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Some Problems in the Calculus of Variations

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Stefania Perrotta

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Thesis submitted for the degree of "Doctor Philosophiæ"

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Introduction

The subject of this thesis is the minimization problem of integral functionals of the form

$$I(u) = \int_\Omega [f(\nabla u(x)) + g(u(x))]\, dx,$$

where Ω is an open and bounded subset of $\mathbb{R}^N$ and $u : \Omega \to \mathbb{R}^M$ belongs to a suitable Sobolev space and satisfies a prescribed boundary condition. Such minimum problems are called scalar when either $N = 1$ or $M = 1$ and vectorial when both N and M are greater than one.

As is well known, the direct method of the Calculus of Variations ensures the existence of minimizers for a functional provided the functional itself is simultaneously (sequentially) lower semicontinuous and coercive with respect to a certain topology. Here, by a coercive functional, we mean a functional whose sublevel sets are (sequentially) compact. In particular, for functionals of the form considered here, provided suitable regularity and growth assumptions are fulfilled by the functions f and g, the sequential lower semicontinuity with respect to the weak topology of $W^{1,p}(\Omega, \mathbb{R}^M)$ (weak* if $p = \infty$) is equivalent to the quasiconvexity of the function f (see [21]). We recall that, following C.B. Morrey ([39]), a function $f : \mathbb{R}^{MN} \to \mathbb{R}$ is said to be *quasiconvex* if

$$f(A) \leq \frac{1}{\text{meas}D} \int_D f(A + \nabla\varphi(x))\, dx,$$

for every open and bounded subset D of $\mathbb{R}^N$, for every matrix $A \in \mathbb{R}^{MN}$ and for every $\varphi \in W_0^{1,\infty}(D, \mathbb{R}^M)$. In the scalar case, quasiconvexity is equivalent to convexity, while, in the vectorial case, quasiconvexity is a strictly weaker property than convexity (see [21] again).

In this thesis, we present some existence results for non convex scalar minimum problems and for non quasiconvex vectorial minimum problems related to I. Moreover, we investigate the issue of the validity of Euler-Lagrange equations for the solutions of the minimum problem for I when f is a convex and extended valued function.

Let us begin by discussing this latter problem, wich is the subject of Chapter 1. As is well known, the Euler-Lagrange equations, in their weak formulation, are classically derived for continuosly differentiable integrands whose derivatives satisfy suitable growth assumptions from above and below ([21] and [29]). These assumtions allow the application of the Lebesgue's dominated convergence theorem to the difference quotient of the functional I along every variation in $C_c^\infty(\Omega)$.
Beyond this classical framework, it is fairly simple to show that the smoothness assumption on f can be replaced by convexity, provided the subdifferential of f satisfies growth conditions analogous to the classical ones. In particular, in the Euler-Lagrange equations, derivatives are replaced by measurable selections of the subdifferentials.
Whenever the integrand f is still convex but the growth assumptions are no longer valid, the classical approach based on the Lebesgue's theorem does not apply anymore. In particular this is true for extended valued integrands. In this case, it may even happen that not all functions in $C_c^\infty(\Omega)$ are admissible variations for the corresponding functional I.
At this point, let us focus the attention on the functional considered in Chapter 1, where $f(\xi) = j_{[0,1]}(\|\xi\|)$ is the indicator function of the closed unit ball of $\mathbb{R}^N$ and u is a minimizer of the corresponding functional I on $u_0 + W_0^{1,\infty}(\Omega)$.
Such functional can be viewed as the limiting case for $p = \infty$ of the functionals

$$\int_\Omega \left[\frac{1}{p}\|\nabla u(x)\|^p + g(u(x))\right] dx.$$

For a linear function g, the asymptotical behaviour (as $p \to \infty$) of the minimizers on $W_0^{1,p}(\Omega)$ of the functionals above is studied in [6]. In [17], we directly face the problem of the validity of the Euler-Lagrange equations for I, by integrating along the lines of steepest descent of u, so as to determine a measurable selection $\alpha(x)$ of $\partial j_{[0,1]}(\|\nabla u(x)\|)$ such that the equation

$$\int_\Omega \alpha(x)\langle\frac{\nabla u(x)}{\|\nabla u(x)\|}, \nabla\eta(x)\rangle\, dx = \int_\Omega -g'(u(x))\eta(x)\, dx$$

holds for all $\eta \in C_c^\infty(\Omega)$. It is worth noticing that this approach requires a preliminary investigation of the regularity of the minimizers of I on $u_0 + W_0^{1,\infty}(\Omega)$, an investigation wich is usually carried out as a consequence of the validity of the Euler-Lagrange equations themselves.

Now, we turn to the issue of the existence of minimizers for I. Most of the papers related to non convex or non quasiconvex minimum problems feature a constructive approach to the basic issue of existence, thus exhibiting a striking difference with respect to the direct method. In most cases, the relaxed problem is considered, i.e. the minimum problem for the largest lower semicontinuous functional $\bar{I}$ laying below the original functional I. When I is coercive, the existence of minimizers for the relaxed functional $\bar{I}$ is ensured by the direct method and its minimum value agrees with the infimum of the original functional. In the scalar case, under suitable hypotheses on f and g again, the relaxed functional $\bar{I}$ is still an integral functional which can be represented as

$$\bar{I}(u) = \int_\Omega [f^{**}(\nabla u(x)) + g(u(x))]\,dx,$$

where f^{**} is the *convex envelope* of f, i.e. the greatest convex and lower semicontinuous function laying below f. All minimizers for I are minimizers for the relaxed functional as well and they are characterized by the property of laying in the set where f and f^{**} coincide, i.e.

$$(*) \qquad f(\nabla u(x)) = f^{**}(\nabla u(x)), \qquad \text{for a.e. } x \in \Omega,$$

whenever u is a minimizer for I. At this stage, one aims at modifying a minimizer of $\bar{I}$ in such a way that it is still a minimizer and simultaneously satisfies $(*)$. This idea has been fruitfully applied to various one dimensional scalar problems ($N = 1$, $M \geq 1$), see for instance [35], [12] and [14]. In all these papers, the derivative of a minimizer of the relaxed functional is suitably modified by means of an explicit construction in [35] and by Lyapunov like theorems in [12] and [14]. A minimizer for I is then given by a primitive of this modified derivative.

This technique can be applied also to radially symmetric scalar problems with $N > 1$. In Chapter 2 (see [15]), a problem of this kind is considered, namely

$$\min\left\{\int_B [h(\|\nabla u(x)\|) + g(u(x))]\,dx :\ u \in W_0^{1,1}(B)\right\}$$

where B is the open unit ball of $\mathbb{R}^N$, $h : [0,\infty) \to \overline{\mathbb{R}}$ is lower semicontinuous and superlinear and $g : \mathbb{R} \to \mathbb{R}$ is convex and monotonic. We first prove that the relaxed functional has at least one radially symmetric minimizer. By using polar coordinates, the gradient of this radially symmetric solution can be modified along the rays of B as in the one dimensional cases to obtain a solution of the original problem.

When $N > 1$ and the problem is not radially symmetric, a further difficulty arises. Indeed, by modifying the gradient of a function, one does not obtain in general the gradient of a function again. Therefore, the approach described above is to be followed with greater care. This is done for instance in [40] and [9], [10] when $g = 0$. In particular, these latter papers provide a necessary and sufficient condition for the existence of solutions to the minimum problem

$$\min\left\{\int_\Omega f(\nabla u(x))\,dx : \; u \in u_0 + W_0^{1,1}(\Omega)\right\}$$

where u_0 is an affine boundary datum. The idea underlying the sufficiency part of the papers is that of constructing a solution to the relaxed problem on a domain enjoying suitable geometrical properties wich satisfies $(*)$. Once such a local solution is available, it is extended to the whole Ω by Vitali's covering theorem.

It is plain that this approach based on finding a local solution to the problem which is then extended by a covering argument cannot work when $g \neq 0$. For instance, consider the problem of minimizing on $W_0^{1,1}(\Omega)$ the functional

$$\int_\Omega [f(\nabla u(x)) + u(x)]\,dx.$$

Here, a patchwork of local solutions having homogeneous boundary values badly behaves with respect to the issue of making the integral $\int_\Omega u\,dx$ as small as possible. To the author's knowledge, problems of this kind have been treated only in cases when a candidate to be a solution has been previously identified somehow. This is done in [11] and [48] for the functional considered above in the case of a non negative and radially symmetric function f, i.e. $f(\xi) = h(|\xi|)$: under suitable assumptions involving h and Ω, the expected solution is, up to a multiplicative constant, a distance function satisfying $(*)$.

A further example of this approach is given in Chapter 3 (see [8]). There, we consider the problem of minimizing the previous functional on $u_0 + W_0^{1,1}(\Omega)$ in the case f is non negative and vanishes on the boundary of a bounded and convex neighbourhood K of the origin and the boundary datum u_0 is Lipschitz continuous. Relying on a result presented in [40], the candidate u to be a solution is identified among the Lipschitz continuous functions in $u_0 + W_0^{1,1}(\Omega)$ whose gradient lies in the boundary of K. By using a technique similar to the one introduced in [11], we prove that such function u is actually a solution by showing that it verifies the Euler-Lagrange equations for a properly chosen convex functional which lies below the original functional and agrees with it on u.

Now, let us turn to the vectorial problems. We shall confine ourselves to the case $g = 0$, so that the functional I reduces to

$$I(u) = \int_\Omega f(\nabla u(x))\, dx.$$

Again, under suitable regularity and growth assumptions on the integrand, the relaxed functional $\bar{I}$ associated with I admits an integral representation of the form

$$\bar{I}(u) = \int_\Omega Qf(\nabla u(x))\, dx,$$

where Qf is the *quasiconvex envelope* of f, i.e. the largest quasiconvex function laying below f. Unfortunately, no pointwise characterization of quasiconvex functions is available, thus adding a further difficulty when dealing with vectorial problems. Indeed, the quasiconvex envelope has been explicitely computed only for very few special functions. Among these, we quote the function

$$f(\xi) = \begin{cases} 1 + \|\xi\|^2 & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0 \end{cases}$$

whose quasiconvex envelope has been computed by R.V. Kohn and G. Strang in [34]. The corresponding minimum problem, originating as an optimal design problem, has been recently solved in [22] where a necessary and sufficient conditon for the existence of minimizers is given. Further existence results for vectorial problems are available in some cases where the computation of the quasiconvex envelope reduces to the computation of a convex envelope. Among these, we quote [40] which deals with an integrand which is a function of the determinant, [18] where the determinant is replaced by a quasiaffine function and [22] again where a more general functional is considered.

We end this short survey by mentioning those vectorial problems whose solution requires the construction of a function whose gradient takes only prescribed values, i.e. those problems which can be reduced to a system of Hamilton-Jacobi equations. We refer to [23] for various existence results on these equations. Among these problems, we mention in particular the so-called problem of potential wells which consists of in minimizing the functional I when f is a non negative function vanishing only at certain potential wells, described by the rotation of a finite number of matrices. In Chapter 4 (see [16]), we solve this problem with a homogeneous boundary datum when Ω is an open subset of $\mathbb{R}^3$ and there are two wells described by the rotations of the identity matrix $\mathbb{1}$ and $-\mathbb{1}$. The argument, which we sketch below, is entirely constructive. We define, on an open cube of $\mathbb{R}^3$, a Lipschitz continuous function u vanishing on the boundary of the cube whose gradient satisfies

$$\nabla u(x) \in SO(3)\mathbb{1} \cup SO(3)(-\mathbb{1}), \quad \text{for a.e. } x$$

This local solution is then extended to the domain Ω by a covering argument. Finally, we mention that a further existence result for a two wells problem in $\mathbb{R}^2$ has been proved in [40].

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Chapter 1
On the Validity of the Euler Lagrange Equations

As is well known (see [29]), problems of the kind:

$$\text{minimize the functional } \int_\Omega \left[\frac{1}{\alpha}\|\nabla u(x)\|^\alpha + g(u)\right] dx$$

on $u \in u_0 + W_0^{1,\alpha}(\Omega)$, $1 < \alpha < \infty$, at once admit solutions (under some suitable assumptions) and solutions do satisfy the Euler Lagrange equation

$$\operatorname{div}(\|\nabla u\|^{\alpha-2}\nabla u) = g'(u).$$

We wish to investigate the limiting case $\alpha = \infty$. More precisely, the problem we consider is the problem of minimizing the functional

$$F(u) = \int_\Omega (j_{[0,1]}(\|\nabla u(x)\|) + g(u))dx$$

for $u \in u_0 + W_0^{1,\infty}(\Omega)$, where $j_{[0,1]}$ is the indicator function of the closed interval $[0,1]$. The map $y \to j_{[0,1]}(\|y\|)$ is convex, lower semicontinuous and extended valued. The coercivity requirement, for the existence of solutions, is obviously satisfied; hence, when the functional F assumes a finite value for at least one function $u \in W_0^{1,\infty}(\Omega)$, the minimization problem admits a solution. Even though the integrand is not differentiable, the convexity of the function $y \to j_{[0,1]}(\|y\|)$ leads one to expect the validity of an Euler-Lagrange equation in the form

$$\operatorname{div} p(x) = g'(u(x)) \quad \text{for} \quad p(x) \in \partial j_{[0,1]}(\|\nabla u(x)\|).$$

Convex analysis, however, is of no help. In fact, the basic assumption needed for the applicability of the theory, namely the continuity of the map $\xi \in L^\alpha(\Omega) \to \int_\Omega j_{[0,1]}(\|\xi(x)\|)dx$ is violated in this case, no matter what α is.

Here, under some assumptions on g (that include the linear case) but essentially without assumptions on Ω and on u_0, we show that the Euler Lagrange equation holds for a solution to the minimum problem. In particular, whenever the functional F is finite only along one function (the boundary function u_0), it follows that u_0 must be a solution to the Euler Lagrange equation. Equivalently, from an optimal control point of view, we are interested in the conditions satisfied by the solutions to the problem of minimizing

$$\int_\Omega g(u)dx$$

for $u \in u_0 + W_0^{1,\infty}(\Omega)$, subject to the Hamilton Jacobi control equation

$$\nabla u(x) = v, \qquad v \in B$$

where B is the unit ball of $\mathbb{R}^N$, i.e. $\{y \in \mathbb{R}^N : \|y\| \leq 1\}$. We show that to a solution u we can associate a map $p \in (L^1(\Omega))^N$ such that, denoting by H the map

$$H(u, p, v) = -g(u) + < p, v >,$$

we have:

$$\nabla u = \nabla_p H; \quad \operatorname{div} p = -\frac{\partial H}{\partial u}$$

and a.e. $H(u(x), p(x), v(x)) = \max_{w \in B}\{H(u(x), p(x), w)\}$, i.e. the solution satisfies the Pontriagin Maximum Principle ([41]).

In contrast with the usual approach, where regularity of the solution is obtained as a consequence of its being a solution to the Euler Lagrange equation, in our case we must prove first some regularity of the solution in order to obtain, from it, the validity of the Euler Lagrange equation.

1.1. Main results

We consider the following problem

$$\text{(P)} \qquad \min_{u_0 + W_0^{1,\infty}(\Omega)} \int_\Omega (j_{[0,1]}(\|\nabla u(x)\|) + g(u(x)))\, dx$$

It is our purpose to prove the following theorem.

Theorem 1.1.1. *Let Ω be an open bounded subset of $\mathbb{R}^N$, let $g : \mathbb{R} \to \mathbb{R}$ be differentiable and strictly monotonic. Let u_0 be in $W^{1,\infty}(\Omega)$. Assume the existence of at least a function in $u_0 + W_0^{1,\infty}(\Omega)$ that makes the functional F finite. Let u be a solution to the minimum problem* (P). *Then u is an integral solution to the Euler Lagrange inclusion*

$$\operatorname{div} p(x) = g'(u(x)) \quad \text{for} \quad p(x) \in \partial j_{[0,1]}(\|\nabla u(x)\|).$$

To prove the theorem we should consider separately the two cases g increasing and g decreasing. We shall present the proof for the case g increasing. We shall use the notation: for $A \subset \mathbb{R}^N$, $\rho(x, A) = \inf_{y \in A}\{\|x - y\|\}$.
The following lemma is a first regularity result on the solution u.

Lemma 1.1.2. *Under the same assumptions as in Theorem 1.1.1, let u be a solution to problem* (P). *Then for every x_0 and $r > 0$ such that $B_r(x_0)$ is contained in Ω, we have*

$$\sup\{u(x) - u(x_0) : \|x - x_0\| = r\} = r.$$

Proof. The map u must be, on $B_r(x_0)$, lipschitzean of Lipschitz constant 1. Hence the supremum above cannot be larger than r. Assume it is equal to ζr, with $\zeta < 1$. Let η be a Lipschitzean function such that

(i) $\eta(x_0) = -r$,

(ii) $\|\nabla \eta\| = 1$,

(iii) $\eta(x) = 0, x \in \Omega \setminus B_r(x_0)$.

Fix $\lambda \in (\zeta, 1)$ and consider the function

$$\eta_\lambda(x) = \lambda\eta(x) - (u(x) - u(x_0)) + \zeta r.$$

We have that: for $x \in \partial B_r(x_0)$, $\eta_\lambda(x) = \zeta r - (u(x) - u(x_0)) \geq 0$ while $\eta_\lambda(x_0) = -\lambda r + \zeta r < 0$. Call E the connected component of the set $\{\eta_\lambda \leq 0\}$ containing x_0 (the measure of E is positive). The map η_λ^- is then defined to be

$$\eta_\lambda^-(x) = \begin{cases} \eta_\lambda(x), & x \in E, \\ 0 & \text{elsewhere.} \end{cases}$$

We have that $\eta_\lambda^-(x) = 0$ for $x \in \partial B_r$ and that $\eta_\lambda^-(x_0) < 0$; moreover

$$\nabla \eta_\lambda^-(x) = \begin{cases} \lambda \nabla \eta(x) - \nabla u(x) & x \in E, \\ 0 & \text{elsewhere,} \end{cases}$$

so that $\|\nabla \eta_\lambda\| \leq 2$. It is our purpose to show that for parameters $t > 0$ sufficiently small, we have $\|\nabla u + t\nabla \eta_\lambda^-\| \leq 1$.

a) Consider first those $x \in E$ such that $\|\nabla u(x)\| > \frac{1+\lambda}{2}$. We have:

$$\langle \nabla u, \nabla \eta_\lambda^- \rangle \leq \lambda \|\nabla u\| \|\nabla \eta\| - \|\nabla u\|^2 = \|\nabla u\|(\lambda - \|\nabla u\|) \leq \|\nabla u\| \frac{\lambda - 1}{2} < 0$$

and

$$|\langle \nabla u, \nabla \eta_\lambda^- \rangle| = -\langle \nabla u, \nabla \eta_\lambda^- \rangle \geq \|\nabla u\| \frac{1-\lambda}{2} \geq \frac{1-\lambda^2}{4}.$$

Hence, for $t \in (0, \frac{1-\lambda^2}{8})$ and a.e. $x \in E$, we have that $2|\langle \nabla u, \nabla \eta_\lambda^- \rangle| > 4t > t\|\nabla \eta_\lambda^-\|^2$. Since

$$\|\nabla u + t\nabla \eta_\lambda^-\|^2 = \|\nabla u\|^2 + t^2 \|\nabla \eta_\lambda^-\|^2 + 2t\langle \nabla u, \nabla \eta_\lambda^- \rangle$$

we obtain

$$\begin{aligned} \|\nabla u + t\nabla \eta_\lambda^-\|^2 &= \|\nabla u\|^2 + t(t\|\nabla \eta_\lambda^-\|^2 + 2\langle \nabla u, \nabla \eta_\lambda^- \rangle) \\ &\leq 1 + t(t\|\nabla \eta_\lambda^-\|^2 - 2|\langle \nabla u, \nabla \eta_\lambda^- \rangle|) < 1. \end{aligned}$$

b) Consider now those $x \in E$ such that $\|\nabla u(x)\| \leq \frac{1+\lambda}{2}$. Then, for t in $(0, \frac{1-\lambda^2}{8})$, we simply have

$$\|\nabla u + t\nabla \eta_\lambda^-\| \leq \|\nabla u(x)\| + t\|\nabla \eta_\lambda^-\| < \frac{1+\lambda}{2} + \frac{1-\lambda}{4} 2 = 1.$$

Hence, from the above, the variation η_λ^- is admissible, in the sense that a.e. in Ω, for all t sufficiently small,

$$\|\nabla u + t\nabla \eta_\lambda^-\| \leq 1.$$

For one such t, since: $j_{[0,1]}(\|\nabla u + t\nabla \eta_\lambda^-\|) = 0$, a.e. in Ω; $u + t\eta_\lambda^- \leq u$, a.e. in Ω; $u + t\eta_\lambda^- < u$, a.e. in E, we have

$$F(u + t\eta_\lambda^-) = \int_\Omega g(u(x) + t\eta_\lambda^-(x))\, dx < \int_\Omega g(u(x))\, dx = F(u)$$

a contradiction. □

As an obvious consequence of the previous lemma, we have the following proposition.

Proposition 1.1.3. *For a.e.* $x \in \Omega$, $\|\nabla u(x)\| = 1$.

Proof of Theorem 1.1.1. We split the proof into five steps.

Step a) From Lemma 1.1.2, and the Lipschitz continuity of u, it follows that the following property holds: to any point $x \in \Omega$ we can associate at least a unit vector (a direction) d^x and (at least) a non vanishing interval $[0, l)$ such that for $t \in [0, l)$, $u(x + td^x) - u(x) = t$. The set of these directions gives rise to a multivalued map $x \rightarrow D(x)$.
Given x and d^x, let b^x be such that $[0, b^x)$ is the largest such interval. Then it is easy to see that $x + b^x d^x \in \partial\Omega$. In fact let y be on this segment. To y we can associate at least one direction d^y with the property stated above. If this direction d^y does not coincide with d^x, in a neighborhood of y we could contradict the fact that u is lipschitzean with constant 1. This in particular shows that d^y is unique whenever there exist $x \in \Omega$, a direction d^x and t in the interval $(0, b)$ such that $y = x + t\, d^x$. For fixed x and d^x, call (a^x, b^x) the largest open interval such that $u(x + t_1 d^x) - u(x + t_2 d^x) = t_2 - t_1$ for $t_2 > t_1$ and t_1 and t_2 in (a^x, b^x). Whenever x belongs to $S(x) = \{x + td^x : t \in (a^x, b^x)\}$, i.e. when $a^x < 0$, we have that d^x is unique and that a^x and b^x depend only on x, i.e. we can consider the univalent maps $x \rightarrow d(x) = d^x$, $x \rightarrow a(x) = a^x$ and $x \rightarrow b(x) = b^x$. It will be convenient to set

$$\mathcal{S} = \cup_{\{x \in \Omega\}} S(x).$$

Step b) For k in $\{1, ...N\}$, let d_k denote the k-th component of the vector d. About the properties of the map $x \rightarrow d(x)$ we have the following claim, a first regularity result on ∇u.

Claim 1. Fix $k \in \{1, ...N\}$ and $\varepsilon > 0$. On

$$E_\varepsilon^k = \{x \in \mathcal{S} : (x - \varepsilon d(x), x + \varepsilon d(x)) \subset S(x);\ d_k(x) \geq \frac{1}{\sqrt{N}};\ \rho(x, \partial\Omega) \geq 3\varepsilon\}$$

the map $x \rightarrow d(x)$ is Lipschitz continuous with constant $2\sqrt{/}\varepsilon$.

Proof of Claim 1. Consider two points P and P' in E_ε^k and set $d = d(P)$, $d' = d(P')$. In the case $\|P - P'\| \geq \frac{\varepsilon}{2\sqrt{n}}\|d - d'\|$, we have

$$\|d(P) - d(P')\| \leq \frac{2\sqrt{N}}{\varepsilon}\|P - P'\|.$$

Hence we consider the case $\|P-P'\| < \frac{\varepsilon}{2\sqrt{N}}\|d-d'\|$. Set r to be $\{P+\lambda d : \lambda \in \mathbb{R}\}$ and r' to be $\{P' + \lambda d' : \lambda \in \mathbb{R}\}$. Let $O \in r$ and $O' \in r'$ be the two points of minimal distance for r and r'; then $\langle O'-O, d\rangle = \langle O'-O, d'\rangle = 0$. When $O \neq O'$ we shall refer to the unique three dimensional space containing r and r' (the case $O = O'$ being similar and simpler). On the plane orthogonal to $O'-O$ and containing r, let r'' be the projection of the line r'. Also let P^* be the nearest point to P on r'', so that $\|P-P'\| \geq \|P-P^*\|$. The point P'' on r'' is defined to be the point having $\|P-O\| = \|P''-O\|$ and lying on the same side (w.r.t. 0) as P^*. By elementary geometry we have

$$\frac{\|P-P''\|}{\|P-O\|} = \frac{\|d-d'\|}{1}.$$

Consider the triangle O, P, P'', and let H be $\frac{1}{2}P + \frac{1}{2}P''$. We obtain

$$\frac{\|P-P^*\|}{\|P-P''\|} = \frac{\|H-O\|}{\|P''-O\|}.$$

Since, by the definition of E_ε^k, we have that $\frac{\|H-O\|}{\|P''-O\|} \geq \frac{1}{\sqrt{N}}$, we obtain

$$\|P-P''\| = \|P-P^*\|\frac{\|P''-O\|}{\|H-O\|} \leq \sqrt{N}\|P^*-P\|$$

so that

$$\|P-O\| = \frac{\|P-P''\|}{\|d-d'\|} \leq \sqrt{N}\frac{\|P-P^*\|}{\|d-d'\|} \leq \sqrt{N}\frac{\|P-P'\|}{\|d-d'\|} \leq \frac{\varepsilon}{2}.$$

For symmetry reasons, also $\|P'-O'\| \leq \frac{\varepsilon}{2}$. Hence we have obtained that both O and O' are in Ω, and u is therefore defined at O and O'. At this point we are free to assume that $u(O) \geq u(O')$.

Let A and D the extremes of a segment on $S(P)$ centered on O and of halflength $\frac{\varepsilon}{2}$ and B', C' be the same on $S(P')$ with respect to O'. We have: $\|P-A\| \leq \varepsilon$, $\|P-D\| \leq \varepsilon$ and

$$\|P-B'\| \leq \|B'-O'\|+\|O'-P'\|+\|P'-P\| \leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}+\frac{\varepsilon}{\sqrt{N}} \leq 2\varepsilon, \quad \|P-C'\| \leq 2\varepsilon.$$

Therefore, all the points A, D, B' and C' lie in the ball $B_{2\varepsilon}(P) \subset \Omega$. On this set, the map u is lipschitzean of Lipschitz constant 1. Let B and C the projections of the points B' and C' on the plane orthogonal to $O'-O$ and containing O. We can assume that

$$u(D)-u(A)=\|D-A\| \quad \text{and} \quad u(C')-u(B')=\|C'-B'\|=\|C-B\|.$$

Hence, we have

$$\begin{aligned}\|B'-D\| &\geq u(D)-u(B')=u(D)-u(O)+u(O)\\ &-u(O')+u(O')-u(B')=\|D-O\|+u(O)-u(O')+\|B'-O'\|\\ &\geq \|D-O\|+\|B'-O'\|,\end{aligned} \tag{1.1.1}$$

while, on the other hand,

$$\begin{aligned}\|B'-D\|^2 &= \|B'-B\|^2+\|B-D\|^2\\ &=\|O'-O\|^2+\|B-O\|^2+\|O-D\|^2+2\langle D-O,O-B\rangle.\end{aligned} \tag{1.1.2}$$

By (1.1.1) and (1.1.2) we obtain

$$\begin{aligned}\|O'-O\|^2+\|B-O\|^2+\|O-D\|^2+2\langle O-D,B-O\rangle&\\ \geq \|D-O\|^2+\|B'-O'\|^2+2\|D-O\|\|B'-O'\|&\end{aligned}$$

hence

$$\begin{aligned}\|O'-O\|^2 &\geq 2\|D-O\|\,\|B-O\|\left(1-\langle\frac{D-O}{\|D-O\|},\frac{B-O}{\|O-B\|}\rangle\right)\\ &=\|D-O\|\,\|B'-O'\|(2-2\langle d,d'\rangle)\|=\left(\frac{\varepsilon}{2}\right)^2\|d-d'\|^2.\end{aligned}$$

It follows then that $\|d-d'\|\leq\frac{2}{\varepsilon}\|O-O'\|\leq\frac{2}{\varepsilon}\|P-P'\|$.
Hence we have

$$\|d(P)-d(P')\|\leq\frac{2\sqrt{N}}{\varepsilon}\|P-P'\|$$

for every P and P' in E^k_ε. This proves the claim.

Step c) Purpose of this step is to define a countable partition of $\mathcal{S}$ consisting of measurable sets.

Consider the set $\mathcal{P}$ of pairs (p,q), where p and q are integers and q is positive, and let $\sigma:\mathbb{N}\to\mathcal{P}$ be a numbering of this set. Denote by (p_n,q_n) the image $\sigma(n)$. To $n\in\mathbb{N}$ and $k\in 1,\dots N$ we associate the two disjoint sets

$$\begin{aligned}E_n^{+,k}=\Big\{y\in S(x): x_k=\frac{p_n}{q_n};\ d_k(x)=\sup_{1\leq i\leq N}|d_i(x)|;\ \rho(x,\partial\Omega)\geq\frac{3}{q_n}&\\ \text{and } x-\frac{1}{q_n}d(x),x+\frac{1}{q_n}d(x)\in S(x)\Big\}&\end{aligned}$$

and

$$E_n^{-,k} = \left\{ y \in S(x) : x_k = \frac{p_n}{q_n};\ d_k(x) = -\sup_{1\le i\le N} |d_i(x)|;\ \rho(x,\partial\Omega) \ge \frac{3}{q_n} \right.$$
$$\left. \text{and } x - \frac{1}{q_n}d(x), x + \frac{1}{q_n}d(x) \in S(x) \right\}$$

In order to obtain a partition of $\mathcal{S}$ we operate in the standard way. Set $\Sigma_1^{+,1} = E_1^{+,1}$ and, in general,

$$\Sigma_1^{+,k+1} = E_1^{+,k+1} \backslash \{\cup_{i=1,\dots k} \Sigma_1^{+,i}\}.$$

Set $\Sigma_{n+1}^{+,1} = E_{n+1}^{+,1} \backslash \{\cup_{i=1,\dots N; m=1,\dots n} \Sigma_m^{+,i}\}$ and

$$\Sigma_{n+1}^{+,k+1} = E_{n+1}^{+,k+1} \backslash \{(\cup_{i=1,\dots N; m=1,\dots n} \Sigma_m^{+,i}) \cup (\cup_{i=1,\dots k} \Sigma_{n+1}^{+,i})\}.$$

An analogous procedure is applied to the family $E_n^{-,k}$ to yield the disjoint family $\{\Sigma_n^{-,k}\}$. This second family is defined so as to be disjoint from $\{\Sigma_n^{+,k}\}$ as well.

We have defined a disjoint family. We wish to show that it covers $\mathcal{S}$.

Claim 2. $\mathcal{S} = \cup_{k=1,\dots N; n\in\mathbb{N}} (\Sigma_n^{+,k} \cup \Sigma_n^{-,k})$.

Proof of Claim 2. We have only to show that

$$\cup_{k=1,\dots N; n\in\mathbb{N}} (\Sigma_n^{+,k} \cup \Sigma_n^{-,k}) \supset \mathcal{S}.$$

Since

$$\cup_{k=1,\dots N; n\in\mathbb{N}} (\Sigma_n^{+,k} \cup \Sigma_n^{-,k}) = \cup_{k=1,\dots N; n\in\mathbb{N}} (E_n^{+,k} \cup E_n^{-,k}),$$

we have to show that, for every $x \in \Omega$, $S(x)$ is contained in the set at the right hand side. Let $x \in S(x')$ for some $x' \in \Omega$. There exists a k such that either $d_k(x) = \sup_{i=1,\dots N} |d_i(x)|$ or $d_k(x) = -\sup_{i=1,\dots N} |d_i(x)|$. Let us consider the first case (the other being analogous). Call $\Delta = \rho(x, \partial\Omega)$. Since $S(x')$ is an open interval, there exists δ, $0 < \delta < \frac{\Delta}{2}$, such that

$$\{x + \lambda d(x) : -\delta \le \lambda \le \delta\} \subset S(x').$$

Let q be a positive integer such that $1/q < \delta/2\sqrt{N}$; there exists an integer p such that $|(p/q) - x_k| \le 1/2q$. The point $y = x + [(\frac{p}{q} - x_k)/d_k(x)]d(x)$ has the following properties: its k-th component y_k equals $\frac{p}{q}$; recalling that $d_k(x) \geq 1/\sqrt{N}$, we have that

$$\|x - y\| = \left| \frac{\frac{p}{q} - x_k}{d_k(x)} \right| \leq \frac{\sqrt{N}}{2q} < \frac{\delta}{4}.$$

As a consequence, an interval (on $S(x')$) centered on y and halflength $\frac{1}{q}$ is contained in $S(y)$ $(= S(x'))$ and contains x. Moreover, $\rho(y, \partial\Omega) \geq \Delta - (\delta/4) \geq (7/4)\delta \geq 3/q$. Hence, setting $n = \sigma^{-1}(p, q)$, we have $x \in E_n^{+,k}$. This proves Claim 2.

Claim 3. The measure of $\Omega \backslash S$ equals zero.

Proof of Claim 3. Since the subset of Ω of those points where u is not differentiable is of measure zero, it is enough to show that it is of measure zero the subset of $\Omega \backslash S$ where u is differentiable. In particular, for x in such a set, we can assume that there exists a unique vector d^x, as defined in Step a), otherwise we would contradict the differentiability at x.
Since $\cup_{k,n}(E_n^{+,k} \cup E_n^{-,k}) = \mathcal{S}$, we shall prove that $m(\Omega \backslash \cup_{k,n} (E_n^{+,k} \cup E_n^{-,k})) = 0$. Assume, on the contrary, that this set is of positive measure and let x_0 be a point of density of it. As it easy to see, the map $x \to D(x)$ as defined in Step a) is upper semicontinuous; it follows then that for every ε there exists δ such that $\|d^x - d(x_0)\| < \varepsilon$ for $\|x - x_0\| < \delta$ and $d^x \in D(x)$. By changing coordinates we shall assume $x_0 = 0$ and $d_N(x_0) = 1$.
Let us consider the (family of) cubes $Q_\ell = \{x : 0 \leq |x_i| \leq \ell, \ i = 1, ...N\}$ and let us choose ℓ so small that, for every $x \in Q_\ell$, we have:

i) $\rho(x, \partial\Omega) \geq \ell$;

ii) $d_N(x) \geq \max\{\frac{1}{\sqrt{N}}, \frac{4}{5}\}$.

Let us consider the subset of Q_ℓ defined by

$$I_\ell = \{x : x_N = 0 \text{ and } |x_i| \leq \frac{\ell}{2}, \ i = 1, ...N - 1\}.$$

The $(N-1)$-dimensional measure of I_ℓ is ℓ^{N-1}, while the N-dimensional measure of Q_ℓ is $(2\ell)^N$. Fix t, $\frac{\ell}{3} \leq t \leq \frac{2}{3}\ell$ and consider, on the hyperplane $\{x_N = t\}$, the subset of Q_ℓ

$$P_t = \left\{ x + t\frac{d(x)}{d_N(x)} : x \in I_\ell \text{ and } d \in D(x) \right\}.$$

Every point y in this set is in the interior of $S(y)$; $D(y) = d(y)$ so that it is possible to define the map $F_t : P_t \to I_\ell$ defined by

$$F_t(y) = y - t\frac{d(y)}{d_N(y)}.$$

Notice that P_t is contained in E_ε^N as defined in Claim 1, with $\varepsilon = \frac{\ell}{3}$, hence the restriction of d to P_t is lipschitzean with constant $\frac{6\sqrt{N}}{\ell}$. For y and y' in P_t we have

$$\left\| \frac{d(y)}{d_N(y)} - \frac{d(y')}{d_N(y')} \right\| \leq \frac{\|d(y) - d(y')\|}{d_N(y)} + \frac{\|d(y')\|}{d_N(y)d_N(y')}|d_N(y) - d_N(y')|$$
$$\leq \sqrt{N}\frac{6\sqrt{N}}{\ell}\|y - y'\| + N\frac{6\sqrt{N}}{\ell}\|y - y'\| \leq \frac{12N\sqrt{N}}{\ell}\|y - y'\|.$$

Hence the map F_t is lipschitzean with constant $1 + t\frac{12N\sqrt{N}}{\ell} \leq 1 + 8N\sqrt{N}$. Considering the $(N-1)$-dimensional measure of a subset A of P_t, we have then $m(F_t(A)) \leq (1 + 8N\sqrt{N})m(A)$. Hence

$$m(P_t) \geq \frac{m(F_t(P_t))}{1 + 8N\sqrt{N}} = \frac{m(I_\ell)}{1 + 8N\sqrt{N}} = \frac{\ell^{N-1}}{1 + 8N\sqrt{N}}.$$

The set $\cup_{\frac{\ell}{3} \leq t \leq \frac{2\ell}{3}} P_t$ is contained in S and, by Fubini's Theorem, its N-dimensional measure is at least $\ell^N/(3 + 24N\sqrt{N})$, a fixed fraction of the total measure of Q_ℓ. Hence x_0 cannot be a point of density. This proves Claim 3.

Claim 4. For every $k = 1, ...N$, for every n, the sets $\Sigma_n^{\pm,k}$ are measurable.

From now up to Step e) we shall fix a choice of either + or −, of k and of n. Hence, for simplicity sake, we will drop ±, n ,k and simply denote $E_n^{\pm,k}$ by E and $\Sigma_n^{\pm,k}$ by Σ. For every $x \in \mathbb{R}^N$ we shall denote by $\hat{x}$ the $(N-1)$-dimensional vector $(x_1, ...x_{k-1}, x_{k+1}, ...x_N)$, and for every $\hat{x} \in \mathbb{R}^{N-1}$ we shall denote by $\pi\hat{x}$ the N-vector $(\hat{x}_1, ...\hat{x}_{k-1}, \frac{p_n}{q_n}, \hat{x}_k, ...\hat{x}_{N-1})$. It is convenient to set $\hat{E}$ to be the subset of $\mathbb{R}^{N-1}$ defined by $\hat{E} = \{\hat{x} : x \in E \cap \{x_k = \frac{p_n}{q_n}\}\}$, and analogously for $\hat{\Sigma}$. Consider $(\hat{x}, t)$, $\hat{x} \in \hat{E}, a(\pi\hat{x}) < t < b(\pi\hat{x})$ and define the map

$$\Xi(\hat{x}, t) = x + td(\pi\hat{x}).$$

Since, by Claim 1, d is Lipschitz continuous on $E \cap \{x_k = \frac{p_n}{q_n}\}$, the map Ξ is uniformly Lipschitz continuous.

Proof of Claim 4. As it is easy to see, both the maps $a(\pi\hat{x})$ and $b(\pi\hat{x})$ are lower semicontinuous on $\hat{E}$, and $\hat{E}$ can be described as the intersection of a closed set with the counterimages through $a\circ\pi$ and $b\circ\pi$ of the interval $[-\frac{1}{q_n},\frac{1}{q_n}]$, hence it is a measurable set. The subset of $\mathbb{R}^N$ described by $\{(\hat{x},t):\hat{x}\in\hat{E}; a(\pi\hat{x})<t<b(\pi\hat{x})\}$ is measurable and so is E, its image through the Lipschitz continuous map Ξ. It follows that Σ is measurable. This proves Claim 4.

Step d) We wish to study the properties of the maps $\Xi(\hat{x},t)$ defined above and of $J\Xi(\hat{x},t)$, the absolute value of $det\nabla\Xi$. For a.e. $(\hat{x},t)\in(\Xi)^{-1}(\Sigma)$ we have that $\nabla\Xi$ exists and a computation shows that it can be obtained as follows. Consider the $N\times N$ matrix $\nabla d(\pi\hat{x})$ and form the matrix $I+t\nabla d(\pi\hat{x})$. Replace the k-th column by the components of $d(\pi\hat{x})$. This is the matrix $\nabla\Xi(\hat{x},t)$. Hence $J\Xi$ is uniformly bounded on Σ. It is also a.e. different from zero. In fact, differentiating the identity $\|d(x)\|=1$, we obtain $(\nabla d(x))\,d(x)=0$, so that $(I+t\nabla d(\pi\hat{x}))\,d(\pi\hat{x})=d(\pi\hat{x})$. By Cramer's Rule, and the above computation of $\nabla\Xi$, we obtain

$$d_k(\pi\hat{x})=\pm\frac{J\Xi(\hat{x},t)}{det(I+t\nabla d(\pi\hat{x}))}.$$

Since (on Σ), $|d_k|>\frac{1}{\sqrt{N}}$, we finally have $J\Xi(\hat{x},t)\neq 0$.
We wish to define a map α and prove it is in $L^1(\Sigma)$. Define first the map β on $\Xi^{-1}(\Sigma)$ setting

$$\beta(\hat{x},t)=\frac{1}{J\Xi(\hat{x},t)}\int_{a(\pi\hat{x})}^{t} g'(u(\Xi(\hat{x},s)))\,J\Xi(\hat{x},s)\,ds.$$

For $x\in\Sigma$ define α as

$$\alpha(x)=\beta(\Xi^{-1}(x)).$$

Claim 5. $\alpha\in L^1(\Sigma)$.

Proof of Claim 5. We recall the change of variables formula ([26], Theorem 2, p.99) that states that, for a function $v\in L^1$ and an invertible and Lipschitzean trasformation Ξ, we can write

$$\int_{\mathbb{R}^N} v(\Xi(\hat{x},t)\,J\Xi(\hat{x},t)\,d(\hat{x},t)=\int_{\mathbb{R}^N} v(x)\,dx.$$

By this formula we obtain

$$\int_\Sigma g'(u(x))\,dx=\int_{(\Xi)^{-1}(\Sigma)} g'(u(\Xi(\hat{x},t)))J\Xi(\hat{x},t)\,d(\hat{x},t)$$

$$=\int_{\hat{\Sigma}}\left(\int_{a(\pi\hat{x})}^{b(\pi\hat{x})} g'(u(\Xi(\hat{x},t)))J\Xi(\hat{x},t)\,dt\right)d\hat{x}.$$

Similarly, by the change of variables formula and applying the definitions of α and β, we have

$$\int_{\Sigma}\alpha(x)\,dx=\int_{(\Xi)^{-1}(\Sigma)}\beta(\hat{x},t)J\Xi(\hat{x},t)\,d(\hat{x},t)=\int_{\hat{\Sigma}}\left(\int_{a(\pi\hat{x})}^{b(\pi\hat{x})}\beta(\hat{x},t)J\Xi(\hat{x},t)\,dt\right)d\hat{x}$$
$$=\int_{\hat{\Sigma}}\left(\int_{a(\pi\hat{x})}^{b(\pi\hat{x})}\int_{a(\pi\hat{x})}^{t}g'(u(\Xi(\hat{x},s)))J\Xi(\hat{x},s)\,ds\,dt\right)d\hat{x}.$$

Integrating by parts we obtain that

$$\int_{a(\pi\hat{x})}^{b(\pi\hat{x})}\int_{a(\pi\hat{x})}^{t}g'(u(\Xi(\hat{x},s)))J\Xi(\hat{x},s)\,ds\,dt$$
$$=-\int_{a(\pi\hat{x})}^{b(\pi\hat{x})}(t-b(\pi\hat{x}))g'(u(\Xi(\hat{x},t)))J\Xi(\hat{x},t)\,dt.$$

Hence

$$\int_{\Sigma}\alpha(x)\,dx\le\int_{\hat{\Sigma}}diam(\Omega)\left(\int_{a(\pi\hat{x})}^{b(\pi\hat{x})}g'(u(\Xi(\hat{x},t)))J\Xi(\hat{x},t)\,dt\right)d\hat{x}$$
$$=diam(\Omega)\int_{\Sigma}g'(u(x))\,dx.$$

Step e) Since the sets $\Sigma_n^{\pm,k}$ are disjoint and (with the addition of a null set) form a partition of Ω, α is actually defined a.e. on Ω and by adding the previous inequalities over $+$ and $-$ and all k and n, we have that $\alpha\in L^1(\Omega)$.
Setting $p(x)=\alpha(x)\frac{\nabla u(x)}{\|\nabla u(x)\|}$, we want to show that the pair $(u(x),p(x))$ is a solution to the Euler Lagrange equation for the minimization problem (P).
Fix arbitrarily ϕ in $C_0^\infty(\Omega)$ and consider

$$\int_{\Omega}\alpha(x)\langle\nabla u(x),\nabla\phi(x)\rangle\,dx.$$

Since $|\langle\nabla u(x),\nabla\phi(x)\rangle|$ is bounded, the integrand is in $L^1(\Omega)$ and the integral over Ω is the sum of the integrals over $\Sigma_n^{\pm,k}$. We fix one such $\Sigma_n^{\pm,k}$, that we denote by Σ and recall the corresponding notations introduced in Step d). Recall that, by the definition of Ξ and the properties of ∇u,

$$\frac{\partial}{\partial t}\Xi(\hat{x},t)=\nabla u(\Xi((\hat{x},t))=d(\pi\hat{x}),$$

independent of t. Hence,

$$\frac{\partial}{\partial t}\phi(\Xi(\hat{x},t)) = \langle \nabla u(\Xi(\hat{x},t)), \nabla \phi(\Xi(\hat{x},t))\rangle.$$

By the change of variables formula we have

$$\begin{aligned}\int_{\Sigma} \alpha(x)\langle \nabla u(x), \nabla\phi(x)\rangle\, dx &= \int_{\hat{\Sigma}} \left(\int_{a(\pi\hat{x})}^{b(\pi\hat{x})} \alpha(\Xi(\hat{x},t))\frac{\partial}{\partial t}\phi(\Xi(\hat{x},t))\, J\Xi(\hat{x},t)\, dt \right) d\hat{x} \\ &= \int_{\hat{\Sigma}} \left(\int_{a(\pi\hat{x})}^{b(\pi\hat{x})} \frac{\partial}{\partial t}\phi(\Xi(\hat{x},t)) \int_{a(\pi\hat{x})}^{t} g'(u(\Xi(\hat{x},s)))\, J\Xi(\hat{x},s)\, ds\, dt \right) d\hat{x}.\end{aligned}$$

Integrating by parts we have

$$\begin{aligned}&\int_{\Sigma} \alpha(x)\langle \nabla u(x), \nabla\phi(x)\rangle\, dx \\ &= \int_{\hat{\Sigma}} \left(\left[\phi(\Xi(\hat{x},t)) \int_{a(\pi\hat{x})}^{t} g'(u(\Xi(\hat{x},s)))J\Xi(\hat{x},s)\, ds \right]_{a(\pi\hat{x})}^{b(\pi\hat{x})} \right. \\ &\qquad \left. - \int_{a(\pi\hat{x})}^{b(\pi\hat{x})} g'(u(\Xi(\hat{x},t)))\phi(\Xi(\hat{x},t))\, J\Xi(\hat{x},t)\, dt \right) d\hat{x}.\end{aligned}$$

The first term at the r.h.s. is zero since $\Xi(\hat{x}, b(\pi\hat{x}))$ belongs to $\partial\Omega$.
In the same way we compute $\int_{\Sigma} g'(u(x))\phi(x)\, dx$. We have:

$$\int_{\Sigma} g'(u(x))\phi(x)\, dx = \int_{\hat{\Sigma}} \left(\int_{a(\pi\hat{x})}^{b(\pi\hat{x})} g'(u(\Xi(\hat{x},t)))\phi(\Xi(\hat{x},t))\, J\Xi(\hat{x},t)\, dt \right) d\hat{x}.$$

We have obtained:

$$\int_{\Sigma_n^{\pm,k}} \alpha(x)\langle \nabla u(x), \nabla\phi(x)\rangle\, dx + \int_{\Sigma_n^{\pm,k}} g'(u(x))\phi(x)\, dx = 0$$

for every $\Sigma_n^{\pm,k}$, hence the same is true on Ω. The pair $(p(x), u(x))$, where $p(x) = \alpha(x)\frac{\nabla u(x)}{\|\nabla u(x)\|}$, is an integral solution to the differential inclusion

$$\operatorname{div} p(x) = g'(u(x)) \quad \text{for} \quad p(x) \in \partial j_{[0,1]}(\|\nabla u(x)\|).$$

□

Notice also the following chain of implications

$$\langle p, v\rangle = \max_{w\in B}\langle p, w\rangle \;\Leftrightarrow\; v = \begin{cases} \dfrac{p}{\|p\|} & \text{if } \|p\| \neq 0 \\ B & \text{if } p = 0 \end{cases} \;\Leftrightarrow\; p \in \partial j_{[0,1]}(\|v\|)$$

Hence, considering problem (P) as the problem of minimizing

$$\int_\Omega g(u)dx$$

for $u \in u_0 + W_0^{1,\infty}(\Omega)$, subject to the Hamilton Jacobi control equation

$$\nabla u(x) = v, \qquad v \in B$$

where B is the unit ball of $\mathbb{R}^N$, i.e. $\{y \in \mathbb{R}^N : \|y\| \leq 1\}$ and introducing the function

$$H(u, p, v) = -g(u) + < p, v >,$$

we have obtained that p and u are solutions to the differential equations

$$\nabla u = \nabla_p; \quad \operatorname{div} p = -\frac{\partial H}{\partial u}$$

and satisfy a.e. the equation

$$H(u(x), p(x), v(x)) = \max_{w\in B}\{H(u(x), p(x), w)\}.$$

Chapter 2
On Minima of Radially Symmetric Functionals

In this chapter we consider the problems of the existence, the uniqueness and the qualitative properties (symmetry) of the minima to the problem

$$\min_{u\in W_0^{1,1}(B)} \int_B [h(\|x\|, \|\nabla u(x)\|) + g(u(x))]dx$$

where B is the unit ball of $\mathbb{R}^N$ and the map $v \to h(r, v)$ is lower semicontinuous but not necessarily convex.

Problems of this kind arise in domains as different as non-linear elasticity, fluidodynamics and shape optimization, and, either for the problem of the existence of solutions or for the properties of the relaxed problem, are considered in [4], [31], [32], [34], [36], [42], [46]. In particular, the very same problem is considered in [46]. Our results present the following features:

a) no smoothness on h or g is required: h is either a normal integrand or a lower semicontinuous function;

b) the case $g \equiv 0$ is allowed: in this case the assumption on h reduce, for the existence of solutions, to h being lower semicontinuous and growing at infinity, as is to be expected; for the uniqueness, in addition, on h^{**} being strictly increasing, as also is to be expected;

c) the case $g = au$ is allowed: for $a \neq 0$ our theorems yield at once existence and uniqueness of solutions with no further assumptions on h besides lower semicontinuity and growth at infinity.

2.1. Basic notations

In what follows we shall assume that: $\mathbb{R}^N$ is endowed with the Euclidean norm

$\|\cdot\|$; B is the unit ball, whose measure is ω_N. The $(N-1)$-dimensional Hausdorff measure of ∂B is $N\omega_N$.
The subgradient of a convex function h is denoted by ∂h.
A map $h : [0,1] \times [0,\infty) \to \overline{\mathbb{R}}$ is termed a normal integrand, [25], if

i) for a.e. $r \in [0,1]$ $h(r,\cdot)$ is l.s.c. on $[0,\infty)$;

ii) there exists a Borel function $\tilde{h}: [0,1] \times [0,\infty) \to \bar{\mathbb{R}}$: $\tilde{h}(r,\cdot) \equiv h(r,\cdot)$ for a.e. $r \in [0,1]$.

Consider $h : [0,1] \times [0,\infty) \to \overline{\mathbb{R}}$. Let $\tilde{h} : B \times \mathbb{R}^N \to \overline{\mathbb{R}}$ be defined by $\tilde{h}(x,\xi) = h(\|x\|, \|\xi\|)$. Whenever the bipolar of $\tilde{h}$, $\tilde{h}^{**}$, is defined, by extension we call bipolar of h, h^{**}, the map defined by $h^{**}(\|x\|, \|\xi\|) = \tilde{h}^{**}(x,\xi)$. Remark that the map $\xi \to h^{**}(r,\xi)$ is increasing. It is known, [25], that $\tilde{h}^{**}$ is a normal integrand whenever so is $\tilde{h}$ and that $\tilde{h}^{**}$ satisfies the same growth assumptions as $\tilde{h}$.

2.2. Main results

We shall consider the following problem (P):

$$\text{(P)} \qquad \min_{u \in W_0^{1,1}(B)} \int_B [h(\|x\|, \|\nabla u(x)\|) + g(u(x))]dx$$

where $B = \{x \in \mathbb{R}^N : \|x\| < 1\}$, $N \geq 2$, $h: [0,1] \times [0,\infty) \to \overline{\mathbb{R}}$ is a normal integrand and $g: \mathbb{R} \to \mathbb{R}$ is convex.
We shall assume throughout the following growth assumption **GA**:

There exist a convex l.s.c. increasing function $\psi : [0,\infty) \to [0,\infty)$ such that $\lim_{s\to\infty} \frac{\psi(s)}{s} = \infty$ and a function $\alpha \in L^1([0,1])$ satisfying $h(r,s) \geq \alpha(r) + \psi(s)$ for all $s \in [0,\infty)$, for a.e. $r \in [0,1]$.

The following result guarantees the existence of at least one radially symmetric solution to the minimum problem (P) associated to a convex function h.

Theorem 2.2.1. *Let h be a normal integrand satisfying assumptions* **GA**. *Assume further that $h^{**} \equiv h$. Let $g : \mathbb{R} \to \mathbb{R}$ be convex. Then problem* (P) *admits at least one radially symmetric solution. Moreover, if g is monotonic and either $h(r,.)$ or g is strictly monotonic, then every solution to* (P) *is radially symmetric.*

Proof. Let u be a solution to problem (P). Consider the function $\bar{u}$ defined by

$$\bar{u}(x) = \frac{1}{N\omega_N} \int_{\|\omega\|=1} u(\omega\|x\|)d\omega. \tag{2.2.1}$$

It is our purpose to show that $\bar{u}$ is a radially symmetric solution to (P). The symmetry comes from the very definition.

a) We claim that

$$\begin{cases} \nabla\bar{u}(x) = \dfrac{1}{N\omega_N}\dfrac{x}{\|x\|}\displaystyle\int_{\|\omega\|=1}\langle\nabla u(\omega\|x\|),\omega\rangle d\omega, & x \neq 0 \\ \nabla\bar{u}(0) = 0. \end{cases} \tag{2.2.2}$$

First remark that the above is true when u is of class C^1. In this case, when $\|x\| \neq 0$ one can differentiate with respect to the parameter x to obtain

$$\frac{\partial\bar{u}}{\partial x_i}(x) = \frac{1}{N\omega_N}\int_{\|\omega\|=1}\langle\nabla u(\omega\|x\|),\omega\rangle\frac{x_i}{\|x\|}d\omega$$

while, for $x = 0$

$$\begin{aligned} \frac{\partial\bar{u}}{\partial x_i}(0) &= \lim_{h_i\to 0}\frac{1}{h_i}\left(\frac{1}{N\omega_N}\int_{\|\omega\|=1}(u(\omega h_i) - u(0))d\omega\right) \\ &= \lim_{h_i\to 0}\frac{1}{h_i}\left(\frac{1}{N\omega_N}\int_{\|\omega\|=1}(h_i\langle\omega,\nabla u(0)\rangle + h_i\|\omega\|\epsilon(h_i\|\omega\|)d\omega\right) \\ &= 0. \end{aligned}$$

To show the validity of the above formula for any u in $W^{1,1}(B)$ let us consider a sequence $\{u_h\}$, each u_h of class C^1 and $u_h \to u$ in $W^{1,1}(B)$. Hence, from the previous result, $\nabla\bar{u}_h$ satisfy (2.2.2). We are going to show first that the functions $\bar{u}_h$ defined by (2.2.1) convege to $\bar{u}$ strongly in L^1.
Set w to be

$$w(x) = \begin{cases} \frac{1}{N\omega_N}\frac{x}{\|x\|}\int_{\|\omega\|=1}\langle\nabla u(\omega\|x\|),\omega\rangle d\omega, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

To prove the claim hence it will be left to show that $\nabla\bar{u}_k$ converges to w strongly in $L^1(B)$. We have

$$\begin{aligned}
\| \bar{u} - \bar{u}_k \|_1 &\leq \int_B \left(\frac{1}{N\omega_N} \int_{\|\omega\|=1} |u(\omega\|x\|) - u_k(\omega\|x\|)| \, d\omega \right) dx \\
&= \frac{1}{N\omega_N} \int_{\|\omega\|=1} \left(\int_B |u(\omega\|x\|) - u_k(\omega\|x\|)| \, dx \right) d\omega \\
&= \frac{1}{N\omega_N} \int_{\|\omega\|=1} \left(N\omega_N \int_0^1 |u(\omega r) - u_k(\omega r)| \, r^{n-1} dr \right) d\omega \\
&= \int_B |u(x) - u_k(x)| \, dx = \| u - u_k \|_1 .
\end{aligned}$$

Through the same steps,

$$\begin{aligned}
&\| w - \nabla \bar{u}_k \|_1 \\
&= \int_B \frac{1}{N\omega_N} \left| \frac{x}{\|x\|} \int_{\|\omega\|=1} \langle \nabla u(\omega\|x\|) - \nabla u_k(\omega\|x\|), \omega \rangle d\omega \right| dx \\
&\leq \int_B \frac{1}{N\omega_N} \int_{\|\omega\|=1} \| \nabla u(\omega\|x\|) - \nabla u_k(\omega\|x\|) \| \, d\omega dx
\end{aligned}$$

and, by applying again Fubini's theorem,

$$\| w - \nabla \bar{u}_k \|_1 \leq \| \nabla u - \nabla u_k \|_1 .$$

The claim is proved.
The above arguments defining $\bar{u}$ out of u are similar to those employed in [13] and [27] for a problem involving the Laplacian.

b) From the convexity of g we obtain

$$\int_B g(u(x))dx \geq \int_B g(\bar{u}(x))dx.$$

In fact

$$\begin{aligned}
\int_B g(\bar{u}(x))dx &\leq \int_B \frac{1}{N\omega_N} \int_{\|\omega\|=1} g(u(\omega\|x\|)) d\omega dx \\
&= \int_{\|\omega\|=1} \int_0^1 g(u(\omega r)) r^{N-1} dr d\omega = \int_B g(u(x))dx.
\end{aligned}$$

In particular the same computation in the case $g(u) = u$ yields

$$\int_B u(x)dx \geq \int_B \bar{u}(x)dx. \tag{2.2.3}$$

To prove that $\bar{u}$ is a solution it is left to show that

$$\int_B h(\|x\|, \|\nabla \bar{u}(x)\|)dx \leq \int_B h(\|x\|, \|\nabla u(x)\|)dx.$$

Since

$$\|\nabla \bar{u}(x)\| \leq \frac{1}{N\omega_N} \int_{\|\omega\|=1} \|\nabla u(\omega\|x\|)\| \, d\omega$$

and $s \mapsto h(r,s)$ is monotonic,

$$\int_B h(\|x\|, \|\nabla \bar{u}(x)\|)dx \leq \int_B h\left(\|x\|, \frac{1}{N\omega_N} \int_{\|\omega\|=1} \|\nabla u(\omega\|x\|)\| \, d\omega\right) dx.$$

By the convexity of $s \mapsto h(r,s)$ and Jensen's inequality,

$$\begin{aligned}
&\int_B h\left(\|x\|, \frac{1}{N\omega_N} \int_{\|\omega\|=1} \|\nabla u(\omega\|x\|)\| \, d\omega\right) dx \\
&\leq \int_B \left(\frac{1}{N\omega_N} \int_{\|\omega\|=1} h\left(\|x\|, \|\nabla u(\omega\|x\|)\|\right) d\omega\right) dx \\
&= \int_{\|\omega\|=1} \int_0^1 h\left(r, \|\nabla u(\omega r)\|\right) r^{N-1} dr d\omega = \int_B h(\|x\|, \|\nabla u(x)\|)dx.
\end{aligned}$$

Hence $\bar{u}$ is a radially symmetric solution to (P).

c) Assume now that g is monotonic and let us prove the result first in the case g monotonic increasing. The map $v \longrightarrow h(r,v)$ is non decreasing; by assumption, either g or h is strictly increasing. Let us show that every solution is radially symmetric.

Let u be any solution and set $\bar{u}$ to be

$$\bar{u}(x) = \frac{1}{N\omega_N} \int_{\|\omega\|=1} u(\omega\|x\|)d\omega.$$

From the above, $\bar{u}$ is a solution. Consider the average $w = \frac{1}{2}(u + \bar{u})$. By the convexity of the problem, w is a solution. In particular,

$$\begin{aligned}
&\int_B h(\|x\|,\|\nabla w(x)\|)dx \\
&= \frac{1}{2}\int_B h(\|x\|, \|\nabla u(x)\|)dx + \frac{1}{2}\int_B h(\|x\|, \|\nabla \bar{u}(x)\|)dx.
\end{aligned} \tag{2.2.4}$$

By the monotonicity and the convexity of h, we have

$$\begin{aligned} h(\|x\|, \|\nabla w(x)\|) &\leq h\left(\|x\|, \frac{\|\nabla u(x)\| + \|\nabla \bar{u}(x)\|}{2}\right) \\ &\leq \frac{1}{2}h(\|x\|, \|\nabla u(x)\|) + \frac{1}{2}h(\|x\|, \|\nabla \bar{u}(x)\|) \end{aligned}$$

so that, by (2.2.4), equality holds:

$$\begin{aligned} h(\|x\|, \|\nabla w(x)\|) &= h\left(\|x\|, \frac{\|\nabla u(x)\| + \|\nabla \bar{u}(x)\|}{2}\right) \\ &= \frac{1}{2}h(\|x\|, \|\nabla u(x)\|) + \frac{1}{2}h(\|x\|, \|\nabla \bar{u}(x)\|). \end{aligned} \tag{2.2.5}$$

Set $T(r)$ to be $T(r) = \sup\{v : h(r,v) - h(r,0) = 0\}$. The supremum is actually a maximum.

We wish to show that for almost every x, $\|\nabla w(x)\| \geq T(\|x\|)$. In the case h strictly increasing, $T = 0$ and there is nothing to prove. Assume that g is strictly increasing. Remark that $v \to h(r,v)$ is strictly increasing for $v \geq T(r)$. Hence, notice the following property of w to be used later: for those x such that the gradient exists, whenever $\|\nabla w(x)\| \geq T(\|x\|)$, there exists $\lambda(x) \geq 0$ such that $\nabla u(x) = \lambda(x)\nabla \bar{u}(x)$. In fact in this case the first equality in (2.2.5) implies that

$$\|\nabla w(x)\| = \frac{1}{2}(\|\nabla u(x)\| + \|\nabla \bar{u}(x)\|)$$

and by the strict convexity of the euclidean norm, this is true only if $\nabla u(x) = \lambda(x)\nabla \bar{u}(x)$.

We claim that the map $r \to T(r)$ is measurable. Fix $\epsilon > 0$. Since $h(r,v)$ is a normal integrand, by Theorem 1.1, p.232 of [25], there exists a compact K_ϵ in $[0,1]$, $m([0,1]\backslash K_\epsilon) < \epsilon$, such that the restriction of h to $K_\epsilon \times \mathbb{R}$ is lower semicontinuous and the restrictions of $r \to h(r,0)$ and of $r \to \alpha(r)$ to K_ϵ are continuous. In particular, there exists M_ϵ such that $h(r,0) - \alpha(r) \leq M_\epsilon$ in K_ϵ. Hence, for every v satisfying $h(r,v) - h(r,0) = 0$ for some r in K_ϵ we have $\psi(v) \leq h(r,0) - \alpha(r) \leq M_\epsilon$ that implies $\|v\| \leq V_\epsilon$ for some V_ϵ. Consider a sequence (r_n) in K_ϵ converging to r^* and set T^* to be the limsup of $T(r_n)$. By taking a subsequence we can assume that $h(r_n, T(r_n))$ converges to y satisfying $y - h(r^*,0) = 0$; hence $h(r^*,T^*) - h(r^*,0) \leq 0$. Being h non decreasing in v, we have $h(r^*,T^*) - h(r^*,0) = 0$, hence $T(r^*) \geq T^*$, i.e. the restriction to K_ϵ of $r \to T(r)$ is upper semicontinuous. By Lusin's theorem this proves the claim.

Let us first show that $\|\nabla\bar{u}(x)\| \geq T(\|x\|)$ for a.e. x. Assume, by contrtadiction, that $m(A) > 0$, where A is the set defined by $A = \{\|x\| : \|\nabla\bar{u}(x)\| < T(\|x\|)\}$. Define $v: [0,1] \to \mathbb{R}$ by $v(\|x\|) = \bar{u}(x)$. Remark that $r \to v(r)$ is locally absolutely continuous in $(0,1]$. In fact apply the change of variable formula to the transformation from Cartesian to polar coordinates, and obtain for the map $\bar{v}(r,\theta) = u(\phi(r,\theta))$ that for almost every $\bar{\theta}$, the map $v : r \to \bar{v}(r,\bar{\theta})$ is locally absolutely continuous in $(0,1]$ and

$$v'(r) = \frac{\partial\bar{v}}{\partial r}(r,\bar{\theta}) = \langle\frac{\partial\phi}{\partial r}, \nabla u(\phi(r,\bar{\theta}))\rangle.$$

Since $\bar{u}$ is radially symmetric, $\left|\langle\nabla\bar{u}(x), \frac{x}{\|x\|}\rangle\right| = \|\nabla\bar{u}(x)\|$; by the chain rule we obtain $v'(\|x\|) = \langle\nabla\bar{u}(x), \frac{x}{\|x\|}\rangle$, and hence

$$|v'(\|x\|)| = \left|\langle\nabla\bar{u}(x), \frac{x}{\|x\|}\rangle\right|.$$

Set $\tilde{v}(r)$ to be $\int_1^r \left(v'(s)\chi_{C(A)}(s) + T(s)\chi_A(s)\right) ds$ and $\bar{v}: B \to \mathbb{R}$ to be $\bar{v}(x) = \tilde{v}(\|x\|)$. Then, from the very definition, $\bar{v}(x) \leq \bar{u}(x)$ and the strict inequality holds on a subset of B of positive measure, hence

$$\int_B g(\bar{v}(x))dx < \int_B g(\bar{u}(x))dx.$$

On the other hand

$$h(\|x\|, \|\nabla\bar{v}(x)\|) = h(\|x\|, \|\nabla\bar{u}(x)\|),$$

so that $\bar{u}$ cannot be a minimum. Hence $\|\nabla\bar{u}(x)\| \geq T(\|x\|)$ a.e.
To show that the same inequality holds for w, remark that by the definitions one also has

$$\bar{u}(x) = \frac{1}{N\omega_N}\int_{\|\omega\|=1} w(\omega\|x\|)d\omega$$

so that

$$\nabla\bar{u}(x) = \frac{1}{N\omega_N}\frac{x}{\|x\|}\int_{\|\omega\|=1} \langle\nabla w(\omega\|x\|), \omega\rangle d\omega$$

and, by the definition and (2.2.3),

$$\int_B \bar{u}(x)dx \leq \int_B w(x)dx.$$

Assume that there exists a subset S of B, $m(S)>0$, such that $\|\nabla w(x)\|<T(\|x\|)$ for x in S. Since $\nabla w=\frac{1}{2}\nabla u+\frac{1}{2}\nabla\bar{u}$, on S we must have $\|\nabla u(x)\|<T(\|x\|)$; moreover $\|\nabla\bar{u}(x)\|$ must be equal to $T(\|x\|)$. In fact, if it is not so,

$$h(\|x\|,\|\nabla w(x)\|)<\frac{1}{2}h(\|x\|,\|\nabla u(x)\|)+\frac{1}{2}h(\|x\|,\|\nabla\bar{u}(x)\|)$$

contradicting (2.2.5).

Set

$$S_r=\{\omega\,:\,\|\nabla w(\omega r)\|<T(r)\}$$

$$m(S)=\int_0^1 r^{N-1}\left(\int_{\|\omega\|=1}\chi_S(\omega r)d\omega\right)dr=\int_0^1 r^{N-1}\int_{\|\omega\|=1}\chi_{S_r}(\omega)d\omega dr$$

so that for r in a subset $E\subset[0,1]$ of positive measure,

$$\int_{\|\omega\|=1}\chi_{S_r}(\omega)d\omega>0.$$

Consider one such r in E. Since, for $\|x\|=r$,

$$T(r)=\|\nabla\bar{u}(x)\|\le\frac{1}{N\omega_N}\int_{\|\omega\|=1}\|\nabla w(\omega\|x\|)\|\,d\omega,$$

on a subset of $\mathcal{C}(S_r)$ of a positive measure we must have $\|\nabla w(\omega r)\|>T(r)$. Again for r in E,

$$\begin{aligned}\int_{\|\omega\|=1}h(r,\|\nabla w(\omega r)\|)d\omega&=\int_{S_r}h(r,T(r))d\omega+\int_{\mathcal{C}(S_r)}h(r,\|\nabla w(\omega r)\|)d\omega\\&>\int_{\|\omega\|=1}h(r,T(r))d\omega=\int_{\|\omega\|=1}h(r,\|\nabla\bar{u}(\omega r)\|)d\omega.\end{aligned}$$

Set X to be $\{x=\omega r\,:\,\|\omega\|=1,r\in E\}$. Then

$$\begin{aligned}\int_X h(\|x\|,\|\nabla w(x)\|)\,dx&=N\omega_N\int_0^1 r^{N-1}\int_{\|\omega\|=1}\chi_X(\omega r)h(r,\|\nabla w(\omega r)\|)d\omega dr\\&=N\omega_N\int_0^1 r^{N-1}\chi_E(r)\left(\int_{\|\omega\|=1}h(r,\|\nabla w(\omega r)\|)d\omega\right)dr\\&>N\omega_N\int_0^1 r^{N-1}\chi_E(r)\left(\int_{\|\omega\|=1}h(r,\|\nabla\bar{u}(\omega r)\|)d\omega\right)dr\\&=\int_X h(\|x\|,\|\nabla\bar{u}(x)\|)dx.\end{aligned}$$

For x not in X, by a previous remark, $\nabla u(x) = \lambda(x)\nabla\bar{u}(x)$, so that $\nabla w(x) = (\lambda(x)+1)\nabla\bar{u}(x)$; hence on $B\backslash X$, w itself is radially symmetric, i.e. w coincides with $\bar{u}$. By Lemma 7.7 in [30], $\nabla w = \nabla\bar{u}$ a.e. in $B\backslash X$. Hence

$$\begin{aligned}\int_B h(\|x\|, \|\nabla w(x)\|)dx &= \int_B h(\|x\|, \|\nabla w(x)\|)[\chi_X(x) + \chi_{B\backslash X}(x)]dx \\ &> \int_B h(\|x\|, \|\nabla\bar{u}(x)\|)dx,\end{aligned}$$

a contradiction, since w is a solution and

$$\int_B g(w(x))dx \geq \int_B g(\bar{u}(x))dx.$$

Then $m(S) = 0$, i.e. for almost every x in B, $\|\nabla w(x)\| \geq T(\|x\|)$. Hence, for almost every x, $\nabla u(x) = \lambda(x)\nabla\bar{u}(x)$.
This proves the theorem in the case g increasing.

d) Finally, notice that the case of a decreasing function g can be reduced to the previous one by setting

$$\tilde{g}(\xi) = g(-\xi), \quad \xi \in \mathbb{R}.$$

Now, $\tilde{g}$ is increasing and each solution to the problem

$$(\tilde{\mathrm{P}}) \qquad \min_{u\in W_0^{1,1}(B)} \int_B [h(\|x\|, \|\nabla u(x)\|) + \tilde{g}(u(x))]dx$$

is radially symmetric.
Since $\tilde{u}$ is a solution to $(\tilde{\mathrm{P}})$ if and only if $-\tilde{u}$ is a solution to (P), we see that every solution to (P) is radially symmetric. □

Remark 2.2.2. To see how the assumptions of the previous theorem are sharp, consider the case $g \equiv 0$ (g is monotonic, but not strictly monotonic) and $h = h(v)$ to be the indicator of the interval $[0, 1]$ (h is convex and monotonic, but not strictly monotonic). Clearly there exist non radially symmetric solutions to (P).

The next results are concerned with existence and uniqueness of solutions for the minimum problem (P) when h is (possibly) non convex. We shall assume that h is independent on $\|x\|$, i.e. $h(r,\xi) = h(\xi)$.

Theorem 2.2.3. *Assume that: $h(r,\xi) = h(\xi)$ is l.s.c. and satsfies* **GA**; *$g : \mathbb{R} \to \mathbb{R}$ is convex and monotonic. Then the minimum problem* (P) *admits at least a solution u in $W_0^{1,1}(B)$.*

Proof. a) As in the proof of the Theorem 2.2.1, it is enough to consider the case where g is monotonic decreasing.
Let $\tilde{u}$ be a radially symmetric solution to the convexified problem

$$(\mathrm{P}^{**}) \qquad \min_{u \in W_0^{1,1}(B)} \int_B [h^{**}(\|\nabla u(x)\|) + g(u(x))]dx$$

Define $u: [0,1] \to \mathrm{I\!R}$ by $u(\|x\|) = \tilde{u}(x)$. Remark that the map $r \to u(r)$ is locally absolutely continuous on $(0,1]$, and $u'(\|x\|) = \langle \frac{x}{\|x\|}, \nabla \tilde{u}(x) \rangle$.
We are going to show that we can as well assume that $|u'|$ does not take its values (for r in a set of positive measure) on any interval (a,b) where h^{**} is affine. This in particular will show that $|u'|$ takes its values where h and h^{**} coincide, proving that u is a solution to the original problem.
Let $g'_+(u)$ be the right derivative of g at u; the map $r \to g'_+(u(r))$ is negative and bounded; consider G defined by

$$G(r) = \int_0^r s^{N-1} g'_+(u(s))ds.$$

Assume $h^{**\prime} = \alpha$ on (a,b). Consider first the case $\alpha = 0$; in this case $(a,b) = (0,T)$, and consider those r such that $|u'(r)| \leq T$. The same reasoning as in Theorem 2.2.1, point c), imply that there exists another radial solution v such that $|v'| \geq T$. Hence we can as well assume $\alpha > 0$.
Set $E_\sigma = \{r : |u'(r)| \in (a+\sigma, b-\sigma)\}$ and assume that for some σ, $m(E_\sigma) > 0$. We are going to show that, under this assumption, it is possible to define a family $\{\tilde{u}_\epsilon\}$ of variations of $\tilde{u}$ such that, for ϵ small, the functional computed at $\tilde{u}_\epsilon$ has a value strictly smaller that the minimum, a contradiction. Call E this E_σ.
Consider $A(r) = (\alpha r^{N-1} - G(r))\chi_E(r)$. Let r_1 and r_2, $r_1 < r_2$, be points of density for E and Lebesgue points for A, so that

$$\frac{1}{\delta}\int_{r_1}^{r_1+\delta} A(r)dr \to A(r_1) = (\alpha r_1^{N-1} - G(r_1)).$$

$$\frac{1}{\delta}\int_{r_2}^{r_2+\delta} A(r)dr \to A(r_2) = (\alpha r_2^{N-1} - G(r_2)).$$

Since A is monotonic, $A(r_2) - A(r_1) = 2\lambda > 0$. Set $\delta^* = \delta^*(\delta)$ to be the measure of the set $[(r_2 - \delta, r_2) \cap E]$. Since r_2 is of density for E, $\delta^*/\delta \to 1$, i.e. $\delta^* > 0$. Let $\bar{\delta}$ be such that $\delta < \bar{\delta}$ implies

$$\left| \frac{1}{\delta}\int_{r_1}^{r_1+\delta} A(r)dr - A(r_1) \right| < \frac{\lambda}{4};$$

$$\left|\frac{1}{\delta}\int_{r_2-\delta}^{r_2} A(r)dr - A(r_2)\right| < \frac{\lambda}{4}.$$

Set η' to be

$$\eta' = \frac{1}{\delta}\chi_{[r_1,r_1+\delta^*]\cap E} - \frac{1}{\delta^*}\chi_{[r_2-\delta,r_2]\cap E}$$

and define η as $\eta(r) = \int_1^r \eta'(s)ds$ so that $\eta(1) = 0$ and, by the choice of δ^*, $\eta(r) \geq 0$. Let us remark that $u'(r)+\epsilon\eta'(r)$ coincides with $u'(r)$ for r not in E, and, for r in E, is in (a,b) whenever ϵ is sufficiently small. Hence $h^{**}(|u'(r)+\epsilon\eta'(r)|)$ is well defined and integrable.

b) Let us consider the family $\{u+\epsilon\eta\}$. By the mean value theorem there exists: $\xi_1(r) \in [0,\epsilon]$ and $h_\epsilon^{**\prime}(r) \in \partial h^{**}(|u'(r)+\xi_1(r)\eta'(r)|)$ such that

$$h^{**}(|u'(r)+\epsilon\eta'(r)|) - h^{**}(|u'(r)|) = \epsilon h_\epsilon^{**\prime}(r)\eta'(r).$$

Moreover, since the subgradient of g is monotonic, there exists $\xi_2(r) \in [0,\epsilon]$ such that

$$g(u(r)+\epsilon\eta(r)) - g(u(r)) \leq \epsilon g'_+(u(r)+\xi_2(r)\eta(r))\eta(r).$$

Then

$$\begin{aligned}\Delta_\epsilon &= \int_0^1 r^{N-1}[h^{**}(u'+\epsilon\eta') + g(u+\epsilon\eta)]dr - \int_0^1 r^{N-1}[h^{**}(u') + g(u)]dr \\ &\leq \epsilon\int_0^1 r^{N-1}[h_\epsilon^{**\prime}(r)\eta'(r) + g'_+(u(r)+\xi_2(r)\eta(r))\eta(r)]dr.\end{aligned}$$

Set G_ϵ to be

$$G_\epsilon(r) = \int_0^r s^{N-1}g'_+(u(s)+\xi_2(s)\eta(s))ds.$$

Since $G_\epsilon(0) = 0$ and $\eta(1) = 0$, we have

$$\int_0^1 r^{N-1}g'_+(u(r)+\xi_2(r)\eta(r))\eta(r)dr = -\int_0^1 G_\epsilon(r)\eta'(r)dr.$$

Hence

$$\begin{aligned}\Delta_\epsilon \leq \epsilon\int_0^1 [r^{N-1}h_\epsilon^{**\prime}(r) - G_\epsilon(r)]\eta'(r)dr &= \epsilon\frac{1}{\delta}\int_{r_1}^{r_1+\delta^*} [r^{N-1}h_\epsilon^{**\prime}(r) - G_\epsilon(r)]\chi_E(r)dr \\ &\quad - \epsilon\frac{1}{\delta^*}\int_{r_2-\delta}^{r_2} [r^{N-1}h_\epsilon^{**\prime}(r) - G_\epsilon(r)]\chi_E(r)dr.\end{aligned}$$

For every fixed s, $u(s)+\xi_2(s)\eta(s)$ converges to $u(s)$ from the right, so that, being the subdifferential of a convex function monotonic, we have $g'_+(u(s)+\xi_2(s)\eta(s)) \to g'_+(u(s))$ as $\epsilon \to 0$. Moreover since g is finite on $\mathbb{R}$, there exists M that bounds all the values of $|g'_+(v)|$ for v in a neighborhood of the image of the solution u. Hence $G_\epsilon(r) \to G(r)$ pointwise and is dominated by a constant.
Moreover, for every $r \in E$, $h_\epsilon^{**\prime}(r) = \alpha$, for every ϵ sufficiently small. By integrating we obtain that, for every ϵ sufficiently small,

$$\left|\frac{1}{\delta}\int_{r_1}^{r_1+\delta^*}[r^{N-1}h_\epsilon^{**\prime}(r) - G_\epsilon(r)]\chi_E(r)dr - \frac{1}{\delta}\int_{r_1}^{r_1+\delta^*}A(r)dr\right| < \frac{\lambda}{4}$$

$$\left|\frac{1}{\delta^*}\int_{r_2-\delta}^{r_2}[r^{N-1}h_\epsilon^{**\prime}(r) - G_\epsilon(r)]\chi_E(r)dr - \frac{1}{\delta^*}\int_{r_2-\delta}^{r_2}A(r)dr\right| < \frac{\lambda}{4}.$$

Finally, for some positive ϵ,

$$\begin{aligned}
\Delta_\epsilon &\le \epsilon\left\{\frac{\lambda}{4} + \frac{1}{\delta}\int_{r_1}^{r_1+\delta^*}A(r)dr + \frac{\lambda}{4} - \frac{1}{\delta^*}\int_{r_2-\delta}^{r_2}A(r)dr\right\} \\
&= \epsilon\left\{\frac{\lambda}{2} + \frac{\delta^*}{\delta}\frac{1}{\delta^*}\int_{r_1}^{r_1+\delta^*}A(r)dr - \frac{\delta}{\delta^*}\frac{1}{\delta}\int_{r_2-\delta}^{r_2}A(r)dr\right\} \\
&\le \epsilon\left\{\frac{\lambda}{2} + \frac{1}{\delta^*}\int_{r_1}^{r_1+\delta^*}A(r)dr - \frac{1}{\delta}\int_{r_2-\delta}^{r_2}A(r)dr\right\} \\
&\le \epsilon\left\{\frac{\lambda}{2} + A(r_1) + \frac{\lambda}{4} - A(r_2) + \frac{\lambda}{4}\right\} \le -\lambda\epsilon < 0
\end{aligned}$$

i.e. for some positive ϵ, the function $\tilde{u}_\epsilon$ defined by $\tilde{u}_\epsilon(x) = u(\|x\|) + \epsilon\eta(\|x\|)$ yields a value for the integral in (P^{**}) less than the value computed at $\tilde{u}$, a contradiction. □

Remark 2.2.4. In the case $g \equiv 0$, the assumptions of the above theorem reduce to lower semicontinuity and growth at infinity for h, as is to be expected.

Theorem 2.2.5. *Let h and g satisfy the same assumption as in Theorem 2.2.3; in addition assume that either h^{**} or g is strictly monotonic. Then problem* (P) *admits one and only one solution.*

Proof. Assume that $u, v : [0,1] \to \mathbb{R}$ are such that the maps $x \to u(\|x\|)$ and $x \to v(\|x\|)$ are two distinct solutions to (P). There exists an interval (r_1, r_2) such that: $u(r) > v(r)$, $r \in (r_1, r_2)$; $u(r_2) = v(r_2)$ and, when $r_1 > 0$, $u(r_1) = v(r_1)$. Set w to be $\frac{1}{2}(u+v)$. The map $x \to w(\|x\|)$ is a further solution to (P**) so that

$$\frac{1}{2}\int_{r_1}^{r_2} r^{N-1}[h^{**}(|v'(r)|) + g(v(r))]dr + \frac{1}{2}\int_{r_1}^{r_2} r^{N-1}[h^{**}(|u'(r)|) + g(u(r))]dr$$
$$= \int_{r_1}^{r_2} r^{N-1}[h^{**}(|w'(r)|) + g(w(r))]dr.$$

The convexity of both h^{**} and g implies, in particular, that

$$\int_{r_1}^{r_2} r^{N-1} g(w(r))dr = \int_{r_1}^{r_2} r^{N-1} g(u(r))dr.$$

Being g monotonic we infer

$$g(w(r)) = g(u(r)), \quad \forall r \in (r_1, r_2). \tag{2.2.6}$$

The above is a contradiction to the existence of u and v in the case g is strictly monotonic.

Assume now h^{**} strictly monotonic. Since g is convex there exists at most an interval I on which g is constant and attains its minimum. Set r^* to be the supremum of the set $\{r \le 1 : u(r) \in I\}$ and consider the map u^* defined by

$$u^*(x) = \begin{cases} u(r^*) & \text{for } r \le r^* \\ u(r) & \text{for } r > r^*. \end{cases}$$

Then both

$$\int_0^1 r^{N-1} g(u^*(r))dr \le \int_0^1 r^{N-1} g(u(r))dr$$

and

$$\int_0^1 r^{N-1} h^{**}(|u^{*\prime}(r)|)dr \le \int_0^1 r^{N-1} h^{**}(|u'(r)|)dr$$

hold, and the last inequality is strict in the case $|u^{*\prime}|$ differs from $|u'|$ for r on a set of positive measure. Since u is a minimum, $u'(r)$ must be 0 on $(0, r^*)$. From (2.2.6) one has that $w(r) \in I$ if and only if $u(r) \in I$. Since w is a solution, the above reasoning implies that also $w'(r) = 0$ on $(0, r^*)$ i.e. $u'(r) = w'(r)$ on $(0, r^*)$. The case $r^* > r_1$ would violate the assumptions on r_1, r_2, u, v. Hence on (r_1, r_2) the inequality $w(r) > u(r)$ implies the inequality $g(w(r)) > g(u(r))$ a contradiction to (2.2.6). □

Remark 2.2.6. Whenever g is linear, g not zero, uniqueness (besides existence) is guaranteed simply by the lower semicontinuity and growth at infinity of h.

Chapter 3
An Existence Result for a Class of Nonconvex Minimum Problems

The variational approach to various problems of shape optimization arising in both solid mechanics and fluid dynamics leads to the problem of minimizing a non convex functional of the form (see [31] and [33])

$$J(v) = \int_\Omega [h(\|\nabla v\|) + v]\ dx, \qquad v \in W_0^{1,1}(\Omega),$$

where Ω is an open and bounded subset of $\mathbb{R}^2$ and $h : [0,\infty) \to \mathbb{R}$ is the minimum between two convex parabolas having the same axis of symmetry.

In the previous chapter, we have seen that, when Ω is an open ball in $\mathbb{R}^N$, such functional features a unique radially symmetric minimizer provided h is only assumed to be lower semicontinuous and superlinear. For Ω a square in $\mathbb{R}^2$, the corresponding convexified minimum problem, i.e. the minimum problem for the functional

$$J^{**}(v) = \int_\Omega [h^{**}(\|\nabla v\|) + v]\ dx, \qquad v \in W_0^{1,1}(\Omega),$$

is studied in [31] and [33] from both an analytical and numerical point of view. In both papers, the authors, relying on numerical experiments concerning the convexified problem, suggest that the minimum problem for the non convex functional J fails to have a solution. In nearly optimal configurations, homogenization occurs.

From an analytical point of view, a closely related class of non convex minimum problems is studied in [11]. In that paper, Ω is an open, bounded and convex subset of $\mathbb{R}^2$ with piecewise smooth boundary and $h : [0,\infty) \to [0,\infty]$ is a lower semicontinuous function which admits a largest minimum point $\rho \geq 0$ where it satisfies $h(t) \geq h(\rho) + \Lambda(t-\rho)$, $t \geq 0$ for some $\Lambda \geq 0$. The existence of solutions of the minimum problem on $W_0^{1,1}(\Omega)$ is proved in [11] provided the slope Λ of

h is at least equal to the width of Ω, i.e. the least upper bound of the radii of the open balls contained in Ω. In such case, up to the multiplicative constant $-\rho$, a solution is given by the distance function from the boundary of Ω. It is noteworth that the gradient of this solution lies in $\rho\partial B$, almost everywhere on Ω, i.e. in the set where the function $\xi \to h(\|\xi\|)$ vanishes. Whenever the above mentioned hypothesis is violated, the minimum problem for the functional J may lack solution. In particular, this occurs when Ω is a square and h is finite only at two points (see [7]).
The result of [11] has inspired further research. Indeed, in [48], the restriction on the dimension of the space is removed while [47] deals with a more general functional, namely

$$I(v) = \int_\Omega [h(\gamma_K(\nabla v)) + v]\, dx, \qquad v \in W_0^{1,1}(\Omega),$$

where Ω is an open, bounded and convex subset of $\mathbb{R}^2$, γ_K is the Minkowski functional of a closed, bounded and convex subset K of $\mathbb{R}^2$ containing the origin in its interior and h satisfies the same hypotheses as in [11]. The dependence of the functional above upon the gradient ∇v can be regarded as a generalization of the radially symmetric dependence. Indeed, the level sets of the function $\xi \in \mathbb{R}^2 \to h(\gamma_K(\xi))$ consist of the union of homothetic copies of the boundary of K. In particular, for K the closed unit ball, the functional above reduces to the one considered in [11].
For Ω an open, bounded and convex subset of $\mathbb{R}^2$ (no regularity assumptions are imposed on the boundary of Ω) and for K a closed, bounded and convex polytope containing the origin in its interior, the existence of minimizers for I is proved in [47] under the same kind of assumptions considered in [11]: the slope Λ of h has to be at least equal to a suitable measure of the width of Ω which involves the convex polytope K. Again, the gradient of the solution defined in [47] lies, almost everywhere on Ω, in the set where the function $h \circ \gamma_K$ vanishes.
In this chapter, we consider the problem of minimizing on $u_0 + W_0^{1,1}(\Omega)$ the functional

$$F(v) = \int_\Omega [f(\nabla v) + v]\, dx, \qquad v \in W^{1,1}(\Omega),$$

where Ω is an open, bounded and convex subset of $\mathbb{R}^N$ and $f : \mathbb{R}^N \to [0, \infty)$ is a possibly non convex, Borel measurable function vanishing on the boundary of a closed, bounded and convex subset K of $\mathbb{R}^N$ containing the origin in its interior

and u_0 is a convex and Lipschitz continuous boundary datum.
In the homogeneous case $u_0 = 0$, the proof essentially follows the same ideas of [11] and [47] combined with a suitable approximation argument. In particular, the existence of solutions is ensured provided the slope of f with respect to γ_K, i.e. the largest Λ such that $f(\xi) \geq \Lambda\,[\gamma_K(\xi) - 1]$ for all $\xi \in \mathbb{R}^N$, exceeds the width of Ω related to K introduced in [47]. Oncemore, the gradient of the solution turns out to be in the boundary of K almost everywhere on Ω.
For the non homogeneous minimum problem, we rely on a result presented in [40], namely the construction of a Lipschitz continuous function u which agrees with the boundary datum u_0 on the boundary of Ω and whose gradient belongs to the boundary of K almost everywhere on Ω. Such function u is the natural candidate to be a minimizer on $u_0 + W_0^{1,1}(\Omega)$ for the functional F. This is actually proved by showing that, up to an additive constant, the function u is the restriction to Ω of a minimizer of F on $W_0^{1,1}(\Omega')$, where Ω' is a suitable convex set containing Ω.
Finally, we wish to point out that, although the result presented here subsumes all the previously mentioned results, its proof is entirely self-contained.

3.1. Notations and statement of the main result

Before stating the main theorem, we recall some elementary definitions and results from convex analysis that will be useful in the sequel. Our definitions and notations mainly agree with those of [44].
Let C be a convex subset of $\mathbb{R}^N$. We denote the set of its interior points by $\mathrm{int}(C)$, its relative interior by $\mathrm{ri}(C)$ and the polar set of C by C°. The normal cone to C at a point $x \in \mathbb{R}^N$ is defined by

$$N_C(x) = \{\xi \in \mathbb{R}^N : < \xi, x - y > \geq 0 \quad \text{for all } y \in C\}$$

and we consider also the tangent cone to C at x, i.e. the set

$$T_C(x) = \{\zeta \in \mathbb{R}^N : < \zeta, \xi > \leq 0 \quad \text{for all } \xi \in N_C(x)\}.$$

We point out that this definition of tangent cone agrees with those given in [11] and [1].

Then, recall that a *polytope* is the closed convex envelope of a finite numbers of points. By an *open polytope*, we mean the set of interior points of a polytope. Recall also that a point $x \in \partial C$ is said to be *exposed*, if there exists a supporting hyperplane to C at x wich meets the closure of C only at the point x itself. An open, bounded and convex set C is said to be *regular* whenever its boundary ∂C is continuously differentiable and every point of its boundary is exposed. It is easy to see that every open, bounded and convex set can be exhausted by an increasing sequence of regular and open convex sets $(C_n)_n$, such that $\overline{C}_n \subset C$ for every n (see for istance [24]).

Whenever the convex set C is a bounded neighbourhood of the origin, we let $\gamma_C : \mathbb{R}^N \to [0, \infty)$ be the Minkowski functional of C, that is

$$\gamma_C(x) = \inf\{t \geq 0 : x \in tC\}, \qquad x \in \mathbb{R}^N.$$

It is well known that it coincides with the supporting function of the polar set of C, that is

$$\gamma_C(x) = \sup\{< \xi, x >: \xi \in C^\circ\}, \qquad x \in \mathbb{R}^N.$$

The function $(x, y) \mapsto \gamma_C(x - y)$ enjoies all properties of a metric on $\mathbb{R}^N$ but symmetry. In fact it is not symmetric unless C is. Nevertheless, in the sequel, we shall refer to it as the metric associated with the convex set C. On account of this agreement for all open, bounded and convex subsets A of $\mathbb{R}^N$, we define the distance function from the set A with respect to the metric γ_C by

$$d_C(x, A) = \inf\{\gamma_C(y - x) : y \in A\}, \qquad x \in \mathbb{R}^N.$$

For C the unit ball centered at zero, we simply write $d(\cdot, A)$. Whenever the set A is the boundary of an open, bounded and convex subset $\mathcal{O}$ of $\mathbb{R}^N$, the function $d_C(\cdot, \partial\mathcal{O})$ is Lipschitz continuous and its gradient belongs to $-C^\circ$ almost every where on $\mathcal{O}$ (see [40]). We define also the width of $\mathcal{O}$ with respect γ_C by

$$W_C(\mathcal{O}) = \sup\{d_C(x, \partial\mathcal{O}) : x \in \mathcal{O}\}.$$

In the sequel, we shall consider the class of functions

$$\mathcal{F}(C) = \{f : \mathbb{R}^N \to [0, +\infty] : f \text{ is Borel measurable and } f(\xi) = 0,\ \xi \in \partial C\}$$

and, for every $f \in \mathcal{F}(C)$, we define the slope of f with respect to γ_C as

$$\Lambda_C(f) = \sup\{m \geq 0 : \; f(\xi) \geq m[\gamma_C(\xi) - 1] \text{ for all } \xi \in \mathbb{R}^N\}.$$

Throughout this paper, we let Ω and K be two bounded and convex subset of $\mathbb{R}^N$, the former open and the latter closed and containing the origin in its interior. For $f \in \mathcal{F}(K)$, we consider the functional

$$F(v) = \int_\Omega [f(\nabla v) + v] \, dx, \qquad v \in W^{1,1}(\Omega),$$

and the associated minimum problem

$$\text{(P)} \qquad \left\{F(u) : \, u \in u_0 + W_0^{1,1}(\Omega)\right\}$$

where the boundary datum u_0 is a given function in $W^{1,1}(\Omega)$. For such kind of problems we shall prove the following existence result.

Theorem 3.1.1. *Let f be in $\mathcal{F}(K)$ and let $u_0 : \overline{\Omega} \to \mathbb{R}$ be such that*

(a) *u_0 is convex and Lipschitz continuous;*

(b) *$\nabla u_0(x) \in K$ for almost every $x \in \Omega$.*

If

$$\text{(3.1.1)} \qquad \sup_{x \in \Omega} \inf_{y \in \partial\Omega} \left\{ \left[\max_{\overline{\Omega}} u_0 - u_0(y)\right] + \gamma_{K^\circ}(y - x) \right\} \leq \Lambda_K(f),$$

then the function

$$\text{(3.1.2)} \qquad u(x) = - \inf_{y \in \partial\Omega} \{-u_0(y) + \gamma_{K^\circ}(y - x)\}, \qquad x \in \Omega$$

is a solution to the minimum problem (P).

We split the proof of the theorem into two parts. In paragraph 3.2 we first give the proof in the case $u_0 = 0$ (see Theorem 3.2.1 ahead) and in the last paragraph we prove the general statement. We postpone to the end of this section some comments on the hypoteses of the theorem above and we begin by examining some consequences of it.

In the case $u_0 = 0$, Theorem 3.1.1 extends some recent results contained in [11], [47] and [48]. In fact, whenever $h : [0, +\infty) \to [0, +\infty]$ vanishes for some positive

ρ, the function $f = h \circ \gamma_K$ belongs to the class $\mathcal{F}(\rho K)$. Such function is the one considered in the quoted papers for K being either the closed unit ball or a polytope. Moreover, as $W_{(\rho K)^\circ}(\Omega) = \rho W_{K^\circ}(\Omega)$ and $\Lambda_{\rho K}(f) = \rho\Lambda$, where

$$\Lambda = \sup\{m \geq 0 : \, h(t) \geq m(t-\rho) \text{ for all } t \geq 0\},$$

the condition (3.1.1) with $u_0 = 0$ reduces to $W_{K^\circ}(\Omega) \leq \Lambda$, the very same condition considered in [11], [47] and [48]. In these papers the case $\rho = 0$ is allowed. Such case is not covered by Theorem 3.1.1 since zero is assumed to belong to the interior of K. However, this limiting case can be easily recovered. Indeed, we prove the following corollary which extends the previously mentioned results by allowing non homogeneous boundary conditions and by removing the smoothness assumption on the boundary of Ω, the geometrical hypotheses on K and the restriction on the dimension of the space.

Corollary 3.1.2. *Let* $h : [0,+\infty) \to [0,+\infty]$ *be a Borel measurable function such that* $h(\rho) = 0$ *for some* $\rho \geq 0$ *and set*

$$\Lambda = \sup\{m \geq 0 : \, h(t) \geq m(t-\rho) \text{ for every } t \geq 0\}.$$

Let $u_0 : \overline{\Omega} \to \mathbb{R}$ *be such that*

(a) u_0 *is convex and Lipschitz continuous;*

(b) $\nabla u_0(x) \in \rho K$ *for almost every* $x \in \Omega$*;*

and assume that either

$$\sup_{x\in\Omega} \inf_{y\in\partial\Omega} \left\{ \left[\max_{\overline{\Omega}} \left(\frac{1}{\rho} u_0 \right) - \left(\frac{1}{\rho} u_0(y) \right) \right] + \gamma_{K^\circ}(y-x) \right\} \leq \Lambda \tag{3.1.3}$$

if $\rho > 0$,

$$W_{K^\circ}(\Omega) \leq \Lambda$$

if $\rho = 0$. *Then, the function*

$$u(x) = \begin{cases} -\rho \inf\left\{ -\frac{1}{\rho} u_0(y) + \gamma_{K^\circ}(y-x) : \, y \in \partial\Omega \right\} & \text{if } \rho > 0, \\ u_0(x) & \text{if } \rho = 0, \end{cases} \tag{3.1.4}$$

is a minimizer on $u_0 + W_0^{1,1}(\Omega)$ *for the functional*

$$I(v) = \int_\Omega [h(\gamma_K(\nabla v)) + v] \, dx \qquad v \in W^{1,1}(\Omega). \tag{3.1.5}$$

Proof. First assume $\rho > 0$ and set $f = h \circ \gamma_K$. Then, $f \in \mathcal{F}(\rho K)$ and, as $\Lambda_{\rho K} f = \rho\Lambda$, it is easy to check that (3.1.3) is equivalent to the hypothesis (3.1.1) for f, u_0 and the convex set ρK. Thus, Theorem 3.1.1 applies and, as $\gamma_{(\rho K)^\circ} = \rho\gamma_{K^\circ}$, it follows that the function u defined by (3.1.4) for $\rho > 0$ is a minimizer for the functional I on $u_0 + W_0^{1,1}(\Omega)$.
Then, let $\rho = 0$, so that u_0 has to be constant due to (b). Hence, it is not restrictive to assume u_0 null on Ω. For $\epsilon > 0$, set

$$h_\epsilon(t) = \begin{cases} 0 & \text{if } 0 \le t \le \epsilon \\ \Lambda(t-\epsilon) & \text{if } t \ge \epsilon \end{cases}$$

and let I_ϵ be the functional defined by (3.1.5) with h replaced by h_ϵ. The first part of the proof ensures that $u_\epsilon(x) = -\epsilon d_{K^\circ}(x, \partial\Omega)$, $x \in \Omega$, is a minimizer for I_ϵ on $W_0^{1,1}(\Omega)$ for each $\epsilon > 0$. Moreover, $h_\epsilon(\gamma_K(\nabla u_\epsilon(x))) = 0$ for almost every $x \in \Omega$ and $u_\epsilon \to 0$ uniformly on Ω. Thus, $I_\epsilon(u_\epsilon) \to I(0)$. As $I_\epsilon \le I$ for each $\epsilon > 0$, the conclusion follows. □

Finally, we briefly discuss the hypotheses of Theorem 3.1.1 in the case $u_0 \neq 0$. As we look for solutions to the minimum problem (P) whose gradient lies almost everywhere in the boundary of K, i.e. in the set where f vanishes, it is natural to assume (b). As far as (3.1.1) is concerned, recall that, as previously mentioned in the introduction, the proof of the theorem in the non homogeneous case consists in considering the homogeneous minimum problem for F on a larger convex set Ω'. The existence of a solution to this problem is ensured provided $W_{K^\circ}(\Omega') \le \Lambda_K(f)$ and the left hand side of (3.1.1) turns out to be the width of Ω' with respect to K°.

3.2. The homogeneous problem

The aim of this section is to prove Theorem 3.1.1 when $u_0 = 0$. For the reader's convenience, we state the theorem in such case.

Theorem 3.2.1. *Let $f \in \mathcal{F}(K)$ and assume that $W_{K^\circ}(\Omega) \le \Lambda_K(f)$. Then, the function*

$$u(x) = -d_{K^\circ}(x, \partial\Omega), \qquad x \in \Omega \tag{3.2.1}$$

is a minimizer on $W_0^{1,1}(\Omega)$ *for the functional*

$$F(v) = \int_\Omega [f(\nabla v) + v]\, dx, \qquad v \in W^{1,1}(\Omega).$$

We prove the theorem by showing that an increasing sequence of open polytopes $(\mathcal{O}_n)_n$ contained in Ω exists with the property that, setting

$$u_n(x) = \begin{cases} -d_{K^\circ}(x, \partial\mathcal{O}_n) & \text{if } x \in \mathcal{O}_n, \\ 0 & \text{if } x \in \Omega \backslash \mathcal{O}_n, \end{cases} \tag{3.2.2}$$

the integrals

$$\int_{\mathcal{O}_n} [f(\nabla u_n) + u_n]\, dx, \qquad n \geq 1,$$

simultaneously converge to $F(u)$ and to $\inf\left\{F(v)\colon\ v \in W_0^{1,1}(\Omega)\right\}$.
In order to do this, we investigate the properties of the distance function associated with K° from the boundary of an open polytope. Therefore, let $\mathcal{O}$ be one such set and let $F_1, \ldots, F_m$ be its $(N-1)$-dimensional faces. The following Lemmas 3.2.2 and 3.2.3 are easy consequence of the definition of $d_{K^\circ}(\cdot, \partial\mathcal{O})$.

Lemma 3.2.2. *Let* $x \in \mathcal{O}$ *and let* $c > 0$. *The following are equivalent:*

(a) $d_{K^\circ}(x, \partial\mathcal{O}) = c$;

(b) the following conditions hold:
- $x + cK^\circ \subset \overline{\mathcal{O}}$;
- *there exist* $y \in \partial\mathcal{O}$ *and* $\xi \in \partial K^\circ$ *such that* $y = x + c\xi$.

For every $x \in \mathcal{O}$, let

$$\Pi(x) = \{y \in \partial\mathcal{O}\colon \gamma_{K^\circ}(y - x) = d_{K^\circ}(x, \partial\mathcal{O})\}$$

be the set of all points on the boundary of $\mathcal{O}$ which lie at minimal distance from x with respect to γ_{K°. Such set is non empty for all $x \in \mathcal{O}$ and $\Pi(x) = \partial\mathcal{O} \cap (x + cK^\circ)$ where $c = d_{K^\circ}(x, \partial\mathcal{O})$.

Lemma 3.2.3. *Let* $y \in \partial\mathcal{O}$, $\xi \in \partial K^\circ$ *and* $c > 0$ *be such that* $y - c\xi \in \mathcal{O}$ *and* $d_{K^\circ}(y - c\xi, \partial\mathcal{O}) = c$. *Then,*

(a) $y \in \Pi(y - c\xi)$;

(b) $d_{K^\circ}(y - b\xi, \partial\mathcal{O}) = b$ *for every* $0 < b \leq c$.

It is plain that the results above hold true when the open polytope $\mathcal{O}$ is replaced by an arbitrary open, bounded and convex set. Conversely, the subsequent lemma fails to be true in such a general case.

Lemma 3.2.4. *Let $y \in \partial\mathcal{O}$ and $\xi \in \partial K^\circ$ be given. Then, the following are equivalent:*

(a) $N_{\mathcal{O}}(y) \subset N_{K^\circ}(\xi)$;

(b) there exists $c > 0$ such that $y - c\xi \in \mathcal{O}$ and $d_{K^\circ}(y - c\xi, \partial\mathcal{O}) = c$.

Proof. Assume that $N_{\mathcal{O}}(y) \subset N_{K^\circ}(\xi)$. It follows that $T_{K^\circ}(\xi) \subset T_{\mathcal{O}}(y)$. By the definition of tangent cone, it follows also that $\lambda(\zeta - \xi) \in T_{K^\circ}(\xi) \subset T_{\mathcal{O}}(y)$ for every $\zeta \in \partial K^\circ$ and $\lambda > 0$. Then, set

$$\lambda_\zeta = \sup\left\{\lambda \geq 0\colon\ y + \lambda(\zeta - \xi) \in \overline{\mathcal{O}}\right\}$$

and notice that

$$\lambda_\zeta \|\zeta - \xi\| \geq \min\{d(y, F_i)\colon\ y \notin F_i\} > 0$$

for every $\zeta \in \partial K^\circ \backslash \{\xi\}$. Hence, λ_ζ is positive and uniformly bounded away from zero with respect to all such ζ. Then, set $c = \frac{1}{2}\inf\{\lambda_\zeta\colon\ \zeta \in \partial K^\circ \backslash \{\xi\}\}$. It follows that $y + c(\zeta - \xi) \in \overline{\mathcal{O}}$ for every $\zeta \in \partial K^\circ \backslash \{\xi\}$ and hence $y - c\xi + cK^\circ \subset \overline{\mathcal{O}}$. Recalling Lemma 3.2.2, we obtain that $d_{K^\circ}(y - c\xi, \partial\mathcal{O}) = c$.
We have thus proved that (b) follows from (a). The other implication is an obvious consequence of the definition of normal cone. □

Now, we aim at proving that, up to a null set, the open polytope $\mathcal{O}$ can be decomposed into as many open sets as its $(N-1)$-dimensional faces with the property that the restriction of the function $d_{K^\circ}(\cdot, \partial\mathcal{O})$ to each of them is affine. To this purpose, set

$$I(y) = \{\xi \in \partial K^\circ\colon\ N_{\mathcal{O}}(y) \subset N_{K^\circ}(\xi)\}, \qquad y \in \partial\mathcal{O},$$

and notice that, if F is any face of $\mathcal{O}$, the set $I(y)$ is independent on the choice of $y \in \mathrm{ri}(F)$. Hence, we write $I_F = I(y)$ for all $y \in \mathrm{ri}(F)$ and in particular I_i when $F = F_i$. In this latter case, the set I_i is non empty.
Then, for every $y \in \mathrm{ri}(F)$ and $\xi \in I_i$, we define

$$c(y, \xi) = \sup\{c > 0\colon\ \Pi(y - t\xi) \subset \mathrm{ri}(F_i) \text{ for all } 0 < t \leq c\}$$

and the set

$$\mathcal{O}_i(\xi) = \{y - c\xi\colon\ y \in \mathrm{ri}(F_i) \text{ and } 0 < c < c(y,\xi)\}.$$

Now, we investigate the properties of the sets $\mathcal{O}_i(\xi)$.

Lemma 3.2.5. *Let $x \in \mathcal{O}$ be such that $\Pi(x) \subset \mathrm{ri}(F_i)$ and let $\xi \in I_i$. Then, there exists $y \in \mathrm{ri}(F_i)$ such that $x = y - c\xi$ and $c = d_{K^\circ}(x, \partial\mathcal{O})$.*

Proof. By Lemmas 3.2.2 and 3.2.4, we have $x = y' - c\xi'$ for some $y' \in \mathrm{ri}(F_i)$ and $\xi' \in I_i$. Assume that $\xi' \neq \xi$ otherwise there is nothing to prove. We claim that $y = y' - c(\xi' - \xi) \in \partial\mathcal{O}$. In fact, were y a point in $\mathcal{O}$, it would follow that $y' + s(\xi' - \xi) \notin \overline{\mathcal{O}}$ for every $s > 0$. Being $\mathcal{O}$ a polytope, we have

$$T_{\mathcal{O}}(y') = \big\{\eta\colon\ y' + t\eta \in \overline{\mathcal{O}} \text{ for all } t > 0 \text{ small enough}\big\}.$$

Hence, it would follow that $\xi' - \xi \notin T_{\mathcal{O}}(y')$. This would lead to a contradiction since $\xi' - \xi \in T_{K^\circ}(\xi)$ and $T_{K^\circ}(\xi) \subset T_{\mathcal{O}}(y')$.
Therefore $y - c\xi = x = y' - c\xi'$ and $y \in \mathrm{ri}(F_i)$ by assumption. □

Remark 3.2.6. In view of Lemma 3.2.5, the sets $\mathcal{O}_i(\xi)$ are independent on the choice of $\xi \in I_i$. Therefore, we write $\mathcal{O}_i = \mathcal{O}_i(\xi)$ for all $\xi \in I_i$. Moreover, relying on the previous lemmas, it is also easy to check that $\mathcal{O}_i = \{x \in \mathcal{O}\colon\ \Pi(x) \subset \mathrm{ri}(F_i)\}$ and that

$$\mathcal{O}_i \cap \{x\colon\ d_{K^\circ}(x\,,\partial\mathcal{O}) = c\} = \{y - c\xi\colon\ y \in \mathrm{ri}(F_i)\ \xi_i \in I_i \text{ and } c \leq c(y,\xi)\}.$$

Hence, the restriction of the function $d_{K^\circ}(\cdot, \partial\mathcal{O})$ to each of the sets $\mathcal{O}_i$ coincides with an affine function that vanishes on F_i. Moreover, for every $x \in \mathcal{O}_i$, the vector $\nabla d_{K^\circ}(x, \partial\mathcal{O})$ generates the cone $-N_{\mathcal{O}}(y)$ for every $y \in \mathrm{ri}(F_i)$.

The next lemma ensures that each set $\mathcal{O}_i$ is open and hence measurable.

Lemma 3.2.7. *Let $\xi \in I_i$. Then, the function $c(\cdot\,,\xi)$ is continuous on* $\mathrm{ri}F_i$.
Proof. Assume by contradiction that $c(\cdot, \xi)$ fails to be continuous at a point $y_0 \in \mathrm{ri}(F_i)$. Hence, there exists a sequence $(y_n)_n \subset \mathrm{ri}(F_i)$ such that $y_n \to y_0$ and $c(y_n, \xi) \to l$ with $c(y_0, \xi) \neq l$.
By the definition of $c(y, \xi)$, the set $\Pi(y - c(y,\xi)\xi)$ contains at least a point $z \in \partial\mathcal{O}\backslash\mathrm{ri}(F_i)$. Let

$$\begin{cases} z_n = y_n - c(y_n,\xi)\,(\xi - \zeta_n)\,, & n \geq 1, \\ z_0 = y_0 - c(y_0,\xi)\,(\xi - \zeta_0)\,, & \end{cases}$$

be such points associated with y_n and y_0 respectively, where ζ_n and ζ_0 are in ∂K°.

Assume first that $c(y_0,\xi) > l$. Up to a subsequence, $z_n \to z_0'$ where $z_0' = y_0 \; -l\,(\xi - \zeta_0') \in \partial\mathcal{O}\backslash \mathrm{ri}(F_i)$ and this yields a contradiction. Then, assume that $c(y_0,\xi) < l$ and consider the points

$$z_n' = y_n - c(y_n,\xi)\,(\xi - \zeta_0)\,, \qquad n \geq 1.$$

Such points are in $\overline{\mathcal{O}}$ and $z_n' \to z_0''$ where $z_0'' = y_0 - l\,(\xi - \zeta_0)$. The point z_0 lies on the segment joining y_0 and z_0''. As the points y_0 and z_0 are in $\partial\mathcal{O}$, the same is true for z_0''. Therefore, the segment $[y_0, z_0'']$ is entirely contained in F_i and its left hand point is in $\mathrm{ri}(F_i)$ by assumption. Hence, the same holds true for z_0, a contradiction. □

We are left to prove that, up to a null set, the open polytope $\mathcal{O}$ is filled up by the open sets $\mathcal{O}_i$. To see this, we define an appropriate change of variables that will be useful also in the proof of the Theorem 3.2.1. For every $i \in \{1,\ldots m\}$, choose a point $y_i \in \mathrm{ri}(F_i)$ and $\xi_i \in I_i$. Let also $\{\nu_1,\ldots \nu_{N-1}\}$ be an orthonormal basis for the subspace of $\mathbb{R}^N$ orthogonal to ξ_i such that the mapping $T_i : \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$T_i(s,t) = s_1\nu_1 + s_2\nu_2 + \ldots + s_{N-1}\nu_{N-1} - t\xi_i + y_i, \qquad (s,t) \in \mathbb{R}^{N-1} \times \mathbb{R},$$

is an orientation preserving change of variables. In particular, $\det \nabla T_i(s,t) = \|\xi_i\|$. Then, let A_i be an open subset of $\mathbb{R}^{N-1}$ and $\varphi_i : \mathbb{R}^{N-1} \to \mathbb{R}$ be an affine function such that

$$T_i\left(\{(s,\varphi_i(s)) \colon \; s \in A_i\}\right) = \mathrm{ri}(F_i)$$

and let also $\Phi_i : \mathbb{R}^N \to \mathbb{R}^N$ and $\Psi_i : \mathbb{R}^N \to \mathbb{R}^N$ be the affine mappings defined by

$$(3.2.3) \qquad \begin{cases} \Phi_i(s,c) = (s, \varphi_i(s) + c), \\ \Psi_i(s,c) = T_i \circ \Phi_i(s,c), \end{cases} \qquad (s,c) \in \mathbb{R}^{N-1} \times \mathbb{R}.$$

It is easy to check that

$$(3.2.4) \qquad \begin{cases} \det\nabla\Psi_i(s,c) = \|\xi_i\| \\ \dfrac{\partial \Psi_i}{\partial c}(s,c) = -\xi_i \end{cases} \qquad (s,c) \in \mathbb{R}^{N-1} \times \mathbb{R}.$$

For the sake of brevity, set $c_i(s) = c(\Psi_i(s,0),\xi_i)$ for all $s \in A_i$ and finally set also $S_i = \Psi_i^{-1}(\mathcal{O}_i)$, $S_i^\epsilon = \{(s,c)\colon\ s \in A_i \text{ and } 0 < c < c_i(s) - \epsilon\}$ for $\epsilon > 0$ and

$$\mathcal{O}_i^\epsilon = \Psi_i(S_i^\epsilon). \tag{3.2.5}$$

Then, denoting the Lebesgue measure on $\mathbb{R}^k$ by $\mathcal{L}^k$, we have the following lemma.

Lemma 3.2.8. *We have* $\lim_{\epsilon\to 0} \mathcal{L}^N\left(\mathcal{O}\setminus\left(\cup_{1\le i\le m}\mathcal{O}_i^\epsilon\right)\right) = 0$.

Proof. The set $\mathcal{O}\setminus\left(\cup_{1\le i\le m}\mathcal{O}_i^\epsilon\right)$ is the disjoint union of the sets $\cup_{1\le i\le m}\left(\mathcal{O}_i\setminus\mathcal{O}_i^\epsilon\right)$ and $\mathcal{O}\setminus\left(\cup_{1\le i\le m}\mathcal{O}_i\right)$. By Remark 3.2.6, this latter set can be further splitted as the union of the sets

$$E_0 = \{x \in \mathcal{O}\colon\ \Pi(x) \cap [\partial\mathcal{O}\setminus\left(\cup_{1\le i\le m}\mathrm{ri}(F_i)\right)] \neq \emptyset\}$$

and E_0' where E_0' consists of those points $x \in \mathcal{O}$ such that $\Pi(x) \subset \cup_{1\le i\le m}\mathrm{ri}(F_i)$ and there exist two distinct indices i_1, i_2 such that $\Pi(x)\cap\mathrm{ri}(F_{i_j}) \neq \emptyset$ for $j = 1,2$. Letting $\xi_i \in I_i$ be as in the changes of variables defined above and setting

$$E_i^\epsilon = \{y - c\xi_i\colon\ y \in \mathrm{ri}(F_i) \text{ and } c(y,\xi_i) - \epsilon \le c \le c(y,\xi_i)\},$$

it is easy to check that $E_0' \cup \left[\cup_{1\le i\le m}\left(\mathcal{O}_i\setminus\mathcal{O}_i^\epsilon\right)\right] \subset \cup_{1\le i\le m}E_i^\epsilon$.

We claim that E_0 is a null set and that $\mathcal{L}^N\left(E_i^\epsilon\right) \to 0$ as $\epsilon \to 0$.

In order to prove the first claim, notice that E_0 is covered by the union of the sets $N_F = \{x \in \mathcal{O}\colon\ \Pi(x)\cap\mathrm{ri}(F) \neq \emptyset\}$ as F ranges through all faces of $\mathcal{O}$ such that $\dim F \le N-2$. Let F be such a face and denote the affine hull of F by $\mathrm{aff}\,(F)$. It is enough to prove that N_F is contained in an affine subspace whose dimension is $\dim F + 1$. By Lemmas 3.2.2 and 3.2.4, we have $N_F \subset \cup_{\zeta\in I_F}\pi_\zeta$ where $\pi_\zeta = \{y + t\zeta\colon\ y \in \mathrm{aff}\,(F) \text{ and } t \in \mathbb{R}\}$. We wish to prove that, for every pair ζ_1, $\zeta_2 \in I_F$, there exists $x \in N_F$ such that $x \in \pi_{\zeta_1}\cap\pi_{\zeta_2}$ which yields $\pi_{\zeta_1} = \pi_{\zeta_2}$. To see this, assume that $\zeta_1 \neq \zeta_2$ and choose $y_1 \in \mathrm{ri}(F)$. By Lemma 3.2.4, there exists $c_1 > 0$ such that $y_1 - c_1\zeta_1 \in N_F$ and $d_{K^\circ}(y_1 - c_1\zeta_1, \partial\mathcal{O}) = c_1$. Arguing as in the proof of Lemma 3,2.5, we obtain that $y_2 = y_1 - c_1(\zeta_1 - \zeta_2)$ is in $\partial\mathcal{O}$. Relying on Lemma 3.2.3 (b) and applying the prevoius argument to the points $y_1 - c(\zeta_1 - \zeta_2)$, $0 < c \le c_1$, we obtain that the segment $[y_1, y_1 - c_1(\zeta_1 - \zeta_2)]$ is entirely contained in $\partial\mathcal{O}$. Now, by Lemma 3.2.4 again, there exists $c_2 > 0$ such that $y_1 - c_2\zeta_2 \in N_F$ and $d_{K^\circ}(y_1 - c_2\zeta_2, \partial\mathcal{O}) = c_2$. The very same argument used above yields that the segment $[y_1, y_1 - c_2(\zeta_2 - \zeta_1)]$ is contained in $\partial\mathcal{O}$ as well. Hence, the whole segment $[y_1 - c_1(\zeta_1 - \zeta_2), y_1 - c_2(\zeta_2 - \zeta_1)]$ lies in F. In particular, $y_2 \in F$ and $y_1 - c_1\zeta_1 = y_1 - c_2\zeta_2$. Setting $y_1 - c_1\zeta_1 = x = y_1 - c_2\zeta_2$, the first claim is proved. Finally, by the change of variables Ψ_i defined by (3.2.3), we obtain that

$$\mathcal{L}^N(E_i^\epsilon) = \epsilon\mathcal{L}^{N-1}(A_i)\|\xi_i\|.$$

Thus the conclusion follows. □

Now, all the tools needed for the proof of Theorem 3.2.1 are available.

Proof of Theorem 3.2.1. Let $(v_n)_n \subset W_0^{1,1}(\Omega)$ be a minimizing sequence for F such that $F(v_n)$ is finite for every n and let $(\varphi_n)_n$ be a sequence in $C_c^\infty(\Omega)$ such that $\| v_n - \varphi_n \|_{1,1} \to 0$. Then, choose an increasing sequence of open polytopes $(\mathcal{O}_n)_n$ such that supp $(\varphi_n) \subset \mathcal{O}_n$, $\Omega = \cup_{n\geq 1}\mathcal{O}_n$ and

$$\lim_{n\to\infty}\int_{\Omega\setminus\mathcal{O}_n} |f(\nabla v_n) + v_n|\, dx = 0, \tag{3.2.6}$$

and let $(u_n)_n$ be the functions defined by (3.2.2). In order to complete the proof, it is enough to show that

(a) $\lim_{n\to\infty}\int_{\mathcal{O}_n}[f(\nabla u_n) + u_n]\, dx = \int_\Omega [f(\nabla u) + u]\, dx$;

(b) $\lim_{n\to\infty}\int_{\mathcal{O}_n}[f(\nabla u_n) + u_n]\, dx = \inf\left\{F(v)\colon\ v \in W_0^{1,1}(\Omega)\right\}$.

(a) Recall that $\nabla d_{K^\circ}(x, \partial\Omega)$ and $\nabla d_{K^\circ}(x, \partial\mathcal{O}_n)$ are in $-\partial K$ almost everywhere on Ω and $\mathcal{O}_n$ respectively. Hence, $f(\nabla u(x)) = 0$ and $f(\nabla u_n(x)) = 0$ for almost every x in Ω and $\mathcal{O}_n$ respectively. As $u_n \to u$ uniformly on Ω, (a) follows.

(b) Let p be a Borel measurable selection of $\partial\gamma_K$ and, for every n, let $\alpha_n \in L^\infty(\mathcal{O}_n)$ be such that $0 \leq \alpha_n \leq \Lambda_K(f)$ almost everywhere on $\mathcal{O}_n$. By the definition of $\lambda_K(f)$, we obtain that

$$\begin{aligned}\int_{\mathcal{O}_n}&[f(\nabla v_n) + v_n]\, dx \\ &\geq \int_{\mathcal{O}_n}[f(\nabla u_n) + u_n]\, dx + \int_{\mathcal{O}_n}[\alpha_n\langle p(\nabla u_n), \nabla v_n - \nabla u_n\rangle + (v_n - u_n)]\, dx\end{aligned}$$

for every n. Recalling (a) and (3.2.6), we see that, in order to prove (b), it is enough to show that the functions α_n can be so chosen that

$$\lim_{n\to\infty}\int_{\mathcal{O}_n}[\alpha_n\langle p(\nabla u_n), \nabla v_n - \nabla u_n\rangle + (v_n - u_n)]\, dx = 0.$$

To this purpose, split each of the terms of the sequence above as

$$\int_{\mathcal{O}_n} [\alpha_n < \langle p(\nabla u_n), \nabla v_n - \nabla \varphi_n \rangle + (v_n - \varphi_n)]\, dx$$
$$+ \int_{\mathcal{O}_n} [\alpha_n \langle p(\nabla u_n), \nabla \varphi_n - \nabla u_n \rangle + (\varphi_n - u_n)]\, dx$$

and notice that the choice of the functions φ_n and the uniform bound for the functions α_n imply that the first term of the expression above approaches zero as $n \to \infty$. Now, we claim that, for every n, we can choose the function α_n with the further property that

$$\int_{\mathcal{O}_n} [\alpha_n \langle p(\nabla u_n), \nabla \varphi_n - \nabla u_n \rangle + (\varphi_n - u_n)]\, dx = 0.$$

In order to simplify the notations, we drop the index n and, following the ideas of [11], [47] and [48], we prove the existence of a function $\alpha \in L^\infty(\mathcal{O})$ such that $0 \le \alpha \le \Lambda_K(f)$ almost everywhere on $\mathcal{O}$ and

$$\text{(3.2.7)} \qquad \int_{\mathcal{O}} [\alpha \langle p(\nabla u), \nabla \eta \rangle + \eta]\, dx = 0, \qquad \forall \eta \in C_c^\infty(\mathcal{O}).$$

By Lemma 3.2.8, we have

$$\int_{\mathcal{O}} [\alpha \langle p(\nabla u), \nabla \eta \rangle + \eta]\, dx = \lim_{\epsilon \to 0} \sum_{1 \le 1 \le m} \int_{\mathcal{O}_i^\epsilon} [\alpha \langle p(\nabla u), \nabla \eta \rangle + \eta]\, dx$$

where the sets $\mathcal{O}_i^\epsilon$ are those defined by (3.2.5). Hence, we address ourselves to the computation of

$$\text{(3.2.8)} \qquad \int_{\mathcal{O}_i^\epsilon} [\alpha \langle p(\nabla u), \nabla \eta \rangle + \eta]\, dx.$$

To this aim, recalling that u is affine on each set $\mathcal{O}_i$, let $\xi_i \in \mathbb{R}^N$ be such that $\xi_i = p(\nabla u(x)) \in \partial \gamma_K (\nabla u(x))$ for every $x \in \mathcal{O}_i$. This implies that $\nabla u(x) \in N_{K^\circ}(\xi_i)$ for every $x \in \mathcal{O}_i$ (see [44], Theorem 23.5). By Remark 3.2.6, it follows also that $N_{\mathcal{O}}(y) \subset N_{K^\circ}(\xi_i)$ for every $y \in \mathrm{ri}(F_i)$. Hence, $\xi_i \in I_i$. Then, consider the changes of variables $\Psi_i : \mathbb{R}^N \to \mathbb{R}$ defined by (3.2.3) and let $\beta_i : \mathbb{R}^N \to \mathbb{R}$ be the functions defined by

$$\beta_i(s,c) = \begin{cases} c_i(s) - c & \text{if } (s,c) \in S_i, \\ 0 & \text{otherwise,} \end{cases} \qquad (s,c) \in \mathbb{R}^{N-1} \times \mathbb{R}.$$

We are going to prove that the function

$$\alpha(x) = \sum_{1\leq i\leq m} \beta_i \circ \Psi_i^{-1}(x)\chi_{\mathcal{O}_i}(x), \qquad x \in \mathcal{O},$$

is such that (3.2.7) holds true. First of all, notice that α is measurable and that the definition of the functions c_i implies that $0 \leq \alpha(x) \leq W_{K^\circ}(\mathcal{O})$ for every $x \in \mathcal{O}$. Then, recalling the properties of Ψ_i and applying the change of variables formula, we see that (3.2.8) is equal to

$$\int_{S_i^\epsilon} [\beta_i(s,c)\langle p(\nabla u(\Psi_i(s,c))), \nabla\eta(\Psi_i(s,c))\rangle + \eta(\Psi_i(s,c))]\, \|\xi_i\|\, d(s,c).$$

Applying Fubini's theorem to the last term in the expression above and integrating by parts, we obtain

$$\int_{S_i^\epsilon} \eta(\Psi_i(s,c))\|\xi_i\|\, d(s,c) = \int_{A_i}\left(\int_0^{c_i(s)-\epsilon} \eta(\Psi_i(s,c))\|\xi_i\|\, dc\right) ds$$
$$= \int_{A_i}\|\xi_i\|\left\{-\epsilon\eta(\Psi_i(s,c_i(s)-\epsilon)) + \int_0^{c_i(s)-\epsilon} [c_i(s)-c]\langle\nabla\eta(\Psi_i(s,c))\,,\,\frac{\partial\Psi_i}{\partial c}(s,c)\rangle dc\right\} ds.$$

Now, recalling (3.2.4), how ξ_i was defined and the definition of β_i, we have

$$\int_{S_i^\epsilon} [\beta_i(s,c)\langle p(\nabla u(\Psi_i(s,c))), \nabla\eta(\Psi_i(s,c))\rangle \,+\, \eta(\Psi_i(s,c))]\, \|\xi_i\|\, d(s,c)$$
$$= -\epsilon\|\xi_i\| \int_{A_i} \eta(\Psi_i(s,c_i(s)-\epsilon))\, ds$$

and the right hand side of the equality above goes to zero as ϵ goes to zero. This concludes the proof of (b) and hence the proof of the theorem as well. □

3.3. The non homogeneous problem

In this section we consider the minimum problem (P) with non homogeneous boundary condition and we give the proof of Theorem 3.1.1. To this purpose, consider the following construction.

Let $\mathcal{O}$ and C be open, bounded and convex sets such that $0 \in C$ and let the function $\varphi_0 : \overline{\mathcal{O}} \to \mathbb{R}$ be such that

(3.3.1) φ_0 is non negative and concave on $\overline{\mathcal{O}}$;

(3.3.2) φ_0 is Lipschitz continuous on $\overline{\mathcal{O}}$ and $\nabla\varphi_0 \in -C^\circ$ a.e. on $\mathcal{O}$.

In [40], it is proved seen that (3.3.2) is equivalent to

$$\varphi_0(x_1) - \varphi_0(x_2) \leq \gamma_{-C}(x_1 - x_2) = \gamma_C(x_2 - x_1), \qquad x_1, x_2 \in \overline{\mathcal{O}}, \tag{3.3.3}$$

and that the function

$$\varphi(x) = \inf_{y \in \partial\mathcal{O}} \{\varphi_0(y) + \gamma_C(y - x)\}, \qquad x \in \overline{\mathcal{O}},$$

agrees with φ_0 on the boundary of $\mathcal{O}$ and is Lipschitz continuous with $\nabla\varphi \in -\partial C^\circ$ almost everywhere on $\mathcal{O}$.

Our aim is to determine a larger convex set $\mathcal{O}'$ containing $\mathcal{O}$, such that the distance function from the boundary of $\mathcal{O}'$ induced by the metric associated with C agrees with φ on $\mathcal{O}$ and hence with φ_0 on the boundary of $\mathcal{O}$. We shall prove that such set is

$$\mathcal{O}' = \mathcal{O} \cup \left[\bigcup_{y \in \partial\mathcal{O}, \varphi_0(y) > 0} (y + \varphi_0(y)C) \right], \tag{3.3.4}$$

a claim wich is an easy consequence of the following properties of the set $\mathcal{O}'$.

Proposition 3.3.1. *Let $\mathcal{O}$ and C be two open, bounded and convex sets such that $0 \in C$, let $\varphi_0 : \overline{\mathcal{O}} \to \mathbb{R}$ satisfy (3.3.1) and (3.3.2) and let $\mathcal{O}'$ be the set defined by (3.3.4). Then,*

(a) *$\mathcal{O}'$ is an open, bounded and convex set;*

(b) *for all $z \in \partial\mathcal{O}'$, there exists $y \in \partial\mathcal{O}$ such that $\gamma_C(z - y) = \varphi_0(y)$;*

(c) *for all $y \in \partial\mathcal{O}$, there exists $z \in \partial\mathcal{O}'$ such that $\gamma_C(z - y) = \varphi_0(y)$;*

(d) *$d_C(y, \partial\mathcal{O}') = \varphi_0(y)$ for all $y \in \partial\mathcal{O}$;*

(e) *$W_C(\mathcal{O}') = \sup_{x \in \mathcal{O}}\{d_C(x, \partial\mathcal{O}')\}$.*

Proof. Throughout the proof, we assume that φ_0 is not identically zero on $\partial\mathcal{O}$, otherwise $\partial\mathcal{O} = \mathcal{O}$ and nothing is left to prove. In particular, this assumption and the concavity of φ_0 ensure that φ_0 is positive on $\mathcal{O}$.

(a) We need only to prove that $\mathcal{O}'$ is convex. Let $x_1, x_2 \in \mathcal{O}'$ and $0 < \lambda < 1$, and set $x = \lambda x_1 + (1-\lambda)x_2$. It is enough to assume that $x_2 \in \mathcal{O}'\backslash\mathcal{O}$. We claim that there exists $y \in \overline{\mathcal{O}}$ such that $x \in y + \varphi_0(y)C$, i.e. $\gamma_C(x-y) < \varphi_0(y)$. To this purpose, assume first that both points x_1 and x_2 are not in $\mathcal{O}$ so that there exists $y_1, y_2 \in \partial\mathcal{O}$ such that $x_i \in y_i + \varphi_0(y_i)C$, for $i = 1, 2$. Set $y = \lambda y_1 + (1-\lambda)y_2$. Hence, $y \in \overline{\mathcal{O}}$ and the concavity of φ_0 yields

$$\begin{aligned}\varphi_0(y) &\geq \lambda\varphi_0(y_1) + (1-\lambda)\varphi_0(y_2) > \lambda\gamma_C(x_1 - y_1) + (1-\lambda)\gamma_C(x_2 - y_2)\\ &\geq \gamma_C(\lambda x_1 + (1-\lambda)x_2 - \lambda y_1 - (1-\lambda)y_2) = \gamma_C(x-y).\end{aligned}$$

Then, assume that x_1 is in $\mathcal{O}$ and let y_2 be as above. Set $y = \lambda x_1 + (1-\lambda)y_2$, so that $y \in \mathcal{O}$ and

$$\varphi_0(y) \geq \lambda\varphi_0(x_1) + (1-\lambda)\varphi_0(y_2) > (1-\lambda)\varphi_0(y_2) \geq (1-\lambda)\gamma_C(x_2 - y_2) = \gamma_C(x-y).$$

This proves the claim. Now, notice that if either $y \in \partial\mathcal{O}$ or $x \in \mathcal{O}$, we are over. Therefore, assume that $y \in \mathcal{O}$, $x \notin \mathcal{O}$ and let $\overline{y}$ be the unique point of the boundary of $\mathcal{O}$ which lies on the segment $[x, y]$. By (3.3.3), we have

$$\varphi_0(\overline{y}) \geq \varphi_0(y) - \gamma_C(\overline{y} - y) > \gamma_C(x-y) - \gamma_C(\overline{y} - y) = \gamma_C(x - \overline{y})$$

and hence $\varphi_0(\overline{y}) > 0$ and $x \in \overline{y} + \varphi_0(\overline{y})C$. Then $x \in \mathcal{O}'$.

(b) Let $z \in \partial\mathcal{O}'$ be fixed. If $z \in \partial\mathcal{O}$, it follows that $\varphi_0(z) = 0$ so that (b) holds true with $y = z$. Otherwise, let $z \in \partial\mathcal{O}'\backslash\partial\mathcal{O}$ and choose a sequence $(z_n)_n \subset \mathcal{O}'\backslash\mathcal{O}$, such that $z_n \to z$. For all n let $y_n \in \partial\mathcal{O}$ be such that $\gamma_C(z_n - y_n) < \varphi_0(y_n)$. Up to a subsequence, we have that $y_n \to y$ with $y \in \partial\mathcal{O}$. Hence, we have $\gamma_C(z-y) \leq \varphi_0(y)$ and, as $z \in \partial\mathcal{O}'$, we actually have equality.

(c) We assume, without loss of generality, that the origin is contained in $\mathcal{O}$, and we split the proof into two steps.

Step 1. In this step, we prove the thesis under the following additional hypotheses:

(3.3.5) $\varphi_0 > 0$ on $\mathcal{O}$;

(3.3.6) $\mathcal{O}$ and C are regular convex sets.

Let $\nu : \partial C \to S^{N-1}$ be exterior normal to the boundary of C. As C is a regular convex set, ν is a continuous bijection of ∂C onto S^{N-1}. Hence, it is a homeomorphism of ∂C onto S^{N-1}.

Now notice that (3.3.5) and (3.3.6) imply that the boundary of $\mathcal{O}'$ is differentiable and hence continuously differentiable as $\mathcal{O}'$ is convex. In fact, given $z \in \partial\mathcal{O}'$ and $y \in \partial\mathcal{O}$ such that $z \in y + \varphi_0(y)\partial C$, the boundary of $\mathcal{O}'$ lies between the set $y + \varphi_0(y)C$ and its supporting hyperplane at z. Therefore, let $n : \partial\mathcal{O}' \to S^{N-1}$ be the exterior normal to the boundary of $\mathcal{O}'$ and let $d : \partial\mathcal{O}' \to S^{N-1}$ be the continuous function defined by

$$d(z) = -\frac{\nu^{-1}(n(z))}{\|\nu^{-1}(n(z))\|}, \qquad z \in \partial\mathcal{O}'.$$

The meaning of the function d can be easily explained: whenever the convex sets $\mathcal{O}'$ and $y + \lambda C$, with $y \in \mathcal{O}'$ and $\lambda > 0$, have the same supporting hyperplane at z, then y lies on the line $\{z + td(z) : t > 0\}$. In particular, whenever $z \in \partial\mathcal{O}'$ and $y \in \partial\mathcal{O}$ is associated with z by (b), we have $y = z + td(z)$ for a suitable $t > 0$. Then, set

$$\ell(z) = \sup\{l > 0 : z + td(z) \notin \mathcal{O} \quad \text{for all } t \in (0, l)\}, \qquad z \in \partial\mathcal{O}',$$

and notice that $\ell(z)$ is a positive real number for all $z \in \partial\mathcal{O}'$. Indeed, the set $\{\ell > 0 : z + td(z) \notin \mathcal{O} \quad \text{forall } t \in (0, \ell)\}$ is non empty by (3.3.5) and bounded from above by (b).

Now consider the map $p : \partial\mathcal{O}' \to \mathbb{R}^N$ defined by

$$p(z) = z + \ell(z)d(z), \qquad z \in \partial\mathcal{O}'. \tag{3.3.7}$$

We claim that p has the following properties:

(i) $p(z) \in \partial\mathcal{O}$ and $\gamma_C(z - p(z)) = \varphi_0(p(z))$ for all $z \in \partial\mathcal{O}'$;

(ii) p is continuous;

(iii) for all $z \in \partial\mathcal{O}'$, we have $\frac{<z, p(z)>}{\|z\|\|p(z)\|} > -1$, i.e. the vectors z and $p(z)$ are never antipodal.

(i) Let $z \in \partial\mathcal{O}'$ and let $y' \in \partial\mathcal{O}$ be associated with z by (b). We have $y' = z + \ell' d(z)$ for some $\ell' > 0$. Set $y = p(z)$ so that $y \in \partial\mathcal{O}$ and $\ell(z) \le \ell'$ by the definition of $\ell(z)$. Recalling (3.3.3) and the inequality $\varphi_0(y) \le \gamma_C(z - y)$, we obtain that

$$\begin{aligned}\varphi_0(y') = \gamma_C(z - y') = \gamma_C(z - y) + \gamma_C(y - y') &\ge \gamma_C(z - y) + [\varphi_0(y') - \varphi_0(y)] \\ &\ge \varphi_0(y) + [\varphi_0(y') - \varphi_0(y)] = \varphi_0(y').\end{aligned}$$

Thus, $\gamma_C(z-y)=\varphi_0(y)$.

(ii) Let $z\in\partial\mathcal{O}'$ and $(z_n)_n\subset\partial\mathcal{O}'$ be such that $z_n\to z$ and set

$$\begin{cases} y=p(z) \\ \ell=\ell(z) \end{cases} \qquad \begin{cases} y_n=p(z_n) \\ \ell_n=\ell(z_n) \end{cases} \qquad n\geq 1.$$

Up to a subsequence, we have $y_n\to y_0$ and $\ell_n\to\ell_0$ with $y_0\in\partial\mathcal{O}$ and $\ell_0\geq 0$. The continuity of d yields $y_0=z+\ell_0 d(z)$ and hence $\ell_0\geq\ell$. We are left to prove that $\ell_0=\ell$. Assume by contradiction that $\ell_0=\ell+\epsilon$, where $\epsilon=|y-y_0|>0$. As the segment (y,y_0) is contained in $\mathcal{O}$ by (3.3.6), we can choose $x_0\in(y,y_0)$ and $\rho>0$ such that $B_\rho(x_0)\subset\mathcal{O}$ and $|y-x_0|\leq\epsilon/2$. Therefore, there exist an increasing sequence of integer $(n_k)_k$ and a sequence of positive numbers $(t_k)_k$ such that $x_k=z_{n_k}+t_k d(z_{n_k})\subset B_{\rho/k}(x_0)$. It follows that $x_k\to x_0$ and

$$t_k=\|x_k-z_{n_k}\|\leq\|x_k-x_0\|+\|x_0-z\|+\|z-z_{n_k}\|.$$

Hence, we have

$$\limsup_{k\to\infty} t_k\leq\|x_0-z\|=\|x_0-y\|+\|y-z\|\leq\ell+\frac{\epsilon}{2}$$

and, recalling that $\ell_{n_k}<t_k$, we conclude that $\lim_{k\to\infty}\ell_{n_k}\leq\ell+\epsilon/2$. Since $\ell_n\to\ell_0=\ell+\epsilon$, we have a contadiction. By the Uryshon's property of convergence, the conclusion follows.

(iii) For all $z\in\partial\mathcal{O}'$, the segment $(z,p(z))$ contains no points of $\mathcal{O}$. Since the origin is contained in $\mathcal{O}$, the vectors z and $p(z)$ are never antipodal.

In order to conclude this step, we have to prove that p is surjective onto $\partial\mathcal{O}$. To this purpose, consider the canonical projection of $\mathbb{R}^N\backslash\{0\}$ onto S^{N-1}, i.e. the mapping $x\mapsto x/\|x\|$, and denote its restrictions to $\partial\mathcal{O}$ and $\partial\mathcal{O}'$ by Ψ and Ψ' respectively. Such mappings are homeomorphism of $\partial\mathcal{O}$ and $\partial\mathcal{O}'$ on to S^{N-1} and they preserve directions. The mapping $\tilde{p}=\Psi\circ p\circ(\Psi')^{-1}$ is not antipodal and hence it is homotopic to the identity map of S^{N-1}. If $\tilde{p}$ were not surjective, it would be homotopic to a constant map and this cannot be since S^{N-1} fails to be contractible. Being $\tilde{p}$ surjective onto S^{N-1}, the same holds true for p onto $\partial\mathcal{O}$.

Step 2. In this step, we remove the additional hypotheses (3.3.5), and (3.3.6). To this purpose, we approximate $\mathcal{O}$ and C by increasing sequences of regular convex sets $(\mathcal{O}_n)_n$ and $(C_n)_n$ with the property that $\overline{\mathcal{O}}_n \subset \mathcal{O}$. The gradient of φ_0 lies almost everywhere in $-C_n^\circ$ and φ_0 is positive on $\mathcal{O}$, and hence, in particular, on the boundary of each set $\mathcal{O}_n$. We set

$$\mathcal{O}'_n = \mathcal{O}_n \cup \left[\bigcup_{y \in \partial \mathcal{O}_n} (y + \varphi_0(y) C_n) \right], \qquad n \geq 1$$

and we claim that

$$\mathcal{O}'_n = \mathcal{O}_n \cup \left[\bigcup_{x \in \overline{\mathcal{O}}_n} (x + \varphi_0(x) C_n) \right], \qquad n \geq 1. \tag{3.3.8}$$

Indeed, assume by contradiction that there exists $z \in x + \varphi_0(x) C_n$ such that $z \notin \mathcal{O}'_n$ for some $x \in \mathcal{O}_n$. Hence, $\gamma_{C_n}(z - x) < \varphi_0(x)$. Then, let $y \in \partial \mathcal{O}_n$ and $z' \in \partial \mathcal{O}'_n$ be the unique points such that $y, z' \in [x, z]$ and, by Step 1, let $z'' \in \partial \mathcal{O}'_n$ be such that $\gamma_{C_n}(z'' - y) = \varphi_0(y)$. Since $z' \notin \mathcal{O}'_n$, we have

$$\gamma_{C_n}(z' - y) \geq \varphi_0(y) = \gamma_{C_n}(z'' - y)$$

and hence

$$\begin{aligned} \varphi_0(x) - \varphi_0(y) &> \gamma_{C_n}(z - x) - \gamma_{C_n}(z'' - y) \\ &= \gamma_{C_n}(z - z') + \gamma_{C_n}(z' - y) + \gamma_{C_n}(y - x) - \gamma_{C_n}(z'' - y) \\ &\geq \gamma_{C_n}(z' - y) + \gamma_{C_n}(y - x) - \gamma_{C_n}(z'' - y) \geq \gamma_{C_n}(y - x), \end{aligned}$$

that is, a contradiction. This proves the claim.
Relying on (3.3.8), it easy to check that $\mathcal{O}'_n \subset \mathcal{O}'_{n+1}$ and that $\mathcal{O}' = \cup_{n \geq 1} \mathcal{O}'_n$.
Now, choose $y \in \partial \mathcal{O}$ and, for all n, let $y_n \in \partial \mathcal{O}_n$ and $z_n \in \partial \mathcal{O}'_n$ be such that $y_n \to y$ and $\gamma_{C_n}(z_n - y_n) = \varphi_0(y_n)$. For each n, let $c_n \in C_n$ be such that $z_n = y_n + \varphi_0(y_n) c_n$ so that, up to a subsequence, we have $z_n \to z$ and $c_n \to c$ where $z \in \partial \mathcal{O}'$ and $c \in \partial C$. It is clear that $\gamma_C(z - y) = \varphi_0(y)$ and this concludes Step 2 and hence the proof of (c) as well.

(d) Choose $y \in \partial \mathcal{O}$ and notice that $\gamma_C(z - y) \geq \varphi_0(y)$ for all $z \in \partial \mathcal{O}'$. By (c), there exists $z' \in \partial \mathcal{O}'$ such that $\gamma_C(z' - y) = \varphi_0(y)$. Therefore, we have

$$d_C(y, \partial \mathcal{O}') = \inf_{z \in \partial \mathcal{O}'} \gamma_C(z - y) = \gamma_C(z' - y) = \varphi_0(y).$$

(e) As in (c), we split the proof into two steps.

Step 1. First, we assume that φ_0 and the convex sets $\mathcal{O}$ and C satisfy the additional hypotheses (3.3.5) and (3.3.6). Let $x' \in \mathcal{O}'$ and $z' \in \partial\mathcal{O}'$ be such that

$$\gamma_C(z'-x') = d_C(x', \partial\mathcal{O}') = \sup_{x\in\mathcal{O}'} d_C(x, \partial\mathcal{O}')$$

and assume by contradiction that $x' \notin \overline{\mathcal{O}}$. Set $y' = p(z')$ where p is the surjective mapping defined in (3.3.7) and notice that $x' \neq y'$ and that the points x', y' and z' lie on the same straight line. Then, either $x' \in [z', y']$ or $y' \in [z', x']$. In the former case, we have

$$d_C(y, \partial\mathcal{O}') = \gamma_C(z'-y') = \gamma_C(z'-x') + \gamma_C(x'-y') > \gamma_C(z'-x') = \sup_{x\in\mathcal{O}'} d_C(x, \partial\mathcal{O}')$$

and hence a contradiction. In the latter case, let $y'' \in \partial\mathcal{O}$ and $z'' \in \partial\mathcal{O}'$ be such that $[y', y''] = [z', x'] \cap \overline{\mathcal{O}}$ and $p(z'') = y''$. The points x', y'' and z'' cannot lie on the same straight line, otherwise a contradiction would follow as in the first case. Therefore, being C regular, they satisfy the strict triangle inequality

$$\gamma_C(z''-x') < \gamma_C(z''-y'') + \gamma_C(y''-x').$$

Hence, we have

$$\begin{aligned}\sup_{x\in\mathcal{O}'} d_C(x, \partial\mathcal{O}') &= \gamma_C(z'-x') = \gamma_C(z'-y'') + \gamma_C(y''-x') \\ &\geq \gamma_C(z''-y'') + \gamma_C(y''-x') > \gamma_C(z''-x') \geq d_C(x', \partial\mathcal{O}')\end{aligned}$$

a contradiction again.

Step 2. We are left to prove the thesis without the additional hypotheses (3.3.5) and (3.3.6).

First, assume only that C is a regular convex set. Let $(\mathcal{O}_n)_n$ be an increasing exhaustion of $\mathcal{O}$ consisting of regular convex sets such that $\overline{\mathcal{O}}_n \subset \mathcal{O}$ for all n, so that φ_0 is positive on the boundary of each set $\mathcal{O}_n$. Then set

$$\mathcal{O}'_n = \mathcal{O}_n \cup \left[\bigcup_{y\in\partial\mathcal{O}_n} (y + \varphi_0(y)C)\right], \qquad n \geq 1$$

and notice that, arguing as in Step 2 of (c), the sequence $(\mathcal{O}'_n)_n$ is non decreasing and $\mathcal{O}' = \cup_{n\geq 1}\mathcal{O}'_n$. The sequence $(\chi_{\mathcal{O}'_n} d_{C_n}(\cdot, \partial\mathcal{O}'_n))_n$ is in turn non decreasing and converges pointwise to $d_C(\cdot, \partial\mathcal{O}')$ on $\overline{\mathcal{O}}'$. Hence the convergence is actually uniform on the same set. Using the result of Step 1 and taking the limit as $n \to \infty$, one obtains that

$$\sup_{x\in\mathcal{O}'} d_C(x, \partial\mathcal{O}') = \sup_{x\in\mathcal{O}} d_C(x, \partial\mathcal{O}').$$

Finally, an argument similar to the previous one, based on the approximation of C by a n increasing sequence of regular convex sets $(C_n)_n$, yields the conclusion. □

At last, we can prove the main theorem.

Proof of Theorem 3.1.1. Set $C = \mathrm{int}(K^\circ)$ and $\varphi_0(x) = \max_{\overline{\Omega}} u_0 - u_0(x)$, $x \in \overline{\Omega}$, so that all the hypoteses of Proposition 3.3.1 are fulfilled with $\mathcal{O} = \Omega$. Consider the set Ω' associated to Ω by (3.3.4), that is

$$\Omega' = \Omega \cup \left[\bigcup_{y\in\partial\Omega,\varphi_0(y)>0} (y + \varphi_0(y)C) \right],$$

and the function

$$v(x) = -d_{K^\circ}(x, \partial\Omega') + \max_{\overline{\Omega}} u_0, \qquad x \in \overline{\Omega'}.$$

We wish to prove that it agrees with the function u defined by (3.1.2) on the closure of Ω. Indeed, choose $x \in \Omega$ and let $z' \in \partial\Omega'$ and $y' \in \partial\Omega$ be such that $d_{K^\circ}(x, \partial\Omega') = \gamma_{K^\circ}(z' - x)$ and $y' \in [z', x]$. We have

$$\begin{aligned} u(x) &= -\inf_{y\in\partial\Omega}\{-u_0(y) + \gamma_{K^\circ}(y - x)\} = -\inf_{y\in\partial\Omega}\{\varphi_0(y) + \gamma_{K^\circ}(y - x)\} + \max_{\overline{\Omega}} u_0 \\ &\geq -[\varphi_0(y') + \gamma_{K^\circ}(y' - x)] + \max_{\overline{\Omega}} u_0 \geq -\gamma_{K^\circ}(y' - x)] + \max_{\overline{\Omega}} u_0 \\ &= -\gamma_{K^\circ}(z' - x) + \max_{\overline{\Omega}} u_0 = -d_{K^\circ}(x, \partial\Omega') + \max_{\overline{\Omega}} u_0. \end{aligned}$$

Conversely, let $y'' \in \partial\Omega$ be such that

$$\inf_{y\in\partial\Omega}\{\varphi_0(y) + \gamma_{K^\circ}(y - x)\} = \varphi_0(y'') + \gamma_{K^\circ}(y'' - x).$$

By Proposition 3.3.1 (d), we have $d_{K^\circ}(y'', \partial\Omega') = \varphi_0(y'')$ and hence

$$\begin{aligned} u(x) &= -\inf_{y\in\partial\Omega}\{\varphi_0(y) + \gamma_{K^\circ}(y - x)\} + \max_{\overline{\Omega}} u_0 \\ &= -[\varphi_0(y'') + \gamma_{K^\circ}(y'' - x)] + \max_{\overline{\Omega}} u_0 \\ &= -[d_{K^\circ}(y'', \partial\Omega') + \gamma_{K^\circ}(y'' - x)] + \max_{\overline{\Omega}} u_0 \\ &\leq d_{K^\circ}(x, \partial\Omega') + \max_{\overline{\Omega}} u_0 = v(x). \end{aligned}$$

Therefore, u and v agree on Ω and hence on the closure of Ω as well. In particular, we have

$$u(y) = v(y) = -d_{K^\circ}(y, \partial\Omega') + \max_{\overline{\Omega}} u_0 = -\varphi_0(y) + \max_{\overline{\Omega}} u_0 = u_0(y), \qquad y \in \partial\Omega.$$

Now, consider the following minimum problem

$$\text{(3.3.9)} \qquad \min\left\{\int_{\Omega'} [f(\nabla w) + w]\, dx \,:\, w \in \max_{\overline{\Omega}} u_0 + W_0^{1,1}(\Omega')\right\}.$$

On account of Theorem 3.2.1, it is clear that v is a solution to problem (3.3.9) provided $W_{K^\circ}(\Omega') \leq \Lambda_K(f)$. This is easily seen to be a consequence of (3.1.1) and Proposition 3.3.1 (e) since we have

$$\begin{aligned} W_{K^\circ}(\Omega') &= \sup_{x \in \Omega'} d_{K^\circ}(x, \partial\Omega') = \sup_{x \in \Omega} d_{K^\circ}(x, \partial\Omega') \\ &= \sup_{x \in \Omega} \inf_{y \in \partial\Omega} \left\{\left[\left(\max_{\overline{\Omega}} u_0 - u_0(y)\right)\right] + \gamma_{K^\circ}(y - x)\right\} \leq \Lambda_K(f). \end{aligned}$$

We have thus proved that u agrees with the boundary datum u_0 on $\partial\Omega$ and it is the restriction to Ω of a solution to the minimum problem (3.3.9). Therefore, u has to be a solution to the problem (P). □

Chapter 4
An Existence Result for a Problem of Potential Wells

Problems in elasticity, crystallography and phase transitions lead to the consideration of energy functionals of the kind

$$\int_\Omega f(\nabla u(x))dx$$

where g is non negative and is zero only on potential wells described by rotations of finitely many matrices $A_1, \ldots, A_r$, i.e.

$$f(F) = 0 \quad \text{for} \quad F \in \bigcup_{i=1}^{r} SO(3)A_i.$$

In general the matrices A_i describe symmetries of the material and are connected by a symmetry group. See, for instance [2], [3], [5], [19], [28] and [38].
Finding a minimizer of the energy satisfying the homogeneous condition at the boundary of Ω : $u|_{\partial\Omega} = 0$, is then equivalent to solving the differential inclusion

$$\nabla u(x) \in \bigcup_{i=1}^{r} SO(3)A_i,$$

with the boundary condition: $u|_{\partial\Omega} = 0$. This needs not always be possible: from a result of Reshetnyak, see [43] and [28], it follows that the problem

$$\begin{cases} \nabla u(x) \in SO(3)\mathbb{I}, \\ u|_{\partial\Omega} = 0, \end{cases}$$

admits no solution on any open and bounded $\Omega \subset \mathbb{R}^3$
In this chapter we aim at showing that for any open and bounded $\Omega \subset \mathbb{R}^3$, the problem

$$\begin{cases} \nabla u(x) \in SO(3)\mathbb{I} \cup SO(3)\mathbb{I}^-, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where

$$\mathbb{I}^- = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

(or any other matrix giving a change of orientation in $\mathbb{R}^3$) does indeed admit a solution, a Lipschitz continuous map $u : \overline{\Omega} \to \mathbb{R}^3$. More precisely, the matrix $\nabla u(x)$ will belong, for a.e x in Ω, to a subset of $O(3) = SO(3)\mathbb{I} \cup SO(3)\mathbb{I}^-$, the set $\mathcal{R}$ of those orthogonal matrices having rows $\pm e_j$, where (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$. Notice that this result is contrary to the intuition: when $\partial\Omega$ is smooth, in case u was smooth as well, the three components of u would have $\partial\Omega$ as a level set, hence their gradients would all be orthogonal to $\partial\Omega$, i.e. parallel to each other. In particular this result shows that the minimum of the functional

$$\int_\Omega f(\nabla u(x))dx$$

with homogeneous boundary condition is zero. Hence the functional is not quasiconvex since the (affine) boundary datum is not a solution to the minimum problem. The boundary datum zero need not be the only case yielding a zero infimum for the minimum problem. Characterizations of such boundary data under different assumptions are presented in [3] and [45].

4.1. Notations and preliminary results

For x in $\mathbb{R}^3$, define the three maps $x \to |X_s|(x)$, $x \to |X_m|(x)$, $x \to |X_i|(x)$, as follows:

$$|X_s|(x_1, x_2, x_3) = \sup\{|x_j| \, : \, j = 1, 2, 3\}.$$

Let $k \in \{1, 2, 3\}$ be such that $|X_s|(x) = |x_k|$ and set

$$|X_m|(x_1, x_2, x_3) = \sup\{|x_j| \, : \, j = 1, 2, 3; \, j \neq k\}.$$

Remark that $|X_m|$ is unambigously defined: in case k_1 and k_2 are such that $|x_{k_1}| = |X_s|(x) = |x_{k_2}|$, then $|X_m|(x) = |X_s|(x)$ independently of the choice of k. Set also

$$|X_i|(x_1, x_2, x_3) = \inf\{|x_j| \, : \, j = 1, 2, 3\}.$$

Proposition 4.1.1.

a) The maps $|X_s|$, $|X_m|$, $|X_i|$ *are continuous.*

b) For every $x \in \mathbb{R}^3$, $|X_s|(x_1, x_2, x_3) = |X_s|(|x_1|, |x_2|, |x_3|)$, *and the same for* $|X_m|$ *and* $|X_i|$.

c) For any permutation $(x_{j_1}, x_{j_2}, x_{j_3})$ *of* $(x_1, x_2, x_3) \in \mathbb{R}^3$, $|X_s|(x_{j_1}, x_{j_2}, x_{j_3}) = |X_s|(x_1, x_2, x_3)$ *and the same is true for* $|X_m|$ *and* $|X_i|$.

Remark 4.1.2. The composition of a continuous function on $\mathbb{R}^3$ with the map $x \to (|X_s|(x), |X_m|(x), |X_i|(x))$, is a continuous function of x, and is invariant under a permutation of (x_1, x_2, x_3).
For x in $\mathbb{R}^3$ and such that $|x_i| \neq |x_j|$, for $i, j = 1, 2, 3$ and $i \neq j$, set $s(x)$, $m(x)$, $i(x)$ to be such that

$$|x_{s(x)}| = |X_s|(x), \quad |x_{m(x)}| = |X_m|(x), \quad |x_{i(x)}| = |X_i|(x).$$

The maps $x \to s(x)$, $x \to m(x)$, $x \to i(x)$, are locally constant on their (open) domains.

We have the following technical proposition.

Proposition 4.1.3. *Let* $E \subset \mathbb{R}^2$ *be defined by*

$$E = \{x : \| x \|_\infty \leq 1, \ |x_1| + |x_2| \leq 1, \ |x_2| \leq |x_1|\}$$
$$\cup \{x : \| x \|_\infty \leq 1, \ |x_1| + |x_2| \geq 1, \ |x_1| \leq |x_2|\}.$$

Then, (x_1, x_2) *belongs to* E *if and only if* $((x_1)_{mod\,1}, (x_2)_{mod\,1})$ *belongs to* E.

Proof. Set $y_1 = (x_1)_{mod\,1}$ and $y_2 = (x_2)_{mod\,1}$. Four cases are possible: $(x_1, x_2) = (y_1, y_2)$, $(x_1, x_2) = (y_1 - 1, y_2)$, $(x_1, x_2) = (y_1, y_2 - 1)$, $(x_1, x_2) = (y_1 - 1, y_2 - 1)$. One verifies easily the claim, separately for each case. □

We wish to have indices i in $\{1, 2, 3\}$. It is convenient to set $(r)_3 = (r-1)_{mod\,3} + 1$, for any integer r.
We shall need three functions f^1, f^2, f^3, from $\mathbb{R}$ to $\mathbb{R}$. On $[0, 1]$ set

$$f^1(y) = \inf\{y, 1 - y\},$$

and consider f^1 on $\mathbb{R}$ to be its extension by periodicity. We have that f^1 is continuous and that $f^1(y) = f^1(|y|)$. Set also

$$f^2(y) = \frac{1}{2} f^1(2y); \quad f^3(y) = \frac{1}{4} f^1(4y).$$

4.2. Main result

It is our purpose to define a function $u : \overline{\Omega} \to \mathbb{R}^3$, Lipschitz continuous on $\overline{\Omega}$, such that $u|_{\partial\Omega} = 0$ and $\nabla u(x)$ is in $\mathcal{R} \subset SO(3)\mathbb{I} \cup SO(3)\mathbb{I}^-$ for a.e. x in Ω.

Theorem 4.2.1. *Let Ω be a bounded open subset of $\mathbb{R}^3$. Then there exists a map $\tilde{u} : \Omega \to \mathbb{R}^3$, Lipschitz continuous with Lipschitz constant one, such that*

i) $\tilde{u}|_{\partial\Omega} = 0$;

ii) $\nabla \tilde{u}(x) \in \mathcal{R}$, *for a.e.* x *in* Ω.

Proof. The proof consists of the following steps:

a) We define first a map u^1 on the sphere $\| x \|_\infty \leq 1$, satisfying the differential inclusion ii) on $\| x \|_\infty < 1$ but not the boundary condition i) at $\| x \|_\infty = 1$.

b) We recursively extend this map, by defining a function u^n on the set of x such that $\sum_{i=0}^{n-2} \frac{1}{2^i} \leq \| x \|_\infty \leq \sum_{i=0}^{n-1} \frac{1}{2^i}$, a Lipschitz continuous map satisfying condition ii) and such that, for all $j \in \{1, 2, 3\}$, $\sup |u_j^n(x)| \leq \frac{1}{2^{n-1}}$.

c) We define a function u satisfying properties i) and ii) for $\overline{\Omega} = B_2$, the sphere $\| x \|_\infty \leq 2$.

d) Exploiting Vitali's covering theorem, we define $\tilde{u}$ on $\overline{\Omega}$, with the properties i) and ii) of the theorem.

a) On B_1, the unit ball $\| x \|_\infty \leq 1$, set:

$$u_1^1(x) = 1 - \| x \|_\infty = 1 - |X_s|(x);$$
$$u_2^1(x) = \inf\{f^1(|X_i|(x)), f^1(|X_m|(x))\};$$
$$u_3^1(x) = \begin{cases} f^2(|X_m|(x)) & \text{on } |X_i|(x) + |X_m|(x) \leq 1 \\ f^2(|X_i|(x)) & \text{on } |X_i|(x) + |X_m|(x) \geq 1. \end{cases}$$

Notice that on the set $\{x : |X_i|(x) + |X_m|(x) = 1\}$, one has $|X_i|(x) = 1 - |X_m|(x)$, hence $f^2(|X_i|(x)) = f^2(1 - |X_m|(x)) = f^2(|X_m|(x) - 1) = f^2(|X_m|(x))$ by the periodicity of f^2. Recalling Proposition 4.1.1, the map u^1 is continuous, actually piecewise affine. In particular consider $u^1(x_1, x_2, 1)$.

Claim 1. $u^1(x_1, x_2, 1) = u^1((x_1)_{mod\,1}, (x_2)_{mod\,1}, 1)$.

Proof of Claim 1. We have $u_1^1(x_1, x_2, 1) = 0$. Moreover,

$$u_2^1(x_1, x_2, 1) = \inf\{f^1(|x_1|), f^1(|x_2|)\} = \inf\{f^1(x_1), f^1(x_2)\}$$
$$= \inf\{f^1((x_1)_{mod\,1}), f^1((x_2)_{mod\,1})\}.$$

Finally consider $u_3^1(x_1, x_2, 1)$. Recalling Proposition 4.1.3, we have

$$(x_1, x_2) \in E \Leftrightarrow ((x_1)_{mod\,1}, (x_2)_{mod\,1}) \in E.$$

Then if $(x_1, x_2) \in E$ we have

$$u_3^1(x_1, x_2, 1) = f^2(|x_1|) = f^2(x_1)$$

and

$$u_3^1((x_1)_{mod\,1}, (x_2)_{mod\,1}, 1) = f^2(|(x_1)_{mod\,1}|) = f^2((x_1)_{mod\,1}).$$

Since $f^2(x_1) = f^2((x_1)_{mod\,1})$ the claim follows in this case. Analogously when (x_1, x_2) belongs to the complement of E. This proves Claim 1.

Moreover, we have: $\sup\{|u_j^1(x)| : x \in B_1,\ j = 1, 2, 3\} = 1$.
Whenever the gradients exist, we have:

$$\nabla u_1^1(x) = -\operatorname{sign}(\mathrm{x}_{\mathrm{s(x)}})\mathrm{e}_{\mathrm{s(x)}}$$

$$\nabla u_2^1(x) = \begin{cases} \operatorname{sign}(\mathrm{x}_{\mathrm{i(x)}})\mathrm{e}_{\mathrm{i(x)}} & \text{on } |x_{i(x)}| + |x_{m(x)}| < 1 \\ -\operatorname{sign}(\mathrm{x}_{\mathrm{m(x)}})\mathrm{e}_{\mathrm{m(x)}} & \text{on } |x_{i(x)}| + |x_{m(x)}| > 1 \end{cases}$$

$$\nabla u_3^1(x) = \begin{cases} f^{2\prime}(x_{m(x)})e_{m(x)} & \text{on } |x_{i(x)}| + |x_{m(x)}| < 1 \\ f^{2\prime}(x_{i(x)})e_{i(x)} & \text{on } |x_{i(x)}| + |x_{m(x)}| > 1. \end{cases}$$

Since $|f^{2\prime}(t)| = 1$, for $t \notin \{(1/4)z : z \,\text{integer}\}$, we have that, a.e. on B_1, $\nabla u^1(x) \in \mathcal{R}$.

b) We begin by defining two auxiliary functions v and ℓ^1. The function ℓ^1 will, in turn, extend u^1 as u^2 on the layer $1 \leq \| x \|_\infty \leq 1 + \frac{1}{2}$. To do so, we have to carefully consider the continuity of u^2 at $\{x : \| x \|_\infty = 1\}$. An induction argument carries this construction to u^n.
We begin by defining the subsets Q_s^3, Q_m^3, Q_i^3 of $B_{\frac{1}{2}} = \{x : \| x \|_\infty \leq \frac{1}{2}\}$ as

$$Q_s^3 = \{x \in B_{\frac{1}{2}} : |x_3| = |X_s|(x)\};$$
$$Q_m^3 = \{x \in B_{\frac{1}{2}} : |x_3| = |X_m|(x)\};$$
$$Q_i^3 = \{x \in B_{\frac{1}{2}} : |x_3| = |X_i|(x)\}.$$

Set $v : B_{\frac{1}{2}} \to \mathbb{R}^3$ to be:

$$v_2(x) = \frac{1}{2} - \| x \|_\infty = \frac{1}{2} - |X_s|(x);$$

and, for x in $Q_s^3 \cup Q_m^3$,

$$v_1(x) = \begin{cases} f^3(|X_m|(x)) & \text{on } |X_i|(x) + |X_m|(x) \leq \frac{1}{2} \\ f^3(|X_i|(x)) & \text{on } |X_i|(x) + |X_m|(x) \geq \frac{1}{2}; \end{cases}$$
$$v_3(x) = \inf\{f^2(|X_i|(x)), f^2(|X_m|(x))\};$$

for x in Q_i^3,

$$v_1(x) = \begin{cases} f^3(|X_i|(x)) & \text{on } |X_i|(x) + |X_m|(x) \leq \frac{1}{2} \\ f^3(|X_m|(x)) & \text{on } |X_i|(x) + |X_m|(x) \geq \frac{1}{2}; \end{cases}$$
$$v_3(x) = \sup\{f^2(|X_i|(x)), f^2(|X_m|(x))\}.$$

The same arguments as used for u^1 show that v is lipschitzean on $B_{\frac{1}{2}}$ and that

$$\sup\{|v_j(x)| \,:\, x \in B_{\frac{1}{2}},\, j = 1,2,3\} = \frac{1}{2}.$$

Notice, for future use, the following properties of v:

α) $v(x_1, x_2, x_3) = v(|x_1|, |x_2|, |x_3|)$;

β) $v(x_1, x_2, x_3) = v(x_2, x_1, x_3)$.

To prove β) above, remark that when (x_1, x_2, x_3) belongs to one of the set Q_s^3, Q_m^3, Q_i^3, so does (x_2, x_1, x_3). Then v is defined through the maps $|X_s|$, $|X_m|$, $|X_i|$ that assume the same values on (x_1, x_2, x_3) and any of its permutations.

The map v is differentiable a.e. on $B_{\frac{1}{2}}$ and, whenever ∇v exists, has the form, for x in $Q_s^3 \cup Q_m^3$:

$$\nabla v_1(x) = \begin{cases} f^{3\prime}(x_{m(x)})e_{m(x)} & \text{on } |x_{i(x)}| + |x_{m(x)}| < \frac{1}{2} \\ f^{3\prime}(x_{i(x)})e_{i(x)} & \text{on } |x_{i(x)}| + |x_{m(x)}| > \frac{1}{2} \end{cases}$$
$$\nabla v_2(x) = -\,\mathrm{sign}(\mathrm{x}_{\mathrm{s(x)}})\mathrm{e}_{\mathrm{s(x)}}$$
$$\nabla v_3(x) = \begin{cases} f^{2\prime}(x_{i(x)})e_{i(x)} & \text{on } |x_{i(x)}| + |x_{m(x)}| < \frac{1}{2} \\ f^{2\prime}(x_{m(x)})e_{m(x)} & \text{on } |x_{i(x)}| + |x_{m(x)}| > \frac{1}{2}. \end{cases}$$

For x in Q_i^3:

$$\nabla v_1(x) = \begin{cases} f^{3\prime}(x_{i(x)})e_{i(x)} & \text{on } |x_{i(x)}| + |x_{m(x)}| < \frac{1}{2} \\ f^{3\prime}(x_{m(x)})e_{m(x)} & \text{on } |x_{i(x)}| + |x_{m(x)}| > \frac{1}{2} \end{cases}$$
$$\nabla v_2(x) = -\,\mathrm{sign}(\mathrm{x}_{\mathrm{s(x)}})\mathrm{e}_{\mathrm{s(x)}}$$
$$\nabla v_3(x) = \begin{cases} f^{2\prime}(x_{m(x)})e_{m(x)} & \text{on } |x_{i(x)}| + |x_{m(x)}| < \frac{1}{2} \\ f^{2\prime}(x_{i(x)})e_{i(x)} & \text{on } |x_{i(x)}| + |x_{m(x)}| > \frac{1}{2}. \end{cases}$$

Hence, a.e. on $B_{\frac{1}{2}}$, $\nabla v(x) \in \mathcal{R}$.
The following properties will be essential to show the continuity of the extension of the map u^1.

Claim 2. For $(x_1, x_2, 1)$ in B_1 we have

$$u^1(x_1, x_2, 1) = v\left((x_1)_{mod\,1} - \frac{1}{2}, (x_2)_{mod\,1} - \frac{1}{2}, 0\right).$$

Proof of Claim 2. We have already proved that

$$u^1(x_1, x_2, 1) = u^1((x_1)_{mod\,1}, (x_2)_{mod\,1}, 1),$$

hence, without loss of generality, we can assume $x_1, x_2 \geq 0$. Set y in $B_{\frac{1}{2}}$ to be $y = (x_1 - \frac{1}{2}, x_2 - \frac{1}{2}, 0)$. Since $y_3 = 0$, y is in Q_i^3, and $|X_i|(y) + |X_m|(y) = |X_m|(y) \leq 1/2$, so that $v_1(y) = f_3(0) = 0$. Moreover, by the very definition, $u_1^1(x_1, x_2, 1) = 0$.
In order to prove the claim for the second and third components, consider the sets

$$\begin{aligned} A &= \{x_1 + x_2 \leq 1\} \cap \{x_2 \leq x_1\}, \\ B &= \{x_1 + x_2 \geq 1\} \cap \{x_2 \geq x_1\}, \\ C &= \{x_1 + x_2 \leq 1\} \cap \{x_2 \geq x_1\}, \\ D &= \{x_1 + x_2 \geq 1\} \cap \{x_2 \leq x_1\}. \end{aligned}$$

Since $v_2(x_1 - \frac{1}{2}, x_2 - \frac{1}{2}, 0) = \frac{1}{2} - \sup\{|x_1 - \frac{1}{2}|, |x_2 - \frac{1}{2}|\}$, we have

$$\begin{aligned} &v_2\left(x_1 - \frac{1}{2}, x_2 - \frac{1}{2}, 0\right) \\ &= x_2\chi_A(x_1, x_2) + (1 - x_2)\chi_B(x_1, x_2) + x_1\chi_C(x_1, x_2) + (1 - x_1)\chi_D(x_1, x_2). \end{aligned}$$

On the other hand,

$$u_2^1(x_1, x_2, 1) = \inf\{f^1(x_1), f^1(x_2)\} = \inf\{x_1, 1 - x_1, x_2, 1 - x_2\}.$$

On A we have $x_2 \leq x_1$, $1 - x_1 \geq x_2$, $x_2 \leq 1 - x_2$, so that $u_2^1(x_1, x_2, 1) = x_2$. Analogously one verifies that $u_2^1(x_1, x_2, 1) = v_2(x_1 - \frac{1}{2}, x_2 - \frac{1}{2}, 0)$, for $(x_1, x_2) \in B \cup C \cup D$.
Consider now the third component. Notice that

$$|X_i|\left(x_1 - \frac{1}{2}, x_2 - \frac{1}{2}, 0\right) = 0$$

and that

$$|X_m|\left(x_1-\frac{1}{2},x_2-\frac{1}{2},0\right)=\inf\left\{\left|x_1-\frac{1}{2}\right|,\left|x_2-\frac{1}{2}\right|\right\},$$

hence, by definition, $v_3(x_1-\frac{1}{2},x_2-\frac{1}{2},0)=f^2(\inf\{|x_1-\frac{1}{2}|,|x_2-\frac{1}{2}|\})$. We have

$$\inf\left\{\left|x_1-\frac{1}{2}\right|,\left|x_2-\frac{1}{2}\right|\right\}=\begin{cases}|x_1-\frac{1}{2}| & \text{on } A\cup B\\ |x_2-\frac{1}{2}| & \text{on } C\cup D\end{cases}$$

so that

$$\begin{aligned}v_3&\left(x_1-\frac{1}{2},x_2-\frac{1}{2},0\right)\\ &=f^2\left(\left|x_1-\frac{1}{2}\right|\right)\chi_{A\cup B}(x_1,x_2)+f^2\left(\left|x_2-\frac{1}{2}\right|\right)\chi_{C\cup D}(x_1,x_2)\\ &=f^2\left(x_1-\frac{1}{2}\right)\chi_{A\cup B}(x_1,x_2)+f^2\left(x_2-\frac{1}{2}\right)\chi_{C\cup D}(x_1,x_2)\\ &=f^2(x_1)\chi_{A\cup B}(x_1,x_2)+f^2(x_2)\chi_{C\cup D}(x_1,x_2).\end{aligned}$$

On the other hand, by definition,

$$\begin{aligned}u_3^1(x_1,x_2,1)&=f^2(|X_m|(x_1,x_2,1))\chi_{A\cup C}(x_1,x_2)+f^2(|X_i|(x_1,x_2,1))\chi_{B\cup D}(x_1,x_2)\\ &=f^2(x_1)\chi_A+f^2(x_2)\chi_C+f^2(x_1)\chi_B+f^2(x_2)\chi_D.\end{aligned}$$

This proves Claim 2.

Claim 3. For ξ_1,ξ_2: $-\frac{1}{2}\le\xi_1\le\frac{1}{2}$, $-\frac{1}{2}\le\xi_2\le\frac{1}{2}$, and for $r=1,2,3$, we have:

$$v_{(r-1)_3}(\xi_1,\xi_2,0)=2v_r\left(\frac{1}{2}\xi_1+\frac{1}{4},\frac{1}{2}\xi_2+\frac{1}{4},\frac{1}{2}\right).$$

Proof of Claim 3. Consider $r=2$. By definition, since $(\xi_1,\xi_2,0)$ is in Q_i^3, we have $v_1(\xi_1,\xi_2,0)=f^3(0)=0$. On the other hand, since v_2 is zero at the boundary of $B_{\frac{1}{2}}$, the claim holds for $r=2$.
Consider $r=3$. We have

$$v_2(\xi_1,\xi_2,0)=\frac{1}{2}-\sup\{|\xi_1|,|\xi_2|\},$$

while, since $(\frac{1}{2}\xi_1+\frac{1}{4},\frac{1}{2}\xi_2+\frac{1}{4},\frac{1}{2})$ is in Q_s^3,

$$\begin{aligned}v_3\left(\frac{1}{2}\xi_1+\frac{1}{4},\frac{1}{2}\xi_2+\frac{1}{4},\frac{1}{2}\right)&=\inf\left\{f^2\left(\frac{1}{2}\xi_1+\frac{1}{4}\right),f^2\left(\frac{1}{2}\xi_2+\frac{1}{4}\right)\right\}\\ &=\inf\left\{\frac{1}{2}f^1\left(\xi_1+\frac{1}{2}\right),\frac{1}{2}f^1\left(\xi_2+\frac{1}{2}\right)\right\}.\end{aligned}$$

Since f^1, for $t \in [0,1]$, can be written as $f^1(t) = \frac{1}{2} - |t - \frac{1}{2}|$,

$$\inf\left\{\frac{1}{2}f^1\left(\xi_1 + \frac{1}{2}\right), \frac{1}{2}f^1\left(\xi_2 + \frac{1}{2}\right)\right\} = \frac{1}{2}\inf\left\{\frac{1}{2} - |\xi_1|, \frac{1}{2} - |\xi_2|\right\}$$
$$= \frac{1}{2}\left(\frac{1}{2} - \sup\{|\xi_1|, |\xi_2|\}\right),$$

and the claim holds in this case as well.
Consider $r = 1$. Since $(\xi_1, \xi_2, 0)$ is in Q_i^3,

$$v_3(\xi_1, \xi_2, 0) = f^2(\inf\{|\xi_1|, |\xi_2|\}).$$

On the other hand,

$$v_1\left(\frac{1}{2}\xi_1 + \frac{1}{4}, \frac{1}{2}\xi_2 + \frac{1}{4}, \frac{1}{2}\right)$$
$$= \begin{cases} f^3\left(\sup\left\{\left|\frac{1}{2}\xi_1 + \frac{1}{4}\right|, \left|\frac{1}{2}\xi_2 + \frac{1}{4}\right|\right\}\right) & \text{on } \left|\frac{1}{2}\xi_1 + \frac{1}{4}\right| + \left|\frac{1}{2}\xi_2 + \frac{1}{4}\right| \le \frac{1}{2} \\ f^3\left(\inf\left\{\left|\frac{1}{2}\xi_1 + \frac{1}{4}\right|, \left|\frac{1}{2}\xi_2 + \frac{1}{4}\right|\right\}\right) & \text{on } \left|\frac{1}{2}\xi_1 + \frac{1}{4}\right| + \left|\frac{1}{2}\xi_2 + \frac{1}{4}\right| \ge \frac{1}{2} \end{cases}$$
$$= \begin{cases} f^3\left(\frac{1}{4} + \frac{1}{2}\sup\{\xi_1, \xi_2\}\right) & \text{on } \xi_1 + \xi_2 \le 0 \\ f^3\left(\frac{1}{4} + \frac{1}{2}\inf\{\xi_1, \xi_2\}\right) & \text{on } \xi_1 + \xi_2 \ge 0 \end{cases}$$
$$= \begin{cases} f^3\left(\frac{1}{2}\sup\{\xi_1, \xi_2\}\right) & \text{on } \xi_1 + \xi_2 \le 0 \\ f^3\left(\frac{1}{2}\inf\{\xi_1, \xi_2\}\right) & \text{on } \xi_1 + \xi_2 \ge 0 \end{cases}$$
$$= \begin{cases} \frac{1}{2}f^2(\sup\{\xi_1, \xi_2\}) & \text{on } \xi_1 + \xi_2 \le 0 \\ \frac{1}{2}f^2(\inf\{\xi_1, \xi_2\}) & \text{on } \xi_1 + \xi_2 \ge 0. \end{cases}$$

Consider the four sets:

$$A = \{(\xi_1, \xi_2) : \xi_1 + \xi_2 \le 0 \,\text{and}\, \xi_2 \ge \xi_1\},$$
$$B = \{(\xi_1, \xi_2) : \xi_1 + \xi_2 \le 0 \,\text{and}\, \xi_2 \le \xi_1\},$$
$$C = \{(\xi_1, \xi_2) : \xi_1 + \xi_2 \ge 0 \,\text{and}\, \xi_2 \le \xi_1\},$$
$$D = \{(\xi_1, \xi_2) : \xi_1 + \xi_2 \ge 0 \,\text{and}\, \xi_2 \ge \xi_1\},$$

so that

$$B \cup D = \{(\xi_1, \xi_2) : |\xi_2| \ge |\xi_1|\}$$

and

$$A \cup C = \{(\xi_1, \xi_2) : |\xi_1| \ge |\xi_2|\}.$$

We have

$$v_1\left(\frac{1}{2}\xi_1 + \frac{1}{4}, \frac{1}{2}\xi_2 + \frac{1}{4}, \frac{1}{2}\right) = \frac{1}{2}f^2(\xi_2)\chi_A + \frac{1}{2}f^2(\xi_1)\chi_B + \frac{1}{2}f^2(\xi_2)\chi_C + \frac{1}{2}f^2(\xi_1)\chi_D$$
$$= \frac{1}{2}f^2(|\xi_1|)\chi_{B\cup D} + \frac{1}{2}f^2(|\xi_2|)\chi_{A\cup C} = \frac{1}{2}f^2(\inf\{|\xi_1|, |\xi_2|\}).$$

Thus Claim 3 is fully proved.

Having proved the properties of the map v described in Claims 2 and 3, we introduce the "layer" function ℓ^1, that will be used to extend the map u^1. On the set $\mathbb{R}^2 \times [-\frac{1}{2}, \frac{1}{2}]$, we define ℓ^1 as

$$\ell^1(x_1, x_2, x_3) = v\left((x_1)_{mod\,1} - \frac{1}{2}, (x_2)_{mod\,1} - \frac{1}{2}, x_3\right).$$

We shall use the following property of ℓ^1:

Claim 4.

$$\ell^1(x_1, x_2, x_3) = \ell^1(|x_1|, |x_2|, |x_3|).$$

Proof of Claim 4. We have

$$\begin{aligned}\ell^1(x_1, x_2, x_3) &= v\left((x_1)_{mod\,1} - \frac{1}{2}, (x_2)_{mod\,1} - \frac{1}{2}, x_3\right)\\ &= v\left(\left|(x_1)_{mod\,1} - \frac{1}{2}\right|, \left|(x_2)_{mod\,1} - \frac{1}{2}\right|, |x_3|\right).\end{aligned}$$

By inspection, one verifies that $\left|(|t|)_{mod\,1} - \frac{1}{2}\right| = \left|(t)_{mod\,1} - \frac{1}{2}\right|$, so that

$$\begin{aligned}\ell^1(x_1, x_2, x_3) &= v\left((|x_1|)_{mod\,1} - \frac{1}{2}, (|x_2|)_{mod\,1} - \frac{1}{2}, |x_3|\right)\\ &= \ell^1(|x_1|, |x_2|, |x_3|).\end{aligned}$$

Claim 4 is proved.

Having introduced ℓ^1, define , for $n \in \mathbb{N}^+$, $\ell^n : \mathbb{R}^2 \times [-\frac{1}{2^n}, \frac{1}{2^n}]$ as

$$\ell^n(x_1, x_2, x_3) = \frac{1}{2^{n-1}}\ell^1(2^{n-1}x_1, 2^{n-1}x_2, 2^{n-1}x_3).$$

Notice that, by Claim 4, $\ell^n(x_1, x_2, x_3) = \ell^n(|x_1|, |x_2|, |x_3|)$.
The analogue of the property expressed by Claim 3 is given by the following Claim 5.

Claim 5. For $m \in \mathbb{N}^+$ and $r = 1, 2, 3$,

$$\ell^{m+1}_{(r-1)_3}(x_1, x_2, 0) = \ell^m_r\left(x_1, x_2, \frac{1}{2^m}\right).$$

Proof of Claim 5.

$$\ell_r^m\left(x_1, x_2, \frac{1}{2^m}\right) = \frac{1}{2^{m-1}}\ell_r^1\left(2^{m-1}x_1, 2^{m-1}x_2, \frac{1}{2}\right)$$
$$= \frac{1}{2^{m-1}}v_r\left((2^{m-1}x_1)_{mod\,1} - \frac{1}{2}, (2^{m-1}x_2)_{mod\,1} - \frac{1}{2}, \frac{1}{2}\right).$$

On the other hand

$$\ell_{(r-1)_3}^{m+1}(x_1, x_2, 0) = \frac{1}{2^m}\ell_{(r-1)_3}^1(2^m x_1, 2^m x_2, 0)$$
$$= \frac{1}{2^m}v_{(r-1)_3}\left((2^m x_1)_{mod\,1} - \frac{1}{2}, (2^m x_2)_{mod\,1} - \frac{1}{2}, 0\right).$$

At this point notice that, by inspection, for t in $\mathbb{R}$, $(2t)_{mod\,1} - [2(t)_{mod\,1} - 1] \in \{0,1\}$. Hence the difference between $((2^m x_1)_{mod\,1} - \frac{1}{2}, (2^m x_2)_{mod\,1} - \frac{1}{2}, 0)$ and $(2(2^{m-1}x_1)_{mod\,1} - 1 - \frac{1}{2}, 2(2^{m-1}x_2)_{mod\,1} - 1 - \frac{1}{2}, 0)$ has 0 or 1 at the two first components; so that, by the periodicity of v when $x_3 = 0$,

$$v_{(r-1)_3}\left((2^m x_1)_{mod\,1} - \frac{1}{2}, (2^m x_2)_{mod\,1} - \frac{1}{2}, 0\right)$$
$$= v_{(r-1)_3}\left(2(2^{m-1}x_1)_{mod\,1} - 1 - \frac{1}{2}, 2(2^{m-1}x_2)_{mod\,1} - 1 - \frac{1}{2}, 0\right),$$

and applying Claim 3,

$$v_{(r-1)_3}\left((2^m x_1)_{mod\,1} - \frac{1}{2}, (2^m x_2)_{mod\,1} - \frac{1}{2}, 0\right)$$
$$= 2v_r\left((2^{m-1}x_1)_{mod\,1} - \frac{1}{2} - \frac{1}{4} + \frac{1}{4}, (2^{m-1}x_2)_{mod\,1} - \frac{1}{2} - \frac{1}{4} + \frac{1}{4}, \frac{1}{2}\right)$$
$$= 2v_r\left((2^{m-1}x_1)_{mod\,1} - \frac{1}{2}, (2^{m-1}x_2)_{mod\,1} - \frac{1}{2}, \frac{1}{2}\right),$$

proving Claim 5.

Set

$$L^1 = \{(x_1, x_2, x_3) \,:\, |x_3| \le 1 \text{ and } \sup\{|x_1|, |x_2|\} \le |x_3|\},$$

and, for $n \ge 2$,

$$L^n = \left\{(x_1, x_2, x_3) \,:\, \sum_{i=0}^{n-2}\frac{1}{2^i} \le |x_3| \le \sum_{i=0}^{n-1}\frac{1}{2^i} \quad \text{and} \quad \sup\{|x_1|, |x_2|\} \le |x_3|\right\}.$$

On L^1 the map u^1 is already defined. For $n \geq 2$ and (x_1, x_2, x_3) in L^n, set

$$u_j^n(x_1, x_2, x_3) = \ell^{n-1}_{(j-(n+1))_3}\left(x_1, x_2, |x_3| - \sum_{i=0}^{n-2} \frac{1}{2^i}\right).$$

Remark that, from property $\beta)$ of the map v and Claim 4, we have, for the map u^n, the analogous properties

$\alpha')$ $u^n(x_1, x_2, x_3) = u^n(|x_1|, |x_2|, |x_3|)$;

$\beta')$ $u^n(x_1, x_2, x_3) = u^n(x_2, x_1, x_3)$.

Notice that it follows that the map $u^1\chi_{L^1} + u^2\chi_{L^2}$ is continuous. To prove this fact, we have to show that

$$u_j^1(x_1, x_2, 1) = \ell_j^1(x_1, x_2, 0),$$

and the validity of this statement is supplied by Claim 2 and by the definition of ℓ^1.

Claim 6

$$u_j^n\left(x_1, x_2, \sum_{i=0}^{n-2} \frac{1}{2^i}\right) = u_j^{n-1}\left(x_1, x_2, \sum_{i=0}^{n-2} \frac{1}{2^i}\right).$$

Proof of Claim 6. We have to show that

$$\ell^{n-1}_{(j-(n+1))_3}(x_1, x_2, 0) = \ell^{n-2}_{(j-n)_3}\left(x_1, x_2, \frac{1}{2^{n-2}}\right),$$

and this follows from Claim 5 setting $m = n-2$ and $r = (j-n)_3$. Thus Claim 6 is proved.

We wish to extend each map u^n to the set $\{x : \sum_{i=0}^{n-2} \frac{1}{2^i} \leq \| x \|_\infty \leq \sum_{i=0}^{n-1} \frac{1}{2^i}\}$. Set

$$u^n(x) = u^n(|X_i|(x), |X_m|(x), |X_s|(x)).$$

It is a true extension: let x be in L^n. Then $|x_3| = |X_s|(x)$, and

$$(|X_i|(x), |X_m|(x), |X_s|(x)) \in \{(|x_1|, |x_2|, |x_3|), (|x_2|, |x_1|, |x_3|)\}.$$

From $\alpha')$ and $\beta')$ it follows then that

$$u^n(x_1, x_2, x_3) = u^n(|X_i|(x), |X_m|(x), |X_s|(x)),$$

so that the new definition coincides with the old. Moreover each u^n is a composition of continuous maps, hence continuous.

We have in addition that, for $j = 1,2,3$ and $n \in \mathbb{N}^+$,

$$\sup \left\{ |u_j^n(x)| : \sum_{i=0}^{n-2} \frac{1}{2^i} \leq \| x \|_\infty \leq \sum_{i=0}^{n-1} \frac{1}{2^i} \right\} \leq \frac{1}{2^{n-1}}.$$

Consider now $\nabla u^n(x)$. Recalling that the maps $i(\cdot)$, $m(\cdot)$ and $s(\cdot)$ are defined on an open set of full measure and are locally constant on it, we see that, given $\overline{x}$, there exist integer values, say $\overline{\imath}$, $\overline{m}$, $\overline{s}$, which are the values of $i(\cdot)$, $m(\cdot)$, $s(\cdot)$ respectively on a neighborhood of $\overline{x}$. For x in this neighborhood

$$u^n(x) = u^n(x_{\overline{\imath}}, x_{\overline{m}}, x_{\overline{s}}).$$

If we consider the gradient with respect to the variables $x_{\overline{\imath}}$, $x_{\overline{m}}$, $x_{\overline{s}}$, we have

$$\begin{aligned}
\nabla_{\overline{\imath},\overline{m},\overline{s}} u_j^n(x_{\overline{\imath}}, x_{\overline{m}}, x_{\overline{s}}) &= \nabla_{\overline{\imath},\overline{m},\overline{s}} \ell^{n-1}_{(j-(n+1))_3} \left(x_{\overline{\imath}}, x_{\overline{m}}, |x_{\overline{s}}| - \sum_{i=0}^{n-2} \frac{1}{2^i} \right) \\
&= \nabla_{\overline{\imath},\overline{m},\overline{s}} \frac{1}{2^{n-2}} \ell^1_{(j-(n+1))_3} \left(2^{n-2} x_{\overline{\imath}}, 2^{n-2} x_{\overline{m}}, 2^{n-2} \left(|x_{\overline{s}}| - \sum_{i=0}^{n-2} \frac{1}{2^i} \right) \right) \\
&= \nabla_{\overline{\imath},\overline{m},\overline{s}} \frac{1}{2^{n-2}} v_{(j-(n+1))_3} \left(2^{n-2} (\xi_{\overline{\imath}}, \xi_{\overline{m}}, \xi_{\overline{s}}) \right),
\end{aligned}$$

Where

$$(\xi_{\overline{\imath}}, \xi_{\overline{m}}, \xi_{\overline{s}}) = \frac{1}{2^{n-2}} \left((2^{n-2} x_{\overline{\imath}})_{mod\,1} - \frac{1}{2}, (2^{n-2} x_{\overline{m}})_{mod\,1} - \frac{1}{2}, 2^{n-2} \left(|x_{\overline{s}}| - \sum_{i=0}^{n-2} \frac{1}{2^i} \right) \right).$$

Except on a set of measure zero, this gradient equals

$$\nabla_{\overline{\imath},\overline{m},\overline{s}} v_{(j-(n+1))_3} (\xi_{\overline{\imath}}, \xi_{\overline{m}}, \xi_{\overline{s}}).$$

Since, a.e.,

$$\nabla_{\overline{\imath},\overline{m},\overline{s}} v (\xi_{\overline{\imath}}, \xi_{\overline{m}}, \xi_{\overline{s}}) \in \mathcal{R}$$

and $\nabla_{\overline{\imath},\overline{m},\overline{s}} u^n(x_{\overline{\imath}}, x_{\overline{m}}, x_{\overline{s}})$ is obtained from it by a permutation of the rows, then it follows that $\nabla_{\overline{\imath},\overline{m},\overline{s}} u^n(x_{\overline{\imath}}, x_{\overline{m}}, x_{\overline{s}})$ belongs to $\mathcal{R}$ as well. Since the columns of $\nabla u^n(x)$ are a permutation of the columns of $\nabla_{\overline{\imath},\overline{m},\overline{s}} u^n(x_{\overline{\imath}}, x_{\overline{m}}, x_{\overline{s}})$,we have

$$\nabla u^n(x) \in \mathcal{R}.$$

c) For x such that $\| x \|_\infty < 2$, set:

$$u(x) = u^\nu(x), \quad \text{when}\, x \in L^\nu.$$

By Claim 6, the map u is unambigously defined and continuous, actually Lipschitz continuous, a.e. $\nabla u(x)$ is in $\mathcal{R}$ and, by the estimate on $|u_j^\nu(x)|$, one has

$$\lim_{\|x\|_\infty \to 2} u(x) = (0,0,0)$$

i.e. u satisfies i) and ii) with $\overline{\Omega} = B_2$.

d) The collection $\{z + rB_2 \,:\, z \in \Omega,\, r \in \mathbb{R}^+,\, r < \frac{1}{2}d(z, \partial\Omega)\}$ is a Vitali covering of Ω. Let z_j and r_j, $j \in \mathbb{N}$, be such that:

(1) $(z_j + r_j B_2)$ are mutually disjoint;

(2) $\Omega = N \cup \left(\bigcup_{j\in\mathbb{N}} (z_j + r_j B_2)\right)$, with N a subset of Ω of zero measure.

For each $j \in \mathbb{N}$, define the vector function $\tilde{u}^j$ on Ω, by setting

$$\tilde{u}^j(x) = r_j u\left(\frac{x - z_j}{r_j}\right) \chi_{z_j + r_j B_2}(x),$$

so that $\nabla \tilde{u}^j(x) \in \mathcal{R}$ for a.e. x in $z_j + r_j B_2$.
Finally set, for x in Ω,

$$\tilde{u}(x) = \sum_{j\in\mathbb{N}} \tilde{u}^j(x).$$

Then $\tilde{u}$ is the required function: $\tilde{u}$ is Lipschtz continuous and satisfies i) and ii) of the theorem. □

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