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SOME NON–CONVEX PROBLEMS OF THE CALCULUS OF VARIATIONS

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SOME NON–CONVEX PROBLEMS OF THE CALCULUS OF VARIATIONS

CONTENTS

INTRODUCTION

We consider the Problem of the existence of solutions to minimum problems of the Calculus of Variations in the scalar case, when the independent variable is a scalar. We are interested in minimizing

$$\text{(P)} \qquad I(x) = \int_0^T f(t, x(t), x'(t))\, dt \,:\, x \in W^{1,p}([0,T], \mathbb{R}^n),\ x(0) = a,\ x(T) = b.$$

The basic reasoning used in the proofs of existence of minima for the functionals $I(x)$ is the following: by imposing conditions on the growth at infinity with respect to x' on the integrand (Hypothesis (H), §I.2) one obtains that the functions that make the integral finite are contained in a weakly compact subset of $W^{1,p}$, so that a weakly convergent sequence (x_n) can be obtained. Now, in order to achieve the proof of the existence of a minimum, one has to show that the value of the functional on the limit point $\bar{x}$ of (x_n) is not larger than the lim inf of the values of the functional along the sequence, i.e one requires to have

$$\text{(1)} \qquad I(\bar{x}) \leq \liminf_n I(x_n).$$

If the above equality holds for every sequence x_n weakly converging to $\bar{x}$, the functional is called weakly lower semi–continuous.

In most of the recent books on the Calculus of Variations, the concept of weak l.s.c. plays a preminent role and the following result is presented in main light (see [**CE**], [**DA**]).

THEOREM. *A necessary and sufficient condition for the weak l.s.c. of the functional I, under suitable regularity and growth conditions, is that the map $\xi \to f(t, x, \xi)$ be convex for each t, x.*

The previous Theorem had the effect that until recently, not much effort has been made in order to provide Theorems guaranteeing existence of solutions without the condition of convexity on the integrand. Nevertheless Tonelli's convexity assumption on x' is far from being necessary for the existence of a solution to (P); in fact there is no reason to be concerned about what happens along sequences that are not minimizing sequences.

The first problem that has been investigated outside the realm of weak l.s.c. has been the problem of minimizing

$$\text{(P1)} \qquad \int_0^T f(t, x'(t))\, dt,\ x(0) = a,\ x(T) = b$$

under the usual regularity and growth assumptions on f but without the condition of convexity. Surprisingly, [**O**] and [**MC**] showed that the above problem always admits at least one solution.
When the integrand f depends on the space variable x, conditions under which (1) holds for each minimizing sequence weakly converging to $\bar{x}$ (l.s.c. along minimizing sequences) have been given in [**A–T1**], [**R1**]. The basic drawback of this approach (when the state variable is a scalar) is that l.s.c. along minimizing sequences is equivalent to impose that each solution to the convexified problem is a solution to the original one, so that these results will never contain as a special case the result for problem (P1).
In [**MC**] and [**C–C**], l.s.c. is not considered: sufficient conditions for the mere existence of a minimum are given. Their results below contain that for (P1).

THEOREM (MARCELLINI). *Let $g : \mathbb{R} \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ be such that $f(x, x') = g(x) + h(x')$ satisfies growth conditions, and g be monotonic. Then Problem (P) admits at least one solution.*

THEOREM (CELLINA–COLOMBO). *Let $g, h : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ be such that $f(t, x, x') = g(t, x) + h(t, x')$ satisfies growth conditions. Assume further that $x \to g(t, x)$ is concave for a.e. t. Then Problem (P) admits at least one solution.*

In the proofs of the above results, a solution is built; this is in fact the main difference in the investigation of conditions for l.s.c. and those for the (naked) existence. In particular, the main tool of the proof of Cellina–Colombo's result is Liapunov's Theorem on the range of vector measures: it allows to substitute to the solution $\tilde{x}$ to the relaxed problem another function $\bar{x}$, a candidate for being a solution to the original problem. This new function is not defined directly; one defines a measurable function $v(t)$ and $\bar{x}(t)$ is the primitive of v. Liapunov's Theorem has been succesfully applied for integrals on a symmetric domain of $\mathbb{R}^n$ in [**C–F**], [**F1**], [**F2**]. In general, for integrals defined on non radially symmetric domains the above reasoning cannot be followed: if our problem involves the gradient and we define a measurable function v, there is no reason for v to be the gradient of some function u. In spite of this, when the boundary datum u^* is affine, i.e. $u^*(x) =< a, x > +b$, a necessary and sufficient condition for the existence of a minimum of

$$\int_\Omega g(\nabla u(x))\, dx\, ,\; u \in u^* + W_0^{1,1}(\Omega)$$

has been given recently (with no assumptions on the domain) in [**C1**], [**C2**].
When no construction is involved, i.e. when the Theorems concern l.s.c. along minimizing sequences, several results for functionals involving the gradient have been presented in [**A–T**], [**R3**].

Presentation of our results

When the integrand $f(t,x,x')$ is not the sum of two functions whose arguments are t,x and t,x' separately, it is not known whether the concavity assumption on the map $x \mapsto f(t,x,x')$ is sufficient for the existence of a solution to Problem (P). Purpose of Ch. II is to consider this problem. In Theorem II.2 we prove that the functional I (under concavity assumption) attains a minimum if we assume further the existence of a solution

$$(\tilde{x}, p_1, \ldots, p_{n+1}, v_1, , v_{n+1})$$

to the associated convexified problem (PR') in the sense of **[E–T]** (§I.2) satisfying

$$\text{(C)} \qquad \bigcap_{i=1}^{n+1} \partial_x \left(-f(t, \tilde{x}(t), v_i(t))\right) \neq \emptyset a.e.$$

$(\partial_x (-f(t,\tilde{x}(t), v_i(t)))$ being the subdifferential at the point $\tilde{x}(t)$ of the convex function $x \to -f(t,x,v_i(t))$, see §I.1).
Obviously, each solution to (P) is a solution to (PR') satisfying (C) (in this case it is enough to set $v_1 = \cdots = v_{n+1} =$ solution to (P)). The cases for which our Theorem can be usefully applied are those where the converse does not hold. For instance, condition (C) is automatically satisfied (for each solution to the convexified problem) when the integrand is the sum of two functions whose arguments are t,x and t,x' separately (although if, in general, there are solutions to (PR') that are not solutions to (P)): in this situation Theorem II.2 yields Cellina–Colombo's existence result. As a further application of our condition, we show (Theorem II.3) that Problem (P) attains a minimum if $f(t,x,x') = g(t,x) + h(t,x)l(t,x')$ and its bipolar $f^{**}(t,x,.)$ is locally constant on each $A(t,x) = \{\xi : f(t,x,\xi) > f^{**}(t,x,\xi)\}$. The main tools are basically the arguments of **[C–C]**: an extension of Liapunov's Theorem (Theorem I.3.1) and a selection Theorem (Lemma II.1).

Non–convex variational problems of slow growth are considered in Ch. III. If we assume, instead of the usual growth assumption (H), the weaker condition that the integrand f satisfies (H_1):

$$f(t,x,x') \geq \alpha(t) - \beta|x| + \gamma|x'| \ (\alpha \in L^1,\, \beta \in \mathbb{R},\, \gamma > 0)$$

then Tonelli's convexity assumption on x' is no more sufficient to ensure existence to (P), so that the convexified problem does not admit, in general, a solution. Assume that $f(t,x')$ does not depend on the space variable x. In this situation the proof of Theorem II.2 show that if the convex hull of the epigraph of $x \to f(t,x)$ is supposed to be closed for a.e. t and if the relaxed problem (PR) admits at least one solution then so does (P) (without being necessarily the same). We prove, in Theorem III.2.2 that the above conditions are satisfied if $f(x')$ is positive homogeneous of degree one (i.e. $f(kx') = kf(x')$ for each $k \geq 0$) so that, under this assumption, the parametric problem

$$\text{minimize} \int_a^b f(x'(t))\, dt$$

among all rectifiable curves $C : x = x(t)$ satisfying prescribed boundary conditions admits at least one solution. Analogously, an application of the above yields a non–convex version of some non–parametric problems of slow growth, treated for instance (in the convex case) in [**CE**, §14.3].

Tonelli, Marcellini, Cellina–Colombo's existence Theorems lead to believe that, for maps of the form $g(x) + h(x')$ (x, $x' \in \mathbb{R}$), the property of yielding existence is strictly related to very special behaviour in x or in x': either convexity in x' or concavity in x or monotonicity in x. Purpose of Ch. IV is to show that this is not so. We consider the class of functions $g(x) + h(x')$ where $g : \mathbb{R} \to \mathbb{R}$ is continuous and $h : \mathbb{R} \to \mathbb{R}$ is l.s.c. and we show that there exists a subset $\mathcal{D}$ of the space of continuous functions, dense for the topology of uniform convergence on compacta, such that, for g in it problem (P) has existence of solutions for every function h satisfying growth conditions.

The main tool here is a one– dimensional version of Liapunov's Theorem (Proposition I.3.1), due to M. Amar and A. Cellina, which cannot be extended in dimension greater or equal than 2 (example I.3.1).

Finally, a weaker 2–dimensional version of the previous Proposition I.3.1 (Proposition I.3.2) provides, under geometrical assumptions on the non–convex integrand $f(x')$, the existence of a solution with fixed end points entirely contained in a prescribed closed half–plane of $\mathbb{R}^2$. The problem whether this result can be generalized (i.e. other geometrical features instead of a half–plane, less conditions on the integrand,...) is still open.

I. ASSUMPTIONS AND PRELIMINARY RESULTS

1. Convex analysis

Let us denote by $< .,. >$ the usual scalar product in $\mathbb{R}^n$. Let f be a function of $\mathbb{R}^n$ into $\mathbb{R}$. If $u^* \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, the continuous affine function $u \to < u^*, u > -\alpha$ is everywhere less than f if and only if

$$\forall u \in \mathbb{R}^n, \ \alpha \geq \ < u^*, u > -f(u)$$

i.e. $\alpha \geq f^*(u^*)$ if we agree to set

$$f^*(u^*) = \sup_{u \in \mathbb{R}^n} \{< u^*, u > -f(u)\}. \tag{1.1}$$

The consideration of the affine minorants of f thus leads to define by (1.1) a function f^* of $\mathbb{R}^n$ into $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

Definition 1.1. *If $f : \mathbb{R}^n \to \bar{\mathbb{R}}$, formula (1.1) defines a function from $\mathbb{R}^n$ into $\bar{\mathbb{R}}$, denoted by f^*, and called the polar function of f.*

We refer to [**E–T**] for the basic properties of f^*.
If we repeat the process, thereby leading to the bipolar

$$f^{**}(u) = \sup_{u^* \in \mathbb{R}^n} \{< u^*, u > -f^*(u^*)\}, \tag{1.2}$$

the comparison between f and f^{**} leads to the following

Proposition 1.1 [**E–T**, Prop. I.4.1]. *Let f be a function of $\mathbb{R}^n$ into $\mathbb{R}$. Then f^{**} is the greater convex lower semi-continuous (l.s.c.) function everywhere less or equal than f.*

Remark 1.1 [**E–T**, §I.3.2]: The epigraph of f^{**} coincides with cl(co(epi f)), the closed convex hull of the epigraph of the function f.

Let f be a mapping of $\mathbb{R}^n$ into $\mathbb{R}$. We say that an affine function l everywhere less than f is exact at the point $u \in \mathbb{R}^n$ if $l(u) = f(u)$. Necessarily, $f(u)$ will be finite and l will have the form:

$$\begin{aligned} l(v) &= < u^*, v - u > +f(u) \\ &= < u^*, v > +f(u) - < u^*, u > . \end{aligned}$$

As a consequence, we have:

$$f(v) - f(u) \geq < u^*, v - u > \text{ for every } v \in \mathbb{R}^n. \tag{1.3}$$

DEFINITION 1.2. *A function f is said to be subdifferentiable at the point $u \in \mathbb{R}^n$ if it has an affine minorant which is exact at u. The slope $u^* \in \mathbb{R}^n$ of such a minorant is called a subgradient of f at u, and the set of subgradients at u is called the subdifferential at u and is denoted $\partial f(u)$.*

Continuous convex functions are subdifferentiable:

PROPOSITION 1.2 [**E–T**, PROP.I.5.2]. *Let f be a convex function of $\mathbb{R}^n$ into $\mathbb{R}$ and continuous at the point $u \in \mathbb{R}$. Then $\partial f(u) \neq \emptyset$.*

2. Relaxation and non–convex variational problems

The integrands used in the calculus of variations are normal integrands:

Definition 2.1. *Let Ω be a measurable subset of $\mathbb{R}^n$, B be a Borel subset of $\mathbb{R}^p$. A mapping f of $\Omega \times B$ into $\bar{\mathbb{R}}$ is termed a normal integrand if:*

$$\text{for a.e. } t \in \Omega, f(t,.) \text{ is l.s.c. on } B \tag{2.1}$$

(2.2) *there exists a Borel function $\tilde{f} : \Omega \times B \to \bar{\mathbb{R}}$ such that $\tilde{f}(t,.) = f(t,.)$ for a.e. $t \in \Omega$.*

A first consequence of this definition is that if u is a measurable mapping of Ω into B, the function $x \mapsto f(t,u(t))$ is measurable on Ω.
Functions measurable in t and continuous in x are normal integrands.

Definition 2.2. *A mapping $f : \Omega \times B \to \bar{\mathbb{R}}$ is said to be a Carathéodory function if*

$$\text{for almost all } t \in \Omega,\ f(t,.) \text{ is continuous on } B, \tag{2.3}$$

$$\text{for all } x \in B,\ f(.,x) \text{ is measurable on } \Omega. \tag{2.4}$$

Proposition 2.1 [**E–T**, Prop. VIII.1.1]. *Every Carathéodory function is a normal integrand.*

Let $f : [0,T] \times (\mathbb{R}^n \times \mathbb{R}^n) \to \bar{\mathbb{R}}$ be a normal integrand. We are interested in

$$\text{(P) minimizing } I(x) = \int_0^T f(t,x(t),x'(t))\,dt \ :\ x \in W^{1,p}([0,T],\mathbb{R}^n),\ x(0) = a,\ x(T) = b.$$

The following growth assumption will be considered:
Hypothesis (H): if $p = 1$, there exist: a convex l.s.c. monotonic function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\left(\lim_{r\to\infty} \frac{\psi(r)}{r} = +\infty\right),$$

a constant β_1 and a function $\alpha_1(\cdot)$ in L^1 satisfying

$$f(t,x,\xi) \geq \alpha_1(t) - \beta_1|x| + \psi(|\xi|) \text{ for each } x,\ \xi \text{ and for a.e. } t.$$

if $p > 1$, there exist: a positive constant γ_p, a constant β_p ($\frac{\beta_p}{\gamma_p}$ being strictly smaller than the best Sobolev constant in $W_0^{1,p}([0,T],\mathbb{R}^n)$), a function $\alpha_p(\cdot)$ in L^1 such that

$$f(t,x,\xi) \geq \alpha_p(t) - \beta_p|x|^p + \gamma_p|\xi|^p \text{ for each } x,\ \xi \text{ and for a.e. } t.$$

The basic existence criterion for solution to Problem (P) is the following:

THEOREM 2.1. *Let f be a normal integrand of $[0,T] \times (\mathbb{R}^n \times \mathbb{R}^n)$ satisfying (H). Moreover assume that*

$$\forall (t,x) \in [0,T] \times \mathbb{R}^n, \ \ f(t,x,.) \text{ is convex on } \mathbb{R}^n. \tag{2.5}$$

Then Problem (P) admits at least one solution.

SKETCH OF THE PROOF: (see [**E–T**, Th. VIII.2.2] for details) Tonelli's convexity assumption (2.5) implies the weak lower semi–continuity of the integral functional $I(x)$, i.e. for each sequence x_n in $W^{1,p}$ converging a.e. to x and such that x'_n converges weakly to x', we have

$$I(x) \leq \liminf_n I(x_n).$$

Let x_n be a minimizing sequence. By (H), the sequence x'_n of the derivatives is contained in a weakly compact subset of L^p and we can extract a subsequence x'_{n_k} converging to $u \in L^p$. Then, if we set $x(t) = a + \int_0^t u(s)\,ds$ the sequence x_{n_k} converges to x a.e. The l.s.c. of I now gives

$$\inf\,(P) \leq I(x) \leq \liminf_k I(x_{n_k}) = \inf\,(P)$$

i.e. x is a solution to (P).

When $f(t,x,\xi)$ is not convex in ξ, it is natural to introduce $f^{**}(t,x,.)$, the bipolar of $f(t,x,.)$. We have the following

PROPOSITION 2.2 [**E–T**, PROP. VIII.2.1]. *If f is a normal integrand of $[0,T] \times (\mathbb{R}^n \times \mathbb{R}^n)$ satisfying (H) then f^{**} is also a normal integrand of $[0,T] \times (\mathbb{R}^n \times \mathbb{R}^n)$ satisfying (H).*

The Problem of minimizing

$$\text{(PR)} \qquad I^{**}(x) = \int_0^T f^{**}(t,x(t),x'(t))\,dt \,:\, x \in W^{1,p}([0,T],\mathbb{R}^n),\ x(0) = a,\ x(T) = b$$

is termed the relaxed problem associated to (P).

REMARK 2.1: Since $f^{**} \leq f$ then min(PR) $\leq$ inf (P).

The relaxed problem (PR) can be rewritten in an equivalent form.

PROPOSITION 2.3 [**E–T**, LEMMA 4.1]. *Let $f : [0,T] \times (\mathbb{R}^n \times \mathbb{R}^n) \to \mathbb{R}$ be a normal integrand satisfying (H). Then, for any $\tilde{x} \in W^{1,p}([0,T])$ such that $f^{**}(t,\tilde{x}(t),\tilde{x}'(t)) < \infty$ a.e. there exist $n+1$ measurable mappings $v_i : [0,T] \to \mathbb{R}^n$ and $n+1$ measurable $p_i : [0,T] \to [0,1] (\sum p_i \equiv 1)$ such that:*

$$\sum_{i=1}^{n+1} p_i(t) v_i(t) = \tilde{x}'(t) \ \text{ a.e.} \tag{2.6}$$

$$(2.7) \qquad \sum_{i=1}^{n+1} p_i(t) f(t, \bar{x}(t), v_i(t)) = f^{**}(t, \bar{x}(t), \bar{x}'(t)) \quad \text{a.e.}$$

SKETCH OF THE PROOF: Assumption (H) implies that coepi $f(t,x,.)$, the convex hull of the epigraph of $f(t,x,.)$ is closed for each t, x (see the proof of [**E–T**, Lemma IX.3.3] for the details). In this situation we have [**E–T**, Lemma IX.3.3]:

$$f^{**}(t,x,\xi) = \min \left\{ \sum_{i=1}^{n+1} \lambda_i f(t,x,\xi_i) \; : \; \sum_{i=1}^{n+1} \lambda_i \xi_i = \xi, \, \lambda_i \geq 0, \, \sum_{i=1}^{n+1} \lambda_i = 1 \right\}.$$

A measurable selection argument yields the conclusion.

We are thus led to reformulate the relaxed problem (PR'):

$$(\text{PR'}) \qquad \begin{cases} \text{minimize} \displaystyle\int_0^T \sum_{i=1}^{n+1} p_i(t) f(t, x(t), v_i(t)) \, dt \\ p_i : [0,T] \to \mathbb{R} \,, \; v_i : [0,T] \to \mathbb{R}^n \text{ measurable} \\ \displaystyle\sum_i p_i(t) = 1 \,, \; p_i \geq 0 \\ x'(t) = \displaystyle\sum_i p_i(t) v_i(t) \in L^p \\ x(0) = a \,, \; x(T) = b. \end{cases}$$

As a consequence of Proposition 2.3, (PR) and (PR') are equivalent.

PROPOSITION 2.4 [**E–T**,§VIII.4.5]. *Under the hypothesis (H), min(PR)=min(PR').*

REMARK 2.2: The proof of Proposition 2.3 shows that in order to have (2.6) and (2.7), it is enough to assume, instead of the growth assumption (H), the more general statement that coepi $f(t,x,.)$, the convex hull of the epigraph of $f(t,x,.)$, is closed for every $(t,x) \in [0,T] \times \mathbb{R}^n$.

Some cases for which coepi $f(t,x,.)$ is closed for every (t,x) under weaker growth conditions on f will be treated in Ch. III.

3. LIAPUNOV'S TYPE THEOREMS

The following extensions of Liapunov's Theorem on the range of a vector measure [**CE**, Ch.16] will be the main tools in the next chapters II, III, IV. The proof of Theorem 3.1 is based on an argument by A. Cellina and G. Colombo [**C–C**].
Let us indicate by χ_E the characteristic function of a set E. Also, by μ we mean the Lebesgue measure on $\mathbb{R}^n$.

THEOREM 3.1 ([**M1**], [**R2**]). *Let Ω be a measurable bounded subset of $\mathbb{R}^n$, $f_1 \dots, f_m$ be vector–valued measurable functions with values in $\mathbb{R}^l$ $(l \geq 1)$. Let $p_1 \dots, p_m$ be real valued, measurable and such that $p_i(\omega) \geq 0$, $\sum_i p_i = 1$, $\sum_i p_i f_i \in L^1(\Omega)$. Let us further assume that there exists an integrable function δ such that $f_j(x) \geq \delta(x)$ for a.e. $x \in \Omega$. Then there exists a measurable partition E_1 ,..., E_m of Ω with the property that $\sum_i f_i \chi_{E_i} \in L^1(\Omega)$ and the following equality holds:*

$$\int_\Omega \sum_i p_i f_i \, d\mu = \sum_i \int_{E_i} f_i \, d\mu \tag{3.1}$$

PROOF: Let us suppose that $l = 1$, the general case being similar. By Lusin's Theorem there exists a sequence $(K_j)_{j \in \mathbb{N}}$ of disjoint compact subsets of Ω and a null set N such that $\Omega = N \cup (\cup_j K_j)$ and the restriction of each of the maps f_i to any K_j is continuous. For any j fixed in $\mathbb{N}$, Liapunov's Theorem on the range of vector measures [**CE**, Ch.16] provides the existence of a measurable partition $(E_i^j)_{i=1,\dots,m}$ of K_j with the property that

$$\int_{K_j} \sum_i p_i f_i \, d\mu = \sum_i \int_{E_i^j} f_i \, d\mu \tag{3.2}$$

Set, for any $\nu \in \mathbb{N}$, the function s_ν to be

$$s_\nu = \sum_{j \leq \nu} \sum_{i=1}^m (f_i - \delta) \chi_{E_i^j} \, .$$

Each term of the right–hand side of the above equality is a sum of non–negative terms, hence the sequence s_ν is monotone non–decreasing. Furthermore, by (3.2) we have:

$$\begin{aligned} \int_\Omega s_\nu \, d\mu &= \sum_{j \leq \nu} \sum_{i=1}^m \int_{E_i^j} (f_i - \delta) \, d\mu \\ &= \sum_{j \leq \nu} \int_{K_j} \sum_i p_i (f_i - \delta) \, d\mu \\ &\leq \int_\Omega (\sum_i p_i f_i - \delta) \, d\mu < \infty. \end{aligned}$$

Moreover, if we set $E_i = \bigcup_{j\in\mathbb{N}}(E_i^j)$, we have

$$\lim_\nu s_\nu = \sum_i f_i\chi_{E_i} - \delta \quad \text{a.e.}$$

Then Beppo Levi's convergence Theorem implies that : $\sum_i f_i\chi_{E_i} \in L^1(\Omega)$ and

$$\begin{aligned}\int_\Omega \sum_i f_i\chi_{E_i}\, d\mu &= \int_\Omega \lim_\nu s_\nu\, d\mu + \int_\Omega \delta\, d\mu \\ &= \lim_\nu \int_\Omega s_\nu\, d\mu + \int_\Omega \delta\, d\mu \\ &= \int_\Omega \sum_i p_i(f_i - \delta)\, d\mu + \int_\Omega \delta\, d\mu \\ &= \int_\Omega \sum_i p_i f_i\, d\mu\,,\end{aligned}$$

which proves (3.1).

A generalization of Theorem 3.1 to the case where Ω is not bounded has been given by F. Flores in [**F**]. The following Corollary to Theorem 3.1 is strictly related to the relaxed formulation (PR') of a variational problem (P).

COROLLARY 3.1. *Let $f : [0,T] \times (\mathbb{R}^n \times \mathbb{R}^n) \to \mathbb{R}$ be a normal integrand satisfying (H). Let $\tilde{x} \in W^{1,p}([0,T],\mathbb{R}^n)$, $p_1,\dots,p_m : [0,T] \to [0,1]$ ($\sum_i p_i \equiv 1$), $v_1,\dots,v_m : [0,T] \to \mathbb{R}^n$ be measurable and such that*

$$\tilde{x}' = \sum_i p_i v_i,\quad \sum_i p_i(t) f(t,\tilde{x}(t),v_i(t)) \in L^1. \tag{3.3}$$

Then there exists a measurable partition $E_1,\dots,E_m$ of $[0,T]$ satisfying

$$\sum_i v_i\chi_{E_i} \in L^p,\ \sum_i f(t,\tilde{x}(t),v_i(t))\chi_{E_i}(t) \in L^1$$

and the following equalities hold:

$$\begin{aligned}\int_0^T \sum_i p_i v_i\, dt &= \int_0^T \sum_i v_i\chi_{E_i}\, dt; \\ \int_0^T \sum_i p_i(t) f(t,\tilde{x}(t),v_i(t))\, dt &= \int_0^T \sum_i p_i(t) f(t,\tilde{x}(t),v_i(t))\chi_{E_i}(t)\, dt.\end{aligned} \tag{3.4}$$

PROOF: Let us denote by v_i^+ (resp. v_i^-) the positive (resp. negative) part of v_i, so that $v_i = v_i^+ - v_i^-$ and $|v_i| = v_i^+ + v_i^-$. Set f_i to be the function defined by:

$$f_i(t) = (v_i^+(t), v_i^-(t), f(t, \tilde{x}(t), v_i(t)).$$

Let us show, in order to apply Theorem 3.1, that $\sum_i p_i f_i \in L^1$. By (3.3), it is enough to prove that $\sum_i p_i|v_i| \in L^1$. For this purpose, let us remark that by (H) we have $\sum_i p_i|v_i|^p \in L^1$. Since $p_i \leq p_i^{1/p}$ ($p_i \leq 1$) then Hölder's inequality leads to

$$\sum_{i=1}^{m} p_i|v_i| \leq \left(\sum_{i=1}^{m} p_i|v_i|^p\right)^{1/p} m^{1-1/p}$$

so that $\sum_i p_i|v_i| \in L^1$, which proves the claim.
Moreover, if we set $\delta(t) = (0, 0, \alpha_p(t) - \beta_p|\tilde{x}(t)|^p)$ then by (H) we have:

$$f_i(t) \geq \delta(t) \text{ a.e.}$$

Theorem 3.1 yields the conclusion.

In the case where $n = 1$ and $f : \mathbb{R} \to \mathbb{R}$ does not depend on (t, x), the measurable partition of Corollary 3.1 can be chosen in a more precise way.

PROPOSITION 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be l.s.c. and satisfy (H), let $p_1, p_2, v_1, v_2 : [0, T] \to \mathbb{R}$ be measurable ($p_i \geq 0, p_1 + p_2 \equiv 1$) and such that:*

$$p_1 v_1 + p_2 v_2 \in L^1, \;\; p_1 f(v_1) + p_2 f(v_2) = f^{**}(p_1 v_1 + p_2 v_2) \in L^1.$$

Then there exists a measurable partition E_1, E_2 of $[0, T]$ such that

$$\sum_i v_i \chi_{E_i} \in L^p, \;\; \sum_i f(t, \tilde{x}(t), v_i(t))\chi_{E_i} \in L^1$$

and the following relations hold:

$$(3.5) \qquad \begin{gathered} \int_0^T \sum_i p_i v_i \, dt = \int_0^T \sum_i v_i \chi_{E_i} \, dt, \\ \int_0^T \sum_i p_i(t) f(v_i(t)) \, dt = \int_0^T \sum_i p_i(t) f(v_i(t)) \chi_{E_i}(t) \, dt \end{gathered}$$

and, for each t,

$$(3.6) \qquad \int_0^t \sum_i v_i \chi_{E_i} \, dt \geq \int_0^t \sum_i p_i v_i \, dt.$$

The proof of Proposition 3.1, due to A. Cellina and M. Amar, is based on Lemmas 3.1 and 3.2 below.

Lemma 3.1 [**A–C**, Lemma AE]. *Let $\Phi : [0,T] \to 2^{\mathbb{R}}$ be a multivalued function with values in the closed intervals of $\mathbb{R}$. Let $K \subset [0,T]$ be a measurable set such that $\Phi(t)\chi_K(t)$ is integrably bounded and let $u'(t)$ be a measurable selection of $\Phi(t)$.*
Then there exists a measurable selection ω, with ω belonging to the extremal points of $\Phi(t)\chi_K(t)$, such that:

$$i) \int_0^T \omega(t)\chi_K(t)\,dt = \int_0^T u'(t)\chi_K(t)\,dt$$

$$ii) \int_0^t \omega(s)\chi_K(s)\,dt \leq \int_0^t u'(s)\chi_K(s)\,dt \quad \text{for every } t \in [0,T].$$

Proof: Let $m(t) = \min\ \Phi(t)\chi_K(t)$ and $M(t) = \max\ \Phi(t)\chi_K(t)$. We consider the multivalued function Ψ defined by

$$\Psi(t) = \int_0^t m(s)\chi_K(s)\,ds + \int_t^T \Phi(s)\chi_K(s)\,ds - \int_0^T u'(s)\chi_K(s)\,ds$$

(see [**O**] for the definition of the integral of a multifunction). Clearly Ψ is continuous with respect to the Hausdorff topology of the multivalued maps (see [**AUB–C**]); moreover

$$\begin{aligned} \inf\ \Psi(t) &= \int_0^t m(s)\chi_K(s)\,ds + \inf\left\{\int_t^T \Phi(s)\chi_K(s)\,ds\right\} - \int_0^T u'(s)\chi_K(s)\,ds \\ &= \int_0^T m(s)\chi_K(s)\,ds - \int_0^T u'(s)\chi_K(s)\,ds \end{aligned}$$

which is independent of the time. Since $u'(t)$ is a measurable selection from $\Phi(t)$ and

$$\Psi(0) = \int_0^T \Phi(s)\chi_K(s)\,ds - \int_0^T u'(s)\chi_K(s)\,ds,$$

we have that $0 \in \Psi(0)$. Let us define $\delta \in [0,T]$ as follows

$$\delta = \sup\ \{\tau \leq T :\ 0 \in \Psi(\tau)\}. \tag{3.7}$$

Since Ψ is continuous and takes values in the closed subsets of $\mathbb{R}$, δ must be a maximum, i.e. $0 \in \Psi(\delta)$.
If $\delta = T$, then

$$0 = \int_0^T (m(s) - u'(s))\chi_K(s)\,ds$$

and, since $m(s) \leq u'(s)$ a.e. in K, it follows that $m(s) = u'(s)$ a.e. in K. Hence, it is clear that $\omega(s) = u'(s)$ is the extremal selection.
Let us consider now the case in which $\delta < T$. If 0 belongs to the interior of $\Psi(\delta)$, then the continuity of Ψ could not agree with the definition of δ given in (3.7). On the other hand, if $0 = \min\, \Psi(\delta)$, then we should have $0 \in \Psi(t)$ for every $t \in [0, T]$, because $\min\, \Psi(t)$ is independent of t, and this again contradicts the definition (3.7). Hence the only possible case is the one in which $0 \in \max\, \Psi(\delta)$, i.e.

$$\int_0^\delta m(s)\chi_K(s)\,ds + \int_\delta^T \Phi(s)\chi_K(s)\,ds - \int_0^T u'(s)\chi_K(s)\,ds = 0. \tag{3.8}$$

If we set $\omega(t) = m(t)\chi_{K\cap[0,\delta]}(t) + M(t)\chi_{K\cap[\delta,T]}(t)$, then clearly ω is an extremal selection and the equation (3.8) gives i).
For ii), we consider

$$\int_0^t \omega(s)\chi_K(s)\,ds - \int_0^t u'(s)\chi_K(s)\,ds. \tag{3.9}$$

If $t \leq \delta$ then (3.9) becomes

$$\int_0^t m(s)\chi_K(s)\,ds - \int_0^t u'(s)\chi_K(s)\,ds \tag{3.10}$$

and, since $m(s) \leq u'(s)$ in K, (3.10) is less or equal than zero, hence ii) holds for every $t \leq \delta$.
If $t > \delta$, then (3.9) becomes

$$\int_0^\delta m(s)\chi_K(s)\,ds + \int_\delta^T M(s)\chi_K(s)\,ds - \int_0^t u'(s)\chi_K(s)\,ds. \tag{3.11}$$

Since for every $t > \delta$ the function $M\chi_{K\cap[\delta,T]} + u'\chi_{K\cap[t,T]}$ is a measurable selection of $\Phi(s)\chi_K(s)$, (3.11) belongs to $\Psi(\delta)$; moreover, since $0 = \max\, \Psi(\delta)$, it follows that (3.11) is less or equal than zero, which gives ii) for every $t > \delta$. The claim is proved.

Let us denote by extr(S) we mean the subset of extreme points of a set S.

LEMMA 3.2 ([**A–C**],[**C–M**]). *Let $f : \mathbb{R} \to \mathbb{R}$ be convex, l.s.c., satisfy*

$$f(x') \geq \psi(|x'|) + \gamma$$

$$(\gamma \in \mathbb{R}\,,\ \psi \geq 0 \text{ being convex, l.s.c. and such that } \lim_{r\to\infty} \frac{\psi(r)}{r} = +\infty).$$

Then, for each measurable $x' : I = [\alpha, \beta] \to \mathbb{R}$ *with the property that* $t \mapsto f(x'(t)) \in L^1$ *there exists an integrable selection* ω *of* $\Phi(t) : t \mapsto (\partial f)^{-1}(\partial f(x'(t)))$ *with values in* $extr(\Phi(t))$ *such that :*

$$\int_I \omega(t)\,dt = \int_I x'(t)\,dt \; ; \; \int_I f(\omega(t))\,dt = \int_I f(x'(t))\,dt$$

and, for each t,

$$\int_\alpha^t \omega(s)\,ds \leq \int_\alpha^t x'(s)\,ds.$$

PROOF: Let us first remark that each inverse image under ∂f of a point $c \in \mathbb{R}$ is closed, convex (f being convex) and bounded, since $\lim_{x' \to \infty} \dfrac{f(x')}{|x'|} = +\infty$. It follows that each $(\partial f)^{-1}(c)$ is either empty or a closed interval. Let $(c_i)_{i \in J \subset \mathbb{N}}$ be the values of ∂f whose inverse image under ∂f is a non–trivial interval $[a_i, b_i]$ ($a_i \neq b_i$) and set $K_i = (x')^{-1}([a_i, b_i])$ for i in J, $K_0 = I \setminus \cup_i K_i$. Then, by Lemma 3.1, there exists a measurable selection ω_i of the set–valued map $\Phi(t)\chi_{K_i}(t)$ with values in $\mathrm{extr}(\Phi(t)\chi_{K_i}(t)) = \{a_i, b_i\}$ such that:

$$\int_I \omega_i(t)\chi_{K_i}(t)\,dt = \int_I x'(t)\chi_{K_i}(t)\,dt\,; \tag{3.12}$$

and, for each t

$$\int_\alpha^t \omega_i(s)\chi_{K_i}(s)\,ds \leq \int_\alpha^t x'(s)\chi_{K_i}(s)\,ds\,. \tag{3.13}$$

For each $i \in J$, we have:

$$\int_I f(x'(t))\chi_{K_i}(t)\,dt = \int_I f(a_i) + c_i(x'(t) - a_i)\chi_{K_i}(t)\,dt$$

hence, by (3.12) :

$$\begin{aligned} \int_I f(x'(t))\chi_{K_i}(t)\,dt &= \int_I f(a_i) + c_i(\omega_i(t) - a_i)\chi_{K_i}(t)\,dt \\ &= \int_I f(\omega_i(t))\chi_{K_i}(t)\,dt \end{aligned} \tag{3.14}$$

since $\omega_i(t) \in \{a_i, b_i\}$. By the growth assumption on f, it follows that, for each $n \in \mathbb{N}$,

$$\begin{aligned} \int_I \psi(|x'|\chi_{K_0} + \sum_{i \leq n} |\omega_i|\chi_{K_i})\,dt &\leq \int_{\bigcup_{i=0}^n K_i} (f(x'(t)) - \gamma)\,dt \\ &\leq \int_I (f(x'(t)) - \gamma)\,dt. \end{aligned}$$

i.e. the functions $x'\chi_{K_0} + \sum_{i\le n} \omega_i \chi_{K_i}$, $n \in \mathbb{N}$, are equi–integrable, thus Vitali's convergence Theorem [**E–T**, VII, Corollary 1.3] yields $\omega = x'\chi_{K_0} + \sum_i \omega_i \chi_{K_i} \in L^1$. By (3.12), (3.13), (3.14), ω has the required properties.

PROOF OF PROPOSITION 3.1: Let $\tilde{x} \in W^{1,p}$ be such that $\tilde{x}'(t) = p_1 v_1 + p_2 v_2$. Clearly $\mathrm{extr}(\partial f^{**})^{-1}(\partial f^{**}(\tilde{x}(t)) = \{v_1(t), v_2(t)\}$ a.e. Then Lemma 3.2 yields the conclusion.

Let us show that Proposition 3.1 does not hold, in general, if either $f(t,x)$ depends on t or $n \ge 2$ (i.e. $x' \in \mathbb{R}^2$).

EXAMPLE 3.1. *Given measurable $v_1, v_2 : [0,1] \to \mathbb{R}$ ($v_1 \ne v_2$ on a set of positive measure), there does not exist a measurable partition E_1, E_2 of $[0,1]$ such that $v_1\chi_{E_1} + v_2\chi_{E_2} \in L^1$ and satisfying:*

$$\int_0^1 \frac{1}{2}v_1 + \frac{1}{2}v_2\,dt = \int_0^1 v_1\chi_{E_1} + v_2\chi_{E_2}\,dt, \tag{3.15}$$

$$\int_0^1 t\left(\frac{1}{2}v_1 + \frac{1}{2}v_2\right) dt = \int_0^1 t\,(v_1\chi_{E_1} + v_2\chi_{E_2})\,dt \tag{3.16}$$

$$\int_0^t \frac{1}{2}v_1 + \frac{1}{2}v_2\,dt \ge \int_0^t v_1\chi_{E_1} + v_2\chi_{E_2}\,dt \text{ for each } t. \tag{3.17}$$

PROOF: Let us assume that such a partition exists.
Set $\tilde{x}(t) = \frac{1}{2}\int_0^t v_1 + v_2\,dt$ and $\bar{x}(t) = \int_0^t v_1\chi_{E_1} + v_2\chi_{E_2}\,dt$ so that, by (3.15), $\tilde{x}(1) = \bar{x}(1)$ whence (3.16), integrated by parts, gives

$$\int_0^1 \tilde{x}(t)\,dt = \int_0^1 \bar{x}(t)\,dt. \tag{3.18}$$

By (3.17)

$$\text{for each } t,\ \tilde{x}(t) \ge \bar{x}(t) \tag{3.19}$$

hence (3.18) implies $\tilde{x}(t) = \bar{x}(t)$ a.e. The differentiation of both terms of the above equality yields to $\tilde{x}'(t) = \bar{x}'(t)$ a.e. By definition this means $v_1 = v_2$ a.e. A contradiction.

If $n = 2$ and $v_1, \ldots, v_m$ are constant vectors of $\mathbb{R}^2$, $p_1, \ldots, p_m$ are measurable ($p_i \ge 0, p_1 + \cdots + p_m \equiv 1$), then the component of $p_1(s)v_1 + \cdots + p_m(s)v_m$ along one fixed direction is given by $p_1(s)a_1 + \cdots + p_m(s)a_m$ for some $a_1, \ldots, a_m \in \mathbb{R}$. We have the following analogue of Proposition 3.1.

PROPOSITION 3.2. *Let $p_1, \dots, p_m : [0,T] \to [0,1]$ be measurable ($p_1 + \dots + p_m \equiv 1$), $I \subset [0,T]$ be measurable, $a_1, \dots, a_m \in \mathbb{R}$. Then there exists a measurable partition $E_1, \dots, E_m$ of I such that*

$$\mu(E_i) = \int_I p_i \, dt \ (i = 1, 2, 3) \tag{3.20}$$

and, for each $t \in [0,T]$,

$$\int_0^t \sum_i a_i \chi_{E_i} \, ds \geq \int_0^t (\sum_i a_i p_i) \chi_I \, ds. \tag{3.21}$$

PROOF: Assume

$$m = 3, \ a_1 \geq a_2 \geq a_3$$

the general case being similar. Let $t_1, t_2 \in [0,T]$ be such that

$$\int_0^{t_1} \chi_I \, dt = \int_I p_1 \, dt \, ; \int_{t_1}^{t_2} \chi_I \, dt = \int_{t_1}^{t_2} p_2 \, dt \, ; \int_{t_2}^{1} \chi_I \, dt = \int_I p_3 \, dt.$$

Such t_1, t_2 exist since $p_1 + p_2 + p_3 \equiv 1$. We claim that the measurable partition of I defined by:

$$E_1 = [0, t_1[\cap I \, ; \ E_2 = [t_1, t_2[\cap I \, ; \ E_3 = [t_2, 1] \cap I$$

satisfies (3.20) and (3.21). Proof of the claim:
First, by the very definition of E_i, (3.20) holds trivially. In order to prove (3.21), fix $t \in [0,T]$ and set

$$I_\chi(t) = \int_0^t \sum_i a_i \chi_{E_i} \, ds \, ; \ I(t) = \int_0^t (\sum_i a_i p_i) \chi_I \, ds.$$

We wish to show that $I_\chi(t) \geq I(t)$.
If $t \leq t_1$ then

$$\begin{aligned} I_\chi(t) = \int_0^t a_1 \chi_{E_1} \, ds &= \int_0^t a_1 \chi_I \, ds \\ &= \int_0^t a_1 (p_1 + p_2 + p_3) \, ds \\ &\geq \int_0^t (\sum_i a_i p_i) \chi_I \, ds = I(t). \end{aligned}$$

If $t_1 \le t \le t_2$ then

$$I_\chi(t) = a_1\mu(E_1) + a_2\int_0^t \chi_{E_2}\,ds \tag{3.22}$$
$$= a_1\int_I p_1\chi_I\,ds + a_2\int_0^t \chi_{E_2}\,ds$$
$$\ge a_1\int_0^t p_1\chi_I\,ds + a_2 J(t)$$

where

$$J(t) = \int_t^T p_1\chi_I\,ds + \int_0^t \chi_{E_2}\,ds.$$

Now,

$$\int_0^t \chi_{E_2}\,ds = \int_0^t (p_1+p_2+p_3)\chi_{E_2}\,ds$$

hence

$$J(t) = \int_{t_1}^T p_1\chi_I\,ds + \int_{t_1}^t (p_2+p_3)\chi_{E_2}\,ds. \tag{3.23}$$

Moreover

$$\int_{t_1}^T p_1\chi_I\,ds = \int_{t_1}^T (1-p_2-p_3)\chi_I\,ds \tag{3.24}$$
$$= \mu(E_2)+\mu(E_3) - \int_{t_1}^T (p_2+p_3)\chi_I\,ds$$
$$= \int_I (p_2+p_3)\,ds - \int_{t_1}^T (p_2+p_3)\chi_I\,ds$$
$$= \int_0^{t_1} (p_2+p_3)\chi_I\,ds$$

so that (3.23) and (3.24) together yield

$$J(t) = \int_0^t (p_2+p_3)\chi_I\,ds. \tag{3.25}$$

By (3.22) and (3.25) we have

$$I_\chi(t) \ge a_1\int_0^t p_1\chi_I\,ds + a_2\int_0^t (p_2+p_3)\chi_I\,ds$$
$$\ge \int_0^t a_1p_1\chi_I\,ds + \int_0^t (a_2p_2+a_3p_3)\chi_I\,ds = I(t).$$

Finally, if $t \geq t_2$ then

$$I_\chi(t) = a_1 \mu(E_1) + a_2 \mu(E_2) + a_3 \int_0^t \chi_{E_3}\, ds \tag{3.26}$$

$$\geq a_1 \int_0^t p_1 \chi_I\, ds + a_2 \int_0^t p_2 \chi_I\, ds + a_3 K(t)$$

where

$$K(t) = \int_t^T (p_1 + p_2)\chi_I\, ds + \int_0^t \chi_{E_3}\, ds.$$

We have:

$$\int_0^t \chi_{E_3}\, ds = \int_0^t (p_1 + p_2 + p_3)\chi_{E_3}\, ds$$

hence

$$\begin{aligned} K(t) &= \int_{t_2}^T (p_1 + p_2)\chi_I\, ds + \int_{t_2}^t p_3 \chi_I\, ds \\ &= \int_{t_2}^T (1 - p_3)\chi_I\, ds + \int_{t_2}^t p_3 \chi_I\, ds \\ &= \mu(E_3) - \int_t^T p_3 \chi_I\, ds \\ &= \int_0^t p_3 \chi_I\, ds. \end{aligned} \tag{3.27}$$

The conclusion follows from (3.26) and (3.27). The claim is proved.

II. EXISTENCE RESULTS FOR NON–CONVEX VARIATIONAL PROBLEMS

In this chapter, we assume that the set–valued map $\Phi : [0,T] \to 2^{\mathbb{R}^n}$ is measurable [**C–V**, Def.III.1.1] with non–empty closed values. In addition we assume that there exists at least one $v \in L^p([0,T],\mathbb{R}^n)$ such that $v(t) \in \Phi(t)$ a.e. and $\int_0^T v(t)\,dt = b-a$. The most important result of the calculus of variations without convexity assumptions (and without regularity assumption on the integrand) is due to A. Cellina and G. Colombo.

THEOREM 1 [**C–C**]. *Let $f(t,x,x') = g(t,x) + h(t,x')$ be Carathéodory and satisfy hypothesis (H). Assume that*

$$x \mapsto g(t,x) \text{ is concave for a.e. } t.$$

Then the problem

$$\text{(P)} \qquad \text{minimize} \int_0^T g(t,x(t)) + h(t,x'(t))\,dt$$

on the subset of $W^{1,p}$ of those functions satisfying $x(0)=a$, $x(T)=b$, $x'(t) \in \Phi(t)$ a.e. in [0,T] admits at least one solution.

We shall prove the following generalization of Cellina–Colombo's Theorem.

THEOREM 2 [**M3**]. *Let $f(t,x,\xi) : [0,T]\times(\mathbb{R}^n\times\mathbb{R}^n) \to \mathbb{R}$ be Carathéodory (i.e. measurable in t, ξ and continuous in x) and satisfy hypothesis (H). Moreover, assume that*

$$x \mapsto f(t,x,\xi) \text{ is concave for each } t,\ \xi.$$

Then, if there exists a solution $(\tilde{x},p_1,\ldots,p_{n+1},v_1,\ldots,v_{n+1})$ to the associated relaxed problem (PR') (§I.2) satisfying

$$\text{(C)} \qquad \bigcap_{i=1}^{n+1} \partial_x\left(-f(t,\tilde{x}(t),v_i(t))\right) \neq \emptyset \text{ a.e.},$$

the problem

$$\text{(P)} \qquad \text{minimize} \int_0^T f(t,x(t),x'(t))\,dt$$

on the subset of $W^{1,p}$ of those functions satisfying $x(0) = a$, $x(T) = b$, $x'(t) \in \Phi(t)$ a.e. in $[0,T]$ admits at least one solution.

REMARK 1: Obviously, each solution to (P) is a solution to (PR') satisfying (C); the cases for which Theorem 2 can be usefully applied are those where the converse does not hold. For instance, when $f(t,x,x') = g(t,x) + h(t,x')$ we have

$$\partial_x(-f(t,x,\xi)) = \partial_x(-g(t,x)) \text{ for each } t,\ x,\ \xi$$

hence each solution to (PR') does trivially satisfy (C) (without being, in general, a solution to (P)) and this proves Theorem 1.

REMARK 2: Note that, when $f(t,x,\xi)$ is differentiable in x, condition (C) reduces to

$$\frac{\partial f}{\partial x}(t,\tilde{x}(t),v_i(t)) = \frac{\partial f}{\partial x}(t,\tilde{x}(t),v_j(t)) \text{ for each } i\,,\ j.$$

The following Lemma will be used in the proof of Theorem 2. For a subset Q of $\mathbb{R}^n$, we write $\|Q\|$ for the set $\{|q| : q \in Q\}$.

LEMMA 1. *Let $f_1,\dots,f_m : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ be Carathéodory and satisfy:*
i) $f_i(t,x) \le \alpha(t) + \beta|x|^p$ ($\beta > 0$, $\alpha \in L^1$, $i=1,\dots,m$);
ii) for each i, $x \mapsto f_i(t,x)$ is convex for a.e. t.
Let $\tilde{x}$ be continuous and such that

$$\Psi(t) = \bigcap_i \partial_x f_i(t,\tilde{x}(t)) \ne \emptyset \text{ a.e.}$$

Then, the set–valued map Ψ is measurable and admits an integrable selection.

SKETCH OF THE PROOF [C–C]: Let us assume $i = 1$, the general case being similar.
a) Ψ is measurable. In fact, fix $\Delta > 0$; then $f(t,x) \le \beta\Delta^p + \alpha(t)$ in $[0,T] \times \Delta\bar{B}$. By the Corollary to Proposition 2.2.6 in [C] we have that

$$\|\partial_x f(t,x)\| \le \frac{2}{\Delta}\beta(2\Delta)^p + \alpha(t) \text{ for a.e. } t \in [0,T], \text{ for all } x \in \Delta\bar{B}. \tag{1}$$

Fix $\epsilon > 0$ and let, by Scorza Dragoni's Theorem, $E_\epsilon \subset [0,T]$ be closed and such that: $\mu([0,T] \setminus E_\epsilon) \le \epsilon$; the restriction of f to $E_\epsilon \times \Delta\bar{B}$ is continuous as well as the restriction of α to E_ϵ. Then the map $(t,x) \to \partial_x f(t,x)$ is upper semi–continuous on $E_\epsilon \times \Delta\bar{B}$. An application of Lusin's Theorem for multivalued maps yields the claim.
b) By the Theorem of Kuratowski–Ryll Nardzewski [**AUB–C**, §I.14] there exists a measurable selection $\delta(t) \in \partial_x(t,\tilde{x}(t))$ which, by (1), is integrable.

REMARK 3: The proof of Lemma 1 points out the fact that an integrable selection of Ψ exists if, instead of ii), we assume that there exists a function $\alpha(\cdot)$ in L^1 and a function $c : \mathbb{R}^+ \to \mathbb{R}$ such that

$$\|\partial_x(f_i(t,x))\| \leq \alpha(t) + c(\Delta) \text{ for each } t,\ |x| \leq \Delta,\ i = 1, \dots, m.$$

PROOF OF THEOREM 2: Let $(\tilde{x}, p_i, \dots, p_{n+1}, v_1, \dots, v_{n+1})$ be a solution to (PR') satisfying condition (C). Let, by Lemma 1, $\delta(\cdot) \in L^1$ be a selection to

$$t \mapsto \bigcap_{i=1}^{n+1} \partial_x\left(-f(t, \tilde{x}(t), v_i(t))\right).$$

Then, for each $y \in \mathbb{R}^n$ and $i \in \{1, \dots, n+1\}$, we have:

$$f(t, \tilde{x}(t), v_i(t)) \geq f(t, y, v_i(t)) + < \delta(t), y - \tilde{x}(t) > \tag{2}$$

($<\cdot,\cdot>$ being the usual scalar product in $\mathbb{R}^n$). Set

$$B(t) = \int_0^T \delta(s)\, ds,\ \ f_i(t) = (v_i(t), f(t, \tilde{x}(t), v_i(t)), < v_i(t), B(t) >).$$

By the arguments of the proof of Corollary I.3.1, the growth assumptions on f (hypothesis (H)) imply that the conditions concerning the functions f_i stated in Theorem I.3.1 are satisfied: let $E_1, \dots, E_{n+1}$ be a measurable partition of $[0,T]$ such that

$$\left\{ \begin{array}{l} \sum_i v_i \chi_{E_i} \in L^p,\ \displaystyle\int_0^T \sum_i p_i(t) v_i(t)\, dt = \int_0^T \sum_i v_i(t) \chi_{E_i}(t)\, dt\,; \\ \displaystyle\int_0^T \sum_i p_i(t) < v_i(t), B(t) > dt = \int_0^T \sum_i < v_i(t), B(t) > \chi_{E_i}(t)\, dt\,; \\ \displaystyle\int_0^T \sum_i p_i(t) f(t, \tilde{x}(t), v_i(t))\, dt = \int_0^T \sum_i f(t, \tilde{x}(t), v_i(t)) \chi_{E_i}(t)\, dt. \end{array} \right. \tag{3}$$

and set $\bar{x}(t) = a + \int_0^t \sum_i v_i(s) \chi_{E_i}(s)\, ds$: I claim that $\bar{x}$ is a solution to (P).

Proof of the claim:
Clearly, by (3), $\tilde{x}(T) = \bar{x}(T) = b$ and $\bar{x} \in W^{1,p}$. Furthermore, by (2):

$$\sum_i f(t, \tilde{x}(t), v_i(t)) \chi_{E_i}(t) \geq \sum_i f(t, \bar{x}(t), v_i(t)) \chi_{E_i}(t) + < \delta(t), \bar{x}(t) - \tilde{x}(t) > .$$

The integration of the above inequality and (3) yield:

$$(4) \qquad \min\,(PR') = \int_0^T \sum_i f(t,\tilde{x}(t),v_i(t))\chi_{E_i}(t)\,dt \geq$$

$$\geq \int_0^T \sum_i f(t,\bar{x}(t),v_i(t))\chi_{E_i}(t)\,dt + \int_0^T < \delta(t),\bar{x}(t)-\tilde{x}(t) > \,dt.$$

Let us remark that

$$\sum_i f(t,\bar{x}(t),v_i(t))\chi_{E_i}(t) = f(t,\bar{x}(t),\bar{x}'(t)) \text{ a.e.}$$

and, by Tonelli–Fubini's Theorem:

$$\int_0^T < \delta(t),\bar{x}(t)-\tilde{x}(t) > \,dt = \int_0^T \sum_i (\chi_{E_i}(t)-p_i(t)) < v_i(t), B(T)-B(t) > \,dt.$$

Then, (3) and (4) together yield:

$$\min\text{ (P)} \geq \min\text{ (PR')} \geq \int_0^T f(t,\bar{x}(t),\bar{x}'(t))\,dt \geq \min\text{ (P)}.$$

the conclusion follows.

We shall now give a further application of Theorem 2. The following hypothesis will be considered.

HYPOTHESIS ($\tilde{H}$): Set $\Lambda(t) = a + \mathrm{co}\{\int_0^T \Phi(s)\,ds\}$. We assume that:

$\tilde{H}_1$) The functions g, h, l : $[0,T]\times\mathbb{R}^n \to \mathbb{R}$ are such that $f(t,x,\xi) = g(t,x) + h(t,x)l(t,\xi)$ is Carathéodory and satisfies hypothesis (H) for a.e. t, for each ξ and for each $x \in \Lambda(t)$;

$\tilde{H}_2$) either

$$\text{for a.e. } t \,:\, h(t,x) > 0 \;\text{ for each } x \in \Lambda(t)$$

or

$$\text{for a.e. } t \,:\, h(t,x) < 0 \text{ for each } \; x \in \Lambda(t)\,;$$

$\tilde{H}_3$) For a.e. t and $x \in \Lambda(t)$, the set $A(t,x) = \{\xi \in \Phi(t) : f^{**}(t,x,\xi) < f(t,x,\xi)\}$ is open and, on it, the function $\xi \mapsto f^{**}(t,x,\xi)$ is locally constant.

$\tilde{H}_4$) There exist: a function $\alpha(\cdot)$ in L^1 and a function $c : \mathbb{R}^+ \to \mathbb{R}$ such that

$$\text{for a.e. } t \;:\; \|\partial_x(f(t,x,\xi))\| \leq \alpha(t) + c(\Delta) \text{ for each } \xi \in \Phi(t)\,,\; x \in \Lambda(t)\,,\; |x| \leq \Delta.$$

EXAMPLE 1: $\Phi(t) = \mathbb{R}^+$, $a = 0$, $f(t,x,\xi) = -\gamma x^2 + (1+x)|\xi - \phi(t)||\xi - \psi(t)|$ ($\phi\,,\,\psi\,\in L^\infty\,,\phi\,,\,\psi \geq 0\,,\gamma$ being strictly smaller than the best Sobolev constant) satisfies hypothesis ($\tilde{H}$).

We have the following

THEOREM 3. *Let g, h, l, Φ satisfy hypothesis $(\tilde{H})$. Then the problem*

$$\text{(P)} \qquad \textit{minimize } I(x) = \int_0^T g(t, x(t))\, dt + \int_0^T h(t, x(t)) l(t, x'(t))\, dt$$

on the subset of $W^{1,p}$ of those $x(\cdot)$ satisfying $x(0) = a$, $x(T) = b$, $x'(t) \in \Phi(t)$ a.e. in $[0, T]$ admits at least one solution.

Lemma 2 below will be used in the proof of Theorem 3.

LEMMA 2. *Let h, $l : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ and set $\phi(t, x, \xi) = h(t, x) l(t, \xi)$. Then for each t, x, ξ :*

$$\phi^{**}(t, x, \xi) = h(t, x) L(t, \xi)$$

where

$$L(t, \xi) = \begin{cases} l^{**}(t, \xi) & \text{if } h(t, x) \geq 0\,; \\ -(-l)^{**}(t, \xi) & \text{if } h(t, x) < 0. \end{cases}$$

PROOF: Let us suppose $h(t, x) < 0$, the other case $(h(t, x) \geq 0)$ being similar. In this situation, the inequality

$$(-l)^{**}(t, \xi) \leq -l(t, \xi)$$

implies

$$-h(t, x)\left((-l)^{**}(t, \xi)\right) \leq h(t, x) l(t, \xi)$$

whence

$$h(t, x) L(t, \xi) \leq \phi^{**}(t, x, \xi). \tag{5}$$

Conversely, let t, x be fixed and ψ be any convex function satisfying $\psi(\xi) \leq h(t, x) l(t, \xi)$ for each ξ. Then

$$\frac{-1}{h(t, x)} \psi(\xi) \leq -l(t, \xi) \quad \text{for each } \xi$$

whence

$$\frac{-1}{h(t, x)} \psi(\xi) \leq (-l)^{**}(t, \xi). \tag{6}$$

In particular, for $\psi(\xi) = \phi^{**}(t, x, \xi)$, (6) yields

$$\phi^{**}(t, x, \xi) \leq h(t, x) L(t, \xi). \tag{7}$$

The conclusion follows from (5) and (7).

PROOF OF THEOREM 3: In order to apply Theorem 2, it is enough to prove the existence of a solution $(\tilde{x}, p_1, \ldots, p_{n+1}, v_1, \ldots, v_{n+1})$ to (PR') satisfying

$$f(t, x, v_i(t)) = f(t, x, v_j(t)) \text{ for each } t,\, x \text{ and } i,\, j \in \{1, \ldots, n+1\}.$$

For this purpose, let $(\tilde{x}, p_1, \ldots, p_{n+1}, w_1, \ldots, w_{n+1})$ be an arbitrary solution to (PR'). Then, by Proposition I.2.4

$$f^{**}(t, \tilde{x}(t), \tilde{x}'(t)) = \sum_i p_i(t) f(t, \tilde{x}(t), w_i(t)). \tag{8}$$

The map $\xi \mapsto f^{**}(t, \tilde{x}(t), \xi)$ being convex, we can assume

$$f^{**}(t, \tilde{x}(t), w_i(t)) = f(t, \tilde{x}(t), w_i(t)) \text{ a.e.} \tag{9}$$

Set $A = \{t : f^{**}(t, \tilde{x}(t), \tilde{x}'(t)) < f(t, \tilde{x}(t), \tilde{x}'(t))\} = \{t : \tilde{x}'(t) \in A(t, \tilde{x}(t))\}$.
By $\tilde{H}_3$), the convex function $f^{**}(t, \tilde{x}(t), \cdot)$ is constant in a neighbourhood of $\tilde{x}'(t)$ for a.e. $t \in A$. As a consequence

$$f^{**}(t, \tilde{x}(t), \xi) \geq f^{**}(t, \tilde{x}(t), \tilde{x}'(t)) \text{ for a.e. } t \in A \text{ and each } \xi \in \mathbb{R}^n.$$

In particular

$$f^{**}(t, \tilde{x}(t), w_i(t)) \geq f^{**}(t, \tilde{x}(t), \tilde{x}'(t)) \text{ for a.e. } t \in A$$

hence, by (8), we can assume

$$f^{**}(t, \tilde{x}(t), w_i(t)) = f^{**}(t, \tilde{x}(t), \tilde{x}'(t)) \text{ for a.e. } t \in A. \tag{10}$$

Equalities (9) and (10) prove that, if we set

$$v_i = w_i \chi_A + \tilde{x}' \chi_{[0,T] \setminus A}$$

then $(\tilde{x}, p_1, \ldots, p_{n+1}, v_1, \ldots, v_{n+1})$ is a solution to (PR') satisfying

$$f(t, \tilde{x}(t), v_i(t)) = f^{**}(t, \tilde{x}(t), \tilde{x}'(t)) \text{ a.e.} \tag{11}$$

By Lemma 2, there exists a function $L : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ such that:

$$f^{**}(t, x, \xi) = g(t, x) + h(t, x) L(t, \xi).$$

Thus (11) yields

$$l(t, v_i(t)) = L(t, \tilde{x}'(t)) \text{ a.e.}$$

The claim is proved.

III. EXISTENCE RESULTS FOR NON–CONVEX VARIATIONAL PROBLEMS OF SLOW GROWTH

1. A Preliminary Result

We assume now that the integrand $f(t,x,x') : [0,T] \times (\mathbb{R}^n \times \mathbb{R}^n) \to \mathbb{R}$ does not depend on the space variable x and that, instead of the usual growth assumption (H), f satisfies the weaker condition

(H_1): $f(t,x,x') \geq \alpha(t) - \beta|x| + \gamma|x'|$ for each t, x, x' ($\alpha \in L^1$, $\gamma > 0$).

As it is well known (see, for instance [**CE**, §14]), (H_1) does not ensure the existence of a solution to (P), neither if $f(t,x,x')$ is convex with respect to x', so that, opposite to what happens usually, the associated relaxed problem (PR) to a non–convex problem (P) does not admit, in general, a solution. Let us further remark that, under assumption (H_1), the conclusion of Proposition I.2.3 (a main tool in the previous chapter) is not true, in general; in fact the convex hull of the epigraph of a function $f(t,x,.)$ satisfying (H_1) is not necessarily closed.

Example 1: $f(x') = x' + |x'|^{1/2}$ is such that $f^{**}(x') = |x'|$ so that, unless $x' = 0$, $f^{**}(x')$ cannot be written as a sum $\lambda_1 f(v_1) + \lambda_2 f(v_2)$ ($\Lambda_i \geq 0$, $\sum \lambda_i = 1$, $\sum_i \lambda_i v_i = x'$). Here

$$\text{coepi } f = \big\{(x,y) \in \mathbb{R}^2 : y > x\big\} \cup \{(0,0)\}$$

is not closed

Assume now that $f(t,x')$ does not depend on the space variable x. In this situation, the proof of Theorem II.2 (and Remark I.2.2) yield the following Proposition.

Proposition 1.1. *Let $f(t,x') : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ be a normal integrand satisfying (H_1). Assume further that*

coepi $f(t,.)$ is closed for a.e. t.

Then if (PR) admits at least one solution, so does (P).

We are thus led to consider those convex problems of slow growth having solutions.

2. Parametric Problems

A parametric curve $\mathcal{C}$ in $\mathbb{R}^n$ is a suitable equivalence class of n–vector continuous maps

$$x = x(t),\ a \le t \le b;\ y = y(s),\ c \le s \le d$$

leaving unchanged the sense in which the curve is travelled. Usually, two continuous maps x and y are said to be equivalent if there is a strictly increasing continuous map

$$s = h(t),\ a \le t \le b,\ h(a) = c,\ h(b) = d$$

such that

$$y(h(t)) = x(t),\ a \le t \le b.$$

For technical reasons a weaker equivalence relation is needed.

Definition 2.1 [**CE**, §14]. *continuous maps x and y as above are said to be Fréchet equivalent if for every $\epsilon \ge 0$ there is some homeomorphism*

$$h : s = h(t),\ a \le t \le b,\ h(a) = c,\ h(b) = d$$

such that

$$|y(h(t)) - x(t)| \le \epsilon,\ a \le t \le b.$$

A class of F–equivalent maps is called a parametric curve or F(réchet)–curve.

It is easily seen that for any given F–curve $\mathcal{C} : x = x(t),\ a \le t \le b$, the subsets

$$[\mathcal{C}] = [x] = \{x(t) : a \le t \le b\} \text{ and } \{x(a)\}\ ,\ \{x(b)\}$$

of $\mathbb{R}^n$ are F–invariant. The same holds for the *Jordan length* $L(\mathcal{C})$ of a Fréchet curve $\mathcal{C}$, which is defined as a total variation,

$$L(\mathcal{C}) = \sup \sum_{i=1}^{N} |x(t_i) - x(t_{i-1})| \tag{2.1}$$

where sup is taken with respect to all subdivisions

$$a = t_0 \le t_1 \le \ldots \le t_N = b \text{ of } [a, b].$$

A F–curve $\mathcal{C}$ is said to be *rectifiable* if $L(\mathcal{C}) < +\infty$. The following Proposition justifies the definition of F–curve.

PROPOSITION 2.1 [**CE**, §14.1.I]. *A rectifiable curve $\mathcal{C}$ posseses A.C. representations. In particular, the arc–length parameter s yields a unique A.C.representation*

$$x = x(s),\ 0 \leq s \leq L(\mathcal{C}),\ |x'(s)| = 1 \text{ a.e. } in\ [0, L].$$

If $x(t)$, $a \leq t \leq b$ is an A.C. representation of $\mathcal{C}$, the Jordan length $L(\mathcal{C})$ is given by

$$L(\mathcal{C}) = \int_a^b |x'(t)|\, dt. \tag{2.2}$$

Let $f : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function, and $\mathcal{C}$ be a rectifiable F–curve, $x(t)$, $a \leq t \leq b$ be any of its A.C. representations. Then the integral

$$I[x] = \int_a^b f(x(t), x'(t))\, dt \tag{2.3}$$

is independent of the chosen A.C. representation if and only if f is a *parametric integrand*, i.e. f does not depend on t and is positive homogeneous of degree one in x' [**CE**, §14.1.B], i.e.

$$\forall k \geq 0 :\ f(x, kx') = kf(x, x').$$

In this situation (2.3) defines the *parametric integral* $I(\mathcal{C})$ for any F–curve $\mathcal{C}$ and for any of its A.C. representations. The following existence Theorem is rather typical of the parametric case.

THEOREM 2.1. *Let B (resp. K) be a closed (resp. compact) subset of $\mathbb{R}^n$ and let $f(x, x') : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be continuous, convex and positive homogeneous of degree one in x', satisfy (H_1).*
Assume further that there exists a positive monotone non–decreasing function ϕ such that

$$L(\mathcal{C}) \leq \phi(I(\mathcal{C})) \tag{2.4}$$

for all F–curves $\mathcal{C}$. Then $I(\mathcal{C}))$ has an absolute minimum in the class Ω of all rectifiable F–curves $\mathcal{C} : x = x(t)$, $a \leq t \leq b$ satisfying the boundary condition $x(a) \in K$, $x(b) \in B$.

PROOF: a lower semi–continuity argument. See, for instance [**CE**, Th. 14.1.iv].

In the case where a non necessarily convex integrand $f(x')$ does not depend on x, our Proposition 1.1 yield the following existence result.

THEOREM 2.2 [M1]. *Let B (resp. K) be a closed (resp. compact) subset of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ be continuous, positive homogeneous of degree one and satisfy (H_1) ($\alpha \equiv 0$, $\beta = 0$). Then the parametric integral*

$$I(C) = \int_a^b f(x'(t)\, dt$$

has an absolute minimum in the class Ω of all rectifiable F–curves $C : x = x(t)$, $a \leq t \leq b$ satisfying the boundary condition $x(a) \in K$, $x(b) \in B$.

EXAMPLE 2.1: $n = 2$,

$$f(x_1, x_2) = \sqrt{|x_1||x_2|} + |x_1| + |x_2|$$

is positive homogeneous of degree one, but not convex since, for instance, $f(1,0) = 1$, $f(0,1) = 1$, $f(\frac{1}{2}(0,1) + \frac{1}{2}(1,0)) = \frac{3}{2} > 1$.

PROOF OF THEOREM 2.2: In order to apply Proposition 1.1, we wish to show that:

i) coepi $f(.)$ is closed;

ii) the bipolar f^{**} is continuous, positive homogeneous of degree one and satisfies (H_1) so that, by Theorem 2.1, the associated relaxed problem (PR) admits at least one solution.

Ad i). Let C be the closed subset of $\mathbb{R}^n$ defined by

$$C = \{\xi : f(\xi) = 1\}.$$

By (H_1), C is bounded; the homogeneity of f proves that the graph of f is given by:

$$\text{graph } f = \{\lambda(\xi, 1) : \lambda \geq 0,\ \xi \in C\}$$

As a consequence, the convex hull of the epigraph of f coincides with the convex cone generated by co(C), i.e.

$$\text{coepi } f = \{(\lambda\xi, \rho) : \lambda \geq 0,\ \xi \in \text{ co } (C),\ \rho \geq \lambda\}. \tag{2.5}$$

Now, C is compact hence so does co(C) [**R**]; (2.5) gives i).

Ad ii). The function $\xi \to |\xi|$ being convex, f^{**} does trivially satisfy (H_1). In order to prove the second part of the claim, let us recall that, in general, the epigraph of the bipolar of f is the closed convex hull of the epigraph of f (Remark I.1.1). In i) we proved that this set is a closed convex cone. It follows that f is continuous and positive homogeneous. Proposition I.1 yields the conclusion.

3. Non–Parametric Problems

Let $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n \to \mathbb{R}$ be a normal integrand satisfying (H_1) whence, by convexity of $\xi \to |\xi|$ and Proposition I.2.2, so does f^{**}. Let us denote by $F(t,p,u)$ the parametric integrand associated to $f^{**}(t,x')$ ([**CE**], Ch.14) defined by:

$$F : [0,T] \times]0,+\infty[\times \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$(t,p,u) \longmapsto p f^{**}(t, \frac{u}{p}) \; .$$

As it is described in [**CE**], the function F is convex and positive homogeneous in (p,u). As a consequence, if f^{**} is supposed to be continuous, if (t,u) is fixed and we allow $p > 0$ to approach zero, then $F(t,p,u)$ must approach a finite limit or $+\infty$. This limit is taken as the definition of $F(t,0,u)$. Since $F(t,kp,ku) = kF(t,p,u)$ for all $k > 0$, we define $F(t,0,0)$ to be zero, so that the homogeneity property holds for $k \geq 0$. It can be shown [**CE**,§14.2] that if F^{**} is continuous in its domain and $F(t,0,u)$ is finite everywhere then $F(t,p,u)$ is continuous in $[0,T] \times [0,+\infty[\times \mathbb{R}^n$.
Let K_1 , K_2 be two compact subsets of $\mathbb{R}^{n+1}$ such that for every $(t_1,x_1,t_2,x_2) \in K_1 \times K_2$ we have $t_1 < t_2$ and set $K = K_1 \times K_2$. We consider the problem of the minimum of the integral

$$I(x) = \int_{t_1}^{t_2} f(t,x'(t))\,dt$$

in the class Ω of all A.C. functions $x(t) = (x^1,\ldots,x^n) \in K$ (we say that these are the admissible trajectories). The following existence Theorems are the nonconvex analogue of [**CE**, Th.14.3.i, Th.14.3.ii]. Their proof is a direct application of our Proposition 1.1.

Theorem 3.1 [**M2**]. *Let f be a normal integrand satisfying (H_1). Assume that the bipolar of f is of class C^1 in its domain and that its associated parametric integrand F is continuous in $[0,T] \times [0,+\infty[\times \mathbb{R}^n$. Moreover, let us assume that*

$$\forall t \in [0,T],\, \forall u \in \mathbb{R}^n,\, |u| = 1 : \frac{\partial F}{\partial p}(t,0,u) = -\infty$$

and there are constants $M_1, M_2, \delta > 0$ such that for all $t \in [0,T]$, $(p,u) \in [0,+\infty[\times \mathbb{R}^n$, $|p| + |u| = 1$ and t^* with $|t^* - t| < \delta$ we have

$$|\frac{\partial F}{\partial t}(t^{**},p,u)| \leq M_1 F(t,p,u) + M_2 .$$

Assume further that coepi $f(t,.)$ is closed for a.e. t.

Then, $I(x)$ has an absolute minimum in the class of all admissible trajectories.

THEOREM 3.2 [**M2**]. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be independent of t and satisfy hypothesis (H_1). Assume that the parametric integrand F associated to the bipolar of f is continuous in $[0, +\infty[\times\mathbb{R}^n$ with continuous partial derivative $\frac{\partial F}{\partial p}$ in $[0, +\infty[\times\mathbb{R}^n \setminus \{0,0\}$ and that for every $u \neq 0$ we have*

$$\frac{\partial F}{\partial p}(p, u) = 0 \text{ if and only if } p = 0.$$

Assume further that coepi $f(.)$ is closed.
Then, $I(x)$ has an absolute minimum in the class Ω of all admissible trajectories.

REMARK 3.1: It doesn't seem reasonable, in order to satisfy the assumptions of Theorems 3.1, 3.2 to require conditions only on f instead of f^{**}. For instance, the function defined by

$$f(\xi) = |\xi|(1 + sin(2\pi\xi))$$

is such that the limit as p approaches zero of its associated parametric integrand $\tilde{F}(p, u) = |u|(1 + sin(2\pi\frac{u}{p}))$ does not exist whereas the parametric integrand F associated to the bipolar of f , given by $F(p, u) = |u|$, is continuously differentiable in $[0, +\infty[\times\mathbb{R}$.

REMARK 3.2: If $n = 1$, coepi $f(.)$ is closed if, for instance, f is continuous and $f = f^{**}$ in the complement of an interval $I = [a, b]$ in which the graph of f^{**} is a line joining the points $[a, f(a)]$ and $[b, f(b)]$. In fact, in this situation coepi $f(.)$ coincides with the epigraph of f^{**}, a closed set.

EXAMPLE 3.1: Let us consider the following nonconvex continuously differentiable function f defined by:

$$f(\xi) = \begin{cases} \sqrt{1 + \xi^2} \text{ if } |\xi| \geq \pi, \\ \sqrt{1 + \xi^2} + cos(\xi) + 1 \text{ otherwise.} \end{cases}$$

Let x_0 be the point in $]0, \pi[$ such that $f'(x_0) = 0$. Then, the bipolar of f is given by

$$f^{**}(\xi) = \begin{cases} f(\xi) \text{ if } |\xi| \geq x_0; \\ f(x_0) \text{ if } |\xi| \leq x_0. \end{cases}$$

We claim that the parametric integrand F associated to the bipolar of f satisfies the conditions of Theorem 3.2.

In fact, $f(\xi) \geq |\xi|$ and by Remark 3.2, coepi $f(.)$ is closed. Furthermore, F is clearly of class C^1, and if we let $u \neq 0$ its partial derivative with respect to p is given by:

$$\frac{\partial F}{\partial p}(p,u) = \begin{cases} \dfrac{p}{\sqrt{p^2+u^2}} \text{ if } \dfrac{|u|}{p} \geq \pi; \\ f(\dfrac{u}{p}) - \dfrac{u}{p} f'(\dfrac{u}{p}) \text{ if } \pi \geq \dfrac{|u|}{p} \geq x_0; \\ f(x_0) \text{ if } \dfrac{|u|}{p} \leq x_0. \end{cases}$$

Since $f(x_0) \neq 0$ and for each $\xi \in [x_0, \pi]$ we have $f'(\xi) \neq \dfrac{f(\xi)}{\xi}$ then $\dfrac{\partial F}{\partial p}(u,p) = 0$ if and only if $p = 0$. The claim is proved.

IV. ON THE DENSITY OF MINIMUM PROBLEMS HAVING EXISTENCE

Let $g : \mathbb{R} \to \mathbb{R}$ be continuous and $h : \mathbb{R} \to \mathbb{R}$ be l.s.c. Assume, as usual, that $f(x,x') = g(x) + h(x')$ satisfies the growth assumptions (H) and consider the problem

$$\text{(P)} \qquad \text{Minimize} \int_0^T g(x(t)) + h(x'(t))\, dt : x(0) = a,\ x(T) = b,\ x \in W^{1,p}.$$

In chapter II, we proved that a solution to (P) exists if either h is convex (Tonelli's classical Theorem I.2.1) or if g is concave (Cellina–Colombo's Theorem II.1); moreover Marcellini proved [MC] that a solution exists under the assumption that $n = 1$ and g is monotonic. Therefore existence seems to be a property related to very special geometric behaviour in x or in x'. We show here that (for x in $\mathbb{R}$) this is not so: there exists a subset $\mathcal{D}$ of the space of continuous functions, dense for the topology of uniform convergence on compacta, such that, for g in $\mathcal{D}$, problem (P) has existence of solutions for every function h satisfying the usual growth conditions.
Let $\mathcal{D}$ be the subset of $\mathcal{C}(\mathbb{R})$ of those continuous functions with the property: for g in $\mathcal{D}$, there exists a partition of $\mathbb{R}$ into countably many intervals $[a_i, b_i[$ with the property that only finitely many meet every finite interval of $\mathbb{R}$ and such that g is affine on $[a_i, b_i[$; moreover at each a_i (and at each b_i) at least one of the one sided limits of the derivative of g is zero. (Hence, given two consecutive intervals, g is constant on at least one).

Proposition 1 [C–M]. *$\mathcal{D}$ is dense in $\mathcal{C}(\mathbb{R})$.*

Proof: Fix $h \in \mathcal{C}(\mathbb{R})$ and $\epsilon > 0$. Fix n in $\mathbb{Z}$; since h is uniformly continuous on $[n, n+1]$, there exists an integer m such that

$$(1) \qquad x, y \in [n, n+1],\ |x - y| \le \frac{1}{m} \Rightarrow |h(y) - h(x)| < \frac{\epsilon}{2}.$$

Define $g_n : [n, n+1] \to \mathbb{R}$ by:

$$g_n(x) = \begin{cases} h(n) \text{ for } x \text{ in } [n, n + \frac{1}{3m}]\,; \\ h(n + \frac{k}{m}) \text{ for } x \text{ in } [n + \frac{k}{m} - \frac{1}{3m}, n + \frac{k}{m} + \frac{1}{3m}]\,,\ k = 1, \dots, m-1\,; \\ h(n+1) \text{ for } x \text{ in } [n + 1 - \frac{1}{3m}, n+1] \end{cases}$$

and continuous affine elsewhere. Let us show that, on $[n, n+1]$, $\|h - g_n\|_\infty < \epsilon$.
Fix x and let k be such that $x \in [n + \frac{k}{m}, n + \frac{k+1}{m}[$. By the very definition (and by (1)),

$$|g_n(x) - h(n + \frac{k}{m})| \leq |h(n + \frac{k+1}{m}) - h(n + \frac{k}{m})| < \frac{\epsilon}{2}$$

so that, by (1) again,

$$|g_n(x) - h(x)| < \epsilon .$$

Set $g(x)$ to be $g_n(x)$ if $x \in [n, n+1[$. Since g is in $\mathcal{D}$, this proves the claim.

THEOREM 1 [**C–M**]. *Let $p \geq 1$ and $g \in \mathcal{D}$ be such that $g(x) \geq \alpha - \beta|x|^p$ for every x (α, $\beta \in \mathbb{R}$). Then, the Problem:*

$$\text{(P)} \qquad \textit{Minimize} \int_0^T g(x(t))\,dt + \int_0^T h(x'(t))\,dt$$

on the subset of $W^{1,p}([0,T], \mathbb{R})$ of those functions satisfying the boundary conditions $x(0) = a, x(T) = b$ admits a solution for every lower semicontinuous function $h(x')$ satisfying the usual growth assumptions, i.e.
if $p = 1$, there exist a convex l.s.c. function ψ, $\gamma \in \mathbb{R}$ and $h(x') \geq \psi(|x'|) + \gamma$; if $p > 1$, there exist $\sigma > 0$ ($\frac{\beta}{\sigma}$ being strictly less than the best Sobolev constant), $\gamma \in \mathbb{R}$ and $h(x') \geq \sigma|x'|^p + \gamma$.

PROOF: Let $\tilde{x}$ be a solution to the relaxed problem associated to (P) and set $\Delta_1 = \min\{\tilde{x}(t) : t \in [0,T]\}$, $\Delta_2 = \max\{\tilde{x}(t) : t \in [0,T]\}$. Let d_1 be the greater discontinuity point of g' less or equal than Δ_1, $d_2 < \cdots < d_{n-1}$ be those inside $]-\Delta_1, \Delta_2[$, d_n be the next after d_{n-1} and set

$$\epsilon = \frac{1}{5} \min\{|d_{i+1} - d_i| \,:\, i = 0, \ldots, n-1\}.$$

a) We claim that $[0,T]$ can be partitioned in a countable union of disjoint intervals I_j ($j \in \mathbb{N}$) such that g is monotonic on $\tilde{x}(I_j)$.
Proof of the claim. Consider the three sets A, V, B defined by:

$$A = \bigcup_{i=1}^{n-1}]d_i + \epsilon, d_{i+1} - \epsilon[\ ,\ V = \{d_i - \epsilon, d_i + \epsilon \,:\, i = 1, \ldots, n\}\ ,\ B = \bigcup_{i=1}^{n}]d_i - 2\epsilon, d_i + 2\epsilon[.$$

By the continuity of $\tilde{x}$, the inverse image of A under $\tilde{x}$ is a countable union of disjoint relatively open subintervals (σ_i, τ_i) of $[0,T]$. The image of each subinterval is contained in one of the open intervals $]d_i + \epsilon, d_{i+1} - \epsilon[$ on which g is affine. By the continuity of $\tilde{x}$, there exists δ such that

$$|t-s| < \delta \Rightarrow |\tilde{x}(t) - \tilde{x}(s)| < \epsilon.$$

Consider those subintervals (σ_i, τ_i) whose diameter $\tau_i - \sigma_i \geq \delta$, say for $i = 1, \ldots, m$. These are the first elements of our partition. Again by continuity, for each i, at least one between $\tilde{x}(\sigma_i)$ and $\tilde{x}(\tau_i)$ is in V (actually both, except for the case $\sigma_i = 0$ or $\tau_i = T$). Consider the finite union of closed subintervals of $[0, T]$ that is the complement of the finite union of open subintervals (σ_i, τ_i), $i = 1, \ldots, m$: they are the intervals $[0, \sigma_1]$, $\ldots$, $[\tau_{m-1}, \sigma_m]$, $[\tau_m, T]$. For t in this complement, $\tilde{x}(t)$ is in B. In fact either t is in $\tilde{x}^{-1}(A) \setminus \bigcup_{i=1}^m (\sigma_i, \tau_i)$ or $\tilde{x}(t)$ is in $\bigcup_i [d_i - \epsilon, d_i + \epsilon]$.
In the first case, from the choice of δ and the remark on the behaviour at the extremes of the intervals, there exists d_i such that $|\tilde{x}(t) - d_i| < 2\epsilon$, i.e. $\tilde{x}(t) \in B$.
The second case holds since each $[d_i - \epsilon, d_i + \epsilon]$ is in B. Since B is open, its counter image $\tilde{x}^{-1}(B)$ is countable collection of open subintervals. Consider the image of any such subinterval: it is contained in one (and only one) $]d_i - 2\epsilon, d_i + 2\epsilon[$. On one of $]d_i - 2\epsilon, d_i[$ or $]d_i, d_i + 2\epsilon[$ g is constant, on the other affine: g is monotonic on $]d_i - 2\epsilon, d_i + 2\epsilon[$. Intersect each subinterval with the finite collection of intervals $[\tau_i, \sigma_{i+1}] : i = 0, \ldots, m$. The union of this countable collection of intervals and of $(\sigma_i, \tau_i) : i = 1, \ldots, m$ is the required partition of $[0, T]$.
b) By Proposition I.2.3 there exist measurable $p_1, p_2, v_1, v_2 : [0, T] \to \mathbb{R}$ ($p_i \geq 0, p_1 + p_2 = 1$) such that

$$\begin{cases} \tilde{x}' = p_1 v_1 + p_2 v_2 \text{ a.e.;} \\ h^{**}(\tilde{x}'(t)) = p_1(t) h(v_1(t)) + p_2(t) h(v_2(t)) \text{ a.e.} \end{cases}$$

hence extr $(\partial h^{**})^{-1}(\partial h^{**}(\tilde{x}'(t))) = \{v_1(t), v_2(t)\}$ a.e. Let $I_j = (\alpha_j, \beta_j)$ be one of the intervals considered in a). Let us explicitely carry out the construction for the case $g' < 0$ on $]d_{i-1}, d_i[$, the other cases being treated similarly. We claim that there exists a measurable partition E_1^j, E_2^j of $[\alpha_j, \beta_j]$ such that:

(i) $v_1 \chi_{E_1^j} + v_2 \chi_{E_2^j} \in L^p(\alpha_j, \beta_j)$;

(ii) $\displaystyle\int_{\alpha_j}^{\beta_j} v_1 \chi_{E_1^j} + v_2 \chi_{E_2^j} \, dt = \int_{\alpha_j}^{\beta_j} p_1 v_1 + p_2 v_2 \, dt$;

(iii) $\displaystyle\int_{\alpha_j}^{t} v_1 \chi_{E_1^j} + v_2 \chi_{E_2^j} \, ds \geq \int_{\alpha_j}^{t} p_1 v_1 + p_2 v_2 \, ds$ for each $t \in (\alpha_j, \beta_j)$;

(iv) $\displaystyle\int_{\alpha_j}^{\beta_j} p_1(t) h(v_1(t)) + p_2(t) h(v_2(t)) \, dt = \int_{\alpha_j}^{\beta_j} h(v_1(t)) \chi_{E_1^j}(t) + h(v_2(t)) \chi_{E_2^j}(t) \, dt$;

(v) $\displaystyle\left| \int_{\alpha_j}^{t} v_1 \chi_{E_1^j} + v_2 \chi_{E_2^j} \, ds - \int_{\alpha_j}^{t} p_1 v_1 + p_2 v_2 \, ds \right| < \epsilon$ for each $t \in (\alpha_j, \beta_j)$.

Proof of the claim. Set $\psi(x') = \gamma_p|x'|^p$ if $p > 1$. Let us first remark that by La Vallée–Poussin's Theorem, the set

$$\mathcal{H} = \{f \in L^1(\alpha_j, \beta_j) : \int_{\alpha_j}^{\beta_j} \psi(|f|) \leq \int_{\alpha_j}^{\beta_j} h^{**}(\tilde{x}'(t))\, dt - \int_{\alpha_j}^{\beta_j} \gamma\, dt\}$$

is equi–integrable: let $\rho > 0$ be such that $\int_A |f|\, dt < \dfrac{\epsilon}{2}$ for each measurable subset A of I_j whose measure is less than ρ and for each $f \in \mathcal{H}$. Let $\alpha_j = \gamma_0 < \gamma_1 < \cdots < \gamma_m = \beta_j$ be a subdivision of I_j such that $\max_i |\gamma_{i+1} - \gamma_i| < \rho$. By Proposition I.3.1 and the above remark on the inverse image of the subdifferential of h^{**}, for each interval $[\gamma_k, \gamma_{k+1}]$, there exists a measurable partition $E_{1,k}, E_{2,k}$ satisfying:

$(i)_k$ $v_1\chi_{E_{1,k}} + v_2\chi_{E_{2,k}} \in L^p(\gamma_k, \gamma_{k+1})$;

$(ii)_k$ $\displaystyle\int_{\gamma_k}^{\gamma_{k+1}} v_1\chi_{E_{1,k}} + v_2\chi_{E_{2,k}}\, dt = \int_{\gamma_k}^{\gamma_{k+1}} p_1 v_1 + p_2 v_2\, dt$;

$(iii)_k$ $\displaystyle\int_{\gamma_k}^{t} v_1\chi_{E_{1,k}} + v_2\chi_{E_{2,k}}\, ds \geq \int_{\gamma_k}^{t} p_1 v_1 + p_2 v_2\, ds$ for each $t \in (\gamma_k, \gamma_{k+1})$;

$(iv)_k$ $\displaystyle\int_{\gamma_k}^{\gamma_{k+1}} \sum_{i=1}^{2} p_i(t) h(v_i(t))\, dt = \int_{\gamma_k}^{\gamma_{k+1}} \sum_{i=1}^{2} h(v_i(t))\chi_{E_{i,k}}(t)\, dt.$

Set $E_1^j = \bigcup_{k=0}^{m-1} E_{1,k}$, $E_2^j = \bigcup_{k=0}^{m-1} E_{2,k}$. Then (i), (ii), (iii), (iv) can be trivially deduced from their corresponding $(i)_k$, $(ii)_k$, $(iii)_k$, $(iv)_k$. In order to prove (v), fix $t \in (\alpha_j, \beta_j)$ and let k be such that $t \in [\gamma_k, \gamma_{k+1}[$. Let us write that

$$(2) \qquad \int_{\alpha_j}^{t} \sum_i (p_i - \chi_{E_i^j}) v_i\, ds = \int_{\alpha_j}^{\gamma_k} \sum_i (p_i - \chi_{E_i^j}) v_i\, dt + \int_{\gamma_k}^{t} \sum_i (p_i - \chi_{E_i^j}) v_i\, ds.$$

By $(ii)_k$, the first term of the right–hand side of the above equality is zero. Furthermore we have

$$\psi(|v_1|\chi_{E_1^j} + |v_2|\chi_{E_2^j}) \leq h(v_1(t))\chi_{E_1^j}(t) + h(v_2(t))\chi_{E_2^j}(t) - \gamma$$

so that, by $(iv)_k$, $v_1\chi_{E_1^j} + v_2\chi_{E_2^j} \in \mathcal{H}$. Let us recall that $|t - \gamma_k| < \rho$ hence by equi–integrability and (2) we have:

$$\left| \int_{\alpha_j}^{t} \sum_i (p_i - \chi_{E_i^j}) v_i\, ds \right| \leq \int_{\gamma_k}^{t} p_1|v_1| + p_2|v_2|\, ds + \int_{\gamma_k}^{t} |v_1|\chi_{E_1^j} + |v_2|\chi_{E_2^j}\, ds$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which proves the claim.

c) Let us denote by $\bar{x}_j : [\alpha_j, \beta_j] = I_j \to \mathbb{R}$ the function defined by:

$$\bar{x}_j(t) = \tilde{x}(\alpha_j) + \int_{\alpha_j}^{t} v_1\chi_{E_1^j} + v_2\chi_{E_2^j}\, ds.$$

Then, by (i), $\bar{x}'_j \in L^p(\alpha_j, \beta_j)$ and by (ii) we have $\bar{x}_j(\beta_j) = \tilde{x}(\beta_j)$. Furthermore, by (iv),

$$(3) \qquad \int_{\alpha_j}^{\beta_j} h^{**}(\tilde{x}'(t))\, dt = \int_{\alpha_j}^{\beta_j} h(\bar{x}'_j(t))\, dt.$$

Since, by definition, $\bar{x}_j(\alpha_j) = \tilde{x}(\alpha_j)$ then, by (iii) $\bar{x}_j(t) \geq \tilde{x}(t)$ for every $t \in I_j$. Moreover, by (v),

$$|\bar{x}_j(t) - \tilde{x}(t)| < \epsilon \text{ for every } t \in I_j,$$

whence $\bar{x}_j(t) \in]d_i - 3\epsilon, d_i + 3\epsilon[$. Then, g being non–increasing on the above interval,

$$(4) \qquad g(\bar{x}_j(t)) \leq g(\tilde{x}(t)) \text{ for every } t \in I_j.$$

Now, (3) and (4) together give:

$$(5) \qquad \int_{I_j} g(\bar{x}_j(t))\, dt + \int_{I_j} h(\bar{x}'_j(t))\, dt \leq \int_{I_j} g(\tilde{x}(t))\, dt + \int_{I_j} h^{**}(\tilde{x}'(t))\, dt.$$

Let $\bar{x} : [0, T] \to \mathbb{R}$ be the function whose restriction to each I_j ($j \in \mathbb{N}$) is $\bar{x}_j$. Then $\bar{x} \in W^{1,p}$ and $\bar{x}(0) = \tilde{x}(0)$, $\bar{x}(T) = \tilde{x}(T)$. Moreover, by (5):

$$\begin{aligned}
\min \text{ (P)} &\leq \int_0^T g(\bar{x}(t))\, dt + \int_0^T h(\bar{x}'(t))\, dt \\
&\leq \int_0^T g(\tilde{x}(t))\, dt + \int_0^T h^{**}(\tilde{x}'(t))\, dt \\
&= \min \text{ (PR) } \leq \min \text{ (P)}.
\end{aligned}$$

It follows that the above inequalities are in fact equalities: $\bar{x}$ is a solution to (P).

V. AN EXISTENCE RESULT FOR NON–CONVEX VARIATIONAL PROBLEMS WITH RESTRICTED ADMISSIBLE TRAJECTORIES

Let $f : [0,T] \times (\mathbb{R}^n \times \mathbb{R}^n) \to \mathbb{R}$ be a normal integrand satisfying (H) and Tonelli's classical convexity condition (I.2.5). Moreover, let C be a closed subset of $\mathbb{R}^n$. Then, a slight modification of the proof of Tonelli's classical existence Theorem I.2.1 shows that the problem

$$\text{(P)} \quad \text{minimize} \int_0^T f(t, x(t), x'(t))\, dt \,:\, x(0) = a \in C,\; x(T) = b \in C,\; x(t) \in C \text{ for each } t$$

admits at least one solution.
As usual, we are led to consider the case where f is not convex in x'. The basic idea is that to extend the arguments of the proof of the density Theorem IV.1, i.e. given a solution $\tilde{x}$ to the convexified problem (PR), try to find a function $\bar{x}$, a candidate for being a solution to (P), satisfying $\int_0^T f(\bar{x}'(t))\, dt = \int_0^T f^{**}(\tilde{x}'(t))\, dt$ and whose "relative position" with respect $\tilde{x}$ ensures that its trajectory is contained in C. We showed, in example I.3.1, that if $n \geq 2$ then Proposition I.3.1, a basic tool in the proof of Theorem IV.1, does not hold, in general. Nevertheless, in the case where $n = 2$ and $C = \Pi^+$, a half–plane in $\mathbb{R}^2$, its weaker 2–dimensional version (Prop. I.3.2), yields Theorem 1 below.

THEOREM 1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be l.s.c. and satisfy (H). Assume further that*

$$K = \{\xi \,:\, f^{**}(\xi) < f(\xi)\}$$

is a finite union of polygons and that, on each of them, the epigraph of f^{**} is contained in a hyperplane of $\mathbb{R}^3$. Then the problem

$$\text{(P)} \quad \textit{minimize} \int_0^T f(x'(t))\, dt \,:\, x(0) = a \in \Pi^+,\; x(T) = b \in \Pi^+,\; x(t) \in \Pi^+ \textit{ for each } t$$

admits at least one solution.

PROOF: Let us denote int (S) the topological interior of a set S and assume that the closure of $K = \{\xi \,:\, f^{**}(\xi) < f(\xi)\}$ coincides with $\operatorname{int}(\operatorname{co}(v_1,\, v_2\, v_3))$ for some $v_1,\, v_2,\, v_3 \in \mathbb{R}^2$, the general case being similar.
Let $\tilde{x} : [0,T] \to \mathbb{R}^2$ be a solution to the relaxed problem (PR) associated to (P):

$$\text{(PR)} \quad \text{minimize} \int_0^T f^{**}(x'(t))\, dt \,:\, x(0) = a \in \Pi^+,\; x(T) = b \in \Pi^+,\; x(t) \in \Pi^+ \text{ for each } t$$

and let I be the measurable subset of $[0,T]$ defined by

$$I = \{t : \tilde{x}'(t) \in K\}.$$

By Proposition I.2.3 there exist $p_1, p_2, p_3 : [0,T] \to [0,1]$ $(p_1 + p_2 + p_3 \equiv 1)$ such that

$$\text{for } t \text{ in } I,\ \tilde{x}'(t) = p_1(t)v_1 + p_2(t)v_2 + p_3(t)v_3.$$

Assume that the half–plane Π^+ is defined by

$$\Pi^+ = \{u \in \mathbb{R}^2 : < u, n >_{\mathbb{R}^2} \geq \rho\}\ (n \in \mathbb{R}^2,\ \rho \in \mathbb{R}).$$

By Proposition I.3.2 there exists a measurable partition E_1, E_2, E_3 of I with the property that:

(1)
$$\mu(E_i) = \int_I p_i\, dt$$

and, for each t:

(2)
$$\int_0^t \sum_i p_i(s) < v_i, n > \chi_I\, ds \geq \int_0^t \sum_i < v_i, n > \chi_{E_i}(s)\, ds.$$

Define

$$\omega = \tilde{x}'\chi_{[0,T]\setminus I} + \sum_i v_i \chi_{E_i}(s)\, ds$$

and

$$\bar{x}(t) = a + \int_0^t \omega(s)\, ds.$$

We claim that $\bar{x}$ is a solution to (P). Proof of the claim:
Clearly, $\bar{x}(0) = a$. Moreover, by (1) we have

$$\sum_i \int_I \chi_{E_i}\, dt = \sum_i v_i \mu(E_i) = \sum_i \int_I p_i\, dt.$$

As a consequence,

$$\begin{aligned}
\bar{x}(T) &= a + \int_{[0,T]\setminus I} \tilde{x}'(t)\, dt + \int_I \sum_i v_i \chi_{E_i}\, dt \\
&= a + \int_{[0,T]\setminus I} \tilde{x}'(t)\, dt + \int_I \sum_i p_i v_i\, dt \\
&= a + \int_0^T \tilde{x}'(t)\, dt \\
&= \tilde{x}(T) = b.
\end{aligned}$$

Analogously, by (2) we have:

$$< \bar{x}(t),\, n > \geq\; < \tilde{x}(t),\, n > \text{ for each } t$$

hence, by the definition of Π^+, $\bar{x}(t) \in \Pi^+$ for each t.
In order to prove that $\int_0^T f^{**}(\tilde{x}'(t))\,dt = \int_0^T f(\bar{x}'(t))\,dt$, let

$$H = \{(z, z_3) \in \mathbb{R}^2 \times \mathbb{R} :\, z_3 =< \lambda, z > +\lambda_3\} \qquad (\lambda \in \mathbb{R}^2,\, \lambda_3 \in \mathbb{R})$$

be the hyperplane containing the graph of f^{**} restricted to K so that, on I,

$$(\tilde{x}'(t), f^{**}(\tilde{x}'(t)) \in H \text{ and } (\bar{x}'(t), f^{**}(\bar{x}'(t)) \in H,$$

i.e., for t in I,

$$f^{**}(\tilde{x}'(t)) = \sum_i p_i(t) < \lambda, v_i > +\lambda_3,$$
$$f(\bar{x}'(t)) = f^{**}(\bar{x}'(t)) = \sum_i \chi_{E_i}(t) < \lambda, v_i > +\lambda_3.$$

As a consequence we have:

$$\text{(3)} \qquad \int_I (f^{**}(\tilde{x}'(t)) - f(\bar{x}'(t))\,dt = \sum_i < \lambda, v_i > \int_I p_i(t)\,dt - \mu(E_i) = 0$$

by (1). Furthermore, on $[0,T] \setminus I$, we have $\tilde{x}' = \bar{x}'$ and $f^{**}(\tilde{x}'(t)) = f(\bar{x}'(t))$ hence

$$\text{(4)} \qquad \int_{[0,T]\setminus I} f^{**}(\tilde{x}'(t))\,dt = \int_{[0,T]\setminus I} f(\bar{x}'(t))\,dt.$$

Then (3) and (4) together yield the conclusion.

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