $\{\phi \mid \phi : B \to \mathbb{G}(k,n)\}$ and $\{(\mathcal{F}, B, p) \mid (\mathcal{F}, B, p)$, flat family of k-planes of $\mathbb{P}^n\}$.

2. Focal Diagram Associated to a Family of Projective Varieties

In this section we will follow **[CC93]** and **[Vio97]**. Let $\mathcal{G} \subset B \times Y$ be a flat (projective) family of closed subschemes of a nonsingular irreducible projective variety Y parametrized by an irreducible nonsingular scheme B. Let $\mathcal{F}$ a desingularization of $\mathcal{G}$. After possibly shrinking B, we may assume that $\mathcal{F}$ is flat over B also. Then, we have the following diagram:

$$\begin{array}{ccc} B \times Y & \xrightarrow{p_2} & Y \\ {\scriptstyle p_1}\downarrow & & \\ B. & & \end{array}$$

And if we restrict ourself to $\mathcal{F}$, we obtain the diagram:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{f} & Y \\ {\scriptstyle p}\downarrow & & \\ B. & & \end{array}$$

1.5. Definition. The family is said to be *non-degenerate* if the map f in the preceding diagram is dominant.

1.6. Definitions. A point $y \in Y$ is called *fundamental point for the family* if its fibre has dimension greater than the dimension of the general one, *i.e.* if

$$\dim f^{-1}(y) > \dim B + \dim p^{-1}(b) - \dim f(\mathcal{F}), \;\; b \in B.$$

The *locus of the fundamental points* is denoted by $\Phi(\mathcal{F})$.

The scheme defined as the preimage of $\Phi(\mathcal{F})$ by f, $V(f) := f^{-1}(\Phi(\mathcal{F}))$, (with the natural scheme structure defined, *i.e.* $(V(f), f^{-1}(\mathcal{O}_{\Phi}(\mathcal{F}))) \cong \Lambda \times_Y \Phi(\mathcal{F}))$ is called *subscheme of the fundamental points*.

Now, we can consider the following commutative and exact diagram of coherent sheaves on $\mathcal{F}$ which will be called *focal diagram*; it was introduced for the first time in **[CS92]**:

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & \mathcal{T}_{(B\times Y/Y)|\mathcal{F}} & \xrightarrow{\pi} & \mathcal{N}_{\mathcal{F}/B\times Y} & & \\ & & & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{T}_{\mathcal{F}} & \longrightarrow & \mathcal{T}_{B\times Y|\mathcal{F}} & \longrightarrow & \mathcal{N}_{\mathcal{F}/B\times Y} & \longrightarrow & 0 \\ & & {\scriptstyle df}\downarrow & & \downarrow & & & & \\ & & f^*\mathcal{T}_Y & = & p_2^*\mathcal{T}_{Y|\mathcal{F}} & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

this is constructed on the exact sequence defining the normal sheaf of $\mathcal{F}$ in $B \times Y$.

In the diagram the sheaf $\mathcal{T}_{(B\times Y/Y)}$ is the relative tangent sheaf of $B \times Y$ with respect to Y; for this we have:

$$p_1^*\mathcal{T}_B \cong \mathcal{T}_{(B\times Y/Y)} = \mathcal{H}om(\Omega^1_{(B\times Y)/Y}, \mathcal{O}_{B\times Y}).$$

The homomorphism π is defined by the commutativity of the diagram, while df is the differential of the map f.

1.7. DEFINITION. The map $\pi : \mathcal{T}_{(B\times Y/Y)|_{\mathcal{F}}} \to \mathcal{N}_{\mathcal{F}/B\times Y}$ is called *global characteristic map* for the family $\mathcal{F}$.

If we restrict the map π to a fibre $\mathcal{F}_b$, we obtain a morphism

$$\pi(b) : T_{B,b} \otimes \mathcal{O}_{\mathcal{F}(b)} \to \mathcal{N}_{\mathcal{F}(b)/Y},$$

—where we define $\mathcal{F}(b) := f(\mathcal{F}_b)$, $b \in B$—which is called *characteristic map of the family relative to b.*

Passing to global sections, we obtain a homomorphism of vector spaces:

$$T_{B,b} \to H^0(\mathcal{F}(b), \mathcal{N}_{\mathcal{F}(b)/Y}).$$

REMARK. We observe that this map is the differential of the functorial map

$$B \to \mathrm{Hilb}_Y^{q(x)}$$

given by the flat family $\mathcal{F}$, since (see **[CS89]**) the tangent space to $\mathrm{Hilb}_Y^{q(x)}$ at the point $\mathcal{F}(b)$ is the first order deformation space of $\mathcal{F}(b)$, which can be identified with $H^0(\mathcal{F}(b), \mathcal{N}_{\mathcal{F}(b)/Y})$.

1.8. DEFINITIONS. The condition

$$\mathrm{rk}(\pi) < \min\{\mathrm{rk}(\mathcal{T}_{(B\times Y/Y)|_{\mathcal{F}}}), \mathrm{rk}(\mathcal{N}_{\mathcal{F}/B\times Y})\}$$

defines a closed subscheme $V(\pi)$ of $\mathcal{F}$, which is called *subscheme of the foci of the first order* of the family, while $F = f(V(\pi)) \subset Y$ is called *locus of the first order foci* in Y.

REMARK. From the diagram (if we think of $\mathcal{T}_{(B\times Y/Y)|_{\mathcal{F}}}$ and $\mathcal{T}_{\mathcal{F}}$ as subsheaves of $\mathcal{T}_{B\times Y|_{\mathcal{F}}}$), it is easy to see that

$$\ker \mathrm{d}f = \ker \pi. \tag{1}$$

So, we can think of $V(\pi)$ and F as the set of branch and ramification points of f respectively.

From this, we easily obtain:

PROPOSITION 1.2. *The following claims are equivalent:*

1. $\mathrm{d}f$ *has maximal rank, i.e. it has cokernel of torsion or* $\ker \mathrm{d}f = 0$*;*
2. π *has maximal rank;*
3. $V(\pi)$ *is a closed proper subscheme of* $\mathcal{F}$.

PROPOSITION 1.3. *The locus of the fundamental points is contained in the locus of the first order foci.*

PROOF. It is a consequence of the fact that F is the set of the ramification points of f. □

COROLLARY 1.4. *The scheme of fundamental points is a subscheme of the scheme of the first order foci.*

3. Cellular Decomposition and Schubert Cycles

We recall some basic facts (without proofs) about cellular decomposition and Schubert cycles on the Grassmannians. We follow [GH78].

Let us consider $\mathbb{C}^n$ with the (canonical) basis $e_1, \dots, e_n$. From this we define the flag $\mathcal{V}$ such that

$$V_i := \langle e_1, \dots, e_i \rangle, \quad V_i \subset V_{i+1}, \quad i = 1, \dots, n.$$

Then, for every element $\Lambda \in \mathrm{G}(k,n) = \mathbb{G}(k-1, n-1)$ we have this other flag

$$\Lambda \cap V_i \subset \Lambda \cap V_{i+1}$$

and from this we set:

$$W_{a_1\cdots a_k} := \{\Lambda \in \mathrm{G}(k,n) \mid \dim(\Lambda \cap V_{n-k+i-a_i}) = i\}.$$

It is not hard to see that its (analytic) closure is

$$\overline{W_{a_1\cdots a_k}} = \{\Lambda \in \mathrm{G}(k,n) \mid \dim(\Lambda \cap V_{n-k+i-a_i}) \geq i\}.$$

It can be shown that these are analytic subvarieties of $\mathrm{G}(k,n)$ and the importance of these is explained by the following

THEOREM 1.5. *The integral homology of the Grassmannian of the k-planes of $\mathbb{C}^n$, $\mathrm{G}(k,n)$, is torsion-free and is freely generated by the cycles*

$$\sigma_{a_1\cdots a_k} := [\overline{W_{a_1\cdots a_k}}]$$

in real codimension $2\sum a_i$ (i.e. $\sigma_{a_1\cdots a_k} \in H_{k(n-k)-2\sum a_i}(\mathrm{G}(k,n),\mathbb{Z})$), where the k-tuple $a_1\cdots a_k$ varies in all the nondecreasing sequences of integers between 0 and $n-k$. In particular, the homology (and so, by Poincaré duality, the cohomology) in $\mathrm{G}(k,n)$ is analytic.

REMARK. Clearly, the definition of the cycles is independent of the choice of the flag $\mathcal{V}$; in fact, if we fix another flag $\mathcal{W}$,

$$W_i \subset W_{i+1}, \qquad \dim W_i = i,$$

we can set as before

$$\sigma_a(\mathcal{W}) = \{\Lambda \in \mathrm{G}(k,n) \mid \dim(\Lambda \cap W_{n-k+i-a_i}) \geq i\}$$

and the homology class of this analytic subvariety is independent of the flag chosen, since we can find a continuous family of linear automorphism of $\mathbb{C}^n$ taking any flag into any other.

1.9. DEFINITION. The analytic subvarieties $\sigma_a(\mathcal{W})$ are called *Schubert cycles* of the Grassmannian.

All this determines the additive structure of the homology (and so, dually, the cohomology) of the Grassmannian; now we will see how to describe its multiplicative structure, *i.e.* how to express the intersection of two (general) Schubert cycles as a linear combination of other Schubert cycles.

The first formula expresses the intersection of two cycles σ_a and σ_b in complementary dimensions, *i.e.* $\sum a_i + \sum b_i = k(n-k)$:

$$\#(\sigma_a \cdot \sigma_b) = \delta_{a_1\cdots a_k}^{n-k-b_k\cdots n-k-b_1}. \tag{2}$$

Then, we have the intersection formulas for the intersection of three cycles σ_a, σ_b and σ_c always in complementary dimensions, *i.e.* $\sum a_i + \sum b_i + \sum c_i = k(n-k)$:

INTERSECTION'S FORMULAS. *Let $\sigma_a, \sigma_b, \sigma_c$ be three Schubert cycles of $\mathrm{G}(k,n)$; then:*

1. *if we have three indices $0 \leq \alpha, \beta, \gamma \leq k$ such that $\alpha + \beta + \gamma = 2k+1$, then*
$$\#(\sigma_a \cdot \sigma_b \cdot \sigma_c)_{\mathrm{G}(k,n)} = \begin{cases} 0 & \text{if } a_\alpha + b_\beta + c_\gamma > n-k \\ \#(\sigma_{a-a_\alpha} \cdot \sigma_{b-b_\beta} \cdot \sigma_{c-c_\gamma})_{\mathrm{G}(k-1,n-1)} & \text{if } a_\alpha + b_\beta + c_\gamma = n-k; \end{cases}$$
2. *if we have three coefficients $a_\alpha, b_\beta, c_\gamma$ such that $a_\alpha + b_\beta + c_\gamma \geq 2(n-k)+1$, then*
$$\#(\sigma_a \cdot \sigma_b \cdot \sigma_c)_{\mathrm{G}(k,n)} = \begin{cases} 0 & \text{if } \alpha + \beta + \gamma > k \\ \#(\sigma_{a_1-1\cdots a_\alpha-1 a_{\alpha+1}\cdots a_k} \cdot \sigma_{b_1-1\cdots b_\beta-1 b_{\beta+1}\cdots b_k} \cdot \\ \quad \cdot \sigma_{c_1-1\cdots c_\gamma-1 c_{\gamma+1}\cdots c_k})_{\mathrm{G}(k,n-1)} & \text{if } \alpha+\beta+\gamma = k. \end{cases}$$

We conclude this exposition with the Pieri's and the Giambelli's formulas; Pieri's formula expresses the intersection of an arbitrary Schubert cycle with a *special* one, *i.e.* a cycle of the type $\sigma_{a0\cdots 0}$:

PIERI'S FORMULA. *If* $a = (a0\cdots 0)$, *then* $\forall b$ *we obtain:*

$$(\sigma_a \cdot \sigma_b) = \sum_{\substack{b_i \leq c_i \leq b_{i-1} \\ \sum c_i = a + \sum b_i}} \sigma_c. \tag{3}$$

Giambelli's formula expresses the general Schubert cycle as the sum of products of special cycles with others which have less indices:

GIAMBELLI'S FORMULA. *Let us consider a Schubert cycle* $\sigma_{a_1\cdots a_k}$; *then, we have:*

$$(-1)^k \sigma_{a_1\cdots a_k} = \sum_{j=1}^{k} \sigma_{a_1\cdots a_{j-1} a_{j+1}-1 \cdots a_k - 1} \cdot \sigma_{a_j + k - j}; \tag{4}$$

from this, we get the following relation (the Giambelli's Formula):

$$\sigma_{a_1\cdots a_k} = \det \begin{pmatrix} \sigma_{a_1} & \sigma_{a_1+1} & \sigma_{a_1+2} & \cdots & \sigma_{a_1+k-1} \\ \sigma_{a_2-1} & \sigma_{a_2} & \sigma_{a_2+1} & \cdots & \sigma_{a_2+k-2} \\ \sigma_{a_3-2} & \sigma_{a_3-1} & \sigma_{a_3} & & \\ \vdots & & & & \\ \sigma_{a_1-k+1} & & \cdots & & \sigma_{a_k} \end{pmatrix}. \tag{5}$$

REMARK. We note that Pieri's and Giambelli's formulas give us an algorithm for evaluating every intersection of cycles. Besides we can observe that the Chow ring (or the cohomology ring) of the Grassmannians is generated by the classes of the special Schubert cycles.

4. Universal Bundles and Plücker Embeddings of the Grassmannians

We recall some basic facts (without proofs) about the universal subbundle, the universal quotient bundle on $G(k,n)$ and the Plücker embedding of $G(k,n)$. We follow **[GH78]**, **[Arr96]** and **[AS92]**; in particular, **[GH78]** for the notations.

Let us consider the product $\mathbb{C}^n \times G(k,n)$; this induces the projection maps on the two factors:

$$\begin{array}{ccc} \mathbb{C}^n \times G(k,n) & \xrightarrow{\ p\ } & \mathbb{C}^n \\ {\scriptstyle q}\downarrow & & \\ G(k,n); & & \end{array}$$

the map q defines trivial vector bundle of rank n over $G(k,n)$. By abuse of notation, we will denote with $\mathbb{C}^n \times G(k,n)$ this trivial vector bundle.

1.10. DEFINITION. We define the *universal subbundle* $S \to G(k,n)$ to be the subbundle of $\mathbb{C}^n \times G(k,n)$ whose fibre at each point $\Lambda \in G(k,n)$ is just the subspace $\Lambda \subset \mathbb{C}^n$.

It is easy to see that S is a holomorphic subbundle of $\mathbb{C}^n \times G(k,n)$.

1.11. DEFINITION. The quotient subbundle $Q := (\mathbb{C}^n \times G(k,n))/S$ is called the *universal quotient subbundle* of $G(k,n)$.

REMARK. Under the identification $* : G(k,n) \to G(n-k,n)$, the universal subbundle of $G(n-k,n)$ corresponds to the dual of the universal quotient bundle of $G(k,n)$, and likewise $Q \to G(n-k,n)$ pulls back to the dual $S^* \to G(k,n)$.

In particular, if we take $k=1$, *i.e.* if we are on $\mathbb{P}^{n-1}$, the universal subbundle is the universal line bundle, *i.e.* the line bundle whose corresponding invertible sheaf is $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$.

We can associate to the three bundles on $G(k,n)$, $\mathbb{C}^n \times G(k,n)$, S and Q their locally free sheaves, which will be, respectively, $\mathbb{C}^n \otimes \mathcal{O}_{G(k,n)}$, $\mathcal{S}$ and $\mathcal{Q}$. By what we have just said, we get the so-called *universal exact sequence*:

$$0 \to \mathcal{S} \to \mathbb{C}^n \otimes \mathcal{O}_{G(k,n)} \to \mathcal{Q} \to 0. \tag{6}$$

An alternative way of constructing the universal bundles is to consider in $\mathbb{P}^{n-1} = \mathbb{P}(\mathbb{C}^n)$ the Euler sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^{n-1}}(-1) \to \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}^{n-1}} \to \mathcal{T}_{\mathbb{P}^{n-1}}(-1) \to 0 \tag{7}$$

then we pull it back to the incidence variety $\mathcal{H}_{k-1,n-1} \subset \mathbb{P}^{n-1} \times G(k,n)$ and we push it forward to $G(k,n)$, we get the universal exact sequence. In particular

$$\mathcal{S} = q_* p^*(\mathcal{O}_{\mathbb{P}^{n-1}}(-1)) \tag{8}$$

$$\mathcal{Q} = q_* p^*(\mathcal{T}_{\mathbb{P}^{n-1}}(-1)), \tag{9}$$

where, by abuse of notation, p is the projection to $\mathbb{P}^{n-1}$.

PROPOSITION 1.6. *We have the following canonical isomorphisms of invertible sheaves on* $G(k,n)$*:*

$$\wedge^{n-k} \mathcal{Q} \cong \wedge^k \mathcal{S}^* \cong \mathcal{O}_{G(k,n)}(1). \tag{10}$$

PROOF. It follows from the universal exact sequence: let us take the highest exterior powers, we get a perfect paring:

$$\wedge^{n-k} \mathcal{Q} \otimes \wedge^k \mathcal{S} \cong \wedge^n \mathbb{C}^n \otimes \mathcal{O}_{G(k,n)} \cong \mathcal{O}_{G(k,n)}$$

(see **[Har77]** for a proof) and so our claim. □

We recall other results:

THEOREM 1.7. *The tangent sheaf of* $G(k,n)$ *is canonically isomorphic to the sheaf* $\mathcal{H}om(\mathcal{S},\mathcal{Q}) \cong \mathcal{S}^* \otimes \mathcal{Q}$ *and the canonical sheaf is* $\omega_{G(k,n)} \cong \mathcal{O}_{G(k,n)}(-n)$.

The *Plücker embedding* of the Grassmannian $G(k,n)$ is the embedding given by the line bundle $L := \det(S^*) \cong \det(Q)$—we recall that the determinant of a bundle is the highest exterior power. In any case, we shall give the Plücker embedding directly. The *Plücker map*

$$\pi : G(k,n) \to \mathbb{P}(\wedge^k \mathbb{C}^n) \cong \mathbb{P}^{\binom{n}{k}-1}$$

simply sends a k-plane $\Lambda = \langle v_1, \dots, v_k \rangle \subset \mathbb{C}^n$ to the multivector $v_1 \wedge \dots \wedge v_k$.

It is easy to see that π is an embedding, takes every Schubert cycle of the form σ_1 into a hyperplane section of $\pi(G(k,n))$ and that $\pi(G(k,n))$ is cut out by a linear system of quadrics: see, for example **[GH78]**, page 211.

5. Focal Diagram Associated to a Family of k-planes of $\mathbb{P}^n$

We start this section by recalling that $\mathbb{G}(k,n)$ is the Hilbert scheme of the flat families of k-planes of $\mathbb{P}^n$, see example 1, and so any flat family of k-planes of $\mathbb{P}^n$, (Λ, B, p), is given by the pull-back of the universal family $\mathcal{H}_{k,n}$ by the functorial map

$$\phi : B \to \mathbb{G}(k,n). \tag{11}$$

Equivalently, we can give a family of k-planes as a (in general singular) subvariety B' of the Grassmannian $\mathbb{G}(k,n)$ and then we can take its desingularization B. So, our family will be the triple $(\Lambda, B, p) := (\phi^* \mathcal{H}_{k,n}, B, p)$.

PROPOSITION 1.8. *The total space Λ of a flat family (Λ, B, p) of k-planes of $\mathbb{P}^n$ with B nonsingular is nonsingular.*

PROOF. It follows from the fact that the total space is the pull-back of the universal family, that is indeed a vector bundle on the nonsingular variety B. □

1.12. DEFINITIONS. Let us consider now a flat family (Λ, B, p) of k-planes of $\mathbb{P}^n$ obtained by the desingularization of a subscheme B' of dimension $n-k$ of the Grassmannian $\mathbb{G}(k,n)$; this family (or better its basis B or also B') is also called *congruence of k-planes of* $\mathbb{P}^n$ or, classically, *complex of k-planes.*

Then, we have

$$\begin{array}{ccc} \Lambda \subset B \times \mathbb{P}^n & \xrightarrow{f} & \mathbb{P}^n \\ {\scriptstyle p}\big\downarrow & & \\ B & & \end{array} \tag{12}$$

where $p := p_{1|_\Lambda}$, $f := p_{2|_\Lambda}$ and so $f(\Lambda_b) =: \Lambda(b)$ is a k-plane of $\mathbb{P}^n$ (as before, $\Lambda_b := p^{-1}(b)$).

As before, we have the *global characteristic map*:

$$\lambda : \mathcal{T}_{B\times\mathbb{P}^n|_\Lambda} \to \mathcal{N}_{\Lambda/B\times\mathbb{P}^n}$$

and the one relative to b:

$$\begin{array}{ccc} \lambda(b) : T_{B,b} \otimes \mathcal{O}_{\Lambda(b)} & \longrightarrow & \mathcal{N}_{\Lambda(b)/\mathbb{P}^n} \\ \cong\big\downarrow & & \cong\big\downarrow \\ \mathcal{O}_{\Lambda(b)}^{n-1} & \longrightarrow & \mathcal{O}_{\Lambda(b)}(1)^{n-k}. \end{array}$$

From this description, we obtain

PROPOSITION 1.9. *On every fibre Λ_b of the family, the subscheme of foci $V(\lambda)$ either coincides with the whole k-plane Λ_b or is a hypersurface of Λ_b of degree $n-k$.*

PROOF. From the preceding isomorphism, we have that the map $\lambda(b)$ can be seen as a $(n-k) \times (n-k)$-matrix with linear entries on $\Lambda(b)$; so the scheme of foci on Λ_b is given by the vanishing of the determinant of this matrix, and our claim follows. □

1.13. DEFINITION. If the k-plane Λ_b is contained in the scheme of the first order foci $V(\lambda)$, then it is called *focal k-plane*.

1.14. DEFINITION. By *locus of the second order foci* we mean the set of ramification points of $g := f|_{V(\lambda)}$.

REMARK. The fundamental locus is contained in the locus of the second order foci, since the fibre of g at the points of F has dimension greater than the general one.

It can be easily shown that the locus of the second order foci is the locus of first order foci of the family given by the scheme of the first order foci $V(\lambda)$. If the family of k-planes does not consist of focal k-planes only, then the family given by $V(\lambda)$, *i.e.* $(p(V(\lambda)), V(\lambda), p|_{V(\lambda)})$ (and if $p(V(\lambda))$ is not smooth we will consider its desingularization) is a family of $(k-1)$-dimensional k-*linear*—*i.e.* contained in a k-plane—schemes of degree $(n-k)$ of $\mathbb{P}^n$ of dimension $\dim(p(V(\lambda))$, by proposition 1.9.

By recurrence, we could clearly define the loci of higher order foci.

THEOREM 1.10. *Let $\ell := f(\Lambda_b)$ be a general k-plane of the congruence Λ. If the locus of second order foci is properly contained in F, then ℓ is tangent to the locus of the first order foci F at its points $\ell \cap F$.*

PROOF. If $P \in (V(\lambda) \cap \Lambda_b)$ is a general point, then we have an epimorphism of vector spaces

$$\mathrm{d}_P f : T_{\Lambda,P} \to T_{F,f(P)} \subset T_{\mathbb{P}^n,f(P)},$$

where $f(P)$ is not a second order focus.

This induces a surjective morphism on the embedded tangent spaces

$$\mathrm{d}_P f : \mathbb{T}_{\Lambda,P} \to \mathbb{T}_{F,f(P)}. \tag{13}$$

Since the line Λ_b is contained in Λ, then our thesis follows from the fact that

$$\Lambda_b \subset \mathbb{T}_{\Lambda,P},$$

and so, $\ell \subset \mathbb{T}_{F,f(P)}$. □

2. EXAMPLE. The most simple example of the preceding Theorem is given in $\mathbb{P}^2$. In this case the congruence is given by a curve C in the dual plane $\mathbb{P}^{2*}$ and the focal locus is the dual curve $C^* \subset \mathbb{P}^2$. Theorem 1.10 states then the fact that the general line of the congruence is in fact tangent to C^*.

REMARK. If the hypothesis that F must properly contain the locus of second order foci fails, then nothing can be said: in fact, we have also that $F = \Phi(\Lambda)$ and, for example, the map (13) has not maximal rank, *i.e.* it is not surjective.

We can see a simple example of this:

3. EXAMPLE. The Schubert cycle $\sigma_{(n-2)1}$, which is

$$\begin{aligned}\sigma_{(n-2)1} &= \{\Lambda \in \mathrm{G}(2, n+1) \mid \dim(\Lambda \cap V_2) \geq 1,\ \dim(\lambda \cap V_n \geq 2\}) \\ &= \{\ell \in \mathrm{G}(1,n) \mid \dim(\ell \cap \ell'),\ \ell, \ell' \subset \mathbb{P}^{n-1}\}),\end{aligned}$$

(*i.e.* the set of lines contained in a hyperplane and meeting a line); in this case, the focal locus is given by the line ℓ', and the lines of the congruence simply intersect it.

CHAPTER 2

First Order Congruences of Lines of $\mathbb{P}^n$

Since for the low dimensional projective spaces there is not much to say, we will suppose, in what follows, that $n \geq 3$.

1. Generalities

The starting point is to consider, as we have done in Section 5 of Chapter 1, a flat family (Λ, B, p) of straight lines in $\mathbb{P}^n$, parametrized by a nonsingular scheme B of dimension $n-1$. In this case, if we identify B with its class in the analytic cohomology of the Grassmannian (and if we identify it with its image B' via the map (11)), we have that $n-1 = \dim B = \operatorname{codim} B$, and therefore, by Theorem 1.5 we can write:

$$B = \sum_{i=0}^{\nu} a_i \sigma_{(n-1-i)i}, \tag{14}$$

where we put $\nu := \left[\frac{n-1}{2}\right]$.

2.1. Definition. We say that *the congruence B has the sequence of degrees (or multidegree, or is a congruence of $(\nu+1)$ – degree)* $(a_0, \ldots, a_\nu)$ if Equation (14) holds.

The first thing one can say is to explain the geometrical meaning of the sequence of degrees:

Theorem 2.1. *Let B be a congruence of lines of $\mathbb{P}^n$ whose sequence of degrees is $(a_0, \ldots, a_\nu)$; then a_j is the number of lines intersecting a general j-plane and contained in a general $(n-j)$-plane of $\mathbb{P}^n$.*

Proof. It is an easy consequence of the Schubert calculus; in fact:

$$\begin{aligned}\sigma_{(n-1-j)j} &= \{\Lambda \in \mathrm{G}(2, n+1) \mid \dim(\Lambda \cap V_{j+1}) \geq 1,\ \dim(\Lambda \cap V_{n+1-j}) \geq 2\} \\ &= \{\ell \in \mathbb{G}(1,n) \mid \emptyset \neq \mathbb{P}^j \cap \ell \subset \mathbb{P}^{n-j}\}.\end{aligned}$$

Then, by the formula (2), we have that:

$$\begin{aligned}B \cdot \sigma_{(n-1-j)j} &= \Big(\sum_{i=0}^{\nu} a_i \sigma_{(n-1-i)i}\Big) \cdot \sigma_{(n-1-j)j} \\ &= a_j.\end{aligned}$$

□

Corollary 2.2. *In the hypothesis of the preceding Theorem, we obtain, in particular, that*

1. *a_0 is the number of straight lines passing through a general point $P \in \mathbb{P}^n$;*
2. *a_ν is the number of lines contained in a general hyperplane H of $\mathbb{P}^n$ and that intersect a general line of H.*

2.2. Definitions. Classically, a_0 is called the *order* of the congruence, and a_ν its *class*. So a first order congruence is a congruence with sequence of degrees $(1, a_1, \ldots, a_\nu)$.

REMARK. Corollary 2.2 says in particular that the map $f : \Lambda \to \mathbb{P}^n$—which, we recall, is defined in the Diagram (12)—is generically $(a : 1)$. So, if $a \geq 1$, we see that the family Λ is non-degenerate; then, by proposition 1.2, $\mathrm{d}f$ has maximal rank and so the scheme of first order foci is a proper subscheme of Λ. We expect that the map $p_1|_{V(\lambda)}$ will be generically $(n-1:1)$ by proposition 1.9, so the "expected" dimension of the locus of the first order foci is $n-1$.

REMARK. We said that the "expected" dimension for F is $n-1$, and on a general line of the congruence the scheme of foci has length $n-1$; so, in general, the "expected" dimension for the fundamental locus is $\leq n-2$, since "in general" it has dimension less than the focal locus. It can happen that there exists a $(n-2)$-(pure) dimensional scheme which (in general) intersects every line of the congruence in—at most—$n-1$ points, or a scheme of dimension $d < n-2$ which intersects every line of the congruence. These are clearly subschemes of the fundamental locus of the family, so we give the following

2.3. DEFINITIONS. *The fundamental $(n-2)$-locus* for the congruence Λ is the component of the fundamental locus of (pure) dimension $(n-2)$ which intersects every line of the congruence (in—at most—$n-1$ points); *the fundamental d-locus*, with $d < n-2$, is the component of the fundamental locus of (pure) dimension d which intersects every line of the congruence.

PROPOSITION 2.3. *If C is a fundamental $(n-2-j)$-locus (with $j > 0$) for a congruence Λ, then the intersection of $f^{-1}(C)$ with a general line of the congruence is given by a 0-dimensional subscheme of length (at least) $j+1$.*

PROOF. Since C intersects every line of the congruence, we must have

$$\dim f^{-1}(C) = n-1,$$

and therefore, if $P \in C$ is a general point, then $\dim f^{-1}(P) = j+1$. Assume that $P \in \Lambda(b)$, where $\Lambda(b)$ is a general line of the congruence.

By proposition 1.2, or, better by formula (1), we obtain that

$$\mathrm{rk}(\mathrm{d}f|_{f^{-1}(C)}) = \mathrm{rk}(\lambda|_{f^{-1}(C)});$$

we have that

$$\mathrm{rk}(\mathrm{d}f|_{f^{-1}(C)}) = n-2-j$$

by hypothesis. The closed set

$$S_{j+1} := \{(P,\ell) \in \Lambda \mid \mathrm{rk}(\mathrm{d}f|_{f^{-1}(C)}) \leq n-2-j\}$$

has a natural subscheme structure, which is defined by a Fitting ideal, *i.e.* the ideal generated by the $(n-1-j)$-minors of $\mathrm{d}f$, see **[Kle77]**; besides $f^{-1}(C)$ is a subscheme of S_{j+1}.

So, $f^{-1}(C) \cap \Lambda_b$ (scheme-theoretically) is a 0-dimensional scheme Z, for which the matrix A associated to the characteristic map relative to b, $\lambda(b)$, has rank (at most) $(n-2-j)$. This means that the determinant of this matrix has (at least) $(j+1)$ roots concentrated in P; in fact, the characteristic polynomial of the matrix $\lambda(b)(P)$ (*i.e.* the matrix $\lambda(b)$ calculated in P), has zero as root of multiplicity (at least) $(j+1)$, since the kernel has dimension $(j+1)$. Then, our thesis follows. ☐

COROLLARY 2.4. *If P is a fundamental 0-locus for a congruence Λ, then Λ is the star of lines through P.*

PROOF. Since P intersects every line of the congruence, we must have

$$\dim f^{-1}(P) = n-1,$$

so $\Lambda = f^{-1}(P)$; but Λ will be contained in the star of lines through P, which has dimension $n-1$, so our thesis follows. ☐

4. EXAMPLE. Consider a non-linear surface S of $\mathbb{P}^4$ and let B be the family of its trisecants lines. B is clearly a flat family. There is a natural surjective map $\phi : \mathrm{Al}^3\,\mathbb{P}^4 \to B$, where $\mathrm{Al}^3(S) \subset \mathrm{Hilb}^3(S)$ is the subscheme of collinear (*i.e.* which are contained in a line) triple points (see [**Bar87**]).

Indeed, $\mathrm{Al}^3(S) \cong \mathrm{Hilb}^3(S) \times_{\mathrm{Hilb}^3(\mathbb{P}^4)} \mathrm{Al}^3(\mathbb{P}^4)$, so either $\mathrm{Al}^3(S)$ is empty or has dimension at least three, because $\dim(\mathrm{Al}^3\,\mathbb{P}^4) = 9$. Hence the same conclusion holds for B, because the general fibre of ϕ is finite (if not, all the trisecants lines of S are contained in S, and S is a plane).

Moreover, we can exclude that $\dim(B) > 3$. In fact, if it had dimension four, then for a general point P of S would pass ∞^2 lines of the congruence. Therefore, this would be the join of S and P and so the general secant line would be a trisecant: this would contradict the trisecant Lemma.

So, if not empty, the family B has dimension three. S is a fundamental surface and is contained in the fundamental locus. The three secancy points of a general trisecant line are its three foci.

But, in general, we can also have a focal locus of dimension three, if through all points of S pass focal lines: in fact, the family of 4-secant lines is empty or has dimension at least two, because $\mathrm{Al}^4(S) \cong \mathrm{Hilb}^4(S) \times_{\mathrm{Hilb}^4(\mathbb{P}^4)} \mathrm{Al}^4(\mathbb{P}^4)$, and $\dim(\mathrm{Al}^4(\mathbb{P}^4)) = 12$. But clearly a 4-secant line is a fundamental line, so "in general"—*i.e.* if the family of the 4-secants is not empty—the closure of the union of 4-secant lines is a focal locus of dimension three.

2. Linear Sections of a Congruence in $\mathbb{G}(1,n)$

We recall (Theorem 2.1) that the class a_ν of a congruence is the degree of the scroll generated by the lines of the congruence which belong to a general $\mathbb{P}^{n-1}$ (*i.e.* as a Schubert cycle, $B \cdot \sigma_{(n-1-\nu)\nu}$).

LEMMA 2.5. *Let $B(\subset \mathbb{G}(1,n))$ be a congruence with the sequence of degrees $(a_0,\dots,a_\nu)$; then we have that*

$$B \cdot \sigma_1 = \sum_{k=1}^{\nu}((\sum_{i=0}^{k} a_i)\sigma_{(n-k)k}). \tag{15}$$

PROOF. We have that $B = \sum_{i=0}^{\nu} a_i \sigma_{(n-1-i)i}$, and

$$\begin{aligned}\sigma_1 &= \{\Lambda \mid \dim \Lambda \cap V_{n-1} \geq 1\}\\ &= \{\ell \mid \ell \cap \Pi \neq \emptyset\}\end{aligned}$$

(where $\Pi \cong \mathbb{P}^{n-2}$) so, applying Pieri's formula to the cycle $G_\Pi := B\cdot\sigma_1$, we obtain

$$\begin{aligned}G_\Pi &= (\sum_{i=0}^{\nu} a_i \sigma_{(n-1-i)i}) \cdot \sigma_1\\ &= \sum_{i=0}^{\nu} a_i (\sigma_{(n-1-i)i} \cdot \sigma_1)\\ &= \sum_{i=0}^{\nu} a_i (\sum_{\substack{(n-1-i)\leq c_{i_1}\\ i\leq c_{i_2}\leq (n-1-i)\\ c_{i_1}+c_{i_2}=n}} \sigma_{c_{i_1} c_{i_2}})\end{aligned}$$

$$= \sum_{i=0}^{\nu} a_i (\sum_{j=1}^{i+1} \sigma_{(n-1-j)j})$$
$$= \sum_{k=1}^{\nu} ((\sum_{i=0}^{k} a_i) \sigma_{(n-k)k}).$$

□

PROPOSITION 2.6. *Let V_Π be the scroll given by the lines of the congruence which meet a (fixed) general $(n-2)$-plane Π. Then V_Π is a hypersurface of $\mathbb{P}^n$ of degree $\sum_i a_i$.*

PROOF. With notation as in Lemma 2.5, note that $p(f^{-1}(V_\Pi)) = G_\Pi$.

Since V_Π is a hypersurface, to obtain its degree we can intersect it with a general line and compute the length of the zero dimensional subscheme so obtained, or, which is the same, calculate the intersection of G_Π with the Schubert cycle σ_{n-2}, *i.e.* the lines which meet a general line; in fact:

$$\sigma_{n-2} = \{\Lambda \mid \dim \Lambda \cap V_2 \geq 1\}$$
$$= \{\ell \mid \ell \cap \ell' \neq \emptyset\}.$$

So, from intersection's formula for complementary Schubert cycles, *i.e.* formula (2), we have

$$G_\Pi \cdot \sigma_2 = \sum_{i=0}^{\nu} a_i.$$

□

REMARK. In the case of first order congruences, if ℓ is a line of the congruence not contained in V_Π, and P is a point of $V_\Pi \cap \ell$, then P is a focus for the congruence, since at least two lines of the congruence pass through it.

3. First Order Congruences

From now on we will consider first order congruences; in any case, some of the results we will obtain will be valid without this hypothesis.

As we have seen in 2.2, a first order congruence is a congruence with sequence of degrees $(1, a_1, \dots, a_\nu)$, *i.e.* through a general point of $\mathbb{P}^n$ passes only one line of the congruence.

The first observation, due to C. Segre in **[Seg88]** is the following:

PROPOSITION 2.7. *The fundamental locus of a first order congruence Λ coincides with the focal locus and has dimension at most $n-2$.*

PROOF. The fact that the two loci coincide is a straightforward consequence of the fact that the map f is generically $(1:1)$. Then, the fundamental locus F cannot have dimension $n-1$; otherwise the subscheme of the first order foci $V(\lambda)$ would coincide with Λ, and this would contradict the fact that we have a $(1:1)$ map. □

REMARK. We can also prove the proposition in another way: we can apply the following

ZARISKI'S MAIN THEOREM. *Let $h : X \to Y$ be a morphism of projective varieties, with Y normal. Then, if h is birational, all the fibres of the morphism h are connected.*

For a proof, see **[Har77]**. In our case, applying the Zariski's Main Theorem to the map $f : \Lambda \to \mathbb{P}^n$ if we have a focal point, *i.e.* a point through which passes more than one line, then there will be infinitely many lines through this point.

Then, we can collect in the next theorem the results of Section 2 of Chapter 2, in the case of a first order congruence of $\mathbb{P}^n$:

THEOREM 2.8. *Let Λ be a first order congruence of $\mathbb{P}^n$. Then the focal locus is a scheme whose components are the fundamental $(n-2-j_i)$-loci C_i (with $i = 1, \dots, s$ and $0 \leq j_i \leq n-2$) for the congruence Λ. The following relation holds:*

$$n - 1 = \sum_{i=1}^{s} (j_i + 1). \tag{16}$$

PROOF. It is a Corollary of proposition 2.3. □

In particular we observe that (apart from the trivial case of a star of lines) in $\mathbb{P}^3$ we can only have curves as fundamental loci, while in $\mathbb{P}^4$ we can have surfaces and curves. Besides, from Equation(16) we see that if there is a fundamental curve, then there must exist a surface also and the general line of the congruence will meet once the curve and the surface. We will precise these things in Theorems 3.2 and 4.3.

The next lemma is technical and will be used in Theorem 2.10:

LEMMA 2.9. *Let $B(\subset \mathbb{G}(1,n))$ be a (general) congruence with sequence of degrees $(a_0, \dots, a_\nu)$; then we have that*

$$B \cdot \sigma_1 \cdot \sigma_1 = \sum_{i=1}^{\nu} ((\sum_{j=1}^{\nu+1-i} j a_{\nu-j}) \sigma_{(n-i)(i+1)}). \tag{17}$$

PROOF. This is, as usual, a calculus with the Schubert cycles on the formula (15); in fact:

$$\begin{aligned} B \cdot \sigma_1 \cdot \sigma_1 &= (B \cdot \sigma_1) \cdot \sigma_1 \\ &= (\sum_{k=1}^{\nu} ((\sum_{i=0}^{k} a_i) \sigma_{(n-k)k})) \cdot \sigma_1 \\ &= \sum_{k=1}^{\nu} ((\sum_{i=0}^{k} a_i)(\sum_{\substack{n-k \leq c_{i_1} \\ k \leq c_{i_2} \leq n-k \\ c_{i_1} + c_{i_2} = n+1}} \sigma_{c_{i_1} c_{i_2}})) \\ &= \sum_{k=1}^{\nu} ((\sum_{i=0}^{k} a_i)(\sum_{j=1}^{k} \sigma_{(n-j)(j+1)})) \\ &= \sum_{i=1}^{\nu} ((\sum_{j=1}^{\nu+1-i} j a_{\nu-j}) \sigma_{(n-i)(i+1)}). \end{aligned}$$

□

THEOREM 2.10. *Let Λ be a first order congruence, and let Π and Π' be general 2-planes of $\mathbb{P}^n$. Then the complete intersection of the hypersurfaces V_Π and $V_{\Pi'}$ is a (reducible) $(n-2)$-dimensional scheme Γ whose components are the locus of fundamental points $\Phi(\Lambda)$ and the scroll Σ given by the lines of the congruence meeting Π and Π', which has degree $\nu + \sum_{j=1}^{\nu-1} j a_{\nu-j}$.*

PROOF. First of all, we observe that if a point P of $V_\Pi \cap V_{\Pi'}$ does not belong to the scroll Σ, then it belongs to the fundamental surface. Indeed in this case

$$P \in \ell \cap \ell', \text{ where } \ell \in G_\Pi,\ \ell \in G_{\Pi'}, \text{ and } \ell \neq \ell'.$$

—Where, as in the proof of the preceding proposition, G_Π and $G_{\Pi'}$ denote the subvarieties of the Grassmannian corresponding to the two scrolls V_Π and $V_{\Pi'}$.

Since Λ is a first order congruence and P belongs to two of the lines of Λ, then it belongs to infinitely many ones, by proposition 2.7.

Then, if $P \in F$ is a general fundamental point, the set of fundamental lines through P, χ_P, is a cone of dimension (at least) two, so its intersection with Π and Π' will not be empty and therefore $P \in V_\Pi \cap V_{\Pi'}$.

Finally it remains to prove that the degree of the scroll is $\nu + \sum_{j=2}^{\nu} ja_{\nu-j}$; but this follows from formula (17):

$$\begin{aligned} G_\Pi \cdot G_{\Pi'} &= B \cdot \sigma_1 \cdot \sigma_1 \\ &= ((\sum_{j=1}^{\nu+1-i} ja_{\nu-j})\sigma_{(n-i)(i+1)}). \end{aligned}$$

Since the scroll Σ has dimension $(n-2)$, to obtain its degree we can intersect it with a general plane and compute the length of the zero dimensional subscheme obtained, or, which is the same, calculate the intersection of $G_\Pi \cdot G_{\Pi'}$ with the Schubert cycle σ_{n-3}, *i.e.* the lines which meet a general plane; in fact:

$$\begin{aligned} \sigma_{n-3} &= \{\Lambda \mid \dim \Lambda \cap V_3 \geq 1\} \\ &= \{\ell \mid \ell \cap \mathbb{P}^2 \neq \emptyset\}. \end{aligned}$$

So, finally, we obtain

$$\begin{aligned} G_\Pi \cdot G_{\Pi'} \cdot \sigma_{n-3} &= (\sum_{i=1}^{\nu}((\sum_{j=1}^{\nu+1-i} ja_{\nu-j})\sigma_{(n-i)(i+1)})) \cdot \sigma_{n-3} \\ &= \sum_{j=1}^{\nu} ja_{\nu-j} \\ &= \nu + \sum_{j=1}^{\nu-1} ja_{\nu-j}. \end{aligned}$$

□

5. EXAMPLE. Let us return to the example 4, adding the hypothesis that S generates a first order congruence. We suppose that the surface S contains a plane curve C of degree at least 3. Let η be the plane of C. Then, every line of η is a line of the congruence, all points of η are fundamental points and all the lines of η are focal lines. So η is a component of the fundamental surface.

This example motivates us to give the following

2.4. DEFINITION. A $(n-2)$-plane η of $\mathbb{P}^n$, is called *i-parasitic of the congruence* (Λ, B, p_1) (or simply *parasitic*) if every line

$$\Lambda(b) \in f^{-1}(\eta)$$

is contained in the focal scheme $V(\lambda)$ with multiplicity (at least) i, with $i \geq 1$, but η is not met by the general line of the congruence.

COROLLARY 2.11. *Let η be a $(n-2)$-plane, then the following are equivalents:*

1. *η is an i-parasitic space;*
2. *for the general line ℓ of η we have that*

$$\deg f^{-1}(\ell) = i, \quad \dim f^{-1}(\ell) \leq 2;$$

3. *if we identify B with its image in the Grassmannian, the $(n-2)$-plane η is such that its correspondent subscheme $p(f^{-1}(\eta)) := \sigma_\eta (= \sigma_{22})$ is contained in B with multiplicity i and B is not contained in the hyperplane section of $\mathbb{G}(1,n)$ given by the lines of $\mathbb{P}^n$. meeting η;*

4. *η is a component of the fundamental locus which is not a fundamental $(n-2)$-locus (see Definitions* 2.3*).*

REMARK. The definition of parasitic plane for a congruence of lines of $\mathbb{P}^4$ was introduced by Ascione in [**Asc97**], and used in [**Mar09b**]. Ascione was interested in classifying the surfaces S with one apparent triple point, *i.e.* the surface such that for a general point of $\mathbb{P}^4$ there passes only one trisecant line of S. Then the parasitic planes comes out naturally as the components of the fundamental surface different from S.

Marletta generalized this concept to general congruences in higher dimensional spaces, in [**Mar27**]; besides, he defined an i-parasitic d-plane, with $d < (n+1)/2$ as a linear space such that a general line ℓ of it is of multiplicity i for the congruence (*i.e.* $\deg(f^{-1}(\ell)) = i$). We will not follow Marletta in this, since—at least for the first order congruences—apart from the linear spaces of codimension two, the others are such that the general line of the congruence meets them (*i.e.* they are not parasitic in Ascione's sense).

REMARK. It is clear that a i-parasitic $(n-2)$-space η is such that $f^{-1}(\eta)$ is a component of the focal scheme $V(\lambda)$ with multiplicity i, then it is a component of the scheme Γ of the preceding Theorem of multiplicity i^2.

NOTATIONS. From now on, we will denote with i the multiplicity of the general focal $(n-2)$-space. We will also set $x := \sum i^2$, where i varies among all the i-parasitic spaces.

Concerning the case of $\mathbb{P}^3$, we must say that the situation is easier, since we do not have parasitic lines, otherwise, by Corollary 2.11, (2) the parasitic line would meet the general line of the congruence.

Indeed, we can have focal lines only; in any case, see Section 3 of Chapter 3 for more details on this point.

2.5. DEFINITION. The union of the components of the fundamental locus F which are not parasitic spaces is called *pure fundamental locus*, or, in what follows, simply *fundamental locus*.

PROPOSITION 2.12. *If F is the pure fundamental locus and η is a i-parasitic space, then $F \cap \eta$ is a hypersurface of η.*

PROOF. F is the pure fundamental locus, so it does not contain η.

Clearly, if the intersection $F \cap \eta$ were proper, we would find a $(n-4)$-dimensional scheme; but every line of the space η is a line of the congruence, so it contains at least $(n-1)$ focal points. If we had a $(n-4)$-dimensional scheme, we could find a line of η which would not intersect it, and then η itself would generate a first order congruence, but this cannot happen since we suppose that $F \neq \emptyset$. □

COROLLARY 2.13. *In the hypothesis of the preceding proposition, we have that*

$$i = \binom{\mu}{n-1} \tag{18}$$

where $\mu := \deg(F \cap \eta)$.

PROOF. A general line ℓ of the parasitic space η will intersect F in μ points, so taking the μ points $(n-1)$ by $(n-1)$ we obtain the formula. □

4. First General Examples of First Order Congruences

Let us start giving some general examples of first order congruences of lines; first of all, we will analyse the examples which come out from sections of the Grassmannian $\mathbb{G}(1, n)$.

4.1. Linear sections of $\mathbb{G}(1,n)$. In this subsection we will analyse the congruences that come out from linear section of the Grassmannian, *i.e.* we will consider the so called, classically, *linear congruences*.

We recall that the Schubert cycle which correspond to a hyperplane section of (the projective embedding of) the Grassmannian is σ_1, so, the following technical lemma gives us the formula for the general intersection of these special Schubert cycles:

LEMMA 2.14. *If $\ell \leq n-1$ and we set $k := [\frac{\ell}{2}]$, then the following formula holds:*

$$\sigma_1^\ell = \sum_{i=0}^{k} \binom{\ell-1}{i} \sigma_{(\ell-i)i} \tag{19}$$

PROOF. Let us prove the proposition by induction;

- for $\ell = 1$ it is obvious;
- let us suppose it is true for $\ell-1$; then, we have that, by inductive hypothesis

$$\sigma_1^{\ell-1} = \sum_{i=0}^{k'} \binom{\ell-2}{i} \sigma_{(\ell-1-i)i},$$

where $k' := [\frac{\ell-1}{2}]$; then we have to prove formula (19). By Pieri's formula, we have

$$\sigma_{(\ell-1-i)i} \cdot \sigma_1 = \sum_{\substack{\ell-1-i \leq c_{i_1} \\ i \leq c_{i_2} \leq \ell-1-i \\ c_{i_1}+c_{i_2}=\ell}} \sigma_{c_{i_1} c_{i_2}} = \begin{cases} \sigma_{(\ell-i)i} & \text{if } \ell-1-i = i \\ \sigma_{(\ell-i)i} + \sigma_{(\ell-1-i)(i+1)} & \text{otherwise,} \end{cases}$$

i.e. if $\ell - 1 - i \neq i$ the multiplication behaves as the multiplication in the develop of the binomial $(x+y)$; in fact we recall that

$$(y^i x^{\ell-1-i} + x^i y^{\ell-1-i})(x+y) = (x^i y^{\ell-i} + y^i x^{\ell-i}) + (x^{i+1} y^{\ell-1-i} + y^{i+1} x^{\ell-1-i}).$$

So, with i from 0 to k, in the formula the develop of σ_1 is equal to the develop of the binomial, while for i between k and ℓ the develop obviously vanishes. So formula (19) holds.

□

THEOREM 2.15. *If Λ is a congruence with sequence of degrees $(a_0, \ldots, a_\nu)$, then B, as subvariety of the Grassmannian has degree*

$$\deg(B) = \sum_{i=0}^{\nu} a_i \binom{n-2}{i}. \tag{20}$$

PROOF. We recall that σ_1 corresponds to the hyperplane section of the Grassmannian; so we have that, by Equation (19)

$$\begin{aligned}\deg(B) &= (\sum_{i=0}^{\nu} a_i \sigma_{(n-1-i)i}) \cdot \sigma_1^{n-1} \\ &= (\sum_{i=0}^{\nu} a_i \sigma_{(n-1-i)i}) \cdot (\sum_{j=0}^{\nu} \binom{n-2}{j} \sigma_{(n-1-j)j}) \\ &= \sum_{i=0}^{\nu} a_i \binom{n-2}{i}.\end{aligned}$$

□

COROLLARY 2.16. *A $(n-1)$-linear section B of the Grassmannian of the lines of $\mathbb{P}^n$ generates a first order congruence Λ with sequence of degrees*

$$(\nu+1)\text{-}\deg(\Lambda) = (1, \ldots, \binom{n-2}{i}, \ldots, \binom{n-2}{\nu});$$

in particular, as a subvariety of the Grassmannian, this is a smooth congruence of degree

$$\deg(B) = \sum_{j=0}^{\nu} \binom{n-2}{j}^2.$$

This corollary gives us a first non-trivial example of a first order congruence. Some general results about fundamental varieties of these congruences are given in [BM]; but for this, we need a "Bertini Type Theorem:"

BERTINI TYPE THEOREM. *Let E and F be two vector bundles on a variety X with ranks m and n respectively. Let $E^* \otimes F$ be generated by its global sections. Then for the generic morphism $\phi : E \to F$ $D_k(\phi)(:= \{x \in X \mid \mathrm{rk}(\phi_x) \leq k\})$ is empty or has the (expected) codimension $(m-k)(n-k)$ and $\mathrm{Sing}(D_k(\phi)) \subset D_{k-1}(\phi)$. In particular, if $\dim(X) < (m-k+1)(n-k+1)$ $D_k(\phi)$ is smooth for a generic ϕ.*

See [**Ott95**] page 27 for a proof of this Bertini type theorem.

THEOREM 2.17. *If Λ is a general linear congruence, then its focal locus is the degeneracy locus F of a general morphism*

$$\phi : \mathcal{O}_{\mathbb{P}^n}^{\oplus(n-1)} \to \Omega_{\mathbb{P}^n}(2) \tag{21}$$

of (coherent) sheaves on $\mathbb{P}^n$. In particular, F is smooth if $\dim(F) \leq 3$.

PROOF. In fact, if we start from the Euler sequence for $\mathbb{P}^n = \mathbb{P}(V)$ twisted by 2

$$0 \to \Omega_{\mathbb{P}^n}(2) \to \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}(1) \to \mathcal{O}_{\mathbb{P}^n}(2) \to 0$$

and we get the global sections, noting that

$$H^0(\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}(1)) \cong V^* \times V^*$$

and that

$$H^0(\mathcal{O}_{\mathbb{P}^n}(2)) \cong \mathrm{Sym}^2(V^*)$$

—where $\mathrm{Sym}^2(V^*)$ denote the symmetric algebra of V^*—we obtain that

$$H^0(\Omega_{\mathbb{P}^n}(2)) \cong (\wedge^2 V)^*. \tag{22}$$

From Equation (22) we can interpret a global section of $\Omega_{\mathbb{P}^n}(2)$ as a bilinear alternating form on V, or as a skew-symmetric matrix of type $(n+1) \times (n+1)$ with entries in the base field.

Then, the general morphism ϕ defined in (21) is assigned by giving $n-1$ general skew-symmetric matrices, $A_1, \dots, A_{n-1}$ and the corresponding degeneracy locus F in $\mathbb{P}^n$ is the set of points P such that

$$\sum_{i=1}^{n-1} \lambda_i A_i[P] = 0 \tag{23}$$

for some $(\lambda_1, \dots, \lambda_{n-1} \neq (0, \dots, 0)$, and $[P]$ denotes the column matrix of the coordinates of P.

To interpret F as the focal locus of a congruence, we consider the Plücker embedding of the Grassmannian, see [**GH78**] or [**Har92**]:

$$\psi : \mathbb{G}(1,n) \hookrightarrow \mathbb{P}(\wedge^2 V).$$

The (dual) space $\mathbb{P}(\wedge^2 V^*)$ parametrizes the hyperplane sections of $\mathbb{G}(1,n)$; then a hyperplane section H is represented by a linear equation in the Plücker coordinates p_{ij}:

$$\sum_{0 \le i < j \le n} a_{ij} p_{ij}. \tag{24}$$

We can associate it a skew-symmetric matrix $A := (a_{ij})_{\substack{i=0,\dots,n \\ j=0,\dots,n}}$ of order $n+1$.

2.6. Definitions. A point $P \in \mathbb{P}^n$ is called *centre* of H if all the lines through P belong to H. The space $\mathbb{P}(\ker(A))$ results to be the set of centres of H: it is called the *singular space* of H.

From this discussion, we see that the focal locus of a linear congruence is the set of centres of a (general) linear system of dimension $n-2$ of hyperplane sections of the Grassmannian.

From the Bertini type theorem, it follows that the singular locus of F is empty if the dimension of F is at most 3; in fact, we have

$$\begin{aligned} \dim(F) + 2 &= n \\ &< (n-1-n+2+1)(n-n+2+1) \\ &= 6. \end{aligned}$$

□

The following result about the focal loci of a linear congruence is proved in [BM]:

Proposition 2.18. *If F is the degeneracy locus of ϕ (defined in* (21)*) then the cohomology groups $H^i(\mathcal{I}_F(p))$, for $i > 0$, are all zero, except for $H^1(\mathcal{I}_F(n-3))$; in particular, F is arithmetically Buchsbaum.*

Besides, they observed, by the interpretation with the skew-symmetric matrices, that

Proposition 2.19. *If F is the focal locus of a general linear congruence of $\mathbb{P}^n$, then*

1. *if n is even, Equation* (23) *has at least one solution and F is a unirational variety parametrized by $\mathbb{P}^{n-2}$;*
2. *if n is odd, then the vanishing of the Pfaffian of the matrix of* (23) *defines a hypersurface Z of degree $(n+1)/2$ in $\mathbb{P}^{n-2}$, in which $\lambda_1, \dots, \lambda_{n-1}$ are the coordinates. Furthermore, if ϕ is general, for a fixed point $[\lambda] \in Z$, we find a line of solutions of equation* (23) *in F, so that F is a scroll over Z.*

In particular F is always not empty. Finally, in low dimensions, we have the following results, if the section is general:

1. *if* $n = 3$, *F is the union of two skew lines;*
2. *if* $n = 4$, *F is a smooth projected Veronese surface;*
3. *if* $n = 5$, *F is a (rational) 3-fold of degree seven, which is a scroll over a cubic surface in* $\mathbb{P}^3$. *It is also known as Palatini scroll (see* **[Ott92]***).*

4.2. Matrices of type $(n-1)\times n$ with linear entries. Following **[DP95]** and **[DES93]**, (see also **[Pop93]**) we construct codimension 2 subvarieties $F \subset \mathbb{P}^n$ as the determinantal loci of maps between (particular) vector bundles, or—which is the same—as degeneracy loci of the corresponding morphisms of coherent sheaves.

First of all, we start with an example:

6. EXAMPLE. Let us consider the rational normal curve C of $\mathbb{P}^3$; it is well known that its ideal is generated by the minors of order two of the following catalecticant matrix:

$$A := \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

where $x_0, \ldots, x_3$ are the projective coordinates of $\mathbb{P}^3$. It is easy to see that a secant line ℓ of C has equations $h_1 = h_2 = 0$ with:

$$h_1 := \sum_{i=0}^{2} \lambda_i x_i \qquad\qquad h_2 := \sum_{i=0}^{2} \mu_i x_{i+1}$$

where $(\lambda_0 : \lambda_1 : \lambda_2) \in \mathbb{P}^2$, *i.e.* a (h_1, h_2) is a (non zero) linear combination of the columns of A.

Then, the secant line ℓ has, as Plücker coordinates, the minors of order two of the matrix

$$B := \begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 & 0 \\ 0 & \lambda_0 & \lambda_1 & \lambda_2 \end{pmatrix}$$

i.e. we can define a map

$$\phi : \mathbb{P}^2 \to \mathbb{G}(1,3)$$

which associates the point $(\lambda_0 : \lambda_1 : \lambda_2)$ the Plücker coordinates of the line ℓ, *i.e.*

$$\phi(\lambda_0 : \lambda_1 : \lambda_2) := (\lambda_0^2 : \lambda_0\lambda_1 : \lambda_0\lambda_2 : \lambda_1^2 - \lambda_0\lambda_2 : \lambda_1\lambda_2 : \lambda_2^2).$$

So, the family of the secant lines of the rational normal curve is a Veronese surface.

After this example, let us consider the two sheaves $\mathcal{O}_{\mathbb{P}^n}^{\oplus(n-1)}$ and $\mathcal{O}_{\mathbb{P}^n}^{\oplus n}(1)$ and a general morphism $\phi \in \mathcal{H}om(\mathcal{O}_{\mathbb{P}^n}^{\oplus(n-1)}, \mathcal{O}_{\mathbb{P}^n}^{\oplus n)}(1))$, whose minors vanish in the expected codimension 2. In this case, $F := V(\phi)$ is a locally Cohen-Macaulay subscheme and the Eagon-Northcott complex **[BE75]**

$$0 \leftarrow \mathcal{O}_F(n) \leftarrow \mathcal{O}_{\mathbb{P}^n}(n) \cong \wedge^{n-1}\mathcal{O}_{\mathbb{P}^n}^{\oplus(n-1)*} \otimes \wedge^n \mathcal{O}_{\mathbb{P}^n}^{\oplus n}(1) \leftarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus n}(1) \xleftarrow{\phi} \mathcal{O}_{\mathbb{P}^n}^{\oplus(n-1)} \leftarrow 0$$

is exact and identifies $\operatorname{coker}\phi$ with the twisted ideal sheaf

$$\operatorname{coker}\phi \cong \mathcal{I}_F(n)$$

since the shifting term is

$$n = c_1\mathcal{O}_{\mathbb{P}^n}^{\oplus n}(1) - c_1\mathcal{O}_{\mathbb{P}^n}^{\oplus(n-1)}.$$

REMARK. Let ϕ_1 and ϕ_2 two elements of $\mathcal{H}om(\mathcal{O}_{\mathbb{P}^n}^{\oplus(n-1)}, \mathcal{O}_{\mathbb{P}^n}^{\oplus n)}(1))$ whose minors vanish in codimension 2. Then $V(\phi_1)$ and $V(\phi_2)$ lie in the same irreducible component of the Hilbert scheme (see, for example, **[BB90]** and **[MDP90]**).

Let us return to the Eagon-Northcott complex; in general a mapping cone between the minimal free resolutions of $\mathcal{O}_{\mathbb{P}^n}^{\oplus(n-1)}$ and $\mathcal{O}_{\mathbb{P}^n}^{\oplus n}(1)$ is a free resolution of $\mathcal{I}_F(n)$. In our case the Eagon-Northcott complex gives us directly a free resolution of our ideal sheaf:

$$0 \leftarrow \mathcal{O}_F(n) \leftarrow \mathcal{O}_{\mathbb{P}^n}(n) \leftarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus n}(1) \xleftarrow{\phi} \mathcal{O}_{\mathbb{P}^n}^{\oplus(n-1)} \leftarrow 0 \tag{25}$$

and normalize it; we obtain an exact sequence of type

$$0 \leftarrow \mathcal{I}_F \leftarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus n}(1-n) \xleftarrow{\phi(-n)} \mathcal{O}_{\mathbb{P}^n}^{\oplus(n-1)}(-n) \leftarrow 0. \tag{26}$$

Riemann-Roch without denominators (see **[Ful84a]**, pages 296–297) gives

$$1 - i_*(c(\mathcal{N}^*_{F/\mathbb{P}^n})) = c(\mathcal{O}_{\mathbb{P}^n}^{\oplus n}(1-n) - \mathcal{O}_{\mathbb{P}^n}^{\oplus(n-1)}(-n)); \tag{27}$$

where $i : \mathcal{F} \hookrightarrow \mathbb{P}^n$ is the inclusion and $\mathcal{O}_{\mathbb{P}^n}^{\oplus n}(1-n) - \mathcal{O}_{\mathbb{P}^n}^{\oplus(n-1)}(-n)$ is the difference in the Grothendieck group.

We can clearly compute the invariants of F by the formula (27), or, better, by computing its Hilbert polynomial by its resolution (26). We get

$$0 \leftarrow \mathcal{O}_F(k) \leftarrow \mathcal{O}_{\mathbb{P}^n}(k) \leftarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus n}(k+1-n) \xleftarrow{\phi} \mathcal{O}_{\mathbb{P}^n}^{\oplus(n-1)}(k-n) \leftarrow 0 \tag{28}$$

and then

$$\begin{aligned}
\mathrm{hilb}(F) &= p(k) && (29)\\
&= \chi(\mathcal{O}_F(k)) && (30)\\
&= \chi(\mathcal{O}_{\mathbb{P}^n}(k)) - \chi(\mathcal{O}_{\mathbb{P}^n}^{\oplus n}(k+1-n)) + \chi(\mathcal{O}_{\mathbb{P}^n}^{\oplus(n-1)}(k-n)) && (31)\\
&= \binom{n+k}{k} - n\binom{k+1}{n} + (n-1)\binom{k}{n} && (32)
\end{aligned}$$

From this we obtain:

PROPOSITION 2.20. *F is smooth if $n \leq 3$. The following formulas hold:*

$$\deg(F) = \binom{n}{2} \tag{33}$$

$$\pi(F) = 1 + \frac{2n-7}{3}\binom{n}{2} \tag{34}$$

where $\pi(F)$ is the sectional genus of F.

PROOF. The smoothness is a consequence of the Bertini type theorem:

$$\begin{aligned}
\dim(F) + 2 &= n\\
&< (n-1-n+2+1)(n-n+2+1)\\
&= 6.
\end{aligned}$$

For getting the formulas, we can consider a 3 dimensional linear section $H \cong \mathbb{P}^3$ of F, *i.e.* $F_H := F \cap H$; then the Hilbert polynomial (29) becomes

$$\begin{aligned}
\mathrm{hilb}(F_H) &= p(k)\\
&= \chi(\mathcal{O}_{F_H}(k))\\
&= \chi(\mathcal{O}_{\mathbb{P}^3}(k)) - \chi(\mathcal{O}_{\mathbb{P}^3}^{\oplus n}(k+1-n)) + \chi(\mathcal{O}_{\mathbb{P}^3}^{\oplus(n-1)}(k-n))\\
&= \binom{k+3}{k} - n\binom{k-n+4}{3} + (n-1)\binom{k-n+3}{3}\\
&= \frac{1}{6}(\binom{n}{2}k - 2n^3 + 9n^2 - 7n).
\end{aligned}$$

So, we obtain that $\dim(F_H) = 1$, (and so $\dim(F) = n-2$) and

$$\begin{aligned}\deg(F) &= \deg(F_H)\\ &= 6 \cdot \frac{1}{6}\binom{n}{2}\\ &= \binom{n}{2}.\end{aligned}$$

Finally, the sectional genus is given by

$$\begin{aligned}\pi(F) &= g(F_H)\\ &= 1 + \chi(\mathcal{O}_{F_H}(0))\\ &= 1 + \frac{2n-7}{3}\binom{n}{2}.\end{aligned}$$

□

We use now the *Liaison of algebraic varieties*, studied for the first time in **[PS74]**; in particular, we use the results of this article for the Liaison of codimension 2.

If $A_1, \dots, A_n$ are the minors of order $(n-1)$ of the matrix $\phi(-n)$, then the equations of a complete intersection X of type $(n-1, n-1)$ containing F are

$$h_1 := \sum_{i=1}^{n} \lambda_i A_i \qquad\qquad h_2 := \sum_{i=1}^{n} \mu_i A_i.$$

Then, the ideal sheaf of the scheme F' linked to F via $X(:= V(h_1, h_2))$ has a free resolution of the type

$$0 \leftarrow \mathcal{I}_{F'} \leftarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus(n-1)}(2-n) \oplus \mathcal{O}_{\mathbb{P}^n}^{\oplus 2}(1-n) \xleftarrow{\psi(-n)} \mathcal{O}_{\mathbb{P}^n}^{\oplus n}(1-n) \leftarrow 0,$$

where the matrix $\psi(-n)$ is defined as follows:

$$\psi(-n) := \begin{pmatrix} & {}^t\phi(-n) & \\ \lambda_1 & \cdots & \lambda_n \\ \mu_1 & \cdots & \mu_n \end{pmatrix}.$$

The resolution is not minimal, but we can minimize it:

(35) $$0 \leftarrow \mathcal{I}_{F'} \leftarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus(n-1)}(2-n) \xleftarrow{\psi(-n)} \mathcal{O}_{\mathbb{P}^n}^{\oplus(n-2)}(1-n) \leftarrow 0.$$

PROPOSITION 2.21. *A variety F defined by a resolution as in (26) (i.e. by the minors of a matrix $(n-1)\times(n)$ of linear entries) is linked in a complete intersection of type $(n-1, n-1)$ to a variety F' defined by the minors of a matrix $(n-2)\times(n-1)$ of linear entries. F' is smooth if $n < 6$ and the following formulas hold:*

(36) $$\deg F' = \binom{n-1}{2}$$

(37) $$\pi(F') = 1 + \frac{2n-9}{3}\binom{n-1}{2}$$

where $\pi(F')$ is the sectional genus of F'.

PROOF. See Proposition 2.20; in fact, if we get a hyperplane section of F', we get a $(n-2) \times (n-1)$ of $\mathbb{P}^{n-1}$. The Bertini type theorem gives, in this case

$$\begin{aligned}\dim(F') + 2 &= n \\ &< (n-2-n+3+1)(n-1-n+3+1) \\ &= 6.\end{aligned}$$

□

PROPOSITION 2.22. *With notations as above, given three general hypersurfaces of degree $n-1$ defined by h_1, h_2 and h_3 through a general point $P \in \mathbb{P}^n$, let us consider their intersection I. Then there is a component F_P of I of dimension $n-3$ and degree $\binom{n-1}{3}$ through P.*

PROOF. Fix a general point $P \in \mathbb{P}^n$. Let us take h_1 and h_2 as above passing through P; then F and F' are the two components of $X := V(h_1, h_2)$. Consider another $(n-1)$-tic h_3 through P and we intersect it with X, we obtain F and another component F'' of dimension $(n-3)$ and degree

$$\begin{aligned}\deg(F'') &= \deg(F')\deg(h_3) \\ &= \binom{n-1}{2}(n-1) \\ &= (n-1)^2\frac{n-2}{2}.\end{aligned}$$

Besides, F and F' intersect in a subscheme $S := F \cap F'$ of dimension $(n-3)$ and degree $(F \cap H) \cdot (F' \cap H)|_{h_1 \cap H}$, where H is a 3-dimensional linear section, which is a component of F'' not passing through P. We want to find the degree of the component F_P of F'' through P.

First of all we calculate $\pi(X)$: if we consider a 3-dimensional linear section H of X, then $X \cap H$ is a complete intersection of two surfaces of degree $(n-1)$; therefore its genus is (see for example **[Har77]**, page 352)

$$\begin{aligned}g(X \cap H) &= \frac{1}{2}(n-1)^2(n-1+n-1-4)+1 \\ &= (n-1)^2(n-3)+1\end{aligned}$$

and so $\pi(X) = (n-1)^2(n-3)+1$. Then, by Propositions 2.20 and 2.21

$$\begin{aligned}\deg(S) &= \pi(X) + 1 - \pi(F) - \pi(F') \\ &= (n-1)^2(n-3)+1+1-1-\frac{2n-7}{3}\binom{n}{2} - 1 - \frac{2n-9}{3}\binom{n-1}{2} \\ &= 2\cdot\binom{n}{3}.\end{aligned}$$

So, we obtain that the component F_P of F'' through P has degree

$$\begin{aligned}\deg F_P &= (n-1)^2\frac{n-2}{2} - 2\cdot\binom{n}{3} \\ &= \binom{n-1}{3}.\end{aligned}$$

□

COROLLARY 2.23. *With notations as above, if $n = 4$, then the trisecant lines of F generate a first order congruence.*

PROOF. In fact, a trisecant line ℓ through P is the variety F_P of the preceding theorem, since $\dim(F_P) = 1 = \deg(F_P)$ and, obviously $F_P \supset \ell$, by degree reasons, as $\deg(h_i) = 3$. $\square$

This way of reasoning does not allow to conclude if $n > 4$.

But, in fact, the $(n-1)$-secant lines do generate a first order congruence: for proving this, we see that the map ϕ give rise to a map

$$\varphi : \mathcal{O}_{\mathbb{G}(1,n)}^{\oplus n} \to (\mathcal{S}^*)^{\oplus(n-1)} \tag{38}$$

where φ is obtained by considering the dual of ϕ twisted by 1 and then pulled back to the incidence variety and finally pushed it forward to $\mathbb{G}(1,n)$, *i.e.* using notations as in Section 4 of Chapter 1, $\varphi = q_*p^*(\phi^*(1))$.

We can now apply the Eagon-Northcott complex to the morphism of coherent sheaves on $\mathbb{G}(1,n)$, $\varphi \in \mathcal{H}om_{\mathcal{O}_{\mathbb{G}(1,n)}}(\mathcal{O}_{\mathbb{G}(1,n)}^{\oplus n}, (\mathcal{S}^*)^{\oplus(n-1)})$ (we follow for the Eagon-Northcott complex, in this case [**GP82**], or [**Ott95**]), getting, since $\mathrm{rk}(\mathcal{S}) = 2$

$$\begin{aligned} 0 \to \mathcal{S}ym^{n-2}(\mathcal{O}_{\mathbb{G}(1,n)}^{\oplus n}) &\to (\mathcal{S}^*)^{\oplus(n-1)} \otimes \mathcal{S}ym^{n-3}(\mathcal{O}_{\mathbb{G}(1,n)}^{\oplus n}) \to \\ &\to \wedge^2((\mathcal{S}^*)^{\oplus(n-1)}) \otimes \mathcal{S}ym^{n-4}(\mathcal{O}_{\mathbb{G}(1,n)}^{\oplus n}) \to \cdots \\ \cdots \to \wedge^{n-2}((\mathcal{S}^*)^{\oplus(n-1)}) &\to \det((\mathcal{S}^*)^{\oplus(n-1)}) \otimes \det(\mathcal{O}_{\mathbb{G}(1,n)}^{\oplus n})^* \to \\ &\to \mathcal{O}_B \otimes \det((\mathcal{S}^*)^{\oplus(n-1)}) \otimes \det(\mathcal{O}_{\mathbb{G}(1,n)}^{\oplus n})^* \to 0 \end{aligned} \tag{39}$$

where $B := D_{n-1}(\varphi)$ and $\dim(B) = n-1$, *i.e.* is a congruence. The complex is exact and identifies $\operatorname{coker}\varphi$ with the twisted ideal sheaf

$$\operatorname{coker}\varphi \cong \mathcal{I}_B(n-1)$$

—where $\mathcal{I}_B$ is the ideal sheaf of B in the Grassmannian—since the shifting term is, recalling (10)

$$n-1 = c_1(\mathcal{S}^*)^{\oplus(n-1)} - c_1\mathcal{O}_{\mathbb{G}(1,n)}^{\oplus n}.$$

The Eagon-Northcott complex (39) gives, knowing (10) and some multilinear algebra

$$\begin{aligned} 0 \to \mathcal{O}_{\mathbb{G}(1,n)}^{\oplus\binom{n-1}{2}} &\to (\mathcal{S}^*)^{\oplus(n-1)\binom{n-2}{2}} \to \\ &\to \wedge^2((\mathcal{S}^*)^{\oplus(n-1)\binom{n-3}{2}}) \to \cdots \\ &\cdots \to \wedge^{n-2}((\mathcal{S}^*)^{\oplus(n-1)}) \to \mathcal{O}_{\mathbb{G}(1,n)}(n-1) \to \mathcal{O}_B(n-1) \to 0. \end{aligned} \tag{40}$$

Tensorizing, we get a free resolution of $\mathcal{I}_B$:

$$\begin{aligned} 0 \to \mathcal{O}_{\mathbb{G}(1,n)}^{\oplus\binom{n-1}{2}}(1-n) &\to (\mathcal{S}(n-1))^{\oplus(n-1)\binom{n-2}{2}} \to \\ &\to \wedge^2((\mathcal{S}(n-1))^{\oplus(n-1)\binom{n-3}{2}}) \to \cdots \\ &\cdots \to \wedge^{n-2}((\mathcal{S}(n-1))^{\oplus(n-1)}) \to \mathcal{O}_{\mathbb{G}(1,n)} \to \mathcal{O}_B \to 0. \end{aligned} \tag{41}$$

We know that the image of B in the Chow ring of the Grassmannian is (see [**GP82**], page 7) $c_{n-1}(\operatorname{coker}\varphi)$; then we have to calculate $c_{n-1}(\mathcal{I}_B(n-1))$. This Chern class can be calculated also from the free resolution of the ideal sheaf of B: in fact, we have that, from (41), the Chern class we want is $c_{n-1}(\wedge^{n-2}((\mathcal{S}(n-1))^{\oplus(n-1)}))$: see for example [**Har77**], page 431.

Perhaps the easiest way of calculating this Chern class is to apply the Giambelli-Thom-Porteous formula: see [**FP98**], page 15; we recall this formula gives the class in the Chow ring of a degeneracy locus of a map as a polynomial in the Chern classes of the two bundles:

GIAMBELLI-THOM-PORTEOUS FORMULA. *Let E and F be two vector bundles on a variety X with ranks m and n respectively. Let $E^* \otimes F$ be generated by its global sections and for the generic morphism $\phi : E \to F$ $D_k(\phi)$ has the (expected) codimension $(m-k)(n-k)$; then the class in the Chow ring of X of $D_k(\phi)$ is given by the formula*

$$[D_k(\phi)] = \Delta_\lambda(c) \qquad \lambda \text{ is the multiindex } \lambda := ((n-k)^{m-k}) \tag{42}$$

where $c := c(E - F) = c(E)/c(F)$ and $\Delta_\lambda(c)$ is the Schur determinant:

$$\Delta_\lambda(c) = \det \begin{pmatrix} c_{n-k} & c_{n-k+1} & \cdots & c_{n+m-2k-1} \\ c_{n-k-1} & c_{n-k} & 1 & cn+m-2k-2 \\ \vdots & & & \\ c_{n-m+1} & \cdots & c_{n-k-1} & c_{n-k} \end{pmatrix}.$$

From this we get that

$$\begin{aligned} [D_{n-1}(\varphi)] &= \Delta_{n-1}(c) \\ &= c_{n-1}(\mathcal{O}_{\mathbb{G}(1,n)}^{\oplus n} - (\mathcal{S}^*)^{\oplus(n-1)}) \\ &= c_{n-1}(\mathcal{S}^{\oplus(n-1)}). \end{aligned}$$

The calculation of this Chern class is not too difficult: fist of all, by the universal exact sequence of the Grassmannian (6)

$$\begin{aligned} c(\mathcal{S}) &= c(\mathcal{Q})^{-1} \\ &= s(\mathcal{Q}) \end{aligned}$$

—where c and s denote, respectively, the total Chern and Segre classes. but $c_i(\mathcal{Q}) = \sigma_i$, see **[Arr96]** and the inverse of a Chern class is given by the formula (see, for example, **[Ful84b]**)

$$c_i^{-1} = (-1)^i \det \begin{pmatrix} c_1 & 1 & 0 & \cdots & 0 \\ c_2 & c_1 & 1 & & \\ \vdots & & & & 0 \\ & & & c_1 & 1 \\ c_i & \cdots & & \cdots & c_1 \end{pmatrix},$$

therefore, applying Giambelli's formula (5) we get

$$c_i(\mathcal{S}) = (-1)^i \sigma_{1\cdots 1}$$

and, since $\mathrm{rk}(\mathcal{S}) = 2$, the total Chern class is

$$c(\mathcal{S}) = 1 - \sigma_1 + \sigma_{11},$$

and then

$$c(\mathcal{S}^{\oplus(n-1)}) = (1 - \sigma_1 + \sigma_{11})^{n-1}. \tag{43}$$

To see the order of the congruence B is therefore sufficient to find the coefficient of σ_{n-1} in (43); it is not hard to see, that by Pieri's formula, the only way to get σ_{n-1} is from the expansion of σ_1^{n-1}; then, by formula (19) (or by Corollary 2.16) we get that B is a first order congruence. So, we get that

THEOREM 2.24. *The $(n-1)$-secant lines of the variety F defined as above form a first order congruence B of lines of $\mathbb{P}^n$. The congruence B is smooth for general ϕ.*

PROOF. The fact that the family of the $(n-1)$-secant lines of the variety F is indeed B, can be seen arguing as in the proof of Proposition 2.22, intersecting by a sufficient number of general h_i, in such a way that for the general $P \in \mathbb{P}^n$ we get a 1-dimensional scheme, *i.e.* $i = 1, \ldots, n-1$.

The fact that the congruence B is smooth is a straightforward consequence of the Bertini type theorem: in fact, in this case we have

$$\begin{aligned}\dim(\mathbb{G}(1,n)) &= 2n-2\\ &< (n-n+1+1)(2n-2-n+1+1)\\ &= 2n.\end{aligned}$$

□

4.3. First order congruences whose focal locus is a linear space. We pass now to analyse the first order congruences whose focal locus is linear.

The following theorem characterize the congruences for which, set-theoretically, the fundamental locus is irreducible and linear. We obtain that there are two possibilities: set-theoretically, the fundamental locus is either a point or a $(n-2)$-plane only.

NOTATIONS. Let us fix some notations: first of all, ℓ_{n-2} will be a fixed $(n-2)$-plane of $\mathbb{P}^n$ and $\mathbb{P}^1_{\ell_{n-2}}$ the set of hyperplanes containing ℓ_{n-2}. Besides, we will denote by ℓ^*_{n-2} the set of all the hyperplanes of ℓ_{n-2}, with ϕ_{n-2} (respectively, ϕ^*_{n-2}) a general map from $\mathbb{P}^1_{\ell_{n-2}}$ to ℓ_{n-2} (respectively, ℓ^*_{n-2}) and with Π_{n-1} a general element of $\mathbb{P}^1_{\ell_{n-2}}$, while we set $\Pi_n := \mathbb{P}^n$.

Then, we define recursively ℓ_i, $i = 0, \ldots, n-3$ as an i-plane element of $\phi^*_{i+1}(\mathbb{P}^1_{\ell_{i+1}})$, with $\mathbb{P}^1_{\ell_{i+1}}$ the pencil of $(i+2)$-planes containing ℓ_{i+1} and contained in a fixed Π_{i+3}, which is a $(i+3)$-plane, element of a $\mathbb{P}^1_{\ell_{i+2}}$; ϕ_{i+1} (respectively, ϕ^*_{i+1}) is a general map from $\mathbb{P}^1_{\ell_{i+1}}$ to ℓ_{i+1} (respectively, ℓ^*_{i+1}).

Finally, $\mathbb{P}^i_{\ell_0}$ is the set of lines through a point ℓ_0 and contained in a $\Pi_{i+1} \in \mathbb{P}^1_{\ell_i}$ (and ℓ_i contains ℓ_0). We observe that $\ell^*_0 = \emptyset$ and $\ell^*_1 = \ell_1$, so we take $\phi_1 = \phi^*_1$.

THEOREM 2.25. *If the focal locus is—set-theoretically—a linear space, then it is either a point ℓ_0, in which case the congruence is a star of lines, or a $(n-2)$-plane ℓ_{n-2}. In the latter case, the congruence can be constructed as follows: there exists an index $i \in \{1, \ldots, n-2\}$ and there exist linear spaces $\ell_{n-2}, \ldots, \ell_i$ and maps ϕ^*_j, $j = i+1, \ldots, n-2$ and ϕ_i as above which can be constants or birational and (at least) ϕ_i is birational, such that*

$$B = \overline{\cup_{\Pi_{n-1}\in\mathbb{P}^1_{\ell_{n-2}}} \cup_{\ell_{n-3}\in\phi^*_{n-2}(\Pi_{n-1})} \cdots} \tag{44}$$
$$\overline{\cdots \cup_{\ell_{i+1}\in\phi^*_{i+2}(\Pi_{i+3})} \cup_{\ell_i\in\phi^*_{i+1}(\Pi_{i+2})} \cup_{\Pi_{i+1}\in\mathbb{P}^1_{\ell_i}} \mathbb{P}^i_{\phi_i(\Pi_{i+1})}}.$$

Before giving the proof of this theorem, let us see some examples:

1. EXAMPLES. Let us suppose $n = 5$, and $i = 2$; then, we suppose that

$$\phi^*_3 : \mathbb{P}^1_{\ell_3} \to C^* \subset \ell^*_3$$
$$\phi_2 : \mathbb{P}^1_{\ell_2} \to C' \subset \ell_2$$

are birational maps and where, indeed, ϕ_2 is a family of maps, since it varies when ℓ_2 varies in C^*. Then we have

$$B = \overline{\cup_{\Pi_4\in\mathbb{P}^1_{\ell_3}} \cup_{\ell_2\in\phi^*_3(\Pi_4)} \cup_{\Pi_3\in\mathbb{P}^1_{\ell_2}} \mathbb{P}^2_{\phi_2(\Pi_3)}},$$

i.e. for each hyperplane Π_4 containing ℓ_3 there is a plane ℓ_2 of ℓ_3 such that $\Lambda|_{\Pi_4}$ is constructed in this way: there is a birational map ϕ_2 between $\mathbb{P}^1_{\ell_2}$—the pencil of 3-planes of Π_3 passing through ℓ_2—and ℓ_2; then the congruence is given by the stars of lines through the point $\ell_0 := \phi_2(\Pi_3)$ contained in Π_3.

If, instead, ϕ_3^* is constant, then $\ell_2 := \phi_3^*(\mathbb{P}^1_{\ell_3})$ is fixed and the family of maps ϕ_2 can be interpreted as an isomorphism

$$\varphi_2 : \mathbb{P}^2_{\ell_2} \to \ell_2$$

—$\mathbb{P}^2_{\ell_2}$ is the set of hyperplanes containing ℓ_2—and the congruence is given as follows:

$$B = \overline{\cup_{\Pi_4 \in \mathbb{P}^2_{\ell_2}} \mathbb{P}^2_{\varphi_2(\Pi_4)}},$$

i.e. for each flag $\Pi_4 \subset \Pi_3 \subset \ell_2$ there is a point $\ell_0 := \varphi(\Pi_4)$ and the congruence is given by the star of lines of through ℓ_0 and contained in Π_3.

PROOF. From Theorem 2.8 we deduce easily that the linear spaces which generate a first order congruence are either a point ℓ_0 , and so, by Corollary 2.4 we have a star of lines $\mathbb{P}^{n-1}_{\ell_0}$, or a $(n-2)$-plane ℓ_{n-2}.

Let us prove the second part of the theorem by induction: for $\mathbb{P}^2$ the theorem is trivially satisfied, while cases $\mathbb{P}^3$ and $\mathbb{P}^4$ are Theorems 3.5, (1b) and 4.7 respectively.

Let us suppose now that the theorem is true $\forall i \leq n-1$; let ℓ_{n-2} be the fundamental locus for the congruence Λ; then, if we consider the general point $Q \in \mathbb{P}^n$ this determine a general hyperplane Π_{n-1} of $\mathbb{P}^1_{\ell_{n-2}}$, and the lines of the congruence contained in this hyperplane form a first order congruence there, since for the general point Q of Π_{n-1} there passes only one line of Λ.

Besides, we note that the focal locus of $\Lambda|_{\Pi_{n-1}}$ is contained in ℓ_{n-2}; in fact it can be either a point $\ell_0 \in \ell_{n-2}$ or a $(n-3)$-plane $\ell_{n-3} \subset \Pi_{n-1}$, by inductive hypothesis.

Then, we have the following possibilities:

1. If for almost every Π_{n-1} we have that the congruence induces a star of lines, then we can construct a map

$$\phi_{n-2} : \mathbb{P}^1_{\ell_{(n-2)(n-2)}} \to \ell_{n-2}$$

which associates to each hyperplane Π_{n-1} the support ℓ_0 of its star of lines. The map ϕ_{n-2} cannot be constant since otherwise we have a star of lines of $\mathbb{P}^n$, then, we apply Riemann-Hurwitz's Theorem getting that the image of ϕ_{n-2} is rational; besides, it must be generically injective because the degree of the map is in fact the order of the congruence and we obtain case (44), with $i = n-2$.

2. If for almost every Π_{n-1} the congruence induces a congruence of $\mathbb{P}^{n-1}$ with a focal $(n-3)$-plane, then we can construct a map

$$\phi^*_{n-2} : \mathbb{P}^1_{\ell_{n-2}} \to \ell^*_{n-2}$$

which associates the hyperplane Π_{n-1} the $(n-3)$-plane ℓ_{n-3} which is fundamental for the congruence restricted to Π_{n-1}. Then, we conclude by the inductive hypothesis on the $\Pi_{n-1} \cong \mathbb{P}^{n-1}$'s.

□

REMARK. We observe that the case of the star of lines is the case in which all the maps ϕ_j^* and ϕ_j are constants. Besides, with this observation, we could avoid using Theorem 2.8 to prove the statement, since everything would follow by inductive hypothesis.

We note, finally, that the congruences with $i = 1$ are particular cases of linear congruences, since they are given by the intersection of the hyperplane section

$\sigma_{\ell_{n-2}} \cong \sigma_1$ with hyperplanes $\sigma_{\ell^j_{n-2}} \cong \sigma_1$ such that $\ell^j_{n-2} \supset \ell_j$. From this we see besides that these congruences are—in general—non-smooth since they are contained in a hyperplane tangent section of $p(f^{-1}(\ell_1))$.

5. General Formulas and Degree Bounds

We will suppose, throughout this section, that the fundamental $(n-2)$-locus $F' \subset F$ is not empty and not linear, *i.e.* if $\Lambda(b)$ is a general line of the congruence, we have that $\text{length}(F' \cap \Lambda(b)) = \text{length}(f^{-1}(F') \cap \Lambda_b)$.

NOTATIONS. Assume that η is an i-parasitic $(n-2)$-plane and $C = \eta \cap F$ is the hypersurface of η of Proposition 4.8. Then we recall Corollary 2.13 and we denote with μ the degree of C; we have that the Equation (57) continues to hold.

Let F_i, $i = 1, \dots, h$ be the $(n-2)$-dimensional irreducible components of F'; we will denote then with m_i the degree of F_i and with k_i the multiplicity of $F_i \cap V_\Pi$ in V_Π. Finally, we denote with $s_i := \text{length}(F_i \cap \Lambda(b))$, where $\Lambda(b)$ is a general line of the congruence.

PROPOSITION 2.26. *The following formula holds:*

$$\sum_{i=1}^{h} s_i k_i \leq 1 + \sum_{j=1}^{\nu} a_j. \tag{45}$$

PROOF. If we take a line ℓ of the congruence not contained in $F \cap V_\Pi$, then, intersecting ℓ with V_Π, we obtain a 0-dimensional scheme of length $1 + \sum_{j=1}^{\nu} a_j$, since, as we have seen in Proposition 2.6, V_Π has degree $1 + \sum_{j=1}^{\nu} a_j$. This scheme contains the schemes $F_i \cap \cap \ell$ which has support in (at most) s_i points, each of them of length k_i, or, in the classical language, we have s_i foci of multiplicity k_i, and so the relation is proved. □

PROPOSITION 2.27. *The following formula holds:*

$$(1 + \sum_{j=1}^{\nu} a_j)^2 = x + \sum_{i=1}^{h} k_i^2 m_i + \nu + \sum_{i=1}^{\nu-1} i a_{\nu-i}. \tag{46}$$

PROOF. We recall that, by Theorem 2.10 the degree of $V_\Pi \cap V_{\Pi'}$ is $(1 + \sum_{j=1}^{\nu} a_j)^2$, and its components are the (pure) fundamental locus with multiplicity $\sum_{i=1}^{h} k_i^2 m_i$, the parasitic planes, which give $x = \sum i^2$, and the scroll Σ which has degree $\nu + \sum_{i=1}^{\nu-1} i a_{\nu-i}$, so

$$(1 + \sum_{j=1}^{\nu} a_j)^2 = x + \sum_{i=1}^{h} k_i^2 m_i + \nu + \sum_{i=1}^{\nu-1} i a_{\nu-i}.$$

□

THEOREM 2.28. *If the focal locus F coincides with the fundamental $(n-2)$-locus, it is irreducible and $m := \deg F$, then*

$$n - 1 < m < (n-1)^2. \tag{47}$$

PROOF. If we substitute formula (45) in this particular case, in formula (46) we obtain

$$(n-1)^2 k^2 - k^2 m - \nu - \sum_{i=1}^{\nu-1} i a_{\nu-i} = x \geq 0, \tag{48}$$

and since $-\nu - \sum_{i=1}^{\nu-1} i a_{\nu-i} < 0$, we deduce $m < (n-1)^2$

We have that $n - 1 < m$ by degree reasons, since the congruence is given by the $(n-1)$-secant lines of F. □

REMARK. The upper limit is not sharp, at least for smooth (or general) varieties: see Corollary 4.19.

CHAPTER 3

First Order Congruences of Lines in $\mathbb{P}^3$

As we saw at the beginning of chapter 2, the first non-trivial case is the one of the first order congruences of lines of $\mathbb{P}^3$, which in this section we will analyse, obtaining the Kummer's classical results ([**Kum75**]) in a different way from the other modern proofs of this in [**Ran86**], [**Gol85**] and [**ZILO**].

1. Generalities

We recall what we said in chapter 2, in the case of $\mathbb{P}^3$: we consider a flat family (Λ, B, p) of straight lines in $\mathbb{P}^3$, parametrized by a nonsingular scheme B of dimension 3. In this case, we can write:

$$B = a_0\sigma_{20} + a_1\sigma_{11}. \tag{49}$$

3.1. DEFINITION. We say that B is a *congruence of lines of bidegree* (a_0, a_1) if Equation (49) holds.

Corollary 2.2 in this case means:

THEOREM 3.1. *If B is a congruence of lines of bidegree (a_0, a_1), then*

1. *a_0 is the number of straight lines passing through a general point $p \in \mathbb{P}^3$;*
2. *a_1 is the number of lines contained in a general plane of $\mathbb{P}^3$.*

REMARK. in this case, the "expected" dimension for F is 2 and for the fundamental locus is 1.

7. EXAMPLE. Consider a curve C of $\mathbb{P}^3$ and let B be the family of its secant lines. It is easy to see that B has dimension 2, if C is not a line.

B is clearly a flat family of dimension 2 and C is the fundamental curve for the family of its secants; in fact it is contained in the fundamental locus.

But, in general, we can also have a focal locus of dimension two, if through all points of C pass focal lines: in fact, it can be shown that the family of 3-secant lines is empty or has dimension at least 1. But clearly a 3-secant line is a fundamental line, so "in general"—*i.e.* if the family of the 3-secants is not empty—we have a focal locus of dimension 2.

2. First Order Congruences

As we have seen in 2.2, a first order congruence is a congruence of bidegree $(1, a)$ and through a general point of $\mathbb{P}^3$ passes only one line of the congruence.

We recall (proposition 2.7) that the fundamental locus of a first order congruence Λ coincides with the focal locus and has dimension at most 1 and then, we can summarize what we have just proven:

THEOREM 3.2. *If (Λ, B, p) is a first order congruence of lines of $\mathbb{P}^3$, then its focal locus can either be*

1. *a curve (possibly reducible) C such that every line of the congruence $\Lambda(b)$ will intersect the scheme of the first order foci $V(\lambda)(= f^{-1}(C))$ in a 0-dimensional scheme of length 2,*

2. *a point, i.e. a star of lines.*

REMARK. A more precise statement will be given at the end of this section in Theorem 3.7.

3. Linear Sections

We recall (Theorem 3.1) that the class a of a congruence is the degree of the scroll given by the lines of the congruence which belong to a general $\mathbb{P}^2$ (*i.e.* as a Schubert cycle, $B \cdot \sigma_{11}$).

Proposition 2.6 in this case means:

PROPOSITION 3.3. *The scroll given by the lines of the congruence which meet a (fixed) general line r is a surface V_r of $\mathbb{P}^3$ of degree $a+1$.*

REMARK. If ℓ is a line of the congruence not contained in V_r, and we have that

$$P \in V_r \cap \ell$$

then P is a focus for the congruence, since at least two lines of the congruence pass through it.

Theorem 2.10 means, in this case:

THEOREM 3.4. *The complete intersection of two general surfaces V_r and $V_{r'}$ is a (reducible) curve Γ whose components are the focal curve C and $(a+1)$ lines, i.e. the lines meeting r and r'.*

REMARK. If a focal line $\ell := p^{-1}(b)$ is a component of the focal scheme $V(\lambda)$ with multiplicity i, then $f(\ell)$ is a component of the curve Γ of the preceding Theorem of multiplicity i^2.

NOTATIONS. From now on, we will denote with i the multiplicity of the general focal line. We will also set $x := \sum i^2$, where i varies among all the focal lines.

3.2. DEFINITION. The curve given by the components of the fundamental curve C which are not (images of) focal lines is called *pure fundamental curve.*

The following Theorem characterize the fundamental curves whose components are lines:

THEOREM 3.5. *If the all the components of the fundamental curve C are lines, then we are in one of the following cases:*

1. *the fundamental curve is a line and the congruence is as in Theorem* 0.1, (1b)*;*
2. *the fundamental curve is given by two skew lines ℓ_1 and ℓ_2 and the congruence is given by the lines meeting both ℓ_1 and ℓ_2.*

PROOF. 1. Let us start with the case in which the fundamental curve is a line ℓ only.

First of all, we observe that for every point $P \in \ell$ we must have only one pencil of lines—contained in a plane Π_P—through it and for every (general) point $Q \in \mathbb{P}^3$ there is only a line of the congruence passing through it—say r_Q—which passes through a point $P \in \ell$. So, for every point $P \in \ell$ there is a plane Π_P and all these planes are distinct; then the map φ which associates Π_P to P is a bijection, hence an isomorphism.

2. Then, we pass to the case in which the fundamental curve C is given by more than one line: $C = \ell_1 \cup \cdots \cup \ell_k$.

Let P be a general point of $\mathbb{P}^3$; then only one line of the congruence passes through it—say this line ℓ, with $P_j = \ell \cap \ell_j$, $j = 1, \ldots, s$—and then the intersection of each two planes $P_j\ell_k$ and $P_k\ell_j$, with $j \neq j$, $j, k = 1, \dots, s$ must be the line ℓ only. So $s = 2$ and ℓ_1 and ℓ_2 are two skew lines.

□

PROPOSITION 3.6. *If η is an focal line and C the pure fundamental curve (with $C \neq \emptyset$ and irreducible), then we have that $C \cap \eta$ has length at least three; if $\mu :=$ length$(C \cap \eta)$, then $i = \binom{\mu}{2}$.*

PROOF. The line η is a focal line of the congruence, so it will intersect the pure fundamental curve in at least three points; in fact if the line did not intersect C, then the line would generate itself a congruence.

C is the pure fundamental curve, so it does not contain η and then the fundamental points of η must be points of C also.

Taking the μ points 2 by 2 we obtain the formula involving i. □

So, we can classify the congruences of lines according to the splitting type of the fundamental curve C:

THEOREM 3.7. *The possible cases of a first order congruence of lines Λ of $\mathbb{P}^3$ are the following:*

1. *there is an irreducible curve C of $\mathbb{P}^3$ such that Λ is given by the lines Λ_b for which*
$$\text{length}(\Lambda_b \cap V(\lambda)) \geq 2,$$
where $V(\lambda) := f^{-1}(C)$;
2. *Λ is given by the lines meeting once each of two irreducible curves C_1 and C_2;*
3. *Λ is a star of lines.*

REMARK. It is clear that the last case is trivial, so it will not be analysed.

3.1. Congruences of the first type. First of all, we will classify the congruence of the first type. A consequence of proposition 3.4 is the following:

LEMMA 3.8. *If C is the pure fundamental curve and it is irreducible, then the possible cases for the degree of C are:*

$$\deg C = 3, 4. \tag{50}$$

PROOF. Clearly we have that $\deg C \geq 3$ for degree reasons.

If we denote with k the multiplicity of C in the two surfaces V_r and $V_{r'}$ of proposition 3.4 and with m the degree of C, then we deduce, by 3.4 and Bézout, that

$$\begin{aligned} \deg(V_r \cap V_{r'}) &= (a+1)^2 \\ &= a + 1 + k^2 m + x \end{aligned}$$

from which, we obtain

$$k^2 m = a(a+1) - x^2. \tag{51}$$

Besides, since a line of the congruence not belonging to the $(a+1)$ lines of 3.4 must intersect the curve C in (at least) two points, we deduce

$$a = 2k - 1. \tag{52}$$

From formulas (51) and (52) we conclude

$$k^2(4-m) - 2k = x^2 \geq 0,$$

from which we obtain the thesis. □

THEOREM 3.9 (Congruences of the first type). *The smooth curves which generate a first order congruence are*

1. *the rational normal curve* C^3 *of* $\mathbb{P}^3$*, in which case the congruence is given by the secant lines of* C^3*; this congruence has bidegree* $(1,3)$*;*
2. *the fundamental curve is a line (which is a focal line for the congruence) and the congruence is as in Theorem* 0.1, (1b)*; this congruence has bidegree* $(1,1)$.

PROOF. We divide the proof in two steps; first we consider the congruences given by the secants of a curve; then the congruences for which $\forall b \in B$, $\Lambda(b) \cap V(\lambda)$ is (at least) a double point.

1. First of all, we consider the case of the secant lines of a curve:
   if C is an irreducible curve of degree m, it is clear that it generates a first order congruence if and only if it has one apparent double point; By Lemma 3.8 $m = 3$ or 4 so, applying the Clebsch formula we easily obtain that the only possibility is to have the rational normal curve C^3.
   Since this curve has degree three, we have that the bidegree is $(1,3)$.
2. Then, we pass at the other case:
   first of all, we observe that the general line r of the congruence meets the focal curve C in only a point, otherwise we are in the preceding case. Then, we have that,—since the congruence has dimension 2—once fixed a point $P \in C$, the lines of the congruence passing through it form a cone C_P (with $\dim C_P = 2$). Besides, if we fix a general point $Q \in \mathbb{P}^3$, the cone QC given by the join between Q and C must intersect the cone C_P in only a line, since we have a first order congruence. Then the two cones C_P and QC are two planes; besides the curve C is in fact a line ℓ (clearly $\ell = C_P \cap C_{P'}$, with P, P' are two general points of the focal curve).
   For finishing the proof we can then apply the first part of Theorem 3.5.
   This congruence has bidegree $(1,1)$, as it is easily seen if we consider the lines of the congruence contained in a general plane.

□

COROLLARY 3.10. *If the pure fundamental curve is irreducible and not empty, then we do not have fundamental lines.*

REMARK. We could prove the first part of the preceding Theorem in another way: if C is the curve we search, then C must have (geometric) genus

$$g = \frac{(d-1)(d-2)}{2} - 1 - \delta = \frac{d(d-3)}{2} - \delta,$$

where

$$\delta \geq \sum_{P \in C} \frac{s_P(s_P - 1)}{2}$$

and $s_P = \text{mult}_C(P)$.

Then, applying Castelnuovo's bound (see, for example [**ACGH84**]) we obtain

$$\frac{d(d-3)}{2} - \delta \leq \begin{cases} \frac{1}{4}d^2 - d + 1 & \text{if } d \text{ is even,} \\ \frac{1}{4}(d^2-1) - d + 1 & \text{if } d \text{ is odd;} \end{cases}$$

and we obtain that $d \leq 3$, so our thesis easily follows.

3.2. Congruences of the second type. Then, we will classify the congruence of the second type.

NOTATIONS. First of all, we set that C_1 and C_2 are the irreducible components of the fundamental curve C, we denote with m_j the degree of C_j and with k_j the multiplicity of C_j in the surface V_r, $j = 1, 2$.

LEMMA 3.11. *If* $C = C_1 \cup C_2$ *is the fundamental curve, then the following formulas hold:*

$$k_1^2 m_1 + k_2^2 m_2 = a(a+1) - x, \tag{53}$$

$$k_1 + k_2 = a + 1. \tag{54}$$

PROOF. We prove the two formulas separately:

1. The first formula is obtained as we did in Lemma 3.8, by proposition 3.4 and Bézout:
$$\begin{aligned}\deg(V_r \cap V_{r'}) &= (a+1)^2 \\ &= a + 1 + k_1^2 m_1 + k_2^2 m_2 + x\end{aligned}$$
from which, we obtain
$$k_1^2 m_1 + k_2^2 m_2 = a(a+1) - x.$$
2. For the second formula, since a line of the congruence not belonging to the $(a+1)$ lines of 3.4 must intersect the curve C in (at least) two points, we deduce
$$a + 1 = k_1 + k_2.$$

□

THEOREM 3.12 (Congruences of the second type). *The congruence is given by the lines meeting both a rational curve* C_1 *of degree* m_1*, which is the pure fundamental curve, and a fundamental line* C_2*, such that* $\text{length}(C_1 \cap C_2) = m_1 - 1$ *(and so we have that* $i = \binom{m_1 - 1}{2}$*). Besides, this is a congruence of bidegree* $(1, m_1)$*.*

Vice versa, for every rational curve C_1 *of degree* m_1 *which possesses a* $(m_1 - 1)$*-secant line* C_2*, the join of* C_1 *and* C_2 *gives a first order congruence.*

PROOF. Let us denote with Z the (at most) 0-dimensional scheme given by $C_1 \cap C_2$ and we set $u = \text{length}(Z)$.

Let P be a general point of $\mathbb{P}^3$; then the cone given by the join PC_j has degree m_j; as usual, by Bézout
$$\deg(PC_1 \cap PC_2) = m_1 m_2.$$
Since we have a first order congruence, we obtain:

$$u = m_1 m_2 - 1. \tag{55}$$

It is not restrictive to suppose that $m_1 \geq m_2$, so it will be supposed in the following.

First of all, let us suppose that C_1 is not a line. Then, if Q is a general point of C_1, there will pass $m_2(m_1 - 1)$ secant lines of C_1 through Q meeting C_2 also; but these lines will pass through the points of Z, since if one of these lines did not intersect Z, then this line would be a focal line, and varying these lines when we vary Q on C_1, we would obtain a focal surface. Then we have that
$$u = (m_1 - 1)m_2,$$
and by Equation (55) we obtain $m_2 = 1$.

So, we obtain that C_2 is a line, and $u = m_1 - 1$; projecting C_1 from a point of C_2 to a plane, we obtain a curve of degree m_1 with (at least) a point of multiplicity $m_1 - 1$, *i.e.* C_1 is a rational curve.

It remains to consider the case in which $m_1 = m_2 = 1$, and this is the case (2) of Theorem 3.5.

The bidegree is—as usual—calculated intersecting with a general plane.

If, finally, it is given a rational curve C_1 of degree m_1 which has a (m_1-1)-secant line C_2, we can reverse the argument just given. □

4. Singularities of First Order Congruences

It is interesting to see which of the congruences found in the previous section correspond to smooth surfaces in $\mathbb{G}(1,3)$.

THEOREM 3.13. *The fundamental loci of the first order congruences in $\mathbb{P}^3$ for which B is smooth are the following:*

1. *F is the rational normal cubic;*
2. *F is given by two skew lines;*
3. *F is given by a line and a conic which meet in a point.*

Moreover, the corresponding smooth B are:

1. *B is a Veronese surface;*
2. *B is a quadric given by a linear section of the Klein's quadric;*
3. *B is a cubic obtained by the Veronese surface projected from a point in it, i.e. it is a rational normal scroll of type $S_{1,2}$ of degree 3.*

PROOF. First of all, we analyse the congruences of the first type:

- In the case of the secant lines of the rational normal cubic, it is known that B, as a subvariety of the Grassmannian is smooth: see, for example [**AG93**].
- In the case in which we have as focal locus a line ℓ only, we have that $\forall b \in B$, $\Lambda(b) \cap f^{-1}(\ell)$ is (at least) a double point. This means that the scheme of focal points $V(\lambda)$ is a not reduced line.

  In particular we can see that the image of $V(\lambda)$ in the Grassmannian is a singular point for $B \subset \mathbb{G}(1,3)$, and in fact B is a 3-dimensional tangent linear section (*i.e.* contained in the embedded tangent space of the point) of the Grassmannian.

We pass then to the congruences of the second type:

- If F is given by two skew lines or by a line and a conic, it is easy to see that the congruence is smooth (see, for example, [**AS92**]).
- If instead $F = C \cup \ell$ and $\deg C \geq 3$, then $\text{length}(C \cap \ell) \geq 2$, so $\deg(f^{-1}(\ell)) \geq 2$ and B is not smooth.

For a proof, that the corresponding B are, respectively, the Veronese surface, a linear section of the Klein's quadric and a scroll of type $(1,2)$, see [**AS92**].

□

CHAPTER 4

First Order Congruences of Lines in $\mathbb{P}^4$

1. Generalities

We recall the results of Chapter 2, in the case of $\mathbb{P}^4$: we consider a flat family (Λ, B, p) of straight lines in $\mathbb{P}^4$, parametrized by a nonsingular scheme B of dimension 3. In this case, we can write:

$$B = a_0\sigma_{30} + a_1\sigma_{21}. \tag{56}$$

4.1. DEFINITION. We say that B is a *congruence of lines of bidegree* (a_0, a_1) if Equation (56) holds.

Corollary 2.2 in this case means:

THEOREM 4.1. *If B is a congruence of lines of bidegree (a_0, a_1), then*

1. *a_0 is the number of straight lines passing through a general point $p \in \mathbb{P}^4$;*
2. *a_1 is the number of lines contained in a general hyperplane of $\mathbb{P}^4$ and that intersect a general line of this hyperplane.*

REMARK. In this case, the "expected" dimension for F is 3 and for the fundamental locus is 2.

In this case, proposition 2.3 means:

PROPOSITION 4.2. *If C is a fundamental curve for a congruence Λ, then the intersection of $f^{-1}(C)$ with a general line of the congruence is given by a 0-dimensional subscheme of length 2.*

2. First Order Congruences

As we have seen in 2.2, a first order congruence is a congruence of bidegree $(1, a)$, *i.e.* through a general point of $\mathbb{P}^4$ passes only one line of the congruence.

We recall (proposition 2.7) that the fundamental locus of a first order congruence Λ coincides with the focal locus and has dimension at most 2. We can summarize what we have proven as follows:

THEOREM 4.3. *The focal locus can either be*

1. *a surface (possibly reducible) F such that every line of the congruence $\Lambda(b)$ intersects the scheme of the first order foci $V(\lambda)$ ($V(\lambda) = f^{-1}(F)$) in a 0-dimensional scheme of length 3,*
2. *a scheme which is the union of a surface F and a curve C (possibly with $C \subset F$), both met by every line of the congruence,*
3. *a point, i.e. Λ is a star of lines.*

REMARK. A more precise statement will be given at the end of this section in Theorem 4.10.

3. Linear Sections

We recall (Theorem 2.1) that the class a of a congruence is the degree of the scroll given by the lines of the congruence which belong to a general $\mathbb{P}^3$ (*i.e.* , as a Schubert cycle, $B \cdot \sigma_{21}$).

Proposition 2.6 becomes, in this case

PROPOSITION 4.4. *The scroll given by the lines of the congruence which meet a (fixed) general 2-plane* Π *is a hypersurface* V_Π *of* $\mathbb{P}^4$ *of degree* $a+1$.

REMARK. If ℓ is a line of the congruence not contained in V_Π, and we have that

$$P \in V_\Pi \cap \ell$$

then P is a focus for the congruence, since at least two lines of the congruence pass through it.

As we did in the case of $\mathbb{P}^3$, the meaning of Theorem 2.10 in this case is the following:

THEOREM 4.5. *The complete intersection of two general hypersurfaces* V_Π *and* $V_{\Pi'}$ *is a (reducible) surface* Γ *whose components are the focal surface* F *and the scroll* Σ *given by the lines meeting* Π *and* Π'*, which has degree* $2a+1$.

We can translate what we said in the general case of $\mathbb{P}^n$ about parasitic spaces:

4.2. DEFINITION. A plane η of $\mathbb{P}^4$ is called *i-parasitic of the first order congruence of lines* (Λ, B, p_1) (or simply *parasitic*) if every line

$$\Lambda(b) \in f^{-1}(\eta)$$

is contained in the focal scheme $V(\lambda)$ with multiplicity (at least) $i \geq 1$ and η in not met by the general line of the congruence.

COROLLARY 4.6. *Let* η *be a plane; then the following are equivalent:*

- η *is an i-parasitic plane;*
- *for the general line* ℓ *of* η *we have that*

$$\deg f^{-1}(\ell) = i \quad \dim f^{-1}(\ell) \leq 2;;$$

- *if we identify* B *with its image in the Grassmannian, the plane* η *is such that its correspondent subscheme* $p(f^{-1}(\eta)) := \sigma_\eta (= \sigma_{22})$ *is contained in* B *with multiplicity* i *and* B *is not contained in the hyperplane section of* $\mathbb{G}(1,4)$ *given by the lines of* $\mathbb{P}^4$ *meeting* η*;*
- η *is a component of the fundamental locus which is not a fundamental 2-locus.*

REMARK. It is clear that a parasitic plane η is such that $f^{-1}(\eta)$ is a component of the focal scheme $V(\lambda)$ with multiplicity i, then it is a component of the scheme Γ of the preceding Theorem of multiplicity i^2.

NOTATIONS. From now on, we will denote with i the multiplicity of the general focal plane. We will also set $x := \sum i^2$, where i varies among all the i-parasitic spaces.

4.3. DEFINITION. The scheme given by the components of the fundamental surface F which are not parasitic planes is called *pure fundamental surface*, or, in what follows, simply *fundamental surface*.

The following theorem characterizes the congruences for which the fundamental surface is set-theoretically a plane only.

THEOREM 4.7. *If the focal surface is set-theoretically a plane η, then the congruence Λ is obtained in one of the following ways:*

1. *Fix a rational curve C in η and a birational map φ between C and $\mathbb{P}^1_\eta$, the pencil of hyperplanes containing η:*

$$\varphi : C \dashrightarrow \mathbb{P}^1_\eta;$$

let $\mathbb{P}^2_P$ be the star of lines passing through P and lying in $\varphi(P)$; then the congruence is formed by the lines of the stars $\mathbb{P}^2_P$ as P varies in C i.e. it is $\overline{\cup_{P\in C}\mathbb{P}^2_P}$; besides, if $\deg(C) = a$, then the bidegree of Λ is $(1, a)$ and if r is the general line of η, then $\deg(f^{-1}(r)) = a$; the components of the fundamental locus are η, which is the fundamental 2-locus and C, which is the fundamental 1-locus.

2. *Fix a line ℓ in η and an isomorphism ψ between ℓ and $\mathbb{P}^1_\eta$, the pencil of hyperplanes containing η:*

$$\psi : \ell \to \mathbb{P}^1_\eta;$$

let $\mathbb{P}^1_P$ be the pencil of planes containing ℓ and lying in $\psi(P)$; if Π_P is a plane of $\mathbb{P}^1_P$, then we denote with $\mathbb{P}^1_{\Pi_P}$ the pencil of lines of Π_P passing through P: the congruence is formed by the lines of the pencils $\mathbb{P}^1_{\Pi_P}$ as P varies in ℓ and Π_P in $\mathbb{P}^1_P$, i.e. it is $\overline{\cup_{P\in\ell}\cup_{\Pi_P\in\mathbb{P}^1_P}\mathbb{P}^1_{\Pi_P}}$; besides, the bidegree of Λ is $(1,1)$ and if r is the general line of η, then $\deg(f^{-1}(r)) = 1$; η is the fundamental locus.

3. *Fix a rational curve C^* in η^*, the set of lines contained in η, and a birational map φ between C^* and $\mathbb{P}^1_\eta$, the pencil of hyperplanes containing η:*

$$\varphi : C^* \dashrightarrow \mathbb{P}^1_\eta;$$

let $\mathbb{P}^1_\ell$ be the pencil of planes containing ℓ and lying in $\varphi(\ell)$; then $\forall \ell \in C^*$ we fix an isomorphism between ℓ and $\mathbb{P}^1_\ell$:

$$\varphi_\ell : \ell \to \mathbb{P}^1_\ell$$

if $\varphi_\ell(P)$ is a plane of $\mathbb{P}^1_\ell$, then we denote with $\mathbb{P}^1_P$ the pencil of lines of $\varphi_\ell(P)$ passing through P: the congruence is formed by the lines of the pencils $\mathbb{P}^1_P$ as P varies in ℓ and ℓ in C^*, i.e. it is $\overline{\cup_{P\in\ell}\cup_{\ell\in C^*}\mathbb{P}^1_P}$; besides, the bidegree of Λ is $(1,1)$ and if r is the general line of η, then $\deg(f^{-1}(r)) = 1$; η is the fundamental locus.

PROOF. First of all we note that, if we consider a general point $Q \in \mathbb{P}^4$ then this determines a general hyperplane H of $\mathbb{P}^1_\eta$, and the lines of the congruence contained in this hyperplane form a first order congruence, since for the general point of H there passes only one line of Λ.

Besides, we note that the focal locus of $\Lambda|_H$ is contained in η; in fact it can be either a point $P \in \eta$ or a line $\ell \subset \eta$, by Corollary 2.4 and Theorem 3.5.

Then, we have the following possibilities:

1. If for almost every H we have that the congruence induces a star of lines, then we can construct a map

$$\phi : \mathbb{P}^1_\eta \to \eta$$

which associates to each hyperplane H the support P_H of its star of lines. The map ϕ cannot be constant since otherwise we have a star of lines of $\mathbb{P}^4$. Then, we apply Riemann-Hurwitz's Theorem getting that the image C of ϕ is rational; besides, ϕ must be generically injective because the degree of the map is in fact the order of the congruence and we obtain case (1).

If $\deg(C) = a$, we can in fact calculate the bidegree of the congruence: the second degree is the number of lines of the congruence contained in a hyperplane H and meeting a line r of it. But $H \cap \eta$ is a line ℓ_H, which meets C in a points and so the thesis follows.

2. If for almost every H the congruence induces a congruence of $\mathbb{P}^3$ with a focal line only, then, this means that, the congruence is given as in Theorem 0.1, (1b); besides, we have two possibilities: either the line is the same for all the hyperplanes containing H or it changes:
   (a) If the line is the same for all the planes of $\mathbb{P}^1_\eta$, then, by Theorem 0.1, (1b), we are in case (2).
   The second degree of the congruence is the number of lines of the congruence contained in a hyperplane H and meeting a line r of it and, as before $H \cap \eta = \ell_H$. But if $P \in r$, then the hyperplane $H' := \overline{P\eta} \in \mathbb{P}^1_\eta$ contains r, since $H \cap H' = \overline{P\ell_H} \supset r$. Then our thesis easily follows.
   (b) If the lines vary, then we set the following map
   $$\phi : \mathbb{P}^1_\eta \to \eta^*$$
   which associate the hyperplane H the line ℓ_H, centre of the pencil of planes defining the congruence. Then, as before, using Riemann-Hurwitz's Theorem, we conclude—by Theorem0.1, (1b)—that we are in case (3).
   The same argument as in the preceding case gives the second degree of the congruence.

□

Proposition 2.12 is, in this case:

PROPOSITION 4.8. *If the pure fundamental surface F is not empty and η is a i-parasitic plane, then $F \cap \eta$ is a curve C of η.*

COROLLARY 4.9. *In the hypothesis of the preceding proposition, we have that*

$$i = \binom{\mu}{3} \tag{57}$$

where $\mu := \deg(F \cap \eta)$.

So, we can classify the congruences of lines according to the splitting type of the (pure) fundamental surface F:

THEOREM 4.10. *The possible cases of a first order congruence of lines Λ of $\mathbb{P}^4$ are the following:*

1. *There exists an irreducible surface F of $\mathbb{P}^4$ such that Λ is the closure of the union of the lines Λ_b such that*
$$\text{length}(\Lambda_b \cap f^{-1}(F)) = 3;$$
2. *There exists two irreducible surfaces F_1 and F_2 of $\mathbb{P}^4$ such that Λ is the closure of the union of the lines Λ_b such that*
$$\text{length}(\Lambda_b \cap f^{-1}(F_1)) = 2,$$
$$\text{length}(\Lambda_b \cap f^{-1}(F_2)) = 1;$$
3. *Λ is given by the lines meeting once each of three irreducible surfaces F_1, F_2 and F_3;*
4. *Λ is given by the lines meeting once an irreducible surface F and a curve C (possibly with $C \subset F$);*
5. *Λ is a star of lines.*

REMARK. In the next sections we will study congruences of the first four cases. Precisely, in case (1) we will obtain a complete classification, assuming that F has only ordinary singularities, while in the remaining cases we will give general results.

4. Congruences of the First Type

In this section we classify the congruences of lines of order one with an irreducible surface F as a (pure) fundamental surface.

REMARK. Since we have analysed the case in which F is a plane in Theorem 4.7, we will suppose in the following that F is non-linear. In this hypothesis, it follows that the congruence is given by the closure of the set of the trisecant lines of the surface F.

NOTATIONS. Assume that η is an i-parasitic plane and $C = \eta \cap F$ is the curve of Proposition 4.8. Then Corollary 4.9 continues to hold: if we call μ the degree of the curve C, we have that the Equation (57) continues to hold.

We will denote with m the degree of F and with k the multiplicity of $F \cap V_\Pi$ in V_Π.

First of all, we recall some general results. Proposition 2.26 gives

PROPOSITION 4.11. *The following formula holds:*

$$3k = a + 1. \tag{58}$$

Proposition 2.27 is

PROPOSITION 4.12. *The following formula holds:*

$$k^2 m = a^2 - x. \tag{59}$$

And so, by Theorem 2.28 (and its proof), we have

COROLLARY 4.13. *The following formula holds:*

$$(9 - m)k^2 - 6k + 1 = x \geq 0, \tag{60}$$

and then the focal surface has degree at most 8.

We recall now some classical definitions:

4.4. DEFINITIONS. A projective variety X of dimension k embedded in $\mathbb{P}^{2k+1}$ is said to have h *apparent double points* if its general projection to a $\mathbb{P}^{2k}$ from a general point not belonging to the variety itself has h improper double points.

By *improper double point* we mean a double point of X which is the origin of two linear branches of X and whose tangent cone consists of two k-planes intersecting transversally at the point.

REMARK. It is easy to see that the general projection of a projective smooth surface S embedded in $\mathbb{P}^N$, with $N \geq 5$ to $\mathbb{P}^4$ has, as singularities, only improper double points. So, we will say that a surface S of $\mathbb{P}^N$ has *ordinary singularities* if it is smooth when $N \geq 5$ or has a finite number of improper double points when $N = 4$.

If we project S to $\mathbb{P}^3$, then, by the "General Projection Theorem", see [**MP97**], the projection will have as singularities a curve Γ, whose singularities are a finite number of ordinary triple points, which are triple also for S, and a finite number of pinch points lying on Γ (see, for more details about this [**MP97**] and [**GH78**]).

REMARK. In the remaining part of this section, we will make the assumption that the fundamental surface F has only ordinary singularities. Under this assumption we will obtain a complete classification.

We recall now a formula, due to Cayley (for a modern proof see [Bar82]) which expresses the number of trisecant lines of a curve of $\mathbb{P}^3$, meeting a line:

THEOREM 4.14. *Let D be a smooth curve of $\mathbb{P}^3$, of degree m and with h apparent double points; then the number of trisecants of D which meet a fixed line is*

$$t(D) = h(m-2) - \binom{m}{3}. \tag{61}$$

In particular, if D is a an irreducible curve of genus g, then

$$t(D) = 2\binom{m-1}{3} - g(m-2). \tag{62}$$

In our situation this gives:

COROLLARY 4.15. *Let C_H be a general hyperplane section of the fundamental surface F. Then, if h is the number of the apparent double points of C_H, we have*

$$a = h(m-2) - \binom{m}{3}. \tag{63}$$

PROPOSITION 4.16. *The following formula holds:*

$$k = h - m + 2. \tag{64}$$

PROOF. Since k is the multiplicity of F in V_Π, this means that through a general point P of F there are k lines of the congruence (*i.e.* trisecant lines of F) that meet Π, and these lines belong to the $\mathbb{P}^3$ spanned by the point P and the plane Π; in other words, k is the number of trisecant lines of F belonging to a general $\mathbb{P}^3$.

The formula (64) can be obtained by computing the (geometric) genus of the generic hyperplane section C_H in two ways with the Clebsch formula: the first by projecting C_H to a plane from a point not belonging to it, and the second by projecting from $P \in F$ to a general plane. we obtain in the first case

$$g = \binom{m-1}{2} - h,$$

in the second case

$$g = \binom{m-2}{2} - k.$$

Eliminating g from these equalities, we get the relation (64). □

REMARK. It is easy to see that the number h of the improper double points of the hyperplane section C_H of the surface F is equal to the degree of the curve Γ, which is the singular locus of the general projection of F to a $\mathbb{P}^3$.

PROPOSITION 4.17. *The following formula holds:*

$$6h(m-5) = (m^2+2m-6)(m-5) \tag{65}$$

PROOF. It is obtained from equalities (58), (63) and (64) eliminating a and k. □

COROLLARY 4.18. *If $m \neq 5$, we have:*

$$h = \frac{m(m+2)}{6} - 1 \tag{66}$$

COROLLARY 4.19. *The possible values for the degree of the fundamental surface F are $m = 4, 5, 6$.*

PROOF. This is due to the fact that for the values $m = 3, 7, 8$ we have non-integer values for h, by formula (66). □

So, the only possible cases to analyse are obtained for $m = 4, 5, 6$.

4.1. Case $m = 4$. We will prove in Theorem 4.26 that the only irreducible surface of $\mathbb{P}^4$ with isolated singularities, whose trisecant lines generate a first order congruence is the projected Veronese surface.

LEMMA 4.20. *If F is a surface of degree 4, focal surface of a first order congruence, then its hyperplane sections are rational curves.*

PROOF. By formula (66) we have that $h = 3$, so the general hyperplane section of F is a rational curve C_H; in fact we have that its (geometric) genus is

$$g = \frac{3 \cdot 2}{2} - 3 = 0.$$

□

We recall now a classical theorem on the surfaces of $\mathbb{P}^4$:

KRONECKER-CASTELNUOVO'S THEOREM. *Let $S \subset \mathbb{P}^4$ be an irreducible surface having at most isolated singularities. If the general tangent hyperplane to S intersects S in a reducible curve, then S is either a scroll or the Veronese surface in $\mathbb{P}^4$.*

For a proof, see [**MP97**].

From this, we can obtain a useful corollary for our situation:

COROLLARY 4.21. *Let $S \subset \mathbb{P}^4$ be an irreducible surface having at most isolated singularities and with rational sections; then S is either a projection of a rational normal scroll or the Veronese surface in $\mathbb{P}^4$.*

PROOF. For proving this corollary, we will show that we are in the hypotheses of the Kronecker-Castelnuovo's theorem. In fact, if P is a general point of S and H_P a general tangent hyperplane, then we have that $H_P \supset \mathbb{T}_{S,P}$, where, as usual, $\mathbb{T}_{S,P}$ is the embedded tangent space. $C_{H_P} := H_P \cap S$ will be a curve of arithmetic genus zero and degree $n := \deg S$ in $H_P \cong \mathbb{P}^3$, and all the lines of $\mathbb{T}_{S,P}$ passing through P will be tangent lines in P. So P will be a singular point for C_{H_P}, and then C_{H_P} is reducible, since if it were irreducible, we would have $0 = p_a(C_{H_P}) \geq p_g(C_{H_P}) \geq 0$, so $0 = p_a(C_{H_P}) = p_g(C_{H_P})$, but $p_a(C_{H_P}) = p_g(C_{H_P})$ if and only if C_{H_P} were smooth. So, the surface S is a scroll, and $C_{H_P} = C \cup F_P$, where $F_P \cong \mathbb{P}^1$ is the fibre of the scroll at the point P and C is the basis of the scroll; so we have to prove that C is rational. In fact, we have, by adjunction:

$$\begin{aligned} p_a(C + F_P) + 1 = 1 & \\ &= p_a(C) + p_a(F_P) + CF_P \\ &= p_a(C) + 0 + 1 \end{aligned}$$

so, $p_a(C) = 0$ and then C is rational. □

We will recall now two theorems which will be useful in the following:

THEOREM 4.22. *Let S be a surface in $\mathbb{P}^4$; if it is contained in a quadric, then the family of its trisecant lines either is empty or is a congruence of order zero.*

PROOF. If ℓ is a trisecant line of S, then ℓ is a trisecant of the quadric, so it is contained in the quadric. □

Before giving the second theorem, we need the definitions of the—so called, classically—elementary projective characters (see [**SR49**]):

4.5. DEFINITIONS. The *projective characters* of a projective surface S of $\mathbb{P}^N$ with ordinary singularities are

1. μ_0, which is the length of $F \cap \Pi$, where Π is a generic $(N-2)$-plane;
2. μ_1, which is the number of tangents of the general hyperplane section C_H of S meeting a fixed $(N-3)$-plane contained in the hyperplane (classically, this is called the *rank* of S);
3. μ_2, which is the number of tangent planes which meet a $(N-2)$-plane along a line;
4. ν_2, which is the number of tangent planes which meet a $(N-4)$-plane in (at least) a point.

REMARK. It is easy to see that these numbers are invariant under a general projection of the surface to a $\mathbb{P}^M$, with $M \geq 3$.

The projective characters can be expressed easily in the following way:

PROPOSITION 4.23. *Let S be a projective surface of degree m with ordinary singularities; then*

$$\mu_0 = m$$
$$\mu_1 = \deg(C_H^*)$$
$$\mu_2 = \deg(S^*),$$

where C_H^* is the dual variety of the general hyperplane section of S and S^* is the dual variety of S.

PROOF. The formulas for μ_0 and μ_1 are straightforward from the definitions. For μ_2, we can easily see that this number is equal to the number of tangent hyperplanes belonging to a general pencil of hyperplanes. □

Finally, we can state the second theorem, which expresses some formulas involving projective characters of a surface.

THEOREM 4.24. *Let S be a projective surface with ordinary singularities of degree m, whose general projection to $\mathbb{P}^4$ has δ improper double points and whose general projection to $\mathbb{P}^3$ has a double curve Γ of degree d with t triple points and h' apparent double points; then*

$$2d = m(m-1) - \mu_1, \tag{67}$$

$$2\delta = m(m-1) - \mu_1 - \nu_2, \tag{68}$$

$$t = \binom{m}{3} - \frac{1}{2}m\mu_1 + \frac{1}{3}(2\mu_1 + 2\mu_2). \tag{69}$$

In particular, if S is smooth and connected and if c_1^2 and c_2 are its Chern numbers and e its hyperplane class, then

$$\mu_1 = 3m - e \cdot c_1, \tag{70}$$

$$\nu_2 = 6m - 4e \cdot c_1 + c_1^2 - c_2, \tag{71}$$

$$2\mu_2 = c_1^2 + c_2 - 12m + 8\mu_1 - \nu_2, \tag{72}$$

$$2h' = d(d - m + 2) - \delta - 3t. \tag{73}$$

See, for example **[SR49]** and **[Bar87]**.

REMARK. It is clear that the number t defined in the preceding Theorem is the number of trisecant lines of a general projection of S to a $\mathbb{P}^4$.

COROLLARY 4.25. *The number of trisecant lines of a smooth surface S in $\mathbb{P}^4$ passing through a general point of the space is*

$$t = \binom{m-1}{3} - \pi(m-3) + 2\chi - 2, \tag{74}$$

where m is the degree of S, π its sectional genus and $\chi = \chi(\mathcal{O}_S)$ its Euler-Poincaré characteristic.

For a proof, see [**Aur88**].

From this we can classify the first order congruences whose fundamental surface has degree four and has only isolated singularities:

THEOREM 4.26. *The only surface F of degree four with isolated singularities that generates a first order congruence is the Veronese surface in $\mathbb{P}^4$.*

Vice versa, the trisecant lines of the projected Veronese surface in $\mathbb{P}^4$ generate a first order congruence.

PROOF. By Corollary 4.21, the possibilities are the projection of the Veronese surface, which has in fact degree four, and three quartic scrolls, projection of the rational normal scrolls of degree four in $\mathbb{P}^5$, which are the cone on the rational normal curve of $\mathbb{P}^4$, $S_{0,4}$, and the two smooth scrolls $S_{1,3}$ and $S_{2,2}$.

REMARK. It is an interesting point that the possible surfaces of degree four are the projections of all the surfaces of minimal degree of $\mathbb{P}^5$. For an account on the varieties of minimal degree and on the rational normal scrolls, see [**EH87**].

Now, if we project a scroll S of degree four (*i.e.* all the varieties of minimal degree but the Veronese surface) of $\mathbb{P}^5$ from two general points P_1 and P_2 (or, which is the same, from the straight line given by the span of the two points), we obtain a quadric Q in $\mathbb{P}^3$. So, the (general) projection of S from one point P_1 is contained in the quadric cone given by the quadric Q and the other point P_2. (This argument was suggested by A. Bruno).

Then the scrolls cannot generate a first order congruence by 4.22.

On the other hand, it is well known that the Veronese surface generates a first order congruence, see [**Cas91**]; in fact, the projection of the Veronese surface in $\mathbb{P}^4$ is smooth by Severi's Theorem and it is of course a rational variety. So, we have that

$$m = 4$$
$$g = 0$$
$$\chi = 1,$$

so, applying Corollary 4.25, we obtain the number of trisecant lines

$$t = \binom{3}{3} - 0(4-3) + 2 - 2$$
$$= 1.$$

□

REMARK. Besides, we have also that, for the Veronese variety,

$$n = 2, \qquad k = 1, \qquad x = 0$$

and so we do not have parasitic planes.

4.2. Case $m = 5$. In this subsection we will prove the following theorem:

THEOREM 4.27. *If F is an irreducible surface of $\mathbb{P}^4$ of degree 5 with ordinary singularities whose trisecant lines generate a first order congruence, then*

1. *either F is a projection of a Del Pezzo surface of $\mathbb{P}^5$*
2. *or F is a projection of a rational normal scroll of type $(1,4)$ projected from a line contained in the linear span of an unisecant curve of degree 4.*

First of all we observe that

PROPOSITION 4.28. *If the surface F has degree 5 and generates a first order congruence, then F cannot be smooth.*

PROOF. It is a consequence of Corollary 4.25: if F were smooth, then the number of trisecant lines of F through a general point of $\mathbb{P}^4$ would be, by formula (74)

$$\begin{aligned} t &= \binom{4}{3} - 2\pi + 2\chi - 2 \\ &= 2(\chi - \pi + 1) \end{aligned}$$

(where π is the sectional genus and χ the Euler-Poincaré characteristic of F), so t is an even number and then it cannot be one. □

THEOREM 4.29. *The only possible values for h are 5 and 6. Besides, we have the following list of possible invariants*

$$h = 5, \qquad k = 2, \qquad a = 5, \qquad x = 5, \tag{75}$$

$$h = 6, \qquad k = 3, \qquad a = 8, \qquad x = 19. \tag{76}$$

PROOF. From (60), solving the inequality, we have that

$$k \geq \frac{3 + \sqrt{5}}{4} > 1$$

and from (64) we deduce that $k = h - 3$, so, finally, that $h > 4$.
On the other hand, by the Clebsch formula,

$$h \leq \frac{4 \cdot 3}{2} = 6.$$

From this we deduce (75) and (76). □

First of all, we consider case (75).
4.2.1. *The case* (75).

LEMMA 4.30. *If F is a surface of case* (75), *then its hyperplane section is an elliptic curve; besides it has* 5 *1-parasitic planes.*

PROOF. From the fact that $h = 5$ and by the Clebsch formula we deduce that the sectional genus of the focal surface F is

$$\pi = \binom{4}{2} - 5 = 1,$$

i.e. F is a surface with elliptic sections. From (57) and from $x = 5$ we deduce that the only possibility is to have 5 1-parasitic planes such that each of them contains a cubic plane curve. □

We recall now the "Double Point Formula:"

THEOREM 4.31. *Let S be a smooth surface of $\mathbb{P}^n$, of degree m; if we denote with K its canonical divisor and with H its hyperplane divisor, then the number of apparent double points δ is given by the formula*

$$2\delta = m^2 - 10m - 5HK - 2K^2 + 12 + 12p_a, \tag{77}$$

where p_a is the arithmetic genus of S.

See [**Har77**], pages 433–434, [**Bar87**], pages 58–59, or [**BS95**], page 217 for details.
Besides, we need the following theorems of classical adjunction theory:

THEOREM 4.32. *Let L be an ample and spanned line bundle on an irreducible normal projective variety X of dimension n. Then $\pi(L) \geq q(X)$, where $\pi(L)$ is the sectional genus and $q(X) := h^1(\mathcal{O}_X)$ the irregularity of X. If in addition X has at worst Cohen-Macaulay singularities and $q(X) \geq 1$, then $\pi(L) = q(X)$ if and only if $(X, L) \cong (\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$ for an ample and spanned vector bundle $\mathcal{E}$ over a smooth curve C of genus $\pi(C) = q(X)$. In particular, if $\pi(C) = q(X) > 0$, X is smooth.*

4.6. DEFINITION. An irreducible non-singular projective surface S polarized with an ample line bundle L is said to be a *Del Pezzo surface* if $L \cong -K$, where K is the canonical line bundle of S.

A proof is in [**BS95**], pages 234–235. In the case of surfaces, this implies:

THEOREM 4.33. *Let L be an ample and spanned line bundle on a smooth connected projective surface S; then*

1. *if $\pi(L) = q(S) > 0$, (S, L) is as described in Theorem 4.32;*
2. *if $\pi(L) = 1$ and $q(S) = 0$, (S, L) is a Del Pezzo surface.*

See [**BS95**], page 262. A classical proof can be found in [**Con39**] chapter III, in which Theorem 4.33 is ascribed to Castelnuovo.

Let us return to our problem:

LEMMA 4.34. *The surface F of case (75) cannot be a scroll.*

PROOF. We can prove the lemma simply applying Theorem 4.32, because, if F is a scroll, then, in this case, $\pi(L) = 1 > 0$ by Lemma 4.30, since we have that $L = \mathcal{O}_F(1)$ and an ordinary singularity is Cohen-Macaulay—in fact it is a local complete intersection. Then F is smooth and we can conclude by proposition 4.28. □

So, finally, we have

THEOREM 4.35. *The surface F of case (75) is a projection of a Del Pezzo surface of $\mathbb{P}^5$ of degree 5.*

Vice versa, the trisecant lines of a projected Del Pezzo surface of $\mathbb{P}^5$, generate a first order congruence.

PROOF. It follows from Theorem 4.33 that the desingularization of F is a Del Pezzo surface; it has degree 5 and so it can only be a Del Pezzo surface of $\mathbb{P}^5$.

It remains to prove that this surface generates a first order congruence. We have that the degree d of the double curve Γ of the general projection of F to $\mathbb{P}^3$ is $5 (= h)$, so, applying Theorem 4.24, we obtain

$$\begin{aligned} \mu_1 &= m(m-1) - 2d \\ &= 5(5-1) - 2 \cdot 5 \\ &= 10. \end{aligned}$$

Since F is a Del Pezzo surface, which is rational, then

$$\begin{aligned} p_a &= 0 \\ K &= -H \\ K^2 &= 5 \\ HK &= -5 \end{aligned}$$

so, by the double point formula (77), we have

$$\begin{aligned}\delta &= \frac{1}{2}(m^2 - 10m - 5HK - 2K^2 + 12 + 12p_a) \\ &= \frac{1}{2}(5^2 - 10 \cdot 5 - 5(-5) - 2 \cdot 5 + 12 + 12 \cdot 0) \\ &= 1.\end{aligned}$$

Then, applying Theorem 4.24

$$\begin{aligned}\nu_2 &= m(m-1) - \mu_1 - 2\delta \\ &= 5(5-1) + 10 - 2 \\ &= 28.\end{aligned}$$

Besides, if we denote by F' the Del Pezzo surface of $\mathbb{P}^5$, we have that

$$\begin{aligned}\chi(F') &= 1 \\ c_1^2 &= 5\end{aligned}$$

and by Noether's formula

$$\begin{aligned}c_2 &= 12 - 5 \\ &= 7.\end{aligned}$$

Again, from Theorem 4.24

$$\begin{aligned}\mu_2 &= \frac{1}{2}(c_1^2 + c_2 - 12m + 8\mu_1 - \nu_2) \\ &= \frac{1}{2}(5 + 7 - 12 \cdot 5 + 8 \cdot 10 - 28) \\ &= 2;\end{aligned}$$

finally, since the projective characters are invariant under projections, we have that the number of trisecants of F passing through a general point of $\mathbb{P}^4$ is

$$\begin{aligned}t &= \binom{m}{3} - \frac{1}{2}m\mu_1 + \frac{1}{3}(2\mu_1 + 2\mu_2) \\ &= \binom{5}{3} - \frac{1}{2}5 \cdot 10 + \frac{1}{3}(2 \cdot 10 + 2 \cdot 2) \\ &= 1.\end{aligned}$$

Therefore, the trisecants of a projected Del Pezzo surface of $\mathbb{P}^5$ generate a first order congruence. □

Then, we consider case (76).

4.2.2. *The case* (76). First of all, we recall a formula, due to Cayley (for a modern proof see again [**Bar82**]) which expresses the number of quadrisecant lines of a curve of $\mathbb{P}^3$:

THEOREM 4.36. *Let D be a smooth curve of $\mathbb{P}^3$, of degree m and with h apparent double points; then the number of quadrisecants of D is*

$$q(D) = \frac{m}{24}(m-2)(m-3)(m-13) - 2h(m-3) + \binom{h}{2}. \tag{78}$$

In particular, if D is a an irreducible curve of genus g, we have that

$$q(D) = \frac{1}{12}(m-2)(m-3)^2(m-4) - \frac{1}{2}g(m^2 - 7m + 13 - g). \tag{79}$$

Then we prove a lemma concerning the rational normal scrolls:

LEMMA 4.37. *Let $S_{a_1,\dots,a_k}$ be a rational normal scroll of type $(a_1,\dots,a_k)$ embedded in $\mathbb{P}^N$ (with $N = \sum_{i=1}^k a_i + k - 1$) and $a_m := \max\{a_1,\dots,a_k\}$. Then any reduced scheme $\Gamma \subset S_{a_1,\dots,a_k}$ of pure dimension $(k-1)$ and of degree d with $d < \sum_{i=1}^k a_i - a_m (= \deg S_{a_1,\dots,a_k} - a_m)$ is given by $\ell_1 \cup \dots \cup \ell_d$, where ℓ_i, with $i = 1,\dots,d$ are $(k-1)$-planes of the ruling, or, in the case of the surfaces, if $a_i = 1$, ($i = 1$ and/or 2) one of the lines can be the unisecant curve C_{a_i}.*

PROOF. Let us prove this lemma by induction on the degree. For $d = 1$ the claim is obvious. Let us suppose now that the claim is true from 1 to $(d-1)$.

Let us prove it by contradiction: let us suppose that the scroll contain an irreducible (it is not restrictive: if it is not irreducible, we can consider its irreducible components) variety Γ of codimension one (*i.e.* a hypersurface) and of degree d with $1 < d < \sum_{i=1}^k a_i - a_m$, then Γ would be contained in a linear subspace of dimension $(d+k-2) < N$, since Γ would be at most a variety of minimal degree.

Besides, for every point of Γ would pass a $(k-1)$-space of the ruling. Then the scroll would contain the scheme of dimension k given by the $(k-1)$-planes of the ruling of the scroll passing through the points of Γ. But this scheme would be contained in the linear space of dimension $M(\le a_m + (d+k-2) + 1 < \sum_{i=1}^k a_k + k - 1 = N)$ (Grassmann) generated by the hypersurface and the rational normal curve C_{a_m} of degree a_m of the scroll.

Then this scheme would be the scroll and it would be contained in a M-plane, which is a contradiction.

□

We can now find which is the surface we are looking for:

THEOREM 4.38. *The only surface F of degree 5 with isolated singularities of the case (76) is a projection of the rational normal scroll $S_{1,4}$ of $\mathbb{P}^6$ from a line contained in the 4-subspace generated by the unisecant quartic C_4 of the scroll.*

The surface F contains 4 parasitic planes, one is the 4-parasitic plane which contains the quartic curve image of C_4, and the other 3 are 1-parasitic. They are the planes containing the (projection of the) unisecant line C_1 and the couples of ruling lines passing through the 3 singular points of the quartic plane curve contained in F.

Vice versa, the trisecants of such a rational scroll generate a first order congruence.

PROOF. From the fact that $h = 6$ and by the Clebsch formula we deduce that the sectional genus of the focal surface F is

$$\pi = \binom{4}{2} - 6 = 0,$$

i.e. F is a surface with rational sections; so, by the Corollary 4.21 F is a projection of a rational normal scroll. Since it has degree 5, it is the projection from a non-secant line of one of the three rational normal scrolls of $\mathbb{P}^6$, *i.e.* the cone $S_{0,5}$ and the two smooth scrolls $S_{1,4}$ and $S_{2,3}$.

So, the general hyperplane section C_H of F is a smooth rational curve of degree 5 in $H \cong \mathbb{P}^3$; then, by formula (79), we have that the number of quadrisecant lines of C_H is

$$\begin{aligned} q(C_H) &= \frac{1}{12}(m-2)(m-3)^2(m-4) - \frac{1}{2}g(m^2 - 7m + 13 - g) \\ &= \frac{1}{12}(5-2)(5-3)^2(5-4) \\ &= 1. \end{aligned}$$

As in the previous case, (57) and $x = 19$ say that either F contains a quartic plane curve and three plane cubics, or 19 plane cubics. But every hyperplane section contains a 4-secant, so we are in the first case: indeed, if the quadrisecant lines were not contained in a plane, then they would generate a family of dimension two and all these lines would be contained in the focal surface; but the only surface with this characteristic is the plane. Then there is a plane rational quartic C contained in F. This curve is clearly the projection of a rational quartic of a linear subspace of dimension at most four contained in $\mathbb{P}^6$, and this curve is contained in the scroll, which therefore cannot be $S_{2,3}$.

Consider first of all $S_{0,5}$: by the Lemma 4.37 the quartic C must be the union of four lines.

Then in this case, C must be the projection of four lines of the ruling of the cone—call these lines $\ell_1, \ldots, \ell_4$—from a line ℓ contained in the $\mathbb{P}^4$ generated by $\ell_1, \ldots, \ell_4$.

Besides, the three other parasitic planes contain three lines of the ruling each, and these planes are the projections of three dimensional spaces of $\mathbb{P}^6$ given by three lines of the cone. Hence, each of the three spaces will intersect the line ℓ in a point.

Clearly, each of these 3-spaces will intersect the 4-space in a line, containing the vertex of the cone P and the point of ℓ of this 3-space.

Now, if we consider the corresponding situation on the rational normal curve of degree 5 which is the unisecant of the cone, we have a 3 dimensional 4-secant space, three 2 dimensional 3-secants and the line ℓ' given by the intersection of the plane generated by ℓ and P, with the hyperplane containing the unisecant.

In this case, we would have that the only 3-secant planes are the three just considered, but the 3-secant variety of the rational normal curve of $\mathbb{P}^5$ is the whole space and so for every point of ℓ' we have a 3-secant line. Then F cannot be the projection of the cone $S_{0,5}$.

The only possibility is that F is a projection of a $S_{1,4}$. As in the preceding case, it is easy to see that this scroll cannot contain an irreducible curve Γ of degree d with $d = 2, 3$.

We have two cases: either C is a projection of a rational normal curve of degree four of the scroll from a line ℓ in its $\mathbb{P}^4$, or it is the union of four lines. One of these is the unisecant line $D \cong \mathbb{P}^1$ and the others are three fibres—say $\ell_1, \ldots, \ell_3$.

First consider the case of the four lines: the three parasitic planes will be generated by three lines of the scroll each. If this were the case, then the $\mathbb{P}^3$ would have in common with the $\mathbb{P}^4$ the unisecant D, since it would contain (at least) two points of it; but $\mathbb{P}^3$ should have a point in common with ℓ also, which would be contained in D, which is a contradiction.

The only possibility is therefore the projection from a line of a $\mathbb{P}^4$ containing a rational normal curve of degree four contained in $S_{1,4}$.

The observations concerning the parasitic planes are straightforward; it remains to prove that this scroll generates a first order congruence. This is given by the computation of the projective characters of F or, which is the same, of $S_{1,4}$. Since these surfaces are rational, we have that

$$\begin{aligned} \chi(S_{1,4}) &= 1 \\ c_1^2 &= (-K_{S_{1,4}})^2 \\ &= 8; \end{aligned}$$

then, by Noether's formula

$$\begin{aligned} c_2 &= 12\chi - c_1^2 \\ &= 12 - 8 \\ &= 4. \end{aligned}$$

If e is the hyperplane class of $S_{1,4}$, then—since $e^2 = \deg F = 5$—by the adjunction formula

$$\begin{aligned} -c_1 e &= 2g(e) - 2 - e^2 \\ &= -2 - 5 \\ &= -7. \end{aligned}$$

From this and from Theorem 4.24 we can calculate the projective characters:

$$\begin{aligned} \mu_1 &= 3m - c_1 e \\ &= 3 \cdot 5 - 7 \\ &= 8, \\ \nu_2 &= 6m - 4c_1 e + c_1^2 - c_2 \\ &= 6 \cdot 5 - 7 \cdot 4 + 8 - 4 \\ &= 6; \end{aligned}$$

from μ_1 and ν_2 we deduce μ_2:

$$\begin{aligned} \mu_2 &= \frac{1}{2}(c_1^2 + c_2 - 12m + 8\mu_1 - \nu_2) \\ &= \frac{1}{2}(8 + 4 - 12 \cdot 5 + 8 \cdot 8) \\ &= 5. \end{aligned}$$

So, finally:

$$\begin{aligned} t(F) &= \binom{m}{3} - \frac{1}{2}m\mu_1 + \frac{1}{3}(2\mu_1 + 2\nu_2 + \mu_2) \\ &= \frac{5 \cdot 4 \cdot 3}{3 \cdot 2} - \frac{1}{2}5 \cdot 8 + \frac{1}{3}(2 \cdot 8 + 12 + 5) \\ &= 1. \end{aligned}$$

This concludes the proof of Theorem 4.38, since the trisecant lines of such a scroll generate a first order congruence. □

4.3. Case $m = 6$: First of all we can deduce that

PROPOSITION 4.39. *If F has degree 6, then we have*

$$h = 7, \qquad k = 3, \qquad a = 8, \qquad x = 10; \tag{80}$$

so F has 10 1-parasitic planes. Besides, F is a surface with sections of genus 3.

Then, we need another result of classical adjunction theory on surfaces; for this we have to recall the Remmert-Stein factorization (see **[Har77]** for a proof):

REMMERT-STEIN FACTORIZATION. *Let $\phi : X \to Z$ be a projective morphism of Noetherian schemes. Then one can factor ϕ into $\phi = s \circ r$, where $r : X \to Y$ is a projective morphism with connected fibres, and $s : Y \to Z$ is a finite morphism.*

THEOREM 4.40. *Let L be a very ample line bundle on a smooth connected projective surface S. Assume that $q(S) < \pi(L) = 3$ and $L^2 \geq 6$; if we denote with ϕ the map associated to $H^0(K + L)$ and $\phi = s \circ r$ the Remmert-Stein factorization of ϕ, then we have the following possibilities:*

1. *(S, L) is a conic fibration over $\mathbb{P}^1$ under ϕ, in which case $L^2 = 6$;*
2. *(S, L) has $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4))$, $r : S \to \mathbb{P}^2$ as the reduction;*
3. *the reduction (Y, L_Y), $r : S \to Y$ of (S, L) is a Del Pezzo surface with $K_Y^2 \approx 2$, $L_Y \approx -2K_Y$, in which case $L^2 = 8$ ($\approx$ denotes the $\mathbb{Q}$-linear equivalence on Weil divisors, see* **[BS95]***, pages 5–6 for more details).*

The preceding Theorem is an interpretation of the results contained in **[BS95]**, pages 259–264, in which this (and much more) is explained.

We note also that the smooth scrolls of $\mathbb{P}^4$ are known:

THEOREM 4.41. *If S is a smooth irreducible scroll of $\mathbb{P}^4$, then S is either a conic bundle or an elliptic quintic scroll.*

See **[Lan80]**, **[Aur87]** and **[DP95]**.

LEMMA 4.42. *The surface F of case* (4.3) *cannot be a scroll.*

PROOF. Let us suppose first that the scroll is an irregular surface (*i.e.* $q(F) > 0$). For proving that this case cannot occur, we could apply simply Theorem 4.40 to the desingularization of F, since there are not irregular surfaces in the list. We can prove it also directly: F is a scroll with $\pi(L) = 3 > 0$, by Proposition 4.39; besides, we have that $L = \mathcal{O}_F(1)$. We see also that an ordinary singularity is Cohen-Macaulay—in fact it is a local complete intersection. Therefore we can apply to F Theorem 4.32, and we see that in particular F is smooth. Then we conclude by Theorem 4.41 or by the fact that there are no scrolls in the list of smooth surfaces of degree 6 in $\mathbb{P}^4$ (see **[Ran88]** for a list).

If we are in case $q(F) = 0$, then we are in case (1) of Theorem 4.40, *i.e.* (projection of) a conic fibration on $\mathbb{P}^1$. Let us calculate the number of trisecant lines for this surface. This surface is a rational ruled surface, and then we have that $K_F^2 = 0$, so we obtain

$$\begin{aligned}\chi(F) &= 1\\ c_1^2 &= (-K_F)^2\\ &= 0;\end{aligned}$$

then, by Noether's formula

$$\begin{aligned}c_2 &= 12\chi - c_1^2\\ &= 12.\end{aligned}$$

If e is the hyperplane class of S, then—since $e^2 = \deg F = 6$—by the adjunction formula

$$\begin{aligned}-c_1 e &= 2g(e) - 2 - e^2\\ &= 4 - 6\\ &= -2.\end{aligned}$$

From this and from Theorem 4.24 we can calculate the projective characters:

$$\begin{aligned}\mu_1 &= 3m - c_1 e\\ &= 3 \cdot 6 - 2\\ &= 16,\\ \nu_2 &= 6m - 4c_1 e + c_1^2 - c_2\\ &= 6 \cdot 6 - 2 \cdot 4 + 0 - 12\\ &= 16;\end{aligned}$$

from μ_1 and ν_2 we deduce μ_2:

$$\begin{aligned}\mu_2 &= \frac{1}{2}(c_1^2 + c_2 - 12m + 8\mu_1 - \nu_2) \\ &= \frac{1}{2}(0 + 12 - 12\cdot 6 + 8\cdot 16 - 16) \\ &= 26.\end{aligned}$$

So, finally:

$$\begin{aligned}t(F) &= \binom{m}{3} - \frac{1}{2}m\mu_1 + \frac{1}{3}(2\mu_1 + 2\nu_2 + \mu_2) \\ &= \frac{6\cdot 5\cdot 4}{3\cdot 2} - \frac{1}{2}6\cdot 16 + \frac{1}{3}(2\cdot 16 + 2\cdot 16 + 26) \\ &= 2.\end{aligned}$$

So, this does not generate a first order congruence. □

THEOREM 4.43. *The only surface F of case (4.3) is the Bordiga surface of $\mathbb{P}^4$, i.e. F is a blow up of $\mathbb{P}^2$ in 10 points $x_1, \ldots, x_{10}$ embedded in $\mathbb{P}^4$ by the linear system*

$$|D| := |\pi^* 4L - E_1 - E_2 - \cdots - E_{10}|$$

where $\pi : F \to \mathbb{P}^2$ is the blow up in the points $x_1, \ldots, x_{10}$, L is a line in $\mathbb{P}^2$ and E_i is the fibre of π over x_i. In fact F contains 10 distinct lines and 10 distinct plane cubics such that each line meets a single cubic. (see **[DP95]** *and* [Bea83] *page 56 for further details)*

Vice versa, the trisecant lines of a Bordiga surface generate a first order congruence.

PROOF. It a straightforward consequence of Theorem 4.40 and Corollary 4.42 that the only possibility left is to the Bordiga surface of $\mathbb{P}^4$.

Then it remains to prove that it generates a first order congruence: the surface is rational and also smooth; then

$$\chi(F) = 1.$$

Therefore, we can apply formula (74):

$$\begin{aligned}t(S) &= \binom{m-1}{3} - g(m-3) + 2\chi - 2 \\ &= \frac{5\cdot 4\cdot 3}{3\cdot 2} - 3(6-3) + 2 - 2 \\ &= 1,\end{aligned}$$

obtaining that the trisecanls of a Bordiga surface generate a first order congruence. □

REMARK. We observe that a Bordiga surface is indeed given by the vanishing of the minors of a 3×4 matrix with linear entries, see **[DP95]**, and so we are in case of Corollary 2.23 and Theorem 2.24.

4.4. Final remarks on the congruences of the first type. Let us make some concluding remarks about the results just proved: we consider now the congruence B as a subvariety of dimension 3 in $\mathbb{G}(1,4)$.

PROPOSITION 4.44. *Let B be the variety parametrizing the trisecant lines of the Veronese surface. Then B is a general linear congruence. In particular, B is smooth.*

See Proposition 2.19, (2).

An interesting characterization of the Bordiga surface is the following:

PROPOSITION 4.45. *The only linearly normal surface in $\mathbb{P}^4$ whose trisecants generate a first order congruence is the Bordiga surface.*

REMARK. The Bordiga surface is given by the vanishing of the minors of a general matrix of type 3×4 of linear entries. We recall that also the rational normal cubic is the only curve of $\mathbb{P}^3$ whose secants generate a first order congruence, see Theorem 0.1 and that this curve is given by the vanishing of the minors of a general matrix of type 2×3 of linear entries.

PROPOSITION 4.46. *The variety B parametrizing the trisecant lines of the Bordiga surface is smooth.*

PROOF. It is Theorem 2.24 with $n = 4$. □

5. Congruences of the Second Type

NOTATIONS. We consider now a first order congruence of the second type, *i.e.* the secants lines of an irreducible surface F_1 which meet another surface F_2 also. So, $F = F_1 \cup F_2$ is our (pure) focal surface. In this case, we will denote with m_i the degree of F_i, $i = 1, 2$; so $m = m_1 + m_2$.

We will assume also for the rest of this section that the surface F_1 is with isolated singularities. With this assumption, we denote with h_1 the number of apparent double points of the general hyperplane section $C_{1,H}$ of F_1.

REMARK. If F_1 has not isolated singularities, many of the results of this section continues to be valid if we suppose that the general hyperplane section $C_{1,H}$ of F_1 has ordinary singularities—and so, we can define h_1—or, even if the general hyperplane section $C_{1,H}$ of F_1 has not ordinary singularities, by setting $h_1 := \binom{m_1-1}{2} - g(C_{1,H})$, where $g(C_{1,H})$ is the geometric genus of $C_{1,H}$.

First of all, we can give some observations on this situation from the point of view of the Schubert calculus: it easy to see that the family of the secant lines, $S(F_1)$ of F_1 is a flat family of dimension 4, so it determines a cycle of codimension 2 in the Grassmannian, such that

$$S(F_1) = c\sigma_{11} + d\sigma_{20}.$$

We can actually calculate the numbers c and d:

PROPOSITION 4.47. *The following formula holds:*

$$S(F_1) = \binom{m_1}{2}\sigma_{11} + h_1\sigma_{20}, \tag{81}$$

where h_1 is the number of apparent double points of a general hyperplane section of the surface F_1.

PROOF. We will compute the two numbers separately:

1. First of all we consider c:

   we have

$$\sigma_{11} = \{\ell \in \mathbb{G}(1,4) \mid \ell \subset \mathbb{P}^3 \subset \mathbb{P}^4\}$$

and its complementary cycle is

$$\sigma_{22} = \{\ell \in \mathbb{G}(1,4) \mid \ell \subset \mathbb{P}^2\}.$$

We have that

$$\begin{aligned} m_1 &= \deg F_1 \\ &= \text{length}(\Pi \cap F_1), \end{aligned}$$

where Π is a general plane of $\mathbb{P}^4$. So

$$S(F_1) \cdot \sigma_{22} = c$$
$$= \binom{m_1}{2},$$

since we have m_1 points in the plane Π, so the number of the lines passing through two of them is $\binom{m_1}{2}$.

2. Then, we can compute d:
we have

$$\sigma_{20} = \{\ell \in \mathbb{G}(1,4) \mid \ell \cap \mathbb{P}^1 \neq \emptyset\}$$

and its complementary cycle is

$$\sigma_{31} = \{\ell \in \mathbb{G}(1,4) \mid P \in \ell \subset \mathbb{P}^3\}.$$

From this observation we see that

$$S(F_1) \cdot \sigma_{31} = d$$
$$= h_1.$$

□

Similarly, the (flat) family $M(F_2)$ of the lines meeting F_2 is a family of dimension 5, and so

$$M(F_2) = e\sigma_{10}.$$

Similarly to the preceding case, we can calculate e:

PROPOSITION 4.48. *The following formula holds:*

$$M(F_2) = m_2\sigma_{10}. \tag{82}$$

PROOF. We have

$$\sigma_{10} = \{\ell \in \mathbb{G}(1,4) \mid \ell \cap \mathbb{P}^2 \neq \emptyset\}$$

and its complementary cycle is

$$\sigma_{32} = \{\ell \in \mathbb{G}(1,4) \mid P \in \ell \subset \mathbb{P}^2\}.$$

So, since $\deg F_2 = \text{length}(F_2 \cap \Pi)$, where Π is a general plane, we have that

$$M(F_2) \cdot \sigma_{32} = e$$
$$= m_2.$$

□

PROPOSITION 4.49. *If the two surfaces F_1 and F_2 properly intersect, then F_2 is a plane and F_1 is a surface of minimal degree of $\mathbb{P}^4$, i.e. a rational normal scroll either of type $S_{1,2}$, which is smooth, or of type $S_{0,3}$, i.e. a cone over a rational normal cubic.*

PROOF. If the two surfaces are in general position *i.e.* if they intersect in a scheme of dimension zero (and not in a curve), we have, as a cycle

$$B = S(F_1) \cdot M(F_2) \tag{83}$$

$$= (\binom{m_1}{2}\sigma_{11} + h_1\sigma_{20}) \cdot (m_2\sigma_{10}) \tag{84}$$

$$= m_2(h_1\sigma_{30} + (\binom{m_1}{2} + h_1)\sigma_{21}). \tag{85}$$

So, in order to obtain a congruence of the first order, we must have $m_2 = 1$, *i.e.* F_2 is a plane and $h_1 = 1$, *i.e.* the sectional genus of F_1 is $\binom{m_1-1}{2} - 1 = \frac{m_1(m_1-3)}{2}$.

From the Castelnuovo's bound for the (non-degenerate) curves of degree m_1 in $\mathbb{P}^3$ we obtain, in this case:

$$\frac{m_1(m_1-3)}{2} \leq \begin{cases} \frac{1}{4}m_1^2 - m_1 + 1 & \text{if } m_1 \text{ is even,} \\ \frac{1}{4}(m_1^2-1) - m_1 + 1 & \text{if } m_1 \text{ is odd;} \end{cases}$$

with an easy count we have that the only possible case is for $m_1 = 3$. □

REMARK. In the case of the preceding proposition, the congruence is given, as a subscheme of the Grassmannian, as the intersection of a codimension two linear section and a tangent linear section.

We give some observations similar to the ones of the preceding section:

PROPOSITION 4.50. *If η is a i-parasitic plane and we call μ_i the degree of the curves $C_i = F_i \cap \eta$, with $i = 1, 2$, we have that*

$$i = \binom{\mu_1}{2}\mu_2. \tag{86}$$

PROOF. It is analogous to the proof of Corollary 4.9, but in this case we have a reducible curve $C = C_1 \cup C_2$ such that any line ℓ of η intersects C_i in μ_i points, and for every choice of two of these μ_1 points of $C_1 \cap \ell$ and one of the μ_2 points of $C_2 \cap \ell$ give ℓ as a line of the congruence. So ℓ has multiplicity $\binom{\mu_1}{2}\mu_2$. □

NOTATIONS. We recall that m_i is the degree of F_i and we denote with k_i the multiplicity of $F_i \cap V_\Pi$ in V_Π, $i = 1, 2$.

Propositions 2.26 and 2.27 are

PROPOSITION 4.51. *The following formula holds:*

$$2k_1 + k_2 = a + 1. \tag{87}$$

PROPOSITION 4.52. *The following formula holds:*

$$k_1^2 m_1 + k_2^2 m_2 = a^2 - x. \tag{88}$$

NOTATIONS. From now on we will denote with C the scheme (of dimension less or equal to one) intersection of F_1 and F_2. The degree of the subscheme (possibly empty) of the components of C of pure dimension one will be indicated with c. So, $c = 0$ if and only if $\dim(C) = 0$.

5.1. Case $m_2 = 1$. First of all we suppose that F_2 is a plane. We will also suppose for the rest of this subsection that F_1 has only isolate singularities.

We can start with the following

PROPOSITION 4.53. *The following formula holds:*

$$k_2 = h_1 - \binom{c}{2}. \tag{89}$$

PROOF. Let H be a general hyperplane of $\mathbb{P}^4$ and we put

$$r := H \cap F_2;$$

then, arguing as in the proof of Proposition 4.16, we see that k_2 is the number of lines of the congruence contained in H that pass through a general point P of r; but this number can be calculated by considering the hyperplane section of F_1, which has h_1 apparent double points; or, in other words, the curve

$$C_{1,H} := H \cap F_1$$

has h_1 secant lines passing through P. This curve has c points belonging to r, since

$$\begin{aligned} C_{1,H} \cap r &= H \cap F_1 \cap F_2 \\ &= C \cap r. \end{aligned}$$

So, the number of the lines of the congruence through P is equal to the number of the secant lines of F_1 through P excluding the line r (with its multiplicity, which is $\binom{c}{2}$), since we are supposing that F_2 is not a parasitic plane; so we are done. □

PROPOSITION 4.54. *The following formula holds:*

$$a = h_1 - \binom{c}{2} + \binom{m_1 - c}{2}. \tag{90}$$

PROOF. As in the preceding proposition, we can consider a hyperplane H. We recall that a is the number of lines of the congruence belonging to H and meeting a line contained in it. For computing a, we consider a plane Π contained in H and containing the line $r = H \cap F_2$; this will intersect the curve $C_{1,H} = H \cap F_1$ in m_1 points, c of them belong to the line r. So, the general line ℓ' of Π will intersect the $\binom{m_1-c}{2}$ lines determined by the $m_1 - c$ points not lying on r and k_1 times the line r. □

PROPOSITION 4.55. *The following formula holds:*

$$k_1 = m_1 - 1 - c. \tag{91}$$

PROOF. First of all, we observe that k_1 is the number of lines of the congruence contained in a hyperplane H and passing through a point P of the curve $C_{1,H} = H \cap F_1$. This number is, in fact, the number of secant lines passing through P and which meet the line $r = H \cap F_2$ in a point not belonging to F_1. This number is clearly obtained considering the number of points of the plane section of $C_{1,H}$ determined by r and P that are different from P and that do not belong to r. This number is then $m_1 - 1 - c$. □

COROLLARY 4.56. *The only possible values for c are $m_1 - 2$ and $m_1 - 3$.*

PROOF. Putting formulae (89), (90) and (91) in formula (87), we obtain

$$\begin{aligned} 2(m_1 - 1 - c) + h_1 - \binom{c}{2} &= h_1 - \binom{c}{2} + \binom{m_1 - c}{2} + 1 \\ 4(s-1) &= s(s-1) + 2 \\ s^2 - 5s + 6 &= 0, \end{aligned}$$

where we have put $s := m_1 - c$. So the only possible values are $s = 3$ or $s = 2$, so our claim follows. □

REMARK. Another proof of this proposition could consider a hyperplane section which contains the plane F_2. This will intersect F_1 in C and another curve C_H of degree $s = m_1 - c$. Now, since the lines of the congruence contained in this hyperplane are the secant lines of the curve C_H, this must have only one apparent double point; so, we can argue as usual by the Castelnuovo's bound:

$$\frac{s(s-3)}{2} \leq \begin{cases} \frac{1}{4}s^2 - s + 1 & \text{if } s \text{ is even,} \\ \frac{1}{4}(s^2 - 1) - s + 1 & \text{if } s \text{ is odd;} \end{cases}$$

so, the inequalities are satisfied only if s is 3. For this value, the curve C_H is clearly a twisted cubic. Besides, C_H can be a pair of non-planar lines, which is a curve with an apparent double point, so s can be 2 also.

Another way of arguing is to say that the congruence restricted to the general hyperplane H containing F_2 is a first order congruence of $H \cong \mathbb{P}^3$.

We analyse now the two possibilities, $m_1 - c = 2$ and $m_1 - c = 3$.

5.1.1. *The case* $m_1 - c = 3$. Let us start with the case $m_1 - c = 3$;

COROLLARY 4.57. *We have that:*

$$k_2 = h_1 - \binom{m_1 - 3}{2} \tag{92}$$

$$a = h_1 - \frac{m_1(m_1 - 7)}{2} - 3 \tag{93}$$

$$k_1 = 2, \tag{94}$$

and:

$$x = 6h_1 - 3m_1^2 + 17m_1 - 27 \geq 0 \tag{95}$$

PROOF. We will prove formula (95), since the rest is immediate. Putting in Equation (88) the formulas (92) we obtain (putting $b = \binom{m_1-3}{2}$):

$$\begin{aligned} x &= -k_1^2 m_1 - k_2^2 m_2 + a^2 \\ &= (h_1 + 3)^2 - 2(h_1 + 3)b + b^2 - 4m_1 - h_1^2 + 2h_1 b - b^2 \\ &= 6h_1 + 9 - 6b - 4m_1 \\ &= 6h_1 - 3m_1^2 + 17m_1 - 27. \end{aligned}$$

□

From now on, until the end of this subsection, we will suppose that the singular points of F_1 are out of F_2, *i.e.* $\mathrm{Sing}(F_1) \cap C = \emptyset$.

THEOREM 4.58. *The following identity*

$$h_1 = 3(m_1 - 3) - t + 1 \tag{96}$$

holds, where t is the length of $C \cap C_H$. So $0 \leq t \leq 3$.

PROOF. The cone χ_P given by the secant lines of F_1 passing through a (general) point P of $\mathbb{P}^4$ has degree h_1 (since its general hyperplane section containing P consists of h_1 lines), and then the hyperplane section of this cone will determine h_1 lines of the ruling of it. If we consider the hyperplane section H with the hyperplane spanned by P and F_2, these lines are: the secant line of the twisted cubic C_H passing through P, and the lines which are common to the join PC of P and C and to the one PC_H of P and C_H, excluded the lines passing through the points of $C \cap C_H$.

Now, it is easy to see that

$$\deg PC = \deg C = m_1 - 3$$
$$\deg PC_H = \deg C_H = 3$$

and putting $t := \deg C \cap C_H$, we are done. □

COROLLARY 4.59. *The only possible cases are the following:*

1. $t = 0 \Rightarrow x = -3m_1^2 + 35m_1 - 75 \geq 0 \Rightarrow 3 \leq m_1 \leq 8;$
2. $t = 1 \Rightarrow x = -3m_1^2 + 35m_1 - 81 \geq 0 \Rightarrow 4 \leq m_1 \leq 8;$
3. $t = 2 \Rightarrow x = -3m_1^2 + 35m_1 - 87 \geq 0 \Rightarrow 4 \leq m_1 \leq 8;$
4. $t = 3 \Rightarrow x = -3m_1^2 + 35m_1 - 93 \geq 0 \Rightarrow 5 \leq m_1 \leq 7$

PROOF. We use Equation (96) in (95) to eliminate h_1, obtaining

$$x = -3m_1^2 + 35m_1 - 6t - 75 \geq 0,$$

and solving it we obtain our claim. □

Using formula (96), formulas (92) and Corollary 4.59 we obtain the list of the possible (general) surfaces F_1 with their invariants:

THEOREM 4.60. *The list of the possible surfaces F_1 with invariants t, m_1, h_1 as above and with sectional genus π is given by Table 1; we have also, for these surfaces, $k_1 = 2$.*

case #	t	m_1	h_1	π	a	c	k_2	x
i	3	5	4	2	6	2	3	7
ii	3	6	7	3	7	3	4	9
iii	3	7	10	5	7	4	4	5
iv	2	4	2	1	5	1	2	5
v	2	5	5	1	7	2	4	13
vi	2	6	8	2	8	3	5	15
vii	2	7	11	4	8	4	5	11
viii	2	8	14	7	7	5	4	1
ix	1	4	3	0	6	1	3	11
x	1	5	6	0	8	2	5	19
xi	1	6	9	1	9	3	6	21
xii	1	7	12	3	9	4	6	17
xiii	1	8	15	6	8	5	5	7
xiv	0	3	1	0	4	0	1	3
xv	0	6	10	0	10	3	7	27
xvi	0	7	13	2	10	4	7	23
xvii	0	8	16	5	9	5	6	13

TABLE 1

PROOF. It is a straightforward consequence of formula (96) and Corollary 4.59. We observe that cases $t = 0$ with $m_1 = 4, 5$ must be excluded since we have $\pi = -1$. □

If, in addition, we suppose that the surface is smooth, since all the smooth surfaces up to degree 10 are classified, (see [**Ran88**] for the complete list and details in degree 10, [**Rot37**] for details in degree less than 7, [**Oko84**], [**Oko86**], for degree 7 and 8, supplemented in [**Ale88**] and [**AR92**] for degree 9) we can analyse which of them exist and generate a first order congruence.

First of all, we recall the Noether's Lemma:

NOETHER'S LEMMA. *An algebraic surface S is rational if and only if it contains an irreducible rational curve C with* $\dim|C| \geq 1$.

For a proof, see [**GH78**]. Then, we pass to the following:

LEMMA 4.61. *If, in the hypothesis of Theorem* 4.60 *we add that F_1 is smooth, then F_1 must be rational.*

PROOF. An element of the pencil of hyperplanes through the plane F_2 of C intersects F_1 in a rational normal cubic (meeting C in t points). Therefore the linear system of these cubics has dimension at least one, so, by Noether's Lemma, we conclude. □

THEOREM 4.62. *If the surface F_1 is smooth, then only the cases* (i), (ii), (iv) and (xiv) *of Table* 1 *are effective.*

PROOF. Let us study each case separately:

1. The case (i) of Table 1 gives, as the possible F_1, a rational quintic with sectional genus 2, linked to a plane in the complete intersection of a quadric and a cubic, *i.e.* a Castelnuovo surface; F_1 is a blow up of $\mathbb{P}^2$ in 8 points $x_1, \ldots, x_8$ embedded in $\mathbb{P}^4$ by the linear system

$$|D| := |\pi^* 4L - 2E_1 - E_2 - \cdots - E_8|$$

where $\pi : F_1 \to \mathbb{P}^2$ is the blow up in the points $x_1, \ldots, x_8$, L is a line in $\mathbb{P}^2$ and E_i is the fibre of π over x_i.

The curve C should be a conic contained in a plane F_2, and a conic in F_1 is given by the proper transform $\tilde{\ell}_{2,3}$ of a line $\ell_{2,3}$ through—for example—x_2 and x_3.

By the adjunction formula, we get, in fact, that a hyperplane through F_2 intersects F_1, out of C, in a rational normal cubic and this cubic intersects C in three points. These rational normal cubics are clearly the proper transforms of the cubics having a double point in x_1 and passing through $x_4, \ldots, x_8$. Then, in fact, these surfaces generate a first order congruence.

We observe, from Proposition 4.50 and from the adjunction formula, that we can have only 7 1-parasitic planes, *i.e.* each of them contains a conic meeting C in two points. For getting these planes, we see that the proper transforms of the 5 conics passing through the points $x_1, x_{i_1} \ldots, x_{i_1}$, where $i_1 < \cdots < i_4 \in \{4, \ldots, 8\}$ give 5 such conics and so 5 of these 1-parasitic planes. The other two are given by the planes of (the conic component of) the proper transforms of the cubics having a double point in x_1 and passing through $x_i, x_4, \ldots, x_8$, with $i = 2, 3$.

2. The case (ii) of Table 1 gives, as the possible F_1, a rational sextic with sectional genus 3, linked to a cubic scroll in the complete intersection of two cubics, *i.e.* a Bordiga surface, which is described in Theorem 4.43

The curve C should be a cubic contained in a plane F_2, and a cubic in F_1 is given by the proper transform of a cubic through 8 points—for example, $x_2, \ldots, x_{10}$.

By the adjunction formula, we get, in fact, that a hyperplane through F_2 intersects F_1, out of C, in a rational normal cubic and this cubic intersects C in three points. These rational normal cubics are clearly the proper transforms of the lines through x_1. Then, in fact, these surfaces generate a first order congruence.

We observe, from Proposition 4.50 and from the adjunction formula, that we can have only 9 1-parasitic planes, *i.e.* each of them contains a conic meeting C in two points. For getting these planes, we see that the proper transforms of the 9 lines passing through the points x_1 and x_i, where $i \in \{2, \ldots, 10\}$ give 9 conics and so 9 1-parasitic planes.

3. The case (iii) of Table 1 gives, as a possible F_1, a surface of degree 7 and sectional genus 5, which is a non-minimal $K3$-surface, linked to a degenerate quadric surface in the complete intersection of two cubics. But this surface cannot generate a first order congruence, since it is not rational. But we can prove it also directly: in fact, the curve C should be a quartic of genus 3 contained in F_2, with three points in common with the curve C_H, element of the pencil of rational normal cubics obtained by the intersections of F_1 by the hyperplanes through F_2 out of C. These points would form a linear series of degree 3 and dimension 1, *i.e.* a g_3^1 on C; but a g_3^1 should be cut out by the lines through a fixed point of C, *i.e.* the three points of the linear system should be contained in a line, which is absurd, because they are three points of a rational normal curve also.

4. The case (iv) of Table 1 gives, as a possible F_1, a quartic with sectional genus 1, which is a Del Pezzo surface, a complete intersection of two quadrics; F_1 is a blow up of $\mathbb{P}^2$ in 5 points $x_1, \dots, x_5$ embedded in $\mathbb{P}^4$ by the linear system
$$|D| := |\pi^* 3L - E_1 - E_2 - \cdots - E_5|$$
where $\pi : F_1 \to \mathbb{P}^2$ is the blow up in the points $x_1, \dots, x_5$, L is a line in $\mathbb{P}^2$ and E_i is the fibre of π over x_i.
   The curve C should be a line, and a line in F_1 is given by the proper transform of either a line through—for example—x_1 and x_2, or one of the exceptional divisors E_i or the proper transform of the conic through $x_1, \dots, x_5$ (see, for example, [**GH78**], pages 545–549). F_2 is then a plane through one of these lines.
   By the adjunction formula, we get that a hyperplane through F_2 intersects F_1, out of C, in a rational normal cubic which meets C in two points; therefore, the line C must be the proper transform of a line through x_1 and x_2. The rational normal cubics are then the proper transforms of the conics through x_3, x_4, x_5 and another point $P \in \mathbb{P}^2$. Then, in fact, these surfaces generate a first order congruence.
   We observe, from Proposition 4.50 and from the adjunction formula, that we can have only 5 1-parasitic planes, *i.e.* each of them contains a conic meeting C in two points. For getting these planes, we see that 3 of them are the planes of the proper transforms of the 3 lines passing through the points P and x_i, where $i \in \{3, 4, 5\}$; the remaining two are the planes of the proper transforms of the two conics through P, x_j, $j = 1, 2$ and x_3, x_4, x_5.
5. The case (v) of Table 1 gives, as the possible F_1, a quintic with sectional genus 1, which is an elliptic scroll. This surface cannot generate a first order congruence, since it is not rational. Indeed, we can prove this also directly: in fact, C should be a plane conic, but F_1 does not contain plane conics.
6. The case (vi) of Table 1 gives, as the possible F_1, a sextic with sectional genus 2, which does not exist.
7. The case (vii) of Table 1 gives, as the possible F_1, a rational surface of degree 7 and with sectional genus 4, linked to an elliptic quintic scroll in a complete intersection of a cubic and a quartic; F_1 is a blow up of $\mathbb{P}^2$ in 12 points $x_1, \dots, x_{12}$ embedded in $\mathbb{P}^4$ by the linear system
$$|D| := |\pi^* 5L - 2E_1 - 2E_2 - E_3 - \cdots - E_{12}|$$
where $\pi : F_1 \to \mathbb{P}^2$ is the blow up in the points $x_1, \dots, x_{12}$, L is a line in $\mathbb{P}^2$ and E_i is the fibre of π over x_i. But this surface cannot generate a first order congruence: the curve C should be a plane quartic, but an easy count shows that F_1 does not contain such a curve.
   Another way of proving this is to see that C should be a quartic of genus 3, with two points in common with every curve of the pencil of rational normal cubics given by the intersections out of C of F_1 with the elements of the pencil of hyperplanes through F_2; but these points form a linear series of degree 2 and dimension 1, *i.e.* a g^1_2 on C; but C cannot be hyperelliptic.
8. The case (viii) of Table 1 gives, as the possible F_1, a surface of degree 8 and with sectional genus 7, which is a regular elliptic surface, linked to a plane in the complete intersection of two cubics; this surface cannot generate a first order congruence, since it is not rational. But we can prove this also directly: C should be a smooth plane quintic of genus 6, with two points in common with every curve of the pencil of rational normal cubics given by the intersections out of C of F_1 with of the elements of the pencil of hyperplanes

through F_2. These points form a linear system of degree 2 and dimension 1, *i.e.* a g_2^1 on C; but C cannot be hyperelliptic, since the canonical divisor of C is given by $C \cdot D$, where D is a conic, *i.e.* it is given by 10 points on a conic. Then, if we impose a conic to pass through two points P and Q on C, this condition imposes two linear independent conditions on the space of the conics; then $\dim|P+Q| = 1$ by Riemann-Roch.

9. We give now a lemma concerning the case in which $t = 1$:

LEMMA 4.63. *If $t = 1$, then C is a rational curve.*

PROOF. Since $t = 1$, C has one point in common with every rational normal cubic given by the intersection out of C of F_1 with a hyperplane section through F_2; but these points form a linear system of degree 1 and dimension 1, *i.e.* a g_1^1 on C; so C is rational. □

COROLLARY 4.64. *If $t = 1$, then $\pi(F_1) = 0$.*

PROOF. It follows from Lemma 4.63 and the adjunction formula. □

COROLLARY 4.65. *The only possible cases with $t = 1$ are* (ix) *and* (x).

(a) The case (ix) of Table 1 gives as F_1 the Veronese surface, which does not generate a first order congruence. In fact, the curve C should be a line, but the Veronese surface does not contain lines.

REMARK. It is easy to see that, in the case (ix), if we suppose that the surface F_1 has ordinary singularities instead of being smooth, we see that this case is effective. In fact, if we take as F_1 the projection of the rational normal scroll $S_{1,3}$ and as F_2 a plane passing through a line C of the ruling, we have that the intersection of a hyperplane passing through C is given by a rational normal cubic out of C and so these surfaces generate a first order congruence. Indeed, by the double point formula we have that F_1 has one double point A.
Besides, we have to see that $x = 11$: there is a plane η through A and the unisecant line D of the scroll which is 3-parasitic, since it contains the cubic whose components are D and the two lines of the ruling through A, and intersects F_2 in a line, since the hyperplane H spanned by F_2 and A contains η, because H contains a point of $D \cap \eta$ and then $D \subset H$.
We have also two 1-parasitic planes, given by the two conics through A obtained by the intersection, out of $C \cup D$ and the lines of the ruling, of F_1 with the embedded tangent cone to F_1 in A.
The other cases in which we have surfaces with ordinary singularities and with rational sections can be studied in the same way.

(b) The case (x) of Table 1 gives, as the possible F_1, a quintic with rational sections, which does not exist.

10. For the case $t = 0$ the only possibility is the case (xiv) of Theorem 4.60 which gives as F_1 the cubic scroll $S_{1,2}$, *i.e.* we are in the case of Proposition 4.49. Besides we have 3 1-parasitic planes.

In fact, since $t = 0$, we get that C cannot be a curve (since otherwise C would be an irreducible component of F_1!). Therefore we have a proper intersection, *i.e.* we are in the case of Proposition 4.49. Finally, the three 1-parasitic planes are those of the conics of the scroll passing through two by two of the three points of C.

□

REMARK. Reverting the arguments of the proof just given, we see that the surfaces of the preceding theorem, with the configurations given in the proof, generate a first order congruence.

5.1.2. *The case $m_1 - c = 2$.* Now, let us consider the case $m_1 - c = 2$;

COROLLARY 4.66. *We have that:*

$$k_2 = h_1 - \binom{m_1 - 2}{2} \tag{97}$$

$$a = h_1 - \frac{m_1(m_1 - 5)}{2} - 2 \tag{98}$$

$$k_1 = 1, \tag{99}$$

and:

$$x = 2h_1 - m_1^2 + 4m_1 - 5 \geq 0 \tag{100}$$

PROOF. we prove only formula (100): the rest is straightforward. Putting in Equation (88) the formulas (97) we obtain (putting $b = \binom{m_1-2}{2}$):

$$\begin{aligned} x &= -k_1^2 m_1 - k_2^2 m_2 + a^2 \\ &= (h_1 - b + 1)^2 - m_1 - (h_1 - b)^2 \\ &= 2(h_1 - b) + 1 - m_1 \\ &= 2h_1 - m_1^2 + 4m_1 - 5. \end{aligned}$$

□

From now on, until the end of this subsection, we will suppose that the singular points of F_1 are out of F_2, *i.e.* $\text{Sing}(F_1) \cap C = \emptyset$.

THEOREM 4.67. *The following identity*

$$h_1 = 2(m_1 - 2) - t + 1 \tag{101}$$

holds, where t is the length of $C \cap C_H$ (and so $0 \leq t \leq 2$).

PROOF. We recall, as in the proof of Equation (96), that the cone χ_P given by the secant lines of F_1 which meet a (general) point P of $\mathbb{P}^4$ also, has degree h_1. Then a hyperplane section of χ_P through P is the union of h_1 lines of the ruling of it.

In particular, if we consider the hyperplane section H with the hyperplane determined by P and F_2, the h_2 lines are: the secant line of the two non-coplanar lines ℓ and ℓ' (which are the irreducible components of C_H) which passes through P, and the lines which are in the intersection of the joins PC—given by the lines meeting P and C—and PC_H—*i.e.* the lines through P and C_H—and not passing through the points of $C \cap C_H$.

Now, it is easy to see that

$$\deg PC = \deg C = m_1 - 2$$

while PC_H is the union of the two planes $P\ell$ and $P\ell'$; then, putting $t := \deg C \cap C_H$, we are done. □

COROLLARY 4.68. *The cases to be considered are the following:*

1. $t = 0 \Rightarrow x = m_1^2 + 8m_1 - 11 \geq 0 \Rightarrow 2 \leq m_1 \leq 6$;
2. $t = 1 \Rightarrow x = m_1^2 + 8m_1 - 13 \geq 0 \Rightarrow 3 \leq m_1 \leq 5$;
3. $t = 2 \Rightarrow x = m_1^2 + 8m_1 - 15 \geq 0 \Rightarrow 3 \leq m_1 \leq 5$.

PROOF. We use Equation (101) in (100) to eliminate h_1, obtaining

$$x = m_1^2 + 8m_1 - 2t - 11 \geq 0,$$

and solving it we obtain our claim. □

As in the preceding case, using formula (101) and Corollary 4.68 we obtain the list of the possible surfaces F_1 with their invariants:

THEOREM 4.69. *The list of the possible (general) surfaces F_1 with invariants t, m_1, h_1 as above and with sectional genus π is given by Table 2; we have also,*

case #	t	m_1	h_1	π	a	c	k_2	x
i	2	3	1	0	2	1	1	0
ii	2	4	3	0	3	2	2	1
iii	2	5	5	1	3	3	2	0
iv	1	5	6	0	4	3	3	2
v	0	6	9	1	4	4	3	1

TABLE 2

for these surfaces, $k_1 = 1$.

PROOF. It is a straightforward consequence of formula (101) and Corollary 4.68. We observe that cases $t = 1$ with $m_1 = 3, 4$, $t = 0$ with $m_1 = 2, 5$ must be excluded since we have $\pi = -1$ and $t = 0$ with $m_1 = 3, 4$ since we have $\pi = -2$. □

As in the preceding case, let us analyse the possible cases of smooth surfaces:

LEMMA 4.70. *If, in the hypothesis of Theorem* 4.69 *we add that F_1 is smooth, then F_1 must be rational.*

PROOF. An element of the pencil of hyperplanes through the plane F_2 of C intersects F_1 in a couple of skew lines ℓ_1 and ℓ_2 (meeting C in t points). Therefore the linear system of—say—ℓ_1 has dimension at least one, so, by Noether's Lemma, we conclude. □

THEOREM 4.71. *If the surface F_1 is smooth, then only the case* (i) *of Table* 2 *is effective.*

PROOF. Let us study each case separately:

1. The case (i) of Table 2 gives, as the possible F_1, the cubic scroll $S_{1,2}$; C should be a line, so it can either be a line of the ruling of the unisecant line of the scroll; besides, F_2 is a plane through C. C cannot be a line of the ruling, since if we consider a hyperplane H containing F_2, then H intersects F_1 out of C in a plane conic.

   Then C can only be the unisecant line and F_2 a plane through it. It is obvious that with this configuration we have a first order congruence, since a general hyperplane containing F_2 intersects F_1 in two (skew) lines of the ruling.
2. The case (ii) of Table 2 gives, as the possible F_1, a (projected) Veronese surface, which cannot generate a first order congruence since the Veronese surface does not contain lines.
3. The case (iii) of Table 2 gives as F_1 an elliptic quintic scroll, which has sectional genus 1 and it is not contained in any quadric, but it is cut out by cubics; but this surface cannot generate a first order congruence, since it is not rational. We can prove this also directly: C should be a cubic smooth plane curve, since it has genus 1 by adjunction. Then C is a unisecant cubic

curve of the scroll. But then if we intersect F_1 with a hyperplane through F_2, we would have a couple of skew lines, which cannot occur, since for the elliptic quintic scroll the hyperplane through a unisecant cubic gives a reducible (plane) conic.

4. The case (iv) of Table 2 gives, as the possible F_1, a quintic with rational sections, which does not exist.
5. The case (v) of Table 2 gives, as the possible F_1, a sextic with sectional genus 1, which does not exist.

□

REMARK. Reverting the argument of the proof just given, we see that the surface of the preceding theorem, with the configuration given in the proof, generates a first order congruence.

5.2. Case $m_2 \neq 1$. We will suppose now that F_2 is not a plane.
Let us start with the following

THEOREM 4.72. *If F_2 is not a plane, then F_1 has degree three. So either F_1 is the rational normal scroll $S_{1,2}$ or it is a cone over a rational normal curve of degree three.*

PROOF. We will denote with u the number of lines passing through a general point of $\mathbb{P}^4$ which meet the (at most) one-dimensional scheme $C := F_1 \cap F_2$ and are secants of F_1.

We have that the cone χ_P of the secant lines of F_1 passing through P has degree h_1, so we have $h_1 m_2$ lines of this cone meeting F_2 too. Only one of these lines does not meet C, since we have a first order congruence (and the lines of the join of C and F_1 give a congruence which is distinct from Λ, since F_2 is not a plane and Λ is, by hypothesis, irreducible), so $u = h_1 m_2 - 1$

On the other hand, since F_2 is not a plane, through a general point $Q \in F_2$ cannot pass infinitely many secant lines of F_1 meeting C also, since these lines would be lines of the congruence and varying the point Q, we would obtain all the lines of the congruence; then, we had a congruence of the fourth type (*i.e.* the lines meeting a curve and a surface).

Then, through Q there pass $h_1(m_2 - 1)$ secant lines of F_1 meeting again F_2, that must be the u secant lines of F_1 passing through Q and that meet C also. This is due to the fact that if one of the $h_1(m_2 - 1)$ lines would meet F_2 outside C, then this would be a focal line, since it contained (at least) four focal points. So, Λ would have a focal hypersurface.

So, we have

$$
\begin{aligned}
u &= h_1 m_2 - 1 \\
&= h_1(m_2 - 1)
\end{aligned}
$$

and so $h_1 = 1$, and, as usual, by Castelnuovo's bound, we obtain $m_1 = 3$. Therefore it is a surface of minimal degree, so it is the rational normal scroll $S_{1,2}$, or the cone $S_{0,3}$. □

REMARK. From this proposition, we can give an alternative proof of the fact that in this case C must be one dimensional.

In fact, if F_1 is the scroll $S_{1,2}$. the cone χ_P of the secant lines of F_1 passing through a general point P will be the pencil of lines through P in the (unique) plane Π which contains P and intersects F_1 in a conic. Since Λ is a first order congruence, $m_2 - 1$ of the m_2 points in common between Π and F_2 must be in C; so C has dimension (at least) one, since by hypothesis we have $m_2 \neq 1$.

If F_1 is the cone $S_{0,3}$, the cone χ_P of the secant lines of F_1 passing through a general point P will be a plane: in fact, if we consider a general plane Π, this will span a hyperplane H with P; then, this hyperplane will intersect F_1 in a rational normal cubic, which has only one apparent double point, *i.e.* there will pass only one secant line of it through P. Then, arguing as before, we conclude that $\dim(C) \geq 1$.

We will suppose now that F_1 is the scroll $S_{1,2}$; clearly analogous results can be proven in the case of the cone.

PROPOSITION 4.73. *If $F_1 = S_{1,2}$, we have that*

$$c \leq 2(m_2 - 1). \tag{102}$$

PROOF. If ℓ is a generic line of the ruling of the scroll F_1, then $\ell \cap C$ has length $\mu \leq (m_2 - 1)$, since for a (general) point P of ℓ, the cone χ_P of secant lines of F_1 through P will be the pencil of lines through P in the plane given by ℓ and the unisecant line of the scroll d, and the $m_2 - 1$ points of intersection of C with this plane will be contained in the lines of the pencil χ_P which meet C also and the line ℓ is an element of χ_P.

So, intersecting C with the hyperplane given by two lines of the ruling ℓ and ℓ' of the scroll, we obtain, as $\ell \cap C$ and $\ell' \cap C$ two zero dimensional schemes of length μ, and so out thesis follows. □

PROPOSITION 4.74. *If $F_1 = S_{0,3}$, we have that*

$$c \leq 3(m_2 - 1). \tag{103}$$

PROOF. If ℓ is a generic line of the ruling of the cone F_1, then $\ell \cap C$ has length $\mu \leq (m_2 - 1)$, since for a (general) point P of ℓ, the cone χ_P of secant lines of F_1 through P will be a pencil of lines through P, and the $m_2 - 1$ points of intersection of C with this plane will be contained in the lines of the pencil χ_P which meet C also and the line ℓ is an element of χ_P.

So, intersecting C with the hyperplane given by three lines of the ruling ℓ, ℓ' and ℓ'' of the scroll, we obtain, as $\ell \cap C$ and $\ell' \cap C$ three zero dimensional schemes of length μ, and so out thesis follows. □

PROPOSITION 4.75. *We have that*

$$k_1 = 2m_2 - c \tag{104}$$

$$k_2 = 1 \tag{105}$$

$$n = 4m_2 - 2c \tag{106}$$

PROOF. Let us start proving formula (105); k_2 is the number of lines of the congruence contained in a hyperplane H that pass through a general point P of the curve $C_{2,H} = H \cap F_2$; to compute this number, we consider the rational normal curve $C_{1,H} = H \cap F_1$, which has one apparent double point, so we have only one secant line of $C_{1,H}$ passing through P, and this line (for P general) cannot intersect C, otherwise we had $C = F_1$.

For proving formula (104), we recall that k_1 is the number of lines of the congruence contained in a hyperplane H that pass through a general point P of the rational normal curve $C_{1,H}$; these lines are the secant lines of $C_{1,H}$ passing through P and that meet $C_{2,H}$ out of C. So we have to intersect a cone of degree 2 and the curve $C_{2,H}$ of degree m_2, obtaining $2m_2$ points, from which we have to subtract the points of C, whose number is c.

The last formula follows from formula (87). □

COROLLARY 4.76. *We have that:*

$$x = (2m_2 - c)^2 - m_2 \geq 0. \tag{107}$$

PROOF. We use the formula (88) in this case

$$\begin{aligned} x &= -3(2m_2 - c)^2 - m_1 + (4m_2 - 2c)^2 \\ &= (2m_2 - c)^2 - m_1. \end{aligned}$$

□

From this we can obtain an improvement of Propositions 4.73 and 4.74:

COROLLARY 4.77. *The following inequality holds:*

$$c \leq 2m_2 - \sqrt{m_2}. \tag{108}$$

THEOREM 4.78. *The surface F_2 is rational.*

PROOF. We can construct a rational map in the following way: given a general plane Π, we can associate to a general point P in this plane, the unique point of F_2 contained in the plane of the secant lines of F_1 passing through P and that is not in C. □

We can summarize what we have proven:

THEOREM 4.79. *If F_2 is not a plane, then F_1 either is the rational normal scroll $S_{1,2}$ or is the cone $S_{0,3}$ and F_2 is a rational surface.*

REMARK. If restrict ourself to smooth surfaces, then, by a Ellingsrud and Peskine's Theorem, see [**EP89**], we have only a finite number of cases.

6. Congruences of the Third Type

NOTATIONS. Let us consider three irreducible surfaces F_1, F_2 and F_3, and the congruence Λ of the lines that meet once each of the three surfaces.

So, $F = F_1 \cup F_2 \cup F_3$ is our (pure) focal surface. In this case, we will denote with m_i the degree of F_i, $i = 1, 2, 3$; so $m = m_1 + m_2 + m_3$.

First of all, we can give, as in the preceding sections, some observations on this situation from the point of view of the Schubert calculus.

We recall from proposition 4.48 that the (flat) family $M(F_i)$ of the lines meeting F_i, $i = 1, 2, 3$ is a family of dimension 5, and that

$$M(F_i) = m_i \sigma_{10}.$$

REMARK. If the three surfaces are in general position *i.e.* they intersect two by two in a scheme of dimension zero (and not in a curve), so we have, as a Schubert cycle

$$B = M(F_1) \cdot M(F_2) \cdot M(F_3) \tag{109}$$

$$= (m_1\sigma_{10}) \cdot (m_2\sigma_{10}) \cdot (m_3\sigma_{10}) \tag{110}$$

$$= m_1 m_2 (\sigma_{20} + \sigma_{11}) \cdot (m_3\sigma_{10}) \tag{111}$$

$$= m_1 m_2 m_3 (2\sigma_{21} + \sigma_{30}), \tag{112}$$

So, in order to obtain a congruence of the first order, we must have $m_i = 1$, $i = 1, 2, 3$, *i.e.*

THEOREM 4.80. *If the three surfaces F_1, F_2 and F_3 meet properly, then these surfaces are three planes and the congruence has bidegree $(1, 2)$. Each of them is a fundamental 2-locus (of multiplicity one), and we have a 1-parasitic plane also: the plane spanned by the 3 points $F_1 \cap F_2$, $F_1 \cap F_3$ and $F_2 \cap F_3$.*

We can repeat the observations given in the preceding sections; the proofs are (almost) the same given before:

PROPOSITION 4.81. *If η is a i-parasitic plane and we call μ_i the degree of the curves $C_{i,\eta} = F_i \cap \eta$, with $i = 1, 2, 3$, we have that*

$$i = \mu_1 \mu_2 \mu_3. \tag{113}$$

NOTATIONS. We recall that m_i is the degree of F_i and we denote with k_i the multiplicity of $F_i \cap V_\Pi$ in V_Π, $i = 1, 2, 3$.

Propositions 2.26 and 2.27 are

PROPOSITION 4.82. *The following formula holds:*

$$k_1 + k_2 + k_3 = a + 1. \tag{114}$$

PROPOSITION 4.83. *The following formula holds:*

$$k_1^2 m_1 + k_2^2 m_2 + k_3^2 m_3 = a^2 - x. \tag{115}$$

NOTATIONS. From now on we will denote with C_{i+j-2} the scheme (of dimension less or equal to one) intersection of F_i and F_j, $1 \leq i < j \leq 3$. The degree of the subscheme (possibly empty) of pure dimension one of C_{i+j-2} will be indicated with c_{i+j-2}.

The first important fact is the following:

THEOREM 4.84. *At least one of the* 3 *surfaces F_1, F_2 and F_3 is a plane.*

PROOF. Assume by contradiction that none of the surfaces F_1, F_2 and F_3 is a plane.

We will denote with u the number of lines passing through a general point P of $\mathbb{P}^4$ which meet the—possibly empty—subscheme Z of $C_2 \cup C_3$ given by the components of $C_2 \cup C_3$ of dimension one, and meet F_1 and F_2 also.

Since we have analysed the case in which the three surface meet properly, we can suppose that—at least—C_1 is in fact a curve. We have that the cone χ_P of the lines passing through P and meeting both F_1 and F_2 and not pass through the points of C_1 has dimension two and degree $m_1 m_2 - c_1$. We have to cut out the points of C_1 since the lines meeting C_1 and C_3 form in fact a congruence distinct from Λ and Λ is irreducible.

We have $(m_1 m_2 - c_1) m_3$ lines of the cone χ_P meeting F_3 too. Only one of these lines does not meet Z, since Λ is a first order congruence (and the lines of the join of Z and F_1 give a congruence which is distinct from Λ, since F_2 or F_3 is not a plane and Λ is, by hypothesis, irreducible), so

$$(m_1 m_2 - c_1) m_3 = u + 1. \tag{116}$$

On the other hand, F_3 is not a plane, so if Q is a general point of F_3, $Z \subset F_3$ and Q are not coplanar. Besides, through a general point $Q \in F_3$ cannot pass infinitely many lines of the join of F_1 and F_2 meeting $Z \subset F_3$, since these lines would be lines of the congruence and varying the point Q, we would obtain all the lines of the congruence; then, we had a congruence of the fourth type (*i.e.* the lines meeting a curve—Z—and a surface, F_3).

Then, through Q there pass $(m_1 m_2 - c_1)(m_3 - 1)$ lines of the join of F_1 and F_2 meeting Z, and so

$$u = (m_1 m_2 - c_1)(m_3 - 1). \tag{117}$$

From Equations (116) and (117) we deduce that $m_1 m_2 = c_1 + 1$. This means that the general projections of F_1 and F_2—under ϕ, the projection from a general point $P \in \mathbb{P}^4$—intersect out of $\phi(C_1)$ in a line ℓ.

Besides, since we can suppose $m_1 \geq m_2 > 1$, it follows that $c_1 \geq m_1 + 1 \geq m_2 + 1$. Therefore, $\deg(\phi(C_1)) = c_1$, $\phi(C_1)$ cannot be a plane curve and so C_1.

$\phi^{-1}(\ell)$ is given by two lines $\ell_1 \subset F_1$ and $\ell_2 \subset F_2$ (non necessarily distinct) and for every choice of $P \in \mathbb{P}^4$ we have two of these lines, so we deduce that the trisecant line of the congruence through P must be in the intersection of the two planes $\overline{P\ell_1}$ and $\overline{P\ell_2}$. Then, ℓ_1 and ℓ_2 meet in (at least) one point, which, by definition, must be contained in $C_1 = F_1 \cap F_2$, but this is impossible, since the general line of the congruence do not intersect C_1. □

Then, either one of the surfaces is a plane or two of them are planes. The case of three planes was just considered. We consider the two cases separately:

6.1. The case in which F_2 and F_3 are planes. Let us suppose then that F_2 and F_3 are planes and F_1 a non-linear surface.

What we said at the beginning of this section in Propositions 4.81 and 4.83, in this case means:

COROLLARY 4.85. *With notations of Propositions 4.81 and 4.83, the following formulas hold:*

$$i = \mu_1, \qquad k_1^2 m_1 + k_2^2 + k_3^2 = a^2 - x. \tag{118}$$

But naturally, we can say more:

PROPOSITION 4.86. *The following formulas hold:*

$$k_1 = 1, \tag{119}$$

$$k_2 = m_1 - c_2, \tag{120}$$

$$k_3 = m_1 - c_1. \tag{121}$$

PROOF. If Σ_H is the scroll given by the lines of the congruence contained in a hyperplane H, then $H \cap F_i$, with $i = 2, 3$ is a line ℓ_i, while $H \cap F_1$ is a curve C_H of degree m_1, with c_{i-1} points in common with ℓ_i. The multiplicity of ℓ_i is k_i since, as usual, k_i is the number of lines of the congruence contained in H that pass through a general point $P_i \in \ell_i$. The cone χ_{P_i} of the lines of the congruence passing through the general point P_i of ℓ_i is given by the lines of the pencil of centre P_i contained in the plane $\overline{P_i\ell_j}$ which meet C_H out of C_{j-1}, with $i \neq j$; so we have formulas (120) and (121).

k_1 is the number of lines of the congruence contained in H that pass through a general point $P \in C_H$; then $k_1 = 1$, since the only line is given by the intersection of the two planes $\overline{P\ell_2}$ and $\overline{P\ell_3}$. □

COROLLARY 4.87. *The following formula, for the second degree a, holds:*

$$a = 2m_1 - c_2 - c_1. \tag{122}$$

PROOF. It is formula (114) knowing formulas (119), (120) and (121). □

COROLLARY 4.88. *The following formula holds:*

$$4m_1^2 - m_1(2c_1 + 2c_2 + 1) - 2c_1c_2 - x = 0. \tag{123}$$

PROOF. It is the second formula of Corollary 4.85 knowing formulas (119), (120),(121) and (122). □

NOTATIONS. Let us fix some other notations: let P be a general point of $\mathbb{P}^4$; the two hyperplanes $\overline{PF_2}$ and $\overline{PF_3}$ will intersect in a plane Π_P which has a line in common with F_i—say it ℓ_i—with $i = 2, 3$. These lines pass through the point C_3 and let ℓ_P the line joining P with C_3. Finally, let us denote with h_3 the multiplicity of C_3 for F_1 and with h_i the multiplicity of C_3 for C_i, $i = 1, 2$.

PROPOSITION 4.89. *The following formula holds:*

$$m_1 - 1 = (c_1 - h_1) + (c_2 - h_2) + h_3; \tag{124}$$

vice versa if Equation (124) *is satisfied with the hypothesis of this section, then we have a first order congruence.*

PROOF. If fact, we have that only one line of the congruence passes through the general point P of $\mathbb{P}^4$; the formula is obtained in this way: consider the cone χ_P of lines through P and meeting F_1. χ_P has dimension 3 and degree m_1. But a line of the congruence must intersect the two planes F_2 and F_3, *i.e.* it is contained in the plane Π_P. $\Pi_P \cap \chi_P$ is given by m_1 lines (counted with multiplicities), and among them there are the line ℓ_P with multiplicity h_3 and the lines joining P and the points of $\Pi_P \cap C_i$, $i = 1, 2$, which are c_i; but among these c_i lines, there is the line ℓ_P also, counted with multiplicity h_i; so formula (124) holds.

Reverting the proof we get the second part of the theorem. □

THEOREM 4.90. *The surface F_1 is rational.*

PROOF. Let Π be a general plane of $\mathbb{P}^4$ and $Q \in \Pi$ a general point of it. Let Π_Q be the plane defined as above and ℓ_2 and ℓ_3 the corresponding intersection of Π_Q with F_2 and F_3 respectively. $\Pi_Q \cap F_1$ will be given by a 0-dimensional scheme of length m_1; but in this intersection, we have that the subscheme which has support in C_3 has length h_3, while C_i, $i = 1, 2$ intersects Π out of C_3 in a scheme of length $(c_i - h_i)$. Then, by formula (124) we have that there is only one point $P_Q \in \Pi_Q \cap F_1$ which is not contained in ℓ_i, $i = 1, 2$. The map

$$\phi : \Pi \to F_1$$

defined by $\phi(Q) := P_Q$ is, in fact, birational. □

PROPOSITION 4.91. *A parasitic plane passes through C_3.*

PROOF. Indeed, a parasitic plane must intersect F_2 and F_3 in a line, so it must pass through C_3. □

6.2. The case in which there is a plane only. Let us suppose then that F_3 is a plane and F_1 and F_2 are not linear surfaces.

What we said at the beginning of this section in Propositions 4.81 and 4.83, in this case means:

COROLLARY 4.92. *With notations of Propositions* 4.81 *and* 4.83, *the following formulas hold:*

$$i = \mu_1\mu_2, \qquad k_1^2 m_1 + k_2^2 m_2 + k_3^2 = a^2 - x. \tag{125}$$

But naturally, we can say more:

PROPOSITION 4.93. *The following formulas hold:*

$$k_1 = m_2 - c_3, \tag{126}$$

$$k_2 = m_1 - c_2, \tag{127}$$

$$k_3 = m_1 m_2 - c_1 - c_2 c_3. \tag{128}$$

PROOF. If Σ_H is the scroll given by the lines of the congruence contained in a hyperplane H, then $H \cap F_3$ is a line ℓ_3, while $H \cap F_i$, with $i = 1, 2$, is a curve $C_{i,H}$ of degree m_i, with c_{i+1} points in common with ℓ_3.

The multiplicity of $C_{i,H}$ is k_i since, as usual, k_i is the number of lines of the congruence contained in H that pass through a general point $P_i \in C_{i,H}$. The lines of the congruence passing through the general point P_i of $C_{i,H}$ is given by the lines of the cone χ_{P_i} obtained by the join of P_i with F_j, , with $i \neq j$, contained in the plane $\overline{P_i\ell_3}$ which meet ℓ_3 out of C_{j+1}; so we have formulas (126) and (127).

k_3 is the number of lines of the congruence contained in H that pass through a general point $P \in \ell_3$; then these lines are the intersection of the two cones χ_i obtained as the join of P with F_i, , with $i = 1, 2$, to which we have to subtract the lines of the cones which pass through $C_2 \cap C_3$ and through C_1. So formula (128) holds.

□

COROLLARY 4.94. *The following formula holds:*

$$a = m_1m_2 - c_3 - c_2 - c_1 - c_1c_2 - 1. \tag{129}$$

PROOF. It is formula (114) knowing formulas (126), (127) and (128). □

COROLLARY 4.95. *The following formula holds:*

$$\begin{aligned}(m_2 - c_3)^2m_1 + (m_1 - c_2)^2m_2 + (m_1m_2 - c_1 - c_2c_3)^2 &= \\ &= (m_1m_2 - c_3 - c_2 - c_1 - c_1c_2 - 1)^2 - x.\end{aligned} \tag{130}$$

PROOF. It is the second formula of Corollary 4.92 knowing formulas (126), (127),(128) and (129). □

PROPOSITION 4.96. *If C is a curve of $\mathbb{P}^3$ of degree m and $M(C)$ is the family of lines meeting C then, if we consider $M(C)$ as a cycle in the Grassmannian, the following formula holds:*

$$M(C) = m\sigma_{10}. \tag{131}$$

PROOF. We are interested in the lines meeting C; it is easy to see that these form a (flat) family $M(C)$ of lines of dimension 3, so it determines a cycle of codimension 1 in the Grassmannian. Then, we have, as Schubert cycle

$$M(C) = b\sigma_{10}.$$

We can actually calculate a:
we recall that

$$\sigma_{10} = \{\ell \in \mathbb{G}(1,3) \mid \ell \cap \ell' \neq \emptyset\}$$

and its complementary cycle is

$$\sigma_{21} = \{\ell \in \mathbb{G}(1,3) \mid P \in \ell \subset \mathbb{P}^2\}.$$

From this observation we see that

$$\begin{aligned}M(C) \cdot \sigma_{21} &= b \\ &= m,\end{aligned}$$

since the general plane section gives m points and the lines of the congruence passing to a general point P of this hyperplane are the lines through P and the m points. □

THEOREM 4.97. *The following formula holds:*

$$a = 2m_1m_2 - (c_1 + c_3m_1 + c_2m_2). \tag{132}$$

PROOF. We recall that the second degree a of the congruence is given by the lines of the congruence contained in a hyperplane H that meet a (fixed) line $\ell \subset H$.

Let H be a general hyperplane and ℓ a line in it. We set $\ell_3 := H \cap F_3$ and $C_{i,H}$ the components of $H \cap F_i$ with $i = 1, 2$, out of F_3. By formulas (126) and (127), we have that $\deg(C_{1,H}) = m_1 - c_2 = k_2$ and $\deg(C_{2,H}) = m_2 - c_3 = k_1$.

We have to find the degree of the scroll Σ_H given by the lines of the congruence contained in the hyperplane H. If the line ℓ meets ℓ_3 in one point, then through this point will pass k_3 lines of the congruence.

We have to find now the lines meeting ℓ, F_1 and F_2: we have seen in Proposition 4.96 that the family of lines meeting a curve C of degree m is, as a Schubert cycle, $m\sigma_{10}$, so, the number of the lines meeting our three curves is given by the following intersection

$$\begin{aligned} M(\ell)\cdot M(F_1)\cdot M(F_2) &= \sigma_{10}(k_1\sigma_{10})(k_2\sigma_{10}) \\ &= 2k_1k_2\sigma_{21}, \end{aligned}$$

i.e. we obtain a scroll of degree $2k_1k_2$. Therefore there are $2k_1k_2$ lines meeting $\ell_3 \subset F_3$ also. But among these $2k_1k_2$ lines, there are k_1k_2 through $P := \ell_3 \cap \ell$: these are the lines in common of the two cones $\chi_{P,C_{1,H}}$ and $\chi_{P,C_{2,H}}$ with vertex P on $C_{1,H}$ and $C_{2,H}$. Then, the number of the lines of the congruence meeting ℓ is given by:

$$\begin{aligned} a &= k_1k_2 + k_3 \\ &= (m_1 - c_2)(m_2 - c_3) + m_1m_2 - c_1 - c_2c_3 \\ &= 2m_1m_2 - (c_1 + c_3m_1 + c_2m_2) \end{aligned}$$

and so the theorem is proved. □

LEMMA 4.98. *If H is a hyperplane containing the plane F_3, then $\Lambda|_H$ is a first order congruence of $H \cong \mathbb{P}^3$.*

PROOF. If H is a hyperplane containing F_3 and P a general point in it, any line of the congruence through P will be contained in H, since it has to meet F_3. *Vice versa,* if Q is a general point of $\mathbb{P}^4$, it will span with F_3 a hyperplane H_Q, and the line of the congruence through Q will be contained in H_Q, since it meets F_3. So the lemma is proved. □

COROLLARY 4.99. *One of the following formulas hold:*

1. *either* $c_2 = m_1 - 1$,
2. *or* $c_3 = m_2 - 1$.

PROOF. By the preceding lemma, the congruence restricted to a hyperplane H containing F_3 is a first order congruence of H. In particular, this congruence is a congruence of the second type of $\mathbb{P}^3$, *i.e.* we are in the hypothesis of Theorem 3.12 and the congruence is given by the lines joining the two curves $C_{i,H}$ where $C_{i,H}$ are the components of $H \cap F_i$ with $i = 1, 2$, out of F_3. In particular, one of these, *e.g.* $C_{1,H}$, is a line. Then, we have, by formula (126):

$$\begin{aligned} \deg(H \cap F_1) &= m_1 \\ &= \deg(C_{1,H}) + c_2 \\ &= 1 + c_2, \end{aligned}$$

i.e. $c_2 = m_1 - 1$. □

NOTATION. Let us suppose for the rest of this section that $c_2 = m_1 - 1$, which is obviously not restrictive.

With this assumption, we deduce, from formula (127) that

$$k_2 = 1, \tag{133}$$

and we can prove the following:

THEOREM 4.100. *The surface F_1 is a rational scroll, possessing an unisecant curve C_2' which is the irreducible component of C_2 formed by the points which do not lie on a focal line; besides, the general line of the ruling intersects F_2 in $m_2 - c_3 - 1$ points.*

PROOF. As we saw in Lemma 4.98, if H is a hyperplane containing F_3, then $\Lambda|_H$ is a first order congruence of H. $\Lambda|_H$ is a congruence of the second type of $\mathbb{P}^3$, *i.e.* we are in the hypothesis of Theorem 3.12 and the congruence is given by the lines joining the two curves $C_{i,H}$ where $C_{i,H}$ are the components of $H \cap F_i$ with $i = 1, 2$, out of F_3 and, by the hypothesis done and by preceding Corollary, $C_{1,H}$, is a line. We denote with C_2' the (irreducible) component of C_2 which is not contained in the join of $C_{1,H}$ and $C_{2,H}$.

Besides, if $\mathbb{P}^1_{F_3}$ is the pencil of hyperplanes containing F_3, and we vary H in this pencil, we obtain that F_1 is a scroll, and the general line $C_{1,H}$ will meet C_2' in a point, since otherwise C_2' would be an isolated component of F_1.

If we fix a (general) hyperplane H, by Identity (133) we deduce that there is only one line of the congruence contained in H that pass through a general point $P \in C_2'$; from what we said above, this is a line of the join of $C_{1,H}$ and $C_{2,H}$. In particular, these lines give a birational map

$$\phi : C_{1,H} \dashrightarrow C_2'$$

from the line $C_{1,H}$ onto its image, and $\phi(Q)$ is defined as the intersection of the cone χ_Q of the lines of the congruence $\Lambda|_H$ passing through Q and the curve C_2'. Then we deduce that F_2 is a rational scroll and C_2' is its unisecant curve.

Finally, since $\Lambda|_H$ is a first order congruence of H, we have that $C_{1,H}$ meets $C_{2,H}$ in $\deg(C_{2,H}) - 1$ points.

From formula (127) we obtain:

$$\begin{aligned} \deg(C_{2,H}) - 1 &= k_1 - 1 \\ &= m_2 - c_3 - 1. \end{aligned}$$

□

COROLLARY 4.101. *The following formula holds:*

$$\text{length}(C_1 \cap F_3) = c_1 - (m_2 - c_3 - 1). \tag{134}$$

PROOF. We recall that the curve C_1 is the intersection of F_1 and F_2. In particular, if H is a general hyperplane, then $\text{length}(C_1 \cap H) = c_1$.

If H is a general hyperplane containing F_3, by Theorem 4.100, we have $m_2 - c_3 - 1$ points of $F_1 \cap F_2$ out of F_3; so formula (134) holds. □

THEOREM 4.102. *The surface F_2 is rational.*

PROOF. Let Π be a general plane of $\mathbb{P}^4$ and $Q \in \Pi$ a general point of it. Let Π_Q be the plane contained in the hyperplane $H := \overline{QF_3}$ and containing the line $C_{1,H}$, *i.e.* the component of $H \cap F_1$ out of F_3. Clearly, $\ell := F_3 \cap \Pi_Q$ is a line. $\Pi_Q \cap F_2$ is a 0-dimensional scheme of length m_2; but in this intersection, the subscheme which has support in $C_{1,H}$ has length $m_2 - c_3 - 1$, by Theorem 4.100, and the subscheme contained in F_3 is $\ell \cap C_3$, which has length c_3. Then, there is only one point $P_Q \in \Pi_Q \cap H$ which is not contained in ℓ or $C_{1,H}$. The map

$$\phi : \Pi \dashrightarrow F_2$$

defined as $\phi(Q) := P_Q$ is, in fact, birational. □

7. Congruences of the Fourth Type

NOTATIONS. Let us consider an irreducible surface F_1 and an irreducible curve C, and the congruence Λ of the lines that meet once the surface and the curve.

So, $F = F_1 \cup C$ is our (pure) focal locus. In this case, we will denote with m_1 the degree of F_1, and with m_2 the degree of C.

As usual, we can give some observations on this situation from the point of view of the Schubert calculus.

We recall from proposition 4.48 that the (flat) family $M(F_1)$ of the lines meeting F_1 is a family of dimension 5, and that

$$M(F_1) = m_1\sigma_{10}.$$

We are interested in the lines meeting C; it is easy to see that these form a (flat) family $M(C)$ of lines of dimension 4, so it determines a cycle of codimension 2 in the Grassmannian. Then, we have, as Schubert cycle

$$M(C) = a\sigma_{11} + b\sigma_{20}.$$

We can actually calculate the numbers a and b:

PROPOSITION 4.103. *The following formula holds:*

$$M(C) = m_2\sigma_{20}. \tag{135}$$

PROOF. We recall that

$$\sigma_{11} = \{\ell \in \mathbb{G}(1,4) \mid \ell \subset \mathbb{P}^3 \subset \mathbb{P}^4\}$$

and its complementary cycle is

$$\sigma_{22} = \{\ell \in \mathbb{G}(1,4) \mid \ell \subset \mathbb{P}^2\}.$$

So, since a general plane will not intersect C, we have that $a = 0$.

It remains to calculate the coefficient of σ_{20}; we have

$$\sigma_{20} = \{\ell \in \mathbb{G}(1,4) \mid \ell \cap \mathbb{P}^1 \neq \emptyset\}$$

and its complementary cycle is

$$\sigma_{31} = \{\ell \in \mathbb{G}(1,4) \mid P \in \ell \subset \mathbb{P}^3\}.$$

From this observation we see that

$$\begin{aligned} M(C) \cdot \sigma_{31} &= b \\ &= m_2, \end{aligned}$$

since the general hyperplane section gives m_2 points and the lines of the congruence passing to a general point P of this hyperplane are the lines through P and the m_2 points. □

REMARK. If the curve and the surface are in general position *i.e.* they do not intersect, we have, as a Schubert cycle

$$B = M(F_1) \cdot M(C) \tag{136}$$

$$= (m_1\sigma_{10}) \cdot (m_2\sigma_{20}) \tag{137}$$

$$= m_1 m_2(\sigma_{30} + \sigma_{21}), \tag{138}$$

So, in order to obtain a congruence of the first order, we must have $m_1 = m_2 = 1$, *i.e.*

THEOREM 4.104. *If F_1 and C meet properly, i.e. $F_1 \cap C = \emptyset$, then F is a plane and C is a line. The congruence has bidegree $(1,1)$, F_1 is the fundamental 2-locus and C the fundamental 1-locus. We do not have parasitic planes.*

REMARK. It is easy to see that in this case the congruence, as a subscheme of the Grassmannian, is the intersection of a hyperplane section (*i.e.* σ_1) and a codimension two section (*i.e.* σ_2). In particular, we see that this is a limit case of the intersection of three hyperplane sections of the Grassmannian, *i.e.* if the corresponding planes meet in a line. See also [**Cas91**].

Then, we consider the general case. First we will consider the case in which C is contained in F_1 and then the other case.

7.1. The case $C \subset F_1$. We have analysed in Theorem 4.7, (1) the case in which F_1 is a plane, so in the following we will suppose that F_1 is not a plane. We start with the following

LEMMA 4.105. *If $C \subset F_1$, then C must be a plane curve.*

PROOF. Let us prove the lemma *ab absurdo:* if $\ell = \overline{PQ}$ is a general secant line of C, we have that $\deg f^{-1}(\ell) = 2$, since P and Q are points both of C and F_1; *i.e.* ℓ is a focal line. Then the (embedded) secant variety of C, which has dimension 3 because C is not plane, is contained in the focal locus, which is absurd. □

THEOREM 4.106. *If $C \subset F_1$ then there are the following possibilities:*

1. *C is a line and F_1 is a rational surface of degree m_1 and, if we suppose that F_1 has only isolated singularities, with sectional genus $m_1 - 2$;*
2. *C is a conic and F_1 is a projection of a rational normal scroll of type $S_{m_1-2k,2k}$, with $m_1 \geq 3$, one of its unisecant curves is C and a general hyperplane through C intersects F_1 in C with multiplicity k and in a line.*

PROOF. Let η be a plane containing C and $\mathbb{P}^1_\eta$ the pencil of 3-planes containing η. If H is a general element of $\mathbb{P}^1_\eta$, then $\Lambda|_H$ is a first order congruence of $H \cong \mathbb{P}^3$; in particular it is a congruence of second type, *i.e.* we are in the situation of Theorem 3.12.

Besides, since C is a plane curve, and F_1 is not a plane, C can either be a line or a conic. In fact, if $m_2 > 2$, then $H \cap F_1$ is given by C and a line ℓ_H, and $\text{length}(C \cap \ell_H) = m_2 - 1 > 1$, by Theorem 3.12. It cannot be $\ell_H \subset H$, because otherwise $\text{length}(C \cap \ell_H) = m_2$. So, C must have a $(m_2 - 1)$-multiple point P and ℓ_H pass through it. So, varying H in $\mathbb{P}^1_\eta$, we obtain that F_1 is a cone with vertex in P. But the general hyperplane section through P must be a line only out of the plane η, and so F_1 should be the union of η and a plane, which is a contradiction.

1. If C is a line, this is the fundamental line of Theorem 3.12; besides, $H \cap F_1$ is a rational curve C_H of degree m_1 and we have that $\text{length}(C_H \cap C) = m_1 - 1$. Therefore F_1 is rational, by the map
$$\phi : \mathbb{P}^2_C \to F_1$$
where $\mathbb{P}^2_C$ is the set of hyperplanes containing C and ϕ associates to H the unique point of $C_H \cap C$ which is not in C. The map is dominant because otherwise the congruence would be given by the join of two curves and it is injective because we have a first order congruence. We can also conclude because an unirational surface is in fact rational. Besides, the sectional genus $\pi(F_1)$ of F_1 is obtained by adjunction:
$$\begin{aligned}\pi(F_1) &= g(C_H) + g(C) - 1 + C_1 \cdot C \\ &= m_1 - 2.\end{aligned}$$
2. If C is a conic, then $H \cap F_1$ must be a line ℓ and $\text{length}(\ell \cap C) = 1$, by Theorem 3.12, and the thesis follows from Corollary 4.21.

□

COROLLARY 4.107. *In the hypothesis of the preceding theorem, if we are in case (2), then F_1 is smooth if and only if is the rational normal scroll $S_{1,2}$ of degree 3 linearly normal in $\mathbb{P}^4$, i.e. for $m_1 = 3$ and $k = 1$ and C is a unisecant conic.*

REMARK. Let us consider case (1) of Theorem 4.106: let us suppose that F_1 is smooth; then, its hyperplane section is a (smooth) curve $C_H \subset H \cong \mathbb{P}^3$ of degree m_1 and genus $m_1 - 2$; we recall that

PROPOSITION 4.108. *There exists a smooth curve C of degree d and genus g in $\mathbb{P}^3$, whose hyperplane section D is non-special (i.e. $h^1(D) = h^0(K_C - D) = 0$), if and only if either*

1. $g = 0$ *and* $d \geq 1$,
2. $g = 1$ *and* $d \geq 3$, *or*
3. $g \geq 2$ *and* $d \geq g + 3$.

See [**Har77**], page 350 for a proof.

Therefore C_H has the hyperplane section special, since it cannot be a plane curve. We recall also that

PROPOSITION 4.109. *If C is a curve in $\mathbb{P}^3$, not lying in any plane, for which the hyperplane section D is special, then $d \geq 6$ and $g \geq \frac{1}{2}d + 1$. Furthermore, the only such curve with $d = 6$ is the canonical curve of genus 4.*

See [**Har77**], pages 350–351 for a proof.

Clearly, we are in the hypothesis of the preceding proposition and we observe that the condition $g = d - 2$ in fact implies, with $g \geq \frac{1}{2}d + 1$, that $d \geq 6$. By Riemann-Roch we have

$$\chi(\mathcal{O}_{C_H}(1)) = \deg(C_H) + 1 - g(C_H) \tag{139}$$

$$= 3 \tag{140}$$

and, since C_H is not a plane curve, we have that $h^0(\mathcal{O}_{C_H}(1)) \geq 4$.

Let us consider then the exact sequence of coherent sheaves on F_1 defining C_H:

$$0 \to \mathcal{O}_{F_1} \xrightarrow{\cdot H} \mathcal{O}_{F_1}(1) \to \mathcal{O}_{C_H}(1) \to 0 \tag{141}$$

and the corresponding long exact sequence of cohomology:

$$\begin{aligned} 0 &\to H^0(\mathcal{O}_{F_1}) \to H^0(\mathcal{O}_{F_1}(1)) \to H^0(\mathcal{O}_{C_H}(1)) \to \\ &\to H^1(\mathcal{O}_{F_1}) \to H^1(\mathcal{O}_{F_1}(1)) \to H^1(\mathcal{O}_{C_H}(1)) \to \\ &\to H^2(\mathcal{O}_{F_1}) \to \cdots \end{aligned}$$

but F_1 is a rational surface and it is not the Veronese surface, since its sectional genus is not zero, and so it is linearly normal; then, we deduce

$$h^0(\mathcal{O}_{F_1}) = 1 \tag{142}$$

$$h^0(\mathcal{O}_{F_1}(1)) = 5 \tag{143}$$

$$h^1(\mathcal{O}_{F_1}) = 0 \tag{144}$$

$$h^2(\mathcal{O}_{F_1}) = 0; \tag{145}$$

from equations (142) and (143) we get that $h^0(\mathcal{O}_{C_H}(1)) = 4$, and then, by (140) we obtain that

$$h^1(\mathcal{O}_{C_H}(1)) = 1. \tag{146}$$

Finally, from (146), (144) and (145), we get $h^1(\mathcal{O}_{F_1}(1))$. We recall (see [**Ale88**]) the following

4.7. DEFINITION. The *speciality* of a rational surface S in $\mathbb{P}^n$ is the number $q(1) := h^1(\mathcal{O}_S(1))$.

So, we obtain that:

PROPOSITION 4.110. *If the surface F_1 of case* (1) *of Theorem* 4.106 *is smooth, then it is a speciality one rational surface of* $\mathbb{P}^4$.

Now, the speciality one rational surfaces of $\mathbb{P}^4$ can have degree $8, 9, 10$ or 11, and are known and classified only the surfaces in degrees $8, 9$ and 10: see [**Ale92**].

In particular, in degree eight we have (see [**Oko86**]) the following

THEOREM 4.111. *Let S be a speciality one rational surface of degree eight in $\mathbb{P}^4$; then it is linked to a Veronese surface in a complete intersection of a cubic and a quartic. S is a blow up of $\mathbb{P}^2$ in* 16 *points $x_1, \dots, x_{16}$ embedded in $\mathbb{P}^4$ by the linear system*

$$|D| := |\pi^* 6L - 2E_1 - \cdots - 2E_4 - E_5 - \cdots - E_{16}|$$

where $\pi : S \to \mathbb{P}^2$ is the blow up in the points $x_1, \dots, x_{16}$, L is a line in $\mathbb{P}^2$ and E_i is the fibre of π over x_i.

See [**Ran88**] and [**Ale92**] for references.

In degree nine we have the following

THEOREM 4.112. *Let S be a speciality one rational surface of degree nine in $\mathbb{P}^4$; then it lies on a net of quartics. In fact, if S is a rational surface of degree* 9 *in $\mathbb{P}^4$ with speciality $q(1) > 0$, then $q(1) = 1$ and the first and the second adjunctions (we recall that the first adjunction map is the map associated to the linear system $|H + K|$, where H is the hyperplane divisor and K the canonical divisor; then the higher order adjunction maps are defined naturally by recurrence: see for example* [**Som79**]*) of S give a canonical sequence of birational morphisms of rational surfaces*

$$S \xrightarrow{f_1} S_1 \xrightarrow{f_2} S_2$$

where S_2 is canonically a cubic surface in $\mathbb{P}^3$. The morphism f_2 blows up in three distinct (closed) points x_1, x_2, x_3, while f_1 blows up in six distinct points $x_4, \dots, x_9$. Let K_1 and K_2 be the inverse images of the canonical divisors of S_1 and S_2 respectively, the linear system of the hyperplane sections of S is given by

$$|H| = |-K - K_1 - K_2|$$

where K is the canonical divisor on S.

See [**Ran88**] and [**Ale92**] for references and proofs.

In degree ten we have the following

THEOREM 4.113. *Let S be a speciality one rational surface of degree ten in $\mathbb{P}^4$; then S is a blow up of $\mathbb{P}^2$ in* 13 *points $x_1, \dots, x_{13}$ embedded in $\mathbb{P}^4$ by the linear system*

$$|D| := |\pi^* 14L - 6E_1 - 4E_2 - \cdots - 4E_{10} - 2F_{11} - E_{12} - E_{13}|$$

where $\pi : S \to \mathbb{P}^2$ is the blow up in the points $x_1, \dots, x_{13}$, L is a line in $\mathbb{P}^2$ and E_i is the fibre of π over x_i.

See [**Ran88**] and [**Ale92**] for proofs.

COROLLARY 4.114. *A speciality one surface of degree m_1, with $8 \le m_1 \le 10$ contains (at least) a line.*

Therefore, we have proven that

THEOREM 4.115. *If we are in case* (1) *of Theorem* 4.106 *and we suppose that F_1 is smooth, then F_1 is a speciality one rational surface, and so F_1 is one of the surfaces of Theorems* 4.111, 4.112 *and* 4.113 *or, if it exists, a speciality one rational surface of degree* 11. *Vice versa, the lines meeting speciality one rational surface and a line in it generate a first order congruence.*

PROOF. It remains to prove that the lines meeting a speciality one surface and a line in it generate a first order congruence; but this can be done reverting the proof of Theorem 4.106. □

7.2. The case $C \not\subset F_1$.

NOTATION. First of all, we set $c := \text{length}(C \cap F_1)$; then, we give the following

LEMMA 4.116. *If C is not a plane curve, then F_1 is a plane.*

PROOF. We will denote with u the number of lines passing through a general point P of $\mathbb{P}^4$ which meet the zero-dimensional scheme $F_1 \cap C$.

We have that the two cones $\chi_{F_1,P}$ and $\chi_{C,P}$ of the lines passing through P and meeting respectively F_1 and C have dimensions 3 and 2 and degrees m_1 and m_2.

Then, they meet in m_1m_2 lines and only one of these lines belongs to Λ, since we have a first order congruence, so $u = m_1m_2 - 1$, as the lines through $F_1 \cap C$ cannot be computed as lines of the congruence; in fact, if $Q \in F_1 \cap C$ the lines through Q form a star of lines, which is a first order congruence.

On the other hand, since C is not a plane curve, given a general point $Q \in C$, the cone $\chi_{Q,C}$ (which has degree $m_2 - 1$) cannot be contained in the cone χ_{Q,F_1}, otherwise all the secant lines of C would meet F_1 and so they would be focal lines.

Then, through Q there pass $m_1(m_2 - 1)$ secant lines of C meeting again F_1, that must be the u lines passing through Q and that meet $F_1 \cap C$ also. This is due to the fact that if one of the $m_1(m_2 - 1)$ lines would meet F_2 outside C, then this would be a focal line, since it contained (at least) four focal points. So, Λ would have a focal hypersurface.

So, we have

$$\begin{aligned} u &= m_1m_2 - 1 \\ &= m_1(m_2 - 1) \end{aligned}$$

and so $m_1 = 1$. □

THEOREM 4.117. *If F_1 is a plane, then C is a rational curve such that $c = m_2 - 1$.*

PROOF. Let $\mathbb{P}^1_{F_1}$ be the pencil of 3-planes containing F_1. If H is a general element of $\mathbb{P}^1_{F_1}$, then $\Lambda|_H$ is a first order congruence of $H \cong \mathbb{P}^3$; besides, $\text{length}(H \cap C) = m_2$, *i.e.* a finite number of points. Therefore, $\Lambda|_H$ must be a star of lines with centre $P_H \in C$; so $c = m_2 - 1$ and C is rational by the birational map

$$\phi : \mathbb{P}^1_{F_1} \to C$$

such that $\phi(H) = P_H$. □

THEOREM 4.118. *If C is a plane curve, then there are the following possibilities:*

1. *C is a line and F_1 is a rational surface and, if we suppose that F_1 has only isolated singularities, with sectional genus $m_1 - 2$; besides, $c = m_1 - 1$;*
2. *C is a rational curve with a point P of multiplicity $m_2 - 1$ and F_1 is a cone with vertex in P and basis a rational curve and the intersection of F_1 and the plane of C is given by $m_1 - 1$ lines (and so, $c \geq (m_1 - 1)m_2$).*

PROOF. Let η be a plane containing C and $\mathbb{P}^1_\eta$ the pencil of 3-planes containing η. If H is a general element of $\mathbb{P}^1_\eta$, then $\Lambda|_H$ is a first order congruence of $H \cong \mathbb{P}^3$; in particular a congruence of second type, *i.e.* we are in Theorem 3.12. Besides, since C is a plane curve, it can either be a focal line or a fundamental curve for $\Lambda|_H$ and with $m_2 > 1$.

1. If C is a fundamental line of Theorem 3.12 then $H \cap F_1$ is a rational curve C_H of degree m_1 and we have that length$(C_H \cap C) = m_1 - 1$. Therefore F_1 is rational, by the map
$$\phi : \mathbb{P}^2_C \to F_1$$
where $\mathbb{P}^2_C$ is the set of hyperplanes containing C and ϕ associates to H the unique point of $C_H \cap C$ which is not in C. The map is dominant because otherwise we had that the congruence is given by the join of two curves and it is injective because we have a first order congruence. We can also conclude because an unirational surface is in fact rational. Besides, the sectional genus $\pi(F_1)$ of F_1 is obtained by adjunction:
$$\begin{aligned}\pi(F_1) &= g(C_H) + g(C) - 1 + C_1 \cdot C\\ &= m_1 - 2.\end{aligned}$$
2. If C is a rational curve not a line, then $H \cap F_1$ must be a line ℓ out the plane η of C and length$(\ell \cap C) = m_2 - 1$, by Theorem 3.12, so F_1 must be a cone with vertex on a point $P \in C$ of multiplicity $m_2 - 1$ for C and $F_1 \cap \eta$ is given by $m_1 - 1$ lines.
   The base of the cone is a rational curve; in fact, called B the basis for the cone, we see that the map
$$\phi : \mathbb{P}^1_\eta \to B$$
—where $\mathbb{P}^1_\Pi$ is the pencil of hyperplanes containing Π—defined associating the hyperplane H to the point $b \in B$ through which there is the unique line $\ell \subset F_1 \cap H$ not contained in η. The map is clearly birational, and so we are done.

□

As we have seen in Proposition 4.110, the surface of Theorem 4.118, case (1), if it is smooth, is a speciality one rational surface of $\mathbb{P}^4$. But such surface must have a $(m_1 - 1)$-secant line. Then surely the surfaces of Theorems 4.111 and 4.112 does not have, respectively, a 7 and a 8-secant line, since are contained in complete intersections of hypersurfaces of degree ≤ 4.

But, in fact, we can see that no one of the speciality one rational surfaces can generate a first order congruence: if we project F_1 from a point of the $(m_1 - 1)$-secant line ℓ, we get a point P of multiplicity $m_1 - 1$ on the projected surface G. Clearly, projecting from this point, or from ℓ, to $\mathbb{P}^2$, we obtain a birational map. In particular, the hyperplane section G_H through P of G is a (plane) rational curve with only one singular point, P. But we see that F_1 is the desingularization of G (or, the blow up of G in P) and taking the proper transform of G_H we get a hyperplane section of F_1, which has genus $m_2 - 2$. But $m_2 > 2$, therefore this situation cannot happen. So we have proved that

THEOREM 4.119. *Smooth surfaces, giving a first order congruence as in Theorem* 4.118, *case* (2), *do not exist.*

Besides, we have that

THEOREM 4.120. *A first order congruence as in Theorem* 4.118, *case* (2) *exists only if* $m_1 = 1$ *and* $m_2 = 2$, *i.e. if* C *is an irreducible conic and* F_1 *a plane meeting* C *in a point (and so,* $c = 1$*).*

Vice versa, given an irreducible conic and a plane meeting it in a point, then the lines meeting both the varieties generate a first order congruence.

PROOF. It is straightforward that these are the only smooth varieties of case (2); besides, reverting the argument of the proof of Theorem 4.118, we get the second part of the theorem. □

7.3. Final remarks on the congruences of the fourth type. Let us consider a congruence B as a subvariety of dimension 3 in $\mathbb{G}(1,4)$; from the article [**ABT94**] we can deduce which are the smooth congruences B of the fourth type. We recall the following result of [**ABT94**]:

THEOREM 4.121. *Let $B \subset \mathbb{G}(1,n)$ a smooth congruence of bidegree (a_0, a_1) with a fundamental 1-locus C. Then one of the following holds:*

1. *$n = 3$ and the congruence consists of the secants of either a twisted cubic or an elliptic quintic; or*
2. *the curve C is a line; then, denote with $\Gamma(\cong \sigma_{n-2})$ the cone given by the lines meeting C, then either*
   (a) *$a_0 = a_1$ and B is the intersection of Γ with a hypersurface of degree a_0; or*
   (b) *$a_0 = a_1 + 1$ and B is linked to a $(n-1)$-fold of degree $n-2$ passing through the vertex of Γ, in the intersection of Γ with a hypersurface of degree $a_0 + 1$; or*
   (c) *$n = 3$, $a_0 = a_1 - 1$ and B is linked under the complete intersection of Γ and a hypersurface of degree a_1 to a plane consisting of all lines passing through a point; or*
3. *the congruence is a scroll, then either*
   (a) *C is a conic and $a_0 = 1$ or 2, $a_1 = 2$; or*
   (b) *C is a plane cubic and $a_0 = a_1 = 3$; or*
4. *C is a plane cubic and $a_0 = 3$, $a_1 = 6$.*

Besides, it is possible to construct all the above congruences.

Therefore, we get

THEOREM 4.122. *The smooth congruences B of the fourth type are only the following:*

1. *B is given by the lines meeting a plane F_1 and a line C not intersecting, i.e. the case of Theorem 4.104;*
2. *the congruence is given by the lines meeting a plane F_1 and a conic C meeting in a point (out of the point in common), i.e. we are in the case (2) of Theorem 4.118 with $m_1 = 1$ and $m_2 = 2$. For this congruence, we have $a = 2$ and $c = 1$.*

PROOF. It is straightforward to see that the only possible cases of first order congruence of $\mathbb{P}^4$ in Theorem 4.121 are (2a) and (3a). The case in which we have a line is straightforward. Concerning the case of the conic, we see that we can exclude case (2) of Theorem 4.106 since in this case B is not a scroll, while in the case (2) of Theorem 4.118, the only way to get a conic with B a scroll is with $m_1 = 1$ and $m_2 = 2$. We could conclude the case of the conic also directly, by quoting the explicit construction given in this case in [**ABT94**], page 54. □

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