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Topological and Integrability Properties of Multi-Matrix Models

Thesis submitted for the degree of "Doctor Philosophiæ"

Candidate
Emil Vinteler

Supervisor
Prof. Loriano Bonora

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SISSA - SCUOLA
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DI STUDI AVANZATI
TRIESTE
Strada Costiera 11

TRIESTE

Părinţilor mei Alla şi Onufrie

Abstract

This thesis is a study of topological and integrability properties of the multi-matrix models. The main method used in deriving these results is the Q-matrices approach.

There are ten chapters. After a general introduction in the first chapter, in chapter two we deal with the Liouville theory. We make special emphasis on the dynamics of the boundaries, due to the recent interest in the p-branes theory. In chapter three we introduce the discrete states and argue that they could be interpreted as excitations of the boundary. In chapter four we pass to the matrix theory and introduce the general Q-matrices approach. After, we take some simple examples where we apply this approach and the classical method of W-constraints. In chapter five we show how the discrete states appear in the $2q$-matrix model and how to calculate the correlation functions. We argue that the multi-matrix models are topological models and that they might accomodate the states of the W_{n+1} minimal models coupled to topological gravity. In the next three chapters we apply the general method to some concrete models. In chapter six we calculate nonzero momentum correlation functions in the $c = 1$-matrix model and show the connection between our approach and the free fermion approach. Chapter seven deals with the star-matrix models which describe the Potts-model on a random surface. Chapter eight discusses the quantum chaos in multi-matrix models. Chapter nine shows that it is possible to classify the reductions of the Toda lattice hierarchy according to the Drinfeld-Sokolov generalized KdV hierarchies. In the last chapter of conclusions we discuss some of the unsolved problems.

The supervisor of the thesis has been Prof. Loriano Bonora.

Preface

The thesis consists of several papers and also unpublished material. The chapters 4, 5 and 6 have been developed further from the initial ideas of the joint paper [84]. The chapter 7 is based on the work [95]. The chapter 8 is a revised version of the paper [103]. The chapter 9 consists mainly of the joint work [119].

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Contents

1 Introduction

"O god, I could be bounded in a nutshell,
and count myself a king of infinite space..."
Hamlet, II, 2

The Physics begins by asking apparently simple questions: "What would you weigh in a falling elevator?", "What is the electrical charge of electron" and so on. I choose these questions because they were the first impulses to build up general relativity and quantum field theory- the two pillars on which stands the modern physics.

The first question gave rise to the equivalence principle- the gravity and inertial forces are locally indistinguishable. If you rush to answer the second question: "$e = 1,6 \times 10^{-19}$ Coulombs", you are cheated. Instead, the right answer is:"It depends", and the fact that the electrical charge depends on the scale, gave us the running coupling and renormalization. From the tricky and dirty method of removing divergences, the renormalization grew up to the basic principle of QFT.

Today we have a theory of gravity and a QFT of electromagnetic, weak and strong interactions. The ultimate quest is to unify these two so different kinds of theories.

At the end of '60's, by accident or fortune, has appeared the candidate- the dual model renamed later as string theory. It was various times abandoned in favour of other theories: asymptotically free gauge theories, finite supersymmetric theories, supergravity. But it resurfaced again and again. And now, the string theory is the established Theory of Everything.

Like in the Shakespeare's verses, the string limited to a size of order 10^{-33} cm, can declare that it explains the infinite diversity of Universe.

Two times the physicists were able to open the stringy, hard nutshell and look at the core. During 1982-1983 Green and Schwarz used the magic word: anomaly and discovered inside 5 differents nuts: open type I $SO(32)$, closed type IIA (non-chiral) and IIB (chiral) and Heterotic $SO(32)$ and $E_8 \times E_8$(the heterotic string was discovered in 1984 by Princeton Quartet : Gross,Harvey, Martinec, Rohm).

The superstring theory is defined in 10 dimensions. To get a description of the real 4D world were proposed different compactifications. The simplest method is to compactify on a torus (proposed by Narain, Sarmadi, Witten), but it gives a 4-dimensional $N = 4$ theory in contradiction with the supposed $N = 1$ spontaneously broken theory.

In 1985 was proposed by Candelas, Horowitz, Strominger and Witten the first phenomenologically viable semi-realistic compactification of the string theory on a Calabi-Yau manifold which gives a 4-dimensional $N = 1$ theory, chiral fermions, the right number of generations and so on.

Due to the difficulty in dealing with the Calabi-Yau compactification, were proposed other compactifications on orbifolds which can be viewed as a limiting case of Calabi-Yau manifolds (by Dixon, Harvey, Vafa, Witten) and orientifolds (by Horava, Bianchi and

Sagnotti).

However, at the beginning of the '90's were already known at least 10.000 Calabi-Yau compactifications giving semi-realistic theories in 4D.

Instead of one Theory of Everything, from the string horn of abundance were flowing thousands new theories of something. The big puzzle was how to relate these disparate theories and how to choose the right one for our 4D world.

The unification principle come a bit later in the face of duality relation.

During the next decade was a silent and slow development of the string theory. Until 1994, when Seiberg and Witten used the magic word (anticipated in string theory in 1990 by Font, Ibanez, Lust, Quevedo) : duality and opened for the second time the nutshell. They, together with many others, discovered that the previous nuts are twins and their mother is the eleventh dimensional membrane M-theory (with the low-energy behaviour of the 11D supergravity) or/and their father- the 12D fundamental F-theory.

The string theory give a logically consistent framework, encompassing both gravity and QFT. At the same time, the conceptual framework in which this should be properly understood, analogous to the principle of equivalence in gravity, hasn't yet emerged.

2 The boundary of Liouville theory

2.1 The cosmological constant

Another important question which string theory must be able to answer is why the cosmological constant is small [1].

The cosmological constant is basically the energy density of empty space, and since energy density is what makes curvature occur in relativity, if there's a big cosmological constant, the universe is very small, very curved up. And so the fact that the universe is so big is a mystery which must be explained in the string theory.

We explain more in detail how the cosmological constant was introduced in the Einstein equation and what consequences for our Universe follow from this idea.

The Einstein equation for gravity with matter is:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

The solutions to this equation are extremely difficult. Only in special cases, it can be solved. One of this cases is the Robertson-Walker metric, for the homogeneous and isotropic Universe in uniform motion (expansion):

$$ds^2 = dt^2 - \frac{R^2(t)}{(1 + kr^2/4)^2}(dr^2 + r^2 d\Omega^2)$$

We plug the explicit form of the Robertson-Walker metric in Einstein equation and we remain with 2 equations for the time dependent curvature $R(t)$ and a constant A:

$$\begin{aligned} 3(\dot{R}/R)^2 + 3A/R^2 &= 8\pi G\rho \\ 2\ddot{R}/R + (\dot{R}/R)^2 + A/R^2 &= -8\pi G p \end{aligned} \tag{2.1}$$

A direct consequence of these equations is:

$$d(\rho R^3) + p dR^2 = 0 \tag{2.2}$$

which is nothing else as the first law of thermodynamics for the entire Universe, which is seen as an adiabatically expanding system.

In our period the matter predominates and the ratio $p/\rho \gg 1$, so we can put $p = 0$ in the last equation and we get the conservation for the mass of Universe:

$$\rho_m R^3 = M = const$$

Using this equation we get the Friedman evolution:

$$\dot{R}^2 = 2GM/R - A$$

For the special case when $A = 0$ we get the Einstein-de Sitter evolution: the curvature grows with the time like $R(t) \sim t^{2/3}$ and the Universe expands.

When Einstein first found this evolution in 1917 nothing was known about expansion of Universe. The Universe was considered as infinite in time and space, in other words a stationary Universe.

But the attractive forces of gravitation could collapse the Universe, the acceleration of collapsing must be balanced by a repulsive acceleration: :

$$a_{rep} = -a_{atr} = -\ddot{R} = GM/R^2$$

To explain this repulsive acceleration a_{rep} Einstein has introduced in his equation the cosmological constant:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G(T_{\mu\nu} + \Lambda g_{\mu\nu})$$

Later, understanding that the Universe expands, he has considered the introduction of the cosmological constant as the biggest mistake in his life.

But in the same year, de Sitter showed that if the radiation dominates over the matter the equation (2.2) is modified to :

$$d(\rho_\gamma R^3) + \frac{1}{3}\rho_\gamma dR^3 = 0 \tag{2.3}$$

The radiation exerts a pressure like a relativistic gas $p_\gamma = \rho_\gamma/3$.

Using this equation we get the following evolution:

$$\dot{R}^2 = 8\pi G\rho/(3R^2)$$

and the curvature grows in this ancient period like $R(t) \sim t^{1/2}$.

If we calculate the acceleration produced by the radiation we get :

$$a = \ddot{R} = -\frac{4\pi G}{3}(3p + \epsilon)R < 0$$

We can see that the radiation, like the matter, gives an attractive acceleration.

To produce the repulsive acceleration we must consider the vacuum which has the strange property that $p_{vac} = -\rho_{vac}$. This property is related to the fact that the vacuum which is filling all the space of Universe can be considered as the absolute ether. We can try, like Michelson, to measure our absolute velocity, by using the energy flux of the ether $\rho_{vac}c^2$ flowing to us. But in the vacuum, this energy flux is balanced by a negative pressure $p_{vac} = -\rho_{vac}c^2$ and the result of our measure is zero: we do not detect any absolute velocity.

Introducing this relation in the previous equation and comparing with the solution of Einstein equation with cosmological constant Λ we obtain:

$$a_{rep} = \ddot{R} = \frac{8\pi G\rho_{vac}}{3}R = \Lambda R/3$$

Hence the cosmological constant describes the density of the vacuum $\Lambda = 8\pi G\rho_{vac}$. And it also explains in a crude manner why we have the inflation, the exponential grow of the Universe in the early stages:

$$R = R_0 \exp(\sqrt{\Lambda/3}t) \tag{2.4}$$

The velocity of expansion $v = \sqrt{\Lambda/3}R$ can be compared with the speed of recession v of a distant galaxy which satisfies the Hubble law $v = HR$.

Introducing explicitly the velocity of light (until now for simplicity we considered $c = 1$) we can relate the Hubble constant with the cosmological constant $\Lambda = 3(H/c)^2$.

The Hubble constant H can be measured experimentally by observing the red-shift of distant galaxies $\Delta\lambda/\lambda = v/c$ and knowing the distance to that galaxy (from the period of Cepheid variable stars or other methods).

The experimental value of Hubble constant is between $H = 50 - 100$ km/s$\times$ Mpc and this data permits the estimation of the cosmological constant which is very small $\Lambda \sim 10^{-120} M_p^2$ (in characteristic hadron masses).

2.2 The unbroken phase of gravity

The creation of the gravity field for the inflationary Universe could be explained by the symmetry breaking mechanism. We have also other mechanisms, but this one appears also in the string theory and the 2D quantum gravity.

This mechanism was introduced by Witten in topological field theories. In this theories, the distance and other local properties have no meaning, only the topology of the manifold and other global properties matter. Because of the success of the topological field theories, it was supposed in 1987 that this mechanism can be extended also to Einstein gravity.

In general relativity we have a covariant action and any metric (apart $g_{\mu\nu} = 0$) is not invariant under diffeomorphisms. By expanding around the flat metric $g_{\mu\nu} = \delta_{\mu\nu}$, we break general covariance down to Poincare symmetry.

In some sense, the massless gravitons should be viewed as Goldstone bosons of spontaneously broken general covariance [2] (see also [3]). In the same way, massless gauge bosons in gauge theory reflect the breaking of the local gauge invariance down to global symmetry. (We understand that this is only a way of thinking, because the first difference is that the Goldstone bosons have spin zero and the second difference is that they are generated as free particles only by symmetry breaking of a global symmetry).

Hence, by analogy with gauge theory, we can define 2 phases of the theory : the unbroken and the broken phase.

The unbroken phase of general covariance is a topological theory defined by:

$$< g_{\mu\nu} >= 0$$

The observables of this theory are global quantities or topological invariants. This phase has no dynamics.

The broken phase has the flat (or Minkowski) metric:

$$< g_{\mu\nu} >= \delta_{\mu\nu}$$

The gravitons in this phase are generated dynamically, by spontaneous breaking of the general covariance.

Hence we can begin with a theory where the action is a topological invariant and so also a covariant invariant. The radiative quantum correction or the anomaly will break this general covariance (more precisely the Weyl symmetry) and the gravity is induced at the quantum level. In 4D we don't know how to implement this mechanism for the Einstein gravity, but it is possible to do this for other models.

In 2D quantum gravity, the gravitons are generated exactly by this mechanism, hence the theory is also called "induced gravity". The theory develops at quantum level a conformal anomaly. It is known that the partition function of this theory is invariant by the direct product of diffeomorphisms (which act only on the coordinates) and Weyl rescalings (which act only on the metric). The conformal transformations change only the coordinates (so they are forming a subgroup of diffeomorphisms), but due to their properties, they change the metric in the same way a Weyl rescaling will do it. Hence the appearance ot the conformal anomaly can be interpreted as breaking of diffeomorphism symmetry, but also of the Weyl symmetry.

If we work in conformal gauge $G_{\mu\nu} = e^{\phi} g_{\mu\nu}$, we can see that in the unbroken phase we can perform the Weyl scaling $\phi \rightarrow \phi + \sigma$, while in the broken phase we have a definite scale, let's call it M. Another point is that the scale M we mentioned acts as an UV cut-off. For momenta bigger than this scale $p \gg M$, the 2 phases, the broken and the unbroken, are undistinguishable. For low energies $p < M$, the broken phase has the massless gravitons and can be distinguished from the unbroken phase.

It is known that the massless particles give rise to infrared divergencies. So we must impose also an IR cut-off by embedding the theory in a box of finite size R. This implies that we must impose a boundary for the theory and in what follows the dynamics of this boundary will play an important role.

The string theory includes gravity, and one of its fields is the space-time metric. The unbroken phase of gravity – a sort of confined phase of gravity– was expected to correspond to the strongly coupled string theory. This theory has large fluctuations of space-time geometry and of the other fields. The duality of the last 2 years has shown that the metric of the strongly coupled string theory behaves more and more classically as the string coupling goes to infinity.

2.3 The induced QED

The CP^{N-1} sigma-model induces the interaction between sources in a similar way with the 2D gravity. The sigma model describes a free particle at classical level. but induces the electromagnetic field at the quantum level. In the same way, the 2D gravity action at classical level is a topological invariant and does not describe the gravity. but it induces the gravity at the quantum level.

We study the sigma-model with the internal global symmetry of the complex projective space CP^{N-1}:

$$S = \frac{1}{2\kappa_0} \int dx^2 \left(|\partial_\mu n|^2 + \lambda(nn^+ - 1) \right) \tag{2.5}$$

where n is the N-dimensional unit vector $nn^+ = 1$ and λ is the Lagrange multiplier.

The calculation of the β-function shows that the theory at the first order of perturbation has $\beta(\kappa) = -((N-2)/\pi)\kappa^2$ and is asymptotic free:

$$\frac{1}{\kappa_R} = \frac{1}{\kappa_0} + \frac{N-2}{\pi} \log \frac{\Lambda_R}{\Lambda_0} \tag{2.6}$$

We can integrate the n-field and we remain with the effective action in the dynamical constraint $\lambda(x)$:

$$S_{eff} = \frac{1}{2\kappa_0} \int d^2x \lambda(x) - \frac{N}{2} \log \mathrm{Det}(-\partial^2 + \lambda(x)) \tag{2.7}$$

For large N we can apply the saddle-point method and we get the equation of motion:

$$1 = 2N\kappa_0 G(x, x; \lambda) = 2N\kappa_0 \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 + \lambda} \tag{2.8}$$

This equation is nothing else than the constraint $< n^2(x) >= 1$ in momentum space (we can observe that $< n_i(x) n_j(y) >= \kappa_0 \delta_{ij} G(x, y; \lambda)$).

In $D = 2$ we have no phase transitions and the physical mass m_{phys} is always non-zero. (Physical mass is proportional to the inverse correlation length, which is infinite at the critical point). The dependence of the physical mass on the coupling κ_0 is exponential:

$$m^2_{phys} = \Lambda^2 \exp(-\frac{2\pi}{N\kappa_0})$$

Due to the strong interaction (imposed by the constraints), the Goldstone bosons are confined in particles with isotopic spin 1.

The induced QED appears as the first correction to the $1/N$ expansion in the large N regime. But because the phase transitions are absent in this model, the large N regime is connected with the finite N regime and they describe the same phase.

If we enhance our global symmetry to a gauge group:

$$n(x) \rightarrow e^{i\Lambda(x)} n(x)$$

we can show that the n-field action induces at the quantum level the 2D QED. The gauged action is:

$$S = \frac{1}{\kappa_0} \int d^2x |(\partial_\mu - iA_\mu)n|^2$$

where $A_\mu = \partial_\mu \Lambda$ is a new field.

We can repeat the derivation of the effective action giving the same expression (2.7), but with $-\partial^2$ replaced by $-(\partial_\mu - iA_\mu)^2$.

The saddle point method gives for large N a similar expression with the ungauged case (2.8):

$$\begin{aligned} <\lambda> &= 0, \quad <A_\mu>=0 \\ 1 &= 2N\kappa_0 G(x, x; \lambda) = 2N\kappa_0 \int \frac{d^p}{(2\pi)^2} \frac{1}{p^2 + m_{phys}^2} \end{aligned}$$

But if we consider the corrections to the $1/N$ expansion we get the quadratic action of QED:

$$S_{II} \sim \frac{N}{m_{phys}^2} \int d^2x F_{\mu\nu}^2$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

The induced electromagnetic field A_μ produces an infinite Coulomb potential for very low energies when $\kappa_0 \to 0$ and $m_{phys} \to \infty$ (the charge of n-field is $q \sim m_{phys}^2/N$). The n-fields are confined, forming neutral pairs nn^+.

2.4 The Liouville model-the induced gravity

The Einstein action is a topological invariant in 2 dimensions:

$$S = -\frac{1}{16\pi G_0} \int_{\mathcal{M}} dx^2 \sqrt{g} R = -\chi/(4G_0) \tag{2.9}$$

where $\chi = 2 - 2g$ is the Euler number of the manifold $\mathcal{M}$.

We consider the Einstein-Hilbert action coupled to some matter fields:

$$S = \int dx^D \sqrt{g}(\Lambda - \frac{1}{16\pi G_0} R) + S_m \tag{2.10}$$

The Einstein gravity is inappropriate in 2 dimensions because the equations of motion:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_0 \Lambda \tag{2.11}$$

yield for $\Lambda \neq 0$ the unacceptable condition $\sqrt{g} g_{\mu\nu} = 0$ (no gravity or the topological phase) and for $\Lambda = 0$ there are no restrictions on the metric at all.

Hence in 2 dimensions we need a new theory of gravity. Because the full information is contained in the curvature scalar R (the curvature tensor in 2 dimensions can be written as $R_{\mu\nu\ \sigma\rho} = (R/2)(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma})$) the most natural choice for gravity equations in vacuum is:

$$R - 8\pi G_0 \Lambda = 0 \tag{2.12}$$

Since all two dimensional manifolds are conformally flat (the metric can be chosen as $g_{\mu\nu} = e^{\phi}\delta_{\mu\nu}$), the previous equation reduces to the Liouville equation:

$$\delta_{\mu\nu}\partial_\mu \phi \partial_\nu \phi = 8\pi G_0 \Lambda e^{\phi}$$

If we simply take the trace in the equations of motions (2.11) of the Einstein-Hilbert action. we get:

$$\frac{(1 - D/2)}{8\pi G_0} R = (T + D\Lambda) \tag{2.13}$$

where $T = T^{\mu}_{\mu}$ is the trace of the energy-momenta tensor of matter fields. For $D = 2$ we observe that the left part is identical equal with zero, and we have no dynamics. no propagating degrees of freedom.

We see that this equation (2.13) does not coincide with the natural generalization (2.12). Teitelboim and Jackiw [4] (see also [5]) have proposed a modified action:

$$S = S = \int dx^D \sqrt{g} Y (\Lambda - \frac{1}{16\pi G_0} R) \tag{2.14}$$

where Y is a new auxiliary field playing the role of Lagrange multiplier. Varying the action with respect to the Y field we get the desired equation of motion (2.12). This equation does not contain the field Y and also the equation of motion for Y does not impose new constraints on the metric $g_{\mu\nu}$.

The interesting point is that the Einstein-Hilbert action like the modified action (2.14) gives the Liouville equation (2.12). If we consider the quantum theory as the path integral over 2D metrics, with the phase given by the Einstein-Hilbert action, then the path integral in the conformal gauge $g_{\mu\nu} = e^{\phi}\delta_{\mu\nu}$ depends only on ϕ with an appropriate measure. Polyakov has shown how to perform the integral over the ϕ measure. Although the measure is formally independent of ϕ, the conformal anomaly contributes a ϕ dependence which is just the classical Liouville action for ϕ.

Hence the equation of motion (2.13) will be modified at the quantum level and will give the Liouville equation. This modification at the quantum level is signalized by the appearance of an UV divergence, which is due to the dependence of the Newton constant on the mass scale μ, $G_0 \sim \mu^{2-D}$.

The behaviour of the Newton constant (with $\epsilon = D - 2$):

$$\epsilon / G_0 = \epsilon \mu^{\epsilon} / G_R - \beta \mu^{\epsilon} + O(\epsilon) \tag{2.15}$$

makes the left part of equation (2.13) nonzero. In 1981, Polyakov [6] has computed the conformal anomaly of the noncritical string:$c = c_m - 26$. Later, the 2D gravity coupled to matter fields was interpreted as a noncritical string and gave the result $\beta = -2c/3$.

At the quantum level, the equation (2.13) becomes $-2\lambda_R =< T^\mu_\mu >$. But the conformal anomaly relates the trace of energy-momentum tensor with the curvature scalar $< T^\mu_\mu >= cR/(24\pi)$.

The conformal anomaly of the 2D gravity has changed the equation (2.13) to the equation of Liouville describing manifolds with negative constant curvature:

$$R = -\mu^2 = 48\pi\lambda_R/c \tag{2.16}$$

where $\lambda_R = \Lambda - M^2$ is the renormalized cosmological constant. The Liouville equation follows as the equation of motion from the non-local action:

$$S = -\frac{c}{48\pi}\int d^2\sigma\sqrt{g(\sigma)}\int d^2\sigma'\sqrt{g(\sigma')}R(\sigma)\frac{1}{\Delta_{\sigma\sigma'}}R(\sigma) + \lambda_R\int d^2\sigma\sqrt{g} \tag{2.17}$$

We can change to conformal gauge $G_{\mu\nu} = e^\phi g_{\mu\nu}$ and reduce this action to the local Liouville action:

$$S = -\frac{c}{48\pi}\int d^2\sigma(\frac{1}{2}(\partial\phi)^2 + \mu^2 e^\phi) \tag{2.18}$$

The equations of motion are simpler:

$$-\partial^2\phi = \mu^2 e^\phi, \mu^2 = -48\pi\frac{\lambda_R}{c} \tag{2.19}$$

We must emphasize that the Liouville theory until now was treated at the classical level, and at this level the theory is still a topological theory, with no dynamics and no propagating degrees of freedom. We must quantize it to have the final 2D quantum gravity.

The semiclassical limit is achieved for $\mu^2 \to 0$, hence when $c \to -\infty$ (we suppose that $\lambda_R > 0$). The curvature is zero in this limit ($R = 0$) and the 2D Universe is stationary. If we perturb the central charge from its limiting value we get an anti-de-Sitter Universe ($R < 0$). To get the true de-Sitter Universe with $R > 0$ we must take the limit $c \to \infty$. But this limit is instable because the kinetic term has the wrong, negative sign and the action is unbounded below.

We can see that μ^2 (or $-1/c$) plays the role of cosmological constant in 2D gravity and the stationarity around $\mu^2 = 0$ is instable. The instability is similar with the 4D Einstein gravity with cosmological constant.

We must now quantize the Liouville theory. One way of doing it is by discretizing the space-time. The quantization is based on the similarity the Liouville theory bears with the QCD. QCD is asymptotically free for high energies and is confined at low energies

Weinberg [7] has shown that in the one-loop the Liouville theory has a negative β-function (for a not too large c_m) and has supposed that the theory is asymptotically free (for high energies). The Liouville theory was supposed to have confined gravitons, because it is a topological theory with no propagating degrees of freedom.

The Newton coupling G plays the role of gluon coupling: it tends to zero at high energies or small distances. The discretized Liouville theory achieves its continuum limit for some value of Newton constant $G = G_c = 1/M^2$, at the scale M. For high energies $p^2 \sim \mu^2 \gg M^2$ the Newton constant behaves as $G_R \sim 1/\mu^2 + O(log\mu)$. For low energies this behaviour changes to $G_R \sim 1/M^2 + O(logM)$ and the gravitons are deconfined. We must tune the cosmological constant $\Lambda \rightarrow \Lambda_c = M^2$ such that the renormalized value tends to zero $\Lambda_R = \Lambda - M^2 = \mu^2 - M^2 \rightarrow 0$ and in order to have massless gravitons and long-range gravity interactions.

The cosmological constant Λ and the Newton constant G are related (we remember the relation in 4D gravity $\Lambda = 8\pi G\rho_{vac}$) and we must not only tune $\Lambda \rightarrow \Lambda_c, G \rightarrow G_c$ but also hold a fixed ratio between Λ and G (the double scaling limit):

$$\frac{1}{G_R} = \frac{1}{G_0} + 2\beta \ln\frac{\lambda_R}{\mu^2} \tag{2.20}$$

2.5 The boundary for abelian field

To explain better the implications of the boundary in gravity, we begin by analizing the simpler case: the boundary of the abelian field.

We introduce the action $S(f) = -(1/4)\int dx^D (F_{\mu\nu} + f_{\mu\nu})^2$ which represents the coupling of the electromagnetic field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ with an antisymmetric external field $f_{\mu\nu}$. The free energy of this system $\exp(-W(f)) = \int DA_\mu \exp(-S(f))$ must be invariant under the transformation [8]:

$$f_{\mu\nu} \rightarrow f_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu . \tag{2.21}$$

because it can be compensated by the "gauge" transformation: $A_\mu \rightarrow A_\mu + \Lambda_\mu$ (the above transformation is more general than the gauge transformation which is a particular case when $\Lambda_\mu = \partial_\mu \Lambda$).

Hence when we are switching-on the constant external field $f_{\mu\nu}$, we are changing the gauge sector of the theory.

Hence we can conclude that the constant external field $f_{\mu\nu}$ has no influence on the system, because under the transformation (2.21) we must have $W(f) = W(0)$. But the equality does not hold always. Why? Because the transformation (2.21) also modifies the boundary conditions of the field $A_\mu|_{bound} = 0$.

We consider the boundary at infinity. In the case of a massive abelian field, we have a Yukawa potential $e^{-\mu r}/r$. The potential decays exponentially and the field is unchanged on the boundary. In the massless case we have a long-range interaction and the field's boundary conditions are modified.

We observe that in the massive case the abelian field has $D-1$ degrees of freedom and is not interacting with the boundary. Instead in the massless case the abelian field looses one degree of freedom and is interacting with the boundary. We can conclude that in passing from the massive to the massless case the abelian vector losses a degree of freedom which becomes a dynamical degree of freedom on the boundary. This field on the boundary is created in the spirit of Kalb-Ramond mechanism [9]. But more precisely how this mechanism works here?

As an example, we consider the $2D$ abelian Yang-Mills theory (not coupled with gravity) coupled with the boundary:

$$S = \int_{\Sigma} dx^2 \left(-\frac{1}{4}F^2_{\mu\nu}\right) - \frac{\theta}{2\pi}\oint_{\partial\Sigma} dx^{\mu} A_{\mu} \tag{2.22}$$

The last term is nothing else as the θ-term which can be rewritten as $-\frac{\theta}{4\pi}\int dx^2 \epsilon^{\mu\nu} F_{\mu\nu}$. Without the θ-term the theory is IR-instable and we must or introduce by hand a small mass μ for the abelian boson or compactify the spatial dimension to a circle of radius $R \sim 1/\mu$.

Instead of compactifying the theory, we introduce a boundary. We show that the 2 problems are equivalent, the coupling constant of the boundary can be expressed in terms of the radius of compactification.

The θ-term solves this problem because it induces a mass $\mu^2 = \theta^2/\pi$ for the abelian boson. It is due to the anomaly of the axial current $J_5^{\mu} = \theta\epsilon^{\mu\nu}A_{\nu}$(see Jackiw Templeton [23]). The relation of the axial current J_5^{μ} to the vector A_{μ} is a peculiarity of the $2D$ space-time. The demonstration is as follows.The equation of motion for the abelian boson are $\Box F_{\mu\nu} = \partial_{\mu} j_{\nu} - \partial_{\nu} j_{\mu}$. On the other hand due to the anomaly we have $\partial_{\mu} j_{\nu} - \partial_{\nu} j_{\mu} = \epsilon_{\mu\nu}\partial_{\alpha} j_5^{\alpha} = -(\theta^2/\pi)F_{\mu\nu}$

We now consider the abelian system and the external field, both fields interacting only with a particle of charge $\theta/2\pi$ moving along the trajectory C (a Wilson loop). The proper time is τ (tangent or locally parallel with C) and σ is a coordinate perpendicular on the curve C. We consider the case when the external field is time-dependent (slowly-varying) and constant in space electric field $E_j = f_{0j} = \partial_0 \Lambda_j(X_0)$.

The action of this model is:

$$S = (-\frac{1}{4})\int dx^D (F^2_{\mu\nu} + f^2_{\mu\nu}) - \frac{\theta}{2\pi}\oint_C d\tau \frac{\partial X^j}{\partial \tau}\Lambda_j(X_0) - \frac{\theta}{2\pi}\oint_C d\sigma \frac{\partial X^j}{\partial \sigma} A_j(X_0) \tag{2.23}$$

We observe that the external field interacting with the boundary gives rise to a Neumann condition for external field (or parallel to the boundary), instead the condition fulfilled by the abelian field is the Dirichlet condition (or perpendicular to the boundary):

$$\partial_{\tau}\Lambda_j(X_0)|_{bound} = 0, \quad \partial_{\sigma} A_j(X_0)|_{bound} = 0 \tag{2.24}$$

But the curve $C = \partial\Sigma$ can be interpreted as the boundary of an open string Σ defined by the coordinates $X_{\mu}(\sigma, \tau)$. The string coupled with the boundary has the action:

$$S_1 = \int_{\Sigma} \partial^2\sigma[(\partial_{\sigma} X_{\mu})^2 - (\partial_{\tau} X_{\mu})^2] - \frac{\theta}{2\pi}\oint_C d\tau \frac{\partial X^j}{\partial \tau}\Lambda_j(X_0) - \frac{\theta}{2\pi}\oint_C d\sigma \frac{\partial X^j}{\partial \sigma} A_j(X_0) \tag{2.25}$$

With this interpretation, the abelian field is nothing else then the boundary coordinate of the open string.

The fact that the correlation functions are equal:

$$< \partial_\tau X_j(\tau)\partial_\tau X_k(\tau') >= - < \partial_\sigma X_j(\tau)\partial_\sigma X_k(\tau') >= \delta_{jk}/(\tau-\tau')^2 \qquad (2.26)$$

shows that the Neumann and Dirichlet degrees of freedom can not be distinguished from the physical point of view. More, they describe free noninteracting bosons. The Neumann and Dirichlet fields can trade the needed degrees of freedom. In special cases, the interaction of the boundary with the external field permits the boundary to have a dynamics of its own (described by the abelian field.

The T-duality transformation $\partial_j X = \epsilon_{jk}\partial^k X$ interchanges the axial abelian A_j and the external Λ_j fields. We can observe that the external field Λ_j is parallel to the boundary. so can be regarded as a function of boundary coordinates $\Lambda_j = \Lambda(x_j|_{bound})$. Instead the abelian field A_j is transverse to the boundary and acts as an axial field $A_j = A(\epsilon_{jk}X^k)$. We see that the T-duality is interchanging the Neumann with Dirichlet conditions on the boundary.

All what we said before can be applied in any dimensions. In 2 dimensions the abelian field has no propagating degrees of freedom. The only remaining degrees of freedom are topological. When the field becomes massless these degrees are transferred to the boundary. They couple with the internal degrees of freedom of the boundary and give rise to the discrete states.

2.6 The boundary of Liouville theory

2.6.1 The Liouville theory as CFT

The Liouville model has a hidden $SL(2,R)$ symmetry [10][11].It permits to express the quantized Liouville field in terms of a free scalar field [12],[13] (for classical Liouville solution see [14][15]). This relation was further confirmed by the calculation of critical exponents using the free field representation [16],[17] (or Coulomb gas representation).

The Liouville action is (we have rescaled the field ϕ to get a standard form for the kinetic term):

$$S = \frac{1}{4\pi}\int d^2\sigma \left(\frac{1}{2}(\partial\phi)^2 + \frac{\mu^2}{\gamma^2}e^{\gamma\phi}\right) \qquad (2.27)$$

In the classical theory $\gamma = \sqrt{12/(26-c_m)}$ and at the quantum level γ is renormalized to the value $\gamma = (\sqrt{25-c_m} - \sqrt{1-c_m})/\sqrt{12}$, where c_m is the central charge of the matter.

Using the equation of motion $-\partial^2\phi = \mu^2/\gamma e^{\gamma\phi}$ we get the following formulas for the improved energy-momenta tensor (when we add to the action specific total derivative terms) :

$$T_{z\bar{z}} = 0$$

$$T_{zz} = -\frac{1}{2}(\partial_z \phi)^2 + \frac{1}{\gamma}\partial_z^2 \phi \tag{2.28}$$

We have a similar relation for $T_{\bar{z}\bar{z}}$, but depending this time on $\partial_{\bar{z}}$.

The first relation is a manifestation of conformal invariance (at classical level), meaning that we can decompose the dynamics into two independent pieces: holomorphic and anti-holomorphic (depending only on z and $\bar{z}$ respectively). However at quantum level the conformal invariance is in general destroyed by a conformal anomaly (the dynamics of the holomorphic and anti-holomorphic fields will be mixed).

The Liouville theory has a hidden $SL(2,R)$ symmetry. It permits to write ϕ at the classical level as a finite sum of holomorphic and anti-holomorphic fields:

$$e^{-j\gamma\phi} = (16/\mu)^{-j} \sum_{m=-j\ldots j} \psi_m^j(z)\psi^{jm}(\bar{z}) \tag{2.29}$$

where $\psi_m^j(z)$ transforms like the spin j representation of $SL(2,R)$. All the fields can be bosonized. The simplest fields $\psi_{\pm 1/2}(z)$ can be written as follows:

$$\psi_{\pm 1/2}(z) = \exp((-\gamma/2)\Phi_{\pm}(z))$$

The two free scalar fields $\Phi_{\pm}$ have the energy-momentum tensor:

$$T(z) = -\frac{1}{2}(\partial\Phi_{\pm}(z))^2 + \frac{1}{\gamma}\partial^2\Phi_{\pm}(z) \tag{2.30}$$

and are not independent. Even though the relations (2.28) and (2.30) seem very similar, the passage from the first set of relations to the second one is completely non-trivial, and involves the quantum group theory.

We can express one field in terms of another,by the so called Bäcklund transformation:

$$\dot{\Phi}_+ = \acute{\Phi}_- + \sqrt{2/\gamma}e^{-\gamma\Phi_+/2}\cosh(\gamma\Phi_-/2)$$
$$\dot{\Phi}_- = \acute{\Phi}_+ + \sqrt{2/\gamma}e^{-\gamma\Phi_-/2}\sinh(\gamma\Phi_+/2)$$

where the derivatives $\dot{\Phi}, \acute{\Phi}$ are with respect to x, respectively to $y (z = x + i\ y)$.

Then, the theory is described by a single free field (the free field representation of Liouville model). (More precisely there are 2 massless free fields from the left and right sectors, completely independent apart the zero mode which they share). If we retain all two free fields $\Phi_{\pm}$ we have an explicit Kac-Moody $SL(2,R)$ symmetry.

The Liouville model appears to be equivalent with a free massless scalar field. It means that the Liouville model can be treated as a conformal theory.

As we will see this is not exactly true. There are subtilities related with the boundary which prevents the Liouville theory from being a conformal theory. The reason is that we have two types of vertex operators $e^{\alpha\phi}$ which behave quite different. These two types are the local (or microscopic) and macroscopic operators [14][18].

The remedy consists in introducing a boundary; the boundary conditions must behave in such way to not spoil the conformal invariance. Hence the right equivalent of the Liouville model is the free scalar field with a boundary.

The correlation functions for fixed area are:

$$< e^{\alpha\phi} >_A = \int D\phi \exp(-S(\phi) + (\mu^2/\gamma^2)A) e^{\alpha\phi} \delta(\int e^{-\gamma\phi} - A) \tag{2.31}$$

If we shift the Liouville field by:

$$\phi \to \phi - \frac{1}{\gamma} \log A$$

we get:

$$< e^{\alpha\phi} >_A \sim A^{-\alpha/\gamma} < e^{\alpha\phi} >_{A=1} \tag{2.32}$$

For $\alpha > 0$ the correlation function is divergent (because $\gamma > 0$). In this case the operators $e^{\alpha\gamma}$ are called microscopic and have an ill behaviour. This divergence arise from the region of small area $A \to 0$ (or $\phi \to \infty$). The vertex operator with $A \to 0$ is the puncture operator and it is not normalizable. For large area we have macroscopic operators which are well-defined and are the analog of Wilson loops, hence they are non-local oparators. Hence we have a critical area A_{crit} beneath which the theory is ill defined.

In conformal theory we create the state $\psi(\phi)$ by cutting a small hole in the surface and performing the functional integral with boundary conditions $\phi(\sigma)$ at the hole. In this way for arbitrary small holes we get the local states. In conformal theory the local operator is local with respect to the metric g_{ab}.

In Liouville theory the local operator is local with respect to the intrinsic metric $g_{\mu\nu}$ and not the full physical metric $G_{\mu\nu} = e^{\gamma\phi} g_{\mu\nu}$. Hence in Liouville theory, the state $\psi(\phi)$ is peaked for small holes ($\phi \to \infty$) and is non-normalizable.

The conclusion is that we must impose a cutoff: or by discretizing the surface as in the random matrix models or by imposing a boundary condition to the Liouville field as in the continuos description.

2.6.2 The Liouville theory as low-energy effective model of 2D string

The string in D dimensions describes gravity coupled to D scalar matter fields. The σ-model action of the 2D string is:

$$S = \int d\sigma^2 \sqrt{g} [T(X) + R^{(2)} \Phi(X) + \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} g^{ab}] \tag{2.33}$$

The lightest fields are the target metric (or massive graviton) $G_{\mu\nu}(X)$, the massive dilaton $\Phi(X)$ and the massless tachyon $T(X)$. All these fields depend on the coordinates $X^\mu(\sigma)$:the embedding dimension $t(\sigma)$ and the Liouville field $\phi(\sigma)$.

We have two interpretation for the string as critical and non-critical. For the 2D non-critical string, the embedding dimension is $D - 1 = 1$ and there is only one degree of freedom- the Liouville field coming from the dynamical metric $g_{ab} = e^{\phi}\tilde{g}_{ab}$. The 2D critical string * is described by a CFT with 2 degrees of freedom: the embedding dimension $t(\sigma)$ and the Liouville field $\phi(\sigma)$. It was shown that [20] that the dynamics of the critical string is described by a free massless tachyon $\delta T(t, \phi)$ (restricted to move in a box of dimension $\bar{\phi}$). The conclusion is that due to the physical equivalence of the critical and non-critical interpretation, the dynamics of Liouville field is equivalent with that of a free massless scalar constrained to a box.

The non-critical string can be viewed as $D-1$ string coordinates coupled to the Liouville field. The effective action of the Liouville field can be obtained by integrating in the path integral over the string coordinates and by computing the Faddeev-Popov determinant of the fixed metrices $\tilde{g}_{ab}$. The result is [16],[17]:

$$S_{eff} = \int d^2x[\frac{1}{2}(\partial\phi)^2 + Q\phi R^{(2)} + \frac{\mu^2}{\gamma^2}e^{\gamma\phi}] \tag{2.34}$$

($Q = 1/\gamma + \gamma/2 = \sqrt{(26 - D)/12}$).

For the critical string, the conformal invariance on the world-sheet for the zero modes translates into the β-equations for the graviton, dilaton and tachyon fields [30].

$$\begin{aligned}
\beta^G_{\mu\nu} &= R_{\mu\nu} - \nabla_\mu \nabla_\nu \Phi + \nabla_\mu T \nabla_\nu T = 0 \\
\beta^\Phi &= \frac{26 - D}{3} + R + 4(\nabla\Phi)^2 - 4\nabla^2\Phi + (\nabla T)^2 + V(T) = 0 \\
\beta^T &= -2\nabla^2 T + 4\nabla\Phi\nabla T + V'(T) = 0
\end{aligned} \tag{2.35}$$

where ∇_μ is covariant derivative and $V(T)$- the tachyon potential.

For the critical string, the linear dilaton background is defined as having a flat metric and a dilaton background related to Liouville field:

$$\tilde{G}_{\mu\nu} = \eta_{\mu\nu}, \quad \tilde{\Phi} = Q\phi, \quad \tilde{T} = T$$

The only dynamics comes from the tachyon, having the effective action [20] (for $D = 2$ the screening charge is $Q = \sqrt{2}$):

$$S_{eff}(T) = \int d^2x e^{-2\sqrt{2}\phi}(\frac{1}{2}T(\partial_\phi^2 - \partial_t^2 + 2\sqrt{2}\partial_\phi)T - \frac{1}{2}V(T)) \tag{2.36}$$

The tachyon satisfies the equation of motion:

$$\partial_\phi^2 T - \partial_t^2 T + 2\sqrt{2}\partial_\phi T + 2T - 2T^2 = 0 \tag{2.37}$$

*The critical $D = 2$ string is a real truncation of the critical $D = 4$, $N = 2$ string. The consistence of this critical theory is controversate and different checks are still in study.

(it was chosen the simplest tachyonic potential $V(T) = -2T^2 + 4T^3/3$). When we want to study ϕ independently of the world-sheet dimension σ we apply the so-called mini-superspace approximation . This corresponds to restricting the Liouville field ϕ to its zero mode ϕ_0, in other words we omit any excitations modes of the Liouville field.

In this approximation the tachyon solution $T_{\phi_0}(\phi) = T(\phi - \phi_0)$ has the following limits:

$$\begin{aligned} T_0(\phi) &\rightarrow 1 - ae^{(2-\sqrt{2})\phi}, \text{ as } \phi \rightarrow -\infty \\ T_0(\phi) &\rightarrow b\phi e^{-\sqrt{2}\phi}, \text{ as } \phi \rightarrow \infty \end{aligned} \tag{2.38}$$

with $a, b > 0$ depending on the parameters of the potential $V(T)$. The solution is a kink centered at ϕ_0, tending asymptotically to 0 for $\phi \rightarrow \infty$, and to 1 for $\phi \rightarrow -\infty$.

The energy of these tachyon solutions diverges as $\phi \rightarrow \infty$. This limit corresponds to short distances on the world-sheet.

We impose a Dirichlet boundary condition on the tachyon, limiting the Liouville field to the semi-infinite pozitive plane $\phi \geq 0$:

$$T(\phi = 0, t) = \Lambda \tag{2.39}$$

where Λ is the bare cosmological constant.

The tachyonic background solution minimizes the energy of the system (with boundary included). We fix the tachyonic background solution $\bar{T}(\phi) = T(\phi + \bar{\phi})$ such that it is centered at $\phi = -\bar{\phi}$ and:

$$\bar{T}(-\bar{\phi}) = \Lambda \tag{2.40}$$

The region $\phi < -\bar{\phi}$ is forbidden.

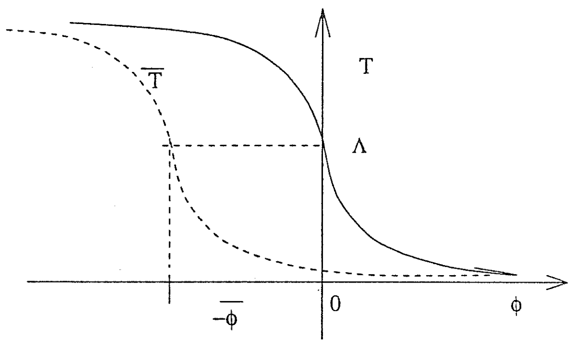


Fig.1. Tachyonic background solution

If we consider $0 < \Lambda \leq 1$ the relation $\bar{T}(0) = T(\bar{\phi}) = \lambda \sim \Lambda$ occurs in the asymptotic region $\bar{\phi} \sim \infty$ and we have:

$$b\bar{\phi}e^{-\sqrt{2}\bar{\phi}} = \Lambda$$

When the cosmological constant $\Lambda \rightarrow 0$, the Liouville field tends to $\bar{\phi} \rightarrow +\infty$.

The conclusion is that the dynamics is restricted by the tachyonic background to a region of space from $-\bar{\phi}$ to $+\infty$. When the cosmological constant $\Lambda \rightarrow 0$, the space is the full line from $-\infty$ to ∞.

Consider now the boundary moving with the momentum p. The perturbation of the tachyonic background $\delta T = T - \bar{T}$ satisfies the linearized version of equations of motion:

$$(\partial_\phi^2 + 2\sqrt{2}\partial_\phi + 2 - 4\bar{T})\delta T = 0$$

The solution of perturbation is:

$$\delta T \sim \frac{\sin p(\phi + \bar{\phi})}{\sin \bar{\phi}} \tag{2.41}$$

The rescaled field $T_1 = e^{-\sqrt{2}\phi} T$ is massless:

$$S_{eff}(T_1) = \int d^2x \left(\frac{1}{2} T_1 (\partial_\phi^2 - \partial_t^2) T_1 - g_{eff} T_1^3/3 + \ldots\right) \tag{2.42}$$

but the self-interaction has a spatially dependent coupling constant:

$$g_{eff} = e^{+\sqrt{2}\phi} \tag{2.43}$$

This coupling grows and becomes infinite at $\phi \to +\infty$. This fact is taken usually as a signal that the linear dilaton background is instable and should be modified in the region $\phi \to -\infty$. This would imply also that the correct vacuum is a condensate $\tilde{T} =< \bar{T} >$.

We can think about this instability as the existence of another tachyon background solution $\bar{T}$- the antikink. the anti-kink $\bar{T} = T(\phi - \bar{\phi})$ is centered at $\phi = \bar{\phi}$ and has the same asymptotics (2.38) but with $\phi \to -\phi$.

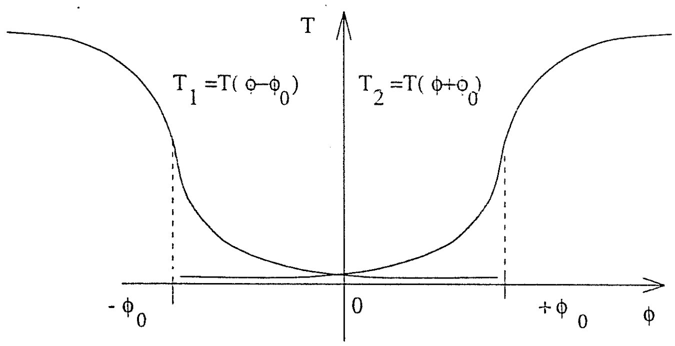


Fig.2. Tachyonic kink and anti-kink vacua

The correct picture of the tachyon background solution will be that of the figure. The coupling constant g_{eff} behaves now like $g_{eff} \sim 1/\cosh(\sqrt{\phi})$ increasing in the region $0 < \phi < -\bar{\phi}$ and decreasing for $\phi < 0$ as we decrease ϕ.

The tachyon perturbation δT is a free particle in the box of length $|\bar{\phi}|$ in both regions $0 < \phi < -\bar{\phi}$ and $\bar{\phi} < \phi < 0$. We have 2 boundaries one at $\phi = \pm\bar{\phi}$ due to the cutoff (depending on the region we choose) and the second at $\phi = 0$ due to the dynamics.

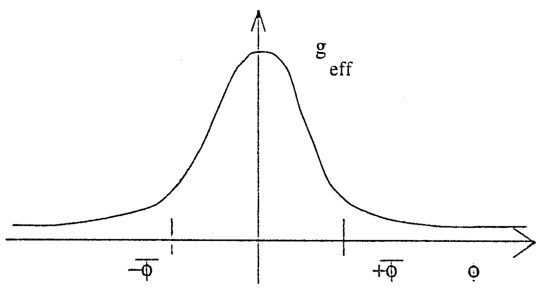


Fig.3. The coupling constant g_{eff}

We can have also large tachyon perturbations which tunnel the wall in the region $\phi \sim 0$ from one tachyon background to another.

3 Discrete states

3.1 The free boson on compact space

The simplest model which mimics what happens in 2D gravity is that of the free boson on circle [21]. It generates the "bear" discrete states (or null states) which became "gravitationally" dressed in 2D gravity. Apart certain subtilities (for example, the shifted energy-momentum tensor, or the need of infrared cut-off) the Liouville field can be considered as a free boson in 2D.

Hence, we arrive to consider the free boson in 2D with the lagrangian:

$$L = \frac{1}{2}(\partial_\mu \Phi)^2$$

The theory is conformal invariant and equivalent with the $c = 1$ conformal field theory. But it is infrared singular. To correct this behaviour we can either compactify the space either impose boundary conditions compatible with conformal invariance. Hence we must introduce new parameters in theory: the radius of compactification R or the variables which parametrize the boundary. When we compactify the boson on the circle of radius R we can do this in two ways [21]:

$$\Phi \rightarrow \Phi + 2\pi R$$

or

$$\Phi \rightarrow \Phi + 2\pi R, \quad \Phi \rightarrow -\Phi$$

giving rise to the Kosterlitz-Thouless (KT) line of the gaussian model and the Ashkin-Teller (AT) line for the boson on the orbifold S^1/Z_2 respectively. We have also other three models without "moduli" which are described by the exceptional Lie groups E_6, E_7, E_8. Their symmetry appears only at quantum level , hence they have no classical interpretation in terms of the compactification of the boson.

Due to the compactification we have 2 kinds of states: states with compact momentum:

$$\frac{1}{2}(p + \bar{p}) = p_m = m/R \tag{3.1}$$

and winding states:

$$\Phi(2\pi) = \Phi(0) + 2\pi n R, \quad p - \bar{p} = p_n = nR \tag{3.2}$$

This means that the $c = 1$ CFT has the primary fields with conformal dimensions ($h = p^2/2, \bar{h} = \bar{p}^2/2$):

$$(h, \bar{h}) = \left(\frac{1}{2}\left(\frac{m}{2R} + nR\right)^2, \frac{1}{2}\left(\frac{m}{2R} - nR\right)^2\right) \tag{3.3}$$

We consider the string theory in D dimensions to understand better how these primary states are made. The string is compactified on a torus of radius R. The previous CFT is describing the low-energy field theory of this compactified string.

The general vertex operator in this string theory is (where $\phi, \bar{\phi}$ are the left and right momving coordinate components with the momenta k, respectively $\bar{k}$):

$$V_{h\bar{h}} = \exp(ik\phi + i\bar{k}\bar{\phi}) \tag{3.4}$$

gives states with the mass:

$$\begin{aligned} M^2 = -k^2 = n^2R^2 + \frac{m^2}{4R^2} + L + \bar{L} - 2 \\ mn + L - \bar{L} = 0 \end{aligned} \tag{3.5}$$

(-2 comes from the Casimir energy correction). L and $\bar{L}$ are the string excitations energies for the left and right sectors.

Apart the massive spectrum, we have also a massless spectrum described by the previous CFT with the primary fields given by (3.3).

In the framework of the string theory we can build a set of vertex operators giving massless states. The first vertex operator is:

$$\partial\phi^\mu\bar{\partial}\phi^\nu \exp(ik\phi + i\bar{k}\bar{\phi}), \quad \mu, \nu \neq 1 \tag{3.6}$$

gives the massless graviton-dilaton and antisymmetric tensor in $D-1$ dimension.

The second kind of operators:

$$(\partial\phi^\mu\bar{\partial}\phi^1 \pm \partial\phi^1\bar{\partial}\phi^\mu) \exp(ik\phi + i\bar{k}\bar{\phi}), \quad \mu \neq 1 \tag{3.7}$$

gives the $D-1$ dimensional vector boson. For the plus sign we have the Kaluza-Klein boson which couples with compact momentum and for the minus sign we have a boson (which we will call the H-boson) which couples with the winding.

The last vertex operator is a $D-1$ dimensional scalar:

$$\partial\phi^1\bar{\partial}\phi^1 \exp(ik\phi + i\bar{k}\bar{\phi}) \tag{3.8}$$

. The interesting point in the compactified string theory is that we have a mechanism similar to the Higgs mechanism in the quantum field theory. For generic radius of compactification R we have a massless spectrum given by the primary fields (3.3). But for special values of R some of the massive states become massless and enrich the massless spectrum. In this way also the symmetry of the effective QFT describing the massless spectrum is enhanced, from the group H (described by the H boson) to the group G. The KK-bosons play the same role as the Higgs scalars belonging to the quotient G/H.

For example, for the self-dual radius $R_0 = 1/\sqrt{2}$ the four vertex operators with $m = n = \pm 1$ and $L + \bar{L} = 1$ are:

$$J_\pm = \exp(\pm i\sqrt{2}\phi), \quad \bar{J}_\pm = exp(\pm i\sqrt{2}\bar{\phi})$$

become massless Kaluza-Klein bosons[22]:

$$M = |R - \frac{1}{2R}| \to 0 \tag{3.9}$$

These KK-bosons couple to the H-bosons (already present in the massless spectrum and given by (3.3) with $m = n = 0$):

$$J_3 = -i\partial\phi, \bar{J}_3 = -i\bar{\partial}\phi \tag{3.10}$$

forming $SU(2)$ gauge bosons [22]:

$$\partial\phi\bar{J}^a \exp(ik\phi + i\bar{k}\bar{\phi}) \quad \text{and} \quad \partial\bar{\phi}J^a \exp(ik\phi + i\bar{k}\bar{\phi}) \tag{3.11}$$

In this case the initial symmetry of the massless spectrum is the $U(1) \times U(1)$ symmetry (for the left and right sectors). At the self-dual radius the generic symmetry of the effective model $U(1) \times U(1)$ is enhanced to $SU(2) \times SU(2)$.

At the self-dual radius, the massless states are classified according to $SU(2)$ representation theory as $\psi_{h,\bar{h}} = |J, m> \times |\bar{J}, \bar{m}>$ with $m = -J, \dots J$ and $\bar{m} = -\bar{J}, \dots \bar{J}$:

$$|Jm> (q) = (\oint \frac{dz}{2\pi i} e^{-i\sqrt{2}\phi(z)})^{J-m} e^{iJ\sqrt{2}\phi(q)}) \tag{3.12}$$

The conformal dimensions are $(h, \bar{h}) = (J^2, \bar{J}^2)$

We can break again the theory from $SU(2) \times SU(2)$ to $U(1) \times U(1)$ by perturbing with the mass term $J_3\bar{J}_3 = -\partial\phi\bar{\partial}\phi$. This operator increases the radius R. However, a rotation by π of one of $SU(2)$'s currents changes the sign of this operator. So increasing R is gauge-equivalent to decreasing it. This is the T-duality: the R theories and $R' = 1/R$ are equivalent if we identify the fields:

$$\phi(R') = \phi(R) - \log(R/\sqrt{2}).$$

When the radius is an integer multiple of the self-dual radius $R = nR_0$ [21] we have additional discrete symmetries. On the KT line this is the cyclic group C_{2n} generated by the element $h = \exp(2\pi i J_3/n)$ (rotation with angle $2\pi/n$). The generator h acts on the currents in the following way:

$$hJ_3h^{-1} = J_3, \quad hJ_\pm h^{-1} = e^{\pm 2\pi i/n} J_\pm \tag{3.13}$$

On the AT line we have dihedral group generated by the elements h and $\tilde{h} = \exp(i\pi J_1)$ (reflection). The generator $\tilde{h}$ acts on the currents in the following way:

$$\tilde{h}J_3\tilde{h}^{-1} = -J_3, \quad \tilde{h}J_\pm\tilde{h}^{-1} = e^{\pm 2\pi i/n} J_\mp \tag{3.14}$$

For the models based on exceptional groups E_6, E_7, E_8 we have additional discrete symmetries related with the tetrahedral, octahedral and icosahedral groups.

Instead of the usual states $|J, m>$ classified according to the $SU(2)$ symmetry in the self-dual point with $R = R_0$, in these special points with $R = nR_0$ we have the equivalence classes invariant with respect to these discrete symmetries.

3.2 The abelian field

In 2D QED, like in 2D gravity, we have a similar phenomenon of creation of discrete states. We show that at special values of the momenta the number of degrees of freedom grows up. These new states are related with the values of the abelian field A_μ at the boundary, hence in some sense these states are non-local. On the other hand we show that these states are related with the θ-term and have explicit topological properties. The field F_{10} is related with the angle θ through the relation (3.20).

We impose the Lorentz condition on the states of 2D QED:

$$p_\mu A_\mu|\phi>=0 \tag{3.15}$$

Using also the gauge transformation $A_\mu \rightarrow A_\mu + p_\mu\Phi$ we can eliminate the longitudinal and temporal components of the states $|\phi>=|\phi_\perp> +|\phi>_{||}=|\phi_\perp>$. In two dimensions we have no local states $|\phi>$ satisfying these two requirements. However for special values of momentum $p_\mu = 0$ we have no constraints and we have an infinity of new states.

The condition (3.15) imposes the cancelation of the longitudinal degrees of freedom. It also means that Φ is a free massless field $p^2\Phi|\phi>=0$. Because in 2D we have no transversal excitations of field A_μ, the physics is described by the longitudinal degree of freedom - the free scalar field Φ with boundary.

We must impose the boundary conditions to solve the eq.(3.15) equation :

$$\begin{aligned} \Box\Phi &= 0, \\ \Phi|_{bound} &= \Phi_0, \quad \text{where} \quad \Phi = A_{long} \end{aligned} \tag{3.16}$$

The value of the solution at the boundary determinates the solution over the entire space-time volume.

If we look at the states related with this field Φ we can see that:

$$<\phi|A_\mu|\phi>=\partial_\mu\Phi \tag{3.17}$$

Hence by choosing the appropriate boundary solution Φ_0 we fix also the corresponding state $|\phi>$. We can say that for the momentum $p_\mu = 0$ we have some non-local states which are related with the properties of the boundary.

The vacuum states $|\phi>$ are known to be parametrized by the famous θ angle [23]. The background field Φ_0 is related to θ through relation (3.20). We can introduce explicitly in the lagrangian a term depending on θ (the analog of Pontryagin index in 2D):

$$L = -\frac{1}{4}F^2_{\mu\nu} - \frac{k}{4\pi}\epsilon^{\mu\nu}F_{\mu\nu} \tag{3.18}$$

At infinity the abelian field $A_\mu \rightarrow g(\theta)\partial_\mu g(\theta)^{-1}$, where $g(\theta)$ is a mapping from S^1 (the boundary of the 2D plane) to $S^1 \sim U(1)$. For the mapping $g(\theta) = e^{in\theta}$ we get the winding number $n = 1/(2\pi)\oint dx_\mu A_\mu$ The states $|\phi>$ parametrized by the angle θ are physically equivalent, in case we do not couple the gauge field to a chiral current. If we introduce chiral fields in the model we have a chiral anomaly which makes the states $|\phi>$ different.

Indeed, we can see that axial current $J_5^\mu = \epsilon^{\mu\nu}A_\nu$ is conserved at the classical level generating the charge Q_5. Different states are related by axial transformation:

$$e^{i\alpha Q_5}|\phi(\theta)>=|\phi(\theta-2\alpha)> \tag{3.19}$$

In case the axial symmetry is destroyed at the quantum level, we have inequivalent vacua which can be decomposed into a linear combination of winding states $|n>$:

$$|\phi(\theta)>=\sum_{n=-\infty}^{\infty} e^{in\theta}|n>$$

The winding states $|n>$, with n integer, are the eigenstates of the hamiltonian having a given Pontryagin index (n2D is the winding number) $n = 1/(4\pi)\int d^2x\epsilon^{\mu\nu}F_{\mu\nu}$.

We can have also solitons which connect two winding states $|n>$ and $|m>$.

When the axial symmetry is anomalous broken, the states $|\phi(\theta)>$ are inequivalent and are characterized by the physical measurable quantities as the energy density:

$$\frac{E_\theta}{L} = a - be^{-S(\theta)}\cos\theta$$

and by the background field F_{10}:

$$<\theta|F_{10}|\theta>=<\theta|\epsilon^{\mu\nu}F_{\mu\nu}|\theta>=2\pi be^{-S(\theta)}\sin\theta \tag{3.20}$$

The field F_{10} is closed in a box of length L, and the action $S(\theta)$ (for the lagrangian (3.18)) corresponds to the boundary solution $A_\mu \to g(\theta)\partial_\mu g(\theta)^{-1}$ extended to the entire box.

3.3 The dilaton+graviton system

We review the lightest states in the string theory.

For the closed string the massless states are given by acting with the following vertex operator $h_{\mu\nu}(X)\partial X^\mu\bar{\partial}X^\nu e^{ikX}$ on the vacuum (where $h_{\mu\nu}$ is the metric):

$$|f(k)>=h_{\mu\nu}\alpha_{-1}^\mu\tilde{\alpha}_{-1}^\nu|0,k>, \quad k^2=0 \tag{3.21}$$

This set can be decomposed under the $SO(D-2)$ transversal rotation group into traceless symmetric tensor (graviton), the trace or invariant (dilaton), and the antisymmetric tensor. In case we have a conserved gauge current j^a (where a is the group index), we can define the gauge boson using the vertex operators $\partial X^\mu \bar{j}^a e^{ikX}$ and $j^a\bar{\partial}X^\mu e^{ikX}$ with $k^2=0$.

For the open string we have the tachyon given by the operator e^{ikX}:

$$|0,k>, \quad k^2=2 \tag{3.22}$$

and the massless vector boson with the polarization e_μ:

$$e_\mu \alpha^\mu_{-1}|0, k>, \quad k^2 = 0, \quad k \cdot e = 0 \tag{3.23}$$

The non-critical 2D string theory is described by the Liouville field ϕ interacting with the matter field x. The energy-momentum tensor of the Liouville field (the holomorphic part) is like that of the free boson, but shifted, having a screening charge at infinity. The total stress tensor of the Liouville and matter field is:

$$T(z) = \frac{1}{2}(\partial\phi)^2 - Q\partial^2\phi + \frac{1}{2}(\partial x)^2$$

with $Q = \sqrt{(25 - c_m)/12}$. This means that the central charge of the Liouville field is not 1 as that of the free boson, but $c_L = 1 + 12Q^2$=25.

We explain the strange point why the Liouville field has the central charge $c_L = 25$. In 2D gravity the only degree of freedom is described by the Liouville field acting as a free boson, hence having the central charge $c_L = 1$. But this happens in the when we interpret 2D gravity as a non-critical string. —n the critical 2D string interpetation we have 2 fields- the matter $x(\sigma)$ and the Liouville field $\phi(\sigma)$, which has now the central charge $c_L = 25$ and a screening charge at infinity $Q = \sqrt{2}$, hence is not a free field. But the only 1 degree of freedom of the 2D critical string is described by the tachyonic field $T(\phi, x)$ which is also a massless and free scalar field.

From the 26D bosonic string we can arrive at the 2D gravity by the Distler-Kawai argument. It says that in order to get the massless spectrum of the bosonic string in d dimensions, we can replace on the world-sheet $26 - d$ degrees of freedom of the gravity wth a single field- the Liouville field having the central charge $c_L = 27 - d$ and the screening charge $Q = \sqrt{(26 - d)/12}$. The Liouville field couples with the conformal matter having the central charge $c_M = d - 1$. For example, in $d = 26$ dimensions the Liouville field has $c_L = 1$ and $Q = 0$ in other words it is a free field which decouples from the matter sector. For $d = 2$ (the 2D gravity) we have the Liouville field with $c_L = 25$ and $Q = \sqrt{2}$ which couples with the matter field $c_M = 1$.

We introduce the primary vertex operators $V_p = \int d\sigma e^{\epsilon\phi(\sigma)} e^{ipx(\sigma)}$. We require that the conformal dimension of the primary fields $|p>= V_p|0>$ is $(1, 1)$:

$$\frac{1}{2}p^2 - \frac{1}{2}\epsilon^2 - Q\epsilon = 1 \tag{3.24}$$

As we will see in a moment, this condition is nothing else than the mass shell condition.

We can redefine the momentum to the shifted momentum:

$$\begin{aligned} p_\mu &= p'_\mu - q_\mu \\ \epsilon &= \epsilon' - Q \end{aligned} \tag{3.25}$$

with $q_\mu^2 = Q^2$. Then the equation (3.24) reduces to the standard form of the mass shell condition $p'^2/2 - \epsilon'^2/2 = 1$.

The dilaton-graviton operator is a symmetric tensor and can be written as the sum of a traceless symmetric part (graviton) and a trace part (dilaton):

$$h_{\mu\nu} = \hat{h}_{\mu\nu} + \tilde{h}_{\mu\nu} = (h_{\mu\nu} + h_{\nu\mu} - \delta_{\mu\nu}h^{\tau}_{\tau})/2 + \delta_{\mu\nu}h^{\tau}_{\tau}/2$$

The condition of conformal invariance $p'^{\mu}H_{\mu\nu} = 0$ can be rewritten on components as:

$$\begin{aligned} (p_\mu + q_\mu)\hat{h}_{\mu\nu}|\hat{f}(p) >&= 0 \\ p_\mu\tilde{h}_{\mu\nu}|\tilde{f}(p) >&= 0 \end{aligned} \tag{3.26}$$

where $|\hat{f}(p) >, |\tilde{f}(p) >$ are the massless states given by relation (3.21) and characterizing the graviton, respectively the dilaton. This condition implies that the theory has only transverse states.

The conformal transformation act on graviton and dilaton operators as follows:

$$\begin{aligned} \hat{h}_{\mu\nu} &\to \hat{h}_{\mu\nu} + (\hat{\Phi}_\mu p_\nu + \hat{\Phi}_\nu p_\mu) \\ \tilde{h}_{\mu\nu} &\to \tilde{h}_{\mu\nu} + (\tilde{\Phi}_\mu(p_\nu + q_\nu) + \tilde{\Phi}_\nu(p_\mu + q_\mu)) \end{aligned} \tag{3.27}$$

We can see that the "gauge" degrees of freedom are described by the fields $\hat{\Phi}_\mu, \tilde{\Phi}_\nu$. These constraints show that the fields $\hat{\Phi}_\mu, \tilde{\Phi}_\mu$ must be orthogonal on the vectors $p_\mu + q_\mu$, respectively on p_μ:

$$\begin{aligned} (p_\mu + q_\mu)\hat{\Phi}_\mu|\hat{f}(p) >&= 0 \\ p_\mu\tilde{\Phi}_\mu|\tilde{f}(p) >&= 0 \end{aligned} \tag{3.28}$$

In 2D (1 spatial coordinate) is impossible to satisfy these requirements for $p \neq q, p \neq 0$. For $p = q$ or $p = 0$ there is a jump in the number of degrees of freedom. At these values one can chose the gauge $q_\mu\hat{\Phi}_\mu = 0$ or $\tilde{\Phi}_0 = 0$. The vertex operator (3.21) has now the form $\int d\sigma \Phi_\mu(p, \epsilon)\Phi_\mu e^{\epsilon\phi(\sigma)}e^{ipX(\sigma)}$ and does not depend on the excitations modes of the Liouville field $\phi(\sigma)$ (by choosing th e direction along the vector Φ_μ we can simplify the expression $\Phi_\mu \partial X^\mu = \Phi\partial x$). This property holds at all the mass levels. Therefore the new degrees of freedom can be found by looking at the Virasoro primary fields of the matter field $x(\sigma)$.

When central charge $c \leq 1$ the momenta q_μ are quantized. For the central charge $c = 1$, the screening operator is:

$$q_\mu \doteq \left(\frac{1}{2R} + R, \frac{1}{2R} - R\right)$$

satisfying the condition $q_\mu^2 = q_0^2 - q_1^2 = Q^2 = 2$

The primary field (m, n) has the momentum:

$$p_{mn} = \left(\frac{m}{2R} + nR, \frac{m}{2R} - nR\right)$$

with m, n integers. The orthogonality conditions (3.28) are satisfied for the quantized momenta $p_\mu = -q_\mu$ and for $p_\mu = 0$. The new states $|\hat{f}(p) >, |\tilde{f}(p) >$ appearing at these special momenta are called primary discrete states [24][25][26][27][28][29]. These primary discrete states are depending of 2 integers m, n: $|f_{mn} >= |f(p_{mn}) >$. For the self-dual radius $R = 1/\sqrt{2}$, these states are massless. In addition, at the self-dual radius these states are representations of the $SU(2)$ algebra. All the states $|f_{n,m} >$ of a given level n can be derived by repeteadly acting with the lowering operator $L_- =: e^{-i\sqrt{2}x}$: to the highest state (the discrete tachyon) $|f_{n,n} >=: e^{in\sqrt{2}x} : |0 >$. The first two states after the highest state are $|f_{n,1} >= \partial x(z)|0 >$ and

$$|f_{n,2} >=: (\partial^2 x(z) - i(\partial x)^2/\sqrt{2})e^{in\sqrt{2}x}(z) : |0 >$$

As we already have said, these primary discrete states describe the matter sector excitations not coupled with the Liouville field. If we couple the matter excitations with the Liouville field we obtain the discrete states given by:

$$F^{\pm}_{n,m} = f_{nm}e^{(1\mp n)\sqrt{2}\phi} \tag{3.29}$$

Instead studying the discrete states ring we can study a simpler ring having the elements in on-to-one correspondence with the discrete states. It is known [28] that the operators $cF^{+}_{n,m}(z)$ with ghost number 1 (b and c are the ghost fields) have partners, the operators $\mathcal{O}_{n,m}(z)$ with ghost number 0. These operators are part of the so-called chiral ground ring which is generated by the operators:

$$\begin{aligned} x &= \mathcal{O}_{1/2,1/2} =: [cb + \frac{i}{\sqrt{2}}(\partial x - i\partial\phi)]e^{i(x+i\phi)/\sqrt{2}} : \\ y &= \mathcal{O}_{1/2,-1/2} =: [cb - \frac{i}{\sqrt{2}}(\partial x + i\partial\phi)]e^{-i(x-i\phi)/\sqrt{2}} : \end{aligned} \tag{3.30}$$

The operators $\mathcal{O}_{n,m}$ are otained by taking powers of the generators:

$$\mathcal{O}_{n,m} = x^{n+m}y^{n-m} \tag{3.31}$$

The ring (3.30) is relevant in the open string theory. In the closed-string theory we need to combine the holomorphic generators with the the anti-holomorphic counterparts, the generators $\bar{x}, \bar{y}$. We get a ground ring generated by 4 operators:

$$a_1 = x\bar{x}, \quad a_2 = y\bar{y}, \quad a_3 = x\bar{y}, \quad a_4 = y\bar{x} \tag{3.32}$$

which satisfy the condition $a_1a_2 - a_3a_4 = 0$. This condition determinates a 3D quadric cone, which was supposed to describe also the conifold singularities.

4 Methods

There are 2 approaches to quantize the 2D gravity: the covariant method and the discretized method (the matrix model method). In the covariant method, the classical measure over the space of metrics in the conformal gauge is rewritten in terms of the Liouville action. In the works [39][40] were given arguments that the measure of the quantized gravity will still retain the form of the Liouville action, but with the renormalized couplings. But there are no strong reasons why in the quantized Liouville action cannot be included other counterterms of different form.

Even though the Lorentz invariance of the theory is lost in the discretized method, this method is more attractive from the mathematical point of view. The matrix method works from beginning with a regularized theory due to the triangulation of the Riemann surfaces. Also the measure of 2D gravity is rewritten as the measure over the space of matrices which is well defined and well studied in mathematics.

In some sense, the same problem of the measure exists in the string formulation of QCD. Even though the right string theory for QCD seems to be that of the rigid string [41], there are still no clues about the form of the measure for rigid string. Because this, few years ago were introduced the matrix versions of QCD [98], but unfortunately these theories were not describing the correct phase of QCD.

4.1 1-matrix model

The large N expansion of matrix-valued field theories was invented more than twenty years ago by 't Hooft [46] as a way to treat four-dimensional QCD with gauge group $SU(N)$. At $N = \infty$ planar diagrams dominate and it was hoped that this fact would lead either to analytic results or to a reformulation of QCD as a string theory. It was slowly understood that $N = \infty$ field theories retain much of the complexity of the generic N case.

Some of this complexity remains even in the zero-dimensional matrix (the 1-matrix model). The partition function of the 1-matrix model is:

$$Z = \int \mathcal{D}M \exp[-N \ \mathrm{Tr}(\Sigma_{r=1}^{\infty} t_r M^r)], \tag{4.1}$$

where M is a $N \times N$ Hermitian matrix and the t_q's are coupling constants parametrizing a general potential. The model is solved [47] by changing variables to the eigenvalues of the matrix M and thereby reducing the number of degrees of freedom from N^2 to N. The latter proves possible due to the invariance of the above action and of the measure under the group $U(N)$.

The model's free energy describes at $N = \infty$ a sum over all planar diagrams G of spherical topology, where vertices v_r of order q (there are $\#v_r$ of them in G) are weighted with a factor t_r:

$$\log Z \equiv F = \sum_{G} \prod_{v_q \in G} t_r^{\# v_r} \tag{4.2}$$

In 1978, the model was considered as a toy-QCD in zero dimensions. A few years later, however, it became clear that the results could be used to obtain the solution of the 2D quantum gravity [48, 49, 50]. These works were inspired by early Regge calculus [51].

By tuning the couplings t_r (with the simplest cubic potential):

$$t_0 \equiv N = \exp(\frac{1}{4G_0}), t_3 = \exp(-\Lambda a^2),\ t_1 = 0, t_2 = \frac{1}{2},\ t_r = 0, r > 3 \tag{4.3}$$

in an appropriate way a continuum limit could be reached at which the planar graphs condense to give a continuum path integral over two-dimensional metrics g_{ab} of spherical topology:

$$F = \int \mathcal{D}g_{ab} \exp -[\int d^2z\sqrt{\mathrm{Det}g}(\Lambda - \frac{1}{16\pi G_0}R(g_{ab}))]. \tag{4.4}$$

The distance from the critical point in the space of the t_r's turns into a continuum cosmological constant λ_R controlling the area of the surfaces. We also wrote the Einstein term, which is however known to be a constant in two dimensions.

The simplest 1- matrix model contains non-trivial physics: it gives the critical points of the unitary conformal matter coupled to gravity. Various generalizations were solved and shown to give new physical information: certain simple multi matrix models describe the coupling of $c \leq 1$ conformal matter ((p, q) minimal models) to 2D gravity.

4.2 From 1-matrix model to multi-matrix models

We give some reasons why the physicists were interested to pass from the study of 1-matrix model to that of multi-matrix models.

For a simple 1-matrix model with the potential $V(M) = tM + M^{m+1}$, the partition function near the m-th critical point (described by unitary (1,m) CFT) behaves like $Z \sim t^{2/(m+1)}$. The obvious generalization was to consider the 2-matrix model with the potential $V(M, N) = M^{p+1} + N^{q+1} + \ldots$ which would give the other critical points described by the minimal (p,q) CFT's. It was also hoped that the partition function will behave like $Z \sim t^{2/(p+q-1)}$.

It is known now that the integrable structure in the infinite parameter space $(t_r, s_r), r = 1 \ldots \infty$ of the 2-matrix model with the potential $V(M, N) = \sum_{r=1}^{\infty} t_r M_r + s_r N_r + cMN$ is described by the Kadomtev-Petiashvili-hierarchy. The reductions of the KP-hierarchy give rise to the n-KdV hierarchy and their generalizations (N, M)-KdV hierarchies [82]. But there is no simple connection between these reductions and the tuning of parameters (t_r, s_r) for the 2-matrix model to achieve the (p,q) critical points. In addition, near these (p,q) critical points, the partition function depends on all parameters $(t_m, s_n), 1 \leq m \leq$

$p+1, 1 \leq n \leq q+1$. There is no natural way to fix these parameters such that they depend of only one variable t and the partition function has the conjectured behaviour $Z \sim t^{2/(p+q-1)}$.

Even though the critical potentials for the 2-matrix model are known [42], the flows of the parameters to these critical values are not known. Using the Q-matrices method we know how to tune the parameters (t_r, s_r) to achieve one of these critical points beginning from an arbitrary point in the parameter space. But this tuning seems to be quite artificial.

The renormalization group method was also applied to study the flows of the 2-matrix model [43]. Instead of the Kadanoff transformation which connects the blocks of spins of different scales, was used the relation between the matrices of different dimensions, $N \times N$ and $(N+1) \times (N+1)$. However, the method was not sensible enough to get in a systematic and general way all minimal critical points.

Another reason to study multi-matrix models was the simple way of getting the critical exponents for different models defined on the random lattice. It is possible to get analytical results for the model on the regular lattice due to the KPZ relation [44] which relates the critical exponents of the model on the regular and random lattice. For example, the Ising model on random surface is described by the 2-matrix model with quartic potential and permits to obtain the critical exponents even in the presence of a magnetic field [101]. Also, the Potts-model on a random lattice is naturally described by the star-matrix model [96] [97](see chapter 7).

Another hope was that more complicated multi-matrix models describe the string theory in higher dimensions $d > 2$. The terms of the type $Tr(M^2)^k$, added to the potential, give a new phase. This phase, called the branched polymer phase [45], was supposed to appear for strings with $d > 2$.

Because one of the critical points of the 2-matrix model is described by the A_5 CFT, it was supposed that by combining five such 2-matrix models in one 10-matrix model will give a critical point described by the conformal field theory $\otimes_{i=1}^{5} A_5^{(i)}$. But this CFT model describes nothing else than the simplest Calabi-Yau manifold. The hope is to recover the structure of the Calabi-Yau manifold from the properties of the 10-matrix model.

The $c = 1$ matrix model describes the dynamics of discrete states which are considered the remnants of higher dimension states of the strings. But it is possible that the $c = 1$ matrix model has a more general appealing because it describes a certain topological sector of the multi-dimensional string. In this sense it could describe the topological properties of a string near the conifold singularity of a Calabi-Yau manifold [35]. In this interpretation, the discrete states are nothing else that the degrees of freedom of the conifold singularities [37].

Still an open problem remains the description of the deformation of a conifold singularity due to the unknown form of the $c = 1$ matrix model for an arbitrary radius (just at the conifold singularity the properties of the string are described by $c = 1$ matrix model at the self-dual radius).

After these arguments in favor of the study of multi-matrix models, we explain why the method of Q-matrices is well suited to these models.

The Q-matrices method which will be described in the next section is a generalization of the orthogonal polynomials method. The Q_α matrices are nothing else than the initial matrices M_α in the basis of the orthogonal polynomials. The form of Q-matrices in terms of matrix potential's parameters is given by solving the equations for Q-matrices (called the coupling conditions). The flow equations contain all the integrable structure of the matrix theory. The reconstruction formulae permit to express all the correlation functions of the theory in terms of the traces of the Q-matrices.

We explain now why the Q-matrices method is more useful then the classical method, the W-constraints. The origin of the W-constraints can be traced back to the Makeenko-Migdal loop equations of the Yang-Mills theory in 4 dimensions. Later the loop equations were written also for the 2D gravity and was discovered that they have a natural integrable structure of the Virasoro algebra. This integrable structure was extended to a W-algebra structure. The problem with the W- structure is that apart simple models, the equations become very complicated and difficult to solve. Also for higher genera, the W- constraints become too involved to permit the calculations of correlation functions by induction.

The Q-matrices method is more useful in these cases because gives an explicit formula in terms of Q-matrices for all possible correlation functions (the form of Q-matrices is derived from the coupling conditions). The Q-matrices method also gives a consistent and systematic treatment of all multi-matrix models.

Also the method gives for granted the integrable structure of the theory in terms of the Toda lattice hierarchy (also called KP hierarchy, but more precisely it is a dimensional reduction to two dimensions of the KP-hierarchy). The method permits a natural reduction of the Toda hierarchy to the n-KdV hierarchies, which are believed to describe the integrable structure of the critical points. But a big unsolved puzzle is the link between the reductions and the more intuitive way of looking directly in the initial matrix action, at the flows to the critical points.

4.3 Multi-matrix models

In this section we intend to analyze matrix models made of q Hermitean $N \times N$ matrices with bilinear couplings between different matrices. Unless otherwise specified, by this we mean an open chain of q matrices, each linearly interacting with the nearest neighbours. These models have been already introduced and partially analyzed in [53] (for other approaches to multi–matrix models, see [54],[56],[57],[58], [59],[60]). The reasons to go beyond two–matrix models are diverse. The extended two–matrix model provides a useful representation of $c = 1$ string theory at the self–dual point, [61]; in particular it naturally incorporates the so–called discrete states, which appear to be organized in sl_2 multiplets. We find it natural to ask ourselves whether such a construction can be generalized. The answer is affirmative: in the extended q–matrix model we do find discrete states organized according to representations of sl_q.

We review here some general results concerning q–matrix models, [53]. The partition function of the q–matrix model is given by

$$Z_N(t,c) = \int dM_1 dM_2 \dots dM_q e^{TrU} \tag{4.5}$$

where $M_1, \dots, M_q$ are Hermitian $N \times N$ matrices and

$$U = \sum_{\alpha=1}^{q} V_\alpha + \sum_{\alpha=1}^{q-1} c_{\alpha,\alpha+1} M_\alpha M_{\alpha+1}$$

with potentials

$$V_\alpha = \sum_{r=1}^{p_\alpha} \bar{t}_{\alpha,r} M_\alpha^r \qquad \alpha = 1, 2 \dots, q \tag{4.6}$$

The p_α's are finite positive integers.

We denote by $\mathcal{M}_{p_1,p_2,\dots,p_q}$ the corresponding q–matrix model. It has become moreover customary to associate to the generic q–matrix model (4.5) the Dynkin diagram A_q. Occasionally we will stick to this convention and speak about nodes and links.

We are interested in computing the correlation functions of the operators

$$\tau_{\alpha,k} = tr M_\alpha^k$$

and possibly of other composite operators (see below). For this reason we complete the above model by replacing (4.6) with the more general potentials

$$V_\alpha = \sum_{r=1}^{\infty} t_{\alpha,r} M_\alpha^r, \qquad \alpha = 1, \dots q \tag{4.7}$$

where $t_{\alpha,r} \equiv \bar{t}_{\alpha,r}$ for $r \le p_\alpha$.

In other words we have embedded the original couplings $\bar{t}_{\alpha,r}$ into infinite sets of couplings. Therefore we have two types of couplings. The first type consists of those couplings (the barred ones) that define the model: they represent the true *dynamical* parameters of the theory; they are kept non-vanishing throughout the calculations. The second type encompasses the remaining couplings, which are introduced only for computational purposes. In terms of ordinary field theory the former are analogous to the interaction couplings, while the latter correspond to external sources (coupled to composite operators). Any correlation function is obtained by differentiating $\ln Z_N$ with respect to the couplings associated to the operators that appear in the correlator and then setting to zero (only) the external couplings.

From now on we will not make any formal distinction between interacting and external couplings. Case by case we will specify which are the interaction couplings and which are the external ones. Finally, it is sometime convenient to consider N on the same footing as the couplings and to set $t_{\alpha,0} \equiv N$.

4.4 Orthogonal polynomials

The most popular procedure to calculate the partition function consists of three steps [62],[63],[64]:

(i). One integrates out the angular parts such that only the integrations over the eigenvalues are left,

$$Z_N(t,c) = \text{const} \int \prod_{\alpha=1}^{q} \prod_{i=1}^{N} d\lambda_{\alpha,i} \Delta(\lambda_1) e^U \Delta(\lambda_q), \tag{4.8}$$

where

$$U = \sum_{\alpha=1}^{q} \sum_{i=1}^{N} V_\alpha(\lambda_{\alpha,i}) + \sum_{\alpha=1}^{q-1} \sum_{i=1}^{N} c_{\alpha,\alpha+1} \lambda_{\alpha,i} \lambda_{\alpha+1,i}, \tag{4.9}$$

and $\Delta(\lambda_1)$ and $\Delta(\lambda_q)$ are Vandermonde determinants.

(ii). One introduces the orthogonal polynomials

$$\xi_n(\lambda_1) = \lambda_1^n + \text{lower powers}, \qquad \eta_n(\lambda_q) = \lambda_q^n + \text{lower powers}$$

which satisfy the orthogonality relations

$$\int d\lambda_1 \dots d\lambda_q \xi_n(\lambda_1) e^{\mu} \eta_m(\lambda_q) = h_n(t,c)\delta_{nm} \tag{4.10}$$

where

$$\mu \equiv \sum_{\alpha=1}^{q} \sum_{r=1}^{\infty} t_{\alpha,r} \lambda_\alpha^r + \sum_{\alpha=1}^{q-1} c_{\alpha,\alpha+1} \lambda_\alpha \lambda_{\alpha+1}. \tag{4.11}$$

(iii). If one expands the Vandermonde determinants in terms of these orthogonal polynomials and using the orthogonality relation (4.10), one can easily calculate the partition function

$$Z_N(t,c) = const\ N! \prod_{i=0}^{N-1} h_i \tag{4.12}$$

Knowing the $h(c,t)$'s amounts to knowing the partition function, up to an N-dependent constant. In turn the information concerning the $h(c,t)$'s can be encoded in suitable *flow equations*, subject to specific conditions, *the coupling conditions*. Before we come to that, however, we recall some necessary notations.

For any matrix M, we define the conjugate $\mathcal{M}$

$$\mathcal{M} = H^{-1}MH, \qquad H_{ij} = h_i\delta_{ij}, \qquad \bar{M}_{ij} = M_{ji}, \qquad M_l(j) \equiv M_{j,j-l}.$$

As usual we introduce the natural gradation

$$deg[E_{ij}] = j - i, \qquad \text{where} \qquad (E_{i,j})_{k,l} = \delta_{i,k}\delta_{j,l}$$

and, for any given matrix M, if all its non–zero elements have degrees in the interval $[a, b]$, then we will simply write: $M \in [a, b]$. Moreover M_+ will denote the upper triangular part of M (including the main diagonal), while $M_- = M - M_+$. We will write

$$\mathrm{Tr}(M) = \sum_{i=0}^{N-1} M_{ii}$$

The latter operation will be referred to as taking the finite trace.

Coupling conditions.

First we introduce the Q–type matrices

$$\int \prod_{\alpha=1}^{q} d\lambda_\alpha \xi_n(\lambda_1) e^{\mu} \lambda_\alpha \eta_m(\lambda_q) \equiv Q_{nm}(\alpha) h_m = \mathcal{Q}_{mn}(\alpha) h_n, \quad \alpha = 1, \ldots, q. \tag{4.13}$$

Among them, $Q(1), \mathcal{Q}(q)$ are Jacobi matrices: their pure upper triangular part is $I_+ = \sum_i E_{i,i+1}$. We will need two P–type matrices, defined by

$$\int \prod_{\alpha=1}^{q} d\lambda_\alpha \Big(\frac{\partial}{\partial \lambda_1} \xi_n(\lambda_1)\Big) e^{\mu} \eta_m(\lambda_q) \equiv P_{nm}(1) h_m \tag{4.14}$$

$$\int d\lambda_1 \ldots d\lambda_q \xi_n(\lambda_1) e^{\mu} \Big(\frac{\partial}{\partial \lambda_q} \eta_m(\lambda_q)\Big) \equiv \mathcal{P}_{mn}(q) h_n \tag{4.15}$$

The matrices (4.13) we introduced above are not completely independent. More precisely all the $Q(\alpha)$'s can be expressed in terms of only one of them and one matrix P. Expressing the trivial fact that the integral of the total derivative of the integrand in eq.(4.10) with respect to $\lambda_\alpha, 1 \leq \alpha \leq q$ vanishes, we can easily derive the constraints or *coupling conditions*

$$P(1) + V_1' + c_{12} Q(2) = 0, \tag{4.16a}$$

$$c_{\alpha-1,\alpha} Q(\alpha - 1) + V_\alpha' + c_{\alpha,\alpha+1} Q(\alpha + 1) = 0, \qquad 2 \leq \alpha \leq q - 1, \tag{4.16b}$$

$$c_{q-1,q} Q(q - 1) + V_q' + \bar{\mathcal{P}}(q) = 0. \tag{4.16c}$$

where we use the notation

$$V_\alpha' = \sum_{r=1}^{p_\alpha} r t_{\alpha,r} Q^{r-1}(\alpha), \qquad \alpha = 1, 2, \ldots, q$$

These conditions explicitly show that the Jacobi matrices depend on the choice of the potentials. In fact they completely determine the degrees of the matrices $Q(\alpha)$. A simple calculation shows that

$$Q(\alpha) \in [-m_\alpha, n_\alpha], \qquad \alpha = 1, 2, \ldots, q$$

where

$$m_1 = (p_q - 1) \ldots (p_3 - 1)(p_2 - 1)$$

$$m_\alpha = (p_q - 1)(p_{q-1} - 1) \ldots (p_{\alpha+1} - 1), \qquad 2 \leq \alpha \leq q - 1$$

$$m_q = 1$$

and

$$
\begin{aligned}
n_1 &= 1 \\
n_\alpha &= (p_{\alpha-1}-1)\dots(p_2-1)(p_1-1), \qquad\qquad 2 \le \alpha \le q-1 \\
n_q &= (p_{q-1}-1)\dots(p_2-1)(p_1-1)
\end{aligned}
$$

Throughout the paper we will refer to the following coordinatization of the Jacobi matrices

$$
Q(1) = I_+ + \sum_i \sum_{l=0}^{m_1} a_l(i) E_{i,i-l}, \qquad\qquad \bar{Q}(q) = I_+ + \sum_i \sum_{l=0}^{m_q} b_l(i) E_{i,i-l} \tag{4.17}
$$

and for the supplementary matrices

$$
Q(\alpha) = \sum_i \sum_{l=-n_\alpha}^{m_\alpha} T_l^{(\alpha)}(i) E_{i,i-l}, \quad 2 \le \alpha \le q-1 \tag{4.18}
$$

Flow equations

The flow equations of the q–matrix model can be expressed by means of the following hierarchies of equations for the matrices $Q(\alpha)$.

$$
\frac{\partial}{\partial t_{\beta,k}} Q(\alpha) = [Q_+^k(\beta), \;\; Q(\alpha)], \qquad 1 \le \beta \le \alpha \tag{4.19a}
$$

$$
\frac{\partial}{\partial t_{\beta,k}} Q(\alpha) = [Q(\alpha), \;\; Q_-^k(\beta)], \qquad \alpha \le \beta \le q \tag{4.19b}
$$

These flows commute and define a multi–component Toda lattice hierarchy, [127],[59].

Reconstruction formulae.

The coupling conditions and the flow equations allow us to calculate the matrix elements of $Q(\alpha)$. From the latter we can reconstruct the partition function as follows. We start from the following main formula

$$
\frac{\partial}{\partial t_{\alpha,r}} \ln Z_N(t,c) = \mathrm{Tr}\Big(Q^r(\alpha)\Big), \qquad 1 \le \alpha \le q \tag{4.20}
$$

It is evident that, by means of the flow equations for $Q(\alpha)$, we can express all the derivatives of $\ln Z_N$ with respect to the couplings $t_{\alpha,k}$ (i.e. all the correlators) as finite traces of commutators of the $Q(\alpha)$'s themselves. In other words, knowing the $Q(\alpha)$'s, we can reconstruct the partition function (up to a constant depending only on N). In particular we can get

$$
\frac{\partial^2}{\partial t_{1,1}\partial t_{\alpha,r}} \ln Z_N(t,c) = \Big(Q^r(\alpha)\Big)_{N,N-1}, \qquad 1 \le \alpha \le q \tag{4.21}
$$

It was already noticed in [53] that this equation leads to the two-dimensional Toda lattice equation.

4.5 Gaussian chain q–matrix models

Let us now concentrate on the most general case (4.11). In particular μ takes the form

$$\mu = \mu(\lambda_1, \ldots, \lambda_q) = \sum_{\alpha=1}^{q} u_\alpha \lambda_\alpha + \sum_{\alpha=1}^{q} t_\alpha \lambda_\alpha^2 + \sum_{\alpha=1}^{q-1} c_\alpha \lambda_\alpha \lambda_{\alpha+1} \tag{4.22}$$

The coupling conditions are

$$P(1) + u_1 + 2t_1 Q(1) + c_1 Q(2) = 0 \tag{4.23a}$$
$$u_\alpha + 2t_\alpha Q(\alpha) + c_\alpha Q(l+1) + c_{\alpha-1} Q(\alpha - 1) = 0, \quad \alpha = 2, \ldots, q-1 \tag{4.23b}$$
$$\bar{\mathcal{P}}(q) + u_q + 2t_q Q(q) + c_{q-1} Q(q-1) = 0 \tag{4.23c}$$

These coupling conditions imply that $Q(\alpha)$ has only three non–vanishing diagonal lines, the main diagonal and the two adjacent lines. Now let us simplify the coordinatization of such matrix as follows

$$Q(\alpha) = \epsilon_+(\alpha) + \epsilon_0(\alpha) + \epsilon_-(\alpha) \tag{4.24}$$

where

$$\epsilon_-(\alpha) = \sum_n g_\alpha(n) E_{n,n-1}, \qquad \epsilon_0(\alpha) = \sum_n s_\alpha(n) E_{n,n}, \qquad \epsilon_+(\alpha) = \sum_n h_\alpha(n) E_{n,n+1}$$

with the understanding that $h_1(n) = 1$ and $g_q(n) = R(n)$. In terms of these coordinates the above coupling equations take the form of the following linear system

$$\begin{aligned} &2t_1 + c_1 h_2(n) = 0 \\ &2t_1 s_1(n) + c_1 s_2(n) + u_1 = 0 \\ &n + 2t_1 g_1(n) + c_1 g_2(n) = 0 \end{aligned} \tag{4.25a}$$

$$\begin{aligned} &2t_\alpha h_\alpha(n) + c_\alpha h_{\alpha+1}(n) + c_{\alpha-1} h_{\alpha-1}(n) = 0, \qquad \alpha = 2, \ldots, q-1 \\ &2t_\alpha s_\alpha(n) + c_\alpha s_{\alpha+1}(n) + c_{\alpha-1} s_{\alpha-1}(n) + u_\alpha = 0, \quad \alpha = 2, \ldots, q-1 \\ &2t_\alpha g_\alpha(n) + c_\alpha g_{\alpha+1}(n) + c_{\alpha-1} g_{\alpha-1}(n) = 0, \qquad \alpha = 2, \ldots, q-1 \end{aligned} \tag{4.25b}$$

$$\begin{aligned} &\frac{n+1}{R(n+1)} + 2t_q h_q(n) + c_{q-1} h_{q-1}(n) = 0 \\ &2t_q s_q(n) + c_{q-1} s_{q-1}(n) = 0 \\ &2t_q R(n) + c_{q-1} g_{q-1}(n) = 0 \end{aligned} \tag{4.25c}$$

The solution of this system is expressed in terms of the matrices X_α and Y_α, defined as follows

$$X_\alpha = \begin{pmatrix} 2t_1 & c_1 & 0 & \ldots & 0 & 0 \\ c_1 & 2t_2 & c_2 & \ldots & 0 & 0 \\ 0 & c_2 & 2t_3 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 2t_{\alpha-1} & c_{\alpha-1} \\ 0 & 0 & 0 & \ldots & c_{\alpha-1} & 2t_\alpha \end{pmatrix} \tag{4.26}$$

and

$$Y_\alpha = \begin{pmatrix} 2t_\alpha & c_\alpha & 0 & \dots & 0 & 0 \\ c_\alpha & 2t_{\alpha+1} & c_{\alpha+1} & \dots & 0 & 0 \\ 0 & c_{\alpha+1} & 2t_{\alpha+2} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2t_{q-1} & c_{q-1} \\ 0 & 0 & 0 & \dots & c_{q-1} & 2t_q \end{pmatrix} \tag{4.27}$$

Of course $Y_1 \equiv X_q$. One finds

$$\begin{aligned} h_\alpha(n) &= (-1)^\alpha (c_1 c_2 \dots c_{\alpha-1})^{-1} \mathrm{Det} X_{\alpha-1} \\ R(n) &= (-1)^q n c_1 c_2 \dots c_{q-1} \left(\mathrm{Det} X_q\right)^{-1} \\ g_\alpha(n) &= (-1)^\alpha n c_1 c_2 \dots c_{\alpha-1} \frac{\mathrm{Det} Y_{\alpha+1}}{\mathrm{Det} X_q} \end{aligned} \tag{4.28}$$

Moreover, if we denote by S and U the vectors $(s_1, s_2, \dots, s_q)^t$ and $(u_1, \dots, u_q)^t$, respectively, we have

$$S = -X_q^{-1} U \tag{4.29}$$

As we have already remarked we can always without loss of generality suppress the linear terms in u_α by constant shifts of M_α. In such a case $S = 0$.

It is now easy to see that, at the cosmological point ($t_\alpha = u_\alpha = 0$), the solution (4.28) is well defined when q is even, while it is singular when q is odd – in the latter case, for example, $\mathrm{Det} X_q = 0$.

In the last part of this section we would like to dispel a seemingly obvious objection to the very content of this paper. Take the generic quadratic model of q matrices with nearest neighbour interactions

$$U = \sum_{\alpha=1}^{q} t_\alpha M_\alpha^2 + \sum_{\alpha=1}^{q-1} c_{\alpha,\alpha+1} M_\alpha M_{\alpha+1} \equiv \sum_{\alpha,\beta}^{q} M_\alpha A_{\alpha\beta} M_\beta. \tag{4.30}$$

The $q \times q$ matrix A is symmetric, and, for the theory of central quadrics, it can be brought to a canonical diagonal form with all ones or minus ones on the diagonal. The signature of A is of course a characteristic of the potential.

Let us see the consequences of this simple remark as far as the matrix model is concerned. The diagonalization of A can be achieved by integrating in the path integral over suitable linear combinations of the matrices M_α, instead of integrating simply over the M_α's. Of course this gives rise to a Jacobian factor, which is however one if one uses only shifts of the M_α's. In this way one brings A to the diagonal form

$$A = \mathrm{Diag}(f_1, \dots, f_q) \tag{4.31}$$

but does not rescale its elements to unity. However this form is sufficient for our subsequent discussion. The initial matrix model appears at this point to be equivalent to the decoupled model with potential

$$U' = \sum_{\alpha} f_\alpha M_\alpha^2.$$

with partition function $Z = \mathrm{Const}(N)(f_1 f_2 ... f_q)^{-N^2/2}$. We remark however that this procedure is of no help if one has to compute correlation functions of composite operators, in that it screws up the definition of the states and renders the calculation of the correlators practically impossible. The procedure followed in this paper, i.e. the use of the generalized Toda lattice hierarchy, has precisely the virtue that it allows the calculation of the exact correlators of significant composite operators.

Finally let us remark that we can easily generalize the results of this subsection to the cases when in the potential are present, beside the terms of (4.30), also interactions of the type $c_{\alpha,\beta} D_\alpha D_\beta$ where $D_\alpha = \mathrm{Diag}\ \mathrm{M}_\alpha$ and $\beta \neq \alpha - 1, \alpha, \alpha + 1$. In such cases the method is the same as in the chain models, with the only difference that the matrices X_α and Y_α will have, at the position (α, β), additional non-vanishing entries $c_{\alpha\beta}$ if the latter are present in the potential.

4.6 Extended q–matrix models

It is important to be able to compute the correlators not only of the states considered above, but also of new states, the *extra states*. To this end we enlarge the q–matrix model by introducing in the potential U new interaction terms, as follows. We change

$$U \to \hat{U} = \sum_{i=1}^{N} \sum_{b_1,...,b_q} g_{b_1,...,b_q} \lambda_{1,i}^{b_1} \dots \lambda_{q,i}^{b_q} \tag{4.32}$$

in (4.5,4.9), and, accordingly,

$$\mu \to \hat{\mu} = \sum_{b_1,...,b_q} g_{b_1,...,b_q} \lambda_1^{b_1} \dots \lambda_q^{b_q} \tag{4.33}$$

in (4.11). Henceforth $a_i, b_i, c_i, ...$ will denote non–negative indices.

We denote by $\chi_{b_1,...,b_q}$ the state specified (classically) by $\sum_{i=1}^N \lambda_{1,i}^{b_1} \dots \lambda_{q,i}^{b_q}$. It is clear that when $b_i = 0$ for all $i \neq \alpha$, this state reduces to τ_{α,b_α}, while the corresponding coupling g boils down to t_{α,b_α}. Moreover the previously introduced bilinear coupling $c_{\alpha,\alpha+1}$ is nothing but the above g when all the $b_i = 0$ except $b_\alpha = b_{\alpha+1} = 1$.

All the couplings and states that do not appear in the original model (4.9) are called *extra*. Exactly as in the original q–matrix model, we can introduce orthogonal monic polynomials $\xi_n(\lambda_1)$ and $\eta_m(\lambda_q)$ and define the $Q(\alpha)$ matrices. This is parallel to what happens in the the extended two-matrix model, [61].

However, unlike the extended two–matrix model, in the extended q–matrix model, we cannot in general define flow equations in matrix form like eqs.(4.19a,4.19b). This is a remarkable difference between extended two– and q–matrix models (with $q > 2$), and, at first sight, seems to spoil integrability and any possibility of exact calculation of the CF's. Fortunately this is not the case. What one has to do is not to calculate the flows of the matrices $Q(\alpha)$, but the multiple derivatives w.r.t. the couplings of $\ln Z_N$, i.e. the multiple derivatives of h_n, and express them in terms of matrices $Q(\alpha)$. One can verify that such 'weak flows' commute, and thus integrability is preserved, although in a weak sense.

The procedure is as follows. We first introduce two series of functions, [53],

$$\xi_n^{(\alpha)}(t,\lambda_\alpha) \equiv \int \prod_{\beta=1}^{\alpha-1} d\lambda_\beta \xi_n(\lambda_1) e^{\mu_\alpha^L}. \tag{4.34}$$

and

$$\eta_n^{(\alpha)}(t,\lambda_\alpha) \equiv \int \prod_{\beta=\alpha+1}^{q} d\lambda_\beta e^{\mu_\alpha^R} \eta_m(\lambda_q). \tag{4.35}$$

where

$$\mu_\alpha^L \equiv \sum_{\beta=1}^{\alpha-1}\sum_{k=1}^{\infty} t_{\beta,k}\lambda_\beta^k + \sum_{\beta=1}^{\alpha-1} c_{\beta,\beta+1}\lambda_\beta\lambda_{\beta+1}.$$

$$\mu_\alpha^R \equiv \sum_{\beta=\alpha+1}^{q}\sum_{k=1}^{\infty} t_{\beta,k}\lambda_\beta^k + \sum_{\beta=\alpha}^{q-1} c_{\beta,\beta+1}\lambda_\beta\lambda_{\beta+1}.$$

Obviously we have

$$\xi_n^{(1)}(t,\lambda_1) = \xi_n(\lambda_1), \qquad \eta_n^{(q)}(t,\lambda_q) = \eta_m(\lambda_q).$$

but for other values of α one sees immediately that $\xi^{(\alpha)}$ and $\eta^{(\alpha)}$ are not polynomials. But they satisfy the orthogonality relations

$$\int d\lambda_\alpha \xi_n^{(\alpha)}(t,\lambda_\alpha) e^{V_\alpha(\lambda_\alpha)} \eta_m^{(\alpha)}(t,\lambda_\alpha) = \delta_{nm} h_n(t,c), \qquad 1 \leq \alpha \leq q. \tag{4.36}$$

Eq.(4.13) provides a definition of the $Q(\alpha)$ matrix in this basis

$$\int d\lambda_\alpha \xi_n^{(\alpha)}(t,\lambda_\alpha)\lambda_\alpha e^{V_\alpha(\lambda_\alpha)} \eta_m^{(\alpha)}(t,\lambda_\alpha) = Q_{nm}(\alpha) h_m(t,c), \qquad \forall 1 \leq \alpha \leq q. \tag{4.37}$$

Therefore the spectral equations follow

$$\lambda_\alpha \xi^{(\alpha)} = Q(\alpha)\xi^{(\alpha)}, \qquad 1 \leq \alpha \leq q. \tag{4.38}$$
$$\lambda_\alpha \eta^{(\alpha)} = \bar{Q}(\alpha)\eta^{(\alpha)}, \qquad 1 \leq \alpha \leq q. \tag{4.39}$$

where ξ^α and η^α represent the infinite vectors with components $\xi_0^\alpha, \xi_1^\alpha, \dots$ and $\eta_0^\alpha, \eta_1^\alpha, \dots$, respectively.

With these bases at hand one differentiates h_n, i.e. (4.10) for $n = m$, w.r.t the appropriate couplings and evaluate the results when the extra couplings vanish. The result contains derivatives of ξ_n and η_n w.r.t to the couplings, which in turn can be evaluated differentiating (4.10) with $n > m$ or $n < m$. Finally one can express the result in terms of elements of the matrices $Q(\alpha)$, by making use of the above defined bases ξ_n^α and η_m^α. Inserting this into the expressions of the correlators, i.e. into the derivatives of $\ln Z_N$ w.r.t. the appropriate couplings, one can express the latter in terms of finite traces of polynomials in the $Q(\alpha)$'s.

In Appendix A we write down the explicit expressions of the correlation functions up to the 3 coupled operators.

4.7 Star-matrix models

4.7.1 Modified orthogonal polynomials method

In this section we show that it is possible to apply a modified Q-matrix approach and to define consistently the Q-matrices. The method gives consistent results for the cases where other methods can be applied :the saddle-point method, Schwinger-Dyson approach, gaussian integration etc.

The Q-matrix approach was introduced for the 1-matrix model [65] and further developed for the two-matrix and the chain multi-matrix models in [53]. Here we are applying this approach to another class of multi-matrix models – the star-matrix models.

The partition function of the q-star model is given by:

$$Z = \int \prod_{\alpha=1}^{q} dM_\alpha dM_0 \exp(V_0 + \sum_{\alpha}^{q} V_\alpha + M_0 \sum_{\alpha=1}^{q} c_\alpha M_\alpha) \tag{4.40}$$

with the potentials $V_\alpha = \sum_{r=1}^{p_\alpha} t_{\alpha,r} M_\alpha^r, \alpha = 0, 1 \ldots q$. It is possible to integrate over the angular degrees of freedom and to remain with integrals only over the eigenvalues:

$$Z = \int \prod_{i=1}^{N} (d\lambda_{0,i} \prod_{\alpha=1}^{q} d\lambda_{\alpha,i})(\Delta(\lambda_0))^{2-q} \prod_{\alpha=1}^{q} \Delta(\lambda_\alpha) e^V \tag{4.41}$$

with

$$V = \sum_{i=1}^{N} (V_0(\lambda_{0,i}) + \sum_{\alpha=1}^{q} V_\alpha(\lambda_{\alpha,i}) + \lambda_{0,i} \sum_{\alpha=1}^{q} c_\alpha \lambda_{\alpha,i})$$

We define the orthogonal functions basis as ξ_n and $q+1$-conjugate functions $\eta_{\alpha,m}$:

$$\int d\lambda_0 \prod_{\alpha=1}^{q} d\lambda_\alpha \xi_n^q(\lambda_0) e^V \prod_{\alpha=1}^{q} \eta_{\alpha,m_\alpha}(\lambda_\alpha) = h_n \delta_{nm} \prod_{\alpha=1}^{q} \delta_{m,m_\alpha} \tag{4.42}$$

The basic functions have the property:

$$\frac{h_{o,m}}{h_{\alpha,m}} \eta_{0,m}(\lambda_0) = \int d\lambda_\alpha e^{V_\alpha + c_\alpha \lambda_0 \lambda_\alpha} \eta_{\alpha,m}(\lambda_\alpha) \tag{4.43}$$

Introducing this relation (4.43) in the orthogonality condition (4.42) we get:

$$\int d\lambda_0 \xi^q_{0,n}(\lambda_0) e^{V_0} \prod_{\alpha=1}^{q} \eta^q_{0,m_\alpha}(\lambda_0) = h_n \prod_{\alpha=1}^{q} \delta_{n,m_\alpha} \tag{4.44}$$

Because the basic functions $\xi_{0,n}(\lambda_0), \eta_{0,n}(\lambda_0)$ at power q are linear combinations of basic functions:

$$\xi^q_{0,n}(\lambda_0) = \xi_{0,nq}(\lambda_0) + \sum_{k=1}^{nq} a_k \xi_{0,nq}(\lambda_0),\ \eta^q_{0,n}(\lambda_0) = \eta_{0,nq}(\lambda_0) + \sum_{k=1}^{nq} \overline{a}_k \eta_{0,nq}(\lambda_0) \tag{4.45}$$

we can show that the integral (4.44) follows from the orthogonal condition :

$$\int d\lambda_o \xi_{0,n}(\lambda_0) e^{V_0} \eta_{0,m}(\lambda_0) = h_n \delta_{n,m} \tag{4.46}$$

and:

$$h_n = n_{0,nq} + \sum_{k=1}^{nq} |a_k|^2 h_{0,nq-k}$$

Inserting the expression of $\eta_{0,m}$ (4.43) in the relation (4.44)we get the orthogonal condition:

$$\int d\lambda_0 d\lambda_\alpha \xi_{0,n}(\lambda_0) e^{V_0+V_\alpha+c_\alpha\lambda_0\lambda_\alpha} \eta^q_{\alpha,m}(\lambda_\alpha) = h_{\alpha,n}\delta_{n,m} \tag{4.47}$$

The property (4.43) gives the possibility to integrate over Vandermonde determinants:

$$\prod_{i=0}^{N-1} \frac{h_{\alpha,i}}{h_{0,i}} \mathrm{Det}_{ij}[\eta_{0,i}(\lambda_{0,j})] = \int d\lambda_\alpha e^{V_\alpha+c_\alpha\lambda_0\lambda_\alpha} \mathrm{Det}_{ij}[\eta_{\alpha,i}(\lambda_{\alpha,j})]$$

and permits to calculate the partition function:

$$Z = const N! \prod_{i=0}^{N-1} h_{0,i}^{1-q} \left(\prod_{\alpha=1}^{q} h_{\alpha,i} \right) \tag{4.48}$$

We introduce the Q-matrices as:

$$\int d\lambda_0 \prod_{\alpha=1}^{q} d\lambda_\alpha \xi^q_n(\lambda_0) \lambda_\alpha e^V \prod_{\alpha=1}^{q} \eta_{\alpha,m_\alpha}(\lambda_\alpha) = h_n Q_{\alpha,nm} \prod_{\alpha=1}^{q} \delta_{m,m_\alpha} \tag{4.49}$$

and the P-matrices as:

$$\int d\lambda_0 \prod_{\alpha=1}^{q} d\lambda_\alpha \xi^{q-1}_n(\lambda_0) \frac{\partial}{\partial\lambda_0} \left(\xi_n(\lambda_0) e^{V_0+\lambda_0 \sum_{\alpha=1}^{q}} \right) e^{\sum_{\alpha=1}^{q} V_\alpha} \prod_{\alpha=1}^{q} \eta_{\alpha,m_\alpha}(\lambda_\alpha) = \tag{4.50}$$

$$= P_{0,nm} h_m \prod_{\alpha=1}^{q} \delta_{m,m_\alpha}$$

$$\int d\lambda_0 \prod_{\alpha=1}^{q} d\lambda_\alpha \xi_n^q(\lambda_0) e^{V-V_\alpha-c_\alpha\lambda_0\lambda_\alpha} \prod_{\alpha=1}^{q} \frac{\partial}{\partial\lambda_\beta}\left(\eta_{\beta,m_\beta}(\lambda_\beta)e^{V_\alpha+c_\alpha\lambda_0\lambda_\alpha}\right) \prod_{\alpha\neq 0,\beta} \eta_{\alpha,m_\alpha}(\lambda_\alpha) =$$
$$= h_n P_{\beta,nm} \prod_{\alpha=1}^{q} \delta_{m,m_\alpha}$$

We can now derive the coupling conditions:

$$\begin{aligned} qP_0 + V_0'(Q_0) + \sum_{\alpha=1}^{q} c_\alpha Q_\alpha &= 0 \\ \overline{P}_\alpha + V_\alpha'(Q_\alpha) + c_\alpha Q_0 &= 0, \quad \alpha = 1, \dots q \end{aligned} \tag{4.51}$$

We consider only the symmetric case when the order of potentials is $p_\alpha = p$. The calculation of degree for matrices gives:

$$\begin{aligned} Q_\alpha &\in [-m, n], \quad \text{for all } \alpha \\ Q_0 &\in [-m_0, n_0], \end{aligned} \tag{4.52}$$

where $m = 1, n = p_0 - 1; m_0 = p - 1, n_0 = 1$.

The notation shows that we have Q_α-matrices with finite band with m lower and n higher diagonals.

We consider only 1-point correlation functions, hence we do not need the flow equations. The 1-point correlation functions can be calculated in the usual way as:

$$< \mathrm{Tr} M_1^{k_1} \dots M_q^{k_q} >= \mathrm{Tr}(Q_1^{k_1} \dots Q_q^{k_q}) \tag{4.53}$$

These 1-point correlation functions can be calculated in every genus h and in principle for arbitrary potentials, not only gaussian.

4.7.2 Gaussian star-matrix models

After the exact solution of $c = 1$ matrix model (or chain q-multimatrix model) with gaussian potential, the star q-multimatrix model is an obvious target for study. Even with a trivial gaussian potential, it could give an interesting string description. For example, due to the additional permutation symmetry S_q, the tachyonic field structure can be quite different from the original chain model.The 3-star matrix model is also the first one in the class of matrix models having the target space- the D_n Dynkin diagramm [66][56].

The gaussian star-matrix model has the partition function:

$$Z = \int \prod_{\alpha=1}^{q} dM_\alpha dM_0 \exp[t_0 M_0^2 + u_0 M_0 + M_0 \sum_{\alpha=1}^{q} c_\alpha M_\alpha + \sum_{\alpha=1}^{q}(t_\alpha M_\alpha^2 + u_\alpha M_\alpha)] \tag{4.54}$$

The coupling conditions are:

$$\begin{aligned} qP_0 + 2t_0Q_0 + u_0 + \sum_{\alpha=1}^{q} c_\alpha Q_\alpha &= 0 \\ \overline{P}_\alpha + 2t_\alpha Q_\alpha + u_\alpha + c_\alpha Q_0 &= 0, \quad \alpha = 1, \dots q \end{aligned} \tag{4.55}$$

With the following parametrization of Q-matrices:

$$\begin{aligned} Q_0 &= I_+ + a_0 I_0 + a_1 \epsilon_- \\ Q_\alpha &= b_\alpha / R_\alpha I_+ + d_\alpha I_0 + R_\alpha \epsilon_-, \quad \alpha = 1, \dots q \end{aligned} \tag{4.56}$$

we arrive at following equations:

$$\begin{aligned} 2t_\alpha R_\alpha + c_\alpha a_1 = 0 \\ 2t_\alpha b_\alpha + n + c_\alpha R_\alpha = 0 \\ 2t_\alpha d_\alpha + u_\alpha + c_\alpha a_0 = 0 \\ 2t_0 + \sum \frac{c_\alpha b_\alpha}{R_\alpha} = 0 \\ 2t_0 a_0 + u_0 + \sum c_\alpha d_\alpha = 0 \\ 2t_0 a_1 + qn + \sum c_\alpha R_\alpha = 0 \end{aligned} \tag{4.57}$$

Solving the coupling conditions we get :

$$\begin{aligned} a_1 &= -\frac{2q}{A}, a_0 = \frac{1}{A}(\sum \frac{c_\alpha u_\alpha}{t_\alpha} - 2u_0) \\ b_\alpha &= -\frac{1}{2t_\alpha^2}(\frac{c_\alpha^2 q}{A} + t_\alpha), R_\alpha = \frac{c_\alpha q}{t_\alpha A} \\ d_\alpha &= \frac{1}{At_\alpha}(c_\alpha u_0 - 2t_0 u_\alpha + u_\alpha \sum \frac{c_\beta^2}{2t_\beta} - c_\alpha \sum \frac{c_\beta u_\beta}{2t_\beta}) \end{aligned}$$

where $A = 4t_0 - \sum c_\alpha^2 / t_\alpha$.

For quadratic potentials V_α, the basic functions are Hermite polynomials:

$$\begin{aligned} \xi_n(\lambda_0) &= \eta_n(\lambda_0) = H_n(\frac{\lambda_0 - a_0}{\sqrt{2a_1}}), \\ \eta_{\alpha,m}(\lambda_\alpha) &= H_n(\frac{\lambda_\alpha - d_\alpha}{\sqrt{2b_\alpha}}), \alpha = 1 \dots q \end{aligned} \tag{4.58}$$

To calculate the partition function we observe that :

$$h_{0,n} = \frac{1}{A}, h_{\alpha,n} = R_\alpha^n \tag{4.59}$$

giving:

$$Z = const \left(\frac{1}{A} \prod_{\alpha=1}^{q} \frac{c_\alpha}{t_\alpha} \right)^{\frac{N^2}{2}} \tag{4.60}$$

const in our case is the exponent $\exp[-1/4(Aa_0^2 + \sum_\alpha u_\alpha^2/t_\alpha)]$ obtained by shifting the matrices M_α so that the linear terms in the potential vanish.

The partition function and the first simplest 1-point correlation functions as $\mathrm{Tr}Q_\alpha^2$. $\mathrm{Tr}Q_\alpha$ can be calculated by direct integration of the original integral (4.54). This is not the case for more complicated 1-point correlation functions as those given by (4.53).Instead with the Q-matrix approach this is easy , using the explicit form of Q-matrices.In the Dyson-Schwinger approach ,this calculation is also possible but only on the sphere.

As an example we give the result for the 1-point correlation function (when $u_0 = u_\alpha = 0$ and $a_0 = d_\alpha = 0$):

$$\mathrm{Tr}Q_\alpha^n Q_\beta^m = \sum_{2k=0}^{n} \sum_{2l=0}^{m} \sum_{i=0}^{n-2k} \sum_{j=0}^{m-2l} \frac{n!m!(-1)^k 2^{-(l+k)}}{i!j!k!l!(n-2k-i)!(m-2l-j)!} \tag{4.61}$$

$$b_\alpha^{n-(i+k)} b_\beta^{m-(j+l)} R_\alpha^{2(i+k)-n} R_\beta^{2(j+l)-m} (i+j)! \begin{pmatrix} N+i+j \\ i+j+1 \end{pmatrix} \delta(\frac{n+m}{2} - (l+k), i+j)$$

In the large N limit we have $l+k=0$ or $l=0, k=0, i+j = \frac{n+m}{2}$ and the previous formula simplifies to:

$$\mathrm{Tr}Q_\alpha^n Q_\beta^m = \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{n!m!}{i!j!(n-i)!(m-j)!} b_\alpha^{n-i} b_\beta^{m-j} R_\alpha^{2i-n} R_\beta^{2j-m} \frac{N^{\frac{n+m}{2}+1}}{\frac{n+m}{2}+1} \tag{4.62}$$

For $m = 0$ and $n = 2r$ we obtain the correlation function in genus 0:

$$\mathrm{Tr}Q_\alpha^{2r} = \frac{(2r)!}{r!(r+1)!} N^{r+1} (b_\alpha)^r \tag{4.63}$$

We have sum over $i = r, j = 0$.

4.8 Orthogonal polynomials in the 2-matrix model

In this section we derive the correlation functions for the 2-matrix models, relying on the expressions for the Q-matrices. This exercise helps to understand better the connection between the Q-matrix approach and classical method of Schwinger-Dyson equations.

4.8.1 Quadratic potential

The form of the Q-matrices is:

$$Q_\alpha = h_\alpha I_+ + g_\alpha \epsilon_- \quad \alpha = 1, 2 \tag{4.64}$$

-with

$$h_1 = 1, \quad g_1 = -\frac{2s_2}{4t_2s_2 - c^2}, \qquad h_2 = -\frac{2t_2}{c}, \quad g_2 = \frac{c}{4t_2s_2 - c^2}$$

We derive in detail the 1-point correlation functions in Appendix A. We start with the one-point function $\left\langle \mathrm{Tr} M_1^{2k} \right\rangle$ (the odd powers vanish):

$$\begin{aligned} \left\langle \mathrm{Tr} M_1^{2k} \right\rangle &= \mathrm{Tr}(Q_1^{2k}) = \alpha_1^k \gamma_1^k \mathrm{Tr}(I_+ + \epsilon_-)^{2k} \\ &= \sum_{l=0}^{2k} \left[\frac{(2k)! 2^{-l} N(N-1)...(N-k+l)}{l!(k-l)!(k-l+1)!} \right] \left(\frac{2s_2}{4t_2s_2 - c^2} \right)^k \end{aligned} \tag{4.65}$$

as one sees, the leading N component is when $l = 0$:

$$< \mathrm{Tr} M_1^{2k} >_0 = \frac{(2k)!}{k!(k+1)!} \left(\frac{2s_2}{4t_2s_2 - c^2} \right)^k N^{k+1}$$

It agrees with $W_{2k,0} N^{k+1}$.

For the two-point function $\left\langle \mathrm{Tr} M_1^k M_2^{k'} \right\rangle$ we take the formula:

$$\left\langle \mathrm{Tr} M_1^k M_2^{k'} \right\rangle = \mathrm{Tr}(Q_1^k Q_2^{k'}) \tag{4.66}$$

Is not clear that this gives the same result of the SD equations, because $\mathrm{Tr}(Q_1^k Q_2^{k'})$ is the correlation function for diagonal-like interaction ($\left\langle \mathrm{Tr} D_1^k D_2^{k'} \right\rangle$, $D = \mathrm{Diag}(M)$).

In fact, we show that only at the critical point when $4t_2s_2 = c^2$, the two different definitions coincide.

Using the form of Q-matrices we can calculate the 2-point correlation functions:

$$\mathrm{Tr} Q_1^n Q_2^m = \sum_{2k=0}^{n} \sum_{2l=0}^{m} \sum_{i=0}^{n-2k} \sum_{j=0}^{m-2l} \frac{n!m!(-1)^k 2^{-(l+k)}}{i!j!k!l!(n-2k-i)!(m-2l-j)!} \left(\frac{4s_2t_2}{c^2} \right)^{i+k}$$
$$(2t_2)^{\frac{m-n}{2}} c^n (4t_2s_2 - c^2)^{-\frac{m+n}{2}} (i+j) \begin{pmatrix} N \\ i+j+1 \end{pmatrix} \delta\left(\frac{n+m}{2} - (l+k), i+j \right) \tag{4.67}$$

In the large N limit we have $l + k = 0$ or $l = 0, k = 0, i + j = \frac{n+m}{2}$ and the previous formula simplifies to:

$$\mathrm{Tr} Q_1^n Q_2^m = \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{n!m!}{i!j!(n-i)!(m-j)!} \left(\frac{4s_2t_2}{c^2} \right)^i (2t_2)^{\frac{m-n}{2}} c^n (4t_2s_2 - c^2)^{-\frac{m+n}{2}} \frac{N^{\frac{n+m}{2}+1}}{\frac{n+m}{2} + 1}$$

For the first simple cases we have the explicit formulas:

$$\begin{aligned} \mathrm{Tr} Q_1^{2k+1} Q_2 &= \frac{(2k+1)!}{k!(k+2)!} N^{k+2} \frac{(2s_2)^k}{(4s_2t_2 - c^2)^{k+1}} \left(c + \frac{4t_2s_2}{c} \right) \\ &= \frac{1}{2} \left(\frac{c}{2s_2} + \frac{2t_2}{c} \right) W_{2k+2} N^{k+2} \end{aligned} \tag{4.68}$$

and:

$$\mathrm{Tr}Q_1^{2k}Q_2^2 = \frac{(2k)!}{k!(k+1)!}N^{k+2}\frac{(2s_2)^{k-1}}{(4s_2t_2-c^2)^{k+1}}\frac{1}{k+2}\left[k\left(\frac{c}{2s_2}+\frac{2t_2}{c}\right)^2+8t_2s_2\right]$$

The interesting point is that the correlation functions coincide with the ones of the Schwinger-Dyson method only if $4t_2s_2 = c^2$. The condition $4t_2s_2 = c^2$ gives us a critical point where the correlation functions have a singular behaviour.

The more complicated case is the correlation function $\mathrm{Tr}(Q_1^kQ_2^lQ_1^{k'}Q_2^{l'})$ which can be calculated by the same method.

4.8.2 The cubic potential

The Q-matrices are:

$$Q_1 = I_+ + a_0I_0 + a_1I_- + a_2I_{-2}, \overline{Q}_2 = I_+ + b_0I_0 + b_1I_- + b_2I_{-2} \tag{4.69}$$

In the limit of large N the coefficients can be expressed in terms of only three variables $R, X_1 = 3t_3a_0 + t_2, X_2 = 3s_3b_0 + s_2$:

$$\begin{aligned} a_2 &= \frac{3s_3}{c}R^2, b_2 = \frac{3t_3}{c}R^2 \\ a_1 &= \frac{2R}{c}X_2, b_1 = \frac{2R}{c}X_1 \end{aligned} \tag{4.70}$$

These three variables satisfy the system:

$$\begin{aligned} X_1X_2 &= \frac{1}{4}(c^2 - \frac{xc}{R}) \\ \frac{X_1^2}{3t_3} - (\frac{c}{3s_3} - \frac{12t_3R}{c})X_2 &= \frac{t_2^2}{3t_3} - \frac{cs_2}{3s_3} \\ \frac{X_2^2}{3s_3} - (\frac{c}{3t_3} - \frac{12s_3R}{c})X_1 &= \frac{s_2^2}{3s_3} - \frac{ct_2}{3t_3} \end{aligned} \tag{4.71}$$

Taking into account that:

$$\begin{aligned} W_1 &= \mathrm{Tr}Q_1 = a_0, W_{0,1} = \mathrm{Tr}Q_2 = b_0, \\ W_2 &= \mathrm{Tr}Q_1^2 = a_0^2 + a_1, W_{0,2} = \mathrm{Tr}Q_2^2 = b_0^2 + b_1, \\ W_3 &= \mathrm{Tr}Q_1^3 = a_0^3 + 6a_0a_1 + a_2, W_{0,3} = \mathrm{Tr}Q_2^3 = b_0^3 + 6b_0b_1 + b_2, \\ W_{1,1} &= \mathrm{Tr}Q_1Q_2 = a_0b_0 + a_1 + b_1 \end{aligned} \tag{4.72}$$

we can show that the last two equations of the system (4.72) are exactly:

$$\begin{aligned} 3t_3W_2 + 2t_2W_1 - cW_{0,1} &= 0 \\ 3s_3W_{0,2} + 2s_2W_{0,1} - cW_1 &= 0 \end{aligned} \tag{4.73}$$

which are the first Schwinger-Dyson equations satisfied by correlation functions.

We can calculate the 1-point and 2-point correlation functions, by taking the trace of the matrices Q_1 and Q_2. In this way we can reobtain the results for the 2-matrix models with cubic potential in the Schwinger-Dyson approach [67].

4.9 Schwinger-Dyson equations

Schwinger-Dyson equations arise from the invariance of the matrix integrals under reparametrization of the matrix variables $M \to f(M)$. We consider the two-matrix model with partition function:

$$Z = \int DM_1 DM_2 \exp(-\mathrm{Tr}(V_1(M_1) + V_2(M_2) + cM_1M_2)) \tag{4.74}$$

where the potentials are:

$$V_1(M_1) = \sum_{k\geq 1} t_k M_1^k, V_2(M_2) = \sum_{k\geq 1} s_k M_2^k$$

The partition function is invariant under the infinitesimal transformations of the matrices:

$$M_1 \to M_1 + \varepsilon_{1,n} M_1^{n+1}, \quad n \geq -1, \varepsilon_{1,n} \text{ infinitesimal}$$
$$M_2 \to M_2 + \varepsilon_{2,n} M_2^{n+1}, \quad n \geq -1, \varepsilon_{2,n} \text{ infinitesimal}$$

Taking into account the contributions from the measure that transforms as:

$$DM_1 \to \begin{cases} DM_1(1+\varepsilon_{1,-1}Nt_1) & (n=-1), \\ DM_1(1+\varepsilon_{1,0}N(N+1)/2) & (n=0), \\ DM_1[1+\varepsilon_{1,n}((N+\frac{n+1}{2}\mathrm{Tr}M_1^n + \sum_{k=0}^{n}\mathrm{Tr}M_1^k\mathrm{Tr}M_1^{n-k})] & (n\geq 1), \end{cases} \tag{4.75}$$

and the transformed potential term:

$$\exp(V_1(M_1)) \to (1 - \varepsilon_{1,n}\mathrm{Tr}V_1'(M_1)M_1^{n+1})\exp(V_1(M_1))$$

we obtain the Dyson-Schwinger equations (for $c = 0$):

$$\mathcal{L}_n^{[1]}(1)Z(t_k, s_k) = 0 \tag{4.76}$$

with:

$$\mathcal{L}_n^{[1]} = \sum_{k=1}^{\infty} kt_k \frac{\partial}{\partial t_{k+n}} + (N + \frac{n+1}{2})\frac{\partial}{\partial t_n} + \sum_{k=1}^{n-1}\frac{1}{2}\frac{\partial}{\partial t_k \partial t_{n-k}}$$
$$+Nt_1\delta_{n,-1} + N(N+1)/2\ \delta_{n,0}, \quad n \geq -1$$

If we take the analitical transformation of general form we get the W_n constraints for the two-matrix model.

$$W_n^{[r]}Z(t_k, s_k, c) = 0, \tilde{W}_n^{[r]}Z(t_k, s_k, c) = 0, \tag{4.77}$$

with

$$W_n^{[r]} = (-c)^n\mathcal{L}_n^{[r]}(1) - \mathcal{L}_n^{[r+n]}(2),$$
$$\tilde{W}_n^{[r]} = (-c)^n\mathcal{L}_n^{[r]}(2) - \mathcal{L}_n^{[r+n]}(1), \tag{4.78}$$

The partition function is also invariant under mixed infinitesimal transformations of the matrices:

$$M_i \to M_i + \varepsilon_{i,nm}M_1^{n+1}M_2^{m+1}, \quad n,m \geq -1, \qquad \varepsilon_{i,nm}, i = 1,2 \text{ infinitesimal} \tag{4.79}$$

4.10 Schwinger-Dyson equations in 2-matrix models

4.10.1 Quadratic potential

For the quadratic action $S = t_2 M_1^2 + s_2 M_2^2 - cM_1 M_2$ (we can always make a shift in M_1, M_2 to put $t_1 = s_1 = 0$) we have the following Schwinger-Dyson equations:

$$\begin{aligned}
2t_2 W_n - cW_{n-1,1} &= \sum_{j=0}^{n-2} W_j W_{n-j-2} \\
2s_2 W_{n,1} - cW_{n+1} &= 0 \\
2t_2 W_{n,1} - cW_{n-1,2} &= \sum_{j=0}^{n-2} W_j W_{n-j-2,1} \\
2t_2 W_{n,m} - cW_{n-1,m+1} &= \sum_{j=0}^{n-2} W_j W_{n-j-2,m}
\end{aligned} \tag{4.80}$$

(Where $W_{n,m} = \langle \mathrm{Tr} M_1^n M_2^m \rangle$ and $W_n = W_{n,0}$). From the first two equations we get :

$$(4t_2 s_2 - c^2) W_n = 2s_2 (W_0 W_{n-2} + \ldots W_{n-2} W_0) \tag{4.81}$$

We introduce the generating function for correlation functions W_n:

$$G(t) = \sum_{k=0}^{\infty} W_k t^k$$

Using eq. (4.81) the generating function satisfies (we set $W_0 = 1$):

$$t^2 \frac{2s_2}{4t_2 s_2 - c^2} G(t)^2 = G(t) - 1 \tag{4.82}$$

with the solution:

$$\begin{aligned}
G(t) = \frac{1}{2t^2} \frac{4t_2 s_2 - c^2}{2s_2} \left(1 - \sqrt{1 - 4t^2 \left(\frac{2s_2}{4t_2 s_2 - c^2}\right)}\right) &= \\
= \sum_{n=0}^{\infty} \frac{(2n)!}{n!(n+1)!} \left(\frac{2s_2}{4t_2 s_2 - c^2}\right)^n t^{2n}
\end{aligned} \tag{4.83}$$

from which we get the non-zero correlation functions:

$$W_{2k,0} =< \mathrm{Tr} M_1^{2k} >= \frac{(2k)!}{k!(k+1)!} \left(\frac{2s_2}{4t_2 s_2 - c^2}\right)^k \tag{4.84}$$

The same for :

$$W_{0,2k} =< \mathrm{Tr} M_2^{2k} >= \frac{(2k)!}{k!(k+1)!} \left(\frac{2t_2}{4t_2 s_2 - c^2}\right)^k . \tag{4.85}$$

From the second, third and fourth eqns. (4.80) we get the $\langle M_1^n M_2 \rangle$, $\langle M_1^n M_2^2 \rangle$ and $\langle M_1^n M_2^3 \rangle$ correlation functions:

$$\begin{aligned} W_{2k+1,1} &= \frac{c}{2s_2} W_{2k+2} \\ W_{2k,1} &= 0 \end{aligned} \tag{4.86}$$

$$\begin{aligned} W_{2k,2} &= \frac{c^2}{4s_2^2} W_{2k+2} + \frac{1}{2s_2} W_0 W_{2k} \\ &= \left(c^2 \frac{3k}{k+2} + 4t_2 s_2\right) \frac{(2k)!}{k!(k+1)!} \frac{(2s_2)^{k-1}}{(4t_2 s_2 - c^2)^{k+1}} \\ W_{2k+1,2} &= 0 \end{aligned} \tag{4.87}$$

$$\begin{aligned} W_{2k-1,3} &= (\frac{c}{2s_2})^3 W_{2k+2} + \frac{c}{2s_2^2} W_0 W_{2k} \\ &= \left(c^2 \frac{k-1}{k+2} + 4t_2 s_2\right) \frac{(2k)!}{k!(k+1)!} \frac{(2s_2)^{k-1}}{(4t_2 s_2 - c^2)^{k+1}} \frac{c}{2s_2} \\ W_{2k,3} &= 0 \end{aligned} \tag{4.88}$$

Also the two point functions can be incorporated in a generating function:

$$G(t,s) = \sum W_{k,l} t^k s^l$$

which can solve the fourth eq. (4.80):

$$G(t,s) = \frac{2t_2 s G(s) - ct G(t)}{2t_2 s - ct - t^2 s G(t)} \tag{4.89}$$

The symmetry $(s \leftrightarrow t)$ is not manifest but can be proved by means of the identity (4.82) for $G(t)$ and $G(s)$.

The Schwinger-Dyson equations for the four-point functions $W_{n,m;n',m'} = < \mathrm{Tr} M_1^n M_2^m M_1^{n'} M_2^{m'} >$ are:

$$\begin{aligned} 2t_2 W_{n,m;n',m'} - cW_{n-1,m+1;n',m'} &= \sum_{j=0}^{n-2} W_j W_{n-2-j,m;n',m'} + \sum_{j=0}^{n'-1} W_{j,m} W_{n+n'-2-j,m'} \\ 2s_2 W_{n',m+m'+1} - cW_{1,m;n',m'} &= \sum_{j=0}^{m-1} W_j W_{n',m+m'-1-j} + \sum_{j=0}^{m'-1} W_j W_{n',m+m'-1-j} \end{aligned} \tag{4.90}$$

In this case the generating function of four-point functions is:

$$G(t,s;t',s') = \sum W_{n,m;n',m'} t^n s^m (t')^{m'} (s')^{n'} == \frac{1}{2t_2 s - ct - t^2 s G(t)} \times$$
$$\left[(c - t' s G(t',s)) \frac{tt' G(t',s') - t^2 G(t,s')}{t - t'} - 2t_2 \frac{ss' G(t',s') - s^2 G(t',s)}{s - s'} \right]$$

The first four-point non-trivial correlation function is :

$$W_{1,1;1,1} = \langle \mathrm{Tr}(M_1 M_2 M_1 M_2) \rangle = \frac{c}{2t_2} W_{1,3} = \frac{2c^2}{(4t_2 s_2 - c^2)^2} \tag{4.91}$$

to be compared with :

$$W_{2,2} = \left\langle \mathrm{Tr}(M_1^2 M_2^2) \right\rangle = \frac{4t_2 s_2 + c^2}{(4t_2 s_2 - c^2)^2} \tag{4.92}$$

4.10.2 Cubic potential

The action is $S = t_3 M_2^3 + t_2 M_1^2 + s_3 M_1^3 + s_2 M_2^2 - c M_1 M_2$ (we can always make a shift in M_1, M_2 to put $t_1 = s_1 = 0$) we have the following Schwinger-Dyson equations:

$$\begin{aligned}
3t_3 W_{n+1} + 2t_2 W_n - c W_{n-1,1} &= \sum_{j=0}^{n-2} W_j W_{n-j-2} \\
3s_3 W_{n,2} + 2s_2 W_{n,1} - c W_{n+1} &= 0 \\
3t_3 W_{n+1,1} + 2t_2 W_{n,1} - c W_{n-1,2} &= \sum_{j=0}^{n-2} W_j W_{n-j-2,1} \\
3t_3 W_{n+1,m} + 2t_2 W_{n,m} - c W_{n-1,m+1} &= \sum_{j=0}^{n-2} W_j W_{n-j-2,m}
\end{aligned} \tag{4.93}$$

Writing in terms of the generating function, we get an algebraic third order equation in $G(t)$:

$$t^6 G^3(t) - (6t_3 + 4t_2 t + \frac{2s_2 c}{3s_3} t^2) t^3 G^2(t) + \alpha G(t) - \beta = 0 \tag{4.94}$$

with:

$$\begin{aligned}
\alpha &= (3t_3 + 2t_2 t)(3t_3 + 2t_2 t + \frac{2s_2 c}{3s_3} t^2) - \frac{c^3}{3s_3} t^3 + (3t_3 W_1 + 2t_2) t^4 + 3t_3 t^3 \\
\beta &= ((3t_3 W_1 + 2t_2) t + 3t_3)(3t_3 + 2t_2 t + \frac{2s_2 c}{3s_3} t^2) - \frac{c^3}{3s_3} t^2 + \\
&\quad + (3t_3 W_{1,1} + 2t_2 W_1) c t^3 + 3t_3 c W_1 t^2
\end{aligned}$$

We recover the generating function for quadratic potential in the limit $3s_3, 3t_3 \to 0$ if we pick up only the singular terms proportional with $1/(3s_3)$.

We can express $W_{1,1}$ in terms of W_2, W_3:

$$c W_{1,1} = 3t_3 W_3 + 2t_2 W_2 - 1$$

From the fourth eq. 4.94 we get the generating function for the 2-point correlation functions:

$$G(t,s) = \frac{(3t_3 + 2t_2 t) s G(s) + 3t_3 s G_1(s) - c s t^2 G(t)}{(3t_3 + 2t_2 t) s - c t^2 - t^3 s G(t)} \tag{4.95}$$

where:

$$G_m(t) = \frac{1}{m!}\partial_s^m G(t,s)|_{s=0}, \quad G(t,s) = \sum_{m=0}^{\infty} G_m(t)s^m \tag{4.96}$$

4.11 The W–constraints

This section is devoted to the derivation of the W–constraints in q–matrix models. From both the coupling equations (4.16c) and consistency conditions (2.15), we get the W-constraints in the form: $Tr(Q^{n+r}(\alpha)\partial^r_{\lambda_\alpha}(*)) = 0$ where $*$ are the relations (4.13) (For another approach see [65]):

$$\int \prod_{\alpha=1}^{q} d\lambda_\alpha \xi_n(\lambda_1)\epsilon^{\mu}\lambda_\alpha\eta_m(\lambda_q) \equiv Q_{nm}(\alpha)h_m = \bar{Q}_{mn}(\alpha)h_n. \quad \alpha = 1,\ldots,q. \tag{4.97}$$

W-constraints have the form:

$$W_n^{[r]}(\alpha)Z_N(t,c) = 0, \; r \geq 0, n \geq -r; \quad \alpha = 1,\ldots q. \tag{4.98}$$

or

$$(\mathcal{L}_n^{[r]}(\alpha) - (-1)^r T_n^{[r]}(\alpha))Z_N(t,c) = 0.$$

involving the interaction operator $T_n^{[r]}$ which depends only on all the couplings $g_{a_1\ldots a_q}$, except $g_{0,\ldots,0,a_\alpha,0,\ldots,0} = t_{\alpha,a_\alpha}$.

For example $T_n^{[1]}$ and $T_n^{[2]}$ are:

$$\begin{aligned} T_n^{[1]}(\alpha) &= a_\alpha g_{a_1\ldots a_q}\frac{\partial}{\partial g_{a_1\ldots,a_\alpha+n,\ldots a_q}} \\ T_n^{[2]}(\alpha) &= a_\alpha a'_\alpha g_{a_1\ldots a_q} g_{a'_1\ldots a'_q}\frac{\partial}{\partial g_{a_1+a'_1\ldots,a_\alpha+a'_\alpha+n,\ldots a_q+a'_q}} + \\ &+ a_\alpha(a_\alpha-1)g_{a_1\ldots a_q}\frac{\partial}{\partial g_{a_1\ldots,a_\alpha+n,\ldots a_q}} \end{aligned} \tag{4.99}$$

The operator $\mathcal{L}_n^{[r]}(1)$ has the same form as that of the two-matrix model:

$$\mathcal{L}_n^{[r]}(1) = \int dz : \frac{1}{r+1}(\partial_z + J)^{r+1} : z^{r+n} \tag{4.100}$$

where :: is the normal ordering and $J(z)$ is the $U(1)$ current:

$$J(z) = \sum_{k=1}^{p_1} kt_{1,k}z^{k-1} + Nz^{-1} + \sum_{k=1}^{\infty} z^{-k-1}\frac{\partial}{\partial t_{1,k}} \tag{4.101}$$

The same expression holds for $\mathcal{L}_n^{[r]}(q)$.

The expression of $\mathcal{L}_n^{[r]}(\alpha), \alpha = 2, \dots q-1$ is different due to the absence of the P-matrix term:

$$\mathcal{L}_n^{[r]}(\alpha) = \int dz : \frac{1}{r}(\partial_z + V'_\alpha)^r P_\alpha : z^{r+n} \tag{4.102}$$

with

$$\begin{aligned} V'_\alpha &= \sum_{k=1}^{p_\alpha} k t_{\alpha,k} z^{k-1} + N z^{-1}, \\ P_\alpha &= N z^{-1} + \sum_{k=1}^{\infty} z^{-k-1} \frac{\partial}{\partial t_{\alpha,k}} \end{aligned} \tag{4.103}$$

The explicit expression of the first terms is:

$$\begin{aligned} \mathcal{L}_n^{[1]}(\alpha) &= \sum_{k=1}^{\infty} k t_{\alpha,k} \frac{\partial}{\partial t_{\alpha,k+n}} + N t_{\alpha,1} \delta_{n,-1} \\ \mathcal{L}_n^{[2]}(\alpha) &= \sum_{k=1}^{\infty} k(k-1) t_{\alpha,k} \frac{\partial}{\partial t_{\alpha,k+n}} + \sum_{k_1,k_2} k_1 k_2 t_{\alpha,k_1} t_{\alpha,k_2} \frac{\partial}{\partial t_{\alpha,k+n}} + \\ &+ N^2 t_{\alpha,1} \delta_{n,-1} + N(t_{\alpha,1}^2 + 2 t_{\alpha,2}) \delta_{n,-2} \end{aligned}$$

As an example we write down the $W_{-1}^{[1]}, W_0^{[1]}$ and $W_1^{[1]}$ constraints for the three matrix model.

$W^{[1]}$:

$$\begin{aligned} \sum k t_k \langle \tau_{k-1} \rangle + N t_1 + c_{12} \langle \lambda_1 \rangle + c_{13} \langle \sigma_1 \rangle &= 0 \\ \sum k u_k \langle \lambda_{k-1} \rangle + N u_1 + c_{12} \langle \tau_1 \rangle + c_{23} \langle \sigma_1 \rangle &= 0 \\ \sum k s_k \langle \sigma_{k-1} \rangle + N s_1 + c_{23} \langle \lambda_1 \rangle + c_{13} \langle \tau_1 \rangle &= 0 \end{aligned}$$

$$\begin{aligned} \sum k t_k \langle \tau_k \rangle + c_{12} \langle \chi_{110} \rangle + c_{13} \langle \chi_{101} \rangle &= -\frac{N(N+1)}{2} \\ \sum k u_k \langle \lambda_k \rangle + c_{12} \langle \chi_{110} \rangle + c_{23} \langle \chi_{011} \rangle &= 0 \\ \sum k t_k \langle \sigma_k \rangle + c_{13} \langle \chi_{101} \rangle + c_{23} \langle \chi_{011} \rangle &= -\frac{N(N+1)}{2} \end{aligned}$$

$$\begin{aligned} \sum k t_k \langle \tau_{k+1} \rangle + (N+1) \langle \tau_1 \rangle + c_{12} \langle \chi_{210} \rangle + c_{13} \langle \chi_{201} \rangle &= 0 \\ \sum k u_k \langle \lambda_{k+1} \rangle \qquad + c_{12} \langle \chi_{120} \rangle + c_{23} \langle \chi_{021} \rangle &= 0 \\ \sum k t_k \langle \sigma_{k+1} \rangle + (N+1) \langle \sigma_1 \rangle + c_{13} \langle \chi_{102} \rangle + c_{23} \langle \chi_{021} \rangle &= 0 \end{aligned}$$

One easily sees from the second group of identities that the limit of pure chain models (cosmological point) does not exists for three–mm. The same thing holds for odd–q matrix models. However, writing down the W constraints for even–q matrix models, one can see that such a limit exists. This confirms the results obtained with other methods.

5 Discrete states in q-matrix models

5.1 2-matrix model and Penner model

The 2-matrix model , characterized by the small phase space $\mathcal{S}_0$ (when all the interactions are set to zero apart bilinear one $\mathrm{Tr}(M_1 M_2)$), has a topological nature. Remarkable evidence of this is provided by its connection with the Penner model [68]. Such connection is to be found in the N–dependent integration constant (4.21)(not determined by the W constraints) :

$$\frac{\partial^2}{\partial t_{1,1}\partial t_{\alpha,r}}\ln Z_N(t,c) = \Big(Q^r(\alpha)\Big)_{N,N-1}, \qquad 1 \le \alpha \le q \tag{5.1}$$

For the 2-matrix model with bilinear interaction only, one can immediately see that the form of Q-matrices is:

$$\Big(Q_+(1)\Big)_{ij} = \delta_{j,i+1} + a_0(i)\delta_{i,j}, \qquad \Big(Q_-(2)\Big)_{ij} = R_i\delta_{j,i-1} \tag{5.2}$$

where $R_{i+1} \equiv h_{i+1}/h_i$. As a consequence of this coordinatization, the equation above (5.1) gives in particular the important relation:

$$\frac{\partial^2}{\partial t_{1,1}\partial t_{2,1}}\ln Z_N(t,g) = R_N \tag{5.3}$$

and

$$\frac{\partial^2}{\partial t_{1,1}\partial t_{2,1}}\ln R_j = R_{j+1} - 2R_j + R_{j-1} \tag{5.4}$$

From these equations we obtain

$$\frac{\partial^2}{\partial t_{1,1}\partial t_{2,1}}\ln R_n = \frac{\partial^2}{\partial t_{1,1}\partial t_{2,1}}(F_{n+1} - 2F_n + F_{n-1}), \tag{5.5}$$

Where $F_n = \ln Z_n$. After integrating twice with respect to $t_{1,1}$ and $t_{2,1}$, we have

$$\ln R_n = F_{n+1} - 2F_n + F_{n-1} \tag{5.6}$$

We remark that this equation can be directly derived from the expression of the partition function (4.12). Possible integration constants have been dropped since they are irrelevant for the present problem (they provide analytic contributions to the free energy).

At this point it is more convenient to shift to a continuum formalism by introducing the continuum quantities

$$x \equiv \frac{n}{N}, \qquad \epsilon \equiv \frac{1}{N}, \qquad F(x,\epsilon) \equiv \frac{F_n}{N^2}, \tag{5.7}$$

as we did in the previous section (from now on the quantities we mention are understood to be the renormalized ones). However now we are interested not only in genus 0, but also in the higher genera. As is well–known the contributions to the free energy corresponding to different genera rescale in different ways. The appropriate genus expansion for the free energy is as follows

$$F(x,\epsilon) = \sum_{h=0}^{\infty} F_h(x)\epsilon^{2h} \tag{5.8}$$

where F_h is the h genus contribution. We also set

$$F_{n\pm 1} = e^{\pm\epsilon\partial_x} F(x,\epsilon)$$

Thus the continuum version of the equation (5.6) is

$$\ln R(x) = \frac{1}{\epsilon^2}\Big[\exp(\epsilon\partial_x) + \exp(-\epsilon\partial_x) - 2\Big]F(x,\epsilon). \tag{5.9}$$

Once we know $R(x)$, we can solve this equation to obtain the free energy.

Now at the particular critical point ($c = -1$) we are considering, $R(x) = x$, which corresponds to the genus 0 contribution, while higher genus contributions of R vanish. Therefore we can easily obtain the recursion relations among the free energy in different genera

$$\partial_x^2 F_0(x) = \ln R(x)$$

and

$$\sum_{l=1}^{n} \frac{1}{(2l)!}\partial_x^{2l} F_{n-l}(x) = 0, \qquad \forall n \geq 1. \tag{5.10}$$

Using these recursion relations we can obtain the free energy for any genus. For example,

$$F_1(x) = -\frac{1}{12}\ln x, \qquad F_2(x) = -\frac{1}{240}x^{-2},$$

and so on. In general we have

$$F_h(x) = \frac{B_{2h}}{2h(2h-2)}x^{2-2h}, \qquad \forall h \geq 2. \tag{5.11}$$

This assertion can be very easily proved by means of the properties of the Bernoulli numbers B_{2l}, in particular

$$\sum_{l=1}^{n-1} \frac{B_{2l}}{2l(2l-2)!(2n-2l)!} = \frac{1}{(2n)!}$$

Eq.(5.11) is the free energy of the Penner model. We see that $F_h(x)(h \geq 2)$ indeed have scales according to the power $2(1-h)$ while F_0 and F_1 exhibit logarithmic scaling violation, which is a typical feature of $c = 1$ string theory coupled to gravity. The coefficients of the powers of x are topological numbers. We recall that the (virtual) Euler characteristics of moduli space of Riemann surfaces with n punctures in genus h was computed by Harer and Zagier [69] and rederived by Penner,[68], with Feynman graph techniques

$$\chi_h^{(n)} = \frac{(-1)^n(2h-3+n)!(2h-1)}{n!(2h)!}B_{2h}$$

Therefore

$$F_h = \chi_h^{(0)} x^{2-2h}, \qquad < Q^n >_h = n!\chi_h^{(n)} x^{2-2h-n} \tag{5.12}$$

In particular we see that Q, the operator coupled to x, is to be interpreted as the puncture operator in the present theory [70].

It is a common belief that $c = 1$ string theory is a topological field theory with primaries, puncture equation, recursion relations and a Landau–Ginzburg formulation [72],[73],[74],[75].

5.2 sl_2 symmetry of the q-matrix models

The purpose of this section is to show that the extended two–matrix model can accommodate *discrete states* organized in finite dimensional sl_2 representations, analogous to the $c = 1$ string theory ones, and to calculate their correlation functions via the W–constraints introduced before.

5.2.1 sl_2 symmetry of the discrete states

We call discrete states the operators $\chi_{r,s}$ *coupled to the* $g_{r,s}$. In this definition $r = s = 0$ is excluded. However, for later use we introduce also $\chi_{0,0} \equiv Q$ as the operator coupled to $g_{0,0} \equiv N$. From the very definition it is apparent that, classically, $\chi_{r,s}$ is represented by $\sum_{i=1}^{N} \lambda_i^r \mu_i^s$, where, to simplify our notation we set $\lambda_1 = \lambda$ and $\lambda_2 = \mu$. As a consequence these states carry a built–in sl_2 structure. For let us define

$$H = \frac{1}{2}\sum_{i=0}^{N}\Big(\lambda_i\frac{\partial}{\partial\lambda_i} - \mu_i\frac{\partial}{\partial\mu_i}\Big), \qquad E_+ = \sum_{i=1}^{N}\lambda_i\frac{\partial}{\partial\mu_i}, \qquad E_- = \sum_{i=1}^{N}\mu_i\frac{\partial}{\partial\lambda_i}$$

Then

$$[H, E_\pm] = \pm E_\pm, \qquad [E_+, E_-] = 2H$$

and

$$H\chi_{r,s} = \frac{1}{2}(r-s)\chi_{r,s}, \qquad E_+\chi_{r,s} = s\chi_{r+1,s-1}, \qquad E_-\chi_{r,s} = r\chi_{r-1,s+1}$$

Therefore the set $\{\chi_{r,s} = \sum_{i=1}^{N}\lambda_i^r\mu_i^s, \;\; r+s = n\}$ form an (unnormalized) representation of this algebra of dimension $n+1$.

We can do even better and introduce the new states

$$\omega_{r,s} = \sqrt{\frac{(r+s)!}{r!s!}}\chi_{r,s}$$

endowed with the scalar product

$$(\omega_{r,s}, \omega_{r',s'}) = \delta_{r,r'}\delta_{s,s'}$$

One can easily verify that

$$\omega_{r,s} = |j, m>, \qquad r = j+m, \qquad s = j-m$$

is the standard basis for finite dimensional sl_2 representations. It is also easy to see that the products $\chi_{r,s}\chi_{r',s'}$ with $r+s = 2j$, $r'+s' = 2j'$ span the tensor product of the representations j and j'. It can be therefore decomposed into irreducible representations

$$\omega_{r_1,s_1}\omega_{r_2,s_2} = \sum_{r,s} C(r_1, s_1, r_2, s_2|r, s)\omega_{r,s} \tag{5.13}$$

where the summation extends to the range $M(r_1 - s_2, r_2 - s_1) \leq r \leq r_1 + r_2$ and $M(s_1 - r_2, s_2 - r_1) \leq s \leq s_1 + s_2$. $M(x, y)$ means the maximum between x and y. Here C are the standard Clebsh–Gordan coefficients expressed in terms of the labels r, s instead of j, m. A specification is in order: when $j = j'$ the symmetric representations of the Clebsh–Gordan series are absent in these classical products.

To end this introduction let us recall that $\chi_{r,0}$ and $\chi_{0,s}$ coincide with the operators τ_r and σ_s introduced in [53],respectively, which, in turn, were identified with the purely tachyonic states $\mathcal{T}_r$ and $\mathcal{T}_{-s}$ of the $c = 1$ string theory. It is therefore natural to try to interpret the $\chi_{r,s}$ as representatives of the discrete states of the $c = 1$ string theory.

5.2.2 Correlation functions of discrete states

The correlation functions of the extended two–matrix model are in general defined by

$$\ll \chi_{r_1,s_1} \cdots \chi_{r_k,s_k} \gg = \frac{\partial}{\partial g_{r_1,s_1}} \cdots \frac{\partial}{\partial g_{r_k,s_k}} \ln Z_N$$

Our main purpose in this paper is to calculate the correlation functions in a very simple *small phase space* $\mathcal{S}_0$: we set $g_{i,j} = 0$ except for $g_{1,1} \equiv c$, which is left undetermined (but in the topological application we will set $c = -1$). As a consequence the CF's will be functions of c and N. We denote by the symbol $< \cdot >$ the CF's calculated in the small phase space †.

Consider the W constraints

$$\left(\mathcal{L}_0^{[r]}(1) - (-1)^r \mathcal{T}_0^{[r]}(1)\right) Z_N(t; c) = 0, \qquad r \geq 1 \tag{5.14}$$

†In the following we often use the expressions *the model* $\mathcal{S}_0$ *or the critical point* $\mathcal{S}_0$ as equivalent to *the small phase space* $\mathcal{S}_0$.

and evaluate it in $\mathcal{S}_0$. We get

$$\mathcal{L}_0^{[r]}(1)Z_N = \frac{1}{r+1}N(N+1)\dots(N+r)Z_N \tag{5.15}$$

$$T_0^{[r]}(1)Z_N = c^r < \chi_{r,r} > Z_N \tag{5.16}$$

This allows us to calculate $< \chi_{r,r} >$ for any genus. Remember that the genus h and genus $h-1$ contributions differ by N^2. Therefore the genus h contribution, $< \chi_{r,r} >_h$, is given by

$$< \chi_{r,r} >_h = \begin{cases} \frac{N^{r+1-2h}}{(r+1)(-c)^r} b_{2h}(r) & r > 2h-1 \\ 0 & r \leq 2h-1 \end{cases} \tag{5.17}$$

where

$$b_k(n) = \sum_{1 \leq r_1 < r_2 < \dots < r_k \leq n} r_1 r_2 \dots r_k, \qquad b_0(n) = 1 \tag{5.18}$$

In genus 0, one can produce very general compact formulas for the CF's. In order to find the W constraints appropriate for genus 0 we proceed as follows. We assign a homogeneity degree $[\cdot]$ to each of the involved quantities

$$[g_{r,s}] = 1, \qquad [N] = 1, \qquad [\ln Z^{(0)}] = 2$$

where the superscript ${}^{(0)}$ denotes the genus 0 contribution. Then we keep only the leading terms in this degree in the W constraints. Henceforth we denote N by x which is associated to the cosmological constant. We will denote the genus zero part of every CF $< \cdot >$ by $< \cdot >_0$.

The simplest result and the first ingredient we need is $< \chi_{n,n} >_0$. This CF we have calculated exactly above, (5.17). The genus 0 contribution is

$$< \chi_{n,n} >_0 = \frac{x^{n+1}}{(-c)^n} \frac{1}{n+1} \tag{5.19}$$

For the remaining CF's the relevant W constraints are

$$\mathcal{L}_{-r}^{[r]}(1)Z(t;g,x) = (-1)^r T_{-r}^{[r]}(1)Z(t;g,x), \qquad r \geq 1 \tag{5.20a}$$

$$\mathcal{L}_n^{[r]}(1)Z(t;g,x) = (-1)^r T_n^{[r]}(1)Z(t;g,x), \qquad r \geq 1, \qquad n \geq 1 \tag{5.20b}$$

and the analogous ones with $1 \to 2$. But we will not have to consider the latter due to Lemma 1. Performing the degree analysis one easily extracts the dispersionless limit. In particular we have

$$T_n^{[r]}(1) = \sum_{\substack{i_1,\dots,i_r \geq 1 \\ j_1,\dots,j_r \geq 1}} i_1 \dots i_r g_{i_1,j_1} \dots g_{i_r,j_r} \frac{\partial}{\partial g_{i_1+\dots+i_r+n, j_1+\dots+j_r}}. \tag{5.21}$$

These generators satisfy the algebra

$$[T^{[r]}_n(1), T^{[s]}_m(1)] = (sn - rm)T^{[r+s-1]}_{n+m}(1), \qquad r, s \geq 1; \quad n \geq -r, \quad m \geq -s \tag{5.22}$$

which characterizes the area–preserving diffeomorphisms. The algebra of the $T^{[r]}_n(2)$ is just a copy of the above one. Similar simplified (but not quite as simple) expressions can be gotten for the $\mathcal{L}$–type generators.

Now, from (5.20a) and (5.20b), respectively, we obtain

$$\mathrm{LHS} = \sum_{\substack{k_1,\dots,k_r \geq 1\\ l_1,\dots,l_r \geq 1}} k_1 \dots k_r g_{k_1,l_1} \dots g_{k_r,l_r} < \chi_{k_1+\dots+k_r-r, l_1+\dots+l_r} >_0 \tag{5.23}$$

and

$$\sum_{l=1}^{r+1} \frac{1}{r+1} \binom{r+1}{l} x^{r-l+1} \sum_{k_1+\dots+k_l=n} < \chi_{k_1,0} >_0 \dots < \chi_{k_l,0} >_0 = \tag{5.24}$$
$$(-1)^r \sum_{\substack{k_1,\dots,k_r \geq 1\\ l_1,\dots,l_r \geq 1}} k_1 \dots k_r g_{k_1,l_1} \dots g_{k_r,l_r} < \chi_{k_1+\dots+k_r+n, l_1+\dots+l_r} >_0$$

The LHS in (5.23) needs not be written down explicitly (see below).

The first step consists in differentiating (5.23) with respect to $g_{n,n-r}$ with $n > r$ and evaluating the result in the small phase space $\mathcal{S}_0$. One gets

$$c^r < \chi_{0,r} \chi_{n,n-r} >_0 + rnc^{r-1} < \chi_{n-r,n-r} >_0 = 0$$

since the LHS of (5.23) does not depend on $g_{n,n-r}$ with $n > r$. Inserting (5.19) we obtain

$$< \chi_{0,r} \chi_{n,n-r} >_0 = \frac{x^n}{(-c)^n} r = < \chi_{r,0} \chi_{n-r,n} >_0 \tag{5.25}$$

Now differentiate (5.24) w.r.t. $g_{n,n+p}$ with $p > 0$, and evaluate the result in $\mathcal{S}_0$. One gets

$$\begin{aligned} x^r < \chi_{p,0} \chi_{n,n+p} >_0 &= (-1)^r \Big(rnc^{r-1} < \chi_{n+p+r-1,n+p+r-1} >_0 \\ &\qquad + c^r < \chi_{p+r,r} \chi_{n,n+p} >_0 \Big) \end{aligned} \tag{5.26}$$

for only the first terms in the LHS of (5.24) contributes, due to Lemma 2. Finally, using (5.25) and (5.19) one gets

$$< \chi_{p+r,r} \chi_{n,n+p} >_0 = \frac{x^{n+p+r}}{(-c)^{n+p+r}} \frac{(p+r)(n+p)}{n+p+r} = < \chi_{r,p+r} \chi_{n+p,n} >_0 \tag{5.27}$$

One can write down this result in a label–independent way as follows

$$< \chi_{r_1,s_1} \chi_{r_2,s_2} >_0 = \frac{x^\Sigma}{(-c)^\Sigma} \frac{M(r_1,s_1)M(r_2,s_2)}{\Sigma} \tag{5.28}$$

where $\Sigma = r_1 + r_2 = s_1 + s_2$ and $M(r,s) = max(r,s)$. This formula also holds when the two labels of χ coincide.

The procedure just outlined for two–point functions works in general. If we denote by χ^l a generic product of l discrete states, one first obtains $< \chi_{r,0}\chi^l >_0$ by suitably differentiating (5.23), then one derives $< \chi_{r+p,r}\chi^l >_0$ by suitably differentiating (5.24). For the generic three point functions one gets

$$< \chi_{r_1,s_1}\chi_{r_2,s_2}\chi_{r_3,s_3} >_0 = \frac{x^{\Sigma-1}}{(-c)^{\Sigma}} M(r_1,s_1)M(r_2,s_2)M(r_3,s_3) \tag{5.29}$$

where $\Sigma = r_1 + r_2 + r_3 = s_1 + s_2 + s_3$.

For the n–point functions with n $>$ 3, there is more than one possibility. To explain this point we will consider for a while only n–point functions $< \chi_{r_1,s_1} \cdots \chi_{r_n,s_n} >_0$ with $r_k \neq s_k$ for $k = 1, \ldots, n$ (the cases of coincident indices of χ are obtained as limiting cases), and define their signature to be (p,q), where p is the number of label $r_k > s_k$ and q is the number of labels $r_k < s_k$. For n–point functions with signature $(1, n-1)$ we have the general formula

$$\begin{aligned} &< \chi_{r_1,s_1} \cdots \chi_{r_n,s_n} >_0 = \frac{x^{\Sigma-n+2}}{(-c)^{\Sigma}} M(r_1,s_1)\ldots M(r_n,s_n)(\Sigma-1)\ldots(\Sigma-n+3) \\ &\Sigma = r_1 + \ldots + r_n = s_1 + \ldots + s_n = j_1 + \ldots + j_n \end{aligned} \tag{5.30}$$

if $\Sigma > n-2$, and vanishes otherwise. Here we have expressed Σ in terms of the more standard sl_2 j labels.

We remark that the above formulas have been derived for states $\chi_{r,s}$ with r and s not simultaneously vanishing. To obtain CF's involving p insertions of $Q \equiv \chi_{0,0}$, one has simply to differentiate p times with respect to x the corresponding CF without Q insertions.

To conclude this section let us remark that CF's of pure tachyonic states in genus 0, calculated with various methods, are numerous in the literature, [76], [77], [56], [78],[126],[80], [81]. The results in the work [61] coincide with these up to overall numerical factors. Also formula (5.30) has a counterpart in the literature, [79].

The important conclusion is that the results [61] either coincide or extend previous ones about discrete states. From this point of view, therefore, the conjecture that $\chi_{r,s}$ are representatives of the discrete states of $c = 1$ string theory is confirmed.

5.3 Topological field theory properties of 2–matrix models

The so–called $c = 1$ string theory is believed to be representable as a topological field theory. In this section are collected the data concerning this conjecture. In order to be able to make such assertion one should be able to identify the puncture operator, the primary fields and the descendants and show that they define an appropriate metric and satisfy a puncture equation and the appropriate recursion relations.

We have already seen that the puncture operator has to be Q. As primary fields we take $T_n \equiv \chi_{n,0}$ and $T_{-n} \equiv \chi_{0,n}$, where n is any natural number and $T_0 \equiv Q$, while all the other $\chi_{n,m}$ are descendants. We recall that T_n and T_{-n} were identified in [53] with the purely tachyonic states $\mathcal{T}_n$ and $\mathcal{T}_{-n}$, respectively, of $c = 1$ string theory.

The metric (in the topological field theory sense) is given by

$$\eta_{k,l} =< QT_kT_l >_0 \tag{5.31}$$

where k and l are integers. The only nonzero elements are (at $g = -1$)

$$\eta_{n,-n} = \eta_{-n,n} =< QT_nT_{-n} >_0 \equiv \frac{\partial}{\partial x} < \chi_{n,0}\chi_{0,n} >_0 = n^2x^{n-1}, \quad \eta_{0,0} = x^{-1}$$

This metric is non-degenerate, the inverse is $\eta^{k,l}$ with

$$\eta^{n,-n} = \eta^{-n,n} = n^{-2}x^{-n+1}, \qquad \eta^{0,0} = x$$

while all the other elements vanish. The associativity condition for the structure constants $C_{i,j,k}$, i, j, k integers,

$$\sum_{k,l} C_{i,j,k}\eta^{k,l}C_{l,p,q} = \sum_{k,l} C_{i,p,k}\eta^{k,l}C_{l,j,q}$$

is easily seen to be satisfied once we notice that the only nonvanishing three–point functions among primaries are

$$\begin{aligned} C_{n,m,-n-m} &= C_{-n,-m,n+m} =< T_{-n}T_{-m}T_{n+m} >_0 = nm(n+m)x^{n+m-1} \\ C_{n,-m,m-n} &= \begin{cases} nm(n-m)x^{n-1}, & n > m \\ nm(m-n)x^{m-1}, & n < m \end{cases} \end{aligned} \tag{5.32}$$

beside $C_{0,n,m} \equiv \eta_{n,-n}\delta_{n+m,0}$. The primary fields form the commutative associative algebra $\mathcal{A}$

$$T_iT_j = \sum_k C_{i,j}{}^kT_k, \qquad C_{i,j}{}^k \equiv \sum_l C_{i,j,l}\eta^{l,k}$$

where i, j, k, l are integers and $T_0 \equiv Q$. To prove it one has to use

$$\begin{aligned} &C_{n,m}{}^{n+m} = \frac{nm}{n+m}, \qquad C_{0,n}{}^n = 1, \quad C_{n,-n}{}^0 = n^2x^n, \qquad C_{0,0}{}^0 = 1 \\ &C_{n,-m}{}^{n-m} = \begin{cases} \frac{nm}{n-m}x^m, & n > m \\ \frac{nm}{m-n}x^n, & n < m \end{cases} \end{aligned}$$

where $n, m \neq 0$, $C_{i,j}{}^k = C_{j,i}{}^k$, and the other structure constants vanish.

Let us come now to the recursion relations and puncture equations. The *recursion relations* in $\mathcal{S}_0$ are

$$< \chi_{r,s}\chi_{r_1,s_1}\chi_{r_2,s_2} >_0 = M(r,s)\sum_{l,k} < \chi_{r-1,s-1}T_l >_0 \eta^{l,k} < T_k\chi_{r_1,s_1}\chi_{r_2,s_2} >_0 \tag{5.33}$$

where the labels k and l are understood to be integers. The proof is very simple. Suppose for example that $r \geq s$. Then

$$\text{LHS} = rM(r_1, s_1)M(r_2, s_2)x^{r+r_1+r_2-1}$$

when $r + r_1 + r_2 = s + s_1 + s_2$ and vanishes otherwise. On the other hand

$$\begin{aligned} \text{RHS} &= r < \chi_{r-1,s-1} T_{s-r} >_0 \eta^{s-r,r-s} < T_{r-s} \chi_{r_1,s_1} \chi_{r_2,s_2} >_0 = \\ &= rM(r_1, s_1)M(r_2, s_2)x^{r+r_1+r_2-1} \end{aligned}$$

when $r + r_1 + r_2 = s + s_1 + s_2$, and vanishes otherwise. The same can be proven for $r \leq s$.

The *puncture equations* are designed to connect the CF's of of the type

$$< Q\chi_{r_1,s_1}\chi_{r_2,s_2} \cdots \chi_{r_n,s_n} >_0,$$

where the χ's are extra states, with CF's including neighboring descendants of them. For dimensional reason the latter can only be $< \chi_{r_1,s_1} \cdots \chi_{r_i-1,s_i-1} \cdots \chi_{r_n,s_n} >_0$. A heuristic relation which does this job for the CF's (5.30) is the following

$$\begin{aligned} &< Q\chi_{r_1,s_1}\chi_{r_2,s_2} \cdots \chi_{r_n,s_n} >_0 = \\ &\qquad \sum_{i=1}^{n} \frac{M(r_i, s_i)}{M(r_i - 1, s_i - 1)} \frac{\Sigma - 1}{n} < \chi_{r_1,s_1} \cdots \chi_{r_i-1,s_i-1} \cdots \chi_{r_n,s_n} >_0 \end{aligned} \tag{5.34}$$

where $\Sigma = r_1 + \ldots + r_n = s_1 + \ldots + s_n$. In fact the LHS is

$$< Q\chi_{r_1,s_1}\chi_{r_2,s_2} \cdots \chi_{r_n,s_n} >_0 = x^{\Sigma-n+1} M(r_1, s_1) \ldots M(r_n, s_n)(\Sigma - 1) \ldots (\Sigma - n + 2)$$

On the other hand the generic term in the RHS of (5.34) contains

$$\begin{aligned} < \chi_{r_1,s_1} \cdots \chi_{r_k-1,s_k-1} \cdots \chi_{r_n,s_n} >_0 &= M(r_1, s_1) \ldots M(r_k - 1, s_k - 1) \ldots M(r_n, s_n) \cdot \\ &\quad \cdot(\Sigma - 2) \ldots (\Sigma - n + 2)x^{\Sigma-n+1} \end{aligned}$$

Summing all the contributions in the RHS of (5.34) we obtain the equality with the LHS.

5.4 sl_q symmetry of the q-matrix models

In this section we show how to calculate various correlation functions of composite operators (or discrete states). To start with let us illustrate a basic property of the latter: in the q matrix model they are organized in finite dimensional representations of sl_q.

5.4.1 sl_q symmetry of the discrete states

We have shown in [84] that we can enlarge the q–matrix model by introducing in the potential terms of the form:

$$g_{a_1,\ldots a_q} \prod_{\alpha=1}^{q} D_{\alpha}^{a_\alpha}, \qquad \text{with } D_\alpha = \text{Diag}(M_\alpha)$$

We call *discrete states* the operators $\chi_{a_1,\dots a_q}$ coupled to $g_{a_1,\dots a_q}$. We introduce also $\chi_{0\dots 0} \equiv Q$ as the operator coupled to $g_{0,\dots 0} \equiv N$. Classically, the operator $\chi_{a_1,\dots a_q}$ is represented by the product of eigenvalues $\sum_{k=1}^{N} \lambda_{1,k}^{a_1} \dots \lambda_{q,k}^{a_q}$. These states carry a built–in sl_q structure. To see this one has to consider the following generators

$$H_i = \frac{1}{2}\sum_{k=0}^{N}\left(\lambda_{i,k}\frac{\partial}{\partial\lambda_{i,k}} - \lambda_{i+1,k}\frac{\partial}{\partial\lambda_{i+1,k}}\right), \qquad 1 \leq i \leq q-1$$

$$E^+_{i,j} = \sum_{i=1}^{N}\lambda_{i,k}\frac{\partial}{\partial\lambda_{j,k}}, \qquad E^-_{i,j} = \sum_{i=1}^{N}\lambda_{j,k}\frac{\partial}{\partial\lambda_{i,k}}, \qquad 1 \leq i < j \leq q$$

H_i form the Cartan subalgebra of sl_q, while $E^+_{i,j}$ and $E^-_{i,j}$ are, respectively, the raising and lowering operators of the Lie algebra sl_q, corresponding to the roots:

$$\alpha_{ij} = \varepsilon_i - \varepsilon_j, \quad i < j$$

in the standard notation. The action on the states is as follows:

$$H_i\chi_{a_1\dots a_q} = \frac{1}{2}(a_i - a_{i+1})\chi_{a_1\dots a_q}, \qquad E^{\pm}_{i,j}\chi_{a_1,\dots,a_i,\dots,a_j,\dots,a_q} = \chi_{a_1,\dots,a_i\pm 1,\dots,a_j\mp 1,\dots,a_q}$$

Therefore the set $\{\chi_{a_1,\dots a_q} = \sum_{i=1}^{N} \lambda_{1,k}^{a_1}\dots\lambda_{q,k}^{a_q}, \ \ \sum_{i=1}^{q} a_i = n\}$ form an (unnormalized) representation of this algebra of dimension $\begin{pmatrix} n+q-1 \\ n \end{pmatrix}$.

Although everything we do here can be repeated for q–matrix model with q odd, we concentrate from now on on the far more interesting case of even q. The main reason for this is the well-definedness of the cosmological point when q is even. This will allow us to give an unambiguous topological field theory interpretation of the corresponding matrix models, while such an interpretation does not seem to be possible for odd q. Therefore, from now on, unless otherwise specified we consider $2q$–matrix models.

5.4.2 General properties of correlators

The correlation functions of the extended multi-matrix model are in general defined by

$$< \chi_{a_1^{(1)},\dots a_{2q}^{(1)}} \cdots \chi_{a_1^{(n)},\dots a_{2q}^{(n)}} >= \frac{\partial}{\partial g_{a_1^{(1)},\dots a_{2q}^{(1)}}} \cdots \frac{\partial}{\partial g_{a_1^{(n)},\dots a_{2q}^{(n)}}} \ln Z_N$$

Our purpose in this section is to calculate the correlation functions in two simple special cases: *the pure chain models* where we set $g_{a_1,\dots a_{2q}} = 0$ except for $g_{0\dots a_\alpha a_{\alpha+1}\dots 0} \equiv c_\alpha$ and *the quadratic models* where we have also the following nonzero coupling constants $g_{0\dots a_\alpha=2\dots 0} \equiv t_\alpha, g_{0\dots a_\alpha=1\dots 0} \equiv u_\alpha$. As a consequence the CF's will be functions of $c_\alpha, t_\alpha, u_\alpha$ and N. The chain models were referred to above as the *cosmological point* of the relevant $2q$–matrix models, while the quadratic models can be considered as quadratic perturbations of the latter. This second terminology is related to the topological field theory interpretation of section 5.4.

To see some general properties of the CF's, it is convenient to use the W–constraints (see section 4.10). We write down the W constraints in terms of them and obtain a set of (overdetermined) algebraic equations which in general one can solve recursively.

The CF's, in the chain models, have the following symmetry property:

$$< \chi_{a_1^{(1)},\dots a_i^{(1)}\dots a_j^{(1)}\dots a_{2q}^{(1)}} \cdots \chi_{a_1^{(n)},\dots a_i^{(n)}\dots a_j^{(n)}\dots a_{2q}^{(n)}} > \\ =< \chi_{a_1^{(1)},\dots a_j^{(1)}\dots a_i^{(1)}\dots a_{2q}^{(1)}} \cdots \chi_{a_1^{(n)},\dots a_j^{(n)}\dots a_i^{(n)}\dots a_{2q}^{(n)}} >$$

This is due to the symmetry of the W constraints and to the invariance of the chain models under the exchange $i \leftrightarrow j$.

In the chain models the CF's satisfy (charge conservation):

$$\sum_{\alpha=1}^{q} [(a_{2\alpha-1}^{(1)} + \dots a_{2\alpha-1}^{(n)}) - (a_{2\alpha}^{(1)} + \dots a_{2\alpha}^{(n)})] < \chi_{a_1^{(1)}\dots a_{2q}^{(1)}} \cdots \chi_{a_1^{(n)}\dots a_{2q}^{(n)}} >= 0 \qquad (5.35)$$

To prove the last statement we rewrite the $W_0^{[1]}$ constraint as follows:

$$\sum_{a_1\geq 1, a_2\geq 0\dots a_{2q}\geq 0} a_1 g_{a_1\dots a_{2q}} < \chi_{a_1\dots a_{2q}} > + \frac{1}{2}N(N+1) = 0$$
$$\sum_{a_1\geq,\dots a_\alpha\geq 1\dots a_{2q}\geq 0} a_\alpha g_{a_1\dots a_{2q}} < \chi_{a_1\dots a_{2q}} >= 0, \qquad 2\leq \alpha \leq 2q-1$$
$$\sum_{a_1\geq 0, a_{2q-1}\geq 0 a_{2q}\geq 1} a_q g_{a_1\dots a_{2q}} < \chi_{a_1\dots a_{2q}} > + \frac{1}{2}N(N+1) = 0$$

We differentiate these equations w.r.t. $g_{a_1^{(1)}\dots a_{2q}^{(1)}}, \dots, g_{a_1^{(n)}\dots a_{2q}^{(n)}}$ and set $g_{a_1\dots a_{2q}} = 0$ except $g_{0\dots a_\alpha a_{\alpha+1} 0} = c_\alpha$. One gets

$$\sum_{k=1}^{n} a_1^{(k)} < \chi_{a_1^{(1)}\dots a_{2q}^{(1)}} \cdots \chi_{a_1^{(n)}\dots a_{2q}^{(n)}} > + c_1 < \chi_{110\dots 0}\chi_{a_1^{(1)}\dots a_{2q}^{(1)}} \cdots \chi_{a_1^{(n)}\dots a_{2q}^{(n)}} >= 0$$
$$\sum_{k=1}^{n} a_\alpha^{(k)} < \chi_{a_1^{(1)}\dots a_{2q}^{(1)}} \cdots \chi_{a_1^{(n)}\dots a_{2q}^{(n)}} > + < (c_{\alpha-1}\chi_{0\dots 1, a_\alpha=1\dots 0} + \\ c_\alpha \chi_{0\dots a_\alpha=1,1\dots 0})\chi_{a_1^{(1)}\dots a_{2q}^{(1)}} \cdots \chi_{a_1^{(n)}\dots a_{2q}^{(n)}} >= 0, \qquad 2\leq \alpha \leq 2q-1$$
$$\sum_{k=1}^{n} a_{2q}^{(k)} < \chi_{a_1^{(1)}\dots a_{2q}^{(1)}} \cdots \chi_{a_1^{(n)}\dots a_{2q}^{(n)}} > + c_{2q} < \chi_{0\dots 011}\chi_{a_1^{(1)}\dots a_{2q}^{(1)}} \cdots \chi_{a_1^{(n)}\dots a_{2q}^{(n)}} >= 0$$

Subtracting the even equations from the odd ones we obtain the result.

The last property partially reflects the sl_{2q} structure of the discrete states as it means, at the cosmological point, the conservation of the eigenvalue of $H = H_1 + H_3 + \dots + H_{2q-1}$.

5.5 Topological field theory properties of $2q$–matrix models

We study in this section the content of $2q$–matrix models in terms topological field theories. The motivation is offered by the example of 2–matrix model, which can be interpreted as a topological field theory with an infinite number of primary fields, [61]. We want to see whether a similar conclusion can be drawn also for multi–matrix models. The easiest way to identify a possible topological field theory (TFT) content is to go to the cosmological point. We have seen previously that such a point is not well defined for odd q multi–matrix models. Consequently, in this section, we concentrate on even q multi–matrix models. To be definite we start with the 4–matrix model. A more detailed description of the topological properties of the 4–matrix model are given in Appendix C.

We recall that the cosmological point is identified by setting all the couplings to zero except the bilinear ones, $c_{\alpha,\alpha+1}$, with reference to eq.(4.5). To simplify things further we set from now on

$$c_{\alpha,\alpha+1} = (-1)^{\alpha}$$

without loss of generality (one can obtain the same results by suitable rescaling the couplings of the discrete states). Finally we replace N by a continuous variable x (i.e. we pass to a continuous formalism by suitably rescaling all the quantities and taking $N \to \infty$; x is the renormalized quantity that replaces N).

After these preliminaries let us concentrate on the 4–matrix models. Among the discrete states, our candidates for primary states are $\{\psi_{a,b}, Q, \omega_{c,d}\}$, where

$$\psi_{a,b} = \chi_{a,0,b,0}, \qquad \omega_{c,d} = \chi_{0,c,0,d}$$

The relevant genus 0 correlators to study the TFT properties can be computed from (A.2) and (A.3)

$$\begin{aligned} < \psi_{a_1,b_1}\psi_{a_2,b_2}\omega_{c,d} > &= \Big((a_1+b_1)(a_2+b_2)(c+d) - c(a_1b_2+a_2b_1+b_1b_2) \\ &\quad - \frac{b_1b_2cd}{c+d-1}\Big)x^{c+d-1}\delta_{a_1+a_2+b_1+b_2,c+d} \qquad (5.36a) \\ < \psi_{a,b}\omega_{c_1,d_1}\omega_{c_2,d_2} > &= \Big((a+b)(c_1+d_1)(c_2+d_2) - b(c_1d_2+c_2d_1+c_1c_2) \\ &\quad - \frac{abc_1c_2}{a+b-1}\Big)x^{a+b-1}\delta_{a+b,c_1+c_2+d_1+d_2} \qquad (5.36b) \end{aligned}$$

and

$$< Q\psi_{a,b}\omega_{c,d} >= (ac+ad+bd)x^{a+b-1}\delta_{a+b,c+d} \qquad (5.37)$$

We will also need $< QQQ >= x^{-1}$, which follows from the fact that the correlators involving only Q are the same as in the 2–matrix model, see [61].

Now, at any level $r = a+b$ let us select an arbitrary state among the $\psi_{a,b}$'s and call it ψ_r, $r > 0$. Let us call C the collection of such choices for any r. Moreover, let us set

$\omega_s \equiv \omega_{0,s}$. Then the states $\{\psi_r, Q, \omega_s\}$ constitute the set of primary states of a TFT with puncture operator either Q or ψ_1 or ω_1. This can be seen as follows. The non–vanishing structure constants are

$$C_{r_1,r_2,\bar{s}} \equiv <\psi_{r_1}\psi_{r_2}\omega_s>= r_1r_2sx^{s-1}\delta_{s,r_1+r_2}$$
$$C_{r,\bar{s}_1,\bar{s}_2} \equiv <\psi_r\omega_{s_1}\omega_{s_2}>= rs_1s_2x^{r-1}\delta_{r,s_1+s_2}$$
$$C_{0,r,\bar{s}} \equiv <Q\psi_r\omega_s>= rsx^{r-1}\delta_{r,s}, \qquad C_{0,0,0} \equiv <QQQ>= x^{-1}$$

together with the ones obtained from these by permutation of the indices. Now, if the puncture operator is Q, the metric is

$$\eta_{r,\bar{s}} = \eta_{\bar{s},r} \equiv <Q\psi_r\omega_s>= rsx^{r-1}\delta_{r,s}, \qquad \eta_{0,0} = x^{-1}, \tag{5.38}$$

If the puncture operator is ψ_1, the metric is

$$\eta_{r,\bar{s}} = \eta_{\bar{s},r} \equiv <\psi_1\psi_r\omega_s>= rsx^r\delta_{s,r+1}, \qquad \eta_{0,\bar{1}} = \eta_{\bar{1},0} = 1 \tag{5.39}$$

The case when the puncture operator is ω_1 is exactly specular to the latter. These three cases, with exactly the same formulas for structure constants and metric, were met in [82], where it was proven that the inverse metric exists and the associativity conditions are satisfied. The TFT obtained with a definite choice $\mathcal{C}$ will be denoted $\mathcal{T}_{\mathcal{C}}$. If necessary one can specify the symbol of the relevant puncture operator.

Similarly, among the states $\omega_{c,d}$, $c + d = s$ let us choose an arbitrary one and let us call it $\bar{\omega}_s$, $s > 0$. Let us call $\bar{\mathcal{C}}$ such a choice for any level s. Moreover, let us set $\psi_{r,0} \equiv \bar{\psi}_r$, $r > 0$. Once again the states $\{\bar{\psi}_r, Q, \bar{\omega}_s\}$ constitute the primary states of a TFT with puncture operator either Q or ψ_1 or ω_1. We do not need to explicitly prove this since the formulas for the structure constants and the metrics are the same as the previous ones with the substitutions $\psi_r \to \bar{\psi}_r$ and $\omega_s \to \bar{\omega}_s$. The TFT obtained with a definite choice $\bar{\mathcal{C}}$ will be denoted $\mathcal{T}_{\bar{\mathcal{C}}}$.

We can think of $\mathcal{T}_{\mathcal{C}}$ and $\mathcal{T}_{\bar{\mathcal{C}}}$ as unperturbed TFT's to which we couple topological gravity. Therefore we are going to have puncture equations and recursion relations. The latter are the same as in [82] and will not be repeated here. The former can be derived from the W constraints. Instead of writing the most general formula, we write down the simplest one for the puncture operator ψ_1

$$<\psi_1\chi_{a_1,a_2,a_3,a_4}>= a_2 <\chi_{a_1,a_2-1,a_3,a_4}> + a_4 <\chi_{a_1,a_2,a_3,a_4-1}>$$

from which one can infer the action of the puncture: ψ_1 lowers the even indices by 1. Therefore, when ψ_1 is the puncture operator, the descendants of $\psi_{a,b}$ are going to be $\chi_{a,n,b,m}$ for positive n and m, while any $\omega_{c,d}$ may be simultaneously primary and descendant, or an isolated primary.

We notice that the situation here is an interesting generalization of the situation in 2–matrix model, [82], where we have an infinity of primary states denoted $\{T_n, Q, T_{-m}\}$, with nonnegative integer n and m, where T_n, T_{-m} are the discrete tachyonic states. Here we have ∞^2 primary states, which depend on two integral indices and could be referred to as colored tachyons. In 2–matrix model, via reduction, one obtains an infinite set of TFT models (the n–KdV models) coupled to topological gravity, whose primary and descendants are to be found among the T_n's (or, symmetrically, among the T_{-n}'s), [83]. Similarly here we expect that, via reduction (see next section), the set $\psi_{a,b}$ with a and b positive, may support a series of matter TFT's coupled to topological gravity (i.e. primaries and descendants). Due to its characteristics – triangular structure of the primaries and relation with the product of two n–KdV models – possible candidates (certainly not the only ones) are, for example, the W_3 topological minimal models coupled to topological gravity, [85],[86].

In general, if we pass to $2q$–matrix models, the set of primaries will be represented by the states $\{\chi_{a_1,0,a_3,0,\dots,a_{2q-1},0}\}$, by Q and by $\{\chi_{0,a_2,0,\dots,0,a_{2q}}\}$: the primaries are ∞^q. This q should be related to the target space dimension in a string interpretation. In analogy with our previous discussion we are lead to speculate that one of the two sets above can accommodate the states of the W_{q+1} minimal models coupled to topological gravity or analogous TFT's. In Appendix D are given more details about such connection.

6 Discrete $c = 1$-matrix model

6.1 Nonzero momentum correlation functions

We calculate the 1-, 2- and 3-point correlation functions in the discrete $c = 1$ matrix model. We show that the 1-point c.f in the momenta space can be represented as a sum of delta functions of the form $\delta(p + 2k\omega_0)$-this means that the action of the operators $\tau_{\alpha,2r} = \mathrm{Tr}M_\alpha^{2r}$ on the vacuum introduce the particles with integer momenta $p = 2k\omega_0$. The 2-point correlation functions are sums over the δ-functions at integer momenta $p = (l-k)\omega_0$, hence we have extended states labeled by 2 indices which are the discrete states of $c = 1$ matrix model. In the Q-matrix approach [84] we have calculated the n-point correlation functions (for $n = 1, 2, 3$) for the q-multimatrix chain model (see Appendix A). The 1-, 2- and 3-point correlation functions are given by:

$$\begin{aligned} < \tau_{\alpha,2r} > &= \mathrm{Tr}Q_\alpha^{2r} \\ < \tau_{\alpha,2r}\tau_{\beta,2s} > &= \mathrm{Tr}[(Q_\alpha^{2r})_+, (Q_\beta^{2s})_-], \quad \alpha \leq \beta \\ < \tau_{\alpha,2r}\tau_{\beta,2s}\tau_{\gamma,2t} > &= \mathrm{Tr}[Q_\alpha^{2r}Q_\beta^{2s}Q_\gamma^{2t} - \quad , \quad \alpha \leq \beta \leq \gamma \\ & -((Q_\alpha^{2r})_- Q_\beta^{2s}Q_\gamma^{2t} + Q_\beta^{2s}Q_\gamma^{2t}(Q_\alpha^{2r})_+ + \text{cyclic perm.}) + \\ & +((Q_\alpha^{2r})_- Q_\beta^{2s}(Q_\gamma^{2t})_+ + \text{perm.}) + 2(Q_\alpha^{2r})_+(Q_\beta^{2s})_+(Q_\gamma^{2t})_+] \end{aligned} \tag{6.1}$$

The relations above are valid for arbitrary potentials in the multi-matrix model. In relation with $c = 1$ matrix model we will restrict ourselves to the gaussian potentials.

The Q-matrices for a gaussian model take the simple form:

$$Q_\alpha = h_\alpha I_+ + g_\alpha \epsilon_- \tag{6.2}$$

In the work [84] (see Appendix B) we have already derived the form of Q_α^{2r} in terms of h_α, g_α :

$$Q_\alpha^{2r} = \sum_{k=0}^{r} \sum_{i=0}^{2r-2k} \frac{(2r)!}{(2r-2k-i)!i!k!}(-1)^k 2^{-k} g^{i+k} h^{2r-(i+k)} I_+^{2r-2k-i} \epsilon_-^i \tag{6.3}$$

For $(Q_\alpha^{2r})_+$ and $(Q_\alpha^{2r})_-$ we have the same formula but the index i in the sum is restricted to take values only from 0 to $r-k$, respectively from $r-k$ to $2r-2k$.

To calculate the traces of Q-matrices in the relations (6.2), we need to know the following traces $\mathrm{Tr}(I_+^n \epsilon_-^m)$, $Tr(I_+^n \epsilon_-^m I_+^p \epsilon_-^q)$ and $\mathrm{Tr}(I_+^n \epsilon_-^m I_+^p \epsilon_-^q I_+^r \epsilon_-^s)$.

The permutation relation:

$$I_+^n \epsilon_-^m = \sum_{v=0}^{m} \epsilon_-^v I_+^{n-m+v} A_v^{(n,m)}, \quad A_v^{(n,m)} = \frac{n!m!}{v!(n-m+v)!(m-v)!} \tag{6.4}$$

can be used to calculate the up-mentioned traces.

We collect the needed formula:

$$\begin{aligned}
\mathrm{Tr}(I_+^n \epsilon_-^m) &= \delta_{nm} n! \begin{pmatrix} N+n \\ n+1 \end{pmatrix} \\
\mathrm{Tr}(I_+^n \epsilon_-^m I_+^p \epsilon_-^q) &= \sum_{v=0}^{m} F(n,m;p.q|v) = \sum_{v=0}^{m} \frac{n!m!(q+v)!}{v!(n-m+v)!(m-v)!} \begin{pmatrix} N+q+v \\ q+v+1 \end{pmatrix} \quad (6.5) \\
& \text{with } \; n > m, n+p = m+q
\end{aligned}$$

$$\mathrm{Tr}(I_+^n \epsilon_-^m I_+^p \epsilon_-^q I_+^r \epsilon_-^s) = \sum_{0 \le l \le q} \sum_{0 \le l' \le m+l} F(n,m;p,q;r.s|l,l') =$$

$$= \sum_{l,l'} \frac{p!q!n!(m+l)!(s+l+l')!}{l!l'!(q-l)!(p-q+l)!(n-m-l+l')!(m+l-l')!} \begin{pmatrix} N \\ s+l+l'+1 \end{pmatrix}$$

$$\text{with } \; n+p+r = m+q+s$$

Inserting the relations (6.3) and (6.5) in the defining expressions (6.2), we write the correlation functions explicitly in terms of h_α, g_α:

$$< \tau_{\alpha,2r} >= \sum_{k=0}^{r} \frac{(2r)!(-1)^k 2^{-k}}{k!(r-k)!} \begin{pmatrix} N+r-k \\ r-k+1 \end{pmatrix} (h_\alpha g_\alpha)^r \tag{6.6}$$

$$< \tau_{\alpha,2r} \tau_{\beta,2s} >= \sum_{k=0}^{r} \sum_{l=0}^{s} \sum_{i=0}^{r-k} \sum_{j=s-l}^{2s-2l} \frac{(2r)!}{i!k!(2r-2k-i)!} \frac{(2s)!}{j!l!(2s-2l-j)!} \tag{6.7}$$

$$(-1)^{k+l} 2^{-(k+l)} (g_\alpha h_\beta)^{i+k} (h_\alpha g_\beta)^{j+l} \left(\frac{h_\alpha}{h_\beta} \right)^{r-s} (\sum_{v=0}^{i} F(2r-2k-i,i;2s-2l-j,j|v) -$$

$$- \sum_{u=0}^{j} F(2s-2l-j,j;2r-2k-i,i|u))$$

with $i+j = r+s-k-l$ and where function F is given by relation (6.5).

We omit to write down the formula for 3-point correlation function because of its lenght. However, we are interested only in the leading, genus 0 term which we will write later.

To simplify the calculations we will consider in what follows only the genus 0 contribution of the correlation functions.

For the 1-point c.f., the genus 0 contribution has the maximum power of N, which is equivalent with $k = 0$ in relation (6.6):

$$< \tau_{\alpha,2r} >_0 = \frac{(2r)!}{r!(r+1)!} N^{r+1} (h_\alpha g_\alpha)^r \tag{6.8}$$

The formula is the same as that of the 1-point c.f. for 1-matrix model in gaussian potential. But the interesting thing is the dependence of $h_\alpha g_\alpha$ of the time coordinate α .

For the 2-point c.f. the genus 0 constraint imposes the equalities $v = i - 1, u = j - 1$ which give contribution to the subleading term proportional with N^{i+j}. The leading term proportional with N^{i+j+1} (when $v = i, u = j$) is zero. For genus zero $i + j$ is maximum when $k = l = 0$ and $i + j = r + s$.We use the notation $i = r - n, j = s + n$.

The above considerations permit to write genus 0 contribution to the 2-point c.f. :

$$< \tau_{\alpha,2r}\tau_{\beta,2s} >_0 = (2r)!(h_\alpha g_\alpha)^r(2s)!(h_\beta g_\beta)^s(-2)\frac{N^{r+s}}{r+s}\sum_{0\leq n\leq min(r,s)} a_n^{(r,s)}\left(\frac{h_\alpha g_\beta}{g_\alpha h_\beta}\right)^n \tag{6.9}$$

with:

$$a_n^{(r,s)} = \frac{1}{(r-n)!(s-n+1)!}\frac{1}{(r+n+1)!(s+n)!}(n(r+s+1)+(s-r)/2) \tag{6.10}$$

The sum is invariant by the transformations $\alpha \leftrightarrow \beta, r \leftrightarrow s, n \leftrightarrow -n$.

In the same way we can calculate the planar contribution to the 3-point c.f. :

$$< \tau_{\alpha,2r}\tau_{\beta,2s}\tau_{\gamma,2t} >_0 = (2r)!(h_\alpha g_\alpha)^r(2s)!(h_\beta g_\beta)^s(2t)!(h_\gamma g_\gamma)^t\frac{N^{r+s+t-1}}{r+s+t-1} \tag{6.11}$$

$$\sum_{\mathcal{D}}\sum_{m,n,p\in\mathcal{D}}\delta_{p+m+n,0}\left(\frac{g_\alpha}{h_\alpha}\right)^m\left(\frac{g_\beta}{h_\beta}\right)^n\left(\frac{g_\gamma}{h_\gamma}\right)^p F_{mnp}^{(rst)}(\mathcal{D})$$

where we have:

$$a_{mnp}^{(rst)} = \frac{1}{(r-m)!(r+m)!}\frac{1}{(s-n)!(s+n)!}\frac{1}{(t-p)!(t+p)!} \tag{6.12}$$

and

$$F_{mnp}^{(rst)} = \sum_{v=0}^{r+s+n} F(r-m, r+m; s-n, s+n; t-p, t+p|s+n-v, r+n+v-2) \tag{6.13}$$

with F-function given by relation (6.5).

For first domain $\mathcal{D}_1$: $m \in [-r, r], n \in [-s, s], p \in [-\min(r+s,t), \min(r+s,t)$ and $F_{mnp}^{(rst)}(\mathcal{D}_1) = F_{mnp}^{(rst)}$. For example for the third term in relation (6.2) we get the domain $\mathcal{D}_3$: $m \in [0, r], n \in [-s, s], p \in [-\min(r+s,t), \min(s,t)]$ and $F_{mnp}^{(rst)}(\mathcal{D}_3) = F_{npm}^{(str)}$.

6.2 Comparation with the free fermion approach

After these general considerations we consider the special case of the discrete $c = 1$ matrix model.This will permit to find the dependence of the n-point c.f. in terms of the time coordinates for puncture operators .

The $c = 1$ model with discrete time has the partition function:

$$Z = \int dM_i \exp\left[-\frac{\beta}{2}Tr\left(\sum_{i=1}^{q-1}\frac{(M_{i+1}-M_i)^2}{\epsilon} + \epsilon\sum_{i=1}^{q}V(M_i)\right)\right]$$

with a quartic potential $V(M) = M^2 - gM^4$. However, only the contribution near saddle point $V'(M_c) = 0$, where the potential is quadratic in the fluctuation ΔM, is essential

$$V(M) = \frac{1}{4g} - 2\frac{(\Delta M)^2}{\beta}, M = M_c + \frac{\Delta M}{\sqrt{\beta}} \tag{6.14}$$

The new partition function is (up to the constant $\exp(-N\beta\epsilon/(8g))$:

$$Z = \int dM_i \exp\left[Tr\left(\sum_{i=1}^{q}\Delta M_i^2(2\epsilon - \frac{1}{\epsilon}) + \frac{1}{\epsilon}\sum_{i=1}^{q-1}\Delta M_i \Delta M_{i+1}\right)\right]$$

It represents a string theory on circle with radius $R \sim \frac{1}{\epsilon}$.

The coefficients h_α, g_α can be expressed in terms of the determinant D_n of the $n \times n$ matrix :

$$h_\alpha = (-1)^\alpha D_{\alpha-1}, g_\alpha = (-1)^\alpha \frac{D_{q-\alpha}}{D_q} \tag{6.15}$$

where:

$$D_n = \begin{vmatrix} u & 1 & 0 & \dots & \dots & 0 \\ 1 & u & 1 & 0 & \dots & 0 \\ 0 & 1 & u & \ddots & . & . \\ . & 0 & 1 & \ddots & 1 & 0 \\ 0 & \dots & 0 & \ddots & u & 1 \\ 0 & & \dots & 0 & 1 & u \end{vmatrix}$$

We have introduced the parameter $u = 2(2\epsilon^2 - 1)$. In the region $-2 \le u \le 2$ the determinant D_n has the simple representation (outside this region the sin–function is replaced by sinh):

$$D_n = \frac{\sin(n+1)\omega}{\sin\omega} \tag{6.16}$$

where $\omega = \arctan\sqrt{(2/u)^2 - 1}$.

In the limit $\epsilon \to 0$, we have $\cos\omega = -1 + 2\epsilon^2, \sin\omega = 2\epsilon$, hence $\omega \sim m\pi - 2\epsilon \to m\pi$. Instead the discrete variable $\alpha = 1 \dots q$ we define the continuos variable $\frac{\alpha}{q} = t \in [0.T]$. Also we must define the rescaled pulsation ω_0 such that $\alpha\omega \to t\omega_0, (q-\alpha)\omega \to (T-t)\omega_0$.

Due to the limit $q\omega \to T\omega_0 = 2\pi$ we can achieve the continuum limit $q \to \infty, \epsilon \to 0$ and maintaining fixed the product $q\epsilon = \pi$. Hence in the continuum limit the period and the pulsation behaves as $T \sim q \to \infty$, $\omega \sim 2\epsilon \to 0$.

We can now calculate:

$$(h_\alpha g_\alpha)^r = \left(\frac{\sin\alpha\omega\sin((q+1)-\alpha)\omega}{\sin\omega\sin(q+1)\omega}\right)^r \tag{6.17}$$

In the continuum limit $\epsilon \longrightarrow 0$ we have:

$$(h_\alpha g_\alpha)^r \longrightarrow \left(\frac{\sin 2t\omega_0}{4\epsilon}\right)^r \tag{6.18}$$

We can make an expansion in periodic functions:

$$(h_\alpha g_\alpha)^r = \sum_{l=0}^{2r} d_l^{(r)} e^{4it\omega_0(r/2-l)} \tag{6.19}$$

$$\text{with } d_l^{(r)} = (8\epsilon)^{-r} \binom{r}{l} (-1)^{r/2-l} \tag{6.20}$$

We perform a Fourier transform to get the 1-point CF in momentum space.

But we have 2 problems. The first problem is that the integration is in the interval $[-T/4, T/4]$ (we assume the contribution from only one top of the inverted harmonic potential), and we have periodic functions which have the period nT. This problem is resolved by observing that in the continuum limit $T = 2\pi/\omega_0 \longrightarrow \infty$, and that in the continuum limit all functions are periodic in the interval $[-T/4, T/4] \longrightarrow (-\infty, -\infty)$.

The second problem is that the coefficient $d_l^{(r)} \sim (-1)^{r/2}$ for r odd is complex. Hence we must distinguish to cases when r is odd and even. When r is even, the operator $\tau_{2r}(t)$ is:

$$\tau_{2r}(t) \sim (8\epsilon)^{-r} \sum_{l=0}^{r} \binom{r}{l} (-1)^{r/2-l} e^{4it\omega_0(r/2-l)}$$

with the real part:

$$\tau_{2r}(t) \sim (8\epsilon)^{-r} \sum_{l=0}^{r} \binom{r}{l} (-1)^{r/2-l} \cos(4t\omega_0(r/2-l))$$

Instead for r odd the operator $\tau_{2r}(t)$ is multiplied with i and the real part is:

$$\tau_{2r}(t) \sim (8\epsilon)^{-r} \sum_{l=0}^{r} \binom{r}{l} (-1)^{r/2-1-l} \sin(4t\omega_0(r/2-l))$$

Now we can use the result (6.20) in the relation (6.8), and the 1-point CF in the momentum space is:

$$< \tau_{2r} >_0 (p) = \int_{-T/4}^{T/4} dt < \tau_{\alpha,2r}(t) >_0 e^{ipt} = \frac{(2r)!}{r!(r+1)!} N^{r+1} \sum_{l=0}^{r} d_l^{(r)} \delta(p + 4\omega_0(\frac{r}{2} - l)) \tag{6.21}$$

This result is correct when r is even. Instead, when r is odd the momentum is shifted by $\pi/2$.

In the free fermion method (see [71]), the $c = 1$ matrix model is considered equivalent with a system of free fermions in the harmonic inverted potential. The Liouville mode is interpreted as the classical time τ of flight variable.

The period of the oscilations of the classical particle moving in the given potential is related with the cosmological constant μ: $T = 2\sqrt{\beta}|\log\mu|$, where β is the string coupling.

The equation of motion at the Fermi surface is:

$$\frac{d\lambda}{d\tau} = \frac{1}{2}\sqrt{g(\mu_F - V(\lambda))}$$

where the potential is quartic $V(\lambda) = \lambda^2 - g\lambda^4$.

Expanding near the saddle point $\lambda = \lambda_c + x/\sqrt{\beta}$ we get the equation:

$$\frac{dx}{d\tau} = \sqrt{g(\frac{2x^2}{\beta} - \mu)}$$

with the solution:

$$x(\tau) = \sqrt{\beta/2}\mu^{1/2}cosh(\tau/\sqrt{g\beta/8})$$

The puncture operator in 2D gravity is in this case:

$$O_r = x^r = (\beta/2)^{r/2}\mu^{r/2}cosh^r(\tau/\sqrt{g\beta/8}) \tag{6.22}$$

We can observe that the operator O_r behaves in the same way as the operator $\tau_r(t)$ because this operator has the correlation function:

$$< \tau_{2r}(t) >= \frac{(2r)!}{r!(r+1)!}x^{r+1}(\frac{\sin 2t\omega_0}{4\epsilon^2})^r \tag{6.23}$$

with $x = n/N$.

We can identify the variables (putting for simplicity $g = 1$):

$$\epsilon \sim 1/\sqrt{\beta}, \mu \sim x \sim N^{-1} \tag{6.24}$$

At the critical point, the cosmological constant $\mu \to 0$, which means that $N \to \infty$.

The time t is related via Lorentz rotation with the flight time $t \sim i\tau$, up to a translation by $T/2$.

To calculate 2-point c.f. we need:

$$h_\alpha^{r+n}g_\alpha^{r-n} = \frac{\sin^{r+n} t\omega_0 \cos^{r-n} t\omega_0}{(2\epsilon)^{r+n}} = \sum_{l=0}^{2r} A_l^{(r,n)} e^{2it\omega_0(r-l)} \tag{6.25}$$

$$A_l^{(r,n)} = (8\epsilon)^{-(r+n)}(\epsilon)^{-n} \sum_{l'=r-n}^{r+n} \binom{r+n}{l'}\binom{r-n}{l-l'}(-1)^{\frac{r+n}{2}-l'}$$

In the same way we can write:

$$h_\beta^{s-n}g_\beta^{s+n} = \sum_{k=0}^{2s}\sum_{k'=s+n}^{s-n} A_k^{(s,-n)}e^{2it'\omega_0(s-k)}$$

Using the expression (6.25) in the relation (6.9) we get the 2-point c.f. :

$$< \tau_{\alpha,2r}\tau_{\beta,2s} > \;=\; -2(2r)!(2s)!\frac{N^{r+s}}{r+s}\sum_{k=0}^{2s}\sum_{l=0}^{2r}\mathcal{A}_{kl}\exp i[2t\omega_0(r-l)+2t'\omega_0(s-k)] \quad (6.26)$$

where :

$$\mathcal{A}_{kl}^{(n)} = \sum_{0\leq n\leq min(r,s)} a_n^{(r,s)}A_l^{(r,n)}A_k^{(s,-n)} \quad (6.27)$$

with $a_n^{(r,s)}$ given by relation (6.10).

The 2-point c.f. can be rewritten:

$$< \tau_{\alpha,2r}\tau_{\beta,2s} >= -2(2r)!(2s)!\frac{N^{r+s}}{r+s}\sum_{k,l}\mathcal{A}_{kl}e^{i\omega_0\Delta t(r-s-(l-k))}e^{i\omega_0\Delta t(r+s-(l+k))}$$

where $\Delta t = t - t'$.

The 2-point c.f. in the momentum space has the form:

$$< \tau_{\alpha,2r}\tau_{\beta,2s} > (p,P) = \iint_{-T/2}^{T/2} d(\Delta t)\, d(t+t') < \tau_{\alpha,2r}(t)\tau_{\beta,2s}(t') > e^{ip\Delta t}e^{iP(t+t')} = \quad (6.28)$$

$$= 4(2r)!(2s)!\frac{N^{r+s}}{r+s}\sum_{l,k}\mathcal{A}_{kl}\delta(p+(r-s-(l-k))\omega_0)\delta(P+(r+s-(l+k))\omega_0)$$

We have extended states at $p = (r-s-(l-k))\omega_0$ (representing discrete states) where $p = -r-s\ldots r+s$. If $t=0$ we have extended states at $p = r - l\omega_0$ (representing pure tachyons) where $p = -r\ldots r$.

In the same way we calculate the 3-point c.f.:

$$< \tau_{\alpha,2r}\tau_{\beta,2s}\tau_{\gamma,2t} >= (2r)!(2s)!(2t)!\frac{N^{r+s+t-1}}{r+s+t-1}\sum_{\mathcal{D}}\sum_{jkl} \quad (6.29)$$
$$\mathcal{A}_{jkl}(\mathcal{D})\exp i\omega_0[t_1(r-j)+t_2(s-k)+t_3(t-l)]$$

where:

$$\mathcal{A}_{jkl}(\mathcal{D}) = \sum_{m,n,p\in\mathcal{D}} a_{mnp}^{(rst)}F_{mnp}^{(rst)}(\mathcal{D})A_j^{(r,m)}A_k^{(s,n)}A_l^{(t,p)} \quad (6.30)$$

with $a_{mnp}^{(rst)}$ given by relation (6.12) and $m+n+p=0$.

The indices take values in the regions: $j = 0,\ldots 2r, k = 0,\ldots 2s, l = 0,\ldots 2t$.

We can rewrite the 3-point c.f. in the form:

$$< \tau_{\alpha,2r}\tau_{\beta,2s}\tau_{\gamma,2t} >= (2r)!(2s)!(2t)!\frac{N^{r+s+t-1}}{r+s+t-1}\sum_{\mathcal{D}}\sum_{jkl}\mathcal{A}_{jkl}(\mathcal{D}) \quad (6.31)$$
$$\exp[2i/3\omega_0\Delta t_1(2(r-j)-(s-k)-(t-l))]\exp[2i\omega_0T((t-l)+(s-k)+(r-j))]$$
$$\exp[2i/3\omega_0\Delta t_2(2(t-l)-(s-k)-(r-j))]$$

with:

$$T = \frac{t_1 + t_2 + t_3}{3}, \Delta t_1 = t_1 - t_2, \Delta t_2 = t_3 - t_2$$

The 3-point c.f. in the momentum space is:

$$< \tau_{\alpha,2r}\tau_{\beta,2s}\tau_{\gamma,2t} > (p_1, p_2, P) = \int\int\int_{-T/2}^{T/2} d(\Delta t_1)\, d(\Delta t_2)\, dT < \tau_{\alpha,2r}(t_1)\tau_{\beta,2s}(t_2)\tau_{\gamma,2t}(t_3) >$$

$$\times e^{ip_1\Delta t_1 + ip_2\Delta t_2} e^{iPT} = (2r)!(2s)!(2t)! \frac{N^{r+s+t-1}}{r+s+t-1} \sum_{\mathcal{D}} \sum_{jkl} \mathcal{A}_{jkl}(\mathcal{D})$$

$$\delta(p_1 + \frac{2}{3}[2(r-j) - (s-k) - (t-l)]\omega_0)\delta(p_2 + \frac{2}{3}\omega_0[2(t-l) - (s-k) - (r-j)]) \times$$

$$\times \delta(P + \omega_0[(t-l) + (s-k) + (r-j)]) \quad (6.32)$$

We considered previously the limit $\epsilon \to 0$, or the plain limit when the radius R goes to infinity.

Now we study the self-dual point where $R \sim 1/R$.This corresponds to the case $\epsilon \to 1$.In this limit $\omega = \pi/2 - 2(\epsilon - 1) \to \pi/2$

Instead of ϵ and ω is better to define ϵ' and ω' as:$\epsilon = 1 + \epsilon', \omega = \pi/2 - \omega'$. We can take the continuos limit $q \to \infty$ such that $q\omega' = \frac{\pi}{2} = \omega_0$.The variable $t = \frac{\alpha - 1}{q}$ is defined as before.

The 1-point c.f remains unchanged at the new limit–the self-dual point.The 2-point c.f. is still given by the formula (6.26) but with $\mathcal{A}_{k,l}$ given by:

$$\mathcal{A}_{k,l} = \sum_{0 \le n \le min(r,s)} a_n^{(r,s)} A_l^{(r,(-1)^{\alpha-1}n)} A_k^{(s,-(-1)^{\beta-1}n)} \quad (6.33)$$

This new restriction gives the same poles as before (6.29) at $p = \pm i(l-k)\omega_0$, with $l-k$ odd, but only if α, β are both even or odd. In the cases when α is odd, β is even or viceversa, we have the same poles but with $l - k$ even.

For consistency with the case $t = 0$ where we have the poles $p = \pm ik\omega_0$ with k even, we could conclude that we must have only poles with $l - k$ even. This condition imposes that α is odd, β is even or viceversa. This is in agreement with the fact that exactly at the self-dual point the 2-point c.f without momentum do not vanish only if α is odd and β is even or viceversa. Hence discrete states appear as poles in the 2-point c.f. only if the free particles of $c = 1$ matrix model belong to 2 distinct classes with α odd, respectively even. But this is equivalent to the choosing of a special sl_2 subalgebra embedding of the larger algebra sl_n which caracterizes the $c = 1$ matrix model. This also could explain why the discrete states satisfy the sl_2 algebra.

6.3 Time in 2-matrix model

It is difficult to imagine how could appear the time in a truly 0-dimensional theory as the 2-matrix model is. We show that the time in the 2-matrix model is intimately related with the nonzero momentum correlation functions.

The 1-point CF's expression contains a delta function which behave like the delta function for momentum conservation:

$$< \chi_{2r,2s} >= \frac{N^{r+s+1}}{r+s+1}\frac{1}{c^2 r}\delta(2r-2s) \tag{6.34}$$

At the cosmological point when $2t_2 = 2s_2 = 0$ we have the $U(1)$-conservation law $r = s$ which gives us a delta function. But we can interpret this condition as the cancelation of the momentum $p = 2(r-s)$ associated with the state $\chi_{2r,2s}$. Out of the cosmological point we have a non-zero momentum, because there is no $U(1)$-conservation law restricting our momentum to zero.

In fact a computation of the 1-point CF for $2t_2 = 2s_2 \neq 0$ gives us:

$$< \chi_{2r,2s} >= \frac{N^{r+s+1}}{r+s+1}\frac{(-2s_2)^r(-2t_2)^s}{(4t_2s_2-c^2)^{r+s}}(2r)!(2s)!$$
$$\sum_{0\leq n\leq min(r,s)} \frac{1}{(r+n)!}\frac{1}{(s+n)!}\frac{1}{(r-n)!}\frac{1}{(s-n)!}(\frac{c^2}{4t_2s_2})^n \tag{6.35}$$

In the limit $2t_2 = 2s_2 \to 0$ we must have the condition $r = n = s$ for a finite correlation function. We reobtain in this limit the equation (6.34).

Hence the 1-point CF (6.34) is in the momentum representation and has the form $< \chi_0(p_0) >= const\delta(p_0)$. As we said this permits the identification of momentum p_0:

$$\chi_0(p_0) \sim \chi_{2r,2s}, \quad p_0 = 2(r-s) \tag{6.36}$$

In the space-time representation we have the state $\chi_0(t) = e^{-2i(r-s)t}$. Away from the cosmological point we have a more general state $\chi(t) = \chi_0(t)e^{ipt}$, giving the 1-point CF in momentum representation:

$$< \chi(p) >= \int dt < \chi_0(t) > e^{ipt} = \delta(p-p_0)$$

For the 2-point CF in the cosmological point we get:

$$< \chi_{r,s}\chi_{r',s'} >= \frac{N^{r+r'}}{(-c)r+r'}\frac{rs'}{r+r'}\delta((r-s)+(r'-s')) \tag{6.37}$$

We can choose the coeficients r, s, r', s' without restricting the generality such that $r \geq s, r' \leq s'$. We can define 2 momenta:

$$p_0 = r - s \geq 0 \quad p_0' = r' - s' = -p_0 \leq 0 \tag{6.38}$$

We can rewrite the 2-point CF as follows:

$$< \chi(p_0)\chi(p_0') >= const\delta(p_0 + p_0')$$

In the space-time representation (with $t < t'$) the 2-point CF becomes:

$$< \chi_{0,in}(t)\chi_{0,out}(t') >= e^{-ip_0t}e^{+ip_0't'} \sim \sin(\frac{p_0+p_0'}{2}\Delta t)$$

with $\Delta t = t - t'$.

The 2-point CF with nonzero momentum is (this time- Laplace transformation):

$$< \chi_0 \chi_0 > (p) = \int d(\Delta t) < \chi_0(t) \chi_0(t') > e^{-p\Delta t} \sim \frac{(p_0 + p_0')/2}{p^2 + ((p_0 + p_0')/2)^2} \tag{6.39}$$

In the limit $p \to 0$ the 2-point CF becomes a delta function:

$$< \chi_0 \chi_0 > (p) \to \frac{1}{|p_0 + p_0'|} \sim \delta(p_0 + p_0') \tag{6.40}$$

For $p \neq 0$ we obtain poles for the 2-point CF that describe discrete states.

7 Star–Matrix Models

The star-matrix models were considered first in [96] [97] in connection with the q-Potts model and percolation problem.Another field of interest is the "induced" QCD which in the large N limit is equivalent with the star-model on Bethe tree. The star-models are also a direct generalization of the $c = 1$ matrix model, which is the particular case for $q = 1$.

The difficulties in solving exactly these models are related with the multiple powers of the Vandermonde determinants in the partition function.In the Q-matrix approach, another problem related with the first , is the choosing of a proper basis in which to define consistently the Q-matrices.

In this chapter we show that it is possible to apply a modified Q-matrix approach (see section 4.6) and to define consistently the Q-matrices. The method gives consistent results for the cases where other methods can be applied :the saddle-point method. Schwinger-Dyson approach, gaussian integration etc. The gaussian models and matrix model on Bethe tree give similar results with the previous ones. However, our method can be applied in all genera, and permits a more precise study of particular parts of the Bethe tree. The chapter is based on the original work [95].

In the last section we consider the matrix formulation of the q-Potts-like model and study it as a polymer on a random surface. It is important that we still get consistent results for this model. It is a non-trivial result, because the coupling conditions are overdetermined: the Q-matrices depend on $3q + 3$ variables and we have $4q + 3$ equations. More generally when the central potential V_0 is gaussian and the lateral potentials V_α are of order n we have $3q + n$ variables and $(n + 1)q + 3$ equations always with combatible solutions.

7.1 The gaussian model on Bethe tree

Kazakov and Migdal [98] have obtained the so-called induced "induced QCD"- a matrix model embedded in the regular D-dimensional lattice. For gaussian potential , the model was solved by Gross [99]. In the limit $N \to \infty$ the Kazakov-Migdal model with generic potential is equivalent to the matrix model with a Bethe tree target space [100].

Because the model was studied only with the saddle-point method, it is interesting to study it in a different framework , that of Q-matrices approach. It gives higher accuracy in studying different regions of Bethe tree and also permits computations in higher genera.

Our model is the inhomogenous version of matrix model on Bethe tree, at every level of branch we assigne a specific partition function and propagator.

The gaussian matrix model on Bethe tree is $(i = 2j)$:

$$Z = \int dM_i \exp \mathrm{Tr}[\sum_i (t_i M_i^2 + u_i M_i) + \sum_{<ij>} c_i M_i M_j] \tag{7.1}$$

where $< ij >$ denotes the permitted links of Bethe tree. and c_i are the coupling constants of the i-th level branch.

The coupling conditions are:

$$\begin{aligned} qP_i + 2t_iQ_i + u_i + c_iQ_{i-1} + (q-1)c_{i+1}Q_{i+1} &= 0 \\ q\overline{P}_{i+1} + 2t_{i+1}Q_{i+1} + u_{i+1} + c_{i+1}Q_i + (q-1)c_{i+2}Q_{i+2} &= 0 \end{aligned} \tag{7.2}$$

Introducing Q-matrices with the form:

$$\begin{aligned} Q_i &= S_iI_+ + a_iI_0 + f_i\epsilon_- \\ Q_{i+1} &= b_i/R_iI_+ + d_iI_0 + R_i\epsilon_-, \end{aligned} \tag{7.3}$$

we get from the coupling conditions the following equations for coefficients:

$$\begin{aligned} 2t_iS_i + c_i\frac{b_{i-1}}{R_{i-1}} + (q-1)c_{i+1}\frac{b_i}{R_i} = 0 \\ 2t_ia_i + u_i + c_id_{i-1} + (q-1)c_{i+1}d_i = 0 \\ 2t_if_i + \frac{qn}{S_i} + c_iR_{i-1} + (q-1)c_{i+1}R_i = 0 \\ 2t_{i+1}R_i + c_{i+1}f_i + (q-1)c_{i+2}f_{i+1} = 0 \\ 2t_{i+1}d_i + u_{i+1} + c_{i+1}a_i + (q-1)c_{i+2}a_{i+1} = 0 \\ 2t_{i+1}\frac{b_i}{R_i} + \frac{qn}{R_i} + c_{i+1}S_i + (q-1)c_{i+2}S_{i+1} = 0 \end{aligned} \tag{7.4}$$

Let suppose that all coefficients for various i are equal $c_i = c, t_i = t, u_i = u$.In this case the equations reduce to the following set of equations:

$$\begin{aligned} f(S_i) + g(R_{i-1}) = 0, \qquad f(R_i) - g(S_i) = 0 \\ f(a_i) + K = 0, \qquad f(d_i) + K = 0 \\ f(\frac{b_i}{R_i}) - \frac{2qnt}{c^2}\frac{1}{R_i} = 0, \qquad f(f_i) - \frac{2qnt}{c^2}\frac{1}{S_i} = 0 \end{aligned} \tag{7.5}$$

where the constant $K = (q - \frac{2t}{c})u$ and the functions f, g are:

$$\begin{aligned} f(x_i) &= x_{i-1} + \left[2(q-1) - 4\left(\frac{t}{c}\right)^2\right] x_i + (q-1)^2 x_{i+1} \\ g(x_i) &= \frac{qn}{c}(\frac{1}{x_i} + \frac{q-1}{x_{i+1}}) \end{aligned} \tag{7.6}$$

7.1.1 Fractal regime

In the scaling limit $R_i \to R, S_i \to S, b_i \to b, f_i \to f, d_i \to d, a_i \to a$ and (we take $n = 1$):

$$\begin{aligned} RS = \frac{q^2c}{4t^2 - (qc)^2}, \\ b = fS = -\frac{qt}{4t^2 - (qc)^2} \\ a = d = \frac{qc - 2t}{4t^2 - (qc)^2}u \end{aligned} \tag{7.7}$$

This case represents the limiting case of a matrix model on the fractal curve of Bethe tree. We can define the free energy on the unit length:

$$F_{frac} = \log RS = \log \frac{q^2 c}{4t^2 - (qc)^2} \tag{7.8}$$

We observe a singularity of the free energy at $(2t/c)^2 = 1$. This critical point is the analog of critical point for $c = 1$ matrix model (or q-multimatrix model) at the self-dual radius $R^2 = 1$ We must consider the physical domain when $2t < qc$ ($t < 0$ to have a well-defined path-integral). We choose $c < 0$. We see that the other region $2t > qc, c > 0$ is not reached.

We can calculate the 1-point correlation function (for simplicity we take $u = 0$ and genus 0)

$$\begin{aligned} \left\langle \mathrm{Tr} M_i^{2k} \right\rangle_0 = \left\langle \mathrm{Tr} M_{i+1}^{2k} \right\rangle_0 &= \mathrm{Tr}(Q^{2k}) = b^k \mathrm{Tr}(I_+ + \epsilon_-)^{2k} \\ &= \frac{(2k)! N^{k+1}}{k!(k+1)!} \left(\frac{-qt}{4t^2 - (qc)^2} \right)^k \end{aligned} \tag{7.9}$$

7.1.2 Asymptotic regime

We have considered the fractal curve or a tiny strip of surface which is filling densily the extremity of the Bethe tree. We can study a larger strip which tends asymptotically to the fractal curve.

In the large N limit we can scale the coefficients $R_i \to \sqrt{N} R_i(x), S_i \to \sqrt{N} S_i(x), b_i \to N b_i(x), f_i \to \sqrt{N} f_i(x), d_i \to d_i(x), a_i \to a_i(x)$, where $x = n/N$. We can see now that all second terms in the equations (7.6) are proportional with $x, 0 \leq x < 1$ and can be considered as perturbations. We neglect the function g. In this case we can solve the recursion relations (7.6) with the result:

$$\begin{aligned} R_i &= r^i, S_i = s^i, \\ a_i &= r^i - \frac{2t}{q^2 c} K, d_i = s^i - \frac{2t}{q^2 c} K, \; K = \frac{qcu}{2t} - u \\ b_i &= b^i r^i, f_i = f^i \end{aligned} \tag{7.10}$$

where $r = r_+, s = r_-$ or viceversa with $r_\pm$ being the solution of second order eq.:

$$(q-1)^2 r_\pm^2 + (2(q-1) - \frac{4t^2}{c^2}) r_\pm + 1 = 0 \tag{7.11}$$

and $b = b_+, f = b_-$ or viceversa with $b_\pm$ being the solution of second order eq.:

$$(q-1)^2 b_\pm^2 + (2(q-1) + \frac{2tqx}{c^2} - \frac{4t^2}{c^2}) b_\pm + 1 = 0 \tag{7.12}$$

We see that the alternating Q_i, Q_{i+1} in the asymptotic regime can be interchanged; hence it does not matter if i is odd or even.

We can define also in this regime the free energy on the unit length:

$$F_{i,i+2} = \log(q-1)^{2i} R_i S_i = 0 \tag{7.13}$$

(q must be bigger than 2 to have at least one branch).The free energy is 0 and is different from F_{frac}. This is understandable because F_{frac} is proportional with x, but we have considered the case when $x = 0$. hence $F_{i,i+2} = 0$. To see if $F_{i,i+2}$ really tends to F_{frac} we must include the perturbation in x.

As we said before, our model differs from the one studied by Gross and Boulatov [99][100]. Their model is homogeneous ;for the Bethe lattice with coordonation number $2D$ their partition function is:

$$Z = \int dM_i \exp \mathrm{Tr}[-\sum_i \frac{m^2}{2} M_i^2 + \sum_{<ij>} M_i M_j] = \int dM Z(M)^{2D} \exp\left(-\frac{m^2}{2}\mathrm{Tr}M^2\right)$$

where the partition function of a branch $Z(M)$ satisfies the equation:

$$Z(M) = \int dM' Z(M')^{2D-1} \exp \mathrm{Tr}(-\frac{m^2}{2} M'^2 + M'M)$$

Our model is inhomogenous. Every branch of different level i has a different partition function $Z_i(M)$. Hence the total partition function is:

$$Z = \int dM Z(M)^q \exp\left(-\frac{t}{c}\mathrm{Tr}M^2\right)$$

and $Z_i(M)$ satisfies the equation:

$$Z_i(M) = \int dM' Z_{i+1}(M')^{q-1} Z_{i-1}(M') \exp \mathrm{Tr}(-\frac{t}{c} M'^2 + M'M) \tag{7.14}$$

We see that the partition function for i-th level branch is expressed not only in terms of higher level branches $(i+1)$, but we have also the back-reaction on the lower level branches $(i-1)$.We also observe that the eq.(7.14) is different for i odd or even.

Solving the equation is equivalent with the first two equations (7.6). Solving them as recursion equations we get the power-like solution (7.11) $Z_j(M) = r^{j\mathrm{Tr}M^2}$, where r satisfies the second order equation (7.11). If we solve (7.6) as differential equations (after proper scaling when $|(i+1) - i| \ll i$) we get the exponential solution $Z_j(M) = \exp(jr\mathrm{Tr}M^2)$ where r is (from equation (7.11)):

$$r_\pm = \frac{2\left(\frac{t}{c}\right)^2 - (q-1) \pm 2\left(\frac{t}{c}\right)\sqrt{\left(\frac{t}{c}\right)^2 - (q-1)}}{(q-1)^2}$$

The signs alternate for i odd and even. This result can be compared with that of homogeneous model if we identify $t/c = m^2/2, q = 2D$. The partition function per branch for homogeneous model behaves as $Z(M) = \exp(-\alpha\mathrm{Tr}M^2)$, where α is:

$$\alpha_\pm = \frac{m^2(D-1) \pm D\sqrt{m^4 - 4(2D-1)}}{2D-1}$$

We have an interesting property in the asymptotic regime:the coefficient R_k near the point $\frac{1}{q-1}(t/c)^2 = 1/2$ has a slow oscilation with the period $T_{\Delta k}$.

The coefficient R_k is directly related with the free energy of branches of level between k and $k+1$:

$$F_{k,k+1} = \log(q-1)^{2k} R_k$$

S_k has a complementary oscilation such that $F_{k,k+1}(R_k) + F_{k+1,k+2}(S_k) = F_{k,k+2} = 0$. This behaviour is typical to an antiferromagnet: R_k, S_k are like spin-up and spin-down configurations which group pair-wise having a total energy zero.

We take for convenience $R_0 = 1, R_1 = 0$. The coefficient R_k is:

$$R_k = \frac{r_+^{k+1} - r_-^{k+1}}{r_+ - r_-}$$

where $r_\pm$ are the solutions of the equation (7.11). We introduce the notation $\beta = \frac{1}{q-1}(t/c)^2$ and consider the region $0 \leq \beta \leq 1$. In this case:

$$r_\pm = \frac{1}{(q-1)^2} e^{\pm\omega}, \quad \arctan\omega = \frac{2\sqrt{\beta(1-\beta)}}{2\beta - 1} \tag{7.15}$$

In this region $R_k = (q-1)^{-2k}\frac{\sinh k\omega}{\sinh\omega}$ has a fast decaying in amplitude (for $q > 2$) and an oscilatory character with pulsation ω . For $\beta \sim \frac{1}{2}$ the expression (7.32) is singular and the pulsation is $\omega \sim \frac{\pi}{2}$. If we take $\beta = \frac{1}{2} + \epsilon$ then $\omega = \frac{\pi}{2} + \Delta\omega$ and $\Delta\omega \sim \epsilon$. This property induces a modulation of the oscilation with the period :

$$T_{\Delta k} \sim \left(1 - \frac{2}{q-1}\left(\frac{t}{c}\right)^2\right)^{-1} \tag{7.16}$$

For $\beta \to 1/2$ the modulation dissapears because the period $T_{\Delta k} \to \infty$.

7.2 The q-Potts-like model

q-state Potts spins are an interesting generalization of the Ising model ($q = 2$) . On planar random lattice they were studied first time by Kazakov [96] [97]. The cases $q \to 0$ and $q \to 1$ represent the models of tree-polymers, respectively that of percolation. The $q = 2$ case , that of Ising model on random lattice, can be expressed in terms of the 2-matrix model.

The partition function of q-state Potts model on a random lattice is:

$$Z(g,\beta,H) = \sum_n g^n \sum_{\{G^{(n)}\}} \sum_{\{\sigma\}} \exp[-\frac{\beta}{2}\sum_{k,j}^{n} G_{kj}^{(n)}(\delta_{\sigma_k\sigma_j} - 1) + H\sum_{k}^{n}(\delta_{1,\sigma_k} - 1)] \tag{7.17}$$

where the summ run over all triangulations with n triangles $\{G^{(n)}\}$ and all spin configurations $\{\sigma\}$. $G_{kj}^{(n)}$- is the adjancency matrix of planar lattice with n vertices, β- inverse temperature, H- magnetic field .

For zero magnetic field, the partition function can be expressed in terms of the matrix model:

$$Z = \int \prod_{\alpha=1}^{q} dM_\alpha \exp[2c \sum_{\alpha>\beta}^{q} M_\alpha M_\beta - \sum_{\alpha=1}^{q}(M_\alpha^2 + g/3M_\alpha^3)] \tag{7.18}$$

The partition functions (7.18) and (7.17) are equal due to the equivalence of the Feynman graphs of matrix model on dual lattice with inverse temperature $\beta_{dual} = \log(1+q(e^{\beta}-1)^{-1})$ and the Boltzmann weights of Potts model on the original lattice.

Introducing a new matrix M_0 the previous integral can be rewritten as:

$$Z = \int \prod_{\alpha=1}^{q} dM_\alpha dM_0 \exp[\tilde{c}M_0 \sum_{\alpha=1}^{q} M_\alpha - M_0^2/2 - \sum_{\alpha=1}^{q}(M_\alpha^2/2 - \tilde{g}/3M_\alpha^3)] \tag{7.19}$$

with $\tilde{c}^2 = c/(1+c), \tilde{g}^2 = g^2/(2(1+c))^3$. The coupling constant c is connected with the inverse temperature β by the formula

$$c^2 = (e^\beta + q - 1)^{-1}$$

We consider the star-matrix model with partition function:

$$Z = \int \prod_{\alpha=1}^{q} dM_\alpha dM_0 \exp[t_0 M_0^2 + u_0 M_0 + M_0 \sum_{\alpha=1}^{q} c_\alpha M_\alpha + \\ + \sum_{\alpha=1}^{q}(s_\alpha M_\alpha^3 + t_\alpha M_\alpha^2 + u_\alpha M_\alpha)] \tag{7.20}$$

The coupling conditions are:

$$\begin{aligned} qP_0 + 2t_0Q_0 + u_0 + \sum_{\alpha=1}^{q} c_\alpha Q_\alpha &= 0 \\ \overline{P}_\alpha + 3s_\alpha Q_\alpha^2 + 2t_\alpha Q_\alpha + u_\alpha + c_\alpha Q_0 &= 0, \alpha = 1, \dots q \end{aligned}$$

With the following parametrization of Q-matrices:

$$\begin{aligned} Q_0 &= I_+ + a_0 I_0 + a_1 I_- + a_2 I_{-2} \\ Q_\alpha &= b_\alpha/R_\alpha I_+ + d_\alpha I_0 + R_\alpha I_-, \alpha = 1, \dots q \end{aligned}$$

we arrive at following equations: .

$$\left.\begin{aligned} 3s_\alpha R_\alpha^2 + c_\alpha a_2 = 0 \\ 6s_\alpha R_\alpha d_\alpha + 2t_\alpha R_\alpha + c_\alpha a_1 = 0 \\ 6s_\alpha b_\alpha d_\alpha + 2t_\alpha b_\alpha + n + c_\alpha R_\alpha = 0 \\ 3s_\alpha(d_\alpha^2 + 2b_\alpha) + 2t_\alpha d_\alpha + u_\alpha + c_\alpha a_0 = 0 \end{aligned}\right\}, \quad \alpha = 1, \dots q \tag{7.21}$$

$$\left.\begin{aligned} 2t_0 + \sum \tfrac{c_\alpha b_\alpha}{R_\alpha} = 0 \\ 2t_0 a_0 + u_0 + \sum c_\alpha d_\alpha = 0 \\ 2t_0 a_1 + qn + \sum c_\alpha R_\alpha = 0 \end{aligned}\right\} \tag{7.22}$$

7.2.1 Symmetric case

We solve the special symmetric case when $s_\alpha = s, t_\alpha = t, u_\alpha = u, c_\alpha = c$. In this case $R_\alpha = R, d_\alpha = d, b_\alpha = b$.

We can express all the coefficients in terms of only two of them R and d:

$$a_2 = -\frac{3s}{c}R^2, a_1 = -\frac{q(n+cR)}{2t_0}$$
$$a_0 = -\frac{u_0+qcd}{2t_0}, b = -\frac{2t_0R}{qc} \tag{7.23}$$

R and d satisfy a system of 2 non-linear equations:

$$\begin{aligned} d &= -\frac{t}{3s} + \frac{cq}{12st_0}(c+\frac{n}{R}) \\ R &= \frac{qc}{4t_0}d^2 + (\frac{qct}{6st_0} - \frac{(qc)^2c}{24st_0^2})d + \frac{qcu}{12st_0} - \frac{(qc)cu_0}{24st_0^2} \end{aligned} \tag{7.24}$$

Hence all coeficients of the Q-matrices can be expressed in terms of the $R_\alpha = R$ coeficient which satisfies a third-order equation:

$$R^3 + R^2\frac{qc}{(12st_0)^2}\left(\frac{(4tt_0-qc^2)^2}{4t_0} + 6s(cu_0 - 2t_0u)\right) - \frac{(qc)^3n^2}{4t_0(12st_0)^2} = 0 \tag{7.25}$$

For symplicity we choose $6s = 2t_0 = 2t = -1, u_0 = 0, c^2 = 1/q$. If we denote $z = R/qc$ we have instead (7.25) the equation:

$$z^3 - z^2u + 1/2 = 0 \tag{7.26}$$

Two roots coincide $z_1 = z_2 = z_* = 1$ at the critical point when the "cosmological constant" $u_* = 3/2$. Near critical point, the variable R related with the free energy will scale as $R - R_* \sim (u-u_*)^{2/(p+q-1)}$ for the (p,q) matter models coupled with the 2d gravity.

Expanding u and z near critical point:

$$u = u_* + \mu\delta^2, z = z_* + Z\delta^2$$

we get in the lowest order of δ the relation $\mu = 3Z^2$. This means that the variable R scales as $R - R_* \sim (u-u_*)^{1/2}$.This critical point (the continuum limit) corresponds to the pure gravity model or ϕ^3 1-matrix model.The results remain true for arbitrary s, t_0, t, u_0, c in the symmetric case.

7.2.2 Non-symmetric case

We can write the system of equations (7.21), (7.22) in a different way which will permit to remain with a single type of variables X_α:

$$X_\alpha = 3s_\alpha d_\alpha + t'_\alpha \tag{7.27}$$

We denote by:

$$u'_\alpha = u_\alpha - \frac{c_\alpha u_0}{2t_0}, c_{\alpha\beta} = -\frac{c_\alpha c_\beta}{4t_0}, t'_\alpha = t_\alpha - \frac{c_\alpha^2}{4t_0}$$

Then we can express the variables R_α in terms of X_α from the system:

$$X_\alpha R_\alpha + \sum_{\beta\neq\alpha} c_{\alpha\beta} R_\beta = \frac{qn}{4t_0}, \quad \alpha = 1 \dots q \tag{7.28}$$

and also the variables b_α in terms of X_α:

$$\frac{2b_\alpha X_\alpha + n}{R_\alpha} + \sum_{\beta\neq\alpha} c_{\alpha\beta}\frac{2b_\beta}{R_\beta} = 0, \quad \alpha = 1 \dots q \tag{7.29}$$

We remain with the system:

$$\frac{X_\alpha^2}{3s_\alpha} + 3s_\alpha(2b_\alpha) + u'_\alpha - \frac{t'^2_\alpha}{3s_\alpha} + \sum_{\beta\neq\alpha} \frac{2c_{\alpha\beta}}{3s_\alpha}(X_\beta - t'_\beta) = 0, \quad \alpha = 1 \dots q \tag{7.30}$$

We have the suplimentary constraint which can be imposed on the variables X_α:

$$\frac{3s_\alpha}{c_\alpha} R_\alpha^2 = const, \quad \alpha = 1 \dots q \tag{7.31}$$

When we tend to the symmetric case with $c_{\alpha\beta} = c, t'_\alpha = 0, 6s_\alpha = 2t_0 = -1, u'_\alpha = u + c^2(q-1)^2$ we get from the system (7.31) the following equation:

$$2(X + c(q-1))^2 - \frac{1}{2(X + c(q-1))} - u = 0$$

With the notation $z = R/q = -1/(2X + 2c(q-1))$ we obtain the equation (7.26).

q=2 case

We argue that the critical behaviour of the case $q = 2$ coincides with the Ising model on random ϕ^3 lattice.

Solving the system (7.29) we get:

$$2b_1 = -\frac{n(X_2 - cr)}{X_1X_2 - c^2}, \quad 2b_2 = -\frac{n(X_1 - c/r)}{X_1X_2 - c^2}$$

where:

$$r = \frac{R_1}{R_2} = \left(\frac{c_1 s_2}{c_2 s_1}\right)^{1/2} \tag{7.32}$$

Because R_1, R_2 are:

$$R_1 = -\frac{(X_2 - c)}{X_1X_2 - c^2}\frac{2n}{4t_0}, \quad R_2 = -\frac{(X_1 - c)}{X_1X_2 - c^2}\frac{2n}{4t_0}$$

the relation (7.32) can be rewritten as:

$$r = \frac{X_2 - c}{X_1 - c}$$

We remain with the system:

$$\begin{aligned} \frac{X_1^2}{3s_1} + \frac{2c}{3s_2}X_2 - 3s_1 n\frac{X_2 - cr}{X_1X_2 - c^2} + u_1' &= \frac{t_1'^2}{3s_1} + 2c\frac{t_2'}{3s_2}, \\ \frac{X_2^2}{3s_2} + \frac{2c}{3s_2}X_1 - 3s_2 n\frac{X_1 - c/r}{X_1X_2 - c^2} + u_2' &= \frac{t_2'^2}{3s_2} + 2c\frac{t_1'}{3s_1} \end{aligned} \tag{7.33}$$

We point out the great similarity between this system and that of the Ising model on ϕ^3 lattice [101] (which corresponds to the case $q = 2$ Potts model). To show this, we integrate the intermediate matrix M_0 in the relation (7.20).We get the two-matrix model:

$$Z = \int dM_1 dM_2 \exp[\sum_{\alpha=1}^{2}(s_\alpha M_\alpha^3 + t'_\alpha M_\alpha^2 + u'_\alpha M_\alpha) + 2cM_1M_2]$$

with the previous notations for $t'_\alpha, u'_\alpha, c = c_{12}$.

Solving the coupling conditions we remain with three equations:

$$\begin{aligned} X_1X_2 &= c^2 + \frac{nc}{2R}, \\ \frac{X_1^2}{3s_1} + (\frac{2c}{3s_2} - \frac{3s_1 4R}{2c})X_2 + u_1' &= \frac{t_1'^2}{3s_1} + 2c\frac{t_2'}{3s_2}, \\ \frac{X_2^2}{3s_2} + (\frac{2c}{3s_2} - \frac{3s_2 4R}{2c})X_1 + u_2' &= \frac{t_2'^2}{3s_2} + 2c\frac{t_1'}{3s_1} \end{aligned} \tag{7.34}$$

Introducing the expression of $2R = nc/(X_1X_2 - c^2)$ in the last two equations ,we see that they differ from the equations (7.34) only by the terms containing the r quantity. We expect that these terms are only an artefact of the different basis of orthogonal polynomials we have chosen and that they do not change the critical behaviour of the free energy. The variable R scales at the critical point as $R - R_* \sim (u - u_*)^{1/3}$ (we take $u_1' = u, u_2' = 0$).

q=3 case

This is the first non-trivial member of the q-Potts-type set of models.It can not be derived from the two-matrix model or other more complicated chain models.It represents the matrix model on the Dynkin diagram D_3.

We have not managed to derive the critical behaviour.However, for further developments, we write down the system of equations which gives the critical scaling.

We introduce the function:

$$Y_{123}(1, r_1, r_2) = X_2X_3 - c_{23}^2 + r_1(c_{13}c_{23} - c_{12}X_3) + r_2(c_{12}c_{23} - c_{13}X_2) \tag{7.35}$$

The indices 123 of the function Y_{123} are related with the indices of the variables $X_\alpha, c_{\alpha\beta}$. The function is symmetric only in the last two indices: $Y_{123}(1, r_1, r_2) = Y_{132}(1, r_2, r_1)$.

We also introduce:

$$Y = X_1X_2X_3 + 2c_{12}c_{13}c_{23} - (c_{13}^2X_2 + c_{23}^2X_1 + c_{12}^2X_3)$$

Then the coupling conditions (7.30) in this case are (for simplicity we consider $t'_\alpha = 0$):

$$\begin{aligned}
\frac{X_1^2}{3s_1} + \frac{2c_{12}}{3s_2}X_2 + \frac{2c_{13}}{3s_3}X_3 - 3s_1n\frac{Y_{123}(1,r_1,r_2)}{Y} + u'_1 &= 0, \\
\frac{X_2^2}{3s_2} + \frac{2c_{12}}{3s_1}X_1 + \frac{2c_{23}}{3s_3}X_3 - 3s_1n\frac{Y_{231}(1,r_2/r_1,1/r_1)}{Y} + u'_2 &= 0, \\
\frac{X_3^2}{3s_3} + \frac{2c_{13}}{3s_3}X_1 + \frac{2c_{23}}{3s_2}X_2 - 3s_1n\frac{Y_{312}(1,r_1/r_2,1/r_2)}{Y} + u'_3 &= 0,
\end{aligned} \tag{7.36}$$

The variables r_1, r_2 can be written in terms of X_α:

$$r_1 = \frac{Y_{123}(1,1,1)}{Y_{231}(1,1,1)}, r_2 = \frac{Y_{123}(1,1,1)}{Y_{312}(1,1,1)} \tag{7.37}$$

This model was also studied by another techique by Daul[102].

7.3 Conclusions

We have studied the inhomogenous matrix model on the Bethe tree and have obtained similar results with the homogeneous model in the saddle-point method.We have two regimes: the fractal and the asymptotic .In the asymptotic regime, we get the exponential behaviour for the partial partition function $Z_j(M) = \exp(jr\mathrm{Tr}M^2)$, where r satisfies a second order equation like in the homogeneous case. For large j- the level of the branch- we expect that the properties of the model become independent of j and is a transition from the asymptotic regime to the fractal regime. Also in the asymptotic regime when $\beta = \frac{1}{q-1)}(t/c)^2 = 1/2$ we have a slow oscilation of the free energy with the period $T \sim (1/2 - \beta)^{-1}$.

For the q-Potts-like model with arbitrary q we write down the general coupling conditions. For the special cases $q = 2$ and $q = 3$ we solve the coupling conditions in terms of only one type of variables X_α. For $q = 2$ we have argued that the system has the critical behaviour of the Ising model on ϕ^3 lattice .

8 Quantum Chaos in Multi–Matrix Models

This chapter is a revised version of the work [103], which includes more explanations and applications of the general method of orthogonal polynomials.

In heavy nuclei,the complicated many-body interactions lead to statistical theories which explain only the average properties. One of these theories is the random matrix hypothesis [104][105](see also a more recent book [106]). It supposes that the nuclear hamiltonian in an arbitrary basis of functions is a $N \times N$ matrix with N large and elements distributed at random. The joint probability function of the eigenvalues $\lambda_1, \dots \lambda_N$ of this matrix model is given by:

$$P(\lambda_1, \dots \lambda_N) = \exp(-\sum_{i=1}^{N} \lambda_i^2) \prod_{i<j} (\lambda_i - \lambda_j)^\beta \tag{8.1}$$

where $\beta = 1, 2, 4$ for orthogonal, hermitean and unitary ensembles respectively.

This probability distribution gives the familiar phenomenon of level repulsion: the likelihood of having neighboring energy levels separated by an energy spacing $\Delta\lambda_{ij} = \lambda_i - \lambda_j$ becomes vanishingly small as $\Delta\lambda_{ij} \to 0$. For two levels λ_1, λ_2, the plot of the probability $P(\Delta\lambda$ in terms of energy spacing $\Delta\lambda = \lambda_1 - \lambda_2$ shows a rise from zero for $\Delta\lambda = 0$, reaches a peak (known as Wigner's surmise) and then decreases rapidly. This behaviour is completely different from the classical one, given by the Poisson distribution $P(\Delta\lambda) = \exp(-\Delta\lambda)$.

Integrating over eigenvalues $\lambda_{k+1} \dots \lambda_N$ we get the joint distribution function for few levels:

$$P(\lambda_1, \dots \lambda_k) = \int d\lambda_{k+1} \dots d\lambda_N P(\lambda_1, \dots \lambda_N) \tag{8.2}$$

All these joint distribution functions can be expressed in terms of the Dyson correlation function $K(\lambda_1, \lambda_2)$ as:

$$P(\lambda_1, \dots \lambda_k) = \sum_{\sigma} (-1)^\sigma K(\lambda_1, \lambda_{\sigma_1}) \dots K(\lambda_k, \lambda_{\sigma_k})$$

where σ is the permutation of k levels.

In the special case $k = 1$ the Dyson correlation function coincides with level density $K(\lambda, \lambda) = P(\lambda)$.

The density of levels for the 1-matrix model satisfies the semicircular law:

$$P(\lambda) = \sqrt{\beta N/2 - \lambda^2}$$

and the Dyson correlation function behaves as :

$$K(\lambda - \frac{1}{2}\sigma, \lambda + \frac{1}{2}\sigma) \simeq \frac{\sin(\pi\sigma P(\lambda))}{\pi\sigma(\beta N/2)} \quad \text{for} \quad \sigma \ll \lambda$$

Many experimental data of level distribution in nuclei confirm the statistical properties of the random matrix theory. There exists a universality of level fluctuation laws , as described by the Dyson correlation function. The fluctuation properties are shared by broad classes of models: several chaotic models [107], mesoscopic systems [108].etc. The random matrix hypothesis is in some respects a disappointing theory: although it predicts beautifully the observed level fluctuations, it fails to describe adequately the density of levels. The semicircular law was never observed in the experiments .

A possible resolution of the problem is to consider instead of one random matrix more random matrices in interaction. As we will see , even a small interaction gives a qualitatively new behaviour for the level density.

An interesting generalization of the random matrix hypothesis is to consider q matrices describing q nuclear systems in interaction.The total action of such system is:

$$S_1 = \sum_{\alpha=1}^{q}\sum_{i=1}^{N}(t_\alpha(\lambda_i^{(\alpha)})^2 + u_\alpha\lambda_i^{(\alpha)}) + \sum_{\alpha=1}^{q-1}\sum_{i=1}^{N} c_\alpha\lambda_i^{(\alpha)}\lambda_i^{(\alpha+1)} \tag{8.3}$$

This system describes a chain of matrices with neighbour interaction.We can add a term describing the two-body interaction of constituent nuclear subsystems:

$$\sum_{|\alpha-\beta|\neq 1}\sum_{i=1}^{N} c_{\alpha,\beta}\lambda_i^{(\alpha)}\lambda_i^{(\beta)}$$

We have different sets of energy levels $\lambda_1^{(\alpha)},\ldots\lambda_N^{(\alpha)}, \alpha = 1\ldots q$ with distribution probability:

$$P(\lambda_1^{(1)},\ldots\lambda_N^{(1)}\ldots\lambda_1^{(q)},\ldots\lambda_N^{(q)}) = \exp(S)\prod_{i<j}(\lambda_i^{(1)} - \lambda_j^{(1)})(\lambda_i^{(q)} - \lambda_j^{(q)}) \tag{8.4}$$

We have level repulsion only for the first and last energy level sets. Hence for this model the intermediate energy level sets are "classical" and interact with "quantum" first and last energy level sets.Integrating over all intermediate matrices we remain with a two-matrix model.

Kharchev and others [109] have considered the so-called conformal matrix models that contain additional repulsion terms also for intermediate matrices .

Another special random matrix model is the star-matrix model having the action:

$$S_2 = \sum_{i=1}^{N}(t_0(\lambda_i^{(0)})^2 + u_0\lambda_i^{(0)}) + \sum_{\alpha=1}^{q}\sum_{i=1}^{N}(t_\alpha(\lambda_i^{(\alpha)})^2 + u_\alpha\lambda_i^{(\alpha)}) + \sum_{\alpha=1}^{q}\sum_{i=1}^{N} c_\alpha\lambda_i^{(\alpha)}\lambda_i^{(0)} \tag{8.5}$$

The joint distribution of this model reduces again to that of the 2-matrix model.

8.1 Quantum Chaos in two-matrix model

In the more wide context of quantum chaos, the one-matrix model describes the statistical properties of the non-integrable model with hamiltonian[110]:

$$H = H_0 + u\phi$$

the hamiltonian H_0 is deterministic and ϕ is an external random perturbation.

For example,the one-matrix model could describe hydrogen in a random magnetic field.

The two-matrix model describes the correlation between the energy spectra of two systems with the hamiltonians:

$$H_\alpha = H_{\alpha 0} + u_\alpha \phi, \quad \alpha = 1, 2$$

For example, it could describe an electron moving in a ring threaded by a magnetic flux (described by the hamiltonians $H_{\alpha 0}$) and with the electron scattering on impurities in the ring (described by the random interaction ϕ). This example is important in the study of mesoscopic systems [111],[112].

In this section we show that the 2-matrix pure probability distributions $P(\lambda, \lambda').P(\mu, \mu')$ do not behave qualitatively different from those of 1-matrix model. The spectra are only shifted and rescaled due to the interaction ϕ.

Instead, the mixed probability distribution $P(\lambda, \mu)$ is specific only to the 2-matrix model. When the two spectra of H_1 and H_2 are rescaled identicaly (the rescalings are equal $a_1 = b_1$ in rel.(8.10) or $\epsilon = 0$ in rel.(8.22)), the hamiltonians $H_1 = H_2$ and reduces to that of the 1-matrix model. In this case $P(\lambda, \mu)$ reduces to the usual semicircular law.

When we have a small asymmetry in the spectra ($\epsilon \neq 0$ in rel. (8.22) we get a qualitatively new picture: $P(\lambda, \mu \sim \lambda)$ presents peaks and valeys as we increase the asymmetry ϵ. The observed behaviour is the quantum analog for chaotical behaviour of two interacting classical oscilators.

For equal frequencies, the two oscilators are resonant and behave as a single oscilator. From the quantum point of view, the spectra are equally spaced and there is no energy transfer between the oscilators. When the frequencies are slightly different the probability $P(\lambda, \mu)$ describes the quantum analog of beating.

We apply the orthogonal polynomial method [105] to our two-matrix model. Other useful approaches (not used here) to study these models are the saddle-point method [113] and the supersymmetric method [114].

We introduce the distribution probability:

$$P(\lambda_1 \dots \lambda_N, \mu_1 \dots \mu_N) = \exp \sum_{i=1}^{N} (V_1(\lambda_i) + V_2(\mu_i) + c_i \lambda_i \mu_i) \prod_{i<j} (\lambda_i - \lambda_j)(\mu_i - \mu_j) \qquad (8.6)$$

with $V_\alpha(\tau) = t_\alpha \tau^2 + u_\alpha \tau, \alpha = 1, 2$ and the joint distribution function :

$$P(\lambda_1 \dots \lambda_i, \mu_1 \dots \mu_j) = \int d\lambda_{i+1} \dots d\lambda_N d\mu_{j+1} \dots d\mu_N P(\lambda_1 \dots \lambda_N, \mu_1 \dots \mu_N) \qquad (8.7)$$

We study the particular case of the probability distributions for the correlation between only 2 energy levels.

8.2 The pure probability distributions $P(\lambda, \lambda')$, $P(\mu, \mu')$

We demonstrate that the level densities $P(\lambda)$, $P(\mu)$ and the joint probability distributions $P(\lambda_1, \lambda_2)$, $P(\mu_1, \mu_2)$ coincide with those of the hermitean 1-matrix model with distribution probability (8.1):

$$P(\lambda) = P_{Herm}(\lambda'), P(\mu) = P_{Herm}(\mu')$$
$$P(\lambda_1, \lambda_2) = P_{Herm}(\lambda'_1, \lambda'_2), P(\mu_1, \mu_2) = P_{Herm}(\mu'_1, \mu'_2) \quad (8.8)$$

The new joint probability distributions $P(\lambda, \mu)$ behave in a different way because we have not energy repulsion between levels of different sets.

If we set from the beginning the coupling $c = 0$ we get two independent orthogonal 1-matrix models and we have:

$$P(\lambda, \mu) = P_{Orth}(\lambda')P_{Orth}(\mu')$$

For $c \neq 0$, $P(\lambda, \mu)$ behaves like the 1-matrix Dyson correlation function:

$$P(\lambda, \mu) \sim K(\lambda, \mu)$$

When $c \to 0$,$P(\lambda, \mu)$ does not split into two orthogonal 1-matrix models.

Here λ', μ' are related with the coefficients of the potentials V_α in relation (8.6):

$$\lambda' = \frac{\lambda - a_0}{\sqrt{2a_1}}, \mu' = \frac{\mu - b_0}{\sqrt{2b_1}} \quad (8.9)$$

through the relations:

$$a_0 = \frac{c_1u_2 - 2t_2u_1}{4t_1t_2 - c_1^2}, b_0 = \frac{c_1u_1 - 2t_1u_2}{4t_1t_2 - c_1^2},$$
$$a_1 = -\frac{2t_2}{4t_1t_2 - c_1^2}, b_1 = -\frac{2t_1}{4t_1t_2 - c_1^2}. \quad (8.10)$$

In the rest of this section we demonstrate the above relations.

Orthogonal polynomials ξ and η:

$$\xi_n(\lambda_1) = \lambda_1^n + \ldots, \; \eta_n(\mu) = \mu^n + \ldots$$

satisfy the orthogonality condition:

$$\int d\lambda d\mu \xi_n(\lambda) e^{V_1+V_2+c\lambda\mu} \eta_m(\mu) = h_n \delta_{nm} \quad (8.11)$$

where $h_n = h_0 R^n$ and $R = c/(4t_2t_1 - c^2)$.

For quadratic potentials as those in (8.6) we have the following recursion relations of the orthogonal polynomials:

$$\begin{aligned} \lambda\xi_n(\lambda) &= \xi_{n+1}(\lambda) + a_0\xi_n(\lambda) + a_1\xi_{n-1}(\lambda) \\ \mu\eta_n(\mu) &= \eta_{n+1}(\mu) + b_0\eta_n(\mu) + b_1\eta_{n-1}(\mu) \end{aligned} \tag{8.12}$$

Solving these recursion relations it follows that ξ, η are Hermite functions:

$$\xi_n(\lambda) = \alpha_n H_n(\lambda'), \eta_m(\mu) = \beta_m H_m(\mu')$$

To get the proportionality coefficients α_n, β_m we use the orthogonality relation and the Gauss transform:

$$(2\pi u)^{-1/2}\int dy e^{-(x-y)^2/(2u)}H_n(y) = (1-2u)^{n/2}H_n((1-2u)^{-1/2}x) \tag{8.13}$$

Writing the action as:

$$\begin{aligned} S &= V_1(\lambda) + V_2(\mu) + c\lambda\mu = t_1\lambda^2 + u_1\lambda + t_2\mu^2 + u_2\mu + c\lambda\mu = \\ &= S_0 + t_2\left(\mu + \frac{u_2 + c\lambda}{2t_2}\right)^2 - \left(\frac{\lambda - a_0}{\sqrt{2a_1}}\right)^2 \end{aligned}$$

with:

$$S_0 = -\frac{t_1u_2^2 + t_2u_1^2 - cu_1u_2}{4t_1t_2 - c^2} \tag{8.14}$$

we have:

$$\begin{aligned} \int d\lambda d\mu \xi_n(\lambda) e^{V_1+V_2+c\lambda\mu}\eta_m(\mu) &= \\ = \alpha_n\beta_m\delta_{nm}e^{S_0}\frac{2\pi(2c)^n n!}{(4t_1t_2 - c^2)^{(n+1)/2}} &= h_0\delta_{nm}\left(\frac{c}{4t_1t_2 - c^2}\right)^n \end{aligned}$$

In conclusion

$$\begin{aligned} \xi_n(\lambda) &= (2\pi n!)^{-1/2}2^{-n/2}(\sqrt{2a_1})^n H_n(\lambda'), \\ \eta_m(\mu) &= (2\pi n!)^{-1/2}2^{-m/2}(\sqrt{2b_1})^n H_m(\mu') \end{aligned} \tag{8.15}$$

and

$$h_0 = (4t_1t_2 - c^2)^{-1/2}\exp(S_0)$$

We can now calculate the joint probability distribution $P(\lambda, \mu)$.Since we can write the two Vandermonde determinants in terms of orthogonal polynomials ξ_n, η_m

$$\Delta(\lambda)\Delta(\mu) = \sum_n \xi_n(\lambda_1)\Xi_n(\lambda_2\ldots\lambda_N)\sum_m \eta_m(\mu_1)\Theta_m(\mu_2\ldots\mu_N)$$

and the algebraic complements satisfy:

$$\int \prod_{i=2}^{N}(d\lambda_i d\mu_i)\Xi_n(\lambda_2 \dots \lambda_N)\Theta_m(\mu_2 \dots \mu_N) = (N-1)!\delta_{nm}$$

we get for the joint probability distribution the following relation:

$$P(\lambda, \mu) = \frac{1}{N} e^S \sum_{n=0}^{N-1} h_n^{-1} \xi_n(\lambda)\eta_n(\mu) \tag{8.16}$$

It is easy to derive the expression for symmetric joint distribution of pairs of eigenvalues in terms of $P(\lambda, \mu)$:

$$P(\lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_k) = \sum_{\sigma}(-1)^{\sigma} P(\lambda_1, \mu_{\sigma_1}) \dots P(\lambda_k, \mu_{\sigma_k}).$$

Integrating in $\lambda_{j+1} \dots \lambda_k$ we obtain the asymmetric joint distribution of eigenvalues:

$$P(\lambda_1, \dots, \lambda_j, \mu_1, \dots, \mu_k) = \sum_{\sigma}(-1)^{\sigma} P(\lambda_1, \mu_{\sigma_1}) \dots P(\lambda_j, \mu_{\sigma_j}) P(\mu_{\sigma_{j+1}}) P(\mu_{\sigma_k}).$$

In the limit of large N we have the usual behaviour of semi-circular law:

$$P(\lambda) = \sqrt{2N - \lambda'^2}, P(\mu) = \sqrt{2N - \mu'^2}.$$

8.3 The mixed probability distribution $P(\lambda, \mu)$

To calculate the joint distribution of two eigenvalues $P(\lambda, \mu)$ in the large N limit we associate it with the quantum mechanical system :

$$[\frac{1}{2}(p_\lambda^2 + p_\mu^2) + V_1(\lambda) + V_2(\mu) + c\lambda\mu]\phi_n(\lambda)\psi_m(\mu) = E_{nm}\phi_n(\lambda)\psi_m(\mu)$$

where $p_\lambda = i\partial/\partial\lambda, p_\mu = i\partial/\partial\mu$ are the usual momenta operators and

$$\begin{aligned} \phi_n(\lambda) &= \exp(-\lambda'^2/2)\eta_n(\lambda) \\ \psi_m(\mu) &= \exp(-\mu'^2/2)\xi_m(\mu) \end{aligned} \tag{8.17}$$

For $c = 0$ we get two decoupled quantum systems:

$$\begin{aligned} (p_\lambda^2 + \lambda'^2)\phi_n(\lambda) &= 2E_{1,n}\phi_n(\lambda) \\ (p_\mu^2 + \mu'^2)\psi_m(\mu) &= 2E_{2,m}\psi_m(\mu) \end{aligned} \tag{8.18}$$

where $E_{nm} = E_{1,n} + E_{2,m}$.

In the large N limit E_{nm} behaves like $\sim N$ and since we are searching for symmetric solutions we have $E_{1,n} = E_{2,m} \sim N/2$.The joint distribution of two eigenvalues $P(\lambda, \mu)$ will be:

$$P(\lambda, \mu) = \sqrt{2E_{1,n} - \lambda'^2}\sqrt{2E_{2,n} - \mu'^2} \tag{8.19}$$

or

$$P(\lambda, \mu) = \sqrt{N - \lambda'^2}\sqrt{N - \mu'^2}$$

We can see that for $c = 0$, $P(\lambda, \mu)$ is the product of density energy levels for orthogonal ensembles. If we integrate the last matrix ,we get the 1-matrix model .In our case this is equivalent with the condition $2E_{2,m} = p_\mu^2 + V_2(\mu) = 0$ in (8.18) or in other words the second system has no contribution in the joint distribution of two eigenvalues. The equation (8.19) is replaced by:

$$P(\lambda) = \sqrt{2N - \lambda'^2}.$$

For $c \neq 0$, after summing relation (8.16) and using the asymptotic formula (n large) for the Hermite polynomial (near the origin):

$$H_n = \epsilon^{x^2} \frac{\Gamma(n+1)}{\Gamma(n/2+1)} \cos(\sqrt{2n+1} - \frac{n\pi}{2}) + O(1/\sqrt{n})$$

we obtain (up the exponent $S + (\lambda'^2 + \mu'^2)/2$):

$$P(\lambda, \mu) \sim \frac{\sin(\sqrt{2N}(\lambda' - \mu'))}{\pi N(\lambda' - \mu')}, \quad \lambda, \mu \text{ near } 0 \tag{8.20}$$

We also get for arbitrary λ, μ with $|\lambda - \mu| \ll \lambda, \mu$:

$$P(\lambda, \mu) \sim \frac{\sin(\sqrt{2N - (\alpha\lambda)^2}(\alpha\lambda)\epsilon)}{\pi N \epsilon(\alpha\lambda)} \tag{8.21}$$

where:

$$\begin{aligned} \epsilon &= \frac{1}{\sqrt{2a_1}} - \frac{1}{\sqrt{2b_1}} \\ \alpha &= \frac{1}{2}\left(\frac{1}{\sqrt{2a_1}} + \frac{1}{\sqrt{2b_1}}\right) \end{aligned} \tag{8.22}$$

This result was also obtained independently by [115] (see also the paper [116]) in the more general case when λ, μ are arbitrary with no other restrictions. ‡

For the asymmetric potential $t_1 = 1/(a+\tau)^2, t_2 = 1/(a-\tau)^2, (\tau \ll a)$ and a small interaction $c \approx 0$, we have $\epsilon \sim \tau/a^2, \alpha \sim 1/(2a)$ and $\epsilon \ll \alpha$. When $\tau \to 0$ (symmetric

‡I would like to thank D'Anna for pointing me out his paper with Brezin and Zee [115].

potential) $P(\lambda, \mu \sim \lambda)$ tends to the level density of hermitean 1-matrix model $P_{Herm}(\lambda)$. The interaction (even a small one) of asymmetric energy levels changes dramatically the level density $P(\lambda, \lambda)$ of the system.

If for $\tau \to 0$ we get the usual semicircular law, a small asymmetry creates some peaks in the level density $P(\lambda, \lambda)$ (see figure 4 on the last page of the thesis).

In figure 4 we represent the level density $P(x, x)$ in terms of the energy $x = a\lambda$ and the asymmetry $y = N\epsilon$.For $y = 0$ we have the semicircular law $P(x, x) = \sqrt{2N - x^2}$ and for small $y \neq 0$ we get the oscilations of level density.(see formula 8.21).

8.4 q-matrix model

As a random q-multimatrix model we choose the model with partition function:

$$Z = \int \prod_{\alpha=1}^{q} \lambda_\alpha \Delta(\lambda_1)\Delta(\lambda_q) \exp(\sum_{\alpha=1}^{q} t_\alpha \lambda_\alpha^2 + \sum_{\alpha=1}^{q-1} c_\alpha \lambda_\alpha \lambda_{\alpha+1}) \tag{8.23}$$

We show that the joint probability is :

$$P(\lambda_\alpha, \lambda_\beta) = P_{Herm}(\lambda'_\alpha, \lambda'_\beta), 1 \leq \alpha \leq \beta \leq q \tag{8.24}$$

where:

$$\lambda'_\alpha = \lambda_\alpha / \sqrt{2a_\alpha},$$

The parameters a_α are the coefficients of the Q-matrices.

To calculate the joint probability $P(\lambda_\alpha, \lambda_\beta)$ we integrate over all other eigenvalues $d\lambda_i^{(\gamma)}, \gamma \neq \alpha, \beta$. In this way we obtain the joint probability of two-matrix model for which we already know the result.

However, the physics of the multi-matrix model is richer than that of the 2-matrix model. But this could be observed only in the joint probabilities with at least 3 energy levels. Another issue is that we can study the interaction between the intermediate " classical" (or Poisson) sets of energy levels (those with $2 \ll \alpha \ll q-1$ and linear potentials $V_\alpha = s_\alpha \lambda_\alpha$) and the "quantum" sets (with $\alpha = 1, q$). We can interpret the q-matrix like an one-dimensional chain of atoms with localized states. Disorder is introduced through the boundary "quantum" atoms. The interesting problem is to compute the necessary conditions to have extended, conducting states. A concrete application of this model is the chain of polyacetilene.

The $Q(\alpha)$ have only three non–vanishing diagonal lines, the main diagonal and the two adjacent lines.

$$Q(\alpha) = b_\alpha I_+ + a_\alpha \epsilon_- \tag{8.25}$$

where in the particular cases we know that $b_1 = 1$ and $a_q = R$. We can write the parameters in terms of the determinants of two matrices (we use the results of the paper [84]):

$$
\begin{aligned}
b_\alpha &= (-1)^\alpha (c_1 c_2 \dots c_{\alpha-1})^{-1} \mathrm{Det} X_{\alpha-1} \\
R &= (-1)^q c_1 c_2 \dots c_{q-1} \left(\mathrm{Det} X_q\right)^{-1} \\
a_\alpha &= (-1)^\alpha c_1 c_2 \dots c_{\alpha-1} \frac{\mathrm{Det} Y_{\alpha+1}}{\mathrm{Det} X_q}
\end{aligned}
\tag{8.26}
$$

The matrices X_α and Y_α, are given by relations (4.26) (4.27).

Besides the usual orthogonal polynomials (which could be introduced as in 2-matrix models case), we introduce also the basic intermediate functions (this was made previously for multi-matrix models in[117]):

$$
\xi_n^{(\alpha)}(\lambda_\alpha) \equiv \int \prod_{\beta=1}^{\alpha-1} d\lambda_\beta \xi_n(\lambda_1) e^{S_\alpha}
$$

$$
\eta_n^{(\alpha)}(\lambda_\alpha) \equiv \int \prod_{\beta=\alpha+1}^{q} d\lambda_\beta e^{S'_\alpha} \eta_m(\lambda_q) \tag{8.27}
$$

where we denote

$$
\begin{aligned}
S_\alpha &= \sum_{\beta=1}^{\alpha-1} t_\beta \lambda_\beta^2 + \sum_{\beta=1}^{\alpha-1} c_\beta \lambda_\beta \lambda_{\beta+1} \\
S'_\alpha &= \sum_{\beta=\alpha+1}^{q-1} t_\beta \lambda_\beta^2 + \sum_{\beta=\alpha}^{q-1} c_\beta \lambda_\beta \lambda_{\beta+1}.
\end{aligned}
$$

Obviously we have:

$$
\xi_n^{(1)}(\lambda_1) = \xi_n(\lambda_1), \qquad \eta_n^{(q)}(\lambda_q) = \eta_m(\lambda_q).
$$

In the general case for arbitrary potentials one sees immediately that $\xi^{(\alpha)}$ and $\eta^{(\alpha)}$ are not polynomials anymore.In the case of gaussian potentials these intermediate functions are again Hermite functions, but with different arguments. In the general case, they still satisfy an orthogonality relation

$$
\int \prod_{\gamma=\alpha}^{\beta} d\lambda_\gamma \xi_n^{(\alpha)}(\lambda_\alpha) e^{S-S_\alpha-S'_\alpha}(\lambda_\alpha) \eta_m^{(\beta)}(\lambda_\beta) = \delta_{nm} h_n, \qquad 1 \le \alpha \le \beta \le q \tag{8.28}
$$

The equations satisfied by basic intermediate functions are:

$$
\begin{aligned}
\lambda_\alpha \xi^{(\alpha)} &= Q_\alpha \xi^{(\alpha)}, \qquad 1 \le \alpha \le q, \\
\lambda_\alpha \eta^{(\alpha)} &= \bar{Q}_\alpha \eta^{(\alpha)}, \qquad 1 \le \alpha \le q.
\end{aligned}
\tag{8.29}
$$

These equations together with the explicit form of Q-matrices permits to find the basic intermediate functions $\xi_n^{(\alpha)}$, $\eta_m^{(\alpha)}$:

$$\begin{aligned} \lambda_\alpha \xi_n^{(\alpha)}(\lambda_\alpha) &= b_\alpha \xi_{n+1}(\lambda_\alpha) + a_\alpha \xi_{n-1}(\lambda_\alpha) \\ \lambda_\alpha \eta_n^{(\alpha)}(\lambda_\alpha) &= (a_\alpha/R)\eta_{n+1}(\lambda_\alpha) + b_\alpha R \eta_{n-1}(\lambda_\alpha) \end{aligned} \tag{8.30}$$

Solving these recursion relations it follows that $\xi^{(\alpha)}, \eta^{(\alpha)}$ are Hermite functions for gaussian potentials:

$$\xi_n^{(\alpha)}(\lambda_\alpha) = (2\pi n!)^{-1/2} 2^{-n/2} (\sqrt{2a_\alpha})^n H_n(\lambda'_\alpha),$$
$$\eta_m^{(\alpha)}(\lambda_\alpha) = (2\pi m!)^{-1/2} 2^{-m/2} (\frac{R}{\sqrt{2a_\alpha}})^m H_m(\mu'_\alpha)$$

Using intermediate basic functions we get for joint probability:

$$\begin{aligned} P(\lambda_\alpha, \lambda_\beta) = \int (\prod_{i=2}^{N} d\lambda_i^{(\alpha)} d\lambda_i^{(\beta)}) (\prod_{i=1}^{N} \prod_{\gamma=\alpha+1}^{\beta-1} d\lambda_i^{(\gamma)}) \times \\ \times \mathrm{Det}_{ij}[\xi_i^{(\alpha)}(\lambda_j^{(\alpha)})] \mathrm{Det}_{ij}[\eta_i^{(\beta)}(\lambda_j^{(\beta)})] e^{S-S_\alpha-S'_\beta} \end{aligned} \tag{8.31}$$

Integrating over eigenvalues $d\lambda_i^{(\gamma)}, \gamma = \alpha+1, \ldots \beta-1$ we obtain the joint probability of two-matrix model for which we already know the result.Hence we get the result (8.24).

All the derivation above is valid also for more general potentials , polynomial-like $V_\alpha(\tau) = \sum_{k=1}^{p_\alpha} t_k \tau^k$ or not. The sufficient ingredients are the coefficients of the Q-matrices.

8.5 Star-matrix model

This model is interesting because it is supposed to describe the q-Potts model on a random lattice [96][97].It was also used to describe the cristal growth. The limiting cases $q \to 0$ and $q \to 1$ describe the tree-polymers, and the percolation respectively.

We study the star-matrix model with partition function:

$$\begin{aligned} Z = \int \prod_{i=1}^{N} (d\lambda_i^{(0)} \prod_{\alpha=1}^{q} d\lambda_i^{(\alpha)}) \prod_{i<j} [(\lambda_i^{(0)} - \lambda_j^{(0)}) \prod_{\alpha=1}^{q} (\lambda_i^{(\alpha)} - \lambda_j^{(\alpha)})] \times \\ \times \exp(\sum_{i=1}^{N} (V_0(\lambda_i^{(0)} + \sum_{\alpha=1}^{q} V_\alpha(\lambda_i^{(\alpha)}) + \sum_{\alpha=1}^{q} c_\alpha \lambda_i^{(0)} \lambda_i^{(\alpha)}) \end{aligned} \tag{8.32}$$

We define the orthogonal polynomial basis as ξ_n and (instead of one conjugate polynomial η_m $q+1$ polynomials $\eta_m^{(\alpha)}$:

$$\begin{aligned} \int d\lambda^{(0)} \prod_{\alpha=1}^{q} d\lambda^{(\alpha)} \xi_n^q(\lambda^{(0)}) e^{V_0 + \sum_{\alpha=1}^{q} (V_\alpha + c_\alpha \lambda^{(0)} \lambda^{(\alpha)})} \prod_{\alpha=1}^{q} \eta_{m_\alpha}^{(\alpha)}(\lambda^{(\alpha)}) = h_n \delta_{nm}, \\ m = m_\alpha, \alpha = 1, \ldots q. \end{aligned} \tag{8.33}$$

This basis is unusual but it works quite well at least for gaussian potentials: $V_\alpha(\tau) = t_\alpha\tau^2 + u_\alpha\tau, \alpha = 0, 1, \dots q$.

We introduce Q-matrices as:

$$\int d\lambda^{(0)} \prod_{\alpha=1}^{q} d\lambda^{(\alpha)} \xi_n^q(\lambda^{(0)})\lambda^{(\alpha)} e^{V_0+\sum_{\alpha=1}^{q}(V_\alpha + c_\alpha \lambda^{(0)}\lambda^{(\alpha)})} \prod_{\alpha=1}^{q} \eta_{m_\alpha}^{(\alpha)}(\lambda^{(\alpha)}) = h_n Q_{\alpha,nm}.$$
$$m = m_\alpha, \alpha = 1, \dots q. \tag{8.34}$$

The coupling conditions are:

$$\begin{aligned} qP_0 + 2t_0Q_0 + u_0 + \sum_{\alpha=1}^{q} c_\alpha Q_\alpha &= 0 \\ \overline{P}_\alpha + 2t_\alpha Q_\alpha + u_\alpha + c_\alpha Q_0 &= 0. \alpha = 1, \dots q. \end{aligned} \tag{8.35}$$

With the following parametrization of Q-matrices:

$$\begin{aligned} Q_0 &= I_+ + a_0 I_0 + a_1 \epsilon_- \\ Q_\alpha &= b_\alpha / R_\alpha I_+ + d_\alpha I_0 + R_\alpha \epsilon_-, \alpha = 1, \dots q \end{aligned} \tag{8.36}$$

we arrive at the equations:

$$\begin{aligned} 2t_\alpha R_\alpha + c_\alpha a_1 = 0 \\ 2t_\alpha b_\alpha + n + c_\alpha R_\alpha = 0 \\ 2t_\alpha d_\alpha + u_\alpha + c_\alpha a_0 = 0 \\ 2t_0 + \sum \frac{c_\alpha b_\alpha}{R_\alpha} = 0 \\ 2t_0 a_0 + u_0 + \sum c_\alpha d_\alpha = 0 \\ 2t_0 a_1 + qn + \sum c_\alpha R_\alpha = 0 \end{aligned} \tag{8.37}$$

By solving the coupling conditions we get :

$$\begin{aligned} a_1 &= -\frac{2q}{A}, a_0 = \frac{1}{A}(\sum \frac{c_\alpha u_\alpha}{t_\alpha} - 2u_0) \\ b_\alpha &= -\frac{1}{2t_\alpha^2}(\frac{c_\alpha^2 q}{A} + t_\alpha), R_\alpha = \frac{c_\alpha q}{t_\alpha A} \\ d_\alpha &= \frac{1}{At_\alpha}(c_\alpha u_0 - 2t_0 u_\alpha + u_\alpha \sum \frac{c_\alpha^2}{2t_\alpha} - c_\alpha \sum \frac{c_\alpha u_\alpha}{2t_\alpha}) \end{aligned}$$

where $A = 4t_0 - \sum c_\alpha^2 / t_\alpha$.

We can obtain the basic functions for star matrix model, in the same way we get them for q-matrix model:

$$\begin{aligned} \xi_n(\lambda^{(0)}) &= H_n(\lambda'^{(0)}), \\ \eta_m^{(\alpha)}(\lambda^{(\alpha)}) &= R_\alpha^n H_n(\lambda'^{(\alpha)}), \alpha = 0, 1 \dots q \end{aligned} \tag{8.38}$$

with:

$$\lambda'^{(0)} = \frac{\lambda^{(0)} - a_0}{\sqrt{2a_1}}, \lambda'^{(\alpha)} = \frac{\lambda^{(\alpha)} - d_\alpha}{\sqrt{2b_\alpha}}. \tag{8.39}$$

Since these basic functions satisfy the relation:

$$\eta_m(\lambda^{(0)}) = \int e^{V_\alpha + c_\alpha \lambda^{(0)}\lambda^{(\alpha)}} \eta_m(\lambda^{(\alpha)}), \tag{8.40}$$

we can integrate over Vandermonde determinants:

$$\mathrm{Det}_{ij}[\eta_i^{(0)}(\lambda_j^{(0)})] = \int e^{V_\alpha + c_\alpha \lambda^{(0)}\lambda^{(\alpha)}} \mathrm{Det}_{ij}[\eta_i^{(\alpha)}(\lambda_j^{(\alpha)})]. \tag{8.41}$$

Then we have for the joint probability of two eigenvalues the simple expression:

$$P(\lambda^{(\alpha)}, \lambda^{(\beta)}) \sim P_{Herm}(\lambda'^{(\alpha)}, \lambda'^{(\beta)}), \;\; \alpha, \beta = 0, 1 \ldots q \tag{8.42}$$

with λ', μ' given by equation (8.39).

8.6 Generalized Calogero-Sutherland model

The connection with Calogero model permits the calculation of the joint distribution functions for random multimatrix models for other ensembles, different from the hermitean one. This relation is interesting due to the fact that such matrix models might be directly related with conformal field theories with the central charge $c = 1 - 6(\sqrt{\beta} - 1/\sqrt{\beta})^2$, as was stated in [118] (where $\beta = 1, 2, 4$ for orthogonal, hermitean and unitary matrices respectively).

We obtain the Calogero model related to the 2-matrix model .The eigenvalue problem for Calogero model follows from the heat equation satisfied by the Itzykson- Zuber integral.

We introduce the kernel:

$$K(X, Y|t) =< X|e^{-tD}|Y >= (2\pi t)^{-N^2/2} \int dU \exp[-\frac{1}{2t} Tr(X - UYU^+)] \tag{8.43}$$

which is related with the Itzykson-Zuber integral $K(X, Y|t = 1) = \exp(-\frac{1}{2t} Tr(X^2 + Y^2))I(X, Y)$:

$$I(X, Y) = \int dU \exp[Tr(XUYU^+)] = \frac{det_{ij}(e^{x_i y_j})}{(\Delta(X)\Delta(Y))^{\beta/2}} \tag{8.44}$$

where $\beta = 1, 2$, or 4 depending on how are the matrices X, Y : orthogonal, hermitean or unitary respectively.

The kernel (8.43) satisfies the heat equation [64][63]:

$$(\frac{\partial}{\partial t} + D_X)\tilde{K}(X, Y|t) = \delta(X, Y) \tag{8.45}$$

where $\tilde{K}(X,Y|t) = (\Delta(X)\Delta(Y))^{\beta/2} K(X,Y|t)$ and the laplacian is:

$$D_X = -\frac{1}{2}\sum_i \frac{\partial^2}{\partial x_i^2} + \frac{\beta}{2}(\frac{\beta}{2} - 1)\sum_{i<j} \frac{1}{(x_i - x_j)^2} \tag{8.46}$$

Solving equation (8.45) gives:

$$\tilde{K}(X,Y|t) = (2\pi t)^{-N^2/2} \sum_\sigma \eta_\sigma \exp[-\frac{1}{2t}\sum_i (x_{\sigma(i)} - y_i)^2] \tag{8.47}$$

from which follows the expression for the Itzykson-Zuber integral (σ is the permutation).

We introduce the function:

$$\Phi(X|t) = \int \tilde{K}(X,Y|t)\Phi(Y)dY \tag{8.48}$$

that fulfills the heat equation with initial condition $\Phi(X|t=0) = \Phi(X)$.

We can search for stationary solutions in the form $\Phi(X|t) = \sum_n \Phi_n(X)\epsilon^{-E_n t}$ where $\Phi_n(X)$ satisfies the Calogero equation (without potential term):

$$\left(-\frac{1}{2}\sum_i \frac{\partial^2}{\partial x_i^2} + \frac{\beta}{2}(\frac{\beta}{2} - 1)\sum_{i<j} \frac{1}{(x_i - x_j)^2}\right)\Phi_n(X) = E_n \Phi_n(X) \tag{8.49}$$

The eigenvalues of matrix X are chosen such that $y_1 < y_2 \ldots < y_N$. These eigenvalues $y_1 \ldots y_N$ are mapped by the kernel $\tilde{K}(X,Y|t)$ into $x_{\sigma(1)} \ldots x_{\sigma(N)}$.

For $t \to 0$, the kernel $\tilde{K}(X,Y|t)$ tends to $\sum_\sigma \eta_\sigma \delta^{(N)}(x_{\sigma(i)} - y_i)$. Hence if we consider $\Psi(X)$ as a particular solution of Calogero model with $x_1 < x_2 \ldots < x_N$,the function $\Phi(X|t=0)$ is the general solution for eigenvalues x_i in arbitrary order, being the linear combination of functions $\Psi(\sigma X)$:

$$\Phi(X|t=0) = \sum_\sigma \Psi(\sigma X), \;\; \Psi(\sigma X) = \eta_\sigma \Psi(X)$$

where σ is the permutation of eigenvalues x_i ; $\eta_\sigma = -1$ for free fermions ($\beta = 2$ for hermitean matrices) and $\eta_\sigma = +1$ for free bosons ($\beta \to 0$ for harmonic oscillator) . We see that for generic value of β the system describes particles with fractional statistics (anyons).

For $t \to \infty$ the dominant contribution is given by the vacuum configuration $\Phi_0(X)$. The kernel $\tilde{K}(X,Y|t)$ plays the role of instanton propagator connecting the initial vacuum configuration $\Psi_0(Y) = (\Delta(Y))^{\beta/2}$ to final vacuum configuration $\Phi_0(X) = (\Delta(X))^{\beta/2}$.

For 2-matrix model we can define the generalized Calogero system:

$$\left(-\frac{1}{2}(\sum_i \frac{\partial^2}{\partial \lambda_i^2} + \sum_i \frac{\partial^2}{\partial \mu_i^2}) + \frac{\beta}{2}(\frac{\beta}{2} - 1)\sum_{i<j}(\frac{1}{(\lambda_i - \lambda_j)^2} + \frac{1}{(\mu_i - \mu_j)^2}) + \right.$$
$$\left. + \sum_i (V_1(\lambda_i) + V_2(\mu_i) + c\lambda_i\mu_i)\right)\Phi_n(\lambda)\Psi_m(\mu) = E_{nm}\Phi_n(\lambda)\Psi_m(\mu) \tag{8.50}$$

This system describes 2 interacting systems of anyons. Hence all results valid in the 2-matrix model with $\beta = 1$can be extended to arbitrary β and given a meaning in terms of the generalized Calogero's anyons.

When $c = 0$ the generalized system splits into two Calogero systems :

$$\left(-\frac{1}{2}\sum_i \frac{\partial^2}{\partial\lambda_i^2} + \frac{\beta}{2}(\frac{\beta}{2}-1)\sum_{i<j}\frac{1}{(\lambda_i-\lambda_j)^2} + \sum \lambda'^2\right)\Phi_n(\lambda) = E_{1,n}\Phi_n(\lambda) \tag{8.51}$$

$$\left(-\frac{1}{2}\sum_i \frac{\partial^2}{\partial\mu_i^2} + \frac{\beta}{2}(\frac{\beta}{2}-1)\sum_{i<j}\frac{1}{(\mu_i-\mu_j)^2} + \sum \mu'^2\right)\Phi_n(\mu) = E_{2,m}\Phi_n(\mu)$$

The ground states can be written in terms of the eigenfunctions (8.16):

$$\Phi_0(\lambda) = (det_{ij}\xi_i(\lambda_j))^{\beta/2}\exp(-\sum_i \lambda_i'^2/2)$$
$$\Psi_0(\mu) = (det_{ij}\eta_i(\mu_j))^{\beta/2}\exp(-\sum_i \mu_i'^2/2) \tag{8.52}$$

We can see that the probability of amplitudes (8.52) is the partition function of the 2-matrix model:

$$Z = \int d\lambda_1 \ldots d\lambda_N |\Phi_0(\lambda)|^2 = \int d\mu_1 \ldots d\mu_N |\Psi_0(\mu)|^2$$

The system (8.52) permits us to calculate the joint probability $P(\lambda, \mu)$ for general ensemble. It coincides with formula (8.19) (for $c = 0$) where we replace N by $\beta N/2$:

$$P(\lambda,\mu) = \sqrt{\beta N/2 - \lambda'^2}\sqrt{\beta N/2 - \mu'^2} \tag{8.53}$$

8.7 Conclusions

These models present interest in the study of quantum chaos for q systems interacting in various ways. The interaction of q subsystems redistribute the energy between the subsystems and change in non-trivial way the joint distribution functions. Different kinds of coupling between the systems (chain or star- type) change in unexpected way the energy levels and with them the associated probabilities.

9 Toda lattice realization of integrable hierarchies

Scalar integrable hierarchies can be introduced in terms of (pseudo)differential operators by means of a formalism first introduced by Gelfand and Dickey (see [120]). This is the most 'disembodied' form in which such hierarchies can appear, and it can be taken as a reference form. One can then consider realizations of these hierarchies in physical systems. A comprehensive realization is the one studied by Drinfeld and Sokolov in terms of linear systems defined on Lie algebras, [121]; let us refer to it as the Drinfeld–Sokolov realization (DSR).

In this chapter, based on the joint work [119], we present a new general realization of integrable hierarchies in terms of the Toda lattice hierarchy (TLH). We call it Toda lattice realization (TLR), and it looks as general as the DSR. While the DSR is contiguous to (reduced) WZNW models and Toda field theories in 2D, the TLR is inspired by matrix models, see [53],[83].

The chapter is organized as follows. In next section we introduce the TLR. We do not give a general proof of it, but in section 2 we verify it on a large number of examples among KP, n–KdV and other classes of hierarchies. Last section is devoted to some comments.

9.1 The method

In the Gelfand–Dickey (GD) formalism an integrable hierarchy can be entirely specified in terms of the Lax operator

$$L = \partial^N + Na_1\partial^{N-2} + Na_2\partial^{N-3} + ... + Na_{N-1} + Na_N\partial^{-1} + ... \tag{9.1}$$

where $\partial = \frac{\partial}{\partial x}$. The operator L may be purely differential, in which case $a_k = 0$ for $k \geq N$, and we get the N-KdV hierarchy. The fields a_k may be either elementary or composite of more elementary fields, as in the case of the (N, M)–KdV hierarchies studied in [122],[123],[124]. If the hierarchy is integrable, the flows are given by

$$\frac{\partial L}{\partial t_k} = [(L^{k/N})_+, L] \tag{9.2}$$

where the subscript $+$ denotes the differential part of a pseudodifferential operator, t_1 is identified with x and k spans a specific subset of the positive integers.

The Toda lattice hierarchy is defined in terms of a semi–infinite Jacobi matrix $\hat{Q}$ [§]. We parametrize it as follows

$$\hat{Q} = \sum_{j=0}^{\infty} \Big(E_{j,j+1} + \sum_{l=0}^{\infty} \hat{a}_l(j) E_{j+l,j}\Big), \qquad (E_{j,m})_{k,l} = \delta_{j,k}\delta_{m,l} \tag{9.3}$$

[§]In this chapter we limit ourselves to a simple version of the TLH, in which only one matrix $\hat{Q}$ and one set of parameters intervene, instead of two or more [127],[53] and consider $\hat{a}_l$ as fields defined on a lattice. The flows are given by

$$\frac{\partial \hat{Q}}{\partial t_k} = [(\hat{Q}^k)_+, \hat{Q}], \qquad k = 1, 2, ... \tag{9.4}$$

where the subscript $+$ denotes the upper triangular part of a matrix, including the main diagonal. (9.4) represents a hierarchy of differential–difference equations for the fields a_l. In particular the first flows are

$$\hat{a}_l(j)' = \hat{a}_{l+1}(j+1) - \hat{a}_{l+1}(j) + \hat{a}_l(j)\Big(\hat{a}_0(j) - \hat{a}_0(j-l)\Big) \tag{9.5}$$

where we have adopted the notation $\frac{\partial}{\partial t_1} f \equiv f' \equiv \partial f$, for any function f. The parameter t_k of the TLH will be identified later on with the corresponding parameter t_k in (9.2) whenever the latter exists; therefore, in particular, t_1 will be identified with x.

Next, integrability permits us to introduce the function $\hat{F}(n, t)$ (the free energy in matrix models) via

$$\frac{\partial^2}{\partial t_k \partial t_l} \hat{F}(n, t) = \mathrm{Tr}\Big([\hat{Q}^k_+, \hat{Q}^l]\Big) \tag{9.6}$$

where $\mathrm{Tr}(X)$ denotes the finite trace $\sum_{j=0}^{n-1} X_{j,j}$. In particular (9.6) leads to

$$\frac{\partial^2}{\partial t_1^2} \hat{F}(n, t) = \hat{a}_1(n) \tag{9.7}$$

It is clear that by means of (9.4) we can compute the derivatives of any order of $\hat{F}$ in terms of the entries of $\hat{Q}$. In general we will denote by $\hat{F}_{k_1,...,k_s}$ the derivative of $\hat{F}$ with respect to $t_{k_1}, ..., t_{k_s}$.

Next we introduce the operator D_0, defined by its action on any discrete function $f(n)$

$$(D_0 f)(n) = f(n+1)$$

For later use we remark that, if $f_0 = 0$, the operation Tr is the inverse of the operation $D_0 - 1$. We will also use the notation e^{∂_0} instead of D_0, with the following difference: D_0 is meant to be applied to the nearest right neighbour, while e^{∂_0} acts on whatever is on its right. Now we can equivalently represent the matrix $\hat{Q}$ by the following operator

$$\hat{Q}(j) = e^{\partial_0} + \sum_{l=0}^{\infty} \hat{a}_l(j) e^{-l\partial_0} \tag{9.8}$$

The contact between (9.8) and (9.3) is made by acting with the former on a discrete function $\xi(j)$; then $\hat{Q}(j)\xi(j)$ is the same as the j–th component of $\hat{Q}\xi$, where ξ is a column vector with components $\xi(0), \xi(1), ...$ We will generally drop the dependence on j in (9.8) and merge the two symbols.

After this short introduction to the GD formalism and the Toda lattice hierarchy, let us come to the presentation of the TLR of the integrable hierarchy defined by the Lax operator (9.1), i.e. to the problem of embedding the latter into the TLH. The prescription consists of several steps.

Step 1. In $\hat{Q}$ we set $\hat{a}_0 = 0$ and replace the first flows (9.5) with

$$D_0\hat{a}_1 = \hat{a}_1, \qquad D_0\hat{a}_i = \hat{a}_i + \hat{a}'_{i-1}, \qquad i = 2,3,... \tag{9.9}$$

Step 2. We compute

$$\frac{\partial \hat{a}_1}{\partial t_k} = \partial \mathrm{Tr}\Big([\hat{Q}_+, \hat{Q}^k]\Big) \equiv \partial \hat{F}_{1,k} \tag{9.10}$$

The right hand side will be a polynomial of the fields $\hat{a}_k$ to which monomials of D_0 and D_0^{-1} are applied. Next we substitute the first flows (9.9) to eliminate the presence of D_0. Examples:

$$\begin{aligned}
&\hat{F}_{1,1} = \hat{a}_1,\\
&\hat{F}_{1,2} = (D_0+1)\hat{a}_2 = 2\hat{a}_2 + \hat{a}'_1,\\
&\hat{F}_{1,3} = (D_0^2 + D_0 + 1)\hat{a}_3 + D_0\hat{a}_1\hat{a}_1 + \hat{a}_1\hat{a}_1 + \hat{a}_1 D_0^{-1}\hat{a}_1 = 3\hat{a}_3 + 3\hat{a}'_2 + \hat{a}''_1 + 3\hat{a}_1^2
\end{aligned} \tag{9.11}$$

and so on.

Next we recall that

$$\hat{F}_{1,k} = \frac{\partial^2}{\partial t_k \partial t_1}\hat{F}$$

Using this and (9.11), we can recursively write all the derivatives of $\hat{a}_l$ with respect to the couplings t_k (and in particular the flows) in terms of derivatives of $\hat{F}$, which, in turn, can be expressed as functions of the entries of $\hat{Q}$. Example:

$$\frac{\partial}{\partial t_k}\hat{a}_2 = \frac{1}{2}\partial \mathrm{Tr}\Big([\hat{Q}_+^2, \hat{Q}^k]\Big) - \frac{1}{2}\frac{\partial}{\partial t_k}\hat{a}'_1$$

In general we will need all $\hat{F}_{k_1,\dots,k_n}$. Here are some general formulas. Let us introduce the symbols $\hat{A}_j^{[k]}$ as follows

$$\hat{Q}^k = e^{k\partial} + k a_1 e^{(k-2)\partial} + \hat{A}_2^{[k]}e^{(k-3)\partial} + \hat{A}_3^{[k]}e^{(k-4)\partial} + \hat{A}_4^{[k]}e^{(k-5)\partial} + \dots \tag{9.12}$$

The explicit form of the first few is:

$$\begin{aligned}
\hat{A}_2^{[k]} &= \binom{k}{2} a'_1 + k\hat{a}_2\\
\hat{A}_3^{[k]} &= \binom{k}{3} a''_1 + \binom{k}{2}\hat{a}'_2 + k\hat{a}_3 + \binom{k}{2} a_1^2\\
\hat{A}_4^{[k]} &= \binom{k}{4} a'''_1 + \binom{k}{3}\hat{a}''_2 + \binom{k}{2}\hat{a}'_3 + k\hat{a}_4 + (3\binom{k}{3} - \binom{k}{2})a_1 a'_1 + 2\binom{k}{2} a_1\hat{a}_2
\end{aligned} \tag{9.13}$$

$$\hat{A}_5^{[k]} = \begin{pmatrix} k \\ 5 \end{pmatrix} a_1^{(4)} + \begin{pmatrix} k \\ 4 \end{pmatrix} \hat{a}_2''' + \begin{pmatrix} k \\ 3 \end{pmatrix} \hat{a}_3'' + \begin{pmatrix} k \\ 2 \end{pmatrix} \hat{a}_4' + k\hat{a}_5 + \begin{pmatrix} k \\ 2 \end{pmatrix} (\hat{a}_2\hat{a}_2 + 2a_1\hat{a}_3)$$
$$+ \begin{pmatrix} k \\ 3 \end{pmatrix} a_1^3 + (3\begin{pmatrix} k \\ 4 \end{pmatrix} - \begin{pmatrix} k \\ 3 \end{pmatrix})a_1'a_1' + (4\begin{pmatrix} k \\ 4 \end{pmatrix} - 2\begin{pmatrix} k \\ 3 \end{pmatrix} + \begin{pmatrix} k \\ 2 \end{pmatrix})a_1a_1'' +$$
$$(3\begin{pmatrix} k \\ 3 \end{pmatrix} - \begin{pmatrix} k \\ 2 \end{pmatrix})a_1\hat{a}_2' + (3\begin{pmatrix} k \\ 3 \end{pmatrix} - 2\begin{pmatrix} k \\ 2 \end{pmatrix})\hat{a}_2a_1'$$

and so on. In terms of these coefficients we can compute all the derivatives of $\hat{F}$. For example

$$\begin{aligned} \hat{F}_{1,k} &= \hat{A}_k^{[k]} \\ \hat{F}_{2,k} &= (D_0+1)\hat{A}_{k+1}^{[k]} \\ \hat{F}_{3,k} &= (D_0^2+D_0+1)\hat{A}_{k+2}^{[k]} + 3a_1\hat{A}_k^{[k]} \\ \hat{F}_{4,k} &= (D_0^3+D_0^2+D_0+1)\hat{A}_{k+3}^{[k]} + 4a_1(D_0+1)\hat{A}_{k+1}^{[k]} + a_2^{(4)}\hat{A}_k^{[k]} \end{aligned} \tag{9.14}$$

This procedure allows us to compute all the derivatives of the fields $\hat{a}_l$ in terms of the same fields and their derivatives with respect to $x \equiv t_1$ – therefore, in particular, the flows.

So far all our moves have been completely general (except for setting $\hat{a}_0 = 0$, but see the comment at the end of this section). The next step is instead a 'gauge choice', that is we make a particular choice for the matrix $\hat{Q}$. The word 'gauge' is not merely colorful. In fact gauge transformations play here a role analogous to gauge transformations in [121]. The relevant gauge transformations in the present case are defined by $\hat{Q} \to G_-\hat{Q}G_-^{-1}$, where G_- is a strictly lower triangular semi-infinite matrix.

Step 3. We fix the gauge by imposing the condition

$$\hat{Q}^N = e^{N\partial_0} + \sum_{l=1} a_l e^{(N-1-l)\partial_0} \tag{9.15}$$

where the a_l are the same as in eq.(9.1). The matrix $\hat{Q}$ that satisfies such condition will be referred to as $\bar{Q}$. It is clear that $\bar{Q}^N$ exactly mimics the Lax operator L. The condition (9.15) recursively determines $\hat{a}_k$ in terms of the fields a_l that appear in L.

$$\hat{a}_k = \bar{a}_k \equiv P_k(a_l)$$

where P_k are differential polynomials of a_l. In particular we always have $\hat{a}_1 = \bar{a}_1 \equiv a_1$.

Step 4. Then we evaluate both sides of the flows found in *Step 2* at $\hat{a}_k = \bar{a}_k$. The order here is crucial. The gauge fixing of the flows must be the last operation.

Now we claim:

Claim. *The flows obtained in this way coincide with the flows (9.2) for corresponding couplings.*

We will substantiate this claim with a large number of examples in the next section.

It is perhaps useful to summarize our method: *start from the TLH flows, use the first flows (9.9) and impose the relevant gauge fixing; the resulting flows are the desired differential integrable flows.*

We would like to end this section with a remark concerning the restriction $\hat{a}_0 = 0$ we imposed at the very beginning. This can be avoided at the price of working with very encumbering formulas. One can keep $\hat{a}_0 \neq 0$ provided one uses the first flows (9.5) instead of (9.9) in *Step 1*. In this way it is possible, in general, to eliminate D_0 in the flows only when it acts over $\hat{a}_l$, $l \neq 0$ (see the last section for an additional comment on this point). We obtain in this way the same equations as above with the addition of terms involving $\hat{a}_0$. We can suppress all these additional terms at the end (*Step 5*) by imposing $\hat{a}_0 = 0$ as part of the gauge choice. The final result is of course the same as before. This justifies our having imposed $\hat{a}_0 = 0$ from the very beginning.

9.2 Examples.

In this section we present a large number of examples in support of the claim of the previous section. Of course for obvious reasons of space we can explicitly exhibit a few cases only, and for each case only a few flows among those we have checked.

The KP hierarchy

The KP case corresponds to $n = 1$ in (9.1). Therefore there is no gauge fixing: $\hat{a}_l = a_l$. The flows obtained with our method are simply those in *Step 3*. Examples:

$$\frac{\partial \hat{a}_1}{\partial t_2} = \left(2\hat{a}_2 + \hat{a}_1'\right)', \qquad \frac{\partial \hat{a}_1}{\partial t_3} = \left(3\hat{a}_3 + 3\hat{a}_2' + \hat{a}_1'' + 3\hat{a}_1^2\right)'$$

$$\frac{\partial \hat{a}_2}{\partial t_2} = \left(2\hat{a}_3 + \hat{a}_2' + \hat{a}_1^2\right)', \qquad \frac{\partial \hat{a}_2}{\partial t_3} = \left(3\hat{a}_4 + 3\hat{a}_3' + \hat{a}_2'' + 6\hat{a}_1\hat{a}_2\right)'$$

and so on. Setting $\hat{a}_l = a_l$, these are exactly the KP flows.

The N-KdV hierarchy case

In [83] we have explicitly shown that our claim is true for the 3–KdV hierarchy. In this section we generalize that result. To start with we pick a generic N. The relevant differential operator is

$$L = D^N + Na_1 D^{N-2} + Na_2 D^{N-3} + \ldots + Na_{N-1} \tag{9.16}$$

We also write

$$L^{k/N} = D^k + ka_1 D^{k-2} + b_2^{[k]} D^{k-3} + \ldots + b_{j-1}^{[k]} D^{k-j} + \ldots \tag{9.17}$$

The coefficients $b_j^{[k]}$ are differential polynomials in a_l, $l = 1, ..., a_{N-2}$.

Working out the commutator in relation (9.2), we can write down the general formula for arbitrary flow t_m:

$$\begin{aligned}\frac{\partial a_{j-1}}{\partial t_m} &= \sum_{k=0}^{m-1}\binom{m}{k} a_{j+k-1}^{(m-k)} - \sum_{k=0}^{m-2}\frac{1}{N}\binom{N}{j+k}(b_{m-k-1}^{(m)})^{(j+k)} + \\ &+ \sum_{k=0}^{m-3}(\binom{m-2}{k} b_{k-1}^{(m)} a_{j-k-1}^{(m-2-k)} - \sum_{l=0}^{m-2}\binom{N-j-k}{l-k} a_{k-1}(b_{m-l-1}^{(m)})^{(l-k)}) \\ &- \sum_{k=2}^{j-1}\sum_{l=0}^{m-2}\binom{N-k}{j-k+l} a_{k-1}(b_{m-l-1}^{(m)})^{(j-k+l)}\end{aligned} \tag{9.18}$$

Now let us pass to the TLR of this hierarchy. We recall eqs.(9.13) and (9.14). We fix the gauge by imposing $\hat{A}_j^{[N]} = Na_j$. We solve the equations for $\hat{a}_j$ in terms of a_j and obtain $\bar{a}_j$. Next we insert back the result in the formulas of the coefficients $\hat{A}_j^{[k]}$ so that they become functions of a_j. We call the result $\bar{A}_j^{[k]}$. Examples:

$$\begin{aligned}\bar{A}_2^{[k]}/k &= a_2 - \frac{N-k}{2}a_1' \\ \bar{A}_3^{[k]}/k &= a_3 - \frac{N-k}{2}a_2' + \frac{(N-2k+3)(N-k)}{12}a_1'' - \frac{N-k}{2}a_1^2 \\ \bar{A}_4^{[k]}/k &= a_4 - \frac{N-k}{2}a_3' + \frac{(N-2k+3)(N-k)}{12}a_2'' - \frac{(N-k+2)(k-2)(N-k)}{24}a_1''' \\ &-(N-k)a_1a_2 + \frac{(N-k+2)(N-k)}{2}a_1a_1'\end{aligned} \tag{9.19}$$

Then, using our recipe, we obtain

$$\begin{aligned}\partial^{-1}\frac{\partial a_1}{\partial t_k} &= \frac{\partial^2 F}{\partial t_1 \partial t_k}|_{\hat{a}=\bar{a}} = \bar{A}_k^{[k]} \\ \partial^{-1}\frac{\partial a_2}{\partial t_k} &= (\frac{1}{2}\frac{\partial^2 F}{\partial t_2 \partial t_k})|_{\hat{a}=\bar{a}} + \frac{N-2}{2}\frac{\partial a_1}{\partial t_k} = \bar{A}_{k+1}^{[k]} + \frac{N-1}{2}(\bar{A}_k^{[k]})' \\ \partial^{-1}\frac{\partial a_3}{\partial t_k} &= (\frac{1}{3}\frac{\partial^2 F}{\partial t_3 \partial t_k})|_{\hat{a}=\bar{a}} + \frac{N-3}{2}\frac{\partial a_2}{\partial t_k} - \frac{(N-3)^2}{12}\frac{\partial a_1'}{\partial t_k} + (N-3)\partial^{-1}(a_1\frac{\partial a_1}{\partial t_k}) \\ &= \bar{A}_{k+2}^{[k]} + \frac{N-1}{2}(\bar{A}_{k+1}^{[k]})' + \frac{(N-1)(N-2)}{6} \times \\ &\times \ (\bar{A}_k^{[k]})'' + a_1\bar{A}_k^{[k]} + (N-3)\partial^{-1}(a_1(\bar{A}_k^{[k]})') \\ \partial^{-1}\frac{\partial a_4}{\partial t_k} &= (\frac{1}{4}\frac{\partial^2 F}{\partial t_4 \partial t_k} + \frac{N-4}{2}\frac{\partial a_3}{\partial t_k})|_{\hat{a}=\bar{a}} - \frac{(N-4)(N-5)}{12}\frac{\partial a_2'}{\partial t_k} + (N-4)\partial^{-1}\frac{\partial(a_1a_2)}{\partial t_k} \\ &- \frac{(N-4)(N-2)}{2}(\frac{1}{6}\frac{\partial a_1''}{\partial t_k} + a_1\frac{\partial a_1}{\partial t_k}) = \bar{A}_{k+3}^{[k]} + \frac{N-1}{2}(\bar{A}_{k+2}^{[k]})' \\ &+ \frac{(3N-11)(N-1)(N-2)}{24}(\bar{A}_k^{[k]})''' - \frac{(N-2)(N-4)}{2}a_1(\bar{A}_k^{[k]})' + \\ &+ \frac{(5N-16)(N-1)}{12}(\bar{A}_{k+1}^{[k]})'' + (N-4)\partial^{-1}(a_1(\bar{A}_{k+1}^{[k]})' +\end{aligned}$$

$$+\frac{N-1}{2}(a_1(\bar{A}_k^{[k]})''+a_2(\hat{A}_k^{[k]})')$$

and so on, where $\hat{a}=\bar{a}$ denotes gauge fixing.

We give a few concrete examples of the second and third flows:

$$\begin{aligned}
\partial^{-1}\frac{\partial a_1}{\partial t_2} &= 2a_2-(N-2)a_1' \\
\partial^{-1}\frac{\partial a_2}{\partial t_2} &= 2a_3+a_2'-\frac{(N-1)(N-2)}{3}a_1''-(N-2)a_1^2 \\
\frac{\partial a_3}{\partial t_2} &= 2a_4'+a_3''-\frac{(N-1)(N-2)(N-3)}{12}a_1^{(4)}- \\
&-(N-2)(N-3)a_1a_1''-2(N-3)a_2a_1'
\end{aligned}$$

$$\begin{aligned}
\partial^{-1}\frac{\partial a_1}{\partial t_3} &= 3a_3-\frac{3}{2}(N-3)a_2'+\frac{(N-3)^2}{4}a_1''-\frac{3}{2}(N-3)a_1^2 \\
\partial^{-1}\frac{\partial a_2}{\partial t_3} &= 3a_4+3a_3'-\frac{N(N-3)}{2}a_2''+\frac{(N-1)(N-2)(N-3)}{8}a_1'''-3(N-3)a_1a_2 \\
\frac{\partial a_3}{\partial t_3} &= 3a_5'+3a_4''+a_3'''-\frac{3}{N}\begin{pmatrix} N \\ 4 \end{pmatrix}a_2^{(4)}+\frac{3(3N-7)}{10N}\begin{pmatrix} N \\ 4 \end{pmatrix}a_1^{(5)}+3a_1a_3' \\
&- 3(N-4)a_1'a_3-3(N-3)a_2a_2'-\frac{3}{2}(N-2)(N-3)a_1a_2''+ \\
&+ \frac{3}{2}(N-3)a_2a_1''+\frac{6}{N}\begin{pmatrix} N \\ 4 \end{pmatrix}a_1a_1'''
\end{aligned}$$

These are flows pertinent to the N–KdV hierarchy with $N>3$. In general the formulas of the N–KdV hierarchy and the corresponding formulas obtained with our method coincide since

$$b_k^{[m]}=\bar{A}_k^{[m]}$$

We have checked these identities case by case up to the 5-KdV and for $m\leq 5$. For the dispersionless case we have verified the correspondence up to the 8–KdV flows.

The DS hierarchies

Drinfeld and Sokolov, [121], introduced a large set of generalized KdV systems in terms of the pair (G, c_m), where G is a classical Kac-Moody algebra and c_m is a vertex of the Dynkin diagram of G. From each choice of the pair (G, c_m) they were able to construct a pseudo–differential operator L which give rise to a hierarchy of integrable equations. We have studied all the examples corresponding to the operator L of orders 3,4,5 and found a complete agreement with our method. For simplicity here we present a few examples of order 4 and 5, corresponding to the cases with a pseudodifferential Lax operator. The cases with a differential Lax operator are restriction of the 4– and 5–KdV hierarchies, and will be omitted.

In each case we give the explicit form of the (pseudo–)differential operator L, the gauge–fixed matrix $\hat{Q}$ and the first significant flows:

Order 4.

Case $B_2^{(1)}$:

$$\begin{aligned} c_0, c_1 \quad : L &= D^4 + 2u_1 D^2 + u_1' D + 2(u_0 + u_1'') - D^{-1}(u_0 + u_1'')' \qquad (9.20) \\ \hat{Q} &= \epsilon^{\hat{\partial}} + \frac{v_1}{4}\epsilon_- - \frac{v_1'}{4}\epsilon^{-2\partial} + (\frac{1}{4}v_0 + \frac{v_1''}{8} - \frac{3}{32}v_1^2)\epsilon^{-3\partial} + \\ &+ (\frac{3}{8}v_1 v_1' - \frac{1}{2}v_0')\epsilon^{-4\partial} + \ldots \\ c_2 \quad : L &= D^4 + 2u_1 D^2 + u_1' D + u_0^2 - u_0 D^{-1} u_0' \end{aligned}$$

For c_2 we have, up to the order $\epsilon^{-4\partial}$, the same expression for $\hat{Q}$ with $v_0 = u_0^2$. The first non–trivial flows are:

$$\begin{aligned} \frac{\partial v_1}{\partial t_3} &= -\frac{1}{2}v_1''' - \frac{3}{4}v_1 v_1' + 3v_0' \\ \frac{\partial v_0}{\partial t_3} &= v_0''' + \frac{3}{4}v_1 v_0' \end{aligned} \qquad (9.21)$$

where for $c_0, c_1 : v_1 = 2u_1, v_0 = 2(u_0 + u_1'')$ and for $c_2 :\ v_1 = 2u_1, v_0 = u_0^2$.

In appendix E are given more examples of DS hierarchies of order 4 and 5.

The (N,M)–KdV hierarchies

The (N, M)–KdV hierarchies are defined by the pseudodifferential operator

$$L = \partial^N + N\sum_{l=1}^{N-1} a_l \partial^{N-l-1} + N\sum_{l=1}^{M} a_{N+l-1}\frac{1}{\partial - S_l}\frac{1}{\partial - S_{l-1}}\cdots\frac{1}{\partial - S_1}, \quad N \geq 1,\ M \geq 0 \qquad (9.22)$$

The case $(N, 0)$ coincides with the N–KdV case. These hierarchies were studied in [122], [123], [124], [126]. In [124] it was shown that they can be embedded in the DS construction. Now we show that this class of integrable hierarchies can be entirely embedded in the TLH. Let us see, for example, the $(2, 1)$ case. The Lax operator is

$$L = \partial^2 + 2a_1 + 2a_2\frac{1}{\partial - S}$$

The gauge fixing gives

$$\bar{a}_1 = a_1, \qquad \bar{a}_2 = a_2 - \frac{1}{2}a_1', \qquad \bar{a}_3 = -\frac{1}{2}a_2' + \frac{1}{4}a_1'' - \frac{1}{2}a_1^2 + a_2 S$$

and so on. It leads, via our recipe, to the following flows

$$\partial^{-1}\frac{\partial a_1}{\partial t_2} = 2a_2, \qquad \partial^{-1}\frac{\partial a_2}{\partial t_2} = a_2' + 2a_2 S, \qquad \partial^{-1}\frac{\partial S}{\partial t_2} = S^2 + 2a_1 - S.'$$

$$\partial^{-1}\frac{\partial a_1}{\partial t_3} = \frac{3}{2}a_2' + \frac{1}{4}a_1'' + \frac{3}{2}a_1^2 + 3a_2 S, \qquad \partial^{-1}\frac{\partial a_2}{\partial t_3} = a_2'' + 3a_1 a_2 + 3a_2' S + 3a_2 S^2$$

and so on. These are exactly the flows of the (2,1)–KdV hierarchy.

9.3 Comments and conclusion

The examples we have considered in the previous section do not exhaust all possible integrable hierarchies (for an updating on this subject see [128]). However they are very numerous and they leave very little doubt that whatever scalar Lax operator (9.1), defining an integrable hierarchy, we may think of, it can be embedded in the Toda lattice hierarchy in the way we showed above. Anyhow, thus far we have not found any counterexample. Therefore our construction looks at least as general as the DS realization. The fact that we are dealing with semi–infinite matrices may suggest additional possibilities.

We also remark that the TLH, in its general formulation, may encompass several $\hat{Q}$ matrices (not only one, as in this chapter). Therefore there is room for 'tensor products of integrable hierarchies in interaction'.

We end the chapter by recalling that in the case of the $(1, M)$–KdV hierarchies there is a variant to the realization of section 2. This was already pointed out in section 6.2 of [83] and, implicitly, in [122]. If one does not set $\hat{a}_0 = 0$ and replaces the first flows (9.5) in the Toda lattice flows, one gets exactly the $(1, M)$ hierarchies if the gauge fixing simply consists of setting $\hat{a}_l = 0$ for $l > M$. It was shown in [122] that (N, M)–KdV hierarchies can then be extracted from the $(1, M)$ via a cascade Hamiltonian reduction. However it is not clear whether this method can be generalized to other hierarchies, and, anyhow, it does not seem to be appropriate to call it a realization of differential hierarchies, at least in the same sense this terminology has been used in this chapter.

10 Conclusions

Any solved problem creates more puzzles.

For example, we know that c=1 matrix model is closely related to the Penner model. But also the simpler 2-matrix model permits the computations of the correlation functions for tachyons and discrete states. However, we lack a deeper understanding of why the 2-matrix model is also connected with the moduli space of riemann surfaces, or in other words what it is the topological meaning of the model.

Also the well-studied c=1 matrix model has its puzzles. If the c=1 matrix model has some relevance to the conifold topology, then we must be able to deform this topology which in the associated matrix model means to change the radius R. But it is not clear how the radius R can be implemented in the action of the c=1 matrix model.

The discrete states appear at the self-dual radius and outside the self-dual point they dissapear and become massive excitations. At this stage we also are not able to calculate the mass of discrete states, using the matrix model formalism.

Very little has been done to understand the target space picture. It is not known how to control the target space background. As we have seen in the specific example of the c=1 matrix model this means that it is not known how to control the value of the radius R. A possible rezolution of the problem is to apply the Green's ideea to matrix models, which consists in supposing that the world sheet of one string theory is the target space of ano ther, dual string. For example the existence of a dual $c = 1$ matrix model might permit to reformulate the target space problems of the c=1 matrix model in terms of the world-sheet fields of the dual theory.

We now know how to obtain the n-KdV hierarchies from the reductions of the KP-hierarchy. In the parameter space of the 2-matrix model, the reduction doesn't mean putting some parameters equal to zero in the KP-hierarchy. This naive approach doesn't work, because the flow equations change when we apply the constraints. In some sense, the flow equations are like the equations coming out from the renormalization group theory. But we don't know how to derive these flows directly from the action of the matrix model. It is however an amazing and we hope an important fact that the reductions of the KP-hierarchy can be classified according to the Drinfeld-Sokolov scheme.

Appendices

Appendix A. Explicit expressions of correlation functions.

From now on, whenever it is not confusing, we use the simplified notation $Q(\alpha) \equiv Q_\alpha$.

The 1–point CF is easily found to be given by

$$< \chi_{a_1,\dots,a_q} >= \mathrm{Tr}\Big(Q_1^{a_1}\cdots Q_q^{a_q}\Big) \tag{A.1}$$

The derivation of the two point functions, by the above procedure, is as follows

$$< \chi_{a_1\dots a_q}\chi_{b_1\dots b_q} >= \sum_{n=0}^{N-1} \frac{\partial^2 \ln h_n}{\partial g_{a_1\dots a_q}\partial g_{b_1\dots b_q}}$$

$$\begin{aligned}\frac{\partial^2 h_n}{\partial g_{a_1\dots a_q}\partial g_{b_1\dots b_q}} &= \int d\lambda \frac{\partial}{\partial g_{b_1\dots b_q}}\xi_n \lambda_1^{a_1}\dots\lambda_q^{a_q}\eta_n + \int d\lambda \xi_n \lambda_1^{a_1+b_1}\dots\lambda_q^{a_q+b_q}\eta_n + \\ &+ \int d\lambda \xi_n \lambda_1^{b_1}\dots\lambda_q^{b_q}\frac{\partial}{\partial g_{b_1\dots b_q}}\eta_n\end{aligned}$$

Then, using

$$\frac{\partial}{\partial g_{b_1\dots b_q}}\xi_n = -\sum_{m=0}^{n-1}(Q_1^{b_1}\dots Q_q^{b_q})_{nm}\xi_m, \qquad \frac{\partial}{\partial g_{b_1\dots b_q}}\eta_n = -\sum_{m=0}^{n-1}\eta_m(Q_1^{b_1}\dots Q_q^{b_q})_{mn}\frac{h_n}{h_m}.$$

we obtain

$$\begin{aligned}< \chi_{a_1,\dots,a_q}\chi_{b_1,\dots,b_q} > &= \mathrm{Tr}\Big[Q_1^{a_1+b_1}\cdots Q_q^{a_q+b_q} - \Big(Q_1^{b_1}\cdots Q_q^{b_q}\Big)_-\Big(Q_1^{a_1}\cdots Q_q^{a_q}\Big) \\ &\quad -\Big(Q_1^{a_1}\cdots Q_q^{a_q}\Big)\Big(Q_1^{b_1}\cdots Q_q^{b_q}\Big)_+\Big]\end{aligned} \tag{A.2}$$

Along the same lines, we get

$$\begin{aligned}< \chi_{a_1,\dots,a_q}\chi_{b_1,\dots,b_q}\chi_{c_1,\dots,c_q} >= \mathrm{Tr}\Big\{ & Q_1^{a_1+b_1+c_1}\dots Q_q^{a_q+b_q+c_q} \\ -\Big[&\Big(Q_1^{a_1}\dots Q_q^{a_q}\Big)_-\Big(Q_1^{b_1+c_1}\dots Q_q^{b_q+c_q}\Big) + \Big(Q_1^{b_1+c_1}\dots Q_q^{b_q+c_q}\Big)\Big(Q_1^{a_1}\dots Q_q^{a_q}\Big)_+ + \text{ c.p.}\Big] \\ +\Big[&\Big(Q_1^{a_1}\dots Q_q^{a_q}\Big)_-\Big(Q_1^{b_1}\dots Q_q^{b_q}\Big)\Big(Q_1^{c_1}\dots Q_q^{c_q}\Big)_+ + \text{ p.}\Big] \\ +2&\Big(Q_1^{a_1}\dots Q_q^{a_q}\Big)_+\Big(Q_1^{b_1}\dots Q_q^{b_q}\Big)_+\Big(Q_1^{c_1}\dots Q_q^{c_q}\Big)_+\Big\}.\end{aligned} \tag{A.3}$$

where p. (c.p.) means permutations (cyclic permutations) of the sets $\{a_1,\dots,a_q\}$, $\{b_1,\dots,b_q\}$ and $\{c_1,\dots,c_q\}$. The RHS's of both (A.2) and (A.3) are symmetric under the exchange of the χ operators. This property together with the fact that the RHS's can be written down in terms of the Q_α's, which are calculable, expresses what we call *weak integrability.*

Appendix B. Explicit derivation of 1–point correlators.

In this Appendix we give the derivation of the 1–point functions promised in section 4. For this we need to know Q^p where $Q = I_+ + a\epsilon_-$;we express it in terms of the normal ordered quantities : Q^l : $(QQ =: QQ : +[QQ])$

$$Q^p = \sum_{k=0}^{[p/2]} : Q^{p-2k} : ([QQ])^k A_k^{(p)} \tag{B.1}$$

where the contractions are $[QQ] = -aI_0$ because $[I_+, \epsilon_-] = I_0$;the normal ordering means that we have expressions with I_+ on left side and ϵ_- on the right side.

The coefficient $A_k^{(p)}$ is the number of ways in which we can choose k pairs from p identical objects:

$$A_k^{(p)} = \frac{1}{k!}\begin{pmatrix} p \\ 2 \end{pmatrix}\begin{pmatrix} p-2 \\ 2 \end{pmatrix}\cdots\begin{pmatrix} p-2k+2 \\ 2 \end{pmatrix} = \frac{p!2^{-k}}{(p-2k)!k!}$$

Using :

$$: Q^{p-2k} := \sum_{i=0}^{p-2k} \begin{pmatrix} p-2k \\ i \end{pmatrix} a^i I_+^{p-2k-i}(\epsilon_-)^i \tag{B.2}$$

We have the result:

$$Q^{2p} = \sum_{k=0}^{2p}\sum_{i=0}^{2p-2k} \begin{pmatrix} 2p-2k \\ i \end{pmatrix} (-1)^k a^{i+k} A_k^{(2p)} I_+^{2p-2k-i}(\epsilon_-)^i \tag{B.3}$$

We define the Q_1 matrix:

$$Q_1 = I_+ + a_0 I_0 + a_1(\epsilon_-) = Q(a_1) + a_0 I_0$$

$$Q_1^r = \sum_{2l=0}^{r} \begin{pmatrix} r \\ 2l \end{pmatrix} Q(a_1)^{2l} a_0^{r-2l} \tag{B.4}$$

Using (B.3) the expression of Q_1^r is:

$$Q_1^r = \sum_{2l=0}^{r}\sum_{k=0}^{2l}\sum_{i=0}^{2l-2k} \begin{pmatrix} r \\ 2l \end{pmatrix}\begin{pmatrix} 2l-2k \\ i \end{pmatrix} (-1)^k A_k^{(2l)} a_1^{i+k} a_0^{r-2l} I_+^{2l-2k-i}(\epsilon_-)^i \tag{B.5}$$

The same expressions are for :

$$Q_\alpha^r = \sum_{2l=0}^{r}\sum_{k=0}^{2l}\sum_{i=0}^{2l-2k} \begin{pmatrix} r \\ 2l \end{pmatrix}\begin{pmatrix} 2l-2k \\ i \end{pmatrix} (-1)^k A_k^{(2l)} f_1(\alpha) I_+^{2l-2k-i}(\epsilon_-)^i \tag{B.6}$$

$$f_1(\alpha) = (g_\alpha/h_\alpha)^{i+k} h_\alpha^{2l} s_\alpha^{r-2l}, \alpha = 1 \dots q$$

Because we have the summation:

$$\mathrm{Tr}(I_+^k(\epsilon_-)^k) = k!\sum_{N=0}^{N-1}\begin{pmatrix} n+k \\ k \end{pmatrix} = k!\begin{pmatrix} N+k \\ k+1 \end{pmatrix}$$

the 1-point correlation function is:

$$<\tau_r>= \mathrm{Tr}Q_1^r = \sum_{2l=0}^{r}\sum_{k=0}^{l}\frac{(-1)^k 2^{-k} r!}{(r-2l)!k!(l-k)!}\begin{pmatrix} N+l-k \\ l-k+1 \end{pmatrix}(h_\alpha g_\alpha)^l s_\alpha^{r-2l} \qquad \text{(B.7)}$$

Appendix C. Topological field theory properties of the 4-matrix model

We expect that the q-multi-matrix models represent topological field theories. We prove this assertion in the chain phase space where all potentials are turned off. We are able to identify the puncture operator, the primary fields and the descendants and show that they define an appropriate metric and satisfy a puncture equation and the appropriate recursion relations.

Puncture operator Q

The puncture operator is identified with $Q = \chi_{0,\dots,0} \equiv \frac{\partial}{\partial N}$ and as primary fields we take $T_{\alpha,n} \equiv \chi_{0,\dots,a_\alpha=n,0}$ while all the other $\chi_{a_1,\dots,a_q}$ will be descendants.

In this identification the metric is given by:

$$\eta_{\alpha r,\beta s} = \langle Q T_{\alpha r} T_{\beta s}\rangle_0 \qquad \text{(C.1)}$$

and the structure constants are:

$$C_{\alpha r,\beta s,\gamma t} = \langle T_{\alpha r} T_{\beta s} T_{\gamma t}\rangle_0 \qquad \text{(C.2)}$$

(where again $T_{\alpha 0} \equiv Q$ is the puncture operator and the cases $\eta_{0,\beta s}$ will be treated separately).

We will check that the associativity condition for the structure constants (that defines a topological field theorie) is fulfilled:

$$\sum_{\gamma t, deltav} C_{\alpha r,\beta s,\gamma t}\eta^{\gamma t,\delta v}C_{\delta v,\epsilon p,\xi q} = C_{\alpha r,\epsilon p,\gamma t}\eta^{\gamma t,\delta v}C_{\delta v,\beta s,\xi q} \qquad \text{(C.3)}$$

For this we need the explicit expressions for metric and structure constants. Using the flow equations we have derived the 2-point correlation function in genus 0:

$$\langle T_{\alpha r} T_{\beta s}\rangle_0 = r(q_\alpha q_\beta)^r \delta_{rs} N^r \Delta_{\alpha\beta} \qquad \text{(C.4)}$$

where the coefficients q_α are:

$$q_\alpha = \frac{c_2}{c_1}\frac{c_4}{c_3}\ldots\frac{1}{c_{\alpha-1}}, \quad \alpha \ odd$$
$$q_\alpha = \frac{c_1}{c_2}\frac{c_3}{c_4}\ldots\frac{c_{\alpha-2}}{c_{\alpha-1}}, \quad \alpha \ even$$

The metric in this case is:

$$\eta_{\alpha r,\beta s} = \frac{\partial}{\partial N}\langle T_{\alpha r}T_{\beta s}\rangle_0 = rs(q_\alpha q_\beta)^r \delta_{rs} N^{r-1}\Delta_{\alpha\beta} \qquad \eta_{\alpha 0,\beta 0} = N^{-1} \tag{C.5}$$

The delta matrix $\Delta_{\alpha\beta}$ is simply $\delta(\alpha+\beta = even)$ if we take in account only the $U(1)$ charge conservation (the fields $T_{\alpha,r}$ with $\alpha = odd$ have $+r$ U(1) charge and $-r$ U(1) charge otherwise). But if we take in account also the flow equations for intermediate matrices (which are different from the ones for first and last matrix) the delta matrix $\Delta_{\alpha\beta}$ becomes:

$$\Delta_{\alpha\beta} = \begin{cases} 0, & \alpha \ even \quad \alpha \geq \beta \\ 0, & \beta \ odd \quad \alpha \geq \beta \\ 0, & \beta \ even \quad \alpha \leq \beta \\ 0, & \alpha \ odd \quad \alpha \leq \beta \\ 1, & otherwise \end{cases} \tag{C.6}$$

For 4-matrix model, the metric is :

$$\eta_{\alpha,\beta} = \begin{pmatrix} 0 & \eta_{1,2} & 0 & \eta_{1,4} \\ \eta_{1,2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta_{3,4} \\ \eta_{1,4} & 0 & \eta_{3,4} & 0 \end{pmatrix} \tag{C.7}$$

(Where each nonzero entry is proportional to a two-matrix-model metric, i.e. takes in account the r,s indices with the cases $r,s=0$) We observe that the metric is symmetric in the indices α,β: $\eta_{\alpha,\beta} = \eta^T_{\alpha,\beta}$ and that is non-degenerate.

For 4-matrix model, the inverse is:

$$\eta^{\alpha,\beta} = \begin{pmatrix} 0 & \eta^{-1}_{1,2} & 0 & 0 \\ \eta^{-1}_{1,2} & 0 & -\eta^{-1}_{1,2}\eta_{1,4}\eta^{-1}_{3,4} & 0 \\ 0 & -\eta^{-1}_{1,2}\eta_{1,4}\eta^{-1}_{3,4} & 0 & \eta^{-1}_{3,4} \\ 0 & 0 & \eta^{-1}_{3,4} & 0 \end{pmatrix} \tag{C.8}$$

After we have applied two times the flow equations we arrive to the three point functions and we get a symmetric expression for the structure constants:

$$C_{\alpha r,\beta s,\gamma t} = rst(q^r_\alpha q^s_\beta q^t_\gamma)N^{max(r,s,t)-1}\delta(r+s+t, 2max(r,s,t))\Delta_{\alpha r,\beta s,\gamma t} \tag{C.9}$$

From the $U(1)$ charge conservation follows that we must fulfill the following conditions for r, s, t (these conditions enter in the $\Delta_{\alpha r,\beta s,\gamma t}$ matrix):

$$\Delta_{\alpha r,\beta s,\gamma t} = \begin{cases} & \quad\alpha \qquad \beta \qquad \gamma \\ \delta(s=r+t), & \begin{cases} even & odd & even \\ odd & even & odd \end{cases} \\ \delta(r=s+t), & \begin{cases} odd & even & even \\ even & odd & odd \end{cases} \\ \delta(r+s=t), & \begin{cases} odd & odd & even \\ even & even & odd \end{cases} \\ 0, & \qquad otherwise \end{cases} \tag{C.10}$$

But from the flow equations we have few more constraints.

We present the structure constants for the 4-matrix model. The first and last structure constants $C_{\alpha,\beta,1}, C_{\alpha,\beta,4}$ have the usual form which follows from the $U(1)$ conservation (only $C_{\alpha,\beta,1}$ has two more null entries) :

$$C_{\alpha,\beta,1} = \begin{pmatrix} 0 & C_{1,2,1}\delta_{t+r,s} & 0 & C_{1,4,1}\delta_{t+r,s} \\ C_{1,2,1}\delta_{s+t,r} & C_{2,2,1}\delta_{r+s,t} & 0 & C_{2,4,1}\delta_{r+s,t} \\ 0 & 0 & 0 & C_{3,4,1}\delta_{t+r,s} \\ C_{1,4,1}\delta_{s+t,r} & C_{2,4,1}\delta_{r+s,t} & C_{3,4,1}\delta_{s+t,r} & C_{4,4,1}\delta_{r+s,t} \end{pmatrix} \tag{C.11}$$

$$C_{\alpha,\beta,4} = \begin{pmatrix} C_{1,1,4}\delta_{r+s,t} & C_{1,2,4}\delta_{t+s,r} & C_{1,3,4}\delta_{r+s,t} & C_{1,4,4}\delta_{t+s,r} \\ C_{1,2,4}\delta_{r+t,s} & 0 & C_{2,3,4}\delta_{r+t,s} & 0 \\ C_{1,3,4}\delta_{r+s,t} & C_{2,3,4}\delta_{+s,r} & C_{3,3,4}\delta_{r+s,t} & C_{3,4,4}\delta_{t+s,r} \\ C_{1,4,4}\delta_{r+t,s} & 0 & C_{3,4,4}\delta_{r+t,s} & 0 \end{pmatrix}$$

The intermediate structure constants $C_{\alpha,\beta,2}, C_{\alpha,\beta,3}$ have more null elements due to the flow equations:

$$C_{\alpha,\beta,2} = \begin{pmatrix} C_{1,1,2}\delta_{r+s,t} & C_{1,2,2}\delta_{t+s,r} & 0 & C_{1,4,2}\delta_{t+s,r} \\ C_{1,2,2}\delta_{r+t,s} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{3,4,2}\delta_{t+s,r} \\ C_{1,4,2}\delta_{r+t,s} & 0 & C_{3,4,2}\delta_{r+t,s} & 0 \end{pmatrix} \tag{C.12}$$

$$C_{\alpha,\beta,3} = \begin{pmatrix} 0 & 0 & 0 & C_{1,4,3}\delta_{t+r,s} \\ 0 & 0 & 0 & C_{2,4,3}\delta_{r+s,t} \\ 0 & 0 & 0 & C_{3,4,3}\delta_{t+r,s} \\ C_{1,4,3}\delta_{s+t,r} & C_{2,4,3}\delta_{r+s,t} & C_{3,4,3}\delta_{s+t,r} & C_{4,4,3}\delta_{r+s,t} \end{pmatrix}$$

Looking at expressions of the structure constants we can see the symmetry under the permutation of indices α, β, γ. This symmetry show that the primary fields form the commutative associative algebra $\mathcal{A}$

$$T_{\alpha r}T_{\beta s} = \sum_{\gamma t} C^{\gamma t}_{\alpha r,\beta s}T_{\gamma t}, \qquad C^{\gamma t}_{\alpha r,\beta s} \equiv \sum_{\delta p} C^{\delta p}_{\alpha r,\beta s}\eta^{\delta p,\gamma t}$$

The new structure constants are:

$$C^{\gamma t}_{\alpha r,\beta s} = \frac{rs}{t}(\frac{q^r_\alpha q^s_\beta}{q^t_\gamma})N^{max(r,s,t)-t}\delta(r+s+t, 2max(r,s,t))\sum_{\delta p}\Delta_{\alpha r,\beta s,\delta p}\Delta_{\delta p,\gamma t} \qquad (C.13)$$

Puncture operator $T_{\alpha,1}$

We can identify the puncture operator in another way with: $T_{\alpha,1} = \chi_{0....0} \equiv \frac{\partial}{\partial t_{\alpha,1}}$. We have two different cases due to the different flow equations for the first and last matrices ,and respectivelly for intermediate matrices.

For concretness we consider the 4-matrix model and the puncture operators $T_{1,1}, T_{2,1}$. The other two operators $T_{4,1}, T_{3,1}$ can be obtained by symmetry.

First we take as puncture operator $T_{1,1}$.

The form of the metric does not change too much , it only shifts with respect to the variables r, s,from $\delta_{r,s}$ to $\delta_{r,s+1}$ or $\delta_{r+1,s}$. The form of the metric in the general case is:

$$\begin{aligned} \eta_{\alpha r,\beta s} &= \frac{\partial}{\partial t_{1,1}}\langle T_{\alpha r}T_{\beta s}\rangle_0 = rs(q^r_\alpha q^s_\beta)\delta(r+s+1, 2max(r,s))N^{min(r,s)}\Delta_{\alpha\beta} \qquad (C.14) \\ \eta_{\alpha 0,\beta 1} &= \begin{cases} q_1 q_\beta, & \beta \ even \\ 0, & otherwise \end{cases} \end{aligned}$$

For the 4-matrix model we have:

$$\eta_{\alpha,\beta} = \begin{pmatrix} 0 & \eta_{1,2}\delta_{r+1,s} & 0 & \eta_{1,4}\delta_{r+1,s} \\ \eta_{1,2}\delta_{s+1,r} & \eta_{2,2}\delta_{s+r,1} & 0 & \eta_{2,4}\delta_{s+r,1} \\ 0 & 0 & 0 & \eta_{3,4}\delta_{r+1,s} \\ \eta_{1,4}\delta_{s+1,r} & \eta_{2,4}\delta_{s+r,1} & \eta_{3,4}\delta_{s+1,r} & \eta_{4,4}\delta_{s+r,1} \end{pmatrix} \qquad (C.15)$$

(Where each nonzero entry is proportional to a two-matrix-model metric, i.e. takes in account the r, s indices with the cases $r = 0, s = 1$ or $r = 1, s = 0$). We observe that the metric is symmetric in the indices α, β: $\eta_{\alpha,\beta} = \eta^T_{\alpha,\beta}$ and that is non-degenerate.

For 4-matrix model, the inverse is:

$$\eta^{\alpha,\beta} = \begin{pmatrix} \eta_{2,2}\eta^{-2}_{1,2}\delta_{s+r,1} & \eta^{-1}_{1,2}\delta_{s+1,r} & A\delta_{s+r,1} & 0 \\ \eta^{-1}_{1,2}\delta_{r+1,s} & 0 & -\eta^{-1}_{1,2}\eta_{1,4}\eta^{-1}_{3,4}\delta_{r+1,s} & 0 \\ A\delta_{s+r,1} & -\eta^{-1}_{1,2}\eta_{1,4}\eta^{-1}_{3,4}\delta_{s+1,r} & B\delta_{s+r,1} & \eta^{-1}_{3,4}\delta_{s+1,r} \\ 0 & 0 & \eta^{-1}_{3,4}\delta_{r+1,s} & 0 \end{pmatrix} \qquad (C.16)$$

where:

$$\begin{aligned} A &= (\eta_{2,2}\eta_{1,4} - \eta_{1,2}\eta_{2,4})\eta^{-2}_{1,2}\eta^{-1}_{3,4}, \\ B &= (2\eta_{1,2}\eta_{1,4}\eta_{2,4} - \eta_{2,2}\eta^2_{1,4} - \eta^2_{1,2}\eta_{4,4})\eta^{-2}_{1,2}\eta^{-2}_{3,4} \end{aligned}$$

We consider now as puncture operator $T_{2,1}$.

The form of the metric in the general case is:

$$
\begin{aligned}
\eta_{\alpha r,\beta s} &= \frac{\partial}{\partial t_{1,1}} \langle T_{\alpha r} T_{\beta s} \rangle_0 = rs(q_\alpha^r q_\beta^s)\delta_{r+s+1,2max(r,s)} N^{min(r,s)} \Delta_{\alpha\beta} \\
\eta_{\alpha 0,\beta 1} &= \begin{cases} q_1 q_\beta, & \beta \ odd \\ 0, & otherwise \end{cases}
\end{aligned} \tag{C.17}
$$

For the 4-matrix model we have:

$$
\eta_{\alpha,\beta} = \begin{pmatrix} \eta_{1,1}\delta_{s+r,1} & \eta_{1,2}\delta_{s+1,r} & 0 & \eta_{1,4}\delta_{s+1,r} \\ \eta_{1,2}\delta_{r+1,s} & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta_{3,4}\delta_{s+1,r} \\ \eta_{1,4}\delta_{r+1,s} & 0 & \eta_{3,4}\delta_{r+1,s} & 0 \end{pmatrix} \tag{C.18}
$$

It is non-degenerate and the inverse is:

$$
\eta^{\alpha,\beta} = \begin{pmatrix} 0 & \eta_{1,2}^{-1}\delta_{r+1,s} & 0 & 0 \\ \eta_{1,2}^{-1}\delta_{s+1,r} & -\eta_{1,1}\eta_{1,2}^{-2}\delta_{r+s,1} & -\eta_{1,2}^{-1}\eta_{1,4}\eta_{3,4}^{-1}\delta_{s+1,r} & 0 \\ 0 & -\eta_{1,2}^{-1}\eta_{1,4}\eta_{3,4}^{-1}\delta_{r+1,s} & 0 & \eta_{3,4}^{-1}\delta_{r+1,s} \\ 0 & 0 & \eta_{3,4}^{-1}\delta_{s+1,r} & 0 \end{pmatrix} \tag{C.19}
$$

The structure constants are unchanged from the previous case with puncture operator Q, but in addition we have also the following new structure constants:

$$
C_{\alpha r,\beta s,\gamma 0} = rs(q_\alpha q_\beta)^r \delta_{rs} N^{r-1} \Delta_{\alpha\beta} \qquad C_{\alpha 0,\beta 0,\gamma 0} = N^{-1} \tag{C.20}
$$

Appendix D. Discrete states in 2q-matrix model and in topological W_{q+1} gravity

We speculate that the discrete states of $2q$ multimatrix model form the same ground ring $\mathcal{R}$ as that of the W_{q+1} topological gravity.

We review the construction of discrete states for the $c = 1$ string theory.This theory describes topological gravity coupled with topological matter with $c_M = 1$ (free boson):we must also add the ghost system $\{b, c\}$:

$$
M^{\text{matter}} \otimes M^{\text{Liouville}} \otimes \{b, c\}
$$

The most well-known physical states are pure tachyons:

$$
T_{r,s} = cV^L_{r,-s} V^M_{r,s}
$$

where $V_{r,s}$ are the vertex operators .We can express them in terms of the free boson operator ϕ : $V_{r,s} = \exp(i\alpha\phi)$ (we have different operators for Liouville and matter sectors). The ghost number of tachyons is one.

For generic matter charge there exist extra physical states with vanishing ghost charge. They form the so called ground ring $\mathcal{R}$. This ring is generated by two generators : $\chi_{1,0}, \chi_{0,1}$ (in the notations of work [85][86]):

$$\mathcal{R} = (\chi_{1,0})^r \otimes (\chi_{0,1})^s, \; r, s = integers$$

These generators can be written in terms of the free Liouville ϕ_L and matter ϕ_M fields and in terms of the ghost fields b, c:

$$\begin{aligned} \chi_{1,0} &= \left[bc - \frac{1}{\sqrt{2}}(\partial\phi_L - i\partial\phi_M) \right] V^L_{1,2} V^M_{1,2} \\ \chi_{1,0} &= \left[bc - \frac{1}{\sqrt{2}}(\partial\phi_L + i\partial\phi_M) \right] V^L_{2,1} V^M_{2,1} \end{aligned} \tag{D.1}$$

In the 2-matrix model we can introduce external sources of the form:

$$c_{r,s} D_1^r D_2^s \;\; with \;\; D_\alpha = diag(M_\alpha), \alpha = 1, 2$$

In the work [61] the operators $\chi_{r,s}$ coupled to the $c_{r,s}$ were identified with the discrete states of the $c = 1$ matrix model.

Because the set of tachyons $T^+_{\pm j}$ of ghost charge 1 can be mapped into the set of operators $\mathcal{O}_{j,\pm j}$ of ghost charge 0 , we can interpret $\chi_{j,0}$ and $\chi_{0,j}$ as pure tachyons with positive, respectively negative momenta:

$$\chi_{j,0} = T^+_{+j}, \;\; \chi_{0,j} = T^+_{-j} \tag{D.2}$$

This is combatible with the fact that $\chi_{m,0}$ acting on $\chi_{j,0}$ will shift $\chi_{j,0}$ the energy and momentum of the tachyon by the same amount m, creating $\chi_{j+m,0}$. Instead the operator $\chi_{m,n}$ acting on $\chi_{j,0}$ will not only shift the energy and momentum of the tachyon, creating $\chi_{j+m,0}$, but also will afect the number of tachyons, creating n tachyons $\chi_{0,1}$.

For the non-critical W string we couple W_{q+1} topological gravity with $c_M = q$ topological matter (q free independent bosons) and q ghost systems:

$$W^{\text{matter}}_{q+1} \otimes W^{\text{Liouville}}_{q+1} \otimes^q_{\alpha=1} \{b_\alpha, c_\alpha\}$$

To construct the tachyon and extra states operators we need the expression of the vertex operators.We write the matter vertex operator:

$$V^M_{a_{2\alpha-1};\ldots a_{2\alpha}} = \exp(i\alpha_{a_{2\alpha-1};a_{2\alpha}} \phi_M) \tag{D.3}$$

where $\phi_M = \{\phi_{M,i}\}$ represents the set of q free boson operators and :

$$\alpha_{a_{2\alpha-1};a_{2\alpha}} = [\alpha_+(a_{2\alpha-1} - 1) + \alpha_-(a_{2\alpha} - 1)]\lambda_\alpha \tag{D.4}$$

with $\lambda_{\alpha,i}, \alpha = 1, \ldots q$ are fundamental weights. In the same manner we define the Liouville vertex operators.

We can now write the expression of tachyon operators:

$$T_{a_1\dots a_{2q}} = \epsilon \prod_{\alpha=1}^{q} V^L_{a_{2\alpha-1};-a_{2\alpha}} V^M_{a_{2\alpha-1};a_{2\alpha}},\ c = \prod_{\alpha=1}^{q} c_\alpha \qquad (D.5)$$

In the same way as was made in the case of the $c = 1$ string theory we can define the extra states of $2q$ multi-matrix models in terms of vertex operators of topological matter coupled to topological W_{q+1}–gravity.The ground ring is generated by the following operators:

$$\mathcal{R} = \prod_{\alpha=1}^{q} (\chi_{0,\dots 2\alpha-1\dots 0})^{a_\alpha} \otimes \prod_{\alpha=1}^{q} (\chi_{0,\dots 2\alpha\dots 0})^{a_\alpha}$$

In the $2q$ multi-matrix model we can introduce the operators coupled to the

$$c_{a_1\dots a_q} D_1^{a_1} \dots D_q^{a_q} \ \ with\ \ D_\alpha = diag(M_\alpha), \alpha = 1, 2$$

that will form the same ground ring $\mathcal{R}$ of the topological W_{q+1}–gravity.

In the case of 4-matrix model which is related with topological W_3-gravity coupled with topological matter we have the following identification of generators:

$$x_1 = \chi_{1,0,0,0}, \gamma_1^0 = \chi_{0,1,0,0}$$
$$x_2 = \chi_{0,0,1,0}, \gamma_2^0 = \chi_{0,0,0,1}$$

where the notations $x_i, \gamma_i^0, i = 1, 2$ are used in the work [85][86].

Appendix E. The DS Hierarchies

Order 4.

Case$C_2^{(1)}$:

$$\begin{aligned} c_1 : \quad & L = D^4 + 4u_0 D^2 + 4u_0' D + 2u_0'' + 4u_0^2 \\ & \hat{Q} = e^{\partial} - \frac{v_0}{2}\epsilon_- + \frac{v_0'}{4} e^{-2\partial} - (\frac{v_0''}{8} + \frac{5}{16} v_0^2) e^{-3\partial} + \\ & + \frac{1}{16}(15 v_0 v_0' + v_0''') e^{-4\partial} + \dots \\ c_0, c_2 : \quad & L = D^4 + 2u_1 D^2 + 2u_1' D + u_1'' + 2u_0 \\ & \hat{Q} = e^{\partial} + \frac{v_1}{4}\epsilon_- - \frac{v_1'}{8} e^{-2\partial} + (\frac{v_0}{4} - \frac{v_1''}{16} - \frac{3}{32} v_1^2) e^{-3\partial} + \\ & + \frac{1}{32}(9 v_1 v_1' + 5 v_1''' - 12 v_0') e^{-4\partial} + \dots \end{aligned} \qquad (E.1)$$

We have the equations for c_1:

$$\frac{\partial v_0}{\partial t_3} = \frac{1}{4} v_1''' + \frac{3}{4} v_0 v_0' \qquad (E.2)$$

with $v_0 = 4u_0$ and for c_0, c_2:

$$\frac{\partial v_1}{\partial t_3} = -\frac{5}{4}v_1''' - \frac{3}{4}v_1 v_1' + 3v_0' \tag{E.3}$$

with $v_1 = 2u_1, v_0 = 2u_0 + u_1''$.

Case $D_2^{(2)}$:

$$\begin{aligned} c_0: \quad & L = D^4 + 2u_0 D^2 + 3u_0' D + u_0'' \\ & \hat{Q} = e^{\partial} + \frac{u_0}{2}\epsilon_- - (\frac{u_0''}{4} + \frac{3}{8}u_0^2)e^{-3\partial} + \frac{1}{4}(3u_0 u_0' + u_0''')e^{-4\partial} + \dots \\ c_1: \quad & L = D^4 + 2u_0 D^2 + u_0' D \\ & \hat{Q} = e^{\partial} + \frac{u_0}{2}\epsilon_- - u_0'\epsilon^{-2\partial} + (\frac{u_0''}{4} - \frac{3}{8}u_0^2)e^{-3\partial} + (\frac{5}{32}u_0 u_0' + \frac{3}{2}u_0''')e^{-4\partial} + \dots \end{aligned} \tag{E.4}$$

The equations are:

$$\frac{\partial u_0}{\partial t_3} = -\frac{1}{2}u_0''' - \frac{3}{2}u_0 u_0' \tag{E.5}$$

Case $D_3^{(1)}$:

$$\begin{aligned} c_0, c_1: \quad L &= D^4 + 2u_2 D^2 + u_2' D + 2u_2'' + 2u_1 - D^{-1}(u_1' + u_2'')' + (D^{-1}u_0)^2 \\ \hat{Q} &= e^{\partial} + \frac{u_2}{2}\epsilon_- - \frac{1}{2}u_2' e^{-2\partial} + (\frac{3}{4}u_2'' - \frac{3}{8}u_2^2 - \frac{1}{2}u_1)e^{-3\partial} + \\ & +(\frac{3}{2}u_2 u_2' - u_1' - u_2''')e^{-4\partial} + \dots \\ c_2, c_3: \quad L &= D^4 + 2u_2 D^2 + 3u_2' D + (2u_1 + 3u_2'') + (u_1' + u_2''')D^{-1} + u_0 D^{-1} u_0 D^{-1} \\ \hat{Q} &= e^{\partial} + \frac{u_2}{2}\epsilon_- + (\frac{1}{4}u_2'' - \frac{3}{8}u_2^2 + \frac{1}{2}u_1)e^{-3\partial} + \\ & +(\frac{3}{4}u_2 u_2' - \frac{1}{2}u_1' - \frac{1}{4}u_2''')e^{-4\partial} + \dots \end{aligned} \tag{E.6}$$

The first non–trivial equations for c_0, c_1 are:

$$\begin{aligned} \frac{\partial u_1}{\partial t_3} &= 3u_0 u_0' - \frac{3}{2}u_2^{(5)} - 2u_1''' + \frac{3}{2}u_1' u_2 + 3u_2 u_2''' + \frac{9}{2}u_2' u_2'' \\ \frac{\partial u_2}{\partial t_3} &= \frac{5}{2}u_2''' - \frac{3}{2}u_2 u_2' + 3u_1' \end{aligned} \tag{E.7}$$

and for c_2, c_3 are

$$\begin{aligned} \frac{\partial u_1}{\partial t_3} &= 3u_0 u_0' - \frac{3}{2}u_2^{(5)} - 2u_1''' - \frac{3}{2}u_1' u_2 + \frac{9}{2}u_2' u_2'' + 3u_2 u_2''' \\ \frac{\partial u_2}{\partial t_3} &= \frac{5}{2}u_2''' - \frac{3}{2}u_2 u_2' + 3u_1' \end{aligned} \tag{E.8}$$

Order 5.

Case $A_4^{(2)}$

$$
\begin{aligned}
c_0: \quad & L = D^5 + 2u_1D^3 + 3u_1'D^2 + (2u_0 + 3u_1'')D + u_0' + u_1''' \qquad \text{(E.9)}\\
& \hat{Q} = \epsilon^{\partial} + 2u_1\epsilon_- - u_1'e^{-2\partial} + (2u_0 + u_1'' - \frac{24}{5}u_1^2e^{-3\partial} + \ldots\\
c_1: \quad & L = D^5 + 4u_0D^3 + 5u_0'D^2 + 4(u_0'' + u_0^2)D + u_0''' + 2u_0u_0'\\
& \hat{Q} = \epsilon^{\partial} + 4u_0\epsilon_- - 3u_0'e^{-2\partial} + (2u_0'' - \frac{76}{5}u_0^2)e^{-3\partial} + \ldots\\
c_2: \quad & L = D^5 + 2u_1D^3 + 2u_1'D^2 + (2u_0 + u_1'')D\\
& \hat{Q} = \epsilon^{\partial} + 2u_1\epsilon_- - 2u_1'e^{-2\partial} + (2u_0 + u_1'' - \frac{24}{5}u_1^2)\epsilon^{-3\partial} + \ldots
\end{aligned}
$$

The equations for c_0 are:

$$
\begin{aligned}
\frac{\partial u_1}{\partial t_3} &= u_1''' + 3u_0' - \frac{12}{5}u_1u_1' \qquad \text{(E.10)}\\
\frac{\partial u_0}{\partial t_3} &= -2u_0''' + \frac{12}{5}u_1u_1''' + \frac{27}{5}u_1u_1'' + \frac{6}{5}(u_1u_0' - u_0u_1') - \frac{3}{5}u_1^{(5)}
\end{aligned}
$$

for c_1:

$$
\frac{\partial u_0}{\partial t_3} = \frac{1}{4}u_0''' + \frac{6}{5}u_0u_0' \qquad \text{(E.11)}
$$

and for c_2 are:

$$
\begin{aligned}
\frac{\partial u_1}{\partial t_3} &= -\frac{1}{2}u_1''' + 3u_0' - \frac{12}{5}u_1u_1' \qquad \text{(E.12)}\\
\frac{\partial u_0}{\partial t_3} &= -\frac{1}{2}u_0''' + \frac{3}{5}u_1u_1''' + \frac{9}{5}u_1u_1'' + \frac{6}{5}(u_1u_0' - u_0u_1') + \frac{3}{20}u_1^{(5)}
\end{aligned}
$$

Case $A_5^{(2)}$:

$$
\begin{aligned}
c_0, c_1 \quad : \quad & L = D^5 + 2u_2D^3 + 2u_2'D^2 + (2u_1 + 4u_2'')D + D^{-1}(2u_0 + u_1'' + u_2''') \quad \text{(E.13)}\\
& \hat{Q} = e^{\partial} + \frac{2}{5}u_2\epsilon_- - \frac{2}{5}u_2'e^{-2\partial} + \frac{2}{5}(u_1 + 2u_2'' - \frac{4}{5}u_2^2)e^{-3\partial} + \ldots\\
c_2 \quad : \quad & L = D^5 + 2(v_0 + u_1)D^3 + (6v_0' + u_1')D^2 + (6v_0'' + u_0^2 + 4v_0u_1)D +\\
& +(2v_0''' - u_0u_0' + 4u_1v_0' + 2v_0u_1') + u_0D^{-1}(u_0'' + 2u_0v_0)\\
& \hat{Q} = e^{\partial} + \frac{2}{5}(u_1 + v_0)\epsilon_- + \frac{1}{5}(2v_0' - 3u_1')e^{-2\partial} +\\
& +\frac{1}{5}(2u_1'' - 2v_0'' + \frac{4}{5}v_0u_1 + u_0^2 - \frac{8}{5}u_1^2 - \frac{8}{5}v_0^2)e^{-3\partial} + \ldots\\
c_3 \quad : \quad & L = D^5 + 2u_2D^3 + 3u_2'D^2 + (2u_1 + 3u_2'')D + u_1' + u_2''' + u_0D^{-1}u_0\\
& \hat{Q} = e^{\partial} + \frac{2}{5}u_2\epsilon_- - \frac{1}{5}u_2'e^{-2\partial} + \frac{1}{5}(2u_1 + u_2'' - \frac{8}{5}u_2^2)e^{-3\partial} + \ldots
\end{aligned}
$$

The equations are for c_0, c_1:

$$\begin{aligned} \frac{\partial u_2}{\partial t_3} &= 4u_2''' + 3u_1' - \frac{12}{5}u_2u_2' \\ \frac{\partial u_1}{\partial t_3} &= -\frac{7}{2}u_1''' + 6u_2u_2''' + \frac{54}{5}u_2'u_2'' + \frac{6}{5}(u_2u_1' - u_1u_2') - \frac{51}{10}u_2^{(5)} + 3u_0' \end{aligned} \tag{E.14}$$

for c_2:

$$\begin{aligned} 5\frac{\partial v_0}{\partial t_3} &= -v_0''' + \frac{3}{2}u_1''' - 12v_0v_0' + 6v_0'u_1 + 12v_0u_1' \\ 5\frac{\partial u_1}{\partial t_3} &= 6v_0''' - 4u_1''' + 15u_0u_0' + 6v_0u_1' + 12v_0'u_1 - 12u_1u_1' \end{aligned} \tag{E.15}$$

and for c_3 are:

$$\begin{aligned} \frac{\partial u_2}{\partial t_3} &= u_2''' + 3u_1' - \frac{12}{5}u_2u_2' \\ \frac{\partial u_1}{\partial t_3} &= -2u_1''' + \frac{27}{5}u_2'u_2'' + \frac{12}{5}u_2u_2''' + \frac{6}{5}(u_2u_1' - u_1u_2') - \frac{3}{5}u_2^{(5)} + 3u_0u_0' \end{aligned} \tag{E.16}$$

References

[1] S. Weinberg, Rev.Mod.Phys.,**61** (1989),1

[2] E. Witten, Com.Math.Phys. 117 (1988) 353

[3] A.Polyakov,*A few projects in string theory*, PUPT-1394,1993

[4] C. Teitelboim, Phys. Lett. **B126** (1983) 41,46,49
R.Jackiw, Nucl. Phys. **B252** (1985) 343

[5] J.Brown,**Lower dimensional gravity**, World Scientific, Singapore, 1988

[6] A.Polyakov,Phys.Lett.**103B**(1981)207

[7] S.Weinberg, in ("**General Relativity, an Einstein centenary survey**", eds. S.W. Hawking and W.Israel, Cambridge University Press, 1979), p.790

[8] A.M.Polyakov,**Gauge fields and strings**, Harwood Academic Publishers. 1987

[9] M.Kalb,P.Ramond,Phys.Rev.**D9**,(1974)2273

[10] A.Polyakov,Mod.Phys.Lett.**A2**(1987)899

[11] V.G.Knizhnik,A.M.Polyakov,A.A.Zamolodchikov, Mod.Phys.Lett.**A3**(1988)819

[12] T.L.Curtright,C.B.Thorn,Phys.Rev.Lett.**48**(1982)1309
E.Brateen,T.L.Curtright,C.B.Thorn,Phys.Lett.**B118**(1982)1309, Ann.Phys. **142** (1983) 365
E.Brateen, T.L.Curtright, G.Ghandour, C.B.Thorn, Phys. Rev. Lett. **51** (1983) 19. Ann. Phys. **153** (1984)147

[13] J.-L.Gervais,A.Neveu,Nucl.Phys. **199** (1982)125; **224** (1983)329; **238** (1984)125,396; Phys. Lett. **151B** (1985)271
J.-L.Gervais,Phys.Lett.**243B**(1990)85; Commun.Math.Phys. **130** (1990)257

[14] N.Seiberg,Prog.Theor.Phys. **102** (1990)319

[15] L.Alvarez-Gaume,C.Gomez,*Topics in Liouville theory* in ("**Trieste Lectures Notes 1991**", World Scientific,1992)

[16] F.David, Mod.Phys.Lett. **A3** (1988)1651

[17] J.Distler, H.Kawai, Nucl.Phys. **B321** (1989)509

[18] G.Moore,N.Seiberg,M.Staudacher, Nucl.Phys. **B362** (1991)665,
T.Banks,M.Douglas,N.Seiberg,S.Shenker,Phys.Lett, **B238**(1990)279

[19] C.G.Callan, D.Friedan, E.Martinec, M.J.Perry, Nucl.Phys. **B262**(1985)593

[20] J.Polchinski,Nucl.Phys.**B346** (1990)253

[21] P.Ginsparg, Nucl.Phys.**B295**(1988) 153
G.Harris Nucl.Phys.**B300**(1988) 588

[22] J.Polchinski, *What is String Theory?*,hep-th/9411028

[23] R.Jackiw, S.Templeton,Phys.Rev.D 23(1981)2291
S.Deser,R.Jackiw, S.Templeton,Phys.Rev.Lett**48**(1982)975, Ann.Phys.**140**(1982)372

[24] A.M.Polyakov, Mod.Phys.Lett. **A6**(1991)635

[25] A.M.Polyakov, I.R.Klebanov, Mod.Phys.Lett. **A6**(1991)3273

[26] I.R.Klebanov, Mod.Phys.Lett. **A7**(1992)723

[27] B.Lian, G.Zuckerman, Phys.Lett.**B 254**(1991)417

[28] E.Witten,Nucl.Phys.**B373**(1992)187

[29] E.Witten,B.Zwiebach, Nucl.Phys.**B377**(1992)55

[30] C.G.Callan, I.R.Klebanov, hep-th/9311092

[31] C.G.Callan, I.R.Klebanov, A.W.Ludwig, J.M.Maldacena, hep-th/9402113

[32] J.Polchinski,L.Thorlacius,hep-th/9404008

[33] E.Witten,*Bound states of strings and p-branes*,hep-th/9510135

[34] D.Ghoshal,D. P. Jatkar,S. Mukhi,Nucl. Phys.**B395**(1993) 144
D.Ghoshal, P. Lakdawala,S. Mukhi,Mod. Phys. Lett.**A8** (1993) 3187

[35] D. Ghoshal, C. Vafa ,Nucl. Phys.**B453**(1995)121, hep-th/9506122

[36] D. Ghoshal, *Notes on a singular Landau-Ginzburg family* ,hep-th/9510233

[37] M. Bershadsky, V. Sadov, C. Vafa, *D-Branes and Topological Field Theories* , hep-th /9511222

[38] Miao Li, *Dirichlet boundary state in linear dilaton background*, hep-th/9512042

[39] F. David, Mod.Phys.Lett.**A3**, (1988), 1651.

[40] J. Distler, H. Kawai, Nucl.Phys.**B 321**, (1989), 509

[41] A. Polyakov,*Confining Strings*,hep-th/9607049

[42] J.M. Daul, V. Kazakov and I. Kostov, Nucl. Phys.**B 409** (1993) 311

[43] E. Brezin, Jean Zinn-Justin,Phys.Lett.**B 288**, (1992), 54.

[44] V.G. Knizhnik, A.M. Polyakov, A.B. Zamolodchikov, Mod.Phys.Lett.**A3**,(1988),819

[45] S.R.Das, A.Dhar, A.M.Sengupta, S.R.Wadia, Mod.Phys.Lett. **A5**(1990) 1041

[46] G.'t Hooft, Nucl.Phys.**B72**(1974) 461

[47] E. Brézin, C.Itzykson, G.Parisi and J.B.Zuber, Commun. Math. Phys. 59 (1978), 35.

[48] F. David, Nucl. Phys. B257 (1985) 45.

[49] V.A.Kazakov, Phys. Lett. B150 (1985) 282.

[50] J.Fröhlich, in: **Lecture Notes in Physics**, Vol. 216, Springer, Berlin, 1985; J.Ambjørn, B. Durhuus and J.Fröhlich, Nucl. Phys. B257[FS14](1985) 433.

[51] T.Regge, Nuovo Cimento **19**(1961)543

[52] L.Alvarez-Gaume, **Random surfaces, statistical mechanics and string theory**, Helvetica Physica Acta, vol.64, Birkhauser Verlag, Basel,1991

[53] L.Bonora and C.S.Xiong, Nucl.Phys.**B405**(1993)191.

[54] M. Douglas, Phys.Lett.**B238** (1990) 176.

[55] P.Di Francesco,D.Kutasov,Nucl.Phys. **B 342**(1990)589

[56] P.di Francesco and D.Kutasov, Phys.Lett. **265B** (1991) 385.

[57] J.De Boer, Nucl.Phys. **B336** (1991)602.

[58] I.Kostov, Phys.Lett. **297B** (1992) 74.

[59] C.Ahn and K.Shigemoto, *Multi–matrix model and 2D Toda multi–component hierarchy* CLNS 91/1054.

[60] S.Kharchev,A.Marshakov,A.Mironov,A.Morozov, Nucl.Phys.**B397**(1993)339

[61] L.Bonora and C.S.Xiong, Nucl.Phys. **B434** (1995) 408.

[62] D. Bessis, C. Itzykson, and J.-B. Zuber, Adv. Appl. Math.**1**(1980) 109.

[63] C.Itzykson and B.Zuber, J.Math.Phys. **21** (1980) 411.

[64] M.L.Mehta, Comm.Math.Phys. 79 (1981) 327. S. Chadha, G. Mahoux and M. Mehta, J. Phys. A: Math. Gen. **14**(1981)579.

[65] H.Itoyama, Y.Matsuo, Phys.Lett.**B262**(1991) 233

[66] I.Kostov,Nucl.Phys**B 326**(1989)583

[67] M.Staudacher, Phys.Lett.**B305**(1993)332

[68] R.C.Penner, J.Diff.Geom. **27** (1988) 35.

[69] J.Harer and D.Zagier, Inv.Math. **85** (1986) 457.

[70] J.Distler and C.Vafa, Mod.Phys.Lett. **A6** (1991) 259.

[71] D.Gross *The $c = 1$ matrix model*, Jerusalem Winter School 1990.
I.Klebanov, *String theory in two dimensions* , Proc. Trieste Spring School 1991.
P.Ginsparg and G.Moore, *Lecture notes on 2d gravity and 2d string theory* hep-th/9304011.
A.Jevicki, *Developments in 2D string theory*, Trieste lectures, April 1993.

[72] K.Takasaki, *Dispersionless Toda hierarchy and Two-dimensional String Theory*, KUCP-0067 , hep-th/9403190.

[73] T.Eguchi and H.Kanno, *Toda Lattice Hierarchy and the Topological Description of the $c = 1$ string theory*, UT-674, hep-th/9404056.

[74] D.Ghoshal and S.Mukhi, *Topological Landau-Ginzburg model of two-dimensional string theory*, hep-th/9312189.

[75] A.Hanany, Y.Oz and M.R.Plesser *Topological Landau-Ginzburg formulation and integrable structure of 2d string theory* hep-th/9401030.

[76] D.J.Gross and I.R.Klebanov, Nucl.Phys. **B359** (1991) 3.

[77] D.J.Gross and U.H.Danielsson, Nucl.Phys.**B366** (1991) 3.

[78] G.Moore, Nucl.Phys. **B368**(1992) 557; G.Moore and R.Plesser, Phys.Rev. **D46** (1992) 1730.

[79] U.H.Danielsson, Nucl.Phys. **B380** (1992) 83.

[80] I.R.Klebanov and A.Pasquinucci, Nucl.Phys. **B393** (1993) 261.

[81] R.Dijkgraaf, R.Moore and R.Plesser, Nucl.Phys. **B394** (1993) 356.

[82] L.Bonora and C.S.Xiong, *Topological field theory interpretation and LG representation of $c = 1$ string theory*, hep-th/9505140. Nucl. Phys. **B 453** (1995) 129

[83] L.Bonora C.P.Constantinidis and C.S.Xiong, *Exact correlators of two-matrix models*, hep-th/9412006,to appear in Jour.Phys.Geom.

[84] L.Bonora F.Nesti E.Vinteler, Int.J.Mod.Phys.**A 11**(1996)1797

[85] W.Lerche, *Chiral Rings and Integrable Systems for Models of Topological Gravity*,hep-th/9401121 and Nucl. Phys. B434 (1995), 445

[86] M.Bershadsky, W.Lerche,D.Nemeshansky, N.P.Warner, Nucl. Phys. B401 (1993), 304

[87] M. Green, Nucl. Phys. B293 (1987), 593.

[88] I. Kogan, Phys. Lett. B256 (1991), 365; Nucl. Phys. B375 (1992), 362;
S. Carlip, Nucl. Phys. B362 (1991), 111;
M.C. Ashworth, Mod. Phys. Lett. A10 (1995), 2749.

[89] M.O'Loughlin.*Chern-Simons from Dirichlet 2-brane instantons*,hep-th/9601179
L.Cooper,I.I.Kogan,*Boundary conditions and heterotic construction in topological membrane theory*,hep-th/9602062

[90] J. Minahan and A. Polychronakos, Phys. Lett. B312 (1993), 155.

[91] V.Kazakov,T. Wynter, Nucl.Phys.**B440**(1995)407

[92] J. Polchinski, Phys. Rev. D50 (1994), 6041, hep-th/9510017;
J. Polchinski, S. Chaudhuri and C. Johnson.*Notes on D-branes* NSF-ITP-96-003 preprint, hep-th/9602052,

[93] C. Callan and I. Klebanov, *D-brane boundary state dynamics* preprint PUPT-1578, hep-th/9511173;
C. Schmidhuber. *D-brane actions* preprint PUPT-1585, hep-th/9601003.

[94] C. Bachas, *D-brane dynamics* preprint NSF-ITP-95-144, CPTH-S388-1195; hep-th/9511043.

[95] E.Vinteler,Mod.Phys.Lett. **A10**(1995)2709

[96] V.A.Kazakov,Nucl.Phys.**B**Proc.Suppl 4(1988)93

[97] V.A.Kazakov,A.A.Migdal,Nucl.Phys.**B311**(1988)171

[98] V.A.Kazakov,A.A.Migdal,Nucl.Phys.**B397**(1993)214

[99] D.J.Gross,Phys.Lett.**B293**(1992)181

[100] D.V.Boulatov.Mod.Phys.Lett. **A8**(1993)557 **A9**(1994)1963

[101] V.A.Kazakov,JETP.Lett.**44**(1986)105,
D.V.Boulatov,V.A.Kazakov,Phys.Lett.**B186**(1987)379

[102] J.M. Daul, *Q-states Potts model on a random planar lattice,* hep-th/9503014, preprint LPTENS

[103] E.Vinteler,*Quantum Chaos in Multi–Matrix Models* , hep-th/9507089

[104] C.E.Porter,**Statistical Theories of Spectra: Fluctuations,a collection of reprints**,(Academic Press, New York and London,1965)

[105] M.L.Mehta,**Random Matrices**,(Academic Press,New York and London,2nd edition,1990)

[106] K.Nakamura,**Quantum chaos** (Cambridge University Press, England,1993)

[107] M.-J.Giannoni, A.Voros,Z.Zinn-Justin (eds.)**Chaos and Quantum Physics**(North-Holland,Amsterdam,1991)

[108] B.Kramer (ed.)**Quantum coherence in mesoscopic systems**, NATO ASI Ser.B , Phys.254 (Plenum,New York,1991)

[109] S.Kharchev et al., Nucl.Phys.**B 380**,181 (1992)

[110] B.D.Simons,P.A.Lee,B.L.Altshuler,Phys.Rev.Lett.**72**,64(1993)

[111] B.D.Simons,B.L.Altshuler,Phys.Rev.Lett.**70**,4063 (1993)
B.D.Simons,P.A.Lee,B.L.Altshuler,Phys.Rev.Lett.**70**,4122 (1993)

[112] Nucl.Phys.**B424**,435 (1994) Nucl.Phys.**B441**,409 (1995)

[113] V.Kazakov,Nucl.Phys.**B354** ,614(1991)

[114] E.Brezin, in **Applications of field theory to statistical mechanics**.Springer Lecture Notes in Physics,vol.216,p.215 (Springer,Berlin,1984).
K.B.Efetov,Adv.Phys.**32**,53 (1983)

[115] J.D'Anna,E.Brezin, A.Zee,Nucl.Phys.**B443**,433 (1995)

[116] E.Brezin,A.Zee,Nucl.Phys.**B402**,613 (1993)

[117] L.Bonora,C.S.Xiong, Nucl. Phys. (1991)

[118] H. Awata, Y. Matsuo, S. Odake, J. Shiraishi, Nucl. Phys.**B449** (1995) 347

[119] L.Bonora,C.P.Constantinidis, E.Vinteler*Toda lattice realization of integrable hierarchies*, hep-th/9511172 to appear in Lett.Math.Phys.

[120] L.A.Dickey, **Soliton Equations and Hamiltonian Systems** World Scientific, Singapore (1991).

[121] V.G.Drinfeld and V.V.Sokolov, *Algebras and equations of Korteweg–De Vries type*, Jour.Sov.Math. **30** (1984) 1975.

[122] L.Bonora and C.S.Xiong, *Multi–field representation of the KP hierarchy and multi–matrix models*, Phys.Lett.**B317** (1993) 329.

[123] L.Bonora and C.S.Xiong, *The* (N, M)*–th KdV hierarchy and the associated algebras*, Jour.Math.Phys. **35** (1994) 5781.

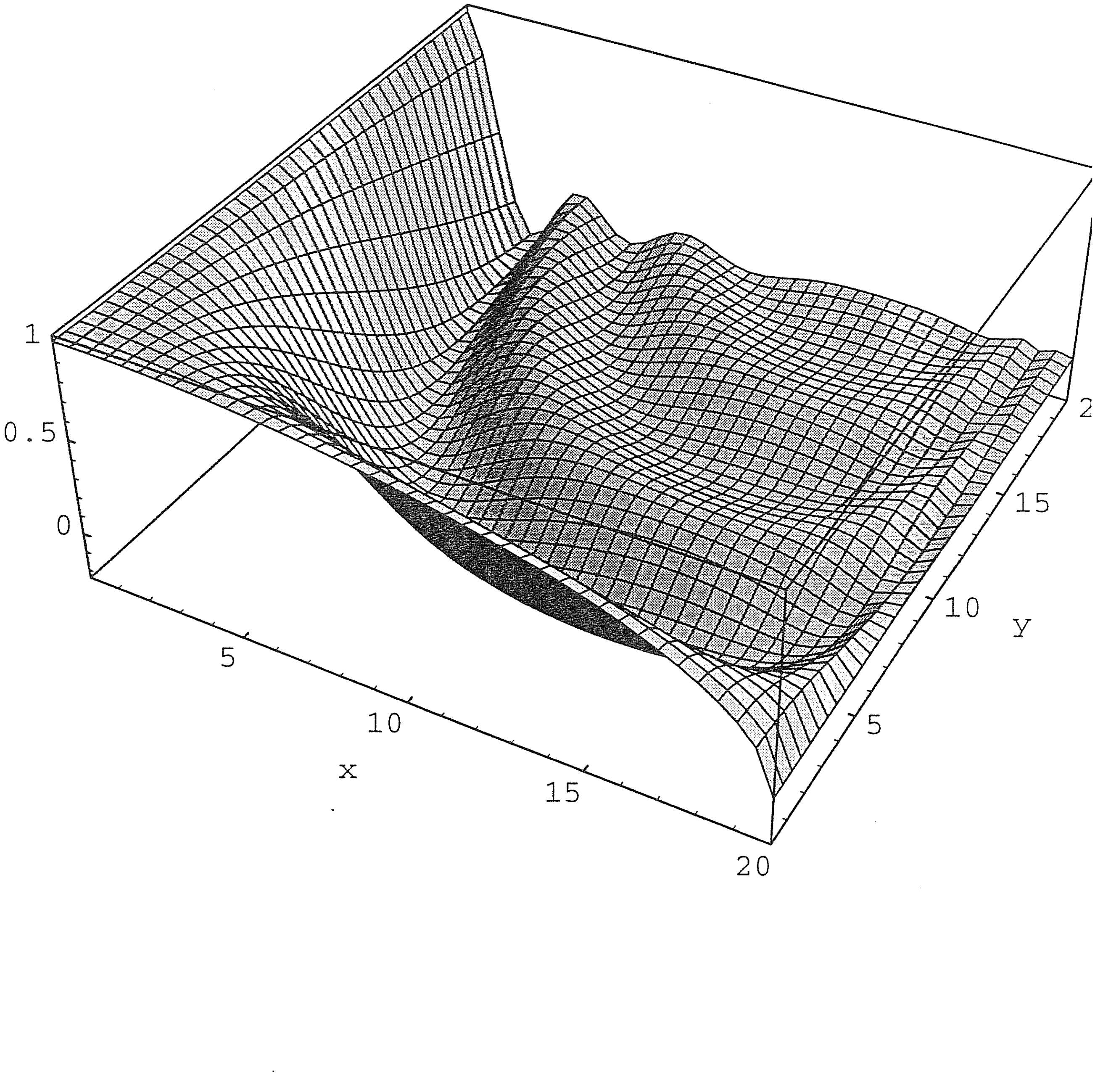
[124] L.Bonora, Q.P.Liu and C.S.Xiong, *The integrable hierarchy constructed from a pair of KdV–type hierarchies and its associated W algebra*, hep-th/9408035, to be published in Comm.Math.Phys.

[125] H.Aratyn, L.A.Ferreira, J.F.Gomes and A.H.Zimerman, *On two–current realization of KP hierarchy*, Nucl.Phys.**B402** (1993) 229.
H.Aratyn, E.Nissimov, S.Pacheva and A.H.Zimerman, Phys.Lett. **341B** (1994) 19.

[126] L.A.Dickey, *On the constrained KP hierarchy.*, Lett.Math.Phys. **34** (1995) 379; **35** (1995) 229.

[127] K.Ueno and K.Takasaki, *Toda Lattice Hierarchy*, Adv.Stud.Pure Math. **4** (1984) 1.

[128] Proc. *W95*, Marseille, July 1995, R.Grimm ed.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