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Models for interface dynamics

Thesis submitted for the degree of

"Doctor Philosophiæ"

CANDIDATE

Paolo Buttà

SUPERVISOR

Prof. Errico Presutti

Academic Year 1994/95

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INTRODUCTION

Interface dynamics describes the evolution of systems after phase segregation. In a quenching experiment a system is initially in thermodynamic equilibrium with a reservoir which is then cooled down below the critical temperature of the system. This process is usually so fast that the macroscopic state of the system does not change significantly. After the cooling it is no longer in equilibrium with the reservoir and one usually models the successive evolution by supposing its state still stationary but unstable. It is therefore very sensitive to external perturbations as those of stochastic nature coming from the reservoir. As soon as the state changes, because of its instability, the deterministic driving forces internal to the system take over, driving it toward the stable phases and the phase separation phenomena take place. Equilibrium is not reached yet at this stage. When the system is spatially extended it reaches locally a thermodynamic stable phase, but since there are several equally accessible phases, there is no reason for the equilibria of faraway regions to coincide. The typical picture at the end of this stage is then a collection of clusters of different phases with interfaces in between. The successive stage of the evolution is the interface dynamics which describes the competition between phases. A rigorous derivation of the whole picture from microscopic models has not yet been carried out in a systematic and rigorous theory as in equilibrium statistical mechanics. In the last years however some results have been obtained in the context of particular and simpler models.

This thesis focuses on interface dynamics referring to two specific models with non conserved order parameter and with two symmetric stable phases. The evolution of the interfaces is ruled by the motion by mean curvature.

The first model is the stochastic Ising model with long range interaction and Glauber dynamics. In chapter 1 we introduce the model and give a list of known results on phase separation and interface dynamics. In this case the order parameter is the magnetization m and the equilibrium phases have magnetization $\pm m_\beta$, $m_\beta > 0$, β the inverse temperature (after the cooling). After phase separation local equilibrium has already established and this state is described by magnetization profiles $m(r)$, the connected regions where $m(r) \approx +m_\beta$ are the $+$ clusters and those where $m(r) \approx -m_\beta$ are the $-$ clusters. Typically the fraction of volume not occupied by the $\pm$ clusters is "very small". This notion of $\pm$ clusters is clearly vague, to have a mathematical theory one considers a scaling limit where the size of the system is let to $+\infty$ and the time is correspondingly scaled (the right way to do it in our case is to scale space and time diffusively). While this procedure does not change the fact that the region occupied by the interface is still as "thick" as before, it however makes the fraction of volume occupied by the interface vanishing. Thus in the macroscopic units, defined so that the size of the system is unchanged under the scaling, the interface has vanishing volume and in the limit it is identified by a regular surface under suitable assumptions of regularity at the initial time. In this frame for a large class of models (including ours), it has been proven that these geometric interfaces remain regular for a finite time at least and evolve by mean curvature. In dimension greater than 2 singularities may develop, however there exists a generalized definition of motion by mean curvature which rules the evolution after the onset of singularities (see §1.4 for more details and references).

It should however be kept in mind that interfaces defined in the previous way are mere geometric objects which do not have an intrinsic physical meaning. They are an artefact of the scaling procedure and of the chosen macroscopic units which allows to identify in the limit a thick region with a surface. In this thesis we show that for our system it is possible to define intrinsically the interface in terms of a local equilibrium property of the magnetization profile, so that the interfaces have a direct physical interpretation. In fact there is a definite magnetization pattern connecting two phases when they coexist at equilibrium and it turns out that the local equilibrium established at the end of the first stage of phase separation does not only ensure that most of the volume is at equilibrium occupied by the clusters of the $\pm$ phases, but also that the magnetization pattern between clusters is (approximately) the same as when the two phases coexist at equilibrium. This statement becomes sharper as the size of the clusters increases and exact in the diffusive limit considered earlier. We prove that in such a limit this sharp local equilibrium structure persists through time so that it is possible to define an interface evolution which we show to be the same as that of the geometrical surfaces described earlier, at least till singularities develop. This is the content of chapter 2 (see the main Theorem 2.1.1) and it is the result of a work made in collaboration with A. De Masi. This same fine structure of the interface has been found in the Allen-Cahn (Ginzburg-Landau) equation by De Mottoni and Schatzman, [33], [34], and Bellettini and Paolini, [4].

Actually we prove all the previous results for a (deterministic) non local evolution equation which describes the magnetization density evolution of the stochastic Ising model in the mean field limit. But this deterministic evolution is a good approximation to the true Glauber dynamics and they coincide in a suitable space-time scaling limit. Then, at least on such space-time scale, the result for the microscopic dynamics follows from our results.

In one dimension there is no motion by mean curvature and in fact there are examples involving deterministic evolutions where, after the phase separation, the system relaxes into a stationary state that is not the true equilibrium. This happens in the Ginzburg-Landau model and we expect a similar mechanism also for the above non local evolution equation. On the other hand in real systems one has to take into account the stochastic forces which are neglected in the deterministic models. For example, as just noticed, the non local mean field equation is a good approximation of the full Glauber dynamics only up to certain times, after which the fluctuations become important.

In this thesis we study the effects of the random forces on the interface motion by perturbing the one dimensional Ginzburg-Landau equation with a white noise of strength $\sqrt{\epsilon}$, with ϵ a small parameter. As in the previous model there is a unique profile which describes the pattern of the layers after phase separation (in one dimension a layer plays the role of an interface). This profile is the instanton solution of the deterministic Ginzburg-Landau equation. We give here some results on the stability properties of the instanton and on its motion. More precisely we prove that if the initial datum is close to an instanton then it remains close to some translated instanton for times that may grow as fast as any inverse power of ϵ and as long as the layer stays away from the boundary of the interval where we consider the equation (we can take the length of the interval equal to $2\epsilon^{-\kappa}$ for any $\kappa \geq 1$). The center of the instanton, suitably renormalized, converges to a Brownian motion and there is an asymptotic coupling of two solutions starting from initial data close to instantons such that, in the limit $\epsilon \to 0^+$, the time of the coupling, suitably renormalized, converges in law to the first encounter of two Brownians starting from the centers of the instantons that approximate the initial data. These results are the content of chapter 3 (see the main Theorems 3.2.1, 3.2.2 and 3.2.3) and are based on a work made in collaboration with S. Brassesco, A. De Masi and E. Presutti.

CHAPTER 1.

THE STOCHASTIC ISING MODEL WITH GLAUBER EVOLUTION AND KAC POTENTIALS

In this chapter we introduce the Ising model with Glauber dynamics and Kac interaction. We describe its behaviour when the range γ^{-1} of the interaction diverges and by looking at the system on different "space-time scales".

The chapter is divided into 5 sections. In the first one we introduce the equilibrium Ising model with long range interaction. In the second one we introduce the Glauber dynamics and describe its mean field behaviour on the "mesoscopic scale". In the last sections we describe the dynamics on a "macroscopic scale" when phase separation and interface dynamics occur.

§1.1 The Ising model with Kac potentials: the Lebowitz-Penrose limit.

In this section we describe the equilibrium properties of the Ising model with Kac potentials and we briefly recall how the limiting (mean-field) theory reproduces exactly the Van der Waals theory of phase transition comprehensive of the Maxwell rule.

We start by recalling the main definitions and properties on the Ising spin systems. These are well established facts so we refer to standard texts for more details (see [55], [57], [58]).

Let $\mathbb{Z}^d$ be the unit square lattice of dimension d. A *spin configuration* σ is a function on $\mathbb{Z}^d$ with values in $\{-1,1\}$, that is an element of $\{-1,1\}^{\mathbb{Z}^d}$. We denote by $\sigma(x)$ the value of σ at $x \in \mathbb{Z}^d$ and by σ_Δ the restriction of σ to $\Delta \subset \mathbb{Z}^d$. Therefore $\sigma(x)$ is a random variable on $\{-1,1\}^{\mathbb{Z}^d}$ and σ_Δ is an element of $\{-1,1\}^\Delta$.

The Kac potentials.

A *Kac potential* is a function $J_\gamma : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$, depending on a (scaling) parameter $\gamma \in (0,1]$, of the form

$$J_\gamma(x,y) = \gamma^d J(\gamma[x-y]), \qquad x,y \in \mathbb{Z}^d \tag{1.1.1}$$

We assume that $J(r)$, $r \in \mathbb{R}^d$, is a nonnegative function depending on $|r|$ with compact support in $\{r : |r| \leq 1\}$ and that $J \in C^\infty(\mathbb{R}^d)$. We suppose also that it is normalized so that

$$\int_{\mathbb{R}^d} dr\, J(r) = 1 \tag{1.1.2}$$

The energy of the spin $\sigma(x)$, when interacting with all the other spins by means of the Kac potential J_γ and in the presence of a magnetic field $h \in \mathbb{R}$, is defined by

$$H_{h,\gamma}(x,\sigma) \doteq -h\sigma(x) - \sigma(x)(J_\gamma \circ \sigma)(x), \qquad (J_\gamma \circ \sigma)(x) \doteq \sum_{\substack{y \neq x \\ y \in \mathbb{Z}^d}} J_\gamma(x,y)\sigma(y) \tag{1.1.3}$$

Therefore the spin $\sigma(x)$ feels an effective magnetic field

$$h_\gamma(x,\sigma) \doteq h + (J_\gamma \circ \sigma)(x) \tag{1.1.4}$$

Given a finite subset Δ of $\mathbb{Z}^d$, the energy of the the spin configuration σ_Δ is defined by

$$H_{h,\gamma}(\sigma_\Delta) \doteq -h \sum_{x \in \Delta} \sigma(x) - \frac{1}{2} \sum_{\substack{x \neq y \\ x,y \in \Delta}} J_\gamma(x,y)\sigma(x)\sigma(y) \tag{1.1.5}$$

while its energy inclusive of the interaction with the spins in the complement Δ^c of Δ is

$$H_{h,\gamma}(\sigma_\Delta | \sigma_{\Delta^c}) \doteq H_{h,\gamma}(\sigma_\Delta) - \sum_{\substack{x \in \Delta \\ y \in \Delta^c}} J_\gamma(x,y)\sigma(x)\sigma(y) \tag{1.1.6}$$

By definition (1.1.1) the interaction between the spins is translational invariant. Notice that the interaction of a spin with any other single one is small as $\gamma \to 0$, but its total interaction with all the other spins remains finite uniformly in γ.

In the Literature the class of Kac potentials is more general, typically one requires only that $J \in L^1(dr, \mathbb{R}^d)$, see [50] and [53]. Some of our assumptions on J have a clear physical meaning. More precisely we consider ferromagnetic, finite range and isotropic interactions. The smoothness assumption on J is more technical and can be relaxed. The condition (1.1.2), introduced for notational convenience, it is not a real restriction since one can always reduces to this case by rescaling the temperature (see below).

Gibbs measures and pressure.

Let $\beta \geq 0$. A probability measure $\mu_{\beta,h,\gamma}$ on $\{-1,1\}^{\mathbb{Z}^d}$ is said to be a *Gibbs measure* (or *Gibbs state*) if it satisfies the DLR equations, that is, for any $x \in \mathbb{Z}^d$ and any spin configuration σ,

$$\mu^{(x)}_{\beta,h,\gamma}(\sigma) = \frac{e^{\beta\sigma(x)h_\gamma(x,\sigma)}}{e^{\beta h_\gamma(x,\sigma)} + e^{-\beta h_\gamma(x,\sigma)}}, \qquad \mu_{\beta,h,\gamma} \text{ almost surely} \tag{1.1.7}$$

where $\mu^{(x)}_{\beta,h,\gamma}(\sigma)$ is the conditional probability

$$\mu^{(x)}_{\beta,h,\gamma}(\sigma) = \mu_{\beta,h,\gamma}(\{\sigma' : \sigma'(x) = \sigma(x)\} | \sigma'(y) = \sigma(y)\ \forall y \neq x) \tag{1.1.8}$$

The parameter β has the physical meaning of an inverse temperature while the Gibbs measures are interpreted as the possible phases of the physical system in the infinite volume limit. We just recall some standard facts about this and refer to [57] and [58] for more details. We denote by $\mathcal{G}_{\beta,h,\gamma}$ the set of all the Gibbs measures for given β, h and γ. For any $\mu_{\beta,h,\gamma} \in \mathcal{G}_{\beta,h,\gamma}$ and any finite subset Δ of $\mathbb{Z}^d$,

$$\mu^{(\Delta)}_{\beta,h,\gamma}(\sigma) = \mu_{\beta,h,\gamma}(\sigma_\Delta | \sigma_{\Delta^c}), \qquad \mu_{\beta,h,\gamma} \text{ almost surely} \tag{1.1.9}$$

where $\mu^{(\Delta)}_{\beta,h,\gamma}(\sigma)$ is the conditional probability

$$\mu^{(\Delta)}_{\beta,h,\gamma}(\sigma) = \mu_{\beta,h,\gamma}(\{\sigma' : \sigma'_\Delta = \sigma_\Delta\} | \sigma'_{\Delta^c} = \sigma_{\Delta^c}) \tag{1.1.10}$$

while $\mu_{\beta,h,\gamma}(\sigma_\Delta | \sigma_{\Delta^c})$ is the "grand canonical ensemble" for the finite system on Δ with boundary condition σ_{Δ^c}, that is the probability measure on $\{-1,1\}^\Delta$ defined by

$$\mu_{\beta,h,\gamma}(\sigma_\Delta | \sigma_{\Delta^c}) \doteq Z^{\beta,h}_\gamma(\sigma_{\Delta^c})^{-1} e^{-\beta H_{h,\gamma}(\sigma_\Delta | \sigma_{\Delta^c})} \tag{1.1.11}$$

where $Z^{\beta,h}_\gamma(\sigma_{\Delta^c})$ is the partition function

$$Z^{\beta,h}_\gamma(\sigma_{\Delta^c}) = \sum_{\{\sigma_\Delta\}} e^{-\beta H_{h,\gamma}(\sigma_\Delta | \sigma_{\Delta^c})}$$

Conversely, any thermodynamic limit of $\{\mu_{\beta,h,\gamma}(\cdot | \sigma_{\Delta^c})\}$ is a Gibbs measure and the closed convex hull of all such limits is the whole $\mathcal{G}_{\beta,h,\gamma}$. The system exhibits phase transition if $\mathcal{G}_{\beta,h,\gamma}$ contains more than one element. By classical results on statistical mechanics we can say that there is no phase transition if $d = 1$ or if $h \neq 0$. If $h = 0$ and $d \geq 2$ there exists a critical inverse temperature β_c, $0 < \beta_c < \infty$, such that there is no phase transition for $\beta < \beta_c$ and there is phase transition for $\beta > \beta_c$. About the dependence of β_c on γ it is known that $\beta_c \geq 1$, [17], $\beta_c > 1$ for $d = 2$, [16], and $\lim_{\gamma \to 0} \beta_c = 1$, [19]. The characterization of the Gibbs states in the presence of phase transition is an open problem. As in the standard ferromagnetic Ising model with nearest-neighbor interaction one expects that in $d = 2$ all the Gibbs states are translational invariant and that the pure phases are only the two ones obtained as the thermodynamic limit with + and − boundary conditions.

The connection with thermodynamics is given by means of the partition function. We define the thermodynamic pressure $p_\gamma(\beta, h)$ with a limiting procedure. Let $\mathcal{M}_\Delta$ the space of the probability measures on $\{-1,1\}^\Delta$. To any $\nu \in \mathcal{M}_\Delta$ we associate the mean energy

$$E^{(\Delta)}_{h,\gamma}(\nu) = \sum_{\{\sigma_\Delta\}} \nu(\sigma_\Delta) H_{h,\gamma}(\sigma_\Delta) \tag{1.1.12}$$

and the entropy

$$S^{(\Delta)}(\nu) = -\sum_{\{\sigma_\Delta\}} \nu(\sigma_\Delta)\log\nu(\sigma_\Delta) \tag{1.1.13}$$

The pressure functional is then defined by

$$P^{(\Delta)}_{\beta,h,\gamma}(\nu) = \frac{1}{|\Delta|}[\beta^{-1}S^{(\Delta)}(\nu) - E^{(\Delta)}_{h,\gamma}(\nu)] \tag{1.1.14}$$

By the variational principle the above functional reaches its maximum on $\mathcal{M}_\Delta$ for $\nu = \nu_{\beta,h,\gamma}$, the grand canonical ensemble

$$\nu_{\beta,h,\gamma}(\sigma_\Delta) = \left(Z^{\beta,h}_{\gamma,\Delta}\right)^{-1} e^{-\beta H_{h,\gamma}(\sigma_\Delta)}, \qquad Z^{\beta,h}_{\gamma,\Delta} = \sum_{\{\sigma_\Delta\}} e^{-\beta H_{h,\gamma}(\sigma_\Delta)} \tag{1.1.15}$$

and

$$P^{(\Delta)}_{\beta,h,\gamma}(\nu_{\beta,h,\gamma}) = \frac{1}{|\Delta|}\log Z^{\beta,h}_{\gamma,\Delta}$$

We define then

$$p_\gamma(\beta,h) \doteq \lim_{\Delta\nearrow\mathbb{Z}^d} \frac{1}{\beta|\Delta|}\log Z^{\beta,h}_{\gamma,\Delta} \tag{1.1.16}$$

when Δ tends to $\mathbb{Z}^d$ in the sense of van Hove. It can be proven that the above limit exists and that $p_\gamma(\beta,h)$ is a positive and convex function of h and β^{-1}. Moreover, for any choice of σ_{Δ^c},

$$\lim_{\Delta\nearrow\mathbb{Z}^d} \frac{1}{\beta|\Delta|}\log Z^{\beta,h}_\gamma(\sigma_{\Delta^c}) = p_\gamma(\beta,h) \tag{1.1.17}$$

By means of the Legendre transform we get all the other thermodynamic potentials. In particular, the free energy is given by

$$F_\gamma(\beta,m) = \sup_h[hm - p_\gamma(\beta,h)] \tag{1.1.18}$$

where m is the magnetization density of the system. We refer to [57] and [58] for details.

The Lebowitz-Penrose limit.

We analyze now the limiting theory when $\gamma \to 0$, which features can be summarized in the following theorem.

1.1.1 Theorem.

Let $p_\gamma(\beta,h)$ be the thermodynamic pressure (see (1.1.16)). Then

$$p(\beta,h) \doteq \lim_{\gamma\to 0} p_\gamma(\beta,h) = \sup_{|m|\le 1}\left(\frac{1}{2}m^2 + \beta^{-1}i(m) + hm\right) \tag{1.1.19}$$

where

$$i(m) = -\frac{1+m}{2}\log\frac{1+m}{2} - \frac{1-m}{2}\log\frac{1-m}{2} \tag{1.1.20}$$

Notice that the quantity $-\beta^{-1}i(m)$ is the free energy of an Ising spin system with no interaction. In fact in this case the only Gibbs state is the product measure $\mu^{(0)}_{\beta,h}$ on $\{-1,1\}^{\mathbb{Z}^d}$ such that

$$\mu^{(0)}_{\beta,h}(\sigma(x)) = \frac{e^{-\beta h\sigma(x)}}{e^{\beta h}+e^{-\beta h}} \tag{1.1.21}$$

where we shorthand $\mu^{(0)}_{\beta,h}(\sigma(x)) = \mu^{(0)}_{\beta,h}(\{\sigma' : \sigma'(x) = \sigma(x)\})$. From the corresponding pressure

$$p^{(0)}(\beta,h) = \log[e^{\beta h}+e^{-\beta h}] \tag{1.1.22}$$

we get the free energy

$$F^{(0)}(\beta,m) = \sup_h[hm - p^{(0)}(\beta,h)] = -\beta^{-1}i(m) \tag{1.1.23}$$

Next we describe the result of the theorem. The limiting free energy is defined by the Legendre transform of the pressure and by (1.1.19) we get

$$F(\beta,m) = CE\{f_\beta(m)\}, \qquad f_\beta(m) \doteq -\frac{1}{2}m^2 - \beta^{-1}i(m) \tag{1.1.24}$$

where $CE\{f\}$ denotes the convex envelope of the function f. If we look at the graph of the function $f_\beta(m)$ we see that for $\beta > 1$ there are two distinct minima at the points $\pm m_\beta$, where m_β is the positive solution of the equation

$$m_\beta = \tanh[\beta m_\beta] \tag{1.1.25}$$

For $m = 0$ there is a local maximum, and at the points $\pm m^*_\beta$, $m^*_\beta = \sqrt{1-\beta^{-1}}$, the curvature changes. So we have exactly the picture of the Van der Waals theory of phase transition, with inverse Lebowitz-Penrose critical temperature $\beta_c^{LP} = 1$. The convex envelope in (1.1.24), which makes flat the free energy in the interval $[-m_\beta, m_\beta]$ corresponds to the Maxwell rule. Notice that $\beta_c^{LP} = 1$ since we impose the normalization condition (1.1.2). Also metastability can be rigorously explained in terms of Kac interactions, [54].

Finally we point out that in the original paper of Lebowitz and Penrose, [53], by working in the canonical ensemble, they proved directly that

$$\lim_{\gamma\to 0} F_\gamma(\beta,m) = F(\beta,m) \tag{1.1.26}$$

with $F(\beta,m)$ given by (1.1.24) (see [31] for a proof of Theorem 1.1.1). Then we recover the same limit theory by working in the two different ensembles (that is the Lebowitz-Penrose limit "commutes" with the Legendre transformation).

§1.2 The Glauber dynamics with Kac potentials: the mesoscopic limit.

In this section we start to treat the nonequilibrium properties of the Ising model with Kac interactions. We let the system evolve with a Glauber (spin-flip) dynamics and we describe this evolution in the limit of Lebowitz and Penrose on the "mesoscopic scale", when the space is scaled with the Kac parameter γ and time is kept fixed. Due to a "mean field effect", the dynamics has a very strong deterministic nature, described by a non local deterministic evolution equation for the magnetization density. We only discuss these results and we refer to [25] for proofs and details.

Given the inverse temperature $\beta > 0$ we define *Glauber dynamics* the unique Markov process with state space $\{-1,1\}^{\mathbb{Z}^d}$ and generator $L_{\beta,h,\gamma}$, where $L_{\beta,h,\gamma}$ is the unique extension of the operator which acts on the cylinder functions f as

$$L_{\beta,h,\gamma}f(\sigma) = \sum_{x\in\mathbb{Z}^d} c_{\beta,h,\gamma}(x,\sigma)[f(\sigma^x) - f(\sigma)] \tag{1.2.1}$$

where σ^x denotes the spin configuration obtained by flipping the spin at x, that is,

$$\sigma^x(y) = \begin{cases} \sigma(y) & \text{if } y \neq x \\ -\sigma(x) & \text{if } y = x \end{cases} \tag{1.2.2}$$

while the "flip rate" $c_{\beta,h,\gamma}(x,\sigma)$ of the spin at x is given by

$$c_{\beta,h,\gamma}(x,\sigma) = \frac{e^{-\beta h_\gamma(x,\sigma)\sigma(x)}}{e^{\beta h_\gamma(x,\sigma)} + e^{-\beta h_\gamma(x,\sigma)}} \tag{1.2.3}$$

where $h_\gamma(x,\sigma)$ is defined in (1.1.4).

The proof of the existence and uniqueness of the above Markov process can be found in [55]. The canonical space of realizations of the process is $D(\mathbb{R}_+,\{-1,1\}^{\mathbb{Z}^d})$, the Skorohod space of cadlag trajectories (continuous from the right and with left limits). We denote by σ_t the spin configuration of the process at time t, whose value at x, $\sigma(x,t) = \sigma_t(x)$, is therefore a random variable on $D(\mathbb{R}_+,\{-1,1\}^{\mathbb{Z}^d})$.

There is a fundamental relation between the Glauber dynamics and the Gibbs states. We observe that the flip rates verify the so called "detailed balance" condition:

$$c_{\beta,h,\gamma}(x,\sigma)e^{-\beta H_{h,\gamma}(x,\sigma)} = c_{\beta,h,\gamma}(x,\sigma^x)e^{-\beta H_{h,\gamma}(x,\sigma^x)} \tag{1.2.4}$$

that is, the function $c_{\beta,h,\gamma}(x,\sigma)\exp[-\beta H_{h,\gamma}(x,\sigma)]$ does not depend on the value of σ at x. From (1.2.4) follows easily that the set $\mathcal{G}_{\beta,h,\gamma}$ of the Gibbs states coincides with the set $\mathcal{R}_{\beta,h,\gamma}$ of all the probability measures on $\{-1,1\}^{\mathbb{Z}^d}$ which are reversible with respect to the Glauber dynamics (recall that $\nu \in \mathcal{R}_{\beta,h,\gamma}$ if $L_{\beta,h,\gamma}$ is self-adjoint in $L^2(\{-1,1\}^{\mathbb{Z}^d},\nu)$).

Moreover, since $J \geq 0$, the Glauber dynamics is attractive and then it is ergodic if and only if there is no phase transition. We refer to the book of Liggett, [55], for details and for the general theory of the stochastic spin systems. We point out that (1.2.4) can be satisfied with other choices of the flip rates. The choice (1.2.3) gives a simpler limiting mesoscopic equation (see below).

The explicit connection between the Glauber dynamics and the Gibbs states makes the former a plausible model to describe the approach to equilibrium phenomena, like escape from unstable states, phase separation and so on.

Next we describe the Glauber dynamics in the mesoscopic regime. The *mesoscopic scale* $(r, t) \in \mathbb{R}^d \times \mathbb{R}_+$ is related to the microscopic one $(x, t) \in \mathbb{Z}^d \times \mathbb{R}_+$, where the Glauber dynamics is defined, by the scaling transformation $(x, t) \to (r, t) = (\gamma x, t)$. Then time is unchanged, while space is measured in interaction range units.

We introduce a block spin transformation. We fix $b \in (0, 1)$ and for any function $f : \mathbb{Z}^d \to \mathbb{R}$ we set

$$\mathcal{A}_{\gamma,b,x}(f) \doteq \frac{1}{|B_{\gamma,b}|} \sum_{y \in B_{\gamma,b,x}} f(y) \tag{1.2.5}$$

where

$$B_{\gamma,b,x} \doteq \{y \in \mathbb{Z}^d : |y - x| \leq \gamma^{-b}\}, \qquad |B_{\gamma,b}| \doteq |B_{\gamma,b,x}| \simeq \gamma^{-bd} \tag{1.2.6}$$

Moreover we define

$$\sigma^{\gamma,b}(r) \doteq \mathcal{A}_{\gamma,b,[\gamma^{-1}r]}(\sigma), \qquad \sigma_t^{\gamma,b}(r) \doteq \mathcal{A}_{\gamma,b,[\gamma^{-1}r]}(\sigma_t) \tag{1.2.7}$$

where we denote by $[v]$, $v \in \mathbb{R}^d$, the vector with components $[v]_i = [v_i]$, $i = 1, ...d$, where $[u]$ is the integer part of $u \in \mathbb{R}$.

In [25] it is proven that, for small γ's, the Glauber dynamics is almost deterministic, described by the so called *mesoscopic equation*, that is the following non local deterministic evolution equation

$$\frac{\partial m}{\partial t}(r, t) = -m(r, t) + \tanh\{\beta[(J \star m)(r, t) + h]\} \tag{1.2.8}$$

where "$\star$" denotes the convolution. In [25] it is proven that the Cauchy problem (1.2.8) with initial datum $m_0 \in L^\infty(\mathbb{R}^d; [1, 1])$ has an unique solution. We will analyze in more details the properties of this equation in the next chapter.

The convergence of the Glauber dynamics to the equation (1.2.8) holds in a very strong sense. Actually one can prove weak convergence of the process, convergence of the correlation functions and convergence in probability of the block spin variables $\sigma_t^{\gamma,b}(r)$.

The physical idea is that some "weak form" of propagation of chaos holds in this case. If the process starts from a product measure on $\{-1,1\}^{\mathbb{Z}^d}$, then for any fixed time t is close (and converges as $\gamma \to 0$) to a product measure with means given by the solution of the mesoscopic equation (1.2.8). We consider a discretized version of (1.2.8)

$$\frac{dm_\gamma}{dt}(x,t) = -m_\gamma(x,t) + \tanh\{\beta[(J \circ m_\gamma)(x,t) + h]\} \tag{1.2.9}$$

where $(x,t) \in \mathbb{Z}^d \times \mathbb{R}_+$ and"$\circ$" denotes the discrete convolution introduced in (1.1.3). If μ is a product measure on $\{-1,1\}^{\mathbb{Z}^d}$, we denote by $m_\gamma(x,t|\mu)$ the solution of (1.2.9) starting from $m_\gamma(x,0|\mu) = \mathbb{E}_\mu[\sigma(x)]$, where $\mathbb{E}_\mu[\cdot]$ means expectation value with respect to the measure μ. In particular $m_\gamma(x,t|\sigma)$ denotes the solution of (1.2.9) with initial condition $m_\gamma(x,0|\sigma) = \sigma(x)$. This equation is related to the Glauber dynamics since for a suitable positive constant c one has

$$\left| \frac{dm_\gamma(x,t|\mu)}{dt} - \mathbb{E}_{\nu_t}\left[L_{\beta,h,\gamma}\sigma(x)\right] \right| \le c\gamma^d \tag{1.2.10}$$

where ν_t is the product measure on $\{-1,1\}^{\mathbb{Z}^d}$ with means $\mathbb{E}_{\nu_t}[\sigma(x)] = m_\gamma(x,t|\mu)$ for all $x \in \mathbb{Z}^d$. If propagation of chaos holds, that is μ_t^γ, the Glauber distribution at time t, remains a product measure, then (1.2.10) allows to determine, to leading order in γ, all the spin correlation functions and then the distribution of the process. But the measure μ_t^γ is not a product measure also if μ_0^γ it is so. What happens is that there is a "distance" between μ_t^γ and ν_t that vanishes in the limit $\gamma \to 0$. This distance is given in terms of the v-functions, special combinations of the spin correlation functions, defined by

$$v_n^\gamma(\underline{x},t|\sigma) = \mathbb{E}_{\mu^\gamma}\left[\prod_{x\in\underline{x}}(\sigma(x,t) - m_\gamma(x,t|\sigma))\right] \tag{1.2.11}$$

where $\underline{x} = (x_1,...,x_n)$ such that $x_i \in \mathbb{Z}^d$ and $x_i \neq x_j$ for $i \neq j$, $i,j = 1,...,n$. The basic result proved in [25] is the following theorem.

1.2.1 Theorem.

There are $K, a > 0$ and, for any n, $c > 0$, such that, for any $t \le a|\log\gamma|$,

$$\sup_\sigma \sup_{\underline{x}} |v_n^\gamma(\underline{x},t)| \le ce^{Knt}\gamma^{dn/2} \tag{1.2.12}$$

We point out that the convergence is guaranteed up to times which diverges as $\gamma \to 0$, but are small in units $|\log\gamma|$. Moreover the rate of convergence increases with n. Notice that this is a property of the v-functions, which are special combinations of the means of all the products of k spins, $k \le n$, not of the mean of the product of n spins. The techniques used to prove (1.2.12) are inspired to the usual cluster expansion of equilibrium statistical mechanics (see also [31] for a survey on this method, which has been used in several other models).

By means of (1.2.12) one obtains all the desired convergence properties of the Glauber dynamics. We just state some of these (see [25] for a complete list of results).

1.2.2 Theorem.

For any $b \in (0,1)$ and $\zeta > 0$ there are $a, b > 0$ and for any $n \geq 1$ and any $k \geq 2$ there is $c > 0$ so that the following holds. For any γ small enough and, given γ, for any $\sigma \in \{-1,1\}^{\mathbb{Z}^d}$ and $m \in L^\infty(\mathbb{R}^d; [-1,1])$ such that

$$\sup_{|r| \leq k\gamma^{-1}} \left| \sigma^{\gamma,b}(r) - m^{\gamma,b}(r) \right| \leq \gamma^\zeta \tag{1.2.13}$$

we have

$$\mathbb{P}^\gamma_\sigma \Big(\sup_{t \leq a|\log\gamma|} \sup_{|r| \leq (k-1)\gamma^{-1}} \left| \sigma^{\gamma,b}_t(r) - m^{\gamma,b}_t(r) \right| > \gamma^b \Big) \leq c\gamma^n \tag{1.2.14}$$

where $\mathbb{P}^\gamma_\sigma$ is the law of the Glauber dynamics when the process starts at time 0 from σ while

$$m^{\gamma,b}(r) \doteq \mathcal{A}_{\gamma,b,[\gamma^{-1}r]}(m_\gamma), \qquad m_\gamma : \mathbb{Z}^d \to [-1,1] \, : \, m_\gamma(x) = m(\gamma x) \tag{1.2.15}$$

and

$$m^{\gamma,b}_t(r) \doteq \mathcal{A}_{\gamma,b,[\gamma^{-1}r]}(m_{\gamma,t}), \qquad m_{\gamma,t} : \mathbb{Z}^d \to [-1,1] \, : \, m_{\gamma,t}(x) = m(\gamma x, t) \tag{1.2.16}$$

where $m(r,t)$ is the solution of (1.2.8) which starts at time 0 from m.

Notice that the choice $m(r) = \sigma([\gamma^{-1}r])$ satisfies automatically (1.2.13). Moreover the sup over $|r| \leq k\gamma^{-1}$ can be replaced by the one over $|r| \leq k\gamma^{-p}$ for any $p > 0$. Finally we do not ask that $m^{\gamma,b}_t(r)$ has a limit for $\gamma \to 0$, since we do not impose this at time 0. To cover this case we introduce particular initial data. We define *macroscopic profile* a family $\{(m_0, \mu^\gamma) : 0 < \gamma \leq 1\}$ where the μ^γ's are probability measures on $\{-1,1\}^{\mathbb{Z}^d}$ such that $\mathbb{E}_{\mu^\gamma}[\sigma(x)] = m_0(\gamma x)$, while $m_0 \in C^1(\mathbb{R}^d)$ with bounded derivatives. In [25] the following theorem is proven.

1.2.3 Theorem.

Let $m(r,t)$ be the solution of (1.2.8) which starts from a macroscopic profile m_0. Then there are a and q positive and for any n and any $k \geq 1$ there is $c > 0$ such that

$$\mathbb{P}^\gamma_{\mu^\gamma} \left(\sup_{t \leq a|\log\gamma|} \sup_{|r| \leq k\gamma^{-1}} \left| \sigma^{\gamma,b}_t(r) - m(r,t) \right| > \gamma^q \right) \leq c\gamma^n \tag{1.2.17}$$

Moreover for any $n \in \mathbb{N}_+$, any distinct $r_1, ..., r_n \in \mathbb{R}^d$ and any $t \geq 0$,

$$\lim_{\gamma \to 0} \mathbb{E}^{\gamma}_{\mu^\gamma} \left[\prod_{i=1}^{n} \sigma([\gamma^{-1} r_i], t) \right] = \prod_{i=1}^{n} m(r_i, t) \tag{1.2.18}$$

where $\mathbb{E}^{\gamma}_{\mu^\gamma}$ and $\mathbb{P}^{\gamma}_{\mu^\gamma}$ stand for expectation value and probability for the Glauber dynamics starting from the distribution μ^γ.

§1.3 Macroscopic limits: phase separation and the development of interfaces.

In the previous section we have described the Glauber evolution in the mesoscopic scale, when the space is scaled with the Kac parameter γ and time is kept fixed. The deterministic behaviour of the dynamics is due to a mean field effect: many spins (infinitely many as $\gamma \to 0$) feel essentially the same potential so that the block spin variables evolve deterministically since a law of large numbers dampens the fluctuations.

In order to see the full effects of the stochastic interaction one has to perform "macroscopic limits", where also the time is scaled with γ. Then each spin variable undergoes, in a time unit, many flips (infinitely many as $\gamma \to 0$) so that it reaches a local equilibrium distribution.

The physical situation one has in mind is a quenching experiment, when the system is rapidly cooled down from high temperature to a temperature below the critical one. Here this situation is realized by considering the Ising model initially in an equilibrium state at infinite temperature with 0 magnetization, i.e. a Bernoulli measure with 0 averages. One then let the system evolve with the Glauber dynamics at a temperature below the Lebowitz-Penrose critical one ($\beta_c^{LP} = 1$) and with external magnetic field $h = 0$. At this temperature the phase with 0 magnetization is thermodynamical instable but stationary for the mesoscopic dynamics given by (1.2.8).

In a first phase of the evolution there are small random perturbations due to the full stochastic dynamics which bring the system out of the stationary state. Then, the deterministic drift drives the system toward the equilibrium phases, corresponding to the magnetizations $\pm m_\beta$ (see (1.1.25)). When the system is spatially extended the game is more complicated: in different subregions the system reaches a stable thermodynamic equilibrium, but there is no reason why the equilibria of far away regions should coincide. Then one observes the development of interfaces, i.e. sharp transition regions which separate regions with different phases. The successive stage of the evolution consists in the competition between phases. In dimension $d \geq 2$, since the Glauber dynamics is of non conservative type, one expects the interfaces move by mean curvature (see in the next section). At very longer times, when the clusters are very large and the relative interfaces very flat, the motion by mean curvature is very slow and the random fluctuations becomes again competitive. After even longer times, with no small probability one can observe tunnelling effects with the appearance of droplets of opposite phase inside the clusters.

It is a natural question whether these effects can be predicted by the long time behaviour of the mesoscopic equation (1.2.8). This happens for example in the development of the interfaces and in the interface dynamics. On the contrary, other effects, like the escape from the instable phase, are intrinsically random and cannot be predicted from (1.2.8).

The phase separation phenomena were studied in [27]. We briefly recall the basic results. Let μ_0 be the Bernoulli measure on $\{-1,1\}^{\mathbb{Z}^d}$ with 0 averages. We denote by μ_t^γ the law of the Glauber process at time t, starting from μ_0, with magnetic field $h=0$ and below the critical temperature, $\beta > \beta_c^{LP}$. The macroscopic scale (ξ,τ) is related to the mesoscopic one (r,t) by setting

$$\xi = \lambda r, \qquad \tau = \lambda^2 t \qquad\qquad \text{where } \lambda = |\log\gamma|^{-1/2} \tag{1.3.1}$$

There is a non random phase separation time τ_c:

$$\tau_c = \frac{d}{2\alpha}, \qquad \alpha = \beta - 1, \qquad (t_c = \lambda^{-2}\tau_c) \tag{1.3.2}$$

and

$$t^* = t_c + |\log\lambda^2|^2 \tag{1.3.3}$$

such that in $[0,t_c]$ the magnetization is still infinitesimal as $\gamma\to 0$, while at time t^* the phases are fully developed. Note that in macroscopic units, since $(t^*-t_c)\lambda^2 \to 0$ as $\gamma \to 0$, the two times coincides when $\gamma\to 0$ and so the time when the phases separate is deterministic and sharp.

The discretized version (1.2.9) of the mesoscopic equation (1.2.8) plays an important role in the analysis. Since $h=0$ it becomes

$$\frac{dm_\gamma}{dt}(x,t) = -m_\gamma(x,t) + \tanh\{\beta(J\circ m_\gamma)(x,t)\} \tag{1.3.4}$$

For any spin configuration $\sigma\in\{-1,1\}^{\mathbb{Z}^d}$, we denote by $m_{\gamma,t_0}(x,t|\sigma)$ the (unique) solution of (1.3.4) for $t\ge t_0$ such that $m_{\gamma,t_0}(x,t_0|\sigma)=\sigma(x)$. We shorthand $m_{\gamma,0}(x,t|\sigma)=m_\gamma(x,t|\sigma)$. Given a distribution μ on $\{-1,1\}^{\mathbb{Z}^d}$, the family of random variables $\{m_{\gamma,t_0}(\cdot,t|\sigma)\}$, solutions of (1.3.4) with σ distributed by μ is called a "statistical solution" of (1.3.4).

Evolution in the time interval $[0,t_c]$.

The statistical solution of (1.3.4) gives a very accurate description of the system except for an initial layer when it is of the same order of the fluctuations. In [27] the following theorem is proven.

1.3.1 Theorem.

There is $\delta > 0$ and, given any $b < 1$ close enough to 1, any $\epsilon > 0$, $R > 0$ and $\tau_0 \in (0, \tau_c)$, there are $c > 0$ and $\gamma_0 \in (0,1]$, so that for any $\gamma \in (0, \gamma_0]$ there is a set $\mathcal{G}_\gamma^{(0)} \subset \{-1,1\}^{\mathbb{Z}^d}$ such that

$$\mu^\gamma_{\lambda^{-2}\tau_0}(\mathcal{G}_\gamma^{(0)}) > 1 - \epsilon \tag{1.3.5}$$

and, for any $\sigma \in \mathcal{G}_\gamma^{(0)}$ and $\tau \in (\tau_0, \tau_c]$,

$$\sup_{\lambda\gamma|x|\le R} |m_{\gamma,\lambda^{-2}\tau_0}(x, \lambda^{-2}\tau|\sigma)| \le c\big[\lambda^{d/2}\gamma^{-\alpha\tau+d/2} + e^{-\lambda^{-2}(\tau-\tau_0)}\big] \tag{1.3.6}$$

Moreover

$$\mathbb{P}^\gamma_{\sigma,\lambda^{-2}\tau_0}\Big(\sup_{\lambda\gamma|x|\le R} \big|\mathcal{A}_{\gamma,b,x}(\sigma_{\lambda^{-2}\tau}) - m_{\gamma,\lambda^{-2}\tau_0}(x, \lambda^{-2}\tau|\sigma)\big| \ge \gamma^{\delta-\alpha\tau+d/2}\Big) \le \epsilon \tag{1.3.7}$$

where $\mathbb{P}^\gamma_{\sigma,t}$ is the law of the Glauber process which starts from σ at time t.

By means of Theorem 1.3.1 and using the estimates (1.2.12) on the v-functions, one gets the following theorem ([27]).

1.3.2 Theorem.

For any $b < 1$ close enough to 1 and any $\delta > 0$ and any $R > 0$

$$\lim_{\gamma\to 0} \mathbb{P}^\gamma_{\mu_0}\Big(\sup_{\tau\le\tau_c} \sup_{|\xi|\le R} \big|\sigma^{\gamma,b}_{\lambda^{-2}\tau}(\lambda^{-1}\xi)\big| > \delta\Big) = 0 \tag{1.3.8}$$

Furthermore for any integer $n \ge 1$ and any $R > 0$,

$$\lim_{\gamma\to 0} \sup_{\substack{x_1\ne\ldots\ne x_n \\ \lambda\gamma|x_i|\le R}} \left|\mathbb{E}^\gamma_{\mu_0}\left(\prod_{i=1}^n \sigma(x_i, \lambda^{-2}\tau)\right)\right| = 0 \qquad \text{for any } 0 \le \tau \le \tau_c \tag{1.3.9}$$

Notice how m_γ is a good approximation of the block spin variable since, by (1.3.6) and (1.3.7), the error is smaller than the magnitude of the m_γ itself. The magnitude of m_γ in (1.3.6) and the critical time t_c can be predicted by the linear theory with the following heuristic argument. The initial datum is not close to $m \equiv 0$ in the sup-norm, but one can argue that the relevant quantity in the evolution is given by $(J_\gamma \circ m_\gamma)$ rather than m_γ. We get from (1.3.4) a closed equation for $u_\gamma(x,t) \doteq (J_\gamma \circ m_\gamma)(x,t)$

$$\frac{du_\gamma}{dt}(x,t) = -u_\gamma(x,t) + (J_\gamma \circ \tanh[\beta u_\gamma])(x,t) \tag{1.3.10}$$

By the independence of the spins and the central limit theorem the initial datum $u_\gamma(x,0) = (J_\gamma \circ \sigma)(x)$ has typical values of order $\gamma^{d/2}$. Then we look at the linearization of (1.3.10) around $u_\gamma \equiv 0$:

$$\begin{aligned}\frac{du_\gamma}{dt}(x,t) &= -u_\gamma(x,t) + \beta(J_\gamma \circ u_\gamma)(x,t) \\ &= \alpha_\gamma u_\gamma(x,t) + \beta(J_\gamma \circ u_\gamma)(x,t) - \beta \hat{J}_{\gamma,0} u_\gamma(x,t)\end{aligned} \tag{1.3.11}$$

where

$$\hat{J}_{\gamma,0} = \sum_{y \in \mathbb{Z}^d} J_\gamma(0,y), \qquad \alpha_\gamma = \beta \hat{J}_{\gamma,0} - 1 \tag{1.3.12}$$

Notice that $\hat{J}_{\gamma,0} \to 1$ and $\alpha_\gamma \to \alpha$ as $\gamma \to 0$. We define

$$\hat{p}_t^\gamma(x,y) = p_t^\gamma(x-y), \qquad p_t^\gamma(x) = e^{-\beta \hat{J}_{\gamma,0} t} \sum_{n \geq 0} \frac{(\beta t)^n}{n!} \sum_{x_1,\dots,x_n} J_\gamma(0,x_1)\dots J_\gamma(x_{n-1},x) \tag{1.3.13}$$

The solution of the linearized equation (1.3.11) is then

$$u_\gamma(x,t) = e^{\alpha_\gamma t}(\hat{p}_t^\gamma \circ u_\gamma)(x,0) \tag{1.3.14}$$

Notice that if $u_\gamma(y,0) \equiv \gamma^{d/2}$, then $u_\gamma(x,t)$ remains finite up to time t_c as $\gamma \to 0$. Now $\hat{p}_t^\gamma(x,\cdot)$ acts on the true $u_\gamma(y,0)$ by averaging it on regions of linear magnitude $\gamma^{-1}\sqrt{t}$. Since the typical value of the average of $[\gamma^{-1}\sqrt{t}]^d$ independent spins is $[\gamma^{-1}\sqrt{t}]^{-d/2}$, we find that up to time t_c the typical value of the quantity $u_\gamma(x,t)$ is

$$e^{\alpha_\gamma t_c}(\gamma^{-1}\sqrt{t})^{-d/2} \simeq t^{-d/4} \simeq \lambda^{d/2}$$

accordingly with the rigorous result (1.3.6). Note moreover that the space dependence of $u_\gamma(x,t_c)$ is ruled by $\hat{p}_t^\gamma$, therefore for $u_\gamma(x,t_c)$ to change significantly, x must vary by $\gamma^{-1}\sqrt{t_c}$, so that the relevant space-time scale is the one defined by (1.3.1).

Next we study the evolution past t_c, when the linear approximation is no longer good.

Evolution in the time interval $[t_c, t^*]$.

We start by introducing the basic notation to describe the interfaces in the macroscopic variables. Denote by $\mathcal{U}$ the set of all the functions $u : \mathbb{R}^d \to \{-m_\beta, m_\beta\}$ such that the discontinuity set Σ of u intersects any ball into finitely many connected regular surfaces of codimension 1. We call *interface* a connected component of Σ. If an interface is closed we call *cluster* the region in its interior.

Elements of $\mathcal{U}$ can be obtained from the statistical solution of the discretized mesoscopic equation (1.3.4) as follows. Let

$$q_t(r) = e^{-\beta t} \sum_{n \geq 1} \frac{(\beta t)^n}{n!} \int dr_1 \dots dr_{n-1} \prod_{i=1}^{n} J(r_i - r_{i-1}), \qquad r_0 = 0,\ r_n = r \tag{1.3.15}$$

Notice that the function $q_t(r)$ is a good approximation of $p_t^\gamma(x)$ for large t as $\gamma \to 0$. By a central limit theorem argument in [27] it is proven that there are $\zeta > 0$ and $c > 0$ such that

$$\left|p_t^\gamma(x) - e^{-\beta \hat{J}_{\gamma,0} t}\delta_{x,0} - \gamma^d q_t(\gamma x)\right| \leq c\gamma^{d+\zeta}$$

Given $0 < \tau_0 < \tau < \tau_c$ and $\sigma \in \{-1,1\}^{\mathbb{Z}^d}$ define, for any $r \in \mathbb{R}^d$ and $\gamma > 0$,

$$\ell_\gamma(r|\sigma) \doteq \lambda^{-d/2}\gamma^{-(\tau_c-\tau)\alpha}\int dr'\, q_{\lambda^{-2}(\tau_c-\tau)}(r-r')m_{\gamma,\lambda^{-2}\tau_0}([\gamma^{-1}r'],\lambda^{-2}\tau|\sigma) \tag{1.3.16}$$

and its macroscopic version

$$\hat{\ell}_\gamma(\xi|\sigma) \doteq \ell_\gamma(\lambda^{-1}\xi|\sigma) \tag{1.3.17}$$

Notice that, by the smoothness of the kernel $q_t(r)$, $\ell_\gamma \in C^\infty(\mathbb{R}^d)$.

For any open bounded subset Λ of $\mathbb{R}^d$, let $C^n(\Lambda)$ the space of functions which have n bounded derivatives in Λ. $C^n(\Lambda)$ is normed with the sup norm of the function and its first n derivatives. For any $n \geq 0$ we define the probability $\mathcal{P}_{\gamma,\tau_0}$ on $C^n(\Lambda)$ as the image of $\mu^\gamma_{\lambda^{-2}\tau_0}$ via the map (1.3.17). Then for any Borel set $A \in C^n(\Lambda)$,

$$\mathcal{P}_{\gamma,\tau_0}(A) = \mu^\gamma_{\lambda^{-2}\tau_0}(\{\sigma : \hat{\ell}_\gamma(\cdot|\sigma) \in A\}) \tag{1.3.18}$$

By the smoothness of $\hat{\ell}_\gamma$ the measure $\mathcal{P}_{\gamma,\tau_0}$ is supported on $\bigcap_{n\geq 0} C^n(\Lambda)$.

In [27] the following theorem is proven.

1.3.3 Theorem.

For any $\epsilon > 0$ there is $\gamma_0 \in (0,1]$ such that the following holds. For any $\gamma \in (0,\gamma_0]$ there is a set $\mathcal{G}_\gamma^{(1)} \subset \{-1,1\}^{\mathbb{Z}^d}$ such that

$$\mu^\gamma_{\lambda^{-2}\tau_0}(\mathcal{G}_\gamma^{(1)}) > 1-\epsilon \tag{1.3.19}$$

and, for any $\sigma \in \mathcal{G}_\gamma^{(1)}$, the function

$$u_\gamma(\xi) \doteq m_\beta \mathrm{sign}\{\hat{\ell}_\gamma(\xi|\sigma)\}, \qquad u_\gamma(\xi) = m_\beta \text{ if } \hat{\ell}_\gamma = 0 \tag{1.3.20}$$

belong to the set $\mathcal{U}$. Finally, for any n and any bounded regular set $\Lambda \subset \mathbb{R}^d$, the probabilities $(C^n(\Lambda), \mathcal{P}_{\gamma,\tau_0})$ converge to $(C^n(\Lambda), \mathcal{P})$, where $\mathcal{P}$ is the law of the Gaussian process with 0 average and covariance

$$\mathcal{E}(X(\xi)X(\xi')) = \left(1+\frac{1}{\alpha}\right)\frac{\alpha^{d/2}}{(\pi d\beta D)^{d/2}}e^{-\alpha(\xi-\xi')^2/(d\beta D)}, \qquad \xi,\xi' \in \mathbb{R}^d$$
$$D = \int dr\, J(r) r^2 \tag{1.3.21}$$

Now we connect the Glauber dynamics with the deterministic evolution up to time t^*. It is the content of the following theorem.

1.3.4 Theorem.

For any $\epsilon > 0$, $b < 1$ close enough to 1, $R > 0$, $0 < \tau_0 < \tau_c$, there is $\gamma_0 \in (0,1]$ so that the following holds. For any $\gamma \in (0,\gamma_0]$ there is a set $\mathcal{G}_\gamma^{(2)} \subset \{-1,1\}^{\mathbb{Z}^d}$ such that

$$\mu^\gamma_{\lambda^{-2}\tau_0}(\mathcal{G}_\gamma^{(2)}) > 1 - \epsilon \tag{1.3.22}$$

and, if $\sigma \in \mathcal{G}_\gamma^{(2)}$, there is $\delta > 0$ so that

$$\sup_{t\in[\lambda^{-2}\tau_c,t^*]} \mathbb{P}^\gamma_{\sigma,\lambda^{-2}\tau_0}\left(\sup_{\lambda\gamma|x|\le R} \left|\mathcal{A}_{\gamma,b,x}(\sigma_t) - \mathcal{A}_{\gamma,b,x}(m_{\gamma,\lambda^{-2}\tau_0}(\cdot,t|\sigma))\right| \ge \gamma^\delta \right) < \epsilon \tag{1.3.23}$$

Then there is a set of good configurations which has large probability and such that if one starts from one of these configurations at time $\lambda^{-2}\tau_0$ then the block spin variable at any time $t \in [\lambda^{-2}\tau_0, t^*]$ is close to the solution of the mesoscopic discretized equation (1.3.4). In order to characterize the development of the interfaces at time t^*, we are left to the analysis of $m_{\gamma,\lambda^{-2}\tau_0}(x,t^*|\sigma)$. First of all we need the following definition.

1.3.5 Definition.

The instanton $\bar{m}(s)$, $s \in \mathbb{R}$, is an antisymmetric and non identically zero solution of the non local mean field equation

$$\bar{m}(s) = \tanh\{\beta\tilde{J} \star \bar{m}(s)\} \tag{1.3.24}$$

where

$$\tilde{J}(s) \doteq \int_{\mathbb{R}^{d-1}} dr\, J((s,r)), \qquad (s,r) \in \mathbb{R}^d \tag{1.3.25}$$

(recall that $J((s,r))$ is a function of $\sqrt{s^2+|r|^2}$).

In [29] it is proven that the instanton exists provided $\beta > 1$, it is unique in the class of functions that are strictly positive (negative) as $s \to +\infty$ ($s \to -\infty$) and that vanish at the origin. Moreover it is strictly increasing and converges exponentially to $\pm m_\beta$, see (1.1.25), as $s \to \pm\infty$.

From the above results it follows that for any unit vector ν the function

$$m^*_\nu(r) = \bar{m}(r\cdot\nu) \tag{1.3.26}$$

is the unique nonhomogeneus stationary solution of the mesoscopic equation (1.2.8) which connects the pure phases $\pm m_\beta$ along the given direction. In §2.2 we will discuss in more details the properties of the instanton. Now we can state the last result.

1.3.6 Theorem.

For any $\epsilon > 0$, $L > 0$ and $0 < \tau_0 < \tau_c$ there is $\gamma_0 \in (0,1]$ such that the following holds. For any $\gamma \in (0,\gamma_0]$ there is a set $\mathcal{G}_\gamma^{(3)} \subset \{-1,1\}^{\mathbb{Z}^d}$ such that

$$\mu^\gamma_{\lambda^{-2}\tau_0}(\mathcal{G}_\gamma^{(3)}) > 1 - \epsilon \tag{1.3.27}$$

There is a positive function R_γ, $\gamma \in (0,\gamma_0]$ so that

$$\lim_{\gamma \to 0} R_\gamma = 0, \qquad \lim_{\gamma\to 0} \lambda^{-1} R_\lambda = \infty \tag{1.3.28}$$

For any $\sigma \in \mathcal{G}_\gamma^{(3)}$ the function

$$u_\gamma(\xi) \doteq m_\beta sign\{\hat{\ell}_\gamma(\xi|\sigma)\}, \qquad u_\gamma(\xi) = 0 \;\; if \;\; \hat{\ell}_\gamma(\xi|\sigma) = 0 \tag{1.3.29}$$

belong to the set $\mathcal{U}$.

Let Σ be an interface of u_γ. Then

$$\left|m_{\gamma,\lambda^{-2}\tau_0}(x, t^*|\sigma) - u_\gamma(\lambda\gamma x)\right| \le \epsilon \tag{1.3.30}$$

for any $\gamma \in (0,\gamma_0]$ and all $|x| \le L(\lambda\gamma)^{-1}$ such that $dist(\lambda\gamma x, \Sigma) \ge R_\gamma$.

Finally, for any $\xi_0 \in \Sigma$, $|\xi_0| \le L$, let $\hat{n}$ be the unit vector normal to Σ at ξ_0 and pointing toward the region where $u_\gamma(\xi) = m_\beta$. Then for all $\gamma \in (0,\gamma_0]$,

$$\left|m_{\gamma,\lambda^{-2}\tau_0}([\gamma^{-1}(\lambda^{-1}\xi_0 + \hat{n}s)], t^*|\sigma) - \bar{m}(s)\right| \le \epsilon, \qquad for\ any\ |s| \le \lambda^{-1}R_\gamma \tag{1.3.31}$$

where $\bar{m}$ is the instanton defined before.

We can summarize all the previous results as follows.

1). There are two times t_c and t^*, depending on γ such that in $[0, t_c]$ the typical values of the magnetization is still infinitesimal as $\gamma \to 0$, while at time t^* the stable phases are completely developed. In the macro-scale the two times converge, as $\gamma \to 0$, to a unique time τ_c which depends only on the inverse temperature β, the interaction J and the dimension d. Then the time when the phases separate is sharp and deterministic in the macro-scale.

2). At time t^* the space is divided into "clusters" where the magnetization is alternatively equal to $\pm m_\beta$, the Lebowitz-Penrose equilibrium magnetizations at inverse temperature β. The typical diameter of the clusters in lattice units is $|\log\gamma|^{1/2}\gamma^{-1}$.

Moreover there is a unique magnetization pattern at the boundaries between different clusters, which is given in the meso-scale by the function $\bar{m}(s)$, s the length parameter along the normal to the boundary.

3). The geometry of the clusters at time t^* is given in terms of a Gaussian field. Moreover, with probability one in the limit $\gamma \to 0$, the actual positions of the clusters at time t^* are completely determined by the spin configuration at any earlier time, except time 0, when times are measured in the macro-scale.

§1.4 Macroscopic limits: interface dynamics and motion by mean curvature.

As explained in the last section, at time t^* the phases are fully developed. The successive stage of the evolution consists in the interface dynamics. As just noticed, the microscopic dynamics is of non conservative type and, in dimension $d \geq 2$, one expects the interfaces move by mean curvature. This is consistent with the experimental observations, according to which the typical size of the clusters should grow as $t^{1/2}$, see [42], [60] and references therein. We restrict the analysis to the case of physical dimension $d \geq 2$ and we refer to §3.1 for some comments on the case $d = 1$.

According to the phenomenological theory, the general picture on the competition between phases is as follows (see [60] and references therein). Let us consider a simple fluid when two pure phases coexist. One represents the interface that separates the two phases by a smooth surface Σ embedded in $\mathbb{R}^d$ (on the macroscopic scale the transition region between distinct phases is infinitely thin). The surface free energy (i.e. the thermodynamical free energy excess due to the presence of the interface) is

$$F = \int_{\Sigma} df \, \sigma(\hat{n}) \tag{1.4.1}$$

where $\hat{n}$ is the local normal to Σ in the surface element df, while $\sigma(\hat{n})$ is the surface tension of a flat interface perpendicular to $\hat{n}$. Phenomenologically it is postulated that, for system with non conservative dynamics, the interface velocity along the local normal is given by

$$v = -\mu \frac{\delta F}{\delta \Sigma} \tag{1.4.2}$$

where μ is called the *mobility* of the interface.

When the system is isotropic both σ and μ do not depend on $\hat{n}$ and equation (1.4.2) reduces to

$$v = \theta \kappa \hat{n} \tag{1.4.3}$$

where κ is $(d-1)$ times the mean curvature of Σ and the constant θ should be related to the mobility μ and to the thermodynamical quantity σ by the "Einstein relation"

$$\theta = \mu \sigma \tag{1.4.4}$$

Since we assume that the Kac interaction $J(r)$ depends only on $|r|$, our model is isotropic. Then, after t^*, the interfaces should move according to equation (1.4.3). We split the rest of the section into two parts. In the first one we give some basic results and notation on the mean curvature flow. In the second one we state the known results on the derivation of the interface dynamics from the microscopic model. In the next section we will prove the validity of the Einstein relation (1.4.4), thus completing the connection with the phenomenological picture.

Motion by mean curvature.

We start with the following definition.

1.4.1 Definition.

The surface Σ_τ evolves according to the classical motion by mean curvature with parameter $\theta > 0$ in the time interval $[0, \tau_0]$ if the following holds.
i) For any $\tau \in [0, \tau_0]$ Σ_τ is the boundary of an open bounded set $\Lambda_\tau \subset \mathbb{R}^d$.
ii) There is a smooth $(d-1)$ dimensional compact manifold S_0 and a smooth map ξ : $[0, \tau_0] \times S_0 \to \mathbb{R}^d$ such that, for any $\tau \in [0, \tau_0]$, $\xi(\tau, \cdot)$ is an embedding of S_0 in $\mathbb{R}^d$,

$$\Sigma_\tau = \{\xi = \xi(\tau, \eta) \,|\, \eta \in S_0\} \tag{1.4.5}$$

and

$$\frac{d\xi}{d\tau} = \theta \kappa \hat{n} \tag{1.4.6}$$

where $\hat{n} = \hat{n}(\xi)$ is the unit vector normal to Σ_τ at ξ and pointing toward the interior of Σ_τ, while κ is $(d-1)$ times the mean curvature of Σ_τ at ξ.

In particular the mean curvature flow starts from the smooth surface Σ_0, the boundary of Λ_0 and, for any $\tau \in [0, \tau_0]$, the surface Σ_τ and the region Λ_τ are diffeomorphic to Σ_0 and Λ_0 respectively.

A local existence and uniqueness theorem for the classical motion by mean curvature follows from general results on parabolic equations (see [38] and [49]). On the other hand it is known that in $d > 2$ singularities may develop in a finite time, while in $d = 2$ the only singularity which can appear is the shrinking to a point of the curve (see [46], [47] and [49]). It is fairly clear that, even if Σ_0 is smooth, a smooth evolution as required in Definition 1.4.1 cannot exist beyond some initial time interval. In fact, after a first time interval the evolved surface Σ_τ would change its topological type and this suggests the problems which come out in the classical differential geometric approach of regarding Σ_0 as a parameterized surface (the parameterization will in general develop singularities).

In the last years alternative descriptions of the evolution of surfaces by mean curvature have been developed, sufficiently general as to allow for the possible onset of singularities and to avoid the topological complications. Among these, we recall the definition of a *generalized motion by mean curvature* as the motion of the zero level set of the unique continuous weak solution of the "level set equation". Given a smooth surface $\Sigma_0 \subset \mathbb{R}^d$, we choose a continuous function $g : \mathbb{R}^d \to \mathbb{R}$ such that

$$\Sigma_0 = \{r \in \mathbb{R}^d \,|\, g(r) = 0\} \tag{1.4.7}$$

We consider the parabolic PDE, called the *level set equation*,

$$\frac{\partial u}{\partial t} = \theta \mathrm{Tr}\left[\left(I - \frac{Du \otimes Du}{|Du|^2}\right) D^2 u\right], \qquad \text{on } \mathbb{R}^d \times \mathbb{R}_+ \tag{1.4.8}$$

with $u = g$ on $\mathbb{R}^d \times \{0\}$.

It can be proven that each level set of u evolves according to its mean curvature at least in regions where u is smooth and its spatial gradient Du does not vanish. Then, selecting for example the set $\{u = 0\}$, we define

$$\Sigma_t \doteq \{r \in \mathbb{R}^d | u(r,t) = 0\}, \qquad \text{for any } t \geq 0 \tag{1.4.9}$$

There are several analytic problems to treat equation (1.4.8). Notice that it is nonlinear, degenerate, and indeed even undefined where $Du = 0$. Moreover it is not clear from (1.4.8) that the definition (1.4.9) does not depend on the choice of g. All these problems were solved by introducing an appropriate definition of *weak solution* for the equation (1.4.8) which has all the desired properties. More precisely there exists a unique continuous weak solution for all times, it does not depend on the choice of g for fixed Σ_0, and finally its level sets move according to the classical motion by mean curvature over any time interval where the latter exists. It has been carried out deep analysis of the geometric properties of the level set surfaces of the weak solution after the singularities. Among these we just recall the possibility of occurrence of the *fattening*: after a singularity, the surface Σ_t defined by (1.4.9) may become fat, e.g. it may develop an interior. We do not enter into these very interesting aspects and we refer to the Literature for details (see [21], [22], [38] and references therein).

Derivation of the interface dynamics.

The first proof that the interfaces evolves according to the mean curvature motion was given in [25]. The analysis covers only times where the motion is classical in the sense of Definition 1.4.1. The extension to infinitely many interfaces, as appear in Theorem 1.3.6, should not bring serious problems. We can summarize this result into the following theorem.

1.4.2 Theorem.

Let us consider a compact domain Λ_0 *whose boundary* Σ_0 *is a smooth surface in* $\mathbb{R}^d$*. Let* $\epsilon = \lambda\gamma$ *and denotes by* μ^ϵ *the product measure on* $\{-1,1\}^{\mathbb{Z}^d}$ *with*

$$\mathbb{E}^{\gamma}_{\mu^\epsilon}[\sigma(x)] = \begin{cases} m_\beta & \text{if } x \in \epsilon^{-1}\Lambda_0 \\ -m_\beta & \text{if } x \in \mathbb{Z}^d \setminus \Lambda_0 \end{cases} \tag{1.4.10}$$

Let Σ_τ *be the mean curvature flow starting from* Σ_0 *with parameter* θ *given below. Let* $\tau^\star$ *such that the motion is classical for any* $\tau \in [0,\tau^\star]$*. Then there is* $\zeta \in (0,1)$ *and for any* $\tau < \tau^\star$ *and any* $n \geq 1$ *there is* c *so that, for* γ *small enough,*

$$\mathbb{P}^{\gamma}_{\mu^\epsilon}\left(\sup_{\tau' \leq \tau} \sup_{\substack{|\xi| \leq \gamma^{-1}\lambda \\ |d(\xi,\Sigma_{\tau'})| \geq \lambda^\zeta}} \left|\sigma^{\gamma,b}_{\lambda^{-2}\tau'}(\lambda^{-1}\xi) \mp m_\beta\right| \leq \lambda^{3/2} \right) \geq 1 - c\gamma^n \tag{1.4.11}$$

where $|d(\xi,\Sigma_{\tau'})|$ *is the distance of the point* ξ *from the surface* $\Sigma_{\tau'}$*; the* $-$ *sign refers to* $\xi \in \Lambda_0$ *and the* $+$ *sign for* ξ *outside* Λ_0*.* $\mathbb{P}^{\gamma}_{\mu^\epsilon}$ *is the law of the Glauber process which starts at time 0 from* μ^ϵ *with* $h = 0$ *and* $\beta > 1$*. The parameter* θ *depends on the interaction* J *and on the inverse temperature* β *by means of the following expression*

$$\theta = N\beta \int dx\, \bar{m}'(x) \int dx' \int_{\mathbb{R}^{d-1}} dy\, J((x',y)-(x,0))\bar{m}'(x')\frac{y_1^2}{2} \tag{1.4.12}$$

where we use the notation $(x',y) = (x', y_1, \dots, y_{d-1}) \in \mathbb{R}^d$ *and*

$$N^{-1} = \int ds \frac{\bar{m}'(s)^2}{1-\bar{m}(s)^2} \tag{1.4.13}$$

The constant θ is well defined since of the nice convergence properties of the instanton $\bar{m}$. Using the strong deterministic behaviour of the Glauber dynamics as $\gamma \to 0$, in particular Theorem 1.2.2, the estimate (1.4.11) follows from an analogous result for the mesoscopic equation (1.2.8). More precisely in [28] it is proven that the singular limit of the mesoscopic equation (1.2.8) with $h = 0$ and $\beta > 1$, under a diffusive scaling, converges to the mean curvature flow. The precise statement is the following theorem.

1.4.3 Theorem.

Let $(\xi,\tau) = (\lambda r, \lambda^2 t)$*,* $\lambda \in (0,1]$*,* $r \in \mathbb{R}^d$*,* $t \in \mathbb{R}_+$*. Define*

$$m^{(\lambda)}(\xi,\tau) = m(\lambda^{-1}\xi, \lambda^{-2}\tau) = m(r,t) \tag{1.4.14}$$

where $m(r,t)$ *solves the equation (1.2.8) with* $h = 0$*,* $\beta > 1$ *and initial datum*

$$m(r,0) = m_0(\lambda r;\lambda) = \bar{m}(\lambda^{-1}d(\xi,\Sigma_0)) \tag{1.4.15}$$

where $\bar{m}$ is the instanton defined in Definition 1.3.5. Σ_0 *is a smooth surface in* $\mathbb{R}^d$*, the boundary of a bounded open region* Λ_0*. Finally* $d(\xi, \Sigma_0)$ *is the signed distance of* ξ *from* Σ_0*, positive when* $\xi \in \Lambda_0$*.*

Let Σ_τ *be the motion by mean curvature flow starting from* Σ_0 *with parameter* θ *as in* (1.4.12)*. Let* $\tau_0 > 0$ *be such that the motion is classical for any* $\tau \in [0, \tau_0]$*. Then there is* λ_0 *so that for any* $\tau < \tau_0$ *and any* $\lambda \in (0, \lambda_0]$*,*

$$|m^{(\lambda)}(\xi,\tau) - sign(d(\xi,\Sigma_\tau))m_\beta| \leq \lambda^{3/2}, \qquad \text{for any } \xi \text{ such that } |d(\xi,\Sigma_\tau)| \geq \lambda^{1/80} \quad (1.4.16)$$

In a recent paper, [52], Katsoulakis and Souganidis have generalized the result of Theorem 1.4.2 by proving that the interface dynamics in the limit is ruled by the generalized motion by mean curvature. Moreover, in this paper, the macroscopic scaling parameter λ has the form $\lambda = \gamma^q$ for $q > 0$ small enough. The global convergence to a motion by mean curvature in $d = 2$, when the only singularity is the disappearance of the cluster, was earlier proven in [13] with λ as in the scaling (1.3.1).

All the previous results are however unsatisfactory in the mesoscopic description, where the space is scaled by a factor λ^{-1} with respect to the macro-scale. In fact the size of the strip around the interface that is not covered in (1.4.11) (as well as in [52]) diverges as $\gamma \to 0$ in the meso-scale, so one losses entirely the structure of the interface. The closeness of the block spin variable to the solution of the mesoscopic equation is good in the meso-scale (see Theorem 1.2.2), while it is the estimate (1.4.16) which is poor in this scale. What one needs then is a more detailed control on the rate of convergence of the rescaled mesoscopic equation to the mean curvature flow. In the next chapter we will solve this problem, by proving that, at least for times when the limiting motion is regular, the interface is localized in the meso-scale and the instanton-like structure of the magnetization pattern at the boundary of the cluster is preserved during the dynamics.

It is clear that all the previous results deal with the evolution of the system just after the phases separate. The structure of the interfaces much after the phase separation should be the same as before if distances are measured in units $t^{1/2}$, since this is the typical size of the clusters. But the analysis of the macroscopic behaviour when $\lambda = \gamma^a$, for generic $a > 0$, or even in the "true hydrodynamic limit", when one first takes $\lambda \to 0$ and then $\gamma \to 0$, should be drastically different, since the stochastic effects one has to consider include events that in our macroscopic scale are large deviations. For example in [26], where the fluctuation theory in the mesoscopic regime is studied, it is proven that in the case $d = 2$ the fluctuations are of order $\gamma t^{1/4}$. Then, on the time scale $t \approx \gamma^{-4}$ the fluctuations produces finite displacement of the interface and there is a competition between fluctuations and the motion by mean curvature. On the scale $t \approx \gamma^a$ with $a > 4$ the initial effect is purely stochastic and only later the curvatures become again important. On these regimes of the evolution there are no results at all.

§1.5 The Einstein relation.

According to the phenomenological picture given at the beginning of the section, the parameter θ defined in (1.4.12) should be related to the surface tension and to the mobility of the interface by the relation (1.4.4). Actually we can prove (1.4.4) rigorously in our model, [12].

The microscopic definition of the surface tension (see [60] and references therein) involves the computation of the logarithm, normalized by the surface area, of the ratio of two partition functions with different boundary conditions. The second one has boundary conditions + on the two opposite faces of a cube and periodic conditions on the other ones; the first one is defined by conditions + and − instead of + and +. This definition is compatible with (1.4.1) under the assumption that the main contribution to the free energy difference in changing the boundary condition comes from the presence of a flat interface, parallel to the opposite faces in the average. So, in the thermodynamic limit, when the fluctuations are depressed, the microscopic definition gives (1.4.1) in the special case of a flat surface. Then this definition is based on a preliminary assumption of validity of (1.4.1), which conceptually should be derived first.

In a recent paper, [2], the validity of (1.4.1) is rigorously proved in the context of the Ising spin system with Kac potential in the limit of Lebowitz and Penrose and an explicit expression for the surface tension is given.

Given $\Lambda \subset \mathbb{R}^d$, we introduce the non local Van der Waals functional

$$\mathcal{F}(\cdot;\Lambda) : L^\infty(\Lambda;[-1,1]) \to [0,+\infty]$$

defined by

$$\mathcal{F}(m;\Lambda) \doteq \int_\Lambda dr\,[f_\beta(m) - f_\beta(m_\beta)] + \frac{1}{4}\int_{\Lambda\times\Lambda} dr\,dr'\,J(r-r')[m(r)-m(r')]^2 \qquad (1.5.1)$$

where f_β is defined in (1.1.24). This functional is closely related to the Gibbs measures of the Ising model in the limit $\gamma \to 0$. For example, in the case of a one dimensional torus $\Lambda = \mathcal{T}$, Eisele and Ellis, [36], have proved that $\mathcal{F}(m,\mathcal{T})$ is the large deviation rate function for the Ising model in the limit $\gamma \to 0$. The result is extended to infinite volumes in [18].

In [2] it is considered the case of a torus of size $\gamma^{-1-\alpha}$, $\alpha \in (0,1)$, in $\mathbb{Z}^d$, $d \geq 2$. Roughly speaking, it is proven that the probability of a "coarse grained" spin configuration behaves like $\exp[-\beta\gamma^{-d}\mathcal{F}(\cdot,\Lambda_\gamma)]$ as $\gamma \to 0$, where Λ_γ is the torus on the meso-scale (actually one would like to characterize the typical Gibbs configurations in the mean field limit, by taking first the limit $\Lambda \to \mathbb{R}^d$ and then $\gamma \to 0$, but for technical reasons one performs the thermodynamic limit together with the Lebowitz-Penrose one in a suitable way). So the problem of finding the magnetization profiles which have the greatest probability reduces to the one of finding the minima of the functional $\mathcal{F}$. Now, as well as for the mesoscopic equation, for any unit vector ν, the function $m^*_\nu(r)$ defined in (1.3.26) is the unique stationary solution of the Euler-Lagrange equation for the functional $\mathcal{F}(\cdot,\mathbb{R}^d)$ connecting the pure phases $\pm m_\beta$ along the given direction. So also at the equilibrium the magnetization pattern near the interface is described by the instanton $\bar{m}$. All these argument are made rigorous in [2] and as a consequence it is proven that the surface tension, in the limit of $\gamma \to 0$, is given by

$$\sigma = \tilde{\mathcal{F}}(\bar{m}) \tag{1.5.2}$$

where $\tilde{\mathcal{F}}$ is the functional (1.5.1) in dimension $d = 1$ with $\Lambda = \mathbb{R}$ and J replaced by $\tilde{J}$ defined in (1.3.25), while $\bar{m}$ is the instanton.

To compute the mobility we use a linear response argument. In the presence of a uniform small magnetic field h, a small deformation of the surface yields an extra free energy excess which, to the first order in h, is $-2m_\beta h\delta\Sigma$, so that the velocity of the interface becomes

$$v = -\mu\left[\frac{\delta F}{\delta\Sigma} - 2m_\beta h\right] \tag{1.5.3}$$

We look then for a planar travelling wave solution of the mesoscopic equation (1.2.8) along the r_1 axis ($r = (r_1, ..., r_d)$)

$$m_h(r,t) = \bar{m}_h(r_1 - v(h)t) \tag{1.5.4}$$

for small h. The existence of a such solutions is proven in [24]. By expanding around $h = 0$ we get

$$v(h) = v_1 h + O(h^2), \qquad \bar{m}_h = \bar{m} + h\psi + O(h^2) \tag{1.5.5}$$

From (1.2.8), at first order in h, we obtain the following identity

$$-v_1\bar{m}' = -\psi + (1-\bar{m}^2)\beta\tilde{J}\star\psi + \beta(1-\bar{m}^2) \tag{1.5.6}$$

We multiply both sides of (1.5.6) by $\bar{m}'(r_1)/(1-\bar{m}(r_1)^2)$ and then integrate; since the derivative of the instanton satisfies the equation

$$(1-\bar{m}^2)\beta\tilde{J}\star\bar{m}' = \bar{m}' \tag{1.5.7}$$

we get

$$v_1 = -2N\beta m_\beta \tag{1.5.8}$$

Equations (1.5.3) and (1.5.8) imply then

$$\mu = N\beta \tag{1.5.9}$$

where N was defined in (1.4.13).

By (1.4.12), (1.5.2) and (1.5.9) the relation (1.4.4) holds if

$$\tilde{\mathcal{F}}(\bar{m}) = \int dx\, \bar{m}'(x) \int dx' \int_{\mathbb{R}^{d-1}} dy\, J((x',y)-(x,0))\bar{m}'(x')\frac{y_1^2}{2} \tag{1.5.10}$$

To prove (1.5.10) we begin by observing that, since $J(r)$ depends only on $|r|$,

$$\frac{\partial}{\partial x'} J((x',y)-(x,0)) = \frac{x'-x}{y_1}\frac{\partial}{\partial y_1} J((x',y)-(x,0)) \tag{1.5.11}$$

Integrating by parts in dx', using (1.5.11), and then integrating by parts in dy_1, we obtain

$$\begin{aligned}\int dx\, \bar{m}'(x) \int dx' \int_{\mathbb{R}^{d-1}} dy\, J((x',y)-(x,0))\bar{m}'(x')\frac{y_1^2}{2}\\ = \frac{1}{2}\int dx dx'(x'-x)\bar{m}'(x)\tilde{J}(x-x')\bar{m}(x')\end{aligned} \tag{1.5.12}$$

Next we consider the left hand side of (1.5.10). Using the condition (1.1.2) we have

$$\int dr dr'\, J(r-r')(m(r)-m(r'))^2 = 2\int dr\, (m^2 - mJ\star m)(r) \tag{1.5.13}$$

so that, using the explicit form (1.1.24) of f_β,

$$\mathcal{F}(m;\mathbb{R}^d) = \int dr\, [g(m)-g(m_\beta)](r) \tag{1.5.14}$$

where

$$g(m) \doteq -\beta^{-1}i(m) - \frac{1}{2}mJ\star m \tag{1.5.15}$$

Then

$$\tilde{\mathcal{F}}(\bar{m}) = \int dx\, [g(\bar{m}(x)) - g(m_\beta)] \tag{1.5.16}$$

By using the identity (1.5.7) we get

$$\frac{d}{dx}g(\bar{m}(x)) = \frac{1}{2}(\bar{m}'\tilde{J}\star\bar{m} - \bar{m}\tilde{J}\star\bar{m}')(x) \tag{1.5.17}$$

Integrating by parts in (1.5.16) we finally obtain

$$\begin{aligned}\tilde{\mathcal{F}}(\bar{m}) &= -\frac{1}{2}\int dx\, x\frac{d}{dx}g(\bar{m}(x))\\ &= \frac{1}{2}\int dx dx'(x'-x)\bar{m}'(x)\tilde{J}(x-x')\bar{m}(x')\end{aligned} \tag{1.5.18}$$

From (1.5.12) and (1.5.18) the equality (1.5.10) follows and hence (1.4.4).

CHAPTER 2.

QUASI-OPTIMAL ERROR ESTIMATES FOR THE MEAN CURVATURE FLOW BY SCALING THE NON LOCAL EQUATION. LOCALIZATION OF THE INTERFACE

In this chapter we study the singular limit, under a diffusive scaling, of the mesoscopic equation in absence of external magnetic field and below the Lebowitz-Penrose critical temperature. We look at this problem as an approximation of the evolution of an interface by its mean curvature. We derive a quasi-optimal error estimate of order $\mathcal{O}(\lambda^2 |\log \lambda|^{20})$, λ the scaling parameter, which is valid before the onset of singularities. Moreover we prove that the instanton-like structure of the magnetization pattern at the interface is persistent, at least until times when the surface is regular. This result improves Theorem 1.4.3 by showing that in the mesoscopic variables the form of the solution remains the same, to leading orders in λ, for times proportional to λ^{-2}, even though the interface during this time moves by distances proportional to λ^{-1}. As just noticed in the last chapter, the strong deterministic behaviour of the Glauber dynamics in the meso-scale allows to obtain, with no serious difficulties, an analogous result for the microscopic dynamics after phase separation, but we do not prove this here.

The chapter is divided into 6 sections. In the first one we state the main notation and the results. In the second one we give a miscellany of known results we will use. In the successive two sections we make formal expansions in λ of the evolution around the interface which we will need in the sequel. In the remaining sections we give the rigorous proofs.

The result proven here (that is Theorem 2.1.1 below) is based on a paper in preparation with A. De Masi, [14].

§2.1 Notation and statement of the results.

Let Σ be a smooth closed manifold of dimension $(n-1)$ embedded in $\mathbb{R}^n$ and denote by $I(\Sigma)$ and $O(\Sigma)$ respectively the interior and the exterior of Σ (here we use the letter n instead of d to denote the physical dimension since we will use the latter to denote the distance of a point from a surface). Let $\hat{n}(\xi, \Sigma)$ be the normal to Σ in $\xi \in \Sigma$ directed toward

$I(\Sigma)$. Call $\kappa_i(\xi,\Sigma)$, $i=1,...,n-1$, the principal curvatures of Σ at $\xi\in\Sigma$, labelled in order of increasing value. We denote by $\underline{\kappa}(\xi,\Sigma)$ the vector in $\mathbb{R}^{n-1}$ whose components are the curvatures $\kappa_i(\xi,\Sigma)$. We set

$$\kappa(\xi,\Sigma)\doteq\sum_{i=1}^{n-1}\kappa_i(\xi,\Sigma),\quad \bar{\kappa}^2(\xi,\Sigma)\doteq\sum_{i=1}^{n-1}\kappa_i(\xi,\Sigma)^2,\quad \hat{\kappa}(\xi,\Sigma)\doteq\sum_{\substack{i,j=1\\ i\neq j}}^{n-1}(\kappa_i\kappa_j)(\xi,\Sigma) \tag{2.1.1}$$

Then $\kappa(\xi,\Sigma)$ is $(n-1)$ times the mean curvature of Σ at $\xi\in\Sigma$. We define the signed distance $d(\xi,\Sigma)$ of a point $\xi\in\mathbb{R}^n$ from Σ by setting

$$d(\xi,\Sigma)\doteq\begin{cases}\text{dist}(\xi,\Sigma) & \text{if } \xi\in I(\Sigma)\\ 0 & \text{if } \xi\in\Sigma\\ -\text{dist}(\xi,\Sigma) & \text{if } \xi\in O(\Sigma)\end{cases} \tag{2.1.2}$$

Given $\epsilon>0$ we define the tubular neighbourhood of Σ

$$\mathcal{T}_\epsilon(\Sigma)\doteq\{\eta\in\mathbb{R}^n\ :\ |d(\eta,\Sigma)|\le\epsilon\} \tag{2.1.3}$$

If ϵ is small enough for any point $\xi\in\mathcal{T}_\epsilon(\Sigma)$ there exists a unique projection $s(\xi,\Sigma)\in\Sigma$ such that

$$\text{dist}(s(\xi,\Sigma),\xi)=|d(\xi,\Sigma)| \tag{2.1.4}$$

Let Σ_τ, $\tau\in[0,\tau_0]$, be the classical mean curvature flow as in Definition 1.4.1 and with parameter θ given by (1.4.12). As discussed in §1.4, for any smooth surface Σ_0, there is a time $\tau_s>0$ such that, for any $\tau_0<\tau_s$, there exists the classical mean curvature flow for $\tau\in[0,\tau_0]$ and starting from Σ_0.

The mesoscopic evolution equation is given in the meso-scale (r,t) related to the macro-scale (ξ,τ) by a diffusive scaling

$$(r,t)=(\lambda^{-1}\xi,\lambda^{-2}\tau),\qquad \lambda\in(0,1] \tag{2.1.5}$$

Notice that the mean curvature evolution is invariant under the scaling (2.1.5). We consider the Cauchy problem

$$\begin{cases}\partial_t m(r,t)=-m(r,t)+\tanh[\beta J\star m](r,t)\\ m(r,0)=m_0(r,\lambda)\end{cases} \tag{2.1.6}$$

where $\beta>1$ and J is the Kac interaction defined in §1.1. Hereafter we prefer to write explicitly the dependence on $|r|$ of the interaction. Then $J=J(s)$ is a smooth nonnegative function with compact support on $[0,1]$ such that

$$\int_{\mathbb{R}^n}dr\ J(|r|)=1 \tag{2.1.7}$$

and, for any function f on $\mathbb{R}^n$,

$$J \star f(r) \doteq \int_{\mathbb{R}^n} dr'\, J(|r-r'|) f(r') \tag{2.1.8}$$

We impose convergence at time $t=0$ by requiring that $m_0(\lambda^{-1}\xi,\lambda)$ converges, as $\lambda \to 0$, to m_β in $I(\Sigma_0)$ and to $-m_\beta$ in $O(\Sigma_0)$. By the results of Theorem 1.3.6 we impose also "local equilibrium" at time $t=0$ and choose

$$m_0(r,\lambda) = \bar{m}(d(r,\lambda^{-1}\Sigma_0)) \tag{2.1.9}$$

where $\lambda^{-1}\Sigma_0$ is the surface Σ_0 in the "mesoscopic space". The function $\bar{m}$ is the instanton (see Definition 1.3.5). In the "macroscopic space" we define $m^{(\lambda)}(\xi,\tau) \doteq m(\lambda^{-1}\xi,\lambda^{-2}\tau)$ which solves

$$\begin{cases} \partial_\tau m^{(\lambda)}(\xi,\tau) = \lambda^{-2}\big(-m^{(\lambda)}(\xi,\tau) + \tanh[\beta J^{(\lambda)} \star m^{(\lambda)}](\xi,\tau)\big) \\ m^{(\lambda)}(\xi,0) = \bar{m}(\lambda^{-1} d(\xi,\Sigma_0)) \end{cases} \tag{2.1.10}$$

where $J^{(\lambda)}(\xi) \doteq \lambda^{-n} J(\lambda^{-1}|\xi|)$.

We have now all the necessary to state the main result of the chapter.

2.1.1 Theorem.

Let $m^{(\lambda)}(\xi,\tau)$ be the solution of the Cauchy problem (2.1.10) and τ_s the first singularity time. Let

$$\Sigma_{\lambda,\tau} \doteq \{\, \xi \in \mathbb{R}^n \,:\, m^{(\lambda)}(\xi,\tau) = 0 \,\} \tag{2.1.11}$$

Then, for any $\tau_0 < \tau_s$ there is $\lambda_0 \in (0,1]$ and a positive constant C, depending on τ_0, Σ_0 and the interaction J such that, for any $\lambda \in (0,\lambda_0]$,

$$\Sigma_{\lambda,\tau} \subseteq \{\, \xi \in \mathbb{R}^n \,:\, dist(\xi,\Sigma_\tau) \le C\lambda^2 |\log\lambda|^{20} \,\} \qquad \forall \tau \in [2\lambda^2|\log\lambda|^2, \tau_0] \tag{2.1.12}$$

$$\Sigma_\tau \subseteq \{\, \xi \in \mathbb{R}^n \,:\, dist(\xi,\Sigma_{\lambda,\tau}) \le C\lambda^2 |\log\lambda|^{20} \,\} \qquad \forall \tau \in [2\lambda^2|\log\lambda|^2, \tau_0] \tag{2.1.13}$$

and

$$\sup_{2|\log\lambda|^2 \le t \le \lambda^{-2}\tau_0} \; \sup_{r \in \mathbb{R}^n} \big| m(r,t) - \bar{m}(d(r,\lambda^{-1}\Sigma_{\lambda^2 t})) \big| \le C\lambda |\log\lambda|^{20} \tag{2.1.14}$$

where $m(r,t)$ is the solution of (2.1.6) with initial datum (2.1.9).

Analogous results are known for the Allen-Cahn equation

$$\frac{\partial u}{\partial t} = \Delta u - V'(u) \tag{2.1.15}$$

where $V(u)$ is a symmetric, double-well potential with minima at $\pm m_\beta$. This equation was introduced first in [3] in order to describe the motion of antiphase boundaries in crystalline solids. Reaction diffusion equations like (2.1.15) can be rigorously derived from microscopic dynamics as well as our non local equation. In [23] an Ising spin system under the combined influence of Glauber and simple exchange (Kawasaki) dynamics is considered. It is proven that, when the conserving exchanges occur on a microscopically fast scale, the magnetization density evolves according to a reaction diffusion equation. Phase separation phenomena for this microscopic model are proven on [30] for $d = 1$ and in [48] for $d \leq 3$, while the interface dynamics, which is ruled by the motion by mean curvature, is studied in [6] and [51].

The rescaled solution

$$u^{(\lambda)}(\xi,\tau) = u(\lambda^{-1}\xi, \lambda^{-2}\tau) \tag{2.1.16}$$

solves the equation

$$\frac{\partial u^{(\lambda)}}{\partial t} = \Delta u^{(\lambda)} - \lambda^{-2}V'(u^{(\lambda)}), \qquad u^{(\lambda)}(\xi,0) = u_0(\xi;\lambda) \tag{2.1.17}$$

The Cauchy problem (2.1.17) with initial datum $u_0(\xi;\lambda) = \bar{u}(\lambda^{-1}d(\xi,\Sigma_0))$ ($\bar{u}$ the instanton solution associated to (2.1.15)) converges to a motion by mean curvature in the sense of Theorem 2.1.1, see [33], [34] (actually the estimate proven there is optimal, that is of order $\mathcal{O}(\lambda^2)$) and, for the more general case when there is a forcing term, [4]. The poorer result like in Theorem 1.4.3 was given in [20] and the convergence to a generalized motion by mean curvature in [37]. In this case the parameter of the limiting motion by mean curvature is $\theta = 1$, independently on the particular form of the potential. As proved by Spohn, [60], this is a general fact. In the Ginzburg-Landau models, also at a microscopic level, the surface tension equals always the inverse of the mobility.

In the context of the Allen-Cahn equation the proof of the analogous of Theorems 1.4.3 or 2.1.1 are based on the construction of appropriate super and sub-solutions that give the desired bounds on the solution of the Cauchy problem (2.1.17) (see [4] and [20]). One of the basic ingredients is that the signed distance function from a surface which moves by its mean curvature is closely related to the heat equation and in the Allen-Cahn equation the laplacian appears explicitly. Conversely the laplacian does not appear in the mean field equation and we recover a diffusion only under the limiting process. This is also related to the form of the transport coefficient θ which, as we shall see, comes from an averaging effect (in time) which takes place during the limiting process.

To avoid these problems in [28] Theorem 1.4.3 is proven by studying separately the evolution for short times and in small neighbourhoods of the interface. Then, using a "patching and iterating" procedure one gets a global solution. The time intervals are taken large enough in the meso-scale so that the averaging takes place and the transport coefficient can be identified. On the other hand, the advantage of working locally is that one can consider perturbations of the planar instanton and, working in an L^∞ setting, use the nice properties of the linear evolution (see §2.2). The analysis in the regions away from the interface can be made separately (by means of the good properties of the evolution, see the barrier lemma in §2.2), and it is easy to do thanks to the attractiveness of the homogenous stationary solutions $\pm m_\beta$. During the iteration the error add up, but they can be absorbed at any step by comparing the evolution with suitable super and sub-solutions. The latter ones are constructed by speed up (down) the motion by mean curvature with a small forcing term depending on the scaling parameter λ. More precisely these are functions with an instanton-like structure around interfaces which move according to suitable h-biased motion by mean curvature. In [28] all the machinery works by choosing $h \sim \lambda^\delta$ for some $\delta < 1$. To localize the interface as in Theorem 2.1.1 we need a similar result with $h \sim \lambda^2 |\log \lambda|^{20}$. As we shall see, this is possible by modifying the "shape" of the instanton-like structure of the super and sub-solutions with appropriate corrections which vanish as $\lambda \to 0$. Clearly also these corrections contribute to the evolution in any time interval and give errors that add up during the iteration. The key point is that these corrections are orthogonal with respect to the 0-eigenvector $\bar{m}'$ of the linear operator L, which describes the linearized evolution around the instanton. Then, by the linear stability of the instanton (see the next section), they contribute only to higher orders in λ that can be absorbed by the shifts.

§2.2 Some basic properties.

In this section we describe some basic properties of the mesoscopic equation and of the instanton. We give only some easy proofs and refer to [25] and [29] for details. We further recall classical results on the mean curvature flow we will need in the proofs.

The evolution equation.

We start by giving some fundamental properties of the mesoscopic evolution, which hold for the equation (1.2.8) in general (without the restriction $h = 0$ and $\beta > 1$).

2.2.1 Theorem.

For any $m_0 \in L^\infty(\mathbb{R}^n; [-1,1])$ there is a unique bounded function $m(r,t)$, differentiable in t in the sup-norm, which solves the Cauchy problem (1.2.8) with initial condition $m(r,0) = m_0(r)$. Moreover $\|m(\cdot,t)\|_\infty \leq 1$ for any $t \geq 0$.

Proof.

We fix $0 < T < \beta^{-1}$ and, for any $t \in [0, T]$, we write the integral version of the Cauchy problem (1.2.8)

$$m(r,t) = e^{-t} m_0(r) + \int_0^t ds\, e^{-(t-s)} \tanh\{\beta[(J \star m)(r,s) + h]\} \tag{2.2.1}$$

It can be thought as a fixed point problem for the map

$$K : L^\infty(\mathbb{R}^n \times [0,T]; [-1,1]) \to L^\infty(\mathbb{R}^n \times [0,T]; [-1,1]) \tag{2.2.2}$$

defined by

$$K(u)(r,t) \doteq e^{-t} m_0(r) + \int_0^t ds\, e^{-(t-s)} \tanh\{\beta[(J \star u)(r,s) + h]\} \tag{2.2.3}$$

By the choice of T it follows easily that the map K is a strictly contraction in the sup-norm and therefore there is a unique fixed point $m(r,t)$. Moreover, since $K(u)(r,t)$ is a differentiable function of t in the sup-norm, we get that $m(r,t)$ solves (1.2.8) for $t \in [0,T]$. Since $m(r,T)$ belongs to the same class of m_0, by iteration of the previous argument we get global existence. Uniqueness follows because any solution of (1.2.8) is also a fixed point for the map K, which is unique. □

2.2.2 The comparison theorem.

Let $m_{0,1}, m_{0,2} \in L^\infty(\mathbb{R}^n; [-1,1])$ such that $m_{0,1}(r) \le m_{0,2}(r)$ for any $r \in \mathbb{R}^n$. Let $m_1(r,t)$ and $m_2(r,t)$ be the solutions of (1.2.8) with initial conditions $m_{0,1}$ and $m_{0,2}$ respectively. Then, for any $(r,t) \in \mathbb{R}^n \times \mathbb{R}_+$,

$$m_1(r,t) \le m_2(r,t) \tag{2.2.4}$$

Proof.

Let $0 < T < \beta^{-1}$ and consider the map

$$G : L^\infty(\mathbb{R}^n \times [0,T]; [-1,1]) \to L^\infty(\mathbb{R}^n \times [0,T]; [-1,1]) \tag{2.2.5}$$

defined by

$$G(u)(r,t) \doteq e^{-t} u(r,0) + \int_0^t ds\, e^{-(t-s)} \tanh\{\beta[(J \star u)(r,s) + h]\} \tag{2.2.6}$$

We observe that $G(u)(r,0) = u(r,0)$ and, since $J \ge 0$, the map is monotonic, that is

$$u \le v \Longrightarrow G(u) \le G(v) \qquad \text{pointwise on } \mathbb{R}^n \times [0,T] \tag{2.2.7}$$

Moreover, as in the proof of Theorem 2.2.1, we easily check that G is a strictly contraction on any subspace of $L^\infty(\mathbb{R}^n \times [0,T]; [-1,1])$ of the functions with the same values at $t=0$. Then, for $i=1,2$,

$$m_i = \lim_{\ell \to \infty} G^\ell(\hat{m}_i), \qquad \hat{m}_i(r,t) \doteq m_i(r) \text{ on } \mathbb{R}^n \times [0,T] \tag{2.2.8}$$

Since $m_{0,1} \le m_{0,2}$, from (2.2.7) we get $G^\ell(\hat{m}_1) \le G^\ell(\hat{m}_2)$ for any $\ell \ge 1$ and then, by (2.2.8), $m_1 \le m_2$ on $\mathbb{R}^n \times [0,T]$. Since the estimate does not depend on the initial datum, by iteration we get (2.2.4). □

2.2.3 The barrier lemma.

There are V and $\bar{C}$, both positive, such that the following holds. If $m_1(r,t)$ and $m_2(r,t)$ solve (1.2.8), $\|m_1(\cdot,0)\|_\infty \le 1$, $\|m_2(\cdot,0)\|_\infty \le 1$ and, for some $T>0$, $m_1(r,0) = m_2(r,0)$ for all $|r| \le VT$, then

$$|m_1(0,t) - m_2(0,t)| \le \bar{C} e^{-T}, \qquad \forall t \in [0,T] \tag{2.2.9}$$

Proof.

Let $q(r,t) \doteq |m_1(r,t) - m_2(r,t)|$ and $J^{\star \ell}$ be the ℓ-fold convolution of J with itself. By (2.2.1) we obtain

$$q(r,t) \le e^{-t} q(r,0) + \int_0^t ds\, e^{-(t-s)} \beta (J \star q)(r,s) \tag{2.2.10}$$

hence, by iteration,

$$q(r,t) \le e^{-t} \sum_{\ell \ge 0} \frac{(\beta t)^\ell}{\ell!} (J^{\star \ell} \star q)(r,0) \tag{2.2.11}$$

We recall that J has compact support in the ball of radius 1, so that, since $q(r,0) = 0$ if $|r| \le VT$, we get

$$(J^{\star \ell} \star q)(0,0) = 0 \qquad \text{if } \ell < [VT] - 1 \tag{2.2.12}$$

By (2.2.11) for $(r,t) = (0,T)$, (2.2.12) and the fact that $q(r,0) \le 2$ for any $r \in \mathbb{R}^n$, the lemma follows immediately by choosing V large enough. □

The instanton.

We describe here in more details the properties of the instanton introduced in §1.3. We start with the mesoscopic equation for $h=0$ and $\beta > 1$

$$\frac{\partial m}{\partial t}(r,t) = -m(r,t) + \tanh\{\beta (J \star m)(r,t)\} \tag{2.2.13}$$

Since J depends on $|r|$ any stationary solution of (2.2.13) with planar symmetry has the form $m^\star(r) = m(r \cdot \nu)$, ν a unit vector, and $m(x)$, $x \in \mathbb{R}$, solves

$$m(x) = \tanh\{(\tilde{J} \star m)(x)\} \tag{2.2.14}$$

where $\tilde{J}$ is defined as in (1.3.25),

$$\tilde{J}(|x|) = \int_{\mathbb{R}^{n-1}} dy\, J\big(\sqrt{x^2 + |y|^2}\big)$$

In [29] the following theorem is proven.

2.2.4 Theorem.

Provided $\beta > 1$ the solution $\bar{m}(x)$ of the problem

$$\bar{m}(x) = \tanh\{(\tilde{J} \star \bar{m})(x)\}, \qquad \lim_{x\to\pm\infty} \bar{m}(x) = \pm m_\beta, \qquad \bar{m}(0) = 0 \tag{2.2.15}$$

exists and has the following properties.

i) $\bar{m}$ is a smooth antisymmetric strictly increasing function, that is $\bar{m} \in C^\infty(\mathbb{R})$ and $\bar{m}'(x) > 0$.

ii) There are constants α and M, both positive, such that

$$\lim_{x\to\pm\infty} e^{\alpha|x|}\Big|\bar{m}(x) \mp \Big[m_\beta - \frac{M}{\alpha} e^{-\alpha|x|}\Big]\Big| = 0 \tag{2.2.16}$$

$$\lim_{|x|\to+\infty} e^{\alpha|x|}\Big|\bar{m}'(x) - M e^{-\alpha|x|}\Big| = 0 \tag{2.2.17}$$

$$\lim_{x\to\pm\infty} e^{\alpha|x|}\Big|\bar{m}''(x) \pm \alpha M e^{-\alpha|x|}\Big| = 0 \tag{2.2.18}$$

iii) $\bar{m}$ is the unique solution of the mean field equation (2.2.14), modulo translation, in the space

$$\mathcal{A}_+ \doteq \{m \in C(\mathbb{R}) : \liminf_{s\to\infty} m(s) > 0, \limsup_{s\to-\infty} m(s) < 0\} \tag{2.2.19}$$

that is, if $m \in \mathcal{A}_+$ and solves (2.2.14), there exists $a \in \mathbb{R}$ such that $m(x) = \bar{m}(x-a)$.

In [29] it is proven that the instanton is globally stable with respect to the evolution (2.2.13) in dimension $n = 1$ and with J replaced by $\tilde{J}$, that is

$$\frac{\partial m}{\partial t}(x,t) = -m(x,t) + \tanh\{\beta(\tilde{J} \star m)(x,t)\} \tag{2.2.20}$$

More precisely any solution of (2.2.20) which starts from an element of $\mathcal{A}_+$ is exponentially attracted by some translation of the instanton. We do not enter into details because we do not need such results. Conversely we will need only the local stability properties of the instanton which we describe next in more details.

Linearization around the instanton.

The linearization of (2.2.20) around $\bar{m}$ is

$$\frac{\partial \phi}{\partial t} = L^{(1)}\phi, \qquad L^{(1)}\phi \doteq -\phi + \beta(1 - \bar{m}^2)\tilde{J} \star \phi \tag{2.2.21}$$

The solution of (2.2.21) with initial datum $\phi(x)$ is

$$\phi(x,t) = e^{L^{(1)}t}\phi(x) \tag{2.2.22}$$

where the semigroup $e^{L^{(1)}t}$ has a kernel $e^{Lt}(x,x')$ whose explicit expression is

$$e^{L^{(1)}t}(x,x') = e^{-t}\delta(x-x') + e^{-t}\sum_{\ell\geq 1}\frac{t^\ell}{\ell!}(H^{(1)})^{\star\ell}(x,x') \tag{2.2.23}$$

$$H^{(1)}(x,x') = \beta(1-\bar{m}(x)^2)\tilde{J}(|x-x'|) \tag{2.2.24}$$

The following theorem gives the required linear stability properties of the instanton.

2.2.5 Theorem.

There is $a>0$ and, for any $|\delta|<\alpha$, there is $c>0$ so that

$$\left\|e^{L^{(1)}t}\phi - C_\phi\bar{m}'\right\|_\delta \leq ce^{-at}\left\|\phi - C_\phi\bar{m}'\right\|_\delta \tag{2.2.25}$$

with α as in Theorem 2.2.4, while

$$\|f\|_\delta \doteq \sup_{x\in\mathbb{R}} e^{-\delta|x|}|f(x)| \tag{2.2.26}$$

$$C_\phi \doteq \int \mu(dx)\,\phi(x)\bar{m}'(x)^{-1} \tag{2.2.27}$$

$$\mu(dx) \doteq N\frac{\bar{m}'(x)^2}{1-\bar{m}(x)^2}dx, \qquad N^{-1} = \int_{\mathbb{R}} dx\,\frac{\bar{m}'(x)^2}{1-\bar{m}(x)^2} \tag{2.2.28}$$

Theorem 2.2.5 is proven in [24], where the existence and stability properties of the travelling fronts of the one dimensional mesoscopic equation are analyzed . It is an infinite version of the Perron-Frobenius theorem. The basic observation is that $L^{(1)}\bar{m}'=0$, as follows immediately by differentiating equation (2.2.14). Therefore the operator $L^{(1)}$ has eigenvalue 0 with eigenvector $\bar{m}'$, which is strictly positive as in the Perron-Frobenius theorem. This allows to transform $L^{(1)}$ into a new operator, $\mathcal{L}^{(1)}$, which is the generator of a Markov process. The mapping is defined by the gauge transformation

$$\phi = \bar{m}'\psi, \qquad \mathcal{L}^{(1)}\psi = \frac{1}{\bar{m}'}L^{(1)}(\bar{m}'\psi), \qquad e^{\mathcal{L}^{(1)}t}(x,x') = \frac{1}{\bar{m}'(x)}e^{L^{(1)}t}(x,x')\bar{m}'(x') \tag{2.2.29}$$

so that

$$\mathcal{L}^{(1)}\psi(x) = \int dx'\,K^{(1)}(x,x')[\psi(x')-\psi(x)] \tag{2.2.30}$$

$$K^{(1)}(x,x') = \beta\frac{1-\bar{m}(x)^2}{\bar{m}'(x)}\tilde{J}(|x-x'|)\bar{m}'(x') \tag{2.2.31}$$

Notice that $K^{(1)}(x,x') \geq 0$ and

$$\int dx' \, K^{(1)}(x,x') = 1, \qquad \text{for any } x \in \mathbb{R} \tag{2.2.32}$$

Therefore $\mathcal{L}^{(1)}$ is the generator of a jump Markov process on $\mathbb{R}$ with $K^{(1)}(x,x')$ as transition rate function. The validity of a Perron-Frobenius theorem is related to the fact that this transition rate function is smooth and has asymptotically a drift toward the origin (notice that, as $|x| \to \infty$, $\beta(1-\bar{m}(x)^2)$ converges to $\beta(1-m_\beta^2) < 1$).

For $n > 1$ the linearization of (2.2.13) around the instanton $\bar{m}(x)$, $x = r \cdot \nu$, can be described similarly in term of the operators L and $\mathcal{L}$ defined by

$$L\phi(r) = -\phi(r) + \beta(1-\bar{m}(x)^2)J \star \phi(r) \tag{2.2.33}$$

$$\mathcal{L}\phi(r) = \int dr' \, K(r,r')[\psi(r') - \psi(r)] \tag{2.2.34}$$

$$K(r,r') = \beta \frac{1-\bar{m}(x)^2}{\bar{m}'(x)} J(|r-r'|)\bar{m}'(x'), \qquad \left(\int_{\mathbb{R}^n} dr' \, K(r,r') = 1 \right) \tag{2.2.35}$$

and the semigroup e^{Lt} admits a kernel $g_t(r,r') = e^{Lt}(r,r')$ given by

$$g_t(r,r') = e^{-t}\delta(r-r') + e^{-t} \sum_{\ell \geq 1} \frac{t^\ell}{\ell!} H^{\star \ell}(r,r') \tag{2.2.36}$$

$$H(r,r') = \beta(1-\bar{m}(x)^2)J(|r-r'|) \tag{2.2.37}$$

We fix coordinates $y = (y_1, ..., y_{n-1})$ in the plane $\{r \cdot \nu = 0\}$ so that $r = (x,y)$. $\mathcal{L}$ is therefore the generator of a jump Markov process on $\mathbb{R}^n$ with a drift toward the plane $\{r \cdot \nu = 0\}$.

The canonical space of realizations of this Markov process is $D(\mathbb{R}^n; \mathbb{R}_+)$, the Skorohod space of cadlag trajectories (continuous from the right and with left limits). We denote by $r_t = (X_t, Y_t) = (X_t, Y_{1,t}, ..., Y_{n-1,t})$ the coordinate mapping on such a space and by $\mathbb{E}_r[\cdot] = \mathbb{E}_{(x,y)}[\cdot]$ the expectation value with respect to the Markov process when it starts from $r = (x,y)$. That is, for any integrable function f on $\mathbb{R}^n$,

$$\mathbb{E}_r[f(r_t)] = \int dr' \, e^{\mathcal{L}t}(r,r') f(r'), \qquad e^{\mathcal{L}t}(r,r') = \frac{1}{\bar{m}'(x)} e^{Lt}(r,r')\bar{m}'(x') \tag{2.2.38}$$

Analogously we realize the one dimensional Markov process with generator $\mathcal{L}^{(1)}$ on $D(\mathbb{R}; \mathbb{R}_+)$ and we denotes by $\mathbb{E}_x^{(1)}[\cdot]$ the expectation value with respect to the process starting from x. By the explicit form of the kernels one gets

$$\int_{\mathbb{R}^{n-1}} dy' \, e^{\mathcal{L}t}((x,y)(x',y')) = e^{\mathcal{L}^{(1)}t}(x,x'), \qquad \text{for any } y \in \mathbb{R}^{n-1} \tag{2.2.39}$$

so that the marginal distribution X_t of the Markov process with generator $\mathcal{L}$ starting from (x, y) is a realization of the one dimensional Markov process with generator $\mathcal{L}^{(1)}$ which starts from x.

The result of Theorem 2.2.5 correspond to an ergodic theorem for the Markov process associated to $\mathcal{L}^{(1)}$ which has $\mu(dx)$, defined in (2.2.28), as unique invariant (and reversible) measure. We give the precise statement since we will use it also in this form.

2.2.6 Theorem.

There are $b \in (0, \alpha)$, α as in Theorem 2.2.4, and $\hat{C} > 0$ such that, for any bounded function f on $\mathbb{R}$,

$$\left|\mathbb{E}_x^{(1)}[f(X_t)] - \mu(f)\right| \leq 2\|f\|_\infty \mathbf{1}_{t \leq |x|} + \hat{C}\|f\|_\infty e^{-b(t-|x|)}\mathbf{1}_{t>|x|} \tag{2.2.40}$$

where $\mathbf{1}_A$ denotes the characteristic function of the set A and

$$\mu(f) \doteq \int \mu(dx)\, f(x) \tag{2.2.41}$$

The dependence on the starting point in the estimate is quite clear. As just noticed the process has asymptotically a drift toward the origin, so that it takes a time proportional to $|x|$ to reach a neighborhood of the origin and then it approaches equilibrium exponentially fast. Notice moreover that, from the definition (2.2.29), all the functions which grow exponentially as $|x| \to \infty$ with a rate less than α are integrable.

The biased motion by mean curvature.

In the course of the proofs we will use the following notion.

2.2.7 Definition.

The surface $\Sigma_\tau^{(h)}$ evolves according to the classical h-biased motion by mean curvature with parameter $\theta > 0$ and forcing term $h \in \mathbb{R}$ in the time $[0, \tau_0]$ if the following holds.
i) For any $\tau \in [0, \tau_0]$ $\Sigma_\tau^{(h)}$ is the boundary of an open bounded set $\Lambda_\tau^{(h)} \subset \mathbb{R}^n$.
ii) There is a smooth $(n-1)$ dimensional compact manifold S_0 and a smooth map $\xi : [0, \tau_0] \times S_0 \to \mathbb{R}^d$ such that, for any $\tau \in [0, \tau_0]$, $\xi(\tau, \cdot)$ is an embedding of S_0 in $\mathbb{R}^n$,

$$\Sigma_\tau^{(h)} = \{\xi = \xi(\tau, \eta) \mid \eta \in S_0\} \tag{2.2.42}$$

and

$$\frac{d\xi}{d\tau} = (\theta\kappa - h)\hat{n} \tag{2.2.43}$$

where $\hat{n} = \hat{n}(\xi, \Sigma_\tau^{(h)})$, i.e. the unit vector normal to $\Sigma_\tau^{(h)}$ at ξ and pointing toward the interior of $\Sigma_\tau^{(h)}$, while κ is $(n-1)$ times the mean curvature of $\Sigma_\tau^{(h)}$ at ξ.

The following theorem, we state without proof, is based on classical results on parabolic equations and can be proven by reasoning as in [49].

2.2.8 Theorem.

Let Σ_τ, $\tau \in [0,\tau_0]$, be a classical motion by mean curvature as in Definition 1.4.1 with parameter θ and parameterization $\xi(\tau,\eta)$, $(\tau,\eta) \in [0,\tau_0] \times S_0$. There are $h_0 > 0$ and $c > 0$, depending on Σ_0 and τ_0, such that the following holds.

i) For any $|h| \le h_0$ there exists the classical h-biased motion by mean curvature $\Sigma_\tau^{(h)}$, $\tau \in [0,\tau_0]$, in the sense of Definition 2.2.7, with the same parameter θ, starting at $\tau = 0$ from $\Sigma_0^{(h)} = \Sigma_0$ and with parameterization $\xi^{(h)}(\tau,\eta)$, $(\tau,\eta) \in [0,\tau_0] \times S_0$, with S_0 the same manifold used for the parameterization of Σ_τ.

ii) For any $|h| \le h_0$, $\tau \in [0,\tau_0]$ and $\eta \in S_0$

$$|\xi^{(h)}(\tau,\eta) - \xi(\tau,\eta)| \le ch \tag{2.2.44}$$

§2.3 Linear approximation and identification of the transport coefficient.

In this section we study the Cauchy problem (2.1.6) with initial datum (2.1.9) in a short time interval $[0,T]$, where $T \sim |\log\lambda|^2$, and in a small neighbourhood of the interface, by analyzing its linear evolution around the planar instanton. Since in this short time interval the evolution produces small changes, the linear approximation is rather accurate. In any case, at this stage we just consider it and we do not carry about the control of the nonlinear terms. We will see that in this approximation the interface moves locally according to the motion by mean curvature with transport coefficient θ as in (1.4.12).

Let $\xi_0 \in \Sigma_0$. We fix a coordinate frame in the meso-scale as follows. We take the origin in $\lambda^{-1}\xi_0$ and choose coordinates $r = (x,y)$, $x \in \mathbb{R}$, $y = (y_1,\dots,y_{n-1}) \in \mathbb{R}^{n-1}$, so that the unit vector $\hat{x}$ coincides with $\hat{n}(\xi_0,\Sigma_0)$, while the unit vectors $\{\hat{y}_i\}_{i=1}^{n-1}$ are directed along the principal axes of $\lambda^{-1}\Sigma_0$ at $\lambda^{-1}\xi_0$. In a small neighbourhood of $\lambda^{-1}\xi_0$, the surface $\lambda^{-1}\Sigma_0$ can be represented as the graph of a function $x = x^*(y)$ which, to the first order in λ, is

$$x^*(y) = -\lambda\omega_1(y,\underline{\kappa}) \tag{2.3.1}$$

having defined, for any $\underline{\kappa} = (\kappa_1,\dots,\kappa_{n-1}) \in \mathbb{R}^{n-1}$,

$$\omega_1(y,\underline{\kappa}) \doteq -\frac{1}{2}\sum_{i=1}^{n-1}\kappa_i y_i^2 \tag{2.3.2}$$

and in (2.3.1) it is computed for $\underline{\kappa} = \underline{\kappa}(\xi_0,\Sigma_0)$.

By expanding the initial datum (2.1.9) up to the first order in λ we get

$$m_0(r,\lambda) = \bar{m}(x) + \lambda\omega_1(y,\underline{\kappa})\bar{m}'(x) \tag{2.3.3}$$

The linear evolution starting from (2.3.3), by using the Markov process introduced in §2.2, is

$$m(r,t) = \bar{m}(x) + \lambda \bar{m}'(x)\mathbb{E}_r[\omega_1(Y_t,\underline{\kappa})] \tag{2.3.4}$$

We are interested in the value of the solution at time T and along the normal to $\lambda^{-1}\Sigma_0$ in $\lambda^{-1}\xi_0$, that is for $(r,t) = ((x,0),T)$. We write

$$\mathbb{E}_{(x,0)}[\omega_1(Y_T,\underline{\kappa})] = \int_0^T dt\, \mathbb{E}_{(x,0)}[(\mathcal{L}\omega_1(\cdot,\underline{\kappa}))(r_t)] \tag{2.3.5}$$

By the explicit form of the rate function (2.2.35),

$$\begin{aligned}(\mathcal{L}\omega_1(\cdot,\underline{\kappa}))((x,y)) &= -\frac{1}{2}\sum_{i=1}^{n-1}\kappa_i \int dx'dy' K((x,y),(x',y'))[(y_i')^2 - y_i^2] \\ &= -\kappa\beta\frac{1-\bar{m}(x)^2}{\bar{m}'(x)}\int dx' \int_{\mathbb{R}^{n-1}} dz\, J(|(x-x')^2 + |z|^2|^{1/2})\bar{m}'(x')\frac{z_1^2}{2} \\ &\doteq \kappa f(x)\end{aligned} \tag{2.3.6}$$

where κ is $(n-1)$ times the mean curvature introduced in (2.1.1). From (2.3.5) and (2.3.6) we finally get

$$\mathbb{E}_{(x,0)}[\omega_1(Y_T,\underline{\kappa})] = -\kappa\mathbb{E}_{(x,0)}\left[Y_{1,T}^2/2\right] = \kappa\int_0^T dt\, \mathbb{E}_x^{(1)}[f(X_t)] \tag{2.3.7}$$

By (2.2.28), (1.4.12) and the definition of $f(x)$ in (2.3.6) it follows that

$$\theta = -\int \mu(dx)\, f(x) \tag{2.3.8}$$

Let

$$F_T(x,\underline{\kappa}) \doteq \mathbb{E}_{(x,0)}[\omega_1(Y_T,\underline{\kappa})] + \kappa\theta T \tag{2.3.9}$$

By Theorem 2.2.6, there is a constant C_1 so that

$$|F_T(x,\underline{\kappa})| \le C_1(1+|x|)|\kappa| \tag{2.3.10}$$

Recalling that $T \sim |\log\lambda|^2$, we have so found that in the linear approximation, to leading orders in λ, it is

$$m((x,0),T) = \bar{m}(x) - \theta\kappa\lambda T\bar{m}'(x) + \lambda F_T(x,\underline{\kappa})\bar{m}'(x) \approx \bar{m}(x - \theta\kappa\lambda T) \tag{2.3.11}$$

Since in macroscopic units the time T becomes $\lambda^2 T$ and the displacement $\theta\kappa\lambda T$ becomes $\theta\kappa\lambda^2 T$, the interface has moved locally with velocity $\theta\kappa$ along the normal. So we recover the mean curvature motion in this approximation.

§2.4 Formal expansions.

As discussed in §2.1 we prove Theorem 2.1.1 by constructing super and sub-solutions. Around interfaces moving by an h-biased motion by mean curvature, these have an instanton-like structure modified by appropriate shape corrections that allow to take the forcing term $h \sim \lambda^2 |\log \lambda|^{20}$. Moreover all the analysis is performed by studying the evolution for short time intervals and separately in small regions of the space. In this section we perform formal expansion of the evolution in order to find these corrections.

As in §2.3 we study the evolution for a short time interval $[0, T]$ and by localizing the analysis around the interface. We consider the Cauchy problem (2.1.6) with an initial datum which is given by (2.1.9) modified by shape corrections up to the second order in λ. We look for a choice of these corrections in order to obtain a structure which is locally preserved by the evolution up to the second order in λ and modulo the "horizontal shift" along the mean curvature flow.

First of all we give a general expression for the expansion of the non local equation around a planar instanton. For any unit vector ν we introduce, as in §2.2, the coordinates $x = r \cdot \nu$ and $y = (y_1, ..., y_{n-1})$ on the plane $\{r \cdot \nu = 0\}$. Let us consider the Cauchy problem (2.2.13) with an initial datum which has the form

$$m_0(r, \lambda) = \bar{m}(x) + \lambda \Omega_1(r) \bar{m}'(x) + \lambda^2 \Omega_2(r) \bar{m}'(x) + U^{(\lambda)}(r) \tag{2.4.1}$$

for some functions $\Omega_i(r)$, $i = 1, 2$. We assume $U^{(\lambda)}(r)$ of higher order in λ and we look for a formal expansion of $m(r, t)$ around $\bar{m}(x)$. Let $u_t(r) \doteq m(r, t) - \bar{m}(x)$. Expanding the equation around $\bar{m}(x)$ up to the second order in λ and using (2.2.15), we easily obtain the following integral equation for $u_t(r)$:

$$u_t(r) = g_t \star u_0^{(\lambda)}(r) + \frac{1}{2} \int_0^t ds\, g_{t-s} \star \left[\Phi\left(J \star (g_s \star u_0^{(\lambda)})\right)^2\right](r) + \int_0^t ds\, g_{t-s} \star R_s(r) \tag{2.4.2}$$

where g_t is the Green kernel defined in (2.2.36), $u_0^{(\lambda)}(r) \doteq m_0(r, \lambda) - \bar{m}(x)$,

$$\Phi(x) \doteq -2\beta^2 \bar{m}(x)[1 - \bar{m}(x)^2], \tag{2.4.3}$$

$$R_s(r) \doteq \tanh\left[\beta J \star (\bar{m} + u_s)\right](r) - \bar{m}(x) - L u_s(r) - \frac{1}{2}\Phi(x)\left[(J \star u_s)^2 - (J \star (g_s \star u_0^{(\lambda)}))^2\right] \tag{2.4.4}$$

Using (2.4.1) and the Markov process with generator $\mathcal{L}$ defined in §2.2, we obtain the following form for $u_t(r)$:

$$\begin{aligned} u_t(r) &= \lambda \Omega_1(r, t) \bar{m}'(x) + \lambda^2 \Omega_2(r, t) \bar{m}'(x) + g_t \star U^{(\lambda)}(r) \\ &+ \frac{1}{2} \int_0^t ds\, g_{t-s} \star B_s[u_0^{(\lambda)}](r) + \int_0^t ds\, g_{t-s} \star R_s(r) \end{aligned} \tag{2.4.5}$$

where

$$\Omega_1(r,t) \doteq \mathbb{E}_r[\Omega_1(r_t)] \tag{2.4.6}$$

$$\Omega_2(r,t) \doteq \mathbb{E}_r[\Omega_2(r_t)] + \frac{1}{2}\int_0^t ds\, \mathbb{E}_r\left[\frac{\Phi}{\bar{m}'}(X_{t-s})\big(J\star(\bar{m}'\mathbb{E}.[\Omega_1(r_s)])\big)^2(r_{t-s})\right] \tag{2.4.7}$$

and

$$B_s[u_0^{(\lambda)}](r) \doteq \Phi(x)\big[\big(J\star(g_s\star u_0^{(\lambda)})\big)^2(r) - \big(J\star(\bar{m}'\mathbb{E}.[\lambda\Omega_1(r_s)])\big)^2(r)\big] \tag{2.4.8}$$

Finally we introduce some functions which, together with ω_1 defined in (2.3.2), will be used in the sequel. For any $\underline{\kappa} = (\kappa_1, ..., \kappa_{n-1}) \in \mathbb{R}^{n-1}$ we define

$$\hat{\omega}_2(r,\underline{\kappa}) \doteq -\frac{1}{2}x\sum_{i=1}^{n-1}\kappa_i^2 y_i^2 \tag{2.4.9}$$

and

$$\omega_2(r,\underline{\kappa}) \doteq \frac{1}{8}\left[\frac{\bar{m}''(x)}{\bar{m}'(x)}\left(\sum_{i=1}^{n-1}\kappa_i y_i^2\right)^2 - 4x\sum_{i=1}^{n-1}\kappa_i^2 y_i^2\right] \tag{2.4.10}$$

Now we consider the Cauchy problem (2.1.6) with an initial datum which, in a tubular neighbourhood of the interface, has the form

$$m_0(r,\lambda) = \bar{m}(d(r)) + \lambda Q_1^{(T)}(d(r),\underline{\kappa}(r))\bar{m}'(d(r)) + \lambda^2 Q_2^{(T)}(d(r),\underline{\kappa}(r))\bar{m}'(d(r)) \tag{2.4.11}$$

where $d(r) = d(r, \lambda^{-1}\Sigma_0)$ and $\underline{\kappa}(r) = \underline{\kappa}(s(\lambda r, \Sigma_0), \Sigma_0)$ (see (2.1.4)). We look for nice functions $Q_i^{(T)}(x,\underline{\kappa})$, $i = 1,2$ with the properties

$$\int \mu(dx)\, Q_i^{(T)}(x,\underline{\kappa}) = 0, \qquad \forall \underline{\kappa} \in \mathbb{R}^{n-1}, \qquad i = 1,2 \tag{2.4.12}$$

and

$$\|\bar{m}' Q_i^{(T)}(\cdot,\underline{\kappa})\|_\infty \le C(\underline{\kappa})T^\rho, \qquad \forall \underline{\kappa} \in \mathbb{R}^{n-1}, \qquad i = 1,2 \tag{2.4.13}$$

for some exponent $\rho \in (0,2)$ and smooth positive function $C(\underline{\kappa})$. Finally we require that, after the evolution in the time interval $[0,T]$, up to the second order in λ, the structure of the solution is locally the same as in (2.4.11), but relative to the interface shifted by the mean curvature flow. As in §2.3 we check all this by working locally in a neighbourhood of a point on the interface and performing formal expansions. The rigorous results and statements will be given in the next section.

We fix $\xi_0 \in \Sigma_0$ and choose coordinate as in §2.3. In a neighbourhood of $\lambda^{-1}\xi_0$ the interface $\lambda^{-1}\Sigma_0$ is the graph of a function $x = x^*(y)$ that has an expansion

$$x^*(y) = -\lambda\omega_1(y, \underline{\kappa}) + \lambda^2 \sum_{i,j,k=1}^{n-1} c_{i,j,k}\, y_i y_j y_k + \mathcal{O}(\lambda^3) \tag{2.4.14}$$

where ω_1 is defined in (2.3.2), $\underline{\kappa} = \underline{\kappa}(\xi_0, \Sigma_0)$, while $c_{i,j,k} = c_{i,j,k}(\xi_0, \Sigma_0)$ are higher order expansion coefficients. By expanding the signed distance function $d(r)$ one easily gets

$$d(r) = x + \lambda\omega_1(y, \underline{\kappa}) + \lambda^2\hat{\omega}_2(r, \underline{\kappa}) - \lambda^2 \sum_{i,j,k=1}^{n-1} c_{i,j,k}\, y_i y_j y_k + \mathcal{O}(\lambda^3) \tag{2.4.15}$$

where $\hat{\omega}_2$ is defined in (2.4.9). Finally we expand the curvatures up to the first order in λ. Since the metric of the surface (2.4.14) is $g_{i,j} = \delta_{i,j} + \lambda^2\kappa_i\kappa_j y_i y_j + \mathcal{O}(\lambda^3)$, we have

$$\underline{\kappa}(r) = \underline{\kappa} + \mathcal{O}(\lambda^2) \tag{2.4.16}$$

where again $\underline{\kappa} = \underline{\kappa}(\xi_0, \Sigma_0)$.

Inserting all the previous expansion in (2.4.11) we get

$$m_0(r, \lambda) = \bar{m}(x) + \lambda\Omega_1(r)\bar{m}'(x) + \lambda^2\Omega_2(r)\bar{m}'(x) + \mathcal{O}(\lambda^3) \tag{2.4.17}$$

with

$$\Omega_1(r) = \omega_1(y, \underline{\kappa}) + Q_1^{(T)}(x, \underline{\kappa}) \tag{2.4.18}$$

and

$$\Omega_2(r) = \omega_2(y, \underline{\kappa}) + Q_2^{(T)}(x, \underline{\kappa}) + \omega_1(y, \underline{\kappa}) \frac{1}{\bar{m}'(x)} \frac{d}{dx}\Big(\bar{m}' Q_1^{(T)}(\cdot, \underline{\kappa})\Big)(x) - \sum_{i,j,k=1}^{n-1} c_{i,j,k}\, y_i y_j y_k \tag{2.4.19}$$

where ω_2 is defined in (2.4.10). As in linear case we look at the solution for $(r, t) = ((x, 0), T)$. To compute the solution up to the second order in λ we use the expansion (2.4.5). We have to analyze the quantities defined in (2.4.6) and (2.4.7) with $\Omega_i(r)$, $i = 1, 2$, as in (2.4.18) and (2.4.19). First of all we note that, since the marginals $\{Y_{i,T}\}_{i=1}^{n-1}$ of the Markov process starting from $(x, 0)$ have symmetric distribution,

$$\mathbb{E}_{(x,0)}\left[\sum_{i,j,k=1}^{n-1} c_{i,j,k}\, Y_{i,T} Y_{j,T} Y_{k,T}\right] = 0 \tag{2.4.20}$$

Moreover, by the assumptions (2.4.12), (2.4.13) and Theorem 2.2.5,

$$\left\|\bar{m}'\mathbb{E}_{(\cdot,0)}[Q_i^{(T)}(X_T, \underline{\kappa})]\right\|_\infty = \left\|e^{L^{(1)}t}(\bar{m}' Q_i^{(T)}(\cdot, \underline{\kappa}))\right\|_\infty \le cC(\underline{\kappa})T^\rho e^{-aT} \tag{2.4.21}$$

By (2.4.20), (2.4.21) and recalling that $T \sim |\log\lambda|^2$, we finally obtain

$$\Omega_1((x,0),T) = \Omega_1^{(T)}(x,\underline{\kappa}) + \mathcal{O}(\lambda), \qquad \Omega_2((x,0),T) = \Omega_2^{(T)}(x,\underline{\kappa}) + \mathcal{O}(\lambda) \tag{2.4.22}$$

where

$$\Omega_1^{(T)}(x,\underline{\kappa}) = \mathbb{E}_{(x,0)}[\omega_1(Y_T,\underline{\kappa})] \tag{2.4.23}$$

and

$$\Omega_2^{(T)}(x,\underline{\kappa}) = \mathbb{E}_{(x,0)}[\omega_2(r_T,\underline{\kappa})] + \mathbb{E}_{(x,0)}\left[\omega_1(Y_T,\underline{\kappa})\frac{1}{\bar{m}'(X_T)}\frac{d}{dx}\Big(\bar{m}'Q_1^{(T)}(\cdot,\underline{\kappa})\Big)(X_T)\right]$$
$$+\frac{1}{2}\int_0^T ds\,\mathbb{E}_r\left[\frac{\Phi}{\bar{m}'}(X_{T-s})\big(J\star(\bar{m}'\mathbb{E}_{\cdot}\left[\omega_1(Y_s,\underline{\kappa}) + Q_1^{(T)}(X_s,\underline{\kappa})\right])\big)^2(r_{T-s})\right] \tag{2.4.24}$$

Then

$$m((x,0),T) = \bar{m}(x) + \lambda\Omega_1^{(T)}(x,\underline{\kappa})\bar{m}'(x) + \lambda^2\Omega_2^{(T)}(x,\underline{\kappa})\bar{m}'(x) + \mathcal{O}(\lambda^3) \tag{2.4.25}$$

Now we have to compare (2.4.25) with the expansion of $m_T((x,0),\lambda)$, having defined

$$m_T(r,\lambda) \doteq \bar{m}(d_T(r)) + \lambda Q_1^{(T)}(d_T(r),\underline{\kappa}_T(r))\bar{m}'(d_T(r)) + \lambda^2 Q_2^{(T)}(d_T(r),\underline{\kappa}_T(r))\bar{m}'(d_T(r)) \tag{2.4.26}$$

where $d_T(r) = d(r,\lambda^{-1}\Sigma_{\lambda^2 T})$ and $\underline{\kappa}_T(r) = \underline{\kappa}(s(\lambda r,\Sigma_{\lambda^2 T}),\Sigma_{\lambda^2 T})$. Since locally the displacement of the interface in the meso-scale is of a quantity $\theta\kappa\lambda T$ along the normal direction $\hat{n}(\xi_0,\Sigma_0)$, one easily gets

$$d_T((x,0)) = x - \theta\kappa\lambda T + \mathcal{O}(\lambda^3 T^2), \qquad \underline{\kappa}_T((x,0)) = \underline{\kappa} + \mathcal{O}(\lambda^2 T) \tag{2.4.27}$$

Then

$$\begin{aligned} m_T((x,0),\lambda) &= \bar{m}(x) - \theta\kappa\lambda T\bar{m}'(x) + \frac{(\theta\kappa\lambda T)^2}{2}\bar{m}''(x) \\ &+ \lambda Q_1^{(T)}(x,\underline{\kappa})\bar{m}'(x) + \lambda^2 Q_2^{(T)}(x,\underline{\kappa})\bar{m}'(x) \\ &- \lambda^2\theta\kappa T\frac{d}{dx}\Big(\bar{m}'Q_1^{(T)}(\cdot,\underline{\kappa})\Big)(x) + \mathcal{O}(\lambda^3 T^2) \end{aligned} \tag{2.4.28}$$

It will be useful to write $\bar{m}''$ in the following manner. By differentiating the identity

$$L^{(1)}\bar{m}'(x) = \bar{m}'(x) \tag{2.4.29}$$

and using the explicit expression of $L^{(1)}$ (see (2.2.21)) we get

$$L^{(1)}\bar{m}''(x) = \beta(1-\bar{m}(x)^2)(\tilde{J}\star\bar{m}')(x) \tag{2.4.30}$$

From the definition (2.4.3) and the identity (1.5.7) we rewrite (2.4.30) as

$$L^{(1)}\bar{m}''(x) = -\Phi(x)(\tilde{J}\star\bar{m}')^2(x) \tag{2.4.31}$$

Since for any $t \geq 0$

$$\bar{m}''(x) - e^{L^{(1)}t}\bar{m}''(x) = -\int_0^t ds\, e^{L^{(1)}(t-s)}(L^{(1)}\bar{m}'')(x), \tag{2.4.32}$$

from (2.4.31) and (2.4.32) we obtain

$$\bar{m}''(x) = e^{L^{(1)}T}\bar{m}''(x) + \int_0^T ds\, e^{L^{(1)}(T-s)}\big(\Phi(\tilde{J}\star\bar{m}')^2\big)(x) \tag{2.4.33}$$

But $\bar{m}''$ is an odd bounded function so that, by Theorem 2.2.5,

$$\left\|e^{L^{(1)}T}\bar{m}''\right\|_\infty \leq c\|\bar{m}''\|_\infty e^{-aT} \tag{2.4.34}$$

(recall that $T \sim |\log\lambda|^2$). Then $m_T((x,0),\lambda)$, can be expanded as follows,

$$\begin{aligned} m_T((x,0),\lambda) &= \bar{m}(x) + \lambda\big(Q_1^{(T)}(x,\underline{\kappa}) - \theta\kappa T\big)\bar{m}'(x) \\ &+ \lambda^2\left(Q_2^{(T)}(x,\underline{\kappa}) - \theta\kappa T\frac{1}{\bar{m}'(x)}\frac{d}{dx}\Big(\bar{m}'Q_1^{(T)}(\cdot,\underline{\kappa})\Big)(x)\right)\bar{m}'(x) \\ &+ \frac{(\theta\kappa\lambda T)^2}{2}\int_0^T ds\,\mathbb{E}_{(x,0)}\left[\frac{\Phi}{\bar{m}'}(X_{T-s})(\tilde{J}\star\bar{m}')^2(X_{T-s})\right]\bar{m}'(x) \\ &+ \mathcal{O}(\lambda^3T^2) \end{aligned} \tag{2.4.35}$$

having used the trivial identity

$$\int_0^T ds\, e^{L^{(1)}(T-s)}\big(\Phi(\tilde{J}\star\bar{m}')^2\big)(x) = \int_0^T ds\,\mathbb{E}_{(x,0)}\left[\frac{\Phi}{\bar{m}'}(X_{T-s})(\tilde{J}\star\bar{m}')^2(X_{T-s})\right]\bar{m}'(x) \tag{2.4.36}$$

By comparing the expression (2.4.25) and (2.4.35) we choose

$$Q_1^{(T)}(x,\underline{\kappa}) \doteq \Omega_1^{(T)}(x,\underline{\kappa}) + \theta\kappa T = F_T(x,\underline{\kappa}) \tag{2.4.37}$$

where $F_T(x,\underline{\kappa})$ is defined in (2.3.9) and

$$\begin{aligned} Q_2^{(T)}(x,\underline{\kappa}) &\doteq \Omega_2^{(T)}(x,\underline{\kappa}) + \theta\kappa T\frac{1}{\bar{m}'(x)}\frac{d}{dx}\Big(\bar{m}'F_T(\cdot,\underline{\kappa})\Big)(x) \\ &- \frac{(\theta\kappa T)^2}{2}\int_0^T ds\,\mathbb{E}_{(x,0)}\left[\frac{\Phi}{\bar{m}'}(X_{T-s})(\tilde{J}\star\bar{m}')^2(X_{T-s})\right] \\ &= G_T(x,\underline{\kappa}) + H_T(x,\underline{\kappa}) \end{aligned} \tag{2.4.38}$$

having defined

$$G_T(x,\underline{\kappa}) \doteq \mathbb{E}_{(x,0)}[\omega_2(r_T,\underline{\kappa})] + \frac{1}{2}\int_0^T ds\, \mathbb{E}_r\left[\frac{\Phi}{\bar{m}'}(X_{T-s})\big(J\star(\bar{m}'\mathbb{E}.[\omega_1(Y_s,\underline{\kappa})])\big)^2(r_{T-s})\right]$$

$$-\frac{(\theta\kappa T)^2}{2}\int_0^T ds\, \mathbb{E}_{(x,0)}\left[\frac{\Phi}{\bar{m}'}(X_{T-s})(\tilde{J}\star\bar{m}')^2(X_{T-s})\right] \tag{2.4.39}$$

and

$$H_T(x,\underline{\kappa}) \doteq Q_2^{(T)}(x,\underline{\kappa}) - G_T(x,\underline{\kappa}) \tag{2.4.40}$$

Then $G_T(x,\underline{\kappa})$ contains only the terms which are generated by the evolution when the initial datum does not contain the shape corrections.

The choice (2.4.37) and (2.4.38) are correct if we verify that the properties (2.4.12) and (2.4.13) are satisfied. Actually in the rigorous proofs we will need stronger properties. These can be summarized in the following theorem, whose proof will be given in §2.6.

2.4.1 Theorem.

Let F_T, G_T and H_T be the functions defined in (2.3.9), (2.4.39) *and* (2.4.40) *with* $|\log\lambda|^2 \le T \le 2|\log\lambda|^2$. *There is $\lambda_0 \in (0,1]$ such that, for any $\lambda\in(0,\lambda_0]$, the following holds.*

1) $F_T(\cdot,\underline{\kappa}) \in C^\infty(\mathbb{R})$ and depends linearly on $\underline{\kappa}$. Moreover there is a positive constant C_1 such that, for any $x\in\mathbb{R}$ and $\underline{\kappa}\in\mathbb{R}^{n-1}$,

$$|F_T(x,\underline{\kappa})| \le C_1(1+|\underline{\kappa}|^2)(1+|x|), \tag{2.4.41}$$

$$\left|\frac{\partial F_T}{\partial x}(x,\underline{\kappa})\right| \le C_1(1+|\underline{\kappa}|^2)(1+|x|), \tag{2.4.42}$$

$$\left|\frac{\partial^2 F_T}{\partial x^2}(x,\underline{\kappa})\right| \le C_1(1+|\underline{\kappa}|^2)(1+|x|) \tag{2.4.43}$$

Finally

$$\int \mu(dx)\, F_T(x,\underline{\kappa}) = 0, \qquad \forall\underline{\kappa}\in\mathbb{R}^{n-1} \tag{2.4.44}$$

2) $G_T(\cdot,\underline{\kappa}), H_T(\cdot,\underline{\kappa}) \in C^\infty(\mathbb{R})$ and are polynomial of degree 2 in $\underline{\kappa}$. Moreover there is a positive constant C_2 such that, for any $x\in\mathbb{R}$ and $\underline{\kappa}\in\mathbb{R}^{n-1}$,

$$|G_T(x,\underline{\kappa})| \le C_2(1+|\underline{\kappa}|^2)(1+|x|)(|\log T|^2\sqrt{T}+|x|^5)T \tag{2.4.45}$$

$$|H_T(x,\underline{\kappa})| \le C_2(1+|\underline{\kappa}|^2)(1+|x|)(|\log T|^2\sqrt{T}+|x|^5)T \tag{2.4.46}$$

$$\left|\frac{\partial G_T}{\partial x}(x,\underline{\kappa})\right| \le C_2(1+|\underline{\kappa}|^2)(1+|x|)(|\log T|^2\sqrt{T}+|x|^5)T \tag{2.4.47}$$

$$\left|\frac{\partial H_T}{\partial x}(x,\underline{\kappa})\right| \le C_2(1+|\underline{\kappa}|^2)(1+|x|)(|\log T|^2\sqrt{T}+|x|^5)T \tag{2.4.48}$$

Finally

$$\int \mu(dx)\, G_T(x,\underline{\kappa}) = 0, \qquad \int \mu(dx)\, H_T(x,\underline{\kappa}) = 0 \qquad \forall \underline{\kappa} \in \mathbb{R}^{n-1} \tag{2.4.49}$$

Actually some of the previous estimates can be improved, but these are sufficient for our purposes.

§2.5 Rigorous Results.

We construct super and sub-solutions $m^{\pm}(r,t)$ for the Cauchy problem (2.1.6) with initial datum (2.1.9). We set

$$T \doteq \chi |\log \lambda|^2, \qquad R_\lambda \doteq R_0 |\log \lambda|, \qquad h = \lambda^2 |\log \lambda|^{20} \tag{2.5.1}$$

where $\chi \in [1,2]$ while R_0 is a constant to be fixed later. We will need the following lemma, we state without proof, which is based on Theorem 2.2.8 and direct compactness arguments.

2.5.1 Lemma.

Let τ_s be the first singularity time for the mean curvature flow Σ_τ starting from Σ_0. Fix $\tau_0 < \tau_s$ and let R_λ as in (2.5.1). Then there is $\lambda_0 \in (0,1]$ such that for any $\lambda \in (0,\lambda_0]$ the following holds.

1) The classical h-biased mean curvature flow $\Sigma_\tau^{(\pm h)}$ starting from Σ_0 and with h as in (2.5.1) exists for $\tau \in [0,\tau_0]$.

2) For any $\tau \in [0,\tau_0]$ and $\xi \in \mathcal{T}_{\lambda R_\lambda}(\Sigma_\tau)$ ($\mathcal{T}_{\lambda R_\lambda}(\Sigma_\tau^{(\pm h)})$), see (2.1.3), the projections $s(\xi,\Sigma_\tau)$ (respectively $s(\xi,\Sigma_\tau^{(\pm h)})$) are uniquely defined.

3) The curvatures $\kappa_i(s(\xi,\Sigma_\tau^{(\pm h)}),\Sigma_\tau^{(\pm h)})$, as functions on

$$\hat{\mathcal{T}}_\lambda \doteq \bigcup_{\tau\in[0,\tau_0]} \mathcal{T}_{\lambda R_\lambda}(\Sigma_\tau^{(\pm h)}) \times \{\tau\} \tag{2.5.2}$$

are smooth and then uniformly bounded with their derivatives on $\hat{\mathcal{T}}_\lambda$.

Let $(r,t) \in \mathbb{R}^n \times [0,\lambda^{-2}\tau_0]$, $\tau_0 < \tau_s$ as in the previous lemma. We set

$$d_\pm = d_\pm(r,t) \doteq d\big(r, \lambda^{-1}\Sigma_{\lambda^2 t}^{(\pm h)}\big) \tag{2.5.3}$$

and, for (r,t) such that $|d_\pm(r,t)| \le R_\lambda$, we define the vector functions $\underline{\kappa}_\pm = \underline{\kappa}_\pm(r,t) \in \mathbb{R}^{n-1}$ setting

$$\underline{\kappa}_\pm(r,t) \doteq \underline{\kappa}\big(s\big(\lambda r, \Sigma_{\lambda^2 t}^{(\pm h)}\big), \Sigma_{\lambda^2 t}^{(\pm h)}\big) \tag{2.5.4}$$

By Lemma 2.5.1 the definition (2.5.4) is well posed for any λ small enough. Now we have all the necessary to construct the super and sub-solutions. We choose the constant R_0 which appears in (2.5.1) such that

$$3 + 1/10 < \alpha R_0 < 3 + 1/5 \tag{2.5.5}$$

with α as in Theorem 2.2.4. For all λ small enough we define

$$m^{\pm}(r,t) \doteq \begin{cases} \bar{m}(d_{\pm}) + \lambda Q_1^{(T)}(d_{\pm},\underline{\kappa}_{\pm})\bar{m}'(d_{\pm}) + \lambda^2 Q_2^{(T)}(d_{\pm},\underline{\kappa}_{\pm})\bar{m}'(d_{\pm}) & \text{if } |d_{\pm}| \le R_\lambda \\ \text{sign}(d_{\pm})m_\beta \pm \lambda^{3+1/10} & \text{if } |d_{\pm}| > R_\lambda \end{cases} \tag{2.5.6}$$

where the functions $Q_i^{(T)}(x,\underline{\kappa})$, $i = 1,2$, were defined in (2.4.37) and (2.4.38). Finally let

$$\hat{m}^{\pm}(r) \doteq m^{\pm}(r,T) - H_T(d_{\pm}(r,T),\underline{\kappa}_{\pm}(r,T))\mathbf{1}_{|d_{\pm}(\cdot,T)|\le R_\lambda}(r) \tag{2.5.7}$$

with $H_T(x,\underline{\kappa})$ as in (2.4.40). Note that the functions $m^{\pm}(r,t)$ and $\hat{m}^{\pm}(r)$ depend on λ and T, that is on λ and χ, but to simplify the notation we omitted this dependence. Now we can state the main theorem of the section.

2.5.2 Theorem.

Let $m(r,t)$ be the solution of the Cauchy problem (2.1.6) *with initial datum* (2.1.9). *Fix $\tau_0 < \tau_s$ and let $m^{\pm}(r,t)$ as in* (2.5.6). *There is $\lambda_0 \in (0,1]$, depending only on τ_0, Σ_0 and the interaction J, such that, for any $\lambda \in (0,\lambda_0]$, $\chi \in [1,2]$ and any integer $j \ge 2$ with $(j+1)T \le \lambda^{-2}\tau_0$,*

$$m^{-}(r,jT) \le m(r,jT) \le m^{+}(r,jT) \tag{2.5.8}$$

The proof of the theorem is based on the following two lemmas. We denote by $S_t(m)$ the solution of the nonlocal equation (2.2.13) starting from m.

2.5.3 Lemma.

Fix $\tau_0 < \tau_s$ and let $m^{\pm}(r,t)$ as in (2.5.6). *There is $\lambda_0 \in (0,1]$, depending on τ_0, Σ_0 and J, so that the following estimates hold. For any $\lambda \in (0,\lambda_0]$, $\chi \in [1,2]$, $j \in \mathbb{N}_+$ such that $(j+1)T \le \lambda^{-2}\tau_0$ and $r \in \mathbb{R}^n$,*

$$S_T(m^{+}(\cdot,jT))(r) \le m^{+}(r,(j+1)T) \tag{2.5.9}$$

and

$$m^{-}(r,(j+1)T) \le S_T(m^{-}(\cdot,jT))(r) \tag{2.5.10}$$

2.5.4 Lemma.

Fix $\tau_0 < \tau_s$ *and let* $m^{\pm}(r,t)$ *as in* (2.5.6) *and* $\hat{m}^{\pm}(r)$ *as in* (2.5.7). *There is* $\lambda_0 \in (0,1]$, *depending on* τ_0, Σ_0 *and* J, *so that the following estimates hold. For any* $\lambda \in (0,\lambda_0]$, $\chi \in [1,2]$ *and* $r \in \mathbb{R}^n$,

$$\hat{m}^-(r) \le S_T\big(\bar{m}(d(\cdot,\lambda^{-1}\Sigma_0))\big)(r) \le \hat{m}^+(r), \tag{2.5.11}$$

$$S_T\big(\hat{m}^+\big)(r) \le m^+(r,2T) \tag{2.5.12}$$

and

$$m^-(r,2T) \le S_T\big(\hat{m}^-\big)(r) \tag{2.5.13}$$

Proof of Lemma 2.5.3.

First of all we recall that by definition (2.5.1), since $\chi \in [1,2]$, it is $|\log\lambda|^2 < T < 2|\log\lambda|^2$. We prove (2.5.9), the proof of (2.5.10) is analogous. We fix j as in the statement of the lemma and we shorthand $\Sigma^{(j)} \doteq \Sigma^{(h)}_{j\lambda^2 T}$. We define

$$W_a^{(j)} \doteq \{r \in \mathbb{R}^n \,:\, |d(r,\lambda^{-1}\Sigma^{(j)})| > 2VT\}, \quad W_c^{(j)} \doteq \{r \in \mathbb{R}^n \,:\, |d(r,\lambda^{-1}\Sigma^{(j)})| \le 3VT\} \tag{2.5.14}$$

where V is the constant that appears in the barrier lemma. We prove (2.5.9) separately into the two regions $W_a^{(j)}$ and $W_c^{(j)}$.

Estimate away from the interface.

Let $r \in \lambda^{-1}I(\Sigma^{(j)}) \cap W_a^{(j)}$. For any λ small enough $VT > R_\lambda$. On the other hand, for any $\bar{r}$ such that $|r-\bar{r}| \le VT$, by triangular inequality, it is $d(\bar{r},\lambda^{-1}\Sigma^{(j)}) \ge VT$ so that, for λ small enough, $m^+(\bar{r},jT) = m_\beta + \lambda^{3+1/10}$. Then, by the barrier lemma,

$$\big|S_T(m^+(\cdot,jT))(r) - S_T(m_\beta + \lambda^{3+1/10})\big| \le \bar{C}e^{-T} \tag{2.5.15}$$

But $m(t) \doteq S_t(m_\beta + \lambda^{3+1/10})$ solves the homogenous equation

$$\frac{dm(t)}{dt} = -m(t) + \tanh\{\beta m(t)\}, \qquad m(0) = m_\beta + \lambda^{3+1/10} \tag{2.5.16}$$

so that, for small λ, it is exponentially attracted by the stable point m_β. Then, for any λ small enough,

$$\big|S_T(m^+(\cdot,jT))(r) - m_\beta\big| \le \lambda^{3+1/10} \tag{2.5.17}$$

and so

$$S_T(m^+(\cdot,jT))(r) \le m_\beta + \lambda^{3+1/10} \tag{2.5.18}$$

On the other hand, the displacement of the h-biased mean curvature flow in the time $\lambda^2 T$ is of order λT. Then, for small λ, $d(r, \lambda^{-1}\Sigma^{(j+1)}) \geq 2VT - O(\lambda T) > R_\lambda$ and so $m^+(r, (j+1)T) = m_\beta + \lambda^{3+1/10}$. Then (2.5.9) for $r \in \lambda^{-1} I(\Sigma^{(j)}) \cap W_a^{(j)}$ follows from (2.5.18). The proof for $r \in \lambda^{-1} O(\Sigma^{(j)}) \cap W_a^{(j)}$ is the same.

Estimate close to the interface.

For any $\xi_0 \in \Sigma^{(j)}$ we define a local frame in the mesoscopic space as in §2.4. Then we take the origin in $\lambda^{-1}\xi_0$ and choose coordinate $r' = (x, y)$, $x \in \mathbb{R}$, $y = (y_1, ..., y_{n-1}) \in \mathbb{R}^{n-1}$ so that the unit vector $\hat{x}$ is directed as $\hat{n}(\xi_0, \Sigma^{(j)})$, while the unit vectors $\{\hat{y}_i\}_{i=1}^{n-1}$ are directed along the principal axes of $\lambda^{-1}\Sigma^{(j)}$ at $\lambda^{-1}\xi_0$.

Let $r \in W_c^{(j)}$. Since $|d(\lambda r, \Sigma^{(j)})| \leq 3V\lambda T$, the projection $s(\lambda r, \Sigma^{(j)})$ is uniquely defined for λ small enough. Moreover, in the local frame relative to $\xi_0 = s(\lambda r, \Sigma^{(j)})$, r is a point of coordinates $(x, 0)$ with $|x| \leq 3VT$. From this observation we conclude that the proof of (2.5.9) in $W_c^{(j)}$ can be done locally by proving that

$$S_T\big(m^+(\cdot, jT)\big)\big((x,0)\big) \leq m^+((x,0),(j+1)T) \qquad \text{for any } x : |x| \leq 3VT \tag{2.5.19}$$

in any local frame constructed as before.

Let then $\xi_0 \in \Sigma^{(j)}$. Since in the proof this point is kept fixed, without risk of confusion we call again $r = (x, y)$ instead of $r' = (x, y)$ the points in the local frame. Denote by $\mathcal{B}_a$ the ball of radius a centered in $\lambda^{-1}\xi_0$. For λ small enough, in the ball $\mathcal{B}_{4VT}$ the surface $\lambda^{-1}\Sigma^{(j)}$ can be represented as the graph of a function $x = x^*(y)$ and there is a constant $c_1 > 0$ such that

$$\left| x^*(y) + \lambda\omega_1(y, \underline{\kappa}) - \lambda^2 \sum_{i,j,k=1}^{n-1} c_{i,j,k}\, y_i y_j y_k \right| \leq c_1 \lambda^3 T^4 \tag{2.5.20}$$

where ω_1 was defined in (2.3.2), $\underline{\kappa} = \underline{\kappa}(\xi_0, \Sigma^{(j)})$ and $c_{i,j,k} = c_{i,j,k}(\xi_0, \Sigma^{(j)})$ are higher order expansion coefficients.

Let $d = d(r) \doteq d(r, \lambda^{-1}\Sigma^{(j)})$. It is not difficult to prove that there is a positive constant c_2 such that, for any λ small enough and any $r \in \mathcal{B}_{4VT}$,

$$\left| d(r) - x - \lambda\omega_1(y, \underline{\kappa}) - \lambda^2 \hat{\omega}_2(r, \underline{\kappa}) + \lambda^2 \sum_{i,j,k=1}^{n-1} c_{i,j,k}\, y_i y_j y_k \right| \leq c_2 \lambda^3 T^4 \tag{2.5.21}$$

where $\hat{\omega}_2$ is defined in (2.4.9).

Let now $\underline{\kappa}(r) = \underline{\kappa}(s(\lambda r, \Sigma^{(j)}), \Sigma^{(j)})$. Analogously to (2.4.16) there is a positive constant c_3 such that, for any $r \in \mathcal{B}_{4VT}$,

$$|\underline{\kappa}(r) - \underline{\kappa}| \leq c_3 \lambda^2 T^2 \tag{2.5.22}$$

Expanding $\bar{m}(d)$ around x and using (2.5.21) together with the properties of the instanton, we get that there is a positive constant c_4 such that, for any $r \in \mathcal{B}_{4\sqrt{T}}$,

$$\Bigg|\bar{m}(d) - \bar{m}(x) - \Bigg[\lambda\omega_1(y,\underline{\kappa})\mathbf{1}_{|y|\le 4\sqrt{T}} + \lambda^2\omega_2(r,\underline{\kappa})\mathbf{1}_{|y|\le 4\sqrt{T}} - \lambda^2 \sum_{i,j,k=1}^{n-1} c_{i,j,k}\, y_i y_j y_k\, \mathbf{1}_{|y|\le 4\sqrt{T}}\Bigg]\bar{m}'(x)\Bigg| \le c_4\lambda^3 T^6 \bar{m}'(x) \tag{2.5.23}$$

where $\omega_2(r,\underline{\kappa})$, is defined in (2.4.10).

We are left with the expansion of $(\lambda Q_1^{(T)} + \lambda^2 Q_2^{(T)})\bar{m}'$. By the properties of the instanton it is not difficult to prove that there is a constant $c_5 > 0$ such that, for any $x \in \mathbb{R}$,

$$\bar{m}''(x) \le c_5\bar{m}'(x), \qquad \bar{m}'''(x) \le c_5\bar{m}'(x) \tag{2.5.24}$$

From (2.5.24) and Theorem 2.4.1 there is a positive constant c_6 so that

$$\left|\frac{d^2}{dx^2}\left(\bar{m}'Q_1^{(T)}(\cdot,\underline{\kappa})\right)(x)\right| \le c_6(1+|\underline{\kappa}|^2)(1+|x|)\bar{m}'(x)$$
$$\left|\frac{d}{dx}\left(\bar{m}'Q_2^{(T)}(\cdot,\underline{\kappa})\right)(x)\right| \le c_6(1+|\underline{\kappa}|^2)(1+|x|)(|\log T|^2\sqrt{T} + |x|^5)T\bar{m}'(x) \tag{2.5.25}$$

From (2.5.25) and (2.5.22) there is $c_7 > 0$ so that, for any $r \in \mathcal{B}_{4\sqrt{T}}$,

$$\Bigg|\bar{m}'(d)Q_1^{(T)}(d,\underline{\kappa}(r)) - \bar{m}'(x)Q_1^{(T)}(x,\underline{\kappa}) - \lambda\omega_1(y,\underline{\kappa})\mathbf{1}_{|y|\le 4\sqrt{T}}\frac{d}{dx}\left(\bar{m}'Q_1^{(T)}(\cdot,\underline{\kappa})\right)(x)\Bigg| \le c_7\lambda^2 T^5\bar{m}'(x) \tag{2.5.26}$$

$$\left|\bar{m}'(d)Q_2^{(T)}(d,\underline{\kappa}(r)) - \bar{m}'(x)Q_2^{(T)}(x,\underline{\kappa})\right| \le c_7\lambda T^9\bar{m}'(x) \tag{2.5.27}$$

Notice that, by Theorem 2.2.8 and Lemma 2.5.1, all the constants which appear in the previous estimates can be chosen independently on $\xi_0 \in \Sigma^{(j)}$ and j as well as $(j+1)T \le \lambda^{-2}\tau_0$.

From (2.5.23), (2.5.26) and (2.5.27) there is $c_8 > 0$ such that, for any $r \in \mathcal{B}_{4\sqrt{T}}$,

$$\Bigg|\bar{m}(d) + \lambda Q_1^{(T)}(d,\underline{\kappa}(r))\bar{m}'(d) + \lambda^2 Q_2^{(T)}(d,\underline{\kappa}(r))\bar{m}'(d) - \bar{m}(x) - \lambda Q_1^{(T)}(x,\underline{\kappa})\bar{m}'(x) - \lambda^2 Q_2^{(T)}(x,\underline{\kappa})\bar{m}'(x) - \Big[\lambda\omega_1(y,\underline{\kappa}) + \lambda^2\omega_2(r,\underline{\kappa}) - \lambda^2\sum_{i,j,k} c_{i,j,k}\, y_i y_j y_k\Big]\mathbf{1}_{|y|\le 4\sqrt{T}}\bar{m}'(x) - \lambda^2\omega_1(y,\underline{\kappa})\mathbf{1}_{|y|\le 4\sqrt{T}}\frac{d}{dx}\left(\bar{m}'Q_1^{(T)}(\cdot,\underline{\kappa})\right)(x)\Bigg| \le c_8\lambda^3 T^9\bar{m}'(x) \tag{2.5.28}$$

We define now

$$n(r,0) \doteq \Big\{ \bar{m}(x) + \Big[c_8\lambda^3 T^9 + \lambda\omega_1(y,\underline{\kappa})\mathbf{1}_{|y|\le 4VT} + \lambda Q_1^{(T)}(x,\underline{\kappa})$$

$$+ \lambda^2 Q_2^{(T)}(x,\underline{\kappa}) + \lambda^2\omega_2(r,\underline{\kappa})\mathbf{1}_{|y|\le 4VT} - \lambda^2 \sum_{i,j,k} c_{i,j,k}\, y_i y_j y_k\, \mathbf{1}_{|y|\le 4VT}\Big]\bar{m}'(x)$$

$$+ \lambda^2\omega_1(y,\underline{\kappa})\mathbf{1}_{|y|\le 4VT}\frac{d}{dx}\Big(\bar{m}' Q_1^{(T)}(\cdot,\underline{\kappa})\Big)(x)\Big\}\mathbf{1}_{|x|\le \bar{R}_\lambda} + \Big[\mathrm{sign}(x) m_\beta + \lambda^{3+1/10}\Big]\mathbf{1}_{|x|>\bar{R}_\lambda} \tag{2.5.29}$$

where $\bar{R}_\lambda \doteq R_\lambda - 1$. We prove that, if λ is small enough,

$$m^+(r,jT) \le n(r,0) \qquad \forall\, r \in \mathcal{B}_{4VT} \tag{2.5.30}$$

We analyze two different cases:
a) Let $r = (x,y) \in \mathcal{B}_{4VT}$ such that $|x| \le \bar{R}_\lambda$. Then $|d| \le |x| + c_9\lambda T^2$ for some constant $c_9 > 0$, so that, for λ small enough, $|d| \le R_\lambda$. By the definition (2.5.6) and the bound (2.5.28) the inequality $m^+(r,jT) \le n(r,0)$ follows immediately.
b) Let $r = (x,y) \in \mathcal{B}_{4VT}$ such that $|x| > \bar{R}_\lambda$. Then $|d| \ge |x| - c_{10}\lambda T^2 \ge R_\lambda - 1 - c_{10}\lambda T^2$ for some $c_{10} > 0$. If $|d| > R_\lambda$ by the definitions (2.5.6) and (2.5.29) $m^+(r,jT) = n(r,0)$. If $R_\lambda - 1 - c_{10}\lambda T^2 \le |d| \le R_\lambda$, using the exponential convergence of the instanton and the bounds (2.4.41), (2.4.45), (2.4.46) we have, for some constant $c_{11} > 0$ and λ small enough,

$$m^+(r,jT) \le \mathrm{sign}(d) m_\beta + c_{11}\lambda^{1+\alpha R_0} T \le \mathrm{sign}(d) m_\beta + \lambda^{3+1/10} = n(r,0)$$

(recall that R_0 satisfies $\alpha R_0 > 3 + 1/10$).

Now we define

$$\tilde{m}(r,0) \doteq m^+(r,jT)\mathbf{1}_{|r|\le 4VT} + n(r,0)\mathbf{1}_{|r|>4VT} \tag{2.5.31}$$

By (2.5.30) $\tilde{m}(r,0) \le n(r,0)$ and $\tilde{m}(r,0) = m^+(r,jT)$ for $r \in \mathcal{B}_{4VT}$. Using the barrier lemma and the comparison theorem we conclude that:

$$S_T\big(m^+(\cdot,jT)\big)(r) \le n(r,T) + \bar{C}e^{-T} \qquad \text{for any } r \in \mathcal{B}_{3VT} \tag{2.5.32}$$

where $n(r,t) \doteq S_t\big(n(\cdot,0)\big)(r)$. We need then an upper bound for $n((x,0),T)$ when $|x| \le 3VT$.

Let $u_t(r) \doteq n(r,t) - \bar{m}(x)$. Then $u_t(r)$ solves the integral equation (2.4.2) with initial datum

$$
\begin{aligned}
u_0^{(\lambda)}(r) = \Big[& \lambda\Big(\omega_1(y,\underline{\kappa})\mathbf{1}_{|y|\le 4VT} + Q_1^{(T)}(x,\underline{\kappa})\Big) \\
& + \lambda^2\Big(\omega_2(r,\underline{\kappa})\mathbf{1}_{|y|\le 4VT} - \sum_{i,j,k} c_{i,j,k}\, y_i y_j y_k \mathbf{1}_{|y|\le 4VT} + Q_2^{(T)}(x,\underline{\kappa})\Big) \\
& + \lambda^2\omega_1(y,\underline{\kappa})\mathbf{1}_{|y|\le 4VT}\frac{1}{\bar m'(x)}\frac{d}{dx}\Big(\bar m' Q_1^{(T)}(\cdot,\underline{\kappa})\Big)(x) + c_8\lambda^3 T^9\Big]\bar m'(x)\mathbf{1}_{|x|\le \bar R_\lambda} \\
& + \Big[\mathrm{sign}(x)m_\beta - \bar m(x) + \lambda^{3+1/10}\Big]\mathbf{1}_{|x|>\bar R_\lambda}
\end{aligned}
\tag{2.5.33}
$$

In analogy with (2.4.1) we write

$$
u_0^{(\lambda)}(r) = \lambda\Omega_1(r)\bar m'(x) + \lambda^2\Omega_2(r)\bar m'(x) + U^{(\lambda)}(r) \tag{2.5.34}
$$

with $\Omega_i(r)$, $i=1,2$, as in (2.4.18), (2.4.19) and

$$
\begin{aligned}
U^{(\lambda)}(r) \doteq\ & c_8\lambda^3T^9\bar m'(x)\mathbf{1}_{|x|\le \bar R_\lambda} + \psi_\lambda(r)\bar m'(x) + \big[\mathrm{sign}(x)m_\beta - \bar m(x) + \lambda^{3+1/10}\big]\mathbf{1}_{|x|>\bar R_\lambda} \\
& - \Big[\lambda\Omega_1(r) + \lambda^2\Omega_2(r)\Big]\bar m'(x)\mathbf{1}_{|x|>\bar R_\lambda}\mathbf{1}_{|y|\le 4VT}
\end{aligned}
\tag{2.5.35}
$$

where

$$
\begin{aligned}
\psi_\lambda(r) \doteq & -\lambda\omega_1(y,\underline{\kappa})\mathbf{1}_{|y|>4VT} - \lambda^2\omega_2(r,\underline{\kappa})\mathbf{1}_{|y|>4VT} + \lambda^2\sum_{i,j,k} c_{i,j,k}\, y_i y_j y_k\mathbf{1}_{|y|>4VT} \\
& - \lambda^2\omega_1(y,\underline{\kappa})\mathbf{1}_{|y|>4VT}\frac{1}{\bar m'(x)}\frac{d}{dx}\Big(\bar m' Q_1^{(T)}(\cdot,\underline{\kappa})\Big)(x)
\end{aligned}
\tag{2.5.36}
$$

Then $u_t(r)$ solves the integral equation (2.4.5). We look for upper bounds of the terms in the right hand side of (2.4.5) when $(r,t) = ((x,0),T)$ and $|x|\le 3VT$.

Bound on $\big[\lambda\Omega_1\big((x,0),T\big) + \lambda^2\Omega_2\big((x,0),T\big)\big]\bar m'(x)$.

By Theorems 2.4.1 and 2.2.5 there is a constant $c_{12}>0$ so that

$$
\Big\|\bar m'\mathbb{E}_{(\cdot,0)}\big[Q_i^{(T)}(X_T,\underline{\kappa})\big]\Big\|_\infty \le c_{12}e^{-aT}T^{3/2}|\log T|^2 \qquad i=1,2 \tag{2.5.37}
$$

We can apply all the computations made in §2.4 and estimate

$$
\begin{aligned}
&\big[\lambda\Omega_1\big((x,0),T\big) + \lambda^2\Omega_2\big((x,0),T\big)\big]\bar m'(x) \\
&\qquad \le \big[\lambda\Omega_1^{(T)}(x,\underline{\kappa}) + \lambda^2\Omega_2^{(T)}(x,\underline{\kappa})\big]\bar m'(x) + c_{12}e^{-aT}T^{3/2}|\log T|^2
\end{aligned}
\tag{2.5.38}
$$

where the functions $\Omega_i^{(T)}(x,\underline{\kappa})$, $i=1,2$, were defined in (2.4.23) and (2.4.24).

Bound on $g_T \star U^{(\lambda)}((x,0))$.

Since $\alpha R_0 > 3+1/10$, by the exponential convergence properties of the instanton, (2.4.41), (2.4.42), (2.4.45) and (2.4.46) there is $c_{13} > 0$ so that, for any λ small enough,

$$\left|\mathrm{sign}(x) m_\beta - \bar{m}(x)\right| \mathbf{1}_{|x|>\bar{R}_\lambda} \le c_{13} \lambda^{\alpha R_0} \mathbf{1}_{|x|>\bar{R}_\lambda} \le \lambda^{3+1/10} \mathbf{1}_{|x|>\bar{R}_\lambda}, \tag{2.5.39}$$

$$\lambda \left|\omega_1(y,\underline{\kappa}) \mathbf{1}_{|y|\le 4\sqrt{T}} + Q_1^{(T)}(x,\underline{\kappa})\right| \mathbf{1}_{|x|>\bar{R}_\lambda} \le c_{13} \lambda^{1+\alpha R_0} T^2 \mathbf{1}_{|x|>\bar{R}_\lambda} \le \lambda^{3+1/10} \mathbf{1}_{|x|>\bar{R}_\lambda} \tag{2.5.40}$$

and

$$\begin{aligned} &\lambda^2 \Bigg| \omega_2(r,\underline{\kappa}) \mathbf{1}_{|y|\le 4\sqrt{T}} - \sum_{i,j,k} c_{i,j,k}\, y_1 y_j y_k \mathbf{1}_{|y|\le 4\sqrt{T}} + Q_2^{(T)}(x,\underline{\kappa}) \\ &+ \omega_1(y,\underline{\kappa}) \mathbf{1}_{|y|\le 4\sqrt{T}} \frac{1}{\bar{m}'(x)} \frac{d}{dx} \Big(\bar{m}' Q_1^{(T)}(\cdot,\underline{\kappa}) \Big)(x) \Bigg| \bar{m}'(x) \mathbf{1}_{|x|>\bar{R}_\lambda} \\ &\le c_{13} \lambda^{2+\alpha R_0} T^4 \mathbf{1}_{|x|>\bar{R}_\lambda} \le \lambda^{3+1/10} \mathbf{1}_{|x|>\bar{R}_\lambda} \end{aligned} \tag{2.5.41}$$

so that, for small λ,

$$U^{(\lambda)}(r) \le c_8 \lambda^3 T^9 \bar{m}'(x) \mathbf{1}_{|x|\le \bar{R}_\lambda} + \psi_\lambda(r) \bar{m}'(x) + 3\lambda^{3+1/10} \mathbf{1}_{|x|>\bar{R}_\lambda} \tag{2.5.42}$$

By Lemma 2.6.3 (see §2.6) there is a constant $c_{14} > 0$ so that

$$\left|\left(g_T \star (m'\psi_\lambda)\right)((x,0))\right| = \bar{m}'(x) \left|\mathbb{E}_{(x,0)}[\psi_\lambda(r_T)]\right| \le c_{14} e^{-T} \tag{2.5.43}$$

and in [28] it is proven that, for some constant $c_{15} > 0$

$$\left|\left(g_T \star \mathbf{1}_{|\cdot|>\bar{R}_\lambda}\right)((x,0))\right| \le c_{15} \lambda^{\alpha R_0} \tag{2.5.44}$$

By (2.5.42), (2.5.43) and (2.5.44), recalling that $L\bar{m}'(x) = 0$, we finally get, for some $c_{16} > 0$,

$$g_T \star U^{(\lambda)}((x,0)) \le c_8 \lambda^3 T^9 \bar{m}'(x) + c_{16} \lambda^{3+1/10+\alpha R_0} \tag{2.5.45}$$

Bound on $1/2 \int_0^T ds\, g_{T-s} \star B_s[u_0^{(\lambda)}]((x,0))$.

We write

$$u_0^{(\lambda)}(r) = \lambda \Omega_1(r) \bar{m}'(x) + \bar{\psi}_\lambda(y) \bar{m}'(x) + I_\lambda(r) \tag{2.5.46}$$

with

$$\bar{\psi}_\lambda(y) \doteq -\omega_1(y,\underline{\kappa}) \mathbf{1}_{|y|>4\sqrt{T}} \tag{2.5.47}$$

and then

$$I_\lambda(r) = u_0^{(\lambda)}(r) - \lambda \Omega_1(r) \bar{m}'(x) - \bar{\psi}_\lambda(y) \bar{m}'(x) \tag{2.5.48}$$

By arguing as made to obtain (2.5.42) and observing that now the second order term $\Omega_2(r)$ appears with the characteristic function $\mathbf{1}_{|x|\le \bar R_\lambda}$ instead of $\mathbf{1}_{|x|>\bar R_\lambda}$ we easily get

$$|I_\lambda(r)| \le c_{17}\lambda^2 T^4 \bar m'(x) + c_{18}\lambda^{3+1/10}\mathbf{1}_{|x|>\bar R_\lambda} \tag{2.5.49}$$

for some positive constants c_{17} and c_{18}.

Recalling the definition (2.4.8) of $B_s[u_0^{(\lambda)}](r)$ and using (2.5.46) we get

$$\frac{1}{2}\int_0^T ds\, g_{T-s} \star B_s[u_0^{(\lambda)}]((x,0)) = \sum_{i=0}^{4} K_\lambda^{(i)}(x) \tag{2.5.50}$$

where

$$\begin{aligned}
K_\lambda^{(0)}(x) &\doteq \frac{1}{2}\int_0^T ds\, g_{T-s}\star\left[\Phi\left(J\star(g_s\star I_\lambda)\right)^2\right]((x,0)),\\
K_\lambda^{(1)}(x) &\doteq \lambda\int_0^T ds\, g_{T-s}\star\left[\Phi\left(J\star(g_s\star I_\lambda)\right)\left(J\star(g_s\star(\bar m'\Omega_1))\right)\right]((x,0)),\\
K_\lambda^{(2)}(x) &\doteq \frac{\lambda^2}{2}\int_0^T ds\, g_{T-s}\star\left[\Phi\left(J\star(g_s\star(\bar m'\bar\psi_\lambda))\right)^2\right]((x,0)),\\
K_\lambda^{(3)}(x) &\doteq \lambda\int_0^T ds\, g_{T-s}\star\left[\Phi\left(J\star(g_s\star I_\lambda)\right)\left(J\star(g_s\star(\bar m'\bar\psi_\lambda))\right)\right]((x,0)),\\
K_\lambda^{(4)}(x) &\doteq \lambda\int_0^T ds\, g_{T-s}\star\left[\Phi\left(J\star(g_s\star(\bar m'\bar\psi_\lambda))\right)\left(J\star(g_s\star(\bar m'\Omega_1))\right)\right]((x,0))
\end{aligned} \tag{2.5.51}$$

In §2.6 we will prove that, for some constant $c_{19}>0$,

$$\begin{aligned}
|K_\lambda^{(0)}(x)| &\le c_{19}\left(\lambda^4 T^{11}\bar m'(x) + \lambda^{6+1/5}T\right),\\
|K_\lambda^{(1)}(x)| &\le c_{19}\lambda^3 T^7(1+|x|)\bar m'(x),\\
\sum_{i=2}^{4}|K_\lambda^{(i)}(x)| &\le c_{19}e^{-T}T
\end{aligned} \tag{2.5.52}$$

Then there is a constant $c_{20}>0$ such that, for $|x|\le 3VT$,

$$\frac{1}{2}\int_0^T ds\, g_{T-s}\star B_s[u_0^{(\lambda)}]((x,0)) \le c_{20}\left(\lambda^3 T^8\bar m'(x) + \lambda^{6+1/5}T\right) \tag{2.5.53}$$

Bound on $1/2\int_0^T ds\, g_{T-s}\star R_s((x,0))$.

We need an upper bound for $u_t(r)$ when $t\in[0,T]$. We observe that $u_t(r)$ solves the integral equation

$$u_t(r) = g_t\star u_0^{(\lambda)}(r) + \int_0^t ds\, g_{t-s}\star\Psi_s(r) \tag{2.5.54}$$

where

$$\Psi_s(r) \doteq \tanh[\beta J \star (\bar{m} + u_s)](r) - \bar{m}(x) - Lu_s(r) \tag{2.5.55}$$

Moreover, by the previous estimates, there are positive constants b_1 and b_2 so that

$$|u_0^{(\lambda)}(r)| \le b_1 \lambda T^2 \bar{m}'(x) + b_2 \lambda^{3+1/10} \tag{2.5.56}$$

Let $c = b_1 \vee b_2 + 1$ and

$$t_c \doteq \inf\{ t \ge 0 \, : \, |u_t(r)| > c(\lambda T^2 \bar{m}'(x) + \lambda^{3+1/10}) \;\; \forall r \in \mathbb{R}^n \} \tag{2.5.57}$$

We are going to prove that for any λ small enough $t_c > T$. Observe that $\|u_s\|_\infty \le c(1 + \|m'\|_\infty)\lambda^2 T$ when $s \le t_c$. So, if λ is small enough, $\|u_s\|_\infty \le 1$. Then, expanding Ψ_s up to the second order in u_s, we get that there is a positive constant $b > 0$ such that, for any λ small enough and $s \le t_c$,

$$|\Psi_s(r)| \le b(J \star u_s)^2(r) \tag{2.5.58}$$

By the positivity of the kernel g_{t-s}, (2.5.54), (2.5.56) and (2.5.58) we get

$$|u_t(r)| \le b_1 \lambda T^2 \bar{m}'(x) + b_2 \lambda^{3+1/10} + b \int_0^t ds\, g_{t-s} \star (J \star u_s)^2(r) \tag{2.5.59}$$

for any λ small enough and any $t \le t_c$. Using the bounds

$$|\tilde{J} \star \bar{m}'(x)| \le c^\star \bar{m}'(x), \qquad \sup_{r \in \mathbb{R}^n} \sup_{t \ge 0} |(g_t \star 1)(r)| \le c_0, \tag{2.5.60}$$

which hold for some positive constants $c^\star$ and c_0, together with the definition of t_c, we easily obtain that, for any small λ,

$$|u_t(r)| \le f_t^{(\lambda)}(x) \qquad \forall\, t \in [0, t_c] \tag{2.5.61}$$

where $f_t^{(\lambda)}(x)$ is a linear function of t of the form

$$f_t^{(\lambda)}(x) = \left[b_1 \lambda T^2 + b_3 \lambda^2 T^4 t\right] \bar{m}'(x) + \left[b_2 \lambda^{3+1/10} + b_4 \lambda^{6+1/5} t\right] \tag{2.5.62}$$

for suitable $b_3, b_4 > 0$. Let us suppose that $T > t_c$. Then for any λ small enough we have $f_{t_c}^{(\lambda)}(x) \le f_T^{(\lambda)}(x) < |u_{t_c}(r)|$, which contradict (2.5.61) by continuity of $u_t(r)$. Then, for any λ small enough, $T \le t_c$ so that

$$|u_t(r)| \le c(\lambda T^2 \bar{m}'(x) + \lambda^{3+1/10}) \qquad \forall\, r \in \mathbb{R}^n, \quad \forall\, t \in [0, T] \tag{2.5.63}$$

Since $\|u_s\|_\infty \le 1$ for $s \le T$, by (2.4.4) we get, for some $c_{21} > 0$,

$$|R_s(r)| \le c_{21}\Big[\big(J \star u_s\big)^2(r) - \big(J \star (g_s \star u_0^{(\lambda)})\big)^2(r) + \big(J \star u_s\big)^3(r)\Big] \tag{2.5.64}$$

We rewrite

$$\begin{aligned}\big(J \star u_s\big)^2(r) - \big(J \star (g_s \star u_0^{(\lambda)})\big)^2(r) &= \big(J \star (u_s - g_s \star u_0^{(\lambda)})\big)^2(r) \\ &+ \big(J \star (u_s - g_s \star u_0^{(\lambda)})\big)(r)\big(J \star (g_s \star u_0^{(\lambda)})\big)(r)\end{aligned} \tag{2.5.65}$$

By (2.5.60), (2.5.63) and the integral equation (2.5.54) there is a positive constants c_{22} such that, for any $s \le T$,

$$\begin{aligned}&\big|(J \star u_s)(r)\big| \le c_{22}(\lambda T^2 \bar{m}'(x) + \lambda^{3+1/10}),\\ &\big|\big(J \star (g_s \star u_0^{(\lambda)})\big)(r)\big| \le c_{22}(\lambda T^2 \bar{m}'(x) + \lambda^{3+1/10}),\\ &\big|(u_s - g_s \star u_0^{(\lambda)})(r)\big| \le c_{22}(\lambda^2 T^5 \bar{m}'(x) + \lambda^{6+1/5})\end{aligned} \tag{2.5.66}$$

and then, by (2.5.64) and (2.5.66), there is $c_{23} > 0$ so that

$$|R_s(r)| \le c_{23}(\lambda^3 T^7 \bar{m}'(x) + \lambda^{9+3/10}) \qquad \forall\, s \in [0,T]$$

and, using the second bound in (2.5.60), we finally get

$$\frac{1}{2}\int_0^T ds\, g_{T-s} \star R_s\big((x,0)\big) \le c_0 c_{23}(\lambda^3 T^8 \bar{m}'(x) + \lambda^{9+3/10} T) \tag{2.5.67}$$

Recalling that $\alpha R_0 > 3 + 1/10$, from all the previous estimates we have proved that

$$S_T\big(m^+(\cdot, nT)\big)\big((x,0)\big) \le M_\lambda(x) \qquad \forall\, |x| \le 3VT \tag{2.5.68}$$

where

$$M_\lambda(x) = \bar{m}(x) + \big[\lambda \Omega_1^{(T)}(x, \underline{\kappa}) + \lambda^2 \Omega_2^{(T)}(x, \underline{\kappa}) + M_1 \lambda^3 T^9\big]\bar{m}'(x) + M_2 \lambda^{6+1/5} T \tag{2.5.69}$$

with M_1 and M_2 suitable positive constants.

Now we need a lower bound for $m^+((x,0),(j+1)T)$ when $|x| \le 3VT$. Let $d_1 = d_1(x) \doteq d((x,0), \lambda^{-1}\Sigma^{(j+1)})$ and $\underline{\kappa}_1 = \underline{\kappa}_1(x) \doteq \underline{\kappa}_+((\lambda x, 0), (j+1)T)$. It is easy to prove that there is a positive constant c_{24} so that, for any λ small enough,

$$|d_1 - x + \lambda T(\theta \underline{\kappa} - h)| \le c_{24} \lambda^3 T^2 \tag{2.5.70}$$

Moreover, by Lemma 2.5.1, we get, for some $c_{25} > 0$,

$$\|\underline{\kappa}_1 - \underline{\kappa}\| \le c_{25}\lambda^2 T \tag{2.5.71)}$$

By arguments similar to the previous ones, one easily proves that there is a positive constant c_{26} so that, for $|x| \le R_\lambda + 1$,

$$\Big| \bar{m}(d_1) + \lambda Q_1^{(T)}(d_1, \underline{\kappa}_1)\bar{m}'(d_1) + \lambda^2 Q_2^{(T)}(d_1, \underline{\kappa}_1)\bar{m}'(d_1) - \bar{m}(x) - \lambda Q_1^{(T)}(x, \underline{\kappa})\bar{m}'(x)$$
$$- \lambda^2 Q_2^{(T)}(x, \underline{\kappa})\bar{m}'(x) + \lambda T(\theta\kappa - h)\bar{m}'(x) - \lambda^2 \frac{(\theta\kappa T)^2}{2}\bar{m}''(x)$$
$$+ \theta\kappa\lambda^2 T \frac{d}{dx}\Big(\bar{m}' Q_1^{(T)}(\cdot, \underline{\kappa})\Big)(x)\Big| \le c_{26}\lambda^3 T^3 \bar{m}'(x) \tag{2.5.72}$$

Using (2.4.33) and (2.4.34) as in §2.4, from (2.5.72) it follows also that, for some $c_{27} > 0$,

$$\Big| \bar{m}(d_1) + \lambda Q_1^{(T)}(d_1, \underline{\kappa}_1)\bar{m}'(d_1) + \lambda^2 Q_2^{(T)}(d_1, \underline{\kappa}_1)\bar{m}'(d_1) - \bar{m}(x) - \lambda Q_1^{(T)}(x, \underline{\kappa})\bar{m}'(x)$$
$$- \lambda^2 Q_2^{(T)}(x, \underline{\kappa})\bar{m}'(x) + \lambda T(\theta\kappa - h)\bar{m}'(x)$$
$$- \frac{(\theta\kappa T\lambda)^2}{2} \int_0^T ds\, \mathbb{E}_{(x,0)}\left[\frac{\Phi}{\bar{m}'}(X_{T-s})(\tilde{J} \star \bar{m}')^2(X_{T-s})\right]\bar{m}'(x)$$
$$+ \theta\kappa\lambda^2 T \frac{d}{dx}\Big(\bar{m}' Q_1^{(T)}(\cdot, \underline{\kappa})\Big)(x)\Big| \le c_{27}\lambda^3 T^3 \bar{m}'(x) \tag{2.5.73}$$

From the estimate (2.5.70) it follows that, for small λ,

$$\begin{cases} |x| \ge R_\lambda + 1 & \Rightarrow |d_1| \ge R_\lambda \\ |x| \le R_\lambda - 1 & \Rightarrow |d_1| \le \bar{R}_\lambda \end{cases} \tag{2.5.74}$$

Using (2.5.73), (2.5.74) and that $\lim_{\lambda\to 0} \lambda^{-(3+1/10)}\bar{m}'(R_\lambda - 1) = 0$, we easily get, for small λ,

$$m^+((x,0),(j+1)T) \ge N_\lambda(x) \tag{2.5.75}$$

where

$$N_\lambda(x) \doteq \Big[\bar{m}(x) - \lambda T(\theta\kappa - h)\bar{m}'(x) + \lambda Q_1^{(T)}(x, \underline{\kappa})\bar{m}'(x) + \lambda^2 Q_2^{(T)}(x, \underline{\kappa})\bar{m}'(x)$$
$$+ \frac{(\theta\kappa T\lambda)^2}{2} \int_0^T ds\, \mathbb{E}_{(x,0)}\left[\frac{\Phi}{\bar{m}'}(X_{T-s})(\tilde{J} \star \bar{m}')^2(X_{T-s})\right]\bar{m}'(x)$$
$$- \theta\kappa\lambda^2 T \frac{d}{dx}\Big(\bar{m}' Q_1^{(T)}(\cdot, \underline{\kappa})\Big)(x) - c_{27}\lambda^3 T^3 \bar{m}'(x)\Big] \mathbf{1}_{|x| \le R_\lambda + 1}$$
$$+ \Big[\mathrm{sign}(x) m_\beta + \lambda^{3+1/10}\Big] \mathbf{1}_{|x| > R_\lambda + 1} \tag{2.5.76}$$

So we have to prove that

$$M_\lambda(x) \le N_\lambda(x) \qquad \forall\ |x| \le 3VT \tag{2.5.77}$$

We prove (2.5.77) separately in the two different cases:

i) $|x| \le R_\lambda + 1$. Comparing the forms of the functions $M_\lambda(x)$ and $N_\lambda(x)$ for these values of x, by the definitions (2.4.37) and (2.4.38) and using the fact that $\sup_{|x|\le R_\lambda+1} m'(x)^{-1} \le \lambda^{-\alpha R_0}$ we get that (2.5.77) is implied by

$$h\lambda T \ge c_{27}\lambda^3 T^3 + M_1 \lambda^3 T^9 + M_2 \lambda^{6+1/5-\alpha R_0} \tag{2.5.78}$$

which is true, for small λ, because $h \ge \lambda^2 T^{10}$ and R_0 is such that $\alpha R_0 < 3 + 1/5$.

ii) $R_\lambda + 1 < |x| \le 3VT$. Now we need, for all small λ,

$$M_\lambda(x) \le \text{sign}(x) m_\beta + \lambda^{3+1/10} \tag{2.5.79}$$

which can be easily proven using the estimates (2.4.41), (2.4.45) and (2.4.46) together with the trivial bound $\sup_{|x|>R_\lambda+1}(1+|x|^p)m'(x) \le \lambda^{\alpha R_0} R_\lambda^p$, $p > 0$, and recalling that R_0 is such that $\alpha R_0 > 3 + 1/10$. □

Proof of Lemma 2.5.4

The proof is very similar to the one of the previous lemma. Again one proves the estimates separately into the regions defined in (2.5.14) with $j = 0$ for the proofs of (2.5.11) and $j = 1$ for the ones of (2.5.12) and (2.5.13). We just make some remarks. The proof of (2.5.11) in $W_a^{(0)}$ follows easily comparing the solution in $r \in W_a^{(0)}$ with the one starting from the function

$$\tilde{m}(r') \doteq \bar{m}(d(r, \lambda^{-1}\Sigma_0))\mathbf{1}_{\mathcal{B}_{VT}(r)}(r') + \bar{m}(\text{sign} d(r, \lambda^{-1}\Sigma_0)VT)\mathbf{1}_{\mathbb{R}^n \setminus \mathcal{B}_{VT}(r)}(r') \tag{2.5.80}$$

The proof of (2.5.11) in $W_c^{(0)}$ is exactly the same as in Lemma 2.5.3 but without the shape corrections $Q_i^{(T)}$, $i = 1, 2$, in the initial datum. The definition (2.5.7) is such that the first and second order powers in λ cancel in this case. The proof of (2.5.12) and (2.5.13) in $W_a^{(1)}$ is exactly the same as in Lemma 2.5.3. Finally, the proof of (2.5.12) and (2.5.13) in $W_c^{(1)}$ is again as in Lemma 2.5.3 with a different second order shape correction in the initial datum (see (2.5.7)), but it does not change the form of the bound M_λ in (2.5.68). □

Proof of Theorem 2.5.2

By means of the comparison theorem, it is an immediate consequence of Lemmas 2.5.3 and 2.5.4.

Proof of Theorem 2.1.1

We will prove (2.1.12), (2.1.13) and (2.1.14) separately in the time grid $\{ j\lambda^2 T \,:\, 2T \le jT \le \lambda^{-2}\tau_0 \}$ for any choice of $T = \chi|\log\lambda|^2$, $\chi \in [1,2]$. We fix then $T = \chi|\log\lambda|^2$ and let $m_\pm^{(\lambda)}(\xi,\tau) \doteq m^\pm(\lambda^{-1}\xi, \lambda^{-2}\tau)$ with $m^\pm(r,t)$ the super and sub-solutions of Theorem 2.5.2. Then there is $\lambda_0 \in (0,1]$ so that, for any $\lambda \in (0,\lambda_0]$ and $\tau = j\lambda^2 T$, $2T \le jT \le \lambda^{-2}\tau_0$,

$$m_-^{(\lambda)}(\xi,\tau) \le m^{(\lambda)}(\xi,\tau) \le m_+^{(\lambda)}(\xi,\tau) \qquad \forall \xi \in \mathbb{R}^n \tag{2.5.81}$$

Proof of (2.1.12): Let $\xi \in \Sigma_{\lambda,\tau}$. By (2.5.81),

$$m_-^{(\lambda)}(\xi,\tau) \le m^{(\lambda)}(\xi,\tau) = 0 \le m_+^{(\lambda)}(\xi,\tau) \tag{2.5.82}$$

But, by (2.2.44), for some positive constant c_1,

$$|d(\xi,\Sigma_\tau^{(-h)}) - d(\xi,\Sigma_\tau^{(h)})| \le c_1\lambda^2|\log\lambda|^{20} \tag{2.5.83}$$

By the definition (2.5.6) and the bounds (2.5.82), (2.5.83) we conclude that $\lambda^{-1}|d(\xi,\Sigma_\tau^{(\pm h)})| = \mathcal{O}(1)$. Then from (2.5.82) we get that there is $c_2 > 0$ so that, for small λ,

$$\bar{m}(\lambda^{-1}d(\xi,\Sigma_\tau^{(-h)})) - c_2\lambda \le 0 \le \bar{m}(\lambda^{-1}d(\xi,\Sigma_\tau^{(h)})) + c_2\lambda \tag{2.5.84}$$

(2.5.83) and (2.5.84) imply that

$$\lambda^{-1}|d(\xi,\Sigma_\tau^{(\pm h)})| \le c_3\lambda|\log\lambda|^{20} \tag{2.5.85}$$

for some $c_3 > 0$. Finally, since

$$d(\xi,\Sigma_\tau^{(-h)}) \le d(\xi,\Sigma_\tau) \le d(\xi,\Sigma_\tau^{(h)}), \tag{2.5.86}$$

we get $|d(\xi,\Sigma_\tau)| \le C\lambda^2|\log\lambda|^{20}$ for some positive constant C.

Proof of (2.1.13): Let $\xi \in \Sigma_\tau$, i.e. $d(\xi,\Sigma_\tau) = 0$. By (2.5.83) and (2.5.86) it follows that

$$d(\xi,\Sigma_\tau^{(-h)}) \le 0 \le d(\xi,\Sigma_\tau^{(h)}), \qquad |d(\xi,\Sigma_\tau^{(\pm h)})| \le c_1\lambda^2|\log\lambda|^{20} \tag{2.5.87}$$

If we define $\Sigma_{\lambda,\tau}^\pm \doteq \{ \xi \in \mathbb{R}^n \,:\, m_\pm^{(\lambda)}(\xi,\tau) = 0 \}$, (2.5.87) implies that

$$|d(\xi,\Sigma_{\lambda,\tau}^\pm)| \le c_4\lambda^2|\log\lambda|^{20} \tag{2.5.88}$$

for some positive constant c_4. Call $\xi_\pm$ two points in $\Sigma_{\lambda,\tau}^\pm$ such that $|\xi - \xi_\pm| \le c_4\lambda^2|\log\lambda|^{20}$. By continuity there exists a point $\xi_0 \in \Sigma_{\lambda,\tau}$ of the form $\xi_0 = q\xi_- + (1-q)\xi_+$ for some $q \in (0,1)$. Then, for some positive constant C, it is $|\xi - \xi_0| \le C\lambda^2|\log\lambda|^{20}$ and so $|d(\xi,\Sigma_{\lambda,\tau})| \le C\lambda^2|\log\lambda|^{20}$.

Proof of (2.1.14): It follows immediately from (2.5.8) and the definitions of the super and sub-solutions, by using (2.5.83) and (2.5.86) together with the smooth properties of the instanton and of the corrections $Q_i^{(T)}$, $i = 1,2$, (see Theorem 2.4.1). □

§2.6 Proofs of Theorem 2.4.1 and estimates (2.5.43) and (2.5.52).

Proof of Theorem 2.4.1.

First of all we rewrite the functions F_T, G_T and H_T in a more explicit form. Analogously to (2.1.1), for any $\underline{\kappa} \in \mathbb{R}^{n-1}$ we set

$$\kappa \doteq \sum_{i=1}^{n-1} \kappa_i, \quad \bar{\kappa}^2 \doteq \sum_{i=1}^{n-1} \kappa_i^2, \quad \hat{\kappa} \doteq \sum_{\substack{i,j=1 \\ i\neq j}}^{n-1} \kappa_i \kappa_j \tag{2.6.1}$$

From the definition (2.3.9) we easily get

$$F_T(x, \underline{\kappa}) = \kappa \mathbb{E}_{(x,0)} \left[\theta T - \frac{Y_{1,T}^2}{2} \right] \tag{2.6.2}$$

We note that the marginals $\{(X_t, Y_{i,t})\}_{i=1}^{n-1}$ of the process r_t starting from $r = (x, 0)$ are equally distributed and, if $\phi(r) = \psi(x) y_i^2$,

$$\mathbb{E}_r [\phi(r_t)] = \mathbb{E}_{(x,0)} [\phi(r_t)] + y_i^2 \mathbb{E}_{(x,0)} [\psi(X_t)] \tag{2.6.3}$$

Using the above observations, after some long but not difficult computations, we get

$$G_T(x, \underline{\kappa}) = \sum_{i=0}^{9} G_T^{(i)}(x, \underline{\kappa}), \qquad H_T(x, \kappa) = \sum_{i=0}^{5} H_T^{(i)}(x, \kappa) \tag{2.6.4}$$

where

$$G_T^{(0)}(x, \underline{\kappa}) \doteq \frac{\bar{k}^2}{8} \mathbb{E}_{(x,0)} [A(X_T) Y_{1,T}^4], \qquad G_T^{(1)}(x, \underline{\kappa}) \doteq \frac{\hat{k}}{8} \mathbb{E}_{(x,0)} [A(X_T) Y_{1,T}^2 Y_{2,T}^2],$$

$$G_T^{(2)}(x, \underline{\kappa}) \doteq -\frac{\bar{k}^2}{2} \mathbb{E}_{(x,0)} [X_T Y_{1,T}^2], \qquad G_T^{(3)}(x, \underline{\kappa}) \doteq \frac{k^2}{8} \int_0^T ds\, \mathbb{E}_{(x,0)} \left[B(X_s) (\hat{J} \star \bar{m}')^2 (X_s) \right],$$

$$G_T^{(4)}(x, \underline{\kappa}) \doteq \frac{k^2}{8} \int_0^T ds\, \mathbb{E}_{(x,0)} \left[B(X_s) (\tilde{J} \star \bar{m}')^2 (X_s) Y_{1,s}^4 \right],$$

$$G_T^{(5)}(x, \underline{\kappa}) \doteq \frac{\hat{k}}{8} \int_0^T ds\, \mathbb{E}_{(x,0)} \left[B(X_s) (\tilde{J} \star \bar{m}') (X_s) Y_{1,s}^2 Y_{2,s}^2 \right],$$

$$G_T^{(6)}(x, \underline{\kappa}) \doteq \frac{k}{4} \int_0^T ds\, \mathbb{E}_{(x,0)} \left[B(X_s) (\hat{J} \star \bar{m}') (X_s) (\tilde{J} \star \bar{m}') (X_s) Y_{1,s}^2 \right],$$

$$G_T^{(7)}(x, \underline{\kappa}) \doteq \frac{k}{4} \int_0^T ds\, \mathbb{E}_{(x,0)} \left[B(X_{T-s}) (\hat{J} \star \bar{m}') (X_{T-s}) \big(\tilde{J} \star (\bar{m}' \mathbb{E}_{(\cdot,0)} [Y_{1,s}^2]) \big) (X_{T-s}) \right],$$

$$G_T^{(8)}(x, \underline{\kappa}) \doteq \frac{k}{4} \int_0^T ds\, \mathbb{E}_{(x,0)} \left[B(X_{T-s}) (\tilde{J} \star \bar{m}') (X_{T-s}) \big(\tilde{J} \star (\bar{m}' \mathbb{E}_{(\cdot,0)} [Y_{1,s}^2]) \big) (X_{T-s}) Y_{1,T-s}^2 \right],$$

$$G_T^{(9)}(x, \underline{\kappa}) \doteq \frac{\kappa}{8} \int_0^T ds\, \mathbb{E}_{(x,0)} \left[B(X_{T-s}) \big(\tilde{J} \star (\bar{m}' \mathbb{E}_{(\cdot,0)} [Y_{1,s}^2]) \big)^2 (X_{T-s}) \right]$$
$$- \frac{(\theta \kappa T)^2}{2} \int_0^T ds\, \mathbb{E}_{(x,0)} \left[B(X_s) (J \star \bar{m}')^2 (X_s) \right] \tag{2.6.5}$$

and

$$H_T^{(0)}(x,\underline{\kappa}) \doteq \theta\kappa T \frac{1}{\bar{m}'(x)}\frac{d}{dx}\Big(\bar{m}'(x)F_T(x,\underline{\kappa})\Big)$$

$$H_T^{(1)}(x,\underline{\kappa}) \doteq -\frac{\kappa}{2}\mathbb{E}_{(x,0)}\left[\frac{1}{\bar{m}'(X_T)}\frac{d}{dx}\Big(\bar{m}'F_T(\cdot,\underline{\kappa})\Big)(X_T)Y_{1,T}^2\right]$$

$$H_T^{(2)}(x,\underline{\kappa}) \doteq \frac{1}{2}\int_0^T ds\, \mathbb{E}_{(x,0)}\left[B(X_{T-s})\big(\tilde{J}\star(\bar{m}'\mathbb{E}_{(\cdot,0)}[F_T(X_s,\underline{\kappa})])\big)^2(X_{T-s})\right]$$

$$H_T^{(3)}(x,\underline{\kappa}) \doteq -\frac{1}{2}\int_0^T ds\, \mathbb{E}_{(x,0)}\left[B(X_{T-s})(\hat{J}\star\bar{m}')(X_{T-s})\big(\tilde{J}\star(\bar{m}'\mathbb{E}_{(\cdot,0)}[F_T(X_s,\underline{\kappa})])\big)(X_{T-s})\right]$$

$$H_T^{(4)}(x,\underline{\kappa}) \doteq$$

$$-\frac{1}{2}\int_0^T ds\, \mathbb{E}_{(x,0)}\left[B(X_{T-s})\big(\tilde{J}\star(\bar{m}'\mathbb{E}_{(\cdot,0)}[Y_{1,s}^2])\big)(X_{T-s})\big(\tilde{J}\star(\bar{m}'\mathbb{E}_{(\cdot,0)}[F_T(X_s,\underline{\kappa})])\big)(X_{T-s})\right]$$

$$H_T^{(5)}(x,\underline{\kappa}) \doteq$$

$$-\frac{1}{2}\int_0^T ds\, \mathbb{E}_{(x,0)}\left[B(X_{T-s})(\hat{J}\star\bar{m}')(X_{T-s})\big(\tilde{J}\star(\bar{m}'\mathbb{E}_{(\cdot,0)}[F_T(X_s,\underline{\kappa})])\big)(X_{T-s})\right] \qquad (2.6.6)$$

with

$$A(x) \doteq \frac{\bar{m}''(x)}{\bar{m}'(x)}, \qquad B(x) \doteq \frac{\Phi(x)}{\bar{m}'(x)}, \qquad \hat{J}(x) \doteq \int_{\mathbb{R}^{n-1}} dy\, J(|x^2+|y|^2|^{1/2})y_1^2 \qquad (2.6.7)$$

To estimate all the previous terms we need the following lemmas.

2.6.1 Lemma.

Let f be a smooth $\mathbb{E}_x^{(1)}[\cdot]$-integrable function on $\mathbb{R}$. For any $p,q\in\mathbb{N}_+$ define

$$\Psi_{p,q}(x,t) \doteq \mathbb{E}_{(x,0)}\left[Y_{1,t}^{2p}Y_{2,t}^{2q}f(X_t)\right] \qquad (2.6.8)$$

Then $\Psi_{p,q}\in C^\infty(\mathbb{R}\times\mathbb{R}_+)$ and, for any $N\geq 0$, there is a positive constant C_N such that, for any $t\geq 0$,

$$\left|\frac{\partial^N\Psi_{p,q}}{\partial x^N}(x,t)\right| \leq C_N \sup_{s\leq t}\ \sup_{p'\leq p,\, q'\leq q}\left|\Psi_{p',q'}(x,s)\right| \qquad (2.6.9)$$

Proof.

Using the explicit form of the kernel $e^{\mathcal{L}t}(r,r')$ in term of the rate transition function $K(r,r')$ (see §2.2) and observing that, for any $N\geq 1$,

$$\frac{\partial^N}{\partial x^N}\left(y_1^{2p}y_2^{2q}f(x)\right)\Big|_{(x,0)} = 0$$

it is easy to verify the following identity

$$\frac{\partial^N\Psi_{p,q}}{\partial x^N}(x,t) = \int_0^t ds\, e^{-(t-s)}\int dr'\,\frac{\partial^N K}{\partial x^N}((x,0),r')\mathbb{E}_{(x',y')}\left[Y_{1,s}^{2p}Y_{2,s}^{2q}f(X_s)\right] \qquad (2.6.10)$$

By using identities similar to (2.6.3), it is easy to verify also that there exists a positive constant $C^\star(p,q)$ such that

$$\sup_{|y'|\le 1} \mathbb{E}_{(x',y')}\left[Y_{1,s}^{2p} Y_{2,s}^{2q} f(X_s)\right] \le C^\star(p,q) \sup_{p'\le p,\, q'\le q} \left|\Psi_{p',q'}(x',s)\right| \tag{2.6.11}$$

Recalling finally that $K((x,0),r')$ is a smooth function, identically zero for $|(x-x')^2+(y')^2| \ge 1$, the lemma follows immediately from (2.6.10) and (2.6.11). □

2.6.2 Lemma.

For any $p,q \in \mathbb{N}_+$ there is a constant $C = C(p,q) > 0$ such that

$$\mathbb{E}_{(x,0)}\left[Y_{1,t}^{2p} Y_{2,t}^{2q}\right] \le C(1+t+|x|)t^{p+q-1} \tag{2.6.12}$$

and, if ϕ is a bounded odd function on $\mathbb{R}$,

$$\mathbb{E}_{(x,0)}\left[Y_{1,t}^{2p} Y_{2,t}^{2q} \phi(X_t)\right] \le \begin{cases} C(1+|\log t|^2+|x|)\|\phi\|_\infty & \text{if } p+q=1 \\ C(1+|x|)(1+|\log t|^2\sqrt{t}+|x|^5)\|\phi\|_\infty & \text{if } p+q=2 \end{cases} \tag{2.6.13}$$

Proof.

First of all we point out that clearly the bounds (2.6.12) and (2.6.13) are poor for small t, but we will use it for t large.

If $p+q=1$, by arguing as in §2.3 (see (2.3.6), (2.3.7) and (2.3.8)), we get

$$\mathbb{E}_{(x,0)}\left[Y_{1,t}^{2p} Y_{2,t}^{2q}\right] = \int_0^t ds\, \mathbb{E}_x^{(1)}[f(X_s)] \tag{2.6.14}$$

where f is defined in (1.3.6). Then, by Theorem 2.2.6,

$$\left|\mathbb{E}_{(x,0)}\left[Y_{1,t}^{2p} Y_{2,t}^{2q}\right]\right| \le \theta t + \bar{c}(1+|x|)\|f\|_\infty \le C(1+t+|x|) \tag{2.6.15}$$

for some positive constants $\bar{c}$ and C. The estimate (2.6.12) for $p+q>1$ is obtained by an induction argument. For simplicity we prove it for $p=2$ and $q=0$, how to extend the argument to the general case will be clear. Using the fact that the kernel $K(r,r')$ of the Markov generator $\mathcal{L}$ depends on $|y-y'|$ one easily compute:

$$\begin{aligned} \mathbb{E}_{(x,0)}\left[Y_{1,t}^4\right] &= \int_0^t ds\, \mathbb{E}_{(x,0)}\left[\int dr'\, K(r_s,r')\left[(y_1')^4 - y_1^4\right]\right] \\ &= \int_0^t ds\, \mathbb{E}_{(x,0)}\left[Y_{1,s}^2 f_1(X_s) + f_2(X_s)\right] \end{aligned} \tag{2.6.16}$$

where $f_1(x)$ and $f_2(x)$ are the bounded functions defined by

$$f_1(x) \doteq 6\int dx'\, dz\, K((x,x'),(z,0))z^2, \qquad f_2(x) \doteq \int dx'\, dz\, K((x,x'),(z,0))z^4 \tag{2.6.17}$$

Then

$$\mathbb{E}_{(x,0)}\left[Y_{1,t}^4\right] \le \int_0^t ds \left(\|f_1\|_\infty \mathbb{E}_{(x,0)}\left[Y_{1,s}^2\right] + \|f_2\|_\infty\right) \tag{2.6.18}$$

From (2.6.12) with $p = 1$, $q = 0$, and (2.6.18) the estimate follows immediately.

Let us now consider the bound (2.6.13). Again we begin with the proof for $p = 1$ and $q = 0$. We restrict ourself to the case $t > 1$ since the estimate for $t \in [0, 1]$ is straightforward. Let $s = |\log t|^2$ and write

$$\begin{aligned}\mathbb{E}_{(x,0)}\left[Y_{1,t}^2\phi(X_t)\right] &= \mathbb{E}_{(x,0)}\left[\mathbb{E}_{r_{t-s}}\left[Y_{1,s}^2\phi(X_s)\right]\right]\\ &= \mathbb{E}_{(x,0)}\left[Y_{1,t-s}^2\mathbb{E}^{(1)}_{X_{t-s}}\left[\phi(X_s)\right]\right] + \mathbb{E}_x^{(1)}\left[\mathbb{E}_{(X_{t-s},0)}\left[Y_{1,s}^2\phi(X_s)\right]\right]\\ &\doteq E_1(x,t) + E_2(x,t)\end{aligned} \tag{2.6.19}$$

In (2.6.19) the second equality is obtained by writing (with some abuse of notation) $Y_s^2 = (Y_s - Y_{t-s})^2 + Y_{t-s}^2 + 2Y_{t-s}(Y_s - Y_{t-s})$ and using the fact that Y_s has symmetric distribution. By (2.6.12),

$$|E_2(x,t)| \le C(1 + s + \mathbb{E}_x^{(1)}[|X_{t-s}|])\|\phi\|_\infty \tag{2.6.20}$$

From Theorem 2.2.6 and the equality

$$\int \mu(dx) \int dx'\, K(x,x')(|x'| - |x|) = 0 \tag{2.6.21}$$

(recall that $\mu(dx)$ is a reversible measure for the process), for some constant $c_1 > 0$,

$$\mathbb{E}_x^{(1)}[|X_{t-s}|] \le c_1|x| \tag{2.6.22}$$

so that

$$|E_2(x,t)| \le C(1 + s + c_1|x|)\|\phi\|_\infty \tag{2.6.23}$$

To bound $E_1(x,t)$ we use first the ergodic theorem for the 1-dimensional Markov process. Since ϕ is an odd function, $\mu(\phi) = 0$ and then, for some constant $c_2 > 0$,

$$|E_1(x,t)| \le c_2\|\phi\|_\infty \mathbb{E}_{(x,0)}\left[Y_{1,t-s}^2\left(e^{-b(s-|X_{t-s}|)}\mathbf{1}_{|X_{t-s}|<s} + \mathbf{1}_{|X_{t-s}|\ge s}\right)\right] \tag{2.6.24}$$

where $b \in (0, \alpha)$ (see Theorem 2.2.6). Using Cauchy-Swartz inequality we get

$$\begin{aligned}|E_1(x,t)| \le c_2\|\phi\|_\infty \sqrt{\mathbb{E}_{(x,0)}\left[Y_{1,t-s}^4\right]}\Bigg(&\sqrt{\mathbb{E}_x^{(1)}\left[e^{-2b(s-|X_{t-s}|)}\mathbf{1}_{|X_{t-s}|<s}\right]}\\ &+ \sqrt{\mathbb{E}_x^{(1)}\left[\mathbf{1}_{|X_{t-s}|\ge s}\right]}\Bigg)\end{aligned} \tag{2.6.25}$$

Using that $b \in (0, \alpha)$ and the asymptotic convergence to zero of the density of the invariant measure $\mu(dx)$ one easily proves that, for some $\hat{c} > 0$,

$$\int \mu(dx)\, e^{-2b(s-|x|)} \mathbf{1}_{|x|<s} \leq \hat{c} e^{-2bs}, \qquad \int \mu(dx)\, \mathbf{1}_{|x|\geq s} \leq \hat{c} e^{-2\alpha s} \tag{2.6.26}$$

By Theorem 2.2.6 and (2.6.12) for $p = 2$, $q = 0$, from (2.6.25) and (2.6.26) there is a constant $c_3 > 0$ so that

$$\begin{aligned} |E_1(x,t)| \leq\ & c_3 \|\phi\|_\infty \sqrt{(1+t-s+|x|)(t-s)} \\ & \times \sqrt{e^{-2bs} + e^{-b(t-s-|x|)} \mathbf{1}_{|x|<t-s} + \mathbf{1}_{|x|\geq t-s}} \end{aligned} \tag{2.6.27}$$

We finally note that, for some constant $c_4 > 0$,

$$\begin{aligned} & (t-s)\mathbf{1}_{|x|>t-s} \leq |x|, \qquad (t-s)^2 \mathbf{1}_{|x|>t-s} \leq x^2, \\ & (t-s)^2 e^{-b(t-s-|x|)} \mathbf{1}_{|x|<t-s} \leq c_4(1+x^2) \end{aligned} \tag{2.6.28}$$

From (2.6.27) and (2.6.28) we get, for some constant $c_5 > 0$,

$$|E_1(x,t)| \leq c_5(1+|x|) \tag{2.6.29}$$

From (2.6.20) and (2.6.29) the bound (2.6.13) for $p = 1$ and $q = 0$ follows. The proof of (2.6.13) for $p + q > 1$ comes from the previous one by an inductive argument. Let us sketch the proof for the case $p = 2$ and $q = 0$. We restrict to the case $t > e$ (so $s > 1$) and write

$$\begin{aligned} \mathbb{E}_{(x,0)}\left[Y_{1,t}^4 \phi(X_t)\right] &= \mathbb{E}_{(x,0)}\left[\mathbb{E}_{\mathcal{F}_{t-s}}\left[Y_{1,s}^4 \phi(X_s)\right]\right] \\ &= \mathbb{E}_{(x,0)}\left[Y_{1,t-s}^4 \mathbb{E}^{(1)}_{X_{t-s}}\left[\phi(X_s)\right]\right] + 6\, \mathbb{E}_{(x,0)}\left[Y_{1,t-s}^2 \mathbb{E}_{(X_{t-s},0)}\left[Y_{1,s}^2 \phi(X_s)\right]\right] \\ &+ \mathbb{E}_{(x,0)}\left[\mathbb{E}_{(X_{t-s},0)}\left[Y_{1,s}^4 \phi(X_s)\right]\right] \\ &\doteq I_1(x,t) + I_2(x,t) + I_3(x,t) \end{aligned} \tag{2.6.30}$$

We treat $I_1(x,t)$ and $I_3(x,t)$ as done before for $E_1(x,t)$ and $E_2(x,t)$ and we get, for some positive constant c_6,

$$|I_1(x,t)| \leq c_6(1+|x|^2)\|\phi\|_\infty, \qquad |I_3(x,t)| \leq c_6(1+s+|x|)s\|\phi\|_\infty \tag{2.6.31}$$

To estimate $I_2(x,t)$ we decompose it as follows:

$$I_2(x,t) = I_2^{(1)}(x,t) + I_2^{(2)}(x,t) \tag{2.6.32}$$

where

$$I_2^{(1)}(x,t) \doteq 6\,\mathbb{E}_{(x,0)}\left[Y_{1,t-s}^2 \mathbf{1}_{|X_{t-s}|\le\sqrt{t}} \mathbb{E}_{(X_{t-s},0)}\left[Y_{1,s}^2\phi(X_s)\right]\right],$$
$$I_2^{(2)}(x,t) \doteq 6\,\mathbb{E}_{(x,0)}\left[Y_{1,t-s}^2 \mathbf{1}_{|X_{t-s}|>\sqrt{t}} \mathbb{E}_{(X_{t-s},0)}\left[Y_{1,s}^2\phi(X_s)\right]\right] \tag{2.6.33}$$

We notice that

$$I_2^{(1)}(x,t) \doteq 6\,\mathbb{E}_{(x,0)}\left[Y_{1,t-s}^2 G(X_{t-s})\right] \tag{2.6.34}$$

where

$$G(x) \doteq \mathbf{1}_{|x|\le\sqrt{t}} \mathbb{E}_{(x,0)}\left[Y_{1,s}^2\phi(X_s)\right]] \tag{2.6.35}$$

$G(x)$ is an odd function and, by using (2.6.13) for $p+q=1$,

$$|G(x)| \le C(1+|\log s|^2+\sqrt{t})\|\phi\|_\infty \tag{2.6.36}$$

We can then apply again (2.6.12) for $p+q=1$ and obtain

$$|I_2^{(1)}(x,t)| \le C(1+|\log(t-s)|^2+|x|)\|G\|_\infty \tag{2.6.37}$$

and so, by (2.6.36), for some constant $c_7>0$,

$$|I_2^{(1)}(x,t)| \le c_7(1+|\log t|^2+|x|)(1+\sqrt{t})\|\phi\|_\infty \tag{2.6.38}$$

Using again (2.6.13) with $p+q=1$, for some constant $c_8>0$,

$$|I_2^{(2)}(x,t)| \le c_8\|\phi\|_\infty \mathbb{E}_{(x,0)}\left[Y_{1,t-s}^2 \mathbf{1}_{|X_{t-s}|>\sqrt{t}}(1+|X_{t-s}|)\right] \tag{2.6.39}$$

(since $|X_{t-s}|>\sqrt{t}$ we bounded the term $|\log s|^2$, which appears applying (2.6.12), by a constant times $|X_{t-s}|$). Using now the explicit expression of the Markov semigroup and the properties of the rate transition function (see §2.2) we get, for some constant $c_9>0$,

$$\begin{aligned} &\mathbb{E}_{(x,0)}\left[Y_{1,t-s}^2 \mathbf{1}_{|X_{t-s}|>\sqrt{t}}(1+|X_{t-s}|)\right] \\ &\quad= e^{-(t-s)}\sum_{n\ge1}\frac{(t-s)^n}{n!}\int dr'\,K^n((x,0),r')(y')^2(1+|x'|)\mathbf{1}_{|x'|>\sqrt{t}} \\ &\quad\le c_9 e^{-(t-s)} \sum_{n\ge\max\{1;\sqrt{t}-|x|\}} \frac{(t-s)^n}{n!}n^2(1+n+|x|) \end{aligned} \tag{2.6.40}$$

It is not difficult to check that, from the bound in (2.6.40), there is a positive constant c_{10} such that

$$\mathbb{E}_{(x,0)}\left[Y_{1,t-s}^2 \mathbf{1}_{|X_{t-s}|>\sqrt{t}}(1+|X_{t-s}|)\right] \le c_{10}(1+x^6) \tag{2.6.41}$$

From (2.6.31), (2.6.38), (2.6.39) and (2.6.41) the bound (2.6.13) for $p = 2$, $q = 0$ follows immediately. $\square$

We have now all the necessary to prove Theorem 2.4.1. We call $F_t(x,\underline{\kappa})$, $G_t^{(i)}(x,\underline{\kappa})$, $i = 0,...,9$, and $H_t^{(j)}(x,\underline{\kappa})$, $j = 0,...,5$, the functions as defined in (2.6.2), (2.6.5), (2.6.6) but computed for $t \in [0,T]$ instead of T. By (2.6.14) and (2.3.8) we get, for some $C_1 > 0$,

$$|F_t(x,\underline{\kappa})| \leq C_1(1+|\underline{\kappa}|^2)(1+|x|) \tag{2.6.42}$$

By (2.6.13) we get, for some $C_2 > 0$,

$$|G_t^{(i)}(x,\underline{\kappa})| \leq \begin{cases} C_2(1+|\underline{\kappa}|^2)(1+|x|)(1+|\log t|^2\sqrt{t}+|x|^5) & \text{if } i=0,1 \\ C_2(1+|\underline{\kappa}|^2)(1+|x|)(1+|\log t|^2\sqrt{t}+|x|^5)t & \text{if } i=4,5 \\ C_2(1+|\underline{\kappa}|^2)(1+|\log t|^2+|x|)t & \text{if } i=6 \end{cases} \tag{2.6.43}$$

where we used the fact that $A(x)$, $B(x)(\tilde{J}\star\bar{m}')(x)$ and $B(x)(\hat{J}\star\bar{m}')(x)$ are bounded odd functions. Using the explicit expression of the Markov semigroup as in (2.6.40) we get, for some $C_3 > 0$,

$$|G_t^{(2)}(x,\underline{\kappa})| \leq C_3(1+|\underline{\kappa}|^2)(1+|x|)t \tag{2.6.44}$$

By Theorem 2.2.6, for some $C_4 > 0$,

$$|G_t^{(3)}(x,\underline{\kappa})| \leq C_4(1+|\underline{\kappa}|^2)(1+|x|)t \tag{2.6.45}$$

Using (2.6.12) and Theorem 2.2.6,

$$\begin{aligned} |G_t^{(7)}(x,\underline{\kappa})| &\leq C_5|\underline{\kappa}|\int_0^t ds\,(1+s)\Big(e^{-b(t-s-|x|)}\mathbf{1}_{t-s>|x|}+\mathbf{1}_{t-s\leq|x|}\Big) \\ &\leq C_5(1+|\underline{\kappa}|^2)(1+|x|)t \end{aligned} \tag{2.6.46}$$

and, using (2.6.13), one easily get

$$|G_t^{(8)}(x,\underline{\kappa})| \leq C_6(1+|\underline{\kappa}|^2)(1+|x|)t|\log t|^2 \tag{2.6.47}$$

for some positive constants C_5 and C_6. $G_t^{(9)}(x,\underline{\kappa})$ can be bounded as $G_t^{(7)}(x,\underline{\kappa})$ and one gets

$$|G_t^{(9)}(x,\underline{\kappa})| \leq C_7(1+|\underline{\kappa}|^2)(1+|x|)t$$

for some $C_9 > 0$ (note that this term grows as t and not as t^2 since, by the definition of θ, in (2.6.5) there is an exact cancellation of the leading term in t). Clearly there is $C_{10} > 0$ so that

$$|H_t^{(0)}(x,\underline{\kappa})| \leq C_{10}(1+|\underline{\kappa}|^2)(1+|x|)t \tag{2.6.48}$$

By (2.6.13), for some $C_{11} > 0$,

$$|H_t^{(1)}(x, \underline{\kappa})| \le C_{11}(1 + |\underline{\kappa}|^2)(1 + |\log t|^2 + |x|) \tag{2.6.49}$$

Using Theorem 2.2.6, for some $C_{12} > 0$,

$$|H_t^{(j)}(x, \underline{\kappa})| \le C_{12}(1 + |\underline{\kappa}|^2)(1 + |x|), \qquad j = 2, 3, 5 \tag{2.6.50}$$

and finally, by arguing as in (2.6.46), we get

$$|H_t^{(4)}(x, \underline{\kappa})| \le C_{13}(1 + |\underline{\kappa}|^2)(1 + |x|)t \tag{2.6.51}$$

for some $C_{13} > 0$.

The proof of the theorem follows easily from all the previous bounds and Lemma 2.6.1. □

Proof of (2.5.43) *and* (2.5.52).

We need the following lemma.

2.6.3 Lemma.

Let $r = (x, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $\psi(r) = \phi(r)\mathbf{1}_{|y| > 4VT}$ *with* $|\phi(r)| \le c\big((1 + |x|)y^2 + |y|^4\big)$ *for some positive constant* c. *Then there is a constant* $C > 0$ *such that*

$$\big|g_T \star (\bar{m}'\psi)((x, 0))\big| \le Ce^{-T} \tag{2.6.52}$$

and, for any bounded function W *on* $\mathbb{R}^n$,

$$\left| \int_0^T ds\, g_{T-s} \star [WJ \star (g_s \star (\bar{m}'\psi))]((x, 0)) \right| \le C\|W\|_\infty Te^{-T} \tag{2.6.53}$$

Proof.

We use again the explicit expression of the Markov semigroup. Recalling the hypothesis on $\phi(r)$ we get, for some positive constant $\hat{C} > 0$,

$$\begin{aligned}
&|g_T \star (\bar{m}'\psi)((x, 0))| = \bar{m}'(x)|\mathbb{E}_{(x,0)}[\psi(r_T)]| \\
&\le c\, e^{-T}\bar{m}'(x) \sum_{n \ge 1} \frac{T^n}{n!} \int dr'\, K^n((x, 0), r')((1 + |x'|)(y')^2 + |y'|^4)\mathbf{1}_{|y'| > 4VT} \\
&\le \hat{C}e^{-T} \sum_{n > 4VT} \frac{T^n}{n!} n^4 \le \hat{C}e^{-T}
\end{aligned} \tag{2.6.54}$$

Analogously we estimate, for some $\tilde{C} > 0$,

$$
\begin{aligned}
&\left| \int_0^T ds\, g_{T-s} \star [WJ \star (g_s \star (\bar{m}'\psi))]((x,0)) \right| \le c\|W\|_\infty \bar{m}'(x) e^{-T} \int_0^T ds \sum_{n,q\ge 1} \frac{(T-s)^n s^q}{n!q!} \\
&\times \int dr\, dr'\, dr''\, K^n((x,0),r) J(r-r') \bar{m}'(x') K^q(r',r'') \big((1+|x''|)(y'')^2 + |y''|^4 \big) \mathbf{1}_{|y''|>4VT} \\
&\le \tilde{C}\|W\|_\infty e^{-T} \int_0^T ds \sum_{N\ge 4VT} \sum_{q\le N} \frac{(T-s)^{N-q} s^q}{(N-q)!q!} N^4 \\
&= \tilde{C}\|W\|_\infty e^{-T} T \sum_{N\ge 4VT} \frac{N^4 T^N}{N!} \le \tilde{C} T e^{-T} \|W\|_\infty \qquad (2.6.55)
\end{aligned}
$$

The lemma is proved. □

The estimate (2.6.52) with $\psi = \psi_\lambda$ gives (2.5.43). The estimate (2.6.53) with $\psi = \bar{\psi}_\lambda$ proves the last bound in (2.5.52). We are left with the bounds on $K_\lambda^{(i)}(x)$ for $i = 0, 1$. These follows easily from the estimate (2.5.50) and the nice properties of the kernel $g_t(r, r')$ (see [28]). We omit the details. □

CHAPTER 3.

THE $D = 1$ GINZBURG–LANDAU EQUATION WITH NOISE. BROWNIAN FLUCTUATIONS OF THE INTERFACE AND COUPLINGS

In this chapter we consider a Ginzburg-Landau equation in the interval $[-\epsilon^{-\kappa}, \epsilon^{-\kappa}]$, $\epsilon > 0$ and $\kappa \geq 1$, perturbed by a white noise of strength $\sqrt{\epsilon}$ and with Neumann boundary conditions. We prove that if the initial datum is close to an instanton then it remains close to some instanton for times that may grow as $\epsilon \to 0^+$ as fast as any inverse power of ϵ and as long as the center of the instanton stays away from the endpoints of the interval. Moreover the center of the instanton, suitably renormalized, converges to a Brownian motion. Finally we construct an asymptotic coupling of two solutions starting from initial data close to instantons such that as $\epsilon \to 0^+$ the time of the coupling converges in laws to the first encounter of two Brownians starting from the centers of the instantons approximating the initial data.

The chapter is divided into 7 sections. In the first one we give general motivations and comments on the results. We refer to the second one for a precise and complete list of results. In section 3 we prove statements concerning the deterministic equation (without noise). In section 4 we extend the results to the case with noise. In section 5 we prove convergence to the Brownian motion and in section 6 we construct the coupling which proves the loss of memory of the initial datum. In the last section we prove a miscellany of lemmas we need in the proofs. A brief outline of the main ideas of the proof is given in section 4 after the proof of Theorem 3.2.1.

The results proven here (that is Theorems 3.2.1, 3.2.2 and 3.2.3 below) are based on a paper written in collaboration with S. Brassesco, A. De Masi and E. Presutti, [10].

§3.1 General motivations and description of the results.

As described in the previous chapters the phase separation and the successive interface dynamics for the Glauber evolution with Kac potentials can be derived from the analogous phenomena for the deterministic mean field equation. There are various examples of deterministic evolutions which give rise to pattern formation and interface dynamics and that can be used to describe the motion of an interface between two fluids. Among these one of the most investigated is the deterministic Ginzburg-Landau (Allen-Cahn) equation, that is the reaction-diffusion equation

$$\frac{\partial m}{\partial t}(r,t) = \Delta m(r,t) - V'(m(r,t)), \qquad (r,t) \in \mathbb{R}^d \times \mathbb{R}_+ \tag{3.1.1}$$

where $V(m)$ is a symmetric double-well potential with minima that, without loss of generality, we assume at $m = \pm 1$. A typical example, which is the one we consider here, is $V(m) = m^4/4 - m^2/2$. The constant solutions $m(r) \equiv \pm 1$ correspond to the equilibrium phases of the system. The phase separation phenomena were investigated in [32]. As already recalled in §2.1, also the derivation of the interface dynamics, that is the convergence to the motion by mean curvature, is enough understood, [4], [20], [33], [34], [37].

As a matter of fact there are examples involving deterministic evolutions where the phase separation even stops and the system fails to reach the true equilibrium, being stuck in some locally stable but spurious equilibrium, see [43]. These effects are more frequent in one dimension and this is the case of the $d = 1$ Ginzburg-Landau equation.

Fusco and Hale, [44], and Carr and Pego, [15], have in fact shown that for the deterministic Ginzburg-Landau equation the motion is extremely slow. They consider the equation on an interval $[-L, L]$, L^{-1} a small parameter, with Neumann boundary conditions. The problem of interface dynamics concerns the evolution of profiles where to the left of some point, say x_0, the state is close to -1 and to the right to $+1$. The system in fact relaxes after a short time to an apparently stationary state with the two ± 1 phases connected by an instanton solution of (3.1.1). This state for our choice of the potential is (very close to) $\bar{m}_{x_0}(x) \doteq \tanh(x - x_0)$ (x_0 is the "center of the instanton"). $\bar{m}_{x_0}$ is not truly stationary, it moves almost without changing shape with speed $\approx e^{-c\ell}$, c a positive "slowly varying" factor, ℓ the distance of the center from the boundary of the region. Thus if ℓ is a fraction of the whole interval, i.e., $\ell \approx L$, the motion is exponentially slow. Actually they consider also the case of more than one layer (which in the one dimensional case plays the role of the interface) and the velocity of each layer is exponentially slow in the minimum distance with the nearest ones. So these patterns persist for times exponentially long in the minimum distances between layers.

One expects that an analogous mechanism holds in the case of the non local mesoscopic equation. So in $d = 1$ there is no motion by mean curvature and the deterministic equation produces significant changes only at times $\exp\{bL_\gamma\}$, for some $b > 0$, where L_γ is the typical size of the clusters which, by the analysis of [27] discussed in §1.3, is of order $|\log \gamma|^{1/2}$. Then the shorter clusters disappear first and the long clusters which have survived are so long that the above mechanism is slow and the fluctuations become competitive. By the analysis of [26], this should happen at times of order γ^{-1}, when the fluctuations are no longer infinitesimal. These questions are currently under investigation as well as the characterization of the distribution of the clusters at times of order γ^{-1}.

Now it is possible to study the effects of the random fluctuations in the context of the Ginzburg-Landau model by perturbing the deterministic equation with a small white noise and this is what we analyze here.

Our model equation is then the $d = 1$ Ginzburg-Landau equation with an additive white noise of strength $\sqrt{\epsilon}$, see (3.2.1) below, where $\epsilon > 0$ is the small parameter of the theory and eventually $\epsilon \to 0^+$. We consider the equation in the interval $\mathcal{T}_{\epsilon,\kappa} = [-\epsilon^{-\kappa}, \epsilon^{-\kappa}]$, $\kappa \geq 1$, imposing Neumann boundary conditions. As just said we choose $V(m) = m^4/4 - m^2/2$, $m \in \mathbb{R}$, with minima at $m = \pm 1$, which represent the equilibrium values of the "order parameter" m. The two equilibrium phases are then the constant functions $m(x) = \pm 1$, $x \in \mathcal{T}_{\epsilon,\kappa}$. If we take into account the stochastic term, the picture initially does not change much: except for small random fluctuations we still see, after a short relaxation time, the same profile $\bar{m}_{x_0}$. However on dramatically shorter times (when compared with the deterministic case discussed before) the instanton moves. At times $t_\epsilon \doteq t\epsilon^{-a}$, $0 < a < 1/3$, $t > 0$, the shape is still the same but the center x_0 has changed by the order of $\sqrt{\epsilon t_\epsilon}$ and on this scale it is a Brownian motion, [8]. This result is extended up to times $t_\epsilon \doteq t\epsilon^{-1}$, $t > 0$, when the displacement is finite and the center of the instanton converges as $\epsilon \to 0^+$ to a Brownian motion b_t, as shown in [11] with $\kappa = 1$. The motion is still Brownian also at longer times, $t_\epsilon \doteq t\epsilon^{-1-\gamma}$, $t > 0$, $\gamma > 0$ small enough, as shown by Funaki, [45], in a somewhat different setup. At much longer times the picture may change, for instance the system may pick up some drift, as it could happen when the potential $V(s)$ is non-symmetric but still with equally deep minima, [9]. Here however we restrict to the symmetric case proving that no drift will appear even at "very long times". Roughly speaking we prove that the process m_t behaves like $\bar{m}_{x_0+\sqrt{\epsilon}b_t}$ (the sup norm of the difference vanishing as $\epsilon \to 0^+$) where b_t has the same law as the previous Brownian motion. Convergence is proven by scaling space and times with the latter "much shorter" than what it takes to a Brownian motion to reach the boundary of $\mathcal{T}_{\epsilon,\kappa}$. The result is far complete, a question of physical interest left open is a precise estimate of the first time when a single phase takes over. We control in this thesis the first time τ when one of the two phases occupies the whole $\mathcal{T}_{\epsilon,\kappa}$ except for a "small interval close to one of the endpoints of $\mathcal{T}_{\epsilon,\kappa}$" of size proportional to ϵ^{-1}. We show that τ has the same law in the limit as $\epsilon \to 0^+$ as a suitable stopping time of b_t (this law can be easily computed). We could prove that also after τ the instanton will move like a Brownian motion, but only till it gets close to the endpoints of $\mathcal{T}_{\epsilon,\kappa}$ by a distance $c \log \epsilon^{-1}$,

$c > 0$. Thereafter (for a critical value of c) the Fusco-Hale drift should in fact be dominant with the minority phase shrinking deterministically till extinction. The time length (after τ) when all this happens should be significantly smaller than τ itself, which is thus a good estimate for the time of relaxation to equilibrium. This is not a true equilibrium, though, and the game is not yet over: at much longer times tunnelling phenomena will appear, but this is really beyond the purposes of the present discussion.

The convergence to a one dimensional process simply described by a Brownian motion holds in a much stronger sense than one could suspect from the above presentation. In fact by extending the work of Mueller, [56], (to the present case where the stable point is replaced by a one-dimensional manifold of equilibria, i.e., the translates of the instanton) we construct, in the limit as $\epsilon \to 0^+$, an asymptotic coupling of processes starting from two different initial data, where the time of coupling is the same in law as that of the first encounter of two independent Brownian motions in $d = 1$. They represent the motion of the centers of the instantons associated to the different initial conditions. Namely it suffices that the centers meet for the whole profiles to be matched completely everafter.

§3.2 Definitions and main results.

We study a Ginzburg-Landau one-dimensional equation with additive "small" noise. More precisely, we consider the stochastic partial differential equation

$$\frac{\partial m}{\partial t} = \frac{1}{2}\frac{\partial^2 m}{\partial x^2} + [m - m^3] + \sqrt{\epsilon}\alpha, \qquad t \geq 0; \quad x \in \mathcal{T}_{\epsilon,\kappa} \doteq [-\epsilon^{-\kappa}, \epsilon^{-\kappa}]$$

$$m(x,0) = m_0(x). \tag{3.2.1}$$

We impose Neumann boundary conditions (N.b.c.) at $\mathcal{T}_{\epsilon,\kappa}$. The noise $\alpha = \alpha(x,t)$ is a standard space-time white noise (see [61] for a precise definition), and ϵ is a small parameter that will go to zero. The standard way to give a precise meaning to (3.2.1) is to consider the corresponding integral equation, in terms of the heat kernels associated with the given boundary conditions. In our case, the construction is as follows. Let $H_t^{(\epsilon)}$ be the Green operator associated to the heat equation on $\mathcal{T}_{\epsilon,\kappa}$ with N.b.c. Let $Z_t(x)$ be the Gaussian process defined by the stochastic integral

$$Z_t(x) \doteq \int_0^t ds \int_{\mathcal{T}_{\epsilon,\kappa}} dy\, \alpha(s,y)\, H_{t-s}^{(\epsilon)}(x-y) \tag{3.2.2}$$

for $x \in \mathcal{T}_{\epsilon,\kappa}$ and extended to $\mathbb{R}$ by successive reflections around $n\epsilon^{-\kappa}$. Call $C^0(\mathbb{R})$ the set of continuous and bounded functions and

$$C_{\epsilon,\kappa}(\mathbb{R}) = \{f : f \in C^0(\mathbb{R}), f \text{ is invariant by reflections around the points } n\epsilon^{-\kappa}, |n| \geq 1\}$$

Given $m_0 \in C^0(\mathbb{R})$ and $Z_t \in C_{\epsilon,\kappa}(\mathbb{R})$ for each t, and continuous in t, we define $m_t =: T_t(m_0; \sqrt{\epsilon} Z)$, $t \geq 0$, as the solution of the equation

$$m_t = H_t \star m_0 - \int_0^t ds H_{t-s} \star (m_s^3 - m_s) + \sqrt{\epsilon} Z_t \tag{3.2.3}$$

where $\star$ denotes the convolution of functions on $\mathbb{R}$, $H_t(x)$ is the heat kernel in $\mathbb{R}$:

$$H_t(x) \doteq \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} \tag{3.2.4}$$

We will consider the case when Z_t is given by (3.2.2), and refer to it as the Ginzburg-Landau process, which is adapted to Z_t. For more details see [11], where $\kappa = 1$ is considered. In particular it is proved in [11] that (3.2.3) has a unique continuous solution (see also [39] where the equation with Dirichlet boundary condition is analyzed). If $m_0 \in C_{\epsilon,\kappa}(\mathbb{R})$ then the process lives in $C_{\epsilon,\kappa}(\mathbb{R})$ at all times. In general one refers to this as the Ginzburg-Landau process on $\mathcal{T}_{\epsilon,\kappa}$ with Neumann boundary conditions, but, for notational convenience, we have extended the range of admissible initial data to the whole $C^0(\mathbb{R})$.

Since the function $\bar{m}(x) \doteq \tanh x$ verifies the identity

$$\frac{1}{2}\frac{d^2\bar{m}}{dx^2} = V'(\bar{m}); \quad V'(m) \doteq \frac{dV(m)}{dm}, \quad V(m) \doteq -\frac{1}{2}m^2 + \frac{1}{4}m^4, \quad m \in \mathbb{R} \tag{3.2.5}$$

$\bar{m}$ is a stationary solution of the deterministic Ginzburg-Landau equation on $\mathbb{R}$; i.e., $\bar{m} = T_t(\bar{m}) \doteq T_t(\bar{m}; 0)$, for any $t \geq 0$. The function $\bar{m}(x)$ is what we call "instanton", and, its translate by $x_0 \in \mathbb{R}$,

$$\bar{m}_{x_0}(x) \doteq \bar{m}(x - x_0), \qquad x \in \mathbb{R} \tag{3.2.6}$$

will be called "the instanton centered at x_0" (thus $\bar{m} \equiv \bar{m}_0$). The manifold

$$\mathcal{M} \doteq \{\bar{m}_{x_0}, x_0 \in \mathbb{R}\} \subset C^0(\mathbb{R}) \tag{3.2.7}$$

is locally attractive under the deterministic flow $T_t(\cdot)$, see Fife and McLeod, [40]–[41]. More precisely, let $\|\cdot\|$ denote the sup norm in $\mathbb{R}$, and, for $\delta \geq 0$, define

$$\mathcal{M}_\delta \doteq \{m \in C(\mathbb{R}) : \text{dist}(m, \mathcal{M}) \doteq \inf_{x_0 \in \mathbb{R}} \|m - \bar{m}_{x_0}\| \leq \delta\} \tag{3.2.8}$$

Then, Fife and McLeod have proved that there exists $\delta_0 \geq 0$ and a real valued function $\zeta(m)$ defined on $\mathcal{M}_{\delta_0}$, such that

$$\lim_{t \to +\infty} T_t(m) = \bar{m}_{\zeta(m)}, \qquad \text{for all } m \in \mathcal{M}_{\delta_0}, \tag{3.2.9}$$

in sup norm and exponentially fast. Thus $\mathcal{M}_{\delta_0}$ is foliated by the submanifolds (transversal to $\mathcal{M}$):

$$\mathcal{V}_{x_0} \doteq \{m \in \mathcal{M}_{\delta_0} : \zeta(m) = x_0\} \tag{3.2.10}$$

one the space translation of the other. $\zeta(m)$ will be called the "true center" of $m \in \mathcal{M}_{\delta_0}$ (in the proofs it will be convenient to work with an approximate center that we call the "linear center", but the above definition has conceptual advantages).

Given $m \in C^0(\mathbb{R})$ we set $m^{\epsilon,\kappa} \doteq m$ if $m \notin C_{\epsilon,\kappa}(\mathbb{R})$, otherwise

$$m^{\epsilon,\kappa}(x) \doteq \begin{cases} m(x) & \text{for } |x| \le \epsilon^{-\kappa} \\ m(\pm\epsilon^{-\kappa}) & \text{for } x \ge \epsilon^{-\kappa}, \text{ respectively } x \le -\epsilon^{-\kappa} \end{cases} \tag{3.2.11}$$

(Observe that if $m \in C_{\epsilon,\kappa}(\mathbb{R})$ then $m \notin \mathcal{M}_\delta$ (if δ is small enough) no matter how close it is to an instanton when restricted to $\mathcal{T}_{\epsilon,\kappa}$, yet $m^{\epsilon,\kappa}$ may be in $\mathcal{M}_\delta$).

We define

$$\zeta^{\epsilon,\kappa}(m) \doteq \begin{cases} x_0 & \text{if } m^{\epsilon,\kappa} \in \mathcal{V}_{x_0} \\ 2\epsilon^{-\kappa} & \text{otherwise} \end{cases} \tag{3.2.12}$$

Given m_t a solution of (3.2.3) we set

$$\zeta_t \doteq \zeta^{\epsilon,\kappa}(m_t) \tag{3.2.13}$$

and, for any $0 < \ell < 1$, we define the stopping time

$$\tau_\epsilon(\kappa,\ell) \doteq \inf\left\{t \ge 0 : |\zeta^{\epsilon,\kappa}(m_t)| \ge \epsilon^{-\kappa} - \ell\epsilon^{-1}\right\} \tag{3.2.14}$$

and the stopped processes

$$\hat{m}_t \doteq m_{t\wedge\tau_\epsilon}, \quad \hat{\zeta}_t \doteq \zeta_{t\wedge\tau_\epsilon}, \qquad \tau_\epsilon \doteq \tau_\epsilon(\kappa,\ell),$$

where $a \wedge b$ stands for the minimum between a and b.

Finally we shall denote by P^ϵ the probability distribution of the various stochastic variables we consider and by E^ϵ the expectation value.

3.2.1 Theorem.

There is $\delta \in (0,\delta_0]$, δ_0 as in (3.2.9), and for any $\kappa \ge 1$, $h \ge 0$, $a \in (0,1/4)$ and $\ell \in (0,1)$ there are $c > 0$ and $p > 0$ so that the following holds.

Let $m \in C^0(\mathbb{R})$, $|\zeta^{\epsilon,\kappa}(m)| < \epsilon^{-\kappa} - \ell\epsilon^{-1}$, then

$$P^\epsilon\Big((m_t)^{\epsilon,\kappa} \in \mathcal{M}_{2\delta}\,\forall t \le (\log\epsilon)^2,\, (m_t)^{\epsilon,\kappa} \in \mathcal{M}_{\epsilon^{1/2-a}}\,\forall t \in [(\log\epsilon)^2, \epsilon^{-h}\wedge\tau_\epsilon]\Big) \ge 1 - ce^{-\epsilon^{-p}} \tag{3.2.15}$$

Our next result states that the process $\hat{\zeta}_t$ converges as $\epsilon \to 0^+$ to a Brownian motion with diffusion coefficient $D \doteq 3/4$.

3.2.2 Theorem. *(Convergence to Brownian motion.)*

Let $\kappa \geq 1$, $h \geq 0$, $t^* > 0$, $\ell \in (0,1)$ *and for all* $\epsilon > 0$ *let* $m_0 \in \mathcal{M}_\delta$, δ *as in Theorem 3.2.1, such that* $|\zeta^{\epsilon,\kappa}(m_0)| \leq \epsilon^{-\kappa} - \ell\epsilon^{-1}$ *We also suppose that there are* $R_\pm \in [-\infty, +\infty]$ *such that*

$$\lim_{\epsilon \to 0+} \epsilon^{h/2}[\pm(\epsilon^{-\kappa} - \ell\epsilon^{-1}) - \zeta^{\epsilon,\kappa}(m_0)] = R_\pm$$

Then $\epsilon^{h/2}\big(\hat{\zeta}_{\epsilon^{-h-1}t} - \zeta^{\epsilon,\kappa}(m_0)\big)$, *as a process on* $C([0,t^*])$, *converges weakly to a Brownian motion with diffusion coefficient* D, *starting from 0 and stopped at* $R_\pm$.

Theorem 3.2.1 thus states that the Ginzburg-Landau process is locally attracted by the manifold $\mathcal{M}$ (when the center is sufficiently far from the endpoints of $\mathcal{T}_{\epsilon,\kappa}$) and Theorem 3.2.2 that it performs a Brownian motion on $\mathcal{M}$. We also have sharper results, see Proposition 3.4.5 and Corollary 3.4.6, on the small deviations of the process transversally to $\mathcal{M}$.

Finally we construct in the same probability space a pair m_t and m'_t of Ginzburg–Landau processes starting from two different initial data. In such space, we define

$$\sigma = \inf\{t : m_t \equiv m'_t\} \tag{3.2.16}$$

(where we are considering the processes in $\mathcal{T}_{\epsilon,\kappa}$). As the process, by Theorem 3.2.2, looks like a Brownian motion on $\mathcal{M}$, a natural guess is that σ has the same (limit) law as the first encounter of two independent Brownians that start from the centers of the initial data.

3.2.3 Theorem. *(Couplings.)*

Let m_0 *and* m'_0 *be in* $C_{\epsilon,\kappa}$ *and let both verify the assumptions of Theorem 3.2.2 with respectively* $R_\pm$ *and* $R'_\pm$. *We suppose that*

$$\lim_{\epsilon \to 0+} \epsilon^{h/2}(x_0 - x'_0) =: r^*$$

Then we can construct m_t *and* m'_t , *Ginzburg Landau processes starting from* m_0 *and* m'_0 *respectively so that, if* σ *is defined by* (3.2.16) , *then* $\epsilon^{h+1}\sigma$ *converges in law to the distribution of the variable* S *defined as follows. Let* b_t *and* b'_t *be independent Brownians with diffusion* D *starting from 0 and stopped respectively at* $R_\pm$ *and* $R'_\pm$. *Then* S *is the first time when* $b'_t - b_t = r^*$, *provided* S *occurs before any of the Brownians is stopped, in that case* $S = +\infty$.

§3.3 The deterministic flow.

In this section we study the deterministic flow $\{T_t(m)\}_{t\geq 0}$, proving that a neighborhood of $\mathcal{M}$ is attracted exponentially by $\mathcal{M}$ and it is therefore foliated by the submanifolds $\{\mathcal{V}_{x_0}, x_0 \in \mathbb{R}\}$ defined in (3.2.10).

If $m \in C^0(\mathbb{R})$ then $m(x,t) \doteq T_t(m)(x)$, $t > 0$, is in $C^2(\mathbb{R})$, is differentiable with respect to t and satisfies the Ginzburg-Landau equation

$$\frac{\partial m}{\partial t}(x,t) = \frac{1}{2}\frac{\partial^2 m}{\partial x^2}(x,t) + m(x,t) - m(x,t)^3 \tag{3.3.1}$$

The instantons $\bar{m}_{x_0}$, $x_0 \in \mathbb{R}$, are stationary solutions of (3.3.1). The linearized flow around $\bar{m}_{x_0}$ is the linear semigroup g_{t,x_0} on $C^0(\mathbb{R})$ whose generator L_{x_0} acts on $C^2(\mathbb{R})$ as

$$L_{x_0} f(x) = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x) + [1 - 3\bar{m}_{x_0}(x)^2] f(x) \tag{3.3.2}$$

The operator L_{x_0} has 0 as an eigenvalue (for any $x_0 \in \mathbb{R}$) since $L_{x_0}\bar{m}'_{x_0} = 0$. Denoting by $\langle \cdot, \cdot \rangle$ the scalar product in $L^2(\mathbb{R})$ we set for any $x_0 \in \mathbb{R}$,

$$\tilde{m}'_{x_0} \doteq \frac{\sqrt{3}}{2}\bar{m}'_{x_0}, \quad \langle \tilde{m}'_{x_0}, \tilde{m}'_{x_0} \rangle = 1 \tag{3.3.3}$$

which is thus a unitary eigenvector in $L^2(\mathbb{R})$ of L_{x_0}. L_{x_0} has a spectral gap:

3.3.1 Theorem.

There are α and c positive so that for any $f \in C^0(\mathbb{R})$ and $x_0 \in \mathbb{R}$

$$\|g_{t,x_0}[f - \langle \tilde{m}'_{x_0}, f\rangle \tilde{m}'_{x_0}]\| \leq c e^{-\alpha t} \| f - \langle \tilde{m}'_{x_0}, f\rangle \tilde{m}'_{x_0} \| \tag{3.3.4}$$

A proof of Theorem 3.3.1 in a L^2 setting may be found in [40], the proof with sup-norms is similar to that in section 4 of [24] and it is omitted.

We will exploit Theorem 3.3.1 by observing that $m(x,t) \doteq T_t(m)(x)$ solves for $t > 0$ the equation:

$$\frac{\partial[m - \bar{m}_{x_0}]}{\partial t}(x,t) = L_{x_0}[m - \bar{m}_{x_0}](x,t) - 3\bar{m}_{x_0}(x)[m - \bar{m}_{x_0}]^2(x,t) - [m - \bar{m}_{x_0}]^3(x,t) \tag{3.3.5}$$

We next define the "linear center" of a function $f \in C^0(\mathbb{R})$. This notion was introduced in [11], where it was called simply "center".

3.3.2 Definition.

The point $x_0 \in \mathbb{R}$ is a linear center of $m \in C^0(\mathbb{R})$ if

$$\langle \tilde{m}'_{x_0}, [m - \bar{m}_{x_0}] \rangle = 0 \tag{3.3.6}$$

Existence and uniqueness of the linear center are stated in the next lemma, whose proof, being essentially the same as that of Proposition 3.2 of [11], is omitted:

3.3.3 Lemma.

There is $\delta_0 > 0$ so that any $m \in \mathcal{M}_{\delta_0}$ has a unique linear center $\xi(m)$. Moreover there is $c_0 > 0$ so that if $m \in C^0(\mathbb{R})$, $y_0 \in \mathbb{R}$ and

$$\|m - \bar{m}_{y_0}\| =: \delta \le \delta_0 \tag{3.3.7}$$

then the linear center x_0 of m is such that

$$|x_0 - y_0| \le c_0\delta; \qquad \left|x_0 - \left[y_0 - \frac{3}{4}\langle \bar{m}'_{y_0}, (m - \bar{m}_{y_0})\rangle\right]\right| \le c_0\delta^2 \tag{3.3.8}$$

Let m and $\tilde{m}$ be in $\mathcal{M}_{\delta_0}$, x_0 and $\tilde{x}_0$, their respective linear centers and $\|m - \tilde{m}\| \le \delta_0$. Then

$$|x_0 - \tilde{x}_0| \le \frac{c_0}{2}|\langle \bar{m}'_{x_0}, m - \tilde{m}\rangle| \le c_0\|m - \tilde{m}\| \tag{3.3.9}$$

The next one is the main result in this section:

3.3.4 Theorem.

There are $\beta > 0$, $c > 0$ and $\delta_1 \in (0, \delta_0]$, ($\delta_0$ as in Lemma 3.3.3) so that any $m \in \mathcal{M}_{\delta_1}$ has a true center $\zeta(m)$, a unique linear center $\xi(m)$ and

$$|\xi(m) - \zeta(m)| \le c\Big[\|m - \bar{m}_{\xi(m)}\| \log\{\|m - \bar{m}_{\xi(m)}\|\}\Big]^2 \tag{3.3.10}$$

Moreover $T_t(m) \in \mathcal{M}_{\delta_0}$ for all $t \ge 0$ and setting $\xi_t \doteq \xi(T_t(m))$,

$$\|T_t(m) - \bar{m}_{\zeta(m)}\| \le c\|m - \bar{m}_{\xi(m)}\|e^{-\beta t}, \quad \|T_t(m) - \bar{m}_{\xi_t}\| \le c\|m - \bar{m}_{\xi(m)}\|e^{-\beta t} \tag{3.3.11}$$

$$\sup_{t\ge 0} |\xi_t - \xi(m)| \le c\|m - \bar{m}_{\xi(m)}\|, \quad \sup_{t \ge 0} \|T_t(m) - \bar{m}_{\xi(m)}\| \le c\|m - \bar{m}_{\xi(m)}\| \tag{3.3.12}$$

We will prove a preliminary lemma, Lemma 3.3.5, and then Theorem 3.3.4. The bound (3.3.10) is not optimal, in fact a bound without the logarithmic factor holds true; the right hand side of the first inequality in (3.3.12) can be replaced by $c\|m - \bar{m}_{\xi(m)}\|^2$. Both refinements are not needed and will not be proved.

3.3.5 Lemma.

There is $c_1 > 0$ so that for any $\lambda \in (0,1]$, $x_0 \in \mathbb{R}$ and $v \in C(\mathbb{R})$ such that $\|v\| \le 1$ and $\langle \bar{m}'_{x_0}, v\rangle = 0$ the following holds. Let

$$m \doteq \bar{m}_{x_0} + \lambda v, \qquad t_\lambda \doteq \max\{1, (\log\lambda)^2\} \tag{3.3.13}$$

and c, α as in Theorem 3.3.1. Then for all $0 \le t \le t_\lambda$

$$\Big|T_t(m) - \bar{m}_{x_0}\Big| \le ce^{-\alpha t}\lambda + c_1\lambda^2 t \tag{3.3.14}$$

Proof.

Let $u_t \doteq T_t(m) - \bar{m}_{x_0}$, $u_0 \doteq \lambda v$. By (3.3.5) and Theorem 3.3.1 there is $c_2 > 0$ so that

$$\|u_t\| \le ce^{-\alpha t}\lambda + c_2 \int_0^t ds \left(\|u_s\|^2 + \|u_s\|^3\right) \tag{3.3.15}$$

Call T the first time such that $\|u_T\| = 2c\lambda$, and suppose that $T \le t_\lambda$. Then by (3.3.15)

$$2c\lambda \le c\lambda + c_2 t_\lambda \left([2c\lambda]^2 + [2c\lambda]^3\right)$$

which does not hold when λ is small enough, say $\lambda \le \lambda_0$, $\lambda_0 \in (0,1]$.

But (3.3.14) follows from (3.3.15) when $\lambda \in (\lambda_0, 1]$, if we use the maximum principle to bound $\|u_s\|$ for all $s \ge 0$. For $\lambda \le \lambda_0$, we have seen that $T > t_\lambda$ so $\|u_t\| < 2c\lambda$ for all $t \le t_\lambda$. With this bound on the right hand side of (3.3.15) we obtain (3.3.14), completing the proof of the lemma. □

Proof of Theorem 3.3.4.

We use the same notation as in Lemmas 3.3.5 and 3.3.3. We take $\delta_1 > 0$ (other requests on δ_1 will be specified later) so that for any $\lambda \in (0, \delta_1]$

$$\sup_{t \le t_\lambda} [ce^{-\alpha t}\lambda + c_1\lambda^2 t] \le \delta_0, \qquad \delta_0 \text{ as in Lemma 3.3.3} \tag{3.3.16}$$

Let $m \in M_{\delta_1}$ and call $x_0 = \xi(m)$. If $m = \bar{m}_{x_0}$, then the theorem follows since $\bar{m}_{x_0}$ is stationary for T_t. We write

$$m = \bar{m}_{x_0} + \frac{m - \bar{m}_{x_0}}{\|m - \bar{m}_{x_0}\|}\|m - \bar{m}_{x_0}\|.$$

We are then in the setting of Lemma 3.3.5, with $\lambda = \|m - \bar{m}_{x_0}\|$ and, by Lemma 3.3.3, (3.3.14) and (3.3.16), the linear center $\xi\big(T_t(m)\big)$ is well defined for $t \le t_\lambda$, and, calling $m^\star \doteq T_{t_\lambda}(m)$ there is a constant $c_2 > 0$ so that,

$$\left|\xi\left(m^\star\right) - x_0\right| \le c_0\|m^\star - \bar{m}_{x_0}\| \le c_2[\lambda \log \lambda]^2 \tag{3.3.17}$$

We will prove that there is a constant $c_3 > 0$ so that

$$\left|\xi(m^\star) - \zeta(m^\star)\right| \le c_3\|m^\star - \bar{m}_{\xi(m^\star)}\| \tag{3.3.18}$$

Proof of (3.3.10).

Since $\zeta(m) = \zeta\big(T_t(m)\big)$, $\zeta(m) = \zeta(m^\star)$. Then by (3.3.17) and (3.3.18)

$$|\zeta(m) - x_0| \le c_2[\lambda \log \lambda]^2 + c_3\|m^\star - \bar{m}_{\xi(m^\star)}\|$$

Recalling that $\bar{m}' \leq 1$,

$$\|m^\star - \bar{m}_{\xi(m^\star)}\| \leq \|m^\star - \bar{m}_{x_0}\| + \|\bar{m}_{x_0} - \bar{m}_{\xi(m^\star)}\| \leq (1+c_0)\|m^\star - \bar{m}_{x_0}\|$$
$$\leq \frac{(1+c_0)c_2}{c_0}[\lambda \log \lambda]^2 \tag{3.3.19}$$

The last inequality uses (3.3.17). Thus, (3.3.10) is proved (provided (3.3.18) holds).

Proof of (3.3.18).

We further specify δ_1 by requiring that for all $\lambda \in (0, \delta_1]$

$$(1+c_0)[ce^{-\alpha t_\lambda}\lambda + c_1\lambda^2 t_\lambda] \leq \delta_1 \tag{3.3.20}$$

By (3.3.19) and (3.3.14), (3.3.20) implies

$$\|m^\star - \bar{m}_{x^\star}\| \leq \delta_1, \quad \text{where} \quad x^\star \doteq \xi(m^\star) \tag{3.3.21}$$

thus reconstructing the initial assumption also at time t_λ.

We define τ so that

$$e^{-\alpha\tau}c(1+c_0) = \frac{1}{4} \tag{3.3.22}$$

We require that δ_1 is such that for all $\lambda \in (0, \delta_1]$

$$(1+c_0)[ce^{-\alpha\tau}\lambda + c_1\lambda^2\tau] \leq \frac{\lambda}{2} \tag{3.3.23}$$

Let $m^{(0)} \doteq m^\star$, $x^{(0)} \doteq x^\star$, $m^{(1)} \doteq T_\tau(m^{(0)})$, $x^{(1)} \doteq \xi(m^{(1)})$ and

$$\lambda_0 \doteq (1+c_0)[ce^{-\alpha t_\lambda}\lambda + c_1\lambda^2 t_\lambda], \quad \lambda \doteq \|m - \bar{m}_{x_0}\|, \quad \|m^\star - \bar{m}_{x^{(0)}}\| \leq \lambda_0$$

and

$$\lambda_1 \doteq (1+c_0)[ce^{-\alpha\tau}\lambda_0 + c_1\lambda_0^2\tau]$$

By iteration we then define $m^{(n)}$, $x^{(n)}$ and λ_n, $n > 1$. We have from (3.3.23)

$$\|T_{n\tau}(m^{(0)}) - \bar{m}_{x^{(n)}}\| \leq \lambda_n \leq 2^{-n}\lambda_0$$

and since $|x^{(n)} - x^{(n-1)}| \leq c_0\|T_\tau(m^{(n-1)}) - \bar{m}_{x^{(n-1)}}\|$ by (3.3.14)

$$|x^{(n)} - x^{(0)}| \leq \sum_{i=1}^{n} \lambda_i \leq 2\lambda_0$$

Since $x^{(n)} \to \zeta(m^{(0)})$ as $n \to +\infty$ this proves (3.3.18) (recall that $m^\star = m^{(0)}$).

Proof of (3.3.12).

We have

$$|\xi_t - \xi(m)| \le |\xi_t - \zeta(m)| + |\zeta(m) - \xi(m)| \tag{3.3.24}$$

The last term is bounded using (3.3.10). For the other one observes that by (3.3.9) for any $t \ge 0$

$$\begin{aligned} &\|\xi_t - \zeta(m)\| = \|\xi(\bar m_{\xi_t}) - \xi(\bar m_{\zeta(m)})\| \\ &\le c_0 \|\bar m_{\xi_t} - \bar m_{\zeta(m)}\| \le c_0 \|\bar m_{\xi_t} - T_t(m)\| + c_0 \|\bar m_{\zeta(m)} - T_t(m)\| \end{aligned} \tag{3.3.25}$$

which by (3.3.11) proves the first inequality in (3.3.12). The second one follows from the first one and (3.3.11). Theorem 3.3.4 is proved. □

§3.4 THE STOCHASTIC FLOW.

In this section we prove Theorem 3.2.1 and some of the key estimates that will be used in §3.5 and §3.6 to prove the other theorems of §3.2. We start with Proposition 3.4.1 where we derive the basic bounds on the Gaussian process Z_t. The proposition is proved in [11] for $\kappa = 1$, its extension to $\kappa > 1$ is not difficult, for completeness we report it in §3.7.

3.4.1 Proposition.

Let $\kappa \ge 1$, $\epsilon > 0$ and Z_t as defined in (3.2.2), *and $T_t(m, \sqrt{\epsilon} Z)$ the solution of* (3.2.3) *starting from m. Then there are positive constants b_0 and b_1 such that, if we set $t_\epsilon \doteq (\log \epsilon)^2$ and*

$$\mathcal{B}_{p,\epsilon} \doteq \Big\{ \sup_{0 \le t \le t_\epsilon} \|\sqrt{\epsilon} Z_t\| \le \epsilon^{\frac{1}{2}-p} \Big\}, \tag{3.4.1}$$

then, for all $\epsilon > 0$ and $p > 0$

$$P^\epsilon(\mathcal{B}_{p,\epsilon}) \ge 1 - b_0 e^{-b_1 |\log \epsilon|^{-1} \epsilon^{-2p}} \tag{3.4.2}$$

and for all $m \in C^0(\mathbb{R})$ with $\|m\| \le 1 + 10^{-2}$ and any $S \ge t_\epsilon$,

$$P^\epsilon\Big(\sup_{0 \le t \le S} \|T_t(m; \sqrt{\epsilon} Z)\| \le 2,\ \|T_S(m; \sqrt{\epsilon} Z)\| \le 1 + 10^{-2} \Big) \ge 1 - S b_0 e^{-b_1 \epsilon^{-1}} \tag{3.4.3}$$

Note how (3.4.3) implies that, for any $N > 0$, the process remains uniformly bounded up to times ϵ^{-N} with probability close to 1 exponentially in ϵ^{-1}.

Let $m \in C^0(\mathbb{R})$, set $m_t \doteq T_t(m; \sqrt{\epsilon} Z)$, $t \geq 0$, then, for any $x_0 \in \mathbb{R}$, $u_t \doteq m_t - \bar{m}_{x_0}$ solves the following integral version of the Ginzburg-Landau stochastic equation (also considered in [11])

$$u_t = g_{t,x_0} u_0 - \int_0^t ds\, g_{t-s,x_0}(3\bar{m}_{x_0} u_s^2 + u_s^3) + \sqrt{\epsilon} \hat{Z}_{t,x_0} \tag{3.4.4}$$

The operator g_{t,x_0} is defined in the beginning of §3.3 and

$$\hat{Z}_{t,x_0} \doteq Z_t + \int_0^t ds\, g_{t-s,x_0} [(3\bar{m}_{x_0}^2 - 1) Z_s] \tag{3.4.5}$$

$\hat{Z}_{t,x_0}$ is also given by the stochastic integral

$$\hat{Z}_{t,x_0} = \int_0^t ds \int_{\mathcal{T}_{\epsilon,\kappa}} dy\, g_{t-s,x_0}^{(\epsilon)}(x,y)\, \alpha(s,y), \tag{3.4.6}$$

where

$$g_{t-s,x_0}^{(\epsilon)}(x,y) \doteq \sum_{j \in \mathbb{Z}} \Big(g_{t-s,x_0}(x, y + 4j\epsilon^{-\kappa}) + g_{t-s,x_0}(x, 4j\epsilon^{-\kappa} + 2\epsilon^{-\kappa} - y) \Big), \tag{3.4.7}$$

and $g_{t,x_0}(\cdot,\cdot)$ stands for the kernel corresponding to the operator g_{t,x_0}. An estimate analogous to that given by the previous proposition follows for $\hat{Z}_{t,x_0}$:

3.4.2 Proposition.

Let $\hat{Z}_{t,x_0}$ as above. Then there are positive constants b_0 and b_1 such that, if we set $t_\epsilon \doteq (\log \epsilon)^2$ and

$$\mathcal{B}_{p,\epsilon,x_0} \doteq \Big\{ \sup_{0 \leq t \leq t_\epsilon} \|\sqrt{\epsilon} \hat{Z}_{t,x_0}\| \leq \epsilon^{\frac{1}{2} - p} \Big\}$$

then

$$P^\epsilon\Big(\mathcal{B}_{p,\epsilon,x_0}\Big) \geq 1 - b_0 e^{-b_1 \epsilon^{-p}} \tag{3.4.8}$$

Proof.

It follows easily from (3.4.2), (3.4.5) and (3.7.32).

Recall from Lemma 3.3.3 that $\xi(m)$ is the linear center of m (see (3.3.6)), and that there is δ_0 so that if $m \in \mathcal{M}_{\delta_0}$ then $\xi(m)$ is uniquely defined. In analogy with (3.2.12), given any $\kappa \geq 1$, we set

$$\xi^{\epsilon,\kappa}(m) \doteq \begin{cases} \xi(m^{\epsilon,\kappa}) & \text{if } m^{\epsilon,\kappa} \in \mathcal{M}_{\delta_0} \\ 0 & \text{otherwise} \end{cases} \tag{3.4.9}$$

and, given any $\ell \in (0,1)$ and $\delta \in (0, \delta_0]$,

$$M_{\kappa,\ell,\delta}^\epsilon \doteq \{ m \in C^0(\mathbb{R}) : m^{\epsilon,\kappa} \in \mathcal{M}_\delta,\ |\xi^{\epsilon,\kappa}(m)| \leq \epsilon^{-\kappa} - \ell\epsilon^{-1} \} \tag{3.4.10}$$

By (3.3.10), Theorem 3.2.1 follows from the analogous statement with the true center replaced by the linear center. Theorem 3.2.1 will then be a consequence of the following proposition.

3.4.3 Proposition.

There is $\delta_2 \in (0, \delta_1]$ *(*δ_1 *as in Theorem 3.3.4) so that the following holds. Let* $a \in (0, 1/2)$, $\ell \in (0,1)$, $\kappa \geq 1$, $\delta \in (0, \delta_2]$ *and* $p \in (0, a/2)$. *Then there are positive constants* b_0, b_1 *and* c *so that for any* ϵ *small enough and for any* $m \in M^\epsilon_{\kappa,\ell,\delta}$, *if* $m_t \doteq T_t(m; \sqrt{\epsilon}Z)$, $\ell' = \ell - \epsilon c(\delta \vee \epsilon^{1/2-a})$, *and* $t_\epsilon = (\log \epsilon)^2$,

$$P^\epsilon\Big(m_t \in M^\epsilon_{\kappa,\ell',c\delta} \text{ for all } t \leq t_\epsilon,\ m_{t_\epsilon} \in M^\epsilon_{\kappa,\ell',\epsilon^{1/2-a}}\Big) \geq 1 - b_0 e^{-b_1 \epsilon^{-p}} \tag{3.4.11}$$

Proof.

Let m be as in the statement and consider first the case $m \in \mathcal{M}_{\delta_1}$. We study $m_t \doteq T_t(m; \sqrt{\epsilon}Z)$ as a perturbation of $m^0_t \doteq T_t(m)$. Let

$$x_0 = \xi(m), \quad u^0_t \doteq m^0_t - \bar{m}_{x_0}, \quad D_t \doteq \|m_t - m^0_t\|, \quad \xi_t \doteq \xi(m_t), \quad \xi^0_t \doteq \xi(m^0_t) \tag{3.4.12}$$

By Theorem 3.3.4 there is a constant $c = c(\delta_1)$ so that for all $t \geq 0$

$$\|m^0_t - \bar{m}_{\xi^0_t}\| \leq c e^{-\beta t}, \quad |\xi^0_t - x_0| \leq c \tag{3.4.13}$$

Next, we write the integral equation for $m_t - m^0_t$ in terms of the operator g_{t,x_0}. Recalling that $V'(m_t) - V'(m^0_t) = (m_t - m^0_t)[3(m^0_t)^2 - 1 + 3\bar{m}^2_{x_0} - 3\bar{m}^2_{x_0}] + 3m^0_t(m_t - m^0_t)^2 + (m_t - m^0_t)^3$, we obtain

$$\begin{aligned} m_t - m^0_t = - \int_0^t ds\, g_{t-s,x_0}\big[&(m_t - m^0_t)3(m^0_t - \bar{m}_{x_0})(m^0_t + \bar{m}_{x_0}) + 3m^0_t(m_t - m^0_t)^2 \\ &+ (m_t - m^0_t)^3\big] + \epsilon^{1/2}\hat{Z}_{t,x_0}, \end{aligned}$$

where $\hat{Z}_{t,x_0}$ is defined in (3.4.5). By (3.3.12), (3.4.12) and (3.7.34), we get that in $\mathcal{B}_{p,\epsilon,x_0}$ (which, by (3.4.8), has probability greater than $1 - b_0 e^{-b_1 \epsilon^{-p}}$),

$$D_t \leq c_3 \int_0^t ds\, \{\|u^0_0\| D_s + D^2_s + D^3_s\} + \epsilon^{1/2-p} \tag{3.4.14}$$

for some positive constant c_3 Then, by Gronwall's Lemma applied in this set, there is $\tau > 0$ so that, calling $t^\star \doteq \tau|\log \epsilon|$,

$$\sup_{t \leq t^\star} D_t \leq \epsilon^{1/2-2p} \tag{3.4.15}$$

By (3.4.12) there is $b = b(\tau)$, $b \in (0, 1/2 - 2p)$, so that calling $y_0 \doteq \xi(m^0_{t^\star})$,

$$\|m^0_{t^\star} - \bar{m}_{y_0}\| \leq \epsilon^b \tag{3.4.16}$$

Then

$$\|m_{t^\star} - \bar{m}_{y_0}\| \leq \epsilon^{1/2-2p} + \epsilon^b \leq 2\epsilon^b \tag{3.4.17}$$

Hence for $\epsilon > 0$ small enough the linear center $x^\star \doteq \xi(m_{t^\star})$ of $m_{t^\star}$ exists and by Lemma 3.3.3 $|x^\star - y_0| \le c_0 2\epsilon^b$. Then

$$|x^\star - x_0| \le |x^\star - y_0| + |y_0 - x_0| \le c_0(1 + 2\epsilon^b) \le c' \tag{3.4.18}$$

for a suitable $c' > 0$. Thus by (3.4.17) and (3.4.18) $m_{t^\star} \in M^\epsilon_{\kappa,\ell_1,\delta}$, $\ell_1 = \ell - \epsilon c'$ and $\delta \doteq 2\epsilon^b$.

Moreover, recalling that $\bar{m}' \le 1$,

$$\begin{aligned} \|m_{t^\star} - \bar{m}_{x^\star}\| &\le \|m_{t^\star} - \bar{m}_{y_0}\| + \|\bar{m}_{y_0} - \bar{m}_{x^\star}\| \le (1 + c_0)\|m_{t^\star} - \bar{m}_{y_0}\| \\ &\le (1 + c_0)2\epsilon^b \end{aligned} \tag{3.4.19}$$

We next consider $m_t \doteq T_t(m_{t^\star}; \sqrt{\epsilon}Z)$. We call $m^0_t \doteq T_t(m_{t^\star})$ and $D_t \doteq \|m_t - m^0_t\|$. Again in $\mathcal{B}_{p,\epsilon,x_0}$, (3.4.15) holds in this setup, but now, by (3.4.19), $\|u^0_0\| \le (1 + c_0)2\epsilon^b$. Hence, by Gronwall's Lemma we get this time that there is $c > 0$ so that $D_t \le c\epsilon^{1/2-p}$ for all $t \le t_\epsilon$.

By the same argument used above we then complete the proof of the proposition under the additional assumption that $m \in \mathcal{M}_{\delta_1}$ and consider next the case $m \in M^\epsilon_{\kappa,\ell,\delta} \cap C_{\epsilon,\kappa}(\mathbb{R})$. Then $m^{\epsilon,\kappa} \in \mathcal{M}_\delta$ and $|\xi^{\epsilon,\kappa}(m)| \le \epsilon^{-\kappa} - \ell\epsilon^{-1}$ and hence (3.4.11) holds for $T_t(m^{\epsilon,\kappa}; \sqrt{\epsilon}Z)$. Let $\mathcal{B}_{p,\epsilon}$ be the set in (3.4.1). Then there are constants c' and V positive so that setting $L_\epsilon \doteq \epsilon^{-\kappa} - Vt_\epsilon$

$$\sup_{0\le t\le t_\epsilon} \sup_{|x|\le L_\epsilon} |T_t(m; \sqrt{\epsilon}Z) - T_t(m^{\epsilon,\kappa}; \sqrt{\epsilon}Z)| \le \hat{c}e^{-t_\epsilon} \tag{3.4.20}$$

(3.4.20) is proved in Proposition 5.3 of [11] (barrier lemma) for $\kappa = 1$, but the proof is also valid for $\kappa > 1$. Let next $x \in [L_\epsilon, \epsilon^{-\kappa}]$ (the proof for $x \in [-\epsilon^{-\kappa}, -L_\epsilon]$, is similar). By assumption there is a constant $c \in (0,1)$ so that

$$\sup_{|x-\epsilon^{-\kappa}|\le 2Vt_\epsilon} |m(x) - 1| \le c \tag{3.4.21}$$

Recalling that $m \in C_{\epsilon,\kappa}(\mathbb{R})$, we define $\hat{m} \in C(\mathbb{R})$ as

$$\hat{m}(x) \doteq \begin{cases} m(x) & \text{if } |x - \epsilon^{-\kappa}| \le 2Vt_\epsilon \\ m(\epsilon^{-\kappa} - 2Vt_\epsilon) & \text{if } x \le \epsilon^{-\kappa} - 2Vt_\epsilon \\ m(\epsilon^{-\kappa} + 2Vt_\epsilon) & \text{if } x \ge \epsilon^{-\kappa} + 2Vt_\epsilon \end{cases} \tag{3.4.22}$$

Using again the barrier lemma there is $c > 0$ so that in $\mathcal{B}_{p,\epsilon}$

$$\sup_{0\le t\le t_\epsilon} \sup_{L_\epsilon < x \le \epsilon^{-\kappa}} |T_t(m; \sqrt{\epsilon}Z) - T_t(\hat{m}; \sqrt{\epsilon}Z)| \le ce^{-t_\epsilon} \tag{3.4.23}$$

Since $m(\cdot) \equiv 1$ is stable (see the proof of Lemma 3.7.2 in §3.7) there is a constant $c > 0$ so that in $\mathcal{B}_{p,\epsilon}$ for any $t \in [0, t_\epsilon]$

$$\|T_t(\hat{m}; \sqrt{\epsilon}Z) - 1\| \le c[e^{-t} + \epsilon^{1/2-p}] \tag{3.4.24}$$

By (3.4.20), (3.4.23) and (3.4.24) there is $c > 0$ so that $\left(T_t(m;\sqrt{\epsilon}Z)\right)^{\epsilon,\kappa}$ is in $\mathcal{M}_{c\delta}$ for all $t \le t_\epsilon$ and in $\mathcal{M}_{c\epsilon^{1/2-a}}$ at time t_ϵ.

It remains to control the position of the center. From (3.4.11), (3.4.20) and (3.4.24), there is a constant $c > 0$ so that

$$\|\left(T_t(m;\sqrt{\epsilon}Z)\right)^{\epsilon,\kappa} - T_t(m^{\epsilon,\kappa};\sqrt{\epsilon}Z)\| \le c\delta \tag{3.4.25}$$

By choosing $\delta_2 > 0$ so small that $c\delta_2 \le \delta_0$, δ_0 as in Lemma 3.3.3 we conclude that

$$\left|\xi\left(\left(T_t(m;\sqrt{\epsilon}Z)\right)^{\epsilon,\kappa}\right) - \xi\left(T_t(m^{\epsilon,\kappa};\sqrt{\epsilon}Z)\right)\right| \le c_0 c\delta \tag{3.4.26}$$

Since $|\xi\left(T_t(m^{\epsilon,\kappa};\sqrt{\epsilon}Z)\right)| \le \epsilon^{-\kappa} - \epsilon^{-1}\ell'$ similar conclusion (with ℓ' replaced by $\ell' - \epsilon(c_0 c\delta)$) holds for $\xi\left(\left(T_t(m;\sqrt{\epsilon}Z)\right)^{\epsilon,\kappa}\right)$, and Proposition 3.4.3 is proved. □

Proof of Theorem 3.2.1.

By iterating (3.4.11) and recalling that by (3.3.10) the true and the linear centers are close we conclude the proof of Theorem 3.2.1. □

Main ideas of the proofs.

The proof of Theorem 3.2.1 is a (simple) perturbative argument that relates $T_t(m;\sqrt{\epsilon}Z)$ and $T_t(m)$; convergence to a Brownian motion is a quite different game, its proof much more delicate.

When times are scaled as $\epsilon^{-1}\tau$, τ in a compact of $\mathbb{R}$, the argument used in [11] applies. It is based on the bound

$$\|T_{t_\epsilon}(m;\sqrt{\epsilon}\underline{Z}) - T_{t_\epsilon}(\bar{m}_{\xi(m)};\sqrt{\epsilon}Z)\| \le c\epsilon^{1-2a} \tag{3.4.27}$$

which holds if $m \in M^\epsilon_{\kappa,\ell,\epsilon^{1/2-a}}$ and $c > 0$ a suitable constant. The bound follows easily from the integral representation (3.4.4) using the estimates of §3.3. The idea then is to split the time into intervals of length t_ϵ (a different value of t_ϵ is actually used in [11]) and to replace at the beginning of each interval the true process by that starting from the instanton with the same linear center. One can see that if a is small then the sum of all the errors, bounded using (3.4.27) vanishes when $\epsilon \to 0^+$ so that we can consider a process that at each interval starts from an instanton. Then except for the (negligible) influence of the boundaries, the increments of the linear center are mutually independent and convergence to a Brownian motion is easily proved.

If times are proportional to ϵ^{-h}, h large, the sum of the errors is no longer negligible and this approach fails even though the bound (3.4.27) is optimal. The way out, at least that is what we do, exploits the fact that there is a much better bound that goes even like $c_n \epsilon^n$ (for any given n) provided we construct the two processes not as simply as when taking just the same noise in the whole interval $[0, t_\epsilon]$. But the most important point is that we only compare the processes modulo translations. We can then conclude that after a time delay t_ϵ the two processes, suitably shifted, are in law very close to each other. The successive increments of the linear centers are independent of the shift (except for the influence of the boundary, controlled by using the barrier lemma) so that they are in law very close to each other.

The crucial bound involving $c_n \epsilon^n$ is proved below, see Proposition 3.4.5 and Corollary 3.4.6, its application to the convergence to a Brownian motion in §3.5.

To investigate the process modulo translation, i.e. its transversal deviations from $\mathcal{M}$ neglecting the localization along $\mathcal{M}$, it is convenient to introduce a function $D_\epsilon(m, m^\star)$ which plays the role of a distance between m and $m^\star$, but it is not a distance.

3.4.4 Definition.

Let $f \in C(\mathbb{R})$, $\epsilon > 0$, x and $y \in \mathbb{R}$, we set

$$\|f\|_\epsilon \doteq \sup_{|x| \le \epsilon^{-1/10}} |f(x)|; \quad \tau_y f(x) \doteq f(x+y) \quad \|f\|_{\epsilon,x} \doteq \|\tau_x f\|_\epsilon \tag{3.4.28}$$

For m and $m^\star$ in $C(\mathbb{R})$, we then define $D_\epsilon(m, m^\star)$ by

$$D_\epsilon(m, m^\star) \doteq \begin{cases} \inf_{r \in \mathbb{R}} \{|r| \vee \|\tau_{x_0} m - \tau_{x_0^\star + r} m^\star\|_\epsilon\} & \text{if } m^{\epsilon,\kappa}, (m^\star)^{\epsilon,\kappa} \in \mathcal{M}_{\epsilon^{1/2-a}} \\ 1 & \text{if not} \end{cases} \tag{3.4.29}$$

where $x_0 \doteq \xi^{\epsilon,\kappa}(m)$, $x_0^\star \doteq \xi^{\epsilon,\kappa}(m^\star)$ (see (3.4.9) for notation) and $a \vee b$ stands for the maximum between a and b.

In general $D_\epsilon(m, m^\star) \ne D_\epsilon(m^\star, m)$. By its definition for any $\delta \in (0,1)$, $D_\epsilon(m, m^\star) \le \delta$ if and only if $m^{\epsilon,\kappa}, (m^\star)^{\epsilon,\kappa} \in \mathcal{M}_{\epsilon^{1/2-a}}$ and there is η such that $|\eta - (x_0^\star - x_0)| \le \delta$ and $\|m - \tau_\eta m^\star\|_{\epsilon, x_0} \le \delta$.

We next prove a contraction property of the evolution with respect to D_ϵ.

3.4.5 Proposition.

Let $\kappa \ge 1$, $\ell \in (0,1)$, $a \in (0, 1/2)$, $n \ge 1$, $b > 0$ and $\gamma \in (0, 1/2 - a)$. Then there are $p > 0$ and $c > 0$ such that for all $0 < \epsilon < 1$ the following holds. For all pairs m and $m^\star \in C(\mathbb{R})$ so that $D_\epsilon(m, m^\star) \le \epsilon^b$, we can construct the processes m_t and $m_t^\star$, solutions of (3.2.1) with *initial data m and $m^\star$ respectively in the same probability space and such that, if we set $t_\epsilon = (\log\epsilon)^2$, then*

$$P^\epsilon\Big(D_\epsilon(m_{t_\epsilon/n}, m^\star_{t_\epsilon/n}) \le \epsilon^{b+\gamma}\Big) \ge 1 - ce^{-\epsilon^{-p}} \tag{3.4.30}$$

Proof.

Notation: for simplicity we consider the case $n = 1$. Let $x_0 \doteq \xi^{\epsilon,\kappa}(m)$, $x_0^\star \doteq \xi^{\epsilon,\kappa}(m^\star)$, $\Delta \doteq x_0^\star - x_0$, $\lambda \doteq \epsilon^{1/2-a}$, $\delta \doteq \epsilon^b$,

$$v(x) \doteq [m(x) - \bar{m}_{x_0}(x)], \qquad u(x) \doteq [m^\star(x) - \bar{m}_{x_0^\star}(x)] \tag{3.4.31}$$

By assumption there is η, $|\eta - \Delta| \le \delta$, so that (recall that $\bar{m}' \le 1$)

$$\|v - \tau_\eta u\|_{\epsilon,x_0} \le 2\delta \tag{3.4.32}$$

We divide the proof into several steps.

STEP 1. Construction of the coupling.

Starting from a white noise process α, consider the processes Z_t and $\hat{Z}_{t,x_0}$ as defined in (3.2.2) and (3.4.5) (or (3.4.6)) respectively.

Next, take a second noise $\bar{\alpha}$ independent of α and set

$$\begin{aligned} Z_t^\star(x) \doteq & \int_0^t ds \int dz\, \mathbf{1}_{\{|z+\Delta-x_0^\star|\le 4\epsilon^{-1/10};\, z+\Delta\in\mathcal{T}_{\epsilon,\kappa}\}}\, H^{(\epsilon)}_{t-s}(x, z+\Delta)\, \alpha(s,z) \\ & + \int_0^t ds \int dy\, \mathbf{1}_{\{|y-x_0^\star|> 4\epsilon^{-1/10};\, y\in\mathcal{T}_{\epsilon,\kappa}\}}\, H^{(\epsilon)}_{t-s}(x,y)\, \bar{\alpha}(s,y) \end{aligned} \tag{3.4.33}$$

($\mathbf{1}_A$ denotes the characteristic function of the set A). It is easy to check (by comparing covariances) that the processes $Z_t^\star$ and Z_t have the same law. Using them, we construct the Ginzburg-Landau processes by setting

$$m_t \doteq T_t(m; \sqrt{\epsilon}Z), \qquad m_t^\star \doteq T_t(m^\star; \sqrt{\epsilon}Z^\star). \tag{3.4.34}$$

Define

$$v_t \doteq m_t - \bar{m}_{x_0}, \quad v_0 \doteq v, \qquad u_t \doteq m_t^\star - \bar{m}_{x_0^\star}, \quad u_0 \doteq u \tag{3.4.35}$$

We also call $v_t^{(\epsilon,\kappa)} \doteq (m_t)^{\epsilon,\kappa} - \bar{m}_{x_0}$ and $u_t^{(\epsilon,\kappa)} \doteq (m_t^\star)^{\epsilon,\kappa} - \bar{m}_{x_0^\star}$.

STEP 2. The good sets.

Let $p \in (0, a/2)$, $\epsilon > 0$, $c > 0$ and

$$\mathcal{B}^{(1)}_{\epsilon,p} \doteq \Big\{ \sup_{0\le t\le t_\epsilon} \{\|\hat{Z}_{t,x_0}\| + \|\hat{Z}^\star_{t,\tilde{x}_0^\star}\|\} \le \epsilon^{-2p}, \sup_{0\le t\le t_\epsilon} \{\|v_t^{(\epsilon,\kappa)}\| + \|u_t^{(\epsilon,\kappa)}\|\} \le c\epsilon^{1/2-a} \Big\} \tag{3.4.36}$$

$$\mathcal{B}_\epsilon^{(2)} \doteq \left\{ \sup_{0\le t\le t_\epsilon} \|Z_{t,x_0} - \tau_\Delta Z^\star_{t,x_0^\star}\|_{x_0,\epsilon} < e^{-\epsilon^{-1/100}} \right\}, \tag{3.4.37}$$

$$\mathcal{B}_\epsilon^{(3)} \doteq \left\{ \sup_{0\le t\le t_\epsilon} \|\hat Z_{t,x_0} - \tau_\Delta \hat Z^\star_{t,x_0^\star}\|_{x_0,\epsilon} < e^{-\epsilon^{-1/100}} \right\} \tag{3.4.38}$$

where $\hat Z_{t,x_0}$ is given by (3.4.5), and $\hat Z^\star_{t,x_0^\star}$ is given by (3.4.5), with $Z_t^\star$ (given by (3.4.33)) in the place of Z_t and $x_0^\star$ in the place of x_0 . That is,

$$\hat Z^\star_{t,x_0^\star} \doteq Z_t^\star + \int_0^t ds g_{t-s,x_0^\star}[(3\bar m^2_{x_0^\star} - 1)Z_s^\star] \tag{3.4.39}$$

We will prove that there is $c > 0$ so that

$$P^\epsilon(\hat{\mathcal{B}}_{\epsilon,p}) \ge 1 - ce^{-\epsilon^{-p}}, \qquad \hat{\mathcal{B}}_{\epsilon,p} \doteq \mathcal{B}^{(1)}_{\epsilon,p} \cap \mathcal{B}^{(2)}_\epsilon \cap \mathcal{B}^{(3)}_\epsilon \tag{3.4.40}$$

Since $D_\epsilon(m, m^\star) \le \epsilon^b < 1$, both m and $m^\star$ are in $\mathcal{M}_{\epsilon^{1/2-a}}$. Then by (3.4.8) and the proof of Proposition 3.4.3 there is a constant c so that

$$P^\epsilon(\mathcal{B}^{(1)}_{\epsilon,p}) \ge 1 - ce^{-\epsilon^{-p}} \tag{3.4.41}$$

A similar bound holds for the probability of $\mathcal{B}_\epsilon^{(i)}$, $i = 2, 3$. (See Lemma 3.7.5 of §3.7 and recall that $p < 1/2$).

STEP 3. Bounds close to the center.

Let $\gamma \in (0, 1/2 - a)$ and $b > 0$, $\bar\gamma \in (\gamma, 1/2 - a)$ and M a positive integer such that $b + \bar\gamma < M(1/2 - a)$. It follows that there exists ϵ_0 so that for $\epsilon < \epsilon_0$

$$[t_\epsilon \epsilon^{1/2-a}]^M \le \delta\epsilon^{\bar\gamma} \tag{3.4.42}$$

We will prove that there are c and p positive so that for $\epsilon < \epsilon_0$, in the set $\hat{\mathcal{B}}_{\epsilon,p}$,

$$\sup_{|x-x_0|\le\epsilon^{-1/10}-M\epsilon^{-1/20}} |v_{t_\epsilon}(x) - \tau_\Delta u_{t_\epsilon}(x)| \le c\delta\epsilon^{\bar\gamma} \tag{3.4.43}$$

Proof of (3.4.43). We set

$$d_t(x) \doteq v_t(x) - \tau_\Delta u_t(x) \tag{3.4.44}$$

For x, y and h in $\mathbb{R}$ and $t > 0$

$$g_{t,x_0+h}(x + h, y + h) = g_{t,x_0}(x, y), \qquad \bar m_{x_0}(x - h) = \bar m_{x_0+h}(x), \tag{3.4.45}$$

hence, for any function $f \in C^0(\mathbb{R})$,

$$\int dy\, g_{t,x_0^\star}(x + \Delta, y)\, \bar m_{x_0^\star}(y)\, f(y) = \int dy\, g_{t,x_0}(x, y)\, \bar m_{x_0}(y)\, \tau_\Delta f(y) \tag{3.4.46}$$

Writing the integral equation (3.4.4) for v_t and u_t (the former with g_{t,x_0}, the latter with g_{t,x_0^*}) we get

$$d_t = \left[g_{t,x_0}v - \tau_\Delta g_{t,x_0^*}u\right] + \int_0^t ds\,\left(g_{t-s,x_0}A_s d_s\right) + \sqrt{\epsilon}[\hat{Z}_{t,x_0} - \tau_\Delta \hat{Z}^*_{t,x_0^*}] \tag{3.4.47}$$

where

$$A_s \doteq -3\bar{m}_{x_0}[v_s + \tau_\Delta u_s] - [v_s^2 + (\tau_\Delta u_s)^2 + v_s\tau_\Delta u_s]$$

For any $k = 0,\dots,M$ we define

$$B_k \doteq \left\{y : |y - x_0| \le \epsilon^{-1/10} - k\epsilon^{-1/20}\right\} \tag{3.4.48}$$

$$D_{k,t} \doteq \sup_{x\in B_k} |d_t(x)| \tag{3.4.49}$$

We next prove that there are $c_1 > 0$, $c_2 > 0$ and for any n there is $c'_n > 0$ so that in $\hat{\mathcal{B}}_{\epsilon,p}$

$$D_{k,t} \le c_1\delta\left[1 + \frac{\epsilon^{1/2-a}}{\sqrt{t}}\right] + c_2\int_0^t ds\,\epsilon^{1/2-a}\,D_{k-1,s} + c'_n\epsilon^n \tag{3.4.50}$$

To prove (3.4.50) we first notice that by (3.4.36) and (3.4.40) there is $c > 0$ so that in $\hat{\mathcal{B}}_{\epsilon,p}$

$$\sup_{t\le t_\epsilon} \|A_t\|_{\epsilon,x_0} \le c\epsilon^{1/2-a} \tag{3.4.51}$$

Furthermore from (3.4.38) and (3.4.40) it follows that for any n there is c_n so that

$$\sqrt{\epsilon}\|\hat{Z}_{t,x_0} - \tau_\Delta\hat{Z}^*_{t,x_0^*}\|_{x_0,\epsilon} \le c_n\epsilon^n \tag{3.4.52}$$

By (3.4.45),

$$(g_{t,x_0}v)(x) - (\tau_\Delta g_{t,x_0^*}u)(x) = \left[g_{t,x_0}(v - \tau_\eta u)\right](x) + \int dy\,\tau_\eta u(y)\left[g_{t,x_0}(x,y) - g_{t,x_0}(x, y - \Delta + \eta)\right] \tag{3.4.53}$$

We then use (3.4.32), that $\|u\| \le 2\epsilon^{1/2-a}$ and (3.7.33) to conclude that for any $k = 0,..,M$

$$\sup_{x\in B_k}\left|g_{t,x_0}v(x) - \tau_\Delta g_{t,x_0^*}u(x)\right| \le c_1\delta\left[1 + \frac{\epsilon^{1/2-a}}{\sqrt{t}}\right] \tag{3.4.54}$$

Furthermore by Theorem 3.3.1 and §3.7, see (3.7.32) and (3.7.34), for any n there is $c_n > 0$ so that for all $k = 0,..,M$

$$\sup_{t\ge 0}\sup_{x\in\mathbb{R}}\int_{\mathbb{R}} dy\,g_{t,x_0}(x,y) \le c_0, \qquad \sup_{t>0}\sup_{x\in B_k}\int_{\mathbb{R}\setminus B_{k-1}} dy\,g_{t,x_0}(x,y) \le c_n\epsilon^n, \tag{3.4.55}$$

so (3.4.50) follows from (3.4.47), (3.4.51), (3.4.52), (3.4.54) and (3.4.55).

By Theorem 3.3.1

$$\|g_{t_\epsilon,x_0}v\| \le ce^{-\alpha t_\epsilon}, \qquad \|g_{t_\epsilon,x_0^\star}u\| \le ce^{-\alpha t_\epsilon} \tag{3.4.56}$$

By (3.4.47), (3.4.56) and (3.4.51) we have that, in the set $\hat{\mathcal{B}}_{\epsilon,p}$

$$D_{M,t_\epsilon} \le 2ce^{-\alpha t_\epsilon} + c_2 \int_0^{t_\epsilon} ds\, \epsilon^{1/2-a} D_{M-1,s} + c_n \epsilon^n$$

Therefore by iterating (3.4.50) we get

$$\begin{aligned} D_{M,t_\epsilon} &\le 2ce^{-\alpha t_\epsilon} \\ &+ \sum_{k=1}^{M} [c_2 \epsilon^{1/2-a}]^k \int_0^{t_\epsilon} ds_1 \cdots \int_0^{s_{k-1}} ds_k c_1 \delta \left(1 + \frac{\epsilon^{1/2-a}}{\sqrt{s_k}}\right) \\ &+ [c_2 \epsilon^{1/2-a}]^{M-1} \int_0^{t_\epsilon} ds_1 \cdots \int_0^{s_{M-1}} ds_M D_{0,s_M} + c_n \epsilon^n \end{aligned}$$

By the choice of M and since $D_{0,s_M} \le \epsilon^{1/2-a}$ we conclude the proof of Step 3.

STEP 4. Bounds away from the center.

We prove that there are c, p and ϵ_0 positive so that if $\epsilon < \epsilon_0$ and the processes m_t and $m_t^\star$ are in $\hat{\mathcal{B}}_{\epsilon,p}$, then

$$\sup_{\epsilon^{-1/10} - M\epsilon^{-1/20} \le |x-x_0| \le 2\epsilon^{-1/10}} |v_t(x) - \tau_\Delta u_t(x)| \le ce^{-t}\lambda, \qquad \text{for all } t \le t_\epsilon \tag{3.4.57}$$

We set

$$r_{\epsilon,M} = \epsilon^{-1/10} - M\epsilon^{-1/20}$$

and consider the case $x - x_0 \in [r_{\epsilon,M}, 2\epsilon^{-1/10}]$. The analysis of the other interval involved in the sup in (3.4.57) is similar and omitted. Given $V > 0$ we define

$$m^+(x) = \begin{cases} m(x) & \text{for } x - x_0 \in \big(r_{\epsilon,M} - Vt_\epsilon, 2\epsilon^{-1/10} + Vt_\epsilon\big) \\ 1 & \text{for } x - x_0 \in \big\{\big(-\infty, r_{\epsilon,M} - 2Vt_\epsilon\big) \cup \big(2\epsilon^{-1/10} + 2Vt_\epsilon, +\infty\big)\big\} \end{cases} \tag{3.4.58}$$

and complete the definition of m^+ in the missing intervals by linear interpolation. $\tilde{m}^+$ is defined similarly with m replaced by $m^\star$ and x_0 by $x_0^\star$.

We set

$$m_t^+ = T_t(m^+, \sqrt{\epsilon}Z), \qquad \tilde{m}_t^+ = T_t(\tilde{m}^+, \sqrt{\epsilon}Z^\star)$$

We choose V in (3.4.58) as the parameter entering in the barrier lemma (Proposition 5.3 of [11]). Then there is $c > 0$ so that in $\hat{\mathcal{B}}_{\epsilon,p}$ for all $t \le t_\epsilon$,

$$\sup_{x - x_0 \in [r_{\epsilon,M}, 2\epsilon^{-1/10}]} |m_t(x) - m_t^+(x)| \le ce^{-t} \tag{3.4.59}$$

The same bound holds for $m_t^\star(x) - \tilde{m}_t^+(x)$ when x_0 is replaced by $x_0^\star$. We define

$$v_t^+ = m_t^+ - 1, \qquad u_t^+ = \tilde{m}_t^+ - 1 \tag{3.4.60}$$

It is not difficult to prove the following a-priori bound for v_t^+ and u_t^+: for $p < a$ there is $c > 0$ so that

$$\|v_t^+\| \le 2\lambda c, \qquad \|u_t^+\| \le 2\lambda c, \qquad \text{for all } t \le t_\epsilon \tag{3.4.61}$$

We consider next the versions of v_t^+ and u_t^+ given by the corresponding solutions of the equations:

$$v_t^+ = e^{-2t} H_t \star v_0^+ + \int_0^t ds\, e^{-2(t-s)} H_{t-s} \star \big(- 3(v_s^+)^2 - (v_s^+)^3 \big) + \sqrt{\epsilon} V_t$$
$$u_t^+ = e^{-2t} H_t \star u_0^+ + \int_0^t ds e^{-2(t-s)} H_{t-s} \star \big(- 3(u_s^+)^2 - (u_s^+)^3 \big) + \sqrt{\epsilon} V_t^\star$$

where

$$V_t(x) \doteq \int_0^t ds \int_{\mathcal{T}_{\epsilon,\kappa}} dy\, e^{-2(t-s)} H_{t-s}^{(\epsilon)}(x,y)\, \alpha(s,y)$$
$$V_t^\star(x) \doteq \int_0^t ds \int dz\, \mathbf{1}_{\{|z+\Delta-x_0^\star| \le 4\epsilon^{-1/10};\; z+\Delta\in\mathcal{T}_{\epsilon,\kappa}\}}\, e^{-(t-s)}\, H_{t-s}^{(\epsilon)}(x, z+\Delta)\, \alpha(s,z)$$
$$+ \int_0^t ds \int dy\, \mathbf{1}_{\{|y-x_0^\star| > 4\epsilon^{-1/10};\; y\in\mathcal{T}_{\epsilon,\kappa}\}}\, e^{-(t-s)}\, H_{t-s}^{(\epsilon)}(x,y)\, \bar{\alpha}(s,y) \tag{3.4.62}$$

We call

$$d_t^+ \doteq \sup_{x-x_0\in(r_{\epsilon,M},\, 2\epsilon^{-1/10})} (v_t^+ - \tau_\Delta u_t^+)$$

From (3.4.61), we get that there is a constant $c' > 0$ such that

$$d_t^+ \le e^{-2t} 2\lambda c + c'\lambda \int_0^t ds\, e^{-2(t-s)} d_s^+ + \sqrt{\epsilon} \|V_t - \tau_\Delta V_t^\star\|_{\epsilon,x_0} \tag{3.4.63}$$

But $\|V_t - \tau_\Delta V_t^\star\|_{\epsilon,x_0} \le \|Z_t - \tau_\Delta Z_t^\star\|_{\epsilon,x_0}$, and, by (3.4.37), a bound like (3.4.52) holds also for $\|Z_t - \tau_\Delta Z_t^\star\|_{\epsilon,x_0}$. Then by (3.4.63) there is $c > 0$ so that

$$d_t^+ \le c e^{-2t}\lambda, \qquad \text{for all } t \le t_\epsilon \tag{3.4.64}$$

Since

$$v_t - \tau_\Delta u_t = (v_t^+ - \tau_\Delta u_t^+) + (m_t - m_t^+) - \tau_\Delta(m_t^\star - \tilde{m}_t^+)$$

from (3.4.64) and (3.4.59), we get that

$$\sup_{x-x_0\in(r_{\epsilon,M},\, 2\epsilon^{-1/10})} |v_t(x) - \tau_\Delta u_t(x)| \le c\big(e^{-2t}\lambda + e^{-t}\big), \qquad \text{for all } t \le t_\epsilon \tag{3.4.65}$$

STEP 5. Conclusion.

By (3.4.43) and (3.4.57) there are c and ϵ_0 so that if $\epsilon < \epsilon_0$ and the processes m_t and $m_t^\star \in \hat{\mathcal{B}}_{\epsilon,p}$, then

$$\sup_{|x-x_0|\le 2\epsilon^{-1/10}} |m_{t_\epsilon}(x) - \tau_\Delta m^\star_{t_\epsilon}(x)| = \sup_{|x-x_0|\le 2\epsilon^{-1/10}} |v_{t_\epsilon}(x) - \tau_\Delta u_{t_\epsilon}(x)| \le \delta\epsilon^{\bar{\gamma}} \tag{3.4.66}$$

We then observe that from Lemma 3.3.3 it follows that in $\hat{\mathcal{B}}_{\epsilon,p}$,

$$|\xi^{\epsilon,\kappa}(m_{t_\epsilon}) - x_0| \le c\epsilon^{1/2-a}$$

so that from (3.4.66) it follows that

$$\|m_{t_\epsilon} - \tau_\Delta m^\star_{t_\epsilon}\|_{\epsilon,\xi(m^{\epsilon,\kappa}_{t_\epsilon})} \le \delta\epsilon^{\bar{\gamma}} \tag{3.4.67}$$

By Lemma 3.3.3

$$|\xi^{\epsilon,\kappa}(m^\star_{t_\epsilon}) - \xi^{\epsilon,\kappa}(m_{t_\epsilon}) - \Delta| \le c_0 |\langle \bar{m}'_{\xi^{\epsilon,\kappa}(m_{t_\epsilon})}, (m_{t_\epsilon})^{\epsilon,\kappa} - \tau_\Delta (m^\star_{t_\epsilon})^{\epsilon,\kappa}\rangle| \tag{3.4.68}$$

By (3.4.67) and the exponential decay at infinity of $\bar{m}'_{\xi(m_{t_\epsilon})}$ there is $c > 0$ so that

$$|\xi^{\epsilon,\kappa}(m^\star_{t_\epsilon}) - \xi^{\epsilon,\kappa}(m_{t_\epsilon}) - \Delta| \le c\delta\epsilon^{\bar{\gamma}} \tag{3.4.69}$$

Since $\bar{\gamma} > \gamma$, we have thus proven that in $\hat{\mathcal{B}}_{\epsilon,p}$, with p small enough,

$$\|m_{t_\epsilon} - \tau_\Delta m^\star_{t_\epsilon}\|_{\epsilon,\xi^{\epsilon,\kappa}(m_{t_\epsilon})} \le \delta\epsilon^{\gamma}, \qquad |\xi^{\epsilon,\kappa}(m^\star_{t_\epsilon}) - \xi^{\epsilon,\kappa}(m_{t_\epsilon}) - \Delta| \le \delta\epsilon^{\gamma} \tag{3.4.70}$$

Proposition 3.4.5 is proved. □

As a corollary of the previous proposition we have a fast decay transversally to $\mathcal{M}$, as we are going to see. We define

$$\psi : C(\mathbb{R}) \longrightarrow C(\mathbb{R}) \qquad \text{such that} \qquad \psi(m)(x) \doteq m^{\epsilon,\kappa}(x + \xi^{\epsilon,\kappa}(m)) \tag{3.4.71}$$

Note that we omit in the expression of ψ the explicit dependence on ϵ and κ.

3.4.6 Corollary.

Let $\kappa \ge 1$, $\ell \in (0,1)$, $a \in (0,1/2)$, $m, m^\star \in M^\epsilon_{\kappa,\ell,\epsilon^{1/2-a}}$. We call

$$\psi \doteq \psi(T_{t_\epsilon}(m;\sqrt{\epsilon}Z)), \qquad \psi^\star \doteq \psi(T_{t_\epsilon}(m^\star;\sqrt{\epsilon}Z))$$

Then for any positive integer N we can construct $T_t(m;\sqrt{\epsilon}Z)$ and $T_t(m^\star;\sqrt{\epsilon}Z)$ in the same probability space and such that there are $c > 0$, and $p > 0$ so that

$$P^\epsilon(\|\psi - \psi^\star\|_\epsilon \le \epsilon^N) \ge 1 - ce^{-\epsilon^{-p}} \tag{3.4.72}$$

Proof.

Let $\gamma \in (0, 1/2 - a)$, $n > 1$, $m_t \doteq T_t(m, \sqrt{\epsilon}Z)$ and $m_t^\star \doteq T_t(m^\star, \sqrt{\epsilon}Z)$ constructed as in Proposition 3.4.5. By iterating (3.4.30) n times we get (for a constant c possibly different from that in (3.4.30))

$$P^\epsilon\big(D_\epsilon(m_{t_\epsilon}, m_{t_\epsilon}^\star) \le \epsilon^{1/2-a}\epsilon^{n\gamma}\big) \ge 1 - ce^{-\epsilon^{-p}} \tag{3.4.73}$$

Then, see right after Definition 3.4.4, there is η such that (we bound $\epsilon^{1/2-a}$ with 1)

$$P^\epsilon\Big(|\eta - \Delta| \le \epsilon^{n\gamma}, \quad \|m_{t_\epsilon} - \tau_\eta m_{t_\epsilon}^\star\|_{\epsilon, x_0} \le \epsilon^{n\gamma}\Big) \ge 1 - ce^{-\epsilon^{-p}} \tag{3.4.74}$$

where $\Delta = x_0^\star - x_0$, with $x_0 = \xi(m_{t_\epsilon}^{\epsilon,\kappa})$ and $x_0^\star = \xi((m_{t_\epsilon}^\star)^{\epsilon,\kappa})$. We note that

$$\|\psi - \psi^\star\|_\epsilon = \|m_{t_\epsilon} - \tau_\Delta m_{t_\epsilon}^\star\|_{\epsilon,x_0} \le \|m_{t_\epsilon} - \tau_\eta m_{t_\epsilon}^\star\|_{\epsilon,x_0} + \|\tau_{\eta-\Delta} m_{t_\epsilon}^\star - m_{t_\epsilon}^\star\|_{\epsilon,x_0^\star} \tag{3.4.75}$$

In §3.7, see Lemma 3.7.3, it is proved the following property of the Ginzburg-Landau process. For any $\delta > 0$, $\alpha \in (0, 1/2)$ there is a constant $c > 0$ so that for any ϵ small enough

$$P^\epsilon\left(\sup_{|x|,|y|\le\epsilon^{-\kappa}} \sup_{\substack{x\neq y\\ |x-y|\le 1}} \frac{|m_{t_\epsilon}(x) - m_{t_\epsilon}(y)|}{|x-y|^\alpha} > \delta\right) \le e^{-c\delta^2} \tag{3.4.76}$$

Applying the previous inequality to $m_t^\star$, we obtain

$$\begin{aligned} &P^\epsilon\left(\sup_{|x|,|y|\le\epsilon^{-\kappa}} \sup_{\substack{x\neq y\\ |x-y|\le|\eta-\Delta|}} \frac{|m_{t_\epsilon}^\star(x) - m_{t_\epsilon}^\star(y)|}{|x-y|^{1/4}} > 1\right) \le \\ &P^\epsilon\left(\sup_{|x|,|y|\le\epsilon^{-\kappa}} \sup_{\substack{x\neq y\\ |x-y|\le 1}} \frac{|m_{t_\epsilon}^\star(x) - m_{t_\epsilon}^\star(y)|}{|x-y|^{3/8}} > |\eta-\Delta|^{-1/8}\right) \le e^{-c|\eta-\Delta|^{-1/4}} \end{aligned} \tag{3.4.77}$$

By (3.4.74), (3.4.75) and (3.4.77), and choosing n such that $\epsilon^{n\gamma/4} < \epsilon^N$ we then derive (3.4.72). Corollary 3.4.6 is proved. □

Corollary 3.4.6 proves that two processes that start from different data become almost equal modulo translations, with probability going to 1 as $\epsilon \to 0^+$ as fast as in (3.4.72). We improve this result in the next proposition in the sense that we have a similar statement without translations. The price we pay is twofold. The processes must start from data with linear centers close to each other and, more importantly, the rate of convergence is not as fast as before. In particular it is not fast enough for what needed in §3.5, where we prove Theorem 3.2.2. In that case we use Corollary 3.4.6 as we can reduce the analysis to events invariant under translations. This is no longer possible when proving Theorem 3.2.3, where however we need only convergence in probability, without bounds on the rate of convergence: for that Proposition 3.4.7 below will suffice.

3.4.7 Proposition.

Let $a \in (0,1/2)$, $b \geq 1-a$, $\gamma \in (0,1/2-a)$, m and $\tilde{m}$ both in $M^{\epsilon}_{\kappa,\ell,\epsilon^{1/2-a}}$ and

$$\|m-\tilde{m}\| \leq \epsilon^b, \qquad |\xi^{\epsilon,\kappa}(m)-\xi^{\epsilon,\kappa}(\tilde{m})| \leq \epsilon^b \tag{3.4.78}$$

Then we can construct the Ginzburg-Landau processes $\{m_t\}_{t\geq 0}$ and $\{\tilde{m}_t\}_{t\geq 0}$ starting respectively from m and $\tilde{m}$ in the same probability space and so that

$$\lim_{\epsilon\to 0+} P^{\epsilon}\Big(\|(m_{t_\epsilon})^{\epsilon,\kappa}-(\tilde{m}_{t_\epsilon})^{\epsilon,\kappa}\| \leq \epsilon^{b+\gamma}, \quad |\xi^{\epsilon,\kappa}(m_{t_\epsilon})-\xi^{\epsilon,\kappa}(\tilde{m}_{t_\epsilon})| \leq \epsilon^{b+\gamma}\Big) = 1 \tag{3.4.79}$$

Proof.

We set $\delta \doteq \epsilon^b$ and $\lambda \doteq \epsilon^{1/2-a}$.

We consider first the case when m and $\tilde{m}$ are not in $C_{\epsilon,\kappa}(\mathbb{R})$. Let $x_0 \doteq \xi(m)$, $\tilde{x}_0 \doteq \xi(\tilde{m})$ and $t^\star_\epsilon \doteq \epsilon^q t_\epsilon$, $q>0$ will be specified later. Let $\{e^{(\epsilon)}_j\}_{j\geq 1}$ be an orthonormal basis of $L^2(\mathcal{T}_{\epsilon,\kappa})$, such that $e^{(\epsilon)}_1 = \sqrt{D_\epsilon}\tilde{m}'_{x_0}$ on $\mathcal{T}_{\epsilon,\kappa}$, D_ϵ the normalization constant, ($D_\epsilon \to D = 3/4$ as $\epsilon \to 0^+$). Set

$$\langle f,g\rangle_{\epsilon,\kappa} \doteq \int_{\mathcal{T}_{\epsilon,\kappa}} dx\, f(x)g(x), \qquad f,g \in L^2(\mathcal{T}_{\epsilon,\kappa}). \tag{3.4.80}$$

Let $\{b_j(t)\}_{j\geq 0}$ be a family of standard independent brownian motions, and consider the Gaussian process

$$\hat{Z}^{(1)}_{t,x_0}(x) \doteq \int_0^t db_1(s) \int_{\mathcal{T}_{\epsilon,\kappa}} dy\, g^{(\epsilon)}_{t-s,x_0}(x,y)\, e^{(\epsilon)}_1(y) + R_t(x) \tag{3.4.81}$$

where

$$R_t(x) \doteq \sum_{j\geq 2} \int_0^t db_j(s) \int_{\mathcal{T}_{\epsilon,\kappa}} dy\, g^{(\epsilon)}_{t-s,x_0}(x,y)\, e^{(\epsilon)}_j(y), \tag{3.4.82}$$

and $g^{(\epsilon)}_{t-s,x_0}$ is defined in (3.4.7) . By comparing covariances, it is easy to check that $\hat{Z}^{(1)}_{t,x_0}$ has the same law of the process $\hat{Z}_{t,x_0}$ defined in (3.4.6). We will construct another Gaussian process $\hat{Z}^{(2)}_{t,x_0}$ with the same law. Consider, for y_0 that will be conveniently chosen later, the process

$$\tilde{b}_1(t) \doteq \begin{cases} b_0(t) & \text{for any } t \in [0,\tau] \\ b_1(t) - y_0 & \text{for any } t \geq \tau \end{cases} \tag{3.4.83}$$

where

$$\tau \doteq \inf\{\, t \geq 0 \,:\, [b_1(t) - b_0(t)] = y_0 \,\} \tag{3.4.84}$$

The process $\tilde{b}_1(t)$ is a Brownian motion, independent of $\{b_j(t)\}_{j\geq 2}$. Finally, let

$$\hat{Z}^{(2)}_{t,x_0}(x) \doteq \int_0^t d\tilde{b}_1(s) \int_{\mathcal{T}_{\epsilon,\kappa}} dy\, g^{(\epsilon)}_{t-s,x_0}(x,y)\, e^{(\epsilon)}_1(y) + R_t(x). \tag{3.4.85}$$

Write the integral equations (3.4.4) for m_t and $\tilde{m}_t$ as in the statement of the proposition, using the Gaussian processes $\hat{Z}^{(1)}_{t,x_0}$ and $\hat{Z}^{(2)}_{t,x_0}$ respectively. Then,

$$m_\tau - \tilde{m}_\tau = g_{\tau,x_0}(m - \tilde{m}) - \int_0^\tau ds\, g_{\tau-s,x_0}\big(A_{x_0}[v_s] - A_{x_0}[u_s]\big) \\ + \sqrt{\epsilon}\int_0^\tau d(b_1 - b_0)(s) \int_{\mathcal{T}_{\epsilon,\kappa}} dy\, g^{(\epsilon)}_{t-s,x_0}(x,y)\, e^{(\epsilon)}_1(y), \tag{3.4.86}$$

where $A_{x_0}[f] \doteq 3\bar{m}_{x_0} f^2 + f^3$, $v_t \doteq m_t - \bar{m}_{x_0}$, $u_t \doteq \tilde{m}_t - \bar{m}_{x_0}$. We multiply both sides by $\bar{m}'_{x_0}$ and integrate over $\mathbb{R}$. We get, in $\{\tau \le t^\star_\epsilon\}$:

$$\langle \bar{m}'_{x_0}, (m_\tau - \tilde{m}_\tau)\rangle = -\langle \bar{m}'_{x_0}, \tilde{m}\rangle - \int_0^\tau ds\, \langle \bar{m}'_{x_0}, (A_{x_0}[v_s] - A_{x_0}[u_s])\rangle \\ + \sqrt{\epsilon}\int_0^\tau d(b_1 - b_0)(s)\, \langle a_\epsilon, e^{(\epsilon)}_1\rangle_{\epsilon,\kappa}, \tag{3.4.87}$$

where

$$a_\epsilon(y) \doteq \langle g^{(\epsilon)}_{\tau-s,x_0}(\cdot,y), \bar{m}'_{x_0}\rangle = \sum_{j\in\mathbb{Z}} \Big(\bar{m}'_{x_0}(x, y + 4j\epsilon^{-\kappa}) + \bar{m}'_{x_0}(x, 4j\epsilon^{-\kappa} + 2\epsilon^{-\kappa} - y)\Big)$$

Choosing now $y_0 = \langle \bar{m}'_{x_0}, \tilde{m}\rangle(\sqrt{\epsilon}\langle a_\epsilon, e^{(\epsilon)}_1\rangle_{\epsilon,\kappa})^{-1}$, from the definition of τ we then get

$$\langle \bar{m}'_{x_0}, m_\tau - \tilde{m}_\tau\rangle = \int_0^\tau ds\, \langle \bar{m}'_{x_0}, A_{x_0}[v_s] - A_{x_0}[u_s]\rangle \tag{3.4.88}$$

By standard results on Brownian motions and since $\lim_{\epsilon\to 0^+}\langle a_\epsilon, e^{(\epsilon)}_1\rangle_{\epsilon,\kappa} = \sqrt{D}$, there is $c > 0$ so that

$$P^\epsilon(\tau \le t^\star_\epsilon) \ge 1 - c\frac{|\langle \bar{m}'_{x_0}\tilde{m}\rangle|}{\sqrt{\epsilon t^\star_\epsilon}} \ge 1 - c\epsilon^{1-a-1/2-q/2}$$

By choosing $q < 1 - 2a$,

$$\lim_{\epsilon\to 0^+} P^\epsilon(\tau \le t^\star_\epsilon) = 1 \tag{3.4.89}$$

The set

$$\mathcal{G} \doteq \Big\{\sup_{t\le t_\epsilon}\Big(\|v_t\| + \|u_t\|\Big) \le 2\delta\Big\}$$

has, by Proposition 3.4.3, probability that goes to 1 as $\epsilon \to 0^+$. There is $c_2 > 0$ so that in $\mathcal{G}$

$$\sup_{0\le t\le t^\star_\epsilon} \|m_t - \tilde{m}_t\| \le c_2\delta \tag{3.4.90}$$

Let $\bar{\gamma} \in (\gamma, 1/2 - a)$, then in $\{\tau \le t^\star_\epsilon\} \cap \mathcal{G}$ and for all $\epsilon > 0$ small enough

$$|\langle \bar{m}'_{x_0}, m_\tau - \tilde{m}_\tau\rangle| \le \delta\epsilon^{\bar{\gamma}} \tag{3.4.91}$$

By Lemma 3.3.3 there is $c_3 > 0$ so that in the same set

$$|\xi(m_\tau) - \xi(\tilde{m}_\tau)| \le c_3 \delta \epsilon^{\bar{\gamma}} \tag{3.4.92}$$

Finally, using the integral equation (3.4.4) in the time interval $[\tau, t^\star_\epsilon]$, by (3.4.90) and (3.4.91), there is $c_4 > 0$ so that in $\{\tau \le t^\star_\epsilon\} \cap \mathcal{G}$

$$|\xi(m_{t^\star_\epsilon}) - \xi(\tilde{m}_{t^\star_\epsilon})| \le c_4 \delta \epsilon^{\bar{\gamma}} \tag{3.4.93}$$

We next consider the time interval $[t^\star_\epsilon, t_\epsilon]$. Let $x^\star_\epsilon \doteq \xi(m_{t^\star_\epsilon})$. We set, for any $t \in [0, \bar{t}_\epsilon]$, $\bar{t}_\epsilon \doteq (1 - \epsilon^q) t_\epsilon$,

$$v^\star_t \doteq m_{t^\star_\epsilon + t} - \bar{m}_{x^\star_\epsilon}, \qquad u^\star_t \doteq \tilde{m}_{t^\star_\epsilon + t} - \bar{m}_{x^\star_\epsilon} \tag{3.4.94}$$

We write (3.4.4) for v_t and u_t relatively to $\bar{m}_{x^\star_\epsilon}$. Setting $\Delta_\star = x^\star_\epsilon - \xi(\tilde{m}_{t^\star_\epsilon})$,

$$g_{\bar{t}_\epsilon, x^\star_\epsilon}(v^\star_0 - u^\star_0) = g_{\bar{t}_\epsilon, x^\star_\epsilon}(m_{t^\star_\epsilon} - \tilde{m}_{t^\star_\epsilon}) = g_{\bar{t}_\epsilon, x^\star_\epsilon}(m_{t^\star_\epsilon} - \tau_{\Delta_\star} \tilde{m}_{t^\star_\epsilon}) + \int dy \left[g_{\bar{t}_\epsilon, x^\star_\epsilon}(x, y - \Delta_\star) - g_{\bar{t}_\epsilon, x^\star_\epsilon}(x, y) \right] \tilde{m}_{t^\star_\epsilon}(y) \tag{3.4.95}$$

Then Theorem 3.3.1, (3.4.93) and (3.7.33) imply that there are c_5 and c_6 so that

$$\lim_{\epsilon \to 0+} P^\epsilon \Big(\| g_{\bar{t}_\epsilon, x^\star_\epsilon}(v^\star_0 - u^\star_0) \| \le c_5 e^{-\alpha \bar{t}_\epsilon} + c_6 \delta \epsilon^{\bar{\gamma}} \Big) = 1 \tag{3.4.96}$$

Using (3.4.96) and the integral equations for $v^\star_t$ and $u^\star_t$, we get, for some constant c_7,

$$\lim_{\epsilon \to 0+} P^\epsilon \Big(\| v^\star_{\bar{t}_\epsilon} - u^\star_{\bar{t}_\epsilon} \| \le c_7 \delta \epsilon^{\bar{\gamma}} \Big) = 1 \tag{3.4.97}$$

By (3.4.97), using Lemma 3.3.3, we have also, for some constant c_8,

$$\lim_{\epsilon \to 0+} P^\epsilon \Big(|\xi(m_{t_\epsilon}) - \xi(\tilde{m}_{t_\epsilon})| \le c_8 \delta \epsilon^{\bar{\gamma}} \Big) = 1 \tag{3.4.98}$$

and so, since $\bar{\gamma} > \gamma$, by (3.4.97) and (3.4.98) we finally obtain

$$\lim_{\epsilon \to 0+} P^\epsilon \Big(\| m_{t_\epsilon} - \tilde{m}_{t_\epsilon} \| \le \delta \epsilon^\gamma, \quad |\xi(m_{t_\epsilon}) - \xi(\tilde{m}_{t_\epsilon})| \le \delta \epsilon^\gamma \Big) = 1 \tag{3.4.99}$$

This proves the proposition for $m, \tilde{m} \notin C_{\epsilon,\kappa}(\mathbb{R})$. By using conveniently the barrier lemma, one easily extends the result to the general case. We omit the details. □

When proving Theorem 3.2.3 we will consider the case when at some time T we have two data, m_T and $\tilde{m}_T$, both in $C_{\epsilon,\kappa}(\mathbb{R})$ and in $M^\epsilon_{\kappa,\ell,\epsilon^{1/2-a}}$, and such that $\xi^{\epsilon,\kappa}(m_T) = \xi^{\epsilon,\kappa}(\tilde{m}_T)$, see the end of §3.6. This case is not directly covered by Proposition 3.4.7, but we will see in the following lemma that if we construct with the same noise the two processes then they will verify with large probability the conditions of Proposition 3.4.7 after a time t_ϵ.

3.4.8 Lemma.

Let m and $\tilde{m}$ both in $M^{\epsilon}_{\kappa,\ell,\epsilon^{1/2-a}}$, $a \in (0,1/2)$, with $x_0 \doteq \xi^{\epsilon,\kappa}(m) = \xi^{\epsilon,\kappa}(\tilde{m})$ and let $p \in (0,a)$. Then for any $\omega \in (2a,1/2)$ and any ϵ small enough, in $\mathcal{B}_{p,\epsilon,x_0}$ the following estimate holds:

$$\|T_{t_\epsilon}(m,\sqrt{\epsilon}Z) - T_{t_\epsilon}(\tilde{m},\sqrt{\epsilon}Z)\| \leq \epsilon^{1-\omega}$$

Proof.

Writing the integral equation (3.4.4) around $\bar{m}_{x_0}$ for the two processes one obtains an equation for the difference like (3.4.47) but with the same center and noises. Then the estimate follows easily. □

We conclude the section with two lemmas consequence of general properties of the Ginzburg-Landau process. For any $a \in (0,1/2)$ and any $\phi \in \mathcal{M}_{\epsilon^{1/2-a}}$, let us denote by E^{ϵ}_{ϕ} the expectation with respect to the Ginzburg–Landau process starting from ϕ. We indicate with m_t the coordinate map on $C(\mathbb{R}_+;C(\mathbb{R}))$ and let $\xi_t = \xi^{\epsilon,\kappa}(m_t)$.

3.4.9 Lemma.

For any $t,s \geq 0$,

$$E^{\epsilon}_{\bar{m}}[E^{\epsilon}_{\psi(m_s)}[\xi_t]] = 0 \tag{3.4.100}$$

where ψ is defined in (3.4.71).

Proof.

Consider the symmetry transformation $\mathcal{R}: C(\mathbb{R}) \to C(\mathbb{R})$ defined by $(\mathcal{R}\phi)(x) \doteq -\phi(-x)$. For any $m \in \mathcal{M}_{\epsilon^{1/2-a}}$, $t \geq 0$, $T_t(m,\sqrt{\epsilon}Z) = \mathcal{R}T_t(\mathcal{R}m,\sqrt{\epsilon}\mathcal{R}\underline{Z})$. Since $\bar{m} = \mathcal{R}\bar{m}$, for any bounded functional F on $C(\mathbb{R}_+;C(\mathbb{R}))$, $E^{\epsilon}_{\bar{m}}[F] = E^{\epsilon}_{\bar{m}}[\mathcal{R}F]$. In particular the law of $\psi(m_s)$ is symmetric with respect to $\mathcal{R}$ so that

$$E^{\epsilon}_{\bar{m}}[E^{\epsilon}_{\psi(m_s)}[\xi_t]] = \frac{1}{2}E^{\epsilon}_{\bar{m}}\left[E^{\epsilon}_{\psi(m_s)}[\xi_t] + E^{\epsilon}_{\mathcal{R}\psi(m_s)}[\xi_t]\right] \tag{3.4.101}$$

On the other hand, by symmetry, if $\xi(m) = 0$ then $E^{\epsilon}_{m}[\xi_t] = -E^{\epsilon}_{\mathcal{R}m}[\xi_t]$. The lemma is proved. □

3.4.10 Lemma.

Let $a \in (0,1/4)$ and $m \in \mathcal{M}_{\epsilon^{1/2-a}}$ such that $\xi(m) = 0$. Then

$$\lim_{\epsilon \to 0+} E^{\epsilon}_{m}[(\epsilon t_\epsilon)^{-1}\xi^2_{t_\epsilon}] = D, \qquad D = 3/4 \tag{3.4.102}$$

Proof.

Given $p \in (0,a)$ let $\mathcal{D}_p$ be the nice set where $\sup_{0\le t\le t_\epsilon} \|\sqrt{\epsilon}\hat{Z}_{t,0}\| \le \epsilon^{1/2-p}$. By Lemma 3.3.3, for small ϵ, in this set $\bar{m} + \sqrt{\epsilon}\hat{Z}_{t_\epsilon,0}$ has a unique linear center ξ_t^Z and it holds

$$|\xi^Z_{t_\epsilon} - \langle \bar{m}', \bar{m} + \sqrt{\epsilon}\hat{Z}_{t_\epsilon,0}\rangle| \le c_0 |\langle \bar{m}', \bar{m} + \sqrt{\epsilon}\hat{Z}_{t_\epsilon,0}\rangle|^2 \tag{3.4.103}$$

Moreover, in $\mathcal{D}_p$,

$$|\langle \bar{m}', \bar{m} + \sqrt{\epsilon}\hat{Z}_{t_\epsilon,0}\rangle| \le \|\sqrt{\epsilon}\hat{Z}_{t_\epsilon,0}\| \le \epsilon^{1/2-p} \tag{3.4.104}$$

On the other hand, looking at the integral version of the Ginzburg-Landau equation, since m is orthogonal to $\bar{m}'$, one easily obtains that, in $\mathcal{D}_p$,

$$\|m_{t_\epsilon} - (\bar{m} + \sqrt{\epsilon}\hat{Z}_{t_\epsilon,0})\| \le t_\epsilon \epsilon^{1/2-a} \tag{3.4.105}$$

and then, by Lemma 3.3.3,

$$|\xi_{t_\epsilon} - \xi^Z_{t_\epsilon}| \le c_0 t_\epsilon \epsilon^{1-2a} \tag{3.4.106}$$

In the proof of Theorem 3.5.2 we will use the following consequence of (3.4.106); in $\mathcal{D}_p$ one has

$$|\xi_{t_\epsilon} - \bar{\xi}_{t_\epsilon}| \le c_0 t_\epsilon \epsilon^{1-2a} \tag{3.4.107}$$

where $\bar{\xi}_{t_\epsilon} = \xi(T_{t_\epsilon}(\bar{m}, \sqrt{\epsilon}Z))$.

Since $P^\epsilon(\mathcal{D}_p)$ converge to 1 as $\epsilon \to 0^+$ faster than any power of ϵ, from (3.4.103), (3.4.104) and (3.4.106),

$$\lim_{\epsilon\to 0^+} E^\epsilon_m[(\epsilon t_\epsilon)^{-1}\xi^2_{t_\epsilon}] = \lim_{\epsilon\to 0^+} E^\epsilon_m[(\epsilon t_\epsilon)^{-1}|\langle \bar{m}', \sqrt{\epsilon}\hat{Z}_{t_\epsilon,0}\rangle|^2] \tag{3.4.108}$$

But one easily compute

$$E^\epsilon_m[(\epsilon t_\epsilon)^{-1}|\langle \bar{m}', \sqrt{\epsilon}\hat{Z}_{t_\epsilon,0}\rangle|^2] = |\langle \bar{m}', e_1^{(\epsilon)}\rangle|^2 \tag{3.4.109}$$

where $e_1^{(\epsilon)}$ is defined in the proof of Proposition 3.4.7 (here is considered as an element of $C_{\epsilon,\kappa}(\mathbb{R})$). The lemma follows then from (3.4.108) and (3.4.109) since $\lim_{\epsilon\to 0^+}\langle \bar{m}', e_1^{(\epsilon)}\rangle = \sqrt{D}$. □

§3.5 Convergence to a Brownian motion.

In this section we prove that the linear center $\xi(m_t)$, suitably normalized, converges to a Brownian motion.

Let $a \in (0, 1/2)$, $\kappa \ge 1$, $\epsilon > 0$, and

$$\mathcal{X} \doteq \left\{\psi \in C(\mathbb{R}) : \xi^{\epsilon,\kappa}(\psi) = 0\right\}; \qquad \mathcal{X}_{\epsilon,a} \doteq \mathcal{X} \cap \mathcal{M}_{\epsilon^{1/2-a}} \tag{3.5.1}$$

As in the previous section, set $t_\epsilon = (\log \epsilon)^2$. We consider an auxiliary Markov chain $(x_n, \psi_n)_{n\in\mathbb{N}}$ with state space $\mathbb{R} \times \mathcal{X}$. We denote by $\mathbb{P}^\epsilon_{(x_n,\psi_n)}$ its transition probabilities, given by:

$$\mathbb{P}^\epsilon_{(x_n,\psi_n)}\Big((x_{n+1}, \psi_{n+1}) = (x_n, \psi_n)\Big) = 1 \quad \text{if } \psi_n \notin \mathcal{X}_{\epsilon,a}$$

If instead $\psi_n \in \mathcal{X}_{\epsilon,a}$ we define, for any given B and A Borel sets in $\mathbb{R}$ and $C^0(\mathbb{R})$ respectively,

$$\mathbb{P}^\epsilon_{(x_n,\psi_n)}\Big((x_{n+1}, \psi_{n+1}) \in B \times A\Big) = P^\epsilon\Big(x_n + \theta \in B, \tau_\theta m_{t_\epsilon} \in A\Big)$$

where

$$m_t \doteq T_t(\psi_n; \sqrt{\epsilon}Z), \quad \theta \doteq \xi^{\epsilon,\kappa}(m_{t_\epsilon})$$

We next introduce some stopping times: for $r > 0$ we set

$$t^0(r) \doteq \inf\{n \in \mathbb{N} : |x_n| \geq r\} \tag{3.5.2}$$

and for $\epsilon > 0$ and $a \in (0, 1/2)$

$$t_{\epsilon,a} \doteq \inf\{n \in \mathbb{N} : \psi_n \notin \mathcal{X}_{\epsilon,a}\} \tag{3.5.3}$$

Finally we define the stopping time $s_{\epsilon,\kappa}(\zeta)$, $\epsilon > 0$, $\kappa \geq 1$, $\zeta > 0$, on $(\mathbb{R} \times \mathcal{X}_\epsilon)^{\mathbb{N}} \times C(\mathbb{R}_+ \times \mathbb{R})$ (i.e. the product of the Markov chain and the Ginzburg-Landau process) in the following way

$$s_{\epsilon,\kappa}(\zeta) \doteq \inf\Big\{n \in \mathbb{N} : |x_n - \xi^{\epsilon,\kappa}(m_{nt_\epsilon})| + \|\psi_n(x) - \tau_{x_n} m_{nt_\epsilon}\|_\epsilon \geq \zeta\Big\} \tag{3.5.4}$$

The seminorm $\|\cdot\|_\epsilon$ is defined in (3.4.28). In the above definitions the stopping times are set equal to $+\infty$ if the sets on the right hand side are empty.

In the next proposition we indeed construct the original Ginzburg-Landau process and the auxiliary Markov chain in the same probability space, and prove lower bounds on $s_{\epsilon,\kappa}(\zeta)$ thus showing that they are close.

3.5.1 Proposition.

Let $\ell \in (0,1)$, $a \in (0, 1/2)$, $\kappa \geq 1$, $h > 0$, $q > 0$. Then there is $c > 0$ so that the following holds. Let $\epsilon > 0$, $m \in C^0(\mathbb{R})$ with $m^{\epsilon,\kappa} \in \mathcal{M}_{\epsilon^{1/2-a}}$,

$$x_0 \doteq \xi(m^{\epsilon,\kappa}), \quad \psi_0(x) \doteq m^{\epsilon,\kappa}(x + x_0), \quad |x_0| < \epsilon^{-\kappa} - \ell\epsilon^{-1} \tag{3.5.5}$$

Then we can construct the Ginzburg-Landau process m_t (that starts from m) and the Markov chain (that starts from (x_0, ψ_0)) in the same probability space so that

$$P^\epsilon\Big(s_{\epsilon,\kappa}(\epsilon^q) \geq t^0(\epsilon^{-\kappa} - \ell\epsilon^{-1}) \wedge \epsilon^{-h} \; ; \; t_{\epsilon,a} \geq \epsilon^{-h}\Big) \geq 1 - c\epsilon^q \tag{3.5.6}$$

Proof.

The proof follows by applying iteratively Corollary 3.4.6 and Proposition 3.4.3. □

We next study the Markov chain (x_n, ψ_n) and prove convergence to a Brownian motion. We set $z_0 \doteq 0$ and for $n \geq 1$

$$z_n \doteq \frac{x_n - x_{n-1}}{\sqrt{\epsilon t_\epsilon}} \tag{3.5.7}$$

We then define

$$Z_n \doteq \sum_{i=0}^{n} z_i = \frac{x_n - x_0}{\sqrt{\epsilon t_\epsilon}} \tag{3.5.8}$$

and given N (eventually we let $N = N(\epsilon)$ and $N(\epsilon) \to +\infty$)

$$X(t) \doteq \frac{1}{\sqrt{N}} Z_n \qquad t = n/N \tag{3.5.9}$$

We finally extend $X(t)$ to $t \in \mathbb{R}_+$ by linear interpolation.

3.5.2 Theorem.

Let $h > 0$, $x_0 \in \mathbb{R}$, $a \in (0, 1/4)$ and $\psi_0 \in \mathcal{X}_{\epsilon,a}$. Let $\mathbb{P}^\epsilon$ be the law on $C(\mathbb{R}_+)$ of the process $X(t)$ induced via (3.5.9) with $N \doteq [\epsilon^{-h}]$ by the Markov chain that starts from (x_0, ψ_0). Then $\mathbb{P}^\epsilon$ converges weakly on the compacts to P as $\epsilon \to 0^+$, where P is the law of a Brownian motion starting from 0 with diffusion equal to 3/4.

Proof.

Without loss of generality we may restrict to $t \in [0,1]$. Tightness on $C([0,1])$ follows from the existence of $c > 0$ for which

$$\mathbb{E}^\epsilon \Big(\sup_{n \leq N} Z_n^2 \Big) \leq cN \tag{3.5.10}$$

$$\mathbb{E}^\epsilon \Big([Z_{n_3} - Z_{n_2}]^2 [Z_{n_2} - Z_{n_1}]^2 \Big) \leq c(n_3 - n_1)^2, \qquad \text{for all } 1 \leq n_1 < n_2 < n_3 \leq N \tag{3.5.11}$$

see [5].

We first prove (3.5.10) and (3.5.11), then a martingale relation for the limit laws that will identify the law P of the theorem.

We call $\mathcal{F}_n$, $n \in \mathbb{N}$, the σ-algebra generated by the coordinates (x_i, ψ_i), $0 \leq i \leq n$, of the Markov chain and denote by $\mathbb{E}_n^\epsilon$, $n \in \mathbb{N}$, the conditional expectation given $\mathcal{F}_n$ (sometimes we write more explicitly $\mathbb{E}^\epsilon_{(x_n, \psi_n)}$). We set

$$\gamma_{1,n} \doteq \mathbb{E}_n^\epsilon (z_{n+1}), \quad \gamma_{1,n}^\star \doteq \mathbb{E}_n^\epsilon (\gamma_{1,n+1}) \quad \text{for } n \geq 0 \text{ and } \gamma_{1,-1}^\star \doteq \gamma_{1,0} \tag{3.5.12}$$

We then have for $n \geq 1$

$$Z_n = \Gamma_{1,n-2}^\star + M_{n-1}^\star + M_n \tag{3.5.13}$$

where

$$\Gamma^\star_{1,n} \doteq \sum_{i=-1}^{n} \gamma^\star_{1,i}, \quad n \geq -1 \tag{3.5.14}$$

$$M^\star_n \doteq \sum_{i=1}^{n} [\gamma_{1,i} - \gamma^\star_{1,i-1}], \quad n \geq 1, \quad M^\star_0 \doteq 0 \tag{3.5.15}$$

$$M_n \doteq \sum_{i=1}^{n} [z_i - \gamma_{1,i-1}], \quad n \geq 1, \quad M_0 \doteq 0 \tag{3.5.16}$$

Observe that $M^\star_n$ and M_n are $\mathcal{F}_n$-martingales. The usual semimartingale representation for Z_n in terms of the compensators $\gamma_{1,i}$ and M_n is not useful in the present context: the time delay in the definition of the compensators $\gamma^\star_{1,i}$ allows in fact to exploit the relaxation properties of the Ginzburg-Landau process stated in Corollary 3.4.6.

The semimartingale representation of M_n^2 is

$$M_n^2 = \Gamma_{2,n-1} + N_n, \quad n \geq 1 \tag{3.5.17}$$

$$\Gamma_{2,n} \doteq \sum_{i=0}^{n} \gamma_{2,i}, \quad \gamma_{2,n} \doteq \mathbb{E}^\epsilon_n\left((z_{n+1} - \gamma_{1,n})^2\right), \quad n \geq 0 \tag{3.5.18}$$

where we set $N_0 = 0$ and for $n \geq 1$

$$N_n \doteq 2 \sum_{1 \leq j < i \leq n} [z_i - \gamma_{1,i-1}][z_j - \gamma_{1,j-1}] + \sum_{i=1}^{n} [(z_i - \gamma_{1,i-1})^2 - \gamma_{2,i-1}] \tag{3.5.19}$$

is a $\mathcal{F}_n$ martingale. For $(M^\star_n)^2$ we have

$$(M^\star_n)^2 = \Gamma^\star_{2,n-1} + N'_n + N''_n \tag{3.5.20}$$

$$\Gamma^\star_{2,n} \doteq \sum_{i=0}^{n} \gamma^\star_{2,i}, \quad \gamma^\star_{2,n} \doteq \mathbb{E}^\epsilon_n\left((\gamma_{1,n+1} - \gamma^\star_{1,n})^2\right), \quad n \geq 0 \tag{3.5.21}$$

$$N'_n \doteq \sum_{i=0}^{n-1} \left[(\gamma_{1,i+1} - \gamma^\star_{1,i})^2 - \gamma^\star_{2,i}\right], \quad n \geq 1, \quad N'_0 = 0 \tag{3.5.22}$$

$$N''_n \doteq 2 \sum_{1 \leq j < i \leq n} [\gamma_{1,i} - \gamma^\star_{1,i-1}][\gamma_{1,j} - \gamma^\star_{1,j-1}], \quad n > 1, \quad N''_1 = N''_0 = 0 \tag{3.5.23}$$

where N'_n and N''_n are $\mathcal{F}_n$-martingales.

Proof of tightness. By (3.5.13)

$$\frac{1}{4}\mathbb{E}^\epsilon\left(\sup_{n \leq N} Z_n^2\right) \leq \mathbb{E}^\epsilon\left(\sup_{n \leq N} (M^\star_{n-1})^2\right) + \mathbb{E}^\epsilon\left(\sup_{n \leq N} (M_n)^2\right) + 2N^2 \sup_{n \leq N} \mathbb{E}^\epsilon\left((\gamma^\star_{1,n})^2\right)$$

and using Doob's inequality

$$\leq \sup_{n\leq N} \mathbb{E}^\epsilon\Big(2N^2(\gamma^\star_{1,n})^2 + 4N\gamma_{2,n} + 4N\gamma^\star_{2,n}\Big) \tag{3.5.24}$$

We set $N(\epsilon) \doteq [\epsilon^{-h}]$, then we obtain

$$\mathbb{E}^\epsilon\Big(\sup_{n\leq N(\epsilon)} Z_n^2\Big) \leq 16N(\epsilon) \sup_{n\leq N(\epsilon)} \mathbb{E}^\epsilon\Big(N(\epsilon)(\gamma^\star_{1,n})^2 + \gamma_{2,n} + \gamma^\star_{2,n}\Big) \tag{3.5.25}$$

Since the chain is stopped once it is not in $\mathcal{X}_{\epsilon,a}$ it follows that if $\psi_n \notin \mathcal{X}_{\epsilon,a}$ then $\gamma^\star_{1,n} = \gamma_{2,n} = \gamma^\star_{2,n} = 0$ so that (3.5.10) holds if there is $c > 0$ so that

$$\sup_{\psi\in\mathcal{X}_{\epsilon,a}} \mathbb{E}^\epsilon_{(0,\psi)}\Big(N(\epsilon)^2(\gamma^\star_{1,0})^2 + \gamma_{2,0} + \gamma^\star_{2,0}\Big) \leq c \tag{3.5.26}$$

We will prove (3.5.26) later.

To prove (3.5.11) we use the same argument after conditioning on $\mathcal{F}_{n_2}$. We call (x_{n_2}, ψ_{n_2}) the state of the chain at time n_2, $n \doteq n_3 - n_2$ and $\bar{n} \doteq n_2 - n_1$. Then

$$\begin{aligned}
&\mathbb{E}^\epsilon\Big([Z_{n_3} - Z_{n_2}]^2[Z_{n_2} - Z_{n_1}]^2\Big) \\
&\qquad\leq \sup_{(x_{n_2},\psi_{n_2})} \mathbb{E}^\epsilon_{n_2}\Big([Z_{n_3} - Z_{n_2}]^2\Big)\mathbb{E}^\epsilon\Big([Z_{n_2} - Z_{n_1}]^2\Big) \\
&\qquad\leq \sup_{\psi\in\mathcal{X}_{\epsilon,a}} \mathbb{E}^\epsilon_{(0,\psi)}\Big(Z_n^2\Big) \sup_{\psi\in\mathcal{X}_{\epsilon,a}} \mathbb{E}^\epsilon_{(0,\psi)}\Big(Z_{\bar{n}}^2\Big)
\end{aligned} \tag{3.5.27}$$

and using (3.5.26) we have, for a suitable constant $c' > 0$,

$$\begin{aligned}
&\sup_{\psi\in\mathcal{X}_{\epsilon,a}} \mathbb{E}^\epsilon_{(0,\psi)}\Big(Z_n^2\Big) \sup_{\psi\in\mathcal{X}_{\epsilon,a}} \mathbb{E}^\epsilon_{(0,\psi)}\Big(Z_{\bar{n}}^2\Big) \\
&\leq 16^2 n\bar{n} \sup_{\psi\in\mathcal{X}_{\epsilon,a}} \mathbb{E}^\epsilon_{(0,\psi)}\Big(n(\gamma^\star_{1,0})^2 + \gamma_{2,0} + \gamma^\star_{2,0}\Big) \sup_{\psi\in\mathcal{X}_{\epsilon,a}} \mathbb{E}^\epsilon_{(0,\psi)}\Big(\bar{n}(\gamma^\star_{1,0})^2 + \gamma_{2,0} + \gamma^\star_{2,0}\Big) \\
&\leq c'(n_3 - n_1)^2
\end{aligned} \tag{3.5.28}$$

having used (3.5.26). (3.5.11) and tightness are proved provided that (3.5.26) holds.

Let $\mathbb{P}$ be a limit law on $C([0,1])$ of $\{\mathbb{P}^\epsilon\}_{\epsilon>0}$. By Levy's characterization theorem it will be sufficient to prove that the coordinate process $X(t)$ in $C([0,1])$ is a square integrable $\mathbb{P}$-martingale and that $X(t)^2 - 3t/4$ is also a $\mathbb{P}$-martingale.

By (3.5.13) and (3.5.24)–(3.5.26)

$$\lim_{\epsilon\to 0+} \frac{1}{N(\epsilon)} \sup_{\psi_0\in\mathcal{X}_{\epsilon,a}} \mathbb{E}^\epsilon_{(0,\psi_0)}\Big(\sup_{n\leq N(\epsilon)} (Z_n - M_n)^2\Big) = 0 \tag{3.5.29}$$

which proves that $X(t)$ is a $\mathbb{P}$-martingale.

By (3.5.17)

$$\Lambda_n \doteq \frac{1}{N(\epsilon)}\Big(Z_n^2 - \Gamma_{2,n-1}\Big) \tag{3.5.30}$$

differs from a $\mathbb{P}^\epsilon$-martingale by the term $[Z_n^2 - M_n^2]$ which by (3.5.29) vanishes in L^1 as $\epsilon \to 0^+$. Thus the proof that $X(t)^2 - 3t/4$ is a $\mathbb{P}$-martingale follows from

$$\lim_{\epsilon\to 0^+} \sup_{\psi_0 \in \mathcal{X}_{\epsilon,a}} \mathbb{E}^\epsilon_{(0,\psi_0)}\Big(\sup_{n\le N(\epsilon)} \frac{1}{N(\epsilon)}\Big|\Gamma_{2,n-1} - \frac{4}{3}\Big|\Big) = 0 \tag{3.5.31}$$

which, by Proposition 3.5.1, is implied by

$$\lim_{\epsilon\to 0^+} \sup_{\psi \in \mathcal{X}_{\epsilon,a}} \Big|\gamma_{2,0}(\psi) - \frac{4}{3}\Big| = 0 \tag{3.5.32}$$

Proof of (3.5.26) *and* (3.5.32). Let $\psi \in \mathcal{X}_{\epsilon,a}$, $\phi \doteq T_{t_\epsilon}\big(\psi; \sqrt{\epsilon}Z\big)$, $\chi \doteq \xi^{\epsilon,\kappa}(\phi)$ and $\psi_{t_\epsilon}(x) \doteq \phi(x+\chi)$. By (3.5.12)

$$\gamma^\star_{1,0}(\psi) = E^\epsilon_{\psi}\Big(\gamma_{1,0}(\psi_{t_\epsilon})\Big), \quad \gamma_{1,0}(\phi) = E^\epsilon_\phi\Big(\xi^{\epsilon,\kappa}(\phi_{t_\epsilon})\Big) \tag{3.5.33}$$

By Lemma 3.4.9 $\gamma^\star_{1,0}(\bar m) = 0$ and $\gamma^\star_{1,0}(\psi) = \gamma^\star_{1,0}(\psi) - \gamma^\star_{1,0}(\bar m)$. We apply Corollary 3.4.6 so that we can construct the processes starting from ψ and $\bar m$ in the same probability space in such a way that for any $q > 0$ there is $c > 0$ so that $\|\psi_{t_\epsilon} - \bar\psi_{t_\epsilon}\|_\epsilon \le \epsilon^q$ with probability larger than $1 - c\epsilon^q$. ($\bar\psi_{t_\epsilon}$ denotes the value of ψ_{t_ϵ} when $\psi = \bar m$). Thus

$$|\gamma^\star_{1,0}(\psi)| \le c\epsilon^q + \sup_{\phi,\bar\phi\in\mathcal{X}_{\epsilon,a}} \sup_{\|\phi-\bar\phi\|_\epsilon \le \epsilon^q} \Big|E^\epsilon_\phi(\xi^{\epsilon,\kappa}(\phi_{t_\epsilon})) - E^\epsilon_{\bar\phi}(\xi^{\epsilon,\kappa}(\bar\phi_{t_\epsilon}))\Big| \tag{3.5.34}$$

By (3.4.70) with $\Delta = 0$ and $\delta = \epsilon^q$ we get $|\gamma^\star_{1,0}(\psi)| \le c\epsilon^q$ (for a suitable constant $c > 0$). (3.5.26) is proved.

By symmetry $\gamma_{1,0}(\bar m) = 0$, so that by (3.4.107)

$$|\gamma_{1,0}(\psi)| \le c t_\epsilon \epsilon^{1/2-2a} \tag{3.5.35}$$

that vanishes as $\epsilon \to 0^+$ by the assumption $a < 1/4$. Finally, by (3.5.18)

$$\gamma_{2,0}(\psi) = \frac{1}{\epsilon t_\epsilon} E^\epsilon_\psi\big(\xi^{\epsilon,\kappa}(\psi_{t_\epsilon})^2\big) - \gamma_{1,0}(\psi)^2$$

By the previous bound the last term vanishes as $\epsilon \to 0^+$ (uniformly on $\mathcal{X}_{\epsilon,a}$) while the first term on the right hand side converges to $3/4$ by Lemma 3.4.10. We have thus proved (3.5.32) and it only remains to prove that $\gamma^\star_{2,0}(\psi)$ is uniformly bounded on $\mathcal{X}_{\epsilon,a}$ which follows from (3.5.21), (3.5.34) and (3.5.35).

The proof of the theorem is complete. □

We next relate the convergence results proved for the auxiliary Markov chain to the Ginzburg-Landau process. We use the same notation as in Theorem 3.5.2 and Proposition 3.5.1. We fix the initial position x_0 in the Markov chain so that $\epsilon^{h/2}x_0 \doteq r_0$ (which is independent of ϵ) with $|r_0| < \epsilon^{-\kappa+h/2}$. We consider the Ginzburg-Landau process whose initial state is related to that of the Markov chain as in Proposition 3.5.1. We call $T_M(r)$, $r \in \mathbb{R}$, the subfix M standing for Markov, the first time when the coordinate process $X(t)$ (that we here suppose starting from r_0) reaches r. The analogous variable in the Ginzburg-Landau process is denoted by $T_{\mathrm{GL}}(r)$. Let $\ell^\star \in (0,1)$ and let $\ell \in (0,\ell^\star)$, call $r_\epsilon^\star \doteq \epsilon^{h/2}[\epsilon^{-\kappa} - \ell^\star\epsilon^{-1}]$, so that (at least for $\epsilon > 0$ small enough) $|r_0| < r_\epsilon^\star$. Then, by Proposition 3.5.1 with ℓ as above, for any $q > 0$ there is $c > 0$ so that

$$P^\epsilon\Big(T_M(r_\epsilon^\star - \epsilon^q) \le T_{\mathrm{GL}}(r_\epsilon^\star) \le T_M(r_\epsilon^\star + \epsilon^q)\Big) \ge 1 - c\epsilon^q \tag{3.5.36}$$

Similar statement holds for $-r_\epsilon^\star$.

For any r the law of $T_M(r)$ converges as $\epsilon \to 0^+$ to the law of the stopping time at $\pm r$ for the limit brownian motion b_t (starting from r_0) because the stopping time for the limit process is almost surely continuous, see [5]. Moreover the probability of $|T_M(r\pm\delta)-T_M(r)| > \zeta\ \delta$ and ζ positive, vanishes as $\delta \to 0^+$, hence by (3.5.36) the law of the stopping time at $\epsilon^{h/2}[\epsilon^{-\kappa} - \ell^\star\epsilon^{-1}]$ in the Ginzburg Landau process converges to the law of the stopping time for the limit Brownian motion at

$$\lim_{\epsilon\to 0+}\{\epsilon^{h/2}[\epsilon^{-\kappa} - \ell\epsilon^{-1}] - r_0\}$$

Recalling that the difference between true and linear centers vanishes in the limit, this together with Theorem 3.5.2 proves Theorem 3.2.2.

§3.6 Asymptotic coupling.

In this section we prove Theorem 3.2.3. By Theorem 3.2.2 and Proposition 3.4.7 we only need, as we will explain later, the following theorem.

3.6.1 Theorem.

Let m, $m^\star \in C_{\epsilon,\kappa}(\mathbb{R})$ (eventually depending on ϵ) such that $\|m\|$, $\|m^\star\| \le 3/2$ and $\|m - m^\star\| \le \epsilon^{2+\kappa}$. Then, we can construct a pair of Ginzburg–Landau processes m_t and $m_t^\star$, starting from m and $m^\star$ respectively, in the same probability space, and so that, if

$$\eta \doteq \inf\{t \ge 0 : \|m_t - m_t^\star\| = 0\}$$

(η *is defined to be infinity if the set above is empty), for any* $\alpha < 1$

$$\lim_{\epsilon \to 0+} P^{\epsilon}\big(\eta > \epsilon^{\alpha}\big) = 0 \tag{3.6.1}$$

Proof.

The proof uses the coupling and the ideas introduced in [56] to prove Theorem 1, but since V' is not monotonic, an extra argument is needed to conclude (3.6.1). Recall that in fact, we do not prove, as in [56] that η is finite with probability 1.

Consider the pair $(m_t, m_t^\star)$ introduced in [56], which satisfies

$$\begin{aligned}
\frac{\partial m_t}{\partial t} &= \frac{1}{2}\frac{\partial^2 m_t}{\partial x^2} - V'(m_t) + \sqrt{\epsilon}\alpha_1 \\
\frac{\partial m_t^\star}{\partial t} &= \frac{1}{2}\frac{\partial^2 m_t^\star}{\partial x^2} - V'(m_t^\star) + \sqrt{\epsilon}[1 - (|m_t - m_t^\star| \wedge 1)^{1/2}\alpha_1 + (|m_t - m_t^\star| \wedge 1)^{1/2}\alpha_2]
\end{aligned} \tag{3.6.2}$$

for α_1 and α_2 two independent space-time white noises and with initial conditions

$$m_0 = m, \qquad m_0^\star = m^\star \tag{3.6.3}$$

Consider the case $m \geq m^\star$. The general case follows from this one as in [56]. If we write the equation for the difference $m_t - m_t^\star$ and approximate the coefficients of the noise by Lipschitz functions as in [56], we can conclude, from Theorem 2.3 of [59], that $m_t \geq m_t^\star \quad \forall x \in \mathbb{R}$, $t \geq 0$. Call $\mathcal{F}_t$ the filtration generated by α_1 and α_2 up to time t. Next, integrate (3.6.2) from 0 to t and over $[-\epsilon^{-\kappa}, \epsilon^{-\kappa}]$. Call

$$U(t) \doteq \int_{-\epsilon^{-\kappa}}^{\epsilon^{-\kappa}} dx\, \big(m_t(x) - m_t^\star(x)\big) \tag{3.6.4}$$

Proceeding as in [56], we obtain for U the equation

$$U(t) = U(0) + \int_0^t ds\, U(s) - \int_0^t ds \int_{-\epsilon^{-\kappa}}^{\epsilon^{-\kappa}} dx\, \big(m_s(x)^3 - m_s^\star(x)^3\big) + M(t) \tag{3.6.5}$$

where M_t is a martingale with respect to $\mathcal{F}_t$, with compensator

$$\langle M\rangle(t) = 2\epsilon \int_0^t ds \int_{-\epsilon^{-\kappa}}^{\epsilon^{-\kappa}} dx\, \frac{|m_s(x) - m_s^\star(x)| \wedge 1}{1 + (1 - |m_s(x) - m_s^\star(x)| \wedge 1)^{1/2}} \tag{3.6.6}$$

Since

$$\frac{d\langle M\rangle(t)}{dt} \geq \epsilon \int_{-\epsilon^{-\kappa}}^{\epsilon^{-\kappa}} dx\, \big(|m_t(x) - m_t^\star(x)| \wedge 1\big) = \epsilon \int_{-\epsilon^{-\kappa}}^{\epsilon^{-\kappa}} dx\, \frac{|m_t(x) - m_t^\star(x)|}{|m_t(x) - m_t^\star(x)| \vee 1} \tag{3.6.7}$$

we have that

$$\frac{d\langle M\rangle(t)}{dt} = U(t)D(t),$$

for some adapted process $D(t)$ satisfying

$$D(t) \geq \frac{\epsilon}{\|m_t - m_t^\star\| \vee 1} \tag{3.6.8}$$

Take

$$\varphi(t) = \int_0^t ds\, D(s) \tag{3.6.9}$$

It is not difficult to see that Lemma 3.3 of [56] also holds in our case and, for each fixed $\epsilon, \varphi(\infty) = \infty$. Then, we can define the time changed process

$$X(t) = U\left(\varphi^{-1}(t)\right) \tag{3.6.10}$$

which satisfies

$$X(t) = U(0) + \int_0^t ds\, \frac{X(s)}{\varphi'(\varphi^{-1}(s))} + \int_0^t ds\, C(s) + \int_0^t dB(s)\, X^{1/2}(s)$$

for some Brownian motion $B(s)$ and nonpositive adapted process $C(s)$. Applying Ito's formula with the function $f(x) = 2x^{1/2}$, we have that, as long as $X(t) \geq 0$,

$$Y(t) \doteq 2\sqrt{X(t)} \tag{3.6.11}$$

satisfies

$$Y(t) = 2\sqrt{U(0)} + \int_0^t ds \left(\frac{2C(s)}{Y(s)} - \frac{1}{2Y(s)} + \frac{Y(s)}{2\varphi'(\varphi^{-1}(s))} \right) + B(t) \tag{3.6.12}$$

Now, let us prove (3.6.1). From (3.6.11) and the definition of the time change, for any positive y

$$P^\epsilon\left(\eta > y\right) \leq P^\epsilon\left(\gamma > \varphi(y)\right) \tag{3.6.13}$$

where

$$\gamma \doteq \inf\{t \geq 0 : Y(t) = 0\}$$

Now, take $\alpha < 1$ as in the statement. From (3.6.13) we can write, for any given positive a,

$$P^\epsilon\left(\eta > \epsilon^\alpha\right) \leq P^\epsilon\left(\gamma > \varphi(\epsilon^\alpha), \varphi(\epsilon^\alpha) > a\right) + P^\epsilon\left(\gamma > \varphi(\epsilon^\alpha), \varphi(\epsilon^\alpha) \leq a\right) \leq$$
$$P^\epsilon\left(\gamma > a\right) + P^\epsilon\left(\varphi(\epsilon^\alpha) \leq a\right)$$

Using (3.6.8) and the a-priori bound on the sup-norm of m_t and $m_t^\star$ (see Proposition 3.4.1), if we take $a = \epsilon^\beta$, for any $\beta > 1 + \alpha$, it is not difficult to prove that this last probability goes to zero as $\epsilon \to 0$, and so, to prove (3.6.1) we only need to show that

$$\lim_{\epsilon \to 0^+} P^\epsilon\big(\gamma \le \epsilon^\beta\big) = 1 \tag{3.6.14}$$

for some β as above. Recall equation (3.6.11) for Y and consider

$$\tau \doteq \inf\{t : B(t) + 2\sqrt{U(0)} = 0 \text{ or } B(t) + 2\sqrt{U(0)} = \epsilon|\log\epsilon|\}$$

and the set S

$$S \doteq \{\varphi'(\varphi^{-1}(s)) \ge \epsilon \quad \forall s \le \epsilon^\beta\}$$

Call

$$E \doteq S \cap \{B(\tau) + 2\sqrt{U(0)} = 0\} \cap \{\tau < \epsilon^\beta\}$$

We shall prove that for all ϵ small enough

$$E \subset \big\{\gamma \le \epsilon^\beta\big\} \tag{3.6.15}$$

Define the stopping time

$$t_0 = \inf\left\{ s \;:\; \frac{1}{2Y(s)} \le \frac{Y(s)}{2\varphi'(\varphi^{-1}(s))} \right\}$$

Take $\omega \in E$ and suppose by contradiction that, for this ω, $Y(t) > 0$ for all $t \le \tau$. Then, equation (3.6.12) holds for $Y(t)$ for any $t \le \tau$, and so, for the ω we are considering,

$$Y(t) \le B(t) + 2\sqrt{U(0)} \quad \forall t \le t_0 \wedge \tau \tag{3.6.16}$$

If $\tau \le t_0$ the evaluation of the previous expression at τ yields $Y(\tau) \le B(\tau) + 2\sqrt{U(0)} = 0$, which is a contradiction. Then $\tau > t_0$ and since $\omega \in E \supset S$,

$$Y(t_0) = \frac{\varphi'(\varphi^{-1}(t_0))}{Y(t_0)} \ge \frac{\epsilon}{Y(t_0)}$$

which implies

$$Y(t_0) \ge \sqrt{\epsilon}$$

and this contradicts (3.6.16) for small ϵ for the definition of τ, which finishes the proof of (3.6.15). To conclude, we only have to show that we can take $\beta > 1 + \alpha$ such that $P^\epsilon(E) \to 1$ as $\epsilon \to 0$. But, if we recall that $U(0) \le 2\epsilon^2$, we obtain

$$P^\epsilon\big(B(\tau) + 2\sqrt{U(0)} = 0\big) = \frac{\epsilon|\log\epsilon| - 2\sqrt{U(0)}}{\epsilon|\log\epsilon|} \ge \frac{\epsilon|\log\epsilon| - 2\sqrt{2}\epsilon}{\epsilon|\log\epsilon|} \longrightarrow 1 \quad \text{as } \epsilon \to 0$$

Also, taking $\beta < 2$, it follows that

$$\lim_{\epsilon \to 0+} P^\epsilon(\tau \leq \epsilon^\beta) = 1$$

To finish, let us prove

$$\lim_{\epsilon \to 0+} P^\epsilon(S) = 1 \tag{3.6.17}$$

First recall that by Proposition 3.4.1 for any $\beta < 2$

$$\lim_{\epsilon \to 0+} P^\epsilon\left(\frac{1}{\|m_s - m_s^*\| \vee 1} > \frac{1}{4} \quad \forall s < \epsilon^\beta\right) = 1$$

so, from (3.6.8) and (3.6.9),

$$\lim_{\epsilon \to 0+} P^\epsilon\left(\varphi(s) > \frac{\epsilon}{4} \quad \forall s < \epsilon^\beta\right) = 1$$

and then

$$\lim_{\epsilon \to 0+} P^\epsilon\left(\varphi^{-1}(s) \leq \frac{4s}{\epsilon} \quad \forall s < \epsilon^\beta\right) = 1 \tag{3.6.18}$$

Recalling $\varphi' = D$, from (3.6.8) and the Proposition 3.4.1

$$\lim_{\epsilon \to 0+} P^\epsilon\left(\varphi'(t) > \frac{\epsilon}{4} \quad \forall t < 4\epsilon^{\beta-1}\right) = 1 \tag{3.6.19}$$

for any $\beta \geq 0$. Finally, (3.6.18) and (3.6.19) imply (3.6.17), and the theorem is proved. □

Proof of Theorem 3.2.3.

The coupling is constructed as follows. The two processes m_t and m'_t are independent of each other till the first time T_1 when $\xi^{\epsilon,\kappa}(m_{T_1}) = \xi^{\epsilon,\kappa}(m'_{T_1})$. Let $a \in (0, 1/4)$, then with probability going to 1 a $\epsilon \to 0^+$, both $(m_t)^{\epsilon,\kappa}$ and $(m'_t)^{\epsilon,\kappa}$ are in $\mathcal{M}_{\epsilon^{1/2-a}}$, we can thus suppose that such a condition is verified. By Lemma 3.4.8 at time $T = T_1 + t_\epsilon$ with large probability we are in the hypothesis of Proposition 3.4.7. We construct the processes in the time interval $[T, T+t_\epsilon]$ using Proposition 3.4.7 with $b = 1-\omega$, $\omega \in (2a, 1/2)$, so that (3.4.79) is verified and we can suppose that the processes at time $T + t_\epsilon$ are in the set which appears on its left hand side. We can thus apply again Proposition 3.4.7 with $b = 1 - \omega + \bar{\gamma}$ and iterate this procedure $N > 1$ times. Then calling $S \doteq T + Nt_\epsilon$:

$$\|(m_S)^{\epsilon,\kappa} - (m'_S)^{\epsilon,\kappa}\| \leq \epsilon^{1/2-a+N\bar{\gamma}}$$

with probability going to 1 as $\epsilon \to 0^+$. Since by assumption m and m' are both in $C_{\epsilon,\kappa}(\mathbb{R})$ the above holds as well for the sup norm (without the cutoff (ϵ, κ)).

We can then apply Theorem 3.6.1 to conclude that if N is large enough there is a coupling before $T+(N+1)t_\epsilon$ with probability going to 1 as $\epsilon \to 0^+$. Thus the time of coupling differs from the time of first encounter of the linear centers by a term bounded by $(N+1)|\log\epsilon|^2$. The law of first encounter of the linear centers converges to that of the brownians limit of the linear centers by Theorem 3.5.2. As the difference between true and linear centers vanishes in the limit, we obtain the proof of Theorem 3.2.3. □

§3.7 Some basic estimates.

3.7.1 Lemma.

Let $Z_t(x)$ be the process defined in §3.2. There are positive constants b_0 and b_1 so that for all $p > 0$,

$$P^\epsilon\Big(\sup_{0\le t\le t_\epsilon}\|\sqrt{\epsilon}Z_t\| > \epsilon^{\frac{1}{2}-p}\Big) \le b_0 e^{-b_1|\log\epsilon|^{-1}\epsilon^{-2p}} \tag{3.7.1}$$

Proof.

The process Z_t is a Gaussian centered process, with bounded and continuous paths a.e. Define

$$D_\epsilon \doteq \{(x,t) : x \in \mathcal{T}_{\epsilon,\kappa}; 0 \le t \le t_\epsilon\} \qquad |||Z||| \doteq \sup_{(x,t)\in D_\epsilon} Z_t(x), \qquad \sigma_\epsilon^2 \doteq \sup_{(x,t)\in D_\epsilon} E^\epsilon[Z_t(x)^2] \tag{3.7.2}$$

Using the explicit form of the covariance of Z_t, see for instance [62], it is not difficult to prove that there exists a constant C_1, independent of ϵ such that

$$E^\epsilon[Z_t(x)^2] \le C_1 t\epsilon^\kappa + C_1\sqrt{t}, \tag{3.7.3}$$

what yields

$$\sigma_\epsilon^2 \le C\sqrt{t_\epsilon} \tag{3.7.4}$$

for some C independent of ϵ. Then, we can apply the following inequality, that follows from a symmetry argument from Theorem 2.1 of [1]: for any $\lambda > E^\epsilon|||Z|||$:

$$P^\epsilon\Big(\sup_{0\le t\le t_\epsilon}\|Z_t\| > \lambda\Big) \le 4\exp\Big[-\frac{(\lambda - E^\epsilon|||Z|||)^2}{2\sigma_\epsilon^2}\Big] \tag{3.7.5}$$

To give an upper bound to $E^\epsilon|||Z|||$, we use Corollary 4.15 of [1]: there exists a universal constant K such that

$$E^\epsilon|||Z||| \le K\int_0^\infty dr\sqrt{\log N_\epsilon(r)}, \tag{3.7.6}$$

where $N_\epsilon(r)$ is the minimal number of balls of radius r needed to cover D_ϵ, with respect to the metric

$$d\big((x,t),(y,s)\big) \doteq \sqrt{E^\epsilon\big[\big(Z_t(x) - Z_s(y)\big)^2\big]} \tag{3.7.7}$$

It can be proven that there are positive constants k_1 and k_2 (independent of ϵ, t, s, x and y) such that, for any $x, y \in \mathbb{R}$ and $t, s \in \mathbb{R}_+$:

$$E^\epsilon\big[\big(Z_t(x) - Z_t(y)\big)^2\big] \le k_1|x-y| \tag{3.7.8}$$

$$E^\epsilon\big[\big(Z_t(x) - Z_s(x)\big)^2\big] \le k_2\sqrt{|t-s|} \tag{3.7.9}$$

(see for example Proposition 4.2 in [62]). From (3.7.8) and (3.7.9) it is easy to check that there is a constant c such that

$$N_\epsilon(r) \le \max\{1, c|\log\epsilon|^2\epsilon^{-\kappa}r^{-3}\} \tag{3.7.10}$$

By (3.7.6) and (3.7.10) it follows that there is a constant K' such that

$$E^\epsilon|||Z||| \le K'\log\epsilon^{-(\kappa+1)} \tag{3.7.11}$$

Using (3.7.4) and (3.7.11), from inequality (3.7.6) with $\lambda = \epsilon^{-p}$ we finally obtain

$$P^\epsilon\Big(\sup_{0\le t\le t_\epsilon}\|Z_t\| > \epsilon^{-p}\Big) \le 4\exp\Big[-\frac{(\epsilon^{-p} - K'\log\epsilon^{-(\kappa+1)})}{2C|\log\epsilon|}\Big] \tag{3.7.12}$$

The bound (3.7.12), which is valid for ϵ small enough, implies the estimate (3.7.1) for some constants b_0 and b_1. The estimate can be then extended to any $\epsilon \in (0,1]$ simply by modifying conveniently the values of b_0 and b_1. The lemma is so proved. □

3.7.2 Lemma.

Let $m \in C^0(\mathbb{R})$, $\|m\| \le 1 + 10^{-2}$ and $t_\epsilon = \chi|\log\epsilon|^2$. Then there are constants c_0 and c_1 so that, for any $\chi \in [1,2]$,

$$P^\epsilon\Big(\sup_{0\le t\le t_\epsilon}\|T_t(m;\sqrt{\epsilon}Z)\| \le 2,\ \|T_{t_\epsilon}(m;\sqrt{\epsilon}Z)\| \le 1 + 10^{-2}\Big) \ge 1 - c_0e^{-c_1\epsilon^{-1}} \tag{3.7.13}$$

Proof.

A comparison theorem holds for the stochastic Ginzburg-Landau equation, see Proposition 5.1 in [11]. So, if $m \in C(\mathbb{R};[-1-10^{-2}, 1+10^{-2}])$, for any $t \ge 0$ it holds

$$m_t^- \le T_t(m;\sqrt{\epsilon}Z) \le m_t^+ \qquad P^\epsilon \text{ - a.s.} \tag{3.7.14}$$

where $m_t^{\pm} \doteq T_t\big(\pm(1+10^{-2}); \sqrt{\epsilon}Z\big)$. It is then sufficient to prove (3.7.13) with $T_t(m;\sqrt{\epsilon}Z)$ replaced by $m_t^{\pm}$. We define $u_{\pm}(x,t) \doteq m_t^{\pm}(x) \mp 1$. Then $u_{\pm}(x,t)$ solve the equations

$$\frac{\partial u_{\pm}}{\partial t} - \frac{1}{2}\frac{\partial^2 u_{\pm}}{\partial x^2} + 2u_{\pm} = \mp 3u_{\pm}^2 - u_{\pm}^3 + \sqrt{\epsilon}\alpha \tag{3.7.15}$$

that is, the integral equations

$$u_{\pm}(t) = e^{-2t}H_t \star u_{\pm}(0) + \int_0^t ds e^{-2(t-s)} H_{t-s} \star [\mp 3u_{\pm}^2 - u_{\pm}^3](s) + \sqrt{\epsilon}V_t \tag{3.7.16}$$

where $e^{-2t}H_t \star u_{\pm}(0) = \pm e^{-2t}[1+10^{-2}]$ and

$$V_t(x) \doteq \int ds dy\, \alpha(s,y) \mathbf{1}_{0\le s\le t} e^{-2(t-s)} H_{t-s}(y-x) \tag{3.7.17}$$

(note that $e^{-2t}H_t(x-y)$ is the Green function for the operator $\partial_t - (1/2)\partial_x^2 + 2Id$). By arguing as in the proof of Lemma 3.7.1, it is easy to prove that for any $b > 0$ there are constants h_0 and h_1 such that

$$P^{\epsilon}\Big(\sup_{0\le t\le t_\epsilon} \|\sqrt{\epsilon}V_t\| > b\Big) \le h_0 e^{-h_1\epsilon^{-1}} \tag{3.7.18}$$

Let $T \doteq \sup\{t \ge 0 : \|u_{\pm}(\cdot,t)\| < 2(10^{-2}+b)\}$. We will prove that there exists b and ϵ_0 such that for all $\epsilon \le \epsilon_0$

$$P^{\epsilon}\Big(\sup_{0\le t\le t_\epsilon} \|m_t^{\pm}\| \le 2,\ \|m_{t_\epsilon}^{\pm}\| \le 1+10^{-2}\Big) \ge P^{\epsilon}\Big(T \ge t_\epsilon,\ \sup_{0\le t\le t_\epsilon} \|\sqrt{\epsilon}V_t\| \le b\Big) \tag{3.7.19}$$

and

$$P^{\epsilon}\Big(T \ge t_\epsilon,\ \sup_{0\le t\le t_\epsilon} \|\sqrt{\epsilon}V_t\| \le b\Big) = P^{\epsilon}\Big(\sup_{0\le t\le t_\epsilon} \|\sqrt{\epsilon}V_t\| \le b\Big) \tag{3.7.20}$$

Clearly, by definition of T and $u_{\pm}$, $\sup_{0\le t\le t_\epsilon} \|m_t^{\pm}\| \le 2$ is implied by $T \ge t_\epsilon$ if b is small enough. Moreover, in the set

$$\{T \ge t_\epsilon,\ \sup_{0\le t\le t_\epsilon} \|\sqrt{\epsilon}V_t\| \le b\},$$

by equation (3.7.16), one has the estimate

$$\|u_{\pm}(\cdot,t_\epsilon)\| \le e^{-2t_\epsilon}10^{-2} + 12(10^{-2}+b)^2 + 8(10^{-2}+b)^3 + b \tag{3.7.21}$$

Now there exists $\epsilon_0 > 0$ and $b_0 > 0$ such that, for any $\epsilon \le \epsilon_0$ and $b \le b_0$, (3.7.21) implies $\|u_{\pm}(\cdot,t_\epsilon)\| \le 10^{-2}$ and hence $\|m_{t_\epsilon}^{\pm}\| \le 1+10^{-2}$. (3.7.19) is then proven. To prove (3.7.20) we note that if $\sup_{0\le t\le t_\epsilon} \|\sqrt{\epsilon}V_t\| \le b$ and $T \le t_\epsilon$ then

$$2(10^{-2}+b) = \|u_{\pm}(\cdot,T)\| \le 10^{-2} + b + 12(10^{-2}+b)^2 + 8(10^{-2}+b)^3 \tag{3.7.22}$$

which gives a contradiction if, for example, $b \le 10^{-2}$. Then, for $b = b_0 \wedge 10^{-2}$, both (3.7.19) and (3.7.20) holds. The estimate (3.7.13) follows then from (3.7.18) for $\epsilon \le \epsilon_0$ (with $c_0 = h_0$ and $c_1 = h_1$) and it extends to any ϵ simply by modifying the values of the constants c_0 and c_1. □

3.7.3 Lemma.

Let $m \in C^0(\mathbb{R})$, $m_t \doteq T_t(m, \sqrt{\epsilon}Z)$. For any $\delta > 0$, $\alpha \in (0, 1/2)$ there is a constant $c > 0$ so that for all ϵ small enough

$$P^\epsilon\left(\sup_{|x|,|y| \le \epsilon^{-\kappa}} \sup_{\substack{x \ne y \\ |x-y| \le 1}} \frac{|m_{t_\epsilon}(x) - m_{t_\epsilon}(y)|}{|x-y|^\alpha} > \delta \right) \le e^{-c\delta^2} \tag{3.7.23}$$

Proof.

We first prove an analogous estimate for the Gaussian process $\sqrt{\epsilon}Z$. We use Theorem 2.1 of [1] applied to the gaussian process

$$G_\epsilon(x,y) \doteq \sqrt{\epsilon}\, \frac{Z_{t_\epsilon}(x) - Z_{t_\epsilon}(y)}{|x-y|^\alpha} \tag{3.7.24}$$

By arguing as in the proof of Lemma 3.7.1 we have, for $\delta > E^\epsilon[\hat{G}_\epsilon]$,

$$P^\epsilon\Big(\sup_{|x|,|y| \le \epsilon^{-\kappa}} \sup_{\substack{x \ne y \\ |x-y| \le 1}} |G_\epsilon(x,y)| > \delta \Big) \le 4 \exp\left[- \frac{(\delta - E^\epsilon[\hat{G}_\epsilon])^2}{2\sigma_\epsilon^2} \right] \tag{3.7.25}$$

where

$$\hat{G}_\epsilon \doteq \sup_{|x|,|y| \le \epsilon^{-\kappa}} \sup_{\substack{x \ne y \\ |x-y| \le 1}} G_\epsilon(x,y), \qquad \sigma_\epsilon^2 \doteq \sup_{|x|,|y| \le \epsilon^{-\kappa}} \sup_{\substack{x \ne y \\ |x-y| \le 1}} E^\epsilon[G_\epsilon(x,y)^2] \tag{3.7.26}$$

Using Corollary 4.15 of [1] as in Lemma 3.7.1 one easily proves that $E^\epsilon[\hat{G}_\epsilon] \le c_1\sqrt{\epsilon}$ for some $c_1 > 0$. To estimate σ_ϵ^2 we note that

$$E^\epsilon[G_\epsilon(x,y)^2] = \frac{\epsilon}{|x-y|^{2\alpha}} \int_0^{2t_\epsilon} ds\, \frac{1}{\sqrt{2\pi s}} \left(1 - e^{-(x-y)^2/2s}\right) \tag{3.7.27}$$

and, for some constant $c_3 > 0$ depending on α,

$$\left|1 - e^{-(x-y)^2/2s}\right| \le c_3 \frac{|x-y|^{2\alpha}}{s^\alpha} \tag{3.7.28}$$

From (3.7.27) and (3.7.28) we obtain $\sigma_\epsilon^2 \le c_4 \epsilon\, t_\epsilon^{1/2-\alpha}$ for some constant $c_4 > 0$. Then by (3.7.25) we recover an estimate like (3.7.23) for the noise $\sqrt{\epsilon}Z$. To prove (3.7.23) we use the integral form (3.2.3) of the Ginzburg–Landau equation and write

$$\begin{aligned} m_{t_\epsilon}(x) - m_{t_\epsilon}(y) &= H_1 \star m_{t_\epsilon - 1}(x) - H_1 \star m_{t_\epsilon - 1}(y) + \\ &\int_0^1 ds\, [H_{1-s}(x - \cdot) - H_{1-s}(y - \cdot)] \star [m_{t_\epsilon - 1 + s} - m^3_{t_\epsilon - 1 + s}] + G_\epsilon(x,y)|x-y|^\alpha \end{aligned} \tag{3.7.29}$$

We consider the intersection of the sets where m_{t_ϵ} is bounded by 2 and where the noise satisfies the bound like (3.7.23) with δ' to be fixed. In this set we can estimate

$$|m_{t_\epsilon}(x) - m_{t_\epsilon}(y)| \leq 2|x-y| + c_5|x-y| + \delta'|x-y|^\alpha \tag{3.7.30}$$

for any $x \neq y$ such that $|x|, |y| \leq \epsilon^{-\kappa}$ and $|x-y| \leq 1$. Choosing δ' small enough (3.7.23) follows easily. □

3.7.4 Lemma.

Let $x_0 \in \mathbb{R}$ and let $g_{t,x_0}(x,y)$ be the fundamental solution of the equation $\partial_t u = L_{x_0} u$. Then the following holds.

$$g_{t,x_0}(x,y) \geq 0 \qquad \textit{for any } x, y \in \mathbb{R} \tag{3.7.31}$$

$$\sup_{t\geq 0}\sup_{x\in\mathbb{R}} \int_{\mathbb{R}} dy\, g_{t,x_0}(x,y) \leq c_0, \qquad c_0 > 0 \tag{3.7.32}$$

$$\sup_{x\in\mathbb{R}} \int_{\mathbb{R}} dy\, |g_{t,x_0}(x,y+d) - g_{t,x_0}(x,y)| \leq c_1 \frac{d}{\sqrt{t}} \qquad c_1 > 0 \tag{3.7.33}$$

$$\sup_{t\leq t_\epsilon}\sup_{x\in B_2} \int_{\mathbb{R}\setminus B_1} dy\, g_{t,x_0}(x,y) \leq c e^{2t_\epsilon} e^{-(r_1-r_2)^2}, \qquad B_i \doteq \{x \in \mathbb{R} : |x - x_0| \leq r_i\},\ r_1 > r_2 \tag{3.7.34}$$

Proof.

From (3.4.45) we can restrict ourselves to the case $x_0 = 0$.

From the Feynman–Kac formula (3.7.31) follows. (3.7.32) follows from Theorem 3.3.1. To prove (3.7.33) we use the following integral equation:

$$g_t(x,y) = H_t(x,y) + \int_0^t \int dz\, H_{t-s}(x-z)[1 - 3\bar{m}^2(z)] g_s(z,y) \tag{3.7.35}$$

We define

$$f_t(d) = \sup_{x\in\mathbb{R}} \int_{\mathbb{R}} dy\, |g_t(x,y) - g_t(x,y+d)|$$

From (3.7.35) we then have

$$f_t(d) \leq c_1 \frac{d}{\sqrt{t}} + c_2 \int_0^t f_s(d) \tag{3.7.36}$$

From (3.7.36) the inequality (3.7.33) follows immediately for $t \in (0,1]$. For $t > 1$ we use the semigroup property getting that $f_t(d) \leq c_0 f_1(d)$ with c_0 as in (3.7.32). (3.7.34) is based on the estimate

$$g_t(x,y) \leq \frac{e^{2t}}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}$$

which follows easily from the Feynman–Kac formula. □

3.7.5 Lemma.

Consider the set $\mathcal{B}_\epsilon^{(i)}$, $i=2, 3$, *defined in* (3.4.37) *and* (3.4.38). *Then, there are* c_0 *and* c_1 *positive constants such that*

$$P^\epsilon\left(\mathcal{B}_\epsilon^{(i)}\right) \geq 1 - c_0 e^{-c_1\epsilon^{-1/2}}, \qquad i=2,3 \tag{3.7.37}$$

Proof.

Recall equations (3.4.5) for $\hat{Z}_t$, and (3.4.39) for $\hat{Z}_t^\star$, in terms of Z_t and $Z_t^\star$. Using (3.4.46), we obtain

$$\begin{aligned}
\hat{Z}_{t,x_0} - \tau_\Delta \hat{Z}^\star_{t,x_0^\star} &= Z_t - \tau_\Delta Z_t^\star + \int_0^t ds\, g_{t-s,x_0}(3\bar{m}^2_{x_0^\star}-1)(Z_{t,x_0} - \tau_\Delta Z^\star_{t,x_0^\star}) \\
&= Z_t - \tau_\Delta Z_t^\star + \int_0^t ds\, g_{t-s,x_0} \mathbf{1}_{|y-x_0|\leq 2\epsilon^{-1/10}}(3\bar{m}^2_{x_0^\star}-1)(Z_{t,x_0} - \tau_\Delta Z^\star_{t,x_0^\star}) \\
&+ \int_0^t ds\, g_{t-s,x_0} \mathbf{1}_{|y-x_0|> 2\epsilon^{-1/10}}(3\bar{m}^2_{x_0^\star}-1)(Z_{t,x_0} - \tau_\Delta Z^\star_{t,x_0^\star}) \\
&\doteq Z_t - \tau_\Delta Z_t^\star + A_1(x,t) + A_2(x,t).
\end{aligned} \tag{3.7.38}$$

We will prove below that

$$P^\epsilon\Big(\sup_{\substack{0\leq t\leq t_\epsilon \\ |x-x_0|\leq 2\epsilon^{-1/10}}} |Z_t - \tau_\Delta Z_t^\star| > e^{-\epsilon^{-1/50}}\Big) \leq c_0 e^{-c_1\epsilon^{-1/2}} \tag{3.7.39}$$

The bound (3.7.37) for $i=2$ follows immediately from this inequality. Moreover, from (3.7.39) and (3.7.36), we can estimate

$$\begin{aligned}
P^\epsilon\Big(\sup_{t\leq t_\epsilon} \|A_1(x,t)\|_{x_0,\epsilon} > \frac{e^{-1/100}}{3}\Big) &\leq P^\epsilon\Big(\sup_{\substack{0\leq t\leq t_\epsilon \\ |x-x_0|\leq 2\epsilon^{-1/10}}} |Z_t - \tau_\Delta Z_t^\star| > \frac{K e^{-\epsilon^{-1/100}}}{t_\epsilon}\Big) \\
&\leq c_0 e^{-c_1\epsilon^{-1/2}}
\end{aligned} \tag{3.7.40}$$

Also, for A_2, we obtain

$$P^\epsilon\Big(\sup_{t\leq t_\epsilon} \|A_2(x,t)\|_{x_0,\epsilon} > \frac{e^{-1/100}}{3}\Big) \leq P^\epsilon\Big(\sup_{\substack{0\leq t\leq t_\epsilon \\ x\in\mathcal{T}_{\epsilon,\kappa}}} |Z_t - \tau_\Delta Z_t^\star| > \epsilon^{-1/2}\Big) \leq b_0 e^{-b_1\epsilon^{-1/2}}$$

Then let us prove (3.7.39) to conclude the proof of the lemma. Recall that

$$\begin{aligned}
&|Z_t - \tau_\Delta Z_t^\star| \leq \\
&\left| \int_0^t ds \int dy \left(\mathbf{1}_{\{y\in\mathcal{T}_{\epsilon,\kappa}\}} H^{(\epsilon)}_{t-s}(x,y) - \mathbf{1}_{\{|y-x_0|\leq 4\epsilon^{-1/10}, y+\Delta\in\mathcal{T}_{\epsilon,\kappa}\}} H^{(\epsilon)}_{t-s}(x+\Delta, y+\Delta)\right) \alpha(y,s) \right| \\
&+ \left| \int_0^t ds \int dy\, \mathbf{1}_{\{|y-x_0^\star|>4\epsilon^{-1/10}; y\in\mathcal{T}_{\epsilon,\kappa}\}}\, H^{(\epsilon)}_{t-s}(x+\Delta, y)\, \bar{\alpha}(y,s) \right| \\
&\doteq |I_1(x,t)| + |I_2(x,t)|
\end{aligned}$$

Both I_1 and I_2 are centered Gaussian processes, for which estimates like (3.7.8) and (3.7.9) are valid. Moreover, recalling that

$$H_t^{(\epsilon)} = \sum_{j\in\mathbb{Z}} \Big(H_t(x, y + 4j\epsilon^{-\kappa}) + H_t(x, 4j\epsilon^{-\kappa} + 2\epsilon^{-\kappa} - y) \Big),$$

it is not difficult to prove that

$$(\sigma_i)^2 \doteq \sup_{\substack{0\le t\le t_\epsilon \\ |x-x_0|\le 2\epsilon^{-1/10}}} E\Big(I_i(x,t)^2\Big) \le e^{-c\epsilon^{-1/16}}$$

Then, proceeding as in the proof of Lemma 3.7.1 , (3.7.39) follows. □

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