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INFRARED STRUCTURES AND SYMMETRY BREAKING IN SIMPLE GAUGE LIKE QUANTUM FIELD THEORY MODELS.

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TRIESTE

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1. GENERAL INTRODUCTION.

In recent years there has been an increasing experimental and theoretical evidence supporting the idea that gauge quantum field theories (GQFT's) are the most relevant for the description of the physics of elementary particles.

A great deal of experiments has indeed given a rather precise agreement for the standard model of electroweak interactions and a rough agreement for quantum chromodynamics, not to speak of the spectacular results concerning quantum electrodynamics.

The formulation of GQFT's along the lines of the Wightman axioms [SW] is not without problems. In particular it has been shown that the introduction of "charged fields" is in conflict with either the locality (microcausality) or the positivity axiom [STR1][STR2][WIG1].

We may formulate gauge theories in several different ways (called gauges) depending on which parts of Wightman theory it is convenient to retain (for the solution of the particular problem at hand).

For instance the Coulomb gauge of quantum electrodynamics is not local but preserves positivity of the metric, while the contrary happens in the Gupta-Bleuler gauge.

On the other hand, the striking success of perturbative renormalization theory [EPS1][BRS] (also in the context of gauge theories) suggests that it may be useful to keep a relation with the wisdom gathered from conventional perturbative approaches, which are local and covariant [MOR1].

Indeed even the (non-perturbtive) solution of the long-standing infrared problem of quantum electrodynamics [MOR2], which is connected with the construction of charged states, has been possible only by exploiting the local structure associated to the Gupta-Bleuler formulation of QED [GUP][BLE][STR1]; this formulation is actually local and covariant and follows the Wightman axiomatic approach to QFT as close as possible.

Moreover, also the more recent developments of QFT, like the

geometrical understanding of anomalies, the covariant quantization of string theories and conformally invariant models, etc., have been possible in a formulation which insists on the locality property as a basic structure for the study of quantum field theories.
Therefore it could be better to retain locality rather than positivity, at least for technical reasons.
Thus it may be of some interest to investigate the general properties of Indefinite Metric Quantum Field Theories, i.e. QFT's satisfying all the Wightman axioms except positivity.
Models of this type are characterized by a set of Wightman functions which are local, covariant, and satisfy the (weak) spectral condition, [GAR1][SW][MOR1].
For such theories the infrared behaviour is much less constrained than in the standard Wightman theories, since the Fourier transform of the matrix elements of the translation operators need not to be measures, (for instance they may contain dipole singularities of the kind $\delta'(k^2)$ whose properties will be studied in subsequent chapters).
This fact, which from a technical point of view may be considered as an unpleasant difficulty, may offer an insight for the confinement problem: indeed this property allows a violation of the cluster property as required by a linearly rising confining potential. Such type of infrared singularities are also necessary forthe spontaneous breaking of global gauge symmetries which otherwise would not be possible , in a standard positive metric Wightman field theory [FER] (see lso chapter 5).

In indefinite metric quantum field theories, in order to get a physical interpretation, the positivity is replaced with a condition which allows the construction of a Hilbert space associated to the given set of correlation functions [MOR1]; this condition is necessary because the ordinary reconstruction theorem [SW] leads us only to a linear space endowed with a sesquilinear form (the local states) and a Hilbert space can be obtained only by adding further information (<u>Hilbert space structure condition</u> [STR2]).

In particular, very important structure properties like the existence of charged states in quantum electrodynamics [MOR2], the spontaneous breaking of gauge symmetry, the existence of ϑ-vacua, etc., crucially depend on the Hilbert structure chosen.

There is an important subclass of Hilbert space structures among the possible ones: those which associate a maximal set of states to the given set of Wightman functions (i.e. a Hilbert space not properly contained in any larger Hilbert space); it is possible to give a characterization of these maximal structures (minimal topologies): they are Krein-Hilbert structures, i.e. Hilbert spaces endowed with a sesquilinear form $<\cdot,\cdot>$ whose metric (or Gram) operator η has a bounded inverse [MOR1][BOGN].[1]

Finally we mention that the physical interpretation of indefinite metric QFT's is usually obtained by giving a subsidiary condition that selects a positive definite subspace of the reconstructed Hilbert space. We stress again that the study of a given subsidiary condition has to be performed only after having completed the local states in a Hilbert topology. The reason of this fact is that physically interesting states are not local (i.e. cannot be obtained by applying local fields to the vacuum) in many important cases (see for instance the construction of charged states in QED_4 [MOR2]).

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A Krein space is a topological vector space endowed with a Hilbert product $(\cdot,\cdot)$ which characterizes its topological properties, and with a sesquilinear form $<\cdot,\cdot>$ which is jointly continuous with respect to the Hilbert norm. There exists a bounded and selfadjoint operator η (the metric or Gram operator) such that

$$<\Phi,\Psi> = (\Phi,\eta\Psi)$$

$$\eta^2 = 1.$$

The sesquilinear form characterize the geometry of the Krein space and is linked to the "physical content" of the Krein space in QFT context (e.g. the transition amplitudes are computed by means of it).

The main aim of the this thesis is the study of a class of models exibithing infrared singularities that are not compatible with the positivity axiom. The models that we will study are so simple that may be exactly solved; we do not pretend that they can reproduce the full complexity of a realistic theory; however they may be used as a theoretical laboratory to isolate mathematical structures and phenomena that may be relevant also for understanding more realistic cases, as well as for checking on them some ideas that are part of the "folklore" of gauge theories.
In particular we will devote our attention to certain models exhibiting the dipole infrared singularity $\delta'(k^2)$ which is believed to play a crucial role in the confinement picture as well as in the Higgs model. Indeed the distribution $\delta'(k^2)$ is a simple example of infrared singularity of the "confining type" [STR2][MOR1], in the sense that it is not a measure in a neighboorhood of the light cone

$C = \{k^2 = k^{\mu}k_{\mu} = 0\}$.

More pictorially, it corresponds to a linearly growing potential between static "quarks" [NAR]. Furthermore, we will prove in chapter 5 that this singularity must necessarily occurr in the correlation functions of the abelian Higgs model (the proof is a corollary of the arguments contained in [FER][STR3]).

We give now a general survey of the content of the thesis. More detailed accounts of our results may be found in the introductions to the single chapters.

1) We start by discussing the mathematical structures of the dipole field model, a scalar field satisfying the equation

$$\Box\Box\,\phi = 0 \qquad (1.1.1)$$

This model attracted the attention of theoretical physicists at different times for different reasons [FRO][LUK][FER][NAR][ZWA][CAP1]. We apply the ideas that we have briefly exposed before in order to characterize the mathematical properties of this model and to discuss some features that have received little or no attention.

The distribution $\delta'(k^2)$ corresponds exactly to the two-point function of the dipole field, as it may be seen by Fourier transforming the equation charcterizing the two-point function:

$$\Box\Box\, W(x) = 0 \longrightarrow k^2 k^2 \tilde{W}(k) = 0$$

$$\tilde{W}(k) \simeq \vartheta(k_0)\delta'(k^2) \tag{1.1.2}$$

$$W(\xi) = -\frac{1}{16\pi^2}\log(-\xi^2 + i\epsilon\xi_0), \quad \xi = x-y, \tag{1.1.3}$$

(actually the (1.1.2) is a formal definition because of the product of singularities at the point k=0). The n-point functions of this model are constructed as in a free field theory [SW].

Our results are the following:

i) we construct a Krein-Hilbert space K^d on which we represent the dipole field as an operator valued distribution.

ii) This Krein-Hilbert space has non conventional features: the most relevant is that it contains "infinitely delocalized states" (or infrared states) which are consequence of the severe infrared singularities of the model and of the minimality of the topology used to control them. These states are Poincare' invariant and have zero η-norm (i.e. if Ψ is such a state we have that $<\Psi,\Psi>=0$); the Wightman vacuum vector is therefore essentially unique [MOR1].

iii) The existence of these infrared states has a counterpart also in the field algebra of the model. Indeed we will show that the strong closure of this algebra contains Poincare' invariant field operators. Operators of this kind have been introduced in literature as extra degrees of freedom to account for the scale transformation of the dipole field [SAL][FUR]; in our framework they are an intrinsic feature of the model and have a well defined mathematical status. The infrared operators also play a role in the study of the global gauge symmetry.

iv) We discuss the possible physical interpretation of the model.

First of all we show that every η-positive definite subspace H' of the (dense) finite particle subspace of K^d, such to be invariant under the translation group is actually a zero definite subspace. This implies that every strictly positive definite subspace of $K^d_{f.p.}$ cannot be invariant under the translation group.
Thus, the dipole infrared singularity implies a sort of breaking of the space-time translations for the physical subspaces. This can be regarded as a simple prototype of a mechanism of confinement.
We further pursue this idea and construct two positive noncovariant quantizations of the dipole field which have in common a gauge invariant content.

2) The construction of the Krein-Hilbert space K^d allows the correct discussion of the structural properties of the dipole field, but it is also the starting point for the introduction of some nonlinear functions of the field itself.
In particular it permits the construction of its Wick-ordered exponential. This construction is an essential step for discussing a class of models [ZWA][FER] in which the dipole interacts with other fields. The Wick-ordered exponential is (formally) defined by the series

$$:\exp(z\phi):(f) = \sum_{n=0}^{\infty} \frac{z^n}{n!} :\phi^n:(f), \quad z\in\mathbf{C}. \tag{1.1.4}$$

The Wick powers $:\phi^n:(x)$ are defined as in the standard case [GAR1][WIG2] and are tempered distributions whose values are operators on the Krein space K^d. The Wick exponential $:\exp z\phi:(f)$ will be constructed by assuring the strong convergence of the series (1.1.4) on a dense domain of K^d. A sufficient condition for this convergence is obtained by restricting the topological space to which the test function f may belong to a suitable nuclear space; this feature characterizes also the Jaffe fields [JAF2] but there is an important difference: in this latter case the restriction is due to ultraviolet reasons while the construction of the Wick exponential of the dipole is

afflicted also by infrared divergences.
It turns out that the test function spaces which allows this construction are certain "spaces of type S", whose properties are studied in detail in the book of Gelfand and Shilov [GEL2].
It is interesting to notice that the Wightman functions of the Wick exponential $:\exp z\phi:(x)$ are well defined tempered distribution. Therefore the exponential $:\exp z\phi:(f)$ as an operator on K^d is a more singular object than its own correlation functions (as distributions), (this feature is expected to occurr also in realistic GQFT models in local gauges).

3) The construction of the Wick ordered exponential of the dipole field then allows a mathematically sound discussion of certain interacting models. The first model that we analyze has been introduced by Zwanziger and is defined by the following equations of motion:

$$(i\gamma^{\mu}\partial_{\mu} - m)\ \psi(x) = g\gamma^{\mu}A_{\mu}(x)\psi(x), \qquad A_{\mu}(x) = \partial_{\mu}\phi(x), \tag{1.1.5}$$

$$\square^{2}\phi(x) = 0. \tag{1.1.6}$$

This model may be considered as a four dimensional version of the two dimensional Schroer model [SCHR1] (actually in the Schroer model the field equation corresponding to the (1.1.6) is $\square\phi = 0$, but the analogy relies in the fact that the two-point function has the same form $W(\xi) \simeq \log(-\xi^2+i\epsilon\xi_0)$ in both the cases) .
The solution of the previous equations is easily written in terms of "building blocks" fields and is given by

$$\psi(x) = :\exp(-ig\phi):\psi_o(x) \tag{1.1.7}$$

where $\psi_0(x)$ is a free Dirac field:

$$(i\gamma^{\mu}\partial_{\mu} - m)\ \psi_o(x). \tag{1.1.8}$$

A reasonable question to ask is whether the interaction of the fermion field with an infrared singular field leads not only to the infraparticle phenomenon [SCHR1] but also costrains or even forbids the appearence of charged states (<u>charge confinement</u>). The main points of our discussion are the careful exploitation of the Krein structure associated to the model and the use of its intrinsic field algebra i.e. that one generated by the fields $\partial_\mu\phi$ and ψ (this field algebra is strictly contained in the algebra generated by the "building blocks" field ϕ and ψ_o).

In particular we show that all the physical states that satisfy the Zwanziger subsidiary condition must have zero "electric" charge; the physical space is therefore equivalent to the zero charge sector of a free fermion field.

We hve also investigted a different subsidiary condition which gives a non trivial positive quantization of the dipole field.

In this case there are charged states that satisfy this new subsidiary condition but the translation group is well defined only in the zero charge sector. Also in this case we may think to the charged states as confined because of the breaking of time translations, a mechanism already noticed in QED_3 and massless QED_4 [MOR2].

The second interacting model that we study has been introduced by Ferrari [FER]. The equations of motion are those corresponding to the electrodynamics of a charged scalar field, and they are made exactly soluble by the derivative coupling ansatz:

$$\Box A_\mu = ie[\chi^*(\partial_\mu + ieA_\mu)\chi - \chi(\partial_\mu - ieA_\mu)\chi^*], \qquad A_\mu = \partial_\mu\phi \tag{1.1.9}$$

$$\Box\chi = -ie\partial^\mu(A_\mu\chi) - ieA^\mu(\partial_\mu + ieA_\mu)\chi \tag{1.1.10}$$

We will show that:

i) a part from the trivial case $\chi=0$, the charged field χ which solves the above field equations has necessarily a non zero order parameter

$$\langle\Psi_0, \chi(x)\Psi_0\rangle \neq 0. \tag{1.1.11}$$

ii) The Wightman functions of the field algebra generated by the fields χ and A_μ are not invariant under the global gauge transformations

$$\phi(x) \longrightarrow \exp(-i\alpha)\phi(x), \qquad A_{\mu}(x) \longrightarrow A_{\mu}(x) \tag{1.1.12}$$

Nevertheless this symmetry is implementable in the Hilbert-Krein space K in which the model is represented, thanks to the existence of the infrared states and operators.

iii) There is no charged state in the physical space which is solution of the suitable subsidiary condition. In this case we speak of <u>charge screening</u>. In contrast with the confinement case, this phenomenon is associated with the non invariance of the Wightman functions under global gauge transformations [KOG][STR2].

Finally, we discussthe infrared structure associated to the generic α-gauge formulation of two-dimensional QED, known as the Schwinger model [SCH]. The importance and significance of the Schwinger model is well known and we do not insist on it. On the other hand the situation is that a mathemtically rigorous study of this model has been worked out only in the Landau gauge [MOR3],which has very peculiar properties. It is therefore of some interest to develop a rigorous treatment of the generic covariant gauge, because even if the physical interpretation of the model must be gauge invariant the mathematical structures characterizing it may depend on the gauge.

It has been shown by Capri and Ferrari that the infrared behaviour of the Schwinger model in a generic covariant gauge is completely accounted by a two-dimensional quantum field satisfying the equation

$$\Box^2 \phi = 0 \tag{1.1.13}$$

This field satisfies again a dipolar equation; however it is much more infrared singular than the four dimensional dipole because of the low dimensionality of this space-time.

We will construct a family of Krein-Hilbert spaces associated with the Wightman functions of the field (1.1.13). These spaces have a very rich infrared structure wich will be explicitely displayed.

Subsequently, we examine also in this case the problem of the construction of the Wick ordered exponential of the field ϕ.

First of all we study the distributional properties of the Wightman functions associated to the formal series (1.1.13). These Wightman functions are non tempered distributions [WIG3].
The ultraviolet problem is absent and their distributional character is accounted by their growth properties for large values of x. Again the use of the Gelfand and Shilov's "spaces of type S" will be crucial for this characterization and also for the construction of the Wick exponential as an operatorial distribution. This construction requires as expected a further restriction of the test function space on which the Wightman functions are defined.

2. THE DIPOLE FIELD MODEL IN FOUR DIMENSIONS.

2.1 INTRODUCTION

In this chapter we present a rigorous treatment of the free dipole field model, that is an hermitian scalar field satisfying the equation

$$\Box^2 \phi = 0 \qquad \Box = \partial^\mu \partial_\mu \tag{2.1.1}$$

The motivations for such analysis are several. This model attracted the interest of theoretical physicists already in the fifties under the influence of the debated paper of Källen and Pauli on the Lee model [KAL][FRO][LUK]. A revival of interest in the model came with the advent of gauge theories and this because the Fourier transform of the two point function of the dipole field has a $\delta'(k^2)$ singularity, $k^2 = k^\mu k_\mu$. This singularity is the quantum field theory version of the linearly growing potential believed to be a crucial feature of the quark-antiquark interaction [NAR]. Besides, as it will be shown in chapter 5, the breaking of the gauge symmetry in the abelian Higgs model requires, in local gauges, this kind of singularity (see chapter 5) Other classes of models which have a dipole field as a building block are the Zwanziger model [ZWA], the conformally invariant models [SAL][FUR] and the supersymmetric models. From a general point of view the model can be regarded as a simple prototype of a four dimensional quantum field theory exhibiting infrared singularities of the confining type [STR2][MOR1], which are not compatible with the axiom of positivity [SW]. Since the lack of positivity is an unavoidable feature of gauge quantum field theories when treated in local (renormalizable) gauges [STR1][STR2], a rigorous treatment of this model will shead light on those general mathematical structures characterizing non positive QFT's in the Wightman framework.

Finally a further motivation for a revisitation of this model is that the previous treatments are not completely satisfactory. The main open problems are:

1) a clear identification of the Hilbert space of states associated to the Wightman functions of this model;

2) the existence of translationally invariant states other than the vacuum state (i.e. the essential uniqueness of the vacuum);

3) the symmetry breaking problem in the model;

4) the possible identification of the physical space and the physical interpretation of the model;

The point is that, as emphasized by Morchio and Strocchi in [MOR1] and by Wightman in [WIG1] the structural questions concerning an indefinite metric QFT cannot be correctly posed and answered without making reference to a Hilbert space realization of the model, and this is the reason for which the previous treatments are not completely satisfactory.

The starting point of the following discussion of the dipole field model is a set of local and covariant Wightman functions which satisfy the weak spectral condition. The lack of positivity imply that the reconstruction theorem [SW][MOR] yields only a linear space $\mathcal{D}$ endowed with a sesquilinear form $< \, , \, >$. To obtain a Hilbert space it is necessary to introduce in $\mathcal{D}$ a Hilbert topology compatible with the intrinsic indefinite product $< \, , \, >$. There are of course many possible ways to introduce a Hilbert structure in $\mathcal{D}$, but the most interesting cases are given by those structures which are maximal, i.e. not properly contained in any other compatible Hilbert structure.

In this case the metric operator η, which represents the sesquilinear form $< \, . \, >$, has the property that $\eta^2=1$ and the corresponding Hilbert space is a Krein space [BOGN].

In section two we will construct a Hilbert-Krein structure associated with the dipole field; we will show that the Hilbert space K in which the model may be represented, contains vectors (different from the vacuum) which are invariant under the Poincare' group (infrared states); the vacuum is however essentially unique [MOR1] i.e. there is no strictly positive (w.r. to $< \, , \, >$) subspace of K consisting of vectors invariant under translations, whose dimension is greater than one. The infrared states have an interesting counterpart in the strong closure of the local field algebra: indeed this closure includes operators that are invariant under the Poincare' group (infrared operators). This property has been already noticed for the massless scalar two-dimensional field [MOR3], and appears naturally when the confining infrared singularities are controled by a maximal Hilbert structure (Krein structure).

In section three we turn our attention to the symmetries of the model; their treatment has unconventional features due to the indefiniteness of the theory. The equation of motion are invariant under the group $\mathcal{G}$ of local gauge transformations $\phi \longrightarrow \phi + \alpha$, where α is a smooth real solution of the equation $\Box\alpha = 0$. The subgroup of global gauge transformations α = const. is not broken in K (in the sense that there exists a generator for this symmetry [STR4]) and its generator is constructed using the infrared operators. Also the scale transformations are implementable in the space K and in fact the translationally invariant operator which was introduced as a new dynamical variable to account for the scale transformations of ϕ by the uthors of [SAL][FUR] is here an intrinsic element of the theory and is exactly the infinitely delocalized limit of ϕ.

In section four we reconsider the problem of the quantization of the dipole field using the canonical formalism. We recover, in an unambiguous way, a (pseudo)-canonical quantization of the dipole field operator.

Finally, in section five, we discuss the physical interpretation of the model. Even if the model is very simple it may be used to test some ideas that may work also in more interesting cases.
The problem of the physical interpretation of this model has already been investigated. In particular the authors of [CAP1] obtain a

Poincare' invariant Hilbert space with positive metric but they represent the field by a non-hermitian operator and give up the relation between the Wightman functions and the scalar product in the physical space (from this point of view their solution has essentially changed the terms of the problem). On the other side a rigid application of the requests of gauge invariance of the fields and Poincare' invariance of the physical space forces the authors of [ZWA][MIN][BOG1] to conclude that the theory has a trivial content.
Finally a positive quantization of the dipole field is constructed in [NAR] but the translation group is not implementable in the corresponding Hilbert space.
We will examine this problem ab initio and first of all we will show that the severe infrared singularities of the theory imply that the Poincare' group must be broken in every non trivial physical Fock space; as a consequence of this fact we have that any non trivial positive quantizations of the dipole field must necessarily be non covariant under space time translations.
In particular we will construct two positive quantizations of the dipole; in the first one the time translations are not a symmetry of the theory while the space translations are an exact symmetry and the contary happens in the second one. However it is possible to define a vacuum sector (connected with gauge invariance) which is the same for the two quantizations and with the property that the whole translation group is a well defined symmetry on it. Part of the contents of this chapter has already been published in [MOS1][MOS2].

2.2 THE HILBERT-KREIN STRUCTURE ASSOCIATED TO THE DIPOLE FIELD. INFRARED STATES AND OPERATORS.

In this section we construct a Krein-Hilbert space associated to the dipole field and contemporarily give the general framework which is suitable for the discussion of local and covariant quantum field theories.

i) Krein Structure.

A local and covariant quantization of the dipole field is characterized by a set of Wightman functions $\{\mathcal{W}_n\}$ satisfying the following axioms:

0 TEMPEREDNESS

$\mathcal{W}_n$ is a distribution belonging to $\mathcal{S}'(R^{4n})$, the dual of the Schwartz space of the rapidly decreasing functions [GEL2][SCHW][REE1][REE2].

I COVARIANCE

For any Poincare' transformation $\{a,\Lambda\}$ the n-point function are invariant:

$$\mathcal{W}_n(\Lambda x_1+ a,\ldots,\Lambda x_n+ a) = \mathcal{W}_n(x_1,\ldots,x_n). \tag{2.2.1}$$

II LOCALITY

If $x_i - x_{i+1} = \xi_i$ is spacelike then

$$\mathcal{W}_n(x_1,..,x_i,x_{i+1},..,x_n) = \mathcal{W}_n(x_1,..,x_{i+1},x_i,..,x_n). \tag{2.2.2}$$

III WEAK SPECTRAL CONDITION

The Fourier transforms $\widetilde{W}(k_1,\ldots,k_{n-1})$ of the distributions $W_n(\xi_1,..,\xi_{n-1})=\mathcal{W}_n(x_1,..,x_n)$, have support contained in the cones

$$C_i^+ = \{ \ (k_i)^\mu (k_i)_\mu \geq 0 \ , \ (k_i)_0 \geq 0 \ \}. \tag{2.2.3}$$

The Fourier transforms of test functions and distributions are defined by the following formulae:

$$\tilde{f}(k_1,..,k_n)=(2\pi)^{-2n}\int \exp(ik_1x_1+...+ik_nx_n)f(x_1,..,x_n)d^4x_1..d^4x_n, \tag{2.2.4}$$

$$\tilde{T}(f) = T(\tilde{f}), \tag{2.2.5}$$

where kx is the Lorentz invariant product, $f\in\mathcal{S}(R^{4n})$ and $T\in\mathcal{S}'(R^4)$. These axioms and the equation of motion (2.1.1) lead us to the following two-point function:

$$\mathcal{W}_2(x_1,x_2) = W(\xi) = -(16\pi^2)^{-1} \ln(-\xi^2+ i\epsilon\xi_0). \tag{2.2.6}$$

We assume that the one-point function and all the truncated n-point functions vanish.

Following the paradigm of [MOR1] we obtain that these Wightman functions define only an "intrinsic" representation of the dipole field (defined through the reconstruction theorem) that is an operator valued distribution on a linear space $\mathcal{D}$ (the quasi-local states): indeed one consider the Borchers algebra $\mathcal{B}$ [BOR] which is the set of finite sequences $\underline{f}=(f_0,..,f_j,..)$ with $f_0\in C$, $f_j\in\mathcal{S}(R^{4j})$; in $\mathcal{B}$ one defines the following inner product:

$$<\underline{f},\underline{g}> = \sum_n \mathcal{W}_n(\underline{f}^* x \ \underline{g})_n \tag{2.2.7}$$

where $(\underline{f}x\underline{g})_n= \sum_{k+1=n} f_k g_1$, $f^*(x_1,..,x_n)=\bar{f}(x_n,..,x_1)$ and the bar means complex conjugation.

Then the linear set $\mathcal{D}$ is defined to be $\mathcal{B}/\mathcal{J}$ where $\mathcal{J}$ is the Wightman ideal:

$$\mathcal{J} = \{ \ \underline{f}\in\mathcal{B} \ : <\underline{f},\underline{g}>=0 \ , \ \forall\underline{g}\in\mathcal{B} \ \} \tag{2.2.8}$$

(it is an ideal of $\mathcal{B}$ w.r. to the product x). Elements of $\mathcal{D}$ are denoted by the symbol $[\underline{f}]$. By construction the inner product (2.2.7) is non degenerate on $\mathcal{D}$. We may define the field operator on $\mathcal{D}$ as follows :

$$\phi(f)[\underline{g}] = [\underline{f} \times \underline{g}] \tag{2.2.9}$$

where a representative for $\underline{f}$ is $(0,f,0,\ldots)$. It is clear that the vacuum vector Ψ_0 , whose representative is $(1,0,\ldots)$, is cyclic w.r. to $\mathcal{F}$, the polynomial algebra generated by the fields $\phi(f)$. There is a linear representation of the Poincare' group on $\mathcal{D}$, defined by

$$U(a,\Lambda)[\underline{f}] = [\underline{f}_{\{a,\Lambda\}}], \tag{2.2.10}$$

where $f_{\{a,\Lambda\}}(x) = f(\Lambda^{-1}(x-a))$. The covariance of the Wightman functions implies that the operators $U(a,\Lambda)$ preserve the inner product (2.2.7) (we will say that they are η-unitary operators).

In the following we will shortly denote by the same symbol f the test function entering in the field $\phi(f)$ aand the corresponding vector obtained by applying that field to the vacuum.

As briefly discussed in the introduction, the lack of positivity of the Wightman functions does not make (naturally) $\mathcal{D}$ a pre-Hilbert space. However to get a full physical interpretation one has to construct a Hilbert space of states: according to the commonly accepted wisdom they are obtained as solutions of a subsidiary condition given in a Hilbert closure of the local states. Thus, a necessary step is to associate a Hilbert space of states to the Wightman functions and in this way obtain a representation of the fields as operators in a Hilbert space.

The following axiom [MOR1], which replaces the standard positivity axiom [SW], guarantees that such Hilbert space construction is possible:

IV HILBERT SPACE STRUCTURE CONDITION

There exists a set of Hilbert seminorms $\{p_n\}$, p_n defined on $\mathcal{S}(R^{4n})$ and $\mathcal{S}$-continuous , such that

$$|\mathcal{W}_{n+m}(f_n^* \times g_m)| \leq p_n(f_n)p_m(g_m). \tag{2.2.11}$$

Without loss of generality we may assume in addition that these seminorms vanish on $\mathcal{J}$. Using standard methods we may now complete $\mathcal{D}$ w.r.t. to the topology induced by the seminorms $\{p_n\}$ and get an Hilbert space H. Furthermore, we can extend the inner product (2.2.7) to the whole H and there exists a bounded and self-adjoint operator η such that [REE1]

$$<\Psi_1,\Psi_2> = (\Psi_1,\eta\Psi_2) \qquad \forall\ \Psi_1,\Psi_2 \in H, \tag{2.2.12}$$

where (,) is the Hilbert scalar product in H, defined by the seminorms $\{p_n\}$. It is worth to point out again that different choices of the seminorms give rise to different Hilbert spaces and whereas in the standard case the Wightman functions uniquely fix the closure of $\mathcal{D}$, in the indefinite metric case different closures are available corresponding to different topologies.

Among the possible Hilbert majorant topologies a distinguished role is played by the so called minimal topolgies; they are characterized as those which associate to the Wightman functions a maximal set of states. Technically, they may be characterized by the property that the metric operator η has a bounded inverse, or equivalently by the property $\eta^2 = 1$ *(Krein topology and Krein structure)*.

We now introduce a Krein structure for the dipole field model.

First of all we notice that the factorization of the n-point functions (free field theory) implies that a possible set of seminorms may be constructed using a single seminorm p defined on $\mathcal{S}(R^4)$. We denote the inner product induced in $\mathcal{S}(R^4)$ by the definition (2.7) by the same symbol < , >. It is possible to choose a $\chi \in \mathcal{S}(R^4)$ such that $\tilde{\chi}(0) = 1$ and $<\chi,\chi> = 0$ [ZWA] . Then one has that [GEL1]

$$<f,g> = \tfrac{1}{2}\pi\int\{(1-D)[\overline{\tilde{f}_0(k)}\tilde{g}_0(k) + \overline{\tilde{f}(0)}\overline{\tilde{\chi}(k)}\tilde{g}_0(k) + \tilde{g}(0)\overline{\tilde{f}_0(k)}\tilde{\chi}(k)]\}|_{C_+}\omega^{-3}d^3k \tag{2.2.13}$$

with $f_0(k) = f(k) - \tilde{f}(0)\chi(k)$, $D\tilde{f}(k) = k_0\partial/\partial k_0\tilde{f}(k)$ and $\omega^2 = k_1^2+k_2^2+k_3^2$.

We now define a Hilbert product in $\mathcal{S}(R^4)$ as follows:

$$(f,g) = \tfrac{1}{2}\pi\int[\overline{F}_1(k)G_1(k)+\overline{F}_2(k)G_2(k)]\omega^{-3}d^3k + \langle f,\chi\rangle\langle\chi,g\rangle + \overline{\tilde{f}(0)}\tilde{g}(0) \tag{2.2.14}$$

with $F_1(k) = [(1-D)\tilde{f}_0(k)]|_{C_+}$ and $F_2(k) = [D\tilde{f}_0(k)]|_{C_+}$. (2.2.15)

It is easy to see that $|\langle f,g\rangle| \leq \|f\|\|g\|$, with $\|f\|^2=(f,f)$. Now we have that the vectors

$$\Psi_{f_1\ldots f_n} = \frac{1}{n!}:\phi(f_1)\ldots.\phi(f_n):\Psi_0 \tag{2.2.16}$$

generate $\mathcal{D}$. The symbol : : denotes the Wick-ordered product defined in terms of Wightman functions [GAR1]. It follows that

$$\langle\Psi^n_{f_1,\ldots,f_n},\Psi^m_{g_1,\ldots,g_m}\rangle = \frac{1}{n!}\,\delta_{n,m}\sum_\pi \langle f_1,g_{i_1}\rangle\ldots\langle f_n,g_{i_n}\rangle, \tag{2.2.17}$$

where $\sum_\pi$ denotes the sum over all the permutations.We may now define a Hilbert product in $\mathcal{D}$ simply by

$$(\Psi^n_{f_1,\ldots,f_n},\Psi^m_{g_1,\ldots,g_m}) = (n!)^{-1}\delta_{n,m}\sum_\pi(f_1,g_{i_1})\ldots(f_n,g_{i_n}) \tag{2.2.18}$$

Denoting by K the Hilbert completion of $\mathcal{D}$ w.r. to the topology induced by the (2.18), it follows that

$$K = \oplus_n K^{(n)} \quad , \quad K^{(n)} = \otimes^n_s K^{(1)} \tag{2.2.19}$$

where $K^{(1)}$ is the Hilbert completion of $\mathcal{S}(R^4)$ and $\otimes_s$ denotes the symmetric tensor product. Therefore the study of $K^{(1)}$ completely fixes the Hilbert space of the theory. The main result of this section consists in the proof that K is a Krein space and therefore the so obtained set of states is maximal.

To prove this we need to study in advance the space $H^{(1)}$, which is the completion of the space

$$\mathcal{S}_0(R^4) = \{\ f\in\mathcal{S}(R^4)\ :\ \tilde{f}(0) = 0\ \} \tag{2.2.20}$$

w.r. to the Hilbert topology induced by the scalar product

$$[f,g] = \frac{1}{2}\pi\int[\overline{F}_1(k)G_1(k) + \overline{F}_2(k)G_2(k)]\omega^{-3}d^3k \qquad (2.2.21)$$

Lemma 2.2.1: it is possible to extend the product (2.13) to the whole $H^{(1)}$ and there exists a bounded and self-adjoint operator η_0 such that

$$\langle f,g\rangle = [f,\eta_0 g] \qquad \forall\ f,g \in H^{(1)} \qquad (2.2.22)$$

and besides $(\eta_0)^2 = 1$, (2.2.23)

i.e. $H^{(1)}$ is a Krein space.

Proof: the first part of the lemma follows from standard theorems of functional analysis [REE1]. We need only to show the (2.2.23). To this end we consider the space $\mathcal{P}_0(R^4)\otimes C^2$ endowed with the products

$$\{F,G\}_{\pm} = \frac{1}{2}\pi\int[\overline{\tilde{f}}_1(k)\tilde{g}_1(k) \pm \overline{\tilde{f}}_2(k)\tilde{g}_2(k)]\,|_{C_+}\,\omega^{-3}d^3k \qquad (2.2.24)$$

and a map $U : \mathcal{P}_0(R^4) \longrightarrow \mathcal{P}_0(R^4)\otimes C^2$ defined by

$$U\tilde{f} = \begin{bmatrix}(1-D)\tilde{f}\\ D\tilde{f}\end{bmatrix} \qquad (2.2.25)$$

It is obvious that we may extend the operator U to an operator $\overline{U}$defined on $H^{(1)}$ with values in the Hilbert completion of ran(U) w.r. to the topology induced by the product $\{\ ,\ \}_+$, which we denote by $\overline{R(U)}$.The operator $\overline{U}$ has the following properties:

$$\{\overline{U}f,\overline{U}g\}_+ = [f,g] \quad , \quad \{\overline{U}f,\overline{U}g\}_- = \langle f,g\rangle \qquad (2.2.26)$$

$\forall\ f,g \in H^{(1)}$. Besides one has that

$$\{\overline{U}f,\overline{U}g\}_- = \{\overline{U}f,\sigma_3\overline{U}g\}_+ \qquad (2.2.27)$$

with $(\sigma_3)_{i,j} = \delta_{i,j}\ (-1)^{i+1}$. Therefore it follows that

$$\{\overline{U}f,\overline{U}\eta_0 g\}_+ = [f,\eta_0 g] = \langle f,g\rangle = \{\overline{U}f,\overline{U}g\}_- = \{\overline{U}f,\sigma_3\overline{U}g\}_+ \qquad (2.2.28)$$

$\forall\ f,g \in H^{(1)}$. It is now possible to show that σ_3 maps $\overline{R(U)}$ into itself; this implies that we may apply eq. (2.28) twice and obtain that

$$[f,(\eta_0)^2 g] = \{\bar{U}f,(\sigma_3)^2\bar{U}g\}_+ = \{\bar{U}f,\bar{U}g\}_+ = [f.g] \qquad (2.2.29)$$

This relation is valid $\forall\ f,g \in H^{(1)}$ and this implies the (2.2.23).##

The fact that D is a differential operator non tangential to the future cone C_+ implies that $\overline{R(U)}$ is isomorphic to the Hilbert space $L^2(C_+ -\{0\},\ \omega^{-3}d^3k)\otimes C^2$, which we briefly denote $L^2\otimes C^2$. The use of the operator $\bar{U}$ makes it possible the proof of the following

<u>Corollary 2.2.2</u>: $H^{(1)}$ is isomorphic to $L^2\otimes C^2$, which is the space of two complex component functions, defined on $\{C_+ -\{0\}\}$, square integrable w.r. to the measure $\omega^{-3}d^3k$. ##

We are now in position to state and prove the main theorem of this section:

<u>Theorem 2.2.3</u>: it is possible to extend the inner product (2.2.13) to the whole $K^{(1)}$ and there exists a bounded and self-adjioint operator $\eta^{(1)}$ such that $\forall\ f,g \in K^{(1)}$ it happens that

$$\langle f,g\rangle = (f,\eta^{(1)}g), \qquad (2.2.30)$$

$$(\eta^{(1)})^2 = 1. \qquad (2.2.31)$$

<u>Proof</u>: as before, we need only to show eq. (2.2.31). Let us define on $K^{(1)}$ the following functional:

$$X(f) = \langle\chi,f\rangle. \qquad (2.2.32)$$

This functional is continuous because $|X(f)| \le \|f\|$; actually it is possible to show that its norm is exactly one: indeed if we take the sequence of elements of $\mathcal{S}_0(R^4)$ defined by

$$\tilde{f}^{\chi}_n(k) = \vartheta_n(\omega)\ \tilde{\chi}(k), \qquad (2.2.33)$$

with $\vartheta(t)$ an infinitely differentiable non decreasing real function, which is zero for $t\le 0$ and is one for $t\ge 1$, and $\vartheta_n(t) = \vartheta(nt)$, we find that

$$|X(f_n)|/\|f_n\| \underset{n\to\infty}{\longrightarrow} 1. \tag{2.2.34}$$

The Riesz lemma implies that there exists a vector $v^+ \in K^{(1)}$ such that $(v^+,v^+) = 1$ and $\forall\ f \in K^{(1)}$

$$<\chi,f> = (v^+,f) \tag{2.2.35}$$

It is not difficult to show that the sequence

$$v^+_n = (<\chi,f^\chi_n>)^{-1}\ f^\chi_n \tag{2.2.36}$$

converges to v^+ in $K^{(1)}$. We may think to $K^{(1)}$ as decomposed into orthogonal subspaces:

$$K^{(1)} = K^{(1)}_0 \oplus V^+ \oplus X \tag{2.2.37}$$

where V^+ and X are the one dimensional subspaces generated by v^+ and χ and

$$K^{(1)}_0 = \{\ f \in K^{(1)}:\ (\chi,f) = (v^+,f) = 0\ \}. \tag{2.2.38}$$

It follows from the formulae (2.2.13,14,38) and the corollary 2.2 that $K^{(1)}_0$ is isomorphic to $L^2 \otimes C^2$.

Now, we want to compute explicitely the action of the metric operator $\eta^{(1)}$ on the subspace $V^+ \oplus X$. We have that for any $f \in K^{(1)}$

$$(v^+,\ f) = <\chi,f> = (\eta^{(1)}\chi\ ,\ f), \tag{2.2.39}$$

and this implies $\eta^{(1)}\chi = v^+$. Let now $f \in \mathcal{S}(R^4)$. We have that

$$(\chi,f) = \tilde{f}(0), \tag{2.2.40}$$

$$<v^+,f> = \lim <v^+_n,f> = \tilde{f}(0) = (\eta^{(1)}v^+,f) \tag{2.2.41}$$

The density of $\mathcal{S}(R^4)$ finally implies that $\eta^{(1)}v^+ = \chi$. We may write

$$f = Pf + (v^+,f)v^+ + (\chi,f)\chi, \qquad \forall\ f \in K^{(1)} \tag{2.2.42}$$

where P is the projector on $K^{(1)}_0$. From eq. (2.42) it follows that

$$<f,g> = (f, P\eta^{(1)}Pg) + (f, v^+)(\chi, g) + (f, \chi)(v^+, g) \tag{2.2.43}$$

Theorem 2.2.1 and the decomposition (2.2.37) imply that $P\eta^{(1)}P = P\eta_0^{(1)}P$.

Defining by $P_\pm$ the projectors on the vectors $2^{-1/2}(v^+ + \chi)$ we obtain that

$$<f,g> = (f, \eta^{(1)}g) = (f, \{P\eta_0^{(1)}P + P_+ - P_-\}g) \tag{2.2.44}$$

and therefore

$$\eta^{(1)} = P\eta_0^{(1)}P + P_+ - P_- . \tag{2.2.45}$$

Eq. (2.2.45) finally implies that $(\eta^{(1)})^2 = 1$. ##

<u>Corollary 2.2.4</u>: $K^{(1)}$ is isomorphic to $(L^2 \otimes C^2) \oplus V^+ \oplus X$

<u>Corollary 2.2.5</u>: K is a Krein space.

ii) <u>Infinitely delocalized states and operators.</u>

The implementers of the Poincare' group are η-unitary but unbounded, and therefore they are defined only on a dense set.

There is an important thing to notice: the sequence (2.2.36) converges pointwise to zero; this means that the vector v^+ is not a function and describe a Poincare' invariant state (infinitely delocalized state or infraredstate); indeed it is easy to prove that v^+ belongs to the domain of the operators $U(a,\Lambda)$; then, $\forall\ f \in \mathcal{S}(R^4)$, we have:

$$<\{U(a,\Lambda)v^+ - v^+\}, f> = <v^+, f_{\{a,\Lambda\}}> - <v^+, f> = \tilde{f}(0) - \tilde{f}(0) = 0. \tag{2.2.46}$$

The density of $\mathcal{S}(R^4)$ and the non degeneracy of the inner product (2.2.13) implies that

$$U(a,\Lambda)v^+ = v^+. \tag{2.2.47}$$

However the vector v^+ has zero η-norm. This implies that the vacuum vector is essentially unique [MOR1], i.e.there is no positive subspace of translationally invariant vectors, whose dimension is greater than one. We may represent the field operator in K by the following formula:

$$(\phi(f)\ \Psi\ )^{(n)} = (n+1)^{-1/2} <\bar{f},\Psi^{(n+1)}>(k_1,..k_n) +$$

$$+ (n)^{-1/2} \sum_{j=1}^{n} \tilde{f}(k_j)\ \Psi^{(n-1)}(k_1,.,k_j,.,k_n) \tag{2.2.48}$$

where the f appearing at the R.H.S. has to be regarded as an element of $K^{(1)}$, $\Psi^{(m+1)} = \Psi^{(m+1)}(k,k_1,...,k_n)$ and k is the "integrated" variable. This formula gives also explicit expressions for the positive and negative frequencies parts of the field operator. Note that with our conventions on the Fourier transforms of distributions, positive frequencies correspond to "creation" operators. Also the covariance of the field follows easily by the (2.2.48). When the smearing function f is real, the operator $\phi(f)$ is essentialy self-adjoint w.r. to the indefinite product (η-self-adjoint). The following estimate may be proven using the representation (2.2.48), exactly as in the ordinary case [REE2] :

$$\|\phi(f)\Psi^{(m)}\| \leq (m+1)^{1/2}(\|f^+\| + \|f^-\|)\|\Psi^{(m)}\| \tag{2.2.49}$$

with $\tilde{f}^{\pm}(k) = \tilde{f}(\pm k)$ We may now define another seminorm on $\mathcal{S}(R^4)$ by

$$q(f)^2 = \|f^+\|^2 + \|f^-\|^2 \tag{2.2.50}$$

The completion of $\mathcal{S}(R^4)$ w.r. to the topology induced by the seminorm q gives, exactly as before, the space

$$(L^2(C_+-\{0\},\omega^{-3}d^3k)\otimes C^2) \oplus (L^2(C_--\{0\},\omega^{-3}d^3k)\otimes C^2) \oplus V^+ \oplus V^- \oplus X \tag{2.2.51}$$

where V^- is the one dimensional subspace generated by the vector $v^- = \lim v_n^-$. By the (2.49) we have that the sequence

$$\{\Psi_{f_{n_1},...,f_{n_k}}\} = \{:\phi(f_{n_1},...,f_{n_k}): \Psi_0\} \tag{2.2.52}$$

converges in K if each of the $\{f_n\}$ converges in the space (2.2.51). It follows that the local field algebra $\mathcal{F}$ has an extension $\mathcal{F}_{ext}$ which

contains the fields $\phi(f)$, with f belonging to the space (2.2.51).
In particular $\mathcal{F}_{ext}$ contains the infrared field operators $\phi(v^+)$ and $\phi(v^-)$, which are invariant under the Poincare' group.
These operators correspond to the operators introduced in [NAK] in the context of the massless scalar two-dimensional field model, as an ill defined integral.
Such integral does not actually exist and our construction provide an alternative mathematically rigorous definition.
We stress that the existence of the infrared states and operators is a consequence of the fact that K is a maximal space associated with the given Wightman functions; this feature is not shared by non Krein realizations of the theory.
As we will see in the next section such infrared operators are exactly those needed to obtain a representation of the scale transformations [SAL][FUR]. It is a virtue of the Krein formulation to give them a sound matemathical status, so that they enter naturally in the theory and need not to be introduced ad hoc from outside, as additional degrees of freedom.

2.3. SYMMETRIES.

i) Introduction.

The construction of the Hilbert-Krein space of the theory allows us a correct discussion of the questions concerning the symmetries of the model. We have already discussed the Poincare' symmetry and we will turn again our attention on it when we will treat the physical interpretation of the model. In this section we will deal with the gauge and the scale symmetry but we begin by recalling briefly some well known facts about symmetries in quantum mechanics, in order to underline the differences that arise in the generalized context of indefinite metric quantum theory.

The following brief exposition is based on [STR4] to which we refer for more details.

An exact (or unbroken) symmetry in quantum mechanics is a transformation of the rays of a Hilbert space H ,

$$T: \ \Psi \longrightarrow \Psi', \tag{2.3.1}$$

which leaves invariant the transition probabilities:

$$|\langle\Phi,\Psi\rangle|^2 = |\langle\Phi',\Psi'\rangle|^2 \ . \tag{2.3.2}$$

The well known and fundamental theorem of Wigner asserts that any such transformation can be described by an operator U in H which is either unitary or antiunitary [WIGN]. This implies that the trnsformation (2.3.1) induces a corresponding transformation of the canonical variables

$$\alpha_T: \ A \longrightarrow A' = UAU^{-1} \tag{2.3.3}$$

which preserves the algebraic relations, including the adjoint operation and the commutation relations, i.e. it is a $*$-automorphism of the canonical algebra $\mathcal{A}$. Since the equations of motion in the Heisenberg picture of quantum mechanics are algebraic relations between elements of the canonical algebra $\mathcal{A}$, it follows that they are invariant under a $*$-automorphism of $\mathcal{A}$ itself.

It is interesting (and extremely important) to ask the opposite question, i.e. whether given an arbitrary $*$-automorphism α of $\mathcal{A}$ it can be represented in the form (2.3.3).

For quantum systems with a finite number of degrees of freedom the answer is affirmative, i.e. any symmetry of the equations of motion (i.e. a $*$-automorphism of the canonical algebra $\mathcal{A}$) defines an exact symmetry of the theory.

As regards infinite quantum systems the situation is drastically different: indeed the existence of inequivalent representations of the canonical algebra implies that not every symmetry of the equations of motion gives rise to a transformation law of the states, which preserves the transition probabilities. If this is the case, one says that the <u>symmetry is spontaneously broken</u>.

A necessary and sufficient condition for which a $*$-automorphism α of the canonical algebra, which commutes with the time translations, defines an exact symmetry in a representation with unique cyclic ground state Ψ_0 , is that all correlation functions are invariant, i.e.

$$\langle\Psi_0,\alpha(A)\Psi_0\rangle = \langle\Psi_0,A\Psi_0\rangle \tag{2.3.4}$$

for any $A\in\mathcal{A}$; this equivalence has lead several authors to assume the opposite of the (2.3.4) as the very definition of a spontaneous breaking of a symmetry, i.e. a symmetry α ($*$-automorphism) of the canonical algebra $\mathcal{A}$ is said spontaneously broken if there is at least an element $A\in\mathcal{A}$ such that

$$\langle\Psi_0,\alpha(A)\Psi_0\rangle \neq \langle\Psi_0,A\Psi_0\rangle \tag{2.3.5}$$

where Ψ_0 is the unique cyclic ground state of the chosen representation of the algebra.

The situation changes again if now enlarge our framework to include quantum systems having an indefinite metric representation space. Indeed in this case the symmetry may be implementable by a η-unitary operator which does not leave the cyclic ground state invariant (the mechanism is that such state may be invariant up to vectors having zero η-norm): this means that a symmetry α may not leave invariant the correlation functions of the canonical algebra (and therefore be "spontneously broken" according to the criterium (2.3.5) and nevertheless it may be implemented by η-unitary operator.

We immediately clarify these facts by discussing the global gauge automorphism of the polynomial algebra $\mathcal{F}$ generated by the dipole field.

ii) Gauge symmetry.

An important symmetry (also for subsequent applications, see chapters 4 and 5) of this model is the gauge symmetry; in fact one sees immediately that the equation of motion (2.1.1) is invariant under the following gauge transformations of the second kind (local gauge transformations)

$$\phi(x) \longrightarrow \phi(x) + \alpha(x) \tag{2.3.6}$$

where $\alpha(x)$ is a real smooth solution of the equation $\Box\alpha = 0$.

We consider the particular solutions of this equation given by $\alpha(x)=\lambda$. These solutions identify the gauge transformations of the first kind (global tranformations); they correspond to the following group of automorphisms of the field algebra:

$$\gamma^{\lambda}:\ \phi(x) \longrightarrow \phi(x) + \lambda. \tag{2.3.7}$$

This group of automorphisms is generated by the conseved current $\partial_\mu\Box\phi(x)$, in the following sense: consider the local charge Q^d_R , given by

$$Q_R = \int \partial_0\Box\ \phi(x)\ f_R(x)\alpha_d(x_0)\ d^4x \tag{2.3.8}$$

where $f(x)$ is an infinitely differentiable function such that $f(x)=1$ if $|x| \leq 1$ and $f(x)=0$ if $|x| \geq 2$, and $\alpha_d(x_0)$ is an infinitely differentiable function of compact support such that

$$\int \alpha_d(x_0)\ dx_0 = 1 \quad , \quad \lim_{d\to 0} \alpha_d(x_0) = \delta(x_0) \tag{2.3.9}$$

Then one has that

$$\frac{d}{d\lambda}\gamma^{\lambda}(\phi(f))|_{\lambda=0} = i \lim_{R\to\infty} [Q_R,\phi(f)] = (2\pi)^2 \tilde{f}(0) = \int f(x)\, d^4x \qquad (2.3.10)$$

Theorem 2.3.1: the automorphism γ^{λ} is implementated in the Krein space K by the operator $\Gamma^{\lambda} = \exp 2\pi^2 i\lambda Q$, with $Q = (2\pi^2)^{-1}\text{w-lim}\, Q_R$.

Proof: define

$$Q = i\,[\phi(v^+) - \phi(v^-)] \qquad (2.3.11)$$

First of all we show that Q_R converges to $(2\pi^2)Q$ as a bilinear form on $\eta\mathcal{D}\times\mathcal{D}$. Indeed we have that

$$\langle\Psi_0,Q_R\phi(g)\Psi_0\rangle = -\frac{1}{2}\pi \int [\,(1-k_0\frac{\partial}{\partial k_0})(-ik_0k^2\, \tilde{f}_R(-k)\tilde{\alpha}_d(-k_0)\tilde{g}(k)]\,|_{C_+}\, \omega^{-3} d^3k =$$

$$-i\pi \int \tilde{f}_R(-k)\tilde{\alpha}_d(-\omega)\; \tilde{g}(\omega,k)\; \omega^{-3}d^3k\ . \qquad (2.3.12)$$

Since $\tilde{f}_R(-k)$ converges to $(2\pi)^{3/2}\delta(k)$ in the sense of distributions as R goes to infinity, and $\tilde{\alpha}_d(0) = (2\pi)^{-1/2}$, we obtain that

$$\lim_{R\to\infty} \langle\Psi_0,Q_R\phi(g)\Psi_0\rangle = 2\pi^2\langle\Psi_0,Q\;\phi(g)\Psi_0\rangle = -2\pi^2 i\;\tilde{g}(0) \qquad (2.3.13)$$

Consequently, the factorization of the Wightman functions implies that

$$\langle\Psi_0,\; \mathcal{P}_1(\phi)\; Q_R\mathcal{P}_2(\phi)\;\Psi_0\rangle = 2\pi^2\;\langle\Psi_0,\; \mathcal{P}_1(\phi)\; Q\; \mathcal{P}_2(\phi)\;\Psi_0\rangle \qquad (2.3.14)$$

and therefore Q_R converges to $2\pi^2 Q$ in the sense of sesquilinear forms on the domain $\eta\mathcal{D}\times\mathcal{D}$.

A direct calculation shows that for any $\Psi\in\mathcal{D}$ the following uniform majorization holds:

$$\|\; \partial_0\Box\;\; \phi\;(f_R\alpha_d)\Psi\;\| \le \text{const.} \qquad (2.3.15)$$

This implies that Q_R is weakly convergent to $2\pi^2 Q$. Now we have that $\mathcal{D}$ is a set of analytic vectors for Q and therefore we may exponentiate it and obtain

$$\Gamma^{\lambda} = \exp\, 2\pi^2 i\; Q \qquad (2.3.16)$$

Thus Γ^{λ} actually implements the symmetry γ^{λ} .##

By the above results, the global gauge symmetry is unbroken in the Krein space (i.e. there exists a one-parameter group of η-unitary operators implementing the global gauge transformations). We remark that this symmetry would have been broken if we had used a non Krein topology (see the analogous mechanism in [MOR3]).
As we have anticipated, here emerges a feature that is not shared by conventional theories: the symmetry is implementable but the vacuum is not invariant under the action of the implementers Γ^{λ} . It is however essentialy invariant: indeed the extra term is a translationally invariant null vector.
Another difference w.r. to the standard case is that the sequence of local charges Q_R converges to $2\pi^2 Q$ in the weak graph limit sense [REE1], a kind of convergence which is forbidden in the standard (positive metric) case [SCHR2].

ii) <u>Scale transformations.</u>

Another very interesting symmetry of the theory is given by the scale transformations $x \longrightarrow sx$. These transformations do not leave invariant the two-point function: it get shifted by the constant $-(8\pi^2)^{-1}\log s$. Let us define the following automorphism of $\mathcal{F}_{ext}$:

$$\alpha_s : \phi(f) \longrightarrow \phi(f_s) + \frac{1}{4}\pi^2 \log s \; \tilde{f}(0)\; \phi(v) \tag{2.3.17}$$

$$\text{with } f_s(x) = s^{-4} f(x/s) \text{ and } \phi(v) = \phi(v^+) + \phi(v^-) \tag{2.3.18}$$

<u>Theorem</u> <u>2.3.2</u>: The automorphism α_s is implementated in the Krein space K by the operators U(s) which are η-unitary and leave the vacuum invariant.

<u>Proof</u>: it is easy to verify that

$$<\Psi_0, \alpha_s(\phi(f))\alpha_s(\phi(g))\Psi_0> = <\Psi_0, \phi(f)\phi(g)\Psi_0> \tag{2.3.19}$$

The action of U(s) is determined by its definition on the one-particle space:

$$U(s)\Psi_f = \Psi_{f_s} + \frac{1}{4}\pi^2 \tilde{f}(0) \log s \ v^+ \tag{2.3.20}$$

and because of the invariance of the Wightman functions one has that

$$U(s)\Psi_0 = \Psi_0 . \ \#\# \tag{2.3.21}$$

Thus, the translationally invariant operator that has been introduced in literature as a new dynamical variable to account for the scale transformations of ϕ in the context of conformally invariant models [SAL][FUR], is exactly $\phi(v)$, the infinitely delocalized limit of ϕ; it is therefore an intrinsic content of the model in the Krein space approach.

2.4. Canonical Quantization.

In this section we want to develop a canonical formalism for the dipole field. To this end we consider again the operator valued distribution ϕ, satisfying the $\Box^2\phi = 0$ and the commutation rules

$$[\phi(x),\phi(y)] = -\frac{i}{8\pi}\,\epsilon(\xi_0)\vartheta(\xi^2), \tag{2.4.1}$$

where $\vartheta(t)$ is the step function and $\epsilon(t) = \vartheta(t) - \vartheta(-t)$; the commutator (2.4.1) follows immediately by the two-point function (2.2.6). General properties of hyperbolic equations [GAR2] allow us to extend the class of test functions which may be used to smear the field ϕ, to distributions of the form $f_t(x) = \delta(x_0 - t)f(\mathbf{x})$ with $f \in \mathcal{S}(R^3)$.
Thank to this property we may introduce the fixed time fields $\phi(t,f) = \int\phi(t,\mathbf{x})f(\mathbf{x})d^3\mathbf{x}$. It follows that the only non zero fixed time commutators are the following ones:

$$[\phi(t,\mathbf{x}),\partial_0\Box\phi(t,\mathbf{y})] = [\Box\phi(t,\mathbf{x}),\partial_0\phi(t,\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y}). \tag{2.4.2}$$

The commutators (2.4.2) are exactly those that one imposes in the canonical quantization of a system of two fields with lagrangian [NAR]

$$\mathcal{L} = \partial^\mu\phi\,\partial_\mu\Lambda + \frac{1}{2}\Lambda^2. \tag{2.4.3}$$

Actually Λ is not independent on ϕ: indeed the equations of motion that follow from (4.3) are

$$\Box\phi = \Lambda \quad , \quad \Box\Lambda = 0 \tag{2.4.4}$$

(which together imply $\Box^2\phi = 0$).
To avoid ambiguities we restrict for the moment test functions to those of $\mathcal{S}_0$. It is then possible to introduce in a standard way [SW] the splitting of the field ϕ into positive and negative frequency parts $\phi = \phi^+ + \phi^-$. We consider then the star algebra $\mathcal{A}$ generated by the fields $\phi^{(\pm)}$, $\partial_0\phi^{(\pm)}$, $\Lambda^{(\pm)}$, $\partial_0\Lambda^{(\pm)}$, taken at t=0 (i.e. smeared with test functions of the form $\delta(x_0)f(\mathbf{x})$). Clearly not all these fields are algebraically independent: indeed they are linked by the four relations given by conjugation and by the two commutators (2.4.3). Taking these relations into account we are led to the following expressions (see

[WIG2] for analogous formulae regarding the canonical free field):

$$\psi_1(0,x) = (-\Delta)^{3/4}\ \phi(0,x) + (-\Delta)^{1/4}\partial_0\phi(0,x) + \frac{i}{2}(-\Delta)^{-3/4}\partial_0\Lambda(0,x), \quad (2.4.5)$$

$$\psi_2(0,x) = (-\Delta)^{3/4}\ \phi(0,x) + i(-\Delta)^{1/4}\partial_0\phi(0,x) - \frac{1}{2}(-\Delta)^{-1/4}\Lambda(0,x), \quad (2.4.6)$$

which, by construction, contain only negative frequencies (consequently, their conjugated contain only positive frequencies); The operator Δ is the Laplacian: $\Delta = -\partial_1^2 - \partial_2^2 - \partial_3^2$.

We obtain two pairs of creation and annihilation operators simply by taking the Fourier transform:

$$a_i(k) = (2\pi)^{-3/2} \int \exp(ikx)\ \psi_i(0,x)\ d^3x\ , \quad (2.4.7)$$

$$a_i^\dagger(k) = (2\pi)^{-3/2} \int \exp(-ikx)\ \psi_i^\dagger(0,x)\ d^3x\ . \quad (2.4.8)$$

These operators satisfy the following pseudo-canonical commutation relations:

$$[a_1(q), a_1^\dagger(k)] = \delta^3(q-k), \quad (2.4.9)$$

$$[a_2(q), a_2^\dagger(k)] = -\delta^3(q-k). \quad (2.4.10)$$

The remaining commutators are zero. We remark that the minus sign at the R.H.S.of (2.4.10) is an unavoidable consequence of the relation between the spectral condition and the splitting of the field operator into positive and negative frequencies.

One may take a combination of these operators to obtain the C.C.R. in the usual form (without the minus sign) but the so obtained operators are no longer related to the positive and negative frequency parts of the field ϕ (we will comment again on this point). The field equation determines the following time evolution of the a's:

$$\tau_t(a_1(k)) = \exp(-i\omega t)\ [(1+i\omega t)\ a_1(k) - i\omega t\ a_2(k)] \quad (2.4.13)$$

$$\tau_t(a_2(k)) = \exp(-i\omega t)\ [i\omega t\ a_1(k) + (1-i\omega t)\ a_2(k)] \qquad (2.4.14)$$

Therefore the time evolution mixes a_1 with a_2; however it is not a (proper) Bogoliubov transformation because it does not mixes creators and annihilators. Using the previous formulae one may obtain the following representation of the field:

$$\phi'(t,x)= (2\pi)^{-3/2}\int \exp(i\omega t-ikx)\ [(1-i\omega t)a_1^\dagger(k) + i\omega t\ a_2^\dagger(k)]\ (2\omega^{3/2})^{-1}d^3k$$

$$+ (2\pi)^{-3/2}\int \exp(-i\omega t+ikx)\ [(1+i\omega t)\ a_1(k) - i\omega t\ a_2(k)](2\omega^{3/2})^{-1}d^3k. \qquad (2.4.15)$$

The commutation relations (2.4.9) and (2.4.10) also imply that the Fock representation for the a's is a space endowed with an indefinite metric. The Fock vacuum is defined by the following conditions:

$$a_1(k)\Psi_0 = a_2(k)\Psi_0 = 0\ . \qquad (2.4.16)$$

The usual methods of Lagrangian field theory lead us to the following four-momentum:

$$H = \int \omega\ [a_2^\dagger(k)(a_1(k) - a_2(k)) + (a_1^\dagger(k) - a_2^\dagger(k))a_2(k)]\ d^3k \qquad (2.4.17)$$

$$P^i = \int k^i[a_1^\dagger(k)a_1(k) - a_2^\dagger(k)a_2(k)]\ d^3k\ . \qquad (2.4.18)$$

The Hamiltonian given in the (4.14) does indeed generate the time evolution τ_t; in fact one has that

$$\frac{d}{dt}\tau_t(a_i(k))|_{t=0} = i\ [H,a_i(k)] \qquad (2.4.19)$$

Therefore the time evolution is implementable in the (indefinite metric) Fock space of the model.
Now we return back and use these results to give an explicit expression of the dipole field as a distribution on $\mathscr{S}(R^4)$ (rather than on $\mathscr{S}_0(R^4)$) with values operators on the Krein space K; this formula takes

into account the two extra degrees of freedom which are connected with the infrared singularities which affect the splitting of the field into positive and negative frequencies parts (tip of the light cone). We obtain the following expression:

$$\phi(f) = \phi'(f_0) + \tilde{f}(0)\phi(\chi) + <\chi,f>\phi(v^+) + <\bar{f},\chi>\phi(v^-) \tag{2.4.20}$$

ϕ', whose explicit expression is given in (2.4.15), here identifies the L^2 part of the field operator (this splitting is obviously a consequence of the structure of the space given in eq.(2.2.51)).
The time evolution gets extra terms; for instance we obtain:

$$\tau_t(a_1^\dagger(k)) = \exp(i\omega t)\ [(1-i\omega t)\ a_1^\dagger(k) + i\omega t a_2^\dagger(k)] +$$

$$+ (\pi/2)^{1/2}\omega^{-3/2}\ [\exp(i\omega t)(1-i\omega t) - 1\ ]\ \tilde{\chi}(\omega,k)\phi(v^+) \tag{2.4.21}$$

$$\tau_t(a_2^\dagger(k)) = \exp(i\omega t)\ [-i\omega t\ a_1^\dagger(k) + (1+i\omega t)\ a_2^\dagger(k)]$$

$$- (\pi/2)^{1/2}\omega^{-3/2}[\exp(i\omega t)(-i\omega t)]\ \tilde{\chi}(\omega,k)\phi(v^-) \tag{2.4.22}$$

We conclude this pargraph by a comparison between our results and the existing positive quantization of the dipole field displayed by Narnhofer and Thirring. As it is clear by the previous derivation, the splitting of the field ϕ into positive and negative frequencies [SW] leads to the introduction of pseudo-canonical operators. We have already said we may obtain "true" canonical operators by taking suitable (frequency-dependent) combinations of them. In particular Narnhofer and Thirring's canonical operators may be obtained by the following Bogoliubov-like transformation:

$$b_1(k) = \frac{1}{2}\ \{[\omega + i(\omega + \frac{1}{2\omega})]a_1(k) + [(-\omega + \frac{1}{2\omega}) - i\omega]a_2(k) +$$

$$+ [\omega - i(\omega - \frac{1}{2\omega})]a_1^\dagger(k) + [(-\omega - \frac{1}{2\omega}) + i\omega]a_2^\dagger(k)\}$$

$$b_2(k) = \frac{1}{2}\{[(-\omega + \frac{1}{2\omega}) - i\omega]a_1(k) + [\omega + i(\omega - \frac{1}{2\omega})]a_2(k) +$$

$$+ [(\omega + \frac{1}{2\omega}) - i\omega]a_1^\dagger(k) + [-\omega + i(\omega + \frac{1}{2\omega})]a_2^\dagger(k)\} \ . \qquad (2.4.23)$$

(These transformtions are infrared singular, but are well defined on a dense subset of L^2).It easy to verify that these operators satisfy the canonical commutation rules. To give an example we compute the following commutators:

$$[b_1(q),b_1^\dagger(k)] = \frac{1}{4}\{(\omega^2 + (\omega + \frac{1}{2\omega})^2)[a_1(q),a_1^\dagger(k)] +$$

$$+ ((-\omega + \frac{1}{2\omega})^2+\omega^2)[a_2(q),a_2^\dagger(k)] + (\omega^2 + (\omega - \frac{1}{2\omega})^2)[a_1^\dagger(q),a_1(k)] +$$

$$+ ((-\omega - \frac{1}{2\omega})^2+\omega^2)[a_2^\dagger(q),a_2(k)]\} = \delta^3(q-k) \qquad (2.4.24)$$

$$[b_1(q),b_2(k)] = \frac{1}{4}\{[\omega + i(\omega + \frac{1}{2\omega})][(\omega + \frac{1}{2\omega}) - i\omega][a_1(q),a_1^\dagger(k)]$$

$$+ [(-\omega + \frac{1}{2\omega}) - i\omega][-\omega + i(\omega + \frac{1}{2\omega})][a_2(q),a_2^\dagger(k)] +$$

$$+ [\omega - i(\omega - \frac{1}{2\omega})][(-\omega + \frac{1}{2\omega}) - i\omega][a_1^\dagger(q),a_1(k)] +$$

$$+ [(-\omega - \frac{1}{2\omega}) + i\omega][\omega + i(\omega - \frac{1}{2\omega})][a_2^\dagger(q),a_2(k)] = 0 \qquad (2.4.25)$$

The remaining commutators may be computed in the same way and we obtain that the only nonzero ones are the following:

$$[b_1(q),b_1^\dagger(k)] = [b_2(q),b_2^\dagger(k)] = \delta^3(q-k)$$

For completeness we give also the inverse formulae:

$$a_1(k) = \frac{1}{2}\{[\omega - i(\omega + \frac{1}{2\omega})]b_1(k) + [(\omega + \frac{1}{2\omega}) - i\omega]b_2(k) +$$

$$+ [-\omega + i(\omega - \frac{1}{2\omega})]b_1^\dagger(k) + [(\omega - \frac{1}{2\omega}) - i\omega]b_2^\dagger(k)\}, \qquad (2.4.26)$$

$$a_2(k) = \frac{1}{2}\{[(\omega - \frac{1}{2\omega}) - i\omega]b_1(k) + [\omega - i(\omega - \frac{1}{2\omega})]b_2(k) +$$

$$+ [-(\omega + \frac{1}{2\omega}) + i\omega]b_1^\dagger(k) + [\omega - i(\omega + \frac{1}{2\omega})]b_2^\dagger(k)\}. \qquad (2.4.27)$$

Finally the dipole field written in yerms of the canonical operators b's hs the following form:

$$\phi(x) = \frac{1}{2(2\pi)^{3/2}}\int\{[(\frac{t}{2} + i\omega)\cos(\omega t-kx)+(\frac{t}{2} - \frac{1}{2\omega}+ i\omega)\sin(\omega t-kx)]\, b_1^\dagger(k)$$

$$+ [(\omega - \frac{it}{2}) \cos(\omega t-kx) - (\omega - \frac{it}{2} - \frac{i}{2\omega}) \sin(\omega t-kx)] \; b_2^\dagger(k)\} \frac{d^3k}{2\omega^{3/2}}$$

$$+ \frac{1}{2(2\pi)^{3/2}}\int \{[(\frac{t}{2} - i\omega) \cos(\omega t-kx) +(\frac{t}{2} - \frac{1}{2\omega} - i\omega) \sin(\omega t-kx)]\, b_1(k)$$

$$+ [(\omega + \frac{it}{2}) \cos(\omega t-kx) - (\omega + \frac{it}{2} + \frac{i}{2\omega}) \sin(\omega t-kx)] \; b_2(k)\} \frac{d^3k}{2\omega^{3/2}}$$

$$(2.4.28)$$

It has been proved by Narnhofer and Thirring that the Fock representation for the operators b's, defined by the condition $b(k)\Psi'_0=0$, leads to non implementable time translations. In our perspective this conclusion appears related to the fact that the operators b are not linked to the splitting of the commutator (2.4.1) into positive and negative frequency parts.

As we have seen there exists a representation, which is non-Fock for the canonical operators b's and that satisfy the Fock condition for the operators $a(k)$, in which the time translations are implementable. In this case the Fock like state is non positive.

2.5 THE PHYSICAL INTERPRETATION.

It is necessary at this point to identify some subspace K' of the Krein space K by means of which constructing the physical space of the theory. K' must satisfy at least the two following requirements: the vacuum vector must belong to K' (i.e. the vacuum is a physical state), and K' must be semidefinite (i.e. $\langle\Psi,\Psi\rangle \geq 0 \quad \forall\ \Psi \in K'$) for the probabilistic interpretation of the theory. It is usual to define the space K' using an operator supplementary condition, as in the Gupta-Bleuler [GUP] [BLE] or B.R.S.T. [KUG] quantization. Then, defining

$$K'' = \{\Psi \in K' : \langle\Psi,\Psi\rangle = 0\} \qquad (2.5.1)$$

we obtain as a candidate for the physical space of the theory the following Hilbert space:

$$K_{phys} = (K'/K'') \ , \qquad (2.5.2)$$

where the completion is taken w.r. to the Hilbert topology induced by the scalar product $\langle\ ,\ \rangle$. Before performing the explicit construction of some possible physical spaces, we state and proof the following theorem by which space-time translations must be broken in every non trivial physical subspace oh $\mathcal{D}$:

Theorem 2.5.1: Every semidefinite subspace of $\mathcal{D}$, invariant under space-time translations is a null subspace of $\mathcal{D}$.

Proof: it is clear that it is enough to show this result at the one particle level and therefore we consider a positive semidefinite subspace of $\mathcal{S}(R^4)$ which contains a certain function f and all its translated f_a; eq. (2.10) imply that $\tilde{f}_a(k) = \exp(ika)\ \tilde{f}(k)$. Call this space $\mathcal{T}_f$. We use at first the invariance of $\mathcal{T}_f$ under time translations. Let $a = (t,0)$ and define

$$T_1(t) = \langle f-f_t, f-f_t\rangle \ . \qquad (2.5.3)$$

By hypothesis $T_1(t) \geq 0$ and it is obvious that $T_1(0) = 0$. Therefore the point t=0 must be a minimum for $T_1(t)$, and this implies that

$$\frac{d^2}{dt^2} T_1(t)|_{t=0} = - \pi \int [(1 + D)|\tilde{f}(k)|^2]|_{C_+} \omega^{-1} d^3k \geq 0 \ . \tag{2.5.4}$$

The same argument may be applied to the function

$$T_n(t) = \langle \sum_{j=0} \binom{n}{j} (-1)^j f_{jt} \ , \ \sum_{j=0} \binom{n}{j} (-1)^j f_{jt} \rangle \ . \tag{2.5.5}$$

By induction one has that

$$\frac{d^m}{dt^m} T_n(t)|_{t=0} = 0 \ , \qquad m \leq 2n-1 \ . \tag{2.5.6}$$

The fact that t=0 must be a minimum for $T_n(t)$ now gives the condition

$$\int \omega^{2n-3}[(1 - 2n - D)|\tilde{f}(k)|^2]|_{C_+} d^3k \geq 0. \tag{2.5.7}$$

Now we exploit the invariance of $\mathcal{T}_f$ under space translations. To illustrate the method let us suppose at first that $f \in \mathcal{S}_0(R^4)$. The non negativity of $\mathcal{T}_f$ implies that

$$<f,f> = \tfrac{1}{2}\pi \int [(1-D)|\tilde{f}(k)|^2]|_{C_+} \omega^{-3} d^3k \geq 0. \tag{2.5.8}$$

The invariance of $\mathcal{T}_f$ under space translations implies that actually it must be

$$[(1-D)|\tilde{f}(k)|]|_{C_+} \geq 0 \tag{2.5.9}$$

pointwise . Indeed let us define

$$\tilde{F}_{\epsilon,N}(k) = (\epsilon/N)^{3/2} \sum_{n_1,n_2,n_3=-N}^{N} \exp(iqn\epsilon) \ \tilde{f}_{n\epsilon}(k) \tag{2.5.10}$$

with $n\epsilon = (0,n_1\epsilon,n_2\epsilon,n_3\epsilon)$ and $\epsilon > 0$. When ϵ and N stay finite $F_{\epsilon,N}$ belongs to $\mathcal{T}_f$ and by hypothesis

$$<F_{\epsilon,N},F_{\epsilon,N}> = \tfrac{1}{2}\pi(\epsilon/N)^3 \sum_{n,m=-N}^{N} \int \exp[i(k-q)(n-m)\epsilon](1-D)|\tilde{f}(k)|^2|_{C_+} \omega^{-3} d^3k \geq 0 \tag{2.5.11}$$

By choosing $\epsilon = N^{-1/2}$ and taking the limit of the last expression for $N \longrightarrow \infty$ we obtain the (5.8). Repeating now the steps that led us to the formula (5.7) we concludethat it must be

$$\{(1-2n-D)|\tilde{f}(k)|^2\}|_{C_+} \geq 0 \quad \forall\ n\geq 1 \ , \tag{2.5.12}$$

where it is no more necessary to suppose that $f\in\mathcal{S}_0(R^4)$. It is now evident that the system of inequalities (2.5.12) may be verified if and only if

$$\tilde{f}(k)|_{C_+} = 0. \tag{2.5.13}$$

Therefore $\mathcal{T}_f$ is contained in the linear space

$$\mathcal{N} = \{f\in\mathcal{S}(R^4) \ : \ \tilde{f}(k)|_{C_+}\} \tag{2.5.14}$$

which is a null subspace of $\mathcal{S}(R^4)$ invariant under the translations group. Therefore $\mathcal{T}_f$ is a null subspace of $\mathcal{S}(R^4)$ ##

<u>Corollary</u> <u>2.5.2</u>: every semidefinite (actually null) translationally invariant subspace of $\mathcal{S}(R^4)$ is contained in $\mathcal{N}$.

Thus, according to corollary (2.5.2) there is no hope of finding a non trivial physical subspace of $K^{(1)}$that be Poincare' invariant: indeed the condition (2.5.13) leads to a physical space that contains only the vacuum vector [ZWA][BOG1][MIN].
(This space is selected by the Zwanziger supplementary condition [ZWA], based on the electrodynamical analogy. We will study in detail this condition in chapter 4)
This fact does not mean that the content of this free field theory is necessarily trivial; it means only that we must construct a physical space in which the Poincare' symmetry is broken. This should not come as a surprise: indeed already in QED_4 the construction of the physical charged sectors requires the breaking of the Lorentz group [FRO][MOR2] (though with a different mechanism).Besides, confinement of charged massless particles in QED_4 and of charged massive particles in QED_3 is a consequence of the breaking of the translation group in the physical

space [24]. Also in the present case the infrared singularities are of the confining type, and lead to the breaking of the translation group in the physical space (mechanism of confinement). Let us see the concrete construction of some possible physical spaces: consider a complex valued infinitely differentiable function $z=z(k)$, such that

$$\mathrm{Re}\, z(k) = \mu < 1/2 \quad , \quad |\, \mathrm{Im}\, z(k)\, | < \mathrm{const}\, |k| . \tag{2.5.15}$$

Each $z(k)$ that satisfies the previous conditions labels a possible physical space $K_{z,phys}$. Indeed, following the procedure briefly illustrated at the beginning of this section, we may define the one particle space $K_z^{(1)\prime}$ as the Krein closure of the dense set

$$\mathcal{D}_z^{(1)} = \{ f \in \mathcal{S}_0(R^4) \; : \; (D-z(k))\tilde{f}(k)|_{C_+} \} . \tag{2.5.16}$$

K'_z may be obtained as the symmetric Fock space over $K_z^{(1)\prime}$, and the physical space $K_{z,phys}$ is given by K'_z/K''_z. It is possible to see that K'_z is a maximal non negative subspace of K (i.e. it is not properly contained in any other non negative subspace of K). Note that the set (5.16) is not stable under the translation group; indeed one has that

$$U(a)\mathcal{D}_z^{(1)} = \mathcal{D}^{(1)}_{z+ik_0 a_0} . \tag{2.5.17}$$

Eq (5.17) implies that K'_z is not stable under time translations while it stable under space translations; in particular the time translations map a dense set of a maximal non negative subspace of K onto a dense set of another maximal non negative subspace of K and therefore define orbits (of maximal non negative subspaces of K).

The same happens for the Lorentz boosts , while purely spatial rotations leave each K'_z invariant.

To get a closer insight into the structure of these spaces we now study in some detail the case $z=0$. We have that

$$K^{(1)\prime}_{z=0} = L^2(C_+ - \{0\}, \omega^{-3} d^3k) \oplus V^+ , \tag{2.5.18}$$

$$K^{(1)\prime\prime}_{z=0} = V^+ ; \tag{2.5.19}$$

$$K^{(1)}_{z=0,\mathrm{phys}} = L^2(C_+ - \{0\}, \omega^{-3} d^3k) \tag{2.5.20}$$

The total physical space $K_{z=0,\mathrm{phys}}$ may be obtained by the usual Fock procedure. The fact that $K'_{z=0}$ is maximal semidefinite may be understood by looking at (2.5.18). Eq. (2.48) implies that the vectors of $K'_{z=0}$ may be characterized by the following Gupta-Bleuler condition:

$$\begin{cases} \phi^-(f)\ \Psi = 0 & \forall f \in \mathcal{S}_0(R^4) \text{ such that } (1-D)\tilde{f}(-k)|_{C_+} = 0 \\ \phi(v^-)\ \Psi = 0 \end{cases} \tag{2.5.21}$$

The next question concerns the definition of the fields on the physical space. We distinguish here two notions of "quotientability" that are similar to those notions of gauge invariance introduced in [STR1]. Let A be a bounded operator in K. A is said <u>quotientable</u> w.r. to K′ if

$$AK' \subseteq K', \quad AK'' \subseteq K''. \tag{2.5.22}$$

A is said <u>weakly quotientable</u> if the matrix elements $\langle \Psi_1, A\Psi_2 \rangle$, $\Psi_1, \Psi_2 \in K'$, depend only on equivalence classes of K'/K''. These definitions may be easily generalizated to cover the case of unbounded operators. There is a unique operator $\hat{A}$ in K_{phys} associated to a quotientable operator A. If A is only weakly quotientable the existence of $\hat{A}$ is guaranteed only in the case in which the space K'/K'' is complete [24]. In this case $\hat{A}$ is constructed using the representation theorem for sesquilinear forms [19]. Let us come back to our concrete case. It is evident that for a generic test function f

$$\phi(f)K'_{z=0} \notin K'_{z=0} \tag{2.5.23}$$

but it is not difficult to show that $\phi(f)$ is weakly quotientable when $f \in \mathcal{S}_0(R^4)$ (the severe infrared singularities of the theory prevent the possibility to extend the quotiented field to functions belonging to $\mathcal{S}(R^4)$). The explicit expression of the quotiented field w.r. to $K'_{z=0}$ is the following:

$$(\hat{\phi}(f)\hat{\Psi})^{(n)} = \tfrac{1}{2}\pi\ (n+1)^{1/2}\int[(1-D)f(-k)]|_{C_+}\hat{\Psi}^{(n+1)}(k,k_1,\ldots,k_{n+1})\omega^{-3}d^3k\ +$$

$$(n)^{-1/2}\sum_{j=0}^{n}[(1-D)f(k_j)]|_{C_+}\hat{\Psi}^{(n)}(k_1,\ldots,k_j,\ldots k_n) \qquad (2.5.24)$$

This expression gives a positive (non covariant) quantization of the dipole field ($\Box\hat{\phi} = 0$) as an operator valued distribution on $\mathcal{S}(R_0^4)$, acting on a Hilbert space with positive metric. The fact that time translations are a broken symmetry becomes now more evident: indeed one may define a transformation γ_t of the polynomial algebra generated by the quotiented field $\hat{\phi}$:

$$\gamma_t(\hat{\phi}(f)) = \hat{\phi}(f_t) \qquad (2.5.25)$$

with $f_t(x) = f(x_0-t,x)$.
it turns out that this transformation may be interpreted as time translations only when restricted to a subalgebra.
Indeed from eq. (2.5.24) one can easily obtain the quotient of the gauge invariant field $\Lambda = \Box\phi$. It is possible to find a non trivial "vacuum sector" by applying polynomials of the quotiented gauge invariant field $\hat{\Lambda}$ to the vacuum vector (and completing w.r. to < , >):

$$K_{vac,z=0} = \overline{\{\mathcal{P}(\hat{\Lambda})\hat{\Psi}_0\}} \qquad (2.5.26)$$

This conclusion is in contrast with those of [ZWA][BOG]. The explanation of this contrast is that while the vacuum expectation values of the polynomial algebra generated by Λ vanish, this is not the case for $\mathcal{P}(\hat{\Lambda})$ at the quotient and this because of the (5.23). It is worth to mention that the transformation γ_t has the meaning of time translation in $K_{vac,z=0}$, and its representation is the usual one; for instance if $\Psi \in K^{(1)}_{vac,z=0}$ one has that the implementer of γ_t is given by

$$V(t)\Psi = \exp(i k_0 t)\Psi \qquad (2.5.27)$$

Therefore, starting from the local and covariant quantization of chapter 2, we have got as a special case a positive quantization somewhat related to that exibited in [NAR]. In this case the time

translations are implementable in the Krein space K, but do not K_{phys} invariant . Here we have another example (see also [MOR2]) supporting the advantages of the strategy of computing the Wightman functions (or equivalently solving the dynamics) in a local and covariant gauge, where a lot symmetries are implementable and the theory has a linear structure. The physical interpretation of the model is then obtained simply by a linear subsidiary condition; it is at this stage that the physical structure, which is in general non symmetric, appears.
The physical space $K_{z=0,phys}$ we have just constructed, may be regained easily using the canonical formalism introduced in the previous section. The Gupta-Bleuler condition now is written as follows:

$$a_2(k)\ |phys\rangle = 0 \qquad (2.5.28)$$

It follows that $\hat{a}_2 = 0$, $\hat{a}_1 = a_1$. The quotiented field is given by

$$\hat{\phi}(t,x) = (2\pi)^{-3/2}\int \exp(i\omega t-ikx)(1-i\omega t)\hat{a}_1^\dagger(k)\ (2\omega^{3/2})^{-1}d^3k +$$

$$(2\pi)^{-3/2}\int \exp(-i\omega t+ikx)(1+i\omega t)\hat{a}_1(k)\ (2\omega^{3/2})^{-1}d^3k \qquad (2.5.29)$$

The Hamiltonian is quotiented to zero . Again the time translations are a symmetry only in the vacuum sector; their generator is the following:

$$H_v = \int \omega\ \hat{a}_1^\dagger(k)\hat{a}_1(k)\ d^3k \qquad (2.5.30)$$

On the other side the space translations are implementable on the whole physical space and their generator is exactly the quotiented momentum:

$$\hat{P}^i = \int k^i\hat{a}_1^\dagger(k)\hat{a}_1(k) \qquad (2.5.31)$$

We note also that the equal time commutators involving quotiented fields which are not gauge invariant may depend explicitely on time.
It would now be intersting to know if there is the possibility of obtaining a positive quantization of the dipole field in which the time translations are an exact symmetry for the whole physical space and not only for the vacuum sector. The Krein space approach allows the possibility to give an answer to this question. Indeed there are many other positive semidefinite subspaces of K and we may try to find some

which are stable under time translations. Therefore let us consider the following construction: let $w = w(k)$ a complex-valued infinitely differentiable function such that

$$\mathrm{Re}\, w(k) = \mu > -1/2 \qquad |\mathrm{Im}\, w(k)| < \mathrm{const.}|k| \tag{2.5.32}$$

The one-particle semidefinite subspace we are looking for is defined as the Krein completion $H_w^{(1)\prime}$ of the dense set

$$\mathcal{G}_w^{(1)} = \{\ f \in \mathcal{P}_0(R^4)\ :\ [(G - w(k))\tilde{f}(k)]|_{C_+} = 0\ \} \tag{2.5.33}$$

where $G = k_1\partial/\partial k_1 + k_2\partial/\partial k_2 + k_3\partial/\partial k_3$. Again H_w' is obtained by constructing the symmetric Fock space over $H_w^{(1)\prime}$ and the physical space $H_{w,phys}$ is given by H_w'/H_w''. In this case we have that

$$U(a)\mathcal{G}_w^{(1)} = \mathcal{G}_{w-ika}^{(1)} \tag{2.5.34}$$

This implies that H_w' is not stable under space translations while it is stable under time translations, which therefore define an exact symmetry in $H_{w,phys}$. As in precedence we give the supplementary condition that characterizes the space $H_{w=0}'$:

$$\begin{cases} \phi^-(f)\Psi = 0 & \forall f \in \mathcal{P}_0(R^4) \text{ such that } [(1-D)\tilde{f}(-k)]|_{C_+} = -G[\tilde{f}(-k)|_{C_+}] \\ \phi(v^-)\Psi = 0 & \end{cases} \tag{2.5.35}$$

Exactly as before the quotiented field may be constructed only for those test functions belonging to $\mathcal{P}_0(R^4)$. We do not give here the complicated expression of the quotiented field but write its two point function:

$$G(x-y) = W(x-y) + (32\pi^3)^{-1} \sum_{i,j=1}^{3} x_i x_j \int k_i k_j \exp[ik(x-y)]|_{C_+} \omega^{-3} d^3k \tag{2.5.36}$$

Thus we have obtained another (non covariant) positive quantization of the dipole field for which the time translations are an exact symmetry of the physical space while the space translations do not leave it invariant.

We remark that this quantization can never be obtained in a formal approach which exploits the usual creation and annihilation operators:

indeed the supplementary condition (2.5.35) cannot be rewritten in terms of them.

Clearly the two quantizations that we have constructed must have the same physical meaning and indeed it is again possible to construct a "vacuum sector" in which the whole translation group is implementable and actually this vacuum sector is isomorphic to the previous one: this may be understood by looking at the Wightman functions of the gauge invariant quotiented fields which are the same in the two cases, as it may be directly verified using formulae (2.5.24) and (2.5.36).

We have come to the following conclusions: a local and covariant quantization of the dipole field model may only be obtained by making use of an indefinite metric space and it turns out that the most natural setting to discuss the model is the Krein space K, whose features has been described in sections two and three. Then the thing to do is to look for a positive semidefinite subspace of K by means of which constructing the physical space (and therefore the physical interpretation) of the model. In our case we have seen that the infrared singularities of the Wightman functions forbid the possibility of constructing a Poincare' invariant physical space different from the vacuum vector. There is however the possibility to find subspaces of K which are not invariant under the Poincare' group and which originate physical spaces exhibiting its breaking (mechanism of confinement). In particular we have constructed two explicit examples: in the first one the time translations are broken while the space translations are an exact symmetry, and the contrary happens in the second one (actually other choices are possible but all exhibiting the breaking of the whole translation group). What is important is the fact that it is possible to construct a "vacuum sector" and this is the same in the two cases; therefore the gauge invariant content of the two positive quantizations that we have constructed is the same.

3. THE WICK ORDERED EXPONENTIAL OF THE DIPOLE FIELD.

3.1 INTRODUCTION.

The discussion of a class of soluble models with an interacting dipole requires the introduction of an important nonlinear function of the dipole field, namely its Wick ordered exponential.
As in the standard case the Wick powers $:\phi^n:$ of the dipole field are well defined fields (except positivity) i.e. it is possible to show that they are tempered distributions with values operators in the Krein space K introduced in the previous chapter (the construction is similar to that exposed by Garding and Wightman in [GAR1]).
It has been known for a long time [GLA][EPS] that yet in the standard (positive metric) case no series of the kind $B(x)=\sum a_n :\rho^n:(x)$, where ρ is a free scalar field and an infinite number of coefficients a_n are different from zero, defines a tempered field operator in a space time of dimension greater than two. However it was also noticed that one may construct QFT's containing objects like these by finding suitable test function spaces allowing the strong convergence of the series defining B(x). In particular B(x) is a well defined operator valued distribution on the test function space $\mathcal{Z}$ [WIG1][SCHR1][KLA]; $\mathcal{Z}$ contains only analytic functions and their Fourier transform are infinitely differentiable functions having a compact support [GEL2]. Fields of this type have been considered in various models of nonrenormalizable and nonlocal interactions [GUT][FRA][EFI].
It was Jaffe [JAF1] to discover that in certain favourable cases these fields admit a formulation of microcausality, i.e. notwithstanding the fact that the growth of the fields in momentum space is faster than any polynomial they may be smeared with test functions having compact support in position space (Jaffe fields). This has lead to some remarkable extensions of the Wightman theory to more general distributional frameworks [CON][NAG].
The main scope of this chapter is to give a sense to the series

$$:\exp(z\phi):(x) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} :\phi^n:(x) \ , \qquad z\in C \tag{3.1.1}$$

The first thing to do is to use the (by now formal) definition (3.1.1) to compute the vacuum expectation values of the "field" $:\exp(z\phi):(x)$. These may be easily computed and one gets the following expression for the n-point function:

$$\langle\Psi_0, \prod_{j=0}^{n} :\exp(z_j\phi):(x_j)\ \Psi_0\rangle = \prod_{i<k} \exp[z_i z_k W(x_i - x_k)] \tag{3.1.2}$$

with $W(\xi) = -(4\pi)^{-2}\ \ln(-\xi^2 + i\epsilon\xi_0)$.

In this way we get a set of tempered distributions, and we want to construct a field (i.e. an operator valued distribution on a Hilbert space H) having these distributions as correlation functions; this operator valued distribution will be called the Wick exponential of the dipole field and denoted with the symbol $:\exp z\phi:(x)$. Let us see how to proceed:

i) having at our disposal the Wightman functions (3.1.2) we may try to use the reconstruction theorem to obtain the field $:\exp(z\phi):(x)$ as an operator valued distribution on a Hilbert space H.

In the standard (positive metric) case this is in principle ever possible (see for example [JAF2]). In the indefinite metric case the generalized reconstruction theorem [MOR1] requires the introduction of a Hilbert majorant topology.

However the direct construction of a set of Hilbert seminorms that majorize the n-point functions (3.1.2) (without passing through the construction of the free dipole field), is very difficult because the truncated Wightman functions of the field $:\exp z\phi:(x)$ do not vanish and the needed Hilbert seminorms cannot have the simple structure found in the previous chapter for the free dipole field.

ii) The alternative way consists in studying the strong convergence of the series (3.1.1) inside the Krein-Hilbert where we have reconstructed the free dipole field. As in the case of Jaffe fields one is compelled to restrict the test function spaces in order to assure this kind of convergence.

There are however some differences w.r. to that case: first of all the strong convergence of the series (3.1.1) depends crucially on the Hilbert majorant topology used to construct the free dipole and in general the control of the strong convergence of the series (3.1.1) is not possible.

Besides the test function spaces which allow the strong convergence of that series may (and actually will) be much more restricted than the fundamental space on which are defined the correlation functions (3.1.2).

The point is that now the construction of the Wick exponential requires the simultaneous control of the ultraviolet and infrared singularities; the latter ones are not shared by standard field theories and, as we have repeatedly said, the control of such infrared singularities requires the introduction of a Hilbert topology. The so constructed field operator Hilbert topology may have infrared and ultraviolet behaviour which is more singular than that of its correlation functions.

These facts lead to introduce what will be called "fields of type S". The dipole field and its Wick exponential provide us an interesting prototype of the general situation described above. It is interesting to note that the methods employed for the actual construction work also in the other singular cases (Jaffe fields, Hyperfunctions, nonlocal fields, etc.) and seem to provide the most general framework for the choice of test functions in quantum field theory. Another important example will be discussed in the sixth chapter.

3.2 CONSTRUCTION OF THE WICK ORDERED EXPONENTIAL

In this chapter we explicitely show that the series (3.1.1) defining the Wick ordered exponential of the dipole field strongly converges on a dense domain of the Krein space introduced in the previous chapter. Let's rewrite the Hilbert product (2.2.14) in a form which is more convenient for our present purposes:

$$(f,g) = \int \bar{f}(x)\ K(x,y)\ g(y) \tag{3.2.1}$$

with

$$K(x,y) = -\ (4\pi)^{-2}\ln(-\xi^2+\ i\epsilon\xi_0) + (2\pi)^{-2}\ x_0y_0(-\xi^2+\ i\epsilon\xi_0)^{-1}+\ (2\pi)^{-4}G_{\chi\chi}$$

$$+\ [W_\chi(x)\ -\ (2\pi)^{-2}][W_\chi(-y)\ -\ (2\pi)^{-2}]\ +\ (2\pi)^{-2}[G_\chi(x)\ +\ G_\chi(-y)] \tag{3.2.2}$$

where, as usual, $\xi = x-y$ and we have defined

$$W_\chi(x)= -(4\pi)^{-2}\int \ln(-\xi^2+\ i\epsilon\xi_0)\ \chi(y)\ d^4y, \qquad G_{\chi\chi}= \int G_\chi(x)\chi(x)d^4x$$

$$G_\chi(x)= (2\pi)^{-2}\int x_0y_0(-\xi^2+\ i\epsilon\xi_0)^{-1}\ \chi(y)\ dy.$$

As we already know from the second chapter, the completion of $\mathcal{S}(R^4)$ in the topology induced by the Hilbert product (3.1.1) yields the one particle Hilbert space $K^{(1)}$, which carries a Krein structure, and the complete Hilbert space of the theory is the Fock space over $K^{(1)}$.

The first step toward the construction of the Wick ordered exponential consists in determinating conditions on the test function f such that the vector $:\exp z\phi:(f)\Psi_0$ belongs to the Krein space K (i.e. has a finite Hilbert norm). It is easy to understand that (at least formally)

$$\|\ :\exp z\phi:(f)\Psi_0\|_K = \int \exp\ [\,|z|^2\ K(x,y)]\ \bar{f}(x)f(y)\ d^4x\ d^4y, \tag{3.2.3}$$

so we have to find conditions such that the integral at the R.H.S. of eq. (3.2.3) is well defined.

The characterization of the distributional aspects of the kernel exp $[K(x,y)]$ will be done in several steps; the method employed is

similar to that applied by Klaiber [KLA1] for the study of a model in which a fermion interacts with a scalar free field (derivative coupling); this method is based on Thirring's formulae expressing the powers of the Pauli-Jordan function [THI]. Some modifications are necessary to deal with the non covariance of the kernel $K(x,y)$.
The estimates necessary to control the norm of the more general vector $:\exp z_1\phi:(f_1):\dots\dots:\exp z_n\phi:(f_n)\Psi_0$ are more complicated and will be given in appendix 3A.2.

The first of thing that we want to do is showing that the following series converges to an analytic function of the complex parameter z, if the function f satisfies suitable conditions:

$$\sum_{n=0}^{\infty} \frac{1}{n!}\int G_{z,n}(x,y)\ \bar{f}(x)\ f(y)\ d^4x\ d^4y \tag{3.2.4}$$

where $G_{z,n}(x,y)$ is the following tempered distribution of $\mathcal{S}'(R^8)$:

$$G_{z,n}(x,y) = [(z/4\pi^2)\ x_0 y_0]^n(-\xi^2+ i\epsilon\xi_0)^{(-z/16\pi^2)-n} \tag{3.2.5}$$

We have that

$$\int G_{z,n}(x,y)\ \bar{f}(x)\ f(y)\ d^4x\ d^4y =$$
$$= (2\pi)^4(z/4\pi^2)^n\int Z_{2n+(z/8\pi^2)}(k)\,|\partial^n/\partial k_0^n\ \tilde{f}(k)|^2\ d^4k$$

Z_s is the Riesz distribution, and is an entire function of the complex parameter s [SCHW]. It follows that $G_{z,n}(x,y)$ is analytic in z for each fixed n. Taking into account the explicit expression of Z_s obtain that

$$\int G_{z,n}(x,y)\ \bar{f}(x)\ f(y)\ d^4x\ d^4y =$$

$$= (2\pi)^3\exp[(1-(z/16\pi^2))\ln 4]\ (z/16\pi^2)^n\ [\Gamma((z/16\pi^2)+n)\ \Gamma((z/16\pi^2)+n-1]^{-1}$$

$$\int \vartheta(k_0)\vartheta(k^2)(k^2)^{(z/16\pi^2)+n-2}\ |\partial^n/\partial k_0^n\ \tilde{f}(k)|^2\ d^4k \tag{3.2.6}$$

where Γ is Euler's gamma function.

In order to study the convergence of the series (3.2.4) we consider the fundamental spaces $\mathcal{S}_{\alpha}^{\beta}(R^4)$, consisting of all infinitely differentiable functions f(x) satisfying the inequalities

$$| x^k \partial^q f(x) | \leq C A^k B^q k^{k\alpha} q^{q\beta} \tag{3.2.7}$$

In this formula we used a multi-indicial notation (i.e. $k = (k_0,..,k_3)$, $A = (A_0,..,A_3)$, $\alpha = (\alpha_0,..,\alpha_3)$, $A^k=(A_1)^{k_1}...(A_n)^{k_n}$,etc.); the constants C, A_i, B_j depend only on the function f and α_i and β_j are real and nonnegative.

The spaces $\mathcal{S}_{\alpha}^{\beta}$ have been introduced by Shilov and are extensively studied in [GEL2]. They are particularly suited for the study of partial differential equation [GEL3] and consequently for QFT.

In particular the spaces $\mathcal{S}_{\infty}^{\beta} \equiv \mathcal{S}^{\beta}$ (with only one index active) have been used for the study of nonrenormalizable and nonlocal field theories [GUT],[CON],[RIE],[FAI].

As we will see, the construction of the Wick ordered exponential of the dipole field, is complicated by infrared and ultraviolet problems occurring contemporarly and requires the use of spaces with two active indices to control both the large x and large k behavior.

The spaces $\mathcal{S}_{\alpha}^{\beta}$ are usually called "spaces of type S" because of their structure very similar to that of the Schwartz space $\mathcal{S}$ of rapidly decreasing functions. They have many common properties. It is very important that the Fourier transformation may be used freely in these spaces: the operators $\partial/\partial x$ and multiplication by x exchange roles under the Fourier tranformation and the spaces $\mathcal{S}_{\alpha}^{\beta}$ transform into each other, i.e. one has that the Fourier transformed of $\mathcal{S}_{\alpha}^{\beta}$ is $\mathcal{S}_{\beta}^{\alpha}$ and the Fourier operator is continuous in the topology of the appropriate fundamental space [GEL2].

The restrictions posed on functions belonging to $\mathcal{S}_{\alpha}^{\beta}$ are the stronger the smaller is the value of $\alpha+\beta$. In particular one gets that $\mathcal{S}_{\alpha}^{\beta}$ consists only of the function identically equal to zero, if for a

certain index j $\alpha_j+\beta_j<1$, while $\mathcal{S}_\infty^\infty = \mathcal{S}$ and $\mathcal{S}_0^\infty = \mathcal{D}$.
Although the inequalities (3.2.7) completely characterize each of these spaces, we may give a more concrete picture of the asymptotic behaviour of the functions that belong to them in terms of the corresponding "indicatrix functions". We give here this kind of characterization in the one dimensional case, the general case being completely analogous.

1) All functions $f(x)\in\mathcal{S}_\alpha(R)$, together with all their derivatives decrease exponentially at infinity, with an order greater or equal than $1/\alpha$, and a type greater or equal than a costant depending on the function f, i.e.

$$|\frac{\partial^q f}{\partial x^q}| \le C_q \exp(-a|x|^{1/\alpha}). \tag{3.2.8}$$

2) Since $\mathcal{S}^\beta(R)$ is the Fourier transformed of $\mathcal{S}_\beta(R)$ it follows that the characterization of the functions belonging to $\mathcal{S}^\beta(R)$ is identical with the previous one if we use the Fourier conjugate variable k:

$$|\frac{\partial^q \tilde{f}}{\partial k^q}| \le C_q \exp(-a|k|^{1/\beta}). \tag{3.2.9}$$

If $\beta<1$ something more can be said. In this case we have that f may be continued analytically in the z=x+iy plane as an entire function of order of growth $1/(1-\beta)$, i.e.

$$|x^k f(x+iy)| \le C_k \exp b|y|^{1/(1-\beta)} \tag{3.2.10}$$

Analogous considerations may be done regarding the space $\mathcal{S}_\alpha^\beta$. For more details about the spaces of type S and their usage see [GEL2].
Let's return now to the series (3.2.4).

<u>Lemma 3.1</u>: If $f \in \mathcal{S}_\alpha^\beta$ with $\alpha_0+\beta_0 < 3/2$ then the series (3.2.4) converges to an analytic function of the complex parameter z.

<u>Proof</u>: suppose that $|z/16\pi^2| < m$, where m is a certain natural number.This immediately implies that $\mathrm{Re}(z/16\pi^2) + m + 1 > 0$. We may

rewrite the series (3.4) as the sum of two terms. The first one is

$$\sum_{n=0}^{m+2} \frac{1}{n!} \int G_{z,n}(x,y)\ \bar{f}(x)\ f(y)\ d^4x\ d^4y \tag{3.2.11}$$

and is obviously an entire function of z, because it is a finite sum of entire functions. The second term that we must control is the following:

$$(2\pi)^3\ \exp[(1-w)\ln 4] \sum_{n=m+3}^{\infty} w^n [\Gamma(w+n)\ \Gamma(w+n-1)]^{-1}\cdot$$
$$\cdot\int \vartheta(k_0)\vartheta(k^2)(k^2)^{w+n-2}\ |\partial^n/\partial k_0^n\ \tilde{f}(k)|^2\ d^4k \tag{3.2.12}$$

with $w=z/16\pi^2$. Since $n\geq m+2$, the complex number $w+n-1$ belongs to the right complex half-plane (because by assumption $|w|\leq m$. We may therefore apply the results of the appendix 3A.1; in particular lemma 3A.4 implies that $\mathcal{O}rd\{[n!\Gamma(w+n)\Gamma(w+n-1)]^{-1}\}=1/3$, and by proposition 3A.2 a sufficient condition for the normal convergence [NAC] of the series (3.4) is the following:

$$\left|\int \vartheta(k_0)\vartheta(k^2)(k^2)^{w+n-2}\ |\partial^n/\partial k_0^n\tilde{f}(k)|^2\ d^4k\ \right| \leq 1/a_n \tag{3.2.13}$$

where $\{a_n\}$ is a positive sequence such that $\mathcal{O}rd\{a_n\} < 1/3$ (see appendix 3A.1). This is guaranteed if the function f belongs to $\mathcal{S}_{\alpha}^{\beta}(R^4)$, with $\alpha_0+\beta_0<3/2$. Indeed after some lenghty calculation one gets the following majorization:

$$\left|\int \vartheta(k_0)\vartheta(k^2)(k^2)^{w+n-2}\ |\partial^n/\partial k_0^n\ \tilde{f}(k)|^2\ d^4k\ \right| \leq$$
$$C'A^{n+m}(n+m)^{2(\alpha_0+\beta_0)(n+m)}, \tag{3.2.14}$$

and the constants C'and A depend only on the function f.
Now we have that

$$\lim_{n\to\infty}\ n\ \log n\ /\{\log[A^{n+m}(n+m)^{2(\alpha_0+\beta_0)(n+m)}]\} = (2\alpha_0+2\beta_0)^{-1} \tag{3.2.15}$$

Thus condition (3.10) implies that if $\alpha_0+\beta_0 < 3/2$ then the series (3.2.4) converges normally on $S_m=\{z\in C\ :\ |z/16\pi^2| \leq m\}$. Since every compact of the complex plane is contained in some S_m the series converges to an entire function of the complex variable z. ##

By polarization we my now give sense to the expression

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int G_{z,n}(x,y)\ \bar{f}(x)\ g(y)\ d^4x\ d^4y \tag{3.2.16}$$

as a functional on $\mathcal{S}_{\alpha}^{\beta}(R^4)\otimes \mathcal{S}_{\alpha}^{\beta}(R^4)$ which depends analytically on the parameter z. Since the spaces $\mathcal{S}_{\alpha}^{\beta}$ are nuclear spaces [GEL4], it follows that the series (3.2.16) uniquely defines a generalized function belonging to $\mathcal{S}_{\alpha}^{\beta\,\prime}(R^8)$ (with an obvious redefinition of the eight-dimensional indices α and β) and depends analitically on z. Therefore we obtain that

$$\frac{d^n}{dz^n} \int G_{z,n}(x,y)\ \bar{f}(x)g(y)\ d^4xd^4y = \int [K_1(x,y)]^n\ \bar{f}(x)g(y)\ d^4xd^4y \tag{3.2.17}$$

$$K_1(x,y) = -\ (4\pi)^{-2}\ln(-\xi^2+i\epsilon\xi_0) + (2\pi)^{-2}\ x_0y_0(-\xi^2+i\epsilon\xi_0)^{-1} \tag{3.2.18}$$

This allows us to write

$$\sum_{n=0}^{\infty} \frac{1}{n!}\ G_{z,n}(x,y) \equiv \exp\ [z(K_1(x,y)] \tag{3.2.19}$$

Obviously the series (3.2.4) converges also on the linear span of the spaces $\mathcal{S}_{\alpha}^{\beta}(R^4)$ such that $\alpha_0+\beta_0<3/2$. This is a linear subset of $\mathcal{S}(R^4)$ symmetric under Fourier transformation, but it is not a fundamental space.

<u>Lemma 3.2</u>: let K(x,y) the distributional kernel defined in eq.(3.1.2). The series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}[K(x,y)]^n \equiv \exp\ [z(K(x,y)] \tag{3.2.20}$$

defines a distribution belonging to $\mathcal{S}'^{\beta}_{\alpha}(R^8)$, $\alpha=(\alpha_0,..,\alpha_3,\alpha_0,..,\alpha_3)$ $\beta=(\beta_0,..,\beta_3,\beta_0,..,\beta_3)$, if the following conditions are satisfied:

$$\alpha_0+\beta_0<3/2, \quad \alpha_0<1, \quad \alpha_i<\infty, \quad \beta_i>1. \tag{3.2.21}$$

This distribution depends analytically on z.

Proof: Eq. (3.2.2) implies that $K(x,y)-K_1(x,y)$ is an infinitely differentiable function. By [GEL2] we know that multiplication by a function $h(x)$, satisfying for any $\epsilon_i>0$ the inequalities

$$|h^{(q)}(x)| \le C_\epsilon \epsilon^q q^{q\beta} \exp(\epsilon_1|x_1|^{1/\alpha_1} + \ldots + \epsilon_n|x_n|^{1/\alpha_n}) \tag{3.2.22}$$

transforms the space $\mathcal{S}_\alpha^\beta(R^n)$ into itself and is a bounded operator. Thus, we must estimate the behavior of the function $\exp(K(x,y)-K_1(x,y))$ and of all its derivatives. The leading term is given by the factor $\exp G_\chi(x)\exp G_\chi(-y)$. We have that

$$G_\chi(x)= (2\pi)^{-2} \int x_0y_0(-\xi^2+i\epsilon\xi_0)\ \chi(y)\ dy =$$

$$= x_0 \int \exp\ (ikx)\vartheta(k_0)\delta(k^2)\partial/\partial k_0\tilde{\chi}(k)\ dk. \tag{3.2.23}$$

Choosing $\tilde{\chi}(k) = \exp\ [-c(k_0^2 + \omega^2)]$, with $c>0$ such that $<\chi,\chi>=0$ we obtain the following estimate:

$$|\partial^n G_\chi(x)| \le C^{|n|} n_0!..n_3!\ (1+|x_0|) \tag{3.2.24}$$

From the recursion formula

$$\partial^{n+1}\exp(f) = \sum \binom{n}{k} (\partial^{n-k+1}f)(\partial^k\exp(f)) \tag{3.2.25}$$

we obtain that if $\beta_i>1$ and $\alpha_0<1$

$$|\partial^n \exp G_\chi(x)| \le C'^{|n|} n_0!..n_3!\exp(c'|x_0|) \le$$

$$C''_\epsilon\ \epsilon^{|n|}(n_0!)^{\beta_0}..(n_3!)^{\beta_3}\exp(\epsilon|x_0|^{1/\alpha_0}) \tag{3.2.26}$$

The other factors are now simply accounted by the condition $\alpha_i<\infty$. Thus if conditions (3.2.21) are verified, the function $\exp[t(K(x,y)-K_1(x,y))]$ is a multiplier on $\mathcal{S}_\alpha^\beta(R^8)$ for any complex t.

Taking into account eq. (3.2.19) and Hartog's theorem [NAC] the lemma is finally proved. ##

The previous lemmas help us in proving that at least the vacuum vector is in the domain of the Wick ordered exponential of the dipole field. Indeed one has the following theorem:

<u>Theorem 3.3</u>: for each $f\in\mathcal{S}_{\alpha}^{\beta}(R^4)$, with $\alpha_0+\beta_0<3/2$, $\alpha_0<1$, $\alpha_i<\infty$, $\beta_i>1$ and each $z\in C$, we have that

$$\lim_{N\to\infty} \|\sum_{n=0}^{N} \frac{z^n}{n!}:\phi^n:(f)\Psi_0\|^2 = \int \exp[|z|^2K(x,y)]\ \bar{f}(x)f(y)\ d^4xd^4y < \infty \quad (3.2.27)$$

Besides, for any $R>|z|$ there exists constant $C_f(R)$ such that

$$\sum_{n=0}^{\infty} \frac{|z|}{n!}\ \|:\phi^n:(f)\Psi_0\| \le C_f(R) \sum_{n=0}^{\infty} (|z|/R)^n \quad (3.2.28)$$

and therefore the series (3.1) is strongly convergent on Ψ_0.

<u>Proof</u>: by lemma 3.2 we have that

$$\frac{d}{dz}\int \exp[z\ K(x,y)]\bar{f}(x)f(y)d^4xd^4y\ \Big|_{z=0} = \int [K(x,y)]^n\ \bar{f}(x)f(y)d^4xd^4y =$$

$$= (n!)^{-1}(:\phi:^n(f)\Psi_0,:\phi:^n(f)\Psi_0)_K \quad (3.2.29)$$

Therefore, if $f\in\mathcal{S}_{\alpha}^{\beta}(R^4)$ with α and β as in (3.2.21) the partial sums

$$\sum_{n,m=0}^{N} (n!m!)^{-1}z^nw^m(:\phi:^n(f)\Psi_0,:\phi:^m(f)\Psi_0)_K =$$

$$\sum_{n,m=0}^{N} (n!)^{-1}z^nw^n \int [K(x,y)]^n\ \bar{f}(x)f(y)d^4xd^4y \quad (3.2.30)$$

converge to $\int\exp[zwK(x,y)]\bar{f}(x)f(y)d^4xd^4y$. Equation (3.2.27) follows by taking $w=\bar{z}$, and the bound (3.2.28) is obtained by Cauchy's integral formula [NAC]. ##

To get a less trivial domain for the field $:\exp z\phi:$ we need to extend the proof of theorem 3.3 to the case of multiple series. As we have already said, the proof of the following theorem is much more difficult and is given in appendix 3A.2.

<u>Theorem 3.4</u>: if $f_i \in \mathcal{S}_\alpha^\beta(R^4)$, with $\alpha_0+\beta_0<3/2$, $\alpha_0<1$, $\alpha_i<\infty$, $\beta_i>1$ we have that for any R>0 it exists $C(f_i,R)$ such that

$$\| :\phi^{n_1}:(f_1):\phi^{n_2}:(f_2)\ldots:\phi^{n_k}:(f_k)\ \Psi_0\| \le C(f_i,R)\ n_1!n_2!\ldots n_k!/(R)^{n_1+n_2+\ldots+n_k}$$

and therefore the multiple series

$$\sum_{n=0}^{\infty}(n_1!n_2!\ldots n_k!)^{-1}z_1^{n_1}z_2^{n_2}\ldots z_k^{n_k}\ :\phi^{n_1}:(f_1):\phi^{n_2}:(f_2)\ldots:\phi^{n_k}:(f_k)\ \Psi_0 \qquad (3.2.31)$$

is strongly convergent.##

Theorem 3.4 imply that the Wick exponentials are operators defined on the dense domain

$$\mathcal{D}_{exp} = \{\mathcal{P}(\ :\exp(z\phi):(f),\ :\phi^n:(g))\Psi_0,\ f,g \in \mathcal{S}_\alpha^\beta(R^4)\ ,\ z\in C\} \qquad (3.2.32)$$

This domain may be enlarged, as the following theorem shows:

<u>Theorem 3.5</u>: the series (3.1.1) is strongly convergent on the dense domain

$$\mathcal{D}_1 = \{\mathcal{P}(:\exp(z\phi):(f),\exp[t\phi(g)])\Psi_0),$$

$$f\in\mathcal{S}_\alpha^\beta(R^4),\ \text{with}\ \alpha_0+\beta_0<3/2,\ \alpha_0<1,\ \alpha_i<\infty,\ \beta_i>1\ g\in\mathcal{S}(R^4),\ t,z\in C\} \qquad (3.2.33)$$

<u>Proof</u>: first of all we notice that the vectors of $\mathcal{D}_{exp}$ are analytic vectors for $\phi(f)$, where f belongs to the space (1.2.51). Indeed consider the projection $\Psi_f^{\{z\},(n)}$ of the vector

$$\Psi_f^{\{z\}} = :\exp z_1\phi:(f_1):\exp\phi z_2:(f_2)\ldots:\exp z_k\phi:(f_k)\ \Psi_0 \qquad (3.2.34)$$

on the n-particle subspace of K. The results of appendix 3.2 imply that

$$\|\Psi_f^{\{z\},(n)}\| \le C_f(R)(|z|/R)^n \qquad (3.2.35)$$

where $|z| = |z_1|+\ldots+|z_n|$. The estimate (2.2.49) now implies that [PIE]

$$\|[\phi(f)]^k\Psi_f^{\{z\}}\| < C_f^k\ (k!)^{1/2} \qquad (3.2.36)$$

and therefore

$$\sum_{k=0}^{\infty}\|[\phi(f)]^k\Psi_f^{\{z\}}\|\ t^k/k! < \infty \qquad (3.2.37)$$

Thus the operators $\mathcal{P}(\exp[\phi(f)])$ may be defined on $\mathcal{D}_{\exp}$ by the corresponding power series.

On $\mathcal{D}_{\exp}$ it holds the following commutation rule:

$$:\exp z\phi:(f)\ \exp t\phi(g) = \exp t\phi(g)\ :\exp z\phi:(Gf)$$

where $G(x)$ is the function $(\exp\text{-}tz[\phi(x),\phi(g)])$ and it may be verified that it is a well defined multiplier for the allowed test function spaces $\mathcal{S}_a^{\beta}$.

Thus we may repeatedly apply to the vacuum ordinary and Wick ordered exponentials and the resulting vectors are well defined vetors of K.##

3.3 Concluding remarks.

The Wick exponential of the dipole field which we have constructed in theorem 3.5 is not a tempered field (operator valued distribution).

The spaces that allow its construction, namely the Gelfand and Shilov spaces $\mathcal{S}_{\alpha}^{\beta}(R^4)$, with $\alpha_0+\beta_0<3/2$, $\alpha_0<1$, $\alpha_i<\infty$, $\beta_i>1$ contain test functions having compact support and therefore the Wick exponential is a localizable field. On the other side the spaces $\mathcal{S}_{\beta}^{\alpha}(R^4)$ which are the Fourier transformed of the previous ones contain only analytic functions.

In any case the correlation functions of the field $:\exp z\phi:(x)$ are tempered distributions and furthermore they satisfy the (weak) spectral property (written in the usual way, i.e. making use of test functions having compact support in momentum space).

Appendix 3.1.

Entire functions and sequences of finite order

Let $f=f(z)$ be an entire function and let $M(r)$ denote the maximum modulus of $f(z)$ for $|z|=r$. We define [BOA]:

$$\mathcal{O}rd\ f = \limsup \log(\log M(r))/\log r = \rho \tag{3A.1.1}$$

$$\mathcal{T}yp\ f = \limsup r^{-\rho} \log M(r) = \tau \tag{3A.1.2}$$

It is well known [BOA] that f has finite order ρ if and only if for any $\epsilon>0$ the quantity $M(r)/\exp(r^{\rho+\epsilon})$ tends to zero for large values of r. Analogously f has finite type τ if and only if for any $\epsilon>0$ the quantity $M(r)/\exp[(\tau+\epsilon)r^{\rho}]$ tends to zero for large values of r.
It holds the following theorem [BOA]:

<u>Theorem 3A1</u>: The entire function $f(z) = \sum a_n z^n$ has finite order ρ and finite type τ if and only if

$$\limsup n \log n / \log(|a_n|^{-1}) = \mu < \infty \tag{3A.1.3}$$

$$\limsup n\ |a_n|^{-\rho/n} = \nu < \infty \tag{3A.1.4}$$

In this case we have that $\mu=\rho$ and $\tau = \nu/e\rho$.

Theorem 3A.1 suggests that eq. (3A.3) and (3A.4) may be used to define the order and the type of a numerical sequence, i.e.

$$\mathcal{O}rd\ \{a_n\} = \limsup n \log n / \log(|a_n|^{-1}) \tag{3A.1.5}$$

and an analogous definition for $\mathcal{T}yp\{a_n\}$ .To give an explicit example consider the following small modification of Euler's gamma function:

$$\Gamma(z,\rho,\tau) = \frac{1}{\rho}\int e^{-u}(u/\tau)^{z/\rho}d(\log u) \tag{3A.1.6}$$

It easily follows that $\Gamma(z,\rho,\tau) = \Gamma(z/\rho)\tau^{-z/\rho}/\rho$.
By using Stirling's formula we obtain that the sequence $\{a_n\} = \{[\Gamma(n,\rho,e\tau)]^{-1}\}$
has order ρ and type τ. The proofs of the following propositions are easy:

Proposition 3A.2: let $\{a_n\}$ and $\{b_n\}$ two real sequences such that $\mathcal{O}rd\{a_n\}=\rho$ and $\mathcal{O}rd\{b_n\}=\eta$. Then

$$\mathcal{O}rd\{a_n b_n\} = \rho\eta/(\rho+\eta) \tag{3A.1.7}$$

The series $\sum a_n z^n/b_n$ converges to an entire function f if $\eta>\rho$. In this case we have

$$\mathcal{O}rd\ f = \rho\eta/(\eta-\rho) \tag{3A.1.8}$$

Proposition 3A.3: let $\mathcal{O}rd\ \{a_n\} = \rho$ and suppose that $a_n>0$. If

$$\lim_{n\to\infty} \log(a_n/a_{n+m})/(n\ \log n) = 0 \tag{3A.1.9}$$

then $\mathcal{O}rd\{a_{n+m}\} = \rho$.

Now we are in position to state and prove the following lemma:

Lemma 3A.4: let $w \in \bar{S}(a,b) = \{z\in C\ :\ |z-a|\le b\}$ where a and b are two positive numbers such that $a>1$ and $a-1\ge b$. We have that

$$\mathcal{O}rd\ \{\Gamma(w+n)^{-1}\} = 1 \tag{3A.1.10}$$

Proof: the recursive formula

$$\Gamma(n+w) = (n+w-1)(n+w-2)\dots\dots\dots w\ \Gamma(w)$$

together with the analyticity of the gamma function in the positive half-plane imply

$$|\Gamma(n+w)| \ge n!\ m(a,b), \qquad m(a,b) = \min_{w\in\bar{S}(a,b)} |\Gamma(w)| \tag{3A.1.11}$$

From the other side we get that

$$|\Gamma(n+w)| \le (n+a+b-1)\dots\dots(a+b-1) \max_{w\in\bar{S}(a,b)} |\Gamma(w)| \le (n+[a+b])!\ M(a,b) \tag{3A.1.12}$$

where [a+b] is the greatest integer less or equal to a+b. Thus we get that

$$\frac{1}{M(a,b)\ (n+[a+b])!} \leq \frac{1}{|\Gamma(n+w)|} \leq \frac{1}{m(a,b)\ n!} \tag{3A.1.13}$$

But by proposition 3A.3 we have that

$$\mathcal{O}rd\{\frac{1}{(n+[a+b])!}\} = 1$$

and therefore the lemma is proved.

Appendix 3.2
Estimate of the n-point norm.

In the paragraph 3.2 we developped a simple charcterization of the two-point norm $\exp[|z|^2K(x,y)]$ by using the explicit expression of the Fourier transform of the Riesz's distribution. This method is very simple and direct but, unfortunately, it may be used only for the two-point norm. The estimate of the behaviour of the n-point norm requires more work; we give a sketch of the methods used in this appendix. In certain respects they are similar to those used in [RIE]. We begin by stating and proving a simple lemma concerning some spaces of type S:

Lemma 3A.4: Let $\mathcal{S}_\alpha^\beta(R^n)$ be isotropic, i.e. $\alpha=(\alpha,\alpha,\ldots\alpha)$ and $\beta=(\beta,\beta,\ldots,\beta)$.
Let $g\in\mathcal{S}_\alpha^\beta(R^n)$ and let A: $R^n \longrightarrow R^n$ linear and invertible.
Then $g'\in\mathcal{S}_\alpha^\beta(R^n)$, where $g'(x) \equiv g(Ax)$.

Proof: let k and q be multiindices. We have to estimate the quantity $x^k\partial^q g'(x)$. First of all observe that

$$y = Ax, \qquad x = A^{-1}y, \qquad \partial_y = (A^t)^{-1}\partial_x, \qquad \partial_x = A^t\partial_y .$$

It follows that

$$(x_i)^{k_i} = \sum_{\sum_{j=1}^{n}\alpha_{ij}=k_i} \frac{k_i!}{\alpha_{i1}!\alpha_{i2}!\ldots\alpha_{in}!}(A^{-1}_{i1}y_1)^{\alpha_{i1}}(A^{-1}_{i2}y_2)^{\alpha_{i2}}\ldots(A^{-1}_{in}y_n)^{\alpha_{in}}$$

Analogously one gets

$$(\partial^{q_i}/\partial x_i^{q_i}) = \sum_{\sum_{j=1}^{n}\beta_{ij}=q_i} \frac{q_i!}{\beta_{i1}!\beta_{i2}!\ldots\beta_{in}!}(A^t_{i1})^{\beta_{i1}}(A^t_{i2})^{\beta_{i2}}\ldots(A^t_{in})^{\beta_{in}} .$$

$$\cdot\ \frac{\partial^{|q|}}{\partial y_1^{\beta_{i1}}\ldots\ldots\partial y_n^{\beta_{in}}}$$

Define

$$\left(\left(\begin{matrix} k \\ \alpha \end{matrix}\right)\right) = \prod_{j=1}^{n} \frac{k_j!}{\alpha_{j1}!\alpha_{j2}!\ldots\alpha_{jn}!}$$

$$\sum_{i=1}^{n} \alpha_{ij}=\mu_j \ , \quad \sum_{i=1}^{n} \beta_{ij}=\nu_j$$

Besides, let B and C be two nxn matrices with integer entries and pose

$$[B]^C = \Pi_{ij} B_{ij}^{C_{ij}} .$$

Then we get the following identity:

$$x^k\partial^q g'(x) = \sum_{\alpha,\beta}\left(\left(\begin{matrix} k \\ \alpha \end{matrix}\right)\right) \left(\left(\begin{matrix} q \\ \beta \end{matrix}\right)\right) [A^{-1}]^{\alpha}[A^t]^{\beta} y_1^{\mu_1} y_2^{\mu_2} \ldots y_n^{\mu_n} \cdot$$

$$\cdot \frac{\partial^{|q|}}{\partial y_1^{\nu_1} \ldots\ldots \partial y_n^{\nu_n}} g(y)\big|_{y=Ax}.$$

where the sum is intended over all matrices α and β such that

$$\sum_{j=1}^{n} \alpha_{ij}=k_i \ , \quad \sum_{j=1}^{n} \beta_{ij}=q_i .$$

Suppose now that $g\in\mathcal{S}_{\alpha}^{\beta}(R^n)$, with $\alpha=(\alpha,\alpha,\ldots\alpha)$ and $\beta=(\beta,\beta,\ldots,\beta)$. It follows that there exist constants C,A,B such that

$$|x^k\partial^q g(x)| \leq C\ A^{|k|}\ B^{|q|}\ k^{k\alpha}\ q^{q\beta}.$$

Therefore we obtain that

$$|x^k\partial^q g'(x)| \leq C'A'^{|k|}\ B'^{|q|} \ \sum_{\alpha,\beta}\left(\left(\begin{matrix} k \\ \alpha \end{matrix}\right)\right) \left(\left(\begin{matrix} q \\ \beta \end{matrix}\right)\right) \mu^{\mu\alpha}\ \nu^{\nu\beta} \leq$$

$$C''A''^{|k|}\ B''^{|q|}\ k^{k\alpha}\ q^{q\beta},$$

and thus also g' belongs to $\mathcal{S}_{\alpha}^{\beta}(R^n)$. ##

The following is the central result of this appendix:

<u>Lemma</u> <u>3A.5</u>: If $f_j \in \mathcal{S}_{\alpha}^{\beta}(R^4)$, with $\alpha_0+\beta_0 < 3/2$, then

$$\int \prod_{i<k}^{n} \exp[G(x_i,x_k)]\ f_1(x_1)\ldots\ldots f_n(x_n) d^4x_1 \ldots d^4x_n < \infty \qquad (3A.2.1)$$

where $G(x,y)=(2\pi)^{-2}\ x_0y_0(-\xi^2+i\epsilon\xi_0)^{-1}$, $\xi=x-y$.

Proof: to show the kind of estimates we have to face we consider at first the case n=2 (two-point norm): we may write [GAR]

$$G(x,y)^r = c^r x_0^r y_0^r \int \exp[-i\xi \sum_{l=1}^{r} k_l] \prod_{l=1}^{r} [\ \vartheta(k_l^0)\delta(k_l^2)\ d^4k_l] \qquad (3A.2.2)$$

where, as usual, $\delta(k_l^2)=\delta(k^{\mu}_{l}k_{\mu l})$. It follows that

$$\int G(x,y)^r\ \bar{f}(x)g(y)d^4xd^4y =$$

$$c_1^r \int [\partial^r/\partial k_0^r\ \bar{\tilde{f}}(\sum_{l=1}^{r} k_l)][\partial^r/\partial k_0^r\ \tilde{g}(\sum_{l=1}^{r} k_l)] \prod_{l=1}^{r} \vartheta(k_l^0)\delta(k_l^2)\ d^4k_l =$$

$$c_2^r \int [\partial^r/\partial k_0^r\ \bar{\tilde{f}}(\sum_{l=1}^{r} k_l)][\partial^r/\partial k_0^r\ \tilde{g}(\sum_{l=1}^{r} k_l)] \prod_{l=1}^{r} (|k_l|)^{-1}d^3k_l \qquad (3A.2.3)$$

where we have in the last expression $k_l^0=|k_l|$. By assumption we know that $\tilde{f}$ and $\tilde{g}$ belong to $\mathcal{S}_{\beta}^{\alpha}(R^4)$ (remember that α and β exchange their roles under Fourier transform). Thus it follows that

$$|(1+(k^0)^m)(\partial^r/\partial k_0^r\ \bar{\tilde{f}}(k^0))(\partial^r/\partial k_0^r\ \tilde{g}(k^0))| \le C\ A^m\ B^r\ m^{m\beta_0}\ r^{2r\alpha_0} \qquad (3A.2.4)$$

Therefore we get:

$$|\int [\partial^r/\partial k_0^r\ \bar{\tilde{f}}(\sum_{l=1}^{r} k_l)][\partial^r/\partial k_0^r\ \tilde{g}(\sum_{l=1}^{r} k_l)] \prod_{l=1}^{r} (k_l^0)^{-1}d^3k_l| \le$$

$$C\ A^m\ B^r\ m^{m\beta_0}\ r^{2r\alpha_0} \int (1+(k^0)^m)^{-1} \prod_{l=1}^{r} (k_l^0)^{-1}d^3k_l =$$

$$C\cdot\Gamma^r\ A^m\ B^r\ m^{m\beta_0}\ r^{2r\alpha_0} \int_0^\infty \ldots \int_0^\infty (1+(k^0)^m)^{-1} \prod_{l=1}^{r} (k_l^0)^{-1}dk_l^0 \qquad (3A.2.5)$$

where integration over angles has been performed in the last step. By changing the integration variables in the following way

$$t_j = \sum_{l=1}^{j} k_l^0 \qquad (3A.2.6)$$

and choosing m=2r+1 we finally obtain that

$$\left|\int G(x,y)^{r}\ \bar{f}(x)g(y)d^4xd^4y\right| \leq C''A^{2r}\ (r!)^{-2}\ r^{2r(\alpha_0+\beta_0)} \qquad (3A.2.7)$$

and again we obtain that if $\alpha_0+\beta_0< 3/2$

$$\int \exp G(x,y)\bar{f}(x)g(y) = \sum \frac{1}{r!}\int G(x,y)^{r}\ \bar{f}(x)g(y)d^4xd^4y < \infty .$$

Now we consider the full expression (3A.2.1):

$$\int \prod_{i<k}^{n} \exp[G(x_i,x_k)]\ f_1(x_1)\ldots\ldots f_n(x_n)d^4x_1\ldots d^4x_n =$$

$$\sum_{\substack{r_{ij}=0\\ 1\leq i<j\leq n}} \frac{1}{r_{12}!r_{13}!\ldots\ldots r_{n-1,n}!}\int G(x_1,x_2)^{r_{12}}\ G(x_1,x_3)^{r_{13}}\ldots$$

$$\ldots\ldots G(x_{n-1},x_n)^{r_{n-1,n}}\quad f_1(x_1)\ldots\ldots f_n(x_n)d^4x_1\ldots d^4x_n =$$

$$\sum_{\substack{r_{ij}=0\\ 1\leq i<j\leq n}} \frac{1}{r_{12}!r_{13}!\ldots\ldots r_{n-1,n}!}\int (x_1^0)^{R_1}\ldots(x_n^0)^{R_n}\ D(\xi_1)^{r_{12}}\ D(\xi_2)^{r_{13}}\ldots$$

$$\ldots\ldots\ D(\xi_{n-1})^{r_{n-1,n}}\quad f_1(x_1)\ldots\ldots f_n(x_n)d^4x_1\ldots d^4x_n \qquad (3A.2.8)$$

where we have adopted the following definitions

$$\xi_i=x_i-x_{i+1}\ ,\ \xi_n=x_n \qquad (3A.2.9)$$

$$r_{ij}=r_{ji}\ ,\ R_i=\sum_{j=1}^{n} r_{ij}\quad ,\ R=\sum_{i=1}^{n} R_i \qquad (3A.2.10)$$

and $D(\xi) = (2\pi)^{-2}\ (-\xi^2+i\epsilon\xi_0)^{-1}$.

Let's pass to difference variables; we have to estimate the quantity

$$\Delta_R = \int[\prod_{1\leq i<j\leq n} D(\xi_i+\ldots+\xi_{j-1})^{r_{ij}}]\ h_R(\xi_1,\ldots\xi_{n-1})\ d\xi_1\ldots d\xi_n \qquad (3A.2.11)$$

where

$$h_R(\xi_1,\ldots,\xi_{n-1}) = \int \prod_{j=1}^{n}[(\sum_{i=j}^{n}\xi_i^0)^{r_{ij}}\ f_j(\sum_{i=j}^{n}\xi_i)]\ d\xi_n \qquad (3A.2.12)$$

Eq.(3A.2.2) then implies that

$$\Delta_R = \int \prod_{1\le i<j\le n} [\exp -i(\xi_i+\ldots+\xi_{j-1}) \sum_{l=1}^{r_{ij}} k_{ij,l})] \prod_{l=1}^{r_{ij}} \vartheta(k^0_{ij,l})\delta(k^2_{ij,l})\ d^4k_{ij,l}$$

$$h_R(\xi_1,\ldots,\xi_{n-1})d\xi_1\ldots d\xi_{n-1} =$$

$$c^R\int \tilde{h}_R(-q_1,\ldots,-q_{n-1}) \prod_{1\le i<j\le n} \prod_{l=1}^{r_{ij}} \vartheta(k^0_{ij,l})\delta(k^2_{ij,l})\ d^4k_{ij,l} \qquad (3A.2.13)$$

where we have defined

$$q_k = \sum_{i=1}^{k} \sum_{j=k+1}^{n} \sum_{l=1}^{r_{ij}} k_{ij,l} \qquad (3A.2.14)$$

Define now

$$f(\xi_1,..\xi_n) = \prod_{j=1}^{n} [f_j(\sum_{i=j}^{n} \xi_i)]$$

$$f_R(\xi_1\ldots\xi_n) = \prod_{j=1}^{n} [(\sum_{i=j}^{n} \xi^0_i)^{R_j}\ \ f_j(\sum_{i=j}^{n} \xi_i)] \qquad (3A.2.15)$$

and let ν be a matrix with integer nonnegative entries, having all zeros below the principal diagonal. Finally, define

$$\left(\!\!\left(\begin{matrix} R \\ \nu \end{matrix}\right)\!\!\right) = \prod_{j=1}^{n} \frac{R_j!}{\nu_{j1}!\nu_{j2}!\ldots\nu_{jn}!}\ , \qquad N_j = \sum_{i=1}^{n} \nu_{ij} \qquad (3A.2.16)$$

(we remember that 0!=1). It follows that

$$\prod_{j=1}^{n} (\sum_{i=j}^{n} \xi^0_i)^{R_j} = \sum_\nu \left(\!\!\left(\begin{matrix} R \\ \nu \end{matrix}\right)\!\!\right)\ (\xi^0_1)^{N_1}\ldots\ldots(\xi^0_n)^{N_n} \qquad (3A.2.17)$$

where the sum is over all the upper triangular matrices with nonnegative integer entries and such that

$$\sum_{j=1}^{n} \nu_{ij} = R_i \qquad (3A.2.18)$$

Thus we get that

$$f_R(\xi_1\ldots\xi_n) = \sum_\nu \left(\!\!\left(\begin{matrix} R \\ \nu \end{matrix}\right)\!\!\right)\ (\xi^0_1)^{N_1}\ldots\ldots(\xi^0_n)^{N_n}\ f(\xi_1,..\xi_n) \qquad (3A.2.19)$$

and therefore, since $\sum R_i = \sum N_j = R$, we have

$$\tilde{h}_R(-q_1,\dots,-q_{n-1}) = c^R \sum_\nu \left(\!\!\left(\begin{matrix} R \\ \nu \end{matrix} \right)\!\!\right) \frac{\partial^R}{(\partial q_1^0)^{N_1} \dots\dots (\partial q_n^0)^{N_n}} \hat{f}(q_1 \dots\dots q_n) \Big|_{q_n}$$

(3A.2.20)

where the symbol ^ indicates Fourier antitransform.

An obvious generalization of the arguments of lemma gives the following estimate:

$$\left| (1+(q_1^0)^{M_1}) \dots (1+(q_{n-1}^0)^{M_{n-1}}) \sum_\nu \left(\!\!\left(\begin{matrix} R \\ \nu \end{matrix} \right)\!\!\right) \frac{\partial^R}{(\partial q_1^0)^{N_1} \dots\dots (\partial q_n^0)^{N_n}} \hat{f}(q_1 \dots\dots q_n) \right| \le$$

$$C\, A_1^{M_1} \dots A_{n-1}^{M_{n-1}} M_1^{M_1 \beta} \dots M_{n-1}^{M_{n-1}\beta} \sum_\nu \left(\!\!\left(\begin{matrix} R \\ \nu \end{matrix} \right)\!\!\right) B_1^{N_1} \dots B_n^{N_n} N_1^{N_1 \alpha} \dots N_n^{N_n \alpha}$$

$$C\, A_1^{M_1} \dots A_{n-1}^{M_{n-1}} M_1^{M_1 \beta} \dots M_{n-1}^{M_{n-1}\beta} (nB_1 \dots B_n)^R\, R^{R\alpha}$$

and thus

$$|\tilde{h}_R(-q_1,\dots,-q_{n-1})| \quad \le \quad C \quad [(1+(q_1^0)^{M_1}) \dots (1+(q_{n-1}^0)^{M_{n-1}})]^{-1}$$

$$A_1^{M_1} \dots A_{n-1}^{M_{n-1}} M_1^{M_1 \beta} \dots M_{n-1}^{M_{n-1}\beta} c^R R^{R\alpha} \qquad (3A.2.21)$$

Notice now the following obvious inequality:

$$q_k^0 = \sum_{i=1}^{k} \sum_{j=k+1}^{n} \sum_{l=1}^{r_{ij}} k_{ij,l}^0 \ge \sum_{j=k+1}^{n} \sum_{l=1}^{r_{ij}} k_{ij,l}^0 = q_k^{0\prime} \qquad (3A.2.22)$$

It then follows that

$$|\Delta_R| \le c^R \int [(1+(q_1^{0\prime})^{M_1}) \dots (1+(q_{n-1}^{0\prime})^{M_{n-1}})]^{-1} \prod_{1 \le i < j \le n} \prod_{l=1}^{r_{ij}} (k_{ij,l}^0)^{-1} d^3 k_{ij,l}$$

$$A_1^{M_1} \dots A_{n-1}^{M_{n-1}} M_1^{M_1 \beta} \dots M_{n-1}^{M_{n-1}\beta} c^R R^{R\alpha} \qquad (3A.2.23)$$

By using a change of variables analogous to that given in (3A.2.6) and choosing

$$M_k = 2(r_{k,k+1}+r_{k,k+2}+\ldots+r_{k,k+n}) + 1 \qquad (3A.2.24)$$

we finally get

$$|\Delta_R| \le c^R \prod_{k=1}^{n-1} \frac{1}{[2(r_{k,k+1}+r_{k,k+2}+\ldots+r_{k,k+n})]!}\, R^{R(\alpha_0+\beta_0)} \qquad (3A.2.25)$$

Thus since R contains twice each $r_{i,j}$, $i<j$, we have that if $\alpha_0+\beta_0< 3/2$ the series (3A.2.8) converges and the lemma is finally proved. ##

Lemma 3A.6: the multiple series

$$\mathcal{F}_f(z_1,..,z_n) =$$

$$\sum \frac{z_1^{k_1}\ldots z_n^{k_n}}{k_1!\ldots k_n!}(:\phi^{k_1}:(f_1)\ldots:\phi^{k_j}:(f_j)\Psi_0,\ :\phi^{k_{j+1}}:(f_{j+1})\ldots:\phi^{k_n}:(f_n)\Psi_0)_K$$

converges for any $f_1,\ldots f_n \in \mathcal{S}_\alpha^\beta$, with $\alpha_0+\beta_0<3/2$, $\alpha_0<1$, $\alpha_i<\infty$, $\beta_i>1$. Besides $\mathcal{F}_f(z_1,..,z_n)$ is an entire function of the variables $z_1,..,z_n$.

Proof: we may rewrite $\mathcal{F}_f(z_1,..,z_n)$ as follows:

$$\mathcal{F}_f(z_1,..,z_n) =$$

$$\int \prod_{i<k}^{n} \exp[z_i z_k W_{ik}(x_i,x_k)] \quad \bar{f}_1(x_1)..\bar{f}_j(x_j)f(x_{j+1})\ldots f_n(x_n)d^4x_1\ldots d^4x_n$$

where

$$W_{ik} = \begin{cases} W(x_i-x_k), & \text{if } 1\le i<k\le j;\quad j+1\le i<k\le n \\ K(x_i,x_k), & \text{if } 1\le i\le j;\quad j+1\le k\le n. \end{cases}$$

W has been defined in eq.(2.1.) and K is the kernel (3.1.2). Using the same trick of lemma 3.2 we get that

$$\mathcal{F}_f(z_1,..,z_n) = \int \prod_{i<k}^{n} \exp[z_i z_k\ W_{1,ik}(x_i,x_k)]\ \cdot$$

$\cdot\ \mathcal{M}(x_1 \ldots, x_n)\ \bar{f}_1(x_1)..\bar{f}_j(x_j) f(x_{j+1}) \ldots f_n(x_n) d^4x_1 \ldots d^4x_n$;

the definition of $W_{1,ik}$ is the same of W_{ik} except that $K(x_i, x_k)$ is substituted by $K_1(x_i, x_k)$ (see eq (3.2.17)); as in lemma 3.2 the function $\mathcal{M}$ is a good multiplier for the space $\mathcal{S}_\alpha^\beta(R^{4n})$ (with the appropriate multiindices α and β). Now estimates analogous to those of lemma 3A.5 give that the kernel

$$\prod_{i<k}^{n} \exp[z_i z_k\ W_{1,ik}(x_i, x_k)]$$

belongs to $\mathcal{S}'^\beta_\alpha(R^{4n})$ and this conclude the proof of this lemma. ##

<u>Proof of theorem 3.4</u>: We have the following formula:

$$\| :\phi^{n_1}:(f_1) :\phi^{n_2}:(f_2) .. :\phi^{n_k}:(f_k)\ \Psi_0 \|^2 =$$

$$\left| \frac{\partial^{2k}}{\partial z_1^{n_1} \ldots\ldots \partial z_k^{n_k}\ \partial z_{k+1}^{n_1} \ldots\ldots \partial z_{2k}^{n_{2k}}} \mathcal{F}_f(z_1, .., z_{2k}) \right|_{z_1 = \ldots = z_n = 0}$$

Therefore, using the multiple Cauchy formula one proofs the bound (3.2..), and consequently the series

$$\sum_{n=0}^{\infty} (n_1! n_2! .. n_k!)^{-1} z_1^{n_1} z_2^{n_2} .. z_k^{n_k}\ :\phi^{n_1}:(f_1) :\phi^{n_2}:(f_2) .. :\phi^{n_k}:(f_k)\ \Psi_0$$

is strongly convergent.##

Appendix 3.3
Some remarks about positive defined generalized functions.

This appendix is devoted to state and prove a certain theorem concerning positive defined generalized functions.
Positive definite functions arise as Fourier transforms of positive summable functions. They find application in many areas of mathematics, like probability theory, the theory of group representations and, obviously, quantum field theory.
The simple theorem we shall proof will allow us to construct certain non trivial positive representations of the gauge invariant subalgebra of the free dipole field, but we believe that it may be useful in showing positive definiteness of other (non free) quantum field theories or stochastic processes.
We start by recalling some definitions and results. We limit ourselves to the discussion of the one dimensional case the general case being analogous.

A continuous function $T(x)$ *is called positive-definite if for any real numbers* $x_1 \ldots x_n$ *and any complex numbers* $\xi_1 \ldots \xi_n$ *one has*

$$\sum_{j,k=1}^{n} T(x_j - x_k)\bar{\xi}_j \xi_k \geq 0 \qquad (3A.3.1)$$

It can be shown that this commonly accepted definition of positive definiteness is equivalent to the following one [GEL4]:

a continuos function $T(x)$ *is called positive definite if for any infinitely differentiable function* $f(x)$ *with compact support one has*

$$\int T(x-y)\overline{f(x)}f(y)dxdy \geq 0 \ . \qquad (3A.3.2)$$

This second definition has the advantage that it may be extended to more general kernels $T(x-y)$; in particular T may be a generalized function.

It is well known the important structure theorem of Bochner-Schwartz that asserts that *the class of positive definite generalized functions on the space $\mathcal{D}$ (distributions) coincides with the class of Fourier transforms of positive tempered* (finite if T is an ordinary continuous function) *measures*.

Thus, according to the Bochner-Schwartz theorem, we may represent a general positive definite generalized function in the following way:

$$"T(x)" = \int \exp(-i qx)\ d\mu(q), \tag{3A.3.3}$$

where μ is a tempered positive measure.

A positive definite generalized function may be used to introduce a pre-Hilbert product in $\mathcal{E}_0^\infty(R)$ (or equivalently $\mathcal{S}(R)$) by means of

$$(f,g)_T = \int T(x-y)\overline{f(x)}g(y)dxdy. \tag{3A.3.4}$$

Let now T and S be two positive definite generalized functions. We may introduce the a pre-Hilbert product in $\mathcal{E}_0^\infty(R)\otimes\mathcal{E}_0^\infty(R)$ in the following way: let f,g,h and k belong to $\mathcal{E}_0^\infty(R)$; we define

$$(f\otimes g,h\otimes k)_{TS} = (f,h)_T(g,k)_S =$$

$$\int T(x-y)\overline{f(x)}h(y)dxdy \int S(z-w)\overline{g(z)}k(w)dzdw.$$

This product is obviously positive semidefinite. Indeed one easily prove the Schwartz inequality on the monomials:

$$|(f\otimes g,h\otimes k)_{TS}|^2 = |(f,h)_T(g,k)_S|^2 \le \|f\|_T^2\ \|h\|_T^2\ \|g\|_S^2\ \|k\|_S^2 =$$

$$\|f\otimes g\|_{TS}^2\|h\otimes k\|_{TS}^2.$$

and this easily implies the positivity of the norm of the linear combinations:

$$\|a(f\otimes g)+b(h\otimes k)\|_{TS}^2\ge 0.$$

Since the set of finite linear combinations of the kind $\sum f_j(x)g_j(x)$ is dense in $\mathcal{E}_0^\infty(R^2)$ it follows that

$$\int T(x-z)S(y-w)\ \overline{f(x,y)}f(z,w)dxdydzdw \ge 0. \tag{3A.3.5}$$

Untill now we said nothing new. The remarks that we have done follow very closely the way in which one constructs the n-point functions of a free field theory starting from the two-point function [SW].
Now we present a result that goes in another direction:

<u>Theorem</u> <u>3A.7</u>: Let $T\in\mathcal{S}'(R)$ be a positive definite distibution and let

$$\mathcal{S}_{bos}(R^2) = \{f\in\mathcal{S}(R^2) : f(x,y)=f(y,x)\}. \tag{3A.3.6}$$

Define in $\mathcal{S}_{bos}(R^2)$ the following sesquilinear form:

$$\langle h_1,h_2\rangle = \int \bar{h}_1(x,y)T(x-z)T(x-w)T(y-z)T(y-w)h_2(z,w)dxdydzdw \tag{3A.3.7}$$

Then the form $\langle\cdot,\cdot\rangle$ defines a pre-Hilbert product in $\mathcal{S}_{bos}(R^2)$, i.e.

$\langle h,h\rangle \geq 0$, for each $h\in\mathcal{S}_{bos}(R^2)$

<u>Proof</u>: Using the Bochner-Schwartz theorem (eq. (3A.3.3)), we may rewrite the expression (3A.3.7) in the following way:

$$\langle h_1,h_2\rangle = \int \bar{\tilde{h}}_1(q+r,k+s)\tilde{h}_2(q+s,k+r)d\mu(q)d\mu(k)d\mu(r)d\mu(s), \tag{3A.3.8}$$

where μ is the tempered measure corresponding to the distribution T.
To begin we consider functions of the following form:

$h_1(x,y)=f(x)f(y)$, $h_2(x,y)=g(x)g(y)$.

We have that

$$\langle h_1,h_1\rangle = \int \bar{\tilde{f}}(q+r)\bar{\tilde{f}}(k+s)\tilde{f}(q+s)\tilde{f}(k+r)d\mu(q)d\mu(k)d\mu(r)d\mu(s) =$$

$$\int d\mu(q)d\mu(k)\left|\int \bar{\tilde{f}}(q+r)\tilde{f}(k+r)d\mu(r)\right|^2 \geq 0 . \tag{3A.3.9}$$

To conclude the proof of the theorem we must show that it holds the following (Schwarz) inequality:

$$|\langle h_1,h_2\rangle|^2\leq \langle h_1,h_1\rangle \langle h_2,h_2\rangle \tag{3A.3.10}$$

By using twice the L^2-Schwarz inequality we obtain the following chain of inequalities:

$$|\langle h_1,h_2\rangle|^2=\left|\int \bar{\tilde{f}}(q+r)\bar{\tilde{f}}(k+s)\tilde{g}(q+s)\tilde{g}(k+r)d\mu(q)d\mu(k)d\mu(r)d\mu(s)\right|^2 =$$

$$\left|\int d\mu(q)d\mu(k)\ \left(\int \bar{\tilde{f}}(q+r)\tilde{g}(k+r)d\mu(r)\right)\ \left(\int \bar{\tilde{f}}(k+s)\tilde{g}(q+s)d\mu(s)\right)\right|^2 \le$$

$$\left|\int d\mu(q)d\mu(k)\ \left|\int \bar{\tilde{f}}(q+r)\tilde{g}(k+r)d\mu(r)\right|^2\right|^2 =$$

$$\left|\int \bar{\tilde{f}}(q+r)\tilde{g}(k+r)\tilde{f}(q+s)\bar{\tilde{g}}(k+s)d\mu(q)d\mu(k)d\mu(r)d\mu(s)\right|^2 =$$

$$\left|\int d\mu(r)d\mu(s)\ \left(\int \bar{\tilde{f}}(r+q)\tilde{f}(s+q)d\mu(q)\right)\ \left(\int \bar{\tilde{g}}(s+k)\tilde{g}(r+k)d\mu(k)\right)\right|^2 \le$$

$$\left(\int d\mu(r)d\mu(s)\ \left|\int \bar{\tilde{f}}(r+q)\tilde{f}(s+q)d\mu(q)\right|^2\right) \cdot$$

$$\cdot\left(\int d\mu(r)d\mu(s)\ \left|\int \bar{\tilde{g}}(s+k)\tilde{g}(r+k)d\mu(k)\right|^2\right) = \quad \langle h_1,h_1\rangle\ \langle h_2,h_2\rangle$$

As consequence of this majorization we have that $\langle\phi,\phi\rangle>0$, with ϕ a finite linear combination of the kind $\phi(x,y) = \sum f_j(x)f_j(y)$.
But these functions constitute a dense subset of $\mathcal{S}_{bos}(R^2)$, and therefore the proof is concluded.##

We remark that we may complete $\mathcal{S}_{bos}(R^2)$ with respect to the Hilbert topology defined by the form (3A.3.7) and get a Hilbert space.
Starting from a certain two-point correlation function T(x-y) we have thus constructed a (nontrivial) four-point function (different from the one whose truncated function vanish). The previous theorem can be generalized to the construction of nontrivial 2n-point functions:

<u>Theorem</u> <u>3A.</u>8: Let $T\in\mathcal{S}'(R)$ be a positive definite distibution and consider the "bosonic" subspace $\mathcal{S}_{bos}(R^n)$ of $\mathcal{S}(R^n)$. Define in $\mathcal{S}_{bos}(R^n)$ the following sesquilinear form:

$$\langle h_1,h_2\rangle = \int \bar{h}_1(x_1,..,x_n)T(x_1-y_1)\cdot\cdot T(x_1-y_n)\cdot\cdot T(x_n-y_1)\cdot\cdot T(x_n-y_n)$$
$$\cdot h_2(y_1,..,y_n)dx_1..dx_n dy_1..dy_n \qquad (3A.3.11)$$

Then the form $\langle\cdot,\cdot\rangle$ defines a pre-Hilbert product in $\mathcal{S}_{bos}(R^n)$, i.e.

$<h,h> \geq 0$, for each $h \in \mathcal{S}_{bos}(R^n)$ ##

Let's show how to apply the theorem 3A.8 to study the the positive definiteness of certain Wightman functions of the field $:\exp i g\phi:(x)$. We illustrate the procedure in the case of the four point function. For convenience we pose $g=4\pi$. We get the following expression:

$$\langle :\exp i g\phi:(f):\exp i g\phi:(f)\Psi_0 \ , \ :\exp i g\phi:(f):\exp i g\phi:(f)\Psi_0 \rangle =$$

$$\int (12)(34)[13][14][23][24]\bar{f}(x_1)\bar{f}(x_2)f(x_3)f(x_4)dx_1dx_2dx_3dx_4$$

We have adopted the following conventions:

$$\xi_{ij}=x_i-x_j, \qquad (ij) = -\xi^2_{ij}, \quad [ij] = (-\xi^2_{ij}+i\epsilon(\xi_0)_{ij})^{-1}.$$

It is impossible to evaluate directly the previous integral, but we may rewrite it in the following way:

$$\langle :\exp i g\phi:(f):\exp i g\phi:(h)\Psi_0 \ , \ :\exp i g\phi:(f):\exp i g\phi:(h)\Psi_0 \rangle =$$

$$\int [13][14][23][24][-(x_1-x_2)^2]\bar{f}(x_1)\bar{f}(x_2)[-(x_3-x_4)^2] \qquad f(x_3)f(x_4)dx_1dx_2dx_3dx_4$$

Now theorem 3A.8 may be directly applied and it follows the positivity of the η-norm of the vector $:\exp i g\phi:(f):\exp i g\phi:(f)\Psi_0$.

More generally we may apply theorem 3A.8 to construct positive definite (coherent) subspaces of the Hilbert space K. This construction involves the introduction of the ϑ-vacua (MOR4) and will not be reproduced here. However the application of theorem 3A.8 constitutes the first step toward the study of the positive definiteness of the Wightman functions of the Wick exponential of the dipole field. We stress the fact that the method of introducing a mass for this study (WIG1) is not applicable here because the nonpositive definiteness of the dipole field is not only due to the infrared problem associated with the tip of the light cone.

4. ZWANZIGER MODEL.

4.1 INTRODUCTION.

In order to clarify the connection between the occurrence of infrared singularities of the so called "confining type" [MOR1] and the actual confinement of charges, it seems worthwhile to consider a simple four dimensional model in which a fermion field interacts with a dipole field.

The model is defined by the following equations:

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi(x) = g\gamma^{\mu}A_{\mu}(x)\psi(x) \qquad (4.1.1)$$

$$\Box^{2}A_{\mu}(x) = 0 \qquad (4.1.2)$$

(for the moment we omit specifying the renormalization required to give a meaning to the R.H.S. of eq. (4.1.1); it will be discussed below).

Following the strategy of the Schroer model we impose the condition

$$A_{\mu}(x) = \partial_{\mu}\phi(x), \qquad (4.1.3)$$

which guarantees its exact solvability. Indeed, in this case the solution of the model may be easily obtained in terms of "building block" fields; in fact we can write

$$\psi(x) = \exp[-ig\phi(x)]\psi_{o}(x) \qquad (4.1.4)$$

and this field solves eq. (4.1.1) if ψ_{o} satisfy the free Dirac equation:

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi_{o}(x) = 0 \qquad (4.1.5)$$

This model can be considered as a four dimensional analogueof the two dimensional Schroer model [SCHR2] and a resonable question to ask is whether the interaction of the fermion field with an infrared singular field (in this case the dipole) leads not only to the infraparticle phenomenon [SCHR2] (disappearance of states with sharp mass) but also costrains or even forbids the appareance of charged states (charge confinement).

The model has been investigated by Zwanziger in connection with quantum electrodynamics, in order to explore the possibility of reformulating the theory in terms of a "grandfather" Hertzian potential, and to verify the compatibility between the existence of physical charged states and the Gupta-Bleuler condition [ZWA].

Because of the occurence of infrared singularities of the confing type (the dipole propagator is proportional to $1/k^4$) the model may also be regarded as a toy model of quantum chromodynamics [MIN], also in view of exploring the infrared structures of the realistic case.

The aim of the analysis of this chapter is twofold. First of all we will revisit the model by careful exploitation of the associated Krein structure. The discussion of the previous chapters allows us to construct and fully control the solution of the model in terms of the Wick exponential of the dipole field, identify the correct charge and gauge automorphisms , discuss their implementation, and solve the subsidiary conditions which identifies the physical states of the theory. Several delicate points overlooked in the literature will emerge, like the characteriztion of the degrees of freedom of the model in term of the intrinsic field algebra $\mathcal{F}$ (defined using only the fields $\partial\phi$ and ψ), the charge content of the Krein space which provides a representation of $\mathcal{F}$, etc..

The focusing of these points will allow us a clear cut identification of the physical space which will have different features w.r, to those claimed in literature.

In particular we find:

i) All physical states have zero "electric" (or fermionic) charge (in contrast with the conclusions of [ZWA][MIN][BOG]).

The point is that the free fermion states $\psi_o(f)\Psi_0$ do not exist in the Krein space in which the intrinsic field algebra $\mathcal{F}$ is represented and therefore such states cannot exist as solution of the subsidiary condition either (this situation is completely analogous to that of the Schwinger model [MOR5] where is found that the bleached states do not belong to the Krein realization of the correspondig intrinsic algebra).

ii) The physical interpretation of the theory gives rise to the following picture: the physical states are generated by application to the vacuum of polynomials of neutral fermion bilinears; the physical space is therefore equivalent to the zero charge sector of a free fermion field.

In the second part of this chapter we further investigate the alternative look at the model initiated in section two. The starting point is the splitting of the dipole field as $\phi = \phi_1+\phi_2$, so that a non-negative physical space H′ can be selected by the supplementary condition which simply rules out the negative definite part of the Krein space of the theory: $\phi_2^-(f)H'=0$

This subsidiary condition is different from the Zwanziger one (which mimic closely the Gupta-Bleuler condition of QED_4); its main virtue is that of giving rise to a non trivial positive realization of the dipole field. In this way we get the following results: the physical states obtained as solutions of the new subsidiary condition are not necessarily neutral; however in the so obtained charged sectors the space-time translations cannot be implementable (by a mechanism similar to that already seen at the level of free dipole field). In this case one may view the confinement of the fermionic charge as due to the breaking of space-time translations, a mechanism which is realized also in the confinement of charged particles in QED_3 and of massless charged particles in QED_4 [MOR2].

In conclusion, the selection of the physical space by means of the Zwanziger's subsidiary condition [ZWA] is essentially related to the idea of giving much emphasis to the gauge invariance and of beeing as close as possible to the Gupta-Bleuler subsidiary condition of QED_4. This excludes the existence of charged states tout court.

In the second approach with non-covariant subsidiary condition one only demands to obtain a non trivial positive realization of the dipole field and the result is the lack of stability under time or space translations. In both cases physical states with well defined time translations must have zero charge.

4.2 Solution of the model and charge content of the associated Hilbert space.

To discuss the solution of our quantum field model we must give a meaning to the non-linear term in (4.1.1) e.g. by using the usual procedure of normal ordering. The interacting fermion field may therefore be written in the following way:

$$\psi(x) = :\exp(-ig\phi):(x)\psi_o(x) \tag{4.2.1}$$

where ψ_o is a free quantum Dirac field. The field $\psi(x)$ satisfies the "renormalized" equation of motion

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = \lim_{y\to x} g\gamma^\mu\partial_\mu(\phi(x)+i\,gW(x-y))\psi(y) \tag{4.2.2}$$

and obviously

$$\Box^2\phi(x) = 0 \tag{4.2.3}$$

$W(\xi) = -(4\pi)^{-2}\ln(-\xi^2+i\epsilon\xi_0)$ is the two-point function of the dipole field.The field algebra $\mathcal{F}$ associated to this model is generated by the fields ψ and $\partial\phi$, and is a proper subalgebra of the algebra $\mathcal{F}^B$, generated by the building blocks ψ_o and ϕ. The field algebra $\mathcal{F}$ contains, as it is usually understood, also operators obtained through Wick products and point splitting regularizations. As we will see, these procedures do not allow the reconstruction of the building block fields.

We may easily compute the Wightman functions of the field algebra $\mathcal{F}$. In particular the Wigthman functions involving the local charged fields have the following form:

$$\langle\Psi_0,\psi_{\alpha_1}(x_1)\ldots\psi_{\alpha_n}(x_n)\bar\psi_{\beta_1}(y_1)\ldots\bar\psi_{\beta_n}(y_n)\Psi_0\rangle =$$

$$= \frac{\prod_{1\le i<j\le n}[-(x_i-x_j)]^{g^2/16\pi^2}\;[-(y_i-y_j)]^{g^2/16\pi^2}}{\prod_{1\le i\le n}\;\prod_{1\le j\le n}[-(x_i-y_j)+i\epsilon(x_i^0-y_j^0)]^{g^2/16\pi^2}}\,.$$

$$\cdot \langle\Psi_0,\psi_{o\alpha_1}(x_1)\ldots.\psi_{o\alpha_n}(x_n)\bar{\psi}_{o\beta_1}(y_1)\ldots.\bar{\psi}_{o\beta_n}(y_n)\Psi_0\rangle \tag{4.2.4}$$

i) <u>A Krein realization of the Zwanziger model.</u>

Knowledge of the Wightman functions of $\mathcal{F}$ identifies a vector space $\mathcal{D}_0$ of local states, endowed with an indefinite inner product (see chapter two): $\mathcal{D}_0 = \mathcal{F}\Psi_0$. Again a topology is needed to close the local states and obtain a Hilbert space. Obviously $\mathcal{D}_0$ is contained in the Hilbert space

$$\overline{\mathcal{F}^B\Psi_0} = K^B = K^d \otimes H^f \tag{4.2.5}$$

where K^d is the Krein space representing the dipole field which we have studied in detail in chapter two, and H^f is the Hilbert space corresponding to a free Dirac field. Since we are interested in representing only the field algebra $\mathcal{F}$, we have to consider only Hilbert seminorms majorizing the Wightman functions of $\mathcal{F}$.

To this end we consider again the distributional kernel $K(x,y)$ defined in eq. (3.2.2). We may obtain the Hilbert seminorms we are looking for simply by taking the Wightman functions of $\mathcal{F}$ and replace the nonpositive distribution $W(x_i-x_j)$ whenever it appears, with the distributional kernel $K(x_i,x_j)$.

For example we get that we may majorize the two-point function of the field ψ in the following way:

$$|\langle\Psi_0,\psi(f)\psi(\bar{f})\Psi_0\rangle| \le \int \exp[g^2K(x,y)]S(x-y)f(x)\bar{f}(y)\ d^4x d^4y. \tag{4.2.6}$$

The conditions for which the integral at the R.H.S. actually converges are exactly the same which allow the existence of the Wick ordered exponential of the dipole field (see chapter 3).

The above seminorms define a (pre)Hilbert majorant topology τ [BOGN] and therefore a Hilbert space

$$H = \overline{\mathcal{F}\Psi_0}^{\tau} \subset K^B. \tag{4.2.7}$$

In H a sesquilinear form $<\cdot,\cdot>$ (the Wightman functional) and a Hilbert product $(\cdot,\cdot)$ are defined, and a bounded and self-adjoint metric operator η representing the sesquilinear form exists.
There are two kind of difficulties that we have to face at this point:
ii) the metric operator may be degenerate.
This problem is easily avoided by redefining the scalar product in H as follows [MOR1]:

$$(\cdot,\cdot) \longrightarrow (\cdot,[1-P_0]\cdot) \tag{4.2.8}$$

where P_0 is the projector on kernel of the metric operator η.
ii) the second problem is concerning the maximality of the so obtained Hilbert space structure.
As it is well known, not every closed subspace of a Krein space is itself a Krein space [BOGN]. The point is that η need not have a bounded inverse.
However starting from H we may reach a Krein space K simply by the following redefinition of the Hilbert product [MOR1]:

$$(\cdot,\cdot) \longrightarrow (\cdot,\cdot)_K = (\cdot,|\eta|\cdot). \tag{4.2.9}$$

(In this case the metric operator may be written as $P_+ - P_-$, where P_+ and P_- are the projectors corresponding to a fundamental decomposition of H; this decomposition exists because H is a non degenerate closed subspace of K^B [BOGN]).

ii) <u>The current operator.</u>

We want to define now the current associated with the local charged field $\psi(x)$. As usual we must define a procedure that allows the ultraviolet limit necessary to construct the bilinear invariants $:\bar{\psi}\gamma^\mu\psi:(x)$. We will use the well known point splitting gauge invariant limit procedure [KLA2][ZIM][LOW] .
First of all we need to evaluate the diverging terms of the expression $\bar{\psi}(x+\epsilon)\gamma^\mu\psi(x)$. We perform the computation for a massless ψ_o the massive case being analogous. We have that

$$\psi_o(x+\epsilon)\gamma^\mu\psi_o(x) = :\psi_o(x+\epsilon)\gamma^\mu\psi_o(x): - i\pi^{-2}\ \epsilon^\mu(\epsilon^\nu\epsilon_\nu)^{-2}; \tag{4.2.10}$$

besides we have that

$$:\exp ig\phi:(x+\epsilon)\ :\exp -ig\phi:(x) \simeq$$

$$Z(\epsilon)\ \{\ 1 + ig\epsilon^\mu\partial_\mu\phi(x) + \frac{i}{2}\ g\ \epsilon^\mu\epsilon^\nu\ \partial_{\mu\nu}\phi(x) + \frac{i}{6}\ g\ \epsilon^\mu\epsilon^\nu\epsilon^\rho\partial_{\mu\nu\rho}\phi(x) +$$

$$- \frac{g^2}{2}\epsilon^\mu\epsilon^\nu:\partial_\mu\phi(x)\partial_\nu\phi(x): - - \frac{g^2}{2}\epsilon^\mu\epsilon^\nu\epsilon^\rho:\partial_{\mu\nu}\phi(x)\partial_\rho\phi(x): +$$

$$- i\ \frac{g^3}{6}\epsilon^\mu\epsilon^\nu\epsilon^\rho:\partial_\mu\phi(x)\partial_\nu\phi(x)\partial_\rho\phi(x):\}. \tag{4.2.11}$$

where $Z(\epsilon) = (-\epsilon^2)^{g^2/16\pi^2}$.

Following Zimmermann [ZIM] we define the following operator:

$$Q^\mu(x,\epsilon) = \overline{\psi}(x+\epsilon)\gamma^\mu\psi(x-\epsilon) - \langle\Psi_0,\overline{\psi}(x+\epsilon)\gamma^\mu\psi(x-\epsilon)\Psi_0\rangle\sum_{n=1}^{3}:\Big[\int_{x-\epsilon}^{x+\epsilon}dy^\mu\partial_\mu\phi(y)\Big]^n: \tag{4.2.12}$$

The current operator may now be obtained by taking the ultraviolet (coinciding points) limit in the following way:

$$J^\mu(x) = Z(\epsilon)^{-1}\ \frac{1}{2}\ \{\ Q^\mu(x,\epsilon) + Q^\mu(x,-\epsilon)\ \}. \tag{4.2.13}$$

We obtain the following result:

$$J^\mu(x) = :\psi_o(x)\gamma^\mu\psi_o(x): \equiv\ J^\mu_{free}(x) \tag{4.2.14}$$

Thus the "electric" charge $Q^{el}_R = \int J^0(x)f_R(\mathbf{x})d^3\mathbf{x}$ here is given by the free fermion charge $Q^f_R = \int J^0_{free}(x)f_R(\mathbf{x})d^3\mathbf{x}$.

The fact that the current relative to the interacting fermion $\psi(x)$ is identical to the free fermion current is a consequence of the derivative coupling; indeed the same result is obtained by applying the previous formulae to compute the current operators of the Schroer [SCHR1] and the Klaiber [KLA1] models.

On the other side this method produce currents different from the free one when applied to models in which the interaction is not so simple (as in the case of the Schwinger [LOW] and Thirring [KLA2] models). Finally notice that we obtained the current operator using only the fields ψ and $\partial\phi$; therefore J^{μ}_{free} belongs to the intrinsic field algebra $\mathcal{F}$.

iii) Charge structure: the gauge transformations of the first kind and the fermionic charge.

We now study in some detail the charge structure associated with the various Hilbert spaces introduced.
First of all we consider the charge operators in the Hilbert space K^{B} associated to the building block fields. Since K^{B} has a tensor product structure, i.e. $K^{B} = K^{d} \otimes H^{f}$ the following analysis may be splitted: indeed since the global gauge transformations are locally generated by the charge Q^{d}_{R} (see eq. (4.2.16)) we may examine the status of the gauge charge in K^{d}. Analogously, since in the last paragraph we identified the fermionic current with the free one, we may study the status of the electric charge in H^{f}; then we will easily extend the so obtained results to K^{B}.
Let's briefly resume the discussion of the gauge symmetry that we began in chapter 2. There we defined the gauge automorphism as

$$\gamma^{\lambda}(\phi(f)) = (\phi + \lambda)(f) \tag{4.2.15}$$

We showed that this automorphism is generated in K^{d} by the local charge

$$Q^{d}_{R} = \int \partial_{0} \Box \phi(x)\ f_{R}(x) \alpha_{d}(x_{0})\ d^{4}x \tag{4.2.16}$$

(for the definition of the test functions f_{R} and α_{d} see chapter 2
Besides we obtained that the local charge Q^{d}_{R} converges in the weak graph limit sense to the operator $2\pi^{2}Q^{d}$, with $Q^{d} = i(\phi(v^{+}) - \phi(v^{-}))$, and that the domain of Q^{d} contains the dense set $\mathcal{D}^{d}_{0} = \mathcal{P}(\phi(f))\Psi_{0}$. Finally, we had that the automorphism γ^{λ} is actually η-unitarily implementable and its generator is exactly $2\pi^{2}Q$.

Now we want to extend these results to a larger domain including also Wick ordered exponentials of the dipole field, and namely we want to prove that the automorphism (4.2.15) is locally generated by Q^d_R also on the algebra $\mathcal{F}^d_1$ which includes Wick ordered exponentials of the dipole field and that w-lim $Q^d_R\ \mathcal{D}^d_1 = 2\pi^2 Q^d\ \mathcal{D}^d_1$

To this purpose we recall that in the previous chapter we obtained that it is possible to construct the Wick ordered exponential of the dipole field as a distribution on the test function space $\mathcal{P}^{\beta}_{\alpha}(R^4)$ (with certain values of α and β) with values operators on the Krein space K^d. The dense domain of definition of the Wick exponentials is the set $\mathcal{D}^d_1$ obtained by applying to the vacuum polynomials in the fields $\exp t\phi(f)$, $\phi(g)$, $:\exp z\phi:(h)$, with $f,g\in\mathcal{P}(R^4)$, $h\in\mathcal{P}^{\beta}_{\alpha}(R^4)$, $t,z\in C$. Besides we had on $\mathcal{D}^d_1$ the following commutation rules:

$$:\exp z\phi:(h)\ \exp t\phi(f) = \exp t\phi(f)\ :\exp z\phi:(h\ \exp[tzC_f]), \tag{4.2.17}$$

with $C_f(x) = [\phi(x),\phi(f)]$.

The gauge automorphism may be extended to the Wick exponentials by strong continuity; we obtain the following action:

$$\gamma^{\lambda}(:\exp z\phi:(f)) = \exp\lambda z\ :\exp z\phi:(f). \tag{4.2.18}$$

Using the previous commutation rules, and the fact that

$$\lim_{R\to\infty} [\phi(x),Q^d_R] = i \tag{4.2.19}$$

we obtain again that

$$i \lim_{R\to\infty} [Q^d_R, :\exp z\phi:(f)] = z\ :\exp z\phi:(f). \tag{4.2.20}$$

and therefore the global gauge symmetry is generated by the local charge Q^d_R also on the algebra $\mathcal{F}^d_1$, which includes polynomials of the fields $\exp t\phi(f)$, $\phi(g)$, $:\exp z\phi:(h)$.

Proposition 4.4.1: the global gauge automorphism γ^{λ} is implementated on $\mathcal{D}^d_1$ by the η-unitary group of operators $V(\lambda) = \exp\ 2\pi^2 i\lambda\ [(\phi(v^+)-\phi(v^-)]$; besides we have that Q^d_R converges weakly to $2\pi^2 Q^d$ on the dense set $\mathcal{D}^d_1$.

Proof: let

$$\Psi_h^z = :\exp z_1\phi:(h_1)\ldots.:\exp z_n\phi:(h_n)\Psi_0 = \sum_{j_1,..j_n} \frac{z^{j_1}\ldots.z^{j_n}}{j_1!\ldots.j_n!}\Psi_h^{(j_1,..j_n)}. \tag{4.2.21}$$

where $\Psi_h^{(j_1,..j_n)} \in K^{(j_1+..+j_n)}$. The results of appendix 3A.2 give us that for any M>0 it exists a constant $C_h(M)$ such that

$$\|\Psi_h^{(j_1,..j_n)}\|_K \le C_h(M)\ j_1!\ldots.j_n!\ M^{-(j_1+\ldots j_n)} \tag{4.2.22}$$

Besides we had that (see eq.(2.2.50)) for each $\Psi^{(m)}\in K^{(m)}$ the following estimate holds:

$$\|\phi(f)\Psi^{(m)}\|_K \le (m+1)^{1/2}\ q(f)\ \|\Psi^{(m)}\|_K\ . \tag{4.2.23}$$

Thus, since $q(\partial_0\Box f_R\alpha_d) \le \text{const}$, we obtain that there exists a constant C depending on Ψ_h^z but not on R such that

$$\|Q_R^d\ \Psi_h^z\ \|_K\le C'q(\partial_0\Box f_R\alpha_d) \le C. \tag{4.2.24}$$

Using again the commutation rules (4.2.12) we obtain that also for a general $\Psi \in \mathcal{D}_1^d$ there exists a $C_\Psi>0$ such that

$$\|Q_R^d\ \Psi\ \|_K \le C_\Psi \tag{4.2.25}$$

Finally, since w-lim $Q_R^d\ \mathcal{D}_0^d = 2\pi^2Q^d\ \mathcal{D}_0^d$ and Q^d is η-symmetric on $\mathcal{D}_1^d$ we obtain that

$$\lim_{R\to\infty}\langle\Phi,Q_R^d\Psi\rangle = \lim_{R\to\infty}\langle Q_R^d\Phi,\Psi\rangle = 2\pi^2\langle Q^d\Phi,\Psi\rangle = 2\pi^2\ \langle\Phi,Q^d\Psi\rangle \tag{4.2.26}$$

for any $\Phi\in\mathcal{D}_1^d$ and any $\Psi\in\mathcal{D}_0^d$.

Thus $\{Q_R^d\Psi\}$ is a bounded sequence of vectors, weakly convergent relatively to a dense set and therefore weakly convergent, i.e.

$$\text{w-}\lim_{R\to\infty}Q_R^d\mathcal{D}_1^d = 2\pi^2Q^d\mathcal{D}_1^d. \tag{4.2.28}$$ ##

Let us briefly revisit now some well known properties of the free Dirac field ψ_o, which is represented on the Fock space H_f. Let $\mathcal{F}^f$ the polynomial algebra generated by ψ_0. It contains (by Wick ordering) the free current $J^\mu_{free}(x) = :\bar\psi_o\gamma^\mu\psi_o:(x)$. We may construct the local charge operator

$$Q^f_R = \int J^0_{free}(x)\ f_R(x)\alpha_d(x_0)\ d^4x\ . \tag{4.2.29}$$

It is well known that the local charge Q^f_R converges neither in the strong nor in the weak graph limit sense when R tends to infinity [SCHR3].

Nevertheless, since

$$\lim <\Phi,Q^f_R\Psi_0> = 0,\ \forall\Phi \in \mathcal{D}^f_0 = \mathcal{P}(\psi_o(f))\Psi_0\ :\ \text{suppf is a compact}, \tag{4.2.30}$$

a charge operator Q^f may be obtained by taking the limit in the sense of the sesquilinear forms [VOL].

Q^f is a symmetric operator, with domain $\mathcal{D}^f_0$, which corresponds to the formally defined integral $\int J^0_{free}(x)d^3\mathbf{x}$. Furthermore one has the following commutation rules on the local states:

$$\lim\ [Q^f_R,\psi_o(f)] = [Q^f,\psi_o(f)] = -\ \psi_o(f) \tag{4.2.31}$$

Since $\lim <\Phi,Q^f_R\Psi_0> = 0$ implies $Q^f\Psi_0 = 0$ one has that

$$Q^f\psi_o(f)\Psi_0 = \lim\ [Q^f_R,\psi_o(f)]\Psi_0 = -\ \psi_o(f)\Psi_0. \tag{4.2.32}$$

Analogously we obtain that $Q^f\ \overline{\psi_o}(f)\ \Psi_0 = \overline{\psi_o}(f)\ \Psi_0$.

More generally, we have that

$$Q^f\ \mathcal{P}(\psi_o(f)\overline{\psi_o}(g)\Psi_0 = \lim\ [Q^f_R,\ \mathcal{P}(\psi_o(f)\overline{\psi_o}(g)]\Psi_0, \tag{4.2.33}$$

where the limit is obtained for finite values of R via the locality property.

Now we are ready to examine the charge structure of the space K^B and the corresponding field algebra $\mathcal{F}^B$.

First of all we notice that we may extend the action of the fermionic charge Q^f to the whole $\mathcal{F}^B$ in the obvious way:

$$[Q^f,\phi(f)] = 0,\ [Q^f,\psi(f)] = -\ \psi(f),\ [Q^f,\overline{\psi}(f)] = \overline{\psi}(f), \tag{4.2.34}$$

with corresponding definitions for the action of Q^f on $\mathcal{D}^B=\mathcal{F}^B\Psi_0$.

On the other side we obtain as an easy consequence of proposition 4.4.1

the fact that Q_R converges weakly to $2\pi^2 Q^d$ on the set $\mathcal{D}^d_1 \otimes \mathcal{D}^f_0$, which is dense in the Hilbert space K^B. An argument entirely similar to that used in proposition 4.4.1 then shows that

$$\text{w-}\lim_{R\to\infty} Q^d_R \mathcal{D}^B = 2\pi^2 Q^d \mathcal{D}^B. \tag{4.2.35}$$

Let us now focus our attention to the "intrinsic" Hilbert space H.

<u>Proposition</u> <u>4.4.2</u>: the charge $Q^T_R = Q^d_R - Q^f$ converges weakly to zero on the set $\mathcal{D} = \mathcal{F}\Psi_0$.

(A word of caution is in order: here weak convergence has obviously to be understood in the sense of the Hilbert space H which is a proper subspace of K^B and therefore weak convergence in H is distinct from weak convergence in K^B).

<u>Proof</u>: let Ψ, $\Psi' \in \mathcal{D}$, with $\Psi' = \mathcal{A}\Psi_0$, $\mathcal{A} \in \mathcal{F}$. We obtain that

$$\lim_{R\to\infty} \langle \Psi, (Q^d_R - Q^f)\Psi' \rangle = \lim_{R\to\infty} \langle \Psi, [(Q^d_R - Q^f), \mathcal{A}]\Psi_0 \rangle + \lim_{R\to\infty} \langle \mathcal{A}^\dagger \Psi, (Q^d_R - Q^f)\Psi_0 \rangle =$$

$$\lim_{R\to\infty} \langle \mathcal{A}^\dagger \Psi, Q^d_R \Psi_0 \rangle, \tag{4.2.36}$$

and this because $Q^f \Psi_0 = 0$ and

$$\lim_{R\to\infty} [Q^T_R, \mathcal{F}] = 0. \tag{4.2.37}$$

Thus we must compute the expression $\lim_{R\to\infty} \langle \Psi, Q^d_R \Psi_0 \rangle$, with $\Psi \in \mathcal{D}$, and it may be directly verified that this matrix element converges to zero. The point is that $\langle \Psi, Q^d_R \Psi_0 \rangle$ may be different from zero only if the state Ψ is generated by applying to the vacuum an equal number of ψ and $\bar{\psi}$; this in turn implies that $\langle \Psi, Q^d_R \Psi_0 \rangle$ converges to zero when the limit is performed. Indeed let's consider the simplest case in which

$$\Psi = \psi_\alpha(f)\bar{\psi}_\beta(g)\Psi_0 \tag{4.2.38}$$

It follows that

$$\lim_{R\to\infty} \langle Q^d_R \Psi_0, \Psi \rangle = \lim_{R\to\infty} \int \langle Q^d_R \Psi_0, [\phi(x) - \phi(y)]\Psi_0 \rangle \; [-(x-y)^2 + i\epsilon(x_0 - y_0)]^{-g^2/16\pi^2}$$

$$S_{\alpha\beta}(x-y)f(x)g(y)d^4xd^4y = 0, \tag{4.2.39}$$

because

$$\lim_{R\to\infty} \langle Q^d_R\Psi_0, [\phi(x)-\phi(y)]\Psi_0\rangle = 0. \tag{4.2.40}$$

It easy to understand that the general matrix element converges to zero for the same reason. Thus the charge Q^T annihilates the vacuum weakly. Since it is true again that $\|Q^d_R\ \Psi\|_H \leq C_\Psi$, it follows that Q^T_R converges weakly to zero on $\mathcal{D}$.

Thus we have the following result: fix an increasing sequence of positive numbers R_j such that $\lim_{j\to\infty} R_j = \infty$. The elements of the sequence

$$\{Q^T_{R_j}\} \quad j\in N \quad D(Q^T_{R_j})=\mathcal{D}, \tag{4.2.41}$$

are unbounded η-symmetric operators on the Hilbert space H. This sequence has a weak graph limit Q^T which is an η-symmetric densely defined operator : indeed we just proved that $Q^T_{R_j}\Psi$ converges weakly for any $\Psi \in \mathcal{D}$. Since $\mathcal{D}$ is dense in H we may apply an easy extension [MOS3] of theorem VIII.28 that may be found in [REE1] and obtain that $D(Q^T)\supset\mathcal{D}$, that Q^T is η-symmetric and therefore closable. Since $Q^T\mathcal{D}=0$ we finally obtain that the closure of Q^T is the zero operator on H:

$$\bar{Q}^T H = 0. \tag{4.2.42}$$ ##

The previous result holds also in the Krein space K , since the definition (4.2.9) implies that

$$\|Q^T_R\ \Psi\|_K \leq \|\eta\|\ \|Q^T_R\Psi\|_H \leq C'_\Psi ,$$

and therefore the charge operator Q^T_Rconverges weakly to zero also in K. These facts have an important consequence: the states $\psi_o(f)\Psi_0$ cannot belong to the space H or K. Indeed if this would be the case we should have

$$\bar{Q}^T\ \psi_o(f)\Psi_0 = Q^f\ \psi_o(f)\Psi_0 = -\ \psi_o(f)\Psi_0 \tag{4.2.43}$$

which contradicts $\bar{Q}^T H = 0$ (or $\bar{Q}^T K = 0$).

Thus the states $\psi_o(f)\Psi_0$, which do not belong to H or K, cannot belong

to any of their subspaces, and there are no free electrons or positron states in the Krein space K.
This implies that it is impossible to get the space of free electrons and positrons as solution of any supplementary condition one imposes to get the physical interpretation of the model as has been claimed in [ZWA][MIN][BOG] (see [MOR4] for a similar conclusion in the Schwinger model).

4.3 ZWANZIGER SUBSIDIARY CONDITION.

As usual in the context of indefinite metric quantum field theory, we must now specify a subsidiary (or supplementary) condition which selects the physical Hilbert space of the theory. We briefly recall that this condition is usually given as an operatorial condition of the form $AH'= 0$, and the physical space is then obtined as the quotient $H_{phys} = H'/H''$, where H'' is the null subspace of H' [MOR1].

The first supplementary condition which we examine is that proposed by Zwanziger [ZWA]. This condition mimic closely the Gupta-Bleuler supplementary condition of QED_4 .
Let's briefly spell in some detail this analogy: the Gupta-Bleuler formulation of QED_4 is given in terms of a local field algebra $\mathcal{F}^{G.B.}$ generated by the vector potential A_μ and the charged fields ψ and $\bar{\psi}$ [MOR3][GUP][BLE]. There is a gauge group acting on $\mathcal{F}^{G.B}$ and leaving the observable subalgebra $\mathcal{A}$ invariant; the guge transformtions of the first kind are generated by the electromagnetic current J_μ which is also the source of the vector potential A_μ:

$$\Box A_\mu(x) = J_\mu(x), \qquad \partial^\mu J_\mu(x) = 0, \tag{4.3.1}$$

$$\lim_{R\to\infty} \int [J_0(x),\psi(y)] \; f_R(x)\alpha_d(x_0) \; d^4x = \lim_{R\to\infty} [Q_R^{el},\psi(y)] = q\psi(y), \tag{4.3.2}$$

$$\lim_{R\to\infty} [Q_R^{el},A_\mu(y)] = 0. \tag{4.3.3}$$

The gauge transformation of the second kind are generated by the conserved current $B_\mu = \partial_\mu \partial^\nu A_\nu$:

$$Q^\Sigma \equiv \int \Sigma(x) \overleftrightarrow{\partial_0}\, \partial^\nu A_\nu(x) \tag{4.3.4}$$

$$[Q^\Sigma, \psi(x)] = q\Sigma(x)\psi(x), \quad [Q^\Sigma, A_\mu(x)] = \partial_\mu \Sigma(x) \tag{4.3.5}$$

where $\Box\Sigma = 0$, $\Sigma(x_0, x) \in \mathcal{S}(R^3_x)$.

The Gupta-Bleuler subsidiary condition, which selects the physical states, is obtained by the request of gauge invariance:

$$Q^\Sigma \Psi = 0 \quad \Leftrightarrow \quad \partial^\nu A^-_\nu\, \Psi = 0. \tag{4.3.6}$$

This condition contemporarly ensures the validity of the Maxwell equations. In our quantum field theory model this condition gives

$$\partial^\nu A^-_\nu\, \Psi = \Box\phi^-\, \Psi = 0 \tag{4.3.7}$$

An important point in this analogy is to regard ϕ as a purely gauge degree of freedom, and therefore the condition of "gauge invariance" of the physical states can be written as the vanishing of the generator of gauge transformations $\partial^\nu A_\nu = \Box\phi \equiv \Lambda$

For the study of the Zwanziger subsidiary condition in the intrinsic Hilbert space H is convenient to think to H as embedded in the space K^B. Let us compute an explicit expression for the operator $\Lambda^-(k)$. Using the results concerning the canonical quantization of the dipole field we obtain the following formula (see chapter 2):

$$\Lambda^-(f) = \langle \Box \bar{f}, \chi \rangle \phi(v^-) +$$

$$+ (2\pi)^{-3/2} \int \exp(i\omega t - ikx)[(1 - i\omega t)a_1(k) + i\omega t a_2(k)] d^3k/2\omega^{-3/2} \Box f(x) d^4x . \tag{4.3.8}$$

By Fourier transformation we then obtain that

$$\Lambda^-(k) = \vartheta(k_0)\, \delta(k^2)\, [2(2\pi)^{1/2}\, \omega^{3/2}\, b(k) + 2\pi\, \tilde{\chi}(\omega, k)\phi(v^-)], \tag{4.3.9}$$

where $b(k) = a_1(k) - a_2(k)$, $\omega = |k|$, and the supplementary condition may be rewritten as

$$\Lambda^{-}(k)H'=0. \tag{4.3.10}$$

First of all we want to show that the auxiliary supplementary condition

$$\Lambda^{-}(k)K^{B'} \tag{4.3.11}$$

implies the condition

$$\phi(v^{-})K^{B'}= 0. \tag{4.3.12}$$

To this end we should prove that in some sense

$$\lim_{\omega\to 0} \omega^{3/2}b(k)K^{B'}=0. \tag{4.3.13}$$

Observe that $b(k)$ acts only on the dipole degrees of freedom of the Hilbert space K^{B} (and of its subspaces), and that

$$(\int b(k)f(k)d^{3}k\ )\ K^{d,(n)} \subset K^{d,(n-1)}. \tag{4.3.14}$$

This implies that we may restrict ourselves to consider the one-particle subspace of K^{d}. Let therefore $h_{n}(k)$ a δ-converging sequence, and let $\Psi\in K^{d,(1)}$. According to the decomposition introduced in chapter 2, we may identify the L^{2} component of the vector Ψ and this is a vector function F whose components f_{1} and f_{2} belong to the Hilbert space $L^{2}(R^{3}-\{0\},\omega^{-3}d^{3}k)$. This fact implies that the components f_{1} and f_{2} should behave near the origin like ω^{ϵ}, with $\epsilon>0$. Therefore we get that

$$(\int\omega^{3/2}b(k)h_{n}(k)d^{3}k\ )\Psi\ = \int h_{n}(k)[f_{1}(k)+f_{2}(k)]d^{3}k\ \xrightarrow[n\to\infty]{} 0, \tag{4.3.15}$$

and this implies that

$$\text{s-}\lim_{n\to\infty}\ [\int h_{n}(k)\ \omega^{3/2}\ b(k)]K = 0 \tag{4.3.16}$$

which is a stronger rigorous version of the eq. (4.3.13). Therefore, since for any $\Psi\in K^{B}$ we have

$$\text{s-}\lim_{n\to\infty}\ [\int h_{n}(k)\ \ [2(2\pi)^{1/2}\ \omega^{3/2}\ b(k)\ +\ 2\pi\ \tilde{\chi}(\omega,k)\phi(v^{-})]d^{3}k]\ \Psi\ =\ 2\pi\ \phi(v^{-})\Psi \tag{4.3.17}$$

it follows that the auxiliary supplementary condition (4.3.12) implies the condition (4.3.13).

Now we may complete our discussion of the Zwanziger supplementary condition and show that no charged state solves this condition.
To this end we notice that

$$\phi(v^-)H \subseteq H \tag{4.3.18}$$

and that the operator $\phi(v^-)$ is proportional to the fermionic charge, when restricted to H. These facts follow easily by the following commutation rules

$$[\phi(v^-),\partial_\mu\phi(x)] = 0 \tag{4.3.19}$$

$$[\phi(v^-),\psi(x)] = -\frac{ig}{4\pi^2}\psi(x) \tag{4.3.20}$$

and by the facts that the field $\phi(v^-)$ annihilates the vacuum. Consequently, since by eq. (4.3.12) it must be $\phi(v^-)H' = 0$, H' cannot contain any charged state and consequently there is no charged state in H_{phys} (the same conclusions hold also for K_{phys}).
Thus we conclude that the physical space of the interacting model is equivalent (modulo vectors of zero η-norm) to the zero-charge sector of a free fermion field. The charged states are therefore confined.

4.4 Non covariant subsidiary condition.

The subsidiary condition which is the object of this paragraph is essentially singled out by the request that the degrees of freedom of the dipole field are not completely swept out from the physical (quotient) space. In this perspective such condition has some connection with the philosophy of Narnhofer and Thirring who exhibited a positive realization of the dipole field.

Clearly this appear as the point of view that one should take if he wants to mimic the case in which the infrared singularities of the confining type characterize the non-gauge part of the "gluon" propagator.

In contrast with the case of the Zwanziger subsidiary condition, the non-covariant subsidiary condition does not decrete from the start that the dipole field describes only non physical degrees of freedom and in fact it gives rise to a non-trivial positive realization of the dipole field, as in Narnhofer and Thirring [NAR].

Let us consider the following construction:

the dipole field is characterized by the two-point function $W(\xi) = -(4\pi)^{-2}\ln(-\xi^2+i\epsilon\xi_0)$. We may introduce two independent noncovariant fields ϕ_1 and ϕ_2, which are characterized by the following two-point functions (in this section the word "field" is used in some loose sense; we will call fields also objects which are not covariant):

$$W_1(x,y) = W(x-y) - \frac{i}{2}\, x_0 y_0 D^-(x-y),$$

$$W_2(x,y) = \frac{i}{2}\, x_0 y_0 D^-(x-y).$$

where D^- is the negative frequency part of the Pauli-Jordan commutator function of mass m=0. It turns out that W_2 is a negative definite distribution while W_1 is not definite because of an infrared singularity of the kind $|k|^{-3}$.

The status of ϕ_1 is closely analogue to that of the massless two-dimensional scalar free field (except for covariance).

Now we consider for convenience a slight modification of the Krein structure introduced for the dipole field in chapter two.

$\langle f,g\rangle_1 = \int W_1(x,y)\ \bar{f}(x)g(y)d^4xd^4y$

$\langle f,g\rangle_2 = \int W_2(x,y)\ \bar{f}(x)g(y)d^4xd^4y$

Obviously $\langle f,g\rangle = \langle f,g\rangle_1 + \langle f,g\rangle_2$. Let now ξ be a test function such that $\tilde{\xi}(0) = 1,\ \langle\xi,\xi\rangle_1 = 0$.

and define as usual $f_o = f - \tilde{f}(0)\xi$. Let

$(f,g)_1 = \langle f_o,g_o\rangle_1 + \langle f,\xi\rangle_1\langle\xi,g\rangle_1 + \bar{\tilde{f}}(0)\tilde{g}(0)$,

$(f,g)_2 = -\langle f,g\rangle_2$.

The usual procedure of completion and quotient gives us the one-particle Hilbert space in the form of a direct sum:

$K^{(1)} = K_1^{(1)} \oplus K_2^{(1)}$,

$K_1^{(1)} = L^2(C_+ - \{0\}) \oplus V \oplus \Xi, \quad K_2^{(1)} = L^2(C_+ - \{0\})$.

This Krein structure coincides with the one introduced in chapter two if $\int x_0 y_0 D(x-y)\bar{\xi}(x)\xi(y) = 0$. In any case this structure makes it transparent that the one dimensional subspaces V and Ξ appear because of the singularities of the field ϕ_1. The usual Fock construction gives the complete Hilbert space where to represent the fields ϕ_1 and ϕ_2. Observe that

$$\langle\Psi_0, Q_R^d\ \phi\ (g_2)\Psi_0\rangle = -\frac{i}{2}\int x_0 y_0 D^-(x-y)\{\partial_0 \Box\ f_R(x)\alpha_d(x_0)]\}g(y)d^4xd^4y =$$

$$-\frac{i}{2}\int \omega\ \{\tilde{f}_R(-k)\tilde{\alpha}_d(-k_0)\tilde{g}(k)\}\Big|_{C_+}\ d^3k \xrightarrow[R\to\infty]{} 0.$$

It follows that

$$i \lim_{R\to\infty} [Q_R^d, \phi_1(f)] = 4\pi^2\tilde{f}(0),$$

$$i \lim_{R\to\infty} [Q_R^d, \phi_2(f)] = 0.$$

Thus we obtain that

$\gamma^{\lambda}(\phi_1(f)) = (\phi_1+\lambda)(f), \quad \gamma^{\lambda}(\phi_2(f)) = \phi_2(f),$

which correctly give $\gamma^{\lambda}(\phi(f)) = (\phi+\lambda)(f)$.

Consider now the intrinsic field algebra $\mathcal{F}$ and the Hilbert space H = $\overline{\mathcal{F}\Psi_0}^{\tau}$. Obviously we may split $\partial\phi$ into $\partial\phi_1+\partial\phi_2$. The first important thing to notice is that we may introduce the field ϕ_2 as an operator valued distribution on the Hilbert space H. This fact may be verified directly. Indeed using the results of chapter two we obtain easily the representation of ϕ_2 on the space K^B. For example we have that

$$\phi_2(f)\Psi_0 = \begin{pmatrix} 0 \\ k_0\partial/\partial k_0 \tilde{f}(k)|_{C_+} \end{pmatrix} .$$

We shall show that this state is available also in H. Indeed it may be constructed as a norm convergent sequence of vectors of the form $\partial_0\phi_2(g_n)\Psi_0$. Consider indeed the following sequence of test functions:

$$\tilde{g}_n(k) = \frac{\partial}{\partial k_0}\tilde{f}(k) - \frac{k^2}{2(\omega+\frac{1}{n})} \frac{\partial^2}{\partial k_0^2}\tilde{f}(k).$$

We have that

$$k_0 \frac{\partial}{\partial k_0}(k_0\tilde{g}_n(k))|_{C_+} = \omega\frac{\partial}{\partial k_0}\tilde{f}(k)|_{C_+} + \frac{\omega^2}{n}(\omega+\frac{1}{n})^{-1}\frac{\partial^2}{\partial k_0^2}\tilde{f}(k)|_{C_+} .$$

Note that

$$\lim_{n\to\infty} \omega\frac{\partial}{\partial k_0}\tilde{f}(k)|_{C_+} + \frac{\omega^2}{n}(\omega+\frac{1}{n})^{-1}\frac{\partial^2}{\partial k_0^2}\tilde{f}(k)|_{C_+} = 0$$

pointwise. Furthermore we have that

$$\int |k_0^2\frac{\partial}{\partial k_0}\tilde{g}_n(k)|^2 \,|_{C_+}\omega^{-3}d^3k = n^{-2}\int \omega^4(\omega+\frac{1}{n})^{-2}|\frac{\partial}{\partial k_0}\tilde{f}(k)|^2|_{C_+}\omega^{-3}d^3k \le$$

$$(2n)^{-1}\int |\frac{\partial}{\partial k_0}\tilde{f}(k)|^2|_{C_+} d^3k \longrightarrow 0.$$

It follows that

$$\lim_{n\to\infty} \|\phi_2(f)\Psi_0 - \partial_0\phi_2(-ig_n)\Psi_0\|_H = 0.$$

Thus it is possible to introduce the field ϕ_2 in the Hilbert space H.

The situation regarding the field ϕ_1 is very different: indeed the infrared singularities of ϕ_1 are much worse and for this reason it cannot be represented as an operator on H. This fact may be verified directly too, but it may also be shown by a different argument.
Indeed in this case it should be $\overline{Q}^T\phi_1(f)\Psi_0 \neq 0$ but we already know that this is impossible.
As a consequence of these results we have that the the intrinsic field algebra $\mathcal{F}$ may be equivalently generated by the fields $\{\partial_\mu\phi_1,\ \phi_2,\ :\exp(-ig\phi_1):\psi_o\}$.
Indeed if A and B are two independent free fields (i.e. quantum fields having vanishing truncated Wightman functions) we have that [GAR1]:

$$:(A+B)^n:(x) = \sum_{j=0}^{n}\binom{n}{j} :A^{n-j}:(x)\ :B^j:(x).$$

Therefore (if the following series converges in some sense) we have that

$$:\exp(A+B):(x) = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j} :A^{n-j}:(x)\ :B^j:(x) =$$

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{1}{(n-j)!j!} :A^{n-j}:(x)\ :B^j:(x) = :\exp A:(x)\ :\exp B:(x).$$

Consequently we have that

$$:\exp z\phi:(x)\psi_o(x) = :\exp z\phi_1:(x)\ :\exp z\phi_2:(x)\psi_o(x)$$

and thus

$$\psi_1(x) = :\exp(-ig\phi_1):(x)\psi_o(x) =$$

$$\lim_{y\to x} Z(y,x)\ :\exp(ig\phi_2):(y)\ :\exp(-ig\phi):(x)\psi_o(x) =$$

$$\lim_{y\to x} Z(y,x)\ :\exp(ig\phi_2):(y)\psi(x).$$

We are now ready to state the supplementary condition that we want to study. It may be written using the field ϕ_2, since we know that it can be introduced as an operator in H:

$$\phi_2^-(f)H' = 0.$$

This supplementary condition may be easily solved and we obtain that

$$H' = \overline{\{\mathcal{P}(\partial\phi_1(f),\ :\exp\text{-}i\,g\phi_1:(h))\Psi_0\}}^{\tau}.$$

We now want to check is that H′ is really a positive subspace of H. As we we already said the situation of $:\exp\text{-}i\,g\phi_1:$ is closely analogue to that of the massless two-dimensional free field; we may try to reply the method by Wightman [WIG] and study the positive definiteness of H′ by intoducing a mass.

First of all we notice that the Fourier transform of the two point function of the dipole field may be (formally) written in the following way:

$$\tilde{W}(k) = (2\pi)^{-1}\vartheta(k_0)\delta(k^2)$$

(we said formally because of the ill-definition of the previous distribution at the tip of the cone k=0). Let us introduce a mass by defining

$$\tilde{W}_m(k) = (2\pi)^{-1}\vartheta(k_0)\delta(k^2-m^2) = -(2\pi)^{-1}\vartheta(k_0)\frac{\partial}{\partial m^2}\ \delta(k^2-m^2).$$

By Fourier transformation it follows that

$$\tilde{W}_m(k) = -(2\pi)^{-3}\ \frac{\partial}{\partial m^2}\int \exp(-ikx)\ \vartheta(k_0)\delta(k^2-m^2)\ d^4k = i\ \frac{\partial}{\partial m^2}\ D^-_m(x-y),$$

where D^-_m is the negative frequency part of the Pauli-Jordan function [BOG2]. In this way we obtain a massive regularization of W_1:

$$W_{1,m}(x,y) = i\Big(\frac{\partial}{\partial m^2} - \frac{1}{2}\,x_0y_0\Big)\ D^-_m(x-y).$$

and a direct calculation shows that

$$\int W_{1,m}(x,y)\ \bar{f}(x)g(y)d^4xd^4y$$

We may give an approximate formula that holds for small values of m: by [BOG2] we know that

$$D^-_m(\xi) \simeq \frac{i}{4\pi^2(-\xi^2+i\,\epsilon\xi_0)} + \frac{i}{16\pi^2}\ m^2\ \ln[m^2(-\xi^2+i\epsilon\xi_0)],$$

and thus we get

$$W_{1,m}(x,y) \simeq -\frac{1}{16\pi^2}\ \{\ \ln[m^2(-\xi^2+i\epsilon\xi\ )] + 1\ \} + \frac{x_0y_0}{8\pi^2(-\xi^2+i\epsilon\xi_0)}.$$

Now using the reconstruction theorem [SW] we may introduce a noncovariant free field $\phi_{1,m}$, whose two-point function is exactly $W_{1,m}$ (We stress that the covariance plays no role in the reconstruction theorem). Besides, we may also introduce the Wick ordered powers of the field $\phi_{1,m}$ and a suitable restriction of the test function space (see chapter 3) allow us to sum the series defining the Wick ordered exponential of $\phi_{1,m}$.
Let us examine the two point function of this Wick exponential:

$$\langle\Psi_0, :\exp(-ig\phi_{1,m}):(x)\ :\exp(ig\phi_{1,m}):(y)\ \Psi_0\rangle = \exp[g^2 W_{1,m}(x,y)] \simeq$$

$$\exp g^2\left(-\frac{1}{16\pi^2} \{ \ln[m^2(-\xi^2+i\epsilon\xi\ )] + 1 \} + \frac{x_0 y_0}{8\pi^2(-\xi_0^2+i\epsilon\xi\ )} \right).$$

By construction this is a positive definite (noncovariant) generalized function. Obviously its positivity is mantained if we multiply it by the factor $\exp(1/16\pi^2)\exp[(\ln m^2)/16\pi^2]$. Taking the limit for $m\longrightarrow 0$ we obtaimn the positiv definiteness of the kernel

$$\exp[g^2 W_1(x,y)] = \exp g^2\left(-\frac{1}{16\pi^2} \{ \ln(-\xi^2+i\epsilon\xi\ ) + \frac{x_0 y_0}{8\pi^2(-\xi_0^2+i\epsilon\xi\ )} \right).$$

Now we may exactly reply Wightman's analysis [WIG] (or Swieca's argument [SWI2]) and it follows the positive semidefinition of the space H'.

We come at the conclusion that the non covariant subsidiary condition admits charged states which solve it. However the time translation are a well defined symmetry only for the (gauge invariant subspace of the) vacuum sector , because of the instability of the charged sectors under time translations(we do not repeat here the detailed analysis of this phenomenon; see chapter 2 for more details).

5. A SIMPLE MODEL OF GAUGE SYMMETRY BREAKING.

5.1 INTRODUCTION.

A characteristic feature of the standard treatment of unified weak and electromagnetic interactions [ABE] is that it relies on the non vanishing expectation value of the Higgs field, i.e. the spontaneous breaking of the global gauge symmetry. Such statement is based on perturbation calculations; a non perturbative analysis of the problem indicates that such breaking is more problematic than one might think [KEN]. Actually the discussion of the Higgs model in local and covariant gauges (the only renormalizable ones) is intrigued by the necessary lack of positivity. Indeed it has been proved quite generally that in the Abelian Higgs model [FER] the breaking of global gauge symmetry requires indefinite metric in the general α-gauge and implies that the space-time translations cannot commute with the metric operator. Actually a slight improvement of the argument shows that global gauge symmetry breaking requires the existence of dipole singularities $\delta'(k^2)$ in the correlation functions of the model.
Thus, also in this case, a careful discussion of the properties of the model, and in particular of the spontaneous symmetry breaking, requires a control of the Hilbert space structures associated to dipole field singularities (vacuum degeneracy, cluster property, etc.).
In order to explore at least at a simple level the breaking of the global gauge symmetry and the role of the dipole singularities we revisit a model first suggested by Ferrari.
This simplified model may shed light on certain aspects of the Higgs model, like the charge screening and the mathematical phenomena connected with spontaneous symmetry breaking.
As we will see, the experience gained in the previous sections allows to completely control the mathematical structure of the model and answer the questions the questions not settled in the previous literature. In particular we will show that in the above simple model

i) the Higgs field must necessarily have a non zero vacuum expectation value (apart from the trivial case) i.e. the solution of the model must show the "breaking" of the global gauge symmetry.

ii) The non invariance of the Wightman functions does not preclude the implementability of the global gauge transformations in the Krein space of the model. This is due to the appearence of the infrared states which makes the vacuum essentially unique but not unique as we already discussed in chapter 2. It is evident that a crucial role is played here by the Krein structure associated to the Wightman functions. This problem could not be decided without making reference to the Hilbert space structure of the model.

iii) The subsidiary condition exclude charged physical states (charge screening) [KOG][STR2][STR3].

In contrast with the confinement case, such screening of the charge is here associated to the non invariance of the Wightman functions w.r. to the global gauge transformations.

Before discussing the model we give the proof of the existence of $\delta'(k^2)$ singularities in the correlation functions of the abelian Higgs model. This model is formally defined by the Lagrangian

$$L = -\frac{1}{2}(\partial_\mu A_\nu)^2 + \frac{\alpha}{2}\partial_\mu A_\nu \partial^\nu A^\mu + (\partial_\mu + ieA_\mu)\phi\,(\partial_\mu - ieA_\mu)\phi^* - \lambda^2|\phi|^4 + $$
$$+ \mu^2|\phi|^2. \tag{5.1.1}$$

This Lagrangian is invariant under the gauge transformations of the second kind:

$$A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu\xi(x) \ , \qquad \phi(x) \longrightarrow \exp(-i\xi(x))\phi(x) \tag{5.1.2}$$

where $\xi(x)$ is a real smooth solution of the equation $\Box\xi = 0$, and one looks for a solution which is not invariant under the gauge transformations of the first kind, $\phi \longrightarrow \exp(i\vartheta)\phi$.

From a more rigorous point of view [STR3] we consider the local field algebra $\mathcal{F}$ generated by the vector potential A_μ and the Higgs field χ.

The fields ϕ and A_μ are operator valued distributions on a Krein-Hilbert space H, and have a common dense domain $\mathcal{D}$. As usual, we denote with $(\cdot,\cdot)$ the Hilbert product and with $<\cdot,\cdot>$ the indefinite product by means of which one computes the transition amplitudes; the metric operator is denoted by η and it is such that $\eta^2=1$.
There exists a non trivial group of local automorphisms α_Ξ of the field algebra $\mathcal{F}$ whose infinitesimal action is given by eq. (5.1.2) and is generated by a local conserved current $J^\Xi_\mu(x)$:

$$\delta\chi(f) = i \lim_{R\to\infty} [Q^\Xi_R , \chi(f)] = i \lim_{R\to\infty} \int \alpha_d(x_0) f_R(x) [Q^\Xi_R , \chi(f)]\, d^4x \quad (5.1.3)$$

where the functions f_R and α_d have been defined in chapter 2.
In particular the current which generates the global gauge transformations is the source of the gauge field $A_\mu(x)$:

$$\Box A_\mu(x) - \alpha\partial_\mu\partial^\nu A_\nu(x) = J_\mu(x). \quad (5.1.4)$$

We come to the result of this section.

<u>Proposition</u> <u>5.1.1</u>: in the (local and covariant) α-gauge formulation of the U(1) Higgs model, the spontaneous breaking of the U(1) gauge symmetry implies the existence of singularities of the kind $\delta'(k^2)$ in the Fourier transform of the correlation function

$$\langle\Psi_0, A_\mu(x)B\,\Psi_0\rangle \quad (5.1.5)$$

where A_μ is the gauge vector field, and B is the field operator whose correlation functions are not U(1) invariant.

<u>Proof</u>: thanks to the generalization of the Goldstone theorem to indefinite metric QFT [STR3], we have that the Fourier transform of the matrix element

$$F_\mu(x) = \langle\Psi_0, J_\mu(x)B\,\Psi_0\rangle \quad (5.1.6)$$

contains a $\delta(k^2)$.
The locality property now implies that the operator

$$J'_\mu(x) = J_\mu(x) - \alpha\partial^\nu F_{\nu\mu}(x), \tag{5.1.7}$$

could be equally well taken as thelocal generator of the global gauge transformations,where $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$,and thus also the spectral function of

$$F'_\mu(x) = \langle\Psi_0, J_\mu(x)B\,\Psi_0\rangle \tag{5.1.8}$$

contains a $\delta(k^2)$. Since $J'_\mu(x) = (1-\alpha)\,\Box A_\mu(x)$ it follows that

$$F'_\mu(x) = (1-\alpha)\Box\langle\Psi_0, A_\mu(x)B\,\Psi_0\rangle \tag{5.1.9}$$

and this implies that the Fourier transform of $G(x) = \langle\Psi_0, A_\mu(x)B\,\Psi_0\rangle$ contains a singularity of the kind $\delta'(k^2)$. ##

This theorem can be seen as a slight improvement of the arguments already contained in [FER] and [STR3]. The same conclusion has already been reched by Ferrari at the first order of perturbation theory, while our proof is non-perturbative and generally valid.

5.2 The Ferrari Model

This model is defined by the following coupled (unrenormalized) field equations, which describe the quantum electrodynamics of a (complex) massless scalar field:

$$\Box A_\mu \equiv J_\mu = ie[\chi^*(\partial_\mu + ieA_\mu)\chi - \chi(\partial_\mu - ieA_\mu)\chi^*] \qquad (5.2.1)$$

$$\Box \chi = - ie\partial^\mu(A_\mu\chi) - ieA^\mu(\partial_\mu + ieA_\mu)\chi \qquad (5.2.2)$$

These equations are invariant under the group of local gauge transformations

$$A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu\alpha(x) \quad , \quad \chi(x) \longrightarrow \exp[-ie\alpha(x)]\chi(x), \qquad (5.2.3)$$

where α is a smooth real solution of the equation $\Box\alpha(x)=0$.

It is clear the (5.2.1) and (5.2.2) cannot be exactly solved unless some additional condition is imposed. We look for the more general solution that may be obtained by the ansatz (derivative coupling)

$$A_\mu(x) = \partial_\mu\phi(x) \quad , \quad \chi(x) = \exp(-ie\phi(x))\rho(x). \qquad (5.2.4)$$

Furthermore we require that J_μ generates the gauge transformations; we write this condition as the following equal-time commutation relations:

$$[J_0(0,x),\chi(0,y)] = -e\delta(x-y)\chi(0,y),$$

$$[J_0(0,x),\dot{\chi}(0,y)] = -e\delta(x-y)\dot{\chi}(0,y). \qquad (5.2.5)$$

First of all we want to know what are the equations of motion for the building block fields $\phi(x)$ and $\rho(x)$. These equations are easily determined by substituting eq. (5.2.4) into the (5.2.1) and (5.2.2), and taking into account the fact that J_μ is a conserved current.
We obtain the following equations:

$$\Box \rho(x) = 0 \ , \tag{5.2.6}$$

$$\Box^2 \phi(x) = 0. \tag{5.2.7}$$

Thus $\rho(x)$ is a solution of the free massless Klein-Gordon equation while $\phi(x)$ solves the dipole equation.
However the fields ρ and ϕ are not independent; indeed they are linked by the equation (5.2.1).
Using again the gauge invariant point-splitting limit procedure which we outlined in chapter 4 we obtain the following equation:

$$\begin{aligned}
\partial_\mu \Box \phi(x) &= J_\mu = ie:[\chi^*(\partial_\mu + ieA_\mu)\chi - \chi(\partial_\mu - ieA_\mu)\chi^*]: = \\
&= ie:[\ :\exp(ie\phi):\rho^* \ (\partial_\mu + ie\partial_\mu\phi) \ :\exp(-ie\phi):\rho \ + \\
&- \ :\exp(-ie\phi):\rho \ (\partial_\mu - ie\partial_\mu\phi) \ :\exp(ie\phi):\rho^*]: = \\
&= ie \ [\ :\rho^*\partial_\mu\rho:(x) \ - \ :\rho \ \partial_\mu\rho^*:(x)] \ .
\end{aligned} \tag{5.2.8}$$

The interacting current is again equal to the free current associated to the field ρ (as we have already said, this feature is shared by all the models in which the coupling is purely derivative) and, by the field equations, it must be equal to the field $\partial_\mu \Box \phi(x)$.
This relation implies that if we want a nontrivial current the field ϕ must be a true dipole (i.e. it must be $\Box \phi(x) \neq 0$) otherwise we would have $J_\mu(x) = 0$. Consequently the two-point function of the field ϕ must contain the distribution $\log(-\xi^2 + i\epsilon\xi_0)$ whose properties have been studied in detail in chapter 2. The most general local and covariant solution for the two-point function of the field ϕ is then the following:

$$\langle \Psi_0, \phi(x)\phi(y)\Psi_0 \rangle = -\frac{c_1}{16\pi^2}\log(-\xi^2 + i\epsilon\xi_0) + \frac{c_2}{-\xi^2 + i\epsilon\xi_0} + c_3. \tag{5.2.9}$$

We obtain that the current $J_\mu(x)$ must have vanishing two-point function: $\langle\Psi_0, J_\mu(x)J_\nu(y)\Psi_0\rangle = 0$.

The commutation relations (5.2.5) can be used to fix the constants in the expression (5.2.9): we obtain that $c_1 = 1$ while c_2 and c_3 may be taken equal to zero without loss of generality. Furthermore we obtain the following commutators:

$$[\partial_0 \Box \phi(0,x), \rho(0,y)] = [\partial_0 \Box \phi(0,x), \dot{\rho}(0,y)] = 0 \qquad (5.2.10)$$

Summarizing we have that the complex field ρ satisfies the massless Klein-Gordon equation $\Box\rho=0$, and has zero equal-time commutators with its own current $J_\mu(x) = ie[:\rho^*\partial_\mu\rho:(x) - :\rho\,\partial_\mu\rho^*:(x)]$.

The free time evolution implies that also for unequal times one gets

$$[J_\mu(x), \rho(y)] = [J_\mu(x), \rho^*(y)] = 0 \qquad (5.2.11)$$

i.e.

$$\Delta_{\rho^*\rho}(x-y) :\partial_\mu\rho:(x) + \partial_\mu \Delta_{\rho\rho}(x-y):\rho^*(x): - \Delta_{\rho\rho}(x-y) :\partial_\mu\rho^*:(x) +$$
$$- \partial_\mu \Delta_{\rho^*\rho}(x-y):\rho(x): = 0,$$

$$\Delta_{\rho^*\rho^*}(x-y) :\partial_\mu\rho:(x) + \partial_\mu \Delta_{\rho\rho^*}(x-y):\rho^*(x): - \Delta_{\rho\rho^*}(x-y) :\partial_\mu\rho^*:(x) +$$
$$- \partial_\mu \Delta_{\rho^*\rho^*}(x-y):\rho(x): = 0 \qquad (5.2.12)$$

where $\Delta_{\rho^*\rho}(x-y) = [\rho^*(x), \rho(y)]$, etc..

There are two possibilities for the solution of eq. (5.2.6): it may be either

$$:\rho:(x) = :\rho^*:(x) \qquad (5.2.13)$$

or all the commutators are equal to zero.

These solutions are in fact the same; indeed if condition (5.2.13) is satisfied, we may write $\rho(x) = \sigma(x) + c$, where $\sigma(x)$ is a real field with vanishing one-point function. Using the equation (5.2.8) we get that

$$\partial_\mu \Box \phi(x) = ie\ [\ :\rho^* \partial_\mu \rho:(x) - :\rho\ \partial_\mu \rho^*:(x)] = ie(c^* - c)\partial_\mu \sigma(x) \qquad (5.2.14)$$

and therefore

$$\rho(x) = \frac{1}{ie(c^* - c)} \Box \phi(x) + c. \qquad (5.2.15)$$

which corresponds to the solution exibithed by Ferrari [FER].

It is clear from the (5.2.15) the fact that also in this case all the commutators involving the fields ρ and ρ^* are actually zero.

In the general case, writing

$$\rho(x) = c_1 \Box \phi(x) + \tau(x) \qquad (5.2.16)$$

where the complex field $\tau(x)$ does not contain terms linear in $\Box\phi(x)$, we obtain

$$\partial_\mu \Box \phi(x) = ie\ [\ :\rho^* \partial_\mu \rho:(x) - :\rho\ \partial_\mu \rho^*:(x)] =$$

$$ie\ [c_1^* \partial_\mu \tau(x) - c_1 \partial_\mu \tau^*(x)] \Box \phi(x) + ie[c_1 \tau^*(x) - c_1^* \tau(x)]\ \partial_\mu \Box \phi(x) +$$

$$+ie[\tau^*(x) \partial_\mu \tau(x) - \tau(x) \partial_\mu \tau^*(x)] \qquad (5.2.17)$$

This implies that $\tau(x) = c_2$ and

$$ie[c_1 c_2^* - c_1^* c_2] = 1 \qquad (5.2.18)$$

Thus we arrive at the most general solution for the charged field:

$$\chi(x) = c_1\ ::\exp(-ie\phi):\Box\phi\ :(x) + c_2\ :\exp(-ie\phi):(x) \qquad (5.2.19)$$

where the complex constants c_1 and c_2 must satisfy condition (5.2.18).

It then follows that

$$\langle \Psi_0, \chi(x) \Psi_0 \rangle = c_2 \qquad (5.2.20)$$

where c_2 cannot be zero becuse of (5.2.18).

Thus the exact solution of eq. (5.2.1) and (5.2.2) that is obtained using the derivative coupling ansatz must necessarily exhibit the spontaneous breking of global gauge symmetry.

We may now shortly conclude the discussion of this model by using the results that have already been obtained in chapter 2 and 3.
In particular we find that:

i) the field algebra of the interacting model is represented on the Krein-Hilbert space K^d which we have constructed in chapter 2. As we know this space contains infrared Poincare' invariant states. The field algebra of the model admit a strong closure (in the topology of the Krein representation space) which contains also the infrared operators $\phi(v^+)$ and $\phi(v^-)$.
ii) The global gauge symmetry which is locally generated by the current (5.2.8) is implementable in the Krein space K^d by the group of η-unitary operators (2.3.16) whose expression we rewrite:

$$\Gamma^\lambda = \exp 2\pi^2 i\ Q \tag{5.2.21}$$

$$Q = i\ [\phi(v^+)-\phi(v^-)]. \tag{5.2.22}$$

Furthermore the charge Q is the the weak graph limit of the local charge Q_R. Also in this case the Wightman functions are not invariant and nevertheless the symmetry is implementable because of the existence of the infrared states. Indeed the vector $\Gamma^\lambda\Psi_0 - \Psi_0$ is Poincare' invariant and has zero η-norm.

iii) The physical space is again identified by the Gupta-Bleuler condition (which assures gauge invariance):

$$\Box\phi^-(x)\ K' = 0 \qquad (5.2.22)$$

There is no charged state that solves this condition and the charge is totally screened. We stress again that this screening is associated to to the non invariance of the Wightman functions w.r. to the global gauge transformations [KOG]. The distinction that we make between the phenomena of "confinement" and "screening" is based on the following difference: a confined charge is associated with an unbroken symmetry while the screening of a charge is associated to a symmetry not shared by the Wightman functions.

VI. Infrared structures in alpha-gauge formulation of two dimensional quantum electrodynamics

6.1 Introduction.

One of the most interesting exactly soluble models is the two dimensional quantum electrodynamics (QED_2), known as the "Schwinger model" because Schwinger was the first to realize that this model could be exactly solved [SCH].

A rigorous tretment of this model in a local and covariant gauge was given in the pioneering work by Lowenstein and Swieca [LOW] who chose to work in the Landau gauge (the one identified by the condition $\partial^\mu A_\mu=0$). A more general discussion in a generic α-gauge has been given by Capri and Ferrari [CAP2][CAP3]. They pointed out that such generic case involves the introduction of a two-dimensional scalar field satisfying the equation

$$\Box^2\phi = 0 \tag{6.1.1}$$

and of its Wick ordered exponential.

However, a full Hilbert space realization of such α-gauges is still lacking (the only one that has been completely characterized is the Landau gauge [MOR5]) and in our opinion this is not a mere academic question since even if the physical interpretation is independent on the chosen gauge, the mathematical structures leading to the physical phenomena of confinement, screening, symmetry breaking, etc. may be different in different gauges.

Such mathematical control on the building block fields needed for the discussion of the generic α-gauge formulation of QED_2 is the subject of this chapter. In particular we discuss the two crucial steps for the Krein realization of QED_2 namely the Hilbert space realization of the field ϕ (eq.(6.1.1)) and the construction of its Wick exponential as a distribution with values operators acting on the same Hilbert space.

This constuction meets two different kinds of problem:

i) first, as discussed in the introduction of chapter 3 it requires making reference to a Krein topology; the latter plays a crucial role in proving the strong convergence of the partial sums

$$\sum_{n=0}^{N} :\phi^n:(f)\Psi \tag{6.1.2}$$

ii) secondly, such strong convergence can be achieved only if the test function f is restricted to belong to a suitable space of type S, (in general more restricted than the Schwarz space of test function of fast decrease).

Another motivation for discussing the mathematical properties of the field ϕ and of its Wick ordered exponential is the fact that the Wightman functions of the latter are not tempered distributions, as noticed by the authors in [CAP3] and especially emphasized by Wightman. Actually in his paper on the choice of test functions in quantum field theory [WIG2] Wightman considers this model as an interesting prototype of a non tempered field theory, as one should expect in local and covariant formulation of realistic gauge quantum field theories.

The points that we have further clarified are the following:

i) the Wightman functions of the Wick ordered exponential of the field ϕ are non tempered distributions and we determinate the class of test function spaces for which such Wightman functions are well defined, actually $\mathcal{S}_\alpha(R^2)$ with $\alpha<\frac{1}{2}$.

ii) These function spaces allow a respectable of quantum fields, since for them the Fourier tranform is well defined (contrary to what stated in [ROT] and [CAP3]) and it maps $\mathcal{S}_\alpha(R^2)$ onto $\mathcal{S}^\alpha(R^2)$. The spaces $\mathcal{S}_\alpha(R^2)$ satisfying the condition $\alpha<\frac{1}{2}$ contain test functions having compact support so that the locality property is defined in the usual way.

On the other side the corresponding Fourier transformed spaces $\mathcal{S}^\alpha(R^2)$ do not contain any function of compact support. It follows that the definition of the spectral condition requires some generalization of the notion of support (carrier) [DER] [FAI].

iii) The construction of the Wick exponential as an operator in the Hilbert space of states associated to the Wightman functions further

restricted the test function spaces allowed to the condition $\alpha<\frac{1}{4}$.
We see here a prototype of a phenomenon which has to be expected in gauge QFT, namely that the Hilbert space realization of the fields involves more severe infrared singularities than those occuring in the Wightman functions.
The Wick exponential of ϕ provides a significant example of field of type S which is not tempered because of its infrared singularities.

As a final remark we notice that our discussion of the mathematical properties of the Wick exponential of the field ϕ also provide the mathematical framework for the fields occuring in the massive Schwinger model, as discussed in [ROT]. In this paper the authors point out that one must expect two point function singularities exactly of the type $\exp\{M^2\xi^2\ln(-\xi^2+i\epsilon\xi_0)\}$. They argue that such singularity is so severe that it prevents the existence of the corresponding quark field. As a result of our discussion, we have that fields which such kind of singularities have a well defined mathematical status as fields of type S.

6.2 ALPHA-GAUGE FORMULATION OF QED$_2$

In this section we follow the exposition given by Wightman [WIG3] of QED$_2$ in a generic gauge. We give for simplicity the classical formulation of the model and then we comment about the its quantum version.
The classical equation of motion corresponding to the two-dimensional quantum electrodynamics are the following:

$$-i\gamma^{\mu}(\partial_{\mu} + iqA_{\mu})\psi(x) = 0 \tag{6.2.1}$$

$$\Box\, A_{\mu}(x) + (\alpha-1)\partial_{\mu}(\partial^{\nu}A_{\nu})(x) = q\,(\bar{\psi}\,\gamma_{\mu}\psi)(x) \tag{6.2.2}$$

where the the Dirac matrices are given by

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \gamma^1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \gamma^5 = \gamma^0\gamma^1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{6.2.3}$$

and ψ is a two-component field on which the gamma matrices act by left multiplication.

As it usual in the context of explicitely soluble models the solution of these equations may be obtained by considering the following ansatz:

$$\psi(x) = \exp(-iq\Omega)(x)\ \psi_o(x) \tag{6.2.4}$$

where $\Omega(x) = c(x) + \gamma^5 d(x)$ and ψ_o satisfies the free massless Dirac equation:

$$-i\gamma^\mu \partial_\mu \psi_o(x) = 0 \tag{6.2.5}$$

Since

$$\gamma^\mu \exp(-iq\Omega(x)) = \exp(-iq\Omega'(x))\gamma^\mu \tag{6.2.6}$$

where Ω' is the field obtained by Ω by replacing d with -d it follows that

$$-i\gamma^\mu \partial_\mu \exp(-iq\Omega(x))\psi_o(x) = -q\gamma^\mu(\partial_\mu c + \epsilon_{\mu\nu}\partial^\nu d)\exp(-iq\Omega(x))\psi_o(x) \tag{6.2.7}$$

where $\epsilon^{\mu\nu}$ is the totally antisymmetric tensor characterized by the condition $\epsilon^{01} = 1$, and we have taken into account the relation $\gamma^\mu\gamma^5\partial_\mu = \gamma_\mu(\epsilon^{\mu\nu}\partial_\nu)$.

We have that the Dirac equation (6.2.1) is satisfied if we choose

$$A^\mu = \partial^\mu c + \epsilon^{\mu\nu}\partial_\nu d. \tag{6.2.8}$$

This implies

$$\partial^\mu A_\mu(x) = \Box c(x) \tag{6.2.9}$$

As for the current we have that

$$\bar\psi(x)\gamma_\mu\psi(x) = \bar\psi_o(x)\ \exp(i\Omega'(x))\ \gamma_\mu \exp(i\Omega(x))\psi_o(x) = \bar\psi_o(x)\gamma_\mu\psi_o(x) \tag{6.2.10}$$

and therefore

$$\alpha\, \partial^{\mu} \Box c + \epsilon^{\mu\nu} \partial_{\nu} \Box d = q\, \bar{\psi}_{o}(x) \gamma_{\mu} \psi_{o}(x) \tag{6.2.11}$$

It is well known [KLA2] that there exist potentials ρ and σ such that

$$q\, \bar{\psi}_{o}(x) \gamma_{\mu} \psi_{o}(x) = (\pi)^{-1/2} \partial_{\mu} \rho = (\pi)^{-1/2} \epsilon_{\mu\nu} \partial^{\nu} \sigma \tag{6.2.12}$$

We may choose

$$\Box c = \frac{\beta}{\alpha} (\pi)^{-1/2} q\, \rho \ , \quad \Box d = (1-\beta)\ (\pi)^{-1/2} q\, \sigma \tag{6.2.13}$$

Thus the quantum version of the Schwinger model leads naturally to introduce quantized fields satisfying the equations

$$\Box^{2} c = 0 \ , \quad \Box^{2} d = 0. \tag{6.2.14}$$

At this point the quantum solution of the model is obtained by substituting classical fields with the corresponding quantum operators. This procedure involves the definition of an appropriate point splitting regularization [KLA2][ZIM][LOW]. As it is well known, an important effect of nonclassical nature is obtained in this way: indeed it happens that equation (6.2.10) does not hold any more: the current obtained with the quantum interacting fermion field is not proportional to that of the free theory and a free massive boson field appears [LOW]. We do not enter further in this aspect of the model; for more details see [KLA2][LOW].

6.2 Krein Structure

In this section we introduce and discuss certain Krein structures associated with a two dimensional scalar field satisfying the equation $\Box^{2}\phi = 0$. We closely follow the general framework of chapter two and we assume that the truncated n-point function vanish.

This implies that we may limit ourselves to discuss the two-point function and the Krein structures associated to it. The complete Hilbert space of the theory will be obtained by the usual Fock procedure. The most general local and covariant distribution which solves the equation $\Box^2 W(\xi) = 0$ may be written in the following way:

$$W(\xi) = (a\xi^2 + b) \log \mu^2(-\xi^2 + i\epsilon\xi_0) \tag{6.3.1}$$

where $\vartheta(t) = \begin{cases} 1 & \text{if } t>0 \\ 0 & \text{if } t<0 \end{cases}$, and $\epsilon(t) = \vartheta(t) - \vartheta(-t)$.

We do not loose in generality by taking b=0. We now give the Fourier transform of this distribution. We have that

$$\tilde{W}(k^0,k^1) = F(u,v) = G(u,v) + G(v,u) \tag{6.3.2}$$

$$G(u,v) = c \frac{\partial^2}{\partial u^2} (\vartheta(u) \log u)\, \delta'(v), \tag{6.3.3}$$

where $u = k^0+k^1$, $v = k^0-k^1$ are the light-cone variables, and c is a certain constant depending on a which we will fix later. We list the obvious transformation rules:

$$2\frac{\partial}{\partial u} = \frac{\partial}{\partial k^0} + \frac{\partial}{\partial k^1}, \qquad 2\frac{\partial}{\partial v} = \frac{\partial}{\partial k^0} - \frac{\partial}{\partial k^1}, \qquad dudv = 2dk^0dk^1.$$

As usual, we use the two-point function to introduce a sesquilinear form in $\mathcal{S}(R^2)$:

$$\langle f,g\rangle = 2\pi \int \tilde{W}(k)\, \bar{\tilde{f}}(k)\tilde{g}(k)\, dk = \pi \int F(u,v)\, \bar{\tilde{f}}(u,v)g(u,v)\, dudv. \tag{6.3.4}$$

We obtain the following explicit expressions:

$$\langle f,g\rangle = -\frac{\pi c}{4}\int dk^1 \log|2k^1|\{(\frac{\partial}{\partial k^0} + \epsilon(k^1)\frac{\partial}{\partial k^1})(\frac{\partial^2}{(\partial k^0)^2} - \frac{\partial^2}{(\partial k^1)^2})\bar{\tilde{f}}(k)\tilde{g}(k)\}|_{C_+} \tag{6.3.5}$$

$$\langle f,g\rangle = -\pi c\int_0^\infty \log u \frac{\partial^2}{\partial u^2}\{\frac{\partial}{\partial v} (\bar{\tilde{f}}(u,v)\; g(u,v))\}|_{v=0}\, du\, +$$
$$-\pi c\int_0^\infty \log v \frac{\partial^2}{\partial v^2} \{\frac{\partial}{\partial u} (\bar{\tilde{f}}(u,v)\; g(u,v))\}|_{u=0} dv \tag{6.3.6}$$

If the functions f and g belong to the space

$$\mathcal{S}_r(R^2) = \{f\in\mathcal{S}(R^2),\ \tilde{f}(0) = \frac{\partial\tilde{f}}{\partial k^0}(0) = \frac{\partial\tilde{f}}{\partial k^1}(0) = 0\}, \tag{6.3.7}$$

we may partially integrate and obtain

$$\langle f,g\rangle = \frac{\pi c}{4}\int dk^1|k^1|^{-2}\{(\frac{\partial}{\partial k^0} - \epsilon(k^1)\frac{\partial}{\partial k^1})\ \bar{\tilde{f}}(k)\tilde{g}(k)\}|_{C_+} =$$

$$\pi c\int_0^\infty u^{-2}\{\frac{\partial}{\partial v}(\bar{f}(u,v)g(u,v))\}|_{v=0}du + \pi c\int_0^\infty v^{-2}\{\frac{\partial}{\partial u}(\bar{f}(u,v)g(u,v))\}|_{u=0}dv \qquad (6.3.8)$$

Now we introduce Krein structures in $\mathcal{S}_r(R^2)$. To do this we use the explicit formula depending on the usual momentum variables (equivalent structures may be obtained using the light cone variables).
We choose c=-1 and define

$$D = -\frac{\partial}{\partial k^0} + \epsilon(k^1)\frac{\partial}{\partial k^1} \qquad (6.3.9)$$

We may rewrite the (6.3.8) in the following way:

$$\langle f,g\rangle = \frac{\pi}{8}\int dk^1|k^1|^{-2}\{[(1+D)\bar{\tilde{f}}(k)][(1+D)\tilde{g}(k)] - [(1-D)\bar{\tilde{f}}(k)](1-D)\tilde{g}(k)\}|_{C_+}. \qquad (6.3.10)$$

Then we introduce the following family of pre-Hilbert products in $\mathcal{S}_r(R^2)$:

$$[f,g]_{\alpha,\pm} = \frac{1}{8}\pi\int\{\alpha(k^1)(\bar{F}_1(k^1)G_1(k^1)+\bar{F}_2(k^1)G_2(k^1)) +$$
$$\pm\ \beta(k^1)(\bar{F}_1(k^1)G_2(k^1)+\bar{F}_2(k^1)G_1(k^1))\ |k^1|^{-2}dk^1 \qquad (6.3.11)$$

$$\text{with } F_1(k^1) = ((1+D)\tilde{f}(k))|_{C_+},\quad F_2(k^1) = ((1-D)\tilde{f}(k))|_{C_+}. \qquad (6.3.12)$$

The function $\alpha(k^1)$ is real-valued and greater than one; β is fixed by the conditions $\beta^2 = \alpha^2-1$, $\beta\geq 0$.
We assume that $\alpha\in\mathcal{C}^\infty(R-\{0\})$; α may be diverging near the origin but its order of divergence cannot be greater than 1 , otherwise the integral (6.3.11) become meaningless:

$$\alpha(k^1) \simeq (k^1)^{-d} \quad , \quad \text{with } 0 \le d < 1 \tag{6.3.13}$$

It is possible to verify that for each $f,g \in \mathscr{S}_r(R^2)$

$$|\langle f,g\rangle|^2 \le [f,f]_{\alpha,\pm}[g,g]_{\alpha,\pm} \tag{6.3.14}$$

i.e. each of the products $[\cdot,\cdot]_{\alpha,\pm}$ defines a pre-Hilbert majorant topology .

We may complete the space $\mathscr{S}_r(R^2)$ in the Hilbert topology induced by each of the previous scalar products. In the following we limit ourselves to characterize the Hilbert spaces that may be obtained by completing $\mathscr{S}_r(R^2)$ the Hilbert products $[\cdot,\cdot]_{\alpha,-}$. To simplify the notation we omit everywhere the minus sign.

The Hilbert spaces associated with the topologies defined by the products $[\cdot,\cdot]_{\alpha,+}$ may be obtained exactly in the same way.

Following the same steps we already walked in chapter two we obtain the family of Hilbert spaces $H^{(1)}_\alpha$, which are the spaces of functions defined on $\{C_+ - \{0\}\}$ with values in C^2, and such that

$$\int\{\alpha(k^1)(|F_1(k^1)|^2 + |F_2(k^1)|^2) - 2\mathrm{Re}\ \beta(k^1)\overline{F}_1(k^1)F_2(k^1)\}|k^1|^{-2}dk^1 < \infty. \tag{6.3.15}$$

Furthermore, there exists a family of bounded and selfadjoint operators η^0_α, each defined on the corresponding $H^{(1)}_\alpha$ and such that

$$\langle\Psi_1,\Psi_2\rangle = [\Psi_1, \eta^0_\alpha \Psi_2]_\alpha \ . \tag{6.3.16}$$

It follows that $(\eta^0_\alpha)^2 = 1$. This may also be understood by looking the explicit expression of η^0_α:

$$\eta^0_\alpha(k^1) = \begin{bmatrix} \alpha(k^1) & , & -\beta(k^1) \\ \beta(k^1) & , & -\alpha(k^1) \end{bmatrix} \tag{6.3.17}$$

We are now ready to introduce a Krein structure in $\mathscr{S}(R^2)$. Let us consider three test functions χ,ξ,λ belonging to $\mathscr{S}(R^2)$ and such that

$$\tilde{\chi}(0) = 1 \ , \quad \frac{\partial\tilde{\chi}}{\partial k^0}(0) = 0 \ , \ \frac{\partial\tilde{\chi}}{\partial k^1}(0) = 0,$$

$$\tilde{\xi}(0) = 0 \ , \quad \frac{\partial\tilde{\xi}}{\partial k^0}(0) = 1 \ , \ \frac{\partial\tilde{\xi}}{\partial k^1}(0) = 0,$$

$$\tilde{\lambda}(0) = 0 \ , \quad \frac{\partial\tilde{\lambda}}{\partial k^0}(0) = 0 \ , \ \frac{\partial\tilde{\lambda}}{\partial k^1}(0) = 1,$$

$$\langle\chi,\chi\rangle = \langle\chi,\xi\rangle = \langle\chi,\lambda\rangle = \langle\xi,\xi\rangle = \langle\xi,\lambda\rangle = \langle\lambda,\lambda\rangle = 0. \tag{6.3.18}$$

Given $f \in \mathcal{S}(R^2)$ we may extract its "regular" part in the following way:

$$f_0 = f - \tilde{f}(0)\chi - \frac{\partial\tilde{f}}{\partial k^0}(0)\xi - \frac{\partial\tilde{f}}{\partial k^1}(0)\lambda \ . \tag{6.3.19}$$

and rewrite the inner product (6.3.5) in the following way:

$$\langle f,g\rangle = \langle f_0,g_0\rangle + \overline{\tilde{f}}(0)\langle\chi,g\rangle + \tilde{g}(0)\langle f,\chi\rangle + \overline{\frac{\partial\tilde{f}}{\partial k^0}}(0)\langle\xi,g\rangle + \frac{\partial\tilde{g}}{\partial k^0}(0)\langle f,\xi\rangle + \overline{\frac{\partial\tilde{f}}{\partial k^1}}(0)\langle\lambda,g\rangle + \frac{\partial\tilde{g}}{\partial k^1}(0)\langle f,\lambda\rangle. \tag{6.3.20}$$

This makes transparent that each of the pre-Hilbert products

$$(f,g)_\alpha = [\,f_0,g_0]_\alpha + \langle f,\chi\rangle\langle\chi,g\rangle + \langle f,\xi\rangle\langle\xi,g\rangle + \langle f,\lambda\rangle\langle\lambda,g\rangle + \overline{\tilde{f}}(0)\tilde{g}(0) + \overline{\frac{\partial\tilde{f}}{\partial k^0}}(0)\frac{\partial\tilde{g}}{\partial k^0}(0) + \overline{\frac{\partial\tilde{f}}{\partial k^1}}(0)\frac{\partial\tilde{g}}{\partial k^1}(0) \tag{6.3.21}$$

defines a pre-Hilbert majorant topology on $\mathcal{S}(R^2)$, i.e.

$$|\langle f,g\rangle|^2 \le (f,f)_\alpha\,(g,g)_\alpha. \tag{6.3.22}$$

Again the usual procedures of completion and quotient give us a family of Hilbert spaces $K_\alpha^{(1)}$ and there exist self-adjoint and bounded operators η_α such that

$$\langle f,g\rangle = (f,\eta_\alpha g)_\alpha \tag{6.3.23}$$

<u>Theorem 6.1</u>: $K_\alpha^{(1)}$ is a Krein space.

<u>Proof</u>: We have to show that $(\eta_\alpha)^2 = 1$.
First of all we consider the following linear functionals defined on

$\mathcal{S}_r(R^2)$:

$$X(f) = \langle\chi,f\rangle, \quad \Xi(f) = \langle\xi,f\rangle, \quad \Lambda(f) = \langle\lambda,f\rangle \ . \tag{6.3.24}$$

We want to show that each of these functionals is discontinuous w.r. to the Hilbert topology defined on $\mathcal{S}_r(R^2)$ by the scalar product $[\cdot,\cdot]_\alpha$.
We consider at first the functional Ξ. Since $\tilde{\xi}(0)$ we may write

$$\langle\xi,f\rangle = \frac{\pi}{4}\int dk^1|k^1|^{-2}\{(\frac{\partial}{\partial k^0} - \epsilon(k^1)\frac{\partial}{\partial k^1})\ \overline{\tilde{\xi}(k)}\tilde{f}(k)\}|_{C_+} \tag{6.3.25}$$

Suppose that Ξ is continuous. The Riesz lemma implies that it must exist a vector $v_\xi \in H_\alpha^{(1)}$ such that

$$[v_\xi,f]_\alpha = \langle\xi,f\rangle \tag{6.3.26}$$

(here f has to be intended as representative of its own equivalence class). This equation gives the following representation for the components of the vector v_ξ:

$$v_{\xi,1}(k^1) = \alpha(k^1)[(1+D)\tilde{\xi}(k)]|_{C_+} - \beta(k^1)[(1-D)\tilde{\xi}(k)]|_{C_+} ,$$

$$v_{\xi,2}(k^1) = \beta(k^1)[(1+D)\tilde{\xi}(k)]|_{C_+} - \alpha(k^1)[(1-D)\tilde{\xi}(k)]|_{C_+} . \tag{6.3.27}$$

In this way we would get that

$$[v_\xi,v_\xi]_\alpha =$$

$$\frac{\pi}{4}\int dk^1|k^1|^{-2}[(\alpha(k^1)-\beta(k^1))|\tilde{\xi}(k)|^2 + (\alpha(k^1)+\beta(k^1))|D\tilde{\xi}(k)|^2]|_{C_+} = \infty.$$

Thus the functional Ξ cannot be continuous. Analogously the functional Λ is not continuous w.r. to the scalar product $[\cdot,\cdot]_\alpha$, and a slight modification of this method gives also the discontinuity of the functional X.
This implies that they exist three sequences of functions of $H_\alpha^{(1)}$ which we denote by $\{f_n^\chi\}$, $\{f_n^\xi\}$, $\{f_n^\lambda\}$ such that

$$\langle\chi,f_n^\chi\rangle = 1, \quad \lim_{n\to\infty}[f_n^\chi,f_n^\chi] = 0, \qquad \langle\xi,f_n^\xi\rangle = 1, \quad \lim_{n\to\infty}[f_n^\xi,f_n^\xi] = 0,$$

$$\langle\lambda,f_n^\lambda\rangle = 1, \quad \lim_{n\to\infty}[f_n^\lambda,f_n^\lambda] = 0. \tag{6.3.28}$$

Now note that the functionals X,Ξ and Λ are linearly independent and that their discontinuity implies that each of the hyperplanes $P(X,c)$, $P(\Xi,c')$ and $P(\Lambda,c'')$ is dense in $H^{(1)}_\alpha$, where

$P(X,c)=\{f\in H^{(1)}_\alpha : \langle\chi,f\rangle = c\}$,

etc.. These facts imply that the we may choose the previous sequences in such a way that

$$\lim_{n\to\infty}\langle\xi,f^\chi_n\rangle = 0, \qquad \lim_{n\to\infty}\langle\lambda,f^\chi_n\rangle = 0, \qquad \lim_{n\to\infty}\langle\chi,f^\xi_n\rangle = 0,$$

$$\lim_{n\to\infty}\langle\lambda,f^\xi_n\rangle = 0, \qquad \lim_{n\to\infty}\langle\chi,f^\lambda_n\rangle = 0, \qquad \lim_{n\to\infty}\langle\xi,f^\lambda_n\rangle = 0.$$

We have that the functionals X,Ξ and Λ are obviously continuous w.r. to any of the Hilbert products (6.3.21). The Riesz lemma implies that there exist three normalized vectors $v_\chi, v_\xi, v_\lambda \in K^{(1)}_\alpha$ such that

$$\langle\chi,f\rangle = (v_\chi,f)_\alpha \ , \quad \langle\xi,f\rangle = (v_\xi,f)_\alpha \ , \quad \langle\lambda,f\rangle = (v_\lambda,f)_\alpha \ , \qquad (6.3.29)$$

for each $f\in K^{(1)}_\alpha$.

Furthermore one may understand that

$$\text{s-}\lim_{n\to\infty} f^\chi_n = v_\chi, \quad \text{s-}\lim_{n\to\infty} f^\xi_n = v_\xi, \quad \text{s-}\lim_{n\to\infty} f^\lambda_n = v_\lambda. \qquad (6.3.30)$$

Indeed

$$\lim_{n\to\infty}(f^\chi_n - v_\chi \ , \ f^\chi_n - v_\chi)_\alpha = \lim_{n\to\infty}(f^\chi_n \ , \ f^\chi_n)_\alpha - 1 = 0 \qquad (6.3.31)$$

and analogously for the other two vectors. It is immediate to note that

$$\eta_\alpha v_\chi = \chi \ , \ \eta_\alpha v_\chi = \xi \ , \ \eta_\alpha v_\lambda = \lambda. \qquad (6.3.32)$$

Since we have that

$$\langle v_\chi,f\rangle = \lim_{n\to\infty}(f^\chi_n,f)_\alpha = \tilde{f}(0) = (\chi,f)_\alpha \qquad (6.3.33)$$

the non-degeneracy of the sesquilinear form $\langle\cdot,\cdot\rangle$ implies that

$$\eta_\alpha\chi = v_\chi \qquad (6.3.34)$$

Analogously we have that

$$\eta_\alpha\xi = v_\xi \ , \ \eta_\alpha\lambda = v_\lambda \qquad (6.3.35)$$

The joint continuity of any Hilbert product now implies that

$$(v_\chi, v_\xi) = (v_\chi, v_\lambda) = (v_\xi, v_\lambda) = 0 \tag{6.3.36}$$

Collecting these results and the fact that $(\eta^0_\alpha)^2 = 1$ we finally get that $(\eta_\alpha)^2 = 1$, and therefore the fact that $K^{(1)}_\alpha$ is a Krein space. ##

Thus we have that each of the Hilbert spaces that we have obtained shows a rich infrared structure; in particular they contain a three dimensional infrared subspace invariant under the Poincare' group which has also a counterpart in the field algebra. These facts have also important consequences for the structure of the gauge group. In particular we have that certain "large" gauge transformations are implementable. We will discuss these facts in a subsequent work.

6.4 The wick ordered exponential of the 2-dim dipole field.

In this section we concern ourselves with the construction of the Wick ordered exponential of the 2-dimensional dipole field.
As we have already said in chapter 3, this problem has two respects: first of all one needs to know the Wightman functions of the field in question and examine their distributional character.
The computation of the Wightman functions of the Wick ordered exponential of a free field is easy and is given by the following expression [WIG1]:

$$\langle \Psi_0, :\exp z_1\phi:(x_1) \ldots\ldots :\exp z_n\phi:(x_n)\Psi_0\rangle = \prod_{i<j} \exp z_i z_j\, W(x_i - x_j) \tag{6.4.1}$$

For instance, in this case we obtain the following two-point function:

$$\langle \Psi_0, :\exp z_1\phi:(x_1):\exp z_2\phi:(x_2)\Psi_0\rangle = \exp \{\frac{z_1 z_2}{4\pi}\, \xi^2 \log(-\xi^2 + i\epsilon\xi_0)\} \tag{6.4.2}$$

A second and more difficult problem consists in reconstructing the Wick-ordered exponential as a distibution whose values are operators

on a certain Hilbert space H.
The method we proposed to follow consists in finding a Krein-Hilbert space K where to reconstruct the free field ϕ, and then determinate a fundamental space $\mathcal{A}$ and a dense domain $\mathcal{D}_{\exp} \subset$ K such that the series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} :\phi:(f)\ \Psi \qquad f\in\mathcal{A}\ ,\quad \Psi\in\mathcal{D}_{\exp} \tag{6.4.3}$$

is strongly convergent.
This procedure leads in general to further restrictions on the test function spaces on which the n-point functions make sense.

Let's begin by examinating the distribution (6.3.2).
The 2-point Wightman function of the 2-dim dipole field is proportional to the distribution $\{\xi^2\log(-\xi^2+i\epsilon\xi_0)\}$, with ξ=x-y. Let us rewrite this distribution more explicitly; we have [GEL1]

$$W(\xi) = c\xi^2\{\log|\xi^2| + i\epsilon(\xi_0)\vartheta(\xi^2)\} \tag{6.4.4}$$

It follows that for any $\epsilon>0$ it exists a positive constant C_ϵ such that

$$|W(\xi)| \le C_\epsilon\ \{1 + |\xi^2|^{1+\frac{\epsilon}{2}}\}. \tag{6.4.5}$$

Since $|\xi^2| = |\xi_0^2 - \xi_1^2| \le \xi_0^2 + \xi_1^2 = \|\xi\|^2$ and $\xi_0^2 = (x_0-y_0)^2 \le 2x_0^2+2y_0^2$,

we obtain that $|\xi^2| \le 2\|x\|^2 + 2\|y\|^2$ and therefore

$$|W(\xi)| \le C'_\epsilon\ \{1 + (\|x\|^2 + \|y\|^2)^{1+\frac{\epsilon}{2}}\} \le C''_\epsilon\{1 + \|x\|^{2+\epsilon} + \|y\|^{2+\epsilon}\}.$$

We finally get that

$$|\exp zW(\xi)| \le \exp|z||W(\xi)| \le \mathrm{const}_\epsilon\ (\exp \|x\|^{2+\epsilon})(\exp \|y\|^{2+\epsilon}). \tag{6.4.6}$$

This implies that the integral $\int \exp zW(\xi)\ \bar{f}(x)\ g(y)\ d^2x\ d^2y$ is well defined when f and g belong to any of the spaces $\mathcal{S}_\alpha(R^2)$ [GEL2] which satisfies the condition $\alpha < \frac{1}{2}$.

These estimates may be replied exactly in the same way also when calculating the asymptotic behaviour of the n-point function; thus we obtain that the Wightman functions of the Wick-ordered exponential of the 2-dimensional dipole field are generalized functions belonging to $\mathcal{S}'_{\alpha}(R^2)$ with $\alpha < \frac{1}{2}$.

We now pass to the construction of the Wick exponential as an operator on a certain Hilbert space K. We do the explicit construction in a Krein structure isomorphic to the one corresponding to the function $\alpha(k^1)\equiv 1$ (regular metric, see the previous chapter) but it is possible to make analogous constructions also in the other cases.

It is more convenient to use the light-cone variables.

The Krein structure we are speaking of corresponds to the introduction of following pre-Hilbert product in $\mathcal{S}_r(R^2)$:

$$[f,g] = -\pi \int_0^\infty \log u \frac{\partial^2}{\partial u^2} \{\bar{f}(u,v)g(u,v) + (\frac{\partial}{\partial v}\bar{f}(u,v))\ (\frac{\partial}{\partial v} g(u,v))\}|_{v=0} du +$$

$$-\pi \int_0^\infty \log v \frac{\partial^2}{\partial v^2} \{\bar{f}(u,v)g(u,v) + (\frac{\partial}{\partial u}\bar{f}(u,v))\ (\frac{\partial}{\partial u} g(u,v))\}|_{u=0} dv \qquad (6.4.7)$$

The complete scalar product may be now obtained as in formula (6.3.21). Our aim is to find an explicit expression for the distributional kernel K(x,y) corresponding to this Krein structure:

$$(f,g) = \int K(x,y)\ \bar{f}(x)\ g(y)\ d^4x\ d^4y \qquad (6.4.8)$$

We have that [GEL1]

$$\int \exp(-i\sigma x)\ \vartheta(x)\ \log x\ dx =$$

$$i[\ (\Gamma'(1) + \frac{i}{2}\pi)(-\sigma + i0)^{-1} - (-\sigma + i0)^{-1}\log(-\sigma + i0)]\ . \qquad (6.4.9)$$

This formula implies that

$$\int du\ dv\ \exp(-\frac{i}{2}ux_- - \frac{i}{2}vx_+)\ \frac{\partial}{\partial u}(\vartheta(u)\ \log u)\ \delta(v) =$$

$$= [\ (\Gamma'(1) + \frac{i}{2}\pi - \log(-\frac{1}{2}x_- + i0)] \qquad (6.4.10)$$

and that

$$\int du\, dv\, \exp\left(-\frac{i}{2}ux_- - \frac{i}{2}vx_+\right) \frac{\partial^2}{\partial u^2}(\vartheta(u) \log u)\, \delta(v) =$$

$$= \frac{i}{2}\, x_- \left[(\Gamma'(1) + \frac{i}{2}\pi - \log\left(-\frac{1}{2}x_- + i0\right)\right] \tag{6.4.11}$$

Thus we have the following equation:

$$\int_0^\infty \log u\, \frac{\partial^2}{\partial u^2} \{\bar{f}(u,v)g(u,v) + (\frac{\partial}{\partial v}\, \bar{f}(u,v))\, (\frac{\partial}{\partial v}\, g(u,v))\}|_{v=0} du =$$

$$\int_0^\infty \log u \frac{\partial^2}{\partial u^2}\{\bar{f}(u,v)g(t,s)+(\frac{\partial}{\partial v}\bar{f}(u,v))(\frac{\partial}{\partial t}\, g(t,s))\}\delta(v)\delta(u-s)\delta(v-t)\, du dv dt ds =$$

$$\frac{1}{2\pi^2} \int \frac{\partial^2}{\partial u^2}(\vartheta(u)\log u)\ \delta(v)\delta(u-s)\delta(v-t)\ \cdot$$

$$\{\int \exp(-\frac{i}{2}ux_- - \frac{i}{2}vx_+)\ \bar{f}(x)\ d^2x \int \exp(\frac{i}{2}sy_- + \frac{i}{2}ty_+)\ g(y)\ d^2y +$$

$$[\frac{\partial}{\partial v}\int \exp(-\frac{i}{2}ux_- -\frac{i}{2}vx_+)\bar{f}(x)d^2x]\, [\frac{\partial}{\partial t}\int \exp(\frac{i}{2}sy_- +\frac{i}{2}ty_+)g(y)d^2y]\, \}du dv ds dt=$$

$$\frac{1}{2\pi^2}\int \frac{\partial^2}{\partial u^2}(\vartheta(u)\log u)\ \delta(v)\ (1 + x_+y_+)\ \exp(-\frac{i}{2}u(x_- -y_-) - \frac{i}{2}v(x_+ -y_+))\ du\ dv$$

$$\cdot\ \bar{f}(x)g(y)\ d^2x d^2y =$$

$$\frac{1}{4\pi^2} i \int (1 + x_+y_+)\ (x_- -y_-)\ [(\Gamma'(1) + \frac{i}{2}\pi - \log[-\frac{1}{2}(x_- -y_-)+ i0]]$$

$$\cdot\ \bar{f}(x)g(y)\ d^2x d^2y\ . \tag{6.4.12}$$

Analogously we obtain that

$$\int_0^\infty \log v\, \frac{\partial^2}{\partial v^2} \{\bar{f}(u,v)g(u,v) + (\frac{\partial}{\partial u}\, \bar{f}(u,v))\, (\frac{\partial}{\partial u}\, g(u,v))\}|_{u=0} dv =$$

$$\frac{i}{4\pi^2} \int (1 + x_-y_-)\ (x_+-y_+)\ [(\Gamma'(1) + \frac{i}{2}\pi - \log[-\frac{1}{2}(x_+-y_+)+ i0]]$$

$$\cdot\ \bar{f}(x)g(y)\ d^2x d^2y\ . \tag{6.4.13}$$

Thus we obtain that the distributional kernel K(x,y) expressing the two-point norm of the Wick ordered exponential of the dipole$_2$ has the following expression:

$$K(x,y) = K_1(x,y) + F(x,y) \tag{6.4.14}$$

$$K_1(x,y) = c\ (1 + x_+y_+)(x_- - y_-)\ [\,(\Gamma'(1) + \tfrac{i}{2}\pi - \log[-\tfrac{1}{2}(x_- - y_-) + i0]] +$$

$$c\ (1 + x_-y_-)\ (x_+ - y_+)\ [\,(\Gamma'(1) + \tfrac{i}{2}\pi - \log[-\tfrac{1}{2}(x_+ - y_+) + i0]] \tag{6.4.15}$$

where F is an infinitely differentiable function whose explicit expression may be calculated by considering eq. (...) and (...):

$$\begin{aligned} F(x,y) = {} & -\frac{1}{2\pi}[\bar{K}_\chi(y) + K_\chi(x)] + \frac{1}{2\pi}i[\bar{K}_\xi(y) + K_\xi(x)] + \frac{1}{2\pi}[\bar{K}_\lambda(y) + K_\lambda(x)] + \\ & + \frac{1}{2\pi^2}K_{\chi\chi} - \frac{i}{2\pi^2}[K_{\chi\xi}x_0 - K_{\xi\chi}y_0] + \frac{i}{2\pi^2}[K_{\chi\lambda}x_1 - K_{\lambda\chi}y_1] + \\ & + \frac{1}{2\pi^2}K_{\xi\xi}\,x_0y_0 - \frac{1}{2\pi^2}[K_{\xi\lambda}x_0y_1 + K_{\xi\chi}y_0x_1] + \frac{1}{2\pi^2}K_{\lambda\lambda}\,x_1y_1 + \\ & + W_\chi(x)\bar{W}_\chi(y) + W_\xi(x)\bar{W}_\xi(y) + W_\lambda(x)\bar{W}_\lambda(y) + \frac{1}{2\pi^2}\{1 + x_0y_0 + x_1y_1\} \end{aligned} \tag{6.4.16}$$

where

$$K_\chi(x) = \int K_1(x,y)\chi(y)d^4y\ , \qquad \bar{K}_\chi(x) = \int K_1(x,y)\bar{\chi}(x)d^4x\ ,$$

$$K_{\chi\chi} = \int K_1(x,y)\bar{\chi}(x)\chi(y)d^4xd^4y\ , \quad \text{etc..} \tag{6.4.17}$$

A little computation shows that for any $\epsilon>0$ it exists a constant $C_\epsilon>0$ so that

$$|K_1(x,y) + F(x,y)\ | \le C_\epsilon\ (\|x\|^{4+\epsilon} + \|y\|^{4+\epsilon}) \tag{6.4.18}$$

This estimate implies that the generalized function $\exp[K_1(x,y)]$ is well defined on each of the Gelfand and Shilov spaces $\mathcal{S}_\alpha(R^4)$ such that $\alpha<\frac{1}{4}$

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