ISAS - INTERNATIONAL SCHOOL
FOR ADVANCED STUDIES

Critical and singular dynamics in the Lorenz equations

Thesis submitted for the degree of
"Doctor Philosophiæ"

CANDIDATE
Stefano Luzzatto

SUPERVISOR
Prof. Jacob Palis

June 1995

SISSA - SCUOLA
INTERNAZIONALE
SUPERIORE
DI STUDI AVANZATI
TRIESTE
Strada Costiera 11

TRIESTE

Scuola Internazionale Superiore di Studi Avanzati

International School for Advanced Studies

Critical and singular dynamics in the Lorenz equations

Thesis submitted for the degree of

"Doctor Philosophiæ"

CANDIDATE

Stefano Luzzatto

SUPERVISOR

Prof. Jacob Palis

June 1995

Author addresses:

SISSA, Via Beirut 4, 34014 Trieste, Italy
Current address: Laboratoire de Probabilités, Univ. Paris VI, 4 Place Jussieu, Paris, France
E-mail address: luzzatto@sissa.it

Abstract. Dynamical systems with complicated orbit structures are best described by suitable invariant measures. Sinai, Ruelle and Bowen showed, in the 70's, that a special class of invariant measures (now called SBR measures) which provide substantial information on the dynamical and statistical properties, can be constructed for uniformly hyperbolic systems. The question arises as to what extent weaker hyperbolicity conditions still guarantee the existence of SBR measures.

We introduce a class of flows in $\mathbb{R}^3$, inspired by a system of differential equation proposed by Lorenz, in which the presence of a singularity and of criticalities constitute obstructions to uniform hyperbolicity. We prove that a weaker form of hyperbolicity exists and is present in a (measure-theoretically) persistent way in one-parameter families. It is expected that such non-uniform hyperbolicity implies the existence of an SBR measure.

Acknowledgments

The scientific guidance of my supervisor Jacob Palis has been a fundamental factor in leading me towards the right problems in the exciting field of non-uniformly hyperbolic dynamics. His mathematical insight, encouragement and enthusiasm have given me the ability, strength and self-confidence to carry out this project. Moreover he has made it possible for me to spend several periods of study at IMPA where I found an exceptionally stimulating, supportive and pressure-free environment in which to work. To him I express my most sincere appreciation and gratitude.

The calm. assured, insightful and thorough, approach to mathematics of my friend and collaborator Marcelo Viana has constituted for me a model and a constant learning experience. To him, for his patience, warmth and friendship I also express my sincere thanks.

Alberto Verjovsky and Graziano Gentili helped me in fundamental ways with their enthusiasm, good spirits, and unprejudiced trust in me. To them I owe a great deal. The same is true for Vinicio Villani who helped me when I returned to Italy after many years of absence.

The friendships which I made both at SISSA and at IMPA during these years have been a constant source of good times and affectionate support In particular I would like to thank Paolo Caldiroli, Piero Montecchiari, Wu He, Lorenzo Diáz and Raul Urés.

Many thanks also to IMPA-CNPq and to SISSA for their financial support and for providing first-rate working environments and allowing me to pursue my somewhat unnorthodox path to a Ph.D. The secreterial and administrative staff at both institutions have always been extremely helpfull and I wish to thank them for their all they have done.

Finally I would like to express my deepest thanks to my parents Paola and Lucio. From them I have learned what it takes. They have stood by me. encouraged me, helped me. loved me and always respected me in everything I have done. Without their example and loving support all this would never have been.

Last but not least I thank my partner Tonya, for being who she is. To her I dedicate this work.

In measure, at first, quality and quantity are immediate, and measure is only their 'relative' identity. But measure shows itself absorbed and superseded in the measureless: yet the measureless, although it be the negation of measure, is itself a unity of quantity and quality. Thus in the measureless the measure is still seen to meet only with itself.

G.W.F. Hegel, *Science of Logic, Part one of the Encyclopedia of the Philosophical Sciences,* § *110*

Life is a mystery
everyone must stand alone
when you call my name
it feels like home....

when you call my name
it's like a little prayer
I'm down on my knees
I want to take you there....

Madonna, *Like a prayer*

Contents

CHAPTER I

Introduction

1. Statement of the problem

An important aim in the theory of differential equations is to describe the solutions of a system of differential equations

$$\dot{x} = f(x, t) \tag{1}$$

defined on some Riemannian manifold M. For the sake of the following discussion we shall always assume that f satisfies sufficient regularity conditions to guarantee the existence of a unique solution through each point $x_0 \in M$ and, in general, we shall restrict ourselves to systems which are periodic or autonomous. For autonomous systems, the set of solutions of (1) can be naturally described by a one-parameter group of transformations (the *flow*),

$$\varphi^t : M \to M \tag{2}$$

where $t \in \mathbb{R}$ represents the time and $\varphi^t(x_0)$ gives the position of the point x_0 after time t under the evolution determined by (1). For time-periodic systems it can also be useful to define a diffeomorphism $\varphi : M \to M$ by $\varphi = \varphi^T$ (the *time-T map*, where T is the period of f). Then many dynamical properties of (2) can be recovered from properties of the discrete system $\{\varphi^t\}_{t \in \mathbb{Z}}$. A natural generalization of flows and diffeomorphisms is constituted by systems in which the time evolution is defined only in the future, i.e. semiflows and iterates of endomorphisms. These systems have the advantage, in some sense, that they can exhibit remarkably complex dynamical features in ambient spaces of lower dimension than is the case for invertible system. For example, one dimensional diffeomorphisms are essentially trivial whereas there exists a rich theory of one-dimensional endomorphisms (see [dMvS93] for an up to date and comprehensive account). The same is true for flows and semiflows, respectively, in two dimensions. A lot of recent results on the dynamics of two dimensional maps (including some of the results presented in this thesis) are based on the ideas and methods of one-dimensional dynamics, even though they clearly also require a non-trivial amount of additional techniques.

1.1. Invariant measures. Returning to the problem of describing the solutions to systems such as (1) or (2), this can be solved completely only in a very limited number of cases, e.g. if the system is integrable or if there exists a unique attracting equilibrium point or periodic orbit of relatively low period to which all trajectories converge. In general this is not the case and the dynamics is much more complicated, at least from a geometrical or analytical point of view. This means that it is very difficult, even in principle, to describe the solutions by such methods. A powerful alternative is to be found in the ideas and methods of ergodic and measure theory. To illustrate the basic style of this approach we consider a group of transformations as in (2) and suppose that some ergodic probability measure μ is preserved. This means that for any measurable set $A \subset M$ we have $\mu(A) = \mu(\varphi^t(A))$ for all $t \in \mathbb{R}$; ergodicity means that any invariant subset B satisfies $\mu(B) = 0$ or $\mu(B) = 1$. Then Birkhoff's ergodic theorem says that the asymptotic distribution of μ-almost every point is determined by the measure μ: given any measurable subset $A \subset M$ the average time which trajectories $\varphi^t(x)$ spend in A is proportional to $\mu(A)$. Moreover other dynamical and statistical features of the dynamics can often be described in terms of corresponding features of the measure μ. This constitutes, therefore, a good way, in some sense the only way, of describing the dynamics of the system.

Thus, our problem can be reformulated in terms of determining the existence and properties of invariant measures. We remark that there are, in general, an infinite number of invariant measures for a given system (e.g. every periodic orbit admits a Dirac measure concentrated on the orbit) each of which captures a different aspect of the dynamics. Ususally it is most desirable to have an invariant measure which is related to the Riemannian volume on M since we would like to describe the dynamics of a set of trajectories which is large in relation to the natural topology or natural measure on M. A Dirac measure on an unstable orbit or other measures which are singular with respect to Lebesgue (Riemannian volume) cannot, a priori, give any information on a positive Lebesgue measure set of points (although sometimes they do, see below).

Some systems (e.g. conservative systems) are known a priori to preserve Lebesgue measure. In this case most of the effort goes into determining the properties of Lebesgue measure with respect to the dynamical system, starting with ergodicity (just the fact that Lebesgue measure is preserved is not sufficient to gain any information since it might have even an uncountable number of ergodic components). This is already a difficult problem and there exists a vast literature (see e.g. [Sin94] and references therein).

For systems which do not preserve Lebesgue measure one would hope, in the first instance, that there exists an invariant measure absolutely continuous with respect to Lebesgue, for the reasons discussed above. In dissipative systems, however, the asymptotic dynamics is concentrated on attractors [Mil85] which contain the support of all invariant measures. By dissipativity these attractors have zero Lebesgue measure and thus all invariant measures are singular with respect to Lebesgue. In these cases it is sometimes possible to show that such an attractor Λ has a *stable foliation* $\mathcal{F}^s$ formed by leaves through points of Λ which reach out into the manifold M. All the points belonging to a given leaf have the same asymptotic distribution: their orbits converge to the orbit of the point $x \in \Lambda$ contained in the leaf, and so, if x is μ-generic then all the points in that leaf will also be μ-generic. Thus if most of the leaves of $\mathcal{F}^s$ correspond to μ-generic points of Λ and if one can show that $\mathcal{F}^s$ has positive Lebesgue meaure then one can conclude that, even though μ is singular with respect to Lebesgue, it is neverteless true that a positive Lebesgue measure set of points is generic with respect to μ.

1.2. Uniform hyperbolicity. Having argued for a measure-theoretic description of the dynamics of complex systems we observe that many of the arguments used in proving ergodicity of invariant measures or in constructing invariant measures for systems in which no a priori measure is preserved or, more generally, in determining the statistical properties of given invariant measures, are often topological and analytical in nature. In their barest essence they usually rely on certain conditions of topological transitivity (existence of dense orbits) and analytical hyperbolicity estimates (exponential expansion and contraction of vectors in the tangent bundle). Moreover a careful analysis of the geometry of the stable and unstable leaves (whose existence is guaranteed by the hyperbolicity estimates) is often required to bring everything together. The first person to have used this kind of approach was Hopf in 1938 who proved the ergodicity of the geodesic flow on surfaces of constant negative curvature. Later Anosov [Ano67] generlized this construction to manifolds of arbitrary dimension and made explicit some of the fundamental properties of the system which made the proof work, in particular that of *uniform hyperbolicity.*

Dissipative systems were studied from this point of view by Sinai, Ruelle and Bowen [Sin70] [Rue76] [Bow75][Bow78]. They showed that under *uniform hyperbolicity* assumptions (and topological transitivity) a measure can be constructed on the attractor which gives considerable dynamical information for almost all points in the basin of attraction (in particular a set of positive Lebesgue measure; see discussion above). These measures are now called Sinai-Bowen-Ruelle, or SBR, measures.

We give below the definition of a uniformly hyperbolic set and then, in the next section discuss two main obstructions to uniform hyperbolicity, namely the presence of singularities and the presence of criticalities. In some cases a weaker form of hyperbolicity can be recovered which is sufficient to guarentee the existence of an SBR measure.

For a discrete dynamical system $\varphi^t : M \to M$ we say that a compact invariant set $\Lambda \subset M$ is uniformly hyperbolic if there exists a *continuous* splitting

$$T\Lambda = E^s + E^u$$

of the tangent bundle over Λ and constants $C > 0$ and $1 > \lambda > 0$ such that for all $n \in \mathbb{N}$ and all $x \in M$ we have

$$\|D\varphi^n(x) \cdot v\| \leq C\lambda^n \|v\| \quad \text{for all } v \in E^s(x)$$

and

$$\|D\varphi^{-n}(x) \cdot v\| \leq C\lambda^n \|v\| \quad \text{for all } v \in E^u(x).$$

The definition is analogous for the case of continuous time. In that case we have a splitting

$$T\Lambda = E^s + E^u + E^0$$

where vectors in E^s and E^u satisfy exponential estimates as above and vectors in E^0 are tangent to the direction of the flow and clearly do not satisfy any exponential estimates.

A natural direction in which the work of Sinai, Ruelle and Bowen is being developed is that of extending their results to other classes of systems, in particular systems which satisfy only weaker hyperbolicity conditions. Two important ways in which systems which appear to have certain hyperbolic structure can fail to be uniformly hyperbolic is through the presence of singularities (in the case of flows this means an equilibrium point, in the case of maps it means a set of discontinuities) and/or the presence of criticalities (e.g. homoclinic tangencies).

1.3. Non-uniform hyperbolicity with singularities. The presence of a singularity in a non-trivial invariant set Λ implies that it is not possible to have a continuous hyperbolic splitting over Λ. For a map this follows simply from the fact that the map is not even continuous, for a flow notice that the hyperbolic decomposition in a regular point includes a neutral direction parallel to the flow whereas this does not occur in the equilibrium point which satisfies exponential estimates in all directions. This fact gives rise to a remarkable number of complications. It turns out, for example, that the stable and unstable manifolds of points of Λ can get cut when the orbits pass close to the singularities and so these invariant manifolds are generally formed of countably many connected components. Moreover there is no uniform lower bound on the size of the connected components. For these reasons the techniques of Bowen-Ruelle-Sinai cannot be applied directly. An important class of systems in which this phenomenon is observed is in Billiard-type dynamical systems. Here there is a natural invariant measure and statistical properties have been extensively studied developing the work started in [Sin70][BS83], see also [Sin90]. For dissipative systems the main example is the geometric Lorenz attractor, a class of flows in $\mathbb{R}^3$ introduced indipendently by Afraimovich, Bykov and Šil'nikov [ABŠ77][ABŠ82] and Guckenheimer and Williams [Guc76][GW79][Wil79]. An important emphasis in these papers was given to the topological and geometrical structure of these attractors. Statistical properties were studied in [BS80], see also [Bun83] and [Bun89]. Pesin and Sataev [Pes86] [AP87] [Pes92] [Sat92] have proved the existence of invariant measures, analogous to the SBR measures discussed above, for a wide class of maps with singularities. Their results apply in particular to the return maps of the geometric Lorenz attractors. A further generalization has been given recently in [JN95] for a class of maps in which certain conditions on the behaviour of the differential near the singularities have been relaxed.

1.4. Non-uniform hyperbolicity with criticalities. A different kind of obstruction to uniform hyperbolicity is given by the presence of criticalities, i.e. homoclinic tangencies or, more generally, non transversal intersections between stable and unstable manifolds. Intuitively a point of non transversal intersection cannot have a hyperbolic splitting because the stable and unstable subspaces E^s and E^u, tangent to the stable and unstable leaves respectively, would then have to coincide implying that $E^s + E^u$ is one dimensional and does not span the tangent space.

The importance of criticalities in the loss of stability and hyperbolicity of dynamical systems has been studied extensively starting from the works of Smale, Palis and others [Pal70][PS70]. It is now apparent that there are a number of important bifurcations associated to homoclinic tangencies and criticalities in general and there exists a substantial body of work on the unfolding of criticalities in parametrized families of maps and on the relation between the existence of criticalities and uniformly hyperbolic dynamics ([New79][NPT83] [PT85][PT87][PV94] [PY93], see also [PT93] for an overview of the theory). Indeed, according to a global research program proposed by Palis, criticalities are on of the main sources of chaotic and non-uniformly hyperbolic dynamics.

An almost canonical example of a family of dynamical systems in which criticalities play an important role is the family of smooth plane diffeomorphisms

$$h_{a,b} : (x, y) \to (1 - ax^2 + y, bx) \tag{Hén}$$

introduced by Hénon and Pomeau [Hén76][HP76]. Numerical studies of this family for a range of parameter values indicate the existence of non trivial attractors and, due to the geometry of the map, the likely occurence of criticalities for a large number of parameter values. However, a rigorous proof of this fact did not appear until Benedicks and Carleson proved, in their ground-breaking paper [BC91] that indeed there exists many parameter values (sets of positive Lebesgue measure) for which the Hénon map exhibits a non-trivial attractor with certain non-uniform hyperbolicity properties. Moreover, such parameter values can be approximated by parameter values for which the corresponding map exhibits criticalities (homoclinic tangencies) [Uré93]. Benedicks and Young [BY93] have shown that these attractors admit SBR invariant measures analogous to the ones which exist for uniformly hyperbolic systems and for systems with singularities.

The powerful ideas introduced in [BC91] are being applied to other classes of systems in which the presence of criticalities constitutes an obstruction to an application of the methods of uniformly hyperbolic dynamics ([DRV94][PRV95]). Higher dimensional systems with criticalities have also been studied succesfully by different methods in [Via95]. Such systems exhibit a remarkable statistical mechanism of persistence as a consequence of which they are fully persistent in a neighbourhood, i.e. nearby systems continue to exhibit non-uniformly hyperbolic attractors.

1.5. The Lorenz equations. Both examples mentioned above, the geometrical Lorenz attractors and the Hénon family of diffeomorphisms were inspired by numerical studies of the following system of differential equations introduced by Lorenz [Lor63] in 1963:

$$\text{(Lor)} \qquad \begin{cases} \dot{x} = -\sigma x + \sigma y \\ \dot{y} = rx - y - xz \\ \dot{z} = -bz + xy \end{cases}$$

as an example of an explicit system with a simple formulation which, nevertheless, appears to exhibit remarkably complex, erratic and nonperiodic behaviour. A rigorous description of the dynamics, for the parameter values $\sigma = 10, b = 8/3$ and $r = 28$ originally considered by Lorenz, remains an open problem up to the present day. Some limited facts can be proved by classical methods. Using the theory of Lyapunov functions, for instance, it can be shown that there exists for all parameter values a neighbourhood of the origin into which all trajectories enter and never leave. Since the Lorenz equations are dissipative this implies that there exists a compact invariant set Λ, of zero Lebesgue measure containing the omega-limit sets of all trajectories. However it seems hard to prove any specific properties of this attractor (see [Spa82] for a thorough discussion of numerical studies and classical approaches). Even the existence, for any parameter value, of a homoclinic orbit to the singularity at the origin seems remarkably difficult to prove. Partial results in this direction include [Rob89][Rob92][Ryc89] where the existence of a homoclinic orbit is proved for systems of differential equations close to that of Lorenz. Recently some computer assisted proofs have also been announced [HT92].

1.6. Geometric models. In some sense the most fruitful approaches to the Lorenz equations have been the geometric models described above which have respresented fundamental examples for the theory of non-uniformly hyperbolic systems.

Geometric Lorenz attractors were introduced to model the dynamics of (Lor) for the parameter values considered by Lorenz himself. For these parameter values a horizontal cross section to the flow can be chosen near the singularity and the return map to this cross section studied numerically. From these studies it appears that the return map has strong hyperbolic properties except for the presence of the discontinuity (which consists of the intersection of the cross section Σ with the two-dimensional stable manifold of the singularity; the trajectories through these points end in the singularity and never return to intersect Σ, thus the return map cannot be defined there).

However, as r is increased to values of around 30 and beyond that the flow begins to twist in such a way that criticalities are formed. Hénon's family was introduced as a simplified model of the dynamics for this range of parameter values. Hénon believed, quite correctly, that it would be easier to concentrate on the effects of the folds and the effect of the criticalities in the context of non-singular smooth maps. A major breakthrough in this direction was accomplished in [BC91] as was already mentioned above.

Our objective, in this thesis, is to recover Hénon's original project and to develop a model for the dynamics exhibited by the Lorenz equations in the region of parameter values in which both criticalities and the singularity are present. We will define a class of one parameter families of vector fields which exhibit, for a certain range of parameter values, geometric Lorenz attractors. As the parameter is varied, a series of bifurcations takes place through which criticalities are formed. On the other side of this sequence of bifurcations we encounter attractors in which dynamical features deriving from the presence of a singularity coexist with dynamical features deriving from the presence of criticalities. We study the way in which these two dynamical phenomena interact and show that a weak form of hyperbolicity still occurs in a measure theoretically persistent way.

Our model is described in detail in the next section where we also give a precise statement of our results. In chapter II we study a 1-dimensional model, which is also of intrinsic interest. Indeed many features of dynamical systems in which singular and critical dynamics coexist are already present in this model. Thus, apart from the present motivation, the family of one-dimensional maps studied below is a rich source of non-smooth dynamics. In chapter III we prove our main theorem.

2. Statement of the results

2.1. Definition of Lorenz-like flows. Let $\mathcal{X}(\mathbb{R}^3)$ denote the space of smooth (C^3) vector fields in $\mathbb{R}^3$. We shall study one-parameter families of vector fields $\{\mathcal{X}_a\}$ in $\mathcal{X}(\mathbb{R}^3)$ defined by the following characteristics:

LL1: Each $\mathcal{X}_a$ has a hyperbolic singularity (equilibrium point) with eigenvalues $\lambda_{ss} < \lambda_s < 0 < \lambda_u$ satisfying $|\lambda_{ss}/\lambda_u| > 1$ and $|\lambda_s/\lambda_u| < 1/2$.

LL2: There exists a two dimensional cross section Σ transversal to the flow such that the first return maps Φ_a to Σ form a family of *Lorenz-like maps* (as defined below).

We call a family of vector fields satisfying **LL1** and **LL2** a family of *Lorenz-like vector fields* or *Lorenz-like flows* or, even, when we want to emphasise the existence of attractors, a family of *Lorenz-like attractors.*

In this section we shal give the precise definition of the family of maps Φ_a through which our flows are defined. We shall obtain Φ_a as a composition $\Phi_a = \Psi_a \circ P$ where P describes the flow near the singularity and Ψ_a describes the flow outside a neighbourhood of the singularity and, as we shall see, exhibits features which are typical of the presence of criticalities. The maps P and Φ are described precisely in 2.2 and 2.3 respectively. In 2.4 we give a precise statement of our results and make some technical remarks.

2.2. Singular dynamics. We introduce here a map describing the dynamics of the flow in a neighbourhood of the singularity. We suppose that the singularity is fixed in the origin of $\mathbb{R}^3$ for all parameters. Let $I_\varepsilon = \{(x, y, z) \in \mathbb{R}^3 : |x|, |y|, |z| \leq \varepsilon\}$ denote an arbitrarily small cube centered at the origin. By a theorem of Wiggins [Wig88], for sufficiently small ε the flow in this cube approximates a linear flow in the sense that the discrete maps describing the flow between one side of the cube and another are close in the C^2 topology to the maps describing a linear flow. In fact the flow in I_ε is smoothly equivalent to a linear flow (i.e. there exists a smooth orbit preserving change of coordinates in a neighbourhood of the origin such that the original flows commutes with a linear flow by this change of coordinates) as long as a finite number of non-resonance conditions on the eigenvalues are satisfied [Ste58]. Moreover one can even assume that for a given interval of parameter values, there exists a smooth family of smooth linearizing coordinates in a neighbourhood of the origin (see [Rov93]). To simplify the exposition we shall assume, without any real loss of generality, that the non-resonance conditions mentioned above are satisifes and therefore that our family $\{\mathcal{X}_a\}$ is smoothly equivalent to a family in which there exists a fixed neighbourhood of the origin where the flow is linear. From now on we shall always work in this linearizing system of coordinates. We remark however that for the purposes of our results it would be quite sufficient to consider a neighbourhood in which the flow was just close to a linear flow, thus avoiding any assumptions whatsoever and working in full generality.

Up to a linear rescaling we can also suppose that the linearizing neighbourhoods always contain the cube

$$I = \{(x, y, z) \in \mathbb{R}^3 : |x|, |y|, |z| \leq 1\}.$$

By standard hyperbolic theory there are submanifolds $W^u_{\text{loc}}, W^s_{\text{loc}}$ and $W^{ss}_{\text{loc}} \subset W^s_{\text{loc}}$ tangent in the origin to the eigenspaces associated to λ_u, $(\lambda_s, \lambda_{ss})$ and λ_{ss} respectively. In the linearizing coordinates W^u_{loc} intersects the boundary of the cube I in $(1,0,0)$, W^{ss}_{loc} in $(0,0,1)$ and W^s_{loc} in the line $\{(0, y, 1), |y| \leq 1$. We let $\Sigma = \{(x, y, 1) : |x|, |y| \leq 1\}$ denote the top of the cube I and $\Sigma_0 = \{(1, y, z) : |y| \leq 1, 0 < z < 1\} \cup \{(-1, y, z) : |y| \leq 1, 0 < z < 1\}$. We want to study the image of Σ under the effect of the flow: for each point in $(x, y) = (x, y, 1) \in \Sigma$ we consider the trajectory of the flow through (x, y) and the first intersection of this trajectory with Σ_0. Notice first of all that the trajectories of points in $\Sigma \cap W^s$ terminate in the origin without ever intersecting Σ_0, trajectories through $\Sigma^+ = \Sigma|_{\{x>0\}}$ intersect $\Sigma_0^+ = \Sigma_0|_{\{x=1\}}$, and trajectories through $\Sigma^- = \Sigma|_{\{x<0\}}$ intersect $\Sigma^-0 = \Sigma_0|_{\{x=-1\}}$. By linearity, the position (x_t, y_t, z_t) after time t of the point $(x, y, 1) \in \Sigma^\pm = \Sigma^+ \cup \Sigma^-$ transported by the flow, is given by

$$(x_t, y_t, z_t) = (xe^{t\lambda_u}, ye^{t\lambda_{ss}}, e^{t\lambda_s}).$$

The "flight time" T that it takes to reach Σ_0 depends only on the norm of the x-coordinate and is given by $|x|e^{T\lambda_u} = 1$ or, equivalently, $T = T(|x|) = (1/\lambda_u)\log(1/|x|)$. From this we easily get $y_T = ye^{T\lambda_{ss}} = ye^{(\lambda_{ss}/\lambda_u)\log(1/|x|)} = y|x|^{|\lambda_{ss}|/\lambda_u}$ and similarly $z_T = |x|^{|\lambda_s|/\lambda_u}$. Let $\lambda = |\lambda_s/\lambda_u|$ and $\sigma = |\lambda_{ss}/\lambda_u|$. Then we define the map

$$P : \Sigma_* \to \Sigma_0$$
$$(x, y, 1) \to (1, u, v) = (1, |x|^\sigma, y|x|^\lambda \text{ sgn } (x)).$$

Notice that there is a natural identification of Σ_0 with Σ_* through which P can be thought of as a two dimensional map of Σ_* into itself by writing $P(x, y) = (v, u)$. To simplify the notation we will sometimes take this point of view without making any further remark. The differential of P can be calculated explicitly to get

$$DP = \begin{pmatrix} \partial_x P_1 & \partial_y P_1 \\ \partial_x P_2 & \partial_y P_2 \end{pmatrix} = \begin{pmatrix} \lambda|x|^{\lambda-1} & 0 \\ \sigma|x|^{\sigma-1}y & |x|^\sigma \end{pmatrix}. \tag{3}$$

The expressions for the map P and for the differential DP capture certain significant features of the dynamics essentially related to the presence of the singularity. Line segments $I_x = \{x = \text{ const.}\} \subset \Sigma_*$ (i.e. colinear with the strong stable local manifold W^{ss}_{loc}) are strongly contracted as they are transported by the flow near the singularity. For small x such segments spend increasing amounts of time near the singularity and thus the effect of the contraction is increased. The overall effect of this process is expressed analytically in the equation $\partial_x P_2 = |x|^\sigma$ which shows that the contraction increases exponentially as x approaches zero giving the characteristic cusp shape to the images $P(\Sigma^+)$ and $P(\Sigma^-)$. On the other hand the equation $\partial_x P_1 = \lambda|x|^{\lambda-1}$ describes the expansivity of line segments contained in $I_y = \{y = \text{ const. } \subset \Sigma_*\}$ (i.e. colinear to the unstable manifold W^u_{loc}). Due to the fact that $\lambda_u > 0$ distances between points in I_y are exponentially stretched. Points which are very close to $x = 0$ spend an arbitrarily large amount of time in a neighbourhood of the singularity and therefore exhibit an arbitrarily large amount of stretching ($|\partial_x P_1| \to \infty$ as $x \to 0$).

2.3. Critical dynamics. The global return maps $\Phi_a : \Sigma_* \to \Sigma$ are obtained by composing the maps $P : \Sigma_* \to \Sigma_0$ described above with a family of maps $\Psi_a : \Sigma_0 \to \Sigma$ describing the flow outside a neighbourhood of the singularity. The strong dissipativity of the Lorenz equations, and of the family of flows we want do describe, leads necessarily to strongly dissipative return maps. This means that, in some sense to be defined precisely below, the family Φ_a is close to a family of one-dimensional maps φ_a.

We begin by describing the families $\{\varphi_a\}$. They are of the form

$$\varphi_a(x) = \begin{cases} \varphi(x) - a & \text{if } x > 0 \\ -\varphi(-x) + a & \text{if } x < 0 \end{cases}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is smooth and satisfies:

L1: $\varphi(x) = \hat{\psi}(x^\lambda)$ for all $x > 0$, where $0 < \lambda < 1/2$ and ψ is a smooth map defined on $\mathbb{R}$ with $\psi(0) = 0$ and $\psi'(0) \neq 0$;

L2: there exists some $c > 0$ such that $\varphi'(c) = 0$;

L3: $\varphi''(x) < 0$ for all $x > 0$.

For small values of the parameter the maximal invariant set of φ_a in the interval $[-a, a]$ is a hyperbolic Cantor set. Under certain natural conditions, implied by L4 and L5 below, the entire interval $[-a, a]$ becomes forward invariant as a crosses some value $a_1 > 0$. This situation persists for a certain range of parameter values and corresponds to the class of maps usually associated to the "Lorenz attractor" (see § 1.3). We are mainly interested in studying the bifurcation which occurs as the parameter crosses the value $a = c$. With this in mind, we add two natural assumptions on φ which ensure that a Lorenz attractor persists for all $a < c$.

Let $x_{\sqrt{2}}$ denote the unique point in $(0, c)$ such that $\varphi'(x_{\sqrt{2}}) = \sqrt{2}$; sometimes we also write $a_2 = x_{\sqrt{2}}$. Then we suppose

L4: $0 < \varphi_a(x_{\sqrt{2}}) < \varphi_a(a) < x_{\sqrt{2}}$ for all $a \in [a_2, c]$.

The last inequality implies that given any y with $|y| \in [x_{\sqrt{2}}, a)$ there exists a unique $x \in [-a, a]$ such that $\varphi_a(x) = y$. Note that x and y have opposite signs. Moreover, the first inequality implies that $|x| < x_{\sqrt{2}}$. Our last assumption is

L5: $|(\varphi_c^2)'(x)| > 2$ for all $x \in [-c, c] \setminus \{0\}$ such that $|\varphi_c(x)| \in [x_{\sqrt{2}}, c]$

Observe that this is automatic if $\varphi_c(x) = x_{\sqrt{2}}$ (because $|x|$ is strictly smaller than $x_{\sqrt{2}}$, by the previous remarks) and also if $\varphi_c(x)$ is close to c (then x is close to zero and so $|(\varphi_c^2)'(x)| \approx |x|^{2\lambda-1} \approx \infty$).

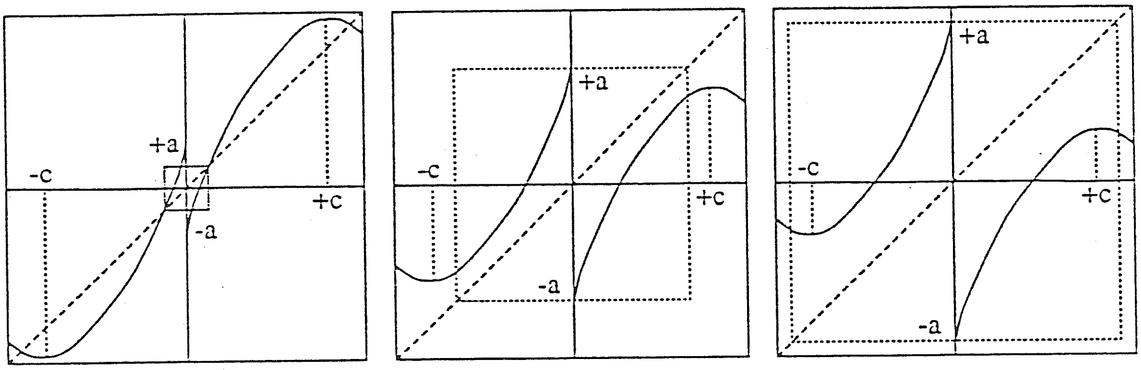

FIGURE I.1. Lorenz-like families with criticalities

It is straightforward to check that L1-L5 are satisfied by a nonempty open set of one-parameter families, where openess is meant with respect to the C^2 topology in the space of real maps $\hat{\psi}$. Notice that the family of one-dimensional maps described above can be embedded into a family of two dimensional singular endomorphisms by defining $\psi : \Sigma_* \to \Sigma$ where we have, for $(x, y) \in \Sigma_*$, $\psi_a \circ P : (x, y) = (\hat{\psi}(|x|^\lambda \text{ sgn } (x), \text{ sgn } (x))$. The maps $\psi \circ P$ correspond to return maps for a class of *semiflows*. In this case the images of the two components Σ_*^+ and Σ_*^- are just line segments and we have $\psi_a \circ P(\Sigma_*^\pm) \subset \{y = \pm 1\} \subset \Sigma$.

We can now define a family of *Lorenz-like maps*

$$\Phi_a : \Sigma_* \to \Sigma$$

as the composition $\Phi_a = \Psi_a \circ P$ where $P(x, y) = (x|^\lambda \text{ sgn } (x), |x|^\sigma y)$ and $\Psi : \Sigma_0 \to \Sigma$ is characterised by the property that

$$\text{(LL)} \qquad \|\Psi_a - \psi_a\|_{C^3} \le b.$$

We always take b small and take (LL) to mean closeness in the C^3 topology with respect to all three variables. i.e.

$$\left\| \frac{\partial \Psi_i^{j+k+l}}{\partial_u^j \partial_v^k \partial_a^l} - \frac{\partial \psi_i^{j+k+l}}{\partial_u^j \partial_v^k \partial_a^l} \right\| \le b$$

where $\Psi_i, \psi_i, i = 1,2$ denote the coordinate components of Ψ_a and of ψ_a respectively, and $1 \leq j + k + l \leq 3$.

2.4. The main theorem. Let $\mathcal{X}$ be a smooth vector field in $\mathbb{R}^3$. We say that $\mathcal{X}$ has a *chaotic attractor* Λ if there exists a compact set $\Lambda \subset \mathbb{R}^3$ with the following properties:

(1) Λ is invariant for the evolution of the flow;
(2) There is a set of points of positive Lebesgue measure for which $\omega(x) \subset \Lambda$, where $\omega(x)$= omega limit set of x;
(3) There exists a point $z \in \Lambda$ whose trajectory is dense in Λ (i.e. $\omega(x) = \Lambda$) and such that z has a positive Lyapunov exponent.

We shall show below (see proposition 1.1) that there is a certain interval of parameter values for which Lorenz-like flows exhibit chaotic attractors of the type of the geometric Lorenz attractor. This occurs for parameter values preceeding the formation of criticalities. Our main theorem concerns the existence (and persistence) of chaotic attractors after the appearence of criticalities.

THEOREM. *Let $\{\mathcal{X}_a\}$ be a family of Lorenz-like flows. Then there exists a set $\mathcal{A}^+$ of parameter values, after the formation of criticalities, such that $\mathcal{X}_a$ exhibits a chaotic attractor for each $a \in \mathcal{A}^+$. These (Lorenz-like) attractors are persistent in the sense that the set $\mathcal{A}^+$ has positive Lebesgue measure.*

The complete proof of this theorem is given in chapter III. For the moment we make some general remarks concerning the result and the ideas and methods used in the proof.

The theorem follows from the analogous result for the return map Φ_a. Our basic approach is inspired by the ideas introduced in [BC91]. Here, however, we have to deal with several additional difficulties deriving from the presence of the singularity and of regions with arbitrarily large derivatives. In particular several estimates (including distortion bounds) which in the smooth case rely on on the boundedness and/or on some Lipschitz continuity properties of the derivative require here a non-trivial reformulation together with a detailed study of the recurrence near the singularity (as well as near the critical region). Even though the expansivity near the singularity should, in principle, help to obtain positive Lyapunov exponents it turns out that it constitutes a serious obstruction at various points of the proof in which bounded distortion estimates play a crucial role. Thus a bounded recurrence condition needs to be imposed and parameters excluded when certain orbits pass too close or too frequently near the discontinuity. The precise formulation of this condition will be given in chapter III, here we just remark briefly on it basic form. Let $\{z_i\}_{i=0}^n$ denote a piece of orbit of the point z and let $\|z_i\|$ denote the distance of z_i from the singularity set. Let $\{z_{i_j}\}_{j=1}^s$ be a subsequence of points belonging to the orbit of z such that z_{i_j} belongs to a small neighbourhood of the discontinuity. The kind of bound which we require on the rate of approach of z_i to the singularities is of the form

$$(*) \qquad \sum_{j=1}^{s} \log \|z_{i_j}\|^{-1} \leq \alpha n$$

for some suitably small α. From the dynamical point of view condition $(*)$ has the effect of bounding the rate of growth of vectors and, therefore, besides the important role it plays in several crucial estimates it also guarantees that the attractors obtained in the proof have finite Lyapunov exponents. It seems interesting to observe that condition $(*)$ is exactly the form of the bounded recurrence condition used to bound the recurrence in the critical region if we let $\|\cdot\|$ denote the distance of z_i to the critical point (or to a suitable critical approximation) instead of its distance to the discontinuities. Indeed it is equivalent (up to some multiplicative factor) to the two conditions (basic assumption and free period assumption) introduced by Benedicks and Carleson. In some sense, condition $(*)$ seems more natural and easier to state, specially since it can be formulated more or less at the beginning of the proof (whereas the free period assumption in [BC91] requires quite a lot of notions to be introduced before it can be even stated).

The proof presented below essentially works for the Hénon map if the existence of the discontinuity (and all the related estimates) are ignored. Indeed most estimates should become significantly simpler. We have tried to give as straightforward and direct a proof as seemed possible, introducing most of the notions and ideas in a simple setting before using them in more general contexts. We begin with a careful analysis of the dynamics outside the critical region (but including the singular region). Here an essentially hyperbolic dynamics takes place and this, together with condition $(*)$, allows us to prove some uniform bounded distortion estimates for the growth of vectors for orbits which remain close (bound) and far from the critical region for a certain (in principle unlimited) amount of time. We then collect in a relatively self-contained section (2) most of the notions and estimates concerning the existence of critical approximations. We also discuss the problem of considering different orbits (corresponding to different critical approximation) at each step of the iteration. We show that new critical approximation are always bound to older ones and that, consequently, the history of the new orbit under consideration is essentially the same as the old one. In section 3 we consider the first time that some critical approximation falls into the critical region. We introduce the notion of binding point, binding period and condition $(*)$ in the critical region. A main advantage in considering a first return is that no further returns occur during the associated binding period. This allows us to prove all the desired binding period estimates (bounded loss of growth, recovery at the end of a binding period, bounded distortion) directly and without requiring too much abstract inductive information. In particular we can highlight some of the differences between the one dimensional and the two dimensional situation and, to some extent, clarify the way in which the additional difficulties in the two dimensional case are dealt with. At this point all the main ideas and methods have been introduced and the general binding period estimates follow without too much trouble in section 4. There we also show that those parameters for which all critical approximations satisfy $(*)$ for all time actually exhibit a chaotic attractor and that $\mathcal{A}^+$ has positive Lebesgue measure. Several estimates in this last section proceed essentially as in the Hénon or quadratic like case [BC91][MV93].

Before going on to the proof of our main theorem, we carry out a detailed analysis of the dynamics of the family of one-dimensional maps $\{\varphi_a\}$ defined above. The precise statement of the theorem in this case and several remarks, as well as a complete proof, are given in the next chapter which is essentially self-contained.

CHAPTER II

The one-dimensional model

1. Introduction and statement of results

Let $\{\varphi_a\}$ be a family of one-dimensional maps satisfying conditions L1-L5 (see subsection 2.3). We begin by observing that these hypotheses imply that φ_a is essentially uniformly expanding for all parameters up to c:

PROPOSITION 1.1. *Given any* $a \in [a_1, c]$,

(1) *the interval* $[-a, a]$ *is forward invariant and* $\varphi|[-a, a]$ *is transitive*

(2) $|(\varphi_a^n)'(x)| \geq \min\{\sqrt{2}, |\varphi_a'(x)|\}(\sqrt{2})^{n-1}$ *for all* $x \in [-a, a]$ *such that* $\varphi_a^j(x) \neq 0$ *for every* $j = 0, 1, \ldots, n-1$.

This relatively strong form of (non-uniform) expansivity replaces, in our context, the Misiurewicz condition which is usually assumed in the case of smooth maps (see remarks following the statement of the theorem). We postpone the prove of 1.1 to the beginning of section 4. After the bifurcation $a = c$ such uniform expansivity is clearly impossible, due to the presence of the critical point in the domain of the map. However, our main result states that nonuniform expansivity persists in a measure theoretic sense after the bifurcation [1]. Let $c_1(a) = \varphi_a(c)$ and for $\sigma > 0$ define $\mathcal{A}^+(\sigma) = \{a > c : |(\varphi_a^j)'(c_1(a))| \geq e^{\sigma j}$ for all $j \geq 1\}$.

THEOREM. *Let* $\{\varphi_a\}$ *be a Lorenz-like family satisfying conditions L1-L5. Then there exists* $\sigma > 0$ *such that* $m\{\mathcal{A}^+(\sigma) > 0$ *where* m *denotes Lebesgue measure on* $\mathbb{R}$.

Measure theoretic persistence of positive Lyapunov exponents (outside the class of uniformly expanding maps) was first proved by Jakobson [Jak81], for maps

[1] This chapter is a revised version of a joint paper with M. Viana: *Positive Lyapunov exponents for Lorenz-like families*, to appear

in the quadratic family $f_a(x) = 1 - ax^2$ close to parameter values $\bar{a}$ satisfying ([Mis81])

$$(4) \qquad \inf_{j \in \mathbb{N}} \left| f_{\bar{a}}^j(c) - c \right| > 0 \qquad (c = \text{critical point} = 0).$$

There exist today many proofs of this theorem, e.g. [CE80], [BC85], as well as generalizations to families of smooth maps with finitely many critical points [TTY92], and to families of maps in which a single discontinuity coincides with the critical point [Rov93]. A number of main differences should be pointed out in this setting, between smooth maps and our Lorenz-like maps.

While all proofs of Jakobson's theorem in the smooth context rely in one way or the other on the nonrecurrence condition (4), here we need no assumption on the orbits of the critical points for $a = c$. Instead, we simply take advantage of the strong expansivity estimates given by Proposition 1.1 for that parameter value.

As we mentioned above there are several technical complications in the proof due to the presence of regions with unbounded derivative. We gain control over these regions by imposing an additional condition on the parameters which bounds the extent of the recurrence which is allowed near the discontinuity. We remark here that the symmetry inherent in the definition of Lorenz-like maps, though partly justified by the symmetry which exists in Lorenz' system of equations, is not necessary for the proof of our theorem. Every part of the proof goes through with a minimal amount of modifications in a non-symmetric case. We carry out the proof in the symmetric case since this allows us to simplify the exposition a great deal. In particular we shall often discuss some construction or result with explicit reference to only one of the critical points with the implicit understanding that the same statements apply to the other one as well.

The proof of our main result is organized as follows. In Section 2 we identify a pair of conditions on the parameter a which ensure that $a \in \mathcal{A}^+$. Sections 3 and 4 are then devoted to showing that the set of parameters for which such conditions are satisfied is large in the sense of the statement of the theorem. The whole global approach is inspired on [BC91].

2. Positive Lyapunov Exponents

We begin by proving Proposition 1.1. In doing this we focus only on $a \in [a_2, c]$: the case $a < a_2$ corresponds to the situation in [GW79], and it also follows from (simpler versions of) these same arguments.

2.1. Proof of Proposition 1.1. The invariance of $[-a, a]$ is an immediate consequence of $\lim_{x \to 0} |\varphi_a(x)| = a$, $|\varphi_a(\pm a)| < x_{\sqrt{2}} \leq a$ (recall L4), and the monotonicity of φ_a on $(-a, 0)$ and $(0, a)$.

Next, let x and $1 \leq j \leq n-1$ be as in part (2). If $|\varphi_a^j(x)| \leq x_{\sqrt{2}}$ then we have $|\varphi_a'(\varphi_a^j(x))| \geq \sqrt{2}$. If $|\varphi_a^j(x)| > x_{\sqrt{2}}$ then, by L4, there exists a unique $z \in [-c, c]$ such that $\varphi_c(z) = \varphi_a^j(x) = \varphi_a(\varphi_a^{j-1}(x))$. Moreover, z and $\varphi_a^{j-1}(x)$ have the same sign (opposite to that of $\varphi_a^j(x)$) and $|z| \geq |\varphi_a^{j-1}(x)|$, because $a \leq c$. Using L5 we get

$$|(\varphi_a^2)'(\varphi_a^{j-1}(x))| = |\varphi'(\varphi_a^{j-1}(x))\, \varphi'(\varphi_a^j(x))| \geq |\varphi'(z)\, \varphi'(\varphi_a^j(x))| = |(\varphi_c^2)'(z)| > 2.$$

Part (2) follows directly from these remarks and it is easy to see that one even gets a somewhat better bound, with $\sqrt{2}$ replaced by some slightly larger constant θ.

To prove transitivity, we let $U_0, V_0 \subset [-a, a]$ be arbitrary open sets and show that $\varphi_a^n(U_0) \cap V_0 \neq \emptyset$ for some $n > 0$. Suppose, without loss of generality, that $0 \notin U_0$ and $U_0 \subset (-x_{\sqrt{2}}, x_{\sqrt{2}})$ (recall L4). As long as $0 \notin \varphi_a^j(U_0)$, write $U_j = \varphi_a^j(U_0)$ and notice that $|U_j| \geq \theta^j |U_0|$. Thus we must have $0 \in \varphi_a(U_{k_1-1})$ for some $k_1 \geq 1$. Let U_{k_1} denote the largest connected component of $\varphi_a(U_{k_1-1}) \setminus \{0\}$ and observe that $|U_{k_1}| \geq \frac{1}{2}|\varphi_a(U_{k_1-1})| \geq \frac{1}{2}\theta^{k_1}|U_0|$. Suppose first that $U_{k_1} \subset (z^-, z^+)$, where $z^- < 0 < z^+$ are the preimages of zero under φ_a; observe that $|z^\pm| < x_{\sqrt{2}}$ as a consequence of the first inequality in L4. Then we proceed as before, with U_0 replaced by U_{k_1}. More precisely, we define $U_{k_1+j} = \varphi_a^j(U_{k_1})$ until the first iterate $k_2 > k_1$ for which $0 \in \varphi_a(U_{k_2-1})$; at that point we take U_{k_2} to be the largest component of $\varphi_a(U_{k_2-1})$ and repeat the whole procedure again. As long as $U_{k_i} \subset (z^-, z^+)$ we have $k_{i+1} \geq k_i + 2$, hence

$$|U_{k_{i+1}}| \geq \frac{1}{2}|\varphi_a(U_{k_i-1})| \geq \frac{1}{2}\theta^{k_{i+1}-k_i}|U_{k_i}| \geq \frac{1}{2}\theta^2|U_{k_i}|$$

grows exponentially with i. Thus, one eventually reaches some $k = k_j$ for which U_k contains either $(z^-, 0)$ or $(0, z^+)$. In the first case $\varphi_a(U_k)$ contains $(0, a) \supset (0, z^+)$ and then $\varphi_a^2(U_k)$ contains $(-a, 0)$, which ensures that either $\varphi_a(U_k)$ or $\varphi_a^2(U_k)$ intersect V_0. The second case is entirely analogous so the proof of the proposition is complete.

Now we fix a number of constants to be used in the sequel of our argument. Recall that $0 < \lambda < 1/2$. We take $\sigma_0 > 0$ and $\sigma > 0$ such that $0 < 2\sigma < \sigma_0 < \log\sqrt{2}$ and also choose $\gamma \in (1, \lambda^{-1} - 1)$ and $\delta > 0$ such that $1 < \gamma + \delta < \lambda^{-1} - 1$. Then, we let $0 < \alpha < \beta$ be small, depending on the previous constants (the precise conditions are stated throughout the proof wherever they are required).

By conditions L1-L3 there exist $\eta_1, \eta_2 > 0$ such that

$$\lim_{x \to 0} \frac{|\varphi(x)|}{|x|^\lambda} = \eta_1 \qquad \text{and} \qquad \lim_{x \to c} \frac{|\varphi(x)|}{|x - c|^2} = \eta_2.$$

For each $i = 1, 2$, we fix constants $\eta_i^- = \eta_i - v$ and $\eta_i^+ = \eta_i + v$, where v is some small positive number (again, precise conditions are to be stated along the way).

Then we have

M1: for all $x \neq 0$ close enough to the origin,

$$\begin{aligned} \eta_1^-|x|^\lambda \le \varphi_a(x) + a \le \eta_1^+|x|^\lambda \qquad &\text{if } x > 0, \\ -\eta_1^+|x|^\lambda \le \varphi_a(x) - a \le -\eta_1^-|x|^\lambda \qquad &\text{if } x < 0, \\ \text{and} \qquad \eta_1^-\lambda|x|^{\lambda-1} \le |\varphi_a'(x)| \le \eta_1^+\lambda|x|^{\lambda-1}; \end{aligned}$$

M2: for all x close enough to the critical point c,

$$\eta_2^-(x-c)^2 \le |\varphi_a(x)-\varphi_a(c)| \le \eta_2^+(x-c)^2 \text{ and } 2\eta_2^-|x-c| \le |\varphi_a'(x)| \le 2\eta_2^+|x-c|,$$

and for all x close enough to $-c$,

$$\eta_2^-(x+c)^2 \le |\varphi_a(x)-\varphi_a(c)| \le \eta_2^+(x+c)^2 \text{ and } 2\eta_2^-|x+c| \le |\varphi_a'(x)| \le 2\eta_2^+|x+c|.$$

Now, for each small $\varepsilon > 0$ we let Δ_+^0, Δ_+^c, Δ_+^{-c} denote the ε^γ- neighbourhooods of the origin and of the critical points, respectively. We define a partition of $\Delta_+^0 \setminus \{0\}$, by writing

$$\Delta_+^0 \setminus \{0\} = \bigcup_{|r|\ge 1} I_r^0$$

where each I_r is an interval of the form $I_r = [\varepsilon^\gamma e^{-r}, \varepsilon^\gamma e^{-r+1})$, for $r \ge 1$, and $I_{-r} = I_r$. We also define analogous partitions

$$\Delta_+^c \setminus \{c\} = \bigcup_{|r|\ge 1} I_r^c \qquad \text{and} \qquad \Delta_+^{-c} \setminus \{-c\} = \bigcup_{|r|\ge 1} I_r^{-c}$$

where $I_r^{\pm c} = I_r^0 \pm c$ are simply the translates of I_r^0. We shall always assume that $\varepsilon > 0$ is small enough so that Δ_+^0 and $\Delta_+^{\pm c}$ are contained in the regions for which M1 and M2 are valid. Moreover, we let $r_\varepsilon = [\delta \log \varepsilon^{-1}]$ (here $[x]$ is the integer part of x) and we consider restricted neighbourhoods

$$\Delta^0 = \{0\} \cup \bigcup_{|r|\ge r_\varepsilon+1} I_r^0 \qquad \text{and} \qquad \Delta^{\pm c} = \{\pm c\} \cup \bigcup_{|r|\ge r_\varepsilon+1} I_r^{\pm c}$$

(of radius $\approx \varepsilon^{\gamma+\delta}$) of the origin and the critical points.

2.2. Breaking the hyperbolic structure. The loss of expansivity occuring after the bifurcation $a = c$ and caused by the critical points entering the domain of the map is, in some sense, local: for $c \le a \le c+\varepsilon$, it occurs only in a neighbourhood of the critical points of size $\varepsilon^{\gamma+\delta} \ll \varepsilon$. More precisely, any piece of orbit that does not intersect $\Delta^{\pm c}$ has an exponentially growing derivative. The proof of this fact requires two preliminary lemmas. First we obtain some estimates on the position and size of the preimage of $\Delta^{\pm c}$ for a convenient range of parameter values. Then we estimate the accumulated derivative of points which pass close to the discontinuity and close to the critical points.

LEMMA 2.1. *For all* $\varepsilon > 0$ *sufficiently small,*
(1) $(\varepsilon/\eta_1^+)^{\frac{1}{\lambda}} \le |\varphi_{c+\varepsilon}^{-1}(c)| \le (\varepsilon/\eta_1^-)^{\frac{1}{\lambda}}$ *and*
(2) *for every* $y \in \Delta_+^c$ *and* $a \in [c+\varepsilon/2, c+\varepsilon]$,

$$\frac{1}{e} \le \frac{|\varphi_a^{-1}(y)|}{|\varphi_{c+\varepsilon}^{-1}(c)|} \le e\,.$$

PROOF. By the second inequality in M1,

$$-\eta_1^+|\varphi_{c+\varepsilon}^{-1}(c)|^\lambda \le c-(c+\varepsilon) \le -\eta_1^-|\varphi_{c+\varepsilon}^{-1}(c)|^\lambda,$$

which immediately gives (1). To prove (2) notice that, for any y and a as in the statement, $y-a=(c-a)-(c-y) \in [-\varepsilon-\varepsilon^\gamma, -\varepsilon/2+\varepsilon^\gamma]$ and so, using M1 in the same way as before,

$$\left(\frac{\varepsilon/2-\varepsilon^\gamma}{\eta_1^+}\right)^{\frac{1}{\lambda}} \le |\varphi_a^{-1}(y)| \le \left(\frac{\varepsilon+\varepsilon^\gamma}{\eta_1^-}\right)^{\frac{1}{\lambda}}.$$

Combining with (1) we get

$$\left(\frac{\eta_1^-}{\eta_1^+}\frac{\varepsilon/2-\varepsilon^\gamma}{\varepsilon}\right)^{\frac{1}{\lambda}} \le \frac{|\varphi_a^{-1}(y)|}{|\varphi_{c+\varepsilon}^{-1}(c)|} \le \left(\frac{\eta_1^+}{\eta_1^-}\frac{\varepsilon+\varepsilon^\gamma}{\varepsilon}\right)^{\frac{1}{\lambda}}.$$

The left hand side is close to $1/2$, and hence larger than $1/e$, if ε is small (and v has been fixed sufficiently small, recall the definition of $\eta_i^\pm$). Analogously, the right hand side is smaller than e if ε and v are small enough. The proof is complete. □

Now we define $r_c = r_c(\varepsilon) \ge 1$ by the condition $\varphi_{c+\varepsilon}^{-1}(c) \in I_{-r_c}^0$. Observe that

$$(5)\qquad \frac{1}{e}\left(\frac{\varepsilon}{\eta_1^+}\right)^{\frac{1}{\lambda}} \le \varepsilon^\gamma e^{-r_c} \le \left(\frac{\varepsilon}{\eta_1^-}\right)^{\frac{1}{\lambda}}$$

by part (1) of the previous lemma. Moreover, part (2) gives

$$(6)\qquad \varphi_a^{-1}(\Delta_+^c) \subset I_{-r_c+1}^0 \cup I_{-r_c}^0 \cup I_{-r_c-1}^0, \quad \text{for every } a \in [c+\varepsilon/2, c+\varepsilon].$$

LEMMA 2.2. *For every* $a \in [c+\varepsilon/2, c+\varepsilon]$ *and* $x \in I_r^0, r \ge 1$,
(1) *if* $\varphi_a(x) \notin \Delta^{\pm c}$ *then* $|(\varphi_a^2)'(x)| \ge e^{2\sigma_0}e^{\beta r}$;
(2) *if* $\varphi_a(x) \in \Delta_\mp^{\pm c}$, *with* $\varphi_a(x) \in I_{\tilde r}^{\pm c}$, *then*

$$|\varphi_a'(x)| \ge e^{\sigma_0}e^{\beta r} \quad \textit{and} \quad |(\varphi_a^2)'(x)| \ge \varepsilon^{\gamma+1-\frac{1}{\lambda}}e^{-\tilde r} \ge e^{2\sigma_0}e^{-\tilde r}.$$

PROOF. We begin by proving (1). Suppose first that $r < r_c - 2$. Then, in view of (6), $|x - \varphi_a^{-1}(c)| \geq |I_{r+1}| = (1 - 1/e)\varepsilon^\gamma e^{-r}$. Thus, by the mean value theorem,

$$\begin{aligned} |\varphi_a(x) - c| &\geq |x - \varphi_a^{-1}(c)| \cdot \inf\{\varphi'(z) : z \in [x, \varphi_a^{-1}(c)]\} \\ &\geq (1 - 1/e)\varepsilon^\gamma e^{-r}\eta_1^- \lambda(\varepsilon^\gamma e^{-r+1})^{\lambda-1} \\ &\geq k_1 \varepsilon^{\gamma\lambda} e^{-r\lambda}, \end{aligned}$$

with $k_1 = (1 - 1/e)\eta_1^- \lambda e^{\lambda-1}$. It follows, using M1, M2,

$$\begin{aligned} |(\varphi_a^2)'(x)| &\geq |\varphi_a'(x)||\varphi_a'(\varphi_a(x))| &&\geq \eta_1^- \lambda(\varepsilon^\gamma e^{-r+1})^{\lambda-1} 2\eta_2^- k_1 \varepsilon^{\gamma\lambda} e^{-r\lambda} \\ &\geq k_2 \varepsilon^{-\gamma(1-2\lambda)} e^{r(1-2\lambda)} &&\geq e^{2\sigma_0} e^{\beta r}, \end{aligned}$$

where $k_2 = 2\eta_1^- \lambda e^{\lambda-1}\eta_2^- k_1$ and, for the last inequality, we suppose $\beta < 1 - 2\lambda$ and ε sufficiently small.

Now suppose that $r \geq r_c - 2$. Since $\varphi_a(x) \notin \Delta^{\pm c}$, we have $|\varphi_a(x) \pm c| \geq \varepsilon^{\gamma+\delta}$ and so we get

$$\begin{aligned} |(\varphi_a^2)'(x)| &\geq |\varphi_a'(x)||\varphi_a'(\varphi_a(x))| &&\geq \eta_1^- \lambda(\varepsilon^\gamma e^{-r+1})^{\lambda-1} 2\eta_2^- \varepsilon^{\gamma+\delta} \\ &\geq k_3 \varepsilon^{\gamma\lambda+\delta} e^{r(1-\lambda)} &&\geq k_3 \varepsilon^{\gamma\lambda+\delta} e^{(r_c-2)(1-\lambda-\beta)} e^{\beta r} \\ &\geq k_4 \varepsilon^{\gamma\lambda+\delta+(1-\lambda-\beta)(\gamma-\frac{1}{\lambda})} e^{\beta r} &&\geq e^{2\sigma_0} e^{\beta r}. \end{aligned}$$

with $k_3 = 2\eta_1^- \lambda e^{\lambda-1}\eta_2^-$ and $k_4 = k_3((\eta_1^-)^{\frac{1}{\lambda}} e^{-2})^{1-\lambda-\beta}$. In the fifth inequality we use (5). For the last one we note that $\gamma\lambda+\delta+(1-\lambda-\beta)(\gamma-\frac{1}{\lambda}) = (\gamma-\frac{1}{\lambda})(1-\beta)+\delta+1 < 0$ if β is small enough (recall the choice of γ and δ above) and we take ε to be sufficiently small.

Finally, suppose that $\varphi_a(x) \in I_{\tilde{r}}^{\pm c} \subset \Delta_+^{\pm c}$. Clearly,

$$|\varphi_a'(x)| \geq \eta_1^- \lambda(\varepsilon^\gamma e^{-r+1})^{\lambda-1} \geq \eta_1^- \lambda e^{\lambda-1}\varepsilon^{-\gamma(1-\lambda)} e^{r(1-\lambda)} \geq e_0^\sigma e^{\beta r}$$

if β and ε are small. Moreover, by (6), $\varepsilon^\gamma e^{-r_c-1} \leq |x| \leq \varepsilon^\gamma e^{-r_c+2}$, which gives

$$\begin{aligned} |(\varphi_a^2)'(x)| &\geq |\varphi_a'(x)||\varphi_a'(\varphi_a(x))| &&\geq \eta_1^- \lambda(\varepsilon^\gamma e^{-r_c+2})^{\lambda-1} 2\eta_2^- \varepsilon^\gamma e^{-\tilde{r}} \\ &\geq k_5 \varepsilon^{\gamma\lambda} e^{(1-\lambda)r_c} e^{-\tilde{r}} &&\geq k_5 \varepsilon^{\gamma\lambda+(1-\lambda)(\gamma-\frac{1}{\lambda})} (\eta_1^-)^{\frac{1}{\lambda}-1} e^{-\tilde{r}} \\ &\geq k_6 \varepsilon^{\gamma-\frac{1}{\lambda}+1} e^{-\tilde{r}} &&\geq e^{2\sigma_0} e^{-\tilde{r}}, \end{aligned}$$

where $k_5 = 2\eta_1^- \lambda e^{2(\lambda-1)}\eta_2^-$ and $k_6 = k_5(\eta_1^-)^{\frac{1}{\lambda}-1}$ and we use the relation (5) in the fourth inequality. □

LEMMA 2.3. *For any $a \in [c + \varepsilon/2, c + \varepsilon]$ and $x \in [-a, a]$,*

$$\textit{if} \quad \{\varphi_a^j(x)\}_{j=0}^{n-1} \cap \Delta^{\pm c} = \emptyset \quad \textit{then} \quad |(\varphi_a^n)'(x)| \geq \min\{e^{\sigma_0}, |\varphi_a'(x)|\} e^{\sigma_0(n-1)}.$$

If moreover $\varphi_a^n(x) \in \Delta_+^{\pm c}$ then $|(\varphi_a^n)'(x)| \geq e^{\sigma_0 n}$.

PROOF. Denote $x_j = \varphi_a^j(x)$, for $0 \le j \le n-1$. We claim that given any $j \ge 1$ either $|\varphi_a'(x_j)| \ge e^{\sigma_0}$ or $|(\varphi_a^2)'(x_{j-1})| \ge e^{2\sigma_0}$. This is obvious if $|x_j| \le x_{\sqrt{2}}$, because we get $|\varphi_a'(x_j)| = |\varphi'(x_j)| > e^{\sigma_0}$. ¿From now on we consider $x_j \ge x_{\sqrt{2}}$, the case $x_j \le -x_{\sqrt{2}}$ being entirely analogous. If $x_{j-1} \in \Delta_+^0$ then $|(\varphi_a^2)'(x_{j-1})| \ge e^{2\sigma_0}$ by part (1) of the previous lemma. Therefore, we may suppose $x_{j-1} \notin \Delta_+^0$, that is $|x_{j-1}| \ge \varepsilon^\gamma$. Then, recall M1, M2, $c - \varphi_c(x_{j-1}) \ge \eta_1^- \varepsilon^{\gamma\lambda}$ and so $|\varphi'(\varphi_c(x_{j-1}))| \ge 2\eta_1^-\eta_2^-\varepsilon^{\gamma\lambda}$. Hence, using also $\varphi_a(x_{j-1}) - \varphi_c(x_{j-1}) = a - c \le \varepsilon$, we get

$$\frac{|(\varphi_a^2)'(x_{j-1})|}{|(\varphi_c^2)'(x_{j-1})|} = \frac{|\varphi'(\varphi_a(x_{j-1}))|}{|\varphi'(\varphi_c(x_{j-1}))|} \ge 1 - \frac{|\varphi'(\varphi_a(x_{j-1})) - \varphi'(\varphi_c(x_{j-1}))|}{|\varphi'(\varphi_c(x))|} \ge 1 - k_7\varepsilon^{1-\lambda\gamma},$$

where $k_7 = k/(2\eta_1^-\eta_2^-)$, with k a Lipschitz constant for φ' on $\{x \ge x_{\sqrt{2}} - \varepsilon_0\}$ (ε_0 is some small constant, we take $\varepsilon \le \varepsilon_0$). Since $1-\lambda\gamma > 0$, the left hand term is larger that $e^{2\sigma_0}/2$ if ε is small enough and then the claim follows from L5. Moreover, the first statement in the lemma is a direct consequence of our claim (cf. the proof of Lemma 2.1).

In order to deduce the second part of the lemma we may suppose $|\varphi_a'(x)| \le e^{\sigma_0}$, for otherwise there is nothing to prove. Observe also that if $\varphi_a^n(x) \in \Delta_+^{\pm c}$ then, by (5), (6), we have $|\varphi_a'(\varphi_a^{n-1}(x))| \ge k_8\varepsilon^{1-\frac{1}{\lambda}}$, with $k_8 = \lambda e^{2(\lambda-1)}(\eta_1^-)^{\frac{1}{\lambda}}$.

Moreover, by hypothesis, $x \notin \Delta^{\pm c}$ and so $|\varphi_a'(x)| \ge \eta_1^-\lambda\varepsilon^{\gamma+\delta}$. Altogether, writing $k_9 = \eta_1^-\lambda k_8$,

$$\begin{aligned} |(\varphi_a^n)'(x)| &\ge |\varphi_a'(x)| e^{\sigma_0(n-2)} |\varphi'(\varphi_a^{n-1}(x))| &&\ge k_9\varepsilon^{\gamma+\delta} e^{\sigma_0(n-2)}\varepsilon^{1-\frac{1}{\lambda}} \\ &\ge k_9\varepsilon^{\gamma+\delta+1-\frac{1}{\lambda}} e^{\sigma_0(n-2)} &&\ge e^{\sigma_0 n}, \end{aligned}$$

if ε is small enough. □

2.3. Recovering expansion. Now we deal with the expansion losses occurring when trajectories pass close to some of the critical points $\pm c$. More precisely, we consider points $x \in \Delta^{\pm c}$. Assuming that the critical trajectories satisfy (exponential) expansivity and bounded recurrence conditions (during a convenient number of iterates, depending on $|x \pm c|$), we show that the small value of $\varphi_a'(x)$ is fully compensated in the subsequent iterates, during which the trajectory of x remains close to that of the critical point (and so exhibits rapidly increasing derivative).

For each $j \ge 0$ let $c_j = c_j(a) = \varphi_a^j(\pm c)$ and denote $d(c_j) = \min\{|c_j|, |c_j \pm c|\}$. In what follows $\varepsilon > 0$ is fixed and we suppose $a \in [c + \varepsilon/2, c + \varepsilon]$.

LEMMA 2.4. *There exists $\theta = \theta(\beta - \alpha) > 0$ such that the following estimates hold. Let $x \in I_r^c$ for some $r \ge r_\varepsilon + 1$. Suppose that there is $n \ge r/\alpha$ such that*

(7) $\qquad d(c_j) \ge \varepsilon^\gamma e^{-\alpha j}$ *and* $|(\varphi_a^j)'(c_1)| \ge e^{\sigma j}$, *for all* $1 \le j \le n-1$.

Then there exists an integer $p = p(x) \ge 1$ such that

(1) *For all* $y_1, z_1 \in [\varphi_a(x), \varphi_a(c)]$ *and for all* $1 \le k \le p$,

$$\frac{1}{\theta} \le \frac{|(\varphi_a^k)'(z_1)|}{|(\varphi_a^k)'(y_1)|} \le \theta$$

(2) $p \le 2(r + \gamma \log \varepsilon) \le n - 1$;
(3) $|(\varphi_a^{p+1})'(x)| \ge e^{(r-r_\varepsilon)(1-2\beta)}$ *and* $|(\varphi_a^{p+1})'(x)| \ge e^{\beta(p+1)}$.

PROOF. Define $p = p(x) \ge 1$ as the maximum integer such that

$$|x_i - c_i| \le \varepsilon^\gamma e^{-\beta i} \quad \text{for all } 1 \le i \le p \tag{8}$$

where $x_i = \varphi_a^i(x)$. Recall that we fix $\beta > \alpha$. Therefore, (8) and the first condition in (7) ensure that the intervals $[x_i, c_i]$, $1 \le i \le p$, do not contain the origin nor any of the critical points $\pm c$. Therefore, $\varphi_a^i : [x_1, c_1] \to [x_{i+1}, c_{i+1}]$ is a diffeomorphism for all $1 \le i \le p$. In particular, given any $y_1, z_1 \in [x_1, c_1]$ we have $y_i, z_i \in [x_i, c_i]$ for $1 \le i \le p$, where $y_i = \varphi_a^i(y)$ and $z_i = \varphi_a^i(z)$. By the chain rule,

$$\left|\frac{(\varphi_a^k)'(z_1)}{(\varphi_a^k)'(y_1)}\right| = \prod_{i=1}^k \left|\frac{\varphi_a'(z_i)}{\varphi_a'(y_i)}\right| = \prod_{i=1}^k \left|1 + \frac{\varphi_a'(z_i) - \varphi_a'(y_i)}{\varphi_a'(y_i)}\right|$$

and so part (1) will follow if we show that

$$\sum_{i=1}^k \left|\frac{\varphi_a'(z_i) - \varphi_a'(y_i)}{\varphi_a'(y_i)}\right| \tag{9}$$

is bounded by some constant depending only on $\beta - \alpha$. Now, by the mean value theorem there exists some $\xi_i \in [z_i, y_i]$ such that

$$\left|\frac{\varphi_a'(z_i) - \varphi_a'(y_i)}{\varphi_a'(y_i)}\right| = \frac{|z_i - y_i|\,|\varphi_a''(\xi_i)|}{|\varphi_a'(y_i)|} \le \varepsilon^\gamma e^{-\beta i}\frac{|\varphi_a''(\xi_i)|}{|\varphi_a'(y_i)|},$$

since $|z_i - y_i| \le |x_i - c_i| \le \varepsilon^\gamma e^{-\beta i}$. In order to estimate the ratio $|\varphi_a''(\xi_i)| \,/\, |\varphi_a'(y_i)|$ we distinguish two cases. Let some small constant $\varepsilon_0 > 0$ be fixed such that M1 holds on $[-2\varepsilon_0, 2\varepsilon_0]$ (we take $\varepsilon \le \varepsilon_0$). Suppose first that $[x_i, c_i] \cap [-\varepsilon_0, \varepsilon_0] = \emptyset$. Then $|\varphi_a''(\xi_i)| \le k$, where k depends on ε_0 but not on ε. Moreover, by M2, (8), and the first part of (7), $|\varphi_a'(y_i)| \ge 2\eta_2^- \varepsilon^\gamma(\varepsilon^{-\alpha i} - e^{-\beta i})$. Thus we get

$$\left|\frac{\varphi_a'(z_i) - \varphi_a'(y_i)}{\varphi_a'(y_i)}\right| \le \frac{k\varepsilon^\gamma e^{-\beta i}}{2\eta_2^- \varepsilon^\gamma(e^{-\alpha i} - e^{-\beta i})} \le \frac{k e^{-(\beta-\alpha)i}}{2\eta_2^-(1 - e^{-(\beta-\alpha)i})} \le \text{const}\, e^{-(\beta-\alpha)i}.$$

Now we show that a similar estimate holds also if $[x_i, c_i] \cap [-\varepsilon_0, \varepsilon_0] \ne \emptyset$. Indeed, since $y_i \in [x_i, c_i] \subset [-2\varepsilon_0, 2\varepsilon_0]$, we have

$$|\varphi_a'(y_i)| \ge \eta_1^- \lambda |y_i|^{\lambda-1} \ge \eta_1^- \lambda(|c_i| + \varepsilon^\gamma e^{-\beta i})^{\lambda-1}.$$

On the other hand, $|\varphi_a''(\xi_i)| \le \text{const}\,|\xi|^{\lambda-2} \le \text{const}\,(|c_i| - \varepsilon^\gamma e^{-\beta i})^{\lambda-2}$, recall L1. In this way we get

$$\left|\frac{\varphi_a'(z_i) - \varphi_a'(y_i)}{\varphi_a'(y_i)}\right| \le \text{const}\,\varepsilon^\gamma e^{-\beta i}\frac{(|c_i| - \varepsilon^\gamma e^{-\beta i})^{\lambda-2}}{(|c_i| + \varepsilon^\gamma e^{-\beta i})^{\lambda-1}} \le \text{const}\,\frac{\varepsilon^\gamma e^{-\beta i}}{|c_i| - \varepsilon^\gamma e^{-\beta i}}$$
$$\le \text{const}\,\frac{e^{-\beta i}}{e^{-\alpha i} - e^{-\beta i}} \qquad \le \text{const}\,e^{-(\beta-\alpha)i},$$

as long as $\varepsilon > 0$ is small enough. Altogether, this shows that (9) is bounded by $\sum_{i\ge 0} \text{const}\,e^{-(\beta-\alpha)i} \le \text{const}\,/(1 - e^{-(\alpha-\beta)})$ which completes the proof of part (1).

Starting the proof of (2), let $q = \min\{p, n\}$. Since $x \in I_r^c$, we have $|x - c| \ge \varepsilon^\gamma e^{-r}$ and so $|x_1 - c_1| \ge \eta_2^- \varepsilon^{2\gamma} e^{-2r}$. Then, in view of the second condition in (7) and the distortion estimate we have just proved, the mean value theorem yields

$$\eta_2^- \varepsilon^{2\gamma} e^{-2r}\theta^{-1} e^{\sigma(q-1)} \le |x_q - c_q| \le \varepsilon^\gamma e^{-\beta q}.$$

Thus

$$q \le \frac{2r + \gamma \log\frac{1}{\varepsilon} + \sigma + \log(\eta_2^-/\theta)}{\sigma + \beta} \le 2(r + \gamma \log\frac{1}{\varepsilon})$$

as long as ε is sufficiently small. Since we also take $\alpha n \ge r \ge r_\varepsilon + 1 \ge \delta \log\frac{1}{\varepsilon}$, we find that $q \le 2(\alpha n + (\gamma/\delta)\alpha n) < n$ (if α is small), so that it must be $q = p$. In this way we have proved that $p \le 2(r + \gamma \log\frac{1}{\varepsilon}) < n$, as claimed in part (2) of the lemma.

Now, by the definition of p we have $|x_{p+1} - c_{p+1}| \ge \varepsilon^\gamma e^{-\beta(p+1)}$. Thus, using part (1) in conjunction with the mean value theorem,

$$|(\varphi_a^p)'(x_1)| \ge \frac{1}{\theta}\frac{|x_{p+1} - c_{p+1}|}{|x_1 - c_1|} \ge \frac{\varepsilon^\gamma e^{-\beta(p+1)}}{\theta\eta_2^- \varepsilon^{2\gamma} e^{-2r+2}} \ge \text{const}\,\varepsilon^{-\gamma} e^{2r-\beta p}.$$

Since $|\varphi_a'(x)| \ge 2\eta_2^- \varepsilon^\gamma e^{-r}$, we find

$$(10) \qquad |(\varphi_a^{p+1})'(x)| \ge \text{const}\,e^{r-\beta p}.$$

Using part (2) and $r \ge r_\varepsilon + 1 \ge \delta \log\frac{1}{\varepsilon}$, we get $r - \beta p \ge r(1 - 2\beta) - 2\beta\gamma \log\frac{1}{\varepsilon}$ and

$$|(\varphi_a^{p+1})'(x)| \ge \text{const}\,e^{(1-2\beta)r - (2\beta\gamma/\delta)r_\varepsilon} \ge e^{(1-2\beta)(r - r_\varepsilon)},$$

as long as we fix $\beta > 0$ small (such that $2\beta\gamma < \delta(1 - 2\beta)$) and take r_ε large (that is, ε small) enough. This proves the first statement in part (3).

Combining (10) with $r \ge \frac{p}{2} - \gamma \log\frac{1}{\varepsilon}$ we also get $|(\varphi_a^p)'(x)| \ge \text{const}\,e^{(\frac{1}{2}-\beta)p}\varepsilon^\gamma$. We now distinguish two cases. Suppose first that $e^{-(\frac{1}{2}-2\beta)p} \le \varepsilon^{\frac{1}{\lambda}-1}$. Then

$$|(\varphi_a^{p+1})'(x)| \ge \text{const}\,e^{(\frac{1}{2}-\beta)p} e^{-(\frac{1}{2}-2\beta)p} \varepsilon^{\gamma - \frac{1}{\lambda}+1} \ge \text{const}\,e^{\beta p}\varepsilon^{\gamma-\frac{1}{\lambda}+1} \ge e^{\beta(p+1)}$$

if ε is small (recall that $\gamma - \frac{1}{\lambda} + 1 < 0$). Now suppose that $e^{(\frac{1}{2}-2\beta)p} \le \varepsilon^{1-\frac{1}{\lambda}}$. Then

$$\beta p \le \beta(\frac{1}{\lambda} - 1)(\frac{2}{1-4\beta}) \log\frac{1}{\varepsilon} \le \beta(\frac{1}{\lambda}-1)(\frac{2}{1-4\beta})\frac{r}{\delta} \le \frac{r}{4}$$

if β is small. It follows from (10) that

$$|(\varphi_a^{p+1})'(x)| \ge \text{const}\, e^{\frac{r}{2}+\beta p} \ge \text{const}\, \varepsilon^{-\frac{\delta}{2}} e^{\beta p} \ge e^{\beta(p+1)}$$

if ε is small. This completes the proof of part (3) and of the lemma. □

2.4. Proving positive Lyapunov exponents. We can now state the main results of this section, asserting that, under two convenient assumptions on the parameter a to be given in CP1, CP2 below, the critical trajectories exhibit exponential growth of the derivative and, in fact, the same is true for most trajectories of φ_a.

As before, we write $c_j = c_j(a) = \varphi_a^j(c)$, for $j \ge 1$. For the time being we fix some $n \ge 1$ and assume that

CP1(n): $d_j = \min\{|c_j|, |c_j \pm c|\} \ge \varepsilon^\gamma e^{-\alpha j}$ for all $1 \le j \le n$.

and

EG($n-1$): $|(\varphi_a^j)'(c_1)| \ge e^{\sigma j}$ for all $1 \le j \le n-1$.

Then we define sequences of integers ν_i, p_i, by $\nu_1 = \inf\{\nu \ge 1 : c_\nu \in \Delta^{\pm c}\}$ and

i) $p_i = p(c_{\nu_i})$, as given by Lemma 2.4;

ii) $\nu_{i+1} = \inf\{\nu > \nu_i + p_i : c_\nu \in \Delta^{\pm c}\}$.

(CP1(n) ensures that $c_{\nu_i} \in I_r^{\pm c}$ for some $r \le \alpha\nu_i \le \alpha n$). We take s maximum such that $\nu_s \le n$. Then either $\nu_s \le n < \nu_s + p_s$ or $\nu_s < \nu_s + p_s \le n$. We define $P_n = (p_1+1)+\cdots+(p_{s-1}+1)$ in the first case and $P_n = (p_1+1)+\cdots+(p_{s-1}+1)+(p_s+1)$ in the second one. Then we further assume that

CP2(n): $P_j \le \frac{j}{2}$ for all $1 \le j \le n$.

LEMMA 2.5. *Suppose that some parameter $a \in [c+\varepsilon/2, c+\varepsilon]$ satisfies CP1(n), CP2(n), and EG($n-1$). Then it also satisfies*

EG(n): $|(\varphi_a^j)'(c_1)| \ge e^{\sigma j}$ *for all* $1 \le j \le n$.

PROOF. We let ν_i, p_i, be as above and define $q_0 = \nu_1 - 1$ and $q_i = \nu_{i+1} - (\nu_i + p_i + 1)$ for $1 \le i \le s-1$. If $n \ge \nu_s + p_s$ we also write $q_s = n - (\nu_s + p_s)$. Then

(11)

$$|(\varphi_a^n)'(c_1)| = |(\varphi_a^{q_0})'(c_1)| \prod_{i=1}^{s-1}(|(\varphi_a^{p_i+1})'(c_{\nu_i})|\,|(\varphi_a^{q_i})'(c_{\nu_1+p_1+1})|)\,|(\varphi_a^{n-\nu_s+1})'(c_{\nu_s})|.$$

The first factor on the right can be estimated as follows. Since $\varphi_a(c_{q_0}) = c_{\nu_1} \in \Delta^{\pm c}$, relations (5) and (6) yield $|\varphi_a'(c_{q_0})| \geq \text{const}\,(\varepsilon^\gamma e^{-r_c})^{\lambda-1} \geq \text{const}\,\varepsilon^{\frac{1}{\lambda}-1}$. Hence, using also the first part of Lemma 2.3,

$$|(\varphi_a^{q_0})'(c_1)| = |(\varphi_a^{q_0-1})'(c_1)||(\varphi_a)'(c_{q_0})| \geq \text{const}\, e^{\sigma_0 q_0}\varepsilon^{1-\frac{1}{\lambda}}$$

(note that the last inequality in L4 implies $|c_1| < x_{\sqrt{2}}$ and so $|\varphi_a'(c_1)| > \sqrt{2} > e^{\sigma_0}$ for all a close to c). On the other hand, Lemma 2.4(3) and the second part of Lemma 2.3 give, for $1 \leq i \leq s-1$,

$$|(\varphi_a^{p_i+1})'(c_{\nu_i})| \geq e^{\beta(p_i+1)} \quad \text{and} \quad |(\varphi_a^{q_i})'(c_{\nu_i+p_i+1})| \geq e^{\sigma_0 q_i}.$$

For estimating the last factor in (11), we distinguish two cases. If $n \geq \nu_s + p_s$ then we use Lemmas 2.4(3) and 2.3 once more and get

$$\begin{aligned}|(\varphi_a^{n-\nu_s+1})'(c_{\nu_s}) &= |(\varphi_a^{p_s+1})'(c_{\nu_s})| \cdot |(\varphi_a^{q_s})'(c_{\nu_s+p_s+1})| \\ &\geq e^{\beta(p_s+1)}\min\{e^{\sigma_0}, |\varphi_a'(c_{\nu_s+p_s+1})|\}e^{\sigma_0(q_s-1)} \\ &\geq \text{const}\, e^{\beta(p_s+1)}\varepsilon^{\gamma+\delta}e^{\sigma_0 q_s}\end{aligned}$$

(the final bound remains valid when $n = \nu_s + p_s$, i.e. $q_s = 0$). Replacing in (11),

$$|(\varphi_a^n)'(c_1)| \geq \text{const}\,\varepsilon^{1-\frac{1}{\lambda}+\gamma+\delta}e^{\left(\sigma_0\sum_{i=0}^s q_i+\beta\sum_{i=1}^s(p_i+1)\right)}. \tag{12}$$

Now, CP2(n) implies (recall that we take $\sigma_0 > 2\sigma$)

$$\sigma_0\sum_{i=0}^s q_i + \beta\sum_{i=1}^s(p_i+1) \geq \sigma_0(n-P_n) + \beta P_n \geq \sigma_0\frac{n}{2} > \sigma n$$

and the lemma follows by replacing this in (12) and assuming ε sufficiently small.

Suppose now that $\nu_s \leq n < \nu_s + p_s$. In this case we can not take advantage of the estimates in Lemma 2.4(3), as we did before. Instead, we use CP1(n), EG($n-1$), and the distortion estimate in Lemma 2.4(1), to conclude that

$$|(\varphi_a^{n-\nu_s+1})'(c_{\nu_s})| = |\varphi'(c_{\nu_s})| \cdot |(\varphi_a^{n-\nu_s})'(c_{\nu_s+1})| \geq \text{const}\,\varepsilon^\gamma e^{-\alpha\nu_s}\,\text{const}\,e^{\sigma(n-\nu_s)}.$$

This gives

$$|(\varphi_a^n)'(c_1)| \geq \text{const}\,\varepsilon^{1-\frac{1}{\lambda}+\gamma}e^{\left(\sigma_0\sum_{i=0}^{s-1} q_i+\beta\sum_{i=1}^{s-1}(p_i+1)-\alpha\nu_s+\sigma(n-\nu_s)\right)}. \tag{13}$$

Now,

$$\begin{aligned}\sigma_0\sum_{i=0}^{s-1} q_i + \beta\sum_{i=1}^{s-1}(p_i+1) - \alpha\nu_s + \sigma(n-\nu_s) &\geq \sigma_0(\nu_s - P_{\nu_s}) - \alpha\nu_s + \sigma(n-\nu_s) \\ &\geq \frac{\sigma_0}{2}\nu_s - \alpha\nu_s + \sigma(n-\nu_s) \geq \sigma n\end{aligned}$$

as long as we take $2\alpha < \sigma_0 - 2\sigma$. Replacing in (13) (and assuming ε small) we get the conclusion of the lemma also in this case. Our argument is complete. □

PROPOSITION 2.6. *Suppose that some parameter $a \in [c+\varepsilon/2, c+\varepsilon]$ satisfies CP1(n) and CP2(n) for all $n \geq 1$. Then $|(\varphi_a^n)'(c_1)| \geq e^{\sigma n}$ for all $n \geq 1$.*

PROOF. This follows directly from the previous lemma, by induction on n. Observe that the step $n = 1$ is an immediate consequence of L4: as we already remarked, it implies $|c_1| < x_{\sqrt{2}}$ and so $|\varphi_a'(c_1)| > \sqrt{2} > e^{\sigma}$. □

3. Partitions and distortion estimates

Our objective in this section is to set up the machinery which will enable us, in the next section, to estimate the size of the set of parameters for which the corresponding maps satisfy the conditions CP1(n) and CP2(n) for all $n \geq 1$.

A main ingredient is a family of maps $\{c_j\}_{j\in\mathbb{N}}$ defined on parameter space and taking values in dynamical space. More precisely we fix some ε sufficently small and let $\omega_0 = [c+\frac{\varepsilon}{\rho}, c+\varepsilon]$ denote a small interval of parameter values (where $\rho > 1$ is chosen as in lemma 2.1). Then we define, for each $j \in \mathbb{N}$ a map

$$\begin{aligned} c_j : \omega_0 &\longrightarrow [-c-\varepsilon, c+\varepsilon] \\ a &\longmapsto \varphi_a^j(c_1). \end{aligned}$$

Our first lemma states that the derivatives of the maps c_j, i.e. the partial derivatives $\partial_a\varphi_a^j(c_1)$ of $\varphi_a^j(c_1)$ with respect to the parameter, are growing exponentially as long as the partial derivatives $\partial_x\varphi_a^j(c_1) = (\varphi_a^j)'(c_1)$ are growing exponentially. This is quite a general fact for parametrized families of maps and will be used in a fundamental way below since it implies that the images $c_j(\omega_0)$ are growing exponentially as long as the derivatives along the critical orbit of the corresponding maps are.

LEMMA 3.1. *There exists a constant $\eta > 1$ such that if $|\partial_x\varphi_a^i(x)| \geq e^{\sigma i}, \leqq 1in$, then*

$$\eta^{-1} \leq \frac{|\partial_x\varphi_a^i(x)|}{|\partial_a\varphi_i^a(x)|} \leq \eta \qquad \leqq 1in$$

PROOF. Using the chain rule we can write

$$\partial_a\varphi_a^n(x) = \partial_a\varphi_a(\varphi_a^{n-1}(x)) + \partial_x\varphi_a(\varphi_a^{n-1}(x))\partial_a\varphi_a^{n-1}(x).$$

Reapplying the same argument several times and using the shorthand notation $\partial_x^i = \partial_x\varphi_a(\varphi_a^i(x))$ and $\partial_a^i = \partial_a\varphi_a(\varphi_a^i(x))$ we get

$$\partial_a\varphi_a^n(x) = \partial_a^{n-1} + \partial_a^{n-2}\partial_x^{n-1} + \cdots + \sum_{i=0}^{n-1}\partial_x^i$$

and, dividing through by $\partial_x\varphi_a^n(x) = \sum_{i=0}^{n-1}$ we get

$$\begin{aligned}\frac{\partial_a\varphi_a^n(x)}{\partial_x\varphi_a^n(x)} &= \frac{\partial_a^{n-1}+\partial_a^{n-2}\partial_x^{n-1}+\cdots+\sum_{i=0}^{n-1}\partial_x^i}{\sum_{i=0}^{n-1}\partial_x^i}\\ &= 1+\frac{1}{\partial_x\varphi_a(x)}+\cdots+\frac{1}{\partial_x\varphi_a^n(x)}\\ &= 1+\sum_{i=1}^{n}\frac{1}{\partial_x\varphi_a^i(x)}.\end{aligned}$$

Using again the fact that $|1-|z|| \le |1+z| \le 1+|z|$ for any real number z, we get

$$\begin{aligned}\left|\frac{\partial_a\varphi_a^n(x)}{\partial_a\varphi_a^n(x)}\right| &\le 1+\left|\sum_{i=1}^{n}\frac{1}{\partial_x\varphi_a^i(x)}\right|\\ &\le 1+\sum_{1=0}^{\infty}e^{-\sigma i}\end{aligned}$$

and

$$\begin{aligned}\left|\frac{\partial_a\varphi_a^n(x)}{\partial_a\varphi_a^n(x)}\right| &\ge \left|1-\sum_{i=1}^{n}\frac{1}{\partial_x\varphi_a^i(x)}\right|\\ &\ge \left|1-\sum_{i=0}^{\infty}e^{-\sigma i}\right|.\end{aligned}$$

The lemma is proved. □

From now on every construction and formula involving ε (e.g. the partitions $\Delta^0, \Delta^{\pm c}$, (CP1), etc.) is defined in terms of the ε chosen above which will remain fixed for the rest of the paper.

3.1. Partitions. We shall now use the family of maps $\{c_j\}$ together with Lemma 3.1 to construct two nested families $\{F_n\}_{n\in\mathbb{N}}$ and $\{E_n\}_{n\in\mathbb{N}}$ of subsets of ω_0. More precisely we shall have

$$\cdots \subseteq E_n \subseteq F_n \subseteq E_{n-1} \subseteq \cdots \subseteq \omega_0.$$

Each F_n will be formed by a finite number of intervals of parameter values and the partition of F_n into such intervals will be denoted by $\mathcal{P}_n$. By construction all parameters belonging to F_n will have the property that the corresponding maps satisfy condition CP1 up to iterate n, i.e. for all $a \in F_n$ the condition

(CP1) $$d(c_j) \ge \varepsilon^{\gamma} e^{-\alpha j}$$

is satisfied for all $1 \leq j \leq n$. E_n is obtained by throwing out those elements of $\mathcal{P}_n$ which do not satisfy CP2 up to n. Thus every parameter $a \in E_n$ has the property that the orbit of the critical point of the corresponding map satisfies

$$\text{(CP2)} \qquad P_n \leq \frac{1}{2} n.$$

Finally the set $\mathcal{A}^+ = \bigcap_{n \in \mathbb{N}} E_n$ will be formed by parameters which satisfy CP1 and CP2 for all time and thus, by Proposition 2.6, the orbit of the critical points of the corresponding maps will have an exponentially growing derivative for all time, i.e. a positive Lyapunov exponent.

The construction of the objects described above is carried out inductively. For the first step of the induction we simply set $E_0 = F_0 = \omega_0$ and $\mathcal{P}_0 = \{\omega_0\}$. Now suppose that E_{n-1}, F_{n-1} and $\mathcal{P}_{n-1}$ have been defined. We shall discuss below how parameters are excluded and the partition $\mathcal{P}_n$ and the sets F_n and E_n are defined. We shall consider seperately the cases $n < \frac{r_\varepsilon}{\alpha}$ and $n \geq \frac{r_\varepsilon}{\alpha}$.

Case 1: $n < \frac{r_\varepsilon}{\alpha}$. Suppose first that for all $1 \leq j \leq n-1$,

$$c_j(\omega_0) \cap \Delta_+ = \emptyset.$$

Then, if also $c_n(\omega_0) \cap \Delta_+ = \emptyset$ we simply set $E_n = F_n = \omega_0$ again and also let $\mathcal{P}_n = \{\omega_0\}$ (in fact in this case we even have $E_j = F_j = \omega_0$ and $\mathcal{P}_j = \omega_0$ for all $1 \leq j \leq n$). If, on the other hand, $c_n(\omega_0) \cap \Delta_+$ then there is a possibility that some parameters parameters fail to satisfy CP1 and therefore need to be thrown out. More precisely we have to exclude any intersection of $c_n(\omega_0)$ with an $\varepsilon^\gamma e^{-\alpha n}$-neighbourhood of the origin and of the critical points. We shall denote such a neighbourhood by $\Delta_{\alpha n}$.

Notice that since $c_j(a) \cap \Delta_+ = \emptyset$, $\forall 1 \leq j \leq n-1$ for all $a \in \omega_0$ we have $|(\varphi_a^j)'(c_1)| \geq e^{\sigma_0 j}$ $\forall 1 \leq j \leq n-1$ by Lemma 2.3 and therefore $|c_j'(a)| \geq \eta e^{\sigma_0 j}$ by Lemma 3.1, and so

$$\begin{aligned} |c_j(\omega_0)| &\geq \eta e^{\sigma_0 j} |\omega_0| \\ &\geq \eta(\rho - 1)\varepsilon \\ &\gamma \;\; \varepsilon^\gamma \\ &\geq \varepsilon^\gamma e^{-\alpha n} \end{aligned}$$

by the mean value theorem. In particular the proportion of excluded parameters at this stage is very small.

Let

$$\tilde{F}_n = F_{n-1} \setminus c_n^{-1}(\Delta_{\alpha n} \cap c_n(\omega_0)).$$

For technical reasons we also want to exclude at this point any connected component of $\tilde{F}_n$ which is too "small" in the sense that it image under c_n is completely contained in Δ_+. Denote by F_n the remaining parameters. Define

$$\mathcal{P}_n = \{ \text{ connected components of } F_n\}.$$

Since n is the first return there have certainly not been any binding periods and so we set $E_n = F_n$. We now continue iterating the elements of $\omega \in \mathcal{P}_n$ and consider their images $c_j(\omega)$ repeating the procedure above if there is any intersection with Δ_+. In fact we shall say that an iterate $j < \frac{r_\varepsilon}{\alpha}$ is a return (for the interval $\omega \in \mathcal{P}_{j-1}$) only if exclusions need to be made since the rest of the time the interval ω is essentially in a free period (see remarks below).

REMARK 3.2. As long as $n < \frac{r_\varepsilon}{\alpha}$ the situation is, in some sense, particularly simple. Indeed this condition implies that the exclusion zone $\Delta_{\alpha n}$ contains $\Delta^0 = (-\varepsilon^{\gamma+\delta}, \varepsilon^{\gamma+\delta})$. In particular it means that we can never have any interesections with $\Delta^{\pm c}_{\mp}$ since $c_n(\omega) \cap \Delta^{\pm c}_{\mp} \neq \emptyset$ would imply $c_{n-1}(\omega) \cap \Delta^0 \neq \emptyset$ which is impossible because any parameters falling into Δ^0 would be thrown out. This means that there are no binding periods and

$$E_n = F_n \qquad \forall\, n < \frac{r_\varepsilon}{\alpha}.$$

REMARK 3.3. Another consequence of remark 3.2 is that for all $n < \frac{r_\varepsilon}{\alpha}$ and $a \in F_n$ we have by Lemma 2.3

$$|(\varphi_a^j)'(c_1)| \geq e^{\sigma_0 j} \qquad \forall 1 \leq j \leq n.$$

REMARK 3.4. Observe also that $\frac{r_\varepsilon}{\alpha}$ can be made arbitrarily large by taking ε or α small.

Case 2: $n \geq \frac{r_\varepsilon}{\alpha}$. We suppose that for each $a \in E_{n-1}$ a sequence

$$0 < \nu_1(a) < \nu_2(a) < \cdots < \nu_s(a), \qquad s = s(a)$$

of return times has been defined. For each $\omega \in \mathcal{P}_{n-1}, \omega \subset E_{n-1} \subset F_{n-1}$ we distinguish two basic possibilities.

1) If

- n belongs to the binding period of some previous return, i.e. if

$$\nu_s(a) + 1 \leq n \leq \nu_s(a) + p_s(a), \quad \forall a \in \omega;$$

or

- $c_n(\omega)$ does not intersect Δ, i.e.

$$c_n(\omega) \cap \Delta = \emptyset$$

or

- the intersection between $c_n(\omega)$ and Δ is very small,i.e.

$$c_n(\omega)\cap\Delta\subset I^*_{r_\varepsilon},\quad *=0,\pm c$$

or

- $c_n(\omega)$ is completely contained in Δ but does not properly contain any subintervals $I^*_r,\quad r\geq r_\varepsilon, *=0,\pm c$

then no exclusions are necessary (see also Lemma 3.7 below). We set $\omega\subset F_n,\omega\in\mathcal{P}_n$. We throw out those $\omega\in\mathcal{P}_n$ which do not satisfy CP2 and define E_n as the union of those remaining.

REMARK 3.5. in the last subcase above we say that n is an *inessential return* for all $a\in\omega$. We shall prove in Lemma 3.7 below that inessential returns always satisfy CP1, thus justifying the fact that we have made no exclusions as explained above. If this return occurs in $\Delta^{\pm c}_{\mp}$ then we define for future reference, a *binding period* of the interval ω by

$$p(\omega)=\min_{a\in\omega}\{p(a)\}.$$

2) If $c_n(\omega)$ contains some I^*_r with $*=0,\pm c$ and $r\geq r_\varepsilon$ then we say that n is an *essential return* for every $a\in\omega$. We start by throwing out the parameters which do not satisfy CP1, i.e. the set

$$c_n^{-1}(c_n(\omega)\cap\Delta_{\alpha n})$$

as well as any connected component $\omega'\subset\omega\setminus c_n^{-1}(c_n(\omega)\cap\Delta_{\alpha n})$ such that $c_n(\omega')$ does not contain at least one I^*_r. Let $\tilde{\omega}$ denote the remaining parameters.

We divide $\tilde{\omega}$ into a finite number of subintervals (which we shall denote again by ω) in such a way that the image $c_n(\omega)$ of each of these subintervals *contains exactly one I^*_r with $r\geq r_\varepsilon$*. These subintervals are by definition elements of $\mathcal{P}_n$ as well as subsets of F_n. Those which satisfy $CP2$ at this point also belong to E_n and those that don't get thrown out. This completes the inductive definition of the sets E_n,F_n and the partitions $\mathcal{P}_n$.

REMARK 3.6. Observe that to each $\omega\in\mathcal{P}_n$ is associated a sequence

$$0<\nu_1(\omega)<\nu_2(\omega)<\cdots<\nu_s(\omega),\qquad s=s(\omega)$$

of return times corresponding to the essential and inessential returns of intervals of parameter values contained in ω_0 and containing ω. These returns can occur either in Δ^0 or in $\Delta^{\pm c}$.

Let $\{\nu_{i_k}\}$ denote the subsequence of return times in which the (essential or inessential) return occurs in $\Delta^{\pm c}$. To this subsequence can be associated a sequence $\{p_k\}$ corresponding to the lengths of the binding periods following each ν_{i_k}.

Let $\{\nu_{i_j}\}$ denote the subsequence formed by the essential returns (to Δ^0 or $\Delta^{\pm c}$). To this sequence can be associated a sequence $\{r^*_j\}$ of (signed) integers with

$|r_j^*| \geq r_\varepsilon$, $\quad * = 0, \pm c$ corresponding to the unique subinterval $I_{r_j}^*$ singled out by the essential return ν_{i_j} as explained above.

Notice that for a given sequence $\{r_j\}$ there can be at most one element $\omega \in \mathcal{P}_n$ to which that sequence can be associated. Concerning this point however, there are a couple of situations which need to be clarified. Notice first of all that all returns occuring before iterate $\frac{r_\varepsilon}{\alpha}$ should be thought of as essential returns. Such returns generally contain more than a single subinterval I_r^* and moreover they contain subintervals with values of $r < r_\varepsilon$. In this case we shall associate to them , by convention, the value r_ε. A similar problem might occur with essential returns $\nu_i \geq \frac{r_\varepsilon}{\alpha}$. Indeed it was specified above that the image under c_{ν_i} of such intervals should contain exactly one element I_r^* with $r \geq r_\varepsilon$. However, if $\omega \in \mathcal{P}_{\nu_i}$ and $c_{\nu_i}(\omega)$ contains I_{r_ε} it will, in general, also contain several other elements I_r with $r < r_\varepsilon$. in this case also we make the convention of assigning the value r_ε to such returns.

LEMMA 3.7. *Inessential returns satisfy (CP1)*

PROOF. Let $\omega \subset F_{n-1}, \omega \in \mathcal{P}_{n-1}$ and suppose that $|(\varphi_a^j)'(c_1)| \geq e^{\sigma j} \leqq 1jn$ and $\forall a \in \omega$. Suppose n is an inessential return for ω and let $\nu < n$ denote the last essential return before n. Then

$$c_\nu(\omega) \supset I_r \qquad \text{for some } r_\varepsilon < r < [\alpha n]$$

and therefore

$$\begin{aligned} |c_\nu(\omega)| &\geq |I_r| \\ &\geq \varepsilon^\gamma(e^{r+1} - e^{-r}) \\ &\geq (e-1)\varepsilon^\gamma e^{-r}. \end{aligned}$$

We shall show below that

$$|c_n(\omega)| \geq 2\varepsilon^\gamma e^{-[\alpha n]+1}$$

implying that

- either all $a \in \omega$ satisfy (CP1) for iterate n;
- or, if some $a \in \omega$ is close enough to the origin or to the critical points to fail to satisfy (CP1) then, due to the length of $c_n(\omega)$ it will certainly contain at least $I_{[\alpha n]-1}$ contradicting the fact that n is an inessential return.

We shall consider seperately the cases $c_\nu(\omega) \subset \Delta^0$ and $c_\nu(\omega) \subset \Delta^{\pm c}$.

(1) $c_\nu(\omega) \subset \Delta^0$.

As was mentioned in remark 2.2 we can suppose that $\varphi_a^{-1}(\Delta_\mp^c) \subset I_{r_c}$ for some r_c with $|r_c| > |r_\varepsilon|$, for all $a \in \omega$. Thus we distinguish two subcases.

(a) $c_\nu(\omega) \supset I_{r_c} \supset \varphi_a^{-1}(\Delta_\mp^c)$;

(b) $c_\nu(\omega) \supset I - r, r \neq r_c$.

The first case cannot actually occur in our present setting since it implies $c_{\nu+1}(\omega) \supset (\Delta_+^c)$ and therefore $n = \nu + 1$ would be an essential return contradicting our hypotheses.
In the second case we have, by the mean value theorem, the distortion estimate in Lemma 4.1 and Lemma 2.4,

$$\begin{aligned} |c_n(\omega)| &\geq \text{const. } \varepsilon^{\gamma(2\lambda-1)} |c_\nu(\omega)| \\ &\geq \text{const. } \varepsilon^{\gamma(2\lambda-1)} \varepsilon^{\gamma} e^{-[\alpha\nu]} \\ &\geq \text{const. } \varepsilon^{\gamma(2\lambda-1)} \varepsilon^{\gamma} e^{-[\alpha\nu]+1} \\ &\geq 2\varepsilon^{\gamma} e^{-[\alpha n]+1} \end{aligned}$$

if $\varepsilon > 0$ is small enough.

(2) $c_\nu(\omega) \subset \Delta^c$

To obtain the desired estimate in this case we define a binding period $p = p(\omega) = \min_{a\in\omega}\{p(a)\}$. By Lemma 3.2 we have, for parameters for which $p(a) = p(\omega)$,

$$|(\varphi_a^p)'(c_\nu(a))| \geq \frac{1}{\theta}\text{const.}\varepsilon^{2\beta\gamma-\delta(1-2\beta)} e^{(r-r_c)(1-2\beta)}$$

and, therefore, by using the bounded distortion estimate we get

$$|(\varphi_a^p)'(c_\nu(a))| \geq \frac{1}{\theta} \text{ const. } \varepsilon^{2\beta\gamma-\delta(1-2\beta)} e^{(r-r_c)(1-2\beta)}.$$

Then the mean value theorem implies

$$\begin{aligned} |c_{\nu+p}(\omega)| &\geq |c_\nu(\omega)| \cdot \inf_{a\in\omega} |(\varphi_a^p)'(c_\nu(a))| \\ &\geq 2\varepsilon^{\gamma} e^{-[\alpha n]+1} \end{aligned}$$

since $2\beta\gamma - \delta(1-2\beta) < 0$. The result follows from the fact that $|c_n(\omega)| \geq |c_{\nu+p}(\omega)|$. □

3.2. Distortion estimates.

LEMMA 3.8. *There exists a constant A such that if $\omega \in \mathcal{P}_n, \omega \subset E_{n-1}$ then*

$$\left| \frac{\partial_a \varphi_a^k(c_1(a))}{\partial_a \varphi_{a'}^k(c_1(a'))} \right| \leq A$$

$\forall 1 \leq k \leq n, \forall a, a' \in \omega$.

Notice that the constant A in the theorem is independent of ω and of n. This result follows by Lemma 3.1 and an analogous result proved below for the partial derivatives with respect to x.

LEMMA 3.9. *There exists a constant $A > 1$ such that if $\omega \in \mathcal{P}_{n-1}\omega \subset E_{n-1}$ then*

$$\left|\frac{(\varphi_a^k)'(c_1(a))}{(\varphi_{a'}^k)'(c_1(a'))}\right| \leq A$$

$\forall 1 \leq k \leq n, \forall a, a' \in \omega$.

PROOF. Observe first of all that by proposition 2.6, $\omega \in E_{n-1}$ implies $|(\varphi_a^k)'(c_1(a)) \geq e^{\sigma k}$ for all $1 \leq k \leq n$. Write $c_i = \varphi_a^j(c)$ and $c_i' = \varphi_{a'}^i(c)$. By the chain rule we can write

$$\begin{aligned}
\left|\frac{\partial_x\varphi_a^k(c_1(a))}{\partial_x\varphi_{a'}^k(c_1(a'))}\right| &= \prod_{i=1}^k \left|\frac{\varphi_a'(c_i)}{\varphi_{a'}'(c_i')}\right| \\
&= \prod_{i=1}^k \left|1 + \frac{\varphi_a'(c_i) - \varphi_a'(c_i')}{\varphi_{a'}'(c_i')}\right| \\
&\leq \prod_{i=1}^k 1 + \left|\frac{\varphi_a'(c_i) - \varphi_a'(c_i')}{\varphi_{a'}'(c_i')}\right| \\
&= \prod_{i=1}^k 1 + |A_i|.
\end{aligned}$$

Thus the proof reduces to showing that the sequence of partial sums $\sum_{i=1}^k |A_i|$ is bounded above by a constant independent of k. By the mean value theorem, for each i, there exists a $\xi_i \in [c_i, c_i']$ such that

$$|\varphi_a'(c_i) - \varphi_{a'}'(c_i')| = \varphi''(\xi_i)\,|c_i - c_i'|$$

and so we can write

$$|A_i| = \left|\frac{\varphi''(\xi_i)\,|c_i - c_i'|}{\varphi_{a'}'(c_i')}\right|.$$

Let $0 < \nu_1 < \nu_2 < \cdots < \nu_s \leq n$ be the returns of ω with corresponding index, as defined above, r_i *strictly greater* than r_ε. For the purposes of this lemma other returns (with $r_i = r_\varepsilon$) can actually be treated as free periods since they all admit the same uniform distorition estimates. Let $p_1, p_2, \ldots, p_{s'}$ denote the lengths of the binding periods corresponding to those returns which occur in $\Delta^{\pm\varepsilon}$. We start by estimating the total accumulated distortion during of free periods.

$$\begin{aligned}
F_j &= \sum_{\nu_{j-1}+p_{j-1}+1}^{\nu_j-1} |A_i| \\
&= \sum_{\nu_{j-1}+p_{j-1}+1}^{\nu_j-1} \left|\frac{\varphi''(\xi_i)\,|c_i - c_i'|}{\varphi_{a'}'(c_i')}\right|.
\end{aligned}$$

By the definition of free period $c_j(\omega)$ is uniformly bounded away from both the origin and the critical points. Notice also that the first and second derivatives in given points are independent of the parameter and we get,

$$|\varphi''(\xi_j)| \leq \left|\varphi''(\varepsilon^{\gamma+\delta})\right|$$

and

$$|\varphi'(c_i')| \geq \left|\varphi'(c-\varepsilon^{\gamma+\delta})\right|.$$

Thus

$$\left|\frac{\varphi''(\xi_i)}{\varphi'(c_i')}\right| \leq \left|\frac{\varphi''(\varepsilon^{\gamma+\delta})}{\varphi'(c-\varepsilon^{\gamma+\delta})}\right| = a_1(\varepsilon)$$

and

$$F_j \leq a_1 \sum_{\nu_{j-1}+p_{j-1}+1}^{\nu_j-1} |c_i - c_i'|.$$

By the inductive hypothesis $|(\varphi_a^j)'(c_1(a))| \geq e^{\sigma j} \quad \leqq 1jn \quad \forall a \in \omega$ and Lemma 4.1 we have

$$|c_i - c_i'| \geq \frac{1}{\theta} e^{\sigma(i-1)} |c_1 - c_1'|$$

and since the sequence of values $|c_i - c_i'|$ is bounded above by $\left|c_{\nu_j} - c_{\nu_j}'\right| \leq \left|c_{\nu_j}(\omega)\right|$ we get

$$\begin{aligned} F_j &\leq a_1 \sum \frac{1}{\theta} e^{-\sigma(\nu_j-1)} \left|c_{\nu_j}(\omega)\right| \\ &\leq a_2 \left|c_{\nu_j}(\omega)\right| \end{aligned}$$

where $a_2 = a_2(\varepsilon) = a_1\theta^{-1}\sum_{i=1}^{\infty} e^{-\sigma i}$.

For returns ν_i we distinguish two cases according as to whether the return occurs near the origin or near the critical point. If ν_i is a return to a neighbourhood of the critical point then $\varphi''(\xi_{\nu_i})$ admits an upper bound independent of ν_i. Indeed we have

$$\begin{aligned} |\varphi''(\xi_{\nu_i})| &\leq \max\{|\varphi''(x)| : x \in [c-\varepsilon^{\gamma+\delta}, c+\varepsilon^{\gamma+\delta}]\} \\ &= a_3(\varepsilon). \end{aligned}$$

Moreover, recall that, by construction, $c_{\nu_j}(\omega)$ is contained in at most three adjacent elements $I_{r_j+1} \cup I_{r_j} \cup I_{r_j-1}$ of the partition of $\Delta^{\pm\varepsilon}$. Thus

$$\left|\varphi'(c_{\nu_j}')\right| \geq 2\eta_1^- \varepsilon^{\gamma} e^{-r_j-1}$$

and

$$\begin{aligned}\left|c_{\nu_j} - c'_{\nu_j}\right| &\leq \left|\varepsilon^\gamma e^{-r_j+2} - \varepsilon^\gamma e^{-r_j-1}\right| \\ &\leq \varepsilon^\gamma e^{-r_j}\left|e^2 - e^{-1}\right|\end{aligned}$$

and, combining these estimates

$$\left|\frac{\varphi''(\xi_{\nu_j})\left|c_{\nu_j} - c'_{\nu_j}\right|}{\varphi'(c_{\nu_j})}\right| \leq a_4 \frac{\left|c_{\nu_j}(\omega)\right|}{\left|I_{r_j}\right|}.$$

If ν_j is a return to Δ^0 then

$$\begin{aligned}\left|\frac{\varphi''(\xi_{\nu_j})}{\varphi'(c'_{\nu_j})}\right| &\leq \frac{\sup\{\varphi''(x) : x \in (\varepsilon^\gamma e^{-r_j+2}, \varepsilon^\gamma e^{-r_j-1})\}}{\inf\{\varphi'(x) : x \in (\varepsilon^\gamma e^{-r_j+2}, \varepsilon^\gamma e^{-r_j-1})\}} \\ &\leq \left|\frac{\varphi''(\varepsilon^\gamma e^{-r_j-1})}{\varphi'(\varepsilon^\gamma e^{-r_j+2})}\right| \\ &\leq \frac{(\lambda-1)\eta_1^+(\varepsilon^\gamma e^{-r_j-1})^{\lambda-2}}{\eta_2^-(\varepsilon^\gamma e^{-r_j+2})^{\lambda-1}} \\ &\leq \frac{(\lambda-1)\eta_1^+ e^{-(\lambda-2)}(\varepsilon^\gamma e^{-r_j})^{\lambda-2}}{\eta_2^- e^{2(\lambda-1)}(\varepsilon^\gamma e^{-r_j})^{\lambda-1}} \\ &\leq a_5\left|I_{r_j}\right|^{-1}\end{aligned}$$

and so we get, as above,

$$\left|\frac{\varphi''(\xi_{\nu_j})\left|c_{\nu_j} - c'_{\nu_j}\right|}{\left|\varphi'(c_{\nu_j})\right|}\right| \leq a_5 \frac{\left|c_{\nu_j}(\omega)\right|}{\left|I_{r_j}\right|}.$$

Finally, consider the contribution of binding periods

$$B_j = \sum_{\nu_j+1}^{\nu_j+p_j}\left|\frac{\varphi''(\xi_i)\left|c_i - c'_i\right|}{\varphi'(c'_i)}\right|.$$

Repeating the arguments in the proof of Lemma 3.2 and keeping in mind that in our case the upper bound for $|c_i - c'_i|$ is $\left|c_{\nu_{j+1}}(\omega)\right|$ (and not $\varepsilon^\gamma e^{-\beta p}$ as in Lemma 3.2), we get

$$|B_j| \leq a_6\left|c_{\nu_j+1}(\omega)\right|$$

where $a_6 \leq$ const. $\sum_{i=1}^{\infty} e^{-\alpha i}$.

Thus we have

$$\begin{aligned}\sum_{i=1}^{k}|A_i| &\le a_2\left|c_{\nu_1}(\omega)\right| + a_5\frac{|c_{\nu_1}(\omega)|}{|I_{r_1}|} + a_6\left|c_{\nu_2}(\omega)\right| + a_2\left|c_{\nu_2}(\omega)\right| + \cdots \\ &\le a_7\sum_{j=1}^{s}\left|c_{\nu_j}(\omega)\right| + a_8\sum_{j=1}^{s}\frac{\left|c_{\nu_j}(\omega)\right|}{\left|I_{r_j}\right|} \\ &\le a_9\sum_{j=1}^{s}\frac{\left|c_{\nu_j}(\omega)\right|}{\left|I_{r_j}\right|}.\end{aligned}$$

since the $\left|c_{\nu_j}(\omega)\right|$ form an exponentially increasing bounded sequence and therefore the corresponding partial sums are uniformly bounded. For this very same reason the last sum above is uniformly bounded. Indeed write

$$\begin{aligned}\sum_{j=1}^{s}\frac{\left|c_{\nu_j}(\omega)\right|}{\left|I_{r_j}\right|} &= \sum_{r>r_\varepsilon}\frac{1}{|I_r|}\sum_{\{i:|r_i|=r\}}|c_{\nu_i}(\omega)| \\ &\le \sum_{r>r_\varepsilon}\frac{\text{const.}}{|I_r|}\end{aligned}$$

since the second sum is bounded above by a constant independent of r. Finally we get

$$\begin{aligned}\sum_{i=1}^{k}|A_i| &\le a_{10}\sum_{r>r_\varepsilon}\frac{1}{|I_r|} \\ &\le a_{10}\sum_{r>r_\varepsilon}\varepsilon^{\gamma}e^{-r} \\ &\le A. \quad \square\end{aligned}$$

4. Parameter exclusions

We now wish to estimate the total measure of the parameters excluded after each iterate. We shall consider seperately exclusions due to (CP1) which are carried out each time some interval of parameter values intersects a small neighbourhood of the origin or of the critical points, and exclusions due to (CP2).

4.1. Exclusions due to (CP1). Recall that, for each n, we need to exclude parameters a for which $c_n(a)$ falls into $\Delta_{\alpha n}$, an $\varepsilon^{\gamma}e^{-\alpha n}$- neighbourhood of the origin or of the critical points. Moreover, if $a\in\omega\in\mathcal{P}_{n-1}$ and $c_n(\omega)\setminus\Delta_{\alpha n}$ contains any small connected components, i.e. components which which do not contain at least one $I_r, r\le[\alpha n]+1$, then these components are also excluded. Thus, letting

$\hat{\omega} \subset \omega$ denote the subset of ω which gets thrown out at the n-th iterate, we have $c_n(\hat{\omega}) \subset \Delta_{[\alpha n]+1}$ and so

$$\frac{|c_n(\hat{\omega})|}{|c_n(\omega)|} \leq \frac{2\varepsilon^{\gamma} e^{-[\alpha n]+1}}{|c_n(\omega)}|$$

and, by the bounded distortion estimates above,

$$\begin{aligned} \frac{|\hat{\omega}|}{|\omega|} &\leq 2A\frac{\varepsilon^{\gamma} e^{[\alpha n]+1}}{|c_n(\omega)|} \\ &\stackrel{def}{=} A_n. \end{aligned}$$

Our objective in this section is to show that $|A_n| \to 0$ exponentially as $n \to \infty$ in order to get, for each n,

$$|F_{n-1} \cap F_n| \geq (1 - A_n)\,|F_n|$$

with $\prod_{i=1}^{\infty}(1 - A_i) > 0$.

LEMMA 4.1. *There exist constants $\theta' > 1$ and $\alpha' > 0$ such that*

$$A_n \leq \varepsilon^{\theta'} e^{-\alpha' n} \qquad \textit{for all } n \geq 1.$$

PROOF. Let $\omega \in \mathcal{P}_{n-1}, \omega \subset F_{n-1}, \omega \subset E_{n-1}$. We have

$$\left|(\varphi_a^j)'(c_1)\right| \geq e^{\sigma j} \quad \leqq 1jn \text{ and for all } a \in \omega.$$

Obviously we can suppose $c_n(\omega) \cap \Delta_+ \neq \emptyset$ since no exclusions would be necessary otherwise. Let $\nu < n$ be the last essential return for ω before n. Then $c_\nu(\omega) \supset I_r$ where $1 \leq |r| \leq [\alpha\nu]$. We shall distinguish four different cases: $c_\nu(\omega) \supset I_r^0$ where i) $r = r_c$, ii) $r < r_c$. iii) $r > r_c$, and iv) $c_\nu(\omega) \supset I_r^{\pm c}$.

i) Suppose first that $r = r_c$. Then clearly $c_{\nu+1} \supset \Delta^c$ and therefore $n = \nu + 1$ is an essential return. Thus we have

$$\begin{aligned} |c_n(\omega)| &\geq |c_\nu(\omega)| \cdot \inf_{x \in c_\nu(\omega)} \{\varphi'(x)\} \\ &\geq (e-1)\varepsilon^{\gamma} e^{-r_c} 2\eta_2^-(\varepsilon^{\gamma} e^{-r_c+2})^{\lambda-1} \\ &\geq 2\eta_2^-(e-1)\varepsilon^{\gamma\lambda-\gamma}\varepsilon^{\gamma} e^{-r_c} e^{-r_c(\lambda-1)} e^{2(\lambda-1)} \\ &\geq 2\eta_2^-(e-1)e^{2(\lambda-1)}\varepsilon^{\gamma\lambda} e^{-\lambda r_c} \\ &\geq 2\eta_2^-(e-1)e^{2(\lambda-1)}(\eta_1^+)^{\frac{1}{\lambda}}\varepsilon^{\gamma\lambda+1-\gamma\lambda} \\ &\geq \text{const. } \varepsilon \end{aligned}$$

where const. $= 2\eta_2^-(e-1)e^{2(\lambda-1)}(\eta_1^+)^{\frac{1}{\lambda}}$ and we use that fact that $r_c \geq (\frac{1}{\lambda} - \gamma)\log\varepsilon^{-1} - \frac{1}{\lambda}\log\eta_1^+$ from lemma 2.1

This gives

$$\begin{aligned} A_n &\le 2A \text{ const. } \frac{\varepsilon^\gamma e^{-[\alpha n]+2}}{\varepsilon} \\ &\le \text{const. } \varepsilon^{\gamma-1} e^{-[\alpha n]} \\ &\le \varepsilon^{\theta'} e^{-\alpha' n} \end{aligned}$$

as long as $\theta' < \gamma - 1$ and $\alpha' < \alpha$.

ii) Now suppose $c_\nu(\omega) \supset I_r^0$ with $r < r_c$. Then

$$\begin{aligned} |c_\nu(\omega)| &\ge |c_{\nu+2}(\omega)| \\ &\ge |c_\nu(\omega)| \cdot \inf_{\substack{x\in c_\nu(\omega)\\ a\in\omega}} \{(\varphi_a^2)'(x)\} \\ &\ge (e-1)\varepsilon^\gamma e^{-r} \text{ const. } (\varepsilon^\gamma e^{-r})^{2\lambda-1} \\ &\ge \text{const. } (\varepsilon^\gamma e^{-r})^{2\lambda} \end{aligned}$$

and therefore

$$\begin{aligned} A_n &\le \text{const. } \frac{\varepsilon^\gamma e^{-[\alpha n]}}{(\varepsilon^\gamma e^{-r})^{2\lambda}} \\ &\le \text{const. } \frac{\varepsilon^\gamma e^{-[\alpha n]}}{(\varepsilon^\gamma e^{-[\alpha n]})^{2\lambda}} \quad \text{since } r < [\alpha\nu] < [\alpha n] \\ &\le \text{const } (\varepsilon^\gamma e^{-[\alpha n]})^{1-2\lambda} \\ &\le \varepsilon^{\theta'} e^{-\alpha' n}. \end{aligned}$$

iii) For $r > r_c$ we have

$$\begin{aligned} |c_\nu(\omega)| &\ge |c_{\nu+2}(\omega)| \\ &\ge |c_\nu(\omega)| \cdot \inf_{\substack{x\in c_\nu(\omega)\\ a\in\omega}} \{(\varphi_a^2)'(x)\} \\ &\ge \text{const. } \varepsilon^\gamma e^{-r} \varepsilon^{\gamma+1+\frac{1}{\lambda}} e^{(1-\lambda)(r-r_c)} \\ &\ge \text{const. } \varepsilon^{2\gamma+1-\frac{1}{\lambda}} e^{-\lambda r} e^{-(1-\lambda)r_c} \end{aligned}$$

and so

$$\begin{aligned} A_n &\le \text{const. } \frac{\varepsilon^\gamma e^{-[\alpha n]}}{\varepsilon^{2\gamma+1-\frac{1}{\lambda}} e^{-\lambda r} e^{-(1-\lambda)r_c}} \\ &\le \text{const. } \varepsilon^{\gamma+1=\frac{1}{\lambda}} e^{-[\alpha n]+\lambda r+(1-\lambda)r_c} \\ &\le \text{const. } \varepsilon^{\gamma+1=\frac{1}{\lambda}} e^{-(\alpha n-r_c)(1-\lambda)} \\ &\le \varepsilon^{\theta'} e^{-\alpha' n} \end{aligned}$$

since $r_c < r < \alpha n$.

iv) Finally suppose the return ν occurs near the critical point and $c_\nu(\omega) \supset I_r^c$ with $r > r_\varepsilon$. Then

$$\begin{aligned} |c_\nu(\omega)| &\geq |c_{\nu+p+1}(\omega)| \\ &\geq |c_\nu(\omega)| \cdot \inf_{\substack{x \in c_\nu(\omega) \\ a \in \omega}} \{(\varphi_a^p)'(x)\} \\ &\geq \text{const. } \varepsilon^\gamma e^{-r} e^{(r-r_\varepsilon)(1-2\beta)}. \end{aligned}$$

Notice that $2\beta\gamma - \delta(1-2\beta) < 0$ by our choice of β and therefore we get

$$\begin{aligned} A_n &\leq \text{const. } \frac{\varepsilon^\gamma e^{-[\alpha n]}}{\varepsilon^{\gamma+2\beta\gamma-\delta(1-2\beta)} e^{(1-2\beta)(r-r_\varepsilon)}} \\ &\leq \varepsilon^{\delta(1-2\beta)-2\beta\gamma} e^{-[\alpha n]-(1-2\beta)(r-r_\varepsilon)} \\ &\leq \varepsilon^{\theta'} e^{-\alpha' n}. \end{aligned}$$

□

4.2. Exclusions due to (CP2). Consider the set F_n of parameters which satisfy $CP1$ up to iterate n and $CP2$ up to iterate $n-1$. Let $\mathcal{P}_n$ denote the partition of F_n as defined in the previous section. We exclude those elements which do not satisfy CP2 up to n and denote the remaining set by E_n. Notice that the exclusions will concern a union of elements of $\mathcal{P}_n$ since all the parameters in any one such element have the same history of returns and bindings. The purpose of this section is to show that most of the parameters in ω_0, as long as they satisfy CP1, also satisfy CP2. We begin by showing, in the next two lemmas, that the total amount of iterates which belong to binding periods between one essential return and the next can be estimated just in terms of the position of the return.

LEMMA 4.2. *Let $\omega \in \mathcal{P}_{\nu_i}$ and $c_{\nu_i}(\omega)$ be an essential return to Δ. Suppose $c_{\nu_i}(\omega) \supset I_{r_i}$ and let ν_{i+1} be the first essential return after ν_i. Then*

$$|c_{\nu_i}(\omega)| \geq \theta |c_{\nu_i}(\omega)| e^{\beta r_i}.$$

PROOF. If ν_i is a return to $\Delta^{\pm c}$ then it is followed by a binding period for which we have (by Lemma 2.4)

$$|(\varphi_a^{p_i})'(c_{\nu_i})| \geq e^{\beta r_i}$$

for all $a \in \omega$, $c_{\nu_i} = c_{\nu_i}(a)$. Following this binding period there may be some free periods and some inessential returns but the overall accumulated derivative is always increasing. Thus, taking into account Lemma 3.8 we obtain our result in this case.

If ν_i is a return to $\Delta^0 \setminus I^0_{r_c}$ then we have at least $\nu_{i+1} - \nu_i \geq 2$ and by Lemma 2.2

$$|(\varphi_a^2)'(c_{\nu_i}) \geq e^{\beta r_i}$$

for each $a \in \omega$. Reasoning as before we obtain the result in this case also. Finally, if ν_i is an essential return to $I^0_{r_c}$ then $\nu_{i+1} = \nu_i + 1$ and by Lemma 2.2 we get the required expansion. □

LEMMA 4.3. *Let $\omega \in \mathcal{P}_{\nu_i}$ and suppose $c_{\nu_i}(\omega)$ is an essential return containing $I^*_{r_i}$ with $* = 0, \pm c$. Let ν_{i+1} be the first essential return after ν_i. Then*

$$\nu_{i+1} - \nu_i \leq \frac{r_i}{\beta}(1 + \frac{\gamma}{\delta}).$$

PROOF. The minimum rate of expansion between a return and the next is ε^3, as follows from Lemmas 2.4, 2.2 and 3.8 . This gives

$$|(\varphi_a^{\nu_{i+1}-\nu_i})'(c_{\nu_i})| \geq e^{\beta(\nu_{i+1}-\nu_i)}$$

and since the lengths of the iterates $c_j(c_{\nu_i}(\omega))$ is bounded above by the length of the domain of our maps, which we can assume to be 1, at which point an essential return is guaranteed to occur, we have

$$1 \geq |c_{\nu_{i+1}-\nu_i}(c_{\nu_i}(\omega))| \geq \varepsilon^{\gamma} e^{-r_i} e^{\beta(\nu_{i+1}-\nu_i)}.$$

This clearly implies

$$\begin{aligned}
\nu_{i+1} - \nu_i &\leq \frac{1}{\beta}(r_i + \gamma \log \frac{1}{\varepsilon}) \\
&\leq \frac{r_i}{\beta}(1 + \frac{\gamma \log \varepsilon^{-1}}{r_i}) \\
&\leq \frac{r_i}{\beta}(1 + \frac{\gamma \log \varepsilon^{-1}}{\delta \log \varepsilon^{-1}}) \\
&\leq \frac{r_i}{\beta}(1 + \frac{\gamma}{\delta})
\end{aligned}$$

since $r_i \geq r_\varepsilon \geq \delta \log \varepsilon^{-1}$, proving the lemma. □

The statement in the lemma clearly implies that the number of iterates belonging to binding periods between one essential return and the next is less than $\frac{r_i}{\beta}(1 + \frac{\gamma}{\delta})$. In particular the number of iterates belonging to binding periods admits a bound depending only on r_j (once the other constants have been fixed). In particular we can reformulate condition CP2 by noting that

$$P_n \leq \frac{1}{\beta}(1 + \frac{\gamma}{\delta}) \sum r_j.$$

Thus if, for some $a \in \omega \in \mathcal{P}_n$ we let $r_1, \dots, r_s$ denote the sequences defined above associated to the sequences $\nu_{i_1} < \cdots < \nu_{i_s}$ (which from now on we shall denote just $\nu_1 < \cdots < \nu_s$) of essential returns of ω, condition CP2 is satisfied by the parameter a if

$$\sum_{j=1}^{s} r_j \leq \frac{\beta}{2(1+\frac{\gamma}{\delta})} n = \tilde{\beta} n$$

where $\tilde{\beta} = \frac{\beta}{2(1+\frac{\gamma}{\delta})}$. Thus we need to show that for each n the proportion of parameters belonging to F_n and not satisfying $\sum r_j(a) \leq \tilde{\beta} n$ is very small.

The strategy is to obtain an estimate on the size of the intervals $\omega \subset F_n$ in terms of the associated sequences $r_1, \dots, r_s$. This will show that intervals with an associated sequence containing many terms or terms with very high values (corresponding to intervals with a history of spending a large proportion of time in binding periods) are very small. Thus the larger elements in $\mathcal{P}_n$ are more likely to satisfy CP2 and the elements which do not satisfy CP2 are more likely to be small. A combinatorial lemma and an analytical lemma are then used to show that there are not enough small elements to compensate for the fact that they are small, i.e. in measure theoretic terms the small elements of $\mathcal{P}_n$ are only a small proportion of the bigger ones.

We shall use the notation $\omega = I(r_1, \dots, r_s)$.

LEMMA 4.4. *Let $I(r_1, \dots, r_s) = \omega \in \mathcal{P}_n, \omega \in F_n$. Then*

$$|\omega| \leq \varepsilon^{1+\delta} e^{-\beta \sum_{j=1}^{s} r_j}.$$

PROOF. Consider the nested sequence of intervals

$$I(r_1, \dots, r_s) \subseteq I(r_1, \dots, r_{s-1}) \subseteq \cdots \subseteq I(r_1) \subseteq I_{r_0} \stackrel{def}{=} \omega_0.$$

For each $0 \leq j \leq s-1$ we have

$$|c_{\nu_j}(I(r_1, \dots, r_j))| \geq \varepsilon^{\gamma} e^{-r_j}$$

and by Lemma 4.2

$$\begin{aligned} |c_{\nu_{j+1}}(I(r_1, \dots, r_j))| &\geq |c_{\nu_j}(I(r_1, \dots, r_j))| e^{\beta r_j} \\ &\geq \varepsilon^{\gamma} e^{-r_j + \beta r_j}. \end{aligned}$$

Thus

$$\begin{aligned}
\frac{|I(r_1,\ldots,r_{j+1})|}{|I(r_1,\ldots,r_j)|} &\approx \frac{|c_{\nu_{j+1}}(I(r_1,\ldots,r_{j+1}))|}{|c_{\nu_{j+1}}(I(r_1,\ldots,r_j))|} \\
&\leq \frac{\varepsilon^{\gamma} e^{-r_{j+1}}}{\varepsilon^{\gamma} e^{-r_j+\beta r_j}} \\
&\leq \frac{e^{-r_{j+1}}}{e^{-r_j}} e^{-\beta r_j}
\end{aligned}$$

and

$$\begin{aligned}
|I(r_1,\ldots,r_s)| &= \frac{|I(r_1,\ldots,r_s)|}{|I(r_1,\ldots,r_{s-1}|}\cdots\frac{|I(r_2)|}{|I(r_1)|}|\omega_0| \\
&\leq (\rho-1)\varepsilon e^{-r_s-r_1} e^{-\beta\sum_{j=1}^{s} r_j} \\
&\leq \varepsilon^{1+\delta} e^{-\beta\sum_{j=1}^{s} r_j}.
\end{aligned}$$

□

Let $\eta(R)$ denote the number of possible sequences $r_1,\ldots,r_s$ with $r_i \geq \delta\log\varepsilon^{-1}$ and $r_i + \cdots + r_s = R$.

LEMMA 4.5. *For each $\xi > 0$ there exists $\varepsilon > 0$ such that*

$$\eta(R) \leq 2e^{\xi R}$$

for all $R \in \mathbb{N}$.

PROOF. The number of ways in which R balls can be distributed in s drawers is $\binom{R}{s}$ and thus we have

$$\eta(R) \leq \frac{R!}{s!(R-s)!}.$$

The crucial ingredient in the following computation and the essential fact which underlies the truth of the lemma is that we have

$$r_i \geq r_\varepsilon \geq \delta\log\varepsilon^{-1}.$$

Consequently the length of the sequences which can occur for any given R is bounded above by a constant depending on ε and on R. More precisely we have

$$s\log\varepsilon^{-1} \leq R$$

or, equivalently,

$$\frac{s}{R} \leq \delta\log\varepsilon.$$

This upper bound can be made arbitrarily small by taking ε small. Using this fact and Stirling's approximation formula for factorials

$$\sqrt{2\pi k}k^k e^k \leq k! \leq \sqrt{2\pi k}k^k e^k(1+\frac{1}{4k})$$

we get

$$\begin{aligned}
\eta(R) &\leq \frac{\sqrt{2\pi R}R^R e^{-R}(1+\frac{1}{4R})}{\sqrt{2\pi s}s^s e^{-s}\sqrt{2\pi(R-s)}(R-s)^{R-s}e^{-(R-s)}} \\
&\leq \frac{R^R}{s^s(R-s)^{R-s}}\sqrt{\frac{r}{s(R-s)}}\frac{(1-\frac{1}{4R})}{\sqrt{2\pi}} \\
&\leq 2\frac{R^R}{s^s(R-s)^{R-s}} \qquad \text{for large } R \text{ and small } \varepsilon > 0 \\
&\leq 2\left(\frac{R}{s}\right)^s\left(\frac{R}{R-s}\right)^{R-s} \\
&\leq 2\left[\left(\frac{R}{s}\right)^{\frac{s}{R}}\left(\frac{R}{R-s}\right)^{1-\frac{s}{R}}\right]^R \\
&\leq 2\left[\left(\frac{1}{\frac{s}{R}}\right)^{\frac{s}{R}}\left(\frac{1}{1-\frac{s}{R}}\right)^{1-\frac{s}{R}}\right]^R \\
&\leq 2\varepsilon^{\xi R}
\end{aligned}$$

since $\left(\frac{1}{\frac{s}{R}}\right)^{\frac{s}{R}}$ and $\left(\frac{1}{1-\frac{s}{R}}\right)^{1-\frac{s}{R}}$ both tend to zero as $\frac{s}{R} \leq \delta \log \varepsilon \to 0$. Thus we can take ε samll enough so that

$$\left(\frac{1}{\frac{s}{R}}\right)^{\frac{s}{R}}\left(\frac{1}{1-\frac{s}{R}}\right)^{1-\frac{s}{R}} < e^{\xi}.$$

The lemma is proved. □

Let $\xi = \frac{\beta}{4}$.

LEMMA 4.6.

$$\int_{F_n} e^{\xi \sum_{j=1}^{s} r_j} d(a) \leq \varepsilon^{1+\delta} \sum_{R \geq \delta \log \varepsilon^{-1}} e^{-\frac{\beta}{2}R}.$$

PROOF.

$$\begin{aligned}
\int_{F_n} e^{\xi\sum_{j=1}^{s} r_j} d(a) &= \sum_{\substack{(r_1,\dots,r_s)\\ \omega=I(r_1,\dots,r_s)}} e^{\xi\sum_{j=1}^{s} r_j}|\omega| \\
&\leq \sum_{(r_1,\dots,r_s)} e^{\xi\sum r_j} e^{-\beta\sum r_j}\varepsilon^{1+\delta} \\
&\leq \varepsilon^{1+\delta}\sum R \sum_{\substack{(r_1,\dots,r_s)\\ r_1+\dots+r_s=R}} e^{(\xi-\beta)\sum r_j} \\
&\leq \varepsilon^{1+\delta}\sum_R \eta(R)e^{(\xi-\beta)R} \\
&\leq \varepsilon^{1+\delta}\sum_R e^{-\frac{\beta}{2}R}
\end{aligned}$$

where $R \geq \delta\log\varepsilon^{-1}$. □

The theorem now follows from Lemmas 4.1 and 4.6. Indeed, by Lemma 4.1 the exclusions due to CP1 are exponentially small in n with a multiplicative factor of $\varepsilon^{\theta'}, \theta' > 1$ which means that proportionately less and less parameters are excluded as we choose initial intervals ω_0 closer to the bifurcation point c. Lemma 4.6 gives

$$\varepsilon^{1+\delta}\sum_{R\geq\delta\log\varepsilon^{-1}} e^{-\frac{\beta}{2}R} \geq e^{\xi}\int_{F_n} e^{\sum_{j=1}^{s} r_j} d(a) \geq e^{\xi}e^{\bar{\beta}n}\cdot m\{a\colon \sum r_j \geq e^{\bar{\beta}n}\}$$

which implies

$$m\{a\colon \sum r_j \geq e^{\bar{\beta}n}\} \leq e^{-\bar{\beta}n}e^{\xi}\varepsilon^{1+\delta}\sum_{R\geq\delta\log\varepsilon^{-1}} e^{-\frac{\beta}{2}R}.$$

Thus the proportion of parameters belonging to F_n (i.e. not excluded by CP1) and excluded by CP2 at each iterate n is also exponentially small with n and admits a multiplicative factor $> \varepsilon^{1+\delta}$. This proves that the set $\mathcal{A}^+$ has positive measure.

CHAPTER III

Lorenz-like flows

We begin the proof of our main theorem by discussing some of the constants which will be used [1]. We also describe briefly the definition of the neighbourhoods of the critical points and the discontinuity. These are straightforward generalizations of the analogous constructions in the one-dimensional case and so we refer the reader to chapter II for more detailed explanations.

Two fundamental constants λ and σ satisfying $0 < \lambda < 1/2$ and $\sigma > 1$ are fixed a priori and depend only on the form of the maps. We choose $c_0 > 0$ and $c > 0$ such that $0 < 2c < c_0 < \log\sqrt{2}$ (c will be a lower bound for the Lyapunov exponents associated the the maps which we construct below). We shall suppose, to simplify certain estimates, that c and c_0 are slightly smaller than the analogous constants σ and σ_0 used in the one-dimensional model. Then there are some "incidental" constants, i.e. constants which do not seem to be intrinsically related to the dynamics but which are necessary for our arguments. We fix γ and δ satisfying $1 < \gamma < \lambda^{-1} - 1$ and $1 < \gamma + \delta < \lambda^{-1} - 1$ and then choose $\beta > \alpha > 0$ sufficiently small. At some point in the proof we fix a small $\varepsilon > 0$ which, on a heuristic level, represents the "distance" between the parameters we are considering and the parameters for which the maps Φ_a exhibit a hyperbolic Lorenz attractor. Generalizing the construction carried out in the one-dimensional context we define the sets $\Delta^0 = \{(x,y) \in \Sigma : x \in (-\varepsilon^{\gamma}, \varepsilon^{\gamma})\}$, $\Delta^0_+ = \{(x,y) \in \Sigma : x \in (-\varepsilon^{\gamma+\delta}, \varepsilon^{\gamma+\delta})\}$ and $\Delta^{\pm c} = \Delta^0 \pm (c,0)$, $\Delta^{\pm c}_+ = \Delta^0_+ \pm (c,0)$. Finally we shall fix some $b > 0$ small which controls the dissipativity of the maps Φ_a. Notice that these constants are chosen in the order $\gamma, \delta, \beta, \alpha, \varepsilon, b$ in the sense that each one has to satisfy certain numerical relations involving only the constants chosen previously. The precise conditions which have to be satisfied will appear explicitly during the course of the proof but, in principle, all the constants could

[1] This chapter is an extract from the paper written in collaboration with M. Viana: *Lorenz-like attractors without invariant foliations*, to appear

be fixed right in the beginning. When there is no possibility of confusion we shall always let z denote a point of Σ and (x, y) its horizontal and vertical coordinates respectively.

We make a final remark on a notation which will be frequently used when considering iterates of points. In general we shall use no subscripts to indicate points which belong to the critical region, e.g. z or ζ. We denote with the subscript 0 the first images of these points, e.g. $z_0 = \Phi(z)$, and the successive iterates are then given by $z_i = \Phi^i(z_0) = \Phi^{i+1}(z)$. Thus, in some sense all iterations begin in the region of the *critical values*. Thus, even when we talk about a certain iterate n of a point z in the critical region what we really mean is the n'th iterate of z_0. This choice of notation might seem confusing at the beginning but it is actually quite usefull.

1. First properties

1.1. The Differential. Several properties of the differential $D\Phi = D\Phi_a$ which are immediate consequences of the definition of Φ will be used repeatedly throughout the proof. For convenience we collect here some of these properties. By the chain rule we have

$$D\Phi(z) = \begin{pmatrix} \partial_x\Phi_1 & \partial_y\Phi_1 \\ \partial_x\Phi_2 & \partial_y\Phi_2 \end{pmatrix} = \begin{pmatrix} \partial_u\Psi_1 & \partial_v\Psi_1 \\ \partial_u\Psi_2 & \partial_v\Psi_2 \end{pmatrix} \begin{pmatrix} \partial_x P_1 & \partial_x P_2 \\ \partial_y P_1 & \partial_y P_2 \end{pmatrix}.$$

Notice that the partial derivatives of P can be calculated explicitly to get

$$DP = \begin{pmatrix} \partial_x P_1 & \partial_y P_1 \\ \partial_x P_2 & \partial_y P_2 \end{pmatrix} = \begin{pmatrix} \lambda|x|^{\lambda-1} & 0 \\ \sigma|x|^{\sigma-1}y & |x|^\sigma \end{pmatrix}.$$

By (LL) the partial derivatives of Ψ satisfy $|\partial_u\Psi_2|, |\partial_v\Psi_1|, |\partial_v\Psi_2| \leq b$ and $|\partial_u\Psi_1 - \partial_u\psi_1| \leq b$. Notice also that $\partial_u\psi_1\partial_x P_1 = \varphi'$ and therefore we can write, to simplify the notation, $\partial_u\Psi_1\partial_x P_1 = \tilde{\varphi}'$ with $|\tilde{\varphi}' - \varphi'| \leq b$. Thus the differential can be written as

$$D\Phi(z) = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} = \begin{pmatrix} \tilde{\varphi}' + \partial_v\Psi_1\sigma|x|^{\sigma-1}y & \partial_v\Psi_1|x|^\sigma \\ \partial_u\Psi_2\lambda|x|^{\lambda-1} + \partial_v\Psi_2\sigma|x|^{\sigma-1}y & \partial_v\Psi_2|x|^\sigma. \end{pmatrix} \tag{14}$$

For future reference we collect below certain estimates concerning the partial derivatives of $D\Phi$ which derive immediately from (LL) and which will be used repeatedly below.

LEMMA 1.1. *There exists constants $\tau > \tau' > 0$ such that:*

$$\begin{aligned} |\varphi'| + \tau b \geq |A(z)| \geq |\varphi'| - \tau' b \ \textit{and} \qquad & \tau \geq |\dot{A}(z)| \geq \tau \\ 2\lambda b|x|^{\lambda-1} \geq |B(z)| \qquad & \textit{and}\ \lambda b |x|^{\lambda-2} \geq |\dot{B}(z)| \\ b \geq |C(z)|, |D(z)| \qquad \textit{and} \qquad & b \geq |\dot{C}(z)|, |\dot{D}(z)| \end{aligned}$$

Finally observe that the definition of Φ_a implies strong area contractiveness. Indeed $\det DP = \partial_x P_1 \partial_y P_2 - \partial_x P_2 \partial_y P_1 = |x|^{\lambda+\sigma-1}$ with $\sigma + \lambda - 1 > 0$, and $\det D\Psi = \partial_u \Psi_1 \partial_v \Psi_2 - \partial_v \Psi_1 \partial_u \Psi_2 \leq b \partial_u \Psi_1 \leq Kb$ for some constant $K \geq \sup\{|\partial_u \Psi_1|\} > 0$. Then

$$\det D\Phi = \det D\Psi \det DP \leq Kb|x|^{\sigma+\lambda-1} \ll 1.$$

1.2. Hyperbolic dynamics outside $\Delta^{\pm c}$.

LEMMA 1.2. *For any $a \in [c + \varepsilon/2, c + \varepsilon]$ and $z_0 \in \Sigma^*$ such that $\{\Phi_a^j(z_0)\}_{j=0}^{n-1} \cap \Delta^{\pm c} = \emptyset$ for some $n \geq 1$, we have, for all $1 \leq j \leq n$, letting $w_j = D\Phi^j(z_0) \cdot (1, 0)$,*

(i) $$\textit{slope}\ (w_j) \leq \sqrt{b}$$

(ii) $$\|w_j\| \geq \varepsilon^{\gamma+\delta} e^{c_0(j-1)}$$

Moreover, if $\Phi_a^{-1}(z_0) \in \Delta_{\mp}^{\pm c}$ and/or $\Phi_a^n(z_0) \in \Delta_{\mp}^{\pm c}$ we actually have, for all $1 \leq j \leq n$,

(iii)) $$\|w_j\| \geq e^{c_0 j}$$

PROOF. We begin by showing the existence of a forward invariant cone field, i.e. we define in each $z \in \Sigma^*$ a cone $C(z) = \{v \in T_z\Sigma^* :\ \text{slope}\ v \leq \sqrt{b}\}$ and show that if $z \in \Sigma^* \setminus \Delta^{\pm c}$ then $D\Phi(z) \cdot C(z) \subset C(\Phi(z))$. Let $z \in \Sigma^* \setminus \Delta^{\pm c}$ and $v = (v_1, v_2) \in C(z)$. Then, clearly, $(1, v_2/v_1) \in C(z)$ since $v_2/v_1 \leq \sqrt{b}$, and $D\Phi(z) \cdot v \in C(\Phi(z))$ if and only if $D\Phi(z) \cdot (1, v_2/v_1) \in C(\Phi(z))$. By (14) we have

$$D\Phi(z) \cdot (1, v_2/v_1) = \begin{pmatrix} \partial_u \Psi_1 \partial_x P_1 + \partial_v \Psi_1 \sigma |x|^{\sigma_1} y + v_2/v_1 \partial_v \Psi_1 |x|^{\sigma} \\ \partial_u \Psi_2 \lambda |x|^{\lambda-1} + \partial_v \Psi_2 \sigma |x|^{\sigma-1} y + v_2/v_1 \partial_v \Psi_2 |x|^{\sigma} \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

By (LL) this gives $|v_2| \leq b\lambda|x|^{\lambda-1} + b\sigma y|x|^{\sigma-1} + \sqrt{b}b|x|^{\sigma} \leq 3b(\varepsilon^{\gamma+\delta})^{\lambda-1}$ and $|v_1| \geq (1 - b)\eta_1^- \varepsilon^{\gamma+\delta}$ where η_1^- is some constant involved in the definition of one-dimensional Lorenz like maps. Then we get

$$\text{slope}\ (v_1, v_2) \leq \frac{3b(\varepsilon^{\gamma+\delta})^{\lambda-1}}{(1-b)\eta_1^- \varepsilon^{\gamma+\delta}} \leq \sqrt{b}$$

if b is sufficiently small. This proves i). To prove ii) and iii) we make the following observations. By i) we have

$$\|w_j\| \geq \|\omega_j\| \geq \|w_j\|(1-b)^2 \tag{15}$$

where ω_j denotes the horizontal component of w_j. Moreover, by (14) and (LL) we have that

$$\|\omega_{j+1}\| \geq \partial_u \Psi_1 \partial_x P_1 \|\omega_j\| \geq (1-b)\varphi'(x_j)\|\omega_j\| \tag{16}$$

where $z_j = (x_j, y_j)$. Finally, from (15)and (16) we get

$$\|w_{j+1}\| \geq (1-b)\varphi'(x_j)\|\omega_j\| \geq (1-b)(1+b)^2\varphi'(x_j)\|w_j\|$$

and the result now follows by taking $b > o$ small and by the analogous result in the one dimensional case, taking $c_0 < \sigma_o$. □

1.3. Controlling recurrence near the discontinuity. We introduce a condition on the rate of approach of orbits to the discontinuity.

DEFINITION 1.3. We say that the point z satisfies condition $(*)$ up to time n if, for all $1 \leq k \leq n$, letting $0 < \nu_1 < \nu_2 \ldots \nu_s < k$ denote the times for which $z_{\nu_i} \in \Delta^0$ with $z_{\nu_i} \in I^0_{r_i}, i = 1, \ldots, s$ we have

$$\sum_{i=1}^{s} r_i \leq \alpha k \tag{$*$}$$

A first important consequence of $(*)$ is that it guarantees a certain minumum number of iterates before entering the critical region.

LEMMA 1.4. *For all $\varepsilon > 0$ sufficiently small and $a \in [c+\varepsilon/2, c+\varepsilon]$ the following fact is true. If z satisfies $(*)$ for some $n \geq 1$ then*

$$z_j \notin \Delta^{\pm c} \qquad \text{for all } 1 \leq j \leq N$$

where $N = \min\{n, \delta/\alpha \log 1/\varepsilon\}$.

PROOF. Notice that $(*)$ implies in particular $|x_{\nu_i}| \geq \varepsilon^\gamma e^{-\alpha\nu_i}$ and therefore it implies that it takes a certain minimum number of iterates before the orbit of z can enter a small given neighbourhood of the discontinuity. In particular, if $n \leq \delta/\alpha \log 1/\varepsilon$, then

$$|x_n| \geq \varepsilon^\gamma e^{-\alpha n} \geq \varepsilon^\gamma e^{-\alpha\delta/\alpha \log 1/\varepsilon} = \varepsilon^{\gamma+\delta}$$

Thus it is sufficient to show that the preimages $\Phi_a^{-1}(\Delta^{\pm c})$ are contained in an $\varepsilon^{\gamma+\delta}$-neighbourhood of the discontinuity for all $a \in [c+\varepsilon/2, c+\varepsilon]$ to imply the desired result. In the one-dimensional case we proved that $\varphi_a^{-1}(\Delta^{\pm c})$ is contained in a neighbourhood of the discontintuity of size $2\varepsilon(\varepsilon/\eta_1^-)^{\frac{1}{\lambda}}$ for every $a \in [c+$

$\varepsilon/2, c+\varepsilon]$. By (LL) this implies that $\Phi_a^{-1}(\Delta^{\pm c}$ is contained in a neighbourhood of the discontinuity of size $3e(\varepsilon/\eta_1^-)^{\frac{1}{\lambda}} \geq 2e(\varepsilon/\eta_1^-)^{\frac{1}{\lambda}} + b$. It remains to show that $3e(\varepsilon/\eta_1^-)^{\frac{1}{\lambda}} < \varepsilon^{\gamma+\delta}$, but this follows since $1/\lambda > \gamma+\delta$ and ε can be chosen small. This completes the proof of the lemma. □

A second very important consequence of (*) is a uniform upper bound on the rate of growth of vectors.

LEMMA 1.5. *For any $a > c$, if z satisfies condition (*) up to n then, for all $1 \leq k \leq n$, we have*

$$\|w_k(z)\| \leq \|D\Phi^k(z)\| \leq (\varepsilon^{\gamma+\delta}e^{-1})^{(\lambda-1)k} \leq \kappa^k$$

for $\kappa = (\varepsilon^{\gamma+\delta}e^{-1})^{(\lambda-1)}$.

PROOF. For $z_j \notin \Delta^0$ we have

$$\|D\Phi(z_j)\| \leq (\varepsilon^{\gamma+\delta})^{\lambda-1}.$$

For $z_{\nu_i} \in \Delta^0$, with $z_{\nu_i} \in I_{r_i}$ we have

$$\|D\Phi(z_{\nu_i})\| \leq (\varepsilon^{\gamma+\delta}e^{-r_i})^{\lambda-1}$$

thus

$$\begin{aligned}\|w_k(z)\| \leq \|D\Phi^k(z)\| &\leq \varepsilon^{(\gamma+\delta)(\lambda-1)k}\prod_{i=1}^{s} e^{-(\lambda-1)r_i} \\ &\leq \varepsilon^{(\gamma+\delta)(\lambda-1)k}e^{-(\lambda-1)\sum_{i=1}^{s} r_i} \leq \varepsilon^{(\gamma+\delta)(\lambda-1)k}e^{-(\lambda-1)k}\end{aligned}$$

proving the lemma. □

1.4. Binding. The notion of binding was already introduced in the context of one-dimensional maps. It plays a significantly greater role in the two dimensional argument. We begin by giving a preliminary definition of binding and proving some simple consequences. Later on we shall give a slightly more sophisticated definition which depends on the points under consideration. However that definition will be more restrictive and thus all the results proved below continue to apply.

DEFINITION 1.6. Two points ξ, η are *bound* up to time n if

$$|\xi_i - \eta_i| \leq 2\varepsilon^{\gamma}e^{-\beta i}, \quad \forall 0 \leq i \leq n$$

where $\xi_0 = \Phi(\xi), \eta_0 = \Phi(\eta)$ and $\xi_i = \Phi_a^i(\xi_0)$ and $\eta_i = \Phi_a^i(\eta_0)$. We let $B^{(n)}(z)$ denote the set of points bound to z up to time n.

Notice from the definition that $\xi \in B^{(n)}(z)$ implies $|\Phi^i(\xi)-\Phi^i(z)| \leq \varepsilon^\gamma e^{-\beta(i-1)}$,for all $0 \leq i \leq n+1$. This seemingly awkward notation will simplify certain calculations later on. The notion of binding is particularly useful in conjunction with condition $(*)$. As a first example of this we give a simple generalization of lemma 1.4.

LEMMA 1.7. *For all $\varepsilon > 0$ sufficiently small and $a \in [c+\varepsilon/2, c+\varepsilon]$ the following fact is true. If z satisfies $(*)$ for some $n \geq 1$ then, for all $\xi \in B^{(n)}(z)$, we have*

$$\xi_j \notin \Delta^{\pm c} \qquad \textit{for all } 1 \leq j \leq N$$

where $N = \min\{n, \delta/\alpha \log 1/\varepsilon\}$.

PROOF. Let ξ_j^x denote the horizontal coordinate of the point ξ_j. By the definition of $B^{(n)}(z)$ and the fact that $\beta > \alpha$ all points $\xi \in B^{(n)}(z)$ satisfy a weakened form of condition $(*)$. More precisely

$$\sum(r_i - 1) \leq \alpha k$$

for all $k \leq n$. This follows easily from the fact that the sets $B_j^{(n)}(z)$ are contained in balls whose radius is decreasing exponentially fast with respect to the rate at which the point z approaches the discontinuity. Thus, by choosing ε small and therefore $\delta/\alpha \log 1/\varepsilon$ large we can guarantee that $|\varepsilon^\gamma e^{-\alpha k} - \varepsilon^\gamma e^{-\beta k}| \leq \frac{1}{4}\varepsilon^\gamma e^{-\alpha k}$ implying that if $z_{\nu_i} \in I_{r_i}$ then $\xi_{\nu_i} \in I_{\tilde{r}_i}$ with $\tilde{r}_i \leq r_i - 1$. Thus the rest of the proof proceeds precisely as above. □

1.5. Bounded distortion outside $\Delta^{\pm c}$. Another important property that the sets $B^{(n)}(z)$ will be proved to satisfy under suitable condition on the orbit of z, is the property of bounded distortion. We begin by proving this result here for some $B^{(n)}(z)$ with z satisfying $(*)$ and $z_j \notin \Delta^{\pm c}\ \forall\ 1 \leq j \leq n$ (notice that we do *not* require $n \leq N = \delta/\alpha \log 1/\varepsilon$). For $\xi \in B^{(n)}(z)$ let $w_j(\xi) = D\Phi_a^j(\xi_0) \cdot (1,0)$.

LEMMA 1.8. *There exist constants $C, C_0 > 0$ with the following properties. Suppose that for some $n \geq 1$ the point $z_0 = \Phi(z)$ satisfies $(*)$ and $z_j \notin \Delta^{\pm c}, \forall\ j \leq n$. Then, for any $\xi, \eta \in B^{(n-1)}(z)$ and any $0 \leq k \leq n$ we have*

$$\text{(i)} \qquad \frac{\|w_k(\xi)\|}{\|w_k(\eta)\|} \leq \prod_{j=0}^{k-1}(1 + Ce^{(\alpha-\beta)j}) \leq C_0 < \infty$$

$$\text{(ii)} \qquad \angle(w_k(\xi), w_k(\eta)) \leq Cbe^{(\alpha-\beta)k}$$

PROOF. Notice first of all that, by lemma 1.2, *slope* $w_j(\xi) \leq \sqrt{b}$ and *slope* $w_j(\eta) \leq \sqrt{b}$ giving $\angle(w_j(\xi), w_j(\eta)) \leq 2\sqrt{b}$. We shall use this fact to prove part i) and then prove the stronger bound in ii). The philosophy behind the proof is similar to that underlying the computations carried out in lemma 1.2: we use in a fundamental way the fact that the vectors are almost horizontal and therefore satisfy 'almost 1-dimensional' estimates. The formal argument, however, is more involved.

For any $\xi \in B^{(n)}(z)$ and any $1 \le j \le n$, let $\omega_j(\xi)$ denote the horizontal component of the vector $w_j(\xi)$, Then we have, by (i) and a simple geometrical argument, that $\|\omega_j(\xi)\|(1+b)^{\frac{1}{2}} \ge \|w_j(\xi)\| \ge \|\omega_j(\xi)\| \ge \|w_j(\xi)\|(1+b)^{-\frac{1}{2}}$. Since these bounds do not depend on ξ nor on n, it will suffice to prove our result for the horizontal components $\omega_j(\xi)$ and $\omega_j(\eta)$. We first prove inductively the following statement. For all $1 \le k \le n$ and any $\xi, \eta \in B^{(n)}(z)$,

$$\frac{\|\omega_k(\xi)\|}{\|\omega_k(\eta)\|} \le \prod_{j=0}^{k-1}(1+B_j) \tag{17}$$

where

$$B_j = \frac{\tilde{\varphi}''(\zeta_j)+b}{\tilde{\varphi}'(\eta_j^x)-b}\varepsilon^{\gamma}e^{-\beta j} \tag{18}$$

and η_j^x denotes the x-coordinate of the point η_j and $\zeta_j \in [\xi_j^x, \eta_j^x]$. Then we show that the product above is bounded by some constant indipendent of n which proves the lemma.

Fix $\xi, \eta \in B^{(n)}(z)$ and let $w_0(\xi) = w_0(\eta) = (1,0)$. Then

$$w_1(\xi) = \begin{pmatrix} \tilde{\varphi}'(\xi^x) + \partial_v\Psi_1\sigma|\xi^x|^{\sigma-1}\xi^y \\ \partial_u\Psi_2\partial_x P_1 + \partial_v\Psi_2\sigma|\xi^x|^{\sigma-1}\xi^y \end{pmatrix}$$

and similarly for $w_i(\eta)$. Thus the corresponding horizontal components $\omega_1(\xi)$ and $\omega_1(\eta)$ satisfy

$$\begin{aligned} |\omega_1(\xi) - \omega_1(\eta)| &\le |\tilde{\varphi}'(\xi^x) - \tilde{\varphi}'(\eta^x)| + \partial_v\Psi_1\sigma|\xi^x|^{\sigma-1}\xi^y - \partial_v\Psi_1\sigma|\eta^x|^{\sigma-1}\eta^y| \\ &\le |\tilde{\varphi}''(\zeta)| \cdot |\xi^x - \eta^x| + b|\xi - \eta| \end{aligned} \tag{19}$$

where $\zeta \in [\xi^x, \eta^x]$ and its existence is given by the mean value theorem. The second term in the last inequality also follows from the mean value theorem and the observation that all the second derivatives involved are small, by (LL). Notice also that

$$\|\omega_1(\eta)\| \ge |\tilde{\varphi}'(\eta^x)| - b$$

and therefore we have

$$\frac{\|\omega_1(\xi)\|}{\|\omega_1(\eta)\|} = 1 + \frac{\|\omega_1(\xi)-\omega_1(\eta)\|}{\|\omega_1(\eta)\|} \le 1 + \frac{(\tilde{\varphi}''(\zeta)+b)|\xi-\eta|}{\tilde{\varphi}'(\eta^x)-b} \le 1 + \frac{\tilde{\varphi}''(\zeta)+b}{\tilde{\varphi}'(\zeta^x)-b}\varepsilon^{\gamma}$$

which completes the first step of the induction. The general inductive step proceeds in the same way. Suppose that

$$\frac{\|\omega_k(\xi)\|}{\|\omega_k(\eta)\|} \leq \prod_{i=0}^{k-1}\left(1+\frac{(\tilde{\varphi}''(\zeta_i)+b)}{\tilde{\varphi}'(\eta_i^x)-b}\varepsilon^{\gamma}e^{-\beta i}\right)$$

for all $k \leq n-1$. Consider the normalized vectors $\omega_{n-1}(\xi)/\|\omega_{n-1}(\xi)\|$ and $\omega_{n-1}(\eta)/\|\omega_{n-1}(\eta)\|$ and (the horizontal components of) their images $\omega_n(\xi)/\|\omega_{n-1}(\xi)\|$ and $\omega_n(\eta)/\|\omega_{n-1}(\eta)\|$. Then, reasoning as in (19)-(20) we get

$$\frac{\|\omega_n(\xi)\|/\|\omega_{n-1}(\xi)\|}{\|\omega_n(\eta)\|/\|\omega_{n-1}(\eta)\|} \leq 1+\frac{\tilde{\varphi}''(\zeta_{n-1})+b}{\tilde{\varphi}'(\xi_{n-1}^x)-b}|\xi_{n-1}-\eta_{n-1}|$$

with $\zeta_{n-1} \in [\xi_{n-1}^x, \eta_{n-1}^x]$. Then, using the inductive assumption, we get

$$\begin{aligned}\frac{\|\omega_n(\xi)\|}{\|\omega_n(\eta)\|} &= \frac{\|\omega_n(\xi)\|/\|\omega_{n-1}(\xi)\|}{\|\omega_n(\eta)\|/\|\omega_{n-1}(\eta)\|}\frac{\|\omega_{n-1}(\xi)\|}{\|\omega_{n-1}(\eta)\|}\\ &\leq \left(1+\frac{\tilde{\varphi}''(\zeta_{n-1})+b}{\tilde{\varphi}'(\xi_{n-1}^x)-b}\varepsilon^{\gamma}e^{-\beta n-1}\right)\left(\prod_{i=0}^{n-2}\left(1+\frac{(\tilde{\varphi}''(\zeta_i)+b)|\xi-\eta|}{\tilde{\varphi}'(\eta_i^x)-b}\varepsilon^{\gamma}e^{-\beta i}\right)\right)\\ &\leq \prod_{i=0}^{n-1}(1+B_i)\end{aligned}$$

which completes the induction. Thus we have reduced the proof of the lemma to proving that the terms B_i tend to zero exponentially fast in i. However the maps involved are now entirely one-dimensional and the proof proceeds exactly as in the one-dimensional case to yield

$$B_i \leq Ce^{(\alpha-\beta)i}$$

for some constant $C > 0$.

The strength of part ii) is that it does *not* require an inductive argument. In fact it follows immediately from the following significantly stronger statement. Let $\xi_j, \eta_j \notin \Delta^{\pm c}$, $|\xi_j - \eta_j| \leq \varepsilon^{\gamma}e^{-\beta j}$ and suppose that *slope* $w_j(\xi), w_j(\eta) \leq b^{\frac{1}{2}}$. Then

$$\angle(w_{j+1}(\xi), w_{j+1}(\eta)) \leq Cbe^{(\alpha-\beta)j}.$$

We have

$$\begin{aligned}w_{j+1}(\xi) &= D\Phi(\xi_j)\cdot w_j = D\Phi(\xi_j)\cdot(\omega_j,\sigma_j) = (\omega_{j+1},\sigma_{j+1})\\ &= \begin{pmatrix}\tilde{\varphi}'(\xi_j^x)\cdot\omega_j + \partial_v\Psi_1\xi^y|\xi^x|^{\sigma-1}\omega_j + \partial_v\Psi_2|\xi^x|^{\sigma}\cdot\sigma_{j+1}\\ \partial_u\Psi_2\lambda|\xi_j^x|^{\lambda-1}\cdot\omega_j + \partial_v\Psi_2\xi^y|\xi^x|^{\sigma-1}\omega_j + \partial_v\Psi_2|\xi^x|^{\sigma}\cdot\sigma_{j+1}\end{pmatrix}\end{aligned}$$

and similarly for $w_{j+1}(\eta)$. Thus

$$\begin{aligned}|\sigma_{j+1}(\xi)-\sigma_{j+1}(\eta)| &\leq b|\xi^x-\eta^x||\omega_j(\xi)-\omega_j(\eta)|(\tilde{\varphi}''(\zeta^x)+2)\\ &\leq b|\xi^x-\eta^x|C_0|\omega_j(\eta)|(\tilde{\varphi}''(\zeta^x)+2)\end{aligned}$$

using the fact that *slope* $w_j < b^{1/2}$ and therefore $\sigma_j \leq b^{1/2}\omega_j$. Moreover

$$|\omega_{j+1}(\eta)| \geq (\tilde{\varphi}'(\eta^x)-b)|\omega_j(\eta)|$$

and thus

$$\begin{aligned}\angle(w_{j+1}(\xi),w_{j+1}(\eta)) &\leq \frac{C_0 b|\xi_j^x-\eta_j^x|(\tilde{\varphi}''(\zeta^x)+2)|\omega_j(\eta)|}{(\tilde{\varphi}'(\eta^x)-b)|\omega_j(\eta)}\\ &\leq C_0 b\varepsilon^{\gamma}e^{-\beta j}\frac{\tilde{\varphi}''(\zeta^x)+2}{\tilde{\varphi}'(\eta^x)-b} \leq \text{const. } e^{(\alpha-\beta)j}\end{aligned}$$

estimating $(\tilde{\varphi}''(\zeta^x)+2)/(\tilde{\varphi}'(\eta^x)-b)$ just like in the one-dimensional case. □

2. Critical Points

One of the additional difficulties in analyzing the dynamics of two-dimensional maps is that there is no, a priori, well defined notion of critical point. We shall show below how to overcome this difficulty by constructing succesive approximations to critical points which are, eventually, defined only for those parameters for which a non-trivial chaotic attractor exists.

2.1. Contractive approximations.

LEMMA 2.1. *Outside* $\Delta^{\pm\varepsilon}$ *there exist two smooth unit vector fields* $f^{(1)}(z)$ *and* $e^{(1)}(z)$ *with the following properties:*

i $f^{(1)}(z)$ *lies in the direction which is most expanded by the action of* $D\Phi(z)$.

ii $e^{(1)}(z)$ *lies in the direction which is most contracted by the action of* $D\Phi(z)$.

iii $e^{(1)}(z)$ *and* $f^{(1)}(z)$ *are orthogonal and*

$$slope(f^{(1)}(z)) \leq \frac{4b|x|^{\sigma+\lambda-1}}{\varepsilon^{2(\gamma+\delta)}}.$$

PROOF. The directions which are most expanded and most contracted by $D\Phi(z)$ are solutions to the differential equation

$$\frac{d}{d\theta}\|D\Phi(z)\cdot(\sin\theta,\cos\theta)\| = 0. \tag{20}$$

Explicit differentiation and some algebraic manipulation yields

$$\tan 2\theta = \frac{2(\partial_x\Phi_1\partial_y\Phi_1+\partial_x\Phi_2\partial_y\Phi_2)}{(\partial_x\Phi_1^2+\partial_x\Phi_2^2)-(\partial_y\Phi_1^2+\partial_y\Phi_2^2)}. \tag{21}$$

By the definition of Lorenz-like maps $|\partial_x\Phi| \le b|x|^\sigma, |\partial_y\Phi_2| \le b|x|^\sigma, |\partial_x\Phi_2| \le 2b\lambda|x|^{\lambda-1}$ and, since $z \notin \Delta^{\pm c}$, $(1-b)\frac{\varepsilon^{\gamma+\delta}}{2} \le |\partial_x\Phi_1| \le 2(1+b)|x|^{\lambda-1}$ and so we get the following estimate

$$\tan 2\theta \le \frac{4b|x|^{\sigma+\lambda-1}}{\varepsilon^{2(\gamma+\delta)}}. \tag{22}$$

This shows that $f^{(1)}(z)$ and $e^{(1)}(z)$ are always orthogonal and that the slope of $f^{(1)}(z)$ is very small ($\le 4b|x|^{\sigma+\lambda-1}$). The smoothness follows clearly from the fact that $f^{(1)}(z)$ and $e^{(1)}(z)$ are defined in terms of a smooth differential equation. Thus the lemma is proved. □

The same calculation can be carried out, in principle, for the matrix of $D\Phi^n(z)$ to estimate the directions of the unit vectors $f^{(n)}(z)$ and $e^{(n)}(z)$ which are respectively most expanded and most contracted by the action of $D\Phi^n(z)$. However the matrix for the action of $D\Phi^n(z)$ becomes unwieldy quite quickly and it is easier to obtain the following result which says that under certain conditions the vectors $e^{(n)}(z)$ and $f^{(n)}(z)$ converge very rapidly. When such directions are defined let $f_n^{(\nu)}(z) = D\Phi^n(z)f^{(\nu)}(z)$ and $e_n^{(\nu)}(z) = D\Phi^n(z)e^{(\nu)}(z)$.

LEMMA 2.2. *Suppose that the point $z = (x,y)$ satisfies condition $(*)$ and exhibits exponential growth up to time n:*

$$\|D\Phi^j(z)\cdot(1,0)\| \ge e^{cj} \qquad 1 \le j \le n \tag{EG(n)}$$

Then, for all $1 \le \mu < \nu \le n$ we have

$$\angle(e^{(\mu)}(z), e^{(\nu)}(z)) \le \frac{b^\mu}{\|f_\mu^{(\mu)}(z)\|} \le Cb^\mu; \tag{i}$$

$$\|D\Phi^\mu(z)\cdot e^{(\nu)}(z)\| \le Cb^{\mu-}. \tag{ii}$$

PROOF. Let $z = (x,y)$ satisfy the hypotheses of the lemma. To simplify the exposition we shall omit reference to the base point z in the notation used below. For $1 \le \nu \le n$ write $e^{(\nu-1)} = \xi e^{(\nu)} + \eta f^{(\nu)}z$. Since $e^{(\nu)}$ and $f^{(\nu)}$ are orthogonal we have $\xi^2+\eta^2 = 1$ as well as $e_\nu^{(\nu-1)} = \xi e_\nu^{(\nu)} + \eta f_\nu^{(\nu)}$ and, in particular $(\xi^2+\eta^2)\|e_\nu^{(\nu-1)}\| = \xi\|e_\nu^{(\nu)}\| + \eta\|f_\nu^{(\nu)}\|$. An algebraic manipulation gives

$$\angle(e^{(\nu-1)}, e^{(\nu)}) = \tan^{-1}\frac{\eta}{\xi} = \sqrt{\frac{\|e_\nu^{(\nu-1)}\|^2 - \|e_\nu^{(\nu)}\|^2}{\|f_\nu^{(\nu)}\|^2 - \|e_\nu^{(\nu-1)}\|^2}}. \tag{23}$$

By $EG(n)$, we have $\|f_\nu^{(\nu)}\| \ge e^{c\nu}$ and $\|f_{\nu-1}^{(\nu-1)}\| \ge e^{c(\nu-1)}$. Using the general fact that $\|f_j^{(j)}\| \cdot \|e_j^{(j)}\| = \det D\Phi^j \le Kb^j$ we obtain $\|e_\nu^{(\nu)}\| \le b^\nu/e^{c\nu}$ and $\|e_{\nu-1}^{(\nu-1)}\| \le$

$b^{\nu-1}/e^{c(\nu-1)}$. The final, and most crucial estimate is an upper bound on $\|e_\nu^{(\nu-1)}\|$. Clearly we have

$$\|e_\nu^{(\nu-1)}\| \leq \|e_{\nu-1}^{(\nu-1)}\| \cdot \|D\Phi(z_{\nu-1})\|. \tag{24}$$

However the norm of the differential is not, a priori, bounded in the maps we are considering. Indeed we have $\|D\Phi\|$ tending to infinity as the line $\{x=0\}$ of discontinuity is approached. Here we use condition $(*)$ which, together with the definition of Lorenz like maps, implies

$$\|D\Phi(z_{\nu-1})\| \leq (1+b)\tau(\varepsilon^\gamma e^{-\alpha(\nu-1)})^{\lambda-1}. \tag{25}$$

From (24) and (25) we get

$$\begin{aligned}\|e_\nu^{(\nu-1)}\| &\leq \frac{b^{\nu-1}}{e^{c(\nu-1)}}(1+b)\tau(\varepsilon^\gamma e^{-\alpha(\nu-1)})^{\lambda-1} \\ &\leq (1+b)\tau\varepsilon^{\gamma(\lambda-1)}(be^{-c}e^{\alpha(1-\lambda)})^{\nu-1} \leq \tau\varepsilon^{\gamma(\lambda-1)}b^{\nu-1}\end{aligned} \tag{26}$$

choosing $\alpha < (1-\lambda)/c$ so that $e^{-c}e^{\alpha(1-\lambda)} < 1$. Thus we have, for some constant $C>0$,

$$\angle(e^{(\nu-1)}, e^{(\nu)}) \leq \sqrt{\frac{(K\varepsilon^{\gamma(\lambda-1)})^2 b^{2(\nu-1)}}{\|f_\nu^{(\nu)}\|/2}} \leq \frac{2K\varepsilon^{\gamma(\lambda-1)}}{2}\frac{b^{\nu-1}}{\|f_\nu^{(\nu)}\|} \leq C\frac{b^{\nu-1}}{\|f_\nu^{(\nu)}\|}.$$

Finally, this gives

$$\angle(e^{(\mu)}, e^{(\nu)}) \leq \sum_{j=\mu}^{\nu-1} \angle(e^{(j)}, e^{(j+1)}) \leq C\sum_{j=\mu}^{\nu-1}\frac{b^j}{\|f_j^{(j)}\|} \leq 2C\frac{b^\mu}{\|f_\mu^{(\mu)}\|}$$

proving (i). To prove (ii) write

$$\begin{aligned}\|D\Phi^\mu(z)\cdot e^{(\nu)}\| &\leq \|D\Phi^\mu(z)\cdot(e^{(\nu)}-e^{(\mu)})\| + \|D\Phi^\mu(z)\cdot e^{(\mu)}\| \\ &\leq \|D\Phi^\mu(z)\|\cdot\|e^{(\nu)}-e^{(\mu)}\| + b^\mu \\ &\leq 2C\|f_\mu^{(\mu)}\|\frac{b^{\mu-1}}{\|f_\mu^{(\mu)}\|} + b^\mu \\ &\leq 2Cb^\mu.\end{aligned}$$

This concludes the proof of the lemma. □

Points which are bound up to time n to some z satisfying $(*)$ also satisfy $|\xi_j^z| \geq \varepsilon^\gamma e^{-\alpha j}(1-e^{(\beta-\alpha)j})$. This estimate is quite sufficient for the entire proof above to go through yielding the following more general result.

LEMMA 2.3. *Suppose that the point* $z = (x, y)$ *satisfies condition* $(*)$ *and all* $\xi \in B^{(n)}(z)$ *exhibit exponential growth up to time* n. *Then, for all* $1 \leq \mu < \nu \leq n$ *we have*

$$(i) \qquad \angle(e^{(\mu)}(z), e^{(\nu)}(z)) \leq \frac{b^{\mu}}{\|f_{\mu}^{(\mu)}(z)\|};$$

$$(ii) \qquad \|D\Phi^{\mu}(z) \cdot e^{(\nu)}(z)\| \leq Cb^{\mu}.$$

Contractive approximations of a given order are actually almost constant functions of the space variable z and of the parameter.

LEMMA 2.4. *There exists a constant* $K_0 > 0$ *such that*

$$\|e^{(\nu)}(z)\|_{C^2(x,y,a)} \leq K_0 b \qquad \textit{for all } 1 \leq \nu \leq n.$$

2.2. Critical approximations. We consider the closure of the unstable manifold W of the hyperbolic periodic orbit $\{p_1, p_2\}$. By considering the iterates of the local unstable manifold of p_1 (resp. p_2) one sees that p_1 (resp. p_2) belongs to a leaf which runs all the way from the discontinuity line $\{x = 0\}$ to the tip of the cusp $\Phi_a^+(0, y) = \lim_{x \to 0^+} \Phi_a(x, y)$ (resp. $\Phi_a^-(0, y) = \lim_{x \to 0^-} \Phi_a(x, y)$). For definiteness let's focus on the leaf containing p_1.

LEMMA 2.5. *Let* $\gamma : x \to z(x) = (x, y(x))$ *parametrize the piece of leaf contained in* Δ^{-c}. *For* $b > 0$ *sufficiently small,* γ *satisfies the following conditions:*

$$|\dot{y}(x)|, |\ddot{y}(x)| \leq b \textit{ for all } z(x) \in \gamma.$$

PROOF. This follows from the smooth dependence of compact parts of invariant manifolds on the map (see e.g. [PT93]). $\square$

Let $\gamma_0 = \Phi_a(\gamma)$ and $z_0(x) = \Phi_a(z(x) \in \gamma$. By the definition of Lorenz-like maps (in particular condition L4 in the definition of one-dimensional Lorenz-like maps with criticalities) we have $\gamma_0 \cap \Delta^{\pm c} = \emptyset$. Thus, by lemma 1.2 we have, for each $z_0 = z_0(x) \in \gamma_0$, $\|w_1(z_0)\| \geq e^{c_0}$ where $w_1(z_0) = D\Phi_a(z_0) \cdot w_0(z_0)$ and $w_0(z_0) = (1, 0)$. In particular the first contractive approximations $e^{(1)}(z_0)$ are defined. Recall from lemmas 2.1 and 2.2 that these contractive approximations are almost vertical: $|\angle(e^{(1)}(z_0), (0, 1))| \leq 4b/\varepsilon^{2(\gamma+\delta)}$, and almost constant: $\|De^{(1)}(z_0)\| \leq K_0 b$. On the other hand the tangent vectors to γ_0, $t_0(z_0) = D\Phi_a(z(x)) \cdot (1, \dot{y}(x))$ are quickly changing direction due to the sharp fold in γ_0.

DEFINITION 2.6. We say that $z^{(1)}(x) = z(x^{(1)})$ is a first order critical approximation if $t_0(z^{(1)})$ is colinear with $e^{(1)}(z^{(1)})$.

LEMMA 2.7. *There exists a unique first order critical approximation* $z^{(1)} \in \gamma$. *Moreover* $|x^{(1)} + c| \leq \tau b/4$.

PROOF. Notice first of all that the formalism introduced above makes sense in the one dimensional situation, i.e. for $b = 0$. In this case γ reduces to the one-dimensional neighbourhood Δ^{-c} of the critical point $-c$, it is trivially b-flat, and we have $t(z_0) = \varphi'(x)$ where $z_0 = z_0(x) = \varphi(x)$. Thus, in particular, $|t(-c_1)| = |\varphi'(-c)| = 0$. Moreover, for $b = 0$, the first order contractive approximations $e^{(1)}(x)$ are always vertical and the first order expanding directions $f^{(1)}(x)$ are defined and horizontal everywhere excpet at $x = \pm c$ (and at the discontinuity). Thus it makes sense to talk about the scalar product $|T(x) \cdot f(x)|$ and we have, by the remarks above, for $b = 0$ that $|t(-c) \cdot f^{(1)}(-c_1)| = 0$. Sonce Φ is close to φ in the C^2 topology we have, for small $b > 0$

$$|t(z_1(-c)) \cdot f^{(1)}(z_0(-c))| \leq b.$$

However we also have

$$|D_x(t(z_0(x)) \cdot f^{(1)}(z_0(x))| = |D_x t(z_0(x)) \cdot f^{(1)}(z_0(x)) + D_x f^{(1)}(z_)(x)) \cdot t(z_1(x))|$$

with $|D_x f^{(1)}(z_0(x)) \cdot t(z_0(x))| \leq K_0 b$ by lemma 2.4 and since $|t(z_0(x))| \leq \varepsilon^{\gamma+\delta} \ll 1$ for $z(x) \in \Delta^{-c}$. For $b = 0$ we have $D_x t(z_0(x)) = (\varphi''(x), 0)$ with $|\varphi''(x)| \geq \tau > 0$. For $b > 0$ small we can write $D_x t(z_0(x)) = (\varphi''(x), 0) + v(x)$ with $\|v\| \leq const.b$. Thus, in particular, $D_x t(z_0(x))$ is almost horizontal and bounded away from zero in norm. Since $f^{(1)}(z_0(x))$ is also almost horizontal and has norm 1 we get $|D_x t(z_0(x)) \cdot f^{(1)}(z_0(x))| \geq \tau/2 > 0$ and therefore

$$|D_x(t(z_0(x)) \cdot f^{(1)}(z_0(x)))| \geq \tau/3 > 0.$$

Thus there exists a unique $x^{(1)}$ such that $z^{(1)}{}_0 = z_0(x^{(1)})$ satisfies

$$|t(z^{(1)}{}_0) \cdot f^{(1)}(z^{(1)}{}_0).$$

In particular, $t(z^{(1)}{}_0)$ is colinear with $e^{(1)}(z^{(1)}{}_0)$ and is the required first order critical approximation. Moreover we have

$$|x^{(1)} + c| \leq b\tau/4$$

and the proof is complete. □

If all points $\xi_0 \in B_0^{(2)}(z^{(1)})$ are expanding up to time 2 then a 2-nd order contractive approximation $e^{(2)}(\xi_0)$ is defined in each such point. we shall show below that, in this case, we can find a 2-nd order critical approximation $z^{(2)} \in B^{(2)}(z^{(1)})$. In fact we can iterate this procedure as long as all points in $B_0^{(i+1)}(z^{(i)})$ exhibit exponential growth up to time $i + 1$.

LEMMA 2.8. *Suppose that critical approximations $z^{(1)}, \ldots, z^{(\nu-1)} \in \gamma$ have been constructed. Suppose that each $z^{(i)}$ satisfies condition $(*)$ and each $\xi_0^{(i)} \in B_0^{(i)}(z^{(i)})$*

exhibits exponential growth up to time $i+1$ (in particular up to time ν). Then there exists a unique critical approximation $z^{(\nu)}$ in $\gamma \cap B^{(\nu)}(z^{(\nu-1)})$.

PROOF. Notice first of all that for each $i = 1, \ldots, \nu - 1$, $B^{(i)}(z^{(i)})$ contains a ball of radius $\varepsilon^{\gamma} e^{-\beta i} \kappa^{-(i+1)}$ centred in $z^{(i)}$. Indeed let $R^{(i)}$ be a straight line segment originating in $z^{(i)}$, with $|R^{(i)} \leq \varepsilon^{\gamma} e^{-\beta i} \kappa^{-(i+1)}$ and, for each $z \in R^{(i)}$, let $t(z)$ be a unit vector tangent to $R^{(i)}$ in z. Then we have

$$\begin{aligned} |R_i^{(i)}| = |\Phi^{i+1}(R)| &= \int_R \|D\Phi^{i+1}(z) \cdot t(z)\| dz \\ &\leq |R|\kappa^{i+1} \leq \varepsilon^{\gamma} e^{-\beta i} \kappa^{-i+1} \kappa^{i+1} \leq \varepsilon^{\gamma} e^{-\beta i} \end{aligned}$$

which implies the claim.Thus it is enough to show the existence of a critical approximation $z^{(\nu)} \in \gamma$ with $|z^{(\nu)} - z^{(\nu-1)}| \leq \varepsilon^{\gamma} e^{-\beta i} \kappa^{-(i+1)}$.

By hypothesis all points in $\gamma_0 \cap B_0^{(\nu-1)}(z^{(\nu-1)})$ are expanding up to time ν and so the contractive approximations $e^{(\nu)}(z_0)$ are defined in each $z_0 \in \gamma \cap B_0^{(\nu-1)}(z^{(\nu-1)})$. By lemma 2.4 we have $\angle(e^{(\nu)}(z_0), e^{(\nu)}(z_0^{(\nu-1)})) \leq b|z_0 - z_0^{(\nu-1)}|$ and, by lemma 2.2 $\angle(e^{(\nu-1)}(z_0^{(\nu-1)}), e^{(\nu)}(z_0^{(\nu-1)})) \leq \frac{b^{\nu}}{\|D\Phi^{\nu-1}(z_0^{(\nu-1)})\|}$ and, therefore,

$$\angle(e^{(\nu-1)}(z_0^{(\nu-1)}). e^{(\nu)}(z_0)) \leq \frac{b^{\nu-}}{\|D\Phi^{\nu-1}(z_0^{(\nu-1)})\|} + b|z_0 - z_0^{(\nu-1)}|$$

Now recall that by the definition of contractive approximation

$$t_0(x^{(\nu-1)}) // e^{(\nu-1)}(z_0^{(\nu-1)})$$

where $z^{(\nu-1)} = (x^{(\nu-1)}. y^{(\nu-1)}) = D\Phi(z^{(\nu-1)}) \cdot t(x^{(\nu-1)})$ and $t_0(x^{(\nu-1)}) = (1, \dot{y}(x^{(\nu-1)}))$ is tangent to γ in $z^{(\nu-1)}$. The direction of $t_0(x)$ is changing quite rapidly as x varies near $x^{(\nu-1)}$. One can see this formally by decomposing $t_0(x)$ into a horizontal component $u(x)$ and a vertical component $v(x)$. Then, by the form of $D\Phi(z(x))$ we have easily that $\dot{v}(x) \leq b$ while $\dot{u}(x) \geq \tau/2$ implying that

$$\frac{d}{dx}(\text{ slope } t_0(x)) \geq \frac{\tau}{3}.$$

Thus the amount of "space" measured along γ required for $t_0(x)$ to swing around from a direction colinear to $e^{(\nu-1)}(z_0^{(\nu-1)})$ to a direction colinear to $e^{(\nu)}(z_0))$ for some $z_0 = \Phi(z(x))$ is of the order of $\angle(e^{(\nu-1)}(z_0^{(\nu-1)}), e^{(\nu)}(z_0))/(\tau/3)$. This implies that there exists a point $z_0^{(\nu)}$ with $t_0(x^{(\nu)})//e^{(\nu)}(()z_0^{(\nu)})$ with

$$\begin{aligned}|z^{(\nu)} - z^{(\nu+1)}| &\leq \frac{3\angle(e^{(\nu-1)}(z_0^{(\nu-1)}), e^{(\nu)}(()z_0))}{\tau} \\ &\leq \frac{3}{\tau}b^{\nu}\|D\Phi^{\nu-1}(z_0^{(\nu-1)})\| + \frac{3}{\tau}b|z_0^{(\nu)} - z_0^{(\nu-1)}| \\ &\leq \frac{3}{\tau}b^{\nu-2}\|D\Phi^{\nu}(z_0^{(\nu-1)})\| + \frac{3}{\tau}b|x^{(\nu)} - x^{(\nu-1)}|\end{aligned}$$

giving

$$\begin{aligned}|z^{(\nu)} - z^{(\nu+1)}| &\leq \frac{3b^{\nu-1}}{(1-3b/\tau)\tau\|D\Phi^{\nu-1}(z_0^{(\nu-1)})} \\ &\leq b^{\nu-2} \leq \varepsilon^{\gamma}e^{-\beta\nu}\kappa^{-(\nu+1)}\end{aligned}$$

if b is small. □

Notice that two critical approximations of different orders, e.g. $z^{(k)}$ and $z^{(k+1)}$, do not, in general, coincide. Strictly speaking therefore we are continually changing the initial point of the orbit we are considering, at each iterate k we are interested in the properties of the orbit of $z^{(k)}$ up to time k. This fact constitutes one of the main motivations behind lemma 2.8 and other lemmas which will be proved below. Indeed $z^{(k+1)}$ might not coincide with $z^{(k)}$ but the two are not completely indipendent either, in lemma 2.3 we show that $z^{(k)} - z^{(k+1)} \leq b^k$ and in particular $z^{(k+1)} \in B^{(k+1)}(z^{(k)})$. Thus there is little difference at this point between considering the orbit of $z^{(k)}$ (up to time $k+1$) or the orbit of $z^{(k+1)}$ up to time $k+1$. In fact we shall see that all points bound to a critical approximation are essentially indistinguishable, at least as far as the expansivity properties which we are interested in are concerned, they satisfy a bounded distortion property and have basically the same itinerary.

Finally we also want to be sure that all points which are bound to $z^{(k+1)}$ up to time $k+1$ are also bound to $z^{(k)}$ up to time k. i.e.

$$B_k^{(k+1)}(z^{(k+1)}) \subset B_k^{(k)}(z^{(k)}). \tag{27}$$

For this reason we introduce below an alternative definition of binding period. Recall that $|z^{(k)} - z^{(k+1)}| \leq b^k$ and therefore $|z_k^{(k)} - z_{k+1}^{(k)}| \leq (\kappa b)^j$. Thus we want to define a binding condition which guarantees that all points satisfying $|\xi_k - z_{k+1}^{(k)}| \leq h_{k+1}\varepsilon^{\gamma}e^{-\beta k}$ also satisfy $|\xi_k - z_k^{(k)}| \leq h_k\varepsilon^{\gamma}e^{-\beta k}$. We have

$$|\xi_k z_k^{(k)}| \leq |\xi_k - z_{k+1}^{(k)}| + |z_{k+1}^{(k)} - z_k^{(k)}| \leq h_{k+1}\varepsilon^{\gamma}e^{-\beta k} + (\kappa b)^k$$

and so $|\xi_k - z_k^{(k)}| \le h_k \varepsilon^\gamma e^{-\beta k}$ if $h_{k+1}\varepsilon^\gamma e^{-\beta k} + (\kappa b)^k \le h_k \varepsilon^\gamma e^{-\beta k}$ which gives

$$h_{k+1} \le h_k - (\kappa e^\beta b)^k \varepsilon^{-\gamma}.$$

This leads us to the following

DEFINITION 2.9. A point ξ is *bound* to a critical approximation $z^{(\nu)}$up to time $k \le \nu$ if

$$|\xi_j - z_k^{(\nu)}| \le h_\nu \varepsilon^\gamma e^{-\beta k} \qquad \text{for all } 1 \le j \le k$$

where

$$h_\nu = 2 - \sum_{i=1}^{\nu} \varepsilon^{-\gamma}(\kappa e^\beta b)^i \in (1,2).$$

2.3. Higher generation critical approximations. So far we have shown how to construct a sequence of critical approximations $z^{(1)}, \dots, z^{(\nu)}$ on a leaf of W containing one of the two periodic points p_1 or p_2 of period 2. We discuss here how these critical approximations induce other, nearby, critical approximations which are, in some sense, more deeply embedded in the attractor. Them we explain what we mean by the *generation* of a critical approximation.

Let $z^{(\nu)}$be one of the critical approximations constructed above lying on $\gamma : x \to z(x) = (x, y(x))$. Recall that $|\dot{y}(x)|, |\ddot{y}(x)| \le b^{\frac{1}{2}}$ for all $x \in \gamma_x$, the domain of the parametrization of γ. Recall also that $B^{(\nu)}(z^{(\nu)})$ contains a ball of radius $\varepsilon^\gamma(\kappa\epsilon^3)^{-\nu}$. Suppose that there exists another piece of W, parametrized by $\tilde{\gamma} : x \to \tilde{z}(x) = (x, \tilde{y}(x))$ for $x \in \tilde{\gamma}_x$ satsifying $|\dot{\tilde{y}}(x)|, |\ddot{\tilde{y}}(x)| \le b^{\frac{1}{2}}$ and $|\tilde{z}(x) - z(x)| \le \varepsilon^{2\gamma}(\kappa\epsilon^3)^{-2\nu}$ for all $x \in \gamma_x \cap \tilde{\gamma}_x$. We suppose that $\tilde{\gamma}$ is sufficiently long and sufficiently well centred with respect to $z^{(\nu)}$: if $z^{(\nu)} = (x^{(\nu)}, y(x^{(\nu)}))$ we suppose that $\tilde{\gamma}_x \supset (x^{(\nu)} - 1/4\varepsilon^\gamma(\kappa e^\beta)^{-\nu}. x^{(\nu)} + 1/4\varepsilon^\gamma(\kappa e^\beta)^{-\nu})$. Then we easily get the following result by using the arguments of lemma 2.8.

LEMMA 2.10. *There exists a unique critical approximation* $\tilde{z}^{(\nu)} \in \tilde{\gamma} \cap B^{(\nu)}(z^{(\nu)})$.

The leaves of W are all classified according to their generation in the following way. We let G_0 denote the connected components of W through the points p_1 and p_2. For $g \ge 1$ let $G_g = \Phi_a^g(G_0) \setminus \varphi_a^{g-1}(G_0)$. Then, if the curve $\tilde{\gamma}$ in the construction above is contained in G_g we say that the critical approximation $\tilde{z}^{(\nu)} \in \tilde{g}g$ is of generation g.

3. The first return

We consider the first time that one of the critical approximations constructed above falls into $\Delta^{\pm c}$. More precisely we define the first return ν as the unique integer such that

$$z_j^{(j)} \notin \Delta^{\pm c} \qquad \forall\, 1 \le j \le \nu - 1$$

and

$$z_\nu^{(\nu)} \in \Delta^{\pm c}.$$

Notice that $\nu > \delta/\alpha \log 1/\varepsilon$ since all critical approximations satisfy $(*)$ by hypothesis (we only consider parameters such that this is the case). By lemma 1.4 condition $(*)$ implies $z_j^{(n)} \notin \Delta^{\pm c}$ for all $0 \le j \le \delta/\alpha \log 1/\varepsilon$. Moreover we also have $\|w_j(z^{(n)})\| \ge e^{c_0 j}$ and *slope* $w_j(z^{(n)}) \le b^{\frac{1}{2}}$ for all $0 \le j \le n \le \delta/\alpha \log 1/\varepsilon$ as discussed above. Analogously to what happens in the one-dimensional case a lot of expansion is lost in the critical region. Here we also have the additional complication of a certain amount of rotation which means that the slopes of the vectors involved are difficult to control. The purpose of this section is to introduce certain key ideas and technical tools to enable to to gain some control over return iterates.

3.1. Binding points and binding periods. Let $z_\nu^{(\nu)}$be a return to $\Delta^{\pm c}$. A *binding point* for $z_\nu^{(\nu)}$is a critical approximation $\zeta^{(\nu)}$ (of order ν) such that $z_\nu^{(\nu)}$and $\zeta^{(\nu)}$ are in *tangential position*: There exists a curve $\gamma : [x^{(\nu)}, x_\nu^{(\nu)}] \to z(x) = (x, y(x))$ with $z(x^{(\nu)}) = \zeta^{(\nu)}$, $z(x_\nu^{(\nu)}) = z_\nu^{(\nu)}$, $|\dot{y}(x)|, |\ddot{y}(x)| \le b^{\frac{1}{2}}$ for all $x \in [x^{(\nu)}, x_\nu^{(\nu)}]$, and satisfying $t(\zeta^{\nu)}) = (1, \dot{y}(x^{(\nu)}))$ colinear to (1,0) and $t(z_\nu^{(\nu)}) = (1, \dot{y}(x_\nu^{(\nu)}))$ colinear to $w_\nu(z^{(\nu)})$. The fact that binding points exist is proved in the second half of the paper where we deal with parameter exclusions. For the moment we suppose that they do. For γ as above, we write $\gamma_0 = \Phi_a(\gamma)$ and $\gamma_i = \Phi^i(\gamma_0)$. Let $|\gamma_i|$ denote the length of γ_i. Then we define the *binding period* associated to the first return $z_\nu^{(\nu)}$ as the interval $[\nu+1, \nu+p]$ where p is the unique integer such that

$$|\gamma_j| \le \varepsilon^\gamma e^{-\beta j} \qquad \text{for all } j \le p-1 \tag{28}$$

and

$$|\gamma_p| \ge \varepsilon^\gamma e^{-\beta p} \tag{29}$$

In this section we shall deal with the case in which there are no further returns during the binding period:

$$z_{\nu+j}^{(\nu)} \notin \Delta^{\pm c},\ \forall\, 1 \le j \le p.$$

This happens for example when $z_\nu^{(\nu)}$is a first return since it implies that $\zeta_j^{(\nu)} \notin \Delta^{\pm c}, \forall\, 1 \le j \le \nu$ and we shall show below that $p < \nu$. In the general case it can (and does) happen that $z_{\nu+j}^{(\nu)}$ falls into $\Delta^{\pm c}$ during a binding period. Such *bound returns* cannot be ignored, as they are in the one dimensional situation, and give rise to higher order binding periods within the main binding period. We shall consider this situation in the next section. There we will give a more general definition of binding period (which coincides with the one given above in the case of the first return considered here) and show that higher level binding periods are nested allowing us to control the cumulative effect. Returns which do not occur within binding periods, such as the first return, are called *free returns*. Any return to $\Delta^{\pm c}$ is either a free return or a bound return.

3.2. Controlling recurrence in the critical region. Let $0 < \nu_1 < \nu_2 < \cdots < \nu_s < n$ be a (maximal) sequence of free returns to $\Delta^{\pm c}$ of the critical approximation $z^{(n)}$ (recall remark in the previous section). For each such returns ν_i let $\zeta^{(\nu_i)}$ be a binding point for $z^{(\nu_i)}$ and let $r_i \in \mathbb{Z}$ be such that

$$\varepsilon^{\gamma} e^{-r_i+2} \geq |\gamma^{(\nu_i)}| \geq |z_{\nu_i}^{(\nu_i)} - \zeta^{(\nu_i)}| \geq |x_{\nu_i}^{(\nu_i)} - x^{(\nu_i)}| \geq \varepsilon^{\gamma} e^{-r_i}. \tag{30}$$

Here $\gamma^{(\nu_i)}$ denotes the curve which binds $z_{\nu_i}^{(\nu_i)}$ to $\zeta^{(\nu_i)}$ in tangential position and r_i is called the *depth* of the return ν_i. We shall always suppose from now on that the following condition is satisfied.

$$\sum_{i=1}^{s} r_i \leq \alpha n \tag{$*$}$$

This is exactly the same condition used to control the recurrence near the discontinuity. We use the same name since from now on we shall always suppose that every critical approximation of order n satisfies condition $(*)$ both near the discontinuity and in the critical region. This is true up to parameter exclusions in the sense that we exclude those parameters for which some critical approximation fails to satisfy $(*)$. Notice that $(*)$ implies in particular

$$|x_{\nu_i}^{(\nu_i)}| \geq \varepsilon^{\gamma} e^{-\alpha \nu_i} \tag{31}$$

where $z_{\nu_i}^{(\nu_i)} = (x_{\nu_i}^{(\nu_i)}, y_{\nu_i}^{(\nu_i)})$.

3.3. Loss of expansivity and recovery.

LEMMA 3.1. *Let ν be a first return, $\zeta = \zeta^{(\nu)}$ a binding point for $z = z_{\nu}^{(\nu)}$, $\gamma = \gamma^{(\nu)}$ the binding curve through z_ν and ζ and $r = r_\nu$ the depth of the return ν, ($|\gamma| \approx \varepsilon^{\gamma} e^{-r}$). Let p be the length of the binding period associated to ν. Then*

$$\frac{r}{\beta \log \kappa} \leq \frac{r}{\gamma(1-\lambda)\log 1/\varepsilon} \leq p \leq 2(1+\gamma/\delta) r \leq \nu - 1 \tag{i}$$

and, moreover,

$$\|w_{\nu+p+1}(z^{(\nu)})\| \geq e^{\beta p} \|w_\nu(z^{(\nu)})\| \tag{ii}$$

$$\text{slope } w_{\nu+p+1}(z^{(\nu)}) \leq b^{\frac{1}{2}} \tag{iii}$$

PROOF. Let $\gamma : [x^{(\nu)}, x_\nu^{(\nu)}] \to z(x) = (1, y(x))$ be the parametrization of the curve which binds $z_\nu^{(\nu)}$ to $\zeta^{(\nu)}$. Let $t(x) = (1, \dot{y}(x))$, $t_0(x) = D\Phi(z(x)) \cdot t(x)$ and, for $1 \leq i \leq p$, $t_i(x) = D\Phi^i(z_0(x)) \cdot t_0(x)$. Let also $\gamma_0 = \Phi(\gamma)$ and $\gamma_i = \Phi^i(\gamma_0)$ and $\gamma_x = [x^{(\nu)}, x_\nu^{(\nu)}]$. Letting $|\gamma_i|$ denote the length of γ_i we have

$$|\gamma| = \int_{\gamma_x} \|t(x)\|dx \text{ and } |\gamma_i| = \int_{\gamma_x} \|t_i(x)\|dx.$$

Notice that we can immediately get a lower bound for p by applying lemma 1.5. Write

$$\varepsilon^\gamma e^{-\beta p} \leq |\gamma_p| = \int_{\gamma_x} \|t_i(x)\|dx \leq \varepsilon^\gamma \varepsilon^{-r} \kappa^p \tag{32}$$

which implies

$$p \geq \frac{r}{\beta \log \kappa}. \tag{33}$$

By the definition of binding. $|\gamma_i| \leq \varepsilon^\gamma e^{-\beta i}$ for all $1 \leq i \leq p-1$ and, in particular, all points in γ are bound to $\zeta^{(\nu)}$ up to time p, i.e. $\gamma \subset B^{(p)}(\zeta^{(\nu)})$. Recall also that, by the definition of contractive approximation, $\zeta_0^{(\nu)}$ exhibits exponential growth up to time ν, that is: $\|w_j(\zeta^{(\nu)})\| \geq e^{c_0 j} \quad \forall\ 0 \leq j \leq \nu$. Moreover, since ν is a first return, $\zeta_\nu^{(\nu)} j \notin \Delta^{\pm c}$ for all $0 \leq j \leq \nu - 1$ and thus, by lemma 1.8 all points in $B_0^{(\nu)}(\zeta^{(\nu)})$ exhibit exponential growth and, moreover, the bounded distortion property is satisfied. We shall prove below that $p < \nu$ implying $B^{(p)}(\zeta^{(\nu)}) \subset B^{(\nu)}(\zeta^{(\nu)})$. However this information is not yet available to us for the present calculations and so we set, for the moment, $\hat{p} = \min\{p, \nu\}$ and use $\hat{p}$ instead of p. In particular we can say that all points in $\gamma \subset B_0^{(\hat{p})}(\zeta^{(\nu)}) \subset B_0^{(\nu)}(\zeta^{(\nu)})$ exhibit exponential growth and satisfy the bounded distortion property.in particular the contractive approximations $e^{(\hat{p})}(z^{(0)})$ are defined for each $z_0 \in \gamma_0$. We summarize below two important facts concerning these contractive approximations. By lemma 2.2 they are "almost vertical": letting $(q, 1), q = q(z_0(x))$ denote a vector in the direction of $e^{(\hat{p})}(z_0(x))$ we have

$$|q| \leq Cb. \tag{34}$$

Moreover, by lemma 2.4 they are almost constant functions of the point $z_0 = z_0(x)$ (and therefore almost constant functions of x), i.e.

$$|\dot{q}| \leq Cb. \tag{35}$$

Now we take, in the tangent space of each $z_0 \in \gamma_0$ a system of coordinates whose base is given by $\{(1,0), e^{(\hat{p})}(z_0)\}$. By what we just said these coordinate systems are almost constant along γ_0. We let

$$t_0(x) = (\beta_0(x), \alpha_0(x))$$

denote the vector $t_0(x)$ the vector $t_0(x)$ in the coordinate system given by $\{(1,0), e^{(\hat{p})}(z_0)\}$. Let also

$$t_{-0}(x) = (u_0(x), v_0(x))$$

denote the vector $t_0(x)$ in the coordinate system given by the horizontal and the vertical coordinate axes. From (34) we have $|v_0(x)| \leq |\alpha_0(x)| \leq (1 + Cb)|v_0(x)|$ and $|v_0(x)| = |\Phi_c(z(x)) + \dot{y}(z(x))\Phi_D(z(x))| \leq Cb$ giving, in particular,

$$|\alpha_0(x)| \leq 2Cb. \tag{36}$$

Moreover, differentiating, with respect to x,

$$|\dot{\alpha}_0(x)| \leq 2Cb \tag{37}$$

Then $\beta_0(x) = u_0(x) - q(x)|\alpha_0(x)|$. Differentiating with respect to x and using (35) we set

$$\tau' \geq \dot{\beta}_0(x) \geq \tau. \tag{38}$$

An explicit estimate for $\beta_0(x)$ is more delicate and constitutes in fact the crucial estimate in the proof. From (38) we have

$$\tau'|x - x^{(\nu)}| \geq |\beta_0(x) - \beta_0(x^{(\nu)})| \geq \tau|x - x^{(\nu)}| \tag{39}$$

and, our aim now, is to show that $\beta_0(z^{(\nu-1)})$ is very small (in relation to $|x - x^{(\nu)}|$) and, from there, to conclude that $\beta_o(x_\nu^{(\nu)})$ is of the order of $|x_\nu^{(\nu)} - x^{(\nu)}| \approx \varepsilon^\gamma e^{-\alpha r}$. to achieve this aim we proceed in the following way. Recall that $\zeta^{(\nu)} = (x^{(\nu)}, y(x^{(\nu)}))$ is a critical approximation and therefore, by definition, $t_0(x^{(\nu)})$ is colinear with $e^{(\nu)}(z_0(x^{(\nu)}))$. By lemma 2.2, $\angle(e^{(\hat{p})}, e^{(\nu)}) \leq b^{\hat{p}}/\|f_{\hat{p}}^{(\hat{p})}(z_0(x^{(\nu)}))$ therefore,

(40)

$$\beta_0(x^{(\nu)}) \leq |u_0(x^{(\nu)}) - \alpha_0(x^{(\nu)})q(x^{(\nu)})| \leq \frac{b^{\hat{p}}}{\|f_{\hat{p}}^{(\hat{p})}(z_0(x^{(\nu)}))\|}(1 + Cb)|\alpha_0| \leq b^{\hat{p}}.$$

It remains to show that $\hat{p}$ is sufficiently large (we want $\beta_0(x^{(\nu)}) \leq b^{\hat{p}} \ll \varepsilon^\gamma e^{-\alpha r}$). From (45) we have

$$|\beta(x^{(\nu)})| \leq \frac{b^{\hat{p}}}{\|f_{\hat{p}}^{(\hat{p})}(z_0(x^{(\nu)}))\|} \ll \varepsilon^\gamma \left(\frac{e^{-\beta}}{\kappa}\right)^p \leq \varepsilon^\gamma e^{-r}. \tag{41}$$

From (39) this gives

$$|\beta(x_\nu^{(\nu)})| \geq \frac{1}{2}\tau|x_\nu^{(\nu)} - x^{(\nu)} \geq \tau\varepsilon^\gamma e^{-r}. \tag{42}$$

The sweaty part of the proof is over. We are at the top of the hill and the view is good. We can now obtain an upper bound for $\hat{p}$ and show that $\hat{p} = p < \nu$.

We begin by writing $t_i(x) = \beta_0(x)w_i(z_0(x)) + \alpha_0(x)e_i^{(\hat{p})}(z_0(x))$ where $e_i^{(\hat{p})}(z_0(x)) = D\Phi^i(z_0(x)) \cdot e^{(\hat{p})}(z_0(x))$. By lemma 2.2 $\|e_i^{(\hat{p})}(z_0(x)) \le Cb^{\hat{p}}$ and therefore we have

$$\varepsilon^{\gamma} e^{-\beta i} \ge |\gamma_i| = \int_{\gamma_x} \|t_i(x)\| dx \ge \int_{\gamma_x} \|\beta_0(x) w_i(z_0(x))\| dx - Cb^{\hat{p}} \tag{43}$$

for all $1 \le i \le \hat{p} - 1$. Moreover we can write

$$\beta_0(x)w_i(z_0(x)) = \beta_0(x^{(\nu)})w_i(z_0(x)) + (\beta_0(x) - \beta_0(x^{(\nu)}))w_i(z_0(x)). \tag{44}$$

By the bounded distortion property $\|w_i(z_0(x))\| \le C\|w_i(\zeta_0^{(\nu)})\|$ and therefore we have $\|\beta_0(x^{(\nu)})w_i(z_0(x))\| \le C\frac{b^i}{\|f_i^{(1)}(\zeta_0^{(\nu)})\|} \le Cb^i$. It follows that $\|\beta_0(x)w_i(z_0(x))\| \ge |\beta_0(x) - \beta_0(x^{(\nu)})| \|w_i(z_0(x))\| - Cb^i$ and so, from (43), letting $i = \hat{p} - 1$,

$$\begin{aligned}
\varepsilon^{\gamma} e^{-\beta(\hat{p}-1)} &\ge |\gamma_i| \ge \int_{\gamma_x} |\beta_0(x) - \beta_0(x^{(\nu)})| \|w_{\hat{p}-1}(z_0(x))\| dx \\
&\ge C\|w_{\hat{p}-1}(\zeta^{(\nu)})\| \int_{\gamma_x} |\beta_0(x) - \beta_0(x^{(\nu)})| dx \\
&\ge Ce^{c_0(\hat{p}-1)}\tau \int_{\gamma_x} |x - x^{(\nu)}| dx \ge C\tau e^{c_0(\hat{p}-1)} |\gamma_x|^2 \ge C\tau e^{c_0(\hat{p}-1)} (\varepsilon^{\gamma} e^{-r})^2.
\end{aligned}$$

From this we get $\varepsilon^{-\gamma} e^{2r} e^{c_0+\beta} C^{-1} \tau^{-1} \ge e^{(c_0+\beta)\hat{p}}$ which gives $\hat{p} \le 2r + \gamma \log 1/\varepsilon + c_0 + \beta - \log C\tau$. Keeping in mind that $r \ge \delta \log 1/\varepsilon$ we can write $\gamma \log 1/\varepsilon \le (\gamma/\delta)r$ and therefore

$$\hat{p} \le (2 + \frac{gg}{\delta})r \le (2 + \gamma/\delta)\alpha\nu \le \frac{1}{2}\nu \tag{45}$$

where the second inequality follows from condition $(*)$ and the third by choosing α small. On the strength of this new piece of information we repeat essentially the scheme used above, in order to obtain (iii). From (39) and the definiton of binding period we have

$$\begin{aligned}
\varepsilon^{\gamma} e^{-\beta p} &\le |\gamma_p| \le \int_{\gamma_x} |\beta_0(x) - \beta_0(x^{(\nu)})| \|w_p(z_0(x))\| dx - 2Cb^p \\
&\le \frac{C\tau}{2} |x^{(\nu)} - x_\nu^{(\nu)}|^2 \|w_p(z_0(x_\nu^{(\nu)}))\|
\end{aligned}$$

which gives

$$\|w_p(z_0(x_\nu^{(\nu)}))\| \ge \frac{2}{C\tau} \frac{\varepsilon^{\gamma} e^{-\beta p}}{\varepsilon^{2\gamma} e^{-2r}} \ge \frac{2}{C\tau} \varepsilon^{-\gamma} e^{2r - \beta p}.$$

Thus

$$\begin{aligned}\|w_{\nu+p+1}(z^{(\nu)})\| &\geq \|w_\nu(z^{(\nu)})\|\beta(x_\nu^{(\nu)})\|w_p(z_{\nu+1}^{(\nu)})\| \\ &\geq \|w_\nu(z^{(\nu)})\|\varepsilon^\gamma e^{-r}\varepsilon^{-\gamma}e^{2r-\beta p} \\ &\geq \|w_\nu(z^{(\nu)})\|e^{r-\beta p}.\end{aligned}$$

Now using $r \geq p/(2+\gamma/\delta)$ from (45) we have

$$e^{r-\beta p} \geq e^{(\frac{1}{2+\gamma/\delta}-\beta)p} \geq e^{\frac{\beta}{2}p}$$

choosing β small. This proves (ii). Now (iii) is relatively straightforward. Notice first of all that (ii) implies $\|w_{\nu+p+1}(z^{(\nu)})\| \geq \|w_\nu(z^{(\nu)})\|$. On the other $w_{\nu+p+1}(z^{(\nu)}) = \omega_{\nu+p+1}(z^{(\nu)}) + \sigma_{\nu+p+1}(z^{(\nu)})$ where $\omega_{\nu+p+1}(z^{(\nu)}) = D\Phi^p(z_{\nu+1}^{(\nu)} \cdot omega_{\nu+1}(z^{(\nu)})$ and $\sigma_{\nu+p+1}(z^{(\nu)}) = D\Phi^p(z_{\nu+1}^{(\nu)} \cdot \sigma_{\nu+1}(z^{(\nu)})$ and $w_{\nu+1}(z^{(\nu)}) = \omega_{\nu+1}(z^{(\nu)}) + \sigma_{\nu+1}(z^{(\nu)})$ is the decomposition of $w_{\nu+1}(z^{(\nu)})$ in the coordinate system given by $\{(1,0), e^{(p)}(z_{\nu+1}^{(\nu)})\}$. Since there are no returns between $\nu+1$ and $\nu+p+1$ we have slope $\omega_{\nu+1}(z^{(\nu)}) \leq b^{1/2}$ and, by lemma, 2.2, $\|\sigma_{\nu+p+1}(z^{(\nu)})\| \leq b^p\|\sigma_{\nu+1}(z^{(\nu)}) \leq b^{p+1}\|w_\nu(z^{(\nu)})\|$ where the last inequality follows from (36). Therefore

$$\text{slope } w_{\nu+p+1}(z^{(\nu)}) \leq b^{\frac{1}{2}} + \frac{b^{p+1}\|w_\nu(z^{(\nu)})\|}{\|w_{\nu+p+1}(x^{(\nu)})}\| \leq 2b^{\frac{1}{2}}.$$

This completes the proof of the lemma. □

REMARK 3.2. The vector $w_{\nu+p+1}(z^{(\nu)})$ actually returns to an almost horizontal position long before the binding period ends. Indeed as we mentioned above, the vector $\omega_{\nu+k+1}(z^{(\nu)}) = w_k(z_{\nu+1}^{(\nu)})$ has a small slope. thus it is sufficient to show that

$$\frac{\|\sigma_{\nu+k+1}(z^{(\nu)})\|\|w_\nu(z^{(\nu)})\|}{\|w_\nu(z^{(\nu)})\|\varepsilon^\gamma e^{-r}\|w_k(z_{\nu+1}^{(\nu)})\|}.$$

We have that $\|sigma_{\nu+k+1}(z^{(\nu)})\| \leq b^k$, $\|w_k(z_{\nu+1}^{(\nu)})\| \geq e^{c_0 k}$ and therefore, $b e^{-c_0} \leq \varepsilon^{-\gamma}e^r$ giving that $w_{\nu+p+1}(z^{(\nu)})$ has a small slope as long as

$$k \geq \frac{r+\gamma\log 1/\varepsilon}{\log 1/b + \log c_0}.$$

This shows that the vector returns to a horizntal position long before the binding period ends since

$$\frac{r+\gamma\log 1/\varepsilon}{\log 1/b+\log c_0} \ll fracr\beta + \log\kappa - 2 \leq p.$$

3.4. Bounded distortion after the first return. Let $z^{(\nu)}$be a critical approximation with a first return ν as above. Notice that we have

$$\|w_{\nu+1}(z^{(\nu)})\| \geq \|w_{\nu-1}(z^{(\nu)})\|\varepsilon^{\frac{1}{\lambda}-1}\varepsilon^{\gamma}e^{-r} \geq e^{c_0(\nu-1)}e^{-\alpha\nu}\varepsilon^{\frac{1}{\lambda}-1+\gamma} \geq e^{c}\nu+1.$$

In particular the point $z_0^{(\nu)}$ is expanding up to time $\nu+1$ and therefore we can construct the critical approximation $z^{(\nu+1)}$. Morover all points in $B_{\nu}^{(\nu)}(z^{(\nu)})$ satisfy the same kind of estimates and exhibit exponential expansion up to iterate $\nu+p+1$. In this section we shall prove that the sets $B^{(\nu+j+1)}(z^{(\nu+j+1)})$ satisfy bounded distortion estimates, for all $0 \leq j \leq p$, analogous to those satisfied by $B^{(j)}(z^{(j)})$ for all $0 \leq j \leq \nu$(lemma 3.1). The proof relies fundamentally on the fact that by the time the return ν occurs, we have $\varepsilon^{\gamma}e^{-\beta\nu} \ll \varepsilon^{\gamma}e^{-\alpha\nu}$. Thus the points of $B_{\nu}^{(\nu)}(z^{(\nu)})$ are all very close together with respect to their distance from the critical approximation (the binding point) and all satisfy essentially the same estimates. More precisely we will prove the following

LEMMA 3.3. *For all $\xi, \eta \in B^{(\nu+k)}(z^{(\nu)})$ and all $1 \leq k \leq p+1$ the following estimates are satisfied:*

$$(i) \qquad \frac{\|w_{\nu+k}(\xi)\|}{\|w_{\nu+k}(\eta)\|} \leq \prod_{j=0}^{\nu+k-1}(1+Ce^{(\alpha-\beta)j}$$

and

$$(ii) \qquad \angle(w_{\nu+k}(\xi), w_{\nu+k}(\eta)) \leq Ce^{(\alpha-\beta)(\nu+k)}$$

PROOF. We begin with the case $k=1$. By lemma 1.8 we have

$$\frac{\|w_{\nu}(\xi)\|}{\|w_{\nu}(\eta)}\| \leq \prod_{j=0}^{\nu-1}(1+Ce^{\alpha-\beta}j).$$

Thus, proceeding as in the proof of that lemma we write

$$\frac{\|w_{\nu+1}(\xi)\|}{\|w_{\nu+1}(\eta)\|} = \frac{\|w_{\nu+1}(\xi)\|/\|w_{\nu}(\xi)\|}{\|w_{\nu+1}(\eta)\|/\|w_{\nu}(\eta)\|}\frac{\|w_{\nu}(\xi)\|}{\|w_{\nu}(\eta)\|}$$

and therefore we just need to prove

$$\frac{\|w_{\nu+1}(\xi)\|/\|w_{\nu}(\xi)\|}{\|w_{\nu+1}(\eta)\|/\|w_{\nu}(\eta)\|} \leq (1+Ce^{\alpha-\beta}\nu).$$

Notice that $\|w_{\nu+1}(\xi)\|/\|w_{\nu}(\xi)\| = \|D\Phi(\xi_{\nu})\hat{w}_{\nu}(\xi)$ and $\|w_{\nu+1}(\eta)\|/\|w_{\nu}(\eta)\| = \|D\Phi(\eta_{\nu})\hat{w}_{\nu}(\eta)$ where $\hat{w}_{\nu}(\xi) = w_{\nu}(\xi)/\|w_{\nu}(\xi)\|$ and $\hat{w}_{\nu}(\xi) = w_{\nu}(\xi)/\|w_{\nu}(\xi)\|$. Let $\hat{w}_{\nu+1}(\xi) = D\Phi(\xi_{\nu})\hat{w}_{\nu}(\xi)$ and $\hat{w}_{\nu+1}(\eta) = D\Phi(\eta_{\nu})\hat{w}_{\nu}(\eta)$. As in the proof of the previous lemma we now need a careful analysis of the norms and slopes of the vectors $\hat{w}_{\nu+1}$ in the coordinate systems $\{(1,0), e^{(p)}(\xi_{\nu+1})\}$ given by the horizontal axis and the contractive approximations of order p. We shall use in a fundamental way the fact that these coordinate systems are *almost orthogonal* (lemma 2.1) and *almost constant* functions of the point (lemma 2.4 , (34)-(35)). To simplify the exposition and highlight the main ideas of the proof we shall carry out the estimates below assuming that these coordinate systems are actually constant functions of the point.

Recall that $|\xi_\nu - \eta_\nu| \leq \varepsilon^\gamma e^{-\beta\nu}$, that $\hat{w}_\nu(\xi)$ and $\hat{w}_\nu(\xi)$ are unit with small slope ($\leq b^{1/2}$), and that $\angle(\hat{w}_\nu(\xi), \hat{w}_\nu(\eta)) \leq Cbe^{(\alpha-\beta)(\nu-1)}$. We write $\hat{w}_{\nu+1}(\xi) = (\hat{\beta}_{\nu+1}(\xi), \hat{\alpha}_{\nu+1}(\xi))$ and $\hat{w}_{\nu+1}(\xi) = (\hat{\beta}_{\nu+1}(\xi), \hat{\alpha}_{\nu+1}(\xi))$ as discussed above. Then we have, by the same arguments used in the proof of lemma 3.1,

$$|\hat{\alpha}_{\nu+1}(\xi) - \hat{\alpha}_{\nu+1}(\eta)| \leq b|\xi_\nu - \eta_\nu| \leq b\varepsilon^\gamma e^{-\beta\nu}$$

and

$$|\hat{\beta}_{\nu+1}(\xi) - \hat{\beta}_{\nu+1}(\eta)| \leq \tau'|\xi_\nu - \eta_\nu| \leq \tau'\varepsilon^\gamma e^{-\beta\nu}.$$

This implies

$$\|\hat{w}_{\nu+1}(\xi) - \hat{w}_{\nu+1}(\eta)\| \leq 2\tau'\varepsilon^\gamma e^{-\beta\nu}.$$

Thus it is sufficient now to show that the $\|\hat{w}_{\nu+1}(\xi)\|$ and $\|\hat{w}_{\nu+1}(\xi)\|$ are large in relation to $\varepsilon^\gamma e^{-\beta\nu}$ (and, in particular, in relation to $\|\hat{w}_{\nu+1}(\xi) - \hat{w}_{\nu+1}(\eta)\|$). We know, from the proof of lemma 3.1 that $|\hat{\beta}_{\nu+1}(\xi)|, |\hat{\beta}_{\nu+1}(\eta)| \geq (3/4)\varepsilon^\gamma e^{-r}$ and therefore, from the fact that the coordinate systems are almost orthogonal, we deduce that

$$\|\hat{w}_{\nu+1}(\xi)\|, \|\hat{w}_{\nu+1}(\eta)\| \geq \frac{1}{2}\varepsilon^\gamma e^{-r} \geq \frac{1}{2}\varepsilon^\gamma e^{-\alpha\nu}.$$

It follows that

$$\frac{\|\hat{w}_{\nu+1}(\xi)\|}{\|\hat{w}_{\nu+1}(\eta)\|} = 1 + \frac{\|\hat{w}_{\nu+1}(\xi) - \hat{w}_{\nu+1}(\eta)\|}{\|\hat{w}_{\nu+1}(\eta)\|} \leq 1 + 4\tau' e^{(\alpha-\beta)\nu}$$

and

$$\angle(w_{\nu+1}(\xi), w_{\nu+1}(\eta)) = \angle(\hat{w}_{\nu+1}(\xi), \hat{w}_{\nu+1}(\eta)) \leq \frac{\|\hat{w}_{\nu+1}(\xi) - \hat{w}_{\nu+1}(\eta)\|}{\|\hat{w}_{\nu+1}(\eta)\|} \leq 4\tau' e^{(\alpha-\beta)\nu}.$$

This proves (i) and (ii). □

4. Positive Lyapunov exponents and parameter exclusions

In this section we finally show that if all critical approximations satisfy condition $(*)$ up to time n there is a vector which is exponentially expanded up to time n. We also deal with the remaining difficulties associated to the presence of the singularity. The estimates on the dependence on the parameter of the various constructions and the estimates on the measure of the set of excluded parameters proceed essentially as in the quadratic-like case [BC91][MV93].

For each $n \geq 1$ we let $\mathcal{C}_n$ denote the set of critical approximations of order n. We recall that the set $\mathcal{C}_n$ is constructed inductively in the following way. For a certain initial number of iterates it is formed by the critical approximations $\{z^{(1)}, \dots\}$ constructed in section 2. Then we suppose that the set $\mathcal{C}_i$ is defined for all $0 \leq i \leq n-1$. By definition each critical approximation $z^{(n-1)}$ satisfies condition $(*)$ and $\|w_j\| e^{cj}$ for all $j \leq n-1$. We then consider those parameters for which condition $(*)$ is satisfied by all critical approximations in $\mathcal{C}_{n-1}$ up to time n. We shall show below that for these parameters all critical approximations also satisfy $\|w_n(z^{(n-1)})\| \geq e^{cn}$. For these parameters then we can also construct higher order critical approximations $z^{(n)} \in B^{(n)}(z^{(n-1)})$ using tha algorithm of lemma 2.8. Then using the algorithm of subsection 2.3 new critical approximation of higher generation can be constructed.

We suppose that for each $z^{(n)} \in \mathcal{C}_n$ a sequence $0 < \nu_1 < \cdots < \nu_s \leq n$ is defined of return times to Δ. Let $r_1, r_2, \dots, r_s$ be the depths of these returns and $p_1, p_2, \dots, p_s$ be the lengths of the associated binding periods (recall that $p_i = 0$ if ν_i is a return to Δ^0). We suppose that estimates analogous to those proved in section 3 are satisfied at the end of each binding period. These are the general inductive assumptions. Section 3 constituted the first step of the induction. The main result of this section is the following lemma that shows that if $z_n^{(n)}$ is a return to $\Delta^{\pm c}$ then a binding period can be defined and the corresponding estimates on expansion and distortion at the end of the binding period can be obtained. It is not difficult, then, to show that condition $(*)$ implies an exponentially growing vector for all time and for all critical value approximations.

LEMMA 4.1. *Suppose that $n \geq \nu_s + p_s + 1$ is a return to $\Delta^{\pm c}$ for the critical approximations $z^{(n)}$ and that $z^{(n)}$ satsifies condition $(*)$ up to time n. In particular suppose there exists a binding point $\zeta^{(n)}$ with $|z^{(n)} - \zeta^{(n)}| \geq \varepsilon^{\gamma} e^{-r} \geq \varepsilon^{\gamma} e^{-\alpha n}$. Then there exists a $p < n$ such that the following estimates are satisfied:*

(i) $$\frac{r}{\beta \log \kappa} \leq \frac{r}{\gamma(1-\lambda)\log 1/\varepsilon} \leq p \leq 2(1+\gamma/\delta)r \leq n-1$$

(ii) $$\|w_{n+p+1}(z^{(n)})\| \geq e^{\beta p}\|w_n(z^{(n)})\|$$

(iii) $$\text{slope } w_{n+p+1}(z^{(n)}) \leq b^{\frac{1}{2}}$$

The main difference between the situation here and that of lemma 3.1 is that the binding point $\zeta^{(n)}$ might have some returns to $\Delta^{\pm c}$ in the interval of time $[1, p]$. If this occurs, also $z^{(n)}$ returns to $\Delta^{\pm c}$ in the interval of time $[n+1, n+p]$. and this is called a *bound return* for $z^{(n)}$. Further loss of growth occurs at iterates corresponding to bound returns and it is also difficult to control the slope of vectors since there is further rotation after bound returns to $\Delta^{\pm c}$. Apart from this (non-trivial) additional difficulty the ideas involved in the proof are the same as those required to prove the analogous estimates in lemma 3.1. Thus we shall limit ourselves here to explaining how to deal with bound returns.

We start by giving a slightly modified definition of the binding period. Let γ be the curve binding $z_n^{(n)}$ to $\zeta^{(n)}$. Let $\tilde{p} \in \mathbb{N}$ be defined by the condition $|\gamma_j| \leq \varepsilon^{\gamma} e^{-\beta j}$ foe all $1 \leq j \leq \tilde{p} - 1$ and $|\gamma_{\tilde{p}}| \geq \varepsilon^{\gamma} e^{-\beta \tilde{p}}$. Then the binding period associated to the return n is the interval $[n+1, n+p]$ where p is the maximum positive integer with $p \leq \tilde{p}$ such that $p+1$ is a free iterate for $\zeta^{(n)}$. Notice that in the case of a first return (see section s:first return) all iterates in $[n+1, n+\tilde{p}]$ are free and therefore $p = \tilde{p}$. Moreover condition $(*)$ and the inductive hypothesis $p_j \leq 2(1+\gamma/\delta)r_j$ for every return ν_j of $\zeta^{(n)}$ implies that

$$\sum_{j=1}^{s} p_j \leq 2(1+\frac{\gamma}{\delta}) \sum_{j=1}^{s} r_j \leq 2(1+\frac{\gamma}{\delta})\alpha \tilde{p} \leq \tilde{\alpha} \tilde{p}$$

where the sum is taken over all returns of $\zeta^{(n)}$ in the interval of time $[1, \tilde{p}]$ and $\tilde{\alpha}$ can be chosen arbitrarily small simply by taking α small. This implies that a proportion greater than or equal to $(1-\tilde{\alpha})$ of iterates of $\zeta^{(n)}$ in the interval of time $[1, \tilde{p}]$ are free. Thus, in particular, we have

$$p \geq (1-\tilde{\alpha})\tilde{p}.$$

This means that p is of the same order as $\tilde{p}$ and all estimates carried out with respect to $\tilde{p}$ are essentially valid when we replace $\tilde{p}$ by p.

Notice that there can be a whole sequence of bound returns within binding periods. An important consequence of the definition of binding periods is that they are always *nested*, i.e. the last binding period to begin is always the first to end. This is the crucial fact which will allow us to mantain some control over the vectors ω_j. We now explain how this is done.

We start by decomposing the vector w_{n+1} into a horizontal component and one which is colinear to $e^{(p)}(z_{n+1}^{(n)})$ as in lemma 3. Thus we have

$$w_{n+1} = \omega_{n+1} + \sigma_{n+1} = \beta_{n+1}(1,0) + \alpha_{n+1} e^{(p)}(z_{n+1}^{(n)}).$$

Notice that $\|\omega_n + 1\| = |\beta_{n+1}|$ and $\|\sigma_{n+1}\| = \alpha_{n+1}$ can be estimated as in lemma 3. Then, for each $1 \leq j \leq p+1$ we write

$$\tilde{\omega}_{n+j+1} = D\Phi(z_{n+j}^{(n)}) \cdot \omega_{n+j}$$

and

$$\tilde{\sigma}_{n+j+1} = D\Phi(z_{n+j}^{(n)}) \cdot \sigma_{n+j}$$

and we distinguish three cases:

1) If j is not a return nor the end of a binding period we simply set

$$\omega_{n+j+1} = \tilde{\omega}_{n+j+1} \text{ and } \sigma_{n+j+1} = \tilde{\sigma}_{n+j+1}.$$

2) If j is a return then we split

$$\tilde{\omega}_{n+j+1} = \tilde{\beta}_{n+j+1}(1,0) + \tilde{\alpha}_{n+j+1} e^{(p)}(z^{(n)}_{n+j+1}) \tag{46}$$

where p is the length of the binding period associated to the (bound) return $n+j+1$ and $e^{(p)}$ is the direction of the p-th contractive approximation. Then we set

$$\omega_{n+j+1} = \tilde{\omega}_{n+j+1} - \tilde{\alpha}_{n+j+1} e^{(p)}(z^{(n)}_{n+j+1})$$

and

$$\sigma_{n+j+1} = \tilde{\sigma}_{n+j+1} + \tilde{\alpha}_{n+j+1} e^{(p)}(z^{(n)}_{n+j+1}).$$

3) If j is the end of a binding period, i.e. $j = \mu_1 + p_1 + 1$ for some previous return μ_1, then we reconstitute the vectors which we had decomposed at the begininning of the binding period (see (46)): let $\omega_{n+j+1} = \tilde{\omega}_{n+j} + \alpha_{\mu_1} D\Phi^{p_1} \cdot e^{(p_1)}$ and $\sigma_{n+j+1} = \tilde{\sigma}_{n+j} - \alpha_{\mu_1} D\Phi^{p_1} \cdot e^{(p_1)}$. In general, if $s \geq 1$ binding periods end at time $n+j$ then we take

$$\omega_{n+j+1} = \tilde{\omega}_{n+j} + \sum_{i=1}^{s} alpha_{\mu_i} D\Phi^{p_i} \cdot e^{(p_i)}$$

and

$$\sigma_{n+j+1} = \tilde{\sigma}_{n+j} - \sum_{i=1}^{s} \alpha_{\mu_i} D\Phi^{p_i} \cdot e^{(p_i)}.$$

It is now easy to see how the arguments used in the proof of lemma 3.1 can be applied in this case also to obtain the desired results in the statement of the lemma.

4.1. Bounded distortion. By using the decomposition above and applying the arguments of lemma 3.3 we also get the following

LEMMA 4.2. *For all $\xi, \eta \in B^{(n+h)}(z^{(n)})$, all $1 \leq h \leq p+1$ and all $1 \leq k \leq n+h$ the following estimates are satisfied:*

$$\frac{\|w_k(\xi)\|}{\|w_k(\eta)\|} \leq \prod_{j=0}^{k-1} (1 + Ce^{(\alpha-\beta)j} \tag{i}$$

and

$$\angle(w_k(\xi), w_k(\eta)) \leq Ce^{(\alpha-\beta)k} \tag{ii}$$

4.2. Positive Lyapunov exponents. We are now ready to show that condition $(*)$ implies positive Lyapunov exponents. Thus completing the proof of the theorem.

LEMMA 4.3. *Let $z^{(n)} \in C_n$ satisfy condition $(*)$ up to time n, and exponential growth up to time $n-1$. Then it also satisfies*

$$\|w_n(z^{(n)})\| \geq e^{cn}.$$

PROOF. Let $Q = \sum q_i$ denote the total number of iterates that $z^{(n)}$ spends in free periods in the interval of time $[1, n]$ and let $P = n - Q = \sum p_i$ denote the total number of iterates belonging to binding periods. By lemma 1.2 and 4.1 we have

$$\|w_n(z^{(n)})\| \geq e^{c_0 Q + \beta P}.$$

By condition $(*)$ we have $P \leq \sum p_i \leq \tilde{\alpha} n$ and therefore $Q \geq (1 - \tilde{\alpha})n$ and we get

$$\|w_n(z^{(n)})\| \geq e^{c_0(1-\tilde{\alpha})n} \geq e^{cn}$$

taking $\tilde{\alpha} < c_o - c$. This concludes the proof. □

Bibliography

[ABŠ77] V.S. Afraimovich, V.V. Bykov, and L.P. Šil'nikov, *On the appearence and structure of the Lorenz attractor*, Dokl. Acad, Sci. USSR **234(2)** (1977), 336–339.

[ABŠ82] V.S. Afraimovich, V.V. Bykov, and L.P. Šil'nikov, *On attracting structurally unstable limit sets of Lorenz type attractors*, Trans. Moscow Math. Soc. **44** (1982), 153–216.

[Ano67] D.V. Anosov, *Geodesic flows on closed Riemannian manifolds with negative curvature*, Proc. Inst. Steklov **90** (1967), 1–235.

[AP87] V.S. Afraimovich and Ya. B. Pesin, *The Dimension of Lorenz type attractors*, Sov. Math.Phys. Rev., vol. 6, Gordon and Breach Harwood Academic, 1987.

[BC85] M. Benedicks and L. Carleson, *On iterations of* $1 - ax^2$ *on* $(-1, 1)$, Annals of Math. **122** (1985), 1–25.

[BC91] M. Benedicks and L. Carleson, *The dynamics of the Hénon map*, Annals of Math. **133** (1991), 73–169.

[Bow75] R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov diffeomorphisms*, Lect. Notes in Math. **470** (1975).

[Bow78] R. Bowen, *On Axiom A Diffeomorfisms*, CBMS Reg. Conf. Ser. **35** (1978).

[BS80] L.A. Bunimovich and Ya.G. Sinai, *Stochasticity of the attractor in the Lorenz model*, Nonlinear Waves, Proc. Winter School, Moscow, Nauka, Moscow, 1980, pp. 212–226.

[BS83] L.A. Bunimovich and Ya.G. Sinai, *On a fundamental theorem in the theory of dispersing billiards*, Math USSR Sb. **19** (1983), 407–423.

[Bun83] L.A. Bunimovich, *Statistical properties ofLorenz attractors*, Nonlinear dynamics And turbulence (Boston MA-London), Pitman, Boston MA-London, 1983, pp. 71–92.

[Bun89] L.A. Bunimovich, *Systems of hyperbolic type with singularities*, Encyclopedia of Mathematics, vol. 2, Springer-Verlag, 1989.

[BY93] M. Benedicks and L.-S. Young, *SBR measures for certain Hènon maps*, Invent. Math. (1993).

[CE80] P. Collet and J.-P. Eckmann, *On the abundance of aperiodic behaviour*, Comm. Math. Phys. **73** (1980), 115–160.

[dMvS93] W. de Melo and S. van Strien, *One dimensional dynamics*, Springer-Verlag, 1993.

[DRV94] J.L. Diaz, J. Rocha, and M. Viana, *Strange attractors in saddle-node cycles: prevalence and globality*, to appear (1994).

[Guc76] J. Guckenheimer, *A strange, strange attractor*, The Hopf bifurcation theorem and its applications (M. J.E. Marsden, McCracken, ed.), Springer-Verlag, 1976, pp. 368–381.

[GW79] J. Guckenheimer and R.F. Williams, *Structural stability of Lorenz attractors*, Publ. Math. IHES **50** (1979), 307–320.

[Hén76] M. Hénon, *A two dimensional mapping with a strange attractor*, Comm. Math. Phys. **50** (1976), 69–77.

[HP76] M. Hénon and Y. Pomeau, *Two strange attractors with a simple structure*, Lect. Notes in Math. **565** (1976), 29–68.

[HT92] S.P. Hastings and W.C. Troy, *A shooting approach to the Lorenz equations*, Bull. A.M.S. **27** (1992), 298–303.

[Jak81] M.V. Jakobson, *Absolutely continuous invariant measures for one-parameter families of one-dimensional maps*, Comm. in Math. Phys. (1981), 39–88.

[JN95] M.V. Jakobson and S. Newhouse, *A two dimensional version of a folklore theorem*, to appear (1995).

[Lor63] E. N. Lorenz, *Deterministic nonperiodic flow*, J. Atmosph. Sci. **20** (1963), 130–141.

[Mil85] J. Milnor, *On the concept of attractor*, Comm. Math. Phys. **99** (1985), 177–195.

[Mis81] M. Misiurewicz, *Absolutely continuous invariant measures for certain interval maps*, Publ. Math. IHES **53** (1981), 17–51.

[MV93] L. Mora and M. Viana, *Abundance of strange attractors*, Acta Mathematica **171** (1993), 1–71.

[New79] S. Newhouse, *The abundance of wild hyperbolic sets and nonsmooth stable sets for diffeomorphisms*, Publ. Math. I.H.E.S. **50** (1979), 5–72.

[NPT83] S. Newhouse, J. Palis, and F. Takens, *Bifurcations and stability of families of diffeomorphisms*, Publ. Math. I.H.E.S. **57** (1983), 101–150.

[Pal70] J. Palis, *A note on Ω-stability*, Global Analysis, Proc. Symp. Pure Math. AMS **14** (1970), 221–222.

[Pes86] Ya. B. Pesin, *Ergodic properties and dimensionlike characteristics of strange attractors that are close to hyperbolic*, Proceedings International Conference of Mathematicians, 1986.

[Pes92] Ya.B. Pesin, *Dynamical systems with generalized hyperbolic attractors; hyperbolic, ergodic and topological properties*, Erg. Th. & Dyn. Syst. **12** (1992), 123–151.

[PRV95] M.J. Pacifico, A. Rovella, and M. Viana, *Global spiral attractors*, preprint (1995).

[PS70] J. Palis and S. Smale, *Structural stability theorems*, Global Analysis, Proc. Symp. Pure Math. AMS **14** (1970).

[PT85] J. Palis and F. Takens, *Cycles and measure of bifurcation sets for two-dimensional diffeomorphisms*, Invent. Math. **82** (1985), 397–422.

[PT87] J. Palis and F. Takens, *Hyperbolicity and the creation of homoclinic orbits*, Annals of Math. **125** (1987), 337–374.

[PT93] J. Palis and F. Takens, *Hyperbolicity and sensitive-chaotic dynamics at homoclinic bifurcations*, Cambridge University Press, 1993.

[PV94] J. Palis and M. Viana, *High dimension diffeomorphisms displaying infinitely many periodic attractors*, Annals of Math. (1994), 207–250.

[PY93] J. Palis and J.-C. Yoccoz, *Homoclinic tangencies for hyperbolic sets of large Hausdorff dimension*, to appear (1993).

[Rob89] C. Robinson, *Homoclinic bifurcation to a transitive attractor of Lorenz type*, Nonlinearity **2** (1989), 495–518.

[Rob92] C. Robinson, *Homoclinic bifurcation to a transitive attractor of Lorenz type, II*, SIAM J. Math. Anal. **23** (1992), 1255–1268.

[Rov93] A. Rovella, *The dynamics of perturbations of the contracting Lorenz attractor*, Bull. Braz. Math. Soc. 24 (1993), 233–259.

[Rue76] D. Ruelle, *A measure associated with Axiom A attractors*, Amer. J. Math. 98 (1976), 98.

[Ryc89] M. Rychlik, *Lorenz attractors through Šil'nikov-type bifurcation. part 1*, Erg. Th. & Dyn. Syst. 10 (1989), 793–821.

[Sat92] E.A. Sataev, *Invariant measures for hyperbolic maps with singularities*, Russ. Math. Surveys 471 (1992), 191–251.

[Sin70] Ya.G. Sinai, *Dynamical systems with elastic reflection*, Russ. Math. Surv. 25 (2) (1970), 137–189.

[Sin90] Ya.G. Sinai, *Hyperbolic billiards*, proceedings of ICM-90, 1990.

[Sin94] Ya. G. Sinai, *Topics in ergodic theory*, Princeton Mathematical Series, vol. 44, Princeton UP, 1994.

[Spa82] C. Sparrow, *The Lorenz equations; bifurcations, chaos and strange attractors*, Applied Mathematical Sciences, vol. 41, Springer-Verlag, 1982.

[Ste58] S. Sternberg, *On the structure of local homeomorophisms of Euclidean n-space ii*, Amer. J. Math. 80 (1958), 623–631.

[TTY92] Thieullen, Tresser, and Young, *Exposant de Lyapunov positif dans des familles à un paramètre d'applications unimodales*, C. R. Acad. Sci. Paris t. 315, Serie I (1992), 69–72.

[Uré93] R. Urés, *Approximation of Hénon-like attractors by homoclinic tangencies*, Ph.D. thesis, IMPA, Rio de Janeiro, 1993.

[Via95] M. Viana, *Multidimensional nonhyperbolic attractors*, preprint (1995).

[Wig88] S. Wiggins, *Global bifurcations and chaos*, Springer-Verlag, 1988.

[Wil79] R.F. Williams, *The structure of the Lorenz attractor*, Publ. Math. IHES. 50 (1979), 321–347.