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Biliaison classes

of reflexive sheaves

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Thesis submitted for the degree of "Doctor Philosophiæ"

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Contents

Introduction

For subschemes in projective space liaison arises from the geometric notion of linkage. Roughly speaking, two subschemes are directly linked if their union is a complete intersection. The equivalence relation generated by linkage is called liaison. One can also define a finer equivalence relation called biliaison (or even liaison): two subschemes are in the same biliaison class if they are related to each other by an even number of direct links. Thus, in some sense, a biliaison class is "half" of a liaison class. It turns out that biliaison classes are the right object to study in the context of liaison.

Actually, at least in codimension two, biliaison theory is now well-established. There are two main aspects of the theory, one is the parametrization of biliaison classes and the other one is a description of the structure of a single class. The former goal was achieved by A. P. Rao in 1981 ([R2]). He showed that biliaison classes of codimension two (equidimensional and locally Cohen-Macaulay) subschemes of $\mathbf{P}^n$, $n \geq 2$, are in bijective correspondence with stable equivalence classes of vector bundles $\mathcal{E}$ on $\mathbf{P}^n$ with $H^1\mathcal{E}(t) = 0$ for any $t \in \mathbf{Z}$ (we set $\mathbf{P}^n = \mathbf{P}^n_K$, where K is an algebraically closed field of arbitrary characteristic, notice also that we say vector bundle as synonym of locally free sheaf).

As a particular case, via the Horrocks' classification of stable equivalence classes ([Ho]), one recovers an earlier result of Rao concerning curves in $\mathbf{P}^3$ ([R1]): biliaison classes of curves in $\mathbf{P}^3$ are in bijective correspondence with finite length graded S-modules, identified up to shift in grading (S denotes the polynomial ring in four indeterminates). The correspondence is obtained associating to a curve C its Hartshorne-Rao module, that is, the first cohomology module of its ideal sheaf: $\bigoplus_{t\in Z} H^1\mathcal{I}_C(t) = H^1_*\mathcal{I}_C$.

This motivates — via the Hartshorne-Serre correspondence ([H]) — our definition

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX of biliaison for rank two reflexive sheaves on $\mathbf{P}^3$. Recall that a reflexive sheaf is a coherent sheaf which is canonically isomorphic to its double dual.

Definition 1. (Def 1.8) Let $\mathcal{F}$ and $\mathcal{F}'$ be rank two reflexive sheaves on $\mathbf{P}^3$. We say that $\mathcal{F}$ and $\mathcal{F}'$ are in the same *biliaison class* if $H^1_*\mathcal{F} \cong H^1_*\mathcal{F}'$.

Notice that we do not allow a shift in the grading of the modules, indeed this would produce equivalence classes too big and not well-behaved. We then parametrize these biliaison classes by strong stable equivalence classes of vector bundles with $H^2_* = 0$ (Thm. 1.10). Let us first set a definition:

Definition 2. (Def. 1.9) Two vector bundles $\mathcal{E}$ and $\mathcal{E}'$ are *strongly stably equivalent* if there exist integers a_i and b_j such that $\mathcal{E} \oplus \bigoplus \mathcal{O}(a_i) \cong \mathcal{E}' \oplus \bigoplus \mathcal{O}(b_j)$.

This equivalence relation is actually stronger than stable equivalence, since no twist of the bundles is allowed.

Theorem 3. (Thm. 1.10) *Biliaison classes of rank two reflexive sheaves on $\mathbf{P}^3$ are in bijective correspondence with strong stable equivalence classes of vector bundles $\mathcal{E}$ on $\mathbf{P}^3$ with $H^2_*\mathcal{E} = 0$ and, equivalently, with isomorphism classes of finite length graded S-modules.*

The second main objective of biliaison theory is to describe the structure of a class. For codimension two subschemes in $\mathbf{P}^n$ this has been done first by E. Ballico, G. Bolondi and J. Migliore ([BBM]) who proved that any biliaison class of codimension two (equidimensional and locally Cohen-Macaulay) non-arithmetically Cohen-Macaulay subschemes in $\mathbf{P}^n$ has the so called Lazarsfeld-Rao property. Roughly, this means that in each class there is a "minimal" element, unique up to deformation, from which the whole class can be built up applying repeatidly a simple process (basic double linkage). For curves in $\mathbf{P}^3$, this has been proved independently by M. Martin-Deschamps and D. Perrin ([MD-P1]) who also provided an algorithm to determine the minimal curve in a given class. This type of results extends to the case of non locally Cohen-Macaulay subschemes of pure codimension two, as has been shown by Nollet in [N]. He has also extended the Rao's correspondence mentioned above to purely two codimensional subschemes on one side and reflexive sheaves $\mathcal{F}$ with $H^1_*\mathcal{F} = 0$ and $\mathcal{E}xt(\mathcal{F}^*, \mathcal{O}) = 0$ on the other side (see [N, Thm. 2.10]).

In Chapter 2 we show that (an adapted version of) the Lazarsfeld-Rao property holds for biliaison classes of rank two reflexive sheaves on $\mathbf{P}^3$ (Thm. 2.24). The key points are the definition of minimal elements and that of a "basic" operation within a class. To do this we use in an essential way work of M. Martin-Deschamps and D. Perrin in [MD-P2]. For a given reflexive sheaf $\mathcal{E}$ they define a finite support function $\chi_{\mathcal{E}} : \mathbf{Z} \to \mathbf{Z}$ such that a general morphism $\bigoplus_{n\in\mathbf{Z}} \mathcal{O}(-n)^{\chi_{\mathcal{E}}(n)} \to \mathcal{E}$ is injective and has a reflexive cokernel. The interest of this function is that it picks up the least possible degrees of sections of $\mathcal{E}$ which degenerate in codimension at least 3, i.e. such that the cokernel is reflexive (it is well known that if we choose $rk(\mathcal{E}) - 2$ sections of large degrees they degenerate in codimension at least 3). The function $\chi_{\mathcal{E}}$ is defined in terms of certain invariants associated with the rank stratification of the subsheaves $\mathcal{E}_{\leq n}$ of $\mathcal{E}$ generated by sections of degree $\leq n$. Furthermore, Martin-Deschamps and Perrin show that if $p : \mathbf{Z} \to \mathbf{Z}$ is any other finite support function such that there is an injection $0 \to \bigoplus_{n\in\mathbf{Z}} \mathcal{O}(-n)^{p(n)} \to \mathcal{E}$ whose cokernel is reflexive, then for any $t \in \mathbf{Z}$ the function p satisfies $\sum_{n\leq t} p(n) \leq \sum_{n\leq t} \chi_{\mathcal{E}}(n)$ plus another condition for "low" values of n (here "low" means not greater than a certain integer depending on $\mathcal{E}$). Since precise statements here require some preliminaries, we ask the reader to refer to Chapter 2.

To be able to apply these results to our situation we first show that any reflexive sheaf $\mathcal{F}$ on $\mathbf{P}^3$ has a special type of locally free resolution connected with a minimal free resolution of its first cohomology module. Let

$$0 \to L_4 \to L_3 \to L_2 \to L_1 \to L_0 \to H^1_*\mathcal{F} \to 0$$

be a minimal graded free resolution and let $\mathcal{N}_0$ be the (locally free) sheaf associated with $ker(L_1 \to L_0)$. Then there is an exact sequence of sheaves on $\mathbf{P}^3$ — that we call the $\mathcal{N}_0$-resolution of $\mathcal{F}$ — of the form:

$$(*) \qquad 0 \to \bigoplus_{n\in\mathbf{Z}} \mathcal{O}(-n)^{p(n)} \to \mathcal{N} \to \mathcal{F} \to 0$$

where $\mathcal{N} \cong \mathcal{N}_0 \oplus$ (direct sum of line bundles). In other words, $\mathcal{N}$ is strongly stably equivalent to $\mathcal{N}_0$ (Prop. 1.1). This is also the way we prove Theorem 3 above.

Indeed, the biliaison class of $\mathcal{F}$ is uniquely determined by the strong stable equivalence class of $\mathcal{N}_0$ and viceversa. Applying now the results of [MD-P2] to $\mathcal{N}_0$ we obtain a description of the biliaison class of $\mathcal{F}$. In particular, we take as minimal elements those given by the function $\chi_{\mathcal{N}_0}$. The "basic" operation, which we call (ascending) elementary biliaison, is also defined in terms of the $\mathcal{N}_0$-resolution (see Def. 2.20). Its effect is to move from a sheaf $\mathcal{F}$ with a resolution (*) to a sheaf $\mathcal{F}'$ that has a resolution

$$0 \to \bigoplus_{n \in \mathbf{Z}} \mathcal{O}(-n)^{p'(n)} \to \mathcal{N} \to \mathcal{F}' \to 0$$

with $\sum_{n \leq t} p'(n) \leq \sum_{n \leq t} p(n)$ for any $t \in \mathbf{Z}$.

With these "ingredients" we are able to prove the following:

Theorem 4. (Thm. 2.24) *Any biliaison class of rank two reflexive sheaves $\mathcal{F}$ on $\mathbf{P}^3$ with $H^1_*\mathcal{F} \neq 0$ has the Lazarsfeld-Rao property. That is, the following conditions hold:*

(1) *there exist minimal elements, which deform to each other with constant cohomology and through sheaves in the same biliaison class;*
(2) *any non minimal sheaf can be obtained from a minimal one by a finite number of ascending elementary biliaison possibly followed by deformation with constant cohomology and through sheaves in the same biliaison class.*

This nice structure allows us to control the behaviour in a class of some invariants associated with a reflexive sheaf. We show in particular that minimal elements have minimal first and third Chern classes. For the first Chern class this is just an easy computation, while for the third Chern class this depends on a result communicated to us by C. Walter (Prop. 3.7). As a corollary we get:

Theorem 5. (Cor. 3.10) *Rank two vector bundles are precisely the minimal elements in their biliaison classes.*

Notice that most biliaison classes do not contain vector bundles. Indeed, the S-modules which are the first cohomology module of some rank two vector bundle on $\mathbf{P}^3$ satisfy quite strong conditions (see [R3] and [D]).

Another invariant that we consider is the minimal integer t such that $\mathcal{F}(t)$ has non-zero sections, we denote it $t_0(\mathcal{F})$ (see Def. 3.14). We prove (Cor. 3.16 and 3.17) that minimal elements have maximal t_0 which depends only on $\mathcal{N}_0$.

In Chapter 4 we analyze the special case of classes associated to a split module. That is, classes of reflexive sheaves $\mathcal{F}$ such that $H^1_*\mathcal{F}$ decomposes as the direct sum of two non-zero graded submodules with disjoint supports (Def. 4.2). In particular, we show that these classes do not contain vector bundles (Thm. 4.7). In other words we have:

Theorem 6. (Thm. 4.7) *The intermediate cohomology modules of a rank two vector bundle on* $\mathbf{P}^3$ *are non-split.*

As a corollary, via the Serre correspondence, we get the analogous result for the Rao module of a subcanonical curve. By curve we mean a locally Cohen-Macaulay equidimensional subscheme X of dimension one. We recall that X is subcanonical if its dualizing sheaf ω_X is isomorphic to a twist of the structural sheaf $\mathcal{O}_X$. We have:

Theorem 7. (Cor. 4.9) *The Rao module of a subcanonical curve in* $\mathbf{P}^3$ *is non-split.*

This provides a complete answer to a question raised a few years ago, namely, whether the Rao module of a subcanonical curve can have gaps in the grading. As far as we know, up to now only partial results were proved ([Da1], [Da2], [B2]), for example, for curves with low speciality index or for vector bundles with Chern classes in a certain range.

We point out that these non-splitness results yield a vanishing criterion for the cohomology groups of a vector bundle and of the ideal sheaf of a subcanonical curve:

Proposition 8. (Cor. 4.8) *Let* $\mathcal{E}$ *be an indecomposable rank two vector bundle on* $\mathbf{P}^3$. *Suppose* $H^i\mathcal{E}(t) = 0$ *for some integer* $t > min\{n|H^i\mathcal{E}(n) \neq 0\}$ *and* $i = 1$ *or* $i = 2$. *Then* $H^i\mathcal{E}(n) = 0$ *for any* $n > t$.

And, similarly:

Proposition 9. (Cor. 4.10) *Let* C *be a non-arithmetically Cohen-Macaulay subcanonical curve in* $\mathbf{P}^3$. *If* $H^1\mathcal{I}_C(t) = 0$ *for some integer* $t > min\{n|H^1\mathcal{I}_C(n) \neq 0\}$, *then* $H^1\mathcal{I}_C(n) = 0$ *for any* $n > t$.

This improves in some sense the well known Castelnuovo-Mumford criterion ([Mu, Lecture 14]) for these particular cases.

Finally, in Chapter 5 we consider the following question:

Let $\mathcal{F}$ be a minimal element in a biliaison class of rank two reflexive sheaves and let C be a curve obtained as zero-locus of a section $s \in H^0\mathcal{F}(t_0)$, where t_0 is the first twist of $\mathcal{F}$ which has non-zero sections. When is C minimal in its biliaison class?

A criterion is given in Prop. 5.1. We then construct a simple example where the curve C is not minimal (Example 5.3). The situation seems more delicate if we assume $\mathcal{F}$ to be a vector bundle. As far as we know, no analogous example is known with $\mathcal{F}$ locally free.

Most of the contents of this thesis appears in two S.I.S.S.A. preprints: Chapters 1, 2 and most of 3 are contained in [B3], while Chapter 4 is [B4].

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Notation

Throughout this thesis we work over an algebraically closed field K of arbitrary characteristic. S denotes the ring of polynomials $K[x_0, \ldots, x_n]$ and $\mathbf{m}$ the maximal ideal $(x_0, \ldots, x_n)$. $\mathbf{P}^n$ is the n-dimensional projective space $\mathbf{P}^n_K = Proj(S)$ and $\mathcal{O}$ stands for the structural sheaf $\mathcal{O}_{\mathbf{P}^n}$.

For short, we call a sheaf $\mathcal{A}$ a **free sheaf** if it is isomorphic to a finite direct sum of line bundles, that is, $\mathcal{A} \cong \bigoplus_{n \in \mathbf{Z}} \mathcal{O}(-n)^{a(n)}$. In this case the finite support function $a : \mathbf{Z} \to \mathbf{Z}$ is called the **characteristic function** of $\mathcal{A}$. As a general rule, we use the same letter to denote a free sheaf (upper case) and its characteristic function (lower case).

If $\mathcal{F}$ is a sheaf on $\mathbf{P}^n$ we write $\mathcal{F}(l)$ for the twisted sheaf $\mathcal{F} \otimes \mathcal{O}(l)$. The cohomology groups $H^i(\mathbf{P}^n, \mathcal{F})$ are denoted simply $H^i\mathcal{F}$ and we write $h^i\mathcal{F}$ for their dimension $dim_K H^i(\mathbf{P}^n, \mathcal{F})$. We also set $H^i_*\mathcal{F} = \bigoplus_{l \in \mathbf{Z}} H^i\mathcal{F}(l)$.

If X is a closed subscheme of $\mathbf{P}^n$ its ideal sheaf is denoted $\mathcal{I}_X$, its saturated homogeneous ideal is $I_X = H^0_*\mathcal{I}_X$, its structural sheaf $\mathcal{O}_X$ and the dualizing sheaf ω_X.

Note that Thm. 1.10 means Theorem 10 in Chapter 1, unless another specific reference is given.

Chapter 0

Biliaison theory for subschemes of projective space: a review

In this introductory Chapter we give a short account of biliaison theory for subschemes of projective space. We focus mainly on those aspects which have a counterpart in biliaison theory for reflexive sheaves that will be developed in the subsequent chapters.

Liaison and biliaison are equivalence relations on subschemes of the same dimension of $\mathbf{P}^n$. They arise from the geometric notion of linkage or residuation in a complete intersection: two subschemes are *(geometrically) linked* if they have no common components and their union is a complete intersection. This idea dates back to the last century but has been developed in the modern setting of algebraic geometry after the fundamental paper of Peskine and Szpiro in 1974 ([PS]). They give an algebraic definition of link which applies also to the case when the two subschemes have common components (and agrees with the geometric definition above when they dont):

Definition 0.1. Let V_1 and V_2 be subschemes of codimension r in $\mathbf{P}^n$. We say that *V_1 is linked to V_2 by the complete intersection X* if X is a global complete intersection of codimension r which contains V_1 and V_2 and such that

$$\mathcal{I}_{V_1}/\mathcal{I}_X \cong \mathcal{H}om(\mathcal{O}_{V_2}, \mathcal{O}_X)$$
$$\mathcal{I}_{V_2}/\mathcal{I}_X \cong \mathcal{H}om(\mathcal{O}_{V_1}, \mathcal{O}_X).$$

We then write $V_1 \sim V_2$.

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The equivalence relation generated by linkage is called *liaison*: two subschemes X and Y are in the same liaison class if there exists a finite sequence of links from X to Y:

$$X \sim V_1 \sim V_2 \sim \cdots \sim Y.$$

If one requires that the number of direct links be even, then the equivalence relation is called *even liaison* or *biliaison*.

The interest of biliaison rather than liaison appears when one looks at the behaviour of the most important invariant of these classes, that is, the collection of deficiency modules $\{H^1_*\mathcal{I}_X, \dots, H^d_*\mathcal{I}_X\}$, where d is the dimension of X. The following theorem says that this sequence is preserved up to duals, shift and re-indexing under liaison, for locally Cohen-Macaulay schemes:

Theorem 0.2. ([R1], [Ch], [Sch], [M1], [H1]) *Let X and Y be locally Cohen-Macaulay subschemes of $\mathbf{P}^n$ of codimension r which are linked by a complete intersection Z. Let $I_Z = (F_1, \dots, F_r)$ and $q = \sum_{i=1}^r degF_i$. Then*

$$H^i_*\mathcal{I}_X \cong (H^{n-r-i+1}_*\mathcal{I}_Y)^\vee(n+1-q) \qquad \text{for each } 1 \le i \le n-r,$$

where $^\vee$ denotes the dual as K-vector space.

In particular, in even liaison duals disappear and the deficiency modules are just shifted all together:

Corollary 0.3. *Let X and Y be locally Cohen-Macaulay subschemes of codimension r in $\mathbf{P}^n$. Assume that X and Y are in the same biliaison class, then there exists an integer p such that $H^i_*\mathcal{I}_X \cong H^i_*\mathcal{I}_Y(p)$ for each $1 \le i \le n-r$.*

This shows that even liaison has a better behaviour than simple liaison. Moreover, it is interesting to notice that the result of Corollary 0.3 still holds if we drop the locally Cohen-Macaulay assumption (this has recently been proved by Hartshorne, see [H1]), while Theorem 0.2 does not (indeed, for a non locally Cohen-Macaulay subscheme the deficiency modules fail to be of finite length but are zero in large degree). This gives further evidence to the fact even liaison classes are the "right" object to study.

Unfortunately, the condition of Corollary 0.3 is not sufficient in general for two subschemes to be in the same biliaison class. For example, there are arithmetically

Cohen-Macaulay curves in $\mathbf{P}^4$ which are not in the biliaison class of a complete intersection. However, the condition is sufficient for curves in $\mathbf{P}^3$. This is a theorem proved by Rao in 1979 ([R1]):

Theorem 0.4. *Let C and C' be locally Cohen-Macaulay equidimensional curves in $\mathbf{P}^3$. Then C is in the even liaison class of C' if and only if $H^1_*\mathcal{I}_C \cong H^1_*\mathcal{I}_{C'}(h)$ for some integer $h \in \mathbf{Z}$. Moreover, for any finite length graded S-module M there exists a nonsingular curve C in $\mathbf{P}^3$ such that $H^1_*\mathcal{I}_C \cong M$ up to shift in grading.*

It follows that even liaison classes of (locally Cohen-Macaulay, equidimensional) curves in $\mathbf{P}^3$ are in bijective correspondence with the isomorphism classes of graded S-modules of finite length, identified up to shift.

This is indeed a particular case — thanks to Horrocks' classification of stable equivalence classes of vector bundles — of a later result of Rao which parametrizes even liaison classes of codimension two (locally Cohen-Macaulay, equidimensional) subschemes in $\mathbf{P}^n$. Let us first recall a definition:

Definition 0.5. Two sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$ on $\mathbf{P}^n$ are called *stably equivalent* if there exist integers a_i, b_j, c such that $\mathcal{E}_1 \oplus \bigoplus_{i=1}^{s} \mathcal{O}(a_i) \cong \mathcal{E}_2(c) \oplus \bigoplus_{j=1}^{t} \mathcal{O}(b_j)$.

Theorem 0.6. ([R2]) *In $\mathbf{P}^n$, $n \geq 2$, the even liaison classes of codimension two, equidimensional and locally Cohen-Macaulay subschemes are in bijective correspondence with the stable equivalence classes of vector bundles $\mathcal{E}$ on $\mathbf{P}^n$ with $H^1_*\mathcal{E} = 0$.*

This result has recently been generalized by Nollet ([N]) to subschemes of pure codimension two (not necessarily locally Cohen-Macaulay) using reflexive sheaves instead than vector bundles.

Thus, at least in codimension two, there is a satisfactory answer to the question of parametrizing even liaison classes.

Another interesting problem is the description of a single class. Here again there is good answer only in codimension two. The structure of an even liaison class of codimension two subschemes of $\mathbf{P}^n$ is described by the so called Lazarsfeld-Rao property. This says roughly that there is a minimal element, unique up to deformation, from which the whole class can be built up applying repeatidly a simple process. The ingredients are essentially two: the existence of a minimal element and a "basic operation" within a class. The latter consists of two consecutive simple links by complete intersections Z_1 and Z_2 defined by homogeneous ideals $I_{Z_1} = (F_1, F_2)$ and $I_{Z_2} = (F_1, AF_2)$, with A a general form of positive degree. If we start with a subscheme X, the result is a subscheme Y which is set-theoretically the union of X and of the complete intersection defined by the ideal (F_1, A). This process is called *basic double link.*

Using this notion one can define a partial ordering on a biliaison class setting $X \leq X'$ if X' can be obtained from X by a finite sequence of basic double links, followed by a deformation which preserves cohomology and biliaison class. Nollet calls it *domination* (see [N], for curves see also [MD-P2,V,1]). The "minimal" element in the sense of the Lazarsfeld-Rao property (or LR-property for short) is precisely a minimal element with respect to this partial ordering. Then one can give a simple formulation of the LR-property (cf. [N]):

Definition 0.7. Let $\mathcal{L}$ be a biliaison class of subschemes of pure codimension two. We say that $\mathcal{L}$ has the *LR-property* if $\mathcal{L}$ has a minimal element with respect to domination.

The name of this property is due to the fact that Lazarsfeld and Rao first showed that certain biliaison classes of curves have this structure (cf. [LR]). Then it was conjecture ([BM1]) and proved ([BBM]) that the same structure holds for every biliaison class of codimension two, locally Cohen-Macaulay and equidimensional subschemes of $\mathbf{P}^n$. In this case the rôle of minimal elements is played by those subschemes which realize the leftmost possible shift of the deficiency modules (see [BM1]). When one deals with non-locally Cohen-Macaulay subschemes the deficiency modules are not of finite length, hence this notion doesn't make sense. In order to prove the LR-property in the more general context of purely two codimensional subschemes, Nollet extends the characterization of minimal curves in $\mathbf{P}^3$ given by Martin-Deschamps and Perrin in [MD-P1]. This is done in terms of the $\mathcal{N}$-type resolutions:

Definition 0.8. (cf. [N]) Let $V \subset \mathbf{P}^n$ be a subscheme of pure codimension two, then an *$\mathcal{N}$-type resolution* for $\mathcal{I}_V$ is an exact sequence

$$0 \to \mathcal{P} \to \mathcal{N} \to \mathcal{I}_V \to 0$$

where $\mathcal{P}$ is a free sheaf and $\mathcal{N}$ is reflexive with $H^1_*\mathcal{N}^\vee = 0$ and $\mathcal{E}xt^1(\mathcal{N}, \mathcal{O}) = 0$

Now, for subschemes of pure codimension two the stable equivalence class of the middle sheaf in an $\mathcal{N}$-type resolution determines uniquely a biliaison class ([N, Cor 2.12]). Moreover, it turns out that the minimal elements are precisely those subschemes for which the map $\mathcal{P} = \bigoplus_{t\in\mathbf{Z}} \mathcal{O}(-t)^{p(t)} \to \mathcal{N}$ in the $\mathcal{N}$-type resolution is given by sections of least possible degrees. This translates in the requirement that the function $p^\#$ — defined by $p^\#(k) = \sum_{t\leq k} p(t)$ — be as large as possible (for precise statements we refer to [MD-P1] and [N]). Then in order to find minimal elements one has to determine, for a given reflexive sheaf $\mathcal{N}$, a finite support function $q : \mathbf{Z} \to \mathbf{Z}$ such that there is an injection $\bigoplus_{t\in\mathbf{Z}} \mathcal{O}(-t)^{q(t)} \hookrightarrow \mathcal{N}$ whose cokernel is the (twisted) ideal sheaf of a subscheme of pure codimension two and such that if $\bigoplus_{t\in\mathbf{Z}} \mathcal{O}(-t)^{r(t)} \hookrightarrow \mathcal{N}$ is any other such injection, then $r^\#(t) \leq q^\#(t)$ for any t. This work has been done by Martin-Deschamps and Perrin ([MD-P1]) for the case of locally Cohen-Macaulay curves in $\mathbf{P}^3$, i.e. when $\mathcal{N}$ is a vector bundle on $\mathbf{P}^3$, but their method applies also to the case when $\mathcal{N}$ is a reflexive sheaf with $H^1_*\mathcal{N}^\vee = 0$ and $\mathcal{E}xt^1(\mathcal{N}, \mathcal{O}) = 0$. The adaption has been carried out by Nollet in the above cited paper. The function q associated with $\mathcal{N}$ is explicitly defined (and can be computed) in terms of certain invariants of the rank stratification of $\mathcal{N}$. The conclusion is the following:

Theorem 0.9. ([BBM], [MD-P1], [N]) *Any biliaison class of non-arithmetically Cohen-Macaulay subschemes of $\mathbf{P}^n$ of pure codimension two has the Lazarsfeld-Rao property.*

It has to be pointed out that [MD-P1] actually provides an explicit algorithm to determine minimal elements in a biliaison class of curves in $\mathbf{P}^3$. This algorithm has recently been implemented using the programmes Macaulay and Maple and concrete examples have been computed ([GLM]).

As we shall see starting from the next chapter, completely analogous results hold for biliaison classes of rank two reflexive sheaves on $\mathbf{P}^3$. Indeed, we define such classes taking into account the characterization of biliaison classes of locally Cohen-Macaulay curves in $\mathbf{P}^3$ given by Thm. 0.4.

The connection between curves and reflexive sheaves is provided by the so called Hartshorne-Serre correspondence which reads as follows:

Theorem 0.10. ([H, Thm. 4.1]) *Let c_1 be a fixed integer. Then there is a bijective correspondence between:*

(1) *pairs $(\mathcal{F}, s)$, where $\mathcal{F}$ is a rank two reflexive sheaf on $\mathbf{P}^3$ with first Chern class c_1 and $s \in H^0(\mathcal{F})$ is a global section which vanishes in codimension two;*

(2) *pairs (Y, ξ), where Y is a locally Cohen-Macaulay curve in $\mathbf{P}^3$ generically complete intersection, and $\xi \in H^0(\omega_Y(4 - c_1))$ is a global section which generates the sheaf $\omega_Y(4 - c_1)$ except at finitely many points.*

Under this correspondence there is an exact sequence

$$(*) \qquad 0 \to \mathcal{O} \to \mathcal{F} \to \mathcal{I}_Y(c_1) \to 0$$

moreover, $c_2 = d$ and $c_3 = 2p_a(Y) - 2 + d(4 - c_1)$, where c_2 and c_3 are the Chern classes of $\mathcal{F}$, d is the degree of Y and $p_a(Y)$ its arithmetic genus.

In particular, this gives a correspondence between rank two vector bundles and subcanonical curves (Serre correspondence).

Notice that sequence (*) implies that $H^1_*\mathcal{I}_Y \cong H^1_*\mathcal{F}(-c_1)$, hence, in view of Theorem 0.4, curves corresponding to the same sheaf $\mathcal{F}$ or to any twist $\mathcal{F}(l)$ belong to the same biliaison class.

Chapter 1

Biliaison classes of rank two reflexive sheaves on $\mathbf{P}^3$

In this Chapter we show that any reflexive sheaf on $\mathbf{P}^3$ admits two special types of locally free resolutions related with a minimal graded free resolution of its first cohomology module. Then we define biliaison classes for rank two reflexive sheaves and parametrize them.

In the following we will work on $\mathbf{P}^3$ and S will denote the ring of polynomials in three indeterminates.

We recall that a reflexive sheaf is a coherent sheaf which is isomorphic to its double dual. In particular, a reflexive sheaf is torsion free and any locally free sheaf is reflexive (we shall use interchangeably the terms vector bundle and locally free sheaf). The rank of a coherent sheaf is the rank at the generic point. For generalities about reflexive sheaves we refer to [H].

Suppose $\mathcal{F}$ is a reflexive sheaf on $\mathbf{P}^3$. Its first cohomology module $H^1_*\mathcal{F}$ is a graded, finite length S-module which has a minimal graded free resolution of the form

$$0 \to L_4 \xrightarrow{\sigma_4} L_3 \xrightarrow{\sigma_3} L_2 \xrightarrow{\sigma_2} L_1 \xrightarrow{\sigma_1} L_0 \to H^1_*\mathcal{F} \to 0.$$

Let $N_0 = ker(\sigma_1)$ and $E_0 = ker(\sigma_2)$, then the associated sheaves $\mathcal{N}_0$ and $\mathcal{E}_0$ are locally free with $H^0_*\mathcal{N}_0 = N_0$, $H^0_*\mathcal{E}_0 = E_0$, $H^1_*\mathcal{N}_0 \cong H^1_*\mathcal{F}$, $H^1_*\mathcal{E}_0 = 0$, $H^2_*\mathcal{N}_0 = 0$ and $H^2_*\mathcal{E}_0 \cong H^1_*\mathcal{F}$ (these properties can be easily proved splitting the resolution above into short exact sequences, sheafifying and taking cohomology, see for example

[MD-P1, II]). It is easy to see that if $H^1_*\mathcal{F} \neq 0$, then $\mathcal{N}_0$ and $\mathcal{E}_0$ have rank at least 3.

Proposition 1.1. *Let $\mathcal{F}$ be a reflexive sheaf on $\mathbf{P}^3$, then $\mathcal{F}$ fits in an exact sequence*

$$(*) \qquad 0 \to \mathcal{A} \to \mathcal{N}_0 \oplus \mathcal{D} \to \mathcal{F} \to 0$$

satisfying the following two conditions:

1. *$\mathcal{A}$ and $\mathcal{D}$ are free sheaves;*
2. *the homomorphism induced on global sections $a : H^0_*\mathcal{A} \to H^0_*(\mathcal{N}_0 \oplus \mathcal{D})$ satisfies $\overline{a^*} = 0$ (where a^* is the dual homomorphism and $\overline{a^*}$ is its reduction modulo $\mathbf{m}$)*

Such a sequence is unique up to isomorphism.

Proof. Set $F = H^0_*\mathcal{F}$ and consider the isomorphism $\mathrm{Ext}^1_S(F, S(-4)) \cong (H^2_*\mathcal{F})'$ — here $'$ stands for dual as K-vector space. Let $G \xrightarrow{g} (H^2_*\mathcal{F})' \to 0$ be a surjection defined by a minimal set of generators of $(H^2_*\mathcal{F})'$, G being a graded free S-module. Then g is an element of $\mathrm{Hom}(G, (H^2_*\mathcal{F})') \cong (H^2_*\mathcal{F})' \otimes G^* \cong \mathrm{Ext}^1_S(F, G^*(-4))$, that is, an extension

$$(\dagger) \qquad 0 \to G^*(-4) \to B \to F \to 0.$$

Sheafifying $(\dagger)$ we get:

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{F} \to 0$$

where $\mathcal{A}$ is the free sheaf associated to $G^*(-4)$ and $\mathcal{B}$ is a torsion free sheaf with $H^1_*\mathcal{B} \cong H^1_*\mathcal{F}$. Applying functor $\mathrm{Hom}(-, S(-4))$ to $(\dagger)$ gives:

$(\ddagger)$

$$0 \to F^*(-4) \to B^*(-4) \xrightarrow{a^*(-4)} G \xrightarrow{g} \mathrm{Ext}^1_S(F, S(-4)) \to \mathrm{Ext}^1_S(B, S(-4)) \to 0,$$

and surjectivity of g implies $\mathrm{Ext}^1_S(B, S(-4)) = 0$, hence $H^2_*\mathcal{B} = 0$. It follows that $\mathcal{B}$ is a vector bundle — since its intermediate cohomology modules are of finite length — and, moreover, by Horrocks' classification of stable equivalence classes of vector bundles ([Ho]), it is of the form $\mathcal{N}_0 \oplus \mathcal{D}$, where $\mathcal{D}$ is a free sheaf. This proves the existence of an exact sequence $(*)$ satisfying (1). Tensoring sequence $(\ddagger)$ with K shows that the minimality of the system of generators given by the image of g —

which means that $\overline{g}$ is an isomorphism — implies $\overline{a^*(-4)} = 0$, hence $\overline{a^*} = 0$. To prove unicity, suppose now that we have two exact sequences of S-modules

$$0 \to A \xrightarrow{a} N_0 \oplus D \xrightarrow{d} F \to 0$$
$$0 \to C \xrightarrow{c} N_0 \oplus E \xrightarrow{e} F \to 0$$

where A,D,C and E are graded free S-modules, $N_0 = H^0_*\mathcal{N}_0$ and $\overline{a^*} = \overline{c^*} = 0$. Since $H^2_*\mathcal{N}_0 = 0$, we have $\mathrm{Ext}^1_S(N_0 \oplus D, C) = 0$ and the homomorphism d lifts to homomorphisms $\phi : N_0 \oplus D \to N_0 \oplus E$ and $\psi : A \to C$ which make a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{a} & N_0 \oplus D & \xrightarrow{d} & F & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\phi} & & \| & & \\ 0 & \longrightarrow & C & \xrightarrow{c} & N_0 \oplus E & \xrightarrow{e} & F & \longrightarrow & 0 \end{array}$$

Dualizing we obtain:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & F^* & \xrightarrow{d^*} & N_0^* \oplus D^* & \xrightarrow{a^*} & A^* & \longrightarrow & \mathrm{Ext}^1_S(F,S) & \longrightarrow & 0 \\ & & \| & & \uparrow{\scriptstyle\phi^*} & & \uparrow{\scriptstyle\psi^*} & & \| & & \\ 0 & \longrightarrow & F^* & \xrightarrow{e^*} & N_0^* \oplus E^* & \xrightarrow{c^*} & C^* & \longrightarrow & \mathrm{Ext}^1_S(F,S) & \longrightarrow & 0 \end{array}$$

The hypothesis $\overline{a^*} = \overline{c^*} = 0$ implies that $\overline{\psi^*}$ is an isomorphism. Thus, by exactness of $- \otimes_S K$ we have $\overline{Coker(\psi^*)} = Coker(\overline{\psi^*}) = 0$ and the graded version of Nakayama's lemma (see for example [MD-P1, p.19]) implies $Coker(\psi^*) = 0$. On the other hand, since C^* is a free S-module, $\overline{Ker(\psi^*)} = Ker(\overline{\psi^*}) = 0$ and, again by Nakayama, $Ker(\psi^*) = 0$. We conclude that ψ^* is an isomorphism, hence ψ and ϕ are also isomorphisms.

Definition 1.2. We say that an exact sequence $0 \to \mathcal{A} \to \mathcal{N}_0 \oplus \mathcal{D} \to \mathcal{F} \to 0$ which satisfies (1) and (2) of Prop. 1.1 is the *$\mathcal{N}_0$-resolution* of $\mathcal{F}$.

Remark 1.3. Condition (2) in the Proposition is a minimality condition for sequence $(*)$ — it corresponds to the minimality of the set of generators chosen for $(H^2_*\mathcal{F})'$ — and implies unicity of the $\mathcal{N}_0$-resolution. It will often be enough for our purposes to consider exact sequences satisfying just (1). Of course, there are infinitely many such sequences, they are all obtained trivially from the $\mathcal{N}_0$-resolution by adding a free direct summand, i.e., assuming now that $(*)$ is the $\mathcal{N}_0$-resolution of $\mathcal{F}$, they look like:

$$0 \to \mathcal{A} \oplus \mathcal{L} \to \mathcal{N}_0 \oplus \mathcal{D} \oplus \mathcal{L} \to \mathcal{F} \to 0$$

with $\mathcal{L}$ a free sheaf. Indeed, suppose $0 \to \mathcal{C} \to \mathcal{N}_0 \oplus \mathcal{E} \to \mathcal{F} \to 0$ is an exact sequence with $\mathcal{C}$ and $\mathcal{E}$ free sheaves, taking global sections and applying functor $\mathrm{Hom}(-, S(-4))$ we get

$$0 \to F^*(-4) \to (N_0^* \oplus E^*)(-4) \xrightarrow{c^*(-4)} C^*(-4) \xrightarrow{\delta} \mathrm{Ext}^1_S(F, S(-4)) \to 0$$

where $C = H^0_* \mathcal{C}$ and $E = H^0_* \mathcal{E}$. If $\overline{c^*(-4)} \neq 0$, the image of δ is a non-minimal system of generators of $\mathrm{Ext}^1_S(F, S(-4))$, then $C^*(-4) \cong G \oplus$ (a free S-module) and $\mathcal{C} \cong \mathcal{A} \oplus \mathcal{L}$, with $\mathcal{L}$ a free sheaf. As in the proof of unicity of the $\mathcal{N}_0$-resolution, since $\mathrm{Ext}^1_S(N_0 \oplus D, A \oplus L) = 0$, we can write an exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "a"] \arrow[d, "i"] & N_0 \oplus D \arrow[r, "d"] \arrow[d, "j"] & F \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \oplus L \arrow[r, "c"] & N_0 \oplus E \arrow[r, "e"] & F \arrow[r] & 0
\end{tikzcd}$$

Taking duals and tensoring with K we see that $\overline{i^*}$ is surjective because $\overline{a^*} = 0$. It follows that i is injective and split and we obtain the following exact commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & A \arrow[r, "a"] \arrow[d, "i"'] & N_0 \oplus D \arrow[r] \arrow[d, "j"'] & F \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \oplus L \arrow[r, "c"] \arrow[d, "p"'] & N_0 \oplus E \arrow[r] \arrow[d] & F \arrow[r] & 0 \\
 & L \arrow[r, "\sim"] \arrow[d] & L \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Since the vertical exact sequence on the left splits, the vertical one in the middle splits too and we have $N_0 \oplus E \cong N_0 \oplus D \oplus L$.

Proposition 1.4. *Let $\mathcal{F}$ be a reflexive sheaf on $\mathbf{P}^3$, then there is an exact sequence*

$$(**)\qquad 0 \to \mathcal{E}_0 \oplus \mathcal{L} \to \mathcal{M} \to \mathcal{F} \to 0$$

such that:

1. *$\mathcal{L}$ and $\mathcal{M}$ are free sheaves;*
2. *the homomorphism induced on global sections $p : H^0_*(\mathcal{E}_0 \oplus \mathcal{L}) \to H^0_*\mathcal{M}$ satisfies $\overline{p} = 0$.*

Such a sequence is unique up to isomorphism.

Proof. Let $F = H^0_*\mathcal{F}$ and let $P \to M \xrightarrow{m} F \to 0$ be a minimal free presentation. Sheafifying we get an exact sequence $0 \to \mathcal{K} \to \mathcal{M} \to \mathcal{F} \to 0$, where $\mathcal{K}$ is the sheafified kernel of m, which is torsion free because $\mathcal{M}$ is a free sheaf. Taking cohomology, we get

$$0 \to H^0_*\mathcal{K} \to M \xrightarrow{m} F \to H^1_*\mathcal{K} \to 0 \to H^1_*\mathcal{F} \to H^2_*\mathcal{K} \to 0 \ldots$$

so we see that $H^1_*\mathcal{K} = 0$ and $H^2_*\mathcal{K} \cong H^1_*\mathcal{F}$. It follows that $\mathcal{K}$ is a vector bundle and it is isomorphic to $\mathcal{E}_0 \oplus \mathcal{L}$, for some free sheaf $\mathcal{L}$. Moreover, minimality of the presentation of F means that $\overline{m}$ is an isomorphism, hence the map $\overline{p} : H^0_*\mathcal{K} \otimes_S K \to M \otimes_S K$ is the zero homomorphism. Suppose now we have two exact sequences of S-modules

$$0 \to E_0 \oplus L \xrightarrow{p} M \xrightarrow{m} F \to 0$$
$$0 \to E_0 \oplus L' \xrightarrow{p'} M' \xrightarrow{m'} F \to 0$$

with L, L', M, M' graded free S-modules and $\overline{p} = \overline{p'} = 0$. Since M is free, m lifts to homomorphisms $\phi : M \to M'$ and $\psi : E_0 \oplus L \to E_0 \oplus L'$ which make a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E_0 \oplus L & \xrightarrow{p} & M & \xrightarrow{m} & F & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\phi} & & \| & & \\
0 & \longrightarrow & E_0 \oplus L' & \xrightarrow{p'} & M' & \xrightarrow{m'} & F & \longrightarrow & 0
\end{array}$$

Tensoring with K we see that $\overline{\phi}$ is an isomorphism, then, by Nakayama's lemma, ϕ is also an isomorphism and so is ψ.

Definition 1.5. We say that an exact sequence $0 \to \mathcal{E}_0 \oplus \mathcal{L} \to \mathcal{M} \to \mathcal{F} \to 0$ which satisfies (1) and (2) of Prop. 1.4 is the *$\mathcal{E}_0$-resolution* of $\mathcal{F}$.

Remark 1.6. As in the case of the $\mathcal{N}_0$-resolution, condition (2) in Prop. 1.4 guarantees the unicity of the $\mathcal{E}_0$-resolution, while there are infinitely many exact sequences satisfying only (1). Again, they all arise from the $\mathcal{E}_0$-resolution adding trivially a free direct summand, that is, assuming (**) is the $\mathcal{E}_0$-resolution of $\mathcal{F}$, they have the form

$$0 \to \mathcal{E}_0 \oplus \mathcal{L} \oplus G \to \mathcal{M} \oplus \mathcal{G} \to \mathcal{F} \to 0$$

for some free sheaf $\mathcal{G}$. The proof is straightforward since $M \xrightarrow{m} F \to 0$ is the beginning of a minimal free presentation of F.

As in the case of the ideal sheaf of a curve, the $\mathcal{N}_0$-resolution and the $\mathcal{E}_0$-resolution of a reflexive sheaf are linked as described in the following:

Proposition 1.7. *Let $\mathcal{F}$ be a reflexive sheaf on $\mathbf{P}^3$ and*

$$0 \to \mathcal{A} \to \mathcal{N}_0 \oplus \mathcal{D} \to \mathcal{F} \to 0$$

$$0 \to \mathcal{E}_0 \oplus \mathcal{L} \to \mathcal{M} \to \mathcal{F} \to 0$$

its $\mathcal{N}_0$-resolution and $\mathcal{E}_0$-resolution respectively. Let $\mathcal{L}_2$ be the free sheaf associated with the module L_2 appearing in the minimal free resolution of $H^1_*\mathcal{F}$. There exist free sheaves $\mathcal{L}_2'$ and $\mathcal{L}_2''$ such that

$$\mathcal{L}_2 \cong \mathcal{L}_2' \oplus \mathcal{L}_2'', \qquad \mathcal{A} \cong \mathcal{L}_2' \oplus \mathcal{L}, \qquad \mathcal{M} \cong \mathcal{L}_2'' \oplus \mathcal{D}$$

and an exact commutative diagram with the middle column split:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathcal{A} \arrow[r, equal] \arrow[d] & \mathcal{A} \arrow[d] & \\
0 \arrow[r] & \mathcal{E}_0 \oplus \mathcal{L} \arrow[r] \arrow[d, equal] & \mathcal{L}_2 \oplus \mathcal{L} \oplus \mathcal{D} \arrow[r] \arrow[d] & \mathcal{N}_0 \oplus \mathcal{D} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{E}_0 \oplus \mathcal{L} \arrow[r] & \mathcal{M} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Proof. The proof goes as in [MD-P1, II, Prop. 6.1]. Applying the global sections functor to the $\mathcal{N}_0$-resolution and to the $\mathcal{E}_0$-resolution of $\mathcal{F}$ we get two exact sequences of S-modules:

$$0 \to A \xrightarrow{a} N_0 \oplus D \xrightarrow{d} F \to 0$$
$$0 \to E_0 \oplus L \xrightarrow{p} M \xrightarrow{m} F \to 0.$$

Consider the surjection $L_2 \oplus D \xrightarrow{\sigma_2' \oplus id} N_0 \oplus D \to 0$, where σ_2' is induced by σ_2 in the minimal free resolution of $H^1_*\mathcal{F}$ (see beginning of Chapter). The map $d \circ (\sigma_2' \oplus id)$ lifts to a morphism $\mu : L_2 \oplus D \to M$ which makes commutative square:

$$\begin{tikzcd}
L_2 \oplus D \arrow[r, "\sigma_2' \oplus id"] \arrow[d, "\mu"] & N_0 \oplus D \arrow[r] \arrow[d, "d"] & 0 \\
M \arrow[r, "m"] & F \arrow[r] \arrow[d] & 0 \\
 & 0 &
\end{tikzcd}$$

Tensoring with K one sees that $\overline{\mu}$ is surjective, hence μ is surjective and has a free kernel L_2'. Let $\gamma : L_2' \to A$ be the morphism induced by $\sigma_2' \oplus id$. Again tensoring with K we see that $\overline{\gamma}$ is injective, hence γ is injective and split. We then have an exact commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & L_2' \arrow[r, "\gamma"] \arrow[d, "\nu"] & A \arrow[d, "a"] & \\
0 \arrow[r] & E_0 \arrow[r] \arrow[d, "\lambda"] & L_2 \oplus D \arrow[r, "\sigma_2' \oplus id"] \arrow[d, "\mu"] & N_0 \oplus D \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & E_0 \oplus L \arrow[r, "p"] & M \arrow[r, "m"] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

where λ is the morphism induced by μ. Then λ is injective and its cokernel is isomorphic to A/L_2', which is free. It follows that λ splits and $A/L_2' \cong L$, hence

$A \cong L_2' \oplus L$. We claim that L_2' is a direct summand of L_2. To prove this consider the projections π and π' of $N_0 \oplus D$ and, respectively, $L_2 \oplus D$ on D. Since by assumption $\overline{a^*} = 0$ we have $\overline{(\pi \circ a)^*} = 0$ and then also $\overline{\pi \circ a} = 0$. Hence $0 = \overline{\pi \circ a \circ \gamma} = \overline{\pi' \circ \nu}$ which means that $Im(\nu) \subset L_2$. Let $L_2 = L_2' \oplus L_2''$, then $M \cong L_2'' \oplus D$. It is now easy to check that there is an exact commutative diagram as in the statement.

We now define the equivalence relation of biliaison for rank two reflexive sheaves on $\mathbf{P}^3$:

Definition 1.8. Let $\mathcal{F}$, $\mathcal{F}'$ be rank two reflexive sheaves on $\mathbf{P}^3$. We say that $\mathcal{F}$ and $\mathcal{F}'$ are in the same *biliaison class* if $H^1_*\mathcal{F} \cong H^1_*\mathcal{F}'$.

Theorem 1.10 below gives a parametrization of these classes. We first need a definition:

Definition 1.9. Two vector bundles $\mathcal{E}$ and $\mathcal{E}'$ are *strongly stably equivalent* if there exist free sheaves $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{E} \oplus \mathcal{A} \cong \mathcal{E}' \oplus \mathcal{B}$.

Warning. Notice that this relation is actually stronger than stable equivalence defined in Def. 0.5 since here we do not allow a twist of the bundles.

Theorem 1.10. *The following three sets are in bijective correspondence:*

(1) *Biliaison classes of rank two reflexive sheaves on* $\mathbf{P}^3$.
(2) *Isomorphism classes of graded S-modules of finite length.*
(3) *Strong stable equivalence classes of vector bundles on* $\mathbf{P}^3$ *with* $H^2_* = 0$.

Proof. We define maps (1) $\to$ (2) $\to$ (3) $\to$ (1): by definition, a biliaison class of sheaves determines an isomorphism class of graded S-modules of finite length, thus $\mathcal{F} \mapsto H^1_*\mathcal{F}$ gives a map (1) $\to$ (2). Now, let M be a finite length S-module and consider a graded free presentation $L_1 \to L_0 \to M \to 0$. The sheaf associated to $ker(L_1 \to L_0)$ is a vector bundle, say $\mathcal{G}$, with $H^1_*\mathcal{G} \cong M$ and $H^2_*\mathcal{G} = 0$. Clearly $\mathcal{G}$ is determined up to a free summand by the isomorphism class of M (the free summand depends on the presentation chosen, note that we do not consider only minimal presentations), thus we have a map (2) $\to$ (3). We now define a map (3) $\to$ (1): let $\mathcal{M}$ be a vector bundle of rank r with $H^2_*\mathcal{M} = 0$, taking $r-2$

general sections of very large degrees $a_1, \ldots, a_{r-2}$ we obtain as a quotient a rank two reflexive sheaf $\mathcal{E}$:

$$0 \to \bigoplus_{i=1}^{r-2} \mathcal{O}(-a_i) \to \mathcal{M} \to \mathcal{E} \to 0.$$

It is clear that $H^1_* \mathcal{M} \cong H^1_* \mathcal{E}$ and since H^1_* modules are invariant in a strong stable equivalence class, any bundle strongly stably equivalent to $\mathcal{M}$ yields a sheaf in the biliaison class of $\mathcal{E}$. It is easy to check that these maps provide bijective correspondences among the three sets.

Remark 1.11. The correspondence between sets (2) and (3) in Thm. 1.10 is actually just part of Horrocks' classification of stable equivalence classes of vector bundles. Then one could obtain the bijection with (1) using Rao's theorem for curves ([R1, Thm. 2.6], see Thm. 0.4) via the Hartshorne-Serre correspondence (Thm. 0.10). We chose to establish directly the correspondences — without mentioning curves — in order to make explicit the relationship between a biliaison class of reflexive sheaves and the corresponding strong stable equivalence class of vector bundles.

Chapter 2

The structure of a class

By the results of the previous Chapter we know that a biliaison class of rank two reflexive sheaves on $\mathbf{P}^3$ is uniquely determined by the minimal element of the corresponding strong stable equivalence class of vector bundles ($\mathcal{E}$ is a minimal element in its strong stable equivalence class if it has minimal rank in the class, i.e. if any other bundle in the same class is isomorphic to the direct sum of $\mathcal{E}$ and a non-zero free sheaf). If the minimal element is $\mathcal{N}_0$, let us denote $Refl(\mathcal{N}_0)$ the corresponding biliaison class of rank two reflexive sheaves. We identify sheaves in this class with their $\mathcal{N}_0$-resolutions.

A natural question then is: for a fixed $\mathcal{N}_0$, which are the possible $\mathcal{A}$ and $\mathcal{D}$ that yield an $\mathcal{F}$ in $Refl(\mathcal{N}_0)$, i.e. $\mathcal{F}$ reflexive of rank two?

A more general question is: given a vector bundle $\mathcal{M}$ of rank $r \geq 3$ on $\mathbf{P}^3$, which are the possible integers $a_1, \ldots, a_{r-2}$ such that a general morphism $0 \to \bigoplus_{i=1}^{r-2} \mathcal{O}(-a_i) \xrightarrow{u} \mathcal{M}$ has a reflexive cokernel?

It is well known that if the degrees $a_1, \ldots, a_{r-2}$ are large enough (i.e., such that $\mathcal{H}om(\bigoplus_{i=1}^{r-2} \mathcal{O}(-a_i), \mathcal{M})$ is globally generated) the map u will drop rank in codimension at least 3, hence the cokernel will be reflexive (this is the argument used in Thm. 1.10). For low degrees one has to analyze the situation more closely. This has been done by M. Martin-Deschamps and D. Perrin in [MD-P2], in an even more general setting: given a reflexive sheaf $\mathcal{E}$ they provide a necessary and sufficient condition for a free sheaf $\mathcal{A}$ in order that there exists a morphism $\mathcal{A} \to \mathcal{E}$ having a reflexive cokernel. To state their result we need some preliminaries.

Let $\mathcal{E}$ be a reflexive sheaf and consider a surjection defined by a minimal system of generators of its global sections: $\bigoplus_{n\in\mathbf{Z}} S(-n)^{l_{\mathcal{E}}(n)} \to H^0_*\mathcal{E} \to 0$. Let $\bigoplus_{n\in\mathbf{Z}} \mathcal{O}(-n)^{l_{\mathcal{E}}(n)} \xrightarrow{f} \mathcal{E} \to 0$ be the corresponding morphism of sheaves; if k is an integer, let $\mathcal{E}_{\leq k}$ be the image of the restricted morphism $f_{\leq k} : \bigoplus_{n\leq k} \mathcal{O}(-n)^{l_{\mathcal{E}}(n)} \to \mathcal{E}$, i.e. $\mathcal{E}_{\leq k}$ is the subsheaf generated by sections of degree $\leq k$.

Definition 2.1. Let $\mathcal{E}$ be a reflexive sheaf on $\mathbf{P}^3$. Define $T_{\mathcal{E}}$ (respectively, $R_{\mathcal{E}}$) as the maximal integer (if it exists, $+\infty$ otherwise) such that:

(1) the morphism $f_{\leq k}$ is injective, i.e. $\mathcal{E}_{\leq k} \cong \bigoplus_{n\leq k} \mathcal{O}(-n)^{l_{\mathcal{E}}(n)}$;

(2) the quotient sheaf $\mathcal{E}/\mathcal{E}_{\leq k}$ is torsion free (respectively, reflexive).

Remark 2.2. Clearly it is $R_{\mathcal{E}} \leq T_{\mathcal{E}}$ and, for any $k \leq T_{\mathcal{E}}$ (resp. $k \leq R_{\mathcal{E}}$), $\mathcal{E}/\mathcal{E}_{\leq k}$ is torsion free (resp. reflexive). Indeed, for any k such that $f_{\leq k}$ is injective one has the following diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & \mathcal{O}(-k)^{l_{\mathcal{E}}(k)} \arrow[d] & \\
0 \arrow[r] & \mathcal{E}_{\leq k-1} \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d, equal] & \mathcal{E}/\mathcal{E}_{\leq k-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{E}_{\leq k} \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] & \mathcal{E}/\mathcal{E}_{\leq k} \arrow[r] \arrow[d] & 0 \\
 & \mathcal{O}(-k)^{l_{\mathcal{E}}(k)} \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd}$$

which shows that if $\mathcal{E}/\mathcal{E}_{\leq k}$ is torsion free (resp., reflexive), then $\mathcal{E}/\mathcal{E}_{\leq k-1}$ is too.

Definition 2.3. Let $\mathcal{E}$ be a reflexive sheaf on $\mathbf{P}^3$. For any integer $k \in \mathbf{Z}$ set:

(1) $\alpha_k(\mathcal{E}) = rk(\mathcal{E}_{\leq k}) = rk(f_{\leq k})$ at a general point;

(2) $\beta_k(\mathcal{E}) = \inf_S\{rk(f_{\leq k})_{|S}\}$, where S describes the set of integral surfaces in $\mathbf{P}^3$;

(3) $\gamma_k(\mathcal{E}) = \inf_C\{rk(f_{\leq k})_{|C}\}$, where C describes the set of integral curves in $\mathbf{P}^3$.

(These invariants can be defined more generally for any coherent sheaf, see [MD-P2, II, 4]).

The following properties are straightforward:

Proposition 2.4. *For any $k \in \mathbf{Z}$ we have inequalities: $\alpha_k(\mathcal{E}) \geq \beta_k(\mathcal{E}) \geq \gamma_k(\mathcal{E})$; if ζ_k denotes any of the invariants $\alpha_k(\mathcal{E})$, $\beta_k(\mathcal{E})$, $\gamma_k(\mathcal{E})$, then for any $k \in \mathbf{Z}$ we have $\zeta_k \leq \zeta_{k+1}$.*

If $g : \mathbf{Z} \to \mathbf{Z}$ is a function with finite support, following notation in [MD-P2] we write $g^{\#}(n) = \sum_{t \leq n} g(t)$. Note that from function $g^{\#}$ one recovers function g by the formula: $g(n) = g^{\#}(n) - g^{\#}(n-1)$.

Remark 2.5. The invariants $T_{\mathcal{E}}$ and $R_{\mathcal{E}}$ associated with a reflexive sheaf $\mathcal{E}$ can be characterized in terms of the $\alpha_k(\mathcal{E})$, $\beta_k(\mathcal{E})$, $\gamma_k(\mathcal{E})$ and of the function $l_{\mathcal{E}}$ as follows:

$$T_{\mathcal{E}} = \sup\{k | l^{\#}_{\mathcal{E}}(k) = \alpha_k(\mathcal{E}) = \beta_k(\mathcal{E})\}$$
$$R_{\mathcal{E}} = \sup\{k | l^{\#}_{\mathcal{E}}(k) = \alpha_k(\mathcal{E}) = \beta_k(\mathcal{E}) = \gamma_k(\mathcal{E})\}$$

(equality $l^{\#}_{\mathcal{E}}(k) = \alpha_k(\mathcal{E})$ means that $f_{\leq k}$ is injective, while $\alpha_k(\mathcal{E}) = \beta_k(\mathcal{E})$ (resp., $\alpha_k(\mathcal{E}) = \beta_k(\mathcal{E}) = \gamma_k(\mathcal{E})$) means that $f_{\leq k}$ drops rank in codimension at least 1 (resp., 2), see [MD-P2, II, 4]). Observe also that for $R_{\mathcal{E}} < k \leq T_{\mathcal{E}}$ we have $l^{\#}_{\mathcal{E}}(k) = \alpha_k(\mathcal{E}) = \beta_k(\mathcal{E}) > \gamma_k(\mathcal{E})$.

Definition 2.6. Given a reflexive sheaf $\mathcal{E}$, we define a function $\chi_{\mathcal{E}} : \mathbf{Z} \to \mathbf{Z}$ setting:

$$\chi^{\#}_{\mathcal{E}}(n) = \begin{cases} \gamma_n(\mathcal{E}), & \text{for } n \leq T_{\mathcal{E}} \\ \inf(\alpha_n(\mathcal{E}) - 2, \beta_n(\mathcal{E}) - 1, \gamma_n(\mathcal{E})), & \text{for } n > T_{\mathcal{E}}. \end{cases}$$

Lemma 2.7. *Let $\mathcal{E}$ be a reflexive sheaf but not a free sheaf, then we have:*

(1) $0 \leq \chi_{\mathcal{E}}(n) \leq l_{\mathcal{E}}(n)$ *for any* $n \in \mathbf{Z}$*, in particular, for* $n \leq R_{\mathcal{E}}$ *we have* $\chi_{\mathcal{E}}(n) = l_{\mathcal{E}}(n)$ *and for* $n = R_{\mathcal{E}} + 1$ *we have* $\chi_{\mathcal{E}}(n) < l_{\mathcal{E}}(n)$*;*

(2) $\chi^{\#}_{\mathcal{E}}(n) = rk(\mathcal{E}) - 2$ *for* $n \gg 0$.

Proof. (1) The inequality $\chi_{\mathcal{E}}(n) \geq 0$ is proven in [MD-P2, III, Prop. 3.7]. We prove $\chi_{\mathcal{E}}(n) \leq l_{\mathcal{E}}(n)$. To simplify notation, we shall drop the $\mathcal{E}$ from the $\alpha_n(\mathcal{E})$, $\beta_n(\mathcal{E})$ and

$\gamma_n(\mathcal{E})$ in the rest of the proof. For $n \leq R_\mathcal{E}$ we have $\chi_\mathcal{E}(n) = \chi^\#_\mathcal{E}(n) - \chi^\#_\mathcal{E}(n-1) = \gamma_n - \gamma_{n-1} = l^\#_\mathcal{E}(n) - l^\#_\mathcal{E}(n-1) = l_\mathcal{E}(n)$ (see Rmks. 2.2 and 2.5). For $n = R_\mathcal{E}+1$ we distinguish two cases: either $R_\mathcal{E} = T_\mathcal{E}$, then we have $\chi_\mathcal{E}(n) = \inf(\alpha_n - 2, \beta_n - 1, \gamma_n) - \gamma_{n-1} \leq \alpha_n - 2 - l^\#_\mathcal{E}(n-1) \leq l^\#_\mathcal{E}(n) - 2 - l^\#_\mathcal{E}(n-1) = l_\mathcal{E}(n) - 2$; or $R_\mathcal{E}+1 \leq T_\mathcal{E}$ and hence $\chi_\mathcal{E}(n) = \gamma_n - \gamma_{n-1} = \gamma_n - \alpha_{n-1} < \alpha_n - \alpha_{n-1} = l^\#_\mathcal{E}(n) - l^\#_\mathcal{E}(n-1) = l_\mathcal{E}(n)$. Suppose now $n > R_\mathcal{E}+1$. We first show that for any n at least one of the expressions $\alpha_n - \alpha_{n-1}$, $\beta_n - \beta_{n-1}$, $\gamma_n - \gamma_{n-1}$ is greater than or equal to $\chi_\mathcal{E}(n)$, then we prove that any of them is smaller than or equal to $l_\mathcal{E}(n)$. If $n \leq T_\mathcal{E}$, then clearly $\chi_\mathcal{E}(n) = \gamma_n - \gamma_{n-1}$. If $n = T_\mathcal{E}+1$, then $\chi_\mathcal{E}(n) = \inf(\alpha_n - 2, \beta_n - 1, \gamma_n) - \gamma_{n-1} \leq \gamma_n - \gamma_{n-1}$. For $n > T_\mathcal{E}+1$ we need to consider separately nine cases. For example, suppose $\inf(\alpha_n - 2, \beta_n - 1, \gamma_n) = \alpha_n - 2$. If $\inf(\alpha_{n-1} - 2, \beta_{n-1} - 1, \gamma_{n-1}) = \alpha_{n-1} - 2$, then clearly $\chi_\mathcal{E}(n) = \alpha_n - \alpha_{n-1}$; if $\inf(\alpha_{n-1} - 2, \beta_{n-1} - 1, \gamma_{n-1}) = \beta_{n-1} - 1$, then $\chi_\mathcal{E}(n) = \alpha_n - 2 - \beta_{n-1} + 1 \leq \beta_n - 1 - \beta_{n-1} + 1 = \beta_n - \beta_{n-1}$; if $\inf(\alpha_{n-1} - 2, \beta_{n-1} - 1, \gamma_{n-1}) = \gamma_{n-1}$, then $\chi_\mathcal{E}(n) = \alpha_n - 2 - \gamma_{n-1} \leq \gamma_n - \gamma_{n-1}$. The remaining six cases are treated in a similar way.

Consider now the surjection $\mathcal{O}(-n)^{l_\mathcal{E}(n)} \to \mathcal{E}_{\leq n}/\mathcal{E}_{\leq n-1} \to 0$. It yields $l_\mathcal{E}(n) \geq rk(\mathcal{E}_{\leq n}/\mathcal{E}_{\leq n-1}) = rk(\mathcal{E}_{\leq n}) - rk(\mathcal{E}_{\leq n-1}) = \alpha_n - \alpha_{n-1}$. For the β's, consider instead the exact sequence $\mathcal{O}(-n)^{l_\mathcal{E}(n)} \to \mathcal{E}/\mathcal{E}_{\leq n-1} \to \mathcal{E}/\mathcal{E}_{\leq n} \to 0$, which remains exact after restriction to any integral surface S. Then we have $l_\mathcal{E}(n) \geq rk(\mathcal{E}/\mathcal{E}_{\leq n-1})|_S - rk(\mathcal{E}/\mathcal{E}_{\leq n})|_S$. Regarding β_k as $rk(\mathcal{E}) - max_S\{rk(\mathcal{E}/\mathcal{E}_{\leq k})|_S\}$ we easily get $l_\mathcal{E}(n) \geq \beta_n - \beta_{n-1}$. The same argument, replacing surfaces with curves, shows that $l_\mathcal{E}(n) \geq \gamma_n - \gamma_{n-1}$ for any n.

(2) The sheaf $\mathcal{E}(n)$ is globally generated for $n \gg 0$, i.e. $f_{\leq n}$ is surjective, hence $\alpha_n(\mathcal{E}) = \beta_n(\mathcal{E}) = \gamma_n(\mathcal{E}) = rk(\mathcal{E})$. It follows that $\chi^\#_\mathcal{E}(n) = \alpha_n(\mathcal{E}) - 2 = rk(\mathcal{E}) - 2$.

Remark 2.8. $\mathcal{A}$ is a free sheaf, of characteristic function a, if and only if $R_\mathcal{A} = T_\mathcal{A} = +\infty$ (one has $l_\mathcal{A} = a$); then $\alpha_k(\mathcal{A}) = \beta_k(\mathcal{A}) = \gamma_k(\mathcal{A}) = a^\#(k)$ for any k and $\chi_\mathcal{A} = a$.

The result which we need is the following:

Theorem 2.9. (Martin-Deschamps – Perrin, [MD-P2]) *Let $\mathcal{E}$ be a reflexive sheaf on $\mathbf{P}^3$, but not a free sheaf. Then there exists a morphism $u : \bigoplus_{n\in\mathbf{Z}} \mathcal{O}(-n)^{\phi(n)} \to \mathcal{E}$*

such that $Coker(u)$ is reflexive if and only if:

(1) $\phi^{\#}(n) \leq \chi_{\mathcal{E}}^{\#}(n)$ *for any* $n \in \mathbf{Z}$;

(2) (FO)-condition ("condition du Facteur Obligatoire"): *if* $\phi^{\#}(n_0) = rk(\mathcal{E}_{\leq n_0})$ *for some integer* $n_0 \leq R_{\mathcal{E}}$, *then* $\bigoplus_{n \leq n_0} \mathcal{O}(-n)^{\phi(n)} \cong \mathcal{E}_{\leq n_0}$; *if* $\phi^{\#}(n_0) = rk(\mathcal{E}_{\leq n_0}) - 1$ *for some integer* $n_0 \leq T_{\mathcal{E}}$, *then* $\bigoplus_{n \leq n_0} \mathcal{O}(-n)^{\phi(n)}$ *is isomorphic to a direct summand of* $\mathcal{E}_{\leq n_0}$.

Proof. See [MD-P2, III, Prop.3.5 and IV, Thm. 2.6].

Remark 2.10. If conditions (1) and (2) above are satisfied, i.e. if there exists a morphism $u_0 : \bigoplus_{n \in \mathbf{Z}} \mathcal{O}(-n)^{\phi(n)} \to \mathcal{E}$ with a reflexive cokernel, then there is a non-empty open subset $U \subset \mathrm{Hom}(\bigoplus_{n \in \mathbf{Z}} \mathcal{O}(-n)^{\phi(n)}, \mathcal{E})$ such that any $u \in U$ has the same property, that is, a general morphism $\bigoplus_{n \in \mathbf{Z}} \mathcal{O}(-n)^{\phi(n)} \to \mathcal{E}$ has a reflexive cokernel.

We now turn back to our biliaison class $Refl(\mathcal{N}_0)$: $\chi_{\mathcal{N}_0}^{\#}$ is the maximal function such that the cokernel of a general morphism $\bigoplus_{n \in \mathbf{Z}} \mathcal{O}(-n)^{\chi_{\mathcal{N}_0}(n)} \to \mathcal{N}_0$ is a rank two reflexive sheaf and all the other possible such functions $\phi_{\mathcal{N}_0}$ are those dominated by $\chi_{\mathcal{N}_0}$ — in the sense of the # function — and such that they satisfy the (FO)-condition and $\phi_{\mathcal{N}_0}^{\#}(n) = rk(\mathcal{N}_0) - 2$ for $n \gg 0$. To deal with other bundles in the strong stable equivalence class of minimal element $\mathcal{N}_0$, i.e. of the form $\mathcal{N}_0 \oplus \mathcal{D}$ with $\mathcal{D}$ a free sheaf, we need the following:

Lemma 2.11. *Let $\mathcal{E}$ be a reflexive sheaf and $\mathcal{A} = \bigoplus_{n \in \mathbf{Z}} \mathcal{O}(-n)^{a(n)}$ a free sheaf, then for any integer k we have:*

(1) $\alpha_k(\mathcal{E} \oplus \mathcal{A}) = \alpha_k(\mathcal{E}) + a^{\#}(k)$;

(2) $\beta_k(\mathcal{E} \oplus \mathcal{A}) = \beta_k(\mathcal{E}) + a^{\#}(k)$;

(3) $\gamma_k(\mathcal{E} \oplus \mathcal{A}) = \gamma_k(\mathcal{E}) + a^{\#}(k)$;

(4) $T_{\mathcal{E} \oplus \mathcal{A}} = T_{\mathcal{E}}$, $R_{\mathcal{E} \oplus \mathcal{A}} = R_{\mathcal{E}}$;

(5) $\chi_{\mathcal{E} \oplus \mathcal{A}}(k) = \chi_{\mathcal{E}}(k) + a(k)$.

Proof. We have $l_{\mathcal{E} \oplus \mathcal{A}} = l_{\mathcal{E}} + a$ and a surjection: $\bigoplus_{n \in \mathbf{Z}} \mathcal{O}(-n)^{l_{\mathcal{E}}(n)} \oplus \mathcal{A} \xrightarrow{f \oplus id_{\mathcal{A}}} \mathcal{E} \oplus \mathcal{A} \to 0$. Then $(f \oplus id_{\mathcal{A}})_{\leq k} = f_{\leq k} \oplus id_{\mathcal{A}_{\leq k}}$ for any integer k and the rank of $(f \oplus id_{\mathcal{A}})_{\leq k}$ at a point is equal to the rank of $f_{\leq k}$ at that point plus $a^{\#}(k)$. One then easily gets equalities (1), (2), (3) and hence (4). In turn they imply $\chi_{\mathcal{E} \oplus \mathcal{A}}^{\#}(k) = \chi_{\mathcal{E}}^{\#}(k) + a^{\#}(k)$ and hence $\chi_{\mathcal{E} \oplus \mathcal{A}}(k) = \chi_{\mathcal{E}}(k) + a(k)$.

As a consequence of Theorem 2.9 and of Lemma 2.11 we get:

Proposition 2.12. *Suppose $\mathcal{N}_0$ is a non-zero vector bundle that is not a free sheaf. The pairs of free sheaves $(\mathcal{A},\mathcal{D})$ such that there exists a morphism $\mathcal{A} \to \mathcal{N}_0 \oplus \mathcal{D}$ whose cokernel is in $Refl(\mathcal{N}_0)$ are exactly those satisfying the following conditions:*

(1) $(a-d)^{\#}(n) = rk(\mathcal{N}_0) - 2$, *for* $n \gg 0$;

(2) $(a-d)^{\#}(n) \leq \chi^{\#}_{\mathcal{N}_0}(n)$ *for any* $n \in \mathbf{Z}$;

(3) (FO)-condition: *if we have* $(a-d)^{\#}(n_0) = rk(\mathcal{N}_{0,\leq n_0})$ *for some* $n_0 \leq R_{\mathcal{N}_0}$, *then* $\mathcal{A}_{\leq n_0} \cong (\mathcal{N}_0 \oplus \mathcal{D})_{\leq n_0}$; *if we have* $(a-d)^{\#}(n_0) = rk(\mathcal{N}_{0,\leq n_0}) - 1$ *for some* $n_0 \leq T_{\mathcal{N}_0}$, *then* $\mathcal{A}_{\leq n_0}$ *is isomorphic to a direct summand of* $(\mathcal{N}_0 \oplus \mathcal{D})_{\leq n_0}$.

Definition 2.13. We say that $(\mathcal{A},\mathcal{D})$ is an *admissible pair* for $\mathcal{N}_0$ if it satisfies conditions (1), (2) and (3) of Prop. 2.12. In such a case we also say that $(a-d)^{\#}$ (or, sometimes, $a-d$) is an *admissible function.*

In the following we assume $\mathcal{N}_0 \neq 0$. We now define a partial ordering in the class $Refl(\mathcal{N}_0)$. Let $\mathcal{F}$, $\mathcal{F}'$ be given by exact sequences:

$$(*) \qquad 0 \to \mathcal{A} \xrightarrow{u} \mathcal{N}_0 \oplus \mathcal{D} \to \mathcal{F} \to 0$$

$$(*') \qquad 0 \to \mathcal{A}' \xrightarrow{u'} \mathcal{N}_0 \oplus \mathcal{D}' \to \mathcal{F}' \to 0$$

respectively, with $\mathcal{A}$, $\mathcal{D}$, $\mathcal{A}'$ and $\mathcal{D}'$ free sheaves (notice that we do not require $(*)$, $(*')$ to be the $\mathcal{N}_0$-resolutions of $\mathcal{F}$ and $\mathcal{F}'$, i.e. to satisfy (2) of Prop. 1.1). Unless otherwise specified we shall stick to this notation in the sequel.

Definition 2.14. $\mathcal{F} \preceq \mathcal{F}'$ if and only if $(a-d)^{\#}(n) \geq (a'-d')^{\#}(n)$ for any $n \in \mathbf{Z}$.

Remark 2.15. Notice that the function $a-d$ is uniquely determined by the sheaf $\mathcal{F}$. Indeed, we know that all pairs $(\mathcal{A},\mathcal{D})$ which yield the same rank two reflexive sheaf $\mathcal{F}$ are of the form $(\mathcal{A}_0 \oplus \mathcal{L}, \mathcal{D}_0 \oplus \mathcal{L})$, where $\mathcal{L}$ is a free sheaf and $(\mathcal{A}_0, \mathcal{D}_0)$ is the pair appearing in the $\mathcal{N}_0$-resolution of $\mathcal{F}$ (see Remark 1.3).

Since $\chi^{\#}_{\mathcal{N}_0}$ is a maximal element for the set of admissible functions $(a-d)^{\#}$, we have in a natural way a notion of minimal element in $Refl(\mathcal{N}_0)$:

Definition 2.16. A sheaf $\mathcal{F}$ is *minimal* in $Refl(\mathcal{N}_0)$ if $(a-d)^{\#}(n) = \chi^{\#}_{\mathcal{N}_0}(n)$ for any $n \in \mathbf{Z}$.

Proposition 2.17. *The* *minimal* *elements in* $Refl(\mathcal{N}_0)$ *are precisely the sheaves* $\mathcal{F}$ *whose* $\mathcal{N}_0$*-resolution has the form*

$$(\dagger) \qquad 0 \to \bigoplus_{n\in \mathbf{Z}} \mathcal{O}(-n)^{\chi_{\mathcal{N}_0}(n)} \to \mathcal{N}_0 \to \mathcal{F} \to 0.$$

Proof. It is clear that any sheaf in $Refl(\mathcal{N}_0)$ with $\mathcal{N}_0$-resolution $(\dagger)$ is minimal. On the other hand, suppose $\mathcal{F}$ is minimal in $Refl(\mathcal{N}_0)$, that is, it fits in an exact sequence $0 \to \mathcal{A} \xrightarrow{u} \mathcal{N}_0 \oplus \mathcal{D} \to \mathcal{F} \to 0$ with $\mathcal{A} \cong \bigoplus_{n\in\mathbf{Z}} \mathcal{O}(-n)^{\chi_{\mathcal{N}_0}(n)} \oplus \mathcal{D}$. Let $t = max\{n | d(n) > 0\}$ and consider the morphism $u_{\leq t} : \mathcal{A}_{\leq t} \to \mathcal{N}_{0,\leq t} \oplus \mathcal{D}$. Write $\mathcal{A}_{\leq t} \cong \bigoplus_{n\leq t-1} \mathcal{O}(-n)^{\chi_{\mathcal{N}_0}(n)+d(n)} \oplus \mathcal{O}(-t)^{\chi_{\mathcal{N}_0}(t)+d(t)}$ and $\mathcal{N}_{0,\leq t} \oplus \mathcal{D} \cong \mathcal{N}_{0,\leq t} \oplus \mathcal{D}' \oplus \mathcal{O}(-t)$ where $\mathcal{D}' = \mathcal{D}_{\leq t-1} \oplus \mathcal{O}(-t)^{d(t)-1}$. If the morphism $\mathcal{O}(-t)^{\chi_{\mathcal{N}_0}(t)+d(t)} \to \mathcal{O}(-t)$ induced by u is zero, then $u_{\leq t}$ factors through $\mathcal{N}_{0,\leq t} \oplus \mathcal{D}'$ — since clearly $\mathrm{Hom}(\mathcal{A}_{\leq t-1}, \mathcal{O}(-t)) = 0$ — and gives a reflexive quotient of $\mathcal{N}_{0,\leq t} \oplus \mathcal{D}'$. By maximality of the function $\chi^{\#}_{\mathcal{N}_{0,\leq t} \oplus \mathcal{D}'}$ we then have: $a^{\#}(t) = \chi^{\#}_{\mathcal{N}_0}(t) + d^{\#}(t) \leq \chi^{\#}_{\mathcal{N}_{0,\leq t} \oplus \mathcal{D}'} = \chi^{\#}_{\mathcal{N}_0}(t) + d^{\#}(t) - 1$, which is absurd. Thus u induces a non-zero morphism $\mathcal{O}(-t)^{\chi_{\mathcal{N}_0}(t)+d(t)} \to \mathcal{O}(-t)$, hence an automorphism $\mathcal{O}(-t) \to \mathcal{O}(-t)$ and we have a commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \bigoplus_{n\in\mathbf{Z}} \mathcal{O}(-n)^{\chi_{\mathcal{N}_0}(n)} \oplus \mathcal{D}' \arrow[r, "u'"] \arrow[d] & \mathcal{N}_0 \oplus \mathcal{D}' \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \bigoplus_{n\in\mathbf{Z}} \mathcal{O}(-n)^{\chi_{\mathcal{N}_0}(n)} \oplus \mathcal{D} \arrow[r, "u"] \arrow[d] & \mathcal{N}_0 \oplus \mathcal{D} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] & 0 \\
 & \mathcal{O}(-t) \arrow[r, "\sim"] \arrow[d] & \mathcal{O}(-t) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

The first row is then a new exact sequence for $\mathcal{F}$ with $rk(\mathcal{D}') = rk(\mathcal{D}) - 1$. Proceeding in this way, after $rk(\mathcal{D})$ steps we obtain an exact sequence of the form $0 \to \bigoplus_{n\in\mathbf{Z}} \mathcal{O}(-n)^{\chi_{\mathcal{N}_0}(n)} \to \mathcal{N}_0 \to \mathcal{F} \to 0$ which is necessarily the $\mathcal{N}_0$-resolution of $\mathcal{F}$.

Proposition 2.18. *With the previous notation, if $\mathcal{F}$, $\mathcal{F}'$ are such that $a - d = a' - d'$, then $\mathcal{F}'$ is a deformation of $\mathcal{F}$ with constant cohomology.*

Proof. We can trivially modify sequences $(*)$ and $(*')$ as follows:

$$0 \to \mathcal{A} \oplus \mathcal{D}' \xrightarrow{u \oplus id_{\mathcal{D}'}} \mathcal{N}_0 \oplus \mathcal{D} \oplus \mathcal{D}' \to \mathcal{F} \to 0$$

$$0 \to \mathcal{A}' \oplus \mathcal{D} \xrightarrow{u' \oplus id_{\mathcal{D}}} \mathcal{N}_0 \oplus \mathcal{D}' \oplus \mathcal{D} \to \mathcal{F}' \to 0$$

From the hypothesis we get $a + d' = a' + d$, that is, the characteristic functions of the free sheaves $\mathcal{A} \oplus \mathcal{D}'$ and $\mathcal{A}' \oplus \mathcal{D}$ are the same. Let us write $\mathcal{P} = \mathcal{A} \oplus \mathcal{D}' \cong \mathcal{A}' \oplus \mathcal{D}$, to simplify notation, and also $\mathcal{M} = \mathcal{N}_0 \oplus \mathcal{D} \oplus \mathcal{D}' \cong \mathcal{N}_0 \oplus \mathcal{D}' \oplus \mathcal{D}$, $v = u \oplus id_{\mathcal{D}'}$ and $v' = u' \oplus id_{\mathcal{D}}$. Consider the family of morphisms

$$f_t = tv + (1-t)v' \in \operatorname{Hom}(\mathcal{P}, \mathcal{M}), \quad t \in \mathbf{A}^1_K,$$

there exists an open subset $U \subset \mathbf{A}^1_K$, containing 1 and 0, such that f_t has a reflexive rank two cokernel for any $t \in U$, hence there is a deformation from $\mathcal{F}$ to $\mathcal{F}'$ within $Refl(\mathcal{N}_0)$. Moreover any rank two reflexive sheaf $\mathcal{E}$ given by an exact sequence

$$(\dagger) \qquad 0 \to \mathcal{P} \to \mathcal{M} \to \mathcal{E} \to 0$$

has the same cohomology. Indeed, from the long exact sequence associated with $(\dagger)$ we get: $h^0\mathcal{E}(n) = h^0\mathcal{M}(n) - h^0\mathcal{P}(n)$ for any $n \in \mathbf{Z}$. By Serre duality and by the isomorphism $\mathcal{E}^* \cong \mathcal{E}(-c_1)$ — where c_1 is the first Chern class of $\mathcal{E}$, which depends only on $\mathcal{P}$ and $\mathcal{M}$ — , we also obtain $h^3\mathcal{E}(n) = h^0\mathcal{M}(-c_1-n-4) - h^0\mathcal{P}(-c_1-n-4)$. Clearly $H^1_*\mathcal{E} \cong H^1_*\mathcal{M} \cong H^1_*\mathcal{N}_0$. Finally, $h^2\mathcal{E}(n) = h^3\mathcal{P}(n) - h^3\mathcal{M}(n) + h^3\mathcal{E}(n) = h^3\mathcal{P}(n) - h^3\mathcal{M}(n) + h^0\mathcal{M}(-c_1 - n - 4) - h^0\mathcal{P}(-c_1 - n - 4)$.

Corollary 2.19. *In $Refl(\mathcal{N}_0)$ there is a unique minimal element up to deformations with constant cohomology.*

Definition 2.20. Let $\mathcal{F} \in Refl(\mathcal{N}_0)$ be given by an exact sequence

$$(*) \qquad 0 \to \mathcal{A} \xrightarrow{u} \mathcal{N}_0 \oplus \mathcal{D} \to \mathcal{F} \to 0$$

with $\mathcal{A}$ and $\mathcal{D}$ free sheaves. Suppose b, c are integers such that $(\mathcal{A} \oplus \mathcal{O}(-b), \mathcal{D} \oplus \mathcal{O}(-c))$ is an admissible pair for $\mathcal{N}_0$ and let $\mathcal{E}$ be the cokernel of a general morphism $v : \mathcal{A} \oplus \mathcal{O}(-b) \to \mathcal{N}_0 \oplus \mathcal{D} \oplus \mathcal{O}(-c)$:

$$0 \to \mathcal{A} \oplus \mathcal{O}(-b) \xrightarrow{v} \mathcal{N}_0 \oplus \mathcal{D} \oplus \mathcal{O}(-c) \to \mathcal{E} \to 0$$

We say that $\mathcal{E}$ is obtained from $\mathcal{F}$ by *elementary biliaison* of type (b, c). If $b > c$ we say that the elementary biliaison is ascending, if $b < c$ we say it is descending.

Remarks 2.21. a) For fixed integers b and c, all sheaves obtained by elementary biliaison of type (b,c) from $\mathcal{F}$ belong to the same family of sheaves in $Refl(\mathcal{N}_0)$ with constant cohomology. This is a consequence of Rmk 2.15 and of Prop 2.18.

b) Notice that even if we start from the $\mathcal{N}_0$-resolution of $\mathcal{F}$, the exact sequence $0 \to \mathcal{A} \oplus \mathcal{O}(-b) \xrightarrow{v} \mathcal{N}_0 \oplus \mathcal{D} \oplus \mathcal{O}(-c) \to \mathcal{E} \to 0$ need not be the $\mathcal{N}_0$-resolution of $\mathcal{E}$.

c) A natural question is: for a given $\mathcal{F}$, which pairs of integers (b,c) yield an admissible pair? i.e., which types of elementary biliaison can we perform starting from $\mathcal{F}$?

Set $\mathcal{A}' = \mathcal{A} \oplus \mathcal{O}(-b)$, $\mathcal{D}' = \mathcal{D} \oplus \mathcal{O}(-c)$ and denote as usual a', respectively, d' their characteristic functions.

Case $b = c$: we have $(a'-d')^{\#}(n) = (a-d)^{\#}(n)$ for any n. Then it is immediate that $(a'-d')^{\#}$ satisfies (1) and (2) of the admissibility condition (see Prop 2.12) and it is easy to check that (3) holds too. Thus an elementary biliaison of type (b,b) is always possible and, by Prop 2.18, it is just a deformation with constant cohomology.

Case $b > c$: we have

$$(a'-d')^{\#}(n) = \begin{cases} (a-d)^{\#}(n), & \text{for } n < c \\ (a-d)^{\#}(n) - 1, & \text{for } c \le n < b \\ (a-d)^{\#}(n), & \text{for } n \ge b. \end{cases}$$

Then $(a'-d')^{\#}(n) \le (a-d)^{\#}(n)$ for any n and equality holds for $n \gg 0$. This guarantees that $(a'-d')^{\#}$ satisfies (1) and (2) of Prop 2.12, since $(a-d)^{\#}$ does. Condition (3) — the (FO)-condition — is somewhat more delicate and is not always fulfilled. Indeed, if one carries out the details of the verification, one can easily see that obstructions arise if $(a'-d')^{\#}(n_0) = rk(\mathcal{N}_{0,\le n_0})$ or $(a'-d')^{\#}(n_0) = rk(\mathcal{N}_{0,\le n_0}) - 1$ for some $n_0 \ge b$. A sufficient condition for (3) to hold is that $b > T_{\mathcal{N}_0}$. Also, the case $b = c+1$ with $c \gg 0$ always works.

Case $b < c$: we have

$$(a'-d')^{\#}(n) = \begin{cases} (a-d)^{\#}(n), & \text{for } n < b \\ (a-d)^{\#}(n) + 1, & \text{for } b \le n < c \\ (a-d)^{\#}(n), & \text{for } n \ge c. \end{cases}$$

Then $(a'-d')^{\#}(n) \ge (a-d)^{\#}(n)$ for any n and since there is an upper bound for admissible functions it is clear that it may not be possible to make a descending elementary biliaison.

d) If $b \neq c$, roughly speaking to perform an elementary biliaison of type (b, c) means to change the degree of one section of the vector bundle in the middle. Actually, we can split the process into two steps: first modify trivially the given sequence $(*)$ to get:

$$0 \to \mathcal{A} \oplus \mathcal{O}(-c) \xrightarrow{u \oplus 1} \mathcal{N}_0 \oplus \mathcal{D} \oplus \mathcal{O}(-c) \to \mathcal{F} \to 0 \tag{c}$$

then change c to b in the left bundle:

$$0 \to \mathcal{A} \oplus \mathcal{O}(-b) \xrightarrow{v} \mathcal{N}_0 \oplus \mathcal{D} \oplus \mathcal{O}(-c) \to \mathcal{E} \to 0. \tag{b}$$

One may observe that we could change the degree of one section in sequence $(*)$, but in that case we should choose an invertible summand of $\mathcal{A}$, while in this way we can *a priori* choose any c. Another advantage of passing through sequence (c) is that (c) and (b) have the same bundle in the middle, thus instead of comparing functions $(a - d)^{\#}$ and $(a' - d')^{\#}$ we can just compare the $\#$ functions of the characteristic functions of $\mathcal{A} \oplus \mathcal{O}(-c)$ and of $\mathcal{A} \oplus \mathcal{O}(-b)$, which are positive and non-decreasing (while $(a - d)^{\#}$ need not be monotone). In the sequel it will be useful to have explicit formulas: set $\mathcal{A}'' = \mathcal{A} \oplus \mathcal{O}(-c)$, then $a - d = a'' - d'$, hence, if $b > c$ we have:

$$a'^{\#}(n) = \begin{cases} a''^{\#}(n), & \text{for } n < c \\ a''^{\#}(n) - 1, & \text{for } c \leq n < b \\ a''^{\#}(n), & \text{for } n \geq b. \end{cases}$$

and, if $b < c$:

$$a'^{\#}(n) = \begin{cases} a''^{\#}(n), & \text{for } n < b \\ a''^{\#}(n) + 1, & \text{for } b \leq n < c \\ a''^{\#}(n), & \text{for } n \geq c. \end{cases}$$

The effect of an ascending (resp., descending) elementary biliaison is then to decrease (resp., increase) of 1 the $a''^{\#}$ function in the interval $[c, b)$ (resp., $[b, c)$). Also, notice that the admissibility condition for $(a' - d')^{\#}$ translates into the following corresponding properties for $a'^{\#}$:

(1*) $a'^{\#}(n) = rk(\mathcal{N}_0 \oplus \mathcal{D}') - 2$ for $n \gg 0$;

(2*) $a'^{\#}(n) \leq \chi^{\#}_{\mathcal{N}_0 \oplus \mathcal{D}'}(n)$ for any n;

(3*) if $a'^{\#}(n_0) = rk(\mathcal{N}_0 \oplus \mathcal{D}')_{\leq n_0}$ for some $n_0 \leq R_{\mathcal{N}_0}$, then $\mathcal{A}'_{\leq n_0} \cong (\mathcal{N}_0 \oplus \mathcal{D}')_{\leq n_0}$; if $a'^{\#}(n_0) = rk(\mathcal{N}_0 \oplus \mathcal{D}')_{\leq n_0} - 1$ for some $n_0 \leq T_{\mathcal{N}_0}$, then $\mathcal{A}'_{\leq n_0}$ is isomorphic to a direct summand of $(\mathcal{N}_0 \oplus \mathcal{D}')_{\leq n_0}$.

Proposition 2.22. *If $\mathcal{F} \preceq \mathcal{F}'$ and $a-d \neq a'-d'$, then we can move from $\mathcal{F}$ to $\mathcal{F}'$ through a finite number of ascending elementary biliaison followed by a deformation with constant cohomology.*

Proof. First we modify trivially sequences $(*)$ and $(*')$ in order to have the same vector bundle in the middle:

$$(\alpha) \qquad 0 \to \mathcal{A} \oplus \mathcal{D}' \xrightarrow{u \oplus id_{\mathcal{D}'}} \mathcal{N}_0 \oplus \mathcal{D} \oplus \mathcal{D}' \to \mathcal{F} \to 0$$

$$(\alpha') \qquad 0 \to \mathcal{A}' \oplus \mathcal{D} \xrightarrow{u' \oplus id_{\mathcal{D}}} \mathcal{N}_0 \oplus \mathcal{D}' \oplus \mathcal{D} \to \mathcal{F}' \to 0$$

Set $\mathcal{P} = \mathcal{A} \oplus \mathcal{D}'$, $\mathcal{P}' = \mathcal{A}' \oplus \mathcal{D}$. From the hypothesis $p^{\#}(n) \geq p'^{\#}(n)$ for any n and $p \neq p'$. Let $k_0 = max\{n | p^{\#}(n) > p'^{\#}(n)\}$ — we know that for $n \gg 0$ $p^{\#}(n) = p'^{\#}(n) = rk(\mathcal{N}_0 \oplus \mathcal{D} \oplus \mathcal{D}') - 2$ — and $k_1 = max\{n \leq k_0 | p(n) > 0\} = min\{n | p^{\#}(n) = p^{\#}(k_0)\}$. We wish to perform an elementary biliaison of type (k_0+1, k_1) in order to diminish of 1 the function $(a-d)^{\#}$ in the interval $[k_1, k_0]$. What we need to do is to check that we obtain an admissible pair. As explained in Remark 2.21, d) above, we perform the elementary biliaison in two steps: let $\mathcal{R} = \mathcal{P} \oplus \mathcal{O}(-k_1)$, $\mathcal{B} = \mathcal{P} \oplus \mathcal{O}(-k_0-1)$, $\mathcal{M} = \mathcal{N}_0 \oplus \mathcal{D} \oplus \mathcal{D}' \oplus \mathcal{O}(-k_1)$ and let also $\mathcal{R}' = \mathcal{P}' \oplus \mathcal{O}(-k_1)$. Now we have three exact sequences:

$$0 \to \mathcal{R} \to \mathcal{M} \to \mathcal{F} \to 0$$

$$0 \to \mathcal{R}' \to \mathcal{M} \to \mathcal{F}' \to 0$$

$$0 \to \mathcal{B} \to \mathcal{M} \to \mathcal{E} \to 0$$

the first two are obtained from (α), (α') respectively, just adding trivially $\mathcal{O}(-k_1)$, while the third one is the effect of an elementary biliaison (k_0+1, k_1) on (α). We compare functions $r^{\#}$, $r'^{\#}$ and $b^{\#}$. Notice that

$$\text{for } n < k_1, \qquad r^{\#}(n) = p^{\#}(n) \quad \text{and} \quad r'^{\#}(n) = p'^{\#}(n)$$

$$\text{for } n \geq k_1, \qquad r^{\#}(n) = p^{\#}(n) + 1 \quad \text{and} \quad r'^{\#}(n) = p'^{\#}(n) + 1$$

and, by Remark 2.21 d),

$$b^{\#}(n) = \begin{cases} r^{\#}(n), & \text{for } n < k_1 \\ r^{\#}(n) - 1, & \text{for } k_1 \leq n < k_0 + 1 \\ r^{\#}(n), & \text{for } n \geq k_0 + 1. \end{cases}$$

hence:

$$\begin{aligned} \text{for } n < k_1, \qquad & r'^{\#}(n) \le b^{\#}(n) = r^{\#}(n) \\ \text{for } k_1 \le n < k_0+1, \qquad & r'^{\#}(n) \le b^{\#}(n) = r^{\#}(n) - 1 \\ \text{for } n \ge k_0+1, \qquad & r'^{\#}(n) = b^{\#}(n) = r^{\#}(n). \end{aligned}$$

Properties (1^*) and (2^*) of the admissibility condition for $b^{\#}$ are then straightforward: $b^{\#}(n) = r^{\#}(n) = rk(\mathcal{M}) - 2$ for $n \gg 0$ and $b^{\#}(n) \le r^{\#}(n) \le \chi^{\#}_{\mathcal{M}}(n)$ for any n. We now check (3^*). Observe first that

$$b(n) = \begin{cases} r(n), & \text{for } n < k_1 \\ r(n) - 1, & \text{for } n = k_1 \\ r(n), & \text{for } k_1 < n < k_0 + 1 \\ r(n) + 1, & \text{for } n = k_0 + 1 \\ r(n), & \text{for } n > k_0 + 1. \end{cases}$$

Recall that with the notation introduced at the beginning of this Chapter we have a surjection $\bigoplus_{n\in \mathbf{Z}} \mathcal{O}(-n)^{l_{\mathcal{M}}(n)} \to \mathcal{M} \to 0$. Then by definition of $T_{\mathcal{M}}$ we have $\mathcal{M}_{\le t} \cong \bigoplus_{n \le t} \mathcal{O}(-n)^{l_{\mathcal{M}}(n)}$ for any $t \le T_{\mathcal{M}}(= T_{\mathcal{N}_0})$ and, by Lemma 2.7, $\chi_{\mathcal{M}}(n) = l_{\mathcal{M}}(\mathrm{n})$ for $n \le R_{\mathcal{M}}(= R_{\mathcal{N}_0})$.

Suppose that $b^{\#}(n_0) = rk(\mathcal{M}_{\le n_0})$ for some $n_0 \le R_{\mathcal{N}_0}$. If $n_0 < k_1$, then $r^{\#}(n_0) = rk(\mathcal{M}_{\le n_0})$, hence $\mathcal{R}_{\le n_0} \cong \mathcal{M}_{\le n_0}$, that is, $r(n) = l_{\mathcal{M}}(n)$ for $n \le n_0$. It follows that $b(n) = l_{\mathcal{M}}(n)$ for $n \le n_0$. The case $n_0 \ge k_1$ cannot occur. Indeed, if $k_1 \le n_0 < k_0 + 1$, then $r^{\#}(n_0) = rk(\mathcal{M}_{\le n_0}) + 1 > \chi^{\#}_{\mathcal{M}}(n_0)$, which is absurd. If $n_0 \ge k_0 + 1$, we have $r'^{\#}(n_0) = r^{\#}(n_0) = rk(\mathcal{M}_{\le n_0})$ hence $r'(n) = r(n) = l_{\mathcal{M}}(n)$ for any $n \le n_0$; since it is also $r'(n) = r(n)$ for $n \ge k_0 + 2$, it turns out that $r = r'$, which contradicts the hypothesis.

Suppose now that for some $n \le T_{\mathcal{N}_0}$ we have $b^{\#}(n_0) = rk(\mathcal{M}_{\le n_0}) - 1$. Again, if $n_0 < k_1$, then $r^{\#}(n_0) = rk(\mathcal{M}_{\le n_0}) - 1$. Hence there exists an integer $t \le n_0$ such that $\mathcal{R}_{\le n_0} \oplus \mathcal{O}(-t) \cong \mathcal{M}_{\le n_0}$, i.e.

$$r(n) = \begin{cases} l_{\mathcal{M}}(n), & \text{for } n \le n_0,\ n \ne t \\ l_{\mathcal{M}}(n) - 1, & \text{for } n = t. \end{cases}$$

This implies:

$$b(n) = \begin{cases} l_{\mathcal{M}}(n), & \text{for } n \le n_0,\ n \ne t \\ l_{\mathcal{M}}(n) - 1, & \text{for } n = t. \end{cases}$$

If $k_1 \leq n_0 < k_0+1$, then $r^{\#}(n_0) = rk(\mathcal{M}_{\leq n_0})$. Hence $r(n) = l_{\mathcal{M}}(n)$ for $n \leq n_0$. It follows that:

$$b(n) = \begin{cases} l_{\mathcal{M}}(n), & \text{for } n < k_1 \\ l_{\mathcal{M}}(n) - 1, & \text{for } n = k_1 \\ l_{\mathcal{M}}(n), & \text{for } k_1 < n \leq n_0 \end{cases}$$

that is, $\mathcal{B}_{\leq n_0} \oplus \mathcal{O}(-k_1) \cong \mathcal{M}_{\leq n_0}$. Finally, if $n_0 \geq k_0+1$, then $r'^{\#}(n_0) = r^{\#}(n_0) = rk(\mathcal{M}_{\leq n_0}) - 1$ and there exist integers $t, t' \leq n_0$ such that $\mathcal{R}_{\leq n_0} \oplus \mathcal{O}(-t) \cong \mathcal{M}_{\leq n_0} \cong \mathcal{R}'_{\leq n_0} \oplus \mathcal{O}(-t')$. In other words:

$$r(n) = \begin{cases} l_{\mathcal{M}}(n), & \text{for } n \leq n_0,\, n \neq t \\ l_{\mathcal{M}}(n) - 1, & \text{for } n = t \end{cases}$$

and

$$r'(n) = \begin{cases} l_{\mathcal{M}}(n), & \text{for } n \leq n_0,\, n \neq t' \\ l_{\mathcal{M}}(n) - 1, & \text{for } n = t'. \end{cases}$$

It is necessarily $t > t'$. Indeed, if $t = t'$, then $r'^{\#}(n) = r^{\#}(n)$ for any $n \leq n_0$, but it is also $r'^{\#}(n) = r^{\#}(n)$ for $n \geq k_0+1$, hence $r = r'$, which is absurd. If $t < t'$, for $t \leq n < t'$ we get $r^{\#}(n) = r'^{\#}(n) - 1$, which contradicts the hypothesis. Thus $t > t'$ and we have

$$r'^{\#}(n) = \begin{cases} r^{\#}(n), & \text{for } n < t' \\ r^{\#}(n) - 1, & \text{for } t' \leq n < t \\ r^{\#}(n), & \text{for } n \geq t. \end{cases}$$

It follows that $k_0 = t-1$. Moreover, $k_1 \geq t'$. Indeed, for $k_1 < t'$ we obtain $r^{\#}(k_1) = r'^{\#}(k_1)$ while we know that $r^{\#}(k_1) > r'^{\#}(k_1)$. In the end we have $t' \leq k_1 < t = k_0+1$ and

$$b^{\#}(n) = \begin{cases} r^{\#}(n), & \text{for } n < k_1 \\ r'^{\#}(n), & \text{for } k_1 \leq n < k_0+1 \\ r^{\#}(n), & \text{for } n \geq k_0+1. \end{cases}$$

Hence:

$$b(n) = \begin{cases} l_{\mathcal{M}}(n), & \text{for } n < k_1 \\ l_{\mathcal{M}}(n) - 1, & \text{for } n = k_1 \\ l_{\mathcal{M}}(n), & \text{for } k_1 < n \leq n_0, \end{cases}$$

i.e. $\mathcal{B}_{\leq n_0} \oplus \mathcal{O}(-k_1) \cong \mathcal{M}_{\leq n_0}$. This shows that the (FO)-condition is satisfied.

Now we repeat the procedure on $b^{\#}$ instead than $p^{\#}$. After a finite number of steps we end up with an exact sequence

$$0 \to \mathcal{C} \to \mathcal{N}_0 \oplus \mathcal{L} \to \mathcal{G} \to 0$$

with $\mathcal{C}$ and $\mathcal{L}$ free sheaves such that $c - l = a' - d'$ and we conclude by Prop 2.18.

Propp 2.18 and 2.22 above allow us to describe the structure of a biliaison class $Refl(\mathcal{N}_0)$. Such a structure is given by the so called LR-property (or Lazarsfeld-Rao property) which, in the case of a biliaison class of reflexive sheaves, reads as follows:

Definition 2.23. We say that a biliaison class of rank two reflexive sheaves on $\mathbf{P}^3$ has the LR-property if the following conditions hold:

(i) given any two minimal elements $\mathcal{F}_0$ and $\mathcal{F}_0'$, there is a deformation from $\mathcal{F}_0$ to $\mathcal{F}_0'$ through minimal elements;

(ii) given a minimal element $\mathcal{F}_0$ and a non-minimal one $\mathcal{F}$, one can move from $\mathcal{F}_0$ to $\mathcal{F}$ through a finite number of ascending elementary biliaison followed by a deformation with constant cohomology.

As a corollary of Propp 2.18 and 2.22 we then obtain:

Theorem 2.24. *Any biliaison class of rank two reflexive sheaves $\mathcal{F}$ on $\mathbf{P}^3$ with $H^1_*\mathcal{F} \neq 0$ has the LR-property.*

Remark 2.25. Any two sheaves $\mathcal{F}$ and $\mathcal{F}'$ in a class $Refl(\mathcal{N}_0)$ are connected one to the other by a finite number of elementary biliaison — not necessarily all ascending or all descending — followed by a deformation with constant cohomology. Indeed, two different situations may occur: either $\mathcal{F}$ and $\mathcal{F}'$ are comparable in the partial ordering defined on $Refl(\mathcal{N}_0)$, or not. The former case is answered by Propp 2.18 and 2.22. The latter requires to pass through some minimal element. Suppose $\mathcal{F}$ and $\mathcal{F}'$ correspond to functions $a-d$ and $a'-d'$ respectively, then $\chi^{\#}_{\mathcal{N}_0}$ is greater than both $(a-d)^{\#}$ and $(a'-d')^{\#}$. By Prop 2.22, there exist two finite sequences of ascending elementary biliaison, one from $\chi^{\#}_{\mathcal{N}_0}$ to $(a-d)^{\#}$ and another from $\chi^{\#}_{\mathcal{N}_0}$ to $(a'-d')^{\#}$. If we perform one of these sequences in the inverse order, followed by the other one we can move from $(a-d)^{\#}$ to $(a'-d')^{\#}$ (or viceversa). We conclude by 2.18.

Chapter 3

Behaviour of invariants in a class

In this Chapter we consider some invariants associated with a reflexive sheaf, namely its Chern classes and the minimal twist which has non-zero sections, and describe their behaviour in a biliaison class. We end the Chapter with some examples.

Let us start with Chern classes. We will show in particular that minimal elements have minimal first and third Chern classes (the latter result is due to C. Walter). As a consequence we prove that in a biliaison class containing vector bundles these are precisely the minimal elements in the class.

We denote $c_i\mathcal{F}$ the i-th Chern class of the sheaf $\mathcal{F}$ and we stick to notation introduced in Chapter 2.

Remark 3.1. Suppose $\mathcal{F}$ is given by an exact sequence $0 \to \mathcal{A} \to \mathcal{N}_0 \oplus \mathcal{D} \to \mathcal{F} \to 0$ with $\mathcal{A}$ and $\mathcal{D}$ free sheaves and $\mathcal{F}'$ is obtained from $\mathcal{F}$ by an elementary biliaison of type (b, c). That is, $\mathcal{F}'$ admits an exact sequence $0 \to \mathcal{A} \oplus \mathcal{O}(-b) \to \mathcal{N}_0 \oplus \mathcal{D} \oplus \mathcal{O}(-c) \to \mathcal{F}' \to 0$. Then a direct computation on these sequences yields:

(1) $c_1\mathcal{F}' = c_1\mathcal{F} + b - c$

(2) $c_2\mathcal{F}' = c_2\mathcal{F} + (b-c)(c_1\mathcal{F} + b)$

(3) $c_3\mathcal{F}' = c_3\mathcal{F} + (b-c)(c_2\mathcal{F} + b^2 + bc_1\mathcal{F})$

As an immediate consequence one gets:

Lemma 3.2. *If $\mathcal{F}'$ is obtained from $\mathcal{F}$ by an ascending elementary bilaison, then $c_1\mathcal{F}' > c_1\mathcal{F}$.*

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Combining this with Prop 2.22 yields:

Corollary 3.3. *If $\mathcal{F} \preceq \mathcal{F}'$, then $c_1\mathcal{F} \leq c_1\mathcal{F}'$. (More precisely, if $\mathcal{F} \preceq \mathcal{F}'$ and $a-d \neq a'-d'$, then $c_1\mathcal{F} < c_1\mathcal{F}'$. Of course, $a-d = a'-d'$ implies $c_1\mathcal{F} = c_1\mathcal{F}'$).*

Remark 3.4. Notice that the converse is not true, that is, the order relation on sheaves induced by the ordering of first Chern classes is coarser than the partial ordering defined in 2.14.

Corollary 3.5. *In $Refl(\mathcal{N}_0)$ minimal elements have minimal first Chern class equal to $c_{1,\mathcal{N}_0}^{min} = c_1\mathcal{N}_0 + \sum_{n\in\mathbf{Z}} n\chi_{\mathcal{N}_0}(n)$.*

Proof. The first statement is clear. Then we can compute the minimal Chern class from sequence (†) in Prop. 2.17.

Remark 3.6. Any integer greater than $c_{1,\mathcal{N}_0}^{min}$ is the first Chern class of some sheaf in $Refl(\mathcal{N}_0)$. Indeed, if we perform an elementary biliaison of type $(b, b-1)$ with $b \gg 0$ on a sheaf $\mathcal{F} \in Refl(\mathcal{N}_0)$ — we need b large enough to obtain an admissible pair, see Remark 2.21 c) — we obtain a sheaf $\mathcal{F}'$ in $Refl(\mathcal{N}_0)$ with first Chern class $c_1\mathcal{F}' = c_1\mathcal{F} + 1$.

The following result is due to C. Walter.

Proposition 3.7. *If $\mathcal{F}'$ is obtained from $\mathcal{F}$ by an ascending elementary biliaison, then $c_3\mathcal{F}' \geq c_3\mathcal{F}$.*

Proof. As usual, suppose $\mathcal{F}$ is given by an exact sequence

$$(*) \qquad 0 \to \mathcal{A} \xrightarrow{u} \mathcal{N}_0 \oplus \mathcal{D} \to \mathcal{F} \to 0$$

with $\mathcal{A}$ and $\mathcal{D}$ free sheaves and $\mathcal{F}'$ is obtained from $\mathcal{F}$ by an elementary biliaison of type (b, c), with $b > c$, that is

$$0 \to \mathcal{A} \oplus \mathcal{O}(-b) \xrightarrow{v} \mathcal{N}_0 \oplus \mathcal{D} \oplus \mathcal{O}(-c) \to \mathcal{F}' \to 0.$$

Let $h = 0$ be the equation of a plane $H \subset \mathbf{P}^3$ which does not intersect the singular loci of $\mathcal{F}$ and $\mathcal{F}'$. Consider the map $u \oplus h^{b-c} : \mathcal{A} \oplus \mathcal{O}(-b) \to \mathcal{N}_0 \oplus \mathcal{D} \oplus \mathcal{O}(-c)$ and define a family of morphisms

$$\phi_t = tv + (1-t)(u \oplus h^{b-c}) \in Hom(\mathcal{A} \oplus \mathcal{O}(-b), \mathcal{N}_0 \oplus \mathcal{D} \oplus \mathcal{O}(-c)), \quad t \in \mathbf{A}^1_K.$$

Then $\phi_1 = v$ degenerates at $c_3\mathcal{F}'$ points, counted with multiplicities — hence so does the generic morphism ϕ_t — while $\phi_0 = u \oplus h^{b-c}$ degenerates on the plane H and on $c_3\mathcal{F}$ points counted with multiplicities and not lying on H. We are going to show that these $c_3\mathcal{F}$ points are specialization of (some of) the $c_3\mathcal{F}'$ points where the generic morphism ϕ_t degenerates.

Let $\pi : \mathbf{P}^3 \times \mathbf{A}^1 \to \mathbf{P}^3$ be the projection, the family $\{\phi_t\}_{t\in\mathbf{A}^1}$ defines a morphism

$$\Phi : \pi^*(\mathcal{A} \oplus \mathcal{O}(-b)) \to \pi^*(\mathcal{N}_0 \oplus \mathcal{D} \oplus \mathcal{O}(-c)).$$

Let Z denote the maximal degeneracy locus of Φ and $I_{Z,(x,t)}$ the ideal defining Z at the point (x,t). Since the rank difference between the bundles is two, at each point $(x,t) \in \mathbf{P}^3 \times \mathbf{A}^1$ we have $height(I_{Z,(x,t)}) \le 3$ ([BV, Thm 2.1]). Thus every component of Z has dimension at least one. On the other hand, by construction, we have $height(I_{Z,(x,t)}) = 3$ for $(x,t) \in \mathbf{P}^3 \times U$, where U is a non-empty open subset of $\mathbf{A}^1$ containing 1 and for $(x,0) \in Sing(\mathcal{F}) \times \{0\}$. Hence, at these points Z has dimension one and is locally Cohen-Macaulay (Thm. 2.6 in [BV]). For any $t \in U$, $Z_t := Z \cap \mathbf{P}^3 \times \{t\}$ is a finite set of points of degree $c_3\mathcal{F}'$, then $Z_U := Z \cap \mathbf{P}^3 \times U$ is flat over U. The flat closure of Z_U over $U \cup \{0\}$ contains $Sing(\mathcal{F}) \times \{0\}$, this implies that $c_3\mathcal{F} \le c_3\mathcal{F}'$.

Corollary 3.8. *If $\mathcal{F} \preceq \mathcal{F}'$, then $c_3\mathcal{F} \le c_3\mathcal{F}'$.*

Proof. It is a consequence of Prop. 3.7 and Prop. 2.22

Corollary 3.9. *In $Refl(\mathcal{N}_0)$, minimal elements have minimal third Chern class, namely,*

$$\begin{aligned} c_{3,\mathcal{N}_0}^{min} = c_3\mathcal{N}_0 - c_2\mathcal{N}_0 c_1(\chi_{\mathcal{N}_0}) + c_1\mathcal{N}_0[c_1(\chi_{\mathcal{N}_0})^2 - c_2(\chi_{\mathcal{N}_0})] \\ + 2c_1(\chi_{\mathcal{N}_0})c_2(\chi_{\mathcal{N}_0}) - c_1(\chi_{\mathcal{N}_0})^3 - c_3(\chi_{\mathcal{N}_0}), \end{aligned}$$

where we set $c_i(\chi_{\mathcal{N}_0}) = c_i(\bigoplus_{n\in\mathbf{Z}} \mathcal{O}(-n)^{\chi_{\mathcal{N}_0}(n)})$.

Proof. We compute $c_{3,\mathcal{N}_0}^{min}$ from the $\mathcal{N}_0$-resolution of a minimal element given in Prop. 2.17.

We point out the following:

Corollary 3.10. *If $Refl(\mathcal{N}_0)$ contains some vector bundles, then they are precisely the minimal elements in the class.*

Proof. If $Refl(\mathcal{N}_0)$ contains vector bundles, then the minimal value of c_3 in the class is zero and, by 3.9, it follows that minimal elements are vector bundles. Suppose now $\mathcal{E} \in Refl(\mathcal{N}_0)$ is a vector bundle but not a minimal element in the class. Then for any minimal element $\mathcal{F} \in Refl(\mathcal{N}_0)$ we have $c_1\mathcal{E} > c_1\mathcal{F}$ (Cor. 3.3). This is a contradiction, because rank two vector bundles on $\mathbf{P}^3$ with the same H^1_* module have the same Chern classes ([R3, Cor. 2.4]).

Remark 3.11. Notice that not every class $Refl(\mathcal{N}_0)$ contains vector bundles. Actually, finite length S-modules which are the first cohomology module of some rank two vector bundle on $\mathbf{P}^3$ satisfy quite stringent conditions ([R3]). Later these modules have been characterized by W. Decker in [D]. Here we have another citerion to determine whether a class $Refl(\mathcal{N}_0)$ contains some vector bundles:

Corollary 3.12. *A class $Refl(\mathcal{N}_0)$ contains vector bundles if and only if $c^{min}_{3,\mathcal{N}_0} = 0$.*

Corollary 3.13. *If $Refl(\mathcal{N}_0)$ contains some vector bundles, then there is an isomorphism $\bigoplus_{n\in\mathbf{Z}} \mathcal{O}(-n)^{\chi_{\mathcal{N}_0}(n)} \cong \mathcal{L}_0^*(c^{min}_{1,\mathcal{N}_0})$.*

Proof. Let $\mathcal{F}$ be a vector bundle in $Refl(\mathcal{N}_0)$, then by Prop. 2.2 in [R3] it is obtained from an exact sequence

$$0 \to \mathcal{L}_0^*(c_1\mathcal{F}) \to \mathcal{N}_0 \to \mathcal{F} \to 0,$$

where $\mathcal{L}_0$ is the free sheaf associated to the module L_0 in a minimal free resolution of $H^1_*\mathcal{F}$ and $\mathcal{L}_0^*$ is its dual. $\mathcal{F}$ is a minimal element in its class (Cor. 3.10), hence $c_1\mathcal{F} = c^{min}_{1,\mathcal{N}_0}$ and its $\mathcal{N}_0$-resolution is $0 \to \bigoplus_{n\in\mathbf{Z}} \mathcal{O}(-n)^{\chi_{\mathcal{N}_0}(n)} \to \mathcal{N}_0 \to \mathcal{F} \to 0$. Comparing with the exact sequence above we get the required isomorphism.

We now consider another important invariant of a reflexive sheaf, that is, the minimal twist which has non-zero sections:

Definition 3.14. If $\mathcal{G}$ is a sheaf we denote $t_0(\mathcal{G})$ the minimal twist of $\mathcal{G}$ which has non-zero sections, i.e. $t_0(\mathcal{G}) := \inf\{n | h^0\mathcal{G}(n) > 0\}$.

The following lemma describes the behaviour of this invariant under elementary biliaison:

Lemma 3.15. *Suppose $\mathcal{F} \in Refl(\mathcal{N}_0)$ and let $\mathcal{F}'$ be obtained from $\mathcal{F}$ by an elementary biliaison of type (b, c). We have:*

(1) *if $b = c$, then $t_0(\mathcal{F}') = t_0(\mathcal{F})$;*
(2) *if $b > c$, then $t_0(\mathcal{F}') \leq t_0(\mathcal{F})$. Moreover, if $c \geq t_0(\mathcal{F})$, then $t_0(\mathcal{F}') = t_0(\mathcal{F})$ and for $c = t_0(\mathcal{F})$ we have $h^0\mathcal{F}'(c) \geq 2$; if $c < t_0(\mathcal{F})$, then $t_0(\mathcal{F}') = c$ and $h^0\mathcal{F}'(c) = 1$;*
(3) *if $b < c$, then $t_0(\mathcal{F}') \geq t_0(\mathcal{F})$. Moreover, if $b > t_0(\mathcal{F})$, then $t_0(\mathcal{F}') = t_0(\mathcal{F})$; if $b \leq t_0(\mathcal{F})$ and $h^0\mathcal{F}(t_0(\mathcal{F})) = 1$, then $t_0(\mathcal{F}') > t_0(\mathcal{F})$.*

Proof. Suppose $\mathcal{F}$ is given by an exact sequence $0 \to \mathcal{A} \to \mathcal{N}_0 \oplus \mathcal{D} \to \mathcal{F} \to 0$,with $\mathcal{A}$ and $\mathcal{D}$ free sheaves, then, by definition, $\mathcal{F}'$ is given by $0 \to \mathcal{A} \oplus \mathcal{O}(-b) \to \mathcal{N}_0 \oplus \mathcal{D} \oplus \mathcal{O}(-c) \to \mathcal{F}' \to 0$. For any n we then have: $h^0\mathcal{F}'(n) = h^0\mathcal{F}(n) + h^0\mathcal{O}(n - c) - h^0\mathcal{O}(n - b)$. Let us consider the different cases separately. (1) is obvious. Suppose $b > c$, then for $n < c$ we have $h^0\mathcal{F}'(n) = h^0\mathcal{F}(n)$, while $h^0\mathcal{F}'(n) > h^0\mathcal{F}(n)$ for $n \geq c$. It follows that $t_0(\mathcal{F}') \leq t_0(\mathcal{F})$. It is also clear that if $c \geq t_0(\mathcal{F})$ we have $t_0(\mathcal{F}') = t_0(\mathcal{F})$ and if $c = t_0(\mathcal{F})$ then $h^0\mathcal{F}'(c) > h^0\mathcal{F}(c) \geq 1$. If $c < t_0(\mathcal{F})$, then $h^0\mathcal{F}'(c) = h^0\mathcal{O} = 1$. This proves (2). Suppose now $b < c$. For $n < b$ we have $h^0\mathcal{F}'(n) = h^0\mathcal{F}(n)$ and for $n \geq b$ we have $h^0\mathcal{F}'(n) < h^0\mathcal{F}(n)$, which implies $t_0(\mathcal{F}') \geq t_0(\mathcal{F})$. Moreover, if $b > t_0(\mathcal{F})$, then $t_0(\mathcal{F}') = t_0(\mathcal{F})$. If $b \leq t_0(\mathcal{F})$ and $h^0\mathcal{F}(t_0(\mathcal{F})) = 1$, then $t_0(\mathcal{F}') > t_0(\mathcal{F})$.

Lemma 3.16. *Let $\mathcal{F}$ be a minimal element in $Refl(\mathcal{N}_0)$. Then $t_0(\mathcal{F}) = R_{\mathcal{N}_0} + 1$ and $h^0\mathcal{F}(R_{\mathcal{N}_0} + 1) = l_{\mathcal{N}_0}(R_{\mathcal{N}_0} + 1) - \chi_{\mathcal{N}_0}(R_{\mathcal{N}_0} + 1)$.*

Proof. By Prop. 2.17 the $\mathcal{N}_0$-resolution of $\mathcal{F}$ has the form

$$0 \to \bigoplus_{n \in \mathbf{Z}} \mathcal{O}(-n)^{\chi_{\mathcal{N}_0}(n)} \xrightarrow{\phi} \mathcal{N}_0 \to \mathcal{F} \to 0.$$

Let $\mathcal{N}_{0,\leq R_{\mathcal{N}_0}}$ denote the subsheaf of $\mathcal{N}_0$ generated by sections of degree $\leq R_{\mathcal{N}_0}$. Then the restriction of ϕ to $\bigoplus_{n \leq R_{\mathcal{N}_0}} \mathcal{O}(-n)^{\chi_{\mathcal{N}_0}(n)}$ factors through $\mathcal{N}_{0,\leq R_{\mathcal{N}_0}}$. We have an exact commutative diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_{n\leq R_{\mathcal{N}_0}} \mathcal{O}(-n)^{\chi_{\mathcal{N}_0}(n)} & \xrightarrow{\phi_{R_{\mathcal{N}_0}}} & \mathcal{N}_{0,\leq R_{\mathcal{N}_0}} & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_{n\leq R_{\mathcal{N}_0}} \mathcal{O}(-n)^{\chi_{\mathcal{N}_0}(n)} & \xrightarrow{\phi_|} & \mathcal{N}_0 & \longrightarrow & \mathcal{Q} & \longrightarrow & 0
\end{array}
$$

and an exact sequence $0 \to \bigoplus_{n>R_{\mathcal{N}_0}} \mathcal{O}(-n)^{\chi_{\mathcal{N}_0}(n)} \to \mathcal{Q} \to \mathcal{F} \to 0$. Then $\mathcal{Q}$ is reflexive, since $\mathcal{F}$ is, and $\mathcal{G}$ is torsion free. Now recall that by definition of $R_{\mathcal{N}_0}$ (Def. 2.1) we have $\mathcal{N}_{0,\leq R_{\mathcal{N}_0}} \cong \bigoplus_{n\leq R_{\mathcal{N}_0}} \mathcal{O}(-n)^{l_{\mathcal{N}_0}(n)}$. Moreover, $l_{\mathcal{N}_0}(n) = \chi_{\mathcal{N}_0}(n)$ for $n \leq R_{\mathcal{N}_0}$ (Lemma 2.7). It follows that $\mathcal{G}$ has rank zero, hence $\mathcal{G} = 0$ and $\phi_{R_{\mathcal{N}_0}}$ is an isomorphism. If we quotient out the $\mathcal{N}_0$-resolution by this isomorphism we obtain a new exact sequence:

$$
(\dagger) \qquad 0 \to \bigoplus_{n>R_{\mathcal{N}_0}} \mathcal{O}(-n)^{\chi_{\mathcal{N}_0}(n)} \to \mathcal{N}_0/\mathcal{N}_{0,\leq R_{\mathcal{N}_0}} \to \mathcal{F} \to 0.
$$

Consider now the following exact commutative diagram of S-modules where $N_0 = H^0_*\mathcal{N}_0$, $N_{0,\leq R_{\mathcal{N}_0}} = H^0_*(\mathcal{N}_{0,\leq R_{\mathcal{N}_0}})$, $N_0/N_{0,\leq R_{\mathcal{N}_0}} = H^0_*(\mathcal{N}_0/\mathcal{N}_{0,\leq R_{\mathcal{N}_0}})$ and $E_0 = H^0_*\mathcal{E}_0$:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \bigoplus_{n\leq R_{\mathcal{N}_0}} S(-n)^{l_{\mathcal{N}_0}(n)} & \xrightarrow{\sim} & N_{0,\leq R_{\mathcal{N}_0}} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_0 & \longrightarrow & \bigoplus_{n\in\mathbf{Z}} S(-n)^{l_{\mathcal{N}_0}(n)} & \xrightarrow{\sigma} & N_0 & \longrightarrow & 0 \\
 & & \downarrow\wr & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_0 & \longrightarrow & \bigoplus_{n>R_{\mathcal{N}_0}} S(-n)^{l_{\mathcal{N}_0}(n)} & \xrightarrow{\tau} & N_0/N_{0,\leq R_{\mathcal{N}_0}} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Since $\bigoplus_{n>R_{\mathcal{N}_0}} S(-n)^{l_{\mathcal{N}_0}(n)}$ has no non-zero components in degree $\leq R_{\mathcal{N}_0}$, it is clear that $H^0(\mathcal{N}_0/\mathcal{N}_{0,\leq R_{\mathcal{N}_0}})(t) = 0$ for any $t \leq R_{\mathcal{N}_0}$. In degree $R_{\mathcal{N}_0} + 1$ we have

$\bigoplus_{n>R_{\mathcal{N}_0}} S(R_{\mathcal{N}_0}+1-n)^{l_{\mathcal{N}_0}(n)} = S^{l_{\mathcal{N}_0}(R_{\mathcal{N}_0}+1)}$ and, by maximality of $R_{\mathcal{N}_0}$ (see Def. 2.1), we know that $l_{\mathcal{N}_0}(R_{\mathcal{N}_0}+1) > 0$. Moreover, since σ is minimal, τ is minimal as well. It follows that there are no relations in degree $R_{\mathcal{N}_0}+1$, i.e. $H^0\mathcal{E}_0(R_{\mathcal{N}_0}+1) = 0$. The conclusion is that $h^0(\mathcal{N}_0/\mathcal{N}_{0,\leq R_{\mathcal{N}_0}})(R_{\mathcal{N}_0}+1) = l_{\mathcal{N}_0}(R_{\mathcal{N}_0}+1)$. Combining this with sequence (†) we finally obtain $h^0\mathcal{F}(t) = 0$ for $t \leq R_{\mathcal{N}_0}$ and $h^0\mathcal{F}(R_{\mathcal{N}_0}+1) = l_{\mathcal{N}_0}(R_{\mathcal{N}_0}+1) - \chi_{\mathcal{N}_0}(R_{\mathcal{N}_0}+1)$.

As a consequence of the Lazarsfeld-Rao property and of Lemma 3.15 we then get:

Corollary 3.17. *For any sheaf $\mathcal{F} \in Refl(\mathcal{N}_0)$ it is $t_0(\mathcal{F}) \leq R_{\mathcal{N}_0}+1$.*

To prove the LR-property for a class $Refl(\mathcal{N}_0)$ we assumed $\mathcal{N}_0 \neq 0$. Actually, for the class $Refl(0)$ we do not have a notion of minimal element (compare with the liaison class of arithmetically Cohen-Macaulay curves in $\mathbf{P}^3$). For sheaves in $Refl(0)$ the two types of resolutions defined in Chapter 1 coincide and are of the form

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{F} \to 0$$

with $\mathcal{A}$ and $\mathcal{B}$ free sheaves. In particular, the class $Refl(0)$ contains all rank two free sheaves $\mathcal{O}(a) \oplus \mathcal{O}(b)$, $a, b \in \mathbf{Z}$. Actually, the only vector bundles in $Refl(0)$ are of this form: by Serre duality, $H^1_*\mathcal{E} = 0$ implies $H^2_*\mathcal{E} = 0$, hence Horrocks' criterion gives $\mathcal{E} \cong \mathcal{O}(a) \oplus \mathcal{O}(b)$ for some integers a and b. An example of a sheaf in $Refl(0)$ which is not a vector bundle can be constructed from a twisted cubic curve using the Hartshorne-Serre correspondence. The sheaf obtained in this way is a stable rank two reflexive sheaf with $c_1 = 0$, $c_2 = 2$ and $c_3 = 4$ (see [H, §4]).

Remark 3.18. Note that if we know the invariants associated with $\mathcal{N}_0$, then automatically we know those associated with any twist $\mathcal{N}_0(h)$, $h \in \mathbf{Z}$. In particular, $R_{\mathcal{N}_0(h)} = R_{\mathcal{N}_0} - h$, $T_{\mathcal{N}_0(h)} = T_{\mathcal{N}_0} - h$ and $\chi_{\mathcal{N}_0(h)}(n) = \chi_{\mathcal{N}_0}(n+h)$. Thus, for example, minimal elements in $Refl(\mathcal{N}_0(h))$ are precisely minimal elements in $Refl(\mathcal{N}_0)$ twisted by h and the whole class is just "shifted".

Let us now consider a few examples:

Example 3.19. We use notation and results from [MD-P2,V 2]. Let f_1, f_2, f_3, f_4 be a regular sequence of homogeneous elements in $\mathbf{m} = (x_0, x_1, x_2, x_3)$ of degrees $n_i = deg(f_i)$; suppose $n_1 \leq n_2 \leq n_3 \leq n_4$ and let $\nu = n_1 + n_2 + n_3 + n_4$. The module $S/(f_1, f_2, f_3, f_4)$ has a minimal free resolution given by the Koszul complex:

$$0 \to S(-\nu) \to \bigoplus_{i=1}^{4} S(n_i - \nu) \to \bigoplus_{i<j} S(-n_i - n_j) \to$$
$$\to \bigoplus_{i=1}^{4} S(-n_i) \to S \to S/(f_1, f_2, f_3, f_4) \to 0$$

then $N_0 = ker(\bigoplus_{i=1}^4 S(-n_i) \xrightarrow{[f_1 f_2 f_3 f_4]} S)$ and the associated sheaf $\mathcal{N}_0$ is locally free of rank three. From [MD-P2,V 2] we know the invariants associated with $\mathcal{N}_0$, in particular, $R_{\mathcal{N}_0} = n_1 + n_2 - 1$, $T_{\mathcal{N}_0} = n_1 + n_3 - 1$ and, setting $\mu = sup(n_1 + n_4, n_2 + n_3)$, we have $\bigoplus_{n \in \mathbf{Z}} \mathcal{O}(-n)^{\chi_{\mathcal{N}_0}(n)} \cong \mathcal{O}(-\mu)$. Thus a general morphism $\mathcal{O}(-\mu) \to \mathcal{N}_0$ gives a reflexive rank two sheaf, which is necessarily minimal, while other sheaves in $Refl(\mathcal{N}_0)$ are obtained as quotients of general morphisms

$$\mathcal{O}(-t) \oplus \mathcal{O}(-a_1) \oplus \cdots \oplus \mathcal{O}(-a_r) \to \mathcal{N}_0 \oplus \mathcal{O}(-d_1) \oplus \cdots \oplus \mathcal{O}(-d_r)$$

for some integers t, $a_1 \leq \cdots \leq a_r$, $d_1 \leq \cdots \leq d_r$, $r \geq 0$, such that $t \geq \mu$ and $a_i \geq d_i$ for $i = 1, \dots r$ (notice that in this example the (FO)-condition is empty since $\mu \geq n_1 + n_3 > T_{\mathcal{N}_0} \geq R_{\mathcal{N}_0}$ and that in order to decrease a function $\chi_{\mathcal{E}}^{\#}$ we need to increase the degrees of sections). The minimal first Chern class in $Refl(\mathcal{N}_0)$ is then $c_{1,\mathcal{N}_0}^{min} = c_1 \mathcal{N}_0 + \mu = -\nu + \mu = -inf(n_1 + n_4, n_2 + n_3)$. On the other hand, Prop. 3.1 in [R3] guarantees that for a module M with just one generator to be the first cohomology module of some rank two bundle is equivalent to having $rk(L_1) = 4$ and $L_1^* \cong L_1(-c)$ for some $c \in \mathbf{Z}$ (L_1 is the free S-module appearing in a minimal free resolution of M). We can apply this criterion to our example to find that $Refl(\mathcal{N}_0)$ contains vector bundles if and only if $n_1 + n_4 = n_2 + n_3$ (just impose the above symmetry condition to $\bigoplus_{i=1}^4 S(-n_i)$).

A well known case when this occurs is for $f_1 = x_0$, $f_2 = x_1$, $f_3 = x_2$, $f_4 = x_3$. Then $n_1 = n_2 = n_3 = n_4 = 1$ and $S/(x_0, x_1, x_2, x_3) \cong K$ has minimal free resolution

$$0 \to S(-4) \to S(-3)^4 \to S(-2)^6 \to S(-1)^4 \to S \to K \to 0.$$

Then $\mathcal{N}_0 \cong \Omega_{\mathbf{P}^3}$ and a minimal element in $Refl(\Omega_{\mathbf{P}^3})$ has $\mathcal{N}_0$-resolution

$$0 \to \mathcal{O}(-2) \to \Omega_{\mathbf{P}^3} \to \mathcal{F} \to 0.$$

$\mathcal{F}$ is a rank two vector bundle with $c_1\mathcal{F} = -2$, $c_2\mathcal{F} = 2$, $t_0(\mathcal{F}) = 2$ and $h^0\mathcal{F}(2) = 5$. Up to deformation with constant cohomology this is the only vector bundle in $Refl(\Omega_{\mathbf{P}^3})$. Twisting by 1 we obtain a null correlation bundle, which is a minimal element in $Refl(\Omega_{\mathbf{P}^3}(1))$.

Example 3.20. We now consider Buchsbaum modules, that is, graded S-modules with trivial structure (i.e. the multiplication maps between graded pieces are all zero). Let us write $M = \bigoplus_{n=0}^{r} K^{\rho(n)}$ and suppose $\rho(0) > 0$ and $\rho(r) > 0$. The minimal free resolution of M is obtained by just adding those of the summands $K^{\rho(n)}$:

$$0 \to \bigoplus_{n=0}^{r} S(-4-n)^{\rho(n)} \to \bigoplus_{n=0}^{r} S(-3-n)^{4\rho(n)} \to$$
$$\to \bigoplus_{n=0}^{r} S(-2-n)^{6\rho(n)} \to \bigoplus_{n=0}^{r} S(-1-n)^{4\rho(n)} \to$$
$$\to \bigoplus_{n=0}^{r} S(-n)^{\rho(n)} \to \bigoplus_{n=0}^{r} K^{\rho(n)} \to 0.$$

We have $N_0 = ker(\bigoplus_{n=0}^{r} S(-1-n)^{4\rho(n)} \to \bigoplus_{n=0}^{r} S(-n)^{\rho(n)})$ and the associated locally free sheaf $\mathcal{N}_0$ has rank $3\sum_{n=0}^{r} \rho(n)$. The invariants of $\mathcal{N}_0$ are $R_{\mathcal{N}_0} = T_{\mathcal{N}_0} = 1$ (see [MD-P2, V, Prop.3.1]) and the function $\chi_{\mathcal{N}_0}$ is:

$$\chi_{\mathcal{N}_0}(t) = \begin{cases} 3\rho(t-2) - 2, & \text{for } t = 2 \\ 3\rho(t-2), & \text{for } 3 \le t \le r+2 \\ 0, & \text{otherwise.} \end{cases}$$

A general morphism

$$\mathcal{O}(-2)^{3\rho(0)-2} \oplus \bigoplus_{n=1}^{r} \mathcal{O}(-2-n)^{3\rho(n)} \to \mathcal{N}_0$$

gives a minimal element in the class $Refl(\mathcal{N}_0)$. The minimal first Chern class is therefore $c_{1,\mathcal{N}_0}^{min} = 2(\rho(0) - 2) + \sum_{n=1}^{r}(6n + 10)\rho(n)$ and a minimal sheaf $\mathcal{F}$ has $t_0(\mathcal{F}) = 2$. Notice that $Refl(\mathcal{N}_0)$ does not contain any vector bundle, unless $\rho(0) = 1$ and $\rho(n) = 0$ for $n \neq 0$, that is, unless $M \cong K$. This is an immediate consequence of a theorem of M. Chang ([C, Prop. 2.1]).

The easiest example is a module concentrated in one degree and of dimension greater than one, say $M \cong K^a$, $a \geq 2$. The corresponding biliaison classes of curves in $\mathbf{P}^3$ have been studied by G. Bolondi and J. C. Migliore (see [BM]). The minimal free resolution of such a module is:

$$0 \to S(-4)^a \to S(-3)^{4a} \to S(-2)^{6a} \to S(-1)^{4a} \to S^a \to K^a \to 0,$$

then $\mathcal{N}_0 \cong a\Omega_{\mathbf{P}^3}$ and minimal elements have $\mathcal{N}_0$-resolution of the form $0 \to \mathcal{O}(-2)^{3a-2} \to a\Omega_{\mathbf{P}^3} \to \mathcal{F} \to 0$, first Chern class equal to $2(a-2)$, $t_0(\mathcal{F}) = 2$ and $h^0\mathcal{F}(2) = 3a+2$.

Another example of Buchsbaum module is a module which has no non-zero consecutive graded components. To fix ideas, let us consider a module with only two non-zero components and a zero component in between, i.e. $M \cong K^a \oplus 0 \oplus K^b$, with $a, b \geq 1$. Its minimal free resolution is:

$$\begin{aligned} 0 \to S(-4)^a \oplus S(-6)^b \to S(-3)^{4a} \oplus S(-5)^{4b} \to S(-2)^{6a} \oplus S(-4)^{6b} \to \\ \to S(-1)^{4a} \oplus S(-3)^{4b} \to S^a \oplus S(-2)^b \to M \to 0. \end{aligned}$$

Here we have $\mathcal{N}_0 \cong a\Omega_{\mathbf{P}^3} \oplus b\Omega_{\mathbf{P}^3}(-2)$ and minimal elements have $\mathcal{N}_0$-resolution:

$$0 \to \mathcal{O}(-2)^{3a-2} \oplus \mathcal{O}(-4)^{3b} \to a\Omega_{\mathbf{P}^3} \oplus b\Omega_{\mathbf{P}^3}(-2) \to \mathcal{F} \to 0,$$

first Chern class equal to $2(a-2+11b)$, $t_0(\mathcal{F}) = 2$ and $h^0\mathcal{F}(2) = 3a+2$.

Chapter 4

Biliaison classes associated to a split module

In this Chapter we consider biliaison classes associated to split modules, i.e. graded S-modules that are the direct sum of two non-zero submodules with disjoint supports. In particular, we show that these classes do not contain any vector bundle. In other words, this says that the intermediate cohomology modules of a rank two vector bundle on $\mathbf{P}^3$ are non-split (Theorem 4.7). As a direct consequence we prove that the Rao module $\bigoplus_{n\in\mathbf{Z}} H^1\mathcal{I}_C(n)$ of a subcanonical curve in $\mathbf{P}^3$ is always non-split. This provides a complete answer to a question raised a few years ago, that is, whether a rank two vector bundle on $\mathbf{P}^3$ can have gaps in its H^1_* module (or, equivalently, whether a subcanonical curve can have gaps in its Rao module), cf. [Da1], [Da2], [B2]. To my knowledge up to now only partial results were obtained, namely, connectedness was proven for vector bundles with Chern classes in a certain range and for curves with "small" speciality index (see above cited papers and more generally for curves [MD-P3] and [B1]). Besides, some of these results were valid only under a characteristic zero assumption.

As a corollary of Theorem 4.7 we also get a vanishing criterion for the cohomology groups $H^i\mathcal{E}(t)$, for $i = 1, 2$, of a rank two vector bundle $\mathcal{E}$ (Cor. 4.8) and for the groups $H^1\mathcal{I}_C(t)$ of a subcanonical curve C (Cor. 4.10). The crucial point in the proof of Theorem 4.7 is Corollary 3.10, which says that a vector bundle is always a minimal element in its biliaison class.

We start with some technical results.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Definition 4.1. Let $M = \bigoplus_{t\in \mathbf{Z}} M_t$ be a finite length graded S-module. We set:

$$\inf M = \inf\{t | M_t \neq 0\}$$
$$\sup M = \sup\{t | M_t \neq 0\}$$

Definition 4.2. Let M be a finite length graded S-module, we say that M is *split* if it is the direct sum of two non-zero graded submodules with disjoint supports, i.e. $M = M' \oplus M''$ with $\sup M' < \inf M''$. We say that M is *non-connected* if $\sup M' + 1 < \inf M''$.

Suppose $M = M' \oplus M''$ is split and let

$$\cdots \to L'_2 \xrightarrow{\sigma'_2} L'_1 \xrightarrow{\sigma'_1} L'_0 \to M' \to 0, \qquad \cdots \to L''_2 \xrightarrow{\sigma''_2} L''_1 \xrightarrow{\sigma''_1} L''_0 \to M'' \to 0$$

be minimal free resolutions of M' and M'' respectively. Their direct sum is then a minimal free resolution of M and $L'_i \oplus L''_i$ is split for any $i = 0, \dots, 4$ (cf. [MD-P4, I, 2]). We denote $N_0 = ker(\sigma'_2 \oplus \sigma''_2) = ker(\sigma'_2) \oplus ker(\sigma''_2) = N'_0 \oplus N''_0$ and $\mathcal{N}_0$, $\mathcal{N}'_0$, $\mathcal{N}''_0$ the associated sheaves, which are locally free. ζ_n (resp. ζ'_n, ζ''_n) will denote any of the invariants α_n, β_n, γ_n attached to $\mathcal{N}_0$ (resp. $\mathcal{N}'_0$, $\mathcal{N}''_0$). For their definition and those of $T_{\mathcal{N}_0}$, $R_{\mathcal{N}_0}$ and of the function $\chi_{\mathcal{N}_0}$ see Chapter 2.

To simplify notation in the following we will write χ, χ', χ'' for $\chi_{\mathcal{N}_0}$, $\chi_{\mathcal{N}'_0}$ and $\chi_{\mathcal{N}''_0}$ respectively. Also, in diagrams we will write $\oplus\mathcal{O}(-n)^{a(n)}$ for $\bigoplus_{n\in\mathbf{Z}} \mathcal{O}(-n)^{a(n)}$.

Lemma 4.3. *Let M be a split S-module, then we have:*

(1)
$$\zeta_n = \begin{cases} \zeta'_n, & \text{if } n < \inf L''_2 \\ rk\mathcal{N}'_0 + \zeta''_n, & \text{if } n \geq \inf L''_2. \end{cases}$$

(2) $T_{\mathcal{N}_0} = T_{\mathcal{N}'_0}$, $R_{\mathcal{N}_0} = R_{\mathcal{N}'_0}$ *and* $R_{\mathcal{N}'_0} \leq T_{\mathcal{N}'_0} < R_{\mathcal{N}''_0} \leq T_{\mathcal{N}''_0}$.

Proof. We have the minimal surjection $f' \oplus f'' : \mathcal{L}'_2 \oplus \mathcal{L}''_2 \to \mathcal{N}'_0 \oplus \mathcal{N}''_0$ which defines $\widetilde{\sigma'_2} \oplus \widetilde{\sigma''_2}$ by composition with the natural inclusion of $\mathcal{N}'_0 \oplus \mathcal{N}''_0$ into $\mathcal{L}'_1 \oplus \mathcal{L}''_1$. Thus the invariants α_n, β_n, γ_n associated with $(f' \oplus f'')_{\leq n}$ can be computed from the matrix $\Sigma_n = \Sigma'_n \oplus \Sigma''_n$ of $(\widetilde{\sigma'_2})_{\leq n} \oplus (\widetilde{\sigma''_2})_{\leq n}$ (see [MD-P2, V, 1, b]). That is, α_n is the rank of Σ_n, β_n is the maximal rank of minors of Σ_n which do not vanish on a same surface and γ_n is the maximal rank of minors of Σ_n which do not vanish on a same curve (and similarly for the ζ'_n and ζ''_n). For $n < \inf L''_2$ we have $(\widetilde{\sigma''_2})_{\leq n} = 0$, then

$\alpha''_n = \beta''_n = \gamma''_n = 0$ and $\zeta_n = \zeta'_n$. For $n \geq \inf L''_2$, we have $f'_{\leq n} = f'$ and $\Sigma'_n = \Sigma'$, then $\alpha'_n = \beta'_n = \gamma'_n = rk\mathcal{N}'_0$ (see Lemma 2.7, (2)). The minors of the matrix Σ_n are products of minors of Σ'_n and of Σ''_n, then clearly $\alpha_n = \alpha'_n + \alpha''_n = rk\mathcal{N}'_0 + \alpha''_n$. For β_n, consider the closed sets V' defined by the β'_n-minors of Σ'_n and V'' defined by the β''_n-minors of Σ''_n. Then $V' \cup V''$ has dimension at most 1 and contains the closed set V defined by the $\beta'_n + \beta''_n$-minors of Σ_n. Thus $\beta_n \geq \beta'_n + \beta''_n$. Consider the $\beta'_n + \beta''_n + 1$-minors of Σ_n. They are product of a t'-minor of Σ'_n and of a t''-minor of Σ''_n with $t' + t'' = \beta'_n + \beta''_n + 1$. If $t' > \beta'_n = rk\Sigma'_n$ there are no non-zero t'-minors of Σ'. If $t' \leq \beta'_n$, then $t'' \geq \beta''_n + 1$ and the t''-minors of Σ''_n all vanish on some surface, hence the same happens for the $t' + t''$-minors of Σ_n. We conclude that $\beta_n = \beta'_n + \beta''_n = rk\mathcal{N}'_0 + \beta''_n$. A similar argument yields $\gamma_n = \gamma'_n + \gamma''_n = rk\mathcal{N}'_0 + \gamma''_n$. This proves (1).

By Rmk 2.5 and using part (1) of this lemma we have:

$$R_{\mathcal{N}_0} = \sup\{n | l_2'^{\#}(n) + l_2''^{\#}(n) = \alpha'_n + \alpha''_n = \beta'_n + \beta''_n = \gamma'_n + \gamma''_n\}.$$

Now, since the invariants α_n, β_n, γ_n do not decrease with n (Prop. 2.4), we have $R_{\mathcal{N}_0} = \inf\{R_{\mathcal{N}'_0}, R_{\mathcal{N}''_0}\}$. The same argument shows that $T_{\mathcal{N}_0} = \inf\{T_{\mathcal{N}'_0}, T_{\mathcal{N}''_0}\}$. On the other hand $T_{\mathcal{N}'_0} < \sup L'_2$ because $\mathcal{N}'_0$ is not a free sheaf, then we get $R_{\mathcal{N}'_0} \leq T_{\mathcal{N}'_0} < \sup L'_2 \leq \inf L''_2 - 1 \leq R_{\mathcal{N}''_0} \leq T_{\mathcal{N}''_0}$ which completes the proof.

Corollary 4.4. *If M is split the function χ $(= \chi_{\mathcal{N}_0})$ is given by:*

$$\chi(n) = \begin{cases} \chi'(n), & \text{if } n < \inf L''_2 \\ h(n), & \text{if } n \geq \inf L''_2 \end{cases}$$

with

$$h(n) = \begin{cases} 0, & \text{if } n < \inf L''_2 \\ \inf(\alpha''_n, \beta''_n + 1, \gamma''_n + 2), & \text{if } n = \inf L''_2 \\ \inf(\alpha''_n - 2, \beta''_n - 1, \gamma''_n) - \inf(\alpha''_{n-1} - 2, \beta''_{n-1} - 1, \gamma''_{n-1}), & \text{if } n > \inf L''_2. \end{cases}$$

Proof. The definition of function $\chi^{\#}$ (Def. 2.6) and the previous lemma give:

$$\chi^{\#}(n) = \begin{cases} \gamma'_n, & \text{if } n \leq T_{\mathcal{N}_0} \\ \inf(\alpha'_n - 2, \beta'_n - 1, \gamma'_n), & \text{if } T_{\mathcal{N}_0} < n < \inf L''_2 \\ rk\mathcal{N}'_0 + \inf(\alpha''_n - 2, \beta''_n - 1, \gamma''_n), & \text{if } n \geq \inf L''_2. \end{cases}$$

For $n \geq \sup L_2'$, $\chi'^{\#}(n) = rk\mathcal{N}_0' - 2$, hence we have:

$$\chi^{\#}(n) = \begin{cases} \chi'^{\#}(n), & \text{if } n < \inf L_2'' \\ \chi'^{\#}(n) + 2 + \inf(\alpha_n'' - 2, \beta_n'' - 1, \gamma_n''), & \text{if } n \geq \inf L_2''. \end{cases}$$

Finally, using the formula $\chi(n) = \chi^{\#}(n) - \chi^{\#}(n-1)$, we obtain:

$$\chi(n) = \begin{cases} \chi'(n), & \text{if } n < \inf L_2'' \\ \inf(\alpha_n'', \beta_n'' + 1, \gamma_n'' + 2), & \text{if } n = \inf L_2'' \\ \inf(\alpha_n'' - 2, \beta_n'' - 1, \gamma_n'') - \inf(\alpha_{n-1}'' - 2, \beta_{n-1}'' - 1, \gamma_{n-1}''), & \text{if } n > \inf L_2''. \end{cases}$$

Proposition 4.5. *If M is split, given a general morphism $\phi : \bigoplus_{n\in\mathbf{Z}} \mathcal{O}(-n)^{\chi(n)} \to \mathcal{N}_0 = \mathcal{N}_0' \oplus \mathcal{N}_0''$ there is an exact commutative diagram:*

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \oplus\mathcal{O}(-n)^{\chi'(n)} \arrow[r] \arrow[d, "\phi'"] & \oplus\mathcal{O}(-n)^{\chi(n)} \arrow[r] \arrow[d, "\phi"] & \oplus\mathcal{O}(-n)^{h(n)} \arrow[r] \arrow[d, "\psi"] & 0 \\ 0 \arrow[r] & \mathcal{N}_0' \arrow[r] \arrow[d] & \mathcal{N}_0' \oplus \mathcal{N}_0'' \arrow[r] \arrow[d] & \mathcal{N}_0'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

where the first two rows split, $\mathcal{F}$ and $\mathcal{F}'$ are minimal elements in $Refl(\mathcal{N}_0)$ and $Refl(\mathcal{N}_0')$ respectively and $\mathcal{G}$ is a torsion sheaf supported on a surface.

Proof. Cor. 4.4 implies $\bigoplus_{n\in\mathbf{Z}} \mathcal{O}(-n)^{\chi(n)} = \bigoplus_{n\in\mathbf{Z}} \mathcal{O}(-n)^{\chi'(n)} \oplus \bigoplus_{n\in\mathbf{Z}} \mathcal{O}(-n)^{h(n)}$, hence we have:

$$\begin{tikzcd} 0 \arrow[r] & \oplus\mathcal{O}(-n)^{\chi'(n)} \arrow[r, "j"'] & \oplus\mathcal{O}(-n)^{\chi(n)} \arrow[r] \arrow[d, "\phi"] & \oplus\mathcal{O}(-n)^{h(n)} \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{N}_0' \arrow[r] & \mathcal{N}_0' \oplus \mathcal{N}_0'' \arrow[r] & \mathcal{N}_0'' \arrow[r] & 0 \end{tikzcd}$$

where the two rows split. Since $\chi'(n) = 0$ for $n > \sup L_2' > \inf L_2''$ and $\mathcal{N}_0''$ has no sections of degree less than $\inf L_2''$, the morphism $\phi \circ j$ factors through $\mathcal{N}_0'$ and we get $\phi' : \bigoplus_{n\in\mathbf{Z}} \mathcal{O}(-n)^{\chi'(n)} \to \mathcal{N}_0'$ which makes a commutative square on the left. Let $\mathcal{F}$

and $\mathcal{F}'$ be the quotients of ϕ and ϕ' respectively. It is clear that they are minimal elements in $Refl(\mathcal{N}_0)$ and $Refl(\mathcal{N}_0')$ respectively. Let $\psi : \bigoplus_{n\in\mathbf{Z}} \mathcal{O}(-n)^{h(n)} \rightarrow \mathcal{N}_0''$ be the morphism induced by ϕ. Again from Corollary 4.4 we obtain:

$$h^{\#}(n) = \begin{cases} 0, & \text{if } n < \inf L_2'' \\ \inf(\alpha_n'', \beta_n'' + 1, \gamma_n'' + 2), & \text{if } n \geq \inf L_2'' \end{cases}$$

thus $h^{\#}(n) \leq \alpha_n''$ for any n and Lemma 4.6 below guarantees that ψ is injective. Let $\mathcal{G}$ be its quotient. We have $rk\mathcal{G} = rk\mathcal{N}_0'' - \sum_{n\in\mathbf{Z}} h(n) = rk\mathcal{N}_0'' - \sum_{n\in\mathbf{Z}} \chi(n) + \sum_{n\in\mathbf{Z}} \chi'(n) = rk\mathcal{N}_0'' - rk\mathcal{N}_0' - rk\mathcal{N}_0'' + 2 + rk\mathcal{N}_0' - 2 = 0$. That is, $\mathcal{G}$ is a torsion sheaf and its support is the surface defined by $det(\psi)$. We complete the diagram by the snake lemma.

Lemma 4.6. *Let $\mathcal{E}$ be a reflexive sheaf and $\mathcal{P} = \bigoplus_{n\in\mathbf{Z}} \mathcal{O}(-n)^{p(n)}$ a free sheaf such that $p^{\#}(n) \leq \alpha_n(\mathcal{E})$ for any n. Then a general morphism $\mathcal{P} \rightarrow \mathcal{E}$ is injective.*

Proof. We go by induction on $rk\mathcal{P}$. If $rk\mathcal{P} = 1$, i.e. $\mathcal{P} = \mathcal{O}(-a)$ for some $a \in \mathbf{Z}$, the hypothesis says that $\alpha_n(\mathcal{E}) \geq 1$ for $n \geq a$. In particular, $\mathcal{E}$ has non-zero sections of degree a, which proves the assertion. Suppose now $rk\mathcal{P} > 1$. Let $a := \sup\{n | p(n) \neq 0\}$, and write $\mathcal{P} = \mathcal{P}' \oplus \mathcal{O}(-a)$. Thus $rk\mathcal{P}' = rk\mathcal{P} - 1$ and by the induction hypothesis a general morphism $u : \mathcal{P}' \rightarrow \mathcal{E}$ is injective. Let $\mathcal{F}$ be the quotient of u:

$$0 \rightarrow \mathcal{P}' \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0.$$

By [MD-P2, II, Cor. 6.5] we have $\alpha_a(\mathcal{F}) = \alpha_a(\mathcal{E}) - rk\mathcal{P}'$, then $\alpha_a(\mathcal{F}) \geq p^{\#}(a) - rk\mathcal{P}' = rk\mathcal{P} - rk\mathcal{P} + 1 = 1$. That is, a general section of degree a of $\mathcal{F}$ is non-zero, which concludes the proof.

Now we come to the non-splitness result for vector bundles.

Theorem 4.7. *Let $\mathcal{F}$ be a rank two vector bundle on $\mathbf{P}^3$. Then the intermediate cohomology modules of $\mathcal{F}$ are not split.*

Proof. By Serre duality $H^2_*\mathcal{F} \cong H^1_*\mathcal{F}(-c_1\mathcal{F} - 4)$, hence we only need to show that $H^1_*\mathcal{F}$ is non-split. Suppose there exists a rank two vector bundle $\mathcal{F}$ with $H^1_*\mathcal{F}$ split and let $Refl(\mathcal{N}_0' \oplus \mathcal{N}_0'')$ be its biliaison class. Since $\mathcal{F}$ is a minimal element in this class (Cor. 3.10) we can assume that there is a diagram as in Prop. 4.5. Then we can also write the following exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \oplus\mathcal{O}(-n)^{\chi'(n)} & \longrightarrow & \oplus\mathcal{O}(-n)^{\chi(n)} & \longrightarrow & \oplus\mathcal{O}(-n)^{h(n)} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\phi} & & \downarrow & & \\
0 & \longrightarrow & \oplus\mathcal{O}(-n)^{\chi'(n)} & \longrightarrow & \mathcal{N}_0'\oplus\mathcal{N}_0'' & \longrightarrow & \mathcal{F}'\oplus\mathcal{N}_0'' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{F} & = & \mathcal{F} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The last column shows that $\mathcal{F}'$ must be a vector bundle as well, then by Cor. 3.13 we have: $\bigoplus_{n\in\mathbf{Z}}\mathcal{O}(-n)^{\chi(n)}\cong\mathcal{L}'^{*}_{0}(c_1\mathcal{F})\oplus\mathcal{L}''^{*}_{0}(c_1\mathcal{F})$ and $\bigoplus_{n\in\mathbf{Z}}\mathcal{O}(-n)^{\chi'(n)}\cong\mathcal{L}'^{*}_{0}(c_1\mathcal{F}')$ (here $\mathcal{L}_0'$, $\mathcal{L}_0''$ are the free sheaves associated with L_0' and L_0'' respectively and * denotes the dual sheaf). On the other hand, from the diagram we get: $c_1\mathcal{F} = c_1\mathcal{F}' + c_1\mathcal{N}_0'' + \sum_{n\in\mathbf{Z}} nh(n)$ where $c_1\mathcal{N}_0'' + \sum_{n\in\mathbf{Z}} nh(n)$ is the degree of the surface which supports $\mathcal{G}$ (see Prop. 4.5). Since $\mathcal{G}$ cannot be the zero-sheaf, this degree is a positive integer, hence $c_1\mathcal{F} > c_1\mathcal{F}'$. This yields a contradiction because $\mathcal{L}'^{*}_{0}(c_1\mathcal{F}')$ is a direct summand of $\mathcal{L}'^{*}_{0}(c_1\mathcal{F})\oplus\mathcal{L}''^{*}_{0}(c_1\mathcal{F})$ but $\sup L'^{*}_{0}(c_1\mathcal{F}') > \sup L'^{*}_{0}(c_1\mathcal{F}) = \sup(L'^{*}_{0}(c_1\mathcal{F})\oplus L''^{*}_{0}(c_1\mathcal{F}))$.

In particular, Thm. 4.7 implies that the H^1_* and H^2_* modules of a rank two vector bundle are connected. As an immediate corollary we get a vanishing criterion, namely:

Corollary 4.8. *Let $\mathcal{F}$ be a rank two vector bundle on $\mathbf{P}^3$ and suppose $\mathcal{F}$ is not a free sheaf. If $H^i\mathcal{F}(t) = 0$ for some $t > min\{n|H^i\mathcal{F}(n)\neq 0\}$, with $1\leq i\leq 2$, then $H^i\mathcal{F}(k) = 0$ for any $k > t$.*

Another direct consequence of Thm. 4.7 concerns subcanonical curves. We recall that the Rao module of a curve C in $\mathbf{P}^3$ is defined by $M(C) := H^1_*\mathcal{I}_C$, where $\mathcal{I}_C$ is the ideal sheaf of the curve. By curve we mean a locally Cohen-Macaulay closed subscheme of pure dimension one. A curve is subcanonical if its dualizing sheaf ω_C is isomorphic to a twist of the structural sheaf $\mathcal{O}_C$, i.e. if there exists $a_C\in\mathbf{Z}$ such that $\omega_C\cong\mathcal{O}_C(a_C)$.

Corollary 4.9. *The Rao module of a subcanonical curve in* $\mathbf{P}^3$ *is non-split.*

Proof. This is just the translation of Prop. 2.1 via the Serre correspondence (Thm. 0.10). Indeed, to any subcanonical curve C we can associate a rank two vector bundle $\mathcal{E}$ on $\mathbf{P}^3$ in such a way that there is an exact sequence

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{I}_C(c_1\mathcal{E}) \to 0.$$

It follows that $H^1_*\mathcal{I}_C$ is isomorphic to $H^1_*\mathcal{E}$ up to shift.

Clearly the analogue of Cor. 4.8 for curves holds, namely:

Corollary 4.10. *Let C be a subcanonical curve in $\mathbf{P}^3$ which is not arithmetically Cohen-Macaulay. Suppose $H^1\mathcal{I}_C(t) = 0$ for some $t > min\{n|H^1\mathcal{I}_C(n) \neq 0\}$, then $H^1\mathcal{I}_C(k) = 0$ for any $k > t$.*

Chapter 5

Minimal sheaves *versus* minimal curves

Any finite length graded S-module M determines a biliaison class of locally Cohen-Macaulay, equidimensional curves in $\mathbf{P}^3$ (Thm. 0.4) — which we denote $Curv(\mathcal{N}_0)$ — and a biliaison class of rank two reflexive sheaves on $\mathbf{P}^3$ (Thm. 1.10) — which is denoted $Refl(\mathcal{N}_0)$ (as usual, $\mathcal{N}_0$ is the sheaf associated with the second syzygies module of M). By the Hartshorne-Serre correspondence (Thm. 0.10) if to a sheaf $\mathcal{F} \in Refl(\mathcal{N}_0)$ we can associate a curve C, then $C \in Curv(\mathcal{N}_0)$ and viceversa, up to twist of the sheaf. Since the correspondence between sheaves and curves in the two classes is not bijective, some natural questions arise. For example, is there any relationship between minimal sheaves and minimal curves?

If we perform an elementary biliaison on a sheaf $\mathcal{F}$ is this equivalent in some sense to a basic double linkage on a corresponding curve?

The first remark to do here is that we have to take into account shifts of the module M. Indeed in a class $Refl(\mathcal{N}_0)$ we do not allow any shift while in a class of curves there is a leftmost possible shift of the module M, which is reached by minimal curves, and any other shift to the right is possible. This means that we may need to twist our sheaves, i.e. to "shift" the whole class $Refl(\mathcal{N}_0)$.

For what concerns minimal elements it seems then reasonable to ask a more precise question:

Let $\mathcal{F} \in Refl(\mathcal{N}_0)$ be a minimal sheaf and t_0 its minimal twist which has non-zero sections (Def. 3.14). Let C be a curve defined as the zero locus of a section

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$s \in H^0\mathcal{F}(t_0)$. Is C minimal in its biliaison class ?

We are going to show that in general the answer is negative.

By [MD-P1] we know that minimal curves in $Curv(\mathcal{N}_0)$ are characterized by having an $\mathcal{N}$-type resolution of the form

$$0 \to \bigoplus_{n \in \mathbf{Z}} \mathcal{O}(-n)^{q_{\mathcal{N}_0}(n)} \to \mathcal{N}_0 \to \mathcal{I}_C(h) \to 0$$

where $q_{\mathcal{N}_0}$ is the function defined by:

$$q^{\#}_{\mathcal{N}_0}(n) = \begin{cases} \alpha_n(\mathcal{N}_0), & \text{for } n \leq T_{\mathcal{N}_0} \\ \inf(\alpha_n(\mathcal{N}_0) - 1, \beta_n(\mathcal{N}_0)), & \text{for } n > T_{\mathcal{N}_0} \end{cases}$$

(for details we refer to [MD-P1].

Recall that if $\mathcal{F}$ is a minimal element in $Refl(\mathcal{N}_0)$, then its minimal twist which has non-zero sections is equal to $R_{\mathcal{N}_0} + 1$ (Lemma 3.16).

Proposition 5.1. *Let $\mathcal{F}$ be a minimal element in $Refl(\mathcal{N}_0)$ and let C be a curve which is the zero locus of a section $s \in H^0\mathcal{F}(R_{\mathcal{N}_0} + 1)$. Then C is minimal in its biliaison class if and only if:*

$$\chi_{\mathcal{N}_0}(n) = \begin{cases} q_{\mathcal{N}_0}(n), & \text{for } n \neq R_{\mathcal{N}_0} + 1 \\ q_{\mathcal{N}_0}(n) - 1, & \text{for } n = R_{\mathcal{N}_0} + 1. \end{cases}$$

Proof. The curve C is given by an exact sequence $0 \to \mathcal{O}(-R_{\mathcal{N}_0} - 1) \to \mathcal{F} \to \mathcal{I}_C(c_1\mathcal{F} + R_{\mathcal{N}_0} + 1) \to 0$. Combining with the $\mathcal{N}_0$-resolution of $\mathcal{F}$ we get:

$$0 \to \bigoplus_{n \in \mathbf{Z}} \mathcal{O}(-n)^{\chi_{\mathcal{N}_0}(n)} \oplus \mathcal{O}(-R_{\mathcal{N}_0} - 1) \to \mathcal{N}_0 \to \mathcal{I}_C(c_1\mathcal{F} + R_{\mathcal{N}_0} + 1) \to 0.$$

Then, by [MD-P1, IV, 4.1], C is minimal if and only if the free sheaf on the left is isomorphic to $\bigoplus_{n \in \mathbf{Z}} \mathcal{O}(-n)^{q_{\mathcal{N}_0}(n)}$.

This condition is quite restrictive and it is not hard to find an example where it is not satisfied. We first prove a lemma:

Lemma 5.2. *Let M be a finite length graded S-module with minimal graded free resolution $\cdots L_2 \xrightarrow{\sigma_2} L_1 \xrightarrow{\sigma_1} L_0 \to M \to 0$. Let $\mathcal{N}_0$ be the sheafified kernel of σ_1 and let $2\mathcal{N}_0 = \mathcal{N}_0 \oplus \mathcal{N}_0$. Then we have:*

(1) $\alpha_n(2\mathcal{N}_0) = 2\alpha_n(\mathcal{N}_0)$;

(2) $\beta_n(2\mathcal{N}_0) = 2\beta_n(\mathcal{N}_0)$;

(3) $\gamma_n(2\mathcal{N}_0) = 2\gamma_n(\mathcal{N}_0)$;

(4) $R_{2\mathcal{N}_0} = R_{\mathcal{N}_0}$ and $T_{2\mathcal{N}_0} = T_{\mathcal{N}_0}$.

Proof. Let $\sigma_2' : L_2 \to H^0_*\mathcal{N}_0 \to 0$ be the minimal surjection induced by σ_2. Then $\sigma_2' \oplus \sigma_2' : L_2 \oplus L_2 \to 2H^0_*\mathcal{N}_0 \to 0$ is also minimal and we can compute the invariants α_n, β_n and γ_n associated with $2\mathcal{N}_0$ from the matrix $\Sigma_n \oplus \Sigma_n$ of the homomorphism $\sigma_{2\leq n} \oplus \sigma_{2\leq n} : L_{2\leq n} \oplus L_{2\leq n} \to L_1 \oplus L_1$ (see [MD-P2, V, 1, b]). Then α_n is simply the rank of the matrix, β_n is the maximal rank of minors which do not vanish on a same surface and γ_n is the maximal rank of minors which do not vanish on a same curve. Clearly we have $\alpha_n(2\mathcal{N}_0) = 2\alpha_n(\mathcal{N}_0)$. Since a p-minor of $\Sigma_n \oplus \Sigma_n$ is the product of a t-minor and an r-minor of Σ_n with $t + r = p$, it is not hard to see that $\beta_n(2\mathcal{N}_0) \geq 2\beta_n(\mathcal{N}_0)$ and $\gamma_n(2\mathcal{N}_0) \geq 2\gamma_n(\mathcal{N}_0)$ (see also [MD-P1, IV, 6.17] or Lemma 4.3). On the other hand to obtain a $2\beta_n(\mathcal{N}_0) + 1$-minor we need either $t \geq \beta_n(\mathcal{N}_0) + 1$ or $r \geq \beta_n(\mathcal{N}_0) + 1$ hence they all have a common factor and we conclude that $\beta_n(2\mathcal{N}_0) = 2\beta_n(\mathcal{N}_0)$. A similar argument shows that $\gamma_n(2\mathcal{N}_0) = 2\gamma_n(\mathcal{N}_0)$. Finally, by Remark 2.5 it is easy to get $R_{2\mathcal{N}_0} = R_{\mathcal{N}_0}$ and $T_{2\mathcal{N}_0} = T_{\mathcal{N}_0}$.

Example 5.3. Consider a so called Koszul module $M = S/(f_1, f_2, f_3, f_4)$ where f_1, f_2, f_3, f_4 is a regular sequence of homogeneous elements of degrees $n_i = deg(f_i)$ with $1 \leq n_1 \leq n_2 \leq n_3 \leq n_4$. Suppose moreover $n_2 < n_3$. The beginning of a minimal free resolution of M is:

$$\cdots \bigoplus_{i<j} S(-n_i - n_j) \xrightarrow{\sigma} \bigoplus_{i=1}^{4} S(-n_i) \to S \to S/(f_1, f_2, f_3, f_4) \to 0.$$

As usual, let $N_0 = Im(\sigma)$ and $\mathcal{N}_0$ be the associated sheaf. Let us consider the sheaf $\mathcal{E} = 2\mathcal{N}_0$. The invariants for $\mathcal{N}_0$ have been computed by Martin-Deschamps and Perrin in [MD-P2,V, 2, a], in particular, we have $R_{\mathcal{N}_0} = n_1 + n_2 - 1$ and $T_{\mathcal{N}_0} = n_1 + n_3 - 1$. Then by Lemma 5.2 we obtain the invariants for $\mathcal{E}$ and we are able to compute the functions $\chi_{\mathcal{E}}$ and $q_{\mathcal{E}}$. Comparing them we find out that:

$$\chi_{\mathcal{E}}(n) = \begin{cases} q_{\mathcal{E}}(n), & \text{for } n \neq R_{\mathcal{N}_0} + 1 \text{ and } n \neq T_{\mathcal{N}_0} + 1 \\ q_{\mathcal{E}}(n) - 2, & \text{for } n = R_{\mathcal{N}_0} + 1 \\ q_{\mathcal{E}}(n) + 1, & \text{for } n = T_{\mathcal{N}_0} + 1. \end{cases}$$

If $\mathcal{F}$ is a minimal element in $Refl(\mathcal{E})$ a section of $\mathcal{F}(R_{\mathcal{N}_0} + 1)$ gives a curve C whose Rao module is shifted $n_3 - n_2$ places to the right with respect to the leftmost shift.

The situation seems more delicate when $Refl(\mathcal{N}_0)$ contains vector bundles. Since vector bundles are automatically minimal elements in their classes, the problem can be formulated as follows:

Problem A: Let $\mathcal{E}$ be a rank two vector bundle on $\mathbf{P}^3$ and let t_0 be its minimal twist which has non-zero sections. If C is a curve defined by $s \in H^0\mathcal{E}(t_0)$, then is C minimal in its biliaison class?

This is equivalent to the following:

Problem B: If a biliaison class contains subcanonical curves, then are the minimal curves in the class subcanonical?

Let us show the equivalence. Suppose that the answer to Problem A is positive. Let X be a subcanonical curve. Then we can associate with X a vector bundle $\mathcal{E}$. Let Y be a curve obtained from a section of $\mathcal{E}(t_0)$, then Y belong to the biliaison class of X, it is minimal and subcanonical. By Lemma 5.4 below it then follows that any minimal curve in the class is subcanonical. Viceversa, suppose that Problem B has positive answer. Let $\mathcal{E}$ be a rank two vector bundle and let X be a curve defined by $0 \to \mathcal{O} \to \mathcal{E}(t_0) \to \mathcal{I}_X(c_1\mathcal{E} + 2t_0) \to 0$. If X is minimal there is nothing to prove. Assume X is not minimal and let Y be a minimal curve in the class of X, then $H^1_*\mathcal{I}_Y \cong H^1_*\mathcal{I}_X(h)$ for some integer $h > 0$. By assumption Y is subcanonical. Let $\mathcal{F}$ be a vector bundle such that $0 \to \mathcal{O} \to \mathcal{F} \to \mathcal{I}_Y(c_1\mathcal{F}) \to 0$. It follows that $H^1_*\mathcal{F} \cong H^1_*\mathcal{E}(c_1\mathcal{F} - c_1\mathcal{E} + h - t_0)$. That is, $\mathcal{F}$ and $\mathcal{E}(c_1\mathcal{F} - c_1\mathcal{E} + h - t_0)$ are in the same biliaison class. In particular, since they are both minimal elements, they have the same cohomology and Chern classes. Hence $h^0\mathcal{E}(c_1\mathcal{F} - c_1\mathcal{E} + h - t_0) = h^0\mathcal{F} > 0$ and $c_1\mathcal{F} = c_1\mathcal{E} + 2(c_1\mathcal{F} - c_1\mathcal{E} + h - t_0)$. Since $h > 0$, this yields $c_1\mathcal{F} - c_1\mathcal{E} + h - t_0 < t_0$, which contradicts minimality of t_0.

Lemma 5.4. *If in a biliaison class there is a minimal curve which is subcanonical, then any other minimal curve in the class is subcanonical of the same level.*

Proof. Suppose X is a minimal curve. Let $\cdots L_2 \xrightarrow{\sigma_2} L_1 \xrightarrow{\sigma_1} L_0 \to M_X \to 0$ be a minimal free resolution of its Rao module and let as usual $N_0 = ker(L_1 \to L_0)$. Set also $A_X = H^0_*\mathcal{O}_X$. The construction of the N-type resolution for the saturated ideal I_X (see [LR] or [MD-P1, II, 4.1]) gives an exact commutative diagram of

S-modules:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[d] & & S \arrow[r] \arrow[d] & S/I_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_2 \arrow[r] \arrow[d] & L_1 \oplus L \arrow[r] \arrow[d, equal] & S \oplus L_0 \arrow[r] \arrow[d] & A_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N_0 \oplus L \arrow[r] \arrow[d] & L_1 \oplus L \arrow[r] & L_0 \arrow[r] \arrow[d] & M_X \arrow[r] \arrow[d] & 0 \\
 & I_X \arrow[d] & & 0 & 0 & \\
 & 0 & & & &
\end{tikzcd}$$

where $0 \to F_2 \to L_1 \oplus L \to S \oplus L_0 \to A_X \to 0$ is a minimal free resolution of A_X and $0 \to F_2 \to N_0 \oplus L \to I_X \to 0$ is by definition the N-type resolution of I_X. Since X is a minimal curve, by [MD-P1, IV, 4.4] we have $L = 0$ and $F_2 \cong \bigoplus_{n \in \mathbf{Z}} S(-n)^{q_{\mathcal{N}_0}(n)}$. It follows that the minimal free resolution of A_X, for any minimal curve X with Rao module M_X, has the form:

$$0 \to \bigoplus_{n \in \mathbf{Z}} S(-n)^{q_{\mathcal{N}_0}(n)} \to L_1 \to S \oplus L_0 \to A_X \to 0.$$

Now, by [Se, 2.5], X is subcanonical of level a if and only if there are isomorphisms:

$$L_1 \cong L_1^*(-4-a)$$
$$\bigoplus_{n \in \mathbf{Z}} S(-n)^{q_{\mathcal{N}_0}(n)} \cong S(-4-a) \oplus L_0^*(-4-a).$$

Since this only depends on M_X, we are done.

As far as we know Problems A and B are still unsolved.

References

[B1] A. Buraggina, *On the connectedness of the Hartshorne-Rao module of curves in* $\mathbf{P}^3$, Arch. Math. **60** (1993), 96–102.

[B2] A. Buraggina, *On the connectedness of the first cohomology module for rank two vector bundles on projective space*, Ann. Univ. Ferrara - Sez. VII - Sc. Mat. **XXXVIII** (1992), 249–262.

[B3] A. Buraggina, *Biliaison classes of rank two reflexive sheaves on* $\mathbf{P}^3$, preprint S.I.S.S.A. Ref. 62/95/M (May 1995).

[B4] A. Buraggina, *Connectedness of cohomology modules of rank two vector bundles on* $\mathbf{P}^3$, preprint S.I.S.S.A. Ref. 122/95/M (October 1995).

[BBM] E. Ballico, G. Bolondi and J. Migliore, *The Lazarsfeld-Rao problem for liaison classes of two codimensional subschemes of* $\mathbf{P}^n$, Amer. J. of Math. **113** (1991), 117–128.

[BM] G. Bolondi and J. Migliore, *Classification of Maximal Rank Curves in the Liaison class* L_n, Math. Ann. **277** (1987), 585–603.

[BM1] G. Bolondi and J. Migliore, *The structure of an even liaison class*, Trans. AMS **316** (1989), 1–37.

[BV] W. Bruns and U. Vetter, *Determinantal Rings*, LNM **1327** (1988).

[C] M. Chang, *Characterization of arithmetically Buchsbaum subschemes of codimension 2 in* $\mathbf{P}^n$, J. Diff. Geom. **31** (1990), 323–341.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

[Ch] N. Chiarli, *Completeness and non-speciality of linear series on space curves and deficiency of the corresponding linear series on normalization*, in The Curves Seminar at Queen's , Queen's Papers in Pure and Applied Mathematics **III** (1984).

[Da1] E. D. Davis, *Some applications of Castelnuovo theory via residuation*, Proceedings Conference on Zero-dimensional schemes, Ravello June 1992, Publisher W. De Gruyter (1993).

[Da2] E. D. Davis, *Gaps in the Rao module of an algebraic space curve and the property "numerically subcanonical"* in The Curves Seminar at Queen's **IX**.

[D] W. Decker, *Monads and cohomology module of rank two vector bundles*, Compositio Math. **76** (1990), 7–17.

[GLM] S. Guarrera, A. Logar and E. Mezzetti, *Algorithms for the computation of minimal curves*, in preparation

[H] R. Hartshorne, *Stable reflexive sheaves*, Math. Ann. **254** (1980), 121–176.

[H1] R. Hartshorne, *Generalized Divisors on Gorenstein Schemes*, K-theory **8** (1994), 287–339.

[Ho] G. Horrocks, *Vector bundles on the punctured spectrum of a local ring*, Proc. Lond. Math. Soc. (3)**14** (1964), 689–713.

[LR] R. Lazarsfeld and A. P. Rao, *Linkage of general curves of large degree*, Algebraic Geometry – Open Problems (Ravello 1982) LNM Springer-Verlag Berlin **997** (1983), 267–289.

[M] J. C. Migliore, *An Introduction to Deficiency Modules and Liaison Theory for Subschemes of Projective Space*, Global Analysis Research Center, Seoul National University, Lecture Notes Series **24** (1994).

[M1] J. C. Migliore, *Liaison of a union of skew lines in* $\mathbf{P}^4$, Pacific J. of Math. **130** (1987), 153–170.

[MD-P1] M. Martin-Deschamps - D. Perrin, *Sur la classification des courbes gauches*, Astérisque, Soc. Math. de France **184-185** (1990).

[MD-P2] M. Martin-Deschamps - D. Perrin, *Construction de courbes lisses : un Théorème à la Bertini*, Rapp. de Recherche du LMENS 92-22, Ecole Normale Supérieure, 45, rue d'Ulm, Paris (1992).

[MD-P3] M. Martin-Deschamps - D. Perrin, *Sur les courbes gauches à modules de Rao non connexe*, C. R. Acad. Sci. Paris, Série I **319** (1994), 233–236.

[MD-P4] M. Martin-Deschamps - D. Perrin, *Courbes gauches et module de Rao*, J. Reine Angew. Math. **439** (1993), 103–145.

[Mu] D. Mumford, *Lectures on curves on an algebraic surface*, Ann. of Math. Studies **59** (1966).

[N] S. Nollet, *Even Linkage classes*, preprint, December 1994.

[PS] C. Peskine and L. Szpiro, *Liaison des variétés algébriques. I*, Inventiones Math. **26** (1974), 271–302.

[R1] A. P. Rao, *Liaison among curves in* $\mathbf{P}^3$, Inventiones math. **50** (1979), 205–217.

[R2] A. P. Rao, *Liaison Equivalence Classes*, Math. Ann. **258** (1981), 169–173.

[R3] A. P. Rao, *A note on cohomology modules of rank two bundles*, Journal of Algebra **86** (1984), 23–34.

[Sch] P. Schenzel, *Notes on Liaison and Duality*, J. Math. Kyoto Univ. **22** (3) (1982), 485–498.

[Se] E. Sernesi, *L'unirazionalità della varietà dei moduli delle curve di genere dodici*, Annali Scuola Norm. Sup. **8**(3) (1981), 405–441.